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Automorphisms of irreducible holomorphic symplectic manifolds and related problems

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Abstract

We study automorphisms and the mapping class group of irreducible holomorphic symplectic (IHS) manifolds. We produce two examples of manifolds of $K3^{[2]}$ type with a symplectic action of the alternating group $\mathcal{A}_7$. Our examples are realized as double EPW-sextics, the large cardinality of the group allows us to prove the irrationality of the associated families of Gushel-Mukai threefolds. We describe the group of automorphisms of double EPW-cubes. We give an answer to the Nielsen realization problem for IHS manifolds in analogy to the case of K3 surfaces, determining when a finite group of mapping classes fixes an Einstein (or Kähler-Einstein) metric. We describe, for some deformation classes, the mapping class group and its representation in second cohomology. We classify non-symplectic involutions of manifolds of OG10 type determining the possible invariant and coinvariant lattices. We study non-symplectic involutions on LSV manifolds that are geometrically induced from non-symplectic involutions on cubic fourfolds.

Contents

Introduction

This thesis is devoted to the study of IHS manifolds, their automorphisms and their mapping class group. The interest in IHS manifolds is quite recent, it rose after the Beauville-Bogomolov decomposition theorem [Bea83, Theorem 2] for compact Kähler manifolds with trivial first Chern class. Any such manifold (up to an étale cover) is the product of a complex torus, Calabi-Yau manifolds and IHS manifolds. A compact Kähler manifold X is an IHS manifold when it is simply connected and there is a holomorphic symplectic form generating $\mathrm{H}^{2,0}(X)$. As a consequence of Yau's proof of Calabi's conjecture, these manifolds correspond exactly to simply connected Riemannian manifolds whose holonomy group is the symplectic group $\mathrm{Sp}(n)$. The IHS manifolds of dimension two are K3 surfaces, which were known for a long time, but the first higher dimensional examples are quite recent and are due to Beauville [Bea83] and Fujiki [Fuj83]. It turned out to be quite hard to construct new examples of IHS manifolds, up to deformation. The first example of higher dimensional IHS manifold is the Hilbert scheme of points on a K3 surface, parametrizing the 0-dimensional subschemes of fixed length n. Manifolds deformation equivalent to such an example are called of $\mathrm{K3}^{[n]}$ type. A similar construction is possible for Abelian surfaces and the subvariety of the Hilbert scheme consisting of points that sum to zero is an IHS manifold. Manifolds deformation equivalent to this example are called of Kum_n type. The discovery of a symplectic form on the moduli space of semistable coherent sheaves on symplectic surfaces due to Mukai [Muk84] led to the hope that new examples of IHS manifolds could be constructed as moduli spaces. Many mathematicians contributed to the development of a good theory which was indeed fruitful, one can refer to [HL10] for a complete treatment. All the smooth moduli spaces of sheaves on symplectic surfaces are deformation equivalent to the already known examples, but O'Grady produced desingularizations of two singular moduli spaces obtaining two new examples, one in dimension ten [O'G99] and one in dimension six [O'G03]. Manifolds that are deformation equivalent to O'Grady's six dimensional example are called of OG6 type and manifolds that are deformation equivalent to O'Grady's ten dimensional example are called of OG10 type. This technique of constructing new examples is somehow saturated, in fact Lehn and Sorger in [LS06] and then with Kaledin in [KLS06] showed that there are no other possibilities for singular moduli spaces to admit a symplectic resolution. Moreover, Perego and Rapagnetta [PR13] proved that the singular moduli spaces that admit a resolution which is an IHS manifold have a resolution which is deformation equivalent to the O'Grady's examples. General facts about moduli spaces of sheaves on symplectic surfaces and their relation with IHS manifolds are outlined in section 2.2.

The main tool for the study of K3 surfaces is the intersection pairing on the second cohomology together with the Hodge decomposition of the complex cohomology. The reason is the global Torelli theorem due to Šapiro and Šafarevič [PSS71] that allows to recover a K3 surface S from its second cohomology $\mathrm{H}^2(S,\mathbb{Z})$ and study Hodge isometries of $\mathrm{H}^2(S,\mathbb{Z})$ that preserve the Kähler cone in order to understand automorphisms of S. A striking fact is that also for a higher dimensional IHS manifold X there is a quadratic form on $\mathrm{H}^2(X,\mathbb{Z})$, called the Beauville-Bogomolov-Fujiki form, that allowed Huybrechts [Huy11], Markman [Mar11] and Vebitsky [Ver13] to define the moduli space of IHS manifolds with its period map and to formulate Torelli theorems for IHS manifolds. They obtain slightly weaker statements, but there are counterexamples to the strongest version of the Torelli theorem, we give an overview of these results in subsection 1.3.3. The Torelli theorem together with the fundational work of Nikulin [Nik76, Nik79] about lattices provide the fundamental theory to study finite groups of automorphisms of IHS manifolds.

The following is an overview of what is known about automorphsims of IHS manifolds. Symplectic automorphisms are automorphisms that preserve the symplectic form, while non-symplectic automorphisms are the ones that do not preserve it. The main techniques that are used to understand automorphisms of IHS manifolds are realizing finite groups of automorphisms as groups of lattice isometries, or determine invariant and coinvariant lattices for finite (prime) order automorphisms. A classification of finite groups of symplectic automorphisms of K3 surfaces is due to Mukai [Muk88], an attempt of classification of finite groups of automorphisms that might not be symplectic is given in [BH21]. There is a classification of prime order symplectic automorphisms in terms of invariant and coinvariant lattices for $\mathrm{K3}^{[2]}$ type in [Cam12] and [Mon12], a classification of symplectic groups of automorphisms $\mathrm{K3}^{[2]}$ type is available in [HM19] with a contribution in [Waw22]. The study of automorphisms of manifolds of Kum_2 type is developed in [BNWS13, MTW18, BC22]. Contributions for the investigation of prime order non-symplectic automorphisms on manifolds of $\mathrm{K3}^{[n]}$ type were given by many authors and can be found in [Bea11, OW13, BCS16, BCMS16, CKKM19, CC20, CCC21]. Manifolds of OG10 type have no finite symplectic automorphisms different from the identity [GGOV22] and for manifolds of OG6 type symplectic automorphisms act trivially in cohomology [GOV23], non-symplectic automorphisms of prime order are classified in [Gro22] for manifolds of OG6 type. Steps towards the classification of non-symplectic automorphisms of prime order for manifolds of OG10 type are given in [BC22], while a classification of non-symplectic involutions is available in chapter 5. Explicit constructions of manifolds with large groups of symplectic automorphisms are given in [BS21] for K3 surfaces and in [DBvGKKW17, Son21, DM22, CDM23], section 3.1 for manifolds of $K3^{[2]}$ type.

Not many explicit families of IHS manifolds are available, we recall some constructions. Beauville and Donagi proved that the variety of lines on a cubic fourfold is a manifold of $K3^{[2]}$ type [Bea83]. O'Grady associated a manifold of $K3^{[2]}$ type called double EPW-sextic to any general Lagrangian subspace of $\bigwedge^3 \mathbb{C}^6$ [O'G13] and Iliev, Kapustka, Kapusta, Ranestad associated to any such Lagrangian space a manifold of $K3^{[3]}$ type called double EPW-cube [IKKR19]. Iliev and Ranestad showed that the variety of sums of ten cubes in six variables is of $K3^{[2]}$ type [IR01]. Debarre and Voisin constructed a manifold of $K3^{[2]}$ associated to a 3-form on a 10-dimensional vector space, called the Debarre-Voisin variety [DV10]. Lehn, Lehn, Sorger and van Straten constructed a family of manifolds of $K3^{[4]}$ type as compactifications of moduli spaces of twisted cubics on cubic fourfolds [LLSVS17]. Laza, Saccà and Voisin constructed a manifold of OG10 type called LSV manifold associated to a cubic fourfold [LSV17], Voisin constructed another manifold of OG10 type called twisted LSV manifold associated to the cubic fourfold [Voi18]. Recently Li, Pertusi and Zhao constructed a family of manifolds of OG10 type as resolutions of singular moduli spaces of Bridgeland stability conditions on the derived category of cubic fourfolds [LPZ22]. All the above families vary in 20 moduli, we recall the definitions and construction of double EPW manifolds in section 2.1 and the definition of LSV and twisted LSV manifolds in section 2.3.

Many of the known explicit constructions of IHS manifold are modular constructions on Fano manifolds, or have a Hodge-theoretical link with Fano manifolds. The most important relation for this thesis is that there are associated families of Gushel-Mukai (GM) manifolds to any EPW-sextic [DK20b]. Gushel-Mukai manifolds are precisely Fano manifolds of dimension $n \in \{3, 4, 5\}$, degree 10, Picard rank 1 and index $n - 2$. It is known that GM fivefolds are rational, the rationality of GM fourfolds is not known but similarly to cubic fourfolds it is expected that the very general GM fourfold is irrational, while the general GM threefold is irrational and there is the belief that any such threefold is. Despite this, there were no explicit examples of irrational GM threefolds before [DM22], where a family of such manifolds was exhibited. We contribute in constructing other two families of irrational GM threefolds in section 3.2, as families associated to very symmetric examples of double EPW-sextics presented in section 3.1. The other construction that we exploit is the one of LSV manifolds associated to cubic fourfolds. In section 5.2 we determine a numerical criterion for a manifold of OG10 type for being bimeromorphic to a twisted LSV manifold, then starting from the classification of involutions on cubic fourfolds [Mar23] we determine the possible non-symplectic involutions of manifolds of OG10 that are induced by a cubic fourfold on the associated LSV manifolds. These involutions are of two different types and constitute geometric examples of the abstrac classification of non-symplectic involutions of manifolds of OG10 given in section 5.1.

A slightly more differential approach in the study of IHS manifolds is to consider the Teichmüller space (parametrizing complex structures on the manifolds) instead of the moduli space. Many questions in this setting lead to the same answers since the Teichmüller space is in fact a covering space of the moduli space, but it is quite natural to study the mapping class group in this context that can be more complicated than the automorphisms group. A natural question about the mapping class group of a manifold is the Nielsen realization problem, which was originally formulated by Nielsen in [Nie42, Section 4], and then affirmatively solved by Kerckhoff in [Ker83]. The question is whether any finite group G of mapping classes of a complex curve can be lifted to a group of diffeomorphisms (which preserve the metric and the complex structure). Equivalently, one wonders if G fixes any point in the Teichmüller space. The answer to the same question for K3 surfaces is given by Farb and Looijenga in [FL21, Theorem 1.2]: their result shows that if S is a K3 surface not every finite subgroup $G \subset \mathrm{Mod}(S) = \pi_0(\mathrm{Diff}^+(S))$ can be lifted, but there is a G-invariant $\mathbf{\Gamma}_G \subseteq \mathrm{H}^2(S, \mathbb{Z})$ which determines if it is possible or not. We contribute giving an answer to the Nielsen realization problem for IHS in section 4.4, in the spirit of the result for K3 surfaces and using a similar condition for the lattice $\mathbf{\Gamma}_G$. Inspired by [BK23] we address other related questions, in section 4.2 we describe the shape of the mapping class group for some classes of IHS manifolds and in section 4.3 we show examples of IHS manifolds in any dimension for which the topological version of the Nielsen realization problem has a different answer than the differential one.

Structure of the thesis

In chapter 1 we give basic definitions and results that will be useful in the following chapters. In the first section we give basic notions and few relevant results in lattice theory. The second section is very brief, we recall basic notions of deformation theory together with a description of the deformation space of manifolds with trivial canonical bundle. The third section is the core of this chapter and gathers all the basic definitions, results and examples about IHS manifolds. We give a broad overview of the following aspects: basic cohomological properties, the period map and the Torelli theorems, the structure of the various cones associated to the manifold, the Teichmüller space, the mapping class group, birationalities an automorphisms.

In chapter 2 we recall some constructions of IHS manifolds that will be used in the following chapters. In the first section we recall the construction of double EPW-sectics and double EPW-cubes, give the basic properties and describe their automorphisms. The automorphism group of the double EPW-sextics was studied by Kuznetsov [DM22, Appendix A], here we get an analogous description for the groups of autormophisms of double EPW-sextics and this is the only innovative part of the chapter. In the second section we give an overview about moduli spaces of sheaves on symplectic surfaces, describing the notion of stability and giving a panoramic of the properties of the moduli spaces depending on the choice of the Mukai vector and the choice of the stability. In the third section we give the definitions of LSV manifold and twisted LSV manifold associated to a cubic fourfold, then we illustrate the Hodge theoretical link of these manifolds with the cubic fourfold.

In chapter 3 we construct two explicit examples of double EPW-sextics with a symplectic action of the alternating group $\mathcal{A}_7$ and prove the irrationality of the associated families of Gushel-Mukai threefolds. According to the classification in [HM19], the group $\mathcal{A}_7$ is one of the maximal groups that can act symplectically on a manifold of $\mathrm{K3}^{[2]}$ type, showing that our examples are among the most symmetric ones. The project is primarily inspired by [DM22], where many ideas have been taken from and adjusted to our case. We point out the related works [Son21, CDM23] where other very symmetric manifolds of $\mathrm{K3}^{[2]}$ type are constructed. The rough idea is simple: if a Lagrangian space has a linear action of $\mathcal{A}_7$, then the double EPW-sextic will also have an action of the group. Our two IHS fourfolds of $\mathrm{K3}^{[2]}$ type are in a sense dual to each other and non-isomorphic as polarized manifolds. Debarre and Kuznetsov associated families of Gushel-Mukai varieties to a Lagrangian space [DK20b]. Moreover, there is an action of the group on the intermediate Jacobian of the associated Gushel-Mukai threefolds. The big cardinality of the group combined with the Clemens-Griffiths criterion allows us to show that any Gushel-Mukai threefold associated to our examples is irrational. The general Gushel-Mukai threefold is irrational [Bea77] but no explicit examples of such irrational threefolds was known before [DM22], we contribute here with two other families of examples.

In chapter 4 we give an answer to the Nielsen realization problem for IHS manifolds, generalizing the result of Farb and Looijenga [FL21, Theorem 1.2] for K3 surfaces. We also give some partial answers to questions related to this problem, we describe for some deformation classes the shape of the mapping class group and its representation in cohomology, in the spirit of [BK23]. In the first section we formulate the various question that we address in the chapter. In the second section we show that for some of the known deformation types the representation map $\mathrm{Mod}(X) \to \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))$ admits a section over its image. In the third section we give an example of a group of order of two mapping classes of a manifold of $\mathrm{K3}^{[n]}$ type that lifts to a group of homeomorphisms but does not lift to a group of diffeomorphisms, using a known example for K3 surfaces. In the fourth section we address the

Nielsen realization problem for IHS manifolds.

In chapter 5 we classify non-symplectic involutions of manifolds of OG10 type, describing their invariant and coinvariant lattices. In the first section we show how we get an exhaustive list for the above lattices and show that any pair of invariant and coinvariant lattices in our list is geometrically realized as invariant and coinvariant on a non-symplectic involution on a manifold of OG10 type. In the second section we study non-symplectic involutions of a LSV manifold induced from non-symplectic involutions on the cubic fourfold, we prove that these involutions are regular and describe their action in cohomology.

In Appendix A we provide more details about some computations concerning the examples of double EPW-sextics with a symplectic action of $\mathcal{A}_7$ of chapter 3, complementing with codes we run with `GAP` [GAP21] and `Macaulay2` [M2].

In Appendix B we collect the tables of lattices that are used in chapter 5, which provide the classification of non-symplectic involutions of manifolds of OG10 type.

Chapter 1

Preliminaries

1.1 Lattice theory

In this section we give basic definitions and some useful classical results about lattices. The main reference is [Nik79] where most of the facts can be found, other valid references are [Huy16] and [MM09].

1.1.1 Basic definitions and examples

Definition 1.1.1. A *lattice* $\mathbf{L}$ is a a free $\mathbb{Z}$-module of finite rank with a non-degenerate integral bilinear form

$$(-,-) : \mathbf{L} \times \mathbf{L} \to \mathbb{Z}.$$

The *signature* (l_+, l_-) of $\mathbf{L}$ is the signature of the real extension of $(-,-)$ on $\mathbf{L}_\mathbb{R} = \mathbf{L} \otimes_\mathbb{Z} \mathbb{R}$. The lattice is *positive-definite* if $l_- = 0$ and *negative-definite* if $l_+ = 0$, otherwise it is called *indefinite*. A lattice is *hyperbolic* if it is indefinite and $l_+ = 1$.

Definition 1.1.2. A lattice $\mathbf{L}$ is called *even* if $x^2 := (x, x) \in 2\mathbb{Z}$ for any $x \in \mathbf{L}$.

The *divisibility* $\mathrm{div}(x)$ of an element $x \in \mathbf{L}$ is the positive generator of the ideal $\{(x, y) | y \in \mathbf{L}\} \subseteq \mathbb{Z}$. There is an obvious notion of direct sum of lattices, moreover a subgroup $\mathbf{N} \subseteq \mathbf{L}$ is a *sublattice* if the restriction of the bilinear form to $\mathbf{N}$ is still non-degenerate. A sublattice $\mathbf{N} \subseteq \mathbf{L}$ is called *primitive* if $\mathbf{L} / \mathbf{N}$ is a free Abelian group. The *saturation* of a sublattice $\mathbf{N} \subseteq \mathbf{L}$ is the smaller primitive sublattice of $\mathbf{L}$ containing $\mathbf{N}$, it can be identified with $(\mathbf{N}^\perp)^\perp \subseteq \mathbf{L}$

A morphism of lattices is a morphism of groups that preserves the bilinear forms, an *isometry* is a bijective morphism of lattices and the group of isometries of a lattice $\mathbf{L}$ is denoted by $\mathrm{O}(\mathbf{L})$. The Cartan-Dieudonné theorem [MM09, Theorem 9.10] guarantees that $\mathrm{O}(\mathbf{L} \otimes_\mathbb{Z} \mathbb{R})$ is generated by reflections with respect to non-isotropic vectors, hence it is possible to give the following

Definition 1.1.3. Let spin : $\mathrm{O}(\mathbf{L} \otimes_{\mathbb{Z}} \mathbb{R}) \to \{\pm 1\}$ be the groups homomorphism that takes value $+1$ on reflections for a vector v with $v^2 < 0$.

This restricts to a map spin : $\mathrm{O}(\mathbf{L}) \to \{\pm 1\}$ called *spinor norm* whose kernel is denoted by $\mathrm{O}^+(\mathbf{L})$ and consists of elements that preserve the orientation of a maximal positive-definite subspace of $\mathbf{L} \otimes_{\mathbb{Z}} \mathbb{R}$.

Consider the *dual lattice*

$$\mathbf{L}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{L}, \mathbb{Z}) \cong \{x \in \mathbf{L} \otimes_{\mathbb{Z}} \mathbb{Q} | (x, l) \in \mathbb{Z} \forall l \in \mathbf{L}\}$$

and observe that $\mathbf{L} \subset \mathbf{L}^\vee$ is a finite index subgroup.

Definition 1.1.4. The *discriminant group* of $\mathbf{L}$ is the finite group $\mathrm{A}_{\mathbf{L}} := \mathbf{L}^\vee / \mathbf{L}$.

The determinant of the bilinear form $\mathrm{disc}(\mathbf{L})$ is called *discriminant* of $\mathbf{L}$ and it equals to $|\,\mathrm{A}_{\mathbf{L}}\,| = [\mathbf{L}^\vee : \mathbf{L}]$. The *length* $l(\mathrm{A}_{\mathbf{L}})$ is the minimum number of generators of $\mathrm{A}_{\mathbf{L}}$. If the lattice is even, then there is a well-defined $\mathbb{Q}$-bilinear form $b_{\mathrm{A}_{\mathbf{L}}} : \mathrm{A}_{\mathbf{L}} \times \mathrm{A}_{\mathbf{L}} \to \mathbb{Q}/\mathbb{Z}$ with associated quadratic form $q_{\mathrm{A}_{\mathbf{L}}} : \mathrm{A}_{\mathbf{L}} \to \mathbb{Q}/2\mathbb{Z}$.

There is a natural map $\mathrm{O}(\mathbf{L}) \to \mathrm{O}(\mathrm{A}_{\mathbf{L}})$ that sends an isometry $\varphi \in \mathrm{O}(\mathbf{L})$ to the induced isometry $\overline{\varphi} \in \mathrm{O}(\mathrm{A}_{\mathbf{L}})$ with respect to the quadratic form $q_{\mathrm{A}_{\mathbf{L}}}$, denote its kernel by $\widetilde{\mathrm{O}}(\mathbf{L})$.

Definition 1.1.5. A lattice $\mathbf{L}$ is called *unimodular* if the group $\mathrm{A}_{\mathbf{L}}$ is trivial. The lattice is called *p-elementary* for a prime number p if $\mathrm{A}_{\mathbf{L}} \cong (\mathbb{Z}/p\mathbb{Z})^k$ for some positive integer k.

Notice that $\mathbf{L}$ is unimodular if and only if $\mathbf{L}^\vee \cong \mathbf{L}$, if and only if $\det(\mathbf{L}) = \pm 1$.

Example 1.1.6. Let $0 \neq n \in \mathbb{Z}$. Denote by $[n]$ the rank one lattice generated by an element x such that $x^2 = n$. In this case, $\mathrm{A}_{[n]} \cong \langle \frac{x}{n} \rangle \cong \mathbb{Z}/n\mathbb{Z}$ and $q_{\mathrm{A}_{[n]}}(\frac{x}{n}) = \frac{1}{n}$. In general, if $\mathbf{L}$ is a lattice, $\mathbf{L}(n)$ denotes the lattice with the same underlying module but where the bilinear form is multiplied by n. Notice that $\mathrm{disc}(\mathbf{L}(n)) = \mathrm{disc}(\mathbf{L}) n^{\mathrm{rk}(\mathbf{L})}$ and there is a short exact sequence

$$0 \to \mathbf{L} / n \, \mathbf{L} \to \mathrm{A}_{\mathbf{L}(n)} \to \mathrm{A}_{\mathbf{L}} \to 0,$$

in particular if $\mathbf{L}$ is unimodular then $\mathrm{A}_{\mathbf{L}(n)} \cong (\mathbb{Z}/n\mathbb{Z})^{\mathrm{rk}(\mathbf{L})}$.

Example 1.1.7. The rank two lattice $\mathbf{U} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an even unimodular hyperbolic lattice called *hyperbolic plane*.

Example 1.1.8. Relevant lattices are the lattices $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ associated to the relative Dynkin diagrams. They can be seen as $\mathbb{Z}$-modules generated by the vertexes $\{e_i\}$ of the Dynkin diagram and bilinear form given by $e_i^2 = 2$, $(e_i, e_j) = -1$ if the two vertexes are connected by an edge and $(e_i, e_j) = 0$ if they are not. The Dynkin diagrams are displayed in Table 1.1 together with the discriminant groups of the associated lattices.

We give the matrix representation of some of the examples that will appear more often in this thesis, notice that $\mathbf{E}_8$ is unimodular.

$$\mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{D}_4 = \begin{pmatrix} 2 & & -1 & \\ & 2 & -1 & \\ -1 & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

$$\mathbf{E}_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & -1 \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & -1 & & & & & 2 \end{pmatrix}$$

An injective morphism of lattices $\mathbf{N} \to \mathbf{L}$ with primitive image is called a *primitive embedding*. Two primitive embeddings $\mathbf{N} \subset \mathbf{L}$ and $\mathbf{N} \subset \mathbf{L}'$ are *isometric* if there is an isometry $\mathbf{L} \to \mathbf{L}'$ that restricts to the identity on $\mathbf{N}$, if the isometry just takes $\mathbf{N}$ to itself we say that the embeddings *detemine isometric primitive sublattices.*

An *overlattice* of $\mathbf{L}$ is a lattice $\mathbf{T}$ for which there is an embedding of finite index $\mathbf{L} \subset \mathbf{T}$, meaning that $\mathbf{L} / \mathbf{T}$ is a finite group. The orthogonal complement $\mathbf{N}^{\perp}$ of a sublattice $\mathbf{N} \subset \mathbf{L}$ is again a sublattice and the finite index embedding $\mathbf{N} \oplus \mathbf{N}^{\perp} \subset \mathbf{L}$ makes $\mathbf{L}$ an overlattice of $\mathbf{N} \oplus \mathbf{N}^{\perp}$, the inclusion is

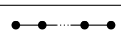
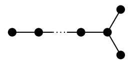
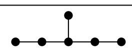
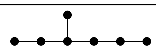
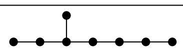
$\mathbf{L}$	graph	$A_{\mathbf{L}}$	
$\mathbf{A}_{n\geq 1}$		$\mathbb{Z}/(n+1)\mathbb{Z}$	
$\mathbf{D}_{n\geq 1}$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	n even
		$\mathbb{Z}/4\mathbb{Z}$	n odd
$\mathbf{E}_6$		$\mathbb{Z}/3\mathbb{Z}$	
$\mathbf{E}_7$		$\mathbb{Z}/2\mathbb{Z}$	
$\mathbf{E}_8$		$\{0\}$	

Table 1.1: ADE lattices

strict in general. To specify that the orthogonal complement of $\mathbf{N}$ is taken inside $\mathbf{L}$ we use the notation $\mathbf{N}^{\perp \mathbf{L}}$.

Consider an even lattice $\mathbf{L}$ and an overlattice $\mathbf{L} \subseteq \mathbf{T}$, then there are embeddings $\mathbf{L} \subseteq \mathbf{T} \subseteq \mathbf{T}^\vee \subseteq \mathbf{L}^\vee$. Set $H_\mathbf{T} = \mathbf{T} \,/\, \mathbf{L}$, so that $H_\mathbf{T} \subseteq \mathbf{T}^\vee \,/\, \mathbf{L} \subseteq \mathbf{L}^\vee \,/\, \mathbf{L} = \mathrm{A}_\mathbf{L}$ is an isotropic subgroup and $(\mathbf{T}^\vee \,/\, \mathbf{L})/H_\mathbf{T} = \mathrm{A}_\mathbf{T}$.

Lemma 1.1.9. *Let $\mathbf{L}$ be an even lattice, then there is a bijective correspondence between overlattices of $\mathbf{L}$ and isotropic subgroups of $\mathrm{A}_\mathbf{L}$. Moreover, if $\mathbf{T}$ is an overlattice and $H_\mathbf{T}$ the associated isotropic group then:*

1. $H_\mathbf{T}^\perp = \mathbf{T}^\vee \,/\, \mathbf{L} \subseteq \mathrm{A}_\mathbf{L}$;

2. $q_{\mathrm{A}_\mathbf{T}} = q_{\mathrm{A}_\mathbf{L} \,|\, \mathrm{A}_\mathbf{T}}$.

Proof. [Nik79, Proposition 1.4.1]. □

Notice that giving a primitive embedding of an even lattice $\mathbf{S}$ into an even lattice $\mathbf{L}$ is equivalent to give $\mathbf{L}$ as an overlattice of $\mathbf{S} \oplus \mathbf{S}^{\perp \mathbf{L}}$ and hence by the previous lemma it amounts to give an isotropic subgroup $H_\mathbf{L} \subseteq \mathrm{A}_\mathbf{S} \oplus \mathrm{A}_{\mathbf{S}^\perp}$. More precisely:

Lemma 1.1.10. *Given even lattices $\mathbf{S}, \mathbf{N}$ and a pair (H, γ) where $H \subseteq \mathrm{A}_\mathbf{S}$ is a subgroup and $\gamma : H \to \mathrm{A}_\mathbf{N}$ an inclusion of groups such that $q_\mathbf{N} \circ \gamma = -q_{\mathbf{S}\,|H}$, then there exists a unique overlattice $\mathbf{T}$ of $\mathbf{S} \oplus \mathbf{N}$ with discriminant form q and*

$$(q_\mathbf{S} \oplus q_\mathbf{N})|_{\Gamma_\gamma^\perp / \Gamma_\gamma} \cong q$$

where $\Gamma_\gamma \subseteq \mathrm{A}_\mathbf{S} \oplus \mathrm{A}_\mathbf{N}$ is the graph of γ.

Two such pairs $(H, \gamma), (H', \gamma')$ determine isometric primitive embeddings of $\mathbf{S}$ in $\mathbf{T}$ if and only if $H = H'$ and the maps γ, γ' are conjugate to each other via some isometry of $\mathbf{N}$, while the pairs define isometric primitive sublattices when there exist $\varphi \in \mathrm{O}(\mathbf{S})$ and $\psi \in \mathrm{O}(\mathbf{N})$ such that $\gamma \circ \overline{\varphi} = \overline{\psi} \circ \gamma'$.

Proof. [Nik79, Proposition 1.5.1]. □

Lemma 1.1.11. *The primitive embedding of an even lattice $\mathbf{S}$ into an even lattice $\mathbf{L}$ is determined by the data $(H_\mathbf{S}, H_\mathbf{L}, \gamma, \mathbf{N}, \gamma_\mathbf{N})$ where $H_\mathbf{S} \subseteq \mathrm{A}_\mathbf{S}$ and $H_\mathbf{L} \subseteq \mathrm{A}_\mathbf{L}$ are subgroups, $\gamma : H_\mathbf{S} \to H_\mathbf{L}$ is an isometry with respect to the restrictions of the quadratic forms $q_{\mathrm{A}_\mathbf{S}}, q_{\mathrm{A}_\mathbf{L}}$, $\mathbf{N}$ is an even lattice of signature $(l_+ - s_+, l_- - s_-)$ and*

$$\gamma_\mathbf{N} : q_\mathbf{N} \cong (q_\mathbf{S} \oplus -q_\mathbf{L})|_{\Gamma_\gamma^\perp / \Gamma_\gamma}$$

is an anti-isometry, where $\Gamma_\gamma \subseteq \mathrm{A}_\mathbf{S} \oplus \mathrm{A}_\mathbf{L}$ is the graph of γ.

Moreover, the data $(H_\mathbf{S}, H_\mathbf{L}, \gamma, \mathbf{N}, \gamma_\mathbf{N}), (H'_\mathbf{S}, H'_\mathbf{L}, \gamma', \mathbf{N}', \gamma'_\mathbf{N})$ determine isometric primitive sublattices if and only if $H'_\mathbf{S} = \overline{\mu}(H_\mathbf{S})$ for some $\mu \in \mathrm{O}(\mathrm{A}_S)$ and there exist $\varphi \in \mathrm{O}(\mathrm{A}_\mathbf{L}), \psi : \mathbf{N} \to \mathbf{N}'$ isometries such that $\gamma' \circ \overline{\psi} = \varphi \circ \gamma$ and $\overline{\mu} \circ \gamma_\mathbf{N} = \gamma'_{\mathbf{N}'} \circ \overline{\psi}$. In the particular case where also $H'_\mathbf{S} = H_\mathbf{S}$, the data determine isometric primitive embeddings.

Proof. [Nik79, Theorem 1.15.1]. □

The following is a specific case of the previous result.

Lemma 1.1.12. *Let $\mathbf{L}$ be an even lattice of signature (l_+, l_-) and $\mathbf{\Lambda}$ an even unimodular lattice of signature (λ_+, λ_-). The existence of a primitive embedding of $\mathbf{L}$ in $\mathbf{\Lambda}$ is equivalent to the existence of a lattice $\mathbf{N}$ of signature (n_+, n_-) such that:*

- $l_+ + n_+ = \lambda_+$ *and* $l_- + n_- = \lambda_-$*;*
- $\mathrm{A}_{\mathbf{L}} \cong \mathrm{A}_{\mathbf{N}}$ *and* $q_{\mathrm{A}_{\mathbf{L}}} \cong -q_{\mathrm{A}_{\mathbf{N}}}$.

Consider $\widetilde{\mathrm{O}}^+(\mathbf{L}) := \widetilde{\mathrm{O}}(\mathbf{L}) \cap \mathrm{O}^+(\mathbf{L})$, we have the following

Lemma 1.1.13 (Eichler's criterion)**.** *Let $\mathbf{L}$ be an even lattice such that $\mathbf{U}^{\oplus 2} \subseteq \mathbf{L}$. Let $x, y \in \mathbf{L}$ and consider the associated classes $\overline{x}, \overline{y} \in \mathrm{A}_{\mathbf{L}}$. Suppose that*

1. $x^2 = y^2$,
2. $\mathrm{div}(x) = \mathrm{div}(y)$,
3. $\overline{x} = \overline{y}$ *in* $\mathrm{A}_{\mathbf{L}}$,

then there exists $\phi \in \widetilde{\mathrm{O}}^+(\mathbf{L})$ such that $\phi(x) = y$.

Proof. [GHS09, Proposition 3.3]. □

1.1.2 Lattices with a prime action, existence and uniqueness

We consider lattices with an isometry of prime order, this often leads to p-elementary lattices. We recall the most useful criteria for existence and uniqueness of lattices with fixed invariants.

Lemma 1.1.14. *Let $\mathbf{\Lambda}$ be a unimodular lattice and $\mathbf{L} \subset \mathbf{\Lambda}$ a primitive sublattice. Then we have $\mathrm{A}_{\mathbf{L}} \cong \mathrm{A}_{\mathbf{L}^\perp} \cong \frac{\mathbf{\Lambda}}{\mathbf{L} \oplus \mathbf{L}^\perp}$ as groups.*

Notice that in this case the only isotropic group associated to the embedding is the trivial group and the embedding is unique.

Consider a lattice $\mathbf{L}$, if $G \subseteq \mathrm{O}(\mathbf{L})$ then the set of fixed points is a sublattice $\mathbf{L}^G$ called the *invariant lattice* and its orthogonal $\mathbf{L}_G = (\mathbf{L}^G)^\perp$ is called the *coinvariant lattice.*

Lemma 1.1.15. *Let $\mathbf{L}$ be a lattice and $G \subset \mathrm{O}(\mathbf{L})$ the group generated by an isometry of prime order p. Then, $m := \mathrm{rk}(\mathbf{L}_G)/(p-1)$ is an integer and*

$$\frac{\mathbf{L}}{\mathbf{L}^G \oplus \mathbf{L}_G} \cong (\mathbb{Z}/p\mathbb{Z})^a$$

as groups, where $a \leq m$. Moreover, there are natural embeddings of $\frac{\mathbf{L}}{\mathbf{L}^G \oplus \mathbf{L}_G}$ in the discriminant groups $\mathrm{A}_{\mathbf{L}^G}$ and $\mathrm{A}_{\mathbf{L}_G}$.

Proof. [Boi12, Lemma 5.3], [MTW18, Lemma 1.8]. □

In particular, when $\mathbf{\Lambda}$ is a unimodular lattice then $\mathbf{L}^G$ and $\mathbf{L}_G$ are p-elementary.

Definition 1.1.16. Let $\mathbf{L}$ be an even lattice, define

$$\delta(\mathbf{L}) := \begin{cases} 0 & \text{if } q_{\mathrm{A}_\mathbf{L}}(x) \in \mathbb{Z}/2\mathbb{Z} \text{ for any } x \in \mathrm{A}_\mathbf{L} \\ 1 & \text{otherwise} \end{cases}.$$

Theorem 1.1.17. *An even 2-elementary lattice $\mathbf{L}$ of signature (l_+, l_-) is determined by the data $(l_+, l_-, l(\mathrm{A}_\mathbf{L}), \delta(\mathbf{L}))$ up to isometry. Moreover, there exists an even 2-elementary lattice of given (l_+, l_-), $l(\mathrm{A}_\mathbf{L}) = a \geq 0$ and $\delta(\mathbf{L}) = \delta \in \{0, 1\}$ if and only if the following conditions are satisfied:*

$$\begin{cases} a \leq l_+ + l_-; \\ l_+ + l_- \equiv a \ (mod\ 2); \\ \textit{if } \delta = 0, \textit{ then } l_+ - l_- \equiv 0 \ (mod\ 4); \\ \textit{if } a = 0, \textit{ then } \delta = 0 \textit{ and } l_+ - l_- \equiv 0 \ (mod\ 8) \\ \textit{if } a = 1, \textit{ then } l_+ - l_- \equiv 1 \ (mod\ 8); \\ \textit{if } a = 1 \textit{ and } l_+ - l_- \equiv 4 \ (mod\ 8), \textit{ then } \delta = 0; \\ \textit{if } \delta = 0 \textit{ and } l_+ + l_- = a, \textit{ then } l_+ - l_- \equiv 0 \ (mod\ 8). \end{cases}$$

Proof. [Nik79, Theorem 3.6.2]. □

Theorem 1.1.18. *There exists an even hyperbolic p-elementary lattice $\mathbf{L}$ with $p \neq 2$, $r = \mathrm{rk}(\mathbf{L})$ and $a = l(\mathrm{A}_\mathbf{L})$ if and only if the following conditions are satisfied:*

$$\begin{cases} a \leq r; \\ r \equiv 0 \ (mod\ 2); \\ \textit{if } a \equiv 0 \ (mod\ 2), \textit{ then } r \equiv 2 \ (mod\ 4); \\ \textit{if } a \equiv 1 \ (mod\ 2), \textit{ then } p \equiv (-1)^{r/2-1} \ (mod\ 4); \\ \textit{if } r \not\equiv 2 \ (mod\ 8), \textit{ then } r > a > 0. \end{cases}$$

The isometry class of the lattice is uniquely determined by (r, a) when $r \geq 3$.

Proof. [RS81, Section 1]. □

The following result helps to reduce to hyperbolic lattices.

Theorem 1.1.19. *Let $\mathbf{L}$ be an even indefinite lattice of signature (l_+, l_-) with $l_+, l_- \geq 1$ such that $\mathrm{rk}(\mathbf{L}) \geq l(\mathrm{A}_\mathbf{L}) + 3$, then $\mathbf{L}$ admits $\mathbf{U}$ as a direct summand. If instead $l_- \geq 8$ and $\mathrm{rk}(\mathbf{L}) \geq l(\mathrm{A}_\mathbf{L}) + 9$, then $\mathbf{L}$ admits $\mathbf{E}_8(-1)$ as a direct summand.*

Proof. [Nik79, Corollary 1.13.5]. □

For some indefinite lattices is still possible to recover the isometry class from the signature and the quadratic form.

Theorem 1.1.20. *Let* $\mathbf{L}$ *be an even indefinite lattice with signature* (l_+, l_-) *and quadratic form* $q_\mathbf{L}$ *such that* $\mathrm{rk}(\mathbf{L}) \geq l(\mathrm{A}_\mathbf{L}) + 2$*, then the data* $(l_+, l_-, q_\mathbf{L})$ *uniquely determines the isometry class of* $\mathbf{L}$.

Proof. [Nik79, Corollary 1.13.3]. □

Let p be a prime number and consider the ring of p-adic integers $\mathbb{Z}_p$, let $\mathbf{L}$ be a lattice. Set $l_p(\mathrm{A}_\mathbf{L}) := l(\mathrm{A}_{\mathbf{L}\otimes\mathbb{Z}_p})$ and

$$(q_{\mathrm{A}_\mathbf{L}})_p := q_{\mathrm{A}_{\mathbf{L}\otimes\mathbb{Z}_p}} : \mathrm{A}_{\mathbf{L}\otimes\mathbb{Z}_p} \times \mathrm{A}_{\mathbf{L}\otimes\mathbb{Z}_p} \to \mathbb{Q}_p/2\mathbb{Z}_p,$$

observe that $\mathbb{Q}_p/2\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$ for $p \neq 2$.

Example 1.1.21. Define the following rank two lattices

$$u(k) := \begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}, v(k) := \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$$

expressed by their intersection matrices. The discriminant forms are respectively given by the following matrices

$$q_{u(k)} = \begin{pmatrix} 0 & \frac{1}{2^k} \\ \frac{1}{2^k} & 0 \end{pmatrix}, q_{v(k)} = \begin{pmatrix} \frac{1}{2^{k-1}} & \frac{1}{2^k} \\ \frac{1}{2^k} & \frac{1}{2^{k-1}} \end{pmatrix}.$$

Theorem 1.1.22. *Let* $\mathbf{L}$ *be an even indefinite lattice with signature* (l_+, l_-)*, such that*

1. $\mathrm{rk}(\mathbf{L}) \geq l_p(\mathrm{A}_\mathbf{L}) + 2$ *for all primes* $p \neq 2$*,*
2. *if* $\mathrm{rk}(\mathbf{L}) = l_2(\mathrm{A}_\mathbf{L})$ *then* $(q_{\mathrm{A}_\mathbf{L}})_2$ *admits either* $(q_{u(1)})_2$ *or* $(q_{v(1)})_2$ *as a direct summand.*

Then the data $(l_+, l_-, q_\mathbf{L})$ *uniquely determines the isometry class of* $\mathbf{L}$ *and the homomorphism* $\mathrm{O}(\mathbf{L}) \to \mathrm{O}(\mathrm{A}_\mathbf{L})$ *is surjective.*

Proof. [Nik79, Theorem 1.14.2]. □

1.2 Deformation theory

In this section we summarize few basic facts about deformations of complex manifolds, our main references are [GHJ12] and [Voi02]. In the whole section X will be a compact complex manifold.

Definition 1.2.1. A *deformation* of X is a smooth proper morphism between connected complex spaces $\mathcal{X} \to B$ with a point $0 \in B$ such that there is an isomorphism $\mathcal{X}_0 \cong X$.

We introduce the deformation-equivalence relation:

Definition 1.2.2. Two compact complex manifolds X_1, X_2 are *deformation equivalent* if there is a smooth proper morphism between connected complex spaces $\mathcal{X} \to B$ and points $b_1, b_2 \in B$ such that $\mathcal{X}_{b_1} \cong X_1$ and $\mathcal{X}_{b_2} \cong X_2$.

By Ehresmann's lemma [Voi02, Proposition 9.3], a deformation is a locally trivial fibration from the differential point of view. Hence, deformation equivalent manifolds share the same topological invariants.

Definition 1.2.3. A deformation $\mathcal{X} \to (B, 0)$ of X is called *universal* if any other deformation $\mathcal{X}' \to (B', 0')$ is isomorphic to the pullback under a uniquely determined morphism $\varphi : S' \to S$ with $\varphi(0') = 0$.

If a universal deformation exists, then it is unique up to isomorphism. We will denote by $\mathcal{X} \to \mathrm{Def}(X)$ the universal deformation of X when it exists. The existence of a universal deformation is guaranteed in the following case:

Theorem 1.2.4 (Kuranishi)**.** *Let X be a compact complex manifold with $\mathrm{H}^0(X, \mathcal{T}_X) = 0$, then a universal deformation exists. Moreover, the universal deformation is universal for any of its fibers.*

It is interesting to know when properties of a manifold like being Kähler or having trivial canonical bundle are preserved when deforming the manifold.

Proposition 1.2.5. *Let X be a compact Kähler manifold and $\mathcal{X} \to B$ a deformation of $X \cong \mathcal{X}_0$, then:*

1. *For $t \in B$ close to 0, the fiber $\mathcal{X}_t$ is compact Kähler.*
2. *If K_X is trivial, then $K_{\mathcal{X}_t}$ is trivial for t close to 0 and the dimension of $\mathrm{H}^1(\mathcal{X}_t, \mathcal{T}_{\mathcal{X}_t})$ does not depend on t.*

Proof. [Voi02, Proposition 9.20 and Proposition 9.23]. □

Lemma 1.2.6. *Let $\mathcal{X} \to \mathrm{Def}(X)$ be the universal deformation of a compact complex manifold with $\mathrm{H}^0(X, \mathcal{T}_X) = 0$. For any t close to $0 \in \mathrm{Def}(X)$ there is a natural isomorphism $\mathcal{T}_t \mathrm{Def}(X) \cong \mathrm{H}^1(\mathcal{X}_t, \mathcal{T}_{\mathcal{X}_t})$.*

Definition 1.2.7. Suppose X admits a universal deformation $\mathcal{X} \to \mathrm{Def}(X)$. We say that deformations are *unobstructed* if $\dim \mathcal{T}_0 \mathrm{Def}(X) = \dim \mathrm{Def}(X)$, i.e. $\mathrm{Def}(X)$ is smooth at 0.

Proposition 1.2.5 readily applies to compact Kähler manifolds X with trivial canonical bundle, implying that if $\mathrm{Def}(X)$ is reduced then deformations are unobstructed and $\mathrm{Def}(X)$ is smooth in a neighborhood of 0. It is a non-trivial but very important fact that in this case the deformation space is reduced:

Theorem 1.2.8 (Bogomolov-Tian-Todorov)**.** *Let X be a compact Kähler manifold with trivial canonical bundle, then deformations are unobstructed.*

1.3 Irreducible holomorphic symplectic manifolds

In this section we give a general overview about irreducible holomorphic symplectic (IHS) manifolds. We recall basic results and the most recent advances, exposing the general theory with supplement of details about known examples and some explicit constructions.

1.3.1 Basic facts and examples

We give the definition of irreducible homolorphic symplectic (IHS) manifold and the one of hyper-Kähler (HK) manifold and recall the equivalence of the two definitions, then we recall the original motivation of study of such manifolds (Theorem 1.3.6) and finally introduce known examples.

Definition 1.3.1. A complex compact Kähler manifold X is called *irreducible homolorphic symplectic* (IHS) if:

1. $\pi_1(X) = \{1\}$,
2. $\mathrm{H}^0(X, \Omega_X^2) = \mathbb{C}\sigma_X$ with σ_X everywhere non-degenerate.

The form σ_X is called *symplectic form*.

The existence of a non-degenerate two-form $\sigma_X \in \mathrm{H}^0(X, \Omega_X^2)$ implies that X has even complex dimension $\dim_{\mathbb{C}} X = 2n$, moreover the form induces an isomorphism $\mathcal{T}_X \cong \Omega_X$ between the tangent and the cotangent bundles. The canonical bundle $K_X = \Omega_X^{2n}$ is trivialized by the form σ_X^n, thus $c_1(X) = 0$.

Definition 1.3.2. A compact Riemannian manifold (M, g) of real dimension $4n$ is called *hyper-Kähler* (HK) if the holonomy group $\mathrm{Hol}(g)$ equals the symplectic group $\mathrm{Sp}(n)$. In this case the metric g is called a hyper-Kähler metric.

If g is an hyper-Kähler metric, then there exist three complex structures I, J, K with $K = IJ = -JI$ such that g is Kähler for any of them. Moreover, for any complex structure $\lambda = aI + bJ + cK$ with a, b, c real numbers such that $a^2 + b^2 + c^2 = 1$, the metric g is Kähler with respect to λ. There are associated Kähler forms $\omega_\lambda = g(\lambda(-), -)$.

Definition 1.3.3. Let X be a compact Kähler manifold. The *Kähler cone* $\mathcal{K}_X \subset \mathrm{H}^{1,1}(X, \mathbb{R})$ is the open convex cone of Kähler classes on X.

The next two results show that the definitions of IHS manifolds and HK manifolds coincide. For details we refer to [GHJ12, Proposition 23.3, Theorem 23.5].

Proposition 1.3.4. *Let (M, g) be a HK manifold, then the complex manifolds $(M, I), (M, J)$ and (M, K) are IHS manifolds.*

The following is a consequence of Yau's Theorem.

Theorem 1.3.5. *Let X be an IHS manifold, then for any $\alpha \in \mathcal{K}_X$ there exists a unique hyper-Kähler metric on the underlying real manifold M such that $X = (M, I)$ and $\alpha = [\omega_I]$.*

In some sense, IHS manifold are one of the building blocks of compact Kähler manifolds with trivial first Chern class, as stated in the next result. This is the original motivation for the interest in this kind of manifolds.

Theorem 1.3.6 (Beauville-Bogomolov decomposition)**.** *Let Z be a compact complex Kähler manifold with $c_1(Z)_{\mathbb{R}} = 0$. Then there exists an étale cover $\widetilde{Z} \to Z$ such that*

$$\widetilde{Z} \cong T \times \prod_i X_i \times \prod_j Y_j$$

where T is a complex torus, X_i are IHS manifolds and Y_j are Calaby-Yau manifolds.

Proof. [Bea83, Theorem 1]. □

The better known examples of such manifolds are $K3$ surfaces:

Definition 1.3.7. A K3 surface is a complex surface S such that $K_S \cong \mathcal{O}_S$ and $\mathrm{H}^1(S, \mathcal{O}_S) = 0$.

It was proved in [Siu83] that every K3 surface is Kähler, moreover one can easily show (cf. [Huy16]) that the definition implies that a K3 surface is simply connected and it has a unique symplectic form up to scalars. It follows that K3 surfaces are examples of IHS manifolds and, conversely, any IHS manifold of dimension 2 is a K3 surface. One could wonder if it is possible to remove the hypothesis to be Kähler in the definition of IHS manifold, but the last is really needed since an example of manifolds not admitting a Kähler structure but satisfying all the other conditions was found in [Gua94]. It is highly non trivial fact that all K3 surfaces are deformation equivalent (see [Kod64]).

Example 1.3.8. We recall some examples among the most classical constructions of K3 surfaces:

1. A double cover $S \to \mathbb{P}^2$ branched along a smooth sextic $C \subset \mathbb{P}^2$.
2. A smooth quartic hypersurface $S \subset \mathbb{P}^3$.
3. A smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$.
4. A smooth complete intersection of three quadrics in $\mathbb{P}^5$.

There are just few other examples of IHS manifolds in higher dimension, up to deformation.

Example 1.3.9 (Hilbert scheme of points on a K3 surface)**.** Let S be a projective K3 and $n \geq 2$ an integer. Denote by $S^{[n]}$ the Hilbert scheme of n points on S, i.e. the space parametrizing zero-dimensional subschemes $(Z, \mathcal{O}_Z)$ with $\dim_{\mathbb{C}} \mathcal{O}_Z = n$. Fogarty proved that the *Hilbert-Chow morphism*

$$\begin{aligned} \rho : S^{[n]} &\to S^{(n)} \\ (Z, \mathcal{O}_Z) &\mapsto \sum_{p \in S} l(\mathcal{O}_{Z,p})p \end{aligned}$$

is a resolution of singularities, which is a blow-up of the diagonal $\Delta \subset S^{(n)}$, whose exceptional divisor E parametrizes non-reduced schemes. Moreover, Beauville proved in [Bea83] that $S^{[n]}$ is a projective IHS manifold of dimension $2n$, its symplectic form comes from the one on S. The space $S^{[n]}$ exists even if S is not projective, it is only a complex space called the *Douady space*. A manifold which is deformation equivalent to $S^{[n]}$ for a K3 surface S is called of $\mathrm{K3}^{[n]}$ type.

A similar approach works for $A^{[n]}$, where A is an Abelian surface, with the difference that $A^{[n]}$ is not symply connected.

Example 1.3.10 (Generalized Kummer manifolds)**.** Let A be an Abelian surface and $n \geq 2$ an integer, consider the composition of the Hilbert-Chow morphism with the summation map

$$\begin{aligned} s : A^{[n+1]} &\to A \\ (Z, \mathcal{O}_Z) &\mapsto \sum_{p \in A} l(\mathcal{O}_{Z,p})p \end{aligned}$$

and set $\mathrm{K}_n(A) = s^{-1}(0)$. The map s happens to be an isotrivial fibration and Beauville proved in [Bea83] that $\mathrm{K}_n(A)$ is an IHS manifold of dimension $2n$, called *generalized Kummer*. A manifold deformation equivalent to $\mathrm{K}_n(A)$, for A an Abelian surface, is called a manifold of Kum_n type.

Notice that for $n = 1$ we get $S^{[1]} = S$ and $\mathrm{K}_1(A) = \mathrm{K}(A)$ which is just the Kummer surface of A (the symplectic resolution of $A/\{\pm 1\}$), suggesting the name for the higher-dimensional analogue.

Example 1.3.11 (O'Grady's manifolds)**.** Two other examples of IHS manifolds can be produced starting from moduli spaces of sheaves on projective symplectic surfaces surfaces. These are resolutions of singular moduli spaces, we remind to Theorem 2.2.11 and the relative section for more detailed description. Manifolds deformation equivalent to these esamples are called respectively of OG10 type and OG6 type.

The previous examples are representatives of the only known deformation families, which are distinct since they present different topological invariants, as we will see in the next section.

1.3.2 Cohomological properties

We resume the most important cohomological properties of an IHS manifold X, as the lattice structure on the second cohomology and its interplay with the Hodge decomposition. An easy consequence of the definition is the following:

Proposition 1.3.12. *Let X be an IHS manifold of dimension $2n$. Then for $0 \leq r \leq n$*

$$\mathrm{H}^0(X, \Omega_X^r) = \begin{cases} \mathbb{C}\sigma_X^{(r/2)} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

In particular, $\chi(X, \mathcal{O}_X) = n + 1$.

Since X is compact Kähler, the Hodge decomposition is available:

$$\mathrm{H}(X, \mathbb{C})^k = \bigoplus_{p+q=k} \mathrm{H}(X)^{p,q}$$

where $\mathrm{H}(X)^{p,q} = \mathrm{H}^q(X, \Omega_X^p)$ and $\mathrm{H}(X)^{p,q} = \overline{\mathrm{H}(X)^{q,p}}$, for k an integer.
In this case, the decomposition for $k = 2$ reads

$$\mathrm{H}(X, \mathbb{C})^2 = \sigma_X \mathbb{C} \oplus \mathrm{H}(X)^{1,1} \oplus \overline{\sigma_X}\mathbb{C}$$

since σ_X is a generator for $\mathrm{H}(X, \mathbb{C})^{2,0}$. Suppose that our choice of the symplectic form is such that $\int (\sigma_X \overline{\sigma_X})^n = 1$.

Definition 1.3.13 (Beauville-Bogomolov-Fujiki form)**.** Define the following quadratic form

$$q_X(\alpha) = (n/2) \int_X \alpha^2 (\sigma_X \overline{\sigma_X})^{n-1} + (1 - n)(\int_X \alpha \sigma_X^{n-1} \overline{\sigma_X}^n)(\int_X \alpha \sigma_X^n \overline{\sigma_X}^{n-1})$$

for $\alpha \in \mathrm{H}^2(X, \mathbb{R})$ and denote by $b_X(-,-)$ the bilinear form associated to q_X.

The following shows that the bilinear form has in fact a topological nature and can be defined on the integral cohomology.

Theorem 1.3.14 (Beauville-Fujiki relation)**.** *There exists a positive real number c_X, called the Fujiki constant, such that*

$$q_X(\alpha)^n = c_X \int_X \alpha^{2n}$$

for all $\alpha \in \mathrm{H}^2(X)$. In particular, q_X can be normalized such that it is a primitive integral quadratic form on $\mathrm{H}^2(X, \mathbb{Z})$.

Proof. [GHJ12, Proposition 23.14]. □

Observe that the universal coefficient theorem with the fact that $\mathrm{H}_1(X, \mathbb{Z}) = 0$ implies that $\mathrm{H}^2(X, \mathbb{Z})$ is torsion-free, hence q_X gives to $\mathrm{H}^2(X, \mathbb{Z})$ a structure of a lattice.

Proposition 1.3.15. *The form q_X on $\mathrm{H}^2(X, \mathbb{R})$ has signature $(3, b_2(X) - 3)$ where $b_2(X)$ is the second Betti number of X, more precisely if $\alpha \in \mathcal{K}_X$ then q_X is positive on $\mathbb{R}\alpha \oplus (\mathrm{H}^{2,0} \oplus \mathrm{H}^{0,2})_{\mathbb{R}}(X)$ and negative on its complement. In the Hodge decomposition of $\mathrm{H}^2(X, \mathbb{C})$, the space $\mathrm{H}^{1,1}(X, \mathbb{C})$ is an orthogonal summand with respect to the form b_X.*

Proof. [GHJ12, Corollary 23.11]. □

We point out that the form q_X and the Fujiki constant c_X are bimeromorphic and deformation invariants.

Definition 1.3.16. The sublattice

$$\mathrm{NS}(X) = \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^{1,1}(X)$$

is called the *Neròn-Severi* lattice.

Using the exponential sequence and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ one finds $\mathrm{NS}(X) = \mathrm{Pic}(X)$ and for this reason when considering IHS manifold the rank $\rho(X)$ of $\mathrm{NS}(X)$ is called *Picard rank*. From the fact that $\mathrm{H}^{1,1}(X)$ is orthogonal to $(\mathrm{H}^{2,0} \oplus \mathrm{H}^{0,2})(X)$, it follows that $\mathrm{NS}(X) = \mathrm{H}^2(X, \mathbb{Z}) \cap \omega^{\perp}$ where ω is the Kähler form.

Definition 1.3.17. The *transcendental* lattice is the sublattice

$$\mathrm{T}(X) = \mathrm{NS}(X)^{\perp} \subset \mathrm{H}^2(X, \mathbb{Z}).$$

There is the following useful projectivity criterion:

Proposition 1.3.18. *Let X be an IHS manifold, then X is projective if and only if there exists $\alpha \in \mathrm{NS}(X)$ with $q_X(\alpha) > 0$.*

Proof. [GHJ12, Proposition 26.13]. □

We report the lattice properties of the known deformation types of IHS manifolds.

Example 1.3.19. If S is a K3 surface, then the Beauvile-Bogomolov-Fujiki form coincides with the intersection paring in the middle cohomology, which forms the following unimodular lattice:

$$(\mathrm{H}^2(S,\mathbb{Z}), q_S) \cong \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3}\,.$$

For the next two examples, the lattices were computed in [Bea83] while information about the Fujiki constant and the Euler characteristic of divisors can be found in [Deb22, Huy16].

Example 1.3.20. Let S be a K3 surface, then for any $n \geq 2$ we have:

$$\mathrm{H}^2(S^{[n]},\mathbb{Z}) = \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta$$

where 2δ is the class of the exceptional divisor E of the Hilbert-Chow morphism. The latter, equipped with the form $q_{S^{[n]}}$, is isometric to the lattice

$$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus [-2(n-1)],$$

this gives $b_2(S^{[n]}) = 23$ and $\mathrm{sign}(\mathrm{H}^2(S^{[n]},\mathbb{Z})) = (3,20)$. The discriminant lattice is

$$\frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \cong \langle \frac{\delta}{2(n-1)} \rangle$$

with discriminant quadratic form given by $(\frac{1}{2(n-1)})$, the Fujiki constant is $c_{S^{[n]}} = \frac{(2n)!}{n!2^n}$.

Example 1.3.21. Let A be an Abelian surface and $n \geq 2$. Similarly to the previous example, in the second cohomology of a manifold of Kummer type there is an extra class coming from the Hilber-Chow morphism:

$$\mathrm{H}^2(\mathrm{K}_n(A),\mathbb{Z}) = \mathrm{H}^2(A,\mathbb{Z}) \oplus \mathbb{Z}\delta$$

which, equipped with $q_{\mathrm{K}_n(A)}$, is isometric to

$$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus [-2(n+1)],$$

so that $b_2(S^{[n]}) = 7$ and $\mathrm{sign}(\mathrm{K}_n(A),\mathbb{Z})) = (3,4)$. The discriminant lattice is

$$\frac{\mathbb{Z}}{2(n+1)\mathbb{Z}} \cong \langle \frac{\delta}{2(n+1)} \rangle$$

with discriminant quadratic form determined by $(\frac{1}{2(n+1)})$, the Fujiki constant is $c_{\mathrm{Kum}_n(A)} = \frac{(2n)!}{n!2^n}(n+1)$.

Example 1.3.22. Let X be the O'Grady's six dimensional example. The example is constructed starting from an Abelian surface A, we have again that $\mathrm{H}^2(A,\mathbb{Z}) \subset \mathrm{H}^2(X,\mathbb{Z})$ but the situation is a bit more complicated and there are two extra classes:

$$(\mathrm{H}^2(X,\mathbb{Z}), q_{OG6}) \cong \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2},$$

it follows that $b_2(X) = 8$ and it is known that $c_X = 60$. For these features we refer to [Rap07].

Example 1.3.23. Let X be the O'Grady's ten dimensional example. The example is constructed starting from a K3 surface S, again $\mathrm{H}^2(S,\mathbb{Z}) \subset \mathrm{H}^2(X,\mathbb{Z})$ and similarly to the previous example there are two extra classes:

$$(\mathrm{H}^2(X,\mathbb{Z}), q_{OG10}) \cong \mathbf{E}_8(-1)^{\oplus 2}\, \mathbf{U}^{\oplus 3} \oplus \mathbf{A}_2(-1),$$

it follows that $b_2(X) = 24$ and it is known that $c_X = 945$. For these features we refer to [Rap08].

Remark 1. *We stress that the second Betti numbers of the previous examples are different, hence they are not deformation equivalent to each other. Moreover, any manifold which is deformation equivalent to one of these examples share the same lattice* $(\mathrm{H}^2(X,\mathbb{Z}), q)$ *and Fujiki constant.*

1.3.3 Period maps and Torelli theorems

We introduce the period domain and the period map, then we give an overview of the Torelli theorem in terms of the period map. We introduce parallel transport operators and recall the Torelli theorem in terms of parallel transport operators and Hodge theory.

Definition 1.3.24. A *marked* IHS manifold (X,η) is an IHS manifold X with an isometry $\eta : \mathrm{H}^2(X,\mathbb{Z}) \cong L$. Two marked IHS manifolds $(X,\eta), (X',\eta')$ are *isomorphic* if there exists an isomorphism $f : X \to X'$ such that $\eta' = \eta \circ f^*$.

By definition, if σ_X is the symplectic form of X, the relations $b_X(\sigma_X,\sigma_X) = 0$ and $b_X(\sigma_X,\overline{\sigma_X}) > 0$ hold. This inspires the definition of the period domain, which is a suitable space where the symplectic forms live.

Definition 1.3.25. The *period domain* associated to $\mathbf{L}$ is the complex space

$$\Omega_{\mathbf{L}} := \{y \in \mathbb{P}(\mathbf{L}_{\mathbb{C}}) | b_X(y,y) = 0, b_X(y,\overline{y}) > 0\}.$$

We remark that Proposition 1.2.5 applies in the case of IHS manifolds, giving that $\dim \mathrm{H}^1(\mathcal{X}_t, \mathcal{T}_{\mathcal{X}_t}) = \dim \mathrm{H}^1(\mathcal{X}_t, \Omega_{\mathcal{X}_t}) = \dim \mathrm{H}^{1,1}(\mathcal{X}_t)$ is constant for a deformation $\mathcal{X}_t$ close to $\mathcal{X}_0 = X$, and a similar proof yields the following:

Proposition 1.3.26. *Let X be an IHS manifold and $\mathcal{X} \to B$ a deformation of X, then any fiber $\mathcal{X}_t$ with t close to $0 \in \mathrm{Def}(X)$ is an IHS manifold.*

For t in a neighborhood of $0 \in \mathrm{Def}(X)$ we know by Ehresmann's lemma that $\mathrm{H}^*(X,\mathbb{Z}) \cong \mathrm{H}^*(\mathcal{X}_t,\mathbb{Z})$, so if we fix a marking $\eta : \mathrm{H}^2(X,\mathbb{Z}) \cong \mathbf{L}$ we have a marking $\eta_t : \mathrm{H}^2(\mathcal{X}_t,\mathbb{Z}) \cong \mathbf{L}$ for any t close to 0. We say that $\mathcal{X}$ with this family of markings is a deformation of the pair (X,η). Moreover, by the previous proposition we know that a Hodge decomposition $\mathrm{H}^2(X,\mathbb{C}) \cong \mathrm{H}^2(\mathcal{X}_t,\mathbb{C}) = \mathrm{H}^{2,0}(\mathcal{X}_t) \oplus \mathrm{H}^{1,1}(\mathcal{X}_t) \oplus \mathrm{H}^{0,2}(\mathcal{X}_t)$ of the fibers is available in this neighborhood, we want to encode the variation of the degree-two Hodge structure: the symplectic form $\sigma_{\mathcal{X}_t}$ recovers all the information.

Definition 1.3.27. Let $\mathcal{X} \to (B,0)$ a deformation of (X,η) and $\mathcal{U} \subset B$ a suitable neighborhood of 0, define the map

$$\mathcal{P} : \mathcal{U} \to \Omega_{\mathrm{L}}$$

by setting $\mathcal{P}(t) := [\eta_t(\sigma_{\mathcal{X}_t})]$ for $t \in \mathcal{U}$. When considering the universal deformation, the map $\mathcal{P} : \mathrm{Def}(X) \to \Omega_{\mathrm{L}}$ is called *local period map.*

The map $\mathcal{P}$ is holomorphic, moreover and the following remarkable fact holds:

Theorem 1.3.28 (Local Torelli)**.** *Let (X,η) be a marked IHS manifold, then the local period map*

$$\mathcal{P} : \mathrm{Def}(X) \to \Omega_{\mathrm{L}}$$

is a local isomorphism.

Proof. [Bea83, Theorem 5]. □

We introduce the moduli space of marked IHS manifolds:

Definition 1.3.29. The *moduli space of marked IHS manifolds* is given by

$$\mathcal{M}_{\mathbf{L}} := \{(X,\eta) | \eta : \mathrm{H}^2(X,\mathbb{Z}) \cong \mathbf{L}\} / \cong$$

where the equivalence relation is the isomorphism of marked IHS manifolds.

A structure of complex analytic space on $\mathcal{M}_{\mathbf{L}}$ is given by Theorem 1.3.28, but unluckily it is well-known that the space is not Hausdorff. We recall that the deformation space is smooth by Theorem 1.2.8, hence $\mathcal{M}_{\mathbf{L}}$ is.

Gluing the local period maps $\mathcal{P} : \mathrm{Def}(X) \to \Omega_{\mathbf{L}}$ one gets:

Definition 1.3.30. The map

$$\mathcal{P} : \mathcal{M}_{\mathbf{L}} \to \Omega_{\mathbf{L}}$$

is called the *global period map.*

Consider a connected component $\mathcal{M}_{\mathbf{L}}^0$ of the moduli space $\mathcal{M}_{\mathrm{L}}$. The global period map is surjective, more precisely Huybrechts proved the following:

Theorem 1.3.31 (Surjectivity of the period map)**.** *The restriction of the period map* $\mathcal{P}_0 : \mathcal{M}_{\mathrm{L}}^0 \to \Omega_{\mathrm{L}}$ *is surjective.*

Proof. [Huy16, Proposition 25.12]. □

In the case of K3 surfaces, the period (S, η) recovers the isomorphism class of S and the general fiber consists of two points (the space has two connected components) given by the choice $\pm\eta$. The next two examples show that in higher dimension the situation is worst.

Example 1.3.32. Let S be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}C$ with C a smooth rational curve, then $X = S^{[2]}$ contains $C^{[2]} \cong \mathbb{P}^2$, let X' be the *Mukai flop* of X along $\mathbb{P}^2$ as in [Muk84]. Then by [Deb84] the two manifolds are not isomorphic, but there are markings of X, X' for which they have the same period.

In the previous case the two manifolds with the same period are still bimeromorphic, but the situation could be even worst.

Example 1.3.33. Consider a non-projective complex torus T which is not isomorphic to the due torus $T^\vee$. Let $X = \mathrm{K}_2(T)$, $X' = \mathrm{K}_2(T^\vee)$ and E, E' the exceptional divisors of the resolution of singularities of the symmetric products of T and $T^\vee$. The two manifolds cannot be bimeromorphic, otherwise E, E' would be and then $T \cong \mathrm{Alb}(E) \cong \mathrm{Alb}(E') \cong T^\vee$. However, by [Nam02] there are markings for which X and X' have the same period and in particular we will see that they must lie on different components of the moduli space.

The following statement was proved by Huybrechts, Markman and Verbitsky, it describes how the period map fails to be injective.

Theorem 1.3.34 (Global Torelli)**.** *The restriction of the period map* $\mathcal{P}_0 : \mathcal{M}_{\mathbf{L}}^0 \to \Omega_{\mathbf{L}}$ *is generically injective. When the injectivity fails, the fiber of a point* $y \in \Omega_{\mathrm{L}}$ *consists of pair-wise inseparable points. Inseparable points are represented by bimeromorphic manifolds.*

Proof. [Mar11, Theorem 2.2]. □

In particular, bimeromorphic IHS manifolds are deformation equivalent.

The concept of parallel transport operator was introduced to keep track of the variations of Hodge structure that avoid changing connected component in the moduli space.

Definition 1.3.35. Let X, Y be IHS manifolds and $\phi : \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(Y, \mathbb{Z})$ a lattice isometry. We say that ϕ is a *parallel transport operator* if there exists a smooth proper family $\pi : \mathcal{X} \to B$ and a continuous path $\gamma : [0,1] \to B$ with $\mathcal{X}_{\gamma(0)} \cong X$, $\mathcal{X}_{\gamma(1)} \cong Y$ and such that ϕ is induced by parallel transport along γ in the local system $R^2\pi_*\mathbb{Z}$.

Two points $(X,\eta),(X',\eta')$ in $\mathcal{M}_{\mathbf{L}}$ belong to the same connected component if and only if $\eta'\circ\eta^{-1}$ is a parallel transport operator.

Definition 1.3.36. A parallel transport operator $\phi : \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z})$ is called a *monodromy operator*, the group generated by such operators is denoted by $\mathrm{Mon}^2(X) \subset \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$. The subgroup of monodromy operators that preserve the Hodge structure is denoted by $\mathrm{Mon}^2_{Hdg}(X) \subset \mathrm{Mon}^2(X)$.

The number of connected components of the moduli space $\mathcal{M}_{\mathbf{L}}$ is computed by the index $[\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z})) : \mathrm{Mon}^2(X)]$ of the subgroup of monodromy operators. That index is finite by [Mar11, Lemma 7.5], hence the number of connected components is finite. Once a marking $\eta : \mathrm{H}^2(X,\mathbb{Z}) \to \mathbf{L}$ is fixed, one can define the monodromy group

$$\mathrm{Mon}^2(\mathbf{L}) := \{\eta\circ\phi\circ\eta^{-1} | \phi \in \mathrm{Mon}^2(X)\} \subset \mathrm{O}(\mathbf{L})$$

of the lattice $\mathbf{L}$ which does not depend on (X,η), but only on the connected component $\mathcal{M}^0_{\mathbf{L}} \subset \mathcal{M}_{\mathbf{L}}$ it belongs to. In case the subgroup $\mathrm{Mon}^2(X) \subset \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$ is normal then $\mathrm{Mon}^2(X)$ is independent of the choice of the connected component.

The Torelli theorem can be then reformulated as follows.

Theorem 1.3.37 (Torelli Hodge-theoretical)**.** *Let X,Y two deformation-equivalent IHS manifolds, then:*

1. *X and Y are bimeromorphic if and only if there is a parallel transport operator $\phi : \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(Y,\mathbb{Z})$ which is an isomorphism of integral Hodge structures.*
2. *Let $\phi : \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(Y,\mathbb{Z})$ be a parallel transport operator which is an isomorphism of integral Hodge structures. There exists an isomorphism $f : Y \to X$ such that $\phi = f^*$ if and only if f sends some Kähler class on X to some Kähler class on Y.*

Proof. [Mar11, Theorem 1.3]. □

The monodromy group of the known deformation types was computed with the contribution of many authors. We introduce some necessary notation and give an overview of the results.

Definition 1.3.38. Let X be an IHS manifold, define the cone of positive classes

$$\widetilde{\mathcal{C}}_X := \{\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R}) | b_X(\alpha,\alpha) > 0\} \subset \mathrm{H}^{1,1}(X,\mathbb{R}).$$

By [Mar11, Lemma 4.1] there is a canonical generator of $\mathrm{H}^2(\widetilde{\mathcal{C}}_X,\mathbb{Z}) \cong \mathbb{Z}$, so it makes sense to talk about orientation-preserving automorphisms of $\widetilde{\mathcal{C}}_X$.

Definition 1.3.39. Denote by

$$\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})) \subset \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$$

the index 2 subgroup consisting of isometries that induce an orientation-preserving automorphism of $\widetilde{\mathcal{C}}_X$.

The subgroup $\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))$ is normal and it can be identified with the kernel of the determinant of the spinor norm (cf. [MM09, Section 10] and Definition 1.1.3).

Example 1.3.40. Let S be a K3 surface, then by [Bor86, Theorem A] we have $\mathrm{Mon}^2(S) = \mathrm{O}^+(\mathrm{H}^2(S,\mathbb{Z}))$ and hence the moduli space of marked K3 surfaces has two connected components identified by the correspondence $(S,\eta) \mapsto (S,-\eta)$.

Definition 1.3.41. Let X be an IHS manifold, define the group

$$\mathcal{W}(X) := \{\phi \in \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})) | \overline{\phi} = \pm \mathrm{id} \in \mathrm{O}(\mathbf{A}_{\mathrm{H}^2(X,\mathbb{Z})})\},$$

and denote by $\chi : \mathcal{W}(X) \to \{\pm 1\}$ the corresponding character.

Example 1.3.42. Let X be an IHS manifold of $\mathrm{K3}^{[n]}$ type. Markman in [Mar11] proved that

$$\mathrm{Mon}^2(X) = \mathcal{W}(X)$$

and that $\mathrm{Mon}^2(X)$ is a normal subgroup of $\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$. It follows that the index of the subgroup is 2^{r-1} where $r = \rho(n-1)$ is the number of distinct primes dividing $n-1$. In particular, $\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))$ when 2^{n-1} is a prime power.

Example 1.3.43. Let X be a IHS manifold of Kum_n type. Mongardi proved in [Mon16a, Theorem 2.3] that

$$\mathrm{Mon}^2(X) = \{\phi \in \mathcal{W}(X) | \det(\phi)\chi(\phi) = 1\}.$$

Example 1.3.44. Let X be an IHS manifold of OG6 type. Mongardi and Rapagnetta proved in [MR21, Theorem 5.4] that the monodromy group is

$$\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})).$$

Example 1.3.45. Let X be an IHS manifold of OG10 type. Onorati proved in [Ono22, Theorem 5.4] that the monodromy group is

$$\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})).$$

1.3.4 Cones and birational geometry

In order to study the bimeromorphic geometry of an IHS manifold X and its automorphisms, it is convenient to introduce several cones contained in $\widetilde{\mathcal{C}}_X \subset \mathrm{H}^{1,1}(X,\mathbb{R})$ that depend on the Beauville-Bogomolov form.

Definition 1.3.46. The *Kähler cone* is the cone $\mathcal{K}_X \subset \mathrm{H}^{1,1}(X,\mathbb{R})$ of Kähler classes. The *positive cone* is the connnected component $\mathcal{C}_X \subset \mathrm{H}^{1,1}(X,\mathbb{R})$ of $\widetilde{\mathcal{C}}_X = \{\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R}) | b_X(\alpha,\alpha) > 0\}$ containing a Kähler class.

The two cones are convex and there is the inclusion $\mathcal{K}_X \subset \mathcal{C}_X$.

Definition 1.3.47. The *birational Kähler cone* is defined as

$$\mathcal{BK}_X := \bigcup_{f:X \dashrightarrow Y} f^*\mathcal{K}_Y$$

where $f : X \dashrightarrow Y$ runs over all the bimeromorphic maps from X to another IHS manifold Y.

There are inclusions $\mathcal{K}_X \subset \mathcal{BK}_X \subset \mathcal{C}_X$. Moreover, the cone $\mathcal{BK}_X$ is not convex:

Theorem 1.3.48 (Boucksom-Huybrechts)**.** *A class $\alpha \in \mathcal{C}_X$ is Kähler if and only if $\int_C \alpha > 0$ for any rational curve $C \subset X$.*

Proof. [Bou01, Theorem 1.2]. □

Remark 2. *One can consider the inclusion $\mathrm{H}_2(X,\mathbb{Z}) \hookrightarrow \mathrm{H}^2(X,\mathbb{Q})$. Since for any $\alpha, \beta \in \mathcal{C}_X$ it holds $b_X(\alpha,\beta) > 0$, for the previous statement it is enough to check rational curves C such that $q_X(C) \leq 0$.*

Corollary 1.3.49. *If $\mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X,\mathbb{Z}) = 0$, then $\mathcal{K}_X = \mathcal{C}_X$. The same holds if X is projective and $\mathrm{Pic}(X) = \mathbb{Z}H$.*

Definition 1.3.50. A *prime* divisor on X is a reduced and irreducible effective divisor E. A prime divisor is called *prime exceptional* if $q_X(E) < 0$. An effective divisor is called *exceptional* if its prime factors are prime exceptional and their intersection matrix is negative-definite. Denote the set of prime exceptional divisors of X by $\mathrm{Pex}(X)$.

Definition 1.3.51. The *fundamental exceptional chamber* of X is

$$\mathcal{FE}_X = \{\alpha \in \mathcal{C}_X | b_X(\alpha, D) > 0 \text{ for any } D \text{ exceptional}\}.$$

By [Mar11, Proposition 5.6] the fundamental exceptional chamber is characterized by classes $\alpha \in \mathcal{C}_X$ with $b_X(\alpha, D) > 0$ for any non-zero uniruled divisor D. For a K3 surface S these are exactly rational curves, hence by Theorem 1.3.48 one has $\mathcal{FE}_S = \mathcal{K}_S$.

Proposition 1.3.52. *Let X, Y be IHS manifolds and $g : \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(Y,\mathbb{Z})$ be a parallel transport operator which is an isomorphism of Hodge structures. Let $\alpha \in \mathcal{FE}_X$, then $g(\alpha) \in \mathcal{FE}_Y$ if and only if there exists a bimeromorphic map $f : Y \dashrightarrow X$ such that $f^* = g$.*

Proof. [Mar11, Corollary 5.7]. □

As a consequence, there are the following inclusions

$$\mathcal{BK}_X \subset \mathcal{FE}_X \subset \overline{\mathcal{BK}}_X$$

as in [Mar11, Proposition 5.6], in particular $\mathcal{K}_X \subset \mathcal{FE}_X$.

Since the positive cone is invariant under the action of $\mathrm{Mon}^2_{Hdg}(X)$, the following definitions are well-posed.

Definition 1.3.53. Let X be an IHS manifold.

1. An *exceptional chamber* of $\mathcal{C}_X$ is $g(\mathcal{FE}_X)$ for $g \in \mathrm{Mon}^2_{Hdg}(X)$.
2. A *Kähler-like chamber* of $\mathcal{C}_X$ is $g(f^*(\mathcal{K}_Y))$ for $g \in \mathrm{Mon}^2_{Hdg}(X)$ and $f : X \dashrightarrow Y$ a bimeromorphic map.

It is a consequence of Theorem 1.3.37 that if two chambers intersect then the chambers must coincide. Moreover, by [Mar11, Lemma 5.11] any Kähler-like chamber is contained in some exceptional chamber.

For a divisor D we consider the reflection $R_D \in \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))$ given by

$$R_D(\alpha) := \alpha - 2\frac{(D,\alpha)}{(D,D)}D,$$

notice that by [Mar11, Proposition 6.2] if E is prime exceptional then $R_E \in \mathrm{Mon}^2_{Hdg}(X)$. Consider the normal subgroup

$$W_{Exc} = \{R_E | E \in \mathrm{Pex}(X)\} \subset \mathrm{Mon}^2_{Hdg}(X)$$

and the subgroup $\mathrm{Mon}^2_{Bir}(X) \subset \mathrm{Mon}^2_{Hdg}(X)$ of all monodromy operators induced by bimeromorphic maps from X to itself (cf. subsection 1.3.6 and Proposition 1.3.74).

The action of these groups on the chambers is described by the following

Theorem 1.3.54. *Let X be an IHS manifold, then it holds:*

1. *The group $\mathrm{Mon}^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers, the group W_{Exc} acts simply-transitively on the set of exceptional chambers.*
2. *Any exceptional chamber is the interior of a fundamental domain for the action of W_{Exc} on $\mathcal{C}_X$*

3. *The* $\mathrm{Mon}^2_{Hdg}(X)$*-stabilizer of* $\mathcal{FE}_X$ *equals to* $\mathrm{Mon}^2_{Bir}(X)$

4. $\mathrm{Mon}^2_{Hdg}(X) = W_{Ecx} \ltimes \mathrm{Mon}^2_{Bir}(X)$

Proof. [Mar11, Theorem 6.18]. □

Definition 1.3.55. A divisor $D \in \mathrm{Pic}(X)$ is called a *wall divisor* if its class is primitive with $q_X(D) < 0$, and for any $g \in \mathrm{Mon}^2_{Hdg}(X)$ one has $g(D)^\perp \cap \mathcal{BK}_X = \emptyset$. The set of wall divisors on X is denoted by $\Delta(X)$.

The orthogonal complements of wall divisors cut $\mathcal{BK}_X$ in Kähler-like chambers, one of the chambers is given by $\mathcal{K}_X$ while the other chambers are Kähler cones of birational models of X. In particular, notice that we have $\mathrm{Pex}(X) \subseteq \Delta(X)$.

Theorem 1.3.56. *Let* X, Y *be IHS manifolds and* $D \in \Delta(X)$*. Consider a parallel transport operator* $g : \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(Y, \mathbb{Z})$ *such that* $g(D) \in \mathrm{Pic}(Y)$*, then* $g(D) \in \Delta(Y)$*.*

Proof. [Mon15, Theorem 1.3]. □

It follows that $\mathrm{Mon}^2_{Hdg}(X)$ preserves the set $\Delta(X)$, and so the wall and chamber structure of $\mathcal{BK}_X$. In the spirit of Theorem 1.3.48, wall divisors can be described as multiples of extremal rational curves up to $\mathrm{Mon}^2_{Hdg}(X)$-action (see [KLCM19, Proposition 2.3], [Mon15, Proposition 1.5]).

It suffices to classify the prime exceptional divisors and wall divisors to parallel transport, this was done for the known deformation types with a numerical criterion.

Proposition 1.3.57. *Let* $n \geq 2$*. There exist the following monodromy-invariant embeddings:*

1. $\mathrm{H}^2(X, \mathbb{Z}) \hookrightarrow \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 4}$ *with orthogonal complement generated by* v *with* $v^2 = 2n - 2$*, if* X *is of* $K3^{[n]}$ *type.*

2. $\mathrm{H}^2(X, \mathbb{Z}) \hookrightarrow \mathbf{U}^{\oplus 4}$ *with orthogonal complement generated by* v *with* $v^2 = 2n + 2$*, if* X *is of* Kum_n *type.*

Proof. [Mar11, Theorem 9.3], [Wie18, Theorem 4.9]. □

Example 1.3.58. Let X be a manifold of $\mathrm{K3}^{[n]}$ type and $D \in \mathrm{Pic}(X)$, denote by $\langle v, D\rangle_{sat}$ the saturation of the lattice generated by v of Proposition 1.3.57 and D. Then by [KLCM19, Theorem 2.9], [BM14] and [Yos12], we have that D is a wall-divisor if and only if there exists a class $s \in \langle v, D\rangle_{sat}$ such that one of the following holds:

1. $0 \leq q_X(s) < b_X(v, s) \leq \frac{q_X(v)+q_X(s)}{2}$

2. $q_X(s) = -2$ and $0 \leq b_X(v, s) \leq \frac{q_X(v)}{2}$.

Example 1.3.59. Let X be a manifold of Kum_n type type and $D \in \mathrm{Pic}(X)$, denote by $\langle v, D\rangle_{sat}$ the saturation of the lattice generated by v of Proposition 1.3.57 and D. Then by [KLCM19, Theorem 2.9], [BM14] and [Yos12], we have that D is a wall-divisor if and only if there exists a class $s \in \langle v, D\rangle_{sat}$ such that:

$$0 \leq q_X(s) < b_X(v,s) \leq \frac{q_X(v)+q_X(s)}{2}$$

Example 1.3.60. Let X be a manifold of OG6 type, then by [MR21, Proposition 6.8] we have:

$$\mathrm{Pex}(X) = \{D \in \mathrm{Pic}(X) | D \text{ primitive with } q_X(D) = -2, \mathrm{div}(D) = 1, 2\}$$
$$\Delta(X) = \mathrm{Pex}(X) \cup \{D \in \mathrm{Pic}(X) | D \text{ primitive with } q_X(D) = -4, \mathrm{div}(D) = 2\}.$$

Example 1.3.61. Let X be a manifold of OG10, then by [MO22, Theorems 3.2, 5.5] we have:

$$\mathrm{Pex}(X) = \{D \in \mathrm{Pic}(X) | D \text{ primitive with } q_X(D) = -2 \text{ or } q_X(D) = -6, \mathrm{div}(D) = 3\}$$
$$\Delta(X) = \mathrm{Pex}(X) \cup \{D \in \mathrm{Pic}(X) | D \text{ primitive with } q_X(D) = -4 \text{ or } q_X(D) = -24, \mathrm{div}(D) = 3\}.$$

1.3.5 Teichmüller spaces and the mapping class group

Some literature about IHS manifolds has a slightly different approach in the study of the period map and the moduli space. The flavour is more differential and the moduli space is replaced with the Teichmüller space, which turns out to be an étale cover of the moduli space. The results are equivalent, but the latter permits the study of a slightly weaker equivalence than holomorphic automorphism. The following definitions and results can be found in [Loo21, Ver13, Ver20].

Let $X = (M, g, I)$ be an IHS manifold, where g is a hyper-Kähler metric and I a complex structure, and fix a marking $\eta : \mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{L}$.

Definition 1.3.62. Consider an IHS manifold $X = (M, g, I)$, the set of all complex structures J that make (M, J) an IHS manifold, up to differential isotopy equivalence, is called the *Teichmüller space* and it is denoted by $\mathcal{T}$.

The choice of a complex structure J determines $\mathrm{H}^{2,0}(M, J)$ independently to the choice of a Kähler metric, hence there is a map $\mathcal{P} : \mathcal{T} \to \Omega_{\mathbf{L}}$ which is called again the period map and it is a local isomorphism by [Ver13, Theorem 1.9], giving to $\mathcal{T}$ the structure of complex space.

One can consider two variants of Teichmüller spaces, with their associated period maps.

Definition 1.3.63. Consider an IHS manifold $X = (M, g, I)$, denote by $\mathcal{T}_{HK}$ the set of hyper-Kähler metrics with unitary volume and a complex structure, up to differential isotopy.

There is a period map $\mathcal{P}_{HK} : \mathcal{T}_{HK} \to \Omega_{\mathbf{L}} \times \mathbb{P}(\mathbf{L}_{\mathbb{R}})$ whose first component is $\mathcal{P}$ and second component consist of the choice of the ray spanned by the Kähler form ω_X associated to the metric. By [Loo21, Corollary 3.7], this map is a local diffeomorphism with its image $\widetilde{\Omega}_{\mathbf{L}}$, that is a proper subset of $\Omega_{\mathbf{L}} \times \mathbb{P}(\mathbf{L}_{\mathbb{R}})$ since $\omega_X \in \mathrm{H}^{1,1}(X, \mathbb{R})$.

Definition 1.3.64. Consider an IHS manifold $X = (M, g, I)$, denote by $\mathcal{T}_{Ein}$ the space of Einstein metrics on M with unitary volume up to differential isotopy.

There is a period map $\mathcal{P}_{Ein} : \mathcal{T}_{Ein} \to \mathrm{Gr}^+(3, \mathbf{L}_X \otimes \mathbb{R})$ that sends (M, h) to the positive-definite real 3-space $P = \langle \sigma_X + \overline{\sigma_X}, i(\sigma_X - \overline{\sigma_X}), \omega_X \rangle$, where $X = (M, h, J)$ and J is any complex structure for which h is Kähler. The map $\mathcal{P}_{Ein}$ is a local diffeomorphism, giving to $\mathcal{T}_{Ein}$ a differentiable structure.

According to [Loo21], there is a commutative diagram with respective vertical period maps

$$\begin{array}{ccccc} \mathcal{T} & \longleftarrow & \mathcal{T}_{HK} & \longrightarrow & \mathcal{T}_{Ein} \\ {\scriptstyle \mathcal{P}}\downarrow & & {\scriptstyle \mathcal{P}_{HK}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_{Ein}} \\ \Omega_{\mathbf{L}} & \longleftarrow & \widetilde{\Omega}_{\mathbf{L}} & \longrightarrow & \mathrm{Gr}^+(3, \mathbf{L}_X \otimes \mathbb{R}) \end{array}$$

where the horizontal maps to the right simply forget the choice of a complex structure and hence are locally trivial 2-sphere bundles, while the horizontal maps to the left forget the choice of a ray in the Kähler cone and consist of locally trivial bundles with contractible fibers.

We now give the definition of mapping class group, state some results about its action on the set of connected components of the Teichmüller space and relate its action in cohomology to the monodromy group. Since all the fibration between the different Teichmüller spaces have connected fibers, the sets of connected components coincide and the results hold for any of them.

Definition 1.3.65. Let $\mathrm{Diff}^+(X)$ be the group of orientation-preserving diffeomorphisms. The *mapping class group* of X is the group $\mathrm{Mod}(X) := \mathrm{Diff}^+(X)/\mathrm{Diff}^+(X)_0 = \pi_0(\mathrm{Diff}^+(X))$ where $\mathrm{Diff}^+(X)_0$ denotes the connected component of $\mathrm{Diff}^+(X)$ containing the identity.

By definition, the group $\mathrm{Mod}(X)$ has a well-defined action on the Teichmüller space via pull-back.

Definition 1.3.66. The *Torelli group* $\mathrm{T}(X)$ is the kernel of the representation map $\rho : \mathrm{Mod}(X) \to \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$.

Theorem 1.3.67. *Consider the action of* $\mathrm{T}(X)$ *on* $\mathcal{T}$. *Then*

1. *An element of* $\mathrm{T}(X)$ *fixing a point in* $\mathcal{T}$ *acts trivially on its connected component.*

2. *The group* $\mathrm{T}(X)$ *acts on* $\pi_0(\mathcal{T})$ *with finitely many orbits and every connected component has finite stabilizer.*

Proof. [Ver20, Theorem 3.1]. □

Following the discussion in [Ver13, Section 1.2], the moduli space of marked IHS manifolds can be reobtained as

$$\mathcal{M}_{\mathbf{L}} = \mathcal{T} / \mathrm{T}(X)$$

where the quotient simply identifies some of the connected components of $\mathcal{T}$.

Proposition 1.3.68. *Let* $\mathcal{C}$ *be a connected component of the Teichmüller space, then* $\rho(\mathrm{Mod}(X)_{\mathcal{C}}) = \mathrm{Mon}^2(X)$ *where* $\mathrm{Mod}(X)_{\mathcal{C}}$ *is the* $\mathrm{Mod}(X)$*-stabilized of the component* $\mathcal{C}$*.*

Proof. [Ver13, Theorem 7.2]. □

We say that a linear form $\delta \in \mathbf{L}^\vee$ is *negative* if its kernel has signature $(3, b_2(X) - 4)$, or equivalently if its image via the embedding $\mathbf{L}^\vee \subset \mathbf{L}_{\mathbb{Q}}$ has negative square. If $\mathcal{C}$ is a connected component of the Teichmüller space, let $\Delta_{\mathcal{C}} \subset \mathbf{L}^\vee$ be the set of indivisible negative forms which are represented by an irreducible rational curve for an hyper-Kähler metric belonging to $\mathcal{C}$.

There is the useful description of the image of the period map of Einstein metrics:

Proposition 1.3.69. *Consider* X *in a connected component* $\mathcal{C}$ *of* $\mathcal{T}_{Ein}$*, then the period map* $\mathcal{P}_{Ein}$ *maps* $\mathcal{C}$ *diffeomorphically onto*

$$\mathrm{Gr}^+(3, \mathbf{L}_{\mathbb{R}})_{\Delta_{\mathcal{C}}} = \mathrm{Gr}^+(3, \mathbf{L}_{\mathbb{R}}) \setminus \bigcup_{\delta \in \Delta_{\mathcal{C}}} \mathrm{Gr}^+(3, \delta^\perp \otimes \mathbb{R}),$$

in particular $\mathcal{C}$ *is simply connected.*

Proof. [Loo21, Corollary 4.4]. □

By Theorem 1.3.48 and the previous results, it is equivalent to consider $\delta \in \Delta_{\mathcal{C}}$ or $\delta \in \Delta(X)$.

1.3.6 Birationalities and automorphisms

Let $\mathrm{Aut}(X)$ be the group of automorphisms of X and $\mathrm{Bir}(X)$ the group of bimeromorphic maps from X to itself, clearly $\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$. We will sometimes refer to elements in $\mathrm{Bir}(X)$ as birationalities.

It is a general fact that for a compact complex manifold

$$\dim(\mathrm{Aut}(X)) = h^0(TX)$$

so that for an IHS manifold $h^0(TX) = h^{0,1}(X) = 0$ and hence $\mathrm{Aut}(X)$ is finite.

There is a well-defined map

$$\nu : \mathrm{Bir}(X) \to \mathrm{O}(\mathrm{H}^2(X, \mathbb{Z})),$$

a remarkable fact is that by [HT10, Theorem 2.1] $\mathrm{Ker}(\nu)$ is a deformation invariant, which was computed for the known deformation classes. Moreover, $\mathrm{Ker}(\nu) \subset \mathrm{Aut}(X)$.

Example 1.3.70. Let X be an IHS manifold of $\mathrm{K3}^{[n]}$ type (including $n = 1$, i.e. K3 surfaces), then $\mathrm{Ker}(\nu) = \{\mathrm{id}\}$ by [Bea83, Lemma 3].

Example 1.3.71. Let X be an IHS manifold of Kum_n type, then $\mathrm{Ker}(\nu) \cong (\mathbb{Z}/n\mathbb{Z})^4 \rtimes \mathbb{Z}/2\mathbb{Z}$ by [BNWS11, Corollary 3.3]. If $X = \mathrm{Kum}_n(A)$ for an Abelian surface A then we have $\mathrm{Ker}(\nu) = A[n] \rtimes \{\pm \mathrm{id}\}$ where $A[n]$ is the group of n-torsion points of A.

Example 1.3.72. Let X be an IHS manifold of OG10 type, then $\mathrm{Ker}(\nu) = \{\mathrm{id}\}$ by [MW17, Theorem 2.1].

Example 1.3.73. Let X be an IHS manifold of OG6 type, then $\mathrm{Ker}(\nu) \cong (\mathbb{Z}/2\mathbb{Z})^8$ by [MW17, Theorem 4.2]. When X is the resolution of a moduli space of sheaves on an Abelian surface A we have $\mathrm{Ker}(\nu) = A[2] \times A^\vee[2]$ where $A^\vee$ is the dual surface.

Recall that we defined $\mathrm{Mon}^2_{Bir} = \nu(\mathrm{Bir}(X))$ in subsection 1.3.4. The following properties are consequence of Theorem 1.3.37 and [Huy03, Proposition 9.1]:

Proposition 1.3.74. *Let X be an IHS manifold, then:*

1. $\nu(\mathrm{Bir}(X)) = \mathrm{Mon}^2_{Bir}(X) \subset \mathrm{Mon}^2_{Hdg}(X)$
2. $\nu(\mathrm{Aut}(X)) = \{g \in \mathrm{Mon}^2_{Hdg}(X) | g(\mathcal{K}_X) = \mathcal{K}_X\}$
3. $\nu^{-1}(\nu(\mathrm{Aut}(X))) = \mathrm{Aut}(X)$
4. $\mathrm{Ker}(\nu) \subset \mathrm{Aut}(X)$
5. $\mathrm{Ker}(\nu)$ *is finite.*

If S is a K3 surface then from subsection 1.3.4 we have $\mathrm{Aut}(S) = \mathrm{Bir}(S)$, but there are examples of IHS manifolds X for which the inclusion $\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$ is strict. Nonetheless, we have the following

Proposition 1.3.75. *Let X be a very general IHS manifold, then* $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.

Proof. Corollary 1.3.49. □

It is worth to notice that if X is projective, then by [BS12, Theorem 2] $\mathrm{Bir}(X)$ is finitely generated. We mention that there are examples of K3 surfaces with automorphisms of infinite order (see [SI77]).

Definition 1.3.76. Let $G \subset \mathrm{Aut}(X)$ a subgroup and fix a marking $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathbf{L}$. The *invariant lattice* $\mathbf{L}_G \subset \mathbf{L}$ is the invariant lattice for the induced action of G, the *coinvariant lattice* is $\mathbf{L}_G = (\mathbf{L}^G)^\perp$.

If $G = \langle f \rangle$ we will sometimes write $\mathbf{L}^f$ and $\mathbf{L}_f$ instead of $\mathbf{L}^G$ and $\mathbf{L}_G$. Consider the morphisms

$$\mathrm{Bir}(X) \to \mathrm{Aut}(\mathrm{H}^0(X,\Omega_X^2)) \cong \mathbb{C}^*$$

$$\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{H}^0(X,\Omega_X^2)) \cong \mathbb{C}^*$$

and denote the kernels respectively by $\mathrm{Bir}^s(X)$ and $\mathrm{Aut}^s(X)$, given by bimeromorphic maps and automorphisms that preserve the symplectic form σ_X.

Definition 1.3.77. Biremorphic maps in $\mathrm{Bir}^s(X)$ and automorphisms in $\mathrm{Aut}^s(X)$ are called *symplectic*, elements that are not symplectic are called *non-symplectic*. Groups $G \subset \mathrm{Bir}^s(X)$ and $G \subset \mathrm{Aut}^s(X)$ are groups of symplectic bimeromorphic maps and symplectic automorphisms of X.

Notice that if $G \subset \mathrm{Bir}(X)$ is a finite group, then there is a short exact sequence

$$0 \to G^s \to G \to \mu_m \to 0$$

where $G^s \subset \mathrm{Bir}^s(X)$ is the symplectic part of the group and μ_m is the cyclic group of order m. This in particular applies when $G \subset \mathrm{Aut}(X)$.

Remark 3. *It is a striking fact that if there is a non-symplectic $f \in \mathrm{Aut}(X)$, then by [Bea83, Proposition 6] X is a projective IHS manifold.*

It is easy to prove the following

Proposition 1.3.78. *Let $G \subset \mathrm{Bir}(X)$ be non-trivial a finite group, then:*

- *If $G \subset \mathrm{Bir}^s(X)$, then $\mathrm{T}(X) \subset \mathbf{L}^G$ and $\mathbf{L}_G \subset \mathrm{NS}(X)$. Moreover, $\mathbf{L}_G$ is negative-definite and does not contain prime exceptional divisors. If in particular $G \subset \mathrm{Aut}^s(X)$ then $\mathbf{L}_G$ does not contain wall divisors.*

- *If $G \cap \mathrm{Bir}^s(X) = \{\mathrm{id}\}$ contains a non-symplectic element, then $\mathbf{L}^G \subset \mathrm{NS}(X)$ and $\mathrm{T}(X) \subset \mathbf{L}_G$. Moreover, $\mathbf{L}^G$ is hyperbolic.*

Remark 4. *If X is very general with $G \subset \mathrm{Aut}(X)$ finite, then by [Nik76, §3] one has $\mathbf{L}^G = \mathrm{T}(X)$ if $G \subset \mathrm{Aut}^s(X)$ and $\mathbf{L}^G = \mathrm{NS}(X)$ if $G \cap \mathrm{Aut}^s(X) = \{\mathrm{id}\}$.*

The following result is due to Nikulin.

Proposition 1.3.79. *Let $f \in \operatorname{Aut}(X)$ be such that $f^*\sigma_X = \xi_m \sigma_X$ with ξ_m a primitive m-th root of unity. Then, $\varphi(m)$ divides $\operatorname{rk}(\mathrm{T}(X))$, where φ is the Euler function. In particular, $\varphi(m) \leq b_2(X) - \operatorname{rk}(\mathrm{NS}(X))$.*

This can be applied and made specific for the known deformation types, one has $\varphi(m) \leq b_2(X) - 1$ and this gives a bound on m.

Proposition 1.3.80. *For a projective IHS manifold, the map*

$$\begin{aligned} \Psi\colon \operatorname{Aut}(X) &\to O(\operatorname{Pic}(X)) \\ f &\longmapsto f^* \end{aligned}$$

has a finite kernel.

Proof. [Deb22, Proposition 4.1]. □

From this, we have the following description of finite groups of automorphisms.

Corollary 1.3.81. *Let X be a projective IHS manifold. A group of automorphisms $G \subset \operatorname{Aut}(X)$ is finite if and only if it fixes an ample class on X.*

Proof. If G is finite, then let $H \in \operatorname{Pic}(X) = \mathrm{NS}(X)$ be any ample class. The class

$$\eta = \sum_{g \in G} g^* H$$

is invariant under G and it is still ample.

Now assume that G fixes an ample class H. Let Ψ be as in Proposition 1.3.80, and put $\Psi_G = \Psi_{|G}$, so that $\ker \Psi_G$ is finite. Define the quotient $\widetilde{G} = G/\ker \Psi_G$ and note that $\widetilde{G}$ acts faithfully on $\mathrm{NS}(X)$. Set $\mathbf{N} = H^{\perp \mathrm{NS}(X)}$ and observe that $\mathbf{N}$ is negative definite since $\mathrm{NS}(X)$ is of index $(1, k)$ for some integer k and $H^2 > 0$ as H is ample, then $\mathbf{N}$ is negative definite. This implies that the group $\mathrm{O}(\mathbf{N})$ is finite, moreover any isometry $f \in O(\mathrm{NS}(X))$ which fixes H is uniquely determined by its restriction to $\mathbf{N}$. In conclusion $|\widetilde{G}| \leq |\,\mathrm{O}(\mathbf{N})|$ which means that $\widetilde{G}$ is finite and also G is. □

We denote by $\operatorname{Aut}_H(X)$ the subgroup of automorphisms that fix the ample class H, and denote by $\operatorname{Aut}^s_H(X)$ the subgroup of symplectic automorphisms that fix the ample class H.

In the following we give a brief overview of the research towards a classifications of finite groups of automorphisms of IHS manifolds. The techniques for doing that are heavily based on lattice theory.

Lemma 1.3.82. *Let X be an IHS manifold deformation equivalent to one of the known examples, then $G \subset \mathrm{Aut}^s(X)$ induces the trivial action on the discriminant group $\mathrm{A}_{\mathrm{H}^2(X,\mathbb{Z})}$.*

Proof. For $\mathrm{K3}^{[n]}$ type we refer to [Mon16b], for Kum_n type we refer to [MTW18, Lemma 5.1], for OG6 type we refer to [GOV23] and for OG10 type to [GGOV22]. □

Definition 1.3.83. The *Leech lattice* L is the unique negative-definite lattice of rank 24 that does not contain any element of square -2.

Definition 1.3.84. Let $Co_1 := \mathrm{O}^+(\mathrm{L})$ where L is the Leech lattice, it is a simple group usually called *Conway's first sporadic group.*

Denote by $W(\mathbf{E}_8)$ the Weyl group of the diagram $\mathbf{E}_8$.

Theorem 1.3.85. *Let $G \subset \mathrm{Mon}^2(X) \cap \nu(\mathrm{Aut}^s(X))$ be a finite group, then:*

1. *If X is of $K3^{[n]}$ type, then G is isomorphic to a subgroup of Co_1.*
2. *If X is of* Kum_n *type, then G is isomorphic to a subgroup of $W(\mathbf{E}_8)$.*
3. *If X is of* OG6 *type or* OG10 *type, then $G = \{\mathrm{id}\}$.*

Proof. The case of manifolds of $\mathrm{K3}^{[n]}$ type is treated in [Mon16b], the case of Kum_n type in [Mon16b], for manifolds of OG6 in [GOV23] and for manifolds of OG10 type in [GGOV22]. □

Notice that for manifolds of OG6 type there are symplectic automorphisms of finite order, but they all act trivially in cohomology.

Chapter 2

Constructions of IHS manifolds

In this chapter we provide a description of some constructions of IHS manifolds. The first section is dedicated to double EPW-sectics and double EPW-cubes, these examples provide locally complete families in the moduli spaces of manifolds of $K3^{[2]}$ type and $K3^{[3]}$ type respectively. We recall the description of the automorphism group of EPW-sextics and we prove a similar description for the automorphism group of EPW-cubes. The second section is dedicated to moduli spaces of semistable sheaves on symplectic surfaces, here a precise definition of manifolds of OG6 type and OG10 type is given and an overview of the properties of these spaces is given. The third section is dedicated to LVS manifolds, a construction of manifolds of OG10 type associated to cubic fourfolds, together with a brief survey on the Hodge theory of cubic fourfolds.

2.1 Double EPW-sextics and double EPW-cubes

In this section we recall two constructions of IHS manifolds, one of $K3^{[2]}$ type and the other of $K3^{[3]}$ type. They are repectively called double EPW-sextics and double EPW-cubes, they form (locally) complete families in the moduli spaces of IHS manifold of repectively manifolds of $K3^{[2]}$ and $K3^{[3]}$ type, meaning that the families are open sets in the respective moduli spaces. The constructions are quite related to each other and both families are parametrized by a Lagrangian vector space. Originally, Eisenbud-Walter-Popescu introduced the EPW-sextics then O'Grady constructed their double cover and showed that it is an IHS manifold of $K3^{[2]}$ type. With a similar construction, Iliev-Kapustka-Kapustka-Ranestad defined EPW-cubes as submanifolds of a Grassmannian and showed that their double cover is an IHS manifold of $K3^{[3]}$ type. We recall the construction, the basic properties and then discuss their automorphisms.

2.1.1 Definitions and basic properties

Fix a complex 6-dimensional vector space V_6 and a volume form $vol : \bigwedge^6 V_6 \cong \mathbb{C}$, this gives a symplectic form $\eta : \bigwedge^3 V_6 \times \bigwedge^3 V_6 \to \mathbb{C}$ where $\eta(\alpha,\beta) = vol(\alpha \wedge \beta)$ for $\alpha, \beta \in \bigwedge^3 V_6$.

Definition 2.1.1. Let $\mathbb{LG}(\bigwedge^3 V_6)$ be the Grassmannian parametrizing Lagrangian subspaces, i.e. maximal isotropic subspaces A of $\bigwedge^3 V_6$.

Set

$$\Sigma := \{A \in \mathbb{LG}(\bigwedge^3 V_6) | \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) \neq \emptyset\}$$

where $\mathrm{Gr}(3, V_6)$ sits in $\mathbb{P}(\bigwedge^3 V_6)$ via the Plücker embedding.

Definition 2.1.2. We say that $A \in \mathbb{LG}(\bigwedge^3 V_6)$ *has no decomposable vectors* if there is no $x \wedge y \wedge z \in A$ for $x, y, z \in V_6$, equivalently $A \notin \Sigma$.

By [O'G12, Proposition 2.1] the locus Σ is a divisor in $\mathbb{LG}(\bigwedge^3 V_6)$. Given a scheme S, we also consider Lagrangian subbundles $\mathscr{A}$ of $\bigwedge^3 V_6 \otimes \mathcal{O}_S$, bundles with the feature that the fiber at any point is a Lagrangian subspace. These are characterized by fitting in an exact sequence

$$0 \to \mathscr{A} \to \bigwedge^3 V_6 \otimes \mathcal{O}_S \to \mathscr{A}^\vee \to 0$$

where $\mathscr{A}^\vee \subset \bigwedge^3 V_6 \otimes \mathcal{O}_S$ is the image of $\mathscr{A}$ via the isomorphism

$$\eta : \bigwedge^3 V_6 \otimes \mathcal{O}_S \xrightarrow{\cong} \bigwedge^3 V_6^\vee \otimes \mathcal{O}_S.$$

Consider two Lagrangian subbundles $\mathscr{A}_1, \mathscr{A}_2 \subset \bigwedge^3 V_6 \otimes \mathcal{O}_S$. Symilarly to above, there is a map $\lambda_{\mathscr{A}_1,\mathscr{A}_2^\vee} : \mathscr{A}_1 \to \mathscr{A}_2^\vee$ and we let

$$S_k = S_k(\mathscr{A}_1, \mathscr{A}_2) \subset S$$

be the corank-k degeneracy locus, $S_k^0 := S_k \setminus S_{k+1}$. Define

$$\mathscr{C}_k = \mathscr{C}_k(\mathscr{A}_1, \mathscr{A}_2) := \mathrm{Coker}(\lambda_{\mathscr{A}_1,\mathscr{A}_2})|_{S_k(\mathscr{A}_1,\mathscr{A}_2)}$$

the restriction of the cokernel of $\lambda_{\mathscr{A}_1,\mathscr{A}_2}$ to the various degeneraci loci and consider

$$\mathscr{R}_k := (\bigwedge^k \mathscr{C}_k)^{\vee\vee}, \tag{2.1}$$

which is a rank 1 reflexive sheaf S_k.

Theorem 2.1.3. *Suppose that S_k is a normal variety, S_{k+1} has codimension at least 2 in S_k, and suppose that S_k^0 is dense in S_k. Then for any line bundle $\mathscr{M}$ on S_k with*

$$(\det(\mathscr{A}_1) \otimes \det(\mathscr{A}_2))|_{S_k} \cong \mathscr{M}(2)$$

there is a double cover

$$f : \widetilde{S}_k = \mathrm{Spec}_{S_k}(\mathcal{O}_{S_k} \oplus (\mathscr{M} \otimes \mathscr{R}_k)) \to S_k$$

such that:

1. *There is an isomorphism $f_*\mathcal{O}_{\widetilde{S}_k} \cong \mathcal{O}_{S_k} \oplus (\mathscr{M} \otimes \mathscr{R}_k)$.*
2. *The morphism f is étale over S_k^0.*

If moreover every element of $\mathrm{H}^0(S_k, \mathcal{O}^*_{S_k})$ is a square, then the cover is unique up to isomorphism.

Proof. [DK20a, Theorem 4.2]. □

The double covers of EPW-sextics and EPW-cubes can be realized as particular covers of Theorem 2.1.3.

Fix a Lagrangian subspace $A \in \mathbb{LG}(\bigwedge^3 V_6)$ and let $F := \bigwedge^3 \mathcal{T}_{\mathbb{P}(V_6)}(-3)$ where $\mathcal{T}_{\mathbb{P}(V_6)}$ is the tangent bundle of the projective space. As the fiber over $[v] \in \mathbb{P}(V_6)$ of the bundle F is given by $F_{[v]} = v \wedge \bigwedge^2 V_6$, it is clear that the bundle is Lagrangian.

Definition 2.1.4. For $k \geq 0$ define

$$Y_A[k] := S_k(A \otimes \mathcal{O}_{\mathbb{P}(V_6)}, F),$$

$Y_A := Y_A[1]$ is called the *EPW-sextic* associated to A.

Any hypersurface which is projectively equivalent to the EPW-sectic associated to a Lagrangian space is called EPW-sextic. One has explicitly

$$Y_A[k] = \{[v] \in \mathbb{P}(V_6) | \dim(A \cap F_{[v]}) \geq k\}$$

and $Y_A = \det(A \otimes \mathcal{O}_{\mathbb{P}(V_6)} \to F^\vee))$.

Set

$$\Delta := \{A \in \mathbb{LG}(\bigwedge^3 V_6) | Y_A[3] \neq \emptyset\},$$

by [O'G13, Proposition 2.2] it is a divisor that does not coincide with Σ.

Theorem 2.1.5. *Suppose A has no decomposable vectors. Then*

1. *The hypersurface Y_A is a normal integral sextic.*
2. *The singular locus of Y_A coincides with $Y_A[2]$, which is an integral normal Cohen-Macaulay surface of degree 40.*

3. *The singular locus of* $Y_A[2]$ *consists of* $Y_A[3]$ *which is finite and smooth, moreover* $Y_A[4] = \emptyset$.

4. *The locus* Δ *is a divisor in* $\mathbb{LG}(\bigwedge^3 V_6)$, *hence for* A *general* $Y_A[3] = \emptyset$.

Proof. The result is a summary of many results in [O'G13, O'G12, O'G15, O'G16], they are also gathered in [DK20a, Theorem 5.1]. □

In this case $\det(A \otimes \mathcal{O}_{\mathbb{P}(V_6)})$ is trivial and $\det(F) \cong \mathcal{O}_{\mathbb{P}(V_6)}(-6)$, so that the line bundle $\det(A \otimes \mathcal{O}_{\mathbb{P}(V_6)}) \otimes \det(F) \cong \mathcal{O}_{\mathbb{P}(V_6)}(-6)$ has a unique square root $\mathcal{O}_{\mathbb{P}(V_6)}(-3)$ and Theorem 2.1.3 applies.

Theorem 2.1.6. *Suppose* A *has no decomposable vectors. Then*

1. *There is a unique double cover* $\pi_A : \widetilde{Y}_A \to Y_A$ *branched along* $Y_A[2]$ *such that*
$$\pi_{A*}\mathcal{O}_{\widetilde{Y}_A} \cong \mathcal{O}_{Y_A} \oplus \mathscr{R}_1(-3).$$
The scheme $\widetilde{Y}_A$ *is integral and normal, smooth out of* $\pi_A^{-1}(Y_A[3])$.

2. *There is a unique double cover* $\pi_A^2 : \widetilde{Y}_A[2] \to Y_A[2]$ *branched along* $Y_A[3]$ *such that*
$$\pi^2_{A*}\mathcal{O}_{\widetilde{Y}_A[2]} \cong \mathcal{O}_{Y_A[2]} \oplus \mathscr{R}_2(-3).$$
The scheme $\widetilde{Y}_A$ *is integral and normal, smooth out of* $(\pi_A^2)^{-1}(Y_A[3])$ *and with ordinary double points along* $(\pi_A^2)^{-1}(Y_A[3])$. *Moreover,* $\mathscr{R}_2 \cong \omega_{Y_A[2]}$.

Proof. [O'G13] and [DK20a, Theorem 5.2]. □

The double cover $\widetilde{Y}_A$ is called *double EPW-sextic*, it carries a canonical polarization $H = \pi_A^*\mathcal{O}_{Y_A}(1)$ and the image of the morphism $\widetilde{Y}_A \to \mathbb{P}(\mathrm{H}^0(\widetilde{Y}_A, H)^\vee)$ is isomorphic to Y_A.

Theorem 2.1.7. *Suppose* $A \in \mathbb{LG}(\bigwedge^3 V_6) \setminus (\Sigma \cup \Delta)$, *then* $\widetilde{Y}_A$ *is a polarized IHS manifold of* $K3^{[2]}$ *type with a polarization of degree* 2 *and divisibility* 1.

Proof. [O'G13, Theorem 4.25]. □

Denote by $\mathcal{U}$ the tautological bundle of $\mathrm{Gr}(3, V_6)$ and set $T := V_6 \wedge \bigwedge^2 \mathcal{U}$, the fiber at a point $U \in \mathrm{Gr}(3, V_6)$ is given by $T_U = V_6 \wedge \bigwedge^2 U$, so that T is a Lagrangian bundle.

Definition 2.1.8. For $k \geq 0$ define
$$Z_A[k] := S_k(A \otimes \mathcal{O}_{\mathrm{Gr}(3,V_6)}, T),$$
$Z_A := Z_A[2]$ is called the *EPW-cube* associated to A.

More explicitely,

$$Z_A[k] = \{U \in \mathrm{Gr}(3, V_6) | \dim(A \cap T_U) \geq k\}$$

via the inclusion $T \hookrightarrow V_6 \otimes \mathcal{O}_{\mathrm{Gr}(3,V_6)}$. Every sixfold in $\mathrm{Gr}(3, V_6)$ which is projectively equivalent to the EPW-cube associated to a Lagrangian space is still called EPW-cube. Set

$$\Gamma := \{A \in \mathbb{LG}(\bigwedge^3 V_6) | Z_A[4] \neq \emptyset\},$$

by [IKKR19, Lemma 3.6] it is a divisor and by [IKKR19, Lemma 3.7] it has no common components with Δ, so that the three divisors Σ, Γ, Δ are different.

Theorem 2.1.9. *Suppose A has no decomposable vectors. Then*

1. *The scheme Z_A is an integral normal Cohen-Macaulay sixfold of degree* 480*, its singular locus coincides with $Z_A[3]$.*
2. *The scheme $Z_A[3]$ is an integral normal Cohen-Macaulay threefold of degree* 4944*, its singular locus coincides with $Z_A[4]$, which is finite and smooth, moreover $Z_A[5] = \emptyset$.*
3. *The locus Γ is a divisor in $\mathbb{LG}(\bigwedge^3 V_6)$, hence for A general $Z_A[4] = \emptyset$.*

Proof. [IKKR19, Propostition 2.6,Corollary 2.10]. □

In this case $\det(A \otimes \mathcal{O}_{\mathrm{Gr}(3,V_6)})$ is trivial, while there is an exact sequence

$$0 \to \bigwedge^3 \mathcal{U} \to V_6 \wedge \bigwedge^2 \mathcal{U} \to (V_6/\mathcal{U}) \otimes \bigwedge^2 \mathcal{U} \to 0$$

so that $\det(T) \cong \det(\bigwedge^3 \mathcal{U}) \otimes \det((V_6/\mathcal{U}) \otimes \bigwedge^2 \mathcal{U}) \cong \mathcal{O}_{\mathrm{Gr}(3,V_6)}(-4)$. The line bundle $\det(A \otimes \mathcal{O}_{\mathrm{Gr}(3,V_6)}) \otimes \det(T) \cong \mathcal{O}_{\mathrm{Gr}(3,V_6)}(-4)$ has a unique square root $\mathcal{O}_{\mathrm{Gr}(3,V_6)}(-2)$ and Theorem 2.1.3 applies again.

Theorem 2.1.10. *Suppose A has no decomposable vectors. Then there is a unique double cover $\pi_A : \widetilde{Z}_A \to Z_A$ branched along $Z_A[3]$ such that*

$$\pi_{A*}\mathcal{O}_{Z_A} \cong \mathcal{O}_{Z_A} \oplus \mathscr{R}_2(-2).$$

The scheme $\widetilde{Z}_A$ is integral and normal, smooth out of $\pi_A^{-1}(Z_A[3])$.

Proof. It follows Theorem 2.1.3. □

By [DK20a, Lemma 5.8] the double cover $\pi_A : \widetilde{Z}_A \to Z_A$ coincides with the one constructed in [IKKR19] when $A \notin \Gamma$, and $\widetilde{Z}_A$ is called *double EPW-cube.* The double EPW-cube carries a canonical polarization $h = \pi_A^*\mathcal{O}_{Z_A}(1)$ and the image of the morphism $\widetilde{Z}_A \to \mathbb{P}(\mathrm{H}^0(\widetilde{Z}_A, h)^\vee)$ is isomorphic to Z_A.

Theorem 2.1.11. *Suppose $A \in \mathbb{LG}(\bigwedge^3 V_6) \setminus (\Sigma \cup \Gamma)$, then $\widetilde{Z}_A$ is a polarized IHS manifold of $K3^{[3]}$ type with polarization of degree* 4 *and divisibility* 2*.*

Proof. [IKKR19, Theorem 1.1]. □

2.1.2 Automorphisms of double EPW-sextics

From now on, we will suppose $A \notin \Sigma$. For such Lagrangian subspaces there is a nice description of the automorphisms of the associated EPW-sextic and its double cover.

The automorphisms of a EPW-sextic Y_A are essentially linear automorphisms of the Lagrangian space:

$$\mathrm{Aut}(Y_A) = \{g \in \mathrm{PGL}(V_6) | (\bigwedge^3 g)(A) = A\} =: \mathrm{PGL}(V_6)_A \tag{2.2}$$

and this is a finite group by [DK18, Proposition B.9].

Every automorphism of Y_A induces an automorphism of the double cover $\widetilde{Y}_A$ that fixes the polarization $H = \pi_{A*}\mathcal{O}_{Y_A}(1)$ (proof of [DK18, Proposition B.8(b)]), conversely any automorphism of $\widetilde{Y}_A$ that fixes H induces an isomorphism $\mathbb{P}(\mathrm{H}^0(\widetilde{Y}_A, H)^\vee) \cong \mathbb{P}(V_6)$ hence descends to an automorphism of Y_A. Denote by $\mathrm{Aut}_H(\widetilde{Y}_A)$ the group of automorphisms that fix the class H and by ι the covering involution of π_A. The discussion above gives a central extension

$$1 \to \langle \iota \rangle \to \mathrm{Aut}_H(\widetilde{Y}_A) \to \mathrm{Aut}(Y_A) \to 1, \tag{2.3}$$

moreover denoting by $\mathrm{Aut}^s_H(\widetilde{Y}_A)$ the subgroup of $\mathrm{Aut}_H(\widetilde{Y}_A)$ consisting of symplectic automorphisms, one gets an extension

$$1 \to \mathrm{Aut}^s_H(\widetilde{Y}_A) \to \mathrm{Aut}_H(\widetilde{Y}_A) \to \mu_r \to 1 \tag{2.4}$$

with μ_r a finite group of order r. Note that the image of ι in μ_r is given by -1.

Consider the embedding $\mathrm{Aut}(Y_A) \hookrightarrow \mathrm{PGL}(V_6)$ and let G be the inverse image of $\mathrm{Aut}(Y_A)$ via the canonical map $\mathrm{SL}(V_6) \to \mathrm{PGL}(V_6)$. It follows that G is an extension of $\mathrm{Aut}(Y_A)$ by the cyclic group $\langle \gamma \rangle$ with $\gamma^6 = \mathrm{id}$, so we have an induced representation of G on $\bigwedge^3 V_6$ and this factors through a representation of $\widetilde{\mathrm{Aut}}(Y_A) := G/\langle \gamma^2 \rangle$. Since A is preserved by this action, we have a morphism of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle \gamma^3 \rangle & \longrightarrow & \widetilde{\mathrm{Aut}}(Y_A) & \longrightarrow & \mathrm{Aut}(Y_A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(A) & \longrightarrow & \mathrm{PGL}(A) & \longrightarrow & 1 \end{array} \tag{2.5}$$

and by [DM22, Lemma A.1] the vertical maps are injective.

Proposition 2.1.12 (Kuznetsov)**.** *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Then the extensions (2.3) and (2.4) are trivial and $r = 2$. In particular there is an isomorphism*

$$\mathrm{Aut}_H(\widetilde{Y}_A) \cong \mathrm{Aut}(Y_A) \times \langle \iota \rangle$$

which splits (2.3) and the factor $\mathrm{Aut}(Y_A)$ *corresponds to the subgroup* $\mathrm{Aut}^s_H(\widetilde{Y}_A)$.

Proof. [DM22, Proposition A.2]. □

Recall that since A has no decomposable vectors there is a canonical connected double covering

$$\widetilde{Y}_A[2] \to Y_A[2],$$

moreover there is a morphism $\mathrm{Aut}(Y_A) \to \mathrm{Aut}(Y_A[2])$ and since as Y_A is not contained in any hyperplane, the morphism is injective. As [DM22, Proposition A.6 (Kuznetsov)] shows, the group of lifts of automorphisms of Y_A to automorphisms of $\widetilde{Y}_A[2]$ is isomorphic to $\widetilde{\mathrm{Aut}}(Y_A)$, hence there is an injection $\widetilde{\mathrm{Aut}}(Y_A) \hookrightarrow \mathrm{Aut}(\widetilde{Y}_A[2])$.
Recall that the analytic representation of a finite group G acting on an Abelian variety X is the composition

$$G \to \mathrm{End}_{\mathbb{Q}}(X) \to \mathrm{End}_{\mathbb{C}}(T_{X,0}). \tag{2.6}$$

We recall the useful

Proposition 2.1.13. *Suppose the surface $Y_A[2]$ is smooth. The restriction of the analytic representation of $\mathrm{Aut}(\widetilde{Y}_A[2])$ on $\mathrm{Alb}(\widetilde{Y}_A[2])$ to the subgroup $\widetilde{\mathrm{Aut}}(Y_A)$ is the injective middle vertical map in the diagram (2.5).*

Proof. [DM22, Proposition A.7]. □

2.1.3 Automorphisms of double EPW-cubes

Consider a double EPW-sextic $\widetilde{Y}_A$ with polarization H_A and a double EPW-cube $\widetilde{Z}_A$ with polarization h_A. Set

$$\Lambda_{\widetilde{Y}_A} := H_A^{\perp} \subset \mathrm{H}^2(\widetilde{Y}_A, \mathbb{Z}) \cong \mathbf{L}_{K3^{[2]}}$$

and

$$\Lambda_{\widetilde{Z}_A} := h_A^{\perp} \subset \mathrm{H}^2(\widetilde{Z}_A, \mathbb{Z}) \cong \mathbf{L}_{K3^{[3]}},$$

then define

$$\Lambda := \mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-1)^{\oplus 2} \oplus [-2]^{\oplus 2}$$

and observe that there are the lattice isometries $\Lambda \cong \Lambda_{\widetilde{Y}_A} \cong \Lambda_{\widetilde{Z}_A}$. If the Lagrangian subspace is general enough, then by [KKM22, Proposition 1.2] there is also an isometry of Hodge structures between $\Lambda_{\widetilde{Y}_A}$ and $\Lambda_{\widetilde{Z}_A}$.

We recall that there are moduli spaces of double EPW-sextics and double EPW-cubes

$$\mathcal{M}_{sex} := (\mathbb{LG}(\bigwedge^3 V_6) \setminus (\Sigma \cup \Delta))// \mathrm{PGL}(V_6),$$

$$\mathcal{M}_{cub} := (\mathbb{LG}(\bigwedge^3 V_6) \setminus (\Sigma \cup \Gamma))// \mathrm{Aut}(\mathrm{Gr}(3, V_6))$$

with respective period domains

$$\Omega_{sex} := \{x \in \Lambda_{\mathbb{C}} | x^2 = 0, x \cdot \overline{x} > 0\} / \operatorname{O}(\Lambda, H),$$
$$\Omega_{cub} := \{x \in \Lambda_{\mathbb{C}} | x^2 = 0, x \cdot \overline{x} > 0\} / \operatorname{O}(\Lambda, h),$$

where $\operatorname{O}(\Lambda, H)$ and $\operatorname{O}(\Lambda, h)$ are the subgroups of $\operatorname{O}(\Lambda)$ obtained by restriction of isometries that respect the inclusions $H^{\perp} = \Lambda \subset \mathbf{L}_{K3^{[2]}}$ and $h^{\perp} = \Lambda \subset \mathbf{L}_{K3^{[3]}}$. There respective period maps

$$\mathcal{P}_{sex} : \mathcal{M}_{sex} \to \Omega_{sex},$$
$$\mathcal{P}_{cub} : \mathcal{M}_{cub} \to \Omega_{cub}$$

associating to the double EPW's their Hodge structures. Notice that $H^2 = 2$ and $\operatorname{div}(H) = 1$, while $h^2 = 4$ and $\operatorname{div}(h) = 2$, so by Lemma 1.1.13 there is only one orbit of elements with those given squares and divisibilities in $\mathbf{L}_{K3^{[2]}}$ and in $\mathbf{L}_{K3^{[3]}}$.

Fix a basis of V_6 and recall there is an element $\delta \in \operatorname{Aut}(\operatorname{Gr}(3, V_6))$ that sends a 3-space to the direct complement determined by sending any 3-vector to its dual with respect to the symplectic form, moreover there is a map that we will call again $\delta \in \operatorname{Aut}(\mathbb{LG}(\bigwedge^3 V_6))$ defined by $\delta(A) = A^{\perp} := \operatorname{Ker}(A^{\vee})$ where $A^{\vee}$ is the dual of A with respect to the symplectic form.

Proposition 2.1.14. *There is a map*

$$p : \mathcal{M}_{sex} \dashrightarrow \mathcal{M}_{cub}$$

which is generically $2:1$*, of degree less or equal than* 2*, which sends* $[\widetilde{Y}_A]$ *to* $[\widetilde{Z}_A]$*. Points with the same image consist of elements* $[\widetilde{Y}_{A_1}], [\widetilde{Y}_{A_2}]$ *with* $A_1, A_2 \in \mathbb{LG}(\bigwedge^3 V_6)$ *in the same* $\langle\delta\rangle \times \operatorname{PGL}(V_6)$*-orbit.*

Proof. The map sends the class of a double EPW-sextic $[\widetilde{Y}_A]$ to the class of the EPW-cube $[\widetilde{Z}_A]$ associated to the same Lagrangian space A. Since we have an inclusion $\operatorname{O}(\Lambda, H) \subset \operatorname{O}(\Lambda, h)$ there is a quotient map $i : \Omega_{sex} \to \Omega_{cub}$ and by [KKM22, Theorem 1.1] the diagram

$$\begin{array}{ccc} \mathcal{M}_{sex} & \overset{p}{\dashrightarrow} & \mathcal{M}_{cub} \\ \downarrow & & \downarrow \\ \Omega_{sex} & \overset{i}{\dashrightarrow} & \Omega_{cub} \end{array}$$

is commutative over an opportune open set. By Lemma 1.1.13 the groups $\operatorname{O}(\Lambda, H)$ and $\operatorname{O}(\Lambda, h)$ can be computed for particular choices of H and h, showing that in this case the index of $\operatorname{O}(\Lambda, H)$ in $\operatorname{O}(\Lambda, h)$ is 2 and hence the degree of the map i is always bounded by 2.

For the last part of the statement, by the proof of [IKKR19, Proposition 5.1] if Z_{A_1}, Z_{A_2} are isomorphic as polarized manifolds then $\exists g \in \operatorname{Aut}(\operatorname{Gr}(3, V_6))$ such that $g(A_1) = A_2$. One concludes using the description $\operatorname{Aut}(\operatorname{Gr}(3, V_6)) \cong \operatorname{PGL}(V_6) \times \langle\delta\rangle$.

□

Observe that by the description of the moduli spaces, the only case when the map is $1:1$ happens when $\delta(A) = A^\perp = A$.

Corollary 2.1.15. *Two Lagrangian spaces* A_1, A_2 *have the same associated EPW-cube* $Z_{A_1} = Z_{A_2} \subset \mathrm{Gr}(3, V_6)$ *if and only if* $A_1 = A_2$ *or* $A_1 = \delta(A_2)$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} \mathbb{LG}(\bigwedge^3 V_6) & \overset{q}{\dashrightarrow} & \mathbb{LG}(\bigwedge^3 V_6)/\langle\delta\rangle \\ \downarrow{\scriptstyle \pi_{sex}} & & \downarrow{\scriptstyle \pi_{cub}} \\ \mathcal{M}_{sex} & \overset{p}{\dashrightarrow} & \mathcal{M}_{cub} \end{array}$$

where the vertical arrows are the quotient maps for the $\mathrm{PGL}(V_6)$-action and the horizontal upper arrow is the quotient by δ. Fibers of the vertical maps are in both cases $\mathrm{PGL}(V_6)$-orbits of some Lagrangian space, hence EPW-cubes associated to different Lagrangian subspaces can only be related by δ since the statement is already known for EPW-sextics. □

Set $\mathrm{Aut}(\mathrm{Gr}(3, V_6))_A := \{g \in \mathrm{Aut}(\mathrm{Gr}(3, V_6)) | g(A) = A\}$, as in the case of EPW-sextics, all the automorphisms are linear:

Corollary 2.1.16. *We have an isomorphism* $\mathrm{Aut}(Z_A) \cong \mathrm{Aut}(\mathrm{Gr}(3, V_6))_A$.

Proof. Clearly $\mathrm{Aut}(\mathrm{Gr}(3, V_6))_A \hookrightarrow \mathrm{Aut}(Z_A)$. Suppose $g \in \mathrm{Aut}(Z_A)$, then g induces an automorphism of $\mathbb{P}(\mathrm{H}^0(Z_A, \mathcal{O}_{Z_A})^\vee) \cong \mathbb{P}(\bigwedge^3 V_6)$ which fixes the locus Z_A and hence the embedding of the Grassmannian by [IKKR19, Lemma 5.2]. In particular, g determines a linear action on $\bigwedge^3 V_6$ up to scalar multiplication and using Corollary 2.1.15 we have either $Y_{g(A)} = Y_A$ or $Y_{g(A)} = Y_{\delta(A)}$. Moreover, we know from [O'G16, Proposition 1.2.1] that this implies either $g(A) = A$ or $g(A) = \delta(A)$, hence $g \in \mathrm{Aut}(\mathrm{Gr}(3, V_6))_A$. □

In the following we generalize the proof of Proposition 2.1.12 in the case of EPW-cubes.

Proposition 2.1.17. *Let* $A \subset \bigwedge^3 V_6$ *a general Lagrangian with no decomposable vectors and* Z_A *the associated EPW-cube. Let* $\widetilde{Z}_A \to Z_A$ *be the associated double EPW-cube, then*

$$\mathrm{Aut}_h(\widetilde{Z}_A) \cong \mathrm{Aut}(Z_A) \times \langle\iota\rangle$$

where ι *is the branching involution and the group* $\mathrm{Aut}(Z_A)$ *corresponds to the subgroup* $\mathrm{Aut}^s_h(\widetilde{Z}_A)$.

Proof. Since $A \notin \Sigma$, by Theorem 2.1.3 and Theorem 2.1.9 we have $\widetilde{Z}_A = \mathrm{Spec}(\mathcal{O}_{Z_A} \oplus \mathscr{R}_2(-2))$, where $\mathscr{R}_2 \cong \omega_{Z_A}(2)$ is defined by (2.1) . It is clear from the construction that there is an inclusion $\mathrm{Aut}(Z_A) \subseteq \mathrm{Aut}(\mathscr{R}_2)$, hence any element of $\mathrm{Aut}(Z_A)$ lifts to an automorphism of $\widetilde{Z}_A$ which fixes the polarization. Viceversa, any element of $\mathrm{Aut}_h(\widetilde{Z}_A)$ induces an automorphism of $\mathbb{P}(\mathrm{H}^0(\widetilde{Z}_A, h)^\vee) \cong \mathbb{P}(\mathrm{H}^0(Z_A, \mathcal{O}_{Z_A})^\vee) \cong \mathbb{P}(\bigwedge^3 V_6)$ which fixes the locus Z_A and again fixes the embedding of the Grassmannian by [IKKR19, Lemma 5.2]. Moreover, the covering involution ι is an involution fixing the polarization, hence there is a central extension

$$1 \to \langle \iota \rangle \to \mathrm{Aut}_h(\widetilde{Z}_A) \to \mathrm{Aut}(Z_A) \to 1.$$

Let the group G be the preimage of $\mathrm{Aut}(Z_A)$ via the map $\mathrm{SL}(V_6) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{PGL}(V_6) \times \mathbb{Z}/2\mathbb{Z}$, so there is an extension

$$1 \to \mu_6 \to G \to \mathrm{Aut}(Z_A) \to 1$$

and observe that Ω_{Z_A} has a structure of G-bundle via the linear action $G \hookrightarrow \mathrm{GL}(\bigwedge^3 V_6)$, moreover the action descends to the quotient $\widetilde{G} = G/\mu_3$. This gives a central extension

$$1 \to \mu_2 \to \widetilde{G} \to \mathrm{Aut}(Z_A) \to 1.$$

The group $\widetilde{G}$ acts on the canonical bundle $\omega_{Z_A} = \bigwedge^6 \Omega_{Z_A}$, where the order two subgroup $\mu_2 = \mu_6/\mu_3$ acts trivially. Since $\mathcal{O}_{Z_A}(1)$ has a linearization where μ_2 acts by -1, then it will act trivially on $\mathcal{O}_{Z_A}(2)$ and hence trivially on $\omega_{Z_A}(2)$. In conclusion, the action of the group G on $\omega_{Z_A}(2) \cong \mathscr{R}_1$ descends to the quotient $G/\mu_6 \cong \mathrm{Aut}(Z_A)$ and using the description $\widetilde{Z}_A \cong \mathrm{Spec}(\mathcal{O}_{Z_A} \oplus \mathscr{R}_1)$ we get an injection $\mathrm{Aut}(Z_A) \to \mathrm{Aut}_h(\widetilde{Z}_A)$ which is a section of $\mathrm{Aut}_h(\widetilde{Z}_A) \to \mathrm{Aut}(Z_A)$.
The action of $\mathrm{Aut}(\widetilde{Z}_A)$ on $\mathrm{H}^2(\widetilde{Z}_A, \mathcal{O}_{\widetilde{Z}_A}) \cong \sigma_{\widetilde{Z}_A}\mathbb{C}$ determines a morphism $\phi : \mathrm{Aut}(\widetilde{Z}_A) \to \mathbb{C}^*$, with finite (cyclic) image, that sends ι to -1. The group G has a trivial action on $\mathrm{H}^2(\widetilde{Z}_A, \mathcal{O}_{\widetilde{Z}_A})$. Indeed, the action factors to the quotient $\mathrm{Aut}(Z_A)$ and the inclusions $\mathrm{Aut}(Z_A) \subset \mathrm{Aut}(\mathrm{Gr}(3, V_6)) \subset \mathrm{PGL}(\bigwedge^3 V_6)$ show that G has no non-trivial character (characters of linear transformations act by their determinant). In conclusion there is the following split exact sequence

$$1 \to \mathrm{Aut}(Z_A) \to \mathrm{Aut}_h(\widetilde{Z}_A) \to \langle \iota \rangle \to 1$$

which is equivalent to

$$1 \to \mathrm{Aut}^s_H(\widetilde{Z}_A) \to \mathrm{Aut}_h(\widetilde{Z}_A) \to \mathrm{Img}(\phi) \to 1.$$

□

2.1.4 EPW manifolds and Gushel-Mukai varieties

Here we explain the relation between the EPW construction and Gushel-Mukai varieties, which is very deeply described in [DK20b] and [Deb20]. We also briefly discuss the rationality problem for threefolds.

Recall that a manifold X is *Fano* if $-K_X$ is ample, where K_X denotes the canonical bundle. In this case the index $i_X = \mathrm{div}(-K_X)$ is the divisibility of $-K_X$ in the Picard lattice $\mathrm{Pic}(X)$.

Let V_5 be a 5-dimensional complex vector space.

Definition 2.1.18. A *Gushel-Mukai manifold* (GM) of dimension $n = 3, 4, 5$ is the smooth complete intersection of the Grassmannian $\mathrm{Gr}(2, V_5) \subset \mathbb{P}(\bigwedge^2 V_5)$ with a linear space $\mathbb{P}^{n+4}$ and a quadric.

Gushel-Mukai manifolds are Fano manifolds with Picard number 1, index $n-2$ and degree 10. Moreover, the converse is also true:

Theorem 2.1.19 (Mukai)**.** *Any Fano manifold of dimension $n = 3, 4, 5$ of Picard rank 1, index $n-2$ and degree 10 is a Gushel-Mukai manifold.*

Proof. [Muk95]. □

The Hodge diamonds of GM varieties can be found in [Deb20, Proposition 4.1]. The relation with the EPW construction is the following:

Theorem 2.1.20. *Let $A \in \mathbb{LG}(\bigwedge^3 V_6) \setminus \Sigma$. There is a bijection between the set of isomorphism classes of GM varieties of dimension n and isomorphism classes of triples (V_6, V_5, A), where $V_5 \subset V_6$ is a hyperplane that satisfies*

$$\dim(A \cap \bigwedge^3 V_5) = 5 - n.$$

Proof. [DK20b, Theorem 3.6]. □

To a GM variety X we can associate an EPW sextic Y_A where (V_6, V_5, A) is the class associated to X by the above correspondence. The other way around, if A is a Lagrangian space of $\bigwedge^3 V_6$, we define a family of GM varieties associated with A consisting of all the GM varieties X such that Y_A is an EPW sextic associated with X. We point out that, by [Deb20, Theorem 2.6] for every $n = 3, 4, 5$ there is a coarse moduli space of GM varieties of dimension n, which is quasi-projective irreducible of dimension $25 - (5-n)(6-n)/2$ with a surjective morphism to the moduli space of EPW-sextics.

Theorem 2.1.21. *Consider a Lagrangian space A and an associated GM variety X_A. Any other GM variety of the same dimension associated either to A or to its dual $\delta(A)$ is bimeromorphic to X_A.*

Proof. [Deb20, Theorem 3.2]. □

The above result shows that the rationality of a GM variety only depends on the associated Lagrangian space.

The picture about rationality of GM varieties is the following. Any GM fivefold is rational by [Deb20, Proposition 3.3]. The rationality of GM fourfolds is not known, there are some rational examples (see [Deb20, Examples 3.4, 3.5, 3.6]) but it is expected that the very general GM fourfolds are irrational. The general GM threefold is known to be irrational by [Bea77, Theorem 5.6(ii)], while there is the belief that any such threefold should be irrational. There were no explicit examples of irrational GM threefolds before [DM22] where a 2-dimensional family of GM threefolds is described, we give other two families of irrational GM threefolds in section 3.2.

An effective technique to prove that a threefold X is irrational is the study of its intermediate Jacobian

$$\mathrm{Jac}(X) := \mathrm{H}^{2,1}(X)^{\vee} / \, \mathrm{H}^3(X, \mathbb{Z})$$

which is a principally polarized Abelian variety.

Theorem 2.1.22 (Clemens-Griffiths criterion)**.** *Let X be a rational projective threefold. The intermediate Jacobian* $\mathrm{Jac}(X)$ *is isomorphic to the product of Jacobians of curves, as principally polarized manifolds.*

The criterion was used by Clemens and Griffiths to prove that any cubic threefold is not rational, as a consequence of the study of the theta divisor. A similar technique is used for proving that the general GM threefold is not rational, but a clear description of the theta divisor is not available yet in this case.

The following gives information about the intermediate Jacobian of a GM threefold, which is a 10-dimensional principally polarized variety.

Theorem 2.1.23. *Let $A \in \mathbb{LG}(\bigwedge^3 V_6) \setminus \Sigma$, consider any GM threefold X_A associated with A and the associated EPW-sextic Y_A. There is a canonical principal polarization θ on the Albanese variety* $\mathrm{Alb}(\widetilde{Y}_A[2])$ *such that there is an isomorphism*

$$(\mathrm{Jac}(X_A), \theta_A) \cong (\mathrm{Alb}(\widetilde{Y}_A[2], \theta))$$

of principally polarized manifolds.

Proof. [DK20b, Theorem 1.1]. □

This allows for example to induce a linear action of a group on A to an action on $\mathrm{Jac}(X_A)$ by means of Proposition 2.1.13, as it is done in [DM22] and in section 3.2.

2.2 Moduli spaces of sheaves on symplectic surfaces

We remind general results about moduli spaces of coherent sheaves on projective symplectic surfaces. These spaces are very interesting because they sometimes give examples of IHS manifolds.

Recall that the dimension of a sheaf is by definition the dimension of its support and a sheaf is called *pure* if any non-trivial proper subsheaf has the same dimension. Consider a projective surface Σ and let $H \in \mathrm{Pic}(\Sigma)$ be an ample line bundle. For any $F \in \mathrm{Coh}(\Sigma)$ and for any $n \in \mathbb{Z}$ consider the Hilbert polynomial

$$P_H(F)(n) = \chi(F(nH))$$

and the reduced Hilbert polynomial

$$p_H(F) = \frac{P_H(F)}{\alpha_H(F)}$$

where $\alpha_H(F)$ is the leading coefficient of $P_H(F)$.

Definition 2.2.1. A sheaf $F \in \mathrm{Coh}(S)$ is called *stable (resp. H-semistable)* if it is pure and if for any $0 \neq E \subsetneq F$ the inequality

$$p_H(E)(n) < p_H(F)(n) \text{ resp. } p_H(E)(n) \leq p_H(F)(n)$$

holds for $n \gg 0$.

A H-semistable sheaf is called *H-polystable* if it is direct sum of H-stable sheaves.

Definition 2.2.2. The *Mukai lattice* of the surface Σ is

$$\widetilde{\mathrm{H}}(\Sigma, \mathbb{Z}) := \mathrm{H}^0(\Sigma, \mathbb{Z}) \oplus \mathrm{H}^2(\Sigma, \mathbb{Z}) \oplus \mathrm{H}^4(\Sigma, \mathbb{Z})$$

with pairing given by

$$(v_0, v_2, v_4) \cdot (w_0, w_2, w_4) := \int_\Sigma (-v_0 w_4 + v_2 w_2 - v_4 w_0).$$

The lattice $\widetilde{\mathrm{H}}(\Sigma, \mathbb{Z})$ is endowed with a Hodge structure of weight 2 by imposing $\mathrm{H}^0(\Sigma, \mathbb{Z})$ and $\mathrm{H}^4(\Sigma, \mathbb{Z})$ being of type $(1,1)$.

Definition 2.2.3. The *Mukai vector* of F is given by

$$v(F) = \mathrm{ch}(F)\sqrt{\mathrm{td}(\Sigma)} \in \widetilde{\mathrm{H}}(\Sigma, \mathbb{Z}),$$

where $\mathrm{ch}(F)$ is the Chern character of F and $\mathrm{td}(\Sigma)$ is the Todd class of Σ. Any vector $v = (v_0, v_2, v_4) \in \widetilde{\mathrm{H}}(\Sigma, \mathbb{Z})$ with $v_2 \in \mathrm{NS}(\Sigma)$ is also called Mukai vector.

Recall that for a K3 surface S and $F \in \mathrm{Coh}(S)$ we have

$$v(F) = (\mathrm{rk}(F), \mathrm{c}_1(F), \mathrm{ch}_2(F)) + \mathrm{rk}(F)),$$

while for an Abelian surface A and $F \in \mathrm{Coh}(A)$ we have

$$v(F) = (\mathrm{rk}(F), \mathrm{c}_1(F), \mathrm{ch}_2(F)).$$

Definition 2.2.4. A vector $v \in \widetilde{\mathrm{H}}(\Sigma, \mathbb{Z})$ is called *primitive* if its components have no common divisors different from one.

It is known that if $F \in \mathrm{Coh}(\Sigma)$ is semi-stable, then there is an associated Jordan-Hölder filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_{k-1} \subset F_k = F$, whose graded factors F_{i+1}/F_i have the same reduced Hilbert polyomial as F and are uniquely determined, up to reordering, by F.

Definition 2.2.5. We say that two sheaves $E, F \in \mathrm{Coh}(\Sigma)$ are *S-equivalent* if they have the same graded factors in the Jordan-Hölder filtration.

Stable sheaves are S-equivalent precisely if they are isomorphic, and in any S-equivalence class there is a polystable sheaf.

Definition 2.2.6. Denote by $\mathrm{M}_v(\Sigma, H)$ the moduli space of H-semistable sheaves with fixed Mukai vector v up to S-equivalence. Denote by $\mathrm{M}_v^s(\Sigma, H)$ the open locus of H-stable sheaves that are not H-semistable.

The moduli space is constructed by Gieseker in [Gie77], where it is shown that $\mathrm{M}_v(\Sigma, H)$ is projective and it is the compactification of $\mathrm{M}_v^s(\Sigma, H)$. The interest in this moduli space is due to the following

Theorem 2.2.7. *Let Σ be a projective symplectic surface. Fix a Mukai vector $v \in \widetilde{\mathrm{H}}(\Sigma, \mathbb{Z})$, then $\mathrm{M}_v^s(\Sigma, H)$ is smooth of dimension $v^2 + 2$. Moreover, $\mathrm{M}_v^s(\Sigma, H)$ admits a symplectic form.*

Proof. [Muk84]. $\square$

The moduli space is well-understood for a generic choice of polarization.

Definition 2.2.8. A polarization H on Σ is called *v-generic* for $v \in \widetilde{\mathrm{H}}(\Sigma, \mathbb{Z})$ if for every H-polystable sheaf E with $v(E) = v$ and every direct summand F of E, we have $v(F) \in v \cdot \mathbb{Q}$.

A geometric characterization of v-genericity is given in [PR13, Section 2], where it is described in terms of a wall and chamber decomposition of the positive cone of Σ.

Observe that by [Saw16, Lemma 2] we have $\mathrm{M}_v^s(\Sigma, H) = \mathrm{M}_v(\Sigma, H)$ in case that v is primitive and the polarization H is v-generic. Many authors studied and described this space, before stating their achievement we need another piece of construction.

Let A be an Abelian surface, fix $F_0 \in \mathrm{M}_v(A, H)$ and consider the map

$$\begin{aligned} a_v : \mathrm{M}_v(A, H) &\longrightarrow A \times A^\vee \\ F &\mapsto (\mathrm{Alb}(\mathrm{c}_2(F)), \det(F) \otimes \det(F_0)^{-1}) \end{aligned}$$

where $\mathrm{Alb} : \mathrm{CH}_0(A) \to A$ is the Albanese homomorphism. The map a_v happens to be isotrivial, set $\mathrm{K}_v(A, H) = a_v^{-1}(0)$.

Theorem 2.2.9. *Consider a projective K3 surface S and an Abelian surface A, both coming with a primitive Mukai vector v and a v-generic polarization H. The followings hold:*

1. $\mathrm{M}_v(S,H)$ *is a reduced point in case* $v^2 = -2$.
2. $\mathrm{M}_v(S,H)$ *is a K3 surface in case* $v^2 = 0$*, moreover there is a Hodge isometry* $v^\perp/\mathbb{Z}v \cong \mathrm{H}^2(\mathrm{M}_v(S,H),\mathbb{Z})$.
3. $\mathrm{M}_v(S,H)$ *is an IHS manifold of* $K3^{[n]}$ *type with* $n = \frac{v^2+2}{2}$ *in case* $v^2 \geq 2$*, moreover there is an Hodge isometry* $v^\perp \cong \mathrm{H}^2(\mathrm{M}_v(S,H),\mathbb{Z})$.
4. $\mathrm{K}_v(A,H)$ *is an IHS manifold of* Kum_n *type with* $n = \frac{v^2-2}{2}$ *in case* $v^2 \geq 6$*, moreover there is an Hodge isometry* $v^\perp \cong \mathrm{H}^2(\mathrm{K}_v(A,H),\mathbb{Z})$.

Proof. [Muk84], [Yos01]. □

Example 2.2.10. Let S be a K3 surface and $v = (1, 0, 1-n) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$, then a pure sheaf $\mathcal{F} \in \mathrm{Coh}(S)$ with $v(\mathcal{F}) = v$ is of rank 1 with $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = n$, in particular it is torsion-free on its support since it consists of points. Suppose $\mathcal{E} \subset \mathcal{F}$ is a non-trivial proper subsheaf, then $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{F}) = 1$, $\mathcal{F}/\mathcal{E}$ is torsion and then

$$p_H(\mathcal{F})(n) - p_H(\mathcal{E})(n) = \chi(\mathcal{F}(nH)) - \chi(\mathcal{E}(nH)) = \chi(\mathcal{F}/\mathcal{E}(nH)) > 0$$

for $n \gg 0$ and for any polarization H. Consider the reflexive sheaf (hence locally free) $\mathcal{F}^{\vee\vee} \cong \mathcal{O}_S$. The cokernel of the inclusion $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$ is the structure sheaf $\mathcal{O}_Z$ of a zero-dimensional subscheme $Z \subset S$, so the sheaf $\mathcal{F}$ is identified with the ideal sheaf of the subscheme $\mathcal{F} \cong \mathcal{I}_Z$ and we have an identification $\mathrm{M}_v(S,H) \cong S^{[n]}$. Similarly, the generalized Kummer varieties can be re-obtained as Albanese fibers of moduli spaces of sheaves on Abelian surfaces.

If v is not primitive, but $v = mw$ with w primitive and $m > 1$, then $\mathrm{M}_w(\Sigma,H)$ can be singular out of $\mathrm{M}^s_w(\Sigma,H)$. One can ask whether there exists a symplectic resolution, a resolution of singularities with a symplectic form that extends the one on $\mathrm{M}^s_w(\Sigma,H)$.

Theorem 2.2.11. *The following hold:*

1. *Consider a projective K3 surface S, $v = (2,0,-2)$ and a v-generic polarization H. There is a symplectic resolution* $\widetilde{\mathrm{M}}_{10} := \widetilde{\mathrm{M}}_v(S,H) \to \mathrm{M}_v(S,H) =: \mathrm{M}_{10}$ *that is an IHS manifold of dimension* 10 *and second Betti number* 24.
2. *Consider an Abelian surface A, $v = (2,0,-2)$ and a v-generic polarization H. Then there is a symplectic resolution* $\widetilde{\mathrm{K}}_6 := \widetilde{\mathrm{K}}_v(A,H) \to \mathrm{Kum}_v(A,H) =: \mathrm{K}_6$ *that is an IHS manifold of dimension* 6 *and second Betti number* 8.

Proof. [O'G99], [Rap08] and [O'G03]. □

As already remarked in Example 1.3.11, these resolutions have different Betti numbers than manifolds of $\mathrm{K3}^{[n]}$ type and Kum_n type, so they are not deformation equivalent. Manifolds deformation equivalent to M_{10} or M_6 are called rispectively of OG10 type or of OG6 type. Numerical invariants of the latter can be found in Example 1.3.22 and Example 1.3.23.

A general picture for choices of non primitive Mukai vector is given by the following:

Theorem 2.2.12. *Let Σ be a projective symplectic surface. Fix a Mukai vector $v = mw$ with w primitive of positive square and $m > 1$. Suppose that $w = (w_0, w_2, w_4)$ is such that either $w_0 > 0$ and $w_2 \in \mathrm{NS}(\Sigma)$, or $w_4 \neq 0$ and $w_2 = \mathrm{c}_1(E)$ with E an effective divisor. Pick a v-generic polarization H. Then:*

1. *If $m = 2$ and $w^2 = 2$ then there is a symplectic resolution $\widetilde{\mathrm{M}}_v(\Sigma, H) \to \mathrm{M}_v(\Sigma, H)$ that is the blow up along the singular locus $\mathrm{M}_v(\Sigma, H) \setminus \mathrm{M}_v^s(\Sigma, H)$ with reduced stucture.*
2. *If $m > 2$ or $m = 2$ and $w^2 > 2$ then $\mathrm{M}_v(\Sigma, H)$ does not admit any symplectic resolution and it has locally factorial singularities.*

Proof. [LS06], [KLS06]. □

The first item of the previous theorem was studied in detail.

Definition 2.2.13. Let Σ be a projective symplectic surface with an ample line bundle H and a Mukai vector v. We say that (Σ, v, H) is an *OLS-triple* if:

1. The polarization H is primitive and v-generic
2. We have $v = 2w$ with w a primitive Mukai vector with $w^2 = 2$
3. If $w = (w_0, w_2, w_4)$ then $w_0 \geq 0$, $w_2 \in \mathrm{NS}(\Sigma)$ and if $w_0 = 0$ then $w_2 = \mathrm{c}_1(E)$ with E an effective divisor.

The triple is after O'Grady, Lehn and Sorger.

Theorem 2.2.14. *Let (Σ, v, H) be a OLS-triple.*

1. *If $\Sigma = S$ is a projective K3 surface, then there is a symplectic resolution $\widetilde{\mathrm{M}}_v(S, H) \to \mathrm{M}_v(S, H)$ which is an IHS manifold of* OG10 *type obtained as the blow up along the singular locus*
2. *If $\Sigma = A$ is an Abelian surface, then there is a symplectic resolution $\widetilde{\mathrm{K}}_v(A, H) \to \mathrm{K}_v(A, H)$ which is an IHS manifold of* OG6 *type obtained as the blow up along the singular locus*

Proof. [PR13, Theorem 1.6]. □

As in the other cases there is a strong relation between the cohomology of the moduli space and the Mukai lattice.

Theorem 2.2.15. *Let (Σ, v, H) be an OLS-triple.*

1. *Suppose $\Sigma = S$ is a projective K3 surface, then $\pi_v^* : \mathrm{H}^2(\mathrm{M}_v, \mathbb{Z}) \to \mathrm{H}^2(\widetilde{\mathrm{M}}_v, \mathbb{Z})$ is injective and there is a Hodge isometry*

$$\lambda_v : v^\perp \to \mathrm{H}^2(\mathrm{M}_v, \mathbb{Z})$$

for the lattice structure and the Hodge structure induced by π_v^*.

2. *Suppose $\Sigma = A$ is an Abelian surface, then $\pi_v^* : \mathrm{H}^2(\mathrm{K}_v, \mathbb{Z}) \to \mathrm{H}^2(\widetilde{\mathrm{K}}_v, \mathbb{Z})$ is injective and there is a Hodge isometry*

$$\nu_v : v^\perp \to \mathrm{H}^2(\mathrm{K}_v, \mathbb{Z})$$

for the lattice structure and the Hodge structure induced by π_v^*.

Proof. [PR13, Theorem 1.7]. □

2.3 Cubic fourfolds and Laza-Saccà-Voisin manifolds

We recall few facts about cubic fourfolds and present two families of manifolds of OG10 type associated to a cubic fourfold.

Let $Y \subset \mathbb{P}^5$ be a cubic fourfold, the intersection product gives to $\mathrm{H}^4(Y, \mathbb{Z})$ a lattice structure and this restricts to the primitive cohomology as

$$\mathrm{H}^4(Y, \mathbb{Z}) \supset (h^2)^\perp = \mathrm{H}^4(Y, \mathbb{Z})_{prim} \cong \mathbf{E}_8^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2$$

where h denotes the class of a hyperplane section.

Theorem 2.3.1. *The map*

$$\mathrm{Aut}(Y) \to \mathrm{O}(\mathrm{H}^4(Y, \mathbb{Z}))$$

is injective.

Proof. [JL17, Proposition 2.12], [MM63]. □

The Hodge decomposition on the middle cohomology reads

$$\mathrm{H}^4(Y, \mathbb{Z}) = \mathrm{H}^{3,1}(Y) \oplus \mathrm{H}^{2,2}(Y) \oplus \mathrm{H}^{1,3}(Y)$$

where $\mathrm{H}^{3,1}(Y)$ has dimension 1, hence many definitions given for IHS manifolds work in this context.

Definition 2.3.2. Let Y be a cubic fourfold, $\phi \in \mathrm{Aut}(Y)$ is called *symplectic* if it acts trivially on the generator of $\mathrm{H}^{3,1}(Y,\mathbb{C})$ and it is called *non-symplectic* otherwise.

An important feature of the Hodge decomposition of a cubic fourfold is that the construction of the period map is very similar to the one of IHS manifolds. Following [Voi86] and [Laz10] we consider the dimension 20 moduli space of cubic fourfolds $\mathcal{M}$ and the perod domain $\mathcal{D}$ with the period map

$$\mathcal{P} : \mathcal{M} \to \mathcal{D}$$

that associates the Hodge structure on the middle cohomology. Similarly to the case of IHS manifolds, the space $\mathrm{H}^{1,3}(Y)$ determines the Hodge decomposition of $\mathrm{H}^4(Y,\mathbb{C})$.

Definition 2.3.3. A vector of square 2 in a lattice is called a *short root* and a vector of square 6 and divisibility 3 in a lattice is called a *long root*.

There is a precise description of the image of the period map in terms of long and short roots:

Theorem 2.3.4 (Torelli Theorem for cubic fourfolds)**.** *The period map of cubic fourfolds*

$$\mathcal{P} : \mathcal{M} \to \mathcal{D}$$

is an isomorphism over its image, consisting of Hodge structures with no long and short roots among the $(2,2)$*-classes.*

Proof. [Laz10, Theorem 1.1]. □

Moreover, we have the following:

Theorem 2.3.5 (Hodge theoretical Torelli Theorem for cubic fourfolds)**.** *Let* Y_1, Y_2 *be cubic fourfolds and* $\phi : \mathrm{H}^4(Y_2,\mathbb{Z}) \to \mathrm{H}^4(Y_1,\mathbb{Z})$ *an isometry of polarized Hodge structures, then there exists a unique isomorphism* $f : Y_1 \to Y_2$ *such that* $f^* = \phi$.

Proof. [Voi86]. □

We say that a *marking* for Y is a primitive lattice $\mathrm{K}_d = \langle h^2, l\rangle \subset \mathrm{H}^{2,2}(Y,\mathbb{Z})$ with h the class of a hyperplane section, $l \in \mathrm{H}^{2,2}(Y,\mathbb{Z})$ and $\mathrm{disc}(\mathrm{K}_d) = d$.

Definition 2.3.6. Let $\mathcal{C}_d \subset \mathcal{M}$ be the set of cubics that admit a marking K_d. The sets $\mathcal{C}_d$ are called *Hasset divisors.*

Hasset divisors are in fact divisors, moreover we have a non-emptiness criterion:

Theorem 2.3.7 (Hassett). *The set $\mathcal{C}_d$ is an irreducible divisor in $\mathcal{M}$, non-empty if and only if*

$$d > 0 \text{ and } d \equiv 0, 2 (\mod 6).$$

Proof. [Has00, Theorem 1.0.1]. □

Consider $Y \subset \mathbb{P}^5$ a smooth cubic fourfold, the dual projective space $(\mathbb{P}^5)^\vee$ parametrizing the hyperplane sections $Y_H = Y \cap H \subset Y$ and the open set $U \subset (\mathbb{P}^5)^\vee$ parametrizing the smooth hyperplane sections. We will often write $\mathbb{P}^5$ instead of $(\mathbb{P}^5)^\vee$, if it does not lead to confusion. Denote by

$$\mathrm{Jac}(Y_H) = \mathrm{H}^1(Y_H, \Omega^2_{Y_H})^\vee / \mathrm{H}_3(Y_H, \mathbb{Z})$$

the intermediate Jacobian of the hyperplane section, which is a principally polarized Abelian fivefold. Over U consider the fibration

$$\pi_U : J_U(Y) \to U$$

whose fiber over the smooth hyperplane section Y_H consists of the intermediate Jacobian $\mathrm{Jac}(Y_H)$. It was proved in [DDFPDM96] that $J_U(Y)$ is quasi-projective and it admits a symplectic form σ_U for which π_U is a Lagrangian fibration.

Following [Voi18] there is another Lagrangian fibration

$$\pi^t_U : J^t_U(Y) \to U$$

whose fibers are given by twisted Jacobians, similarly to the previous case.

It is not easy to find reasonable compactifications, but this was done in the following:

Theorem 2.3.8. *Let Y be a smooth cubic fourfold. There exist smooth projective compactifications $J(Y), J^t(Y)$ of $J_U(Y), J^t_U(Y)$ with projective flat morphisms $\pi : J(Y) \to \mathbb{P}^5, \pi^t : J^t(Y) \to \mathbb{P}^5$ extending π_U, π^t_U. Moreover, $J(Y), J^t(Y)$ are manifolds of* OG10 *type.*

Proof. [LSV17], [Sac23, Theorem 1], for the twisted case [Voi18]. □

The compactification $J(Y)$ is called the *LSV manifold* associated to Y, while $J^t(Y)$ is called the *twisted LSV manifold* associated to Y. There is an effective relative theta divisor $\Theta \subset J(Y)$ obtained as the closure of the union of theta divisors of the smooth fibers, it has the property that $q_{J(Y)}(\Theta) = -2$. There is another class $L = \pi^*\mathcal{O}_{\mathbb{P}^5}(1)$, that together with Θ span a hyperbolic lattice $\langle L, \Theta \rangle = \mathbf{U}_Y \subset \mathrm{NS}(J(Y))$. For a very general cubic fourfold Y one has $\mathrm{NS}(J(Y)) = \mathbf{U}_Y$, in particular the family can not be locally complete since there are always two algebraic classes in the LSV manifolds. Similarly, in the twisted case there are classes $L^t, \Theta^t \in \mathrm{NS}(J^t(Y))$ spanning a lattice $\langle L^t, \Theta^t \rangle = \mathbf{U}^t_Y \cong \mathbf{U}(3)$.

Proposition 2.3.9. *There is a morphism of Hodge structures*

$$\alpha : \mathrm{H}^4(Y,\mathbb{Z})_{prim} \to \mathbf{U}_Y^{\perp} \subset \mathrm{H}^2(J(Y),\mathbb{Z})$$

and there exists an integer $N > 0$ such that $-N(x.y) = q_{J(Y)}(\alpha(x),\alpha(y))$ for any $x, y \in \mathrm{H}^4(Y,\mathbb{Z})_{prim}$. The same statement holds replacing $J(Y)$ with $J^t(Y)$ and $\mathbf{U}_Y$ with $\mathbf{U}_Y^t$.

Proof. [Ono22, Proposition 4.1], for the twisted case [MO22, Lemma 7.1]. □

Consider an automorphism $\phi \in \mathrm{Aut}(Y)$, this acts on the universal family of hyperplane sections $\mathcal{Y}_U \to U$ and on the fibrations $J_U(Y) \to U$, $J_U^t(Y) \to U$, inducing bimeromorphic maps that we call $\widetilde{\phi} \in \mathrm{Bir}(J(Y))$ and $\widetilde{\phi}^t \in \mathrm{Bir}(J^t(Y))$. From the fact that $\widetilde{\phi}, \widetilde{\phi}^t$ preserve the classes L, L^t and Θ, Θ^t, it follows that $\widetilde{\phi}$ acts trivially on $\mathbf{U}_Y \subset \mathrm{NS}(J(Y))$ and $\widetilde{\phi}^t$ acts trivially on $\mathbf{U}_Y^t \subset \mathrm{NS}(J^t(Y))$.

Proposition 2.3.10. *Let Y be a cubic fourfold such that the fibers of $\pi : J(Y) \to \mathbb{P}^5$ are irreducible. Then any birational morphism $\tau \in \mathrm{Bir}(J(Y))$ that fixes the class L extends to a regular automorphism $\tau \in \mathrm{Aut}(J(Y))$.*

Proof. [Sac23, Proposition 3.11]. □

Notice that by [LSV17], the hypothesis of the proposition is satisfied whenever Y is general.

Chapter 3

Very symmetric double EPW-sextics and irrational GM threefolds

We construct two examples of projective IHS fourfolds of $K3^{[2]}$ type with an action of the alternating group $\mathcal{A}_7$, making them some of the most symmetric IHS manifold fourfolds according to the classification in [HM19]. They are realized as double EPW sextics and this allows us to construct an explicit family of irrational Gushel-Mukai threefolds.

The structure of this chapter is as follows: in the first section we outline the construction of the IHS manifold fourfolds and in the second section we prove that for any of the two sextics we construct, each member of the associated family of GM threefolds is irrational. In Appendix A we give more details about some computations we performed, including codes that we run with `GAP` [GAP21] and `Macaulay2` [M2].

3.1 Double EPW-sextics with an action of $\mathcal{A}_7$

The general idea is to find a Lagrangian space which is invariant under the action of the group $\mathcal{A}_7$, to get an invariant EPW-sextic. Our first attempt, the most naïve way to proceed, is to consider the natural representation of $\mathcal{A}_7$ on a 7-dimensional space and quotient out by the trivial subrepresentaton (the one generated by the sum of basis vectors). This leads to an irreducible 6-dimensional representation and with exactly the same construction as it follows, one gets an invariant EPW-sextic. This is in fact three times a quadric so we had to discard it and look for a reduced sextic.

According to [WCN85], there exists a group (going by the notation from the atlas) $3.\mathcal{A}_7$ such that $3.\mathcal{A}_7/\langle\omega\rangle \cong \mathcal{A}_7$ with ω an element of order 3. This group has a unique irreducible representation $\rho : 3.\mathcal{A}_7 \to \mathbb{C}^6$ that we call $\overline{V}_6$

and is generated by the elements ([ATLAS])

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1+\xi_3 & 0 & 1-\xi_3 & -\xi_3 & \xi_3 & 1 \\ 2 & 0 & -1 & -1 & 0 & -1 \end{bmatrix},$$

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -\xi_3 & 0 & \xi_3 & 1 \end{bmatrix},$$

where ξ_3 is a primitive third root of unity. Note that this induces a representation on $\bigwedge^3 \overline{V}_6$. Moreover, since ω has order 3 the representation $\overline{V}_6$ induces an action of the quotient $\mathcal{A}_7$ on $\bigwedge^3 \overline{V}_6$, we denote this (faithful) representation of $\mathcal{A}_7$ by W.
The irreducible complex representations of $\mathcal{A}_7$ of dimension smaller or equal than 20 have dimensions $1, 6, 10, 10, 14, 14$ and 15 and can be read in Table A.1, Appendix A. We point out that the two 10-dimensional representations are not isomorphic.

Lemma 3.1.1. *The representation W decomposes as the direct sum of the only two irreducible 10-dimensional representations $R_1 = (A_1, \rho_1)$ and $R_2 = (A_2, \rho_2)$ of the group $\mathcal{A}_7$, moreover the underlying vector spaces $A_1, A_2 \subset \bigwedge^3 V_6$ of those representations are Lagrangian.*

Proof. The fact that W has the mentioned decomposition is just a computation of characters (we used `GAP`), the subrepresentations being Lagrangian is easily checked with computer algebra. We remind to section A.1 and Table A.1 for more details. □

As a consequence of (2.2), setting $\mathbb{A} = A_1, A_2$ leads to an EPW-sextic $Y_{\mathbb{A}} \subset \mathbb{P}^5$ which is invariant under the action of $\mathcal{A}_7$. The representations R_1 and R_2 are dual to each other, so the manifolds Y_{A_1} and Y_{A_2} are projectively dual to each other and hence $\delta(A_1) = A_2$ (cf. [O'G06, Section 3]). From now on $\mathbb{A}$ will denote one of the two specific Lagrangian spaces.

Proposition 3.1.2. *The Lagrangian space $\mathbb{A}$ has no decomposable vectors, the degeneracy locus $Y_A[3]$ is empty, so $\mathbb{A} \in \mathbb{LG}(\bigwedge^3 V_6) \setminus (\Sigma \cup \Delta)$, and in consequence the EPW-sextic $Y_{\mathbb{A}}$ is singular along the degree 40 smooth surface $Y_{\mathbb{A}}[2]$. Hence the double cover $\widetilde{Y}_{\mathbb{A}} \to Y_{\mathbb{A}}$ is a smooth IHS fourfold.*

Proof. From Theorem 2.1.7 it suffices to prove that $\mathbb{A}$ does not belong to Σ and Δ. According to [O'G15] the singular locus is given by the union of the 40-degree surface $Y_{\mathbb{A}}[2]$ with planes $\mathbb{P}(U)$ where U is a three-dimensional subspace of W such that $\bigwedge^3 U \subset \mathbb{A}$.

Our computation with `Macaulay2` shows that the singular locus has degree 40 (see section A.2) so it must coincide with $Y_{\mathbb{A}}[2]$ (cf. Theorem 2.1.5), thus there are no decomposable vectors in $\mathbb{A}$.

We also compute the singular locus $Y_{\mathbb{A}}[3]$ of $Y_{\mathbb{A}}[2]$ (see section A.3), and it turns out to be empty, completing the proof. □

Corollary 3.1.3. *The fourfold $\widetilde{Y}_{\mathbb{A}}$ has a symplectic action of the group $\mathcal{A}_7$ and the action fixes the polarization H (i.e. $\mathcal{A}_7 \hookrightarrow \mathrm{Aut}^s_H(\widetilde{Y_{\mathbb{A}}})$).*

Proof. Use Proposition 2.1.12 and (2.2). □

Lemma 3.1.4. *The group $\mathrm{Aut}^s_H(\widetilde{Y_{\mathbb{A}}})$ is finite.*

Proof. Follows from Corollary 1.3.81 as (by definition) it fixes H. We can also prove it in a somehow more direct way. We know from Proposition 2.1.12 that $\mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}}) \cong \mathrm{Aut}(Y_A)$, Proposition 3.1.2 ensures that $\mathbb{A}$ has no decomposable vectors and [DK18, Proposition B.9] guarantees that the last group is finite. □

Proposition 3.1.5. *There is an isomorphism $\mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}}) \cong \mathcal{A}_7$.*

Proof. Using the fact that $\mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}})$ is finite combined with the fact that $\mathcal{A}_7$ is maximal [HM19, Theorem A and Table 6], one concludes that the inclusion $\mathcal{A}_7 \hookrightarrow \mathrm{Aut}^s_H(\widetilde{Y}_{\mathbb{A}})$ is in fact an isomorphism. □

Now we are ready to show that the two examples we found $\widetilde{Y}_{A_1}$ and $\widetilde{Y}_{A_2}$ are not isomorphic as polarized manifolds. We will need the following lemma.

Lemma 3.1.6. *There are no $f \in \mathrm{GL}(V_6)$ such that $\bigwedge^3 f(A_1) = A_2$.*

Proof. Set $h = \bigwedge^3 f$ and denote the non isomorphic representations $R_i = (A_i, \rho_i)$ for $i = 1, 2$. Notice that $h : A_1 \to A_2$ defines an isomorphism of representations, and so a faithful representation

$$(A_2, h \circ \rho_1 \circ h^{-1}) \cong R_1$$

which is then not isomorphic to R_2. This means that one has the inclusions

$$\mathcal{A}_7 \subset \langle (h \circ \rho_1 \circ h^{-1})(g), \rho_2(g) | g \in \mathcal{A}_7 \rangle \subset \mathrm{Aut}(Y_{A_2})$$

where the first one is strict and the second follows from (2.2) since all the automorphisms of the middle group are expressed by third wedges of automorphisms of V_6 which preserve the Lagrangian A_2. We conclude again using the isomorphisms

$$\mathrm{Aut}(Y_{A_2}) \cong \mathrm{Aut}^s_H(\widetilde{Y}_{A_2}) \cong \mathcal{A}_7$$

from Proposition 2.1.12 and Proposition 3.1.5 to get a contradiction. □

Proposition 3.1.7. *The manifolds* $(\widetilde{Y}_{A_1}, H_1)$ *and* $(\widetilde{Y}_{A_2}, H_2)$ *are not isomorphic as polarized manifolds where* $H_i = \pi^*_{A_i}\mathcal{O}_{Y_{A_i}}(1)$ *for* $i = 1, 2$.

Proof. By [O'G15, page 486], if $A_1, A_2 \in \mathbb{LG}(\bigwedge^3 V_6)^0$ are not in the same orbit of $\mathrm{PGL}(\bigwedge^3 V_6)$, then $\widetilde{Y}_{A_1}$ and $\widetilde{Y}_{A_2}$ have different periods, so they cannot be isomorphic. Lemma 3.1.6 finishes the proof. □

We also obtain the following information on the constructed manifolds.

Proposition 3.1.8. *The transcendental lattice is given by*

$$\mathrm{T}(\widetilde{Y}_{\mathbb{A}}) \cong \begin{pmatrix} 6 & 0 \\ 0 & 70 \end{pmatrix}.$$

Proof. Based on [Waw22, Table 1], a projective IHS fourfold of $\mathrm{K3}^{[2]}$ type admitting an action of a group extension of $\mathcal{A}_7$ and fixing a primitive ample vector H with $H^2 = 2$ in $\mathrm{NS}(X) \subset \mathrm{H}^2(X, \mathbb{Z})$ must have the transcendental lattice from the statement. Conclude recalling that by Theorem 2.1.7 the polarization $H_{\mathbb{A}}$ satisfies $H^2_{\mathbb{A}} = 2$. □

3.2 Irrational Gushel-Mukai threefolds

In this section we give the main application of our construction: any element in the families of GM threefolds associated to the two Lagrangian spaces described in the previous sections is irrational.

Let $X_{\mathbb{A}}$ be any GM threefold associated with $\mathbb{A}$ and let $\mathrm{Jac}(X_{\mathbb{A}})$ be its intermediate Jacobian. Recall that $\mathbb{A} \notin \Sigma$ and so by Theorem 2.1.23 there are a canonical principal polarization θ on the Albanese variety $\mathrm{Alb}(\widetilde{Y}_{\mathbb{A}}[2])$ and a canonical isomorphism

$$(\mathrm{Jac}(X_{\mathbb{A}}), \theta_{X_{\mathbb{A}}}) \cong (\mathrm{Alb}(\widetilde{Y}_{\mathbb{A}}[2]), \theta) \tag{3.1}$$

between principally polarized Abelian varieties. Furthermore, the tangent spaces at the origin of these varieties are isomorphic to $\mathbb{A}$. Explicitly,

$$T_{\mathrm{Alb}(\widetilde{Y}_{\mathbb{A}}[2]),0} \cong T_{\mathrm{Jac}(X_{\mathbb{A}}),0} \cong \mathbb{A}. \tag{3.2}$$

The action of $\mathcal{A}_7$ on $\mathrm{Jac}(X_{\mathbb{A}})$ gives the following feature

Proposition 3.2.1. *The principally polarized variety* $(\mathrm{Jac}(X_{\mathbb{A}}), \theta_{X_{\mathbb{A}}})$ *is indecomposable.*

Proof. Suppose it was isomorphic to a product of $m \geq 2$ nonzero indecomposable principally polarized Abelian varieties.
Since $\mathcal{A}_7 \cong \mathrm{Aut}(Y_{\mathbb{A}})$, the diagram (2.5) reads:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle \gamma^3 \rangle & \longrightarrow & \widetilde{\mathcal{A}}_7 & \xrightarrow{\psi} & \mathcal{A}_7 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \rho_a & \overset{\rho}{\swarrow} & \downarrow & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(\mathbb{A}) & \xrightarrow{\pi} & \mathrm{PGL}(\mathbb{A}) & \longrightarrow & 1 \end{array} \tag{3.3}$$

where $\widetilde{\mathcal{A}}_7$ is an extension of $\mathcal{A}_7$ by the group of order two, ρ_a is the analytic representation $\widetilde{\mathcal{A}}_7 \to \mathrm{GL}(T_{\mathrm{Jac}(X_{\mathbb{A}}),0})$ by Proposition 2.1.13 and ρ is the irreducible representation $\mathbb{A}$. Now $\rho_a \neq \rho \circ \psi$ since both representations are faithful, but the equality $\pi \circ \rho_a = \pi \circ \rho \circ \psi$ holds by construction (using the commutativity of the diagram (2.5)). This means that the two actions on $\mathbb{A}$ differ by scalar multiplication, hence if one of the representations decomposes then the other must do so as well. We supposed that the Jacobian is a product and so the analytic representation decomposes in the sum of the tangent spaces of the components, but this is a contradiction since $\mathbb{A}$ was irreducible as $\mathcal{A}_7$-representation.

□

Combining this property with the group having a big cardinality, we get the result sought after:

Theorem 3.2.2. *Any smooth GM threefold associated with the Lagrangian* $\mathbb{A}$ *is irrational.*

Proof. The proof is inspired by the one of [DM22, Theorem 5.2]: we want to use the Clemens-Griffiths criterion (Theorem 2.1.22.
Since $(\mathrm{Jac}(X_{\mathbb{A}}), \theta_{X_{\mathbb{A}}})$ is indecomposable, we can reduce to treat the case where $(\mathrm{Jac}(X_{\mathbb{A}}), \theta_{X_{\mathbb{A}}}) \cong (\mathrm{Jac}(C), \theta_C)$ for C a curve of genus 10. We have a faithful action of $\mathcal{A}_7$ on the Jacobian, by the Torelli theorem the group of automorhpisms of that Abelian variety is either $\mathrm{Aut}(C)$ or $\mathrm{Aut}(C) \times \mathbb{Z}/2\mathbb{Z}$. In conclusion $\mathcal{A}_7$ embeds in one of those two groups, but this is a contradiction since $|\mathcal{A}_7| = 2520$ and $|\mathrm{Aut}(C)| < 756$ by Hurwitz's bound [Mir95, Theorem 3.7].

□

Chapter 4

The Nielsen realization problem for IHS manifolds

We give an answer to the Nielsen realization problem for IHS manifolds in terms of the same invariant used for K3 surfaces. Moreover, we address some related questions: we determine that, for some of the known deformation types the representation of the mapping class group on the second cohomology admits a section on its image, and we show that for manifolds of $\mathrm{K3}^{[n]}$ type the problem of lifting diffeomorphisms has a different answer than the case of homeomeorphisms.

4.1 Formulation of the problem and known results for K3 surfaces

Let X be a IHS manifold with $\mathbf{L}_X = \mathrm{H}^2(X, \mathbb{Z})$ and consider the following diagram

$$\begin{array}{ccccc} \mathrm{Diff}^+(X) & \longrightarrow & \mathrm{Mod}(X) & \xrightarrow{\rho} & \mathrm{O}^+(\mathbf{L}_X) \\ \downarrow & & \downarrow & \swarrow & \\ \mathrm{Homeo}(X) & \longrightarrow & O(\mathbf{L}_X) & & \end{array},$$

the two following statements hold when S is a K3 surface:

Theorem 4.1.1 ([BK23],Theorem 1.1)**.** *Let S be a K3 surface. There is a section $s : \mathrm{O}^+(\mathbf{L}_S) \to \mathrm{Mod}(S)$ of $\rho : \mathrm{Mod}(S) \to \mathrm{O}^+(\mathbf{L}_S)$.*

Theorem 4.1.2 ([BK23],Theorem 1.2)**.** *Let S be a K3 surface. There is a subgroup of $\mathrm{Mod}(S)$ of order 2 which does not lift to a subgroup of order 2 of $\mathrm{Diff}^+(S)$. The image of the subgroup in $\mathrm{O}^+(\mathbf{L}_S)$ is non-trivial and it lifts to an order 2 subgroup of $\mathrm{Homeo}(S)$.*

We consider an IHS manifold X and G a finite subgroup of $\mathrm{Mod}(X)$, we can ask the following:

Problem (Nielsen realization)**.** *Does there exists an Einstein metric g on X such that G is realizable as a subgroup of* $\mathrm{Isom}(X, g)$*? Can the metric g be chosen to be also Kähler?*

We want to generalize to the situation where X is a higher dimensional IHS manifold, answer to the Nielsen realization problem (as done in [FL21, Theorem 1.2]) and give analogues of Theorem 4.1.1 and Theorem 4.1.2.

With the above setting, in section 4.2 we determine that, for X of type $\mathrm{K3}^{[n]}$ type with $n-1$ a prime power or OG10 type, the map $\rho : \mathrm{Mod}(X) \to \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$ admits a section. In section 4.3 we show that a similar example of order two group of mapping classes can be produced for IHS manifolds of $\mathrm{K3}^{[n]}$ type. In section 4.4 we define an invariant analogous to the invariant $\mathbf{\Gamma}_G$ used for K3 surfaces and conclude that a similar condition gives an answer to the Nielsen realization problem.

4.2 Sections of the representation map

Denote by Γ the image of $\mathrm{Mod}(X)$ via the representation map ρ, there are inclusions $\mathrm{Mon}^2(X) \subseteq \Gamma \subseteq \mathrm{O}^+(\mathbf{L}_X)$.

In the case of K3 surfaces, $\mathrm{Mod}(S)_{\mathcal{C}}$ maps isomorphically onto $\mathrm{O}^+(\mathbf{L}_S) = \mathrm{Mon}^2(S)$ via ρ giving the isomorphism

$$\mathrm{Mod}(S) \cong \mathrm{T}(S) \rtimes \mathrm{O}^+(\mathbf{L}_S)$$

which implies Theorem 4.1.1. In this particular case, the moduli space of marked K3 surfaces $\mathcal{M}_{\mathbf{L}_S} = \mathcal{T} / \mathrm{T}(S)$ is connected and $\mathrm{T}(S)$ permutes transitively the connected components of $\mathcal{T}_{HK}$.

Remark 5. *If X is an IHS manifold of dimension bigger than* 2 *then* $\mathrm{Mod}(X)$ *could be just an extension of* $\mathrm{T}(X)$ *and* Γ*, similarly* $\mathrm{Mod}(X)_{\mathcal{C}}$ *could be an extension of* $\mathrm{Mon}^2(X)$ *and* $\mathrm{T}(X) \cap \mathrm{Mod}(X)_{\mathcal{C}}$*, but by [Ver20, Remark 2.5] the intersection* $\mathrm{T}(X) \cap \mathrm{Mod}(X)_{\mathcal{C}}$ *is always finite. Moreover,* $\mathrm{T}(X)$ *acts on* $\pi_0(\mathcal{T}_{HK})$ *with finitely many orbits, each connected component has finite stabilizer and an element of* $\mathrm{T}(X)$ *which fixes an element $g \in \mathcal{T}$ fixes the entire connected component of g ([Ver20, Theorem 3.1]). In general, the moduli space of marked IHS manifold manifolds* $\mathcal{M}_{\mathbf{L}_X} = \mathcal{T} / \mathrm{T}(X)$ *could have more connected components, but each one is simply connected.*

Rephrasing what we said before, $\mathrm{Mon}^2(X) \cong \mathrm{Mod}(X)_{\mathcal{C}}$ precisely when $\mathrm{T}(X) \cap \mathrm{Mod}(X)_{\mathcal{C}}$ is trivial and $\mathrm{Mod}(X) \cong \mathrm{T}(X) \rtimes \Gamma$ exactly when ρ admits a section on its image. There could be a section of ρ over its image even if $\mathrm{Mon}^2(X)$ is a proper subgroup of $\mathrm{O}^+(\mathbf{L}_X)$ and on the other hand a priori there is still the possibility that $\mathrm{Mon}^2(X) = \Gamma = \mathrm{O}^+(\mathbf{L}_X)$ but $\mathrm{Mod}(X)$ is a not the semidirect product of $\mathrm{T}(X)$ and the stabilizer of a component.

Here we generalize the proof of [BK23, Theorem 1.1] in some cases. Note that in this setting most of the groups we consider are discrete. If G is a group we denote by $\mathrm{E}\,G$ the *universal bundle* of G and by $\mathrm{B}\,G$ the *classifying space* of G.

Let $X = (M, g, I)$ and recall that the kernel of the map $\mathrm{Aut}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ is a deformation invariant, hence it is the same for any complex structure that makes g a Kähler metric.

Lemma 4.2.1. *If X is such that $\mathrm{Aut}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ is injective, then $\mathrm{T}(X)$ acts freely on $\mathcal{T}_{Ein}$. In particular, the projection $\mathcal{T}_{Ein} \to \mathcal{T}_{Ein}/\,\mathrm{T}(X) =: \mathcal{M}_{Ein}$ is a principal $\mathrm{T}(X)$-bundle.*

Proof. Suppose that there are $[\varphi] \in \mathrm{T}(X)$ and $[g] \in \mathcal{T}_{Ein}$ such that $\varphi(g) = g'$ is isotopic to g, then the path connecting g' to g connects φ to a diffeomorphism φ' which fixes g. The diffeomorphism φ' acts as an orientation-preserving isometry on the 2-sphere of complex structures associated to g so it must preserve a complex structure and hence $\varphi' \in \mathrm{Aut}(X)$ is an automorphism acting trivially in cohomology, in conclusion $\varphi' = \mathrm{id}$ and this implies $[\varphi] = [\mathrm{id}]$. □

We set $\mathrm{M} = \mathcal{T}_{Ein} \times_{\mathrm{Mod}(X)} \mathrm{E}\,\mathrm{Mod}(X)$.

Lemma 4.2.2. *Suppose X is such that $\mathrm{Aut}(X) \to \mathrm{O}^+(\mathbf{\Lambda}_X)$ is injective, then there is a homotopy equivalence*

$$\mathrm{M} \cong \mathcal{M}_{Ein} \times_{\Gamma} \mathrm{E}\,\Gamma.$$

Proof. Since $\mathrm{E}\,\mathrm{Mod}(X) \times \mathrm{E}\,\Gamma$ is a model for $\mathrm{Mod}(X)$ we have

$$\mathrm{M} \cong \mathcal{T}_{Ein} \times_{\mathrm{Mod}(X)} (\mathrm{E}\,\mathrm{Mod}(X) \times \mathrm{E}\,\Gamma),$$

which has a fibration over $\mathcal{T}_{Ein} \times_{\mathrm{Mod}(X)} \mathrm{E}\,\Gamma$ with contractible fiber $\mathrm{E}\,\mathrm{Mod}(X)$. The base space has the homotopy type of a CW-complex hence the fibration is a homotopy equivalence and since the action of $\mathrm{T}(X)$ is free on $\mathcal{T}_{Ein}$ by Lemma 4.2.1, then $\mathcal{T}_{Ein} \times_{\mathrm{Mod}(X)} \mathrm{E}\,\Gamma = \mathcal{M}_{Ein} \times_{\Gamma} \mathrm{E}\,\Gamma$. □

Proposition 4.2.3. *Let X be an IHS manifold such that $\mathrm{Aut}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ is injective, then $\rho : \mathrm{Mod}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ has a section over its image Γ.*

Proof. The long exact sequence of homotopy groups associated to the fibration $\mathcal{M}_{Ein} \to \mathrm{M} \to \mathrm{B}\,\Gamma$ implies that

$$\pi_1(\mathcal{M}_{Ein}) \to \pi_1(\mathrm{M}) \to \pi_1(\mathrm{B}\,\Gamma) \to \pi_0(\mathcal{M}_{Ein})$$

is exact. Moreover, the connected components of the Teichmüller space have the same topology of the ones of the moduli space because $\mathrm{T}(X)$ simply permutes some components and hence $\pi_1(\mathcal{M}_{Ein}) = 1$ by Proposition 1.3.69. The natural projection map $\mathrm{M} \to \mathrm{B}\,\Gamma$ induces an injection $\pi_1(\mathrm{M}) \hookrightarrow \pi_1(\mathrm{B}\,\Gamma) = \Gamma$ and from the description of M, the map $\mathrm{M} \to \mathrm{B}\,\Gamma$ must factor as

$$\mathrm{M} \to \mathrm{B}\,\mathrm{Mod}(X) \to \mathrm{B}\,\Gamma.$$

In conclusion, the induced map $s : \pi_1(\mathrm{M}) \to \pi_1(\mathrm{B}\,\mathrm{Mod}(X)) = \pi_0(\mathrm{Mod}(X)) = \mathrm{Mod}(X)$ is a splitting of $\rho : \mathrm{Mod}(X) \to \Gamma \subseteq \mathrm{O}^+(\mathbf{L}_X)$. □

Corollary 4.2.4. *If X is of $K3^{[n]}$ type with $n-1$ a power of a prime, or X is of* OG10 *-type, then $\rho : \mathrm{Mod}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ has a section.*

Proof. In these cases $\mathrm{Mon}^2(X) = \Gamma = \mathrm{O}^+(\mathbf{L}_X)$ and $\mathrm{Aut}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ is injective. □

Question 1. *What can be said about the other known deformation types?*

The group $\mathrm{Mon}^2(X)$ is available for all the known deformation types. If it is maximal, then $\Gamma = \mathrm{O}^+(X)$, but if it is a proper subgroup of $\mathrm{O}^+(\mathbf{L}_X)$, then Γ is not known by the author.
Moreover, in the case $\mathrm{Aut}(X) \to \mathrm{O}^+(\mathbf{L}_X)$ is not injective the argument given above does not work, but it is a priori not clear if the same result might hold or not.

4.3 Lift of an order 2 subgroup

We now consider the Hilbert scheme of points $S^{[n]}$ of a K3 surface S and its symmetric product $S^{(n)}$. Notice that for $f \in \mathrm{Diff}(S)$ the induced map $f^{(n)}$ fixes the singular locus $\Delta = \{(x_1, \dots, x_n) \in S^{(n)} | \exists i \neq j; x_i = x_j\} \subset S^{(n)}$ and hence it lifts via the resolution

$$S^{[n]} \to S^{(n)}$$

to an element $f^{[n]} \in \mathrm{Diff}^+(S^{[n]})$ which fixes the exceptional locus, by the proof of [Boi12, Lemme 1]. This gives the inclusion

$$\Psi : \mathrm{Diff}^+(S) \hookrightarrow \mathrm{Diff}^+(S^{[n]})$$

since two elements $f, g \in \mathrm{Diff}^+(S)$ such that $f^{[n]} = g^{[n]}$ must coincide: by contracting the exceptional divisor $f^{(n)} = g^{(n)}$ and then restricting to the small diagonal $S \cong \{(x_1, \dots, x_n) \in S^{(n)} | x_1 = \dots = x_n\} \subset \Delta \subset S^{(n)}$ one gets $f = g$. With a similar argument, there is an injection

$$\mathrm{O}^+(\mathbf{L}_S) \hookrightarrow \mathrm{O}^+(\mathbf{L}_{S^{[n]}})$$

with a retraction again given by contraction and restriction.
If two elements f and g lie in the same path connected component of $\mathrm{Diff}^+(S)$, applying Ψ gives a path from $f^{[n]}$ to $g^{[n]}$ so there is also a well-defined map

$$\mathrm{Mod}(S) \to \mathrm{Mod}(S^{[n]})$$

and a commutative diagram

$$\begin{array}{ccccc} \mathrm{Diff}^+(S) & \longrightarrow & \mathrm{Mod}(S) & \longrightarrow & \mathrm{O}^+(\mathbf{L}_S) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{Diff}^+(S^{[n]}) & \longrightarrow & \mathrm{Mod}(S^{[n]}) & \longrightarrow & \mathrm{O}^+(\mathbf{L}_{S^{[n]}}) \end{array} . \tag{4.1}$$

We recall the following construction from [BK23, Section 3]: S is topologically homeomorphic to $3(M \times M)\sharp 2(N)$, where N denotes the compact and simply-connected topological 4-manifold with intersection form the negative E_8-lattice and M is the 2-sphere. Let $f_0 : M \times M \to M \times M$ be given by $f(x, y) = (y, x)$. Consider the equivariant connected sum $3(M \times M)$, the sum of three copies of $(M \times M, f_0)$, remembering that f_0 has fixed points. Attaching two copies of N, we get a continuous involution $f : S \to S$.

Theorem 4.3.1. *Let X be an IHS manifold of $K3^{[n]}$ type. There is a subgroup of* $\mathrm{Mod}(X)$ *of order* 2 *which does not lift to an order* 2 *subgroup of* $\mathrm{Diff}^+(X)$*. The image of this group in* $\mathrm{O}^+(\mathbf{L}_X)$*, which is not trivial, lifts to a subgroup of order* 2 *in* $\mathrm{Homeo}(X)$*.*

Proof. Let $f \in \mathrm{Homeo}(S)$ be the topological involution described above and consider the induced action $\phi \in \mathrm{O}^+(\mathbf{L}_S) \subset \mathrm{O}^+(\mathbf{L}_{S^{[n]}})$. Clearly we can put $\widetilde{\phi} = s(\phi) \in \mathrm{Mod}(S^{[n]})$, where $s : \mathrm{O}^+(\mathbf{L}_S) \to \mathrm{Mod}(S^{[n]})$ is obtained by composing the section of $\mathrm{Mod}(S) \to \mathrm{O}^+(\mathbf{L}_S)$, which exists by Theorem 4.1.1, with the middle vertical arrow in diagram (4.1). We observe that $\widetilde{\varphi}$ is non-trivial because by construction it has non-trivial action in cohomology.

Using the commutativity of (4.1) we can choose a lift of $\widetilde{\phi}$ of the form $h^{[n]} \in \mathrm{Diff}^+(S^{[n]})$ for some $h \in \mathrm{Diff}(S)$. Then $h^{[n]}$ cannot be an involution, since $h \in \mathrm{Diff}^+(S)$ acts in cohomology as ϕ and this is a contradiction to [BK23, Theorem 3.1].

The statement for general X follows by Ehresmann's Lemma.

□

This provides an example of order two subgroup of $\mathrm{Mod}(X)$ which does not admit a lift to $\mathrm{Diff}^+(X)$, but whose representation in second cohomology lifts to $\mathrm{Homeo}(X)$.

4.4 Nielsen realization for IHS manifolds

Let X be an IHS manifold and let G be a finite subgroup of $\mathrm{Mod}(X)$, by abuse of notation its image in $\mathrm{O}^+(\mathbf{L}_X)$ will be sometimes denoted again by G. We want to give an answer to the Nielsen realization problem in terms of a lattice which is invariant, as done in [FL21, Theorem 2.1] for K3 surfaces.

Since $\mathrm{Gr}^+(3, \mathbf{L}_\mathbb{R})$ is the symmetric space of $\mathrm{O}^+(\mathbf{L}_\mathbb{R})$, it is non-positively curved and then G must fix a point P. This means that P is a G-invariant positive 3-space, and hence there is a linear representation $G \to \mathrm{SO}(P)$ of P. Let $\mathbf{I}_G$ be the sum of all the irreducible G-subrepresentations of $\mathbf{L}_\mathbb{R}$ which are isomorphic to any of the ones appearing in P.

Definition 4.4.1. Let $\mathbf{\Gamma}_G = \mathbf{I}_G^\perp \cap \mathbf{L}_X$.

Remark 6. *Notice that in [FL21] this is denoted by $\mathbf{L}_G$ but this might lead to confusion because $\mathbf{L}_G$ sometimes denotes the coinvariant lattice $(\mathbf{L}^G)^\perp$ where $\mathbf{L}^G = \{v \in \mathbf{L} \,|\, g(v) = v \forall g \in G\}$, but the coinvariant lattice and $\mathbf{\Gamma}_G$ in fact differ in general. For example, $\mathbf{\Gamma}_G$ is always negative definite but if G comes from the action of non-symplectic automorphisms then the coinvariant lattice $\mathbf{L}_G$ is not definite.*

Recall that if $\mathcal{C}$ is a connected component of the Teichmüller space, $\Delta_\mathcal{C} \subset \mathbf{L}_X^\vee$ denotes the set of indivisible negative forms which are represented by an irreducible rational curve for an IHS manifold metric belonging to $\mathcal{C}$.

Theorem 4.4.2. *Let G be a finite subgroup of* $\mathrm{Mod}(X)$.

1. *G lifts to a group of isometries of an Einstein metric if and only if G fixes a connected component $\mathcal{C}$ of $\mathcal{T}_{Ein}$ and $\mathbf{\Gamma}_G$ does not contain any element of $\Delta_\mathcal{C}$.*

2. *G lifts to a group of automorphisms if and only if G fixes a connected component $\mathcal{C}$ of $\mathcal{T}_{Ein}$, $\mathbf{\Gamma}_G$ does not contain any element of $\Delta_\mathcal{C}$ and $\mathbf{\Gamma}_G^\perp$ contains the trivial representation (in this case the metric can be chosen so that X is projective and G acts by algebraic automorphisms).*

Similarly, a finite subgroup of $\mathrm{O}^+(\mathbf{L}_X)$ *lifts under the same conditions when it is contained in* $\mathrm{Mon}^2(X)$.

Proof. From the description in Proposition 1.3.69, each connected component $\mathcal{C}$ of the Teichmüller space is mapped diffeomorphically onto

$$\mathrm{Gr}^+(3, \mathbf{L}_\mathbb{R})_{\Delta_\mathcal{C}} = \mathrm{Gr}^+(3, \mathbf{L}_\mathbb{R}) - \bigcup_{\delta \in \Delta_\mathcal{C}} \mathrm{Gr}^+(3, \delta^\perp \otimes \mathbb{R})$$

which is connected (and simply connected). This in particular means that if G comes from a group of isometries for an Einstein metric, then the image P via the period map is G-invariant and not orthogonal to any $\delta \in \Delta_\mathcal{C}$, hence $\mathbf{\Gamma}_G$ does not contain any δ. If G preserves a metric which is also Kähler, then the positive cone must be preserved by G and we can find a G-invariant Kähler class which spans the trivial representation in $\mathbf{\Gamma}_G^\perp$.

Suppose now that G is a subgroup of $\mathrm{Mod}(X)$ which preserves a connected component of the Teichmüller space and for which $\mathbf{\Gamma}_G$ does not contain any

element in $\Delta_{\mathcal{C}}$. We argue as in the proof of [FL21, Theorem 1.2]: among the G-invariant 3-spaces $P \subset \mathbf{\Gamma}_G^{\perp} \otimes \mathbb{R}$, the ones such that $P^{\perp} \cap \mathbf{L}_X = \mathbf{\Gamma}_G$ are dense, so we can find a positive-definite $P \subset \mathbf{\Gamma}_G^{\perp} \otimes \mathbb{R}$ such that $P^{\perp} \cap \mathbf{L}_X = \mathbf{\Gamma}_G$. Now, since P does not lie in any $\delta^{\perp}$ for $\delta \in \Delta_{\mathcal{C}}$, the surjectivity of the period map in Proposition 1.3.69 ensures that there exists a IHS manifold $X = (M, g, I)$ with period P and such that $g \in \mathcal{C}$. By hypothesis $G \subseteq \mathrm{Mod}(X)_{\mathcal{C}}$ hence its action in cohomology consists of monodromy operators by Proposition 1.3.68 and then there is a lift of G (possibly an extension) in $\mathrm{Diff}^{+}(X)$. By construction P is fixed by G and hence g is preserved, so that G consists of isometries for the metric g. Lastly, having the trivial representation in $\mathbf{\Gamma}_G^{\perp}$ means that G fixes a positive class $0 \neq k \in P$ and hence the orientation determines a complex structure on $k^{\perp} \subset P$ which, again by surjectivity of the period map, is achieved by a complex structure on X that makes g a Kähler metric.
The trivial representation is spanned by a positive integral $(1,1)$-class, so we can conclude using Huybrechts' projectivity criterion Proposition 1.3.18.

□

The situation could be much more complicated than for K3 surfaces: as already noticed in [Mar11, Question 10.5] the stabilizer $\mathrm{Mod}(X)_{\mathcal{C}}$ could depend on the component $\mathcal{C}$ and it could intersect nontrivially the Torelli group, so it could happen that not every subgroup of $\mathrm{O}^{+}(\mathbf{L}_X)$ is the image of some stabilizer of a component and even those which are could have elements acting trivially on $\mathbf{L}_X$. In case $G \subseteq \mathrm{Mon}^2(X)$ is the image of a group which intersects non-trivially $\mathrm{T}(X)$, then a lift could be found but it would be an extension of G.

Chapter 5

Non-symplectic involutions of manifolds of OG10 type

In the first section of this chapter we classify the non-symplectic involutions on manifolds of OG10 type. Our classification is lattice-theoretical and consists of determining the involutions by their invariant and coinvariant lattices. In the second section we study the induced transformations of a non-symplectic involution of a cubic fourfold on the associated LSV manifold.

5.1 Classification of non-symplectic involutions

In this section we let $\mathbf{L} := \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus \mathbf{A}_2(-1)$ be the abstract lattice isometric to the second cohomology of a manifold of OG10 type, recall that there is a unique embedding (up to isometry) $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$ in the unimodular lattice $\mathbf{\Lambda} := \mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 5}$ with orthogonal complement given by $\mathbf{L}^\perp \cong \mathbf{A}_2$. We classify the non-symplectic involutions of manifolds of OG10 by listing the possible invariant and coinvariant lattices of their action in cohomology, this is achieved passing to the classification of invariant and coinvariant lattices of an involution on $\mathbf{\Lambda}$.

5.1.1 Admissible invariant and coinvariant sublattices of $\mathbf{\Lambda}$

First of all we list pairs of invariant lattices $\mathbf{\Lambda}^G$ and coinvariant lattices $\mathbf{\Lambda}_G$ of $\mathbf{\Lambda}$ with prescribed signature, where G is generated by an involution.

Proposition 5.1.1. *Let $G \subset \mathrm{O}(\mathbf{\Lambda})$ be a subgroup of order 2. If $\mathrm{sgn}(\mathbf{\Lambda}_G) = (2, \mathrm{rk}(\mathbf{\Lambda}_G) - 2)$ then the pairs $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ appear in Table B.1. If $\mathrm{sgn}(\mathbf{\Lambda}_G) = (3, \mathrm{rk}(\mathbf{\Lambda}_G) - 3)$ then the pairs $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ appear in Table B.1 where the roles of $\mathbf{\Lambda}^G$ and $\mathbf{\Lambda}_G$ are inverted.*

Proof. Since G is cyclic of order 2 and $\mathbf{\Lambda}$ is unimodular, then $\mathbf{\Lambda}^G$ and $\mathbf{\Lambda}_G$ must be 2-elementary lattices and their discriminant groups are anti-isometric by

Lemma 1.1.15, in particular they have the same length. Use Theorem 1.1.17 to get all the possible isometry classes of such lattices by varying the signature, the length a and the invariant δ. Any such pair of lattices are invariant and coinvariant lattices for the isometry that acts trivially on the invariant lattice and as -1 on the coinvariant lattice. □

Lemma 5.1.2. *Consider the primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$. If $\varphi \in \mathrm{O}(\mathbf{L})$ is an isometry such that $\overline{\varphi} = \mathrm{id} \in \mathrm{O}(\mathrm{A}_{\mathbf{L}})$ then it extends to and element $\tilde{\varphi} \in \mathrm{O}(\mathbf{\Lambda})$ acting trivially on $\mathbf{L}^{\perp} \subset \mathbf{\Lambda}$. If $\varphi \in \mathrm{O}(\mathbf{L})$ is an isometry such that $\overline{\varphi} = -\mathrm{id} \in \mathrm{O}(\mathrm{A}_{\mathbf{L}})$, then φ extends to an isometry $\tilde{\varphi} \in \mathrm{O}(\mathbf{\Lambda})$ that acts on $\mathbf{L}^{\perp}$ permuting the generators of $\mathbf{L}^{\perp} \subset \mathbf{\Lambda}$.*

Proof. Let a, b be generators of $\mathbf{A}_2(-1) \subset \mathbf{L}$ and consider the generator $[\frac{a-b}{3}] = [\frac{a+2b}{3}]$ of $\mathbf{L} \cong \mathbb{Z}/3\mathbb{Z}$. If $\varphi \in \mathrm{O}(\mathbf{L})$ is such that $\overline{\varphi} = \mathrm{id}$ then $\overline{\varphi}([\frac{a-b}{3}]) = [\frac{a-b}{3}]$ hence $\varphi(a-b) = a - b + 3w$ with $w \in \mathbf{L}$. Let c, d be generators of $\mathbf{L}^{\perp} \cong \mathbf{A}_2$, its discriminant group is also $\mathbb{Z}/3\mathbb{Z}$ and it is generated by $[\frac{c-d}{3}]$ with discriminant form given by $q(\frac{c-d}{3}) = 2/3$. Notice that $\mathbf{L} \oplus \mathbf{A}_2$ has an overlattice isometric to $\mathbf{\Lambda}$ which is generated by $\mathbf{L}$, $\frac{a-b+c-d}{3}$ and $\frac{a+2b+c+2d}{3}$. We extend φ to $\mathbf{L} \oplus \mathbf{A}_2$ by imposing $\varphi(c) = c$ and $\varphi(d) = d$ and we obtain an extension $\tilde{\varphi}$ of φ on $\mathbf{\Lambda}$ as follows:

$$\tilde{\varphi}(\frac{a-b+c-d}{3}) = \frac{\varphi(a-b)+c-d}{3}$$

and

$$\tilde{\varphi}(\frac{a+2b+c+2d}{3}) = \frac{\varphi(a+2b)+c+2d}{3}.$$

If $\varphi \in \mathrm{O}(\mathbf{L})$ is such that $\overline{\varphi} = -\mathrm{id}$ then $\overline{\varphi}([\frac{a-b}{3}]) = [\frac{b-a}{3}]$ hence we extend φ to $\mathbf{L} \oplus \mathbf{A}_2$ by imposing $\varphi(c) = d$ and $\varphi(d) = c$ and we obtain an extension $\tilde{\varphi}$ of φ on $\mathbf{\Lambda}$ as follows:

$$\tilde{\varphi}(\frac{a-b+c-d}{3}) = \frac{\varphi(a-b)+d-c}{3}$$

and

$$\tilde{\varphi}(\frac{a+2b+c+2d}{3}) = \frac{\varphi(a+2b)+d+2c}{3}.$$

□

Proposition 5.1.3. *Let $G \subset \mathrm{O}(\mathbf{L})$ be a subgroup and consider its image $\overline{G} \subset \mathrm{O}(\mathrm{A}_{\mathbf{L}})$. Consider the primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$, and let c and d be the generators of $\mathbf{A}_2 = \mathbf{L}^{\perp} \subset \mathbf{\Lambda}$.*

- *If $|\overline{G}| = 1$ there exists a subgroup $G' \subset \mathrm{O}(\mathbf{\Lambda})$ such that G' restricts to G on $\mathbf{L}$ and $\mathbf{L}_G = \mathbf{\Lambda}_{G'}$. In particular $\mathrm{sgn}(\mathbf{L}_G) = \mathrm{sgn}(\mathbf{\Lambda}_{G'})$ and $\mathrm{sgn}(\mathbf{L}^G) = \mathrm{sgn}(\mathbf{\Lambda}^{G'}) - (2, 0)$.*

- *If* $|\overline{G}| = 2$ *there exists a subgroup* $G' \subset \mathrm{O}(\mathbf{\Lambda})$ *such that* G' *restricts to* G *on* $\mathbf{L}$ *and* $\mathbf{L}_G = (c-d)^\perp \subset \mathbf{\Lambda}_{G'}$*,* $\mathbf{L}^G \cong (c+d)^\perp \subset \mathbf{\Lambda}^{G'}$*. In particular* $\mathrm{sgn}(\mathbf{L}_G) = \mathrm{sgn}(\mathbf{\Lambda}_{G'}) - (1,0)$ *and* $\mathrm{sgn}(\mathbf{L}^G) = \mathrm{sgn}(\mathbf{\Lambda}^{G'}) - (1,0)$.

Proof. Direct consequence of Lemma 5.1.2. □

Notice that for a subgroup $G \subset \mathrm{O}(\mathbf{L})$ of prime order p, the situation were $|\overline{G}| = 2$ will happen only for $p = 2$.

5.1.2 Invariant and coinvariant lattices of L

Consider an isometry $\varphi \in \mathrm{O}(\mathbf{L})$ and let $G \subset \mathrm{O}(\mathbf{L})$ the subgroup generated by φ. We say that $\varphi \in \mathrm{O}(\mathbf{L})$ is *non-symplectic* if $\mathrm{sgn}(\mathbf{L}^G) = (1, \mathrm{rk}(\mathbf{L}) - 1)$ and $\mathrm{sgn}(\mathbf{L}_G) = (2, \mathrm{rk}(\mathbf{L}) - 2)$.

Let X be a manifold of OG10 type with a marking $\eta : \mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{L}$ and let $G \subset \mathrm{Aut}(X)$ be a group generated by a non-symplectic automorphism of order p, then by Proposition 1.3.78 the group $G \subset \mathrm{O}(\mathbf{L})$ is generated by a non-symplectic isometry in the above sense. Viceversa:

Proposition 5.1.4. *Let* $G \subset \mathrm{O}(\mathbf{L})$ *be a group of prime order* p *generated by a non-symplectic isometry, then there exists an irreducible holomorphic symplectic manifold* X *of* OG10 *type with a marking* $\eta : \mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{L}$ *such that* $G \subset \mathrm{Aut}(X)$ *and* G *is generated by a non-symplectic automorphism.*

Proof. The proof is analogous to the one of [Gro22, Proposition 3.4]. A generator φ of G is non-symplectic and hence one can endow $\mathbf{L}_\mathbb{C}$ with a weight-two Hodge structure such that $\mathbf{L}^G = \mathbf{L}_\mathbb{C}^{1,1} \cap \mathbf{L}$. By the surjectivity of the period map there exists a manifold X of OG10 type and a marking $\mathrm{H}^2(X, \mathbb{Z}) \cong \mathbf{L}$ which is an isomorphism of Hodge structures. By construction G consists of Hodge isometries, moreover since all the algebraic classes are fixed then the positive cone is point-wise fixed and so the Kähler cone is. In this case we know that $\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathbf{L})$, we want to prove that $G \subset \mathrm{O}^+(\mathbf{L})$. This is clear when $p \neq 2$ since it is odd and $\varphi^p = \mathrm{id}$, while if $p = 2$ we have $\mathrm{spin}(\varphi) = +1$ using [Gro22, Lemma 2.4] with $\mathrm{sign}(\mathbf{L}_G) = (2, \mathrm{rk}(\mathbf{L}_G) - 2)$, so that $\varphi \in \mathrm{O}^+(\mathbf{L})$ in any case. Since $G \subset \mathrm{Mon}^2_{Hdg}(X)$ and a Kähler class is preserved by G we can conclude by Theorem 1.3.37, as the representation map on the second cohomology is injective for manifolds of OG10 type by [MW17, Theorem 2.1]. In particular, X is projective by Remark 3. □

Theorem 5.1.5. *Let* X *be a manifold of* OG10 *type and let* $G \subset \mathrm{Aut}(X)$ *be a subgroup of order* 2 *generated by a non-symplectic involution, then the pair* $(\mathbf{L}^G, \mathbf{L}_G)$ *appears either in Table B.2 or in Table B.3. Viceversa, any such pair consist of the invariant and coinvariant lattices for a non-symplectic involution on a manifold of* OG10 *type.*

Proof. Consider the induced action $G \subset \mathrm{O}(\mathbf{L})$ and extend it to an action $\widetilde{G} \subset \mathrm{O}(\mathbf{\Lambda})$ according to Lemma 5.1.2. The two different extensions lead to the possible cases where $\mathbf{L}_G = \mathbf{\Lambda}_{\widetilde{G}}$ or there are inclusions $\mathbf{L}_G \subset \mathbf{\Lambda}_{\widetilde{G}}$ and $\mathbf{L}^G \subset \mathbf{\Lambda}^{\widetilde{G}}$ with complement of rank 1. When the pair $(\mathbf{L}^G, \mathbf{L}_G)$ is determined, one can endow $\mathbf{L}$ with a Hodge structure that makes $\mathbf{L}^G$ and $\mathbf{L}_G$ the invariant and coinvariant lattices of a non-symplectic involution, then conclude using Proposition 5.1.4.

In case the induced action on the discriminant group is trivial then by Lemma 5.1.2 we have a primitive embedding $\mathbf{A}_2 \subset \mathbf{\Lambda}^{\widetilde{G}}$ and we have $\mathbf{L}^G = \mathbf{A}_2^{\perp \mathbf{\Lambda}^{\widetilde{G}}}$ since the orthogonal complement of the embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$ is isometric to $\mathbf{A}_2$. When there is a primitive embedding $\mathbf{A}_2 \hookrightarrow \mathbf{\Lambda}^{\widetilde{G}}$ it is unique up to isometries by Lemma 1.1.11 because $\mathbf{A}_2$ is 3-elementary and since the orthogonal complement satisfies Theorem 1.1.22.

In case the induced action on the discriminant group is non-trivial we consider the unique primitive embedding $\mathbf{A}_2 \hookrightarrow \mathbf{\Lambda}$ and observe that in this case $[2] = \langle a + b \rangle$ is $\widetilde{G}$-invariant, where a, b are generators of $\mathbf{A}_2 = \mathbf{L}^{\perp} \subset \mathbf{\Lambda}$, since $\widetilde{G}$ permutes them. Consider the lattices $\mathbf{\Lambda}^{\widetilde{G}}$ in Table B.1 that admit [2] as a primitive sublattice. We compute the list of possible $\mathbf{L}^G = [2]^{\perp \mathbf{\Lambda}^{\widetilde{G}}}$ for all the primitive embeddings $[2] \hookrightarrow \mathbf{\Lambda}^{\widetilde{G}}$. Then we obtain $\mathbf{L}_G$ as the orthogonal complement of the primitive embedding $\mathbf{L}^G \hookrightarrow \mathbf{L}$ when such an embedding exists.

Orthogonal complements of the previous embeddings are uniquely determined up to isometry because of Theorem 1.1.20 in most of the cases, by Theorem 1.1.22 in all the other cases. When applying the latter result, recall that $q_{\mathrm{A}_{\mathbf{D}_4(-1)}} \cong q_{v(1)}$ and $q_{\mathrm{A}_{\mathbf{U}(2)}} \cong q_{u(1)}$.

□

5.2 Induced non-symplectic involutions on Laza-Sacca-Voisin manifolds

We study the bimeromorphic involutions induced by non-symplectic involutions of a cubic fourfold on the LSV manifold, with the help of the Hodge relation of the cubic fourfold with the associated twisted LSV manifold.

Analogously to the case of IHS manifolds we have that if $\phi \in \mathrm{Aut}(Y)$ is symplectic then $(\mathrm{H}^4(Y,\mathbb{Z})_{prim})_\phi \subseteq \mathrm{H}^{2,2}(Y,\mathbb{Z})_{prim}$ and if ϕ is non-symplectic we have $(\mathrm{H}^4(Y,\mathbb{Z})_{prim})^\phi \subseteq \mathrm{H}^{2,2}(Y,\mathbb{Z})_{prim}$. In this section, a cubic fourfold Y with an automorphism $\phi \in \mathrm{Aut}(Y)$ is called *general* if one of the above inclusions is an equality, accordingly to Remark 4. A stronger version of Proposition 2.3.9 holds in the twisted case, but unfortunately the isometry of the following proposition has no direct geometric interpretation, contrarly to the previous isogeny.

Proposition 5.2.1. *Let Y be a cubic fourfold, then there is an Hodge isometry* $\mathrm{H}^4(Y,\mathbb{Z})_{prim}(-1) \cong \mathbf{U}_Y(3)^\perp \subset \mathrm{H}^2(J^t(Y),\mathbb{Z})$.

Proof. Recall that by [LPZ22] there is a LPZ manifold associated to the cubic fourfold Y that we denote by $\widetilde{\mathrm{M}}_\sigma(2(\lambda_1+\lambda_2),\mathcal{A}_Y)$, which is the resolution of a moduli space of Bridgeland semistable objects in the Kuznestov component $\mathcal{A}_Y$ of $\mathrm{D}^b(Y)$, Mukai vector $2(\lambda_1+\lambda_2)$ and stability condition σ. We know by [GGO22, Example 2.13] that there is a Hodge-isometric embedding $\mathrm{H}^4(Y,\mathbb{Z})_{prim}(-1) \hookrightarrow \mathrm{H}^2(\widetilde{\mathrm{M}}_\sigma(2(\lambda_1+\lambda_2),\mathcal{A}_Y),\mathbb{Z})$ with orthogonal complement of type $(1,1)$ and isometric to $\mathbf{U}(3)$. The manifold $\widetilde{\mathrm{M}}_\sigma(2(\lambda_1+\lambda_2),\mathcal{A}_Y)$ is birational to $J^t(Y)$ by [LPZ22, Theorem 1.3], hence composing the Hodge isometries we get the following Hodge isometry

$$\mathrm{H}^4(Y,\mathbb{Z})_{prim}(-1) \xrightarrow{\cong} \mathbf{U}_Y(3)^\perp \subset \mathrm{H}^2(J^t(Y),\mathbb{Z}).$$

□

We give a numerical criterion for a manifold of OG10 type to be bimeromorphic to a twisted LSV.

Proposition 5.2.2. *Let X be a manifold of* OG10 *type, there exists a cubic fourfold Y such that X is bimeromorphic to $J^t(Y)$ if and only if*

- *There a primitive embedding* $\mathbf{U}(3) \hookrightarrow \mathrm{NS}(X)$.
- *The lattice* $\mathbf{U}(3)^\perp(-1) \subset \mathrm{NS}(X)(-1)$ *has no short or long roots.*

Proof. If X and $J^t(Y)$ are bimeromorphic, then there is a Hodge isometry $\mathrm{H}^2(J^t(Y),\mathbb{Z}) \cong \mathrm{H}^2(X,\mathbb{Z})$, so the embedding $\mathbf{U}_Y^t \subset \mathrm{NS}(J^t(Y)) \cong \mathrm{NS}(X)$ induces the embedding of $\mathbf{U}(3)$ in $\mathrm{NS}(X)$. We know from Proposition 5.2.1 that the lattice $(\mathbf{U}_Y^t)^\perp \subset \mathrm{H}^2(J^t(Y),\mathbb{Z})$ is Hodge-isometric to $\mathrm{H}^4(Y,\mathbb{Z})_{prim}(-1)$. The description of the image of the period map of cubic fourfolds in Theorem 2.3.4 ensures that there are no long or short roots in $\mathrm{H}^4(Y,\mathbb{Z})_{prim}$. Viceversa, if there are no short or long roots in $\mathbf{U}(3)^\perp(-1) \subset \mathrm{NS}(X)(-1)$ then again by Theorem 2.3.4 we know that $\mathbf{U}(3)^\perp(-1) \subset \mathrm{H}^2(X,\mathbb{Z})(-1)$ is Hodge isometric to $\mathrm{H}^4(Y,\mathbb{Z})_{prim}$ for a cubic fourfold Y, which is also Hodge isometric to $\mathbf{U}(3)^\perp(-1) \subset \mathrm{H}^2(J^t(Y),\mathbb{Z})(-1)$. We can extend the Hodge isometry to the entire lattice $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathrm{H}^2(J^t(Y),\mathbb{Z})$ using [Nik79, Corollary 1.5.2] with the fact that all the isometries of the discriminant groups are induced by isometries of the lattices in this case. We conclude that X and $J^t(Y)$ are bimeromorphic, this is possible since $\mathbf{U}(3)$ is primitive and of type $(1,1)$ in both cases. □

Recall that an automorphism of the cubic Y induces a bimeromorphism of the LSV manifold $J(Y)$.

Lemma 5.2.3. *Let Y be a cubic fourfold, $\phi \in \mathrm{Aut}(Y)$ is symplectic if and only if $\widetilde{\phi} \in \mathrm{Bir}(J(Y))$ is.*

Proof. Proposition 2.3.9 or [Sac23, Lemma 3.2] □

Notice that composing the bimeromorphism $\widetilde{\phi}$ with the involution τ that acts as (-1) on the fibers of $J_U(Y) \to U$ turns it from symplectic to non-symplectic and viceversa. Moreover, $\widetilde{\phi}$ acts trivially on the discriminant group, while τ acts as $-\mathrm{id}$.

Lemma 5.2.4. *Let Y be a cubic fourfold, then there exists $N > 0$ such that there is the following finite index embedding of lattices*

$$\mathbf{U}_Y \oplus \mathrm{H}^{2,2}(Y, \mathbb{Z})_{prim}(-N) \subseteq \mathrm{NS}(J(Y)).$$

Proof. We know that the hyperbolic lattice $\mathbf{U}_Y$ is a lattice of $(1,1)$ type, and from Proposition 2.3.9 we know that there is a finite index embedding $\mathrm{H}^{2,2}(Y,\mathbb{Z})_{prim}(-N) \subseteq \mathbf{U}_Y^{\perp_{NS(J(Y))}}$ for some $N > 0$. □

Lemma 5.2.5. *Let Y be a cubic fourfold, $\phi \in \mathrm{Aut}(Y)$ an automorphism of finite order and $\widetilde{\phi} \in \mathrm{Bir}(J(Y))$ the induced bimeromorphism. Then $\mathbf{U}_Y \subseteq \mathrm{H}^2(J(Y),\mathbb{Z})^{\widetilde{\phi}}$ and there are inclusions*

$$(\mathrm{H}^4(Y,\mathbb{Z})_{prim})^{\phi}(-N) \subseteq \mathbf{U}_Y^{\perp} \subset \mathrm{H}^2(J(Y),\mathbb{Z})^{\widetilde{\phi}}$$

for some $N > 0$.

Proof. The anti-isogeny

$$\alpha\colon \mathrm{H}^4(Y,\mathbb{Z})_{prim} \to \mathbf{U}_Y^{\perp} \subset \mathrm{H}^2(J(Y),\mathbb{Z})$$

of Proposition 2.3.9 is an isomorphism of rational Hodge structures. Recall from [MO22, Lemma 7.1] that the map α is the restriction of the map $[Z]_* \circ q^* : \mathrm{H}^4(Y,\mathbb{Z}) \to \mathrm{H}(J(Y),\mathbb{Z})$ where $q : \mathcal{U}_Y \to Y$ is the inclusion of linear sections and $[Z]_*(x) = \pi_{1*}(\pi_2^* x.Z)$ where $Z \in \mathrm{CH}^2(J(Y) \times_{\mathbb{P}^5} \mathcal{U}_Y)_{\mathbb{Q}}$ is a distinguished cycle and π_1, π_2 the respective projections.

Clearly, replacing Z with $\widetilde{Z} = \frac{1}{\sqrt{\mathrm{ord}(\phi)}} \sum_{n\geq 0} \phi^n(Z)$ in the above definition, one gets an ϕ-invariant map

$$\widetilde{\alpha}\colon \mathrm{H}^4(Y,\mathbb{Z})_{prim} \to \mathbf{U}_Y^{\perp} \subset \mathrm{H}^2(J(Y),\mathbb{Z})$$

which is an anti-isometry of rational Hodge structures. Then one concludes that there is $N > 0$ such that $(\mathrm{H}^4(Y,\mathbb{Z})_{prim})^{\phi}(-N) \subseteq \mathbf{U}_Y^{\perp} \subset \mathrm{H}^2(X,\mathbb{Z})^{\widetilde{\phi}}$. □

We can recover a cubic fourfold from a certain action on a manifold of OG10 type.

Proposition 5.2.6. *Let X be a manifold of* OG10 *with a marking* $\mathrm{H}^2(X,\mathbb{Z}) \cong \mathbf{L}$ *and let $k = 1, 3$. Let $f \in \mathrm{Aut}(X)$ be a non-symplectic automorphism of prime order with a primitive embedding $\mathbf{U}(k) \hookrightarrow \mathbf{L}^f \subseteq \mathrm{NS}(X)$ such that such that putting $\mathbf{T} := \mathbf{U}(k)^{\perp \mathrm{NS}(X)} \subset \mathrm{NS}(X)$ the lattice $\mathbf{T}(-1)$ contains no short and long roots. Then there is an associated cubic fourfold Y with an automorphism $\phi \in \mathrm{Aut}(Y)$ such that:*

- $(\mathrm{H}^4(Y,\mathbb{Z})_{prim})^{\phi}(-1) \cong \mathbf{U}(k)^{\perp \mathbf{L}^f}$ *if* $\overline{f} = \mathrm{id} \in \mathrm{O}(A_{\mathbf{L}})$,
- $(\mathrm{H}^4(Y,\mathbb{Z})_{prim})^{\phi}(-1) \cong \mathbf{L}_f$ *if* $\overline{f} \neq \mathrm{id} \in \mathrm{O}(A_{\mathbf{L}})$.

Proof. The lattice $\mathbf{N} := \mathbf{U}(k)^{\perp \mathbf{L}} \subset \mathbf{L}$ is abstractly anti-isometric to the lattice of primitive middle cohomology of a cubic fourfold for $k = 1, 3$. Moreover, $\mathbf{N}$ inherits a Hodge structure such that $\mathbf{N}^{2,2} = \mathbf{T}$. By hypothesis $\mathbf{T}(-1)$ has no short or long roots, so by Theorem 2.3.5 there exists a cubic fourfold Y such that $\mathrm{H}^4(Y,\mathbb{Z})_{prim} \cong \mathbf{N}(-1)$ as Hodge structures. The lattice $\mathbf{N}(-1)$ has an induced action from the one of f that preserves the Hodge structure, we want to extend it to an action on $\mathrm{H}^4(Y,\mathbb{Z})$ that preserves the square of an hyperplane class $\langle h^2 \rangle = \mathrm{H}^4(Y,\mathbb{Z})^{\perp}_{prim} \subset \mathrm{H}^4(Y,\mathbb{Z})$. This can be done whenever f acts trivially on the discriminant group $\mathrm{A}_{\mathbf{L}(-1)} \cong \mathrm{A}_{\mathbf{N}}$.

If f acts trivially on the discriminant group, then we have an extension of f on $\mathrm{H}^4(Y,\mathbb{Z})$ fixing h^2, so by Theorem 2.3.5 there is a unique automorphism $\phi \in \mathrm{Aut}(Y)$ inducing f, in this case $\mathbf{N}^f(-1) \cong \mathbf{U}(k)^{\mathbf{L}^f}$ and $\mathbf{N}_f(-1) \cong \mathbf{L}_f$. The case where f acts as $-\mathrm{id}$ on the discriminant group is possible only when f is an involution, hence we can replace f with $g := -f$ so that there exists a unique $\phi \in \mathrm{Aut}(Y)$ inducing g, in this case $\mathbf{N}^g(-1) \cong \mathbf{L}_f$ and $\mathbf{N}_g(-1) \cong \mathbf{U}(k)^{\mathbf{L}^f}$.

□

Remark 7. *An automorphism on a cubic fourfold Y induces a natural bimeromorphism on $J(Y)$, but it is a priori not clear how to relate its action in cohomology with the action on the cohomology of the cubic fourfold, because of the index N of Lemma 5.2.4. In particular, the induced action on $J(Y)$ will satisfy the hypothesis of Proposition 5.2.6 but then there seems to be no reason to conclude that the associated cubic fourfold is Y itself. All the information about the transformation which is induced geometrically is given by Lemma 5.2.4 and Lemma 5.2.5, the other associated transformations exist only for Hodge-theoretical reasons.*

The involutions of a cubic fourfold were classified in [Mar23, Theorem 1.1] where their action is cohomology is described, there are three types ϕ_1, ϕ_2, ϕ_3, where ϕ_2 is symplectic and ϕ_1, ϕ_3 are not. Moreover, cubic fourfolds with the involution ϕ_3 belong to $\mathcal{C}_{14} \cap \mathcal{C}_8$ and cubic foufolds with the involution ϕ_1 belong to $\mathcal{C}_8$. The Hasset divisor $\mathcal{C}_{14}$ consists of the closure of the locus of Pfaffian cubic fourfolds. It is known that for a Pfaffian cubic fourfold Y the manifolds $J(Y)$ and $J^t(Y)$ are isomorphic, while by [GGO22, Proposition 4.3] together with [LPZ22, Theorem 1.3] we have that for a cubic fourfold Y in $\mathcal{C}_8$ the manifolds $J(Y)$ and $J^t(Y)$ are bimeromorphic.

The description of the invariant and coinvariant lattices of an involution on a cubic fourfold according to [Mar23, Theorem 1.1] involves the following lattice:

Definition 5.2.7. The lattice $\mathbf{M}$ is the lattice given by the following matrix

$$\mathbf{M} = \begin{pmatrix} 6 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & \\ 2 & 4 & -2 & & & & & & & \\ -2 & -2 & 4 & -2 & & & & & & \\ 2 & & -2 & 4 & -2 & & & & & \\ -2 & & & -2 & 4 & -2 & & & & \\ 2 & & & & -2 & 4 & -2 & & & \\ -2 & & & & & -2 & 4 & -2 & & \\ 2 & & & & & & -2 & 4 & -2 & -2 \\ -2 & & & & & & & -2 & 4 & \\ & & & & & & & -2 & & 4 \end{pmatrix}$$

that can also be described as the unique index 2 overlattice of $\mathbf{D}_9(2) \oplus [24]$. Notice that we have the isometry $\mathbf{U} \oplus \mathbf{M}(-1) \cong [2] \oplus [-2] \oplus \mathbf{E}_6(-2) \oplus \mathbf{D}_4(-1)$.

The above classification allows us to describe the induced bimeromorphic non-symplectic involutions of manifolds of OG10 induced by an involutions of cubic fourfolds via the LSV construction, and prove that they are regular in the general case.

Proposition 5.2.8. *The non-symplectic involutions of a cubic fourfold Y induce the following bimeromorphisms on $J(Y)$:*

- ϕ_1 *induces an non-symplectic involution* $f_1 \in \mathrm{Bir}(J(Y))$ *such that* $\mathbf{L}^{f_1} = \mathbf{U} \oplus \mathbf{E}_6(-2)$ *and* $\mathbf{L}_{f_1} = \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 3}$*,*
- ϕ_3 *induces a non-symplectic involution* $f_3 \in \mathrm{Bir}(J(Y))$ *such that* $\mathbf{L}^{f_3} = \mathbf{U} \oplus \mathbf{M}(-1)$ *and* $\mathbf{L}_{f_3} = \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9}$*,*

where $\mathbf{L} = \mathrm{H}^2(J^t(Y), \mathbb{Z})$*.*

Proof. By [Mar23, Theorem 1.1] we have a classification of possible invariant and coinvariant sublattices of $\mathrm{H}^4(Y, \mathbb{Z})_{prim}$ for non-symplectic involutions on a general Y. Furthermore, we know that if Y is a cubic fourfold with a non-symplectic involution then $Y \in \mathcal{C}_8$ and $J(Y)$ is bimeromorphic to $J^t(Y)$, so that $\mathrm{NS}(J(Y)) \cong \mathrm{NS}(J^t(Y))$. Denote by Y_i a cubic fourfold with the involution ϕ_i for $i = 1, 3$. Using Proposition 5.2.1 we have embeddings $\mathrm{NS}(J(Y_1)) \supset \mathbf{U}_{Y_1}(3)^{\perp} \cong \mathbf{E}_6(-2)$ and $\mathrm{NS}(J(Y_3)) \supset \mathbf{U}_{Y_3}(3)^{\perp} \cong \mathbf{M}(-1)$, hence $\mathrm{NS}(J(Y_1)) = \mathbf{U} \oplus \mathbf{E}_6(-2)$ and $\mathrm{NS}(J(Y_3)) = \mathbf{U} \oplus \mathbf{M}(-1)$. We know that since the involutions are non-symplectic $\mathbf{L}^{f_i} \subseteq \mathrm{NS}(J(Y_i))$ must hold for $i = 1, 3$ and by Lemma 5.2.5 the inclusion is a finite index embedding, hence $\mathbf{L}^{f_1}$ appears in Table B.2 with signature $(1, 7)$ and $\mathbf{L}^{f_3}$ with signature $(1, 11)$. We exclude all cases apart from the ones in the statement, since for those there are vectors of square two that would produce short roots in $\mathrm{H}^{2,2}(Y_i, \mathbb{Z})_{prim}$ by Proposition 5.2.1 and this is a contradiction to Theorem 2.3.4. □

Corollary 5.2.9. *The general involution $\phi_i \in \mathrm{Aut}(Y)$ induces a regular automorphism $f_i \in \mathrm{Aut}(J(Y))$ for $i = 1, 3$. Moreover, for any general involution $f \in \mathrm{Aut}(X)$ on X of* OG10 *type with $\mathrm{H}^2(X, \mathbb{Z})^f = \mathbf{U} \oplus \mathbf{E}_6(-2)$ or $\mathrm{H}^2(X, \mathbb{Z})^f = \mathbf{U} \oplus \mathbf{M}(-1)$ there is a cubic fourfold Y such that X is bimeromorphic to $J(Y)$ and f is bimeromorphically induced by the involution $f_1 \in \mathrm{Aut}(J(Y))$ or $f_3 \in \mathrm{Aut}(J(Y))$.*

Proof. From Proposition 5.2.8 we know that $\mathrm{NS}(J(Y)) = \mathrm{H}^2(J(Y), \mathbb{Z})^{f_i}$, hence all the algebraic classes are fixed so the ample cone and the Kähler cone are, then using Theorem 1.3.37 we conclude that $f_i \in \mathrm{Aut}(J(Y))$. For the second part, the existence of the cubic fourfold Y follows from Proposition 5.2.2, the same argument as the proof of Proposition 5.2.6 shows that the action in cohomology is compatible with the Hodge isometry of Proposition 5.2.1. □

Remark 8. *The classification of possible invariant and coinvariant lattices for involutions on manifold of* OG10 *type together with lattice computation for involutions induced by a cubic fourfold allow us to exclude some values of the constant N of Proposition 2.3.9, for example $N \neq 2, 5, 6$ and many others can be checked.*

We believe that it is not a coicidence that ϕ_1, ϕ_3 induce regular involutions on $J(Y)$ and that the actions in cohomology are predicted by the Hodge isometry in Proposition 5.2.1, in fact we expect this would happen also for higher order automorphisms. It would be reasonable to expect a equivariant version of the Hodge isometry in Proposition 5.2.1, for this reason probably $N = 1$ would hold in Proposition 2.3.9.

Appendix A

Computations about the very symmetric examples

We provide codes for the computations we ran using computer algebra. For the computations of characters we used `GAP` [GAP21], and `Macaulay2` [M2] for the other computations.

This appendix consists of three sections, the aim is to find the Lagrangian spaces $\mathbb{A}$ and justify computations about the singular locus of $Y_{\mathbb{A}}$, in particular conclude that it consists of a smooth surface completing the proof of Proposition 3.1.2. In section A.1 we compute the Lagrangian spaces. In section A.2 we compute equations for the EPW-sextic and its singular locus over an appropriate finite field, so that those objects can sit as fibers in a flat family whose central fiber is the solution to the equation on complex numbers. In section A.3 we compute the locus $Y_{\mathbb{A}}[3]$ and conclude that dimension and degree of the spaces over the complex numbers are the same as the ones over the finite field, proving that $\mathbb{A} \in \mathbb{LG}(\bigwedge^3 V_6) \setminus (\Sigma \cup \Delta)$.

A.1 Finding the Lagrangians subspaces

The following is the scheme of the `GAP` code that computes the bases of two Lagrangian subspaces of $\bigwedge^3 V_6$ represented as $\mathbb{C}^{20}$ (more precisely $\mathbb{Q}[\xi_{21}]$ in the code) respectively invariant under two non-isomorphic 10-dimensional representations R_1 and R_2 of $\mathcal{A}_7$. We compute the the invariant subspaces using the formula for a projection

$$P_i = \sum_{g \in \mathcal{A}_7} \chi_{R_i}(g)\phi_W(g),$$

where χ_{R_i} is the character of the representation R_i ($i = 1, 2$) and $\phi_W \colon \mathcal{A}_7 \to \mathrm{GL}(\mathbb{C}^{20})$ is the 20-dimensional representation. For more details on the characters, see Table A.1.

```
InducedMapOnWedge := function(Mat)
#computes the matrix of the linear map induced by the matrix Mat on
#the third Exterior Power of the underlying space in the basis of
#lexographically ordered simple vectors
#obtainted through multiplication of the canonical basis
        #...
end;;

CheckLagrangianWedge3_6 := function(L)
#checks if a subspace of Wedge^3V_6 is a Lagrangian space
        #...
end;;

OrbitSpace := function(vec, Gr, F)
#returns the space spanned by the orbit of
#the vector vec by the group Gr over the field F
        #...
end;;

#http://brauer.maths.qmul.ac.uk/Atlas/alt/A7/gap0/3A7G1-Ar6B0.g
#6-dimensional representation of 3.A7
#defined in the ring Z extended by 21st root of unity
Gens_3A7_6 := [
    #...
];;

#induced representation on Wedge^3C^6; it acts as A7 now
Gens_A7_20 := [InducedMapOnWedge(Gens_3A7_6[1]),
    InducedMapOnWedge(Gens_3A7_6[2])
];;

b := E(7)+E(7)^2+E(7)^4;;
B := -1-b;;
CC := ConjugacyClasses(A7_20);
ImportantCC := []; #only the classes of nonzero are important for computation
for C in CC do
        if Trace(Representative(C)) <> 0 then
                Add(ImportantCC, C);
        fi;
od;
#below we mark which classes have traces -b and -B in 10 dimensional representations
#first one has trace -b under representation and B under the dual,
#the second one the same in reverse
#the rest have a trace equal half the trace from the one on the Wedge^3 C^6
for ind in [1..Length(ImportantCC)] do
        if Trace(Representative(ImportantCC[ind])) = -1 then
                if flag then
                        ind1 := ind;
                        flag := false;
                else
                        ind2 := ind;
                fi;
        fi;
```

```
od;
e1 := CanonicalBasis(Rationals^20)[1];
#almost any nonzero vector would suffice

#P_a will be the projection on a subspace invariant under A7_20
#it will give a 10 dimensional representation
P_1 := 0 * IdentityMat(20,20);;
for ind in [1..Length(ImportantCC)] do
        Paux := 0 * IdentityMat(20);
        for g in ImportantCC[ind] do
                Paux := Paux + g;
        od;
        if ind = ind1 then
                P_1 := P_1 + b * Paux;
        elif ind = ind2 then
                P_1 := P_1 + B * Paux;
        else
                P_1 := P_1 + Trace(Representative(ImportantCC[ind]))/2 * Paux;
        fi;
od;
A_1 := OrbitSpace(v_1, A7_20, CF(21));
CheckLagrangianWedge3_6(R_1); #returns true
#the computation as above for A_2 follows
```

Below is the character for $\mathcal{A}_7$ including the 20-dimensional reducible representation we obtained on the exterior power space.

Conj. class	id	$[ab^{-1}ab]$	$[a]$	$[a^{-1}bab]$	$[a^{-1}bab^2]$	$[b]$	$[ababab^2]$	$[ab]$	$[a^{-1}b]$
V_0	1	1	1	1	1	1	1	1	1
V_6	6	2	3	0	0	1	-1	-1	-1
V_{10}	10	-2	1	1	0	0	1	$-\frac{1}{2}(1-i\sqrt{7})$	$-\frac{1}{2}(1+i\sqrt{7})$
V'_{10}	10	-2	1	1	0	0	1	$-\frac{1}{2}(1+i\sqrt{7})$	$-\frac{1}{2}(1-i\sqrt{7})$
V_{14}	14	2	2	-1	0	-1	2	0	0
V'_{14}	14	2	-1	2	0	-1	-1	0	0
V_{15}	15	-1	3	0	-1	0	-1	1	1
W	20	-4	2	2	0	0	2	-1	-1

Table A.1: Character table of $\mathcal{A}_7$

A.2 Local equations of the EPW and its singular locus

The following `Macaulay2` code computes equations over a finite field for the EPW sextic associated to a Lagrangian space, checks that the sextic is irreducible and has the right degree. Lastly, it computes the singular locus

of the EPW and check it is a surface of degree 40. The two Lagrangian subrepresentations we take into account are defined over the field $\mathbb{Q}[\xi_{21}]$, where ξ_{21} is a 21-st primitive root of unity.
We choose $p = 127$ so that the 21-st cyclotomic polynomial decomposes as

$$\Phi_{21}(v) \equiv_{127} \prod_{j=1}^{12} g_j(v)$$

in the polynomial ring $\mathbb{F}_{127}[v]$ in exactly as many factors as $[\mathbb{Q}[\xi_{21}] : \mathbb{Q}] = 12$. It follows that the decomposition of the ideal (127) in the ring of integers $\mathbb{Z}[\xi_{21}]$ is given by

$$(127) = \prod_{j=1}^{12} \mathfrak{q}_k$$

where $\mathfrak{q}_j = (127, g_j(\xi_{21}))$ and hence the residue field of $\mathbb{Z}[\xi_{21}]$ at any prime $\mathfrak{q}$ in the decomposition is exactly $\mathbb{F}_{127}$. Set D as the DVR obtained by localizing $\mathbb{Z}[\xi_{21}]$ at any such $\mathfrak{q}$. To be more explicit we can put $\mathfrak{q} = (127, \xi_{21} - 25)$ which is one of the factors in the decomposition.

In the following code we read the roots of unity in $\mathbb{F}_{127}$ via the left-down association in the following commutative diagram where we chose $\mathfrak{q} = (127, \xi_{21} - 25)$

$$\begin{array}{ccccc}
\mathbb{Z}[x] & \longrightarrow & \mathbb{Z}[x]/(\Phi_{21}(x)) & \longrightarrow & \mathbb{Z}[\xi_{21}] \\
\downarrow & & & & \downarrow \\
\mathbb{F}_{127}[x] & \longrightarrow & \mathbb{F}_{127}[x]/(x-25) & = & \mathbb{F}_{127}
\end{array}$$

which is in fact the same procedure as taking the residue field at the prime $\mathfrak{q}$ of D.

```
p = 127;
F = ZZ/p;
R = F[v];
I = ideal (v^12-v^11+v^9-v^8+v^6-v^4+v^3-v+1); --21st cyclotomic polynomial
J = decompose(I);
length J --returns 12
K = toField(R/J_0);
P = K[x,y,z,t,u,w];
--Matrix of coordinates
M = matrix {
    { 0,0,0,0,0,0,0,u,-t,z},
    {0,0,0,0,0,-u,t,0,0,-y},
    { 0,0,0,0,u,0,-z,0,y,0},
    {0,0,0,0,-t,z,0,-y,0,0},
    { 0,0,0,0,0,0,0,0,0,x},
    { 0,0,0,0,0,0,0,0,-x,0},
    { 0,0,0,0,0,0,0,x,0,0},
    { 0,0,0,0,0,0,0,0,0,0},
```

```
    { 0,0,0,0,0,0,0,0,0,0,0},
    { 0,0,0,0,0,0,0,0,0,0,0}
};
MM = M + transpose ( M );
--Mat is the symmetric matrix associated to the basis of the lagrangian A
Lambda = Mat-MM; --The EPW is given by
d = det(Lambda); --polynomial of degree 6, the equation for the EPW
I = ideal d;
degree(I) --returns 6
s = ideal singularLocus I; --singular locus of the EPW over the finite field
dim(s) --returns 3, so the projective dim is 2
degree(s) --returns 40
```

The code for computing the equations for EPW-sextics is based on the Appendix of [KKM22].

A.3 The singular locus is smooth

Consider the proper map $\mathbb{P}^5_D \to \operatorname{Spec} D$, this induces a map $V(I) \to \operatorname{Spec} D$ where $V(I)$ is the zero locus of our homogeneous equation in $D[x_0, \dots, x_5]$. In this setting, the fiber over the ideal (0) is the solution to the equation with coefficients in $\mathbb{Q}[\xi_{21}]$ and the fiber over the ideal $\mathfrak{q}$ is the solution to the equation with coefficients in the residue field $\mathbb{F}_{127}$. Since the map is proper, the image must be either the closed point $\mathfrak{q}$ or the entire scheme $\operatorname{Spec} D$, in particular if the fiber $X_{\mathfrak{q}}$ is empty then the fiber $X_{(0)}$ must be empty as well. We compute the locus $Y_{\mathbb{A}}[3]$ as the zero locus of the ideal generated by the 8×8 minors of the symmetric matrix whose zero locus is the EPW. The outcome of the computation is that the locus $Y_{\mathbb{A}}[3]$ is empty over the finite field and this implies that it is also empty over the complex numbers, proving that $\mathbb{A} \notin \Delta$.

The solutions over the two fields lie in a flat family and then the singular locus of $Y_{\mathbb{A}}$ has dimension 2 and degree 40 over the complex numbers. In conclusion the singular locus of the EPW coincides with the smooth surface $Y_{\mathbb{A}}[2]$, since $Y_{\mathbb{A}}[2]$ has already degree 40 [O'G12, Corollary 1.10] and is already contained in the singular locus. This gives $\mathbb{A} \notin \Sigma$

```
--The ideal generated by the 8x8 minors describes the locus Y_A[3]
J = minors(8,Lambda);
-- Dimesion of Y_A[3] (affine chart)
dim(J)
--Homogeneous ideal whose associated variety is Y_A[3]
Jh = saturate homogenize(J,x);
--Projective variety Y_A[3]
Z = Proj(P/Jh);
dim(Z1) --returns -2
```

Variables that are not defined here are the same as above.

Appendix B

Tables of lattices

This appendix contains the tables of lattices classifying non-symplectic involutions on manifold of OG10.

Table B.1: Pairs $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ for $G \subset \mathrm{O}(\mathbf{\Lambda})$ of prime order $p = 2$ and $\mathrm{sgn}(\mathbf{\Lambda}_G) = (2, \mathrm{rk}(\mathbf{\Lambda}_G) - 2)$.

No.	$\mathrm{rk}(\mathbf{\Lambda}^G)$	$\mathbf{\Lambda}_G$	$\mathbf{\Lambda}^G$	$\mathrm{sgn}(\mathbf{\Lambda}_G)$	a	δ
1	4	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2}$	$(2, 20)$	2	1
2	4	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$[2]^{\oplus 3} \oplus [-2]$	$(2, 20)$	4	1
3	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{U}^{\oplus 2} \oplus [2]$	$(2, 19)$	1	1
4	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	$(2, 19)$	3	1
5	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	$(2, 19)$	5	1
6	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{U}^{\oplus 3}$	$(2, 18)$	0	0
7	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	$(2, 18)$	2	0
8	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	$(2, 18)$	2	1
9	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$(2, 18)$	4	0
10	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$(2, 18)$	4	1
11	6	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{U}(2)$	$\mathbf{U}(2)^{\oplus 3}$	$(2, 18)$	6	0
12	6	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2] \oplus [-2]$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	$(2, 18)$	6	1
13	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]$	$(2, 17)$	1	1
14	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	$(2, 17)$	3	1
15	7	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$(2, 17)$	5	1
16	7	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 4}$	$(2, 17)$	7	1
17	8	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	$(2, 16)$	2	1
18	8	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 3}$	$(2, 16)$	4	1
19	8	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$(2, 16)$	6	1
20	8	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 7}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 5}$	$(2, 16)$	8	1
21	9	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 3}$	$(2, 15)$	3	1
22	9	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [2]$	$(2, 15)$	5	1
23	9	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$(2, 15)$	7	1
24	9	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 6}$	$(2, 15)$	9	1
25	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$(2, 14)$	4	0
26	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 4}$	$(2, 14)$	4	1
27	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$(2, 14)$	6	0
28	10	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 5}$	$(2, 14)$	6	1
29	10	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U}(2)^{\oplus 3} \oplus \mathbf{D}_4(-1)$	$(2, 14)$	8	0
30	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 6}$	$(2, 14)$	8	1
31	10	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 5}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 7}$	$(2, 14)$	10	1
32	11	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3}$	$(2, 13)$	3	1
33	11	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 5}$	$(2, 13)$	5	1
34	11	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 6}$	$(2, 13)$	7	1
35	11	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 7}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$(2, 13)$	9	1
36	11	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 8}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 8}$	$(2, 13)$	11	1
37	12	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2}$	$(2, 12)$	2	1
38	12	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]$	$(2, 12)$	4	1
39	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 6}$	$(2, 12)$	6	1

Continues on next page

Table B.1, follows from previous page

No.	rk($\mathbf{\Lambda}^G$)	$\mathbf{\Lambda}_G$	$\mathbf{\Lambda}^G$	sgn($\mathbf{\Lambda}_G$)	a	δ
40	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 6}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 7}$	$(2, 12)$	8	1
41	12	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 8}$	$(2, 12)$	10	1
42	12	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 11}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 9}$	$(2, 12)$	12	1
43	13	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2]$	$(2, 11)$	1	1
44	13	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	$(2, 11)$	3	1
45	13	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	$(2, 11)$	5	1
46	13	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 5}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 7}$	$(2, 11)$	7	1
47	13	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 8}$	$(2, 11)$	9	1
48	13	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$(2, 11)$	11	1
49	13	$[2]^{\oplus 2} \oplus [-2]^{\oplus 11}$	$[2]^{\oplus 3} \oplus [-2]^{\oplus 10}$	$(2, 11)$	13	1
50	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3}$	$(2, 10)$	0	0
51	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	$(2, 10)$	2	0
52	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	$(2, 10)$	2	1
53	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$(2, 10)$	4	0
54	14	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$(2, 10)$	4	1
55	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)^{\oplus 3}$	$(2, 10)$	6	0
56	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	$(2, 10)$	6	1
57	14	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$\mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$(2, 10)$	8	0
58	14	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 8}$	$(2, 10)$	8	1
59	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(-2)$	$\mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{U}(2)^{\oplus 3}$	$(2, 10)$	10	0
60	14	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9}$	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	$(2, 10)$	10	1
61	14	$\mathbf{E}_8(-2) \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-2) \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$(2, 10)$	12	0
62	14	$[2]^{\oplus 2} \oplus [-2]^{\oplus 10}$	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 10}$	$(2, 10)$	12	1
63	15	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus [-2]$	$(2, 9)$	1	1
64	15	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	$(2, 9)$	3	1
65	15	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$(2, 9)$	5	1
66	15	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 4}$	$(2, 9)$	7	1
67	15	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 9}$	$(2, 9)$	9	1
68	15	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$\mathbf{U}^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$(2, 9)$	11	1
69	16	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	$(2, 8)$	2	1
70	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2}$	$(2, 8)$	4	1
71	16	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$(2, 8)$	6	1
72	16	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 7}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 5}$	$(2, 8)$	8	1
73	16	$[2]^{\oplus 2} \oplus [-2]^{\oplus 8}$	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 10}$	$(2, 8)$	10	1
74	17	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 3}$	$(2, 7)$	3	1
75	17	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 4}$	$(2, 7)$	5	1
76	17	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 6}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$(2, 7)$	7	1
77	17	$[2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 6}$	$(2, 7)$	9	1
78	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 3} \oplus \mathbf{D}_4(-1)$	$(2, 6)$	2	0
79	18	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$(2, 6)$	4	0
80	18	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$(2, 6)$	4	1
81	18	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$(2, 6)$	6	0
82	18	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$(2, 6)$	6	1
83	18	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 3}$	$(2, 6)$	8	1
84	19	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 3}$	$(2, 5)$	3	1
85	19	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2, 5)$	5	1
86	19	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$(2, 5)$	7	1
87	20	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]^{\oplus 2}$	$(2, 4)$	2	1
88	20	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 3} \oplus [-2]$	$(2, 4)$	4	1
89	20	$[2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(2, 4)$	6	1
90	21	$\mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2]$	$(2, 3)$	1	1
91	21	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	$(2, 3)$	3	1
92	21	$[2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	$(2, 3)$	5	1
93	22	$\mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3}$	$(2, 2)$	0	0
94	22	$\mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	$(2, 2)$	2	0
95	22	$\mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	$(2, 2)$	2	1
96	22	$\mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$(2, 2)$	4	0
97	22	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$(2, 2)$	4	1
98	23	$\mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus [-2]$	$(2, 1)$	1	1
99	23	$[2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	$(2, 1)$	3	1
100	24	$[2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	$(2, 0)$	2	1

Table B.2: Pairs $(\mathbf{L}^G, \mathbf{L}_G)$ for $G \subset \mathrm{O}(\mathbf{L})$ of prime order $p = 2$, trivial action on the discriminant group.

No.	rk($\mathbf{\Lambda}^G$)	$\mathbf{L}_G$	$\mathbf{L}^G$	sgn($\mathbf{L}_G$)	$a(\mathbf{L}_G)$	$\delta(\mathbf{L}_G)$
1	4	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$[2] \oplus [-6]$	$(2,20)$	2	1
2	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{A}_2(-1) \oplus [2]$	$(2,19)$	1	1
3	5	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$[2] \oplus [-2] \oplus [-6]$	$(2,19)$	3	1
4	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1)$	$(2,18)$	0	0
5	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	$(2,18)$	2	0
6	6	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	$(2,18)$	2	1
7	6	$\mathbf{E}_8(-1) \oplus [-2]^{\oplus 2} \oplus [2]^{\oplus 2}$	$[2] \oplus [-2]^{\oplus 2} \oplus [-6]$	$(2,18)$	4	1
8	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]$	$(2,17)$	1	1
9	7	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2,17)$	3	1
10	7	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$[2] \oplus [-2]^{\oplus 3} \oplus [-6]$	$(2,17)$	5	1
11	8	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 2}$	$(2,16)$	2	1
12	8	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(2,16)$	4	1
13	8	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2}$	$[2] \oplus [-2]^{\oplus 4} \oplus [-6]$	$(2,16)$	6	1
14	9	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 3}$	$(2,15)$	3	1
15	9	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{A}_2(-1) \oplus [-2]^{\oplus 4} \oplus [2]$	$(2,15)$	5	1
16	9	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$[2] \oplus [-2]^{\oplus 5} \oplus [-6]$	$(2,15)$	7	1
17	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{U}(2) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1)$	$(2,14)$	4	0
18	10	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 4}$	$(2,14)$	4	1
19	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$\mathbf{U} \oplus \mathbf{E}_6(-2)$	$(2,14)$	6	0
20	10	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	$(2,14)$	6	1
21	10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 4}$	$[2] \oplus [-2]^{\oplus 6} \oplus [-6]$	$(2,14)$	8	1
22	11	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2]$	$(2,13)$	3	1
23	11	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 5}$	$(2,13)$	5	1
24	11	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$(2,13)$	7	1
25	11	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 7}$	$[2] \oplus [-2]^{\oplus 7} \oplus [-6]$	$(2,13)$	9	1
26	12	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-6]$	$(2,12)$	2	1
27	12	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	$(2,12)$	4	1
28	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 6}$	$(2,12)$	6	1
29	12	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 6}$	$\mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$(2,12)$	8	1
30	12	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$[2] \oplus [-2]^{\oplus 8} \oplus [-6]$	$(2,12)$	10	1
31	13	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2]$	$(2,11)$	1	1
32	13	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2] \oplus [-6]$	$(2,11)$	3	1
33	13	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2,11)$	5	1
34	13	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 5}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 7}$	$(2,11)$	7	1
35	13	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$\mathbf{U} \oplus [-2]^{\oplus 8} \oplus [-6]$	$(2,11)$	9	1
36	13	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$[2] \oplus [-2]^{\oplus 9} \oplus [-6]$	$(2,11)$	11	1
37	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1)$	$(2,10)$	0	0
38	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	$(2,10)$	2	0
39	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	$(2,10)$	2	1
40	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$(2,10)$	4	0
41	14	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	$(2,10)$	4	1
42	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(2,10)$	6	1
43	14	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$\mathbf{D}_4(-1) \oplus \mathbf{U} \oplus \mathbf{E}_6(-2)$	$(2,10)$	8	0
44	14	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 8}$	$(2,10)$	8	1
45	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(-2)$	$\mathbf{D}_4(-1) \oplus \mathbf{U}(2) \oplus \mathbf{E}_6(-2)$	$(2,10)$	10	0
46	14	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9}$	$[2] \oplus [-2] \oplus \mathbf{E}_6(-2) \oplus \mathbf{D}_4(-1)$	$(2,10)$	10	1
47	14	$\mathbf{U}(2)^{\oplus 2} \oplus [-2]^{\oplus 8}$	$\mathbf{U}(2) \oplus [-2]^{\oplus 9} \oplus [-6]$	$(2,10)$	12	1
48	15	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]$	$(2,9)$	1	1
49	15	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2,9)$	3	1
50	15	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 3} \oplus [-6]$	$(2,9)$	5	1
51	15	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$(2,9)$	7	1
52	15	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 9}$	$(2,9)$	9	1
53	15	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$\mathbf{A}_2(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 9}$	$(2,9)$	11	1
54	16	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-6]$	$(2,8)$	4	1
55	16	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 4}$	$(2,8)$	6	1
56	16	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 7}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	$(2,8)$	8	1
57	16	$[2]^{\oplus 2} \oplus [-2]^{\oplus 8}$	$\mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 10}$	$(2,8)$	10	1
58	17	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 3}$	$(2,7)$	3	1
59	17	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$(2,7)$	5	1
60	17	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 5} \oplus [-6]$	$(2,7)$	7	1
61	17	$[2]^{\oplus 2} \oplus [-2]^{\oplus 7}$	$\mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$(2,7)$	9	1
62	18	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1)$	$(2,6)$	2	0
63	18	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)$	$(2,6)$	4	0
64	18	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$(2,6)$	4	1
65	18	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{E}_6(-2)$	$(2,6)$	6	0
66	18	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	$(2,6)$	6	1
67	18	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6}$	$\mathbf{D}_4(-1) \oplus \mathbf{D}_6(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(2,6)$	8	1
68	19	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2]$	$(2,5)$	3	1

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Table B.2, follows from previous page

No.	$\mathrm{rk}(\mathbf{\Lambda}^G)$	$\mathbf{L}_G$	$\mathbf{L}^G$	$\mathrm{sgn}(\mathbf{L}_G)$	$a(\mathbf{L}_G)$	$\delta(\mathbf{L}_G)$
69	19	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2,5)$	5	1
70	19	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3} \oplus [-6]$	$(2,5)$	7	1
71	20	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-6]$	$(2,4)$	2	1
72	20	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	$(2,4)$	4	1
73	20	$[2]^{\oplus 2} \oplus [-2]^{\oplus 4}$	$\mathbf{E}_8(-1) \oplus \mathbf{A}_2(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(2,4)$	6	1
74	21	$\mathbf{U}^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus [2]$	$(2,3)$	1	1
75	21	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2] \oplus [-6]$	$(2,3)$	3	1
76	21	$[2]^{\oplus 2} \oplus [-2]^{\oplus 3}$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_6(-1) \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2,3)$	5	1
77	22	$\mathbf{U}^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{A}_2(-1)$	$(2,2)$	0	0
78	22	$\mathbf{U} \oplus \mathbf{U}(2)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus \mathbf{U}(2)$	$(2,2)$	2	0
79	22	$\mathbf{U} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]$	$(2,2)$	2	1
80	22	$\mathbf{U}(2)^{\oplus 2}$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$(2,2)$	4	0
81	22	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	$(2,2)$	4	1
82	23	$\mathbf{U} \oplus [2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]$	$(2,1)$	1	1
83	23	$[2]^{\oplus 2} \oplus [-2]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(2,1)$	3	1
84	24	$[2]^{\oplus 2}$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 2}$	$(2,0)$	2	1

Table B.3: Pairs $(\mathbf{L}^G, \mathbf{L}_G)$ for $G \subset \mathrm{O}(\mathbf{L})$ of prime order $p = 2$, non-trivial action on the discriminant group.

No.	$\mathrm{rk}(\mathbf{L}_G)$	$\mathbf{L}_G = (\mathbf{L}^G)^{\perp \mathbf{L}}$	$\mathbf{L}^G = [2]^{\perp \mathbf{\Lambda}^G}$	$\mathrm{sgn}(\mathbf{L}^G)$	$a(\mathbf{L}^G)$	$\delta(\mathbf{L}^G)$
1	3	$[2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [-2]^{\oplus 3}$	$(1,20)$	3	1
2	4	$[2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [-2]^{\oplus 2}$	$(1,19)$	2	1
3	4	$[2]^{\oplus 2} \oplus [-2] \oplus [-6]$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 3}$	$(1,19)$	3	1
4	5	$\mathbf{U} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [-2]$	$(1,18)$	1	1
6	5	$[2]^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	$(1,18)$	3	1
7	5	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]$	$(1,18)$	5	1
8	6	$\mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}$	$(1,17)$	0	0
9	6	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}(2)$	$(1,17)$	2	0
11	6	$\mathbf{U} \oplus [2] \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2] \oplus [-2]$	$(1,17)$	2	1
12	6	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$(1,17)$	4	0
13	6	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [-2]^{\oplus 2}$	$(1,17)$	4	1
14	6	$[2]^{\oplus 2} \oplus [-2]^{\oplus 3} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	$(1,17)$	6	1
15	7	$\mathbf{U}^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]$	$(1,16)$	1	1
16	7	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [-2]$	$(1,16)$	3	1
17	7	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$(1,16)$	5	1
18	7	$[2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 7}$	$(1,16)$	7	1
19	8	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_6(-1)$	$(1,15)$	2	0
20	8	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$(1,15)$	4	1
21	8	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2] \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(1,15)$	6	1
22	8	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$(1,15)$	8	1
23	9	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$(1,14)$	3	1
24	9	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(1,14)$	5	1
25	9	$[2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 6}$	$(1,14)$	7	1
26	9	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 5}$	$(1,14)$	9	1
27	10	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus \mathbf{U}(2)$	$(1,13)$	4	0
28	10	$\mathbf{U} \oplus [2] \oplus [-2] \oplus \mathbf{D}_4(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]$	$(1,13)$	4	1
29	10	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 3}$	$(1,13)$	6	0
30	10	$\mathbf{D}_6(-1) \oplus [2]^{\oplus 3} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	$(1,13)$	6	1
31	10	$[2]^{\oplus 2} \oplus [-2]^{\oplus 6} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 4}$	$(1,13)$	8	1
32	10	$\mathbf{U}(2)^{\oplus 2} \oplus \mathbf{E}_6(-2)$	$\mathbf{U} \oplus \mathbf{E}_8(-2) \oplus \mathbf{D}_4(-1)$	$(1,13)$	10	0
33	10	$[2]^{\oplus 2} \oplus [-2]^{\oplus 7} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 8}$	$(1,13)$	10	1
34	11	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 3}$	$(1,12)$	3	1
35	11	$\mathbf{U} \oplus 2 \oplus [-2]^{\oplus 6} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 4}$	$(1,12)$	5	1
36	11	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 3}$	$(1,12)$	7	1
37	12	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 7}$	$(2,12)$	9	1
38	11	$[2]^{\oplus 2} \oplus [-2]^{\oplus 8} \oplus [-6]$	$\mathbf{U} \oplus [-2]^{\oplus 11}$	$(1,12)$	11	1
39	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]^{\oplus 2}$	$(1,11)$	2	1
40	12	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [2] \oplus [-2]$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 3}$	$(1,11)$	4	1
41	12	$\mathbf{D}_4(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]^{\oplus 2}$	$(1,11)$	6	1
42	12	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 6}$	$(1,11)$	8	1
43	12	$\mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus [-6]$	$\mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 7}$	$(1,11)$	10	1
44	12	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9} \oplus [-6]$	$[2] \oplus [-2]^{\oplus 11}$	$(1,11)$	12	1

Continues on next page

Table B.3, follows from previous page

No.	rk($\mathbf{L}_G$)	$\mathbf{L}_G = (\mathbf{L}^G)^{\perp \mathbf{L}}$	$\mathbf{L}^G = [2]^{\perp \mathbf{\Lambda}^G}$	sgn($\mathbf{L}^G$)	$a(\mathbf{L}^G)$	$\delta(\mathbf{L}^G)$
45	13	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [-2]$	$(1, 10)$	1	1
46	13	$\mathbf{E}_8(-1) \oplus \oplus [2] \oplus [-2]^{\oplus 2}\ \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]^{\oplus 2}$	$(1, 10)$	3	1
47	13	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus [-2]$	$(1, 10)$	5	1
48	13	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 5}$	$(1, 10)$	7	1
49	13	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 8} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 9}$	$(1, 10)$	9	1
50	13	$[2]^{\oplus 2} \oplus [-2]^{\oplus 9} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 10}$	$(1, 10)$	11	1
51	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}$	$(1, 9)$	0	0
52	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus \mathbf{U}(2)$	$(1, 9)$	2	0
53	14	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2] \oplus [-2]$	$(1, 9)$	2	1
54	14	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-6]$	$\mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [-2]^{\oplus 2}$	$(1, 9)$	4	1
55	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}(2) \oplus \mathbf{D}_4(-1)^{\oplus 2}$	$(2, 9)$	6	0
56	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 4}$	$(1, 9)$	6	1
57	14	$\mathbf{U}^{\oplus 2} \oplus \mathbf{E}_8(-2) \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{E}_8(-2)$	$(1, 9)$	8	0
58	14	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$\mathbf{D}_4(-1) \oplus [2] \oplus [-2]^{\oplus 5}$	$(1, 9)$	8	1
59	14	$\mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(-2) \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-2) \oplus \mathbf{U}(2)$	$(1, 9)$	10	0
60	14	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 9} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 9}$	$(1, 9)$	10	1
61	15	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$\mathbf{E}_8(-1) \oplus [2]$	$(1, 8)$	1	1
62	15	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 3}$	$(1, 8)$	5	1
63	15	$\mathbf{U} \oplus \mathbf{D}_6(-1) \oplus [2] \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 7}$	$(1, 8)$	7	1
64	15	$\mathbf{D}_6(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 8}$	$(1, 8)$	9	1
65	16	$\mathbf{E}_8(-1) \oplus \mathbf{D}_4(-1) \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]^{\oplus 2}$	$(1, 7)$	4	1
66	16	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 6}$	$(1, 7)$	6	1
67	16	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus [-6]$	$[2] \oplus [-2]^{\oplus 7}$	$(1, 7)$	8	1
68	17	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus \mathbf{D}_4(-1) \oplus [-2]$	$(1, 6)$	3	1
69	17	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 5}$	$(1, 6)$	5	1
70	17	$\mathbf{E}_8(-1) \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 6}$	$(1, 6)$	7	1
71	18	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 4} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 4}$	$(1, 5)$	4	1
72	18	$\mathbf{E}_8(-1) \oplus \mathbf{U} \oplus [2] \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 5}$	$(1, 5)$	6	1
73	19	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-6]$	$\mathbf{U} \oplus [-2]^{\oplus 3}$	$(1, 4)$	3	1
74	19	$\mathbf{E}_8(-1) \oplus \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 5} \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 4}$	$(1, 4)$	5	1
75	20	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]^{\oplus 2}$	$(1, 3)$	2	1
76	20	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2] \oplus [-6]$	$[2] \oplus [-2]^{\oplus 3}$	$(1, 3)$	4	1
77	21	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus \mathbf{A}_2(-1)$	$\mathbf{U} \oplus [-2]$	$(1, 2)$	1	1
78	21	$\mathbf{E}_8(-1)^{\oplus 2} \oplus [2]^{\oplus 2} \oplus [-2] \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]^{\oplus 2}$	$(1, 2)$	3	1
79	22	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$	$\mathbf{U}$	$(1, 1)$	1	1
80	22	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U} \oplus [2] \oplus [-2] \oplus \mathbf{A}_2(-1)$	$[2] \oplus [-2]$	$(1, 1)$	2	1
81	23	$\mathbf{E}_8(-1)^{\oplus 2} \oplus \mathbf{U}^{\oplus 2} \oplus [2] \oplus \mathbf{A}_2(-1)$	$[2]$	$(1, 0)$	1	1

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