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Spectral Asymptotic Properties of Semi-Regular Non-Commutative Harmonic Oscillators

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A tutte le persone che mi hanno accompagnato
nel percorso formativo,
con affetto e fiducia,
ma anche con severità e rigore,
permettendomi di crescere e maturare
per affrontare questo cammino...

To all the people who accompanied me
on the training path,
with affection and cofindence,
but also with severity and rigor,
enabling me to grow up
to face this journey ...

Abstract

The study carried out in this thesis is devoted to spectral analysis of systems of PDEs related also with quantum physics models. Namely, the research deals with classes of systems that contain certain quantum optics models such as Jaynes-Cummings, Rabi and their generalizations that describe light-matter interaction.

First we investigate the spectral Weyl asymptotics for a class of semiregular systems, extending to the vector-valued case results of Helffer and Robert [18], and more recently of Doll, Gannot and Wunsch [11]. Actually, the asymptotics by Doll, Gannot and Wunsch is more precise (that is why we call it *refined*) than the classical result by Helffer and Robert, but deals with a less general class of systems, since the authors make an hypothesis on the measure of the subset of the unit sphere on which the tangential derivatives of the X-Ray transform of the semiprincipal symbol vanish to infinity order.

Next, we give a meromorphic continuation of the spectral zeta function for semiregular differential systems with polynomial coefficients, generalizing the results by Ichinose and Wakayama [30] and Parmeggiani [45].

Finally, we state and prove a quasi-clustering result for a class of systems including the aforementioned quantum optics models and we conclude the thesis by showing a Weyl law result for the Rabi model and its generalizations.

Sommario

Lo studio condotto in questa tesi è dedicato all'analisi spettrale di sistemi di PDE legati a modelli di fisica quantistica. In particolare, la ricerca si occupa di classi di sistemi che contengono alcuni modelli di ottica quantistica tra i quali i sistemi di Jaynes-Cummings, di Rabi e le loro generalizzazioni che descrivono l'interazione luce-materia.

In primo luogo, studiamo l'asintotica spettrale di Weyl per una classe di sistemi semiregolari, estendendo al caso vettoriale i risultati di Helffer e Robert [18] e, più recentemente, di Doll, Gannot e Wunsch [11]. In vero, l'asintotica di Doll, Gannot e Wunsch è più precisa (per questo la chiamiamo *raffinata*) del risultato classico di Helffer e Robert, ma tratta una classe di sistemi meno generale, poiché viene fatta un'ipotesi sulla misura del sottoinsieme della sfera unitaria su cui le derivate tangenziali della trasformata a raggi X del simbolo semiprincipale si annullano all'ordine infinito.

Forniamo, poi, una continuazione meromorfa della funzione zeta spettrale per sistemi differenziali semiregolari a coefficienti polinomiali, generalizzando i risultati di Ichinose e Wakayama [30] e Parmeggiani [45].

Infine enunciamo e dimostriamo un risultato di quasi-clustering per una classe di sistemi che include i modelli di ottica quantistica sopra citati e concludiamo la tesi provando una formula di Weyl per il modello di Rabi e le sue generalizzazioni.

Contents

Introduction

In this thesis we deal with asymptotic spectral properties of a class of pseudodifferential systems containing models which are relevant in Quantum Optics like the Jaynes-Cummings model (see Subsection 2.1.1) and the Rabi model (see Chapter 10). The main class analyzed consists of Semiregular Metric Global Elliptic Systems (SMGES). Namely, we are considering those global semiregular systems (see Section 1.2) with scalar elliptic principal symbol such that there is not only a positively homogeneous of order 0 unitary matrix-valued function whose conjugation diagonalizes both the principal and semiprincipal symbol, but also separation of the eigenvalues for the semiprincipal symbol.

First of all, we will establish a result about the asymptotics of the spectral Weyl spectral counting function of a class of semiregular globally elliptic pseudodifferential $N \times N$ systems (of order 2) which is a subclass of the SMGES class and contains the important model of Jaynes-Cummings that describes the interaction of light and matter (see [57]). The class we consider extends to a semiregular case (with scalar principal symbol) that of Non-Commutative Harmonic Oscillators (NCHOs) introduced by Parmeggiani and Wakayama in [47, 48] (see also [45, 46]). Namely, while the pseudodifferential class considered in [45] had a step $-2j$ in the homogeneity of the jth-term in the asymptotic expansion of the symbol, we consider here a step $-j$. An example of such a scaling in homogeneity is, in fact, the symbol of the Jaynes-Cummings Hamiltonian (we call *semiregular* this kind of classical symbols).

In the scalar case, this kind of asymptotics for global operators was initially established by Chazarain [8] (in a semiclassical setting) and then generalized by Shubin [58] (see also Hörmander [25]), and Helffer and Robert [18] (see also [52] and Helffer's book [20], and references therein), and, more recently, made more precise by Doll, Gannot and Wunsch in [11] (see also Doll and Zelditch [12] for a precise study of the singularities of the trace of the Schrödinger propagator).

The asymptotics of the Weyl spectral counting function will be given in terms of the symbol of the system, and, more precisely, in terms of the principal part, the semiprincipal part and the subprincipal part (respectively, the terms of order 2, 1 and 0 in the asymptotic expansion of the symbol). We will show that one can blockwise diagonalize (through a decoupling theorem) the system, so as to be able to implement the scalar results mentioned above. This is, however, not straightforward, since we have to compose certain Fourier integral operators and ψdos in the Weyl calculus keeping track of the (matrix) symbols.
We will be retaining the notation relative to the Hörmander-Weyl pseudodifferential calculus (also in the semiclassical case) as in Parmeggiani [45] (see also [42] and [46]).

Next, we will study the spectral zeta function associated to a second order differential operator with polynomial coefficients (which contains the Jaynes-Cummings and Rabi models). The spectral zeta function of an operator is an import invariant of the point spectrum of an operator. In fact, when the spectrum of a linear operator $P : H \to H$ (H being an Hilbert space) is discrete, we can define the spectral zeta function of P as

$$\zeta_P(s) := \sum_{\lambda \in \operatorname{Spec} P} \lambda^{-s},$$

for any given complex number s for which it makes sense. In particular, if P is an elliptic, self-adjoint and positive global pseudodifferential operator of order $\mu > 0$ on $\mathbb{R}^n$, then $s \mapsto \zeta_P(s)$ is holomorphic for $\operatorname{Re} s > 2n/\mu$, since there the defining series is absolutely convergent. Actually, to have more precise information on the spectrum of P and since, as we will recall, the spectral zeta function of the scalar (and fundamental) harmonic oscillator is deeply connected with the Riemann zeta function: we want to extend it meromorphically. That is why we will state and prove a result of continuation of the spectral zeta function to obtain a meromorphic function whose poles are real and accumulate at $-\infty$. Namely, we will give the continuation as a linear combination of meromorphic functions modulo a function that is holomorphic on a complex half-plane.

Once obtained fundamental properties on the asymptotics of the spectrum and behavior of its observables, we will provide a more precise property of the spectrum of a subclass of the SMGES containing again the aforementioned quantum optics models. Namely, we will determine a spectral quasi-clustering result, that is, the concentration of the spectrum of a positive self-adjoint ψdo within the union of certain intervals with centers on a sequence determined through invariants of the symbol and width decreasing as the centers go to infinity.

Finally, we will complete this thesis by stating and proving a refined Weyl law for the Rabi model and a classical (two-coefficient) Weyl law for its generalizations. It describes the interaction of a 2-level atom and one cavity-mode electromagnetic field for a bigger class of quantum systems with respect to the Jaynes-Cummings model. In fact, it provides a good description of the quantum system even when the field is not near resonance with the atomic transition and the coupling strength is not weak. Actually, it can be seen as the model which leads to the Jaynes-Cummings model after the rotating waves approximation, which is valid if the field is near resonance with the atomic transition and the coupling strength is weak.

The plan of this thesis is the following. First of all we briefly recall in Chapter 1 the concepts of NCHOs, of the class of semiregular symbols and their main properties that we will be using in this work. We then define the class of systems we will be concerned in the sequel.

In Chapter 2, we recall the Jaynes-Cummings model and its variants, to encompass also atoms with several energy levels. We show that it is possible to associate with such systems coming from physics, geometrical models related to complexes of vector-valued differential forms. This is interesting in our opinion, for it shows that, very likely, higher Lie groups of symmetries are allowed in the theory.

In Chapter 3, we state and prove the decoupling theorem, which shows that, for the class we consider here, it is possible to obtain a pseudodifferential block-reduction of the system. This is fundamental in the study of a parametrix of the Schrödinger flow associated with our system, which, in turn, is the basic object to study for obtaining the Weyl asymptotics we are interested in. The decoupling theorem will be stated both in the semiclassical case as well as in the semiregular case, and the proof will be given in the semiclassical setting (in fact, it will be useful, for future projects, to have also the semiclassical version). Since the subprincipal part enters the picture, we discuss in Chapter 4 the transformation properties of the subprincipal symbol of the system, along with its transformation law when changing the "gauge" (that is, when changing the unitary symbol which diagonalizes the semiprincipal part).

In Chapter 5 we investigate, for the sake of completeness, also the structure and the transformation laws of the semi-subprincipal symbol under different diagonalizers for the semiprincipal part.

In Chapter 6 we study conditions that are necessary and/or sufficient for having that the X-ray transform $\mathsf{R}(\lambda^{\pm}_{\mu-1})$ of the eigenvalues $\lambda^{\pm}_{\mu-1}$ of the semiprincipal part $a_{\mu-1}$ are Morse-Bott functions (see Proposition 6.2.1).

In Chapter 7 (Section 7.1), we will state and prove the Weyl asymptotic results, the first one extending to our class of systems the asymptotics due to Helffer and Robert [18], and the second one presenting a better error term when the zero-set of the determinant of the semiprincipal part has small dimension (see Theorems 7.1.7 and 7.1.8). The results are based on the construction of a reduced propagator, following the approach of Doll, Gannot and Wunsch [11], and it is here that the diagonalization theorem plays a fundamental role. The extension to systems, however, is not for free, for we have to control the conjugation of the Fourier integral operators (FIOs) with quadratic phases by the pseudodifferential diagonalizers, without losing the symbol-calculus properties. This point is very delicate and we follow here the approach of Doll and Zelditch [12], having, however, to adapt it to our case. Then, in Section 7.2, we shall show the resulting asymptotics in the 2×2 Jaynes-Cummings system, and of a perturbation of the 3×3

Jaynes-Cummings system and its 6×6 geometric counterpart.

In Chapter 8 we will give a meromorphic continuation of the spectral zeta function for semiregular differential systems with polynomial coefficients. As an application of our results, we first compute the meromorphic continuation of the Jaynes-Cummings model (JC-model) spectral zeta function. Next, we compute the spectral zeta function of the JC-model generalization to a 3-level atom and a 2 cavity-mode electromagnetic field. For both of them we show that it has only one pole in $s = 1$.

In Chapter 9 we will provide a spectral quasi-clustering spectral estimate for a subclass of the SMGES class. At first, we consider systems with principal symbol given by the harmonic oscillator p_2, semiprincipal symbol with matrix invariants that are functions of the harmonic oscillator and subprincipal symbol of its diagonalized which is constant on the bicharacteristics of p_2. This is a relevant case, since the Jaynes-Cummings model and its generalizations in Chapter 2.1 with $\alpha_k = \alpha$ for all k satisfy this property for all N. Then, we extend the result to be able to include also the case with $\alpha_k \neq \alpha_{k'}$ for some $k \neq k'$.

Finally, in Chapter 10 we will study the asymptotics of the counting function of the Rabi model and we will obtain a refined Weyl law for it (since the Rabi Hamiltonian operator for a 2-level atom with one cavity-mode of the electromagnetic field is a SMGES) and a (classical) two-term Weyl law for its generalization to an N-levels atom ($N \geq 3$) with $N - 1$ cavity-modes of the electromagnetic field by a perturbation argument and operator inequalities.

Chapter 1

Mathematics and physics settings

In this chapter we provide the mathematical an physical prerequisites to describe and analyze the quantum optical Jaynes-Cummings model (JC-model) and give a mathematical formulation of the spectral problems we are going to treat.
More in detail, in Section 1.1 we introduce the class of the Non-Commutative Harmonic Oscillators and in Section 1.2 the one of the semiregular symbols and SMGES that will be central in the study that follows. Then, we present and treat the JC-model from a mathematical and physical point of view in Section 1.3.

1.1 Non-Commutative Harmonic Oscillators (NCHOs)

A **Non-Commutative Harmonic Oscillator (NCHO)** is the Weyl-quantization $a^{\mathrm{w}}(x, D)$ of an $N \times N$ system of the form

$$a(x,\xi) = a_2(x,\xi) + a_0, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n = T^*\mathbb{R}^n,$$

where $a_2(x,\xi)$ is an $N \times N$ matrix whose entries are *homogeneous polynomials of degree* 2 *in the* (x,ξ) *variables*, and a_0 is a constant $N \times N$ matrix. (These systems have been introduced by A. Parmeggiani and M. Wakayama in [47, 48].)

It can also be said that an NCHO comes from the Weyl-quantization of a matrix-valued quadratic form in (x,ξ) adding a constant matrix term.

Remark. *A. Parmeggiani and M. Wakayama choose the name NCHO for two main reasons:*

– the fact that a scalar harmonic oscillator is a single quadratic form in (x,ξ)*;*

– the two levels of non-commutativity that we have to deal with when studying these systems: one due to the matrix-valued nature of the symbol of the system, and the other to the Weyl-quantization rule

$$x_k\xi_j \leftrightarrow (x_k D_{x_j} + D_{x_j} x_k)/2, \quad (\textit{where } D = -i\partial),$$

reflected through symplectic geometry by the Poisson-bracket relations

$$\{\xi_j, x_k\} = \delta_{jk}, \quad 1 \le j,k \le n.$$

Definition. A NCHO $a^{\mathrm{w}}(x, D)$ is said to be *elliptic* when

$$a_2 \text{ is a } N \times N \text{ matrix and } \det a_2(x,\xi) \text{ behaves exactly like } (|x|^2 + |\xi|^2)^N$$

for $|(x,\xi)|$ large.

When a_2 and a_0 are *Hermitian matrices*, the operator $a^{\mathrm{w}}(x, D)$ is "formally self-adjoint" (i.e. symmetric on $\mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$). Moreover, if in addition $a^{\mathrm{w}}(x, D)$ is *positive elliptic* (i.e. the matrix $a_2(x, \xi)$ is positive definite for $(x, \xi) \neq (0, 0)$), then it is *self-adjoint as an unbounded operator* $A : D(A) \subset L^2(\mathbb{R}^n; \mathbb{C}^N) \to L^2(\mathbb{R}^n; \mathbb{C}^N)$ *with a discrete real spectrum,* where

$$D(A) := \{u \in L^2(\mathbb{R}^n);\ Au = a^{\mathrm{w}}(x, D)u \in L^2(\mathbb{R}^n)\} = B^2(\mathbb{R}^n),$$

with $B^2(\mathbb{R}^n)$ the Shubin space of order 2.

Remark 1.1.1. *We note that, while scalar harmonic oscillators have been deeply studied, very little has been investigated about the spectral properties of selfadjoint elliptic systems, even in the basic case of NCHOs.*

The system written below is an especially important example of NCHO:

$$Q^{\mathrm{w}}_{(\alpha,\beta)}(x, D) = \begin{bmatrix} \alpha\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) & -\left(x\partial_x + \frac{1}{2}\right) \\ x\partial_x + \frac{1}{2} & \beta\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) \end{bmatrix}, \qquad x \in \mathbb{R},\ \alpha, \beta \in \mathbb{C}.$$

This is the Weyl-quantization of the matrix

$$Q_{(\alpha,\beta)}(x, \xi) = \begin{bmatrix} \alpha\left(\frac{\xi^2 + x^2}{2}\right) & -ix\xi \\ ix\xi & \beta\left(\frac{\xi^2 + x^2}{2}\right) \end{bmatrix}, \quad (x, \xi) \in \mathbb{R} \times \mathbb{R},$$

introduced by A Parmeggiani and M.Wakayama[47, 48]. When $\alpha, \beta > 0$ with $\alpha\beta > 1$, the system *is positive elliptic*, self-adjoint, and so it has a discrete spectrum in $L^2(\mathbb{R}; \mathbb{C}^2)$, and a very rich and remarkable structure.

It is worth remarking that in [48] the eigenvalues are described in terms of a *scalar three-term recurrence*, that is, in terms of a continued fraction (nevertheless, it is very difficult to get a direct and explicit expression of them).

In addition we mention that when $\alpha = \beta > 1$, $Q^{\mathrm{w}}_{(\alpha,\alpha)}(x, D)$ is *unitarily* equivalent [47, 48] to a scalar harmonic oscillator times the identity 2×2 matrix. Hence, its spectral properties are governed by the tensor product of the oscillator representation and the 2-dimensional trivial representation of

$\mathfrak{sl}_2(\mathbb{R})$ [24], i.e. one has matrix-valued creation/annihilation operators that can be used to “construct” the spectrum.
Therefore, when $\alpha, \beta > 0$ and $\alpha\beta > 1$, we have that $Q^{\mathrm{w}}_{(\alpha,\beta)}(x, D)$ can be seen as *a matrix-valued deformation of the scalar harmonic oscillator*. In the case $\alpha \neq \beta$ and $\alpha, \beta > 0$ it was proved by Parmeggiani in [42] (Theorem 4.4, pp. 351-353) that $Q^{\mathrm{w}}_{(\alpha,\beta)}(x, D)$ does not admit creation/annihilation operators.

Finally, we remark that a motivation for investigating systems like $Q^{\mathrm{w}}_{(\alpha,\beta)}(x, D)$ originates from PDEs, that is, from the study of a-priori lower bound estimates, such as Melin’s or Hörmander’s or Fefferman-Phong’s, for pseudodifferential systems (see [39], and also [43, 44] and the references therein).

1.2 Semiregular symbols and our class

We give in this section the definition of semiregular symbols that we will be considering in the sequel, their basic properties and then introduce the class of systems we consider here.

In order to prepare the ground also to the study of extensions of this kind of systems to more general classes of systems, we will be using the following notation for the Hörmander metric and admissible weight (see Hörmander [26]): with $X = (x, \xi)$, $Y = (y, \eta)$, etc., belonging to the phase-space $\mathbb{R}^n \times \mathbb{R}^n$, and $m(X) := \langle X \rangle = (1 + |X|^2)^{1/2}$ the usual "Japanese bracket", we consider the Hörmander metric $g_X = |dX|^2/m(X)^2$. Then, m is an admissible function (and so is m^μ for any given $\mu \in \mathbb{R}$), and we may exploit the full power of the Weyl-Hörmander pseudodifferential calculus. We will also write $\dot{\mathbb{R}}^{2n}$ for $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$.

Definition 1.2.1. *Let M_N denote the algebra of $N \times N$ complex matrices. A symbol $a \in S(m^\mu, g; \mathsf{M}_N)$ is said to be classical (see Remark 3.2.4 of [45]) if it possesses an asymptotic expansion $\sum_{j\geq 0} a_{\mu-2j}$ in isotropic (i.e. positively homogenous and smooth outside the origin) terms $a_{\mu-2j}$ positively homogeneous of degree $\mu - 2j$. We write $a \in S_{\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$.*

We say that $a \in S(m^\mu, g; \mathsf{M}_N)$ is semiregular (see Remark 3.2.4 of [45]) if $a = a^{(0)} + a^{(1)}$, where $a^{(0)} \in S_{\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ and $a^{(1)} \in S_{\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N)$.

We write $a \in S_{\mathrm{sreg}}(m^{\mu}, g; \mathsf{M}_N)$. In other words, a symbol a is semiregular if it possesses an asymptotic expansion $\sum_{j\geq 0} a_{\mu-j}$ in isotropic terms, that is, $a_{\mu-j}$ is positively homogeneous of degree $\mu - j$ and smooth outside the origin, $j \geq 0$.
Equivalently, a matrix-symbol a of order μ is semiregular if $a = a^{(0)} + a^{(1)}$, where $a^{(0)} \in S_{\mathrm{cl}}(m^{\mu}, g; \mathsf{M}_N)$ and $a^{(1)} \in S_{\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N)$. In such chase, we write $a \in S_{\mathrm{sreg}}(m^{\mu}, g; \mathsf{M}_N)$.
The terms a_{μ}, $a_{\mu-1}$ and $a_{\mu-2}$ are called the principal symbol, the semiprincipal symbol and the subprincipal symbol, respectively, of the operator $a^{\mathrm{w}} = a^{\mathrm{w}}(x, D)$.

More explicitly, $a \in S_{\mathrm{sreg}}(m^{\mu}, g; \mathsf{M}_N)$ if and only if there exists a sequence $(a_{\mu-j})_{j\geq 0} \subset C^{\infty}(\dot{\mathbb{R}}^{2n}; \mathsf{M}_N)$ where $a_{\mu-j}$ is positively homogeneous of degree $\mu - j$ in X and, for an excision function χ,

$$a - \chi \sum_{j=0}^{N} a_{\mu-j} \in S(m^{\mu-(N+1)}, g; \mathsf{M}_N), \quad \forall N \in \mathbb{Z}_+.$$

As usual, in such case we write

$$a \sim \sum_{j\geq 0} a_{\mu-j}.$$

Now we are going to prove that a semiregular pseudodifferential system, i.e. the Weyl-quantization of a symbol in $S_{\mathrm{sreg}}(m^{\mu}, g; \mathsf{M}_N)$, has a two-sided parametrix if it is elliptic. One consequence of this result is that the unbounded operator defined by the pseudodifferential operator on its maximal domain is self-adjoint and, if the domain is compactly embedded into L^2, with discrete spectrum which is semibounded from below when the main elliptical part is positive. According to Proposition 3.2.15 in [45], a preliminary result to prove the existence of a two-sided parametrix is the following one giving a symbol with prescribed WKB expansion.

Proposition 1.2.2. *Let $\mu_j \searrow -\infty, \mu_j > \mu_{j+1}$, $j \in \mathbb{N}$, be a monotone strictly decreasing sequence of real numbers. Let $a_j \in S(m^{\mu_j}, g)$, $j \in \mathbb{N}$. Then, there* *exists* $a \in S(m^{\mu_1}, g)$ *such that*

$$a \sim \sum_{j \geq 1} a_j,$$

that is, for all $r \in \mathbb{N}$ *we have*

$$a - \sum_{j=1}^{r} a_j \in S(m^{\mu_{r+1}}, g).$$

If another a' *has the same property, then* $a - a' \in \mathscr{S}(\mathbb{R}^{2n})$.

Proof. Let χ be an excision function, namely, $\chi \in C^\infty(\mathbb{R}^{2n})$, $0 \leq \chi \leq 1$, such that $\chi(X) = 0$ if $|X| \leq 1/2$ and $\chi(X) = 1$ if $|X| \geq 1$. In the first place, we show that we can choose a monotone strictly increasing sequence of positive numbers $R_j \to +\infty$, increasing so quickly as $j \to +\infty$ that, for any given $j \geq 2$ and for all $\alpha \in \mathbb{Z}_+^{2n}$ with $|\alpha| \leq j$,

$$|\partial_X^\alpha \left(\chi(X/R_j) a_j(X)\right)| \leq 2^{-j} m(X)^{\mu_j + 1 - |\alpha|}. \tag{1.2.1}$$

To see this, note that

$$|\partial_X^\alpha \left(\chi(X/R)\right)| \leq C_\alpha m(X)^{-|\alpha|}, \quad \text{for } R \geq 1. \tag{1.2.2}$$

In fact,

$$\partial_X^\alpha \left(\chi(X/R)\right) = R^{-|\alpha|} (\partial_X^\alpha \chi)(X/R),$$

and

$$|\alpha| \geq 1,\ X \in \operatorname{supp}(\partial_X^\alpha \chi)(\cdot/R) \Longrightarrow R/2 \leq |X| \leq R,$$

from which (1.2.2) follows, and shows that $\chi(\cdot/R) \in S(1, g)$ **uniformly** in $R \geq 1$. Then, for all $\alpha \in \mathbb{Z}_+^{2n}$,

$$|\partial_X^\alpha \left(\chi(X/R) a_j(X)\right)| \leq C_{j,\alpha} m(X)^{\mu_j - |\alpha|}, \tag{1.2.3}$$

if $R \geq 1$, whence

$$\chi(\cdot/R)a_j \in S(m^{\mu_j}, g), \quad j \in \mathbb{N}, \quad R \geq 1.$$

Of course, this is seen also by noting that $\chi(X/R)a_j(X) = a_j(X)$ for X large, for

$$\operatorname{supp}\left(\chi(\cdot/R)a_j\right) \subset \left\{X \in \mathbb{R}^{2n}; |X| \geq R/2\right\}, \quad \forall j \in \mathbb{N}. \tag{1.2.4}$$

Now, given any $j \geq 1$, if $R \geq$ **and** $|\alpha| \leq j$ we have

$$|\partial_X^\alpha \left(\chi(X/R)a_j(X)\right)| \leq \max_{|\alpha|\leq j} \{C_{j,\alpha}\}\, m(X)^{\mu_j - |\alpha|} =: C_j m(X)^{\mu_j - |\alpha|}. \tag{1.2.5}$$

On the other hand,

$$m(X)^{\mu_j - |\alpha|} \leq \varepsilon m(X)^{\mu_j + 1 - |\alpha|},$$

where X is such that

$$m(X) = (1 + |X|^2)^{1/2} \geq \frac{1}{\varepsilon}.$$

So, to satisfy (1.2.1), it suffices to choose $\varepsilon = 1/(2^j C_j)$, and take

$$(1 + R_j^2/4)^{1/2} \geq 2^j C_j, \quad j \geq 2.$$

That is, it suffices to take

$$R_j \geq 2^{j+1} C_j, \quad j \geq 2.$$

Hence, we may choose

$$R_1 = 1,\ R_j = 2^{j+1}(C_j + 1) + R_{j-1}, \quad j \geq 2.$$

Now, (1.2.4) and $R_j \nearrow +\infty$ yield that the sum

$$a(X) := \sum_{j\geq 1} \chi(X/R_j)a_j(X)$$

is locally finite, hence $a \in C^\infty$. On the other hand, given any $r \in \mathbb{N}$ and any $\alpha \in \mathbb{Z}_+^{2n}$, we may find $N \in \mathbb{N}$ so large that $|\alpha| \leq N+1$ and $\mu_{N+1}+1 \leq \mu_r$. Hence

$$\left|\partial_X^\alpha \Big(a(X) - \sum_{j=1}^N \chi(X/R_j)a_j(X)\Big)\right| = \left|\partial_X^\alpha \Big(\sum_{j=N+1}^\infty \chi(X/R_j)a_j(X)\Big)\right| \tag{1.2.6}$$

$$\leq \sum_{j=N+1}^\infty \frac{1}{2^j} m(X)^{\mu_j+1-|\alpha|}$$

$$\leq 2^{-N} m(X)^{\mu_r-|\alpha|} \tag{1.2.7}$$

(recall that $\mu_j > \mu_{j+1}$ and $\mu_j \to -\infty$). Therefore, by choosing $r=1$ we obtain that for any given $\alpha \in \mathbb{Z}_+^{2n}$ (with N large depending on α as above)

$$\partial_X^\alpha a(X) = \partial_X^\alpha \Big(a(X) - \sum_{j=1}^N \chi(X/R_j)a_j(X)\Big) + \partial_X^\alpha \Big(\sum_{j=1}^N \chi(X/R_j)a_j(X)\Big),$$

whence, by (1.2.3) and (1.2.6),

$$|\partial_X^\alpha a(X)| \leq \frac{1}{2^N} m(X)^{\mu_1-|\alpha|} + \sum_{j=1}^N C_{j,\alpha} m(X)^{\mu_j-|\alpha|} \leq C_\alpha m(X)^{\mu_j-|\alpha|},$$

that is, $a \in S(m^{\mu_1}, g)$.

On the other hand, we also obtain from (1.2.6) that for any given $r \in \mathbb{N}$ and any given $\alpha \in \mathbb{Z}_+^{2n}$ with $N \geq r+1$ so large depending on α and r that $|\alpha| \leq N+1$ and $\mu_{N+1}+1 \leq \mu_{r+1}$,

$$\left|\Big(\partial_X^\alpha a(X) - \sum_{j=1}^r a_j(X)\Big)\right| \leq \left|\partial_X^\alpha \Big(a(X) - \sum_{j=1}^N \chi(X/R_j)a_j(X)\Big)\right|$$

$$+ \sum_{j=1}^r \left|\partial_X^\alpha \Big((1-\chi(X/R_j))a_j(X)\Big)\right|$$

$$+ \sum_{j=r+1}^r \left|\partial_X^\alpha \Big(\chi(X/R_j)a_j(X)\Big)\right|$$

$$\leq C_{\alpha,r} m(X)^{\mu_{r+1}-|\alpha|}$$

for we have

$$(1-\chi(\cdot/R_j))a_j \in S(m^{-\infty}, g), \quad \forall j \in \mathbb{N},$$

and

$$\chi(\cdot/R_j)a_j \in S(m^{\mu_{r+1}}, g), \quad \forall j \geq r+1.$$

This shows that $a \sim \sum_{j\geq 1} a_j$.

Finally, if $a' \in S(m^{\mu_1}, g)$ has this last property, then, for all $r \in \mathbb{N}$,

$$a - a' = \Big(a - \sum_{j=1}^{r} a_j\Big) - \Big(a' - \sum_{j=1}^{r} a_j\Big) \in S(m^{\mu_{r+1}}, g),$$

that is, $a - a' \in S(m^{-\infty}, g) = \mathscr{S}(\mathbb{R}^{2n})$, which concludes the proof.

□

Moreover, we give a notion of ellipticity for systems.

Definition 1.2.3. *A symbol $a \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ is said to be globally elliptic when its principal part a_μ satisfies*

$$|\det(a_\mu(X))| \approx |X|^{\mu N}, \quad \forall X \in \dot{\mathbb{R}}^{2n}.$$

When a is globally elliptic, we will say that the corresponding ψdo $a^{\mathrm{w}}(x, D)$ is globally elliptic.

Now, we are ready to prove the existence of a two-sided parametrix for an elliptic semiregular pseudodifferential system.

Theorem 1.2.4. *Let $A \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ be elliptic. Then, there exists $B \in S_{\mathrm{sreg}}(m^{-\mu}, g; \mathsf{M}_N)$ such that*

$$A^{\mathrm{w}} B^{\mathrm{w}} = \mathrm{I} + R, \; B^{\mathrm{w}} A^{\mathrm{w}} = I + R^{'},$$

where R, S are smoothing operators and $I := \mathrm{id} \otimes I_N$.

Proof. In the first place, it suffices to see that $B^{\mathrm{w}} A^{\mathrm{w}} = I + R$ and $A^{\mathrm{w}} \tilde{B}^{\mathrm{w}} = \mathrm{I} + \tilde{R}$, for some B and $\tilde{B}$. In fact, we then have

$$B^{\mathrm{w}} A^{\mathrm{w}} \tilde{B}^{\mathrm{w}} = B^{\mathrm{w}}(I + \tilde{R}) = (I + R)\tilde{B}^{\mathrm{w}},$$

that is, $B^{\mathrm{w}} = \tilde{B}^{\mathrm{w}} + (R\tilde{B}^{\mathrm{w}} - B^{\mathrm{w}}\tilde{R})$ and $R\tilde{B}^{\mathrm{w}} - B^{\mathrm{w}}\tilde{R}$ is smoothing. We hence prove that we may find B^{w} as in the statement, such that $B^{\mathrm{w}}A^{\mathrm{w}} = I + R$ (i.e. B^{w} is a left-parametrix). The construction of a right-parametrix is completely analogous.

Let χ be an excision function. Let

$$b_{-\mu} = \frac{\chi}{a_\mu} \in S_{\mathrm{sreg}}(m^{-\mu}, g; \mathsf{M}_N).$$

Then

$$b^{\mathrm{w}}_{-\mu}A^{\mathrm{w}} = I + r_1^{\mathrm{w}},$$

where $r_1 \in S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$. We hence "Neumann-invert" $I + r_1^{\mathrm{w}}$ as follows. For any given $N \in \mathbb{Z}_+$ we have

$$\left(\sum_{j=0}^{N}(-r_1^{\mathrm{w}})^j\right) b^{\mathrm{w}}_{-\mu}A^{\mathrm{w}} = I - (-r_1^{\mathrm{w}})^{N+1}.$$

If we denote by $r_1^{(j)}$ the symbol of $(r_1^{\mathrm{w}})^j$, that is, $r_1^{(0)} = 1$, and $r_1^{(j)} = \underbrace{r_1\#\ldots\#r_1}_{j \text{ times}}$, where $\#$ denotes the symbol composition, then by the symbol calculus (see also Proposition 3.2.15 of [45]) there is a symbol $s \in S_{\mathrm{sreg}}(1, g; \mathsf{M}_N)$ such that $s \sim \sum_{j=0}^{N}(-1)^j r_1^{(j)}$. Hence, $s^{\mathrm{w}}b^{\mathrm{w}}_{-\mu}A^{\mathrm{w}} = I + R$ with R smoothing. We conclude the proof by setting $B := s\#b_{-\mu}$.

□

Comment on the notation. Helffer in [20] and the authors of [11] and of [12] use Γ_{cl} for the set of semiregular symbols. We decided to adopt our notation S_{sreg} because the natural homogeneity of the Poisson bracket of homogeneous symbols is the sum of the orders minus 2. Whence, it is natural in the global calculus to call "regular" those symbols whose asymptotic expansion is made of homogeneous symbols for which the j-th term has order $\mu - 2j$ where μ is the order of the principal term. This is indeed parallel to the use of "semiregular" appearing in the paper by Boutet De Monvel on the hypoellipticity of the $\bar{\partial}$ operator.

Remark 1.2.5. *It is clear that composition of semiregular symbols yields a semiregular symbol.*

Of course, when the symbol $a \in S(m^\mu, g; \mathsf{M}_N)$ is Hermitian, then the corresponding pseudodifferential operator $a^{\mathrm{w}}(x, D)$, obtained by Weyl-quantizing $a(X)$, is formally self-adjoint. We write $\Psi(m^\mu, g; \mathsf{M}_N)$, respectively $\Psi_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$, for the ψdos obtained by Weyl-quantization of symbols in $S(m^\mu, g; \mathsf{M}_N)$, resp. $S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$.

As usual, for $A, B > 0$, we write $A \lesssim B$ when there is $C > 0$ such that $A \leq CB$, and write $A \approx B$ when there are $C, C' > 0$ such that $CA \leq B \leq C'A$.

When $\mu > 0$ and $a = a^* \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ is globally elliptic (hence, $a^*_{\mu-j} = a_{\mu-j}$ for all $j \geq 0$ and a_μ is globally elliptic), the existence of a (semiregular) two-sided parametrix yields that $a^{\mathrm{w}}(x, D)$, realized as an unbounded operator on $L^2(\mathbb{R}^n; \mathbb{C}^N)$ with maximal domain the Shubin Sobolev space $B^\mu(\mathbb{R}^n; \mathbb{C}^N)$ (see [58], or [20] or [41]), is *self-adjoint* with a discrete spectrum. When furthermore $a_\mu > 0$ (as a Hermitian matrix), then $a^{\mathrm{w}}(x, D)$ is semibounded and hence has a spectrum bounded from below.

We are now in the position to introduce the class of systems we are interested in.

Definition 1.2.6. *We say that an $N \times N$ symbol $a \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ is a* ***semiregular metric globally elliptic system*** *(SMGES for short) of order μ, when*

$$a(X) = a(X)^* = p_\mu(X) I_N + a_{\mu-1}(X) + a_{\mu-2}(X) + S_{\mathrm{sreg}}(m^{\mu-3}, g; \mathsf{M}_N), \quad X \neq 0,$$

where:

- $p_\mu \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ *is positively homogeneous of degree μ and such that $|X|^\mu \approx p_\mu(X)$ for all $X \neq 0$;*
- $a_{\mu-1} = a^*_{\mu-1}$ *is such that there exists $r \geq 1$ and $e_0 \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathsf{M}_N)$* ***unitary*** *and positively homogeneous of degree 0 such that*

$$e_0(X)^* a_{\mu-1}(X) e_0(X) = \mathsf{diag}(\lambda_{\mu-1,j}(X) I_{N_j}; 1 \leq j \leq r), \quad X \neq 0$$

where $N = N_1 + N_2 + \ldots + N_r$ *and* $\lambda_{\mu-1,j} \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$ *are positively homogeneous of degree* $\mu - 1$ *and such that*

$$j < k \Longrightarrow \lambda_{\mu-1,j}(X) < \lambda_{\mu-1,k}(X), \quad \forall X \neq 0.$$

1.3 The Jaynes-Cummings model

The JC-model is a fully solvable *quantum mechanical model* of an atom in a field. The JC-model, introduced in 1963 [1], has served as a theoretical description of the light-matter interaction and has continued to fulfil in unanticipated ways the objectives of its originators, making it possible to examine the basic properties of quantum electrodynamics. The relative simplicity of the JC-model and the ease with which it can be extended through analytic expressions or numerical computations continue to motivate attention.

More in detail, the JC-model was first introduced to study the classical aspects of spontaneous emission (SE) and to reveal the existence of *Rabi oscillations*[1]in atomic excitation probabilities for fields with sharply defined energy (or photon number). In case of fields with a statistical distribution of photon numbers, the oscillations collapse to an expected steady value. In the original formulation [1], the Jaynes-Cummings model (JC-model) considered a ***single two-state atom (molecule) interacting with a single near-resonant quantized cavity mode of the electromagnetic field*** (Fig. 1). Thus, it can be stated that:

[1]If light interacts with a two-level system (e.g. an atom or ion with a ground state and an excited state), this can lead to a periodic exchange of energy between the light field and the two-level system. These oscillations are called **Rabi oscillations** (with reference to the Nobel Prize winner Isidor Isaac Rabi). They are associated with oscillations of the quantum mechanical expectation values of level populations and photon numbers. They can be interpreted as a periodic change between absorption and stimulated emission of photons. A competing process, which can prevent these oscillations, is spontaneous emission.

The Jaynes-Cummings model consists of a single two-level atom coupled to a quantized single-mode field, represented as a harmonic oscillator (HO). The coupling between atom and field is characterized by a Rabi frequency Ω_1. *Loss of excitation in the atom appears as a gain in excitation of the oscillator.*

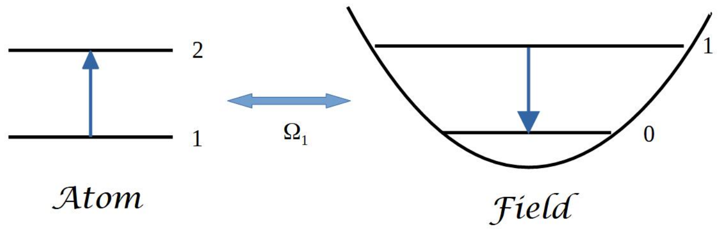

Fig. 1

In 1980 it was found that, under appropriate initial conditions (e.g. a near-classical field), the Rabi oscillations would eventually revive, only to collapse and revive repeatedly in a complicated pattern. Indeed, in JC-model analytic solutions these revivals are present. Their existence provided proof of the truly *quantum nature of radiation*, as it gave direct evidence for discreteness of field excitation (photons).

Further non-classical properties of the JC-model field, such as a tendency of the photons to antibunch (see Section 1.3.1.1), was revealed by subsequent studies.

In the early 90s it has been discovered the existence of the atom and field in a macroscopic superposition state (a Schrödinger cat) during the quiescent intervals of collapsed Rabi oscillations. This provided the opportunity to use the JC-model to enlighten the basic properties of *quantum correlation* (*entanglement*) and, indeed, to investigate in further depth the *relationship between classical and quantum physics*.

As reported in [57], during the years there has arisen a strong scientific effort aimed at exploiting and extending the JC-model. A first significant motivation of this great interest lies in the discovery of many relevant properties of the model, dealing with the possibility of finding solutions (often exact) to fundamental models of a quantum theory of interacting fields and atoms. A second notable motivation stays in the considerable advance in cavity quantum electrodynamics (QED) experiments involving single atoms (usually Rydberg atoms) within single-mode cavities (the micromaser). As a matter of fact, through this the theory has turned from an academic curiosity into a testable enterprise. All the reasons mentioned above, combined with a considerable ease of computational implementation, are at the bases of the still alive strong interest in extension and generalization of the original formulation of the Jaynes-Cummings model.

The ***JC-model Hamiltonian operator*** is:

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + E_1\hat{S}_{11} + E_2\hat{S}_{22} + \frac{\hbar}{2}\Omega_1(\hat{a}^\dagger\hat{S}_{12} + \hat{a}\hat{S}_{21}) \tag{1.3.1}$$

where:

- ω is the *frequency of the mono-modal field*,
- E_k is the *energy of the atomic state* ψ_k,
- Ω_1 is the *atom-field coupling constant*,
- $\hat{S}_{jk}$ is a transition operator acting on atomic states (Section 1.3.2.2).

Now, we introduce some notions that constitute the basis of the *theoretical foundation* for the JC-model (Section 1.3.1): the *harmonic oscillator* (bosonic states), the *two-level system* (fermionic states), *coherent coupling.*

Then, we present the *mathematical formulation* of the JC-model building the *Hamiltonian* (Section 1.3.2).

1.3.1 JC-model theoretical foundation

In this section we introduce some basic concepts that we will follow in the rest of our work: the *quantum harmonic oscillator* (HO), *the two-level system* (2LS), *coherent coupling.*

The approach in Subsection 1.3.1.1 will be based on the works [50, 10], while the one in Subsections 1.3.1.2 and 1.3.1.3 will be based on the work [10].

1.3.1.1 Harmonic Oscillator (HO) - Bosonic states

Light-matter interaction can be satisfactorily described through the semi-classical approach. Nevertheless, not all important effects can be explained by such approach. For instance, that is not possible for emission of excited atom, despite it lies in the fundamentals of many physical systems. For understanding this and other purely quantum effects, it is necessary to *quantize the field*, which requires introducing photons, that is, a quantum of electromagnetic field.

▶ The Hamiltonian of a bosonic quantum field as a sum of HOs

In this section we present a first step in the direction of electromagnetic field quantization. Namely, we show that, starting from the wave nature of electromagnetic field, its Hamiltonian can be represented as a sum of harmonic oscillators energies.

Let us recall that, in the absence of charges and currents, the *Maxwell equations in vacuum* have the form

$$\begin{cases} \mathrm{rot}\, \underline{\mathbf{E}} = -\frac{1}{c}\frac{\partial \underline{\mathbf{H}}}{\partial t}, \\ \mathrm{rot}\, \underline{\mathbf{H}} = \frac{1}{c}\frac{\partial \underline{\mathbf{E}}}{\partial t}, \\ \mathrm{div}\, \underline{\mathbf{E}} = 0, \\ \mathrm{div}\, \underline{\mathbf{H}} = 0. \end{cases} \tag{1.3.2}$$

We will deal with the *vector potential* $\mathbf{A}$, which can be defined as follows:

$$\underline{\mathbf{H}} = \mathrm{rot}\, \mathbf{A}, \tag{1.3.3}$$

$$\underline{\mathbf{E}} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi, \tag{1.3.4}$$

where φ is the *scalar electric potential*, that is, a smooth function which could be smoothly extended to the whole space $\mathbb{R}^3$ for any fixed time $t \geq 0$.

Note that the vector and scalar potential can be defined in non-unique way up to the gradient of an arbitrary real function and time derivative of the same function, often called *gauge freedom*. We eliminate this uncertainty in $\mathbf{A}$ and φ by applying an additional restriction (*Lorentz gauge*):

$$\operatorname{div} \mathbf{A} = 0. \tag{1.3.5}$$

Now, by replacing the electric field expression into the second equation of (1.3.2), we have

$$\operatorname{rot}\operatorname{rot} \mathbf{A} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t} \tag{1.3.6}$$

and, since

$$\operatorname{rot}\operatorname{rot} \mathbf{A} = \operatorname{grad}\operatorname{div} \mathbf{A} - \operatorname{div}\operatorname{grad} \mathbf{A} = -\Delta\, \mathbf{A}, \tag{1.3.7}$$

we get the Helmholtz equation for the vector potential:

$$\Delta\, \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t}. \tag{1.3.8}$$

Through applying the divergence operation to both sides of (1.3.4), we obtain the equation for the scalar potential:

$$\underbrace{\operatorname{div} \mathbf{E}}_{=0 \text{ by } (1.3.2)} = -\tfrac{1}{c}\tfrac{\partial}{\partial t}\underbrace{\operatorname{div} \mathbf{A}}_{=0 \text{ by } (1.3.5)} - \Delta\varphi \implies \Delta\varphi = 0. \tag{1.3.9}$$

Hence, φ is an harmonic function satisfying periodic boundary conditions which could be smoothly extended to $\mathbb{R}^3$ for any fixed time $t \geq 0$. Therefore, φ is constant, which means that

$$\nabla\varphi = 0. \tag{1.3.10}$$

Thus, in a free space, we can simplify the conditions for $\mathbf{A}$ into

$$\begin{cases} \Delta\, \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \tag{1.3.11}$$

Now, having set the equation for the vector potential, we can determine its solution for a very simple, yet very important, case. Let us consider a *cube box* with edge length L, with periodic boundary conditions and take a *wave* with wave vector k. Such a box can figure a free space when $L \longrightarrow \infty$.

We may write the solution of (1.3.11) as a sum of all eigensolutions, which are the plane waves in Cartesian system. Considering periodic boundary and calling $\mathbf{r}$ the position vector in the box, we get the following expression for the solution

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{r}\rangle} \mathbf{A}_{\mathbf{k}}(t), \qquad \mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}, \; k_\alpha = \frac{2\pi n_\alpha}{L}, \; n_\alpha \in \mathbb{Z}, \; \alpha = x, y, z, \tag{1.3.12}$$

where:

– the vector potential $\mathbf{A}(\mathbf{r},t)$ is *real valued* and this provides the condition

$$\mathbf{A}_{\mathbf{k}}(t) = \overline{\mathbf{A}_{-\mathbf{k}}(t)}; \tag{1.3.13}$$

– defining

$$k := \|\mathbf{k}\|, \quad \omega_{\mathbf{k}} = c\,k = c\sqrt{k_x^2 + k_y^2 + k_z^2},$$

the temporal dependence of the vector potential is described, thanks to condition (1.3.13), by two oscillating terms

$$\mathbf{A}_{\mathbf{k}}(t) = \tilde{\boldsymbol{A}}_{\mathbf{k}} \left(e^{-i\omega_{\mathbf{k}} t} \mathbf{c}_{\mathbf{k}} + e^{i\omega_{\mathbf{k}} t} \overline{c_{-\mathbf{k}}} \right), \quad \text{where } \tilde{\boldsymbol{A}}_{\mathbf{k}} \in \mathbb{R}; \tag{1.3.14}$$

– the Lorentz gauge leads to the fact that the *waves are transverse*, that is

$$\operatorname{div} \mathbf{A} = 0 \iff \sum_{\mathbf{k}} \langle \mathbf{k}, \mathbf{A}_{\mathbf{k}}(t) \rangle \, e^{i\langle \mathbf{k},\mathbf{r}\rangle} = 0 \iff \langle \mathbf{k}, \mathbf{A}_{\mathbf{k}}(t) \rangle = 0. \tag{1.3.15}$$

Since, by Maxwell equations, there are two independent polarizations, we introduce two *transverse polarization vectors* $\mathbf{e}_{\mathbf{k}1}$, $\mathbf{e}_{\mathbf{k}2}$. Note that the three vectors $(\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}; \mathbf{k}/k)$ form a *right-handed orthonormal basis* which implies[2]:

$$\begin{cases} \langle \mathbf{k}, \mathbf{e}_{\mathbf{k}s} \rangle = 0, \\ \mathbf{e}_{\mathbf{k}1} \times \mathbf{e}_{\mathbf{k}2} = \mathbf{k}/k, \qquad \text{where } s, s' \in \{1,2\}. \\ \langle \mathbf{e}_{\mathbf{k}s}, \mathbf{e}_{\mathbf{k}s'} \rangle = \delta_{ss'}. \end{cases} \tag{1.3.16}$$

In addition, for suitably chosen terms $c_{\mathbf{k}s}$, we have

$$\mathbf{c}_{\mathbf{k}} = \sum_{\mathbf{k}} c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s}, \tag{1.3.17}$$

since $\mathbf{A}_{\mathbf{k}}(t)$ is a plane wave with wave vector $\mathbf{k}$.

Therefore, we can rewrite the decomposition of $\mathbf{A}$ as follows

$$\begin{aligned} \mathbf{A}(\mathbf{r},t) &= \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{r}\rangle} (e^{-i\omega_{\mathbf{k}} t} c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} + e^{i\omega_{\mathbf{k}} t} \overline{c_{-\mathbf{k}s}}\, \overline{\mathbf{e}_{-\mathbf{k}s}} e^{i\omega_{\mathbf{k}} t}) && (1.3.18) \\ &= \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{r}\rangle} e^{-i\omega_{\mathbf{k}} t} c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} + \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{r}\rangle} e^{i\omega_{\mathbf{k}} t} \overline{c_{-\mathbf{k}s}}\, \overline{\mathbf{e}_{-\mathbf{k}s}} && (1.3.19) \\ &\underbrace{=}_{(1.3.14)} \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{r}\rangle} e^{-i\omega_{\mathbf{k}} t} c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} + \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} e^{-i\langle \mathbf{k},\mathbf{r}\rangle} e^{i\omega_{-\mathbf{k}} t} \overline{c_{\mathbf{k}s}}\, \overline{\mathbf{e}_{\mathbf{k}s}} && (1.3.20) \\ &= \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} (e^{i\langle \mathbf{k},\mathbf{r}\rangle} u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} + e^{-i\langle \mathbf{k},\mathbf{r}\rangle} \overline{u_{\mathbf{k}s}(t)}\, \overline{\mathbf{e}_{\mathbf{k}s}}), && (1.3.21) \end{aligned}$$

where $u_{\mathbf{k}s}(t) = e^{-i\omega_{\mathbf{k}} t} c_{\mathbf{k}s}$.

Now we can write the fields $\mathbf{E}$ and $\mathbf{H}$ as follows

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c} \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} \omega_{\mathbf{k}} (e^{i\langle \mathbf{k},\mathbf{r}\rangle} u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} - e^{-i\langle \mathbf{k},\mathbf{r}\rangle} \overline{u_{\mathbf{k}s}(t)} \overline{\mathbf{e}_{\mathbf{k}s}}), \tag{1.3.22}$$

[2]In this work the notations $\langle \cdot, \cdot \rangle$ and $\cdot \times \cdot$ stand, respectively, for the *scalar product* and the *vector product* for *complex* valued vectors. Moreover, denoting by $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ and $\cdot \times_{\mathbb{R}} \cdot$, respectively, the scalar product and the vector product for *real* valued vectors, one has that

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \langle \mathbf{v}, \overline{\mathbf{w}} \rangle_{\mathbb{R}}, \\ \mathbf{v} \times \mathbf{w} &= \mathbf{v} \times_{\mathbb{R}} \overline{\mathbf{w}}. \end{aligned}$$

$$\mathbf{H} = \operatorname{rot} \mathbf{A} = i \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}} (e^{i\langle \mathbf{k},\mathbf{r}\rangle} u_{\mathbf{k}s}(t)[\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}] - e^{-i\langle \mathbf{k},\mathbf{r}\rangle} \overline{u_{\mathbf{k}s}(t)}[\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}]). \quad (1.3.23)$$

This plane-wave expansion let us get a simple picture of the electromagnetic field as an ensemble of oscillators.

Moreover, it is very explanatory and relevant to consider the energy of electromagnetic field inside the box, which is

$$\mathscr{H} = \frac{1}{8\pi} \int (\|\mathbf{H}\|^2 + \|\mathbf{E}\|^2)\, dV. \quad (1.3.24)$$

This can be further simplified by using a pair of important relations:

− the first is the *orthogonality of the modes*, that is

$$\int_{L^3} e^{i\left\langle \mathbf{k}-\mathbf{k}',\mathbf{r}\right\rangle}\, dV = L^3 \delta_{\mathbf{k}\mathbf{k}'}, \quad (1.3.25)$$

feature that makes the terms

$$\tilde{\boldsymbol{A}}_{\mathbf{k}} \tilde{\boldsymbol{A}}_{\mathbf{k}'} \omega_{\mathbf{k}} \omega_{\mathbf{k}'} \overline{u_{\mathbf{k}s}(t)} u_{\mathbf{k}'s}(t) u_{\mathbf{k}'s}(t) e^{i\left\langle \mathbf{k}'-\mathbf{k},\mathbf{r}\right\rangle} \left\langle \mathbf{e}_{\mathbf{k}s}, \mathbf{e}_{\mathbf{k}'s}\right\rangle, \mathbf{k} \neq \mathbf{k}'$$

vanishing;

− the second is

$$\langle \mathbf{e}_{\mathbf{k}s}, \mathbf{e}_{\mathbf{k}s'}\rangle = \delta_{ss'} \quad , \quad (1.3.26)$$

which implies

$$\langle \mathbf{k} \times \mathbf{e}_{\mathbf{k}s}, \mathbf{k} \times \overline{\mathbf{e}_{\mathbf{k}s}}\rangle = \langle \mathbf{k}, \mathbf{k}\rangle \langle \mathbf{e}_{\mathbf{k}s}, \mathbf{e}_{\mathbf{k}s'}\rangle = \mathbf{k}^2 \delta_{ss'}.$$

Hence, we have

$$\mathscr{H} = \frac{L^3}{4\pi} \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}}^2 \left(\underbrace{\frac{\omega_{\mathbf{k}}^2}{c^2} |u_{\mathbf{k}s}(t)|^2}_{\text{from } \|\mathbf{E}\|^2} + \underbrace{k^2 |u_{\mathbf{k}s}(t)|^2}_{\text{from } \|\mathbf{H}\|^2} \right). \quad (1.3.27)$$

Then, since $k^2 = \frac{\omega_{\mathbf{k}}^2}{c^2}$ is true for each mode, one can write

$$\mathscr{H} = \frac{L^3}{2\pi} \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}}^2 k^2 |u_{\mathbf{k}s}(t)|^2, \tag{1.3.28}$$

which shows that the electric and magnetic counterparts give *equal contribution* to the total electromagnetic energy.

Now let us split the real and imaginary parts of the mode amplitude $|u_{\mathbf{k}s}(t)|$ by introducing the new variables

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + \overline{u_{\mathbf{k}s}(t)}, \tag{1.3.29}$$

$$p_{\mathbf{k}s}(t) = -i\omega_{\mathbf{k}}(u_{\mathbf{k}s}(t) - \overline{u_{\mathbf{k}s}(t)}). \tag{1.3.30}$$

It is clear that

$$u_{\mathbf{k}s}(t) = \tfrac{1}{2} q_{\mathbf{k}s}(t) - \tfrac{1}{2i\omega_{\mathbf{k}}} p_{\mathbf{k}s}(t) \implies |u_{\mathbf{k}s}(t)|^2 = \tfrac{1}{4\omega_{\mathbf{k}}^2} \left(p_{\mathbf{k}s}(t)^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}s}(t)^2 \right).$$

Moreover, substituting this last relation in (1.3.28), one gets the following expression for the energy

$$\mathscr{H} = \frac{L^3}{4\pi c^2} \sum_{\mathbf{k},s} \tilde{\boldsymbol{A}}_{\mathbf{k}}^2 \left(\frac{{p_{\mathbf{k}s}}^2}{2} + \frac{\omega_{\mathbf{k}}^2 q_{\mathbf{k}s}(t)^2}{2} \right). \tag{1.3.31}$$

Finally, assuming $\tilde{\boldsymbol{A}}_{\mathbf{k}} = \sqrt{4\pi c^2/L^3}$, we obtain the following important result

$$\boxed{\mathscr{H} = \sum_{\mathbf{k},s} \left(\frac{{p_{\mathbf{k}s}}^2}{2} + \frac{\omega_{\mathbf{k}}^2 q_{\mathbf{k}s}(t)^2}{2} \right), \qquad (1.3.32)}$$

which, indeed, represents the Hamiltonian of the electromagnetic field as a sum of harmonic oscillators energies, as stated at the beginning of this section.

▶ **Harmonic Oscillator (HO)**

The **quantum harmonic oscillator (HO)** is the most fitting representation of field excitations: it consists in the ceaseless possibility to create particles through a creation (or ladder) operator $\hat{a}^\dagger$, that means that it is the perfect match for *bosons*:

bosons *are particles, quasi-particles or composite particles that have an integer total spin and are allowed to occupy the same state.*

HO exactly models the electromagnetic field, composed of photons.

Let us have a look at the basic properties of the HO and its possible realizations. We remark that in this section we focus on *a singe mode of the electromagnetic field.*

To start with, the **1-PARTICLE STATE** is simply defined as the application of a creation operator $\hat{a}^\dagger$ on the vacuum,

$$|1\rangle = \hat{a}^\dagger |0\rangle ,$$

and lies in the Hilbert space L^2 and the ***n*-PARTICLE STATE** (or **FOCK STATE**) is obtained through recursive creations

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle , \tag{1.3.33}$$

where $1/\sqrt{n!}$ is a normalization prefactor depending on the state of the field. It must be remarked that$(|n\rangle)_{n\in\mathbb{N}}$ provides an orthonormal basis of the Hilbert space of the photons states. From a mathematical point of view.

The Hermitian conjugate $\hat{a}$ of $\hat{a}^\dagger$ annihilates a particle, hence these operators act on the ***number states*** $|\boldsymbol{n}\rangle$ (with n particles, n an *integer*), as:

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle , \tag{1.3.34}$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \tag{1.3.35}$$

$$\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle . \tag{1.3.36}$$

The composition $\hat{a}^\dagger\hat{a}$ is named *number operator.*

Hence, the most compact expression of **Hamiltonian operator of a free, single-mode, field** is

$$\boxed{\hat{H}_a = \hbar\omega_a \hat{a}^\dagger \hat{a}, \qquad (1.3.37)}$$

where ω_a is the *frequency of the monomodal field.*

Remark. *Comparing equation (1.3.37) with equation (1.3.1), it follows that (1.3.37) is the first term in the right-hand side of (1.3.1).*

We deal here with the *Schrödinger picture*, where states carry the temporal dynamics and operators are time-independent[3]. In this description, from the *commutation rules of bosons*

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

additional relations result, as the ones concerning the operators normal ordering (for which the moving of all creation operators to the left is needed):

$$\hat{a}\hat{a}^{\dagger n} = \hat{a}^{\dagger n}\hat{a} + n\hat{a}^{\dagger n-1} \qquad (1.3.38)$$

$$\hat{a}^n\hat{a}^\dagger = \hat{a}^\dagger\hat{a}^n + n\hat{a}^{n-1}. \qquad (1.3.39)$$

To deeper analyse HO interesting states, one can think of an ideal detector absorbing field particles of all frequencies one by one.

Remark. *With "absorption" we mean removing one particle from the initial field state $|i\rangle$ to get the final state $\hat{a}\,|i\rangle$.*

As outlined by Glauber [14] (1963), the probability per unit time of absorbing a particle regardless of the final state $|f\rangle$, is given by

$$\text{Probability}(1) = \sum_f |\langle f|\hat{a}|i\rangle|^2\,, \qquad (1.3.40)$$

[3]Heisenberg picture, where *operators* (and not states) evolve with time, is more suitable when we deal with two-time correlations.

which, under the assumption that the final states forms a *complete system*, is equal to the *mean number of particles*

$$\langle n_a \rangle = \left\langle i \left| \hat{a}^\dagger \hat{a} \right| i \right\rangle ,$$

as

$$\left| \langle f \left| \hat{a} \right| i \rangle \right|^2 = \left\langle i \left| \hat{a}^\dagger \right| f \right\rangle \langle f \left| \hat{a} \right| i \rangle .$$

Summing over f we get the result and obtain that the probability of counting a particle per unit of time is proportional to the *intensity of the field.*

This idea can be generalized to the probability of counting M photons at the same time getting the following relation

$$\text{Probability}(M) = \sum_f \left| \left\langle f | \hat{a}^M | i \right\rangle \right|^2 = \left\langle i \left| \hat{a}^{\dagger M} \hat{a}^M \right| i \right\rangle . \tag{1.3.41}$$

This enlightens the importance of normal ordering and how it is related to observable quantities when photon counting experiments are performed. Among these the most celebrated one is the two-particle coincidence experiment developed by Hanbury Brown and Twiss [17] with photons: taken at zero delay with photons, the two photons detection probability gives information about the statistics of the particle number distribution, that is an outstanding property of the quantum state of the field. As a matter of fact, a broadly considered quantity is the ***degree of second-order coherence*** (or *second-order correlation function at zero delay*)

$$g^{(2)} = \frac{\left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \right\rangle}{\left\langle \hat{a}^\dagger \hat{a} \right\rangle^2}. \tag{1.3.42}$$

This is related to the variance (or second *cumulant*) $\Delta n_0^2 = \left\langle (n_a - \langle n_a \rangle)^2 \right\rangle$ of the particles distribution:

$$g^{(2)} = 1 + \frac{\Delta n_a^2 - \langle n_a \rangle}{\langle n_a \rangle^2}. \tag{1.3.43}$$

In general, the ***degree of M^{th}-order coherence*** can be written as

$$g^{(M)} = \frac{\langle \hat{a}^{\dagger M} \hat{a}^M \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^M} = \frac{\langle n_a(n_a-1)(n_a-2)\dots(n_a-M+1)\rangle}{\langle n_a \rangle^M}. \tag{1.3.44}$$

The *Fock state* already introduced has a completely determined zero variance around the mean number of particles n. This turns into

$$g^{(2)} = 1 - 1/n = \begin{cases} 0, & n = 1 \\ \frac{1}{2}, & n = 2 \text{ (corresponding to a two-photon observable)}, \end{cases}$$

from which it follows that $g^{(2)}$ is always below 1:

$$g^{(2)} < 1.$$

This property of $g^{(2)}$ is linked to some kind of *quantum behaviour* (i.e. *antibunching*).

Remark. *Note that $|n\rangle$ is a very "quantum" state, in the sense that each quantum counts: the change in number results some strong effect, contrary to what happens in a classical continuous field, where a photon would be a minimal contribution, whose subtraction or addition has no impact, as we will see soon.*

In the case of detection of one photon from an initial state $|i\rangle = |1\rangle$, no further photon can be expected as it gets projected into vacuum $|f\rangle = |0\rangle$ when measuring the first. With reference to the number states, as photons are detected the probability of emission decreases: at high numbers, one particle less or more does not result in a relevant difference ($n \approx n \pm 1$). At this point a *classical* description and insights of the state starts to be effective, since $g^{(2)}$ tends to 1. One can find similar behaviour for higher orders of coherence

$$g^{(M)} = \frac{n!}{(n-M)!n^M}. \tag{1.3.45}$$

The probability of p particles present in the field can be stated as a *Kronecker* *delta*

$$\mathscr{P}_p = |\langle p|n\rangle|^2 = \delta_{n,p}.$$

The **COHERENT STATE** $|\boldsymbol{\alpha}\rangle$, determined for the first time in 1926 by Schrödinger, but fully developed in its quantum optical context in 1963 by Glauber [15], is another interesting state. It has the peculiarity of being the *eigenstate* of the annihilator operator a:

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{1.3.46}$$

with complex number eigenvalue

$$\alpha = |\alpha|e^{i\phi}.$$

Equation (1.3.46) shows that

the removed particle does not change the coherent state.

This is a basically *classical property*, where all detections are statistically independent, and it is in harsh contrast to the case of the number state. It follows that for the classical monochromatic wave the coherent state provide a good quantum description.

Let us take one mode of a transversal free electromagnetic field as an illustrative example of this point. The *electric field operator* E is composed of photons (bosons) and, at some point in space, one can write it (omitting constants) as a sum of two contributes

$$E = E^{(+)} + E^{(-)} = \frac{1}{2}(ae^{-i\omega_a t} + a^\dagger e^{i\omega_a t}). \tag{1.3.47}$$

This can be also regarded as the expression of a general bosonic field. The expectation value of the *electric field*, the *intensity operator* and the *field variance* in a coherent state, respectively, are

$$\langle E\rangle = \langle\alpha|E|\alpha\rangle = |\alpha|\cos(\omega_a t - \phi), \tag{1.3.48}$$

$$\left\langle E^2\right\rangle = \left\langle\alpha|E^2|\alpha\right\rangle = \langle E\rangle^2 + \frac{1}{4}, \tag{1.3.49}$$

$$\Delta E^2 = \left\langle E^2\right\rangle - \langle E\rangle^2 = \frac{1}{4}. \tag{1.3.50}$$

This mainly states that

the quantum fluctuations of the field ΔE *are independent of its intensity* $\langle E \rangle$ *and become negligible at large* $|\alpha|$*, since the amplitude of oscillations of* $\langle E \rangle$ *becomes really bigger than* ΔE^2*.*

In this regime *the coherent state can be considered a classical wave.*

However, in the case of *number states* the situation is deeply different:

$$\langle E \rangle = 0, \tag{1.3.51}$$

$$\langle E^2 \rangle = \frac{1}{2}(\frac{1}{2} + n), \tag{1.3.52}$$

$$\Delta E^2 = \langle E^2 \rangle, \tag{1.3.53}$$

since one has no electric mean field but quantum fluctuations.

In a *coherent state*, the variance of the particle number distribution is equal to the mean number

$$\langle n_a \rangle = \Delta n_a^2 = |\alpha|^2. \tag{1.3.54}$$

Indeed, at all orders all cumulants of the distribution converge to this value and the state is *coherent* (in Glauber's sense):

$$g^{(M)} = 1, \quad \text{for all } M.$$

This can be verified by obtaining the explicit expression of the coherent state in terms of number states, namely

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{1.3.55}$$

and, through the analysis of the distribution of the particle number, we obtain

$$\mathscr{P}_p = |\langle p|\alpha\rangle|^2 = e^{-\langle n_a \rangle} \frac{\langle n_a \rangle^p}{p!}, \tag{1.3.56}$$

which is a *Poissonian distribution* (see Fig. 2). If $g^{(2)} < 1$, as in the case of the number state, the distribution is called *subpoissonian*, while if $g^{(2)} > 1$, the distribution is named *superpoissonian*.

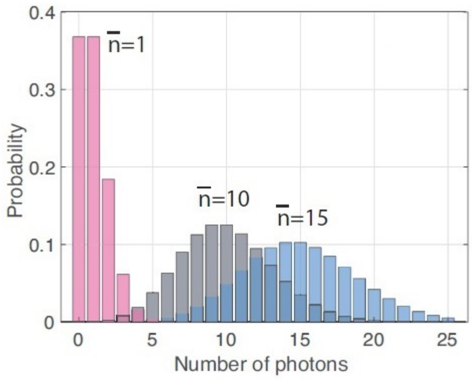

Fig. 2: The Probability distribution of photons in the coherent states for different average number of photons $\bar{\bar{n}}$, by [50].

Note that $|n\rangle$ and $|\alpha\rangle$ are completely described by a wavefunction (one ket): this kind of states are known as **PURE STATES**. They provide a good description for a field in some limit cases where it has good isolation from the environment, and it experiences only *coherent dynamics* given by a Hamiltonian. For instance, a wavefunction

$$\left|e^{-i\omega_a t}\alpha\right\rangle$$

always completely determines the evolution of $|\alpha\rangle$ through the free Hamiltonian (1.3.37) (a phase rotation in its complex parameter).
In general, anyway, we should consider the contamination of this dynamics imputable to the field as it is hopelessly in contact with the external world.

Remark. *Ideally, one could take into account a more complete Hamiltonian, inclusive of the sum of all processes affecting the field a, to model the totality of possible interactions with the environment. Obviously, this task would be impossible (implying the modelling of the whole universe ...), extremely difficult even in case of strong approximations. It is neither possible nor willing to take into account all the degrees of freedom which affect the field. Such a low level of interest in the outside world outcomes in decoherence for our system.*

In the previous example of a coherent state evolution, the field states can be thought of as influenced by an *incoherent process* that ceases their coherent free evolution (e.g. a measuring process that randomizes its phase). Actually, concerning this process we are interested only in preserving its effect on our system, that means the rate at which the perturbation occurs. After some time t_e, when the probability that a first event has happened is $\mathscr{P}_e$, the state of the system is no longer given by

$$\left|e^{-i\omega_a t_e}\alpha\right\rangle .$$

What is known is only that *the state of the system* is

$$\begin{cases} |e^{-i\omega_a t_e}\alpha\rangle & \text{with probability } 1-\mathscr{P}_e, \\ |\alpha\rangle & \text{with probability } \mathscr{P}_e. \end{cases} \tag{1.3.57}$$

Hence, a *mixture of two wavefunctions* is required instead of only one as for the pure state. According to this idea, the dynamics of the system can be seen as a sequence of coherent periods and incoherent random (from our "physiological" ignorant perspective) events, that drive the wavefunction into a given state. These are the ***quantum jumps***. It may be guessed that, after a while and a mixture of *quantum trajectories*, one completely loses track of the state phase. The meaning of this is that the *steady state* (SS) of this system is supposed to be a *mixture of coherent states* with equal probability for all possible phases. Explicitly,

$$\mathscr{P}(\phi)d\phi = 1/(2\pi).$$

This situation, and the most general description of the state of the system, can be consistently expressed by the use of the ***density matrix*** operator $\boldsymbol{\rho}$. As a rule, the density matrix can always be diagonalized as a linear superposition of projectors[4],

[4] A *rank 1 projector* is an operator such as $|\Psi\rangle\langle\Psi|$, that, when applied to $|\xi\rangle$, gives back the state $|\Psi\rangle$ with its weight in $|\xi\rangle$. This is zero if $|\xi\rangle$ and $|\Psi\rangle$ are orthogonal.

$$\rho = \sum_i \mathscr{P}_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| , \tag{1.3.58}$$

where $\{\mathscr{P}_i\}$ stand for the probabilities for the field to be in the states of a given basis $\{|\Psi_i\rangle\}$ in the Hilbert space.

The **pure state** is a particular case where

- $\rho = |\Psi_1\rangle \langle\Psi_1|$,
- all eigenvalues of ρ are zero except one ($\mathscr{P}_1 = 1$ and $\mathscr{P}_{i\neq 1} = 0$).

In this case, it easily follows that

$$\rho^2 = \rho.$$

Conversely, a **mixed state** is characterized by

$$\rho^2 \neq \rho \quad \Rightarrow \quad \mathrm{Tr}(\rho^2) < \mathrm{Tr}(\rho) = 1.$$

These properties do not depend on the choice of base. The same is true for other properties such as $\mathrm{Tr}(\rho) = 1$ (normalization) or $\rho = \rho^\dagger$ (Hermiticity). Considering any basis other than the one of the eigenstates, the off-diagonal elements of ρ give an account of the interplay or *coherence* between each pair of pure states. For instance, all off-diagonal terms of the density matrix of a coherent state

$$\rho^a_\alpha = |\alpha\rangle \langle\alpha| = e^{-|\alpha|^2} \sum_{m,n} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} |m\rangle \langle n| \tag{1.3.59}$$

are in the number state basis. On the other hand, in the case above of a mixture of coherent states with a random phase (see (1.3.57)), we can construct the SS density matrix as follows:

$$\rho^a_{|\alpha|} = \int_0^{2\pi} \frac{1}{2\pi} \left||\alpha|e^{i\phi}\right\rangle \left\langle|\alpha|e^{i\phi}\right| \, d\phi. \tag{1.3.60}$$

Now, applying (1.3.59) to $\left||\alpha|e^{i\phi}\right\rangle \left\langle|\alpha|e^{i\phi}\right|$, we have

$$\left||\alpha|e^{i\phi}\right\rangle \left\langle|\alpha|e^{i\phi}\right| = e^{-|\alpha|^2} \sum_{m,n} \frac{|\alpha|^{n+m} e^{i(n-m)\phi}}{\sqrt{m!n!}} |m\rangle \langle n| \, .$$

Therefore, since $\int_0^{2\pi} e^{i(n-m)\phi} d\phi = 2\pi\delta_{n,m}$, we get

$$\rho^a_{|\alpha|} = e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} |n\rangle \langle n| \, . \tag{1.3.61}$$

Note that in each basis we can see *two aspects of the decoherence* that the coherent state of (1.3.59) has suffered.

– In the first basis, the most direct consequence of the phase randomization results in the lack of off-diagonal elements between states with different phases.

– The second basis of number states shows that the particle number distribution is still Poissonian but also that the off-diagonal elements between number states have become zero. As it is the case for any mixture diagonal in the number state basis, the *average of the field* is zero, that is

$$\langle E \rangle = 0,$$

and its *intensity* is *time independent*,

$$\langle E^2 \rangle = \frac{1}{2}(\frac{1}{2} + \langle n_a \rangle). \tag{1.3.62}$$

It can be noted that these results are closer to those of a number state (1.3.51) than of a coherent state (1.3.48). Anyway, the state still results coherent at all orders.

Finally, the **THERMAL MIXTURE** is a further important state to discuss: the bosonic excitations (the particles of the field) of this state are thermally spread among the energy levels.

In this state, given a mode ω_a, the *density matrix* can be derived from the *Bose-Einstein statistics* as

$$\rho^a_{th} = \frac{e^{-\frac{H}{k_B T}}}{\mathrm{Tr}(e^{-\frac{H}{k_B T}})} = \frac{e^{-\frac{\hbar\omega_a a^\dagger a}{k_B T}}}{1/(1-e^{-\frac{\hbar\omega_a}{k_B T}})}, \tag{1.3.63}$$

where:

- k_B is the *Boltzmann constant*,

- the denominator is the *partition function*.

In addition, the thermal density matrix is *diagonal* in the number basis

$$\rho^a_{th} = \sum_n \frac{\langle n_a \rangle^n}{(1 + \langle n_a \rangle)^{1+n}} |n\rangle \langle n| , \tag{1.3.64}$$

and the average occupation is the *Bose-Einstein distribution*

$$\langle n_a \rangle = \frac{1}{e^{\frac{\hbar\omega_a}{k_B T}} - 1}. \tag{1.3.65}$$

M. Planck suggested this formula in 1900 to match the experiments on blackbody radiation. Later, Bose derived it from a statistical argument for photons, requiring only the particles to be indistinguishable. Indeed, as the system is in thermal equilibrium with a bosonic bath, their average occupation at the frequency ω_a are the same:

$$\langle n_a \rangle = \bar{n}_T. \tag{1.3.66}$$

Remark. *Indeed in the basic picture, analysed in this section, also matter excitations, such as excitons in semiconductors (that are composite bosons in the regime $a_B d \ll 1$, where a_B denotes the exciton Bohr radius and d the density of excitons), can be well described. In fact, in very low density case their energy levels are far from saturation and one can neglect the Pauli effects due to the fermionic components (electrons and holes). How to deal with matter excitations in presence of important fermionic effects is shown in next section.*

1.3.1.2 Two-Level System (2LS) - Fermionic states

HO cannot describe excitons in all regimes. If density is high enough to push together multiple electrons or holes in the same state, the *Pauli Exclusion Principle* comes into play. It is then the perfect match for *fermions.*

Fermions *are particles or composite particles that follow Fermi–Dirac statistics, so they obey the Pauli exclusion principle, and generally have half odd integer spin (1/2, 3/2, etc.).*

In these cases, the system can only populate a finite number of levels with a maximum of one excitation. The most appropriate description is in terms of the projector operators (see footnote 4 on p. 29)

$$|\psi_i\rangle\langle\psi_i| \tag{1.3.67}$$

for each level (with corresponding energy E_i) and their ladder counterparts,

$$\hat{S}_{ji} = |\psi_j\rangle\langle\psi_i|, \tag{1.3.68}$$

the raising (if $E_i < E_j$) and lowering (if $E_i > E_j$) operators. Relation (1.3.68) describes the promotion from state i to $j \neq i$ by the creation of a matter field excitation, similarly to the action of $\hat{a}^\dagger$ in case of bosonic field. The difference is mainly that, since only one excitation is allowed in each level, $\hat{S}_{ji}$ cannot be applied twice [5]:

$$\langle\psi_i|\psi_j\rangle = \delta_{ij}. \tag{1.3.69}$$

For these levels the free Hamiltonian operator is simply

$$\hat{H}_{levels} = \sum_i E_i |\psi_i\rangle\langle\psi_i|. \tag{1.3.70}$$

Consider

– *two* of these *levels* with an *energy difference*

$$\hbar\omega_{el} = E_2 - E_1, \tag{1.3.71}$$

[5] Operator $\hat{a}^\dagger$ had implicit in its expression

$$\hat{a}^\dagger = \sum_n \sqrt{n+1}\,|n+1\rangle\langle n|$$

the possibility of being repeatedly applicable.

– *operators* of *creation* and *annihilation* $\hat{S}_{21}$ and $\hat{S}_{12}$, respectively.

This ***two-level system*** (2LS) covers the *Fermi statistics* in the similar way as the HO covers Bose statistics. Together, they not only describe a great deal of physical situations but also, most relevantly, they constitute the reference model for the study of light-matter interaction. For our work, the 2LS provides a good approximation for an exciton in a small quantum dot. The two levels involved are:

– the *ground state* $|\psi_1\rangle$, in the *absence of an exciton*,

– the *excited state* $|\psi_2\rangle = \hat{S}_{21} |\psi_1\rangle$, in *presence of an exciton.*

The $\hat{S}_{ij}$-operators

$$S_{21} = |\psi_2\rangle \langle\psi_1| \, , \hat{S}_{12} = |\psi_1\rangle \langle\psi_2| \, , \hat{S}_{21}\hat{S}_{12} = |\psi_2\rangle \langle\psi_2| \, , \hat{S}_{12}\hat{S}_{21} = |\psi_1\rangle \langle\psi_1| \tag{1.3.72}$$

can be writtem in terms of the pseudo-spin operators or *Pauli matrices*, $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_1 = \hat{S}_{12} + \hat{S}_{21}, \tag{1.3.73}$$

$$\sigma_2 = i(\hat{S}_{12} - \hat{S}_{21}), \tag{1.3.74}$$

$$\sigma_3 = \hat{S}_{21}\hat{S}_{12} - \hat{S}_{12}\hat{S}_{21} = \left[\hat{S}_{21}, \hat{S}_{12}\right], \tag{1.3.75}$$

used in case of 1/2-spin dynamics.

Remark. *The fermionic properties of the 2LS algebra are summarised by anti-commutation rule*

$$\left[\hat{S}_{21}, \hat{S}_{12}\right] = I_2. \tag{1.3.76}$$

The Hamiltonian operator in equation (1.3.70) can be written as

$$\boxed{\hat{H}_{el} = \hbar\omega_{el} \underbrace{\hat{S}_{21}\hat{S}_{12}}_{=:\hat{S}_{22}} = \hbar\omega_{el}\hat{S}_{22}. \qquad (1.3.77)}$$

Remark. *Actually, condition (1.3.71) is equivalent to assuming* 0 *the energy of ground state and* $\hbar\omega_{el}$ *the energy of the excited state, i.e considering* $E_1 = 0$ *and* $E_2 = \hbar\omega_{el}$*. By (1.3.70), equation (1.3.77) comes from the general form* *of* $\hat{H}_{el}$

$$\hat{H}_{el} = E_1 \underbrace{\hat{S}_{11}}_{:=\hat{S}_{12}\hat{S}_{21}} + E_2\hat{S}_{22}, \tag{1.3.78}$$

Hence comparing equation (1.3.78) with equation (1.3.1), it follows that (1.3.77) is the second and third term in the right-hand side of (1.3.1).

A **GENERAL STATE**, namely a quantum state that can also be non-pure, is described by the 2-dimensional density matrix that is is characterized by two numbers:

– the excitation probability, which is also the average occupation $P_1 = \left\langle \hat{S}_{21}\hat{S}_{12} \right\rangle = \langle n_{el} \rangle$;

– the coherence between the two levels, ρ^{el}_{12}.

This matrix is

$$\rho^{\sigma} = \begin{pmatrix} 1 - \langle n_{el} \rangle & \rho^{el}_{12} \\ \left(\rho^{el}_{12}\right)^{\star} & \langle n_{el} \rangle \end{pmatrix}. \tag{1.3.79}$$

For a **PURE STATE** of the form

$$\sqrt{1 - \langle n_{el} \rangle}\, |\psi_1\rangle + e^{i\phi_1}\sqrt{\langle n_{el} \rangle}\, |\psi_2\rangle ,$$

we have

$$\rho^{\sigma}_{12} = \sqrt{\langle n_{el} \rangle \left(1 - \langle n_{el} \rangle\right)} e^{-i\phi}.$$

Conversely, for a system in **THERMAL EQUILIBRIUM** with some bath at temperature T, we have a *thermal mixture* as it was the case with bosons. Now, to computing the density matrix of equation (1.3.63), the *Fermi-Dirac statistics* should be considered:

$$\rho^{el}_{th} = \frac{e^{-\frac{\hbar\omega_{el}\hat{S}_{21}\hat{S}_{12}}{k_B T}}}{1 + e^{-\frac{\hbar\omega_{el}}{k_B T}}} = \left(1 - \langle n_{el} \rangle\right) |\psi_1\rangle \langle\psi_1| + \langle n_{el} \rangle |\psi_2\rangle \langle\psi_2| , \tag{1.3.80}$$

where $\langle n_{el} \rangle$ is the *Fermi-Dirac distribution*

$$\langle n_{el} \rangle = \frac{1}{e^{\frac{\hbar\omega_{el}}{k_B T}} + 1}. \tag{1.3.81}$$

Remark. For infinite temperature, the maximum value that this probability can take is 1/2.

1.3.1.3 Coherent Coupling

The processes that can be written as a Hamiltonian operator H (always Hermitian) and included in the Schrödinger equation

$$\frac{d\rho}{dt} = \frac{i}{\hbar}\left[\rho, \hat{H}\right] \tag{1.3.82}$$

are called ***coherent processes.***

The free evolution of the bosonic and fermionic fields has already been present in equations (1.3.37) and (1.3.77). In the same point of space two fields a and b can *interact linearly* with a Hamiltonian operator that has the form

$$\hat{H}_{ab} = g(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger). \tag{1.3.83}$$

Remark. *Comparing equation (1.3.83) with equation (1.3.1), it follows that (1.3.83) is third term in the right-hand side of (1.3.1) for*

$$g = \frac{\hbar}{2}\Omega_1, \quad \hat{b} = \hat{S}_{12}, \quad \hat{b}^\dagger = \hat{S}_{21}.$$

During the dynamics of H_{ab}, in case of *detuning between the modes*

$$\Delta = \omega_a - \omega_b \tag{1.3.84}$$

small as compared to the coupling, an a-particle is annihilated while a b-particle is created and vice-versa.

It is assumed

$$\omega_{a,b} \gg g, \Delta \ ,$$

i.e. the *frequencies* are considered to be *much larger* than the coupling and detuning between the modes so that the *Rotating Wave Approximation*[6] holds.

It must be emphasized that the number of particles a and b are not conserved separately by the Hamiltonian operator

$$\hat{H} = \hat{H}_a + \hat{H}_b + \hat{H}_{ab},$$

since they experience a mutual conversion in the form of *Rabi oscillations*: the particles whose number is conserved are the *eigenstates* of $\hat{H}$. However, we must specify the *nature of the fields* in order to diagonalize $\hat{H}$.

It is also worth remarking:

– about field a: in this work we assume it to be an *electromagnetic field inside a cavity*, where one mode with frequency ω_a is selected;

– about field b: it represents what we call *emitter* and, depending on the model for the material excitation, it is described by, typically, another HO, giving rise to the *linear model* (LM) developed by Hopfield [23], or by a 2LS, giving rise to the JC-model. These are the most fundamental cases since they describe material fields with *Bose* and *Fermi statistics*, respectively. Possible extensions are a collection of HOs or of many 2LS or three-level system, etc..

The parameter g deals with the properties of both the cavity and the emitter. More in detail, it depends on

– the *effective cavity volume* V,

– the *oscillator strength of the emitter* f,

since for g one has

$$g \sim (f/V)^{-1/2}.$$

Hence, to achieve strong coupling experimentally, the cavity must have a *small* effective volume V and a *high* quality factor Q [49]. One has that

$$Q^{-1} \sim \gamma_a,$$

[6]In this context the Rotating Wave Approximation allows to neglect the energy non-conserving terms $\hat{a}\hat{b}$ and $\hat{a}^\dagger\hat{b}^\dagger$, i.e. to write the coupling as equation (1.3.83), since these term are related to processes with a much greater energy than detuning.

where γ_a is the *effective rate of excitations loss of the system*: $\gamma_a = \kappa_a(1+\overline{n}_T)$, with κ_a spontaneous emission rate at $T = 0$.

1.3.2 JC-model mathematical formulation

In this section first we present how the JC-model can be derived from the model describing an atom interacting with a EM-field. Than we introduced the analytical study of both the JC-model Hamiltonian and Atomic JC-model Hamiltonian.
The approach in Subsection 1.3.2.1 will be based on the work [34] while in Subsections 1.3.2.2 and 1.3.2.3 it will be based on the paper [57].

1.3.2.1 JC-model derivation

This section is devoted to finding the correct Hamiltonian to describe the dynamics of a single atom interacting with the field inside the cavity. Here, the most simple conditions are assumed, that is they are taken into account:

− just one *atomic transition*,

− *a single quantized mode* of the cavity field [7].

We underline that in this study we assume the *difference between the energies of the two atomic states* to be equal to

$$\hbar\Omega. \tag{1.3.85}$$

The *full Hamiltonian operator* describing an atom interacting with a EM-field is

$$\hat{H}_{a-f} = \frac{1}{2m}\left[\hat{p} - q\hat{A}(x)\right]^2 + U(x) + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hat{H}_{el}, \tag{1.3.86}$$

where:

- q, m are the *charge* and the *mass* of the atom, respectively,
- $\hat{p}$, $\hat{x} = x$ are the operators, respectively, of the *momentum* and the

[7] The deep procedure of quantization of the field is omitted for the sake of brevity, as it can be found in any book on quantum optics.

position of the atom,

• ω is the *frequency* of the mono-modal field,

• $a^\dagger$ and a are *the symbols of the creation and annihilation operators* for the field mode,

• U is the external atomic potential,

• H_{el} is the *Hamiltonian describing the electronic states of the atom.*

Moreover, the *vector potential operator* is given by

$$\hat{A}(x) = A_0(\bar{\varepsilon} f(x)\hat{a} + \bar{\varepsilon}^* \overline{f(x)}\hat{a}^\dagger), \tag{1.3.87}$$

where:

• $\bar{\varepsilon}$ is the *polarization vector*,

• $f(x)$ is a *complex-valued function* that describes the field along the mode in the cavity,

• A_0 is a *constant* whose expression is

$$A_0 = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}},$$

where V is the *effective mode volume* as

$$V = \int_{\mathbb{R}^3} |f(x)|^2 \, dx_1 dx_2 dx_3. \tag{1.3.88}$$

Remark. Under the assumption that *only one atomic transition couples to the mode*, the electronic states may be labelled $|\psi_1\rangle$ and $|\psi_2\rangle$ since there are only two states[8].

Thus, the *free electron Hamiltonian* in (1.3.86) can be written as

$$\hat{H}_{el} = \frac{\hbar\Omega}{2}\sigma_3, \tag{1.3.89}$$

where Ω is the *transition frequency* and σ_3 is the *Pauli z-operator* (or 3^{rd}-*operator*):

$$\sigma_3 |\psi_i\rangle = (\delta_{i,2} - \delta_{i,1}) |\psi_i\rangle$$

[8]We underline that, in general, we will use natural numbers starting from 0 for the ground state.

(where δ is the Kronecker Delta) and the free choice of initial energy level is showed in Fig. 3.

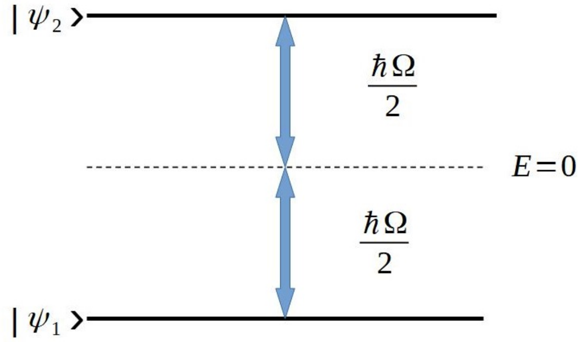

Fig. 3: A free choice of initial energy level.

Now, by neglecting multi-photon processes (leave out the A^2-term) and assuming the external atomic potential to be zero ($U = 0$), we get that the Hamiltonian operators becomes

$$\hat{H}_{a-f} = \hat{H}_{at} + \hat{H}_{field} + \hat{H}_{int},$$

where

$$\begin{aligned} \hat{H}_{at} &= \frac{\hat{p}^2}{2m} + \frac{\hbar\Omega}{2}\sigma_3, \\ \hat{H}_{field} &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \\ \hat{H}_{int} &= -\frac{q}{m}A_0\left[(\hat{p}\cdot\bar{\varepsilon})f(x)\hat{a} + (\hat{p}\cdot\bar{\varepsilon}^*)f(x)^*\hat{a}^\dagger\right]. \end{aligned} \tag{1.3.90}$$

Adopting matrix representation within the atomic basis states, the *interaction Hamiltonian operator* can be written [38]

$$\hat{H}_{int} = g(x)(\hat{a}^\dagger\hat{S}_{12} + \hat{a}\hat{S}_{21}), \tag{1.3.91}$$

where:

- $\hat{S}_{12}$, $\hat{S}_{21}$ are, respectively, the *raising* and *lowering operators* for the atom,
- $g(x)$ has the form

$$g(x) = -\frac{q}{m} \langle\psi_2|(\hat{p}\cdot\bar{\varepsilon})|\psi_1\rangle\, f(x)\sqrt{\frac{2}{\hbar\varepsilon_0\omega V}}.$$

Remark. In deriving (1.3.91) we adopted the *rotating wave approximation*, that means neglected the fast oscillating terms corresponding to virtual processes (for correction terms, see [3]), and we adjusted the phases of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $g(x)$ is *real valued.*

Further, assuming that $g(x)$ is to a good approximation independent of x, the atomic kinetic energy operator is a constant of motion and can be omitted, as may the constant vacuum term $\hbar\omega/2$.
The resulting ***JC-model Hamiltonian operator*** is

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\hbar\Omega}{2}\sigma_3 + g(\hat{a}^\dagger\hat{S}_{12} + \hat{a}\hat{S}_{21}), \qquad (1.3.92)$$

which defines the Jaynes-Cummings model with the free choice of initial energy level showed in Fig. 3. For the general form see (1.3.1) and the discussion in Section 1.3.1.2.

1.3.2.2 JC-model Hamiltonian analytical study

In the *most general case* the original ***JC-model Hamiltonian operator*** can be expressed in the form (1.3.1) that we recall here

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + E_1\hat{S}_{11} + E_2\hat{S}_{22} + \frac{\hbar}{2}\Omega_1(\hat{a}^\dagger\hat{S}_{12} + \hat{a}\hat{S}_{21}) \qquad (1.3.93)$$

where

- ω is the is the *frequency of the mono-modal field,*
- E_k is the *energy of atomic state* ψ_k,
- Ω_1 is the *atom-field coupling constant* i.e. the vacuum (or single-photon) Rabi coupling,
- $\hat{S}_{jk}$ are *transition operators acting on atomic states* defined as

$$\hat{S}_{jk}|\psi_n\rangle = \delta_{kn}|\psi_j\rangle \quad \text{or} \quad \hat{S}_{jk}\hat{S}_{nm} = \delta_{kn}\hat{S}_{jm}. \qquad (1.3.94)$$

In case of restriction to two states, as in JC-model, these atomic operators are commonly express in terms of *Pauli (spin) matrices*:

$$\sigma_1 = \hat{S}_{12} + \hat{S}_{21}, \quad \sigma_2 = i(\hat{S}_{12} - \hat{S}_{21}), \quad \sigma_3 = \hat{S}_{22} - \hat{S}_{11}, \tag{1.3.95}$$

in order to highlight the close association between a two-state atom and a spin-$\frac{1}{2}$ system.

The *photon creation* and *annihilation operators* $\hat{a}^\dagger$ and $\hat{a}$, with commutator

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \tag{1.3.96}$$

act on photon number states $|n\rangle$, eigenstates of the photon number operator $\hat{a}^\dagger \hat{a}$:

$$\hat{a}^\dagger \hat{a} \, |n\rangle = n \, |n\rangle \, , \quad \hat{a}^\dagger \, |n\rangle = \sqrt{n+1} \, |n+1\rangle \, , \quad \hat{a} \, |n\rangle = \sqrt{n} \, |n-1\rangle \, . \tag{1.3.97}$$

Remark. *Indeed, in the theory, the field frequency ω, the atomic energies E_k, and the (vacuum) Rabi frequency Ω_1, appear as arbitrary parameters, even if in applications they are fixed by physical considerations (as, e.g., it happens for the cavity volume V and the atomic transition moment d in the relation $|\Omega_1|^2 = 4d^2\omega/\hbar V \varepsilon_0$).*

Remark. *In the JC-model Hamiltonian of equation (1.3.1) are not included such effects as cavity loss, multiple cavity modes, atomic sublevel degeneracy and atomic polarizability (leading to dynamic Stark shifts).*

1.3.2.3 Atomic JC-model Hamiltonian analytical study

The ***JC-model Hamiltonian operator*** is constructed so that *each photon creation accompanies an atomic de-excitation*, and *each photon annihilation accompanies atomic excitation.* Hence, in addition to the terms related to *conservation of atomic probability*,

$$\left\langle \hat{S}_{11} \right\rangle + \left\langle \hat{S}_{22} \right\rangle = 1, \tag{1.3.98}$$

there occurs a term related to *conservation of excitation*

$$\left\langle \hat{a}^\dagger \hat{a} \right\rangle + \left\langle \hat{S}_{22} \right\rangle = \text{constant}. \tag{1.3.99}$$

Remark. *It follows, as pointed out by Jaynes and Cummings, that in this case the problem arises of having an infinite set of uncoupled two-state Schrödinger equations, each pair identified by the number of photons that are present when the atom is in the lowest-energy state. In this sense, the JC-model is closely related to the classic Lee model of quantum field theory [4].*

We underline that in the study that will be developed in the following part of this section we chose

$$(E_2 - E_1)/2 =: \hbar\omega_0$$

as reference point for the scale of energy, i.e. $E_1 = -\frac{\hbar}{2}\omega_0$ *and* $E_2 = \frac{\hbar}{2}\omega_0$ *in (1.3.1). Moreover, we consider the evolution of the creation/annihilation operators to be fixed by the unperturbed electromagnetic radiation Hamiltonian*

$$\hat{H}_F = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right).$$

Therefore, the JC-model Hamiltonian becomes

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - \frac{\hbar}{2}\omega_0 \hat{S}_{11} + \frac{\hbar}{2}\omega_0 \hat{S}_{22} + \frac{\hbar}{2}\Omega_1 (\hat{a}^\dagger \hat{S}_{12} + \hat{a}\hat{S}_{21}). \tag{1.3.100}$$

Now, to study how the atom-field states evolve in time we can consider an ***atom-field state-vector***. It describes the evolution in time of a the state of a system once given the initial state in which the system is. Note that, for specified photon number, the atom-field state-vector can be written as a combination of two basis states. Namely, the state-vector we take into account is the one that describes the time evolution of a system where the atom is initially in the state $|\psi_1\rangle$ and the field is in the state $|n\rangle$. In fact, this state-vector is the time-dependent Schrödinger equation solution of a system with Hamiltonian operator $\hat{H}$ (see (1.3.100)) and initial state $|n\rangle \otimes |\psi_1\rangle$, and

$$\mathbf{\Psi}(n,t) = e^{-i(n+1)\omega t + i\omega_0 t} \left[C_1(n,t)\phi_1(n) + C_2(n,t)\phi_2(n) \right], \tag{1.3.101}$$

where $\phi_k(n)$ is the atom-field product state

$$\phi_1(n) = |n\rangle \otimes |\psi_1\rangle \,, \quad \phi_2(n) = |n-1\rangle \otimes |\psi_2\rangle \,. \tag{1.3.102}$$

We can observe that from a physical point of view, by the definitions of ϕ_1 and ϕ_2, the coefficient $C_1(n,t)$ and $C_2(n,t)$ (to be determined) are linked, respectively, to

– the probability of finding at time t an electron in the ground state and n photons,

– the probability of finding at time t an electron in the excited state and $n-1$ photons.

Now we are going to prove that the 2×2 *Hamiltonian matrix* of such a pair has the form

$$\boldsymbol{H}(n) = \frac{\hbar}{2} \begin{bmatrix} 2n\omega - \Delta & \Omega_1\sqrt{n} \\ \Omega_1\sqrt{n} & 2n\omega + \Delta \end{bmatrix}, \tag{1.3.103}$$

involving as parameters the *cavity-atom detuning* Δ and, in the case of $\Delta = 0$, the *n-photon Rabi frequency* $\Omega(n) := \Omega_1\sqrt{n}$ (we will see later in this section that $\Omega_1\sqrt{n}$ is indeed the n-photon Rabi frequency for $\Delta = 0$)

$$\hbar\Delta = E_2 - E_1 - \hbar\omega = \hbar(\omega_0 - \omega). \tag{1.3.104}$$

First of all, $\boldsymbol{\Psi}$ could be rewritten in the basis given by $\{\phi_1, \phi_2\}$ as

$$\boldsymbol{\Psi}(n,t) = e^{-i(n+1)\omega t + i\omega_0 t} \begin{bmatrix} C_1(n,t) \\ C_2(n,t) \end{bmatrix}, \tag{1.3.105}$$

and using this last equation we compute, in the same basis, the *matrix-form of the JC-model Hamiltonian*

$$\begin{aligned}\hat{H}\mathbf{\Psi}(n,t) =& e^{-i(n+1)\omega t+i\omega_0 t}\left[\hbar\left(n+\frac{1}{2}\right)\omega C_1(n,t)\phi_1(n)\right.\\&-\frac{\hbar}{2}\omega_0 C_1(n,t)\phi_1(n)\\&+\frac{\hbar}{2}\Omega_1 C_2(n,t)\sqrt{n}\phi_1(n)+\hbar(n-\frac{1}{2})\omega\phi_2(n)C_2(n,t)\\&\left.+\frac{\hbar}{2}\omega_0 C_2(n,t)\phi_2(n)+\frac{\hbar}{2}\Omega_1 C_1(n,t)\sqrt{n}\phi_2(n)\right],\end{aligned}$$

i.e. in vector form

$$\begin{aligned}\hat{H}\mathbf{\Psi}(n,t) =& \begin{bmatrix} \hbar n\omega - \underbrace{\frac{\hbar}{2}(\omega_0-\omega)}_{=\frac{\hbar}{2}\Delta} & \frac{\hbar}{2}\Omega_1\sqrt{n} \\ \frac{\hbar}{2}\Omega_1\sqrt{n} & \hbar n\omega + \underbrace{\frac{\hbar}{2}(\omega_0-\omega)}_{=\frac{\hbar}{2}\Delta}\end{bmatrix} e^{-i(n+1)\omega t+i\omega_0 t}\begin{bmatrix} C_1(n,t)\\ C_2(n,t)\end{bmatrix}\\ =& \frac{\hbar}{2}\begin{bmatrix} 2n\omega-\Delta & \Omega_1\sqrt{n}\\ \Omega_1\sqrt{n} & 2n\omega+\Delta\end{bmatrix} e^{-i(n+1)\omega t+i\omega_0 t}\begin{bmatrix} C_1(n,t)\\ C_2(n,t)\end{bmatrix}.\end{aligned}$$

Thus, we have

$$\boldsymbol{H}(n)=\frac{\hbar}{2}\begin{bmatrix} 2n\omega-\Delta & \Omega_1\sqrt{n}\\ \Omega_1\sqrt{n} & 2n\omega+\Delta\end{bmatrix}$$

as we asserted previously.

Now, we have all the elements to show the *Rabi oscillations* in a system, described by JC-model, in which one photon carries enough energy to make the electron transit from the ground state to the excited state, that means having $E_2-E_1-\hbar\omega=0$ i.e. $\Delta=0$ since $E_2-E_1-\hbar\omega=\hbar\Delta$.

First of all, we take into account the Schrödinger equation for $\mathbf{\Psi}$, using the JC-model Hamiltonian operator,

$$i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}(n,t)=\hat{H}\mathbf{\Psi}(n,t)$$

and we look for its solution with initial condition $\mathbf{\Psi}(n,0) = \phi_1(n)$, i.e.

$$\begin{cases} C_1(n,0) = 1, \\ C_2(n,0) = 0 \end{cases}$$

under the assumption $\Delta = 0$.

The expression in the left-hand side term of the Schrödinger equation is

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}(n,t) =& i\hbar(-i(n+1)\omega + i\omega_0)e^{-i(n+1)\omega t + i\omega_0 t}\left[C_1(n,t)\phi_1(n)\right. \\ & \left.+C_2(n,t)\phi_2(n)\right] + i\hbar e^{-i(n+1)\omega t + i\omega_0 t}\left[\dot{C}_1(n,t)\phi_1(n)\right. \\ & \left.+\dot{C}_2(n,t)\phi_2(n)\right] \end{aligned}$$

i.e. in vector form

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\mathbf{\Psi}(n,t) =& i\hbar e^{-i(n+1)\omega t + i\omega_0 t}\left((-i(n+1)\omega + i\omega_0)\begin{bmatrix} C_1(n,t) \\ C_2(n,t) \end{bmatrix}\right) \\ & + i\hbar e^{-i(n+1)\omega t + i\omega_0 t}\left(\begin{bmatrix} \dot{C}_1(n,t) \\ \dot{C}_2(n,t) \end{bmatrix}\right), \end{aligned}$$

while the expression in the right-hand side is

$$\hat{H}\mathbf{\Psi}(n,t) = \frac{\hbar}{2}\begin{bmatrix} 2n\omega - \Delta & \Omega_1\sqrt{n} \\ \Omega_1\sqrt{n} & 2n\omega + \Delta \end{bmatrix} e^{-i(n+1)\omega t + i\omega_0 t}\begin{bmatrix} C_1(n,t) \\ C_2(n,t) \end{bmatrix}.$$

After the simplification of $e^{-i(n+1)\omega t + i\omega_0 t}$ and by isolating the terms $\dot{C}_1$ and $\dot{C}_2$ to the left in the Schrödinger equation for $\mathbf{\Psi}$, we get

$$\begin{aligned} i\hbar\dot{C}_1(n,t)\phi_1(n) + i\hbar\dot{C}_2(n,t)\phi_2(n) =& \frac{\hbar}{2}\sqrt{n}\Omega_1\left(C_2(n,t)\phi_1(n) + C_1(n,t)\phi_2(n)\right) \\ & + \hbar\Delta C_2(n,t)\phi_2(n), \end{aligned}$$

i.e. in vector form

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} C_1(n,t) \\ C_2(n,t) \end{bmatrix} = \frac{\hbar}{2}\sqrt{n}\Omega_1\begin{bmatrix} C_2(n,t) \\ C_1(n,t) \end{bmatrix} + \hbar\Delta\begin{bmatrix} 0 \\ C_2(n,t) \end{bmatrix},$$

which implies

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} C_1(n,t) \\ C_2(n,t) \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{n}\Omega_1 \\ \sqrt{n}\Omega_1 & 2\Delta \end{bmatrix}\begin{bmatrix} C_2(n,t) \\ C_1(n,t) \end{bmatrix}.$$

Hence, recalling that we assumed $\Delta = 0$, we get

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} C_1(n,t) \\ C_2(n,t) \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{n}\Omega_1 \\ \sqrt{n}\Omega_1 & 0 \end{bmatrix}\begin{bmatrix} C_2(n,t) \\ C_1(n,t) \end{bmatrix}.$$

Thus $C_1(n,t)$ and $C_2(n,t)$ satisfy

$$\begin{cases} \dot{C}_1(n,t) = -\frac{i}{2}\sqrt{n}\Omega_1 C_2(n,t), \\ \dot{C}_2(n,t) = -\frac{i}{2}\sqrt{n}\Omega_1 C_1(n,t). \end{cases}$$

Now, let us remark that we are considering as initial condition of the Cauchy problem $\mathbf{\Psi}(n,0) = \phi_1(n)$, i.e.

$$\begin{cases} C_1(n,0) = 1, \\ C_2(n,0) = 0 \end{cases}$$

leading to the solutions

$$\begin{cases} C_1(n,t) = \cos(\frac{1}{2}\sqrt{n}\Omega_1 t), \\ C_2(n,t) = -i\sin(\frac{1}{2}\sqrt{n}\Omega_1 t). \end{cases} \tag{1.3.106}$$

(Note that any other admissible condition would give the same solution with a different phase factor or amplitudes multiplied by i.)

Remark. *The squared amplitudes of the coefficients ($|C_1(n,t)|^2$ and $|C_2(n,t)|^2$)*

have the real physical meaning of the probability of occupation on ground and excited states respectively.

Therefore, from (1.3.106), it follows that there is a oscillatory phenomenon at the base of the transformation of $\phi_1(n)$ into $\phi_2(n)$ and vice versa: this phenomenon is called *Rabi oscillations.*

Finally, we have to calculate the frequency of Rabi oscillations. We note that the period of $\sin(t)^2$ and of $\cos(t)^2$ is π and

$$|C_1(n,t)|^2 = \cos(\frac{1}{2}\sqrt{n}\Omega_1 t)^2,$$
$$|C_2(n,t)|^2 = \sin(\frac{1}{2}\sqrt{n}\Omega_1 t)^2.$$

(See Fig. 4.) Hence, the *period* of the Rabi oscillations is

$$\frac{\pi}{\frac{1}{2}\sqrt{n}\Omega_1} = \frac{2\pi}{\sqrt{n}\Omega_1}$$

i.e. the Rabi *frequency* for a system of n photons with detuning $\Delta = 0$ is $\sqrt{n}\Omega_1$, as we claimed earlier.

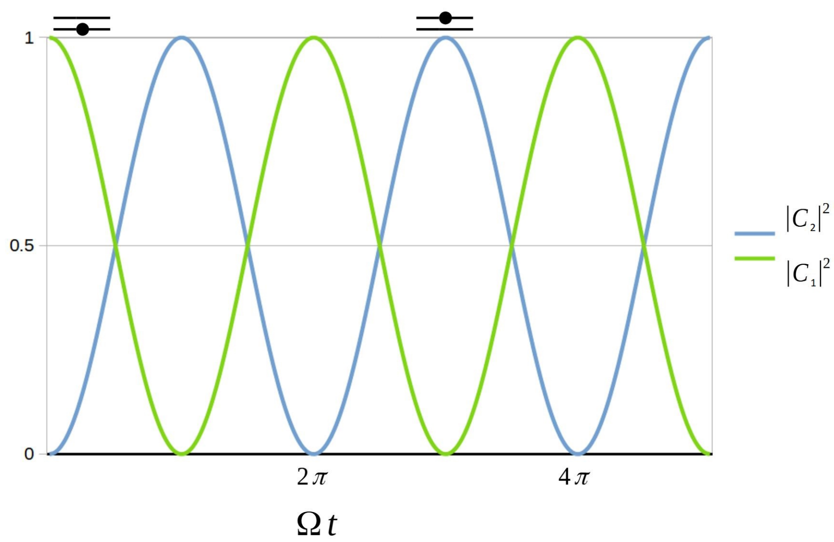

Fig. 4

Chapter 2

Generalizations of the JC-model by semiregular NCHOs

In this chapter, we introduce a new mathematical formulation of JC-model by the use of semiregular NCHOs and extend it to related models. We then show that they can indeed be set within a geometric framework, giving rise to connections on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$ which are in general non-flat.

Namely, in Section 2.1 we introduce the mathematical formulation of the classical JC-model and its extension to systems of a $N \geq 3$ energy level atom and $N-1$ cavity-modes of the electromagnetic field. Next, in Section 2.2 we describe the geometrical setting in which these model can be studied. This is quite an interesting point of view, since it shows that, very likely, higher Lie groups of symmetries are allowed in the theory.

2.1 JC-model by semiregular NCHOs and generalizations

We give here a few examples of semiregular NCHOs in the class SMGES, relevant to Quantum Optics (see [57]), that serve as a model of the class we consider in this work. Then it will be proved that actually there is a geometric framework enclosing them: such geometric setting will be defined through in general non-flat connections on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$.

It will be convenient to use the following notation. We denote by $\boldsymbol{\sigma}_j$, $j = 0, \dots, 3$, the Pauli-matrices, i.e.

$$\boldsymbol{\sigma}_0 = I_2, \quad \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$\boldsymbol{\sigma}_\pm = \frac{1}{2}(\boldsymbol{\sigma}_1 \pm i\boldsymbol{\sigma}_2).$$

Let $\langle \cdot, \cdot \rangle$ be the canonical scalar Hermitian product in $\mathbb{C}^N$, and $e_1, \dots, e_N$ be the canonical basis of $\mathbb{C}^N$. Let

$$E_{jk} := e_k^* \otimes e_j, \quad 1 \le j, k \le N,$$

be the basis of $\mathsf{M}_N(\mathbb{C}) = \mathfrak{gl}(N, \mathbb{C})$, where E_{jk} acts on $\mathbb{C}^N$ as

$$E_{jk} w = \langle w, e_k \rangle e_j, \quad w \in \mathbb{C}^N.$$

Hence, we have the relation

$$E_{jk}E_{hm} = (e_k^* \otimes e_j)(e_m^* \otimes e_h) = e_k^*(e_h)(e_m^* \otimes e_j) = \langle e_h, e_k \rangle (e_m^* \otimes e_j) = \delta_{hk} E_{jm}.$$

We also let, for $X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$,

$$\psi_j(X) := \frac{x_j + i\xi_j}{\sqrt{2}}, \quad 1 \le j \le n,$$

so that $\psi_j^{\mathrm{w}}(x, D)$ is the annihilation operator and $\psi_j^{\mathrm{w}}(x, D)^* = (\bar{\psi}_j)^{\mathrm{w}}(x, D)$ is the creation operator, with respect to the j-th variable. Hence, with $p_2(X) = |X|^2/2$ being the (standard) harmonic oscillator,

$$\sum_{j=1}^n \psi_j^{\mathrm{w}}(x, D)^* \psi_j^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) - \frac{n}{2}.$$

We will also have to consider $2N \times 2N$ matrices of the form $\boldsymbol{\sigma}_j \otimes E_{jk}$, in which case the product is given by

$$(\boldsymbol{\sigma}_j \otimes E_{hk})(\boldsymbol{\sigma}_{j'} \otimes E_{h'k'}) = \boldsymbol{\sigma}_j\boldsymbol{\sigma}_{j'} \otimes E_{hk}E_{h'k'},$$

and the action on a vector $w \in \mathbb{C}^{2N}$, written as

$$w = \sum_{j=1}^{N} \begin{bmatrix} w_{2j-1} \\ w_{2j} \end{bmatrix} \otimes e_j,$$

is given by

$$(\boldsymbol{\sigma}_m \otimes E_{hk})w = \sum_{j=1}^{N} (\boldsymbol{\sigma}_m \begin{bmatrix} w_{2j-1} \\ w_{2j} \end{bmatrix}) \otimes (E_{hk}e_j).$$

We next list a few important models due to Jaynes and Cummings.

2.1.1 The JC-model by semiregular NCHOs

This is the model of a two-level atom in one cavity, given by the 2×2 system in one real variable $x \in \mathbb{R}$

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D)I_2 + \alpha\Big(\boldsymbol{\sigma}_+\psi^{\mathrm{w}}(x, D)^* + \boldsymbol{\sigma}_-\psi^{\mathrm{w}}(x, D)\Big) + \gamma\boldsymbol{\sigma}_3, \quad \gamma > 0, \alpha \in \mathbb{R},$$

where the atom levels are given by $\pm\gamma$.

2.1.2 The JC-model for one atom with N levels and one cavity-mode in the Ξ-*configuration*

In this case we consider, for $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \ldots \gamma_N \in \mathbb{R}$ with $\gamma_1 < \gamma_2 < \ldots < \gamma_N$, the $N \times N$ system in $\mathbb{R}$ given by

$$\begin{aligned} A^{\mathrm{w}}(x, D) =& p_2^{\mathrm{w}}(x, D)I_N + \frac{1}{2}\sum_{k=1}^{N-1} \alpha_k \Big(\psi^{\mathrm{w}}(x, D)^* E_{k,k+1} + \psi^{\mathrm{w}}(x, D)E_{k+1,k}\Big) \\ &+ \sum_{k=1}^{N} \gamma_k E_{kk}. \end{aligned}$$

Here, the atom levels are given by the γ_k.

2.1.3 The JC-model for an N-level atom and $n = N-1$ cavity-modes in the Ξ-*configuration*

In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \le \gamma_2 \le \dots \le \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$\begin{aligned} A^{\mathrm{w}}(x, D) &= p_2^{\mathrm{w}}(x, D) I_N \\ &+ \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x, D)^* E_{k,k+1} + \psi_k^{\mathrm{w}}(x, D) E_{k+1,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}. \end{aligned}$$

Here, the levels of the atom are given by 0 and the γ_k.

In this configuration, through the absorbtion (or emission) of a single photon in the cavity, the electron can move from an energy level to the next higher (or the next lower) one. Namely, the electron can move from the k-th energy level to the $k+1$-st or $k-1$-st one by, respectively, absorbing a photon of the k-th mode (as represented by the annihilation operator $\psi_k^{\mathrm{w}}(x, D)$) or emitting a photon of the $k-1$-th mode (as represented by the creation operator $\psi_{k-1}^{\mathrm{w}}(x, D)^*$).

2.1.4 The JC-model for an N-level atom and $n = N-1$ cavity-modes in the $\bigwedge$-*configuration*

In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \le \gamma_2 \le \dots \le \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$\begin{aligned} A^{\mathrm{w}}(x, D) &= p_2^{\mathrm{w}}(x, D) I_N \\ &+ \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x, D)^* E_{k,N} + \psi_k^{\mathrm{w}}(x, D) E_{N,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}. \end{aligned}$$

Here, the levels of the atom are given by 0 and the γ_k.

In this configuration, through the absorbtion (or emission) of a single photon in the cavity, the electron can move from an energy level to the highest energy level (and viceversa). Namely, the electron can move from the k-th energy level to the N-th or viceversa, respectively, absorbing a photon of the k-th mode (as represented by the annihilation operator $\psi_k^{\mathrm{w}}(x,D)$) or emitting a photon of the k-th mode (as represented by the creation operator $\psi_k^{\mathrm{w}}(x,D)^*$).

2.1.5 The JC-model for an N-level atom and $n = N-1$ cavity-modes in the so-called $\bigvee$-*configuration*

In this case, for $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R}\setminus\{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \le \gamma_2 \le \dots \le \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$A^{\mathrm{w}}(x,D) = p_2^{\mathrm{w}}(x,D) I_N \\ + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x,D)^* E_{1,k+1} + \psi_k^{\mathrm{w}}(x,D) E_{k+1,1} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

Here, the levels of the atom are given by 0 and the γ_k.

In this configuration, through the absorbtion (or emission) of only one photon in the cavity, the electron can move from an energy level to the lowest energy level (and viceversa). Namely, the electron can move from the k-th energy level to the 1-st or viceversa, respectively, absorbing a photon of the kth mode (as represented by the annihilation operator $\psi_k^{\mathrm{w}}(x,D)$) or emitting a photon of the kth mode (as represented by the creation operator $\psi_k^{\mathrm{w}}(x,D)^*$).

2.1.6 The diagonalizability of the first-order part in the above JC-models

We next show that the first-order part of the above JC-models may be diagonalized, so that the Jaynes-Cummings models all belong to the class of systems we consider in this work. The result of the 2×2 system is straightforward. We consider therefore only the 3×3 and the $N \times N$ cases..

Lemma 2.1.1. *The JC-model for a 3-level atom and 2 cavity-modes in the Ξ-configuration, for a N-level atom and $N-1$ cavity-modes in the $\bigwedge$-configuration and in the $\bigvee$-configuration may all be smoothly diagonalized.*

Proof. Let $A_1(X)$ for the first order part of the system. We compute the characteristic polynomial $p(\lambda; X) = \det(\lambda - A_1(X))$ for each of the models in the statement, that we call for short JC-3-Ξ, JC-$\bigwedge$ and JC-$\bigvee$ respectively.

- As for JC-3-Ξ we have

$$p(\lambda; X) = \lambda\Big(\lambda^2 - (\alpha_1^2|\psi_1(X)|^2 + \alpha_2^2|\psi_2(X))|^2)\Big), \quad X \in \mathbb{R}^4.$$

This follows by computing the determinant by Laplace's expansion. We start from the first column: deleting the first row and column and multiplying by the $(1,1)$ entry (i.e. λ) we get the term $\lambda\,(\lambda^2 - |\psi_2(X)|^2)$, and deleting the second row and first column and multiplying by the $(2,1)$ entry (i.e. $-\psi_1(X)$) we get the term $\psi_1(X)(-\overline{\psi_1(X)}\lambda)$. Therefore, adding the terms one get the expression for $p(\lambda; X)$. Hence, there are three eigenvalues

$$\lambda_0(X) \equiv 0, \quad \lambda_\pm(X) = \pm\sqrt{\alpha_1^2|\psi_1(X)|^2 + \alpha_2^2|\psi_2(X)|^2},$$

that may be ordered as

$$\lambda_-(X) < \lambda_0(X) < \lambda_-(X), \quad X \in \dot{\mathbb{R}}^4.$$

Since their pairwise differences in absolute value are bounded from below by $|X|$, the diagonalization Theorem 3.1.1 below can be applied.

- As for JC-$\bigwedge$ we have

$$p(\lambda; X) = \lambda^{N-2}\left(\lambda^2 - \sum_{j=1}^{N-1}\alpha_j^2|\psi_j(X)|^2\right), \quad X \in \mathbb{R}^{2n}.$$

In fact, the expression above for $p(\lambda; X)$ can be obtained by induction on (the number of atomic levels) N. The formula holds for $N = 2$ by a direct computation of the determinant of

$$\begin{bmatrix} \lambda & -\overline{\psi_1(X)} \\ -\psi_1(X) & \lambda \end{bmatrix}.$$

Now, by the inductive hypothesis the characteristic polynomial of an $(N-1)$-level system with $N-2$ cavity-modes in the Λ configuration is

$$p_1(\lambda; X) = \lambda^{N-3}\left(\lambda^2 - \sum_{j=1}^{N-2} |\psi_j(X)|^2\right), \quad X \in \mathbb{R}^{2n}.$$

We wish to show that the formula for characteristic polynomial of an N-level system with $N-1$ cavity-modes in the Λ configuration is

$$p(\lambda; X) = \lambda^{N-2}\left(\lambda^2 - \sum_{j=1}^{N-1} |\psi_j(X)|^2\right), \quad X \in \mathbb{R}^{2n}.$$

We use Laplace's expansion, starting from the $N-1$-st column. On the one hand, deleting the $N-1$-st row and column and multiplying by the $(N-1, N-1)$ entry (i.e. λ) we get the term $\lambda \det B_1(\lambda; X)$ where

$$B_1(\lambda; X) := \begin{bmatrix} \lambda & 0 & \cdots & 0 & -\overline{\psi_1(X)} \\ 0 & \lambda & \cdots & 0 & -\overline{\psi_2(X)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & -\overline{\psi_{N-2}(X)} \\ -\psi_1(X) & -\psi_2(X) & \cdots & -\psi_{N-2}(X) & \lambda \end{bmatrix}.$$

Note that $\det B_1(\lambda; X)$ is the characteristic polynomial of the JC-model for an $(N-1)$-level atom and $N-2$ cavity-modes in the Λ configuration since $\lambda I_N - B_1(\lambda; X)$ is the first-order matrix term of the JC-model for an $(N-1)$-level atom and $N-2$ cavity-modes in the Λ configuration.

On the other hand, deleting the N-th row and $N-1$-st column, multiplying by the $(N, N-1)$ entry (i.e. $-\psi_{N-1}(X)$) and taking into account the sign of $(-1)^{N+N-1} = -1$ we get the term $\psi_{N-1}(X) \det \tilde{B}_1(\lambda; X)$ where

$$\tilde{B}_1(\lambda; X) := \begin{bmatrix} \lambda & 0 & \cdots & 0 & -\overline{\psi_1(X)} \\ 0 & \lambda & \cdots & 0 & -\overline{\psi_2(X)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & -\overline{\psi_{N-2}(X)} \\ 0 & 0 & \cdots & 0 & -\overline{\psi_{N-1}(X)} \end{bmatrix}.$$

Therefore, $\det \tilde{B}_1(\lambda; X) = \lambda^{N-2}(-\overline{\psi_{N-1}(X)})$ since $\tilde{B}_1(\lambda; X)$ is triangular. Thus, adding the terms just computed one gets

$$p(\lambda; X) = \lambda \det B_1(\lambda; X) - \lambda^{N-2}|\psi_{N-1}(X)|^2. \tag{2.1.1}$$

Now, by the inductive hypothesis one has $\det B_1(\lambda; X) = \lambda^{N-3}\left(\lambda^2 - \sum_{j=1}^{N-2} |\psi_j(X)|^2\right)$ whence by (2.1.1) we obtain

$$p(\lambda; X) = \lambda^{N-2}\left(\lambda^2 - \sum_{j=1}^{N-1} |\psi_j(X)|^2\right), \quad X \in \mathbb{R}^{2n}.$$

Hence, there are N eigenvalues

$$\lambda_0(X) \equiv 0, \quad \lambda_\pm(X) = \pm\Big(\sum_{j=1}^{N-1} \alpha_j^2 |\psi_j(X)|^2\Big)^{1/2},$$

that we may order as

$$\lambda_-(X) < \lambda_1(X) = \ldots = \lambda_{N-2}(X) =: \lambda_0(X) < \lambda_+(X), \quad X \in \dot{\mathbb{R}}^{2n}.$$

Thus, Theorem 3.1.3 can be applied with respect to the blockwise diagonalization with blocks

$$\lambda_{1,1} = 0_{N-2} \quad \text{and} \quad \lambda_{2,1} = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix}.$$

• As for JC-$\bigvee$, Theorem 3.1.3 can be applied because the blockwise diagonalization is the same as in the previous case, since the characteristic polynomial of this model is the same as that of JC-$\bigwedge$ above, and have the same structure. Moreover, we can get the characteristic polynomial of this model in a similar way to the computation for JC-Λ above, and we have the same result $p(\lambda;X) = \lambda^{N-2}\left(\lambda^2 - \sum_{j=1}^{N-1}|\psi_j(X)|^2\right)$, that can be proved again by induction on (the number of atomic levels) N.

The formula holds for $N = 2$ by direct computation of the determinant of

$$\begin{bmatrix} \lambda & -\overline{\psi_1(X)} \\ -\psi_1(X) & \lambda \end{bmatrix}.$$

Now, by the inductive hypothesis we have that the characteristic polynomial of an $(N-1)$-level system with $N-2$ cavity-modes in the Λ configuration is

$$p_1(\lambda;X) = \lambda^{N-3}\left(\lambda^2 - \sum_{j=1}^{N-2}|\psi_j(X)|^2\right), \quad X \in \mathbb{R}^{2n}. \tag{2.1.2}$$

On the one hand, deleting the second row and column and multiplying by the $(2,2)$ entry (i.e. λ) we get the term $\lambda \det B_1(\lambda;X)$ where

$$B_1(\lambda;X) := \begin{bmatrix} \lambda & -\overline{\psi_2(X)} & \cdots & -\overline{\psi_{N-1}(X)} \\ -\psi_2(X) & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\psi_{N-1}(X) & 0 & \cdots & \lambda \end{bmatrix}.$$

Note, also in the present case, that $\det B_1(\lambda;X)$ is the characteristic polynomial of the JC-model for an $(N-1)$-level atom and $N-2$ cavity-modes in the V configuration with annihilation operators symbols ψ_j with $j = 2,\dots,N-1$ since $\lambda I_N - B_1(\lambda;X)$ is the first-order matrix term of the JC-model for an $(N-1)$-level atom and $N-2$ cavity-modes in the V configuration with annihilation operator symbols ψ_j with $j = 2,\dots,N-1$.

On the other hand, deleting the first row and the second column, multiplying by the $(1,2)$ entry (i.e. $-\psi_1(X)^*$) and taking into account the sign of

$(-1)^{1+2} = -1$ we get the term $\psi_{N-1}(X) \det \tilde{B}_1(\lambda; X)$ where

$$\tilde{B}_1(\lambda; X) := \begin{bmatrix} -\psi_1(X) & 0 & \cdots & 0 & 0 \\ -\psi_2(X) & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\psi_{N-2}(X) & 0 & \cdots & \lambda & 0 \\ -\psi_{N-1}(X) & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Therefore, $\det \tilde{B}_1(\lambda; X) = \lambda^{N-2}(-\psi_1(X))$ since $\tilde{B}_1(\lambda; X)$ is triangular. Thus, adding the terms just computed one gets

$$p(\lambda; X) = \lambda \det B_1(\lambda; X) - \lambda^{N-2} |\psi_{N-1}(X)|^2. \qquad (2.1.3)$$

As before, by the inductive hypothesis $\det B_1(\lambda; X) = \lambda^{N-3} \left(\lambda^2 - \sum_{j=2}^{N-1} |\psi_j(X)|^2 \right)$ (this is, indeed, (2.1.2) by relabeling the annihilation operator symbols) whence, by using (2.1.1), we have

$$p(\lambda; X) = \lambda^{N-2} \left(\lambda^2 - \sum_{j=1}^{N-1} |\psi_j(X)|^2 \right), \quad X \in \mathbb{R}^{2n}.$$

Therefore $p(\lambda; X) = \lambda^{N-2} \left(\lambda^2 - \sum_{j=1}^{N-1} |\psi_j(X)|^2 \right)$, whence there are N eigenvalues that we may order as $\lambda_-(X) < \lambda_1(X) = \ldots = \lambda_{N-2}(X) =: \lambda_0(X) < \lambda_+(X)$, for all $X \in \mathbb{R}^{2n} \setminus \{0\}$, where

$$\lambda_-(X) = -\Big(\sum_{j=1}^{N-1} |\psi_j(X)|^2 \Big)^{1/2}, \; \lambda_0(X) \equiv 0, \; \lambda_+(X) = \Big(\sum_{j=1}^{N-1} |\psi_j(X)|^2 \Big)^{1/2}.$$

Thus, Theorem 3.1.3 below can be applied with respect to the blockwise diagonalization with blocks

$$\lambda_{1,1} = 0_{N-2} \quad \text{and} \quad \lambda_{2,1} = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix}.$$

□

2.1.7 Possible extensions

In this case, for $\alpha_1, \ldots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \ldots, \gamma_N \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_N$, we consider the following $2N \times 2N$ systems in $\mathbb{R}^n$, with $n = N-1$, given by

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_{2N}$$
$$+ \sum_{k=1}^{N} \sum_{j=1}^{n} \alpha_k \Big(\psi_j^{\mathrm{w}}(x, D)^* \boldsymbol{\sigma}_- \otimes E_{kk} + \psi_j^{\mathrm{w}}(x, D) \boldsymbol{\sigma}_+ \otimes E_{kk} \Big) + \sum_{k=1}^{N} \gamma_k \boldsymbol{\sigma}_3 \otimes E_{kk},$$

and by

$$A^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_{2N}$$
$$+ \sum_{k=1}^{N-1} \alpha_k \Big(\psi_k^{\mathrm{w}}(x, D)^* \boldsymbol{\sigma}_- \otimes E_{k,k+1} + \psi_k^{\mathrm{w}}(x, D) \boldsymbol{\sigma}_+ \otimes E_{k+1,k} \Big) + \sum_{k=1}^{N} \gamma_k \boldsymbol{\sigma}_3 \otimes E_{kk}.$$

2.2 Geometric examples generalizing the JC-model for an N-level atom and $n = N-1$ cavity-modes

Let $\Omega^k(\mathbb{R}^n)$ be the space of smooth (C^∞) k-differential forms over $\mathbb{R}^n$. We will denote by $\Omega^k(\mathbb{R}^n; \mathbb{C}^N) = \Omega^k(\mathbb{R}^n) \otimes \mathbb{C}^N$. Consider the exterior derivative operator d_k acting on k-forms, and its adjoint d_k^* acting on $k+1$-forms which has the expression $d_k^* = (-1)^{nk+1} \star d \star$, where $\star$ is the Hodge-$\star$ operator induced by the Euclidean metric. We may hence define the operators

$$D = D_k := \frac{1}{\sqrt{2}} \Big(d_k + \sum_{j=1}^{n} x_j dx_j \wedge \Big) \colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n), \tag{2.2.1}$$

and its $\star$-adjoint

$$D^* = D_k^* := \frac{1}{\sqrt{2}} \Big(d_k^* + \sum_{j=1}^{n} x_j i_{\partial/\partial x_j} \Big) \colon \Omega^{k+1}(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n).$$

One has

$$\Box_k := D_k^* D_k + D_{k-1} D_{k-1}^* = \left(p_2^{\mathrm{w}}(x, D) + k - \frac{n}{2} \right) \mathbf{1}_k \colon \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n),$$

where $\mathbf{1}_k$ stands for the identity operator on $\bigwedge^k(\mathbb{R}^n)$. We consider ordered multiindices of length k, $I = (i_1, i_2, \dots, i_k)$ where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. The set of all such multiindices is denoted by $\mathsf{I}(n, k)$. We say that $j \in I$ if j appears as one of the entries of I. We also put $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$, so that the dx_I, for $I \in \mathsf{I}(n, k)$, form a basis of $\bigwedge^k(\mathbb{R}^n)$. We have the following set of formulae.

Proposition 2.2.1. *Let* $\omega = \omega_I dx_I \in \Omega^k(\mathbb{R}^n)$, $I \in \mathsf{I}(n, k)$. *We have:*

1. *For* $1 \leq j \leq n$,

$$D_{k-1}(i_{\partial/\partial x_j} \omega) = \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D) \omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I);$$

2. *For* $1 \leq j \leq n$,

$$dx_j \wedge D_{k-1}^* \omega = \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)^* \omega_I dx_j \wedge i_{\partial/\partial x_j}(dx_I);$$

3. *For* $1 \leq j \leq n$,

$$i_{\partial/\partial x_j}(D_k \omega) = \psi_j^{\mathrm{w}}(x, D) \omega_I dx_I - \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D) \omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I);$$

4. *For* $1 \leq j \leq n$,

$$D_k^*(dx_j \wedge \omega) = \psi_j^{\mathrm{w}}(x, D) \omega_I dx_I - \sum_{h=1}^{n} \psi_h^{\mathrm{w}}(x, D)^* \omega_I dx_j \wedge i_{\partial/\partial x_h}(dx_I).$$

Proof. The proof is based on the following elementary formula

$$\star(dx_h \wedge \star(dx_j \wedge dx_I)) = (-1)^{nk} i_{\partial/\partial x_h}(dx_j \wedge dx_I). \tag{2.2.2}$$

Since $d(dx_J) = 0$ we have

$$D_{k-1}(i_{\partial/\partial x_j}\omega) = \frac{1}{\sqrt{2}}\sum_{h=1}^{n}\Big(\frac{\partial\omega_I}{\partial x_h}dx_h \wedge i_{\partial/\partial x_j}(dx_I) + x_h\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I)\Big)$$

$$= \sum_{h=1}^{n}\psi_h^{\mathrm{w}}(x,D)\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I),$$

and this prove *1.*.

Next, using once more the fact that dx_J is closed and using (2.2.2) gives

$$dx_j\wedge D_{k-1}^*\omega = \frac{1}{\sqrt{2}}\sum_{h=1}^{n}\Big((-1)^{n(k-1)+1}\frac{\partial\omega_I}{\partial x_h}dx_j\wedge\star(dx_h\wedge\star dx_I)+x_h\omega_I dx_j\wedge i_{\partial/\partial x_h}(dx_I)\Big)$$

$$= \frac{1}{\sqrt{2}}\sum_{h=1}^{n}\Big(-\frac{\partial\omega_I}{\partial x_h}dx_j \wedge i_{\partial/\partial x_h}(dx_I) + x_h\omega_I dx_j \wedge i_{\partial/\partial x_h}(dx_I)\Big)$$

$$= \sum_{h=1}^{n}\psi_h^{\mathrm{w}}(x,D)^*\omega_I dx_j \wedge i_{\partial/\partial x_j}(dx_I),$$

which proves *2.*.

To prove *3.*, we just note that

$$i_{\partial/\partial x_j}(D_k\omega) = \sum_{h=1}^{n}\psi_h^{\mathrm{w}}(x,D)\omega_I i_{\partial/\partial x_j}\Big(dx_h \wedge dx_I\Big)$$

$$= \psi_j^{\mathrm{w}}(x,D)\omega_I dx_I - \sum_{h=1}^{n}\psi_h^{\mathrm{w}}(x,D)\omega_I dx_h \wedge i_{\partial/\partial x_j}(dx_I).$$

Finally, to prove *4.*, we compute

$$D_k^*(dx_j\wedge\omega) = \frac{1}{\sqrt{2}}\sum_{h=1}^{n}\Big((-1)^{nk+1}\frac{\partial\omega_I}{\partial x_h}\star(dx_h\wedge\star(dx_j\wedge dx_I))+x_h\omega_I i_{\partial/\partial x_h}(dx_j\wedge dx_I)\Big)$$

(by (2.2.2))

$$= \sum_{h=1}^{n}\psi_h^{\mathrm{w}}(x,D)^*\omega_I\Big(\delta_{jh}dx_I - dx_j \wedge i_{\partial/\partial x_h}(dx_I)\Big),$$

which completes the proof. □

Remark 2.2.2. *By convention, if ω is a 0-form then $i_{\partial/\partial x_j}\omega = 0$, for every j.*

2.2.1 The geometric N-level atom in the Ξ-configuration

Next, let $N \geq 2$ be a fixed positive integer and let $n = N - 1$. We define, for $\alpha_1, \dots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, the following connection D on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$:

$$\mathsf{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,j+1}.$$

The connection D extends to the following covariant exterior operator and adjoint covariant exterior operator

$$\mathsf{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,j+1} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N),$$

$$\mathsf{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} \otimes E_{j+1,j} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k}(\mathbb{R}^n; \mathbb{C}^N),$$

where D_k was defined in (2.2.1).
The connection D is non-flat, as the following lemma shows.

Lemma 2.2.3. *For the curvature $F_{\mathsf{D}} = \mathsf{D}^2 \in \Omega^2(\mathbb{R}^n; \mathsf{M}_N)$ of the covariant exterior operator D we have*

$$F_{\mathsf{D}} = \sum_{j=1}^{N-2} \alpha_j \alpha_{j+1} (dx_j \wedge dx_{j+1} \wedge\ ) \otimes E_{j,j+2}.$$

(We put by definition $E_{j,N+1} = 0$, for every j.)

Proof. We have

$$\mathsf{D}^2 = D^2 \otimes I_N + \sum_{h=1}^{N-1} \alpha_h \Big(D_{k+1}(dx_h \wedge\ ) + dx_h \wedge D \Big) \otimes E_{h,h+1}$$

$$+\sum_{j,h=1}^{N-1} \alpha_j\alpha_h(dx_j \wedge dx_h\wedge\ ) \otimes \underbrace{E_{j,j+1}E_{h,h+1}}_{=\delta_{j+1,h}E_{j,h+1}} = F_{\mathsf{D}},$$

for the first and second term vanish. In fact, $D^2 = 0$ and for $\omega = \omega_I dx_I$,

$$D(dx_h\wedge\omega)+dx_h\wedge D\omega = \sum_{j=1}^{N-1} \psi_j^{\mathrm{w}}(x,D)\omega_I\Big(dx_h\wedge dx_j\wedge dx_I+dx_j\wedge dx_h\wedge dx_I\Big) = 0.$$

This concludes the proof.

□

Corollary 2.2.4. *In particular, for $k = 0$, D_0 defines a connection on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$ whose curvature is the vector-valued 2-form*

$$F_{\mathsf{D}_0} = \sum_{j=1}^{N-2} \alpha_j\alpha_{j+2}(dx_j \wedge dx_{j+1}) \otimes E_{j,j+2}.$$

(Recall that, by convention, if ω is a 0-form then $i_{\partial/\partial x_j}\omega = 0$, for every j.)

We next consider the associated Laplacian

$$\Box_k^{(N)} = \mathsf{D}_k^*\mathsf{D}_k + \mathsf{D}_{k-1}\mathsf{D}_{k-1}^* \colon \Omega^k(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n;\mathbb{C}^N).$$

Lemma 2.2.5. *We have*

$$\Box_k^{(N)} = \Big(p_2^{\mathrm{w}}(x,D) + k - \frac{n}{2}\Big)\,\mathbf{1}_k\otimes I_N+\sum_{j=1}^{N-1} \alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\,\mathbf{1}_k\otimes E_{j,j+1}+\psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k\otimes E_{j+1,j}\Big)$$

$$+\sum_{j=1}^{N-1} \alpha_j^2\mathbf{1}_k \otimes E_{j+1,j+1} + \sum_{j=1}^{N-1} \alpha_j^2 dx_j \wedge i_{\partial/\partial x_j}\mathbf{1}_k \otimes (E_{j,j} - E_{j+1,j+1}).$$

Proof. One has

$$\Box_k^{(N)} = (D_k^*D_k+D_{k-1}D_{k-1}^*)\otimes I_N+\sum_{j=1}^{N-1} \alpha_j D_k^*(dx_j\wedge\ )\otimes E_{j,j+1}+\sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} D_k\otimes E_{j+1,j}$$

$$+\sum_{j=1}^{N-1}\alpha_j D_{k-1} i_{\partial/\partial x_j}\otimes E_{j+1,j}+\sum_{j=1}^{N-1}\alpha_j dx_j\wedge D_{k-1}^{*}\otimes E_{j,j+1}$$

$$+\sum_{h,j=1}^{N-1}\alpha_j\alpha_h\Big(i_{\partial/\partial x_h}(dx_j\wedge\ )\otimes E_{h+1,h}E_{j,j+1}+dx_j\wedge i_{\partial/\partial x_h}\otimes E_{j,j+1}E_{h+1,h}\Big),$$

from which the lemma follows by virtue of the formulae of Proposition 2.2.1. $\square$

Corollary 2.2.6. *When $k=0$ we have*

$$\square_0^{(N)}=\Big(p_2^{\mathrm{w}}(x,D)-\frac{n}{2}\Big)\otimes I_N$$
$$+\sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^{*}\otimes E_{j,j+1}+\psi_j^{\mathrm{w}}(x,D)\otimes E_{j+1,j}\Big)+\sum_{j=1}^{N-1}\alpha_j^2\ \mathbf{1}_0\otimes E_{j+1,j+1}.$$

Hence the JC-Ξ model is related to the Laplacian $\square_0^{(N)}$.

Lemma 2.2.7. *The term of order 1 of $\square_1^{(3)}$ can be blockwise-diagonalized with three blocks.*

Proof. Fix the basis $\{dx_i\otimes e_j\}_{i=1,2;j=1,2,3}$. We have that the semiprincipal symbol $A_1(X)$ of $\square_1^{(3)}$,

$$A_1(X)=\sum_{j=1}^{2}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^{*}\,\mathbf{1}_k\otimes E_{j,j+1}+\psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k\otimes E_{j+1,j}\Big),$$

can be rewritten in the above basis as

$$A_1(X):=\left[\begin{array}{c|c|c} 0_2 & \overline{A_{11}(X)} & 0_2\\ \hline A_{11}(X) & 0_2 & \overline{A_{12}(X)}\\ \hline 0_2 & A_{12}(X) & 0_2\end{array}\right],$$

where $A_{1j}(X)=\alpha_j\begin{bmatrix}\psi_j(X) & 0\\ 0 & \psi_j(X)\end{bmatrix}$, $j=1,2$, and where 0_2 is the 2×2 zero-matrix. Now, we compute the characteristic polynomial $p(\lambda;X)=$

$\det(\lambda - A_1(X))$ of $A_1(X)$ following the scheme below where the pairs of numbers in the underbrackets indicate the position of that element in the matrix $\lambda I_3 - A_1(X)$.

$$
\begin{aligned}
p(\lambda;X) =& \underbrace{\lambda}_{(1,1)}\underbrace{\lambda}_{(2,2)}\underbrace{\lambda}_{(3,3)}\Big(\lambda^3 + (\underbrace{-\alpha_2\psi_2(X)}_{(6,4)})(-\lambda(-\overline{\alpha_2\psi_2(X)}))\Big) \\
&+ \underbrace{\lambda}_{(1,1)}\underbrace{\lambda}_{(2,2)}(\underbrace{-\alpha_2\psi_2(X)}_{(5,3)})(-\underbrace{(-\overline{\alpha_2\psi_2(X)})}_{(3,5)})(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
&+ \underbrace{\lambda}_{(1,1)}(\underbrace{-\alpha_1\psi_1(X)}_{(4,2)})(-\underbrace{(-\overline{\alpha_1\psi_1(X)})}_{(2,4)})\underbrace{\lambda}_{(6,6)}(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
&+ \underbrace{(-\alpha_1\psi_1(X))}_{(3,1)}(-\underbrace{(-\overline{\alpha_1\psi_1(X)})}_{(1,3)})\underbrace{\lambda}_{(5,5)}\underbrace{(\lambda^3 - (|\alpha_1\psi_1(X)|^2 + |\alpha_2\psi_2(X)|^2)\lambda)}_{=\det\begin{bmatrix}\lambda & -\overline{\alpha_1\psi_1(X)} & 0 \\ -\alpha_1\psi_1(X) & \lambda & -\overline{\alpha_2\psi_2(X)} \\ 0 & -\alpha_2\psi_2(X) & \lambda\end{bmatrix}} \\
=&\lambda^2\Big((\lambda^4 - |\alpha_2\psi_2(X)|^2\lambda^2) - |\alpha_2\psi_2(X)|^2(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
&- |\alpha_1\psi_1(X)|^2(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
&- |\alpha_1\psi_1(X)|^2(\lambda^2 - (|\alpha_1\psi_1(X)|^2 + |\alpha_2\psi_2(X)|^2))\Big) \\
=&\lambda^2\Big((\lambda^2 - |\alpha_2\psi_2(X)|^2)(\lambda^2 - |\alpha_1\psi_1(X)|^2 - |\alpha_2\psi_2(X)|^2) \\
&- |\alpha_1\psi_1(X)|^2(\lambda^2 - |\alpha_1\psi_1(X)|^2 - |\alpha_2\psi_2(X)|^2)\Big) \\
=&\lambda^2\Big(\lambda^2 - (|\alpha_1\psi_1(X)|^2 + |\alpha_2\psi_2(X)|^2)\Big)^2.
\end{aligned}
$$

Hence, the zeros of $p(\lambda;X)$ are given by

$$0, \quad \lambda_\pm(X) = \pm\sqrt{\alpha_1^2|\psi_1(X)|^2 + \alpha_2^2|\psi_2(X)|^2},$$

with multiplicity 2 each (recall that $\alpha_1, \alpha_2 \in \mathbb{R}\setminus\{0\}$).

Thus, Theorem 3.1.3 can be applied with respect to the blockwise diagonalization with three blocks

$$\lambda_{1,1} = 0_2 \quad , \quad \lambda_{2,1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_+ \end{bmatrix} \quad \text{and} \quad \lambda_{3,1} = \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_- \end{bmatrix}.$$

□

2.2.2 The geometric N-level atom in the $\bigwedge$-configuration

Next, let $N \geq 2$ be a fixed positive integer and let $n = N - 1$. We define, for $\alpha_1, \dots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, the following connection D on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$

$$\mathsf{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,N}.$$

The connection D extends to the following covariant exterior operator and adjoint covariant exterior operator

$$\mathsf{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{j,N} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N),$$

$$\mathsf{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} \otimes E_{N,j} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

The connection D is flat as the following lemma shows.

Lemma 2.2.8. *The operators D_k form a complex. Hence the curvature of D vanishes.*

Proof. We have to to prove that $\mathsf{D}_k \mathsf{D}_{k+1} = 0$. We have

$$\mathsf{D}_{k+1}\mathsf{D}_k = D_{k+1}D_k \otimes I_N + \sum_{h=1}^{N-1} \alpha_h \Big(D_{k+1}(dx_h \wedge\ ) + dx_h \wedge D_k \Big) \otimes E_{h,N}$$

$$+ \sum_{j,h=1}^{N-1} \alpha_j \alpha_h (dx_j \wedge dx_h \wedge\ ) \otimes \underbrace{E_{j,N} E_{h,N}}_{=\delta_{h,N} E_{j,N} = 0} = 0,$$

for the first and second term, as before, vanish. This concludes the proof.

$\square$

We next consider the associated Laplacian

$$\square_k^{(N)} = \mathsf{D}_k^*\mathsf{D}_k + \mathsf{D}_{k-1}\mathsf{D}_{k-1}^* \colon \Omega^k(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n;\mathbb{C}^N).$$

Lemma 2.2.9. *We have*

$$\square_k^{(N)} = \Big(p_2^{\mathrm{w}}(x,D) + k - \frac{n}{2}\Big)\,\mathbf{1}_k\otimes I_N + \sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\,\mathbf{1}_k\otimes E_{j,N} + \psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k\otimes E_{N,j}\Big)$$

$$+\sum_{j=1}^{N-1}\alpha_j^2\mathbf{1}_k\otimes E_{N,N} - \sum_{j=1}^{N-1}\alpha_j^2 dx_j\wedge i_{\partial/\partial x_j}\mathbf{1}_k\otimes E_{N,N} + \sum_{j,h=1}^{N-1}\alpha_j\alpha_h dx_j\wedge i_{\partial/\partial x_h}\mathbf{1}_k\otimes E_{j,h}.$$

Proof. One has

$$\square_k^{(N)} = (D_k^*D_k + D_{k-1}D_{k-1}^*)\otimes I_N + \sum_{j=1}^{N-1}\alpha_j D_k^*(dx_j\wedge\ )\otimes E_{j,N} + \sum_{j=1}^{N-1}\alpha_j i_{\partial/\partial x_j}D_k\otimes E_{N,j}$$

$$+\sum_{j=1}^{N-1}\alpha_j D_{k-1}i_{\partial/\partial x_j}\otimes E_{N,j} + \sum_{j=1}^{N-1}\alpha_j dx_j\wedge D_{k-1}^*\otimes E_{j,N}$$

$$+\sum_{h,j=1}^{N-1}\alpha_j\alpha_h\Big(i_{\partial/\partial x_h}(dx_j\wedge\ )\otimes E_{N,h}E_{j,N} + dx_j\wedge i_{\partial/\partial x_h}\otimes E_{j,N}E_{N,h}\Big),$$

from which the lemma follows by virtue of the formulae of Proposition 2.2.1.

$\square$

Corollary 2.2.10. *When $k=0$ we have*

$$\square_0^{(N)} = \Big(p_2^{\mathrm{w}}(x,D) - \frac{n}{2}\Big)\otimes I_N$$

$$+\sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\otimes E_{j,N} + \psi_j^{\mathrm{w}}(x,D)\otimes E_{N,j}\Big) + \Big(\sum_{j=1}^{N-1}\alpha_j^2\Big)\mathbf{1}_0\otimes E_{N,N}.$$

Hence the JC-$\bigwedge$ model is related to the Laplacian $\square_0^{(N)}$.

2.2.3 The geometric N-level atom in the $\bigvee$-configuration

Next, let $N \geq 2$ be a fixed positive integer and let $n = N-1$. We define, for $\alpha_1, \dots, \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, the following connection D on the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N \to \mathbb{R}^n$

$$\mathsf{D} := D \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{1,j+1}.$$

The connection D extends to the following covariant exterior operator and adjoint covariant exterior operator

$$\mathsf{D}_k := D_k \otimes I_N + \sum_{j=1}^{N-1} \alpha_j (dx_j \wedge\ ) \otimes E_{1,j+1} \colon \Omega^k(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N),$$

$$\mathsf{D}_k^* := D_k^* \otimes I_N + \sum_{j=1}^{N-1} \alpha_j i_{\partial/\partial x_j} \otimes E_{j+1,1} \colon \Omega^{k+1}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n; \mathbb{C}^N).$$

The connection D is flat, by the following Lemma 2.2.11.

Lemma 2.2.11. *The operators D_k form a complex. Hence the curvature of D vanishes.*

Proof. We have to to prove that $\mathsf{D}_k \mathsf{D}_{k+1} = 0$. We have

$$\begin{aligned} \mathsf{D}_{k+1}\mathsf{D}_k = D_{k+1}D_k \otimes I_N &+ \sum_{h=1}^{N-1} \alpha_h \Big(D_{k+1}(dx_h \wedge\ ) + dx_h \wedge D_k \Big) \otimes E_{1,j+1} \\ &+ \sum_{j,h=1}^{N-1} \alpha_j \alpha_h (dx_j \wedge dx_h \wedge\ ) \otimes \underbrace{E_{1,j+1}E_{1,h+1}}_{=\delta_{j+1,1}E_{1,h+1}=0} = 0, \end{aligned}$$

for the first and second term, once more, vanish. This concludes the proof. □

We next consider the associated Laplacian

$$\square_k^{(N)} = \mathsf{D}_k^*\mathsf{D}_k + \mathsf{D}_{k-1}\mathsf{D}_{k-1}^* \colon \Omega^k(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \Omega^k(\mathbb{R}^n;\mathbb{C}^N).$$

Lemma 2.2.12. *We have*

$$\square_k^{(N)} = \Big(p_2^{\mathrm{w}}(x,D) + k - \frac{n}{2}\Big)\,\mathbf{1}_k\otimes I_N + \sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\,\mathbf{1}_k\otimes E_{1,j+1} + \psi_j^{\mathrm{w}}(x,D)\mathbf{1}_k\otimes E_{j+1,1}\Big)$$

$$+\sum_{j=1}^{N-1}\alpha_j^2\mathbf{1}_k\otimes E_{j+1,j+1} + \Big(\sum_{j=1}^{N-1}\alpha_j^2 dx_j\wedge i_{\partial/\partial x_j}\mathbf{1}_k\Big)\otimes E_{1,1} - \sum_{j,h=1}^{N-1}\alpha_j\alpha_h dx_j\wedge i_{\partial/\partial x_h}\otimes E_{h+1,h+1}.$$

Proof. In fact,

$$\square_k^{(N)} = (D_k^*D_k + D_{k-1}D_{k-1}^*)\otimes I_N + \sum_{j=1}^{N-1}\alpha_j D_k^*(dx_j\wedge\ )\otimes E_{1,j+1} + \sum_{j=1}^{N-1}\alpha_j i_{\partial/\partial x_j}D_k\otimes E_{j+1,1}$$

$$+\sum_{j=1}^{N-1}\alpha_j D_{k-1}i_{\partial/\partial x_j}\otimes E_{j+1,1} + \sum_{j=1}^{N-1}\alpha_j dx_j\wedge D_{k-1}^*\ \otimes E_{1,j+1}$$

$$+\sum_{h,j=1}^{N-1}\alpha_j\alpha_h\Big(i_{\partial/\partial x_h}(dx_j\wedge\ )\otimes E_{h+1,1}E_{1,j+1} + dx_j\wedge i_{\partial/\partial x_h}\otimes E_{1,j+1}E_{h+1,1}\Big),$$

from which the lemma follows once more by virtue of the formulae of Proposition 2.2.1.

□

Corollary 2.2.13. *When $k = 0$ we have*

$$\square_0^{(N)} = \Big(p_2^{\mathrm{w}}(x,D) - \frac{n}{2}\Big)\otimes I_N$$

$$+\sum_{j=1}^{N-1}\alpha_j\Big(\psi_j^{\mathrm{w}}(x,D)^*\otimes E_{1,j+1}+\psi_j^{\mathrm{w}}(x,D)\otimes E_{j+1,1}\Big)+\sum_{j=1}^{N-1}\alpha_j^2\ \mathbf{1}_0\otimes E_{j+1,j+1}.$$

Hence the JC-$\bigvee$ model is related to the Laplacian $\square_0^{(N)}$.

Remark 2.2.14. *Note that, therefore, the JC-models possess extensions to states that are vector-valued k-forms. Loosely speaking, one may think of this mathematical generalization as a transposition to a fermionic, or more generally supersymmetric, picture.*

Lemma 2.2.15. *The semiprincipal term of $\square_1^{(3)}$ can be blockwise-diagonalized with three blocks.*

Proof. Fix the basis $\{dx_i \otimes e_j\}_{i=1,2;j=1,2,3}$. We have for the semiprincipal symbol $A_1(X)$ of $\square_1^{(3)}$,

$$A_1(X)=\sum_{j=1}^{2}\alpha_j\Big(\overline{\psi_j(X)}\mathbf{1}_k\otimes E_{1,j+1}+\psi_j(X)\mathbf{1}_k\otimes E_{j+1,1}\Big),$$

that it may be rewritten in the above basis as

$$A_1(X)=\left[\begin{array}{c|c|c} 0_2 & \overline{A_{12}(X)} & \overline{A_{13}(X)} \\ \hline A_{11}(X) & 0_2 & 0_2 \\ \hline A_{12}(X) & 0_2 & 0_2 \end{array}\right],$$

where $A_{1j}(X)=\begin{bmatrix}\alpha_j\psi_j(X) & 0\\ 0 & \alpha_j\psi_j(X)\end{bmatrix}$, $j=1,2$, and where 0_2 is the 2×2 zero-matrix.

Now, we compute the characteristic polynomial $p(\lambda;X)=\det(\lambda-A_1(X))$ of $A_1(X)$ following the scheme below where the pairs of numbers in the underbrackets indicate the position of that element in the matrix $\lambda I_3-A_1(X)$.

We have, starting from the first column,

$$
\begin{aligned}
p(\lambda;X) = \underbrace{\lambda}_{(1,1)} \Bigg(& \underbrace{\lambda}_{(2,2)} \lambda^4 + (-\underbrace{\alpha_1\psi_1(X)}_{(4.2)})(-\underbrace{\lambda}_{(3,3)})\underbrace{\lambda}_{(5,5)}(\lambda(-\overline{\alpha_1\psi_1(X)})) \\
& + (-\underbrace{\alpha_2\psi_2(X)}_{(6,2)})(-\underbrace{\lambda}_{(3,3)})(-\underbrace{\lambda}_{(4,4)})(-\lambda\overline{\alpha_2\psi_2(X)})\Bigg) \\
& \underbrace{-\alpha_1\psi_1(X)}_{(3,1)}\Bigg(\underbrace{\lambda}_{(5,5)}\underbrace{\lambda}_{(4,4)}(-\underbrace{\overline{\alpha_1\psi_1(X)}}_{(1,3)})(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
& + \underbrace{\lambda}_{(5,5)}(-\underbrace{\alpha_1\psi_1(X)}_{(4,2)})\underbrace{\lambda}_{(6,6)}\overline{\alpha_1\psi_1(X)\alpha_1\psi_1(X)}\Bigg) \\
& \underbrace{-\alpha_2\psi_2(X)}_{(5,1)}\Bigg((\underbrace{-\lambda}_{(3,3)})(\underbrace{-\lambda}_{(4,4)})(\underbrace{-\overline{\alpha_2\psi_2(X)}}_{(1,5)})(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
& \underbrace{-\lambda}_{(3,3)}(-\underbrace{\alpha_1\psi_1(X)}_{(4,2)})\underbrace{\lambda}_{(6,6)}\overline{\alpha_1\psi_1(X)\alpha_2\psi_2(X)}\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
= \lambda^2\Bigg(& \lambda^4 - |\alpha_1\psi_1(X)|^2\lambda^2 - |\alpha_2\psi_2(X)|^2\lambda^2 - |\alpha_1\psi_1(X)|^2(\lambda^2 - |\alpha_2\psi_2(X)|^2) \\
& + |\alpha_1\psi_1(X)|^2|\alpha_2\psi_2(X)|^2 - |\alpha_2\psi_2(X)|^2(\lambda^2 - |\alpha_2\psi_2(X)|^2) + |\alpha_1\psi_1(X)|^2|\alpha_2\psi_2(X)|^2\Bigg)
\end{aligned}
$$

$$
= \lambda^2\Big(\lambda^4 - 2\big(|\alpha_1\psi_1(X)|^2 + |\alpha_1\psi_1(X)|^2\big)\lambda^2 + \big(|\alpha_1\psi_1(X)|^2 + |\alpha_1\psi_1(X)|^2\big)^2\Big).
$$

Hence, the zeros of $p(\lambda;X)$ are given by

$$
0, \quad \lambda_\pm(X) = \pm\sqrt{\alpha_1^2|\psi_1(X)|^2 + \alpha_2^2|\psi_2(X)|^2},
$$

each with constant multiplicity 2 (for $X \neq 0$).

Thus, the diagonalization Theorem 3.1.3 (see the next section) can be applied to obtain a blockwise diagonalization with three blocks

$$\lambda_{1,1} = 0_2 \quad , \quad \lambda_{2,1} = \lambda_+ I_2, \quad \text{and} \quad \lambda_{3,1} = \lambda_- I_2.$$

□

Chapter 3

The Decoupling Theorem

In this chapter we prove a decoupling theorem for classes of semiregular global pseudodifferential systems from our class SMGES (that is, of the Jaynes-Cummings kind). For future purposes, we prove the theorem in the semiclassical case (hence $\sharp_h$ will denote the semiclassical composition of symbols in the semiclassical setting, see [45]) and then state the corresponding version valid for the semiregular case. The proof follows the lines of the decoupling theorem in [45], but it has a main twist due to the fact that the terms a_μ and $a_{\mu-1}$ may interact in the composition formula due to the conjugation of the symbol of the diagonalizer, but can be simultaneously blockwise diagonalized. Recall that $S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ stands for the set of *polyhomogeneous semiclassical symbols* (see Point 2 of Definition 9.1.9 of [45]), that is, they are h-dependent symbols that admit an asymptotic expansion in half-powers of h, with the h^j-coefficient which is an h-independent symbol of order equal to the order of a decreased by $2j$. A semiclassical symbol A then belongs to $S^0_{0,\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ if it can be written in the form $A_\mu + h^{1/2}A_{\mu-1}$, where $A_\mu \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N)$ and $A_{\mu-1} \in S^0_{0,\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N)$.

3.1 Statement of the theorem and proof

Below the decoupling theorem is stated and proved following the approach of Theorem 9.2.1 in [45].

Theorem 3.1.1. *Let* $\mu > 0$ *and let* $A = A^* = A_\mu + h^{1/2} A_{\mu-1} \in S^0_{0,\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ *where*

$$A_\mu \sim \sum_{j\geq 0} h^j a_{\mu-2j} \in S^0_{0,\mathrm{cl}}(m^\mu, g; \mathsf{M}_N), \quad A_{\mu-1} \sim \sum_{j\geq 0} h^j a_{\mu-1-2j} \in S^0_{0,\mathrm{cl}}(m^{\mu-1}, g; \mathsf{M}_N),$$

with $a_{-k} = a^*_{-k} \in S(m^{-k}, g, \mathsf{M}_N)$. *Moreover, suppose* $a_\mu = p_\mu I_N$ *with* $p_\mu \in S(m^\mu, g)$, *and that* $a_{\mu-1}$, *for some* $e_0 \in S(1, g; \mathsf{M}_N)$ *such that* $e_0 e_0^* = e_0^* e_0 = I_N$, *can be written as*

$$a_{\mu-1} = e_0 b_{\mu-1} e_0^*, \quad \textit{where } b_{\mu-1} = b^*_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right],$$

where the $\lambda_{j,\mu-1} \in S(m^{\mu-1}, g; \mathsf{M}_{N_j})$, $j = 1, 2$ *and* $N = N_1 + N_2$, *are such that*

$$d_{\lambda_1,\lambda_2}(X) \gtrsim m(X)^{\mu-1}, \quad \forall X \in \mathbb{R}^{2n}, \tag{3.1.1}$$

with

$$d_{\lambda_1,\lambda_2}(X) = \inf\{|\zeta_1 - \zeta_2|;\ \zeta_j \in \mathrm{Spec}(\lambda_{\mu-1,j}(X)),\ j = 1, 2\}.$$

Then, there exists $E \in S^0_{0,\mathrm{sreg}}(1, g; \mathsf{M}_N)$ *with* $E \sim \sum_{j\geq 0} h^{j/2} e_{-j}$ *(with* $e_{-k} \in S(m^{-k}, g, \mathsf{M}_N)$*) and principal symbol* e_0 *such that*

$$E^{\mathrm{w}}(x, hD)^* E^{\mathrm{w}}(x, hD) - I, \quad E^{\mathrm{w}}(x, hD) E^{\mathrm{w}}(x, hD)^* - I \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N), \tag{3.1.2}$$

and

$$E^{\mathrm{w}}(x, hD)^* A^{\mathrm{w}}(x, hD) E^{\mathrm{w}}(x, hD) - B^{\mathrm{w}}(x, hD) \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N), \tag{3.1.3}$$

where the symbol $B \sim \sum_{j\geq 0} h^{j/2} b_{\mu-j} \in S^0_{0,\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ *is blockwise diagonal, with*

$$b_{\mu-j}(X) = \left[\begin{array}{c|c} b_{\mu-j,1}(X) & 0 \\ \hline 0 & b_{\mu-j,2}(X) \end{array}\right], \quad \forall X \in \mathbb{R}^{2n}, \forall j \geq 0,$$

the blocks $b_{\mu-j,k}$ being of sizes $N_k \times N_k$, $k = 1, 2$, with

$$b_\mu = a_\mu = p_\mu I_N, \quad b_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right].$$

Remark 3.1.2. *We shall call B an h^∞-(blockwise)diagonalization of A. Notice that B depends on A and e_0.*

Proof. We immediately observe that once $E^{\mathrm{w}}(x, hD)$ has been constructed with the property that

$$E^{\mathrm{w}}(x, hD)^* E^{\mathrm{w}}(x, hD) = I + r^{\mathrm{w}}(x, hD), \text{ with } r \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N),$$

then by the ellipticity of $E^{\mathrm{w}}(x, hD)^*$ (namely, the existence of a parametrix) we also get

$$E^{\mathrm{w}}(x, hD) E^{\mathrm{w}}(x, hD)^* = I + s^{\mathrm{w}}(x, hD), \text{ with } s \in S^{-\infty}(m^{-\infty}, g; \mathsf{M}_N).$$

Hence it suffices to prove the existence of E and B with the required properties. We show that for every integer $N_0 \in \mathbb{Z}_+$ there exist

$$e_{-k} \in S(m^{-k}, g; \mathsf{M}_N), \quad 0 \le k \le N_0,$$

and

$$b_{\mu-k,j} \in S(m^{\mu-k}, g; \mathsf{M}_j), \quad j = 1, 2, \quad 0 \le k \le N_0,$$

such that, with $E_{N_0}(X) := \sum_{k=0}^{N_0} h^{k/2} e_{-k}(X))$,

$$E_{N_0}^* \#_h E_{N_0} = I + h^{(N_0+1)/2} S_0^0(m^{-(N_0+1)}, g; \mathsf{M}_N),$$

and

$$E_{N_0}^* \#_h A \#_h E_{N_0} = \sum_{k=0}^{N_0} h^{k/2} b_{\mu-k} + h^{(N_0+1)/2} S_0^0(m^{\mu-(N_0+1)}, g; \mathsf{M}_N),$$

where $b_{\mu-k} = \left[\begin{array}{c|c} b_{\mu-k,1} & 0 \\ \hline 0 & b_{\mu-k,2} \end{array}\right]$. We shall then take $E \sim \sum_{k\geq 0} h^{k/2} e_{-k}$.

First of all, we have that $e_0 \in S(1, g; \mathsf{M}_N)$ is such that $e_0^* a_\mu e_0$ and $e_0^* a_{\mu-1} e_0$ are diagonal matrices and e_0 satisfies the unitarity condition (note that a_μ and $a_{\mu-1}$ commute since a_μ is a scalar matrix).

We proceed by induction on N_0, and start by proving that the assertion is true for $N_0 = 1$ and for $N_0 = 2$. (We will omit the dependence on (x, hD) and write e_k^{w} in place of $e_k^{\mathrm{w}}(x, hD)$.) Hence, we look for $e_{-1} \in S(m^{-1}, g; \mathsf{M}_N)$ such that

$$(e_0 + h^{1/2} e_{-1})^* \#_h (e_0 + h^{1/2} e_{-1}) - I \in h S_0^0(m^{-2}, g; \mathsf{M}_N). \tag{3.1.4}$$

Now, the coefficient s_{-1} of $h^{1/2}$ in $e_0^* \#_h e_0$ is zero (because of the step-decrease of the global calculus), whence for the coefficient of $h^{1/2}$ in (3.1.4) we have

$$s_{-1} + e_0^* e_{-1} + e_{-1}^* e_0 = e_0^* e_{-1} + e_{-1}^* e_0 = 0. \tag{3.1.5}$$

Equation (3.1.5) has a general solution

$$e_{-1} = e_0 \alpha_{-1}, \tag{3.1.6}$$

where $\alpha_{-1} \in S(m^{-1}, g; \mathsf{M}_N)$ and

$$\alpha_{-1}^* + \alpha_{-1} = 0. \tag{3.1.7}$$

We next look for α_{-1} in such a way that $b_{\mu-1}$ is blockwise diagonal. Hence, we write

$$(e_0^{\mathrm{w}} + h^{1/2} e_{-1}^{\mathrm{w}})^* A^{\mathrm{w}} (e_0^{\mathrm{w}} + h^{1/2} e_{-1}^{\mathrm{w}}) = (e_0^{\mathrm{w}})^* A^{\mathrm{w}} e_0^{\mathrm{w}} + h^{1/2} \Big((e_{-1}^{\mathrm{w}})^* A^{\mathrm{w}} e_0^{\mathrm{w}} + (e_0^{\mathrm{w}})^* A^{\mathrm{w}} e_{-1}^{\mathrm{w}} \Big) + h r_{\mu-2}^{\mathrm{w}},$$

where $r_{\mu-2} \in S_0^0(m^{\mu-2}, g, \mathsf{M}_N)$.

Now, recalling the definition of A, we have

$$(e_0 + h^{1/2}e_{-1})^* \#_h A \#_h (e_0 + h^{1/2}e_{-1})$$

$$= e_0^* \#_h a_\mu \#_h e_0 + h^{1/2}\Big(e_{-1}^* \#_h a_\mu \#_h e_0 + e_0^* \#_h a_\mu \#_h e_{-1} + e_0^* \#_h a_{\mu-1} \#_h e_0\Big) + h r_{\mu-2},$$

with $r_{\mu-2} \in S_0^0(m^{\mu-2}, g, \mathsf{M}_N)$.

It follows that, since a_μ is a scalar matrix and hence it commutes with every other matrix, we look for e_{-1} such that

$$q_{\mu-1} + a_\mu(e_{-1}^* e_0 + e_0^* e_{-1}) + e_0^* a_{\mu-1} e_0 \tag{3.1.8}$$

is diagonal, where $q_{\mu-1}$ is the coefficient of $h^{1/2}$ in $e_0^* \#_h A_\mu \#_h e_0$ and in this case $q_{\mu-1} = 0$. We have that $e_{-1}^* e_0 + e_0^* e_{-1} = 0$ and that $e_0^* a_{\mu-1} e_0$ is already diagonal by the hypothesis on e_0. Hence (3.1.8) is blockwise diagonal *without* any further conditions on α_{-1}, which is therefore only required to be skew-hermitian. However, further constraints on it will arise in the next step.

This completes the case $N_0 = 1$.

Next we look for $e_{-2} \in S(m^{-2}, g; \mathsf{M}_N)$ such that

$$(e_0 + h^{1/2}e_{-1} + he_{-2})^* \#_h (e_0 + h^{1/2}e_{-1} + he_{-2}) - I \in h^{3/2} S_0^0(m^{-3}, g; \mathsf{M}_N).$$

Hence, since

$$e_0^* \#_h e_0 - I = hs_{-2}, \quad s_{-2} = s_{-2}^* \in S_0^0(m^{-2}, g; \mathsf{M}_N),$$

we require that e_{-2} be a solution of

$$s_{-2} + e_0^* e_{-2} + e_{-2}^* e_0 + e_{-1}^* e_{-1} = 0. \tag{3.1.9}$$

Equation (3.1.9) has as a general solution

$$e_{-2} = -\frac{1}{2} e_0 \big(s_{-2} + \underbrace{e_{-1}^* e_{-1}}_{=\alpha_{-1}^* \alpha_{-1}}\big) + e_0 \alpha_{-2},$$

where $\alpha_{-2} \in S(m^{-2}, g; \mathsf{M}_N)$ and

$$\alpha_{-2}^* + \alpha_{-2} = 0. \tag{3.1.10}$$

We next determine α_{-2} so as to have $b_{\mu-2}$ in blockwise diagonal form with the diagonal blocks $b_{j,\mu-2}$, $j = 1, 2$. Write

$$\begin{aligned}
&(e_0 + h^{1/2}e_{-1} + he_{-2})^* \#_h A \#_h (e_0 + h^{1/2}e_{-1} + he_{-2}) \\
&= e_0^* \#_h A \#_h e_0 + h^{1/2}\Big(e_{-1}^* \#_h A \#_h e_0 + e_0^* \#_h A \#_h e_{-1}\Big) \\
&\quad + h\Big(e_0^* \#_h A_\mu \#_h e_{-2} + e_{-2}^* \#_h A_\mu \#_h e_0 + e_0^* \#_h A_{\mu-1} \#_h e_{-1} + e_{-1}^* \#_h A_{\mu-1} \#_h e_0 \\
&\qquad + e_{-1}^* \#_h A_\mu \#_h e_{-1}\Big) + h^{3/2} S_0^0(m^{\mu-3}, g; \mathsf{M}_N).
\end{aligned}$$

Because of the form of A, we have

$$\begin{aligned}
&(e_0 + h^{1/2}e_{-1} + he_{-2})^* \#_h A \#_h (e_0 + h^{1/2}e_{-1} + he_{-2}) \\
&\quad = e_0^* \#_h a_\mu \#_h e_0 + h^{1/2}\Big(e_{-1}^* \#_h a_\mu \#_h e_0 + e_0^* \#_h a_\mu \#_h e_{-1} + e_0^* \#_h a_{\mu-1} \#_h e_0\Big) \\
&\quad + h\Big(e_0^* \#_h a_\mu \#_h e_{-2} + e_0^* \#_h a_{\mu-1} \#_h e_{-1} + e_0^* \#_h a_{\mu-2} \#_h e_0 + e_{-1}^* \#_h a_\mu \#_h e_{-1} \\
&\qquad + e_{-1}^* \#_h a_{\mu-1} \#_h e_0 + e_{-2}^* \#_h a_\mu \#_h e_0\Big) + h^{3/2} r_{\mu-3}, \quad r_{\mu-3} \in S_0^0(m^{\mu-3}, g, \mathsf{M}_N).
\end{aligned}$$

Hence, we look for e_{-2} such that the coefficient of h in

$$(e_0 + h^{1/2}e_{-1} + he_{-2})^* \#_h A \#_h (e_0 + h^{1/2}e_{-1} + he_{-2}),$$

given by

$$q_{\mu-2} + e_0^* a_\mu e_{-2} + e_0^* a_{\mu-1} e_{-1} + e_{-1}^* a_\mu e_{-1} + e_{-1}^* a_{\mu-1} e_0 + e_{-2}^* a_\mu e_0, \tag{3.1.11}$$

is diagonal. Now, (3.1.11) can be rewritten, by using (3.1.5), as

$$q_{\mu-2} + a_\mu(e_0^* e_{-2} + e_{-2}^* e_0) + (e_0^* a_{\mu-1} e_0)\alpha_{-1} + a_\mu \alpha_{-1}^* \alpha_{-1} + \alpha_{-1}^*(e_0^* a_{\mu-1} e_0). \tag{3.1.12}$$

Since by (3.1.9) and (3.1.6)

$$e_0^* e_{-2} + e_{-2}^* e_0 = -s_{-2} - \alpha_{-1}^* \alpha_{-1},$$

we obtain that (3.1.12) becomes

$$q_{\mu-2} - a_\mu s_{-2} + (e_0^* a_{\mu-1} e_0)\alpha_{-1} + \alpha_{-1}^*(e_0^* a_{\mu-1} e_0). \tag{3.1.13}$$

We now split (3.1.13) into two (Hermitian) parts (note that $q_{\mu-2} = q_{\mu-2}^*$ and $s_{-2} = s_{-2}^*$). The first part is given by

$$q_{\mu-2} - a_\mu s_{-2} = \left[\begin{array}{c|c} u_1 & \gamma \\ \hline \gamma^* & u_2 \end{array}\right], \tag{3.1.14}$$

where $u_j = u_j^*$ are blocks of sizes $N_j \times N_j$. The second part is given by

$$(e_0^* a_{\mu-1} e_0)\alpha_{-1} + \alpha_{-1}^*(e_0^* a_{\mu-1} e_0) = [e_0^* a_{\mu-1} e_0, \alpha_{-1}]$$

(recall that α_{-1} is skew-Hermitian by (3.1.7)). We therefore look for α_{-1} in the form

$$\alpha_{-1} = \left[\begin{array}{c|c} 0 & \delta \\ \hline -\delta^* & 0 \end{array}\right]. \tag{3.1.15}$$

Using the fact that $e_0^* a_{\mu-1} e_0$ is blockwise diagonal with blocks $\lambda_{\mu-1,1}$ and $\lambda_{\mu-1,2}$, in order to make (3.1.13) blockwise diagonal, we are led to the equation

$$\lambda_{\mu-1,1}\delta - \delta\lambda_{\mu-1,2} = -\gamma, \tag{3.1.16}$$

which imposes a condition on α_{-1}. By Lemma 9.2.2 in [45], equation (3.1.16) has a solution and this completes the case $N_0 = 2$.

It is important to note at this point that the only condition that α_{-2} must satisfy by now is that it be skew-Hermitian, that is $\alpha_{-2} + \alpha_{-2}^* = 0$.

Now, we proceed by induction. So, suppose we have already constructed the symbols $e_0, e_{-1}, ..., e_{-N_0}$, and $b_\mu, b_{\mu-1}, ..., b_{\mu-N_0}$, independent of h, with the required properties. Moreover, suppose that we have constructed

$$e_{-N_0} = -\frac{1}{2}e_0\Big(s_{-N_0} + e^*_{-1}e_{-(N_0-1)} + e^*_{-(N_0-1)}e_{-1}\Big) + e_0\alpha_{-N_0}, \qquad (3.1.17)$$

where $s_{-N_0} = s^*_{-N_0} \in S^0_0(m^{-N_0}, g, \mathsf{M}_N)$ is the coefficient of $h^{N_0/2}$ in $E_{N_0-2}\#_h E_{N_0-2}$, and the only condition that α_{-N_0} must satisfy is

$$\alpha_{-N_0} + \alpha^*_{-N_0} = 0. \qquad (3.1.18)$$

Proceeding as in the case $N_0 = 2$, we look for $e_{-(N_0+1)}$ such that

$$\Big(E_{N_0} + h^{\frac{N_0+1}{2}}e_{-(N_0+1)}\Big)^* \#_h \Big(E_{N_0} + h^{\frac{N_0+1}{2}}e_{-(N_0+1)}\Big) = I + h^{\frac{N_0+2}{2}}S^0_0(m^{-(N_0+2)}, g; \mathsf{M}_N).$$

Thus, using the symbol-composition formula $\#_h$ and (part of) the inductive hypothesis, that is, $E^*_{N_0}\#_h E_{N_0} = I + h^{(N_0+1)/2}S^0_0(m^{-(N_0+1)}, g; \mathsf{M}_N)$ (the other part of the inductive hypothesis being relative to the diagonal form of the conjugated operator), we are led to the equation

$$s_{-(N_0+1)} + e^*_0 e_{-(N_0+1)} + e^*_{-(N_0+1)}e_0 + e^*_{-1}e_{-N_0} + e^*_{-N_0}e_{-1} = 0, \qquad (3.1.19)$$

where $s_{-(N_0+1)} = s^*_{-(N_0+1)}$ is the coefficient of $h^{(N_0+1)/2}$ in $E^*_{N_0-1}\#_h E_{N_0-1}$. Equation (3.1.19) has as a general solution

$$e_{-(N_0+1)} = -\frac{1}{2}e_0\big(s_{-(N_0+1)} + e^*_{-1}e_{-N_0} + e^*_{-N_0}e_{-1}\big) + e_0\alpha_{-(N_0+1)},$$

where $\alpha_{-(N_0+1)} \in S(m^{-(N_0+1)}, g; \mathsf{M}_N)$ and

$$\alpha^*_{-(N_0+1)} + \alpha_{-(N_0+1)} = 0. \qquad (3.1.20)$$

Since

$$\left(E_{N_0} + h^{(N_0+1)/2} e_{-(N_0+1)}\right)^* \#_h A \#_h \left(E_{N_0} + h^{(N_0+1)/2} e_{-(N_0+1)}\right)$$
$$= \sum_{k=0}^{N_0} h^{k/2} b_{\mu-k} + h^{(N_0+1)/2} S_0^0(m^{\mu-(N_0+1)}, g; \mathsf{M}_N),$$

with $b_{\mu-k}$ diagonal, we next look for $e_{-(N_0+1)}$ such that the $h^{(N_0+1)/2}$-coefficient in the symbol composition $E^*_{N_0+1} \#_h A \#_h E_{N_0+1}$, given by

$$\begin{aligned} &q_{\mu-(N_0+1)} + e_0^* a_\mu e_{-(N_0+1)} + e^*_{-(N_0+1)} a_\mu e_0 \\ &+ e_0^* a_{\mu-1} e_{-N_0} + e^*_{-N_0} a_{\mu-1} e_0 + e^*_{-1} a_\mu e_{-N_0} + e^*_{-N_0} a_\mu e_{-1}, \end{aligned} \tag{3.1.21}$$

is diagonal. Now, we rewrite (3.1.21) as

$$\begin{aligned} &q_{\mu-(N_0+1)} + a_\mu e_0^* e_{-(N_0+1)} + a_\mu e^*_{-(N_0+1)} e_0 + (e_0^* a_{\mu-1} e_0) e_0^* e_{-N_0} \\ &+ e^*_{-N_0} e_0 (e_0^* a_{\mu-1} e_0) + a_\mu e^*_{-1} e_{-N_0} + a_\mu e^*_{-N_0} e_{-1}, \end{aligned} \tag{3.1.22}$$

so that (3.1.22) becomes (using the fact that a_μ is scalar)

$$\begin{aligned} &q_{\mu-(N_0+1)} + a_\mu \underbrace{(e_0^* e_{-(N_0+1)} + e^*_{-(N_0+1)} e_0 + e^*_{-1} e_{-N_0} + e^*_{-N_0} e_{-1})}_{=-s_{-(N_0+1)} \text{ by (3.1.19)}} \\ &+ (e_0^* a_{\mu-1} e_0) e_0^* e_{-N_0} + e^*_{-N_0} e_0 (e_0^* a_{\mu-1} e_0). \end{aligned} \tag{3.1.23}$$

Next, using (3.1.17), we write

$$\begin{cases} e_0^* e_{-N_0} = -\frac{1}{2} s_{-N_0} + \alpha_{-N_0} =: \tau + \alpha_{-N_0}, \\ e^*_{-N_0} e_0 = \tau - \alpha_{-N_0}, \end{cases} \tag{3.1.24}$$

where

$$\tau = -\frac{1}{2} s_{-N_0} = \tau^*, \quad \alpha_{-N_0} = -\alpha^*_{-N_0}.$$

Hence, (3.1.23) can be rewritten as

$$q_{\mu-(N_0+1)}-a_\mu s_{-(N_0+1)}+(e_0^*a_{\mu-1}e_0)\tau+\tau(e_0^*a_{\mu-1}e_0)+(e_0^*a_{\mu-1}e_0)\alpha_{-N_0}+\alpha_{-N_0}^*(e_0^*a_{\mu-1}e_0).$$

As before, we next split (3.1.13) into two (Hermitian) parts, where the first part is given by

$$q_{\mu-(N_0+1)} - a_\mu s_{-(N_0+1)} + (e_0^*a_{\mu-1}e_0)\tau + \tau(e_0^*a_{\mu-1}e_0) = \left[\begin{array}{c|c} \tilde{u}_1 & \tilde{\gamma} \\ \hline \tilde{\gamma}^* & \tilde{u}_2 \end{array}\right],$$

where $u_j = u_j^*$ are blocks of sizes $N_j \times N_j$, and the second one by

$$(e_0^*a_{\mu-1}e_0)\alpha_{-N_0} + \alpha_{-N_0}^*(e_0^*a_{\mu-1}e_0) = \left[e_0^*a_{\mu-1}e_0, \alpha_{-N_0}\right]$$

(recall that α_{-N_0} is skew-Hermitian by (3.1.18)). Hence, we look for α_{-N_0} in the form

$$\alpha_{-N_0} = \left[\begin{array}{c|c} 0 & \tilde{\delta} \\ \hline -\tilde{\delta}^* & 0 \end{array}\right], \tag{3.1.25}$$

and, using as before the fact that $e_0^*a_{\mu-1}e_0$ is blockwise diagonal with blocks $\lambda_{\mu-1,1}$ and $\lambda_{\mu-1,2}$, we are therefore led to the equation

$$\lambda_{\mu-1,1}\tilde{\delta} - \tilde{\delta}\lambda_{\mu-1,2} = -\tilde{\gamma}. \tag{3.1.26}$$

As before, the equation has a solution and we complete the proof by induction. Once more, it is important to note that the only condition that $\alpha_{-(N_0+1)}$ must satisfy in the N_0+1-st step of the induction is $\alpha_{-(N_0+1)} + \alpha_{-(N_0+1)}^* = 0$. $\square$

We state (omitting the proof, since it follows the lines of the foregoing one) the blockwise diagonalization theorem in the case of semiregular symbols.

Theorem 3.1.3. *Let* $\mu > 0$, *and let* $A = A^* \sim \sum_{j\geq 0} a_{\mu-j} \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$. *Suppose* $a_\mu = p_\mu I_N$ *with* $p_\mu \in S_{\mathrm{cl}}(m^\mu, g)$, *and that* $a_{\mu-1}$, *for some* $e_0 \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_N)$ *positively homogeneous of degree* 0 *and such that* $e_0e_0^* =$

$e_0^* e_0 = I_N$, $X \neq 0$, *can be written as*

$$a_{\mu-1} = e_0 b_{\mu-1} e_0^*, \quad \text{where } b_{\mu-1} = b_{\mu-1}^* = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right], \quad X \neq 0,$$

where $\lambda_{\mu-1,j} \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_{N_j})$, $j = 1, 2$ *and* $N = N_1 + N_2$, *are positively homogeneous of degree* $\mu - 1$, *and are such that*

$$\mathrm{Spec}(\lambda_{\mu-1,1}(X)) \cap \mathrm{Spec}(\lambda_{\mu-1,2}(X)) = \emptyset, \quad \forall X \in \mathbb{R}^{2n},\ |X| = 1.$$

Then there exists $E \in S_{\mathrm{sreg}}(1, g; \mathsf{M}_N)$ *with* $E \sim \sum_{j \geq 0} e_{-j}$ *and principal symbol* e_0 *(hence* $e_{-k} \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_N)$ *is positively homogeneous of degree* $-k$*) such that*

$$E^w(x, D)^* E^w(x, D) - I, \quad E^w(x, D) E^w(x, D)^* - I \in S(m^{-\infty}, g; \mathsf{M}_N), \quad (3.1.27)$$

and

$$E^w(x, D)^* A^w(x, D) E^w(x, D) - B^w(x, D) \in S(m^{-\infty}, g; \mathsf{M}_N), \qquad (3.1.28)$$

where the symbol $B \sim \sum_{j \geq 0} b_{\mu-j} \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$ *is blockwise diagonal, with*

$$b_{\mu-j}(X) = \left[\begin{array}{c|c} b_{\mu-j,1}(X) & 0 \\ \hline 0 & b_{\mu-j,2}(X) \end{array}\right], \forall X \neq 0, \forall j \geq 0,$$

the blocks $b_{\mu-j,k}$ *being of sizes* $N_k \times N_k$, $k = 1, 2$, *with*

$$b_\mu = a_\mu = p_\mu I_N, \quad b_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1,1} & 0 \\ \hline 0 & \lambda_{\mu-1,2} \end{array}\right], \ X \neq 0.$$

Chapter 4

The subprincipal symbol

In spectral asymptotics, the subprincipal symbol of a pseudodifferential system plays an important role. In this chapter we study its structure and the transformation laws under different diagonalizers for the semiprincipal part. From the decoupling theorems (semiclassical case as well as semiregular one) we obtain the following general form for the subprincipal term of the diagonalized symbol.

Proposition 4.0.1. *For the subprincipal part $b_{\mu-2}$ of the h^∞-diagonalization given in Theorem 3.1.1, or in the semiregular case by Theorem 3.1.3, one has, by (3.1.22), the formula (recall that $a_\mu = p_\mu I_2$):*

$$\begin{aligned} b_{\mu-2} =& e_{-2}^* e_0 b_\mu + b_\mu e_0^* e_{-2} + e_0^* a_{\mu-2} e_0 - \frac{i}{2}\left(e_0^*\left\{a_\mu, e_0\right\} + \left\{e_0^*, a_\mu e_0\right\}\right) \\ &+ e_{-1}^* a_\mu e_{-1} + b_{\mu-1} e_0^* e_{-1} + e_{-1}^* e_0 b_{\mu-1}, \end{aligned}$$

where

$$e_{-2} = \frac{i}{4} e_0 \left\{e_0^*, e_0\right\} - \frac{1}{2} e_0 \alpha_{-1}^* \alpha_{-1} + e_0 \alpha_{-2},$$

with $\alpha_{-2}^* = -\alpha_{-2}$ and $\alpha_{-1}^* = -\alpha_{-1}$ determined by (3.1.25) and (3.1.26).

Remark 4.0.2. *It is important to note in e_{-2} the presence of the term $-e_0 \alpha_{-1}^* \alpha_{-1}/2$ which depends on the symbol of order $\mu - 1$ (i.e. the semiprincipal part) of our system.*

To study the structure of the subprincipal term, for the sake of clarity we will be considering in the first place the case $N = 2$ and afterwards the case of a general N.

All of the results below hold also true in the *semiregular* case of Theorem 3.1.3 above, the only change being that where in the case of the h^∞-diagonalization we have $X \in \mathbb{R}^{2n}$, in the semiregular case we have $X \neq 0$.

4.1 The case $N = 2$

Suppose hence that $N = 2$, that $a_\mu = a_\mu^* = p_\mu I_2$, $p_\mu \in S(m^\mu, g)$ scalar, and that $a_{\mu-1} = a_{\mu-1}^*$. Let (by slightly changing notation) $\lambda_{\mu-1}^+, \lambda_{\mu-1}^- \in S(m^{\mu-1}, g)$ be the eigenvalues of $a_{\mu-1}$, and suppose that

$$|\lambda_{\mu-1}^+(X) - \lambda_{\mu-1}^-(X)| \gtrsim m(X)^{\mu-1}, \forall X \in \mathbb{R}^{2n}, \tag{4.1.1}$$

whence the existence of a smooth unitary matrix e_0 such that

$$e_0(X)^* a_{\mu-1}(X) e_0(X) = \begin{bmatrix} \lambda_{\mu-1}^+(X) & 0 \\ 0 & \lambda_{\mu-1}^-(X) \end{bmatrix}, \quad \forall X \in \mathbb{R}^{2n}.$$

We have the following corollary.

Corollary 4.1.1. *Suppose that $a_\mu = a_\mu^* = p_\mu I_2$ is a scalar matrix and that $a_{\mu-1} = a_{\mu-1}^*$ possesses smooth eigenvalues $\lambda_{\mu-1}^\pm$ satisfying (4.1.1). Let $\{w_+, w_-\}$ be the* ***canonical*** *basis of $\mathbb{C}^2$, namely $w_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so that for the semiprincipal symbol $b_{\mu-1}$ of the (h^∞-)diagonalization we have $b_{\mu-1}(X) w_\pm = \lambda_{\mu-1}^\pm(X) w_\pm$, $\pm$-respectively, for all $X \in \mathbb{R}^{2n}$. Then for the subprincipal symbol $b_{\mu-2} = \begin{bmatrix} b_{\mu-2}^+ & 0 \\ 0 & b_{\mu-2}^- \end{bmatrix}$ we have with $j = \pm$*

$$\begin{aligned} b_{\mu-2}^{(j)} &= \langle b_{\mu-2} w_j, w_j \rangle \\ &= \langle e_0^* a_{\mu-2} e_0 w_j, w_j \rangle + \frac{1}{2} \mathrm{Im}\left(\langle \{e_0^*, p_\mu\} e_0 w_j, w_j \rangle\right) + \frac{1}{2} \mathrm{Im}\left(\langle e_0^* \{p_\mu, e_0\} w_j, w_j \rangle\right). \end{aligned}$$

In addition, as for the term δ determined by equation (3.1.16), one has

$$\delta = -\frac{1}{\lambda^+_{\mu-1} - \lambda^-_{\mu-1}} \langle (e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\}) w_-, w_+ \rangle. \tag{4.1.2}$$

Proof. Recall that $b_\mu = a_\mu = p_\mu I_2$. We write the subprincipal term $b_{\mu-2}$ as

$$b_{\mu-2} = b'_{\mu-2} + b''_{\mu-2},$$

where

$$b'_{\mu-2} \coloneqq e^*_{-2} e_0 b_\mu + b_\mu e_0^* e_{-2} + e_0^* a_{\mu-2} e_0 - \frac{i}{2} \left(e_0^* \{p_\mu, e_0\} + \{e_0^*, p_\mu e_0\} \right),$$

and

$$b''_{\mu-2} \coloneqq e^*_{-1} a_\mu e_{-1} + b_{\mu-1} e_0^* e_{-1} + e^*_{-1} e_0 b_{\mu-1}.$$

As for $b'_{\mu-2}$ we have, by Corollary 9.2.6 in [45] (used to deal with the terms coming from the "regular" step in the order of the symbols), that for $j = \pm$, respectively,

$$\begin{aligned} \langle b'_{\mu-2} w_j, w_j \rangle = & \langle e_0^* a_{\mu-2} e_0 w_j, w_j \rangle + \frac{1}{2} \mathrm{Im} \left(\langle \{e_0^*, p_\mu\} e_0 w_j, w_j \rangle \right) \\ & + \frac{1}{2} \mathrm{Im} \left(\langle e_0^* \{p_\mu, e_0\} w_j, w_j \rangle \right) - p_\mu \langle \alpha^*_{-1} \alpha_{-1} w_j, w_j \rangle \end{aligned}$$

(the last term in the above expression is due to the form of e_{-2}, see Remark 4.0.2).

As for $b''_{\mu-2}$, on the other hand, we have (since $e_{-1} = e_0 \alpha_{-1}$)

$$\left\langle e^*_{-1} a_\mu e_{-1} w_j, w_j \right\rangle = \left\langle \alpha^*_{-1} b_\mu \alpha_{-1} w_j, w_j \right\rangle = p_\mu \left\langle \alpha_{-1} w_j, \alpha_{-1} w_j \right\rangle = \left\langle p_\mu \alpha^*_{-1} \alpha_{-1} w_j, w_j \right\rangle,$$

and

$$\left\langle \left(b_{\mu-1} e_0^* e_{-1} + e^*_{-1} e_0 b_{\mu-1} \right) w_j, w_j \right\rangle = \left\langle \left(b_{\mu-1} \alpha_{-1} + \alpha^*_{-1} b_{\mu-1} \right) w_j, w_j \right\rangle = 0,$$

because $b_{\mu-1} \alpha_{-1} w_\pm = r w_\mp$ with $r \in C^\infty(\mathbb{R}^{2n}; \mathbb{C})$ (the same happens for

$\alpha^*_{-1}b_{\mu-1}$ for a different r) and $\langle w_\mp, w_\pm\rangle = 0$ ($\pm$-respectively). It follows that

$$\langle b''_{\mu-2}w_j, w_j\rangle = p_\mu\langle \alpha^*_{-1}\alpha_{-1}w_j, w_j\rangle, \quad j = \pm.$$

Therefore, adding the expressions for $\langle b'_{\mu-2}w_j, w_j\rangle$ and for $\langle b''_{\mu-2}w_j, w_j\rangle$, we obtain the formula for the subprincipal part $b_{\mu-2}$.

We finally prove (4.1.2). By (3.1.16) we have that $\delta = -\gamma/(\lambda^+_{\mu-1} - \lambda^-_{\mu-1})$. Therefore we have to compute γ by means of equation (3.1.14). Hence, recalling that $\alpha_{-1} = \begin{bmatrix} 0 & \delta \\ -\delta^* & 0 \end{bmatrix}$ and that $e_{-1} = e_0\alpha_{-1}$, we have

$$\begin{aligned}\gamma &= \Big\langle \Big(e_0^* a_{\mu-2}e_0 - \frac{i}{2}\left(e_0^*\left\{a_\mu, e_0\right\} + \left\{e_0^*, a_\mu e_0\right\}\right) - p_\mu\big(-\frac{i}{2}\{e_0^*, e_0\}\Big)w_-, w_+\Big\rangle \\ &= \langle (e_0^* a_{\mu-2}e_0 - ie_0^*\{p_\mu, e_0\})w_-, w_+\rangle,\end{aligned}$$

which gives (4.1.2) and concludes the proof.

□

We must now study (still remaining in the 2×2 case) the transformation properties of the subprincipal term depending on the choice of e_0. We have the following proposition.

Proposition 4.1.2. *Suppose that $a_\mu = a_\mu^* = p_\mu I_2$ is a scalar matrix and that $a_{\mu-1} = a^*_{\mu-1}$ possesses smooth eigenvalues $\lambda^+_{\mu-1}$ and $\lambda^-_{\mu-1}$ satisfying (4.1.1). Let e_0 and $\tilde e_0$ be smooth, unitary 2×2 matrices in $S(1, g; \mathsf{M}_2)$ such that*

$$e_0^* a_{\mu-1}e_0 = \tilde e_0^* a_{\mu-1}\tilde e_0 = b_{\mu-1} = \begin{bmatrix} \lambda^+_{\mu-1} & 0 \\ 0 & \lambda^-_{\mu-1}\end{bmatrix}.$$

Denote by $b_{\mu-2}$ and $\tilde b_{\mu-2}$, respectively, the subprincipal terms given in Corollary 4.1.1, associated respectively with e_0 and $\tilde e_0$. Let hence $f \in S(1, g; \mathsf{M}_2)$ be the unitary matrix such that $e_0 = \tilde e_0 f$, so that (since $\lambda^+_{\mu-1} \neq \lambda^-_{\mu-1}$)

$$f = \begin{bmatrix} f_+ & 0 \\ 0 & f_- \end{bmatrix}$$

with $f_j \in S(1, g)$ *and* $|f_j(X)| = 1$, *for all* $X \in \mathbb{R}^{2n}$, $j = \pm$. *Then, with* $\{w_+, w_-\}$ *the canonical basis of* $\mathbb{C}^2$ *as before,*

$$b^{(j)}_{\mu-2} = \langle b_{\mu-2} w_j, w_j \rangle = \left\langle \tilde{b}_{\mu-2} w_j, w_j \right\rangle + \operatorname{Im}\left(f_j \left\{\bar{f}_j, p_\mu\right\}\right), j = \pm. \tag{4.1.3}$$

Moreover, one has

$$\delta = \bar{f}_+ f_- \tilde{\delta} \tag{4.1.4}$$

where $\tilde{\delta}$ *is determined by equation (3.1.16) with* $\tilde{e}_0$ *in place of* e_0.

Proof. By Corollary 4.1.1 and by the proof of Proposition 9.2.7 in [45] we have (4.1.3). Hence, we only need to show that $\delta = \bar{f}_+ f_- \tilde{\delta}$.
On the one hand,

$$\begin{aligned} \langle e_0^* \{a_\mu, e_0\} w_-, w_+ \rangle &= \bar{f}_+ \langle \{a_\mu, \tilde{e}_0 f\} w_-, \tilde{e}_0 w_+ \rangle \\ &= \bar{f}_+ f_- \langle \{p_\mu, \tilde{e}_0\} w_-, \tilde{e}_0 w_+ \rangle + \bar{f}_+ \langle \{p_\mu, f_-\} \tilde{e}_0 w_-, \tilde{e}_0 w_+ \rangle \\ &= \bar{f}_+ f_- \langle \tilde{e}_0^* \{p_\mu, \tilde{e}_0\} w_-, w_+ \rangle + \bar{f}_+ \langle \{p_\mu, f_-\} \tilde{e}_0 w_-, \tilde{e}_0 w_+ \rangle . \end{aligned} \tag{4.1.5}$$

On the other hand, again from the proof of Proposition 9.2.7 in [45],

$$\bar{f}_+ \langle \{p_\mu, f_-\} \tilde{e}_0 w_-, \tilde{e}_0 w_+ \rangle = \bar{f}_+ \{p_\mu, f_-\} \langle \tilde{e}_0 w_-, \tilde{e}_0 w_+ \rangle = 0. \tag{4.1.6}$$

Now, by (4.1.5) and (4.1.6)

$$\begin{aligned} \delta &= -\frac{1}{\lambda^+_{\mu-1} - \lambda^-_{\mu-1}} \langle (e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\}) w_-, w_+ \rangle \\ &= -\frac{\bar{f}_+ f_-}{\lambda^+_{\mu-1} - \lambda^-_{\mu-1}} \langle (\tilde{e}_0^* a_{\mu-2} \tilde{e}_0 - i \tilde{e}_0^* \{p_\mu, \tilde{e}_0\}) w_-, w_+ \rangle \\ &= \bar{f}_+ f_- \tilde{\delta}, \end{aligned}$$

which concludes the proof. □

4.2 The case of blockwise matrices

We pass in this subsection to the study of the subprincipal symbol in the more general case of a diagonalization into 2 blocks, with $N > 2$.

Suppose now that:

(i) $a_\mu = a_\mu^* = p_\mu I_N > 0$ is a scalar matrix with $\mu > 0$, $p_\mu \in S(m^\mu, g)$;

(ii) $a_{\mu-1} = a_{\mu-1}^*$ is such that (as in Theorem 3.1.1) there exists a smooth unitary matrix $e_0 \in S(1, g; \mathsf{M}_N)$ such that

$$e_0(X)^* a_{\mu-1}(X) e_0(X) = \left[\begin{array}{c|c} \lambda_{\mu-1}^+(X) & 0 \\ \hline 0 & \lambda_{\mu-1}^-(X) \end{array}\right], \quad \forall X \in \mathbb{R}^{2n}, \tag{4.2.1}$$

where, writing $N = N_+ + N_-$, we have that $\pm$-respectively $\lambda_{\mu-1}^\pm \in S(m^{\mu-1}, g, \mathsf{M}_{N_\pm})$, with

$$\inf\left\{|\zeta_1 - \zeta_2|; \zeta_1 \in \operatorname{Spec}(\lambda_{\mu-1}^+), \zeta_2 \in \operatorname{Spec}(\lambda_{\mu-1}^-)\right\} \gtrsim m(X)^{\mu-1}, \forall X \in \mathbb{R}^{2n}. \tag{4.2.2}$$

We have the following corollary.

Corollary 4.2.1. *Suppose that a_μ and $a_{\mu-1}$ satisfy the conditions (i) and (ii) above. Consider, $\pm$-respectively, the orthogonal projectors $\pi_\pm \colon \mathbb{C}^N \longrightarrow \mathbb{C}^N = \mathbb{C}^{N_+} \oplus \mathbb{C}^{N_-}$ onto $\mathbb{C}^{N_+} \oplus \{0\}$ and $\{0\} \oplus \mathbb{C}^{N_-}$ respectively (that is, $\pi_+ = [I_{N_+} | 0_{N_-}]$ and $\pi_- = [0_{N_-} | I_{N_+}]$), so that for the semiprincipal symbol $b_{\mu-1}$ of the h^∞-diagonalization we have $\pi_\pm b_{\mu-1}(X) \pi_\pm^* = \lambda_{\mu-1}^\pm(X)$, $\pm$-respectively, for all $X \in \mathbb{R}^{2n}$. Then, for the subprincipal symbol $b_{\mu-2} = \left[\begin{array}{c|c} b_{\mu-2}^+ & 0 \\ \hline 0 & b_{\mu-2}^- \end{array}\right]$ we have, with $j = \pm$,*

$$
\begin{aligned}
b^{(j)}_{\mu-2} =& \pi_j b_{\mu-2}\pi_j^* \\
=& \pi_j e_0^* a_{\mu-2} e_0 \pi_j^* - \frac{i}{2}\pi_j\{e_0^*, a_\mu\}e_0\pi_j^* - \frac{i}{2}\pi_j e_0^*\{a_\mu, e_0\}\pi_j^* \\
=& \pi_j e_0^* a_{\mu-2} e_0 \pi_j^* - i\pi_j e_0^*\{p_\mu, e_0\}\pi_j^*.
\end{aligned}
$$

In addition, for δ *(see (3.1.16)) one has* $\lambda^+_{\mu-1}\delta - \delta\lambda^-_{\mu-1} = -\gamma$ *where*

$$
\gamma = \pi_+ \left(e_0^* a_{\mu-2} e_0 - i e_0^*\{p_\mu, e_0\}\right)\pi_-^*. \tag{4.2.3}
$$

Proof. By Proposition 4.0.1, the terms in $\pi_j b_{\mu-2}\pi_j^*$ for $j = \pm$ are (recall that $b_\mu = a_\mu = p_\mu I_N$)

(1st)

$$
\pi_j\left(e^*_{-2}e_0 b_\mu + b_\mu e_0^* e_{-2}\right)\pi_j^* = \frac{i}{4}\pi_j\left(b_\mu\{e_0^*, e_0\} - \{e_0^*, e_0\}^* b_\mu\right)\pi_j^*
$$

$$
+\pi_j\left(-\frac{1}{2}\alpha^*_{-1}\alpha_{-1} + \alpha^*_{-2}\right)b_\mu\pi_j^* + \pi_j b_\mu\left(-\frac{1}{2}\alpha^*_{-1}\alpha_{-1} + \alpha_{-2}\right)\pi_j^*
$$

(since $\{e_0^*, e_0\}^* = -\{e_0^*, e_0\}$ and $\alpha^*_{-2} = -\alpha_{-2}$)

$$
= \frac{i}{2}\pi_j\left(b_\mu\{e_0^*, e_0\}\right)\pi_j^* - b_\mu\pi_j\alpha^*_{-1}\alpha_{-1}\pi_j^*,
$$

(2nd) $$\pi_j\{e_0^*, a_\mu e_0\}\pi_j^* = \pi_j\{e_0^*, e_0\}p_\mu\pi_j^* + \pi_j\{e_0^*, p_\mu\}e_0\pi_j^*,$$

(3rd) $$\pi_j e^*_{-1} a_\mu e_{-1}\pi_j^* = p_\mu\pi_j\alpha^*_{-1}\alpha_{-1}\pi_j^*,$$

and finally

(4th) $$\pi_j\left(b_{\mu-1}e_0^* e_{-1} + e_{-1}e_0 b_{\mu-1}\right)\pi_j^* = \pi_j b_{\mu-1}\alpha_{-1}\pi_j^* + \pi_j\alpha^*_{-1}b_{\mu-1}\pi_j^* = 0,$$

since $b_{\mu-1}\alpha_{-1}$ is blockwise anti-diagonal. Summing the above terms gives the expression of $b^{(j)}_{\mu-2}$.

We next show (4.2.3). By (3.1.16) we have that $\lambda^+_{\mu-1}\delta - \delta\lambda^-_{\mu-1} = -\gamma$.

Therefore, we just need to compute γ by means of equation (3.1.14). Hence, recalling that $\alpha_{-1} = \left[\begin{array}{c|c} 0 & \delta \\ \hline -\delta^* & 0 \end{array}\right]$ and that $e_{-1} = e_0\alpha_{-1}$, we have

$$\begin{aligned}\gamma &= \pi_+ \left(e_0^* a_{\mu-2} e_0 - \frac{i}{2} \left(e_0^* \{a_\mu, e_0\} + \{e_0^*, a_\mu e_0\} \right) - a_\mu \big(-\frac{i}{2} \{e_0^*, e_0\} \big) \right) \pi_-^* \\ &= \pi_+ \big(e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\} \big) \pi_-^*,\end{aligned}$$

which gives (4.2.3) and concludes the proof.

□

As before, we must now study the transformation properties of the subprincipal terms depending on the choice of e_0 (still remaining in the $N \times N$ case, $N > 2$, with 2 blocks). We have the following proposition.

Proposition 4.2.2. *Suppose that a_μ and $a_{\mu-1}$ satisfy the above conditions (i) and (ii). Let e_0 and $\tilde{e}_0$ be smooth, unitary $N \times N$ matrices in $S(1, g; \mathsf{M}_N)$ such that*

$$e_0^* a_{\mu-1} e_0 = \tilde{e}_0^* a_{\mu-1} \tilde{e}_0 = b_{\mu-1} = \left[\begin{array}{c|c} \lambda_{\mu-1}^+ & 0 \\ \hline 0 & \lambda_{\mu-1}^- \end{array}\right],$$

with the blocks $\lambda_{\mu-1}^\pm$ satisfying (4.2.2). Denote by $b_{\mu-2}$ and $\tilde{b}_{\mu-2}$, respectively, the subprincipal terms given in Corollary 4.1.1, associated respectively with e_0 and $\tilde{e}_0$. Let hence $f \in S(1, g; \mathsf{M}_N)$ be the unitary matrix such that $e_0 = \tilde{e}_0 f$, so that (by the spacing property of the spectra of $\lambda_{\mu-1}^\pm$)

$$f = \left[\begin{array}{c|c} f_+ & 0 \\ \hline 0 & f_- \end{array}\right]$$

with the $f_\pm \in S(1, g; \mathsf{M}_{N_\pm})$ being themselves unitary matrices. As before, consider $\pi_\pm$ the projectors of $\mathbb{C}^N = \mathbb{C}^{N_+} \oplus \mathbb{C}^{N_-}$ respectively onto $\mathbb{C}^{N_+} \oplus \{0\}$ and $\{0\} \oplus \mathbb{C}^{N_-}$. Then, for $j = \pm$,

$$b_{\mu-2}^{(j)} = \pi_j b_{\mu-2} \pi_j^* = f_j^* \pi_j \tilde{b}_{\mu-2} \pi_j^* f_j - \frac{i}{2} \left(f_j^* \{p_\mu, f_j\} - \{p_\mu, f_j\}^* f_j \right) \qquad (4.2.4)$$

$$= f_j^* \pi_j \tilde{b}_{\mu-2} \pi_j^* f_j + \mathsf{Im}(f_j^* \{p_\mu, f_j\})$$

(where, for a matrix A, we put $2i\,\mathsf{Im}(A) = A - A^*$ for its skew-Hermitian part). Moreover,

$$\lambda_{\mu-1}^+ \tilde{\delta} - \tilde{\delta} \lambda_{\mu-1}^- = f_+ \left(\lambda_{\mu-1}^+ \delta - \delta \lambda_{\mu-1}^- \right) f_-^* \tag{4.2.5}$$

where $\tilde{\delta}$ is determined by equation (3.1.16) with $\tilde{e}_0$ in place of e_0.

Proof. We prove (4.2.4) by following the scheme of proof of Corollary 9.2.6 in [45]. One has

$$\pi_j e_0^* a_{\mu-2} e_0 \pi_j^* = \underbrace{\pi_j f^*}_{=f_j^* \pi_j} \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 \underbrace{f \pi_j^*}_{=\pi_j^* f_j} = f_j^* \pi_j \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 \pi_j^* f_j,$$

$$\begin{aligned}
\pi_j \{e_0^*, a_\mu\} e_0 \pi_j^* =& \pi_j \{f^* \tilde{e}_0^*, p_\mu\} \tilde{e}_0 f \pi_j^* \\
=& f_j^* \pi_j \{\tilde{e}_0^*, p_\mu\} \tilde{e}_0 \pi_j^* f_j + \underbrace{\{f_j^*, p_\mu\}}_{=-\{p_\mu, f_j\}^*} \underbrace{\pi_j \tilde{e}_0^* \tilde{e}_0 \pi_j^*}_{I_{N_j}} f_j \\
=& f_j^* \pi_j \{\tilde{e}_0^*, p_\mu\} \tilde{e}_0 \pi_j^* f_j - \{p_\mu, f_j\}^* f_j,
\end{aligned}$$

$$\begin{aligned}
\pi_j e_0^* \{a_\mu, e_0\} \pi_j^* =& \pi_j f^* \tilde{e}_0^* \{p_\mu, \tilde{e}_0 f\} \pi_j^* \\
=& f_j^* \pi_j \tilde{e}_0^* \{p_\mu, \tilde{e}_0\} \pi_j^* f_j + f_j^* \underbrace{\pi_j \tilde{e}_0^* \tilde{e}_0 \pi_j^*}_{I_{N_j}} \{p_\mu, f_j\} \\
=& f_j^* \pi_j \tilde{e}_0^* \{p_\mu, \tilde{e}_0\} \pi_j^* f_j + f_j^* \{p_\mu, f_j\}.
\end{aligned}$$

Hence, Corollary 4.2.1 implies (4.2.4).

We next prove (4.2.5). By (3.1.16) we have that $\lambda_{\mu-1}^+ \delta - \delta \lambda_{\mu-1}^- = -\gamma$ and by (4.2.3) that

$$\gamma = \pi_+ \left(e_0^* a_{\mu-2} e_0 - i e_0^* \{p_\mu, e_0\} \right) \pi_-^*.$$

We have therefore to study the transformation properties of γ. By (4.2.3) one has

$$\gamma = \pi_+\big(f^*\tilde{e}_0^* a_{\mu-2}\tilde{e}_0 f - if^*\tilde{e}_0^*\{p_\mu, \tilde{e}_0 f\}\big)\pi_-^*$$

$$= f_+^*\pi_+\big(\tilde{e}_0^* a_{\mu-2}\tilde{e}_0 - i\tilde{e}_0^*\{p_\mu, \tilde{e}_0\}\big)\pi_-^* f_- - if_+^* \underbrace{\pi_+\tilde{e}_0^*\tilde{e}_0\pi_-^*}_{=0}\{p_\mu, f_-\} = f_+^*\tilde{\gamma} f_-,$$

whence (4.2.5). This concludes the proof.

$\square$

Chapter 5

The semi-subprincipal symbol

In this chapter we will study the structure and the transformation laws of the semi-subprincipal symbol of a pseudodifferential system (that is, the term of homogeneity $\mu - 3$ of the asymptotics of the total symbol where μ is the order of the ψdo) under different diagonalizers for the semiprincipal part.

From the decoupling theorem we will obtain the following general form for the semi-subprincipal term of the diagonalized symbol.

Proposition 5.0.1. *For the semi-subprincipal part $b_{\mu-3}$ of the h^∞-diagonalization given in Theorem 3.1.1, recalling that $a_\mu = p_\mu I_2$, one has, by (3.1.22), the formula:*

$$\begin{aligned} b_{\mu-3} =& e^*_{-3}e_0b_\mu + b_\mu e^*_0e_{-3} + e^*_{-1}e_0b_\mu e^*_0e_{-2} + e^*_{-2}e_0b_\mu e^*_0e_{-1} + e^*_{-1}e_0b_{\mu-1}e^*_0e_{-1} \\ & + b_{\mu-1}e^*_0e_{-2} + e^*_{-2}e_0b_{\mu-1} + e^*_{-1}a_{\mu-2}e_0 + e^*_0a_{\mu-2}e_{-1} + e^*_0a_{\mu-3}e_0 \\ & - \frac{i}{2}\Big(e^*_0\{a_{\mu-1}, e_0\} + \{e^*_0, a_{\mu-1}e_0\} + e_{-1}\{a_\mu, e_0\} + \{e^*_{-1}, a_\mu e_0\} \\ & + e^*_0\{a_\mu, e_{-1}\} + \{e^*_0, a_\mu e_{-1}\}\Big), \end{aligned}$$

where

$$e_{-3} = \frac{i}{4}e_0\left(\{e^*_0, e_{-1}\} + \{e^*_{-1}, e_0\}\right) - \frac{1}{2}e_0\left(\alpha^*_{-1}\alpha_{-2} + \alpha^*_{-2}\alpha_{-1}\right) + e_0\alpha_{-3},$$

with $\alpha^*_{-2} = -\alpha_{-2}$ and $\alpha^*_{-1} = -\alpha_{-1}$ determined by equations (3.1.25) and

(3.1.26), and α_{-3} *skew-Hermitian.*

Remark 5.0.2. *It is important to note in* e_{-3} *the presence of the term*

$$e_0\left(\alpha_{-1}^*\alpha_{-2}+\alpha_{-2}^*\alpha_{-1}\right)/2,$$

which depends on the symbol of order $\mu-2$ *of our system.*

5.1 The case $N = 2$

Suppose hence that $N = 2$ and that the hypotheses of 4.1 are verified.

To study the structure of the semi-subprincipal term, for the sake of clarity we will be considering in the first place the case $N = 2$ and afterwards the case of a general N.
All of the results below hold also true in the *classical semiregular* case of Theorem 3.1.3 above.

Then, we have the following corollary.

Corollary 5.1.1. *Under the hypothesis and notations of Corollary 4.1.1, for the semi-subprincipal symbol* $b_{\mu-3}=\begin{bmatrix} b^+_{\mu-3} & 0 \\ 0 & b^-_{\mu-3}\end{bmatrix}$ *we have*

$$\begin{aligned}
b^{(j)}_{\mu-3} &= \langle b_{\mu-3}w_j, w_j\rangle \\
&= \langle e_0^* a_{\mu-3} e_0 w_j, w_j\rangle \\
&\quad + \frac{1}{2}\mathsf{Im}\left(\left\langle \{e_0^*, \lambda^{(j)}_{\mu-1}\}e_0 w_j, w_j\right\rangle\right) + \frac{1}{2}\mathsf{Im}\left(\langle e_0^*\{a_{\mu-1}, e_0\}w_j, w_j\rangle\right) \\
&\quad + \left(\lambda^{(-j)}_{\mu-1} - \lambda^{(j)}_{\mu-1}\right)|\delta|^2 - \frac{1}{2}\mathsf{Im}\left(p_\mu \bar{\hat{\delta}}_j \left\langle \{e_0^*, e_0\}\, w_j, w_{-j}\right\rangle\right) \\
&\quad + 2\mathsf{Re}\left(\bar{\hat{\delta}}_j \left\langle e_0^* a_{\mu-2} e_0 w_j, w_{-j}\right\rangle\right) + 2\mathsf{Im}\left(\bar{\hat{\delta}}_j \left\langle \{e_0^*, p_\mu\} e_0 w_j, w_{-j}\right\rangle\right), \quad j=\pm,
\end{aligned} \tag{5.1.1}$$

where $\hat{\delta}_j := \begin{cases} -\bar{\delta} & , j=+, \\ \delta & , j=-. \end{cases}$

Proof. Recall that $b_\mu = a_\mu = p_\mu I_2$. We write the semi-subprincipal term $b_{\mu-3}$ as

$$b_{\mu-3} = b'_{\mu-3} + b''_{\mu-3},$$

where

$$b'_{\mu-3} := g^*_{-2} e_0 b_{\mu-1} + b_{\mu-1} e^*_0 g_{-2} + e^*_0 a_{\mu-3} e_0 - \frac{i}{2}\left(e^*_0 \{a_{\mu-1}, e_0\} + \{e^*_0, a_{\mu-1} e_0\}\right),$$

and

$$\begin{aligned} b''_{\mu-3} :=& e^*_{-3} e_0 b_\mu + b_\mu e^*_0 e_{-3} + e^*_{-1} e_0 b_\mu e^*_0 e_{-2} + e^*_{-2} e_0 b_\mu e^*_0 e_{-1} \\ &+ \alpha^*_{-1} b_{\mu-1} \alpha_{-1} - \frac{1}{2} b_{\mu-1} \alpha^*_{-1} \alpha_{-1} - \frac{1}{2} \alpha_{-1} \alpha^*_{-1} b_{\mu-1} \\ &+ e^*_{-1} a_{\mu-2} e_0 + e^*_0 a_{\mu-2} e_{-1} \\ &- \frac{i}{2}\left(e^*_{-1}\{a_\mu, e_0\} + \{e^*_{-1}, a_\mu e_0\} + e^*_0\{a_\mu, e_{-1}\} + \{e^*_0, a_\mu e_{-1}\}\right), \end{aligned}$$

with

$$g_{-2} := \frac{i}{4} e_0 \{e^*_0, e_0\} + e_0 \alpha_{-2}.$$

Then, following the scheme of the proof of Corollary 9.2.6 in [45] (used to deal with the terms coming from the "regular" step in the order of the symbols) we have for $j = \pm$

$$\begin{aligned} \left\langle b'_{\mu-3} w_j, w_j \right\rangle =& \langle e^*_0 a_{\mu-3} e_0 w_j, w_j \rangle + \frac{1}{2} \mathsf{Im}\left(\left\langle \{e^*_0, \lambda^{(j)}_{\mu-1}\} e_0 w_j, w_j \right\rangle\right) \\ &+ \frac{1}{2} \mathsf{Im}\left(\langle e^*_0 \{a_{\mu-1}, e_0\} w_j, w_j \rangle\right). \end{aligned}$$

Now we compute $\left\langle b''_{\mu-3} w_j, w_j \right\rangle$.

First of all, we rewrite the terms in $\left\langle b''_{\mu-3} w_j, w_j \right\rangle$ for $j = \pm$ using $\hat{\delta}_j$. Recalling that $a_\mu = b_\mu = p_\mu I_2$, we have

(i)

$$\begin{aligned}\left\langle\left(b_\mu e_0^* e_{-3}+e_{-3}^* e_0 b_\mu\right) w_j, w_j\right\rangle = & 2\frac{i}{4}\left\langle p_\mu\{e_0^*, e_{-1}\} w_j, w_j\right\rangle \\ & + 2\frac{i}{4}\left\langle\{e_{-1}^*, e_0\} p_\mu w_j, w_j\right\rangle \\ & + 2\mathsf{Re}\left(-\frac{1}{2}\left\langle\left(\alpha_{-1}^*\alpha_{-2}+\alpha_{-2}^*\alpha_{-1}\right) p_\mu w_j, w_j\right\rangle\right) \\ & + 2\mathsf{Re}\left(\left\langle\underbrace{\left(\alpha_{-3}+\alpha_{-3}^*\right)}_{=0} p_\mu w_j, w_j\right\rangle\right),\end{aligned} \tag{5.1.2}$$

(ii)

$$\begin{aligned}\left\langle -\frac{i}{2} e_{-1}^*\{a_\mu, e_0\} w_j, w_j\right\rangle & = -\frac{i}{2}\left\langle\{p_\mu, e_0\} w_j, e_{-1} w_j\right\rangle \\ & = -\frac{i}{2}\left\langle\{p_\mu, e_0\} w_j, \hat{\delta}_j e_0 w_{-j}\right\rangle \\ & = -\frac{i}{2}\bar{\hat{\delta}}_j \underbrace{\left\langle e_0^*\{p_\mu, e_0\} w_j, w_{-j}\right\rangle}_{=\left\langle\{e_0^*, p_\mu\} e_0 w_j, w_{-j}\right\rangle},\end{aligned} \tag{5.1.3}$$

(iii)

$$\begin{aligned}\left\langle\left(e_{-1}^* e_0 b_\mu e_0^* e_{-2}+e_{-2}^* e_0 b_\mu e_0^* e_{-1}\right) w_j, w_j\right\rangle = & \underbrace{\left\langle\left(\alpha_{-1}^*\alpha_{-2}+\alpha_{-2}^*\alpha_{-1}\right) p_\mu w_j, w_j\right\rangle}_{=-2\mathrm{Re}\left(-\frac{1}{2}\left\langle\left(\alpha_{-1}^*\alpha_{-2}+\alpha_{-2}^*\alpha_{-1}\right) b_\mu w_j, w_j\right\rangle\right)} \\ & + 2\mathsf{Re}\left\langle\left(\frac{i}{4}\alpha_{-1}^*\left\{e_0^*, e_0\right\}-\frac{1}{2}\alpha_{-1}^*\alpha_{-1}^*\alpha_{-1}\right) p_\mu w_j, w_j\right\rangle \\ = & -2\mathsf{Re}\left(-\frac{1}{2}\left\langle\left(\alpha_{-1}^*\alpha_{-2}+\alpha_{-2}^*\alpha_{-1}\right) p_\mu w_j, w_j\right\rangle\right) \\ & -\frac{1}{2}\mathsf{Im}\underbrace{\left\langle\alpha_{-1}^*\left\{e_0^*, e_0\right\} p_\mu w_j, w_j\right\rangle}_{=p_\mu\bar{\hat{\delta}}_j\left\langle\left\{e_0^*, e_0\right\} w_j, w_{-j}\right\rangle} \\ & -\mathsf{Re}\left(-p_\mu|\delta|\bar{\hat{\delta}}_j\underbrace{\left\langle w_j, w_{-j}\right\rangle}_{=0}\right),\end{aligned} \tag{5.1.4}$$

(iv)

$$\left\langle \left(-\frac{1}{2} b_{\mu-1}\alpha_{-1}^*\alpha_{-1} - \frac{1}{2}\alpha_{-1}^*\alpha_{-1}b_{\mu-1} + \alpha_{-1}^*b_{\mu-1}\alpha_{-1} \right) w_j, w_j \right\rangle = \left(\lambda_{\mu-1}^{(-j)} - \lambda_{\mu-1}^{(j)} \right) |\delta|^2, \tag{5.1.5}$$

(v)

$$\begin{aligned}
\left\langle -\frac{i}{2}\{e_{-1}^*, a_\mu e_0\} w_j, w_j \right\rangle &= -\frac{i}{2} \left\langle \{e_{-1}^*, e_0\} p_\mu w_j, w_j \right\rangle + -\frac{i}{2} \left\langle \{e_{-1}^*, p_\mu\} e_0 w_j, w_j \right\rangle \\
&= -\frac{i}{2} \left\langle \{e_{-1}^*, e_0\} p_\mu w_j, w_j \right\rangle \\
&\quad -\frac{i}{2} \left\langle \left(\alpha_{-1}^* \{e_0^*, p_\mu\} e_0 + \{\alpha_{-1}^*, p_\mu\} \right) w_j, w_j \right\rangle \\
&= \underbrace{-\frac{i}{2} \left\langle \{e_{-1}^*, e_0\} p_\mu w_j, w_j \right\rangle}_{=-2\left(\frac{i}{4}\left\langle \{e_{-1}^*, e_0\} p_\mu w_j, w_j \right\rangle\right)} - \frac{i}{2} \Big(\bar{\hat{\delta}}_j \left\langle \{e_0^*, p_\mu\} e_0 w_j, w_{-j} \right\rangle \\
&\quad + \{\bar{\hat{\delta}}_j, p_\mu\} \underbrace{\langle w_j, w_{-j} \rangle}_{=0} \Big),
\end{aligned} \tag{5.1.6}$$

(vi)

$$\left\langle (e_{-1}^* a_{\mu-2} e_0 + e_0^* a_{\mu-2} e_{-1}) w_j, w_j \right\rangle = 2\mathrm{Re} \left(\hat{\delta}_j \left\langle e_0^* a_{\mu-2} e_0 w_{-j}, w_j \right\rangle \right), \tag{5.1.7}$$

(vii)

$$\begin{aligned}
\left\langle -\frac{i}{2} e_0^* \{a_\mu, e_{-1}\} w_j, w_j \right\rangle &= -\frac{i}{2} \left\langle \left(e_0^* \{p_\mu, e_0\} \hat{\delta}_j + e_0^* \{p_\mu, \hat{\delta}_j\} e_0 \right) w_{-j}, w_j \right\rangle \\
&= -\frac{i}{2} \left\langle \left(e_0^* \{p_\mu, e_0\} \hat{\delta}_j + e_0^* \{p_\mu, \hat{\delta}_j\} e_0 \right) w_{-j}, w_j \right\rangle \\
&= -\frac{i}{2} \hat{\delta}_j \left\langle e_0^* \{p_\mu, e_0\} w_{-j}, w_j \right\rangle \\
&\quad - \frac{i}{2} \underbrace{\left\langle e_0^* \{p_\mu, \hat{\delta}_j\} e_0 w_{-j}, w_j \right\rangle}_{=\{p_\mu, \hat{\delta}_j\} \left\langle e_0^* e_0 w_{-j}, w_j \right\rangle = 0},
\end{aligned} \tag{5.1.8}$$

(viii)

$$\left\langle -\frac{i}{2}\{e_0^*, a_\mu e_{-1}\}w_j, w_j \right\rangle = -\frac{i}{2}\left\langle \{e_0^*, p_\mu e_0 \alpha_{-1}\}w_j, w_j\right\rangle$$
$$= -\frac{i}{2}\hat{\delta}_j \underbrace{\left\langle \{e_0^*, p_\mu\}e_0 w_{-j}, w_j\right\rangle}_{=\left\langle e_0^*\{p_\mu, e_0\}w_{-j}, w_j\right\rangle}$$
$$-\frac{i}{2}\left\langle p_\mu \{e_0^*, e_{-1}\}w_j, w_j\right\rangle. \tag{5.1.9}$$

We then sum all the above terms and get

$$\begin{aligned} b^{(j)}_{\mu-3} &= \langle b_{\mu-3}w_j, w_j\rangle \\ &= \langle e_0^* a_{\mu-3}e_0 w_j, w_j\rangle + \frac{1}{2}\mathsf{Im}\left(\left\langle\{e_0^*, \lambda^j_{\mu-1}\}e_0 w_j, w_j\right\rangle\right) \\ &\quad + \frac{1}{2}\mathsf{Im}\left(\langle e_0^*\{a_{\mu-1}, e_0\}w_j, w_j\rangle\right) + 2\mathsf{Re}\left(\hat{\delta}_j\left\langle e_0^* a_{\mu-2}e_0 w_{-j}, w_j\right\rangle\right) \\ &\quad + \mathsf{Re}\left(\left(\lambda^{(-j)}_{\mu-1} - \lambda^{(j)}_{\mu-1}\right)|\delta|^2\right) - \frac{1}{2}\mathsf{Im}\left(p_\mu \bar{\hat{\delta}}_j \left\langle\{e_0^*, e_0\}\, w_j, w_{-j}\right\rangle\right) \\ &\quad - i\bar{\hat{\delta}}_j\left\langle\{e_0^*, p_\mu\}e_0 w_j, w_{-j}\right\rangle - i\hat{\delta}_j\left\langle e_0^*\{p_\mu, e_0\}w_{-j}, w_j\right\rangle,\ j = \pm. \end{aligned}$$

Finally, by unitarity of e_0,

$$\begin{aligned} \overline{-i\hat{\delta}_j\left\langle e_0^*\{p_\mu, e_0\}w_{-j}, w_j\right\rangle} &= i\bar{\hat{\delta}}_j\overline{\langle w_{-j}, -\{e_0^*, p_\mu\}e_0 w_j\rangle} \\ &= -i\bar{\hat{\delta}}_j\left\langle\{e_0^*, p_\mu\}e_0 w_j, w_{-j}\right\rangle \end{aligned}$$

means that

$$-i\bar{\hat{\delta}}_j\left\langle\{e_0^*, p_\mu\}e_0 w_j, w_{-j}\right\rangle - i\hat{\delta}_j\left\langle e_0^*\{p_\mu, e_0\}w_{-j}, w_j\right\rangle = 2\mathsf{Im}\left(\bar{\hat{\delta}}_j\left\langle\{e_0^*, p_\mu\}e_0 w_j, w_{-j}\right\rangle\right)$$

which completes the proof of the statement.

□

We must now study the transformation properties of the semi-subprincipal terms depending on the choice of e_0. More precisely, we have the following proposition.

Proposition 5.1.2. *Suppose that* $a_\mu = a_\mu^* = p_\mu I_2 > 0$ *is a positive scalar* *matrix and that $a_{\mu-1} = a^*_{\mu-1}$ possesses smooth eigenvalues $\lambda^{\pm}_{\mu-1}$ satisfying (4.1.1). Let e_0 and $\tilde{e}_0$ be smooth, unitary 2×2 matrices in $S(1, g; \mathsf{M}_2)$ such that*

$$e_0^* a_{\mu-1} e_0 = \tilde{e}_0^* a_{\mu-1} \tilde{e}_0 = b_{\mu-1} = \begin{bmatrix} \lambda^+_{\mu-1} & 0 \\ 0 & \lambda^-_{\mu-1} \end{bmatrix}.$$

Denote by $b_{\mu-3}$ and $\tilde{b}_{\mu-3}$ the semi-subprincipal terms given in Corollary 4.1.1, associated, respectively, with e_0 and $\tilde{e}_0$. Let hence $f \in S(1, g; \mathsf{M}_2)$ be the unitary matrix

$$f = \begin{bmatrix} f_+ & 0 \\ 0 & f_+ \end{bmatrix}, \text{ such that } f^* f = f f^* = I_2 \text{ and } e_0 = \tilde{e}_0 f,$$

so that the $f_j \in C^\infty(\mathbb{R}^{2n}; \mathbb{C})$ belong to $S(1, g)$ and $|f_j(X)| = 1$, for all $X \in \mathbb{R}^{2n}$, $j = \pm$. Then, with $\{w_+, w_-\}$ the canonical basis of $\mathbb{C}^2$ as before,

$$\begin{aligned} b^{(j)}_{\mu-3} &= \langle b_{\mu-3} w_j, w_j \rangle \\ &= \left\langle \tilde{b}_{\mu-3} w_j, w_j \right\rangle + \mathsf{Im}\left(f_j \left\{ \bar{f}_j, \lambda^{(j)}_{\mu-1} \right\} \right) \\ &\quad + \frac{1}{2} \mathsf{Im}\left(p_\mu \bar{\tilde{\bar{\delta}}}_j \bar{f}_j \left\langle \{ \tilde{e}_0^*, f_j \} \tilde{e}_0 w_j, w_{-j} \right\rangle \right) \\ &\quad + \frac{1}{2} \mathsf{Im}\left(p_\mu f_{-j} \bar{\tilde{\bar{\delta}}}_j \left\langle \tilde{e}_0^* \left\{ \bar{f}_{-j}, \tilde{e}_0 \right\} w_j, w_{-j} \right\rangle \right), \quad j = \pm. \end{aligned}$$

Proof. Recall that $b_\mu = a_\mu = p_\mu I_2$. In Corollary 5.1.1 we already wrote the semi-subprincipal term $b_{\mu-3}$ as

$$b_{\mu-3} = b'_{\mu-3} + b''_{\mu-3}.$$

Then, following the scheme of the proof of Proposition 9.2.7 in [45] (p. 137) we have for $j = \pm$

$$\left\langle b'_{\mu-3} w_j, w_j \right\rangle = \left\langle \tilde{b}'_{\mu-3} w_j, w_j \right\rangle + \mathrm{Im}\left(f_j \left\{ \bar{f}_j, \lambda^{(j)}_{\mu-1} \right\} \right),$$

where $\tilde{b}'_{\mu-3}$ is defined as $b'_{\mu-3}$ with $\tilde{e}_0$ in place of e_0. Moreover, by (4.1.4) (which means $\hat{\delta}_j = \bar{f}_{-j}\tilde{\hat{\delta}}_j f_j$)

$$\begin{aligned}
\mathrm{Re}\left(\bar{\hat{\delta}}_j \left\langle e_0^* a_{\mu-2} e_0 w_j, w_{-j}\right\rangle\right) =&\mathsf{Re}\left(\overline{\hat{\delta}_j \left\langle e_0^* a_{\mu-2} e_0 w_{-j}, w_j\right\rangle}\right)\\
=&\mathsf{Re}\left(\hat{\delta}_j \left\langle e_0^* a_{\mu-2} e_0 w_{-j}, w_j\right\rangle\right)\\
=&\mathsf{Re}\left(\bar{f}_{-j}\tilde{\hat{\delta}}_j f_j f_{-j}\bar{f}_j \left\langle \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 w_{-j}, w_j\right\rangle\right)\\
=&\mathsf{Re}\left(\tilde{\hat{\delta}}_j \left\langle \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 w_{-j}, w_j\right\rangle\right)\\
=&\mathsf{Re}\left(\bar{\tilde{\hat{\delta}}}_j \left\langle \tilde{e}_0^* a_{\mu-2} \tilde{e}_0 w_j, w_{-j}\right\rangle\right),
\end{aligned}$$

where $\tilde{\hat{\delta}}_j := \begin{cases} -\bar{\tilde{\delta}} & , j=+, \\ \tilde{\delta} & , j=-, \end{cases}$ and $\left(\lambda_{\mu-1}^{(-j)} - \lambda_{\mu-1}^{(j)}\right)|\delta|^2 = \left(\lambda_{\mu-1}^{(-j)} - \lambda_{\mu-1}^{(j)}\right)|\tilde{\delta}|^2$, since $|f_\pm| = 1$.

Moreover,

$$\begin{aligned}
\mathsf{Im}\left(p_\mu \bar{\hat{\delta}}_j \left\langle \left\{e_0^*, e_0\right\} w_j, w_{-j}\right\rangle\right) =&\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \bar{f}_j \left\langle \left\{f^* e_0^*, e_0 f\right\} w_j, w_{-j}\right\rangle\right)\\
=&\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \bar{f}_j \left\langle \left\{f^* \tilde{e}_0^*, \tilde{e}_0 f\right\} w_j, w_{-j}\right\rangle\right)\\
=&\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \bar{f}_j \left\{\bar{f}_j, f_j\right\} \underbrace{\left\langle w_j, w_{-j}\right\rangle}_{=0}\right)\\
&+\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \bar{f}_j \bar{f}_{-j} f_j \left\langle \left\{\tilde{e}_0^*, \tilde{e}_0\right\} w_j, w_{-j}\right\rangle\right)\\
&+\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \bar{f}_j \bar{f}_{-j} \left\langle \left\{\tilde{e}_0^*, f_j\right\} \tilde{e}_0 w_j, w_{-j}\right\rangle\right)\\
&+\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \bar{f}_j f_j \left\langle \tilde{e}_0^* \left\{\bar{f}_{-j}, \tilde{e}_0\right\} w_j, w_{-j}\right\rangle\right)\\
=&\mathsf{Im}\left(p_\mu \bar{\tilde{\hat{\delta}}}_j \left\langle \left\{\tilde{e}_0^*, \tilde{e}_0\right\} w_j, w_{-j}\right\rangle\right)\\
&+\mathsf{Im}\left(p_\mu \bar{\tilde{\hat{\delta}}}_j \bar{f}_j \left\langle \left\{\tilde{e}_0^*, f_j\right\} \tilde{e}_0 w_j, w_{-j}\right\rangle\right)\\
&+\mathsf{Im}\left(p_\mu f_{-j}\bar{\tilde{\hat{\delta}}}_j \left\langle \tilde{e}_0^* \left\{\bar{f}_{-j}, \tilde{e}_0\right\} w_j, w_{-j}\right\rangle\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{Im}\left(\bar{\hat{\delta}}_j\left\langle\{e_0^*,p_\mu\}e_0w_j,w_{-j}\right\rangle\right)=&\mathsf{Im}\left(f_{-j}\bar{\hat{\bar{\delta}}}_j\bar{f}_j\left\langle\{f^*\tilde{e}_0^*,p_\mu\}\tilde{e}_0fw_j,w_{-j}\right\rangle\right)\\
=&\mathsf{Im}\left(f_{-j}\bar{\hat{\bar{\delta}}}_j\bar{f}_j\bar{f}_{-j}f_j\left\langle\{\tilde{e}_0^*,p_\mu\}\tilde{e}_0w_j,w_{-j}\right\rangle\right)\\
&+\mathsf{Im}\left(f_{-j}\bar{\hat{\bar{\delta}}}_j\bar{f}_jf_j\{\bar{f}_{-j},p_\mu\}\underbrace{\left\langle\tilde{e}_0^*\tilde{e}_0w_j,w_{-j}\right\rangle}_{=0}\right)\\
=&\mathsf{Im}\left(\bar{\hat{\bar{\delta}}}_j\left\langle\{\tilde{e}_0^*,p_\mu\}\tilde{e}_0w_j,w_{-j}\right\rangle\right),
\end{aligned}$$

whence the result.

□

5.2 The case of blockwise matrices

In this section we move to the study of the semi-subprincipal symbol in the more general case of a diagonalization into 2 blocks, with $N > 2$.

With the same hypotheses on a_μ and $a_{\mu-1}$ which we have in Section 4.2, we get the following corollary.

Corollary 5.2.1. *Under the hypotheses of Corollary 4.2.1, for the semi-subprincipal symbol* $b_{\mu-3}=\begin{bmatrix} b^+_{\mu-3} & 0\\ 0 & b^-_{\mu-3}\end{bmatrix}$ *we have*

$$\begin{aligned}
b^{(j)}_{\mu-3}=&\pi_jb_{\mu-3}\pi_j^*\\
=&\pi_je_0^*a_{\mu-3}e_0\pi_j^*+\frac{i}{4}\pi_j\left(b_{\mu-1}\{e_0^*,e_0\}-\{e_0^*,e_0\}b_{\mu-1}\right)\pi_j^*\\
&+\frac{i}{2}\pi_je_0^*\{e_0,b_{\mu-1}\}\pi_j^*-\frac{i}{2}\pi_je_0^*\{a_{\mu-1},e_0\}\pi_j^*\\
&-\frac{1}{2}\left(b^{(j)}_{\mu-1}\hat{\delta}_j^*\hat{\delta}_j+\hat{\delta}_j^*\hat{\delta}_jb^{(j)}_{\mu-1}-2\hat{\delta}_j^*b^{(-j)}_{\mu-1}\hat{\delta}_j\right)\\
&-\frac{1}{2}\mathsf{Im}\left(p_\mu\hat{\delta}_j\pi_{-j}\{e_0^*,e_0\}\pi_j^*\right)+2\mathsf{Re}\left(\hat{\delta}_j\pi_{-j}e_0^*a_{\mu-2}e_0\pi_j^*\right)\\
&+2\mathsf{Im}\left(\hat{\delta}_j^*\pi_{-j}e_0^*\{p_\mu,e_0\}\pi_j^*\right),\quad j=\pm, \qquad (5.2.1)
\end{aligned}$$

where, for a matrix A, we put $2\mathsf{Re}(A) = A + A^*$ for its Hermitian part, $2\mathsf{Im}(A) = A - A^*$ for its skew-Hermitian part, and set $\hat{\delta}_j := \begin{cases} -\delta^* & , j = +, \\ \delta & , j = -. \end{cases}$

Proof. Recall that $b_\mu = a_\mu = p_\mu I_N$. First of all, we rewrite the terms in $\pi_j^* b_{\mu-3}\pi_j$ for $j = \pm$ using $\hat{\delta}_j$. When adding all the terms, those underbraced by the bullets will cancel respectively out. We have, in the first place,

$$\begin{aligned}\pi_j^*\left(b_{\mu-1}e_0^*e_{-2} + e_{-2}^*e_0b_{\mu-1}\right)\pi_j =& \frac{i}{4}\pi_j\left(b_{\mu-1}\{e_0^*, e_0\} + \{e_0^*, e_0\}b_{\mu-1}\right)\pi_j^* \\ &- \frac{1}{2}\underbrace{\pi_j b_{\mu-1}\alpha_{-1}^*}_{=b_{\mu-1}^{(j)}\hat{\delta}_j^*\pi_{-j}}\underbrace{\alpha_{-1}\pi_j^*}_{=\pi_{-j}^*\hat{\delta}_j} - \frac{1}{2}\underbrace{\pi_j\alpha_{-1}^*}_{=\hat{\delta}_j^*\pi_{-j}}\underbrace{\alpha_{-1}b_{\mu-1}\pi_j^*}_{=\pi_{-j}^*\hat{\delta}_j b_{\mu-1}^{(j)}} \\ &+ \underbrace{\pi_j b_{\mu-1}\alpha_{-2}\pi_j^*}_{=M'\underbrace{\pi_j\pi_{-j}^*}_{=0}} + \underbrace{\pi_j\alpha_{-2}^*b_{\mu-1}\pi_j^*}_{=M''\underbrace{\pi_j\pi_{-j}^*}_{=0}},\end{aligned} \tag{5.2.2}$$

where $M', M'' \in C^\infty(\mathbb{R}^{2n}; M_{N_j})$. Next, we have

(i)

$$\begin{aligned}\pi_j\left(e_{-1}^*a_{\mu-2}e_0 + e_0^*a_{\mu-2}e_{-1}\right)\pi_j^* =& \pi_j\left(\alpha_{-1}^*e_0^*a_{\mu-2}e_0 + e_0^*a_{\mu-2}e_0\alpha_{-1}\right)\pi_j^* \\ =& \hat{\delta}_j^*\pi_{-j}e_0^*a_{\mu-2}e_0\pi_j^* + \pi_j e_0^*a_{\mu-2}e_0\pi_{-j}^*\hat{\delta}_j,\end{aligned} \tag{5.2.3}$$

(ii)

$$\begin{aligned}-\frac{i}{2}\pi_j e_0^*\left(\{a_{\mu-1}, e_0\} + \{e_0^*, a_{\mu-1}e_0\}\right)\pi_j^* \underbrace{=}_{e_0^*\partial e_0 = -(\partial e_0^*)e_0} & - \frac{i}{2}\pi_j e_0^*\{a_{\mu-1}, e_0\}\pi_j^* \\ & - \frac{i}{2}\pi_j\{e_0^*, e_0\}b_{\mu-1}\pi_j^* \\ & + \frac{i}{2}\pi_j e_0^*\{e_0, b_{\mu-1}\}\pi_j^*,\end{aligned} \tag{5.2.4}$$

(iii)

$$\begin{aligned}\pi_j\left(b_\mu e_0^* e_{-3} + e_{-3}^* e_0 b_\mu\right)\pi_j^* =& \underbrace{\frac{i}{2}\pi_j\left(b_\mu\{e_{-1}^*, e_0\}\right)\pi_j^*}_{(\bullet)} \\ &+ \underbrace{\frac{i}{2}\pi_j\{e_0^*, e_0\}\alpha_{-1}p_\mu\pi_j^*}_{(\bullet\bullet)} - \underbrace{\frac{i}{2}\pi_j e_0^*\{e_0, \alpha_{-1}\}p_\mu\pi_j^*}_{(\bullet\bullet\bullet\bullet)} \\ &- \underbrace{\pi_j\left(\alpha_{-1}^*\alpha_{-2} + \alpha_{-2}^*\alpha_{-1}\right)p_\mu\pi_j^*}_{(\bullet\bullet\bullet)} \\ &+ \pi_j\underbrace{\left(\alpha_{-3}^* + \alpha_{-3}\right)}_{=0}p_\mu\pi_j^*, \end{aligned} \tag{5.2.5}$$

(iv)

$$\begin{aligned}-\frac{i}{2}\pi_j\left(e_{-1}^*\{a_\mu, e_0\}\right)\pi_j^* &= -\frac{i}{2}\pi_j e_{-1}^*\{p_\mu, e_0\}\pi_j^* \\ &= -\frac{i}{2}\hat{\delta}_j^*\pi_{-j}e_0^*\{p_\mu, e_0\}\pi_j^*, \end{aligned} \tag{5.2.6}$$

(v)

$$\pi_j\left(\alpha_{-1}^* b_{\mu-1}\alpha_{-1}\right)\pi_j^* = \hat{\delta}_j^* b_{\mu-1}^{(-j)}\hat{\delta}_j, \tag{5.2.7}$$

(vi)

$$\begin{aligned}\pi_j\left(e_{-1}^* e_0 b_\mu e_0^* e_{-2} + e_{-2}^* e_0 b_\mu e_0^* e_{-1}\right)\pi_j^* =& \pi_j\left(\alpha_{-1}^*\alpha_{-2} + \alpha_{-2}^*\alpha_{-1}\right)p_\mu\pi_j^* \\ &+ \pi_j\left(\frac{i}{4}\alpha_{-1}^*\{e_0^*, e_0\} - \frac{1}{2}\alpha_{-1}^*\alpha_{-1}^*\alpha_{-1}\right)p_\mu\pi_j^* \\ &+ \pi_j\left(\frac{i}{4}\{e_0^*, e_0\}\alpha_{-1} - \frac{1}{2}\alpha_{-1}^*\alpha_{-1}\alpha_{-1}\right)p_\mu\pi_j^* \\ =& \underbrace{\pi_j\left(\alpha_{-1}^*\alpha_{-2} + \alpha_{-2}^*\alpha_{-1}\right)p_\mu\pi_j^*}_{(\bullet\bullet\bullet)} \\ &+ \frac{i}{4}p_\mu\hat{\delta}_j^*\pi_{-j}\{e_0^*, e_0\}\pi_j^* + \frac{i}{4}p_\mu\pi_j\{e_0^*, e_0\}\pi_{-j}^*\hat{\delta}_j, \end{aligned} \tag{5.2.8}$$

(vii)

$$
\begin{aligned}
&-\frac{i}{2}\pi_j\{e^*_{-1}, a_\mu e_0\}\pi^*_j = -\frac{i}{2}\pi_j\{e^*_{-1}, e_0\}p_\mu\pi^*_j - \frac{i}{2}\pi_j\{e^*_{-1}, p_\mu\}e_0\pi^*_j \\
&= -\frac{i}{2}\pi_j\{e^*_{-1}, e_0\}p_\mu\pi^*_j - \frac{i}{2}\pi_j\alpha^*_{-1}\{e^*_0, p_\mu\}e_0\pi^*_j - \frac{i}{2}\pi_j\{\alpha^*_{-1}, p_\mu\}\pi^*_j \\
&= \underbrace{-\frac{i}{2}p_\mu\pi_j\{e^*_{-1}, e_0\}\pi^*_j}_{(\bullet)} - \frac{i}{2}\hat{\delta}^*_j\pi_{-j}\{e^*_0, p_\mu\}e_0\pi^*_j + \{\hat{\delta}^*_j, p_\mu\}\underbrace{\pi_{-j}\pi^*_j}_{=0},
\end{aligned}
\tag{5.2.9}
$$

(viii)

$$
\begin{aligned}
-\frac{i}{2}\pi_j e^*_0\{a_\mu, e_{-1}\}\pi^*_j &= -\frac{i}{2}\pi_j e^*_0\{p_\mu, e_0\}\pi^*_{-j}\hat{\delta}_j + \pi_j \underbrace{e^*_0 e_0}_{=I_N} \pi^*_{-j}\{p_\mu, \hat{\delta}_j\} \\
&= -\frac{i}{2}\pi_j e^*_0\{p_\mu, e_0\}\pi^*_{-j}\hat{\delta}_j + \underbrace{\pi_j\pi^*_{-j}}_{=0}\{p_\mu, \hat{\delta}_j\},
\end{aligned}
\tag{5.2.10}
$$

(ix)

$$
\begin{aligned}
-\frac{i}{2}\pi_j\{e^*_0, a_\mu e_{-1}\}\pi^*_j &= -\frac{i}{2}\pi_j\{e^*_0, e_0 p_\mu \alpha_{-1}\}\pi^*_j \\
&= \underbrace{-\frac{i}{2}\pi_j\{e^*_0, e_0\}p_\mu\alpha_{-1}\pi^*_j,}_{(\bullet\bullet)} \\
&\quad -\frac{i}{2}\pi_j\{e^*_0, p_\mu\}e_0\pi^*_{-j}\hat{\delta}_j \\
&\quad \underbrace{+\frac{i}{2}\pi_j p_\mu e^*_0\{e_0, \alpha_{-1}\}\pi^*_j}_{(\bullet\bullet\bullet\bullet)}.
\end{aligned}
\tag{5.2.11}
$$

Then, by summing all the terms and simplifying (we used the dots to point at the terms that semplify one another in the sum), we find

$$\begin{aligned}
b_{\mu-3}^{(j)} =&\pi_j b_{\mu-3}\pi_j^* \\
=&\pi_j e_0^* a_{\mu-3} e_0 \pi_j^* + \frac{i}{4}\pi_j \left(b_{\mu-1}\{e_0^*, e_0\} + \{e_0^*, e_0\} b_{\mu-1}\right)\pi_j^* \\
&- \frac{i}{2}\pi_j \{e_0^*, e_0\} b_{\mu-1}\pi_j^* + \frac{i}{2}\pi_j e_0^* \{e_0, b_{\mu-1}\}\pi_j^* - \frac{i}{2}\pi_j e_0^* \{a_{\mu-1}, e_0\}\pi_j^* \\
&- \frac{1}{2}\left(b_{\mu-1}^{(j)}\hat{\delta}_{-j}\hat{\delta}_j + \hat{\delta}_j^*\hat{\delta}_{-j}^* b_{\mu-1}^{(j)} + \hat{\delta}_j^* b_{\mu-1}^{(-j)}\hat{\delta}_j\right) \\
&+ \frac{i}{4}p_\mu \hat{\delta}_j^* \pi_{-j}\{e_0^*, e_0\}\pi_j^* + \frac{i}{4}p_\mu \pi_j \{e_0^*, e_0\}\pi_{-j}^*\hat{\delta}_j \\
&- \frac{i}{2}\hat{\delta}_j^* \pi_{-j} e_0^* \{p_\mu, e_0\}\pi_j^* - \frac{i}{2}\pi_j e_0^* \{p_\mu, e_0\}\pi_{-j}^*\hat{\delta}_j \\
&+ \hat{\delta}_j^* \pi_{-j} e_0^* a_{\mu-2} e_0 \pi_j^* + \pi_j e_0^* a_{\mu-2} e_0 \pi_{-j}^* \hat{\delta}_j \\
&- \frac{i}{2}\hat{\delta}_j^* \pi_{-j}\{e_0^*, p_\mu\} e_0 \pi_j^* - \frac{i}{2}\pi_j \{e_0^*, p_\mu\} e_0 \pi_{-j}^* \hat{\delta}_j, \;\; j = \pm.
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
-\frac{i}{2}\hat{\delta}_j^* \pi_{-j} e_0^* \{p_\mu, e_0\}\pi_j^* &= -\frac{i}{2}\left(\{p_\mu, e_0\}^* e_0 \pi_{-j}^* \hat{\delta}_j\right)^* \pi_j^* \\
&= -\frac{i}{2}\left(-\underbrace{\{e_0^*, p_\mu\} e_0}_{=e_0^*\{p_\mu, e_0\}} \pi_{-j}^* \hat{\delta}_j\right)^* \pi_j^* \\
&= \left(-\frac{i}{2}\pi_j e_0^* \{p_\mu, e_0\}\pi_{-j}^* \hat{\delta}_j\right)^*,
\end{aligned}$$

$$-\frac{i}{2}\hat{\delta}_j^* \pi_{-j} e_0^* \underbrace{\{p_\mu, e_0\}}_{=-\{e_0, p_\mu\}} \pi_j^* \underbrace{=}_{e_0^*\partial e_0 = -(\partial e_0^*) e_0} -\frac{i}{2}\hat{\delta}_j^* \pi_{-j}\{e_0^*, p_\mu\} e_0 \pi_j^*,$$

$$-\frac{i}{2}\pi_j e_0^* \underbrace{\{p_\mu, e_0\}}_{=-\{e_0, p_\mu\}} \pi_{-j}^* \hat{\delta}_j \underbrace{=}_{e_0^*\partial e_0 = -(\partial e_0^*) e_0} -\frac{i}{2}\pi_j \{e_0^*, p_\mu\} e_0 \pi_{-j}^* \hat{\delta}_j,$$

which gives (5.2.1). The proof is complete.

□

Chapter 6

The X-ray transform

In this chapter we study conditions that are necessary and/or sufficient for having that the X-ray transform $\mathsf{R}(\lambda^{\pm}_{\mu-1})$ of the eigenvalues $\lambda^{\pm}_{\mu-1}$ of the semiprincipal part $a_{\mu-1}$ are Morse-Bott functions. Recall that in this chapter the X-ray transform of a functions is the integral of the function on the bicharacteristics of $a_\mu = p_\mu I_2$, hence of p_μ, that will be supposed all periodic with constant period $T > 0$ (e.g., the case of $\mu = 2$ and p_μ the quantum scalar harmonic oscillator).

6.1 Homogeneity property

We start by considering the following slightly more general setup.
Let $p_\mu \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R}_+)$ be positively homogeneous of degree $\mu > 1$. Note then that the bicharacteristic flow associated with p_μ is globally defined (since $p_\mu \to +\infty$ as $|X| \to +\infty$ all the energy hypersurfaces are compact, whence the flow is globally defined on each of them). Let $\Phi\colon \mathbb{R} \times \dot{\mathbb{R}}^{2n} \ni (t, X) \longmapsto \Phi_t(X) \in \dot{\mathbb{R}}^{2n}$ be the bicharacteristic flow associated with p_μ.

Suppose that the following condition holds:

(H) *For all $X \in \dot{\mathbb{R}}^{2n}$ the flow is periodic with minimal period $T_X > 0$ and the map $X \longmapsto T_X$ is C^∞.*

Recall that we therefore have that $\Phi_t(X) \neq X$ for all $0 < t < T_X$, for all $X \in \dot{\mathbb{R}}^{2n}$.

First of all, we prove the following proposition about homogeneity of the X-ray transform.

Proposition 6.1.1. *Suppose condition (H) above holds. Let* $p_{\mu-1} \in C^\infty(\dot{\mathbb{R}}^{2n};\mathbb{C})$ *be positively homogeneous of degree* $\mu - 1$. *Consider the X-ray transform of* $p_{\mu-1}$, *denoted by*

$$\dot{\mathbb{R}}^{2n} \ni X \longmapsto \mathsf{R}(p_{\mu-1})(X) := \int_0^{T_X} p_{\mu-1}(\Phi_t(X))dt.$$

Then, $\mathsf{R}(p_{\mu-1})$ *is positively homogeneous of degree* 1.

Proof. First of all, for any $s > 0$ and any $X \in \dot{\mathbb{R}}^{2n}$ we have, by homogeneity of p_μ,

$$p_\mu(sX) = s^\mu p_\mu(X), \quad s > 0,\ X \neq 0.$$

Moreover, since the Hamiltonian field associated with p_μ is

$$H_{p_\mu}(X) = \sum_{j=1}^n \left(\frac{\partial p_\mu}{\partial \xi_j}(X)\frac{\partial}{\partial x_j} - \frac{\partial p_\mu}{\partial x_j}(X)\frac{\partial}{\partial \xi_j} \right),$$

we have that $H_{p_\mu}(sX) = s^{\mu-1}H_{p_\mu}(X)$ for $s > 0$ and $X \neq 0$, that is, H_{p_μ} is homogeneous of order $\mu - 1$.

Hence, denoting by Φ the flow associated with H_{p_μ},

$$\begin{cases} \dfrac{d}{dt}\Phi_t(X) = H_{p_\mu}(\Phi_t(X)), \\ \Phi_t(X)\big|_{t=0} = X, \end{cases}$$

for $s > 0$ consider $\tilde{\Phi}_t(X) := s\Phi_{s^{\mu-2}t}(X)$. Then,

$$\begin{cases} \dfrac{d}{dt}\tilde{\Phi}_t(X) = ss^{\mu-2}\dfrac{d}{dt'}\Phi_{t'}(X)\Big|_{t'=s^{\mu-2}t} & = ss^{\mu-2}H_{p_\mu}(\Phi_{s^{\mu-2}t}(X)) \\ & = H_{p_\mu}(s\Phi_{s^{\mu-2}t}(X)) \\ & = H_{p_\mu}(\tilde{\Phi}_t(X)), \\ \tilde{\Phi}_t(X)|_{t=0} = sX, \end{cases}$$

so that

$$\Phi_t(sX) = s\Phi_{s^{\mu-2}t}(X), \quad \forall s > 0,\ \forall X \neq 0.$$

Since T_X is minimal, using periodicity and homogeneity we obtain

$$\Phi_{s^{2-\mu}T_X}(sX) = s\Phi_{s^{\mu-2}s^{2-\mu}T_X}(X) = s\Phi_{T_X}(X) = sX,$$

whence

$$T_{sX} = s^{2-\mu}T_X, \quad \forall s > 0,\ \forall X \neq 0.$$

Now, recall that $p_{\mu-1}$ is positively homogeneous of degree $\mu-1$. Consider the X-ray transform

$$\mathsf{R}(p_{\mu-1})(X) = \int_0^{T_X} \Phi_t^* p_{\mu-1}(X)\,dt = \int_0^{T_X} (p_{\mu-1} \circ \Phi_t)(X)\,dt.$$

Thus,

$$\begin{aligned} \mathsf{R}(p_{\mu-1})(sX) &= \int_0^{T_{sX}} (p_{\mu-1} \circ \Phi_t)(sX)\,dt = \int_0^{s^{2-\mu}T_X} p_{\mu-1}(s\Phi_{s^{\mu-2}t}(X))\,dt \\ &= \int_0^{s^{2-\mu}T_X} s^{\mu-1} p_{\mu-1}(\Phi_{s^{\mu-2}t}(X))\,dt \underbrace{=}_{\tau = s^{\mu-2}t} s\int_0^{T_X} p_{\mu-1}(\Phi_\tau(X))\,d\tau \\ &= s\mathsf{R}(p_{\mu-1})(X). \end{aligned}$$

Therefore,

$$\mathsf{R}(p_{\mu-1})(sX) = s\mathsf{R}(p_{\mu-1})(X), \quad \forall s > 0,\ \forall X \neq 0.$$

□

6.2 The zero-set of the X-ray transform

We are interested to focus on the case $\mu = 2$ because of the study that we will carry out in Section 7.1. We next wish to understand when $\mathsf{R}(p_1)$ is a Morse-Bott function (that is, a smooth function whose critical set is a smooth manifold at which the Hessian is nondegenerate in normal directions).

Proposition 6.2.1. *Suppose condition (H) above holds. Let p_1 be smooth, positively homogeneous of degree* 1*, and let C_{cr} denote the critical set of* $\mathsf{R}(p_1)$*. Then, C_{cr} is a conic set contained in the zero-set of* $\mathsf{R}(p_1)$*.*

Moreover, any given $X \in \dot{\mathbb{R}}^{2n}$ belongs to $\operatorname{Ker}\operatorname{Hess}(\mathsf{R}(p_1))(X)$*. Hence* $\operatorname{Hess}(\mathsf{R}(p_1))(X)$ *is always degenerate for every $X \in \dot{\mathbb{R}}^{2n}$.*

Proof. The property of C_{cr} follows immediately from the homogeneity of p_1, the homogeneity of $\mathsf{R}(p_1)$ (see Proposition 6.1.1), and Euler's relation.

Next we consider the Hessian

$$\operatorname{Hess}(\mathsf{R}(p_1))(X) = \begin{bmatrix} {}^t(\nabla_X \frac{\partial}{\partial X_1}\mathsf{R}(p_1)(X)) \\ \vdots \\ {}^t(\nabla_X \frac{\partial}{\partial X_{2n}}\mathsf{R}(p_1)(X)) \end{bmatrix}, \quad X \in \dot{\mathbb{R}}^{2n},$$

where $\frac{\partial}{\partial X_j}\mathsf{R}(p_1)$, $1 \le j \le 2n$, are positively homogeneous of degree 0 since $\mathsf{R}(p_1)$ is positively homogeneous of degree 1. Therefore, for every $X \in \dot{\mathbb{R}}^{2n}$ (now seen as a tangent vector to $\mathbb{R}^{2n}$) we have

$$\left\langle \nabla_X \frac{\partial}{\partial X_j}\mathsf{R}(p_{\mu-1})(X), X \right\rangle = 0\frac{\partial}{\partial X_j}\mathsf{R}(p_{\mu-1})(X) = 0,$$

for $1 \le j \le 2n$, once more by Euler's relation. Thus,

$$\operatorname{Hess}(\mathsf{R}(p_{\mu-1}))(X)X = \begin{bmatrix} \left\langle \nabla_X \frac{\partial}{\partial X_1}\mathsf{R}(p_{\mu-1})(X), X \right\rangle \\ \vdots \\ \left\langle \nabla_X \frac{\partial}{\partial X_{2n}}\mathsf{R}(p_{\mu-1})(X), X \right\rangle \end{bmatrix} = 0.$$

This concludes the proof.

□

Remark 6.2.2. $\nabla\mathsf{R}(p_{\mu-1})$ *vanishes to infinite order on a set of nonzero measure if and only if it vanishes on some open set since* $\mathsf{R}(p_{\mu-1})$ *is smooth. Now, from Proposition 6.2.1, we then have that* $\nabla\mathsf{R}(p_{\mu-1})$ *vanishes on an open set if and only if* $\mathsf{R}(p_{\mu-1})$ *vanishes on an open set. By homogeneity, we hence have that if there is a relatively open set $V \subset \mathbb{S}^{2n-1}$ at which* $\nabla\mathsf{R}(p_{\mu-1})\big|_V = 0$*, then* $\operatorname{Hess}(\mathsf{R}(p_{\mu-1}))(X)$ *is degenerate at all $X \in \dot{\mathbb{R}}^{2n}$ of the* *form* $X = r\omega$, $\omega \in V$ *and* $r > 0$.

However, the critical set C_{cr} *in Proposition 6.1.1 is a smooth manifold (necessarily conic) of* $\dot{\mathbb{R}}^{2n}$ *if there is a smooth submanifold* S *of* $\mathbb{S}^{2n-1}$ *such that* $C_{\text{cr}} = \{r\omega;\ r > 0,\ \omega \in S\}$. *The normal directions of* C_{cr} *at some* $X = r\omega \in C_{\text{cr}}$ *(hence* $\omega \in S$*) are exactly given by tangent vectors at* ω *to the sphere that are normal to* S *within the sphere. In fact, for all* $r\omega \in C_{\text{cr}}$ *we have that* $T_{r\omega}C_{\text{cr}} = T_\omega C_{\text{cr}} = T_\omega S \oplus N_\omega \mathbb{S}^{2n-1}$ *where* $T_\omega S \oplus N_\omega S = T_\omega \mathbb{S}^{2n-1}$ *and* $\oplus$ *denotes orthogonal sum in* $\mathbb{R}^{2n}$ *(here we consider, as customary, the sphere* $\mathbb{S}^{2n-1}$ *as embedded in* $\mathbb{R}^{2n}$*, with the induced Riemannian metric).*

Chapter 7

The Weyl law for semiregular systems

7.1 The Weyl law computation

In this section, we prove for a system $A \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, a semiregular metric globally elliptic system of the kind introduced in Definition 1.2.6 (i.e., an SMGES), a "classical Weyl law" and a "refined Weyl law" result of the kind proved for *scalar* semiregular operators, respectively, by Helffer and Robert [18] and by Doll, Gannot and Wunsch [11]. We follow the approach in [11] for both the results. As is classical situations, the approach is based on the construction of an FIO (Fourier integral operator) parametrix of the Schrödinger unitary group generated by A^{w}. We will hence have to exploit our diagonalization result (in the semiregular setting) developed in the previous chapters. In fact, we construct a parametrix for the diagonalized system and thus obtain a parametrix by conjugating with the operator $E^{\mathrm{w}}(x, D)$ constructed in Chapter 3. However, because of that conjugation we need to have a better control on the compositions occurring in conjugations. Hence, it will be convenient to construct a parametrix following the idea of Doll and Zelditch in [12], that is, by exploiting the fact that the parametrix FIO can in fact be written as a Weyl-quantization. Having the parametrix for $e^{-itA^{\mathrm{w}}}$, we then follow the classical approach, in that we will be able to consider the trace of its Schwartz distribution kernel and obtain our results through the asymptotics of the convolution of the counting function with a suitable scalar function (with compactly supported Fourier transform) and classical Tauberian arguments.

Throughout the section ds denotes the Riemannian metric induced on $\{p_2 = 1\}$ or on $\{p_2 = \lambda\}$ (it will be clear from the context) by the Euclidean one with $\lambda > 0$, and $ds/|\nabla p_2|$ denotes the associated Leray-Liouville measure.

In the proof of Proposition 7.1.2 and Theorem 7.1.7 will be fundamental that the tangential derivatives of order 1 of the X-ray transform of the eigenvalues of the semiprincipal symbol vanish to infinity order on a subset of zero measure of the sphere $\mathbb{S}^{2n-1}$. Namely, if we denote by ∂_ω^α ($\alpha \in \mathbb{N}^{2n-1} \setminus 0$) the $\mathbb{S}^{2n-1}$-tangential derivatives of order $|\alpha|$ and by $\lambda_{1,j}$ ($1 \leq j \leq r$) the eigenvalues of the semiprincipal symbol of the SMGES under study, the corresponding subset of $\mathbb{S}^{2n-1}$,

- **Condition DGW:**

$$\Pi_{2\pi,j} := \left\{\omega \in \mathbb{S}^{2n-1};\ \partial_\omega^\alpha(\mathsf{R}(\lambda_{1,j}))(\omega) = 0,\ \forall \alpha \in \mathbb{N}^{2n-1} \setminus 0\right\} \text{ has zero measure, } \forall j \tag{7.1.1}$$

Remark 7.1.1. *We notice that to impose* ***Condition DGW*** *(7.1.1) we need the explicit knowledge of the eigenvalues of the semiprincipal symbol of the SMGES under study. Hence, we need a way to determine explicitly those eigenvalues to make* ***Condition DGW*** *a useful condition when studying a concrete example of SMGES. It can be done in the following way by relying on the Rouché Theorem. In fact, by the Definition 1.2.6 of SMGES, for all* $\omega_0 \in \mathbb{S}^{2n-1}$ *and all* $1 \leq j \leq r$*, there is a disc on the complex plane* $B_{\omega_0,j}$ *containing* $\lambda_j(\omega_0)$ *and no* $\lambda_{j'}(\omega_0)$ *with* $j' \neq j$*. Hence, by Rouché Theorem, there is an (relatively) open neighbourhood* U_{ω_0} *of* ω_0 *on the sphere* $\mathbb{S}^{2n-1}$ *such that*

$$\lambda_j(\omega) := \frac{1}{2\pi i N_j}\int_{\partial B_{\omega_0,j}} \lambda \frac{\partial_\lambda P(\omega;\lambda)}{P(\omega;\lambda)}\, d\lambda,\ \forall \omega \in U_{\omega_0},\ \forall j, \tag{7.1.2}$$

where $\partial B_{\omega_0,j}$ *denotes the boundary of* $B_{\omega_0,j}$*,* N_j *is the multiplicity of* λ_j *and* $P(\omega;\lambda) := \det(a_1(\omega) - \lambda I_N)$ *(*a_1 *being the semiprincipal symbol of the SMGES* *under study). Thus, we can give a local representation of λ_j around every $\omega \in \mathbb{S}^{2n-1}$ and, therefore, by compactness of $\mathbb{S}^{2n-1}$, there is a finite open cover $\{U_{\omega_k}\}_{k=1,\dots,\bar{k}}$ of $\mathbb{S}^{2n-1}$ such that on every open set of the cover (7.1.2) holds. Finally, a partition of unity argument gives $\mathbb{S}^{2n-1} \ni \omega \mapsto \lambda_j(\omega)$ for all j.*

*Hence, the **Condition DGW** (7.1.1) can be imposed as a condition on the logarithmic derivative of the characteristic polynomial P of the semiprincipal symbol of the SMGES studied.*

For clarity of exposition, we first prove a result in the fully *diagonal* case which serves as a guide to guess what the result should look like in the more general, *nondiagonal* case.

Proposition 7.1.2. *Let $B = B^* \sim \sum_{j\geq 0} b_{2-j} \in S_{\rm sreg}(m^2, g; \mathsf{M}_N)$ be a **diagonal** SMGES symbol. Hence, in particular, $b_2 = p_2 I_N$ with $p_2 \in S(m^2, g)$ the scalar harmonic oscillator. Let $\mathbb{R} \ni \lambda \longmapsto \mathsf{N}(\lambda)$ denote the spectral counting function associated with $B^{\rm w}$. We have the following asymptotics*

$$\mathsf{N}(\lambda) = \left(\frac{N}{(2\pi)^n}\int_{p_2\leq 1} dX\right)\lambda^n - \left((2\pi)^{-n}\int_{p_2=1} \mathsf{Tr}\,(b_1)\,\frac{ds}{|\nabla p_2|}\right)\lambda^{n-1/2} + O(\lambda^{n-1}), \tag{7.1.3}$$

as $\lambda \to +\infty$.
*Furthermore, if **Condition DGW** (7.1.1) is satisfied, then (7.1.3) can be refined to*

$$\mathsf{N}(\lambda) = (2\pi)^{-n}\left(\sum_{j=1}^{N}\left(\int_{p_2+b_{1,j}\leq\lambda} dX\right) - \int_{p_2=\lambda} \mathsf{Tr}\,(b_0)\,\frac{ds}{|\nabla p_2|}\right) + o(\lambda^{n-1}), \tag{7.1.4}$$

as $\lambda \to +\infty$, where $b_{1,j}$ is the j-th term of the diagonal of b_1, $j = 1, \dots, N$.

Proof. Of course, we may write the counting function as

$$\mathbb{R} \ni \lambda \longmapsto \mathsf{N}(\lambda) = \sum_{j=1}^{N} \mathsf{N}_j(\lambda),$$

where N_j is the counting function given by the jth diagonal term of $B^{\rm w}$.

Applying then the scalar results by Doll, Gannot and Wunsch [11] to get the asymptotics of each of the contributions in the two cases of the statement, we sum up the asymptotics of all contributions to get the asymptotics of $\mathsf{N}(\lambda)$.

To obtain (7.1.3) for each $1 \leq j \leq N$, let $\rho \in \mathscr{S}(\mathbb{R})$ such that $\hat{\rho}$ has compact support in $(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $\rho = 1$ on a neighborhood of 0. We have, by [11], Proposition 6.1,

$$(\mathsf{N}_j * \rho)(\lambda) = (2\pi)^{-n} \left(\int_{p_2 + b_{1,j} \leq \lambda} dX - \int_{p_2 = \lambda} \mathsf{Tr}\,(b_0) \frac{ds}{|\nabla p_2|} \right) + O(\lambda^{n-3/2}), \tag{7.1.5}$$

as $\lambda \to +\infty$. Since

$$\mathrm{Vol}(\{p_2 + b_{1,j} \leq \lambda\}) = \lambda^n \mathrm{Vol}(\{p_2 + \lambda^{-1/2} b_{1,j} \leq 1\}),$$

a Taylor-expansion in powers of $\lambda^{-1/2}$ and Lemma IV.7 of [18] give the asymptotics

$$\mathsf{N}_j(\lambda) = (2\pi)^{-n} \Big(\Big(\int_{p_2 \leq 1} dX \Big) \lambda^n - \Big(\int_{p_2 = 1} b_{1,j} \frac{ds}{|\nabla p_2|} \Big) \lambda^{n-1/2} \Big) + O(\lambda^{n-1}),\ \lambda \to +\infty.$$

Therefore, as $\lambda \to +\infty$,

$$\begin{aligned} \mathsf{N}(\lambda) &= \sum_{j=1}^{N} \mathsf{N}_j(\lambda) \\ &= \sum_{j=1}^{N} \left(\left((2\pi)^{-n} \int_{p_2 \leq 1} dX \right) \lambda^n - \left((2\pi)^{-n} \int_{p_2 = 1} b_{1,j} \frac{ds}{|\nabla p_2|} \right) \lambda^{n-1/2} \right) + O(\lambda^{n-1}) \\ &= \left(\frac{N}{(2\pi)^n} \int_{p_2 \leq 1} dX \right) \lambda^n - \left((2\pi)^{-n} \int_{p_2 = 1} \mathsf{Tr}(b_1) \frac{ds}{|\nabla p_2|} \right) \lambda^{n-1/2} + O(\lambda^{n-1}), \end{aligned}$$

which gives (7.1.3).
We next prove (7.1.4). In fact, the assumption of **Condition DGW** (7.1.1)

implies that we may apply Theorem 1.2 of [11] to each diagonal term of B to obtain that for $1 \leq j \leq N$,

$$\mathsf{N}_j(\lambda) = (2\pi)^{-n} \left(\left(\int_{p_2 + b_{1,j} \leq \lambda} dX \right) - \int_{p_2 = \lambda} b_{0,j} \frac{ds}{|\nabla p_2|} \right) + o(\lambda^{n-1}), \; \lambda \to +\infty,$$

whence

$$\begin{aligned} \mathsf{N}(\lambda) =& \sum_{j=1}^{N} \mathsf{N}_j(\lambda) \\ =& (2\pi)^{-n} \left(\sum_{j=1}^{N} \int_{p_2 + b_{1,j} \leq \lambda} dX - \int_{p_2 = \lambda} \mathsf{Tr}\,(b_0) \frac{ds}{|\nabla p_2|} \right) + o(\lambda^{n-1}), \; \lambda \to +\infty, \end{aligned}$$

which concludes the proof.

□

As already mentioned, the fundamental tool to obtain the Weyl law for the class of semiregular ψdo *systems* we are interested in, is a parametrix of the unitary group $t \mapsto e^{-itA^{\mathrm{w}}}$. In our vector-valued situation, by the diagonalization result Theorem 3.1.3, this goes through the construction of the parametrix in the case of a semiregular system with scalar principal part, *blockwise scalar semiprincipal part*, and a full blockwise subprincipal part.

For the parametrix construction in the diagonal case, we will first construct a parametrix of the *reduced propagator* (see Lemma 7.1.3 below) and will then compose the latter with the unitary group of the harmonic oscillator (which is the Weyl-quantization of an exponential, see Hörmander in [28]). The main advantage of such a construction is that, following the approach of Doll and Zelditch [12], the parametrix is a Weyl-quantization. This is crucial, for we have to compose the FIOs by the diagonalizers to obtain a parametrix for $t \mapsto e^{-itA^{\mathrm{w}}}$, and this is a *delicate* point.

We next follow the approach as in the scalar case by Doll, Gannot and Wunsch [11] (which is in turn inspired by Hörmander [27]), which gives a result that generalizes their Proposition 6.1, hence yielding an asymptotics for $\mathsf{N} * \rho$ for a suitable localizing function ρ (belonging to $\mathscr{S}(\mathbb{R})$ such that $\hat{\rho}$ has compact support in $(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $\hat{\rho} = 1$ in a neighborhood of 0). The refined Weyl law estimate will then follow by a Tauberian argument.

We consider at first the construction of the reduced propagator in the case of a system B with *scalar principal and semiprincipal symbols* (note that we allow a matrix-valued subprincipal symbol and lower order terms).

Note that we will have to consider Weyl-quantizations of the kind $(e^{i\phi_1}\alpha)^{\mathrm{w}}$, where $\alpha \in S_{\mathrm{sreg}}(1, g; \mathsf{M}_N)$ and ϕ_1 is an isotropic symbol of order 1. This is done according to the Weyl-Hörmander calculus with metric $|dX|^2$ whose Planck constant is 1.

Lemma 7.1.3. *Let* $B = B^* \sim \sum_{j\geq 0} b_{2-j} \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, *where the* $b_j = b_j^*$ *are positively homogeneous of degree* j *and* b_2 *and* b_1 *are scalar:* $b_2 = p_2 I_N$ *and* $b_1 = p_1 I_N$, *where* p_2 *is the harmonic oscillator and* p_1 *is homogeneous of degree* 1. *For* $t \in \mathbb{R}$ *consider*

$$P(t) := e^{itp_2^{\mathrm{w}}}(B^{\mathrm{w}} - p_2^{\mathrm{w}})e^{-itp_2^{\mathrm{w}}}.$$

Let H_{p_2} *be the Hamilton field of* p_2 *and* $t \mapsto \exp(tH_{p_2})(X)$ *be its bicharacteristic flow. Consider the phase-function*

$$\mathbb{R}_t \times \mathbb{R}^{2n}_X \ni (t, X) \mapsto \tilde{\phi}_1(t, X) := -\int_0^t p_1 \circ \exp(sH_{p_2})(X)\, ds. \tag{7.1.6}$$

Then, there is $\tilde{\alpha} \in C^\infty(\mathbb{R}_t; S_{\mathrm{sreg}}(1, g; \mathsf{M}_N))$ *such that* $\mathbb{R} \ni t \mapsto \tilde{F}(t) := (e^{i\tilde{\phi}_1(t)}\tilde{\alpha}(t))^{\mathrm{w}}$ *solves*

$$(i\partial_t - P)\tilde{F} \in C^\infty(\mathbb{R}_t; \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathsf{M}_N),\ \tilde{F}|_{t=0} = I_N + R,$$

where R *is smoothing.*

Proof. As usual, we make a WKB construction, the main point being that the eikonal equation and the transport equations are globally solvable in time. Note that in the transport equations we have a matrix term of order zero (generated by the in general non-scalar subprincipal part b_0), but this is harmless in solving them.

Observe that, since H_{p_2} is linear, $X \mapsto \exp(tH_{p_2})(X)$ is a global linear diffeomorphism for all t, so that, by Egorov's Theorem (or Hörmander's theorem on the invariance of the Weyl calculus through linear symplectomorphisms), we have that the Weyl symbol of $P(t)$ is given by $(B - p_2) \circ \exp(tH_{p_2})$. Therefore, the principal part of $P(t)$ is

$$\tilde{p}_1(t) := p_1 \circ \exp(tH_{p_2}), \quad t \in \mathbb{R},$$

and the semiprincipal one is

$$\tilde{b}_0(t) := b_0 \circ \exp(tH_{p_2}), \quad t \in \mathbb{R}.$$

The eikonal equation is

$$\begin{cases} \partial_t \tilde{\phi}_1 + \tilde{p}_1 = 0, \\ \tilde{\phi}_1|_{t=0} = 0, \end{cases}$$

and it is solved for all t and X by $\tilde{\phi}_1$ given in (7.1.6).

As for the terms of the WKB expansion of $\tilde{\alpha} \sim \sum_{j\geq 0} \tilde{\alpha}_{-j}$, we have a sequence of transport equations, the first of which has the form

$$\begin{cases} \partial_t \tilde{\alpha}_0 = (\tilde{b}_0 - \frac{1}{2}\{\tilde{p}_1, \tilde{\phi}_1\} I_N)\tilde{\alpha}_0, \\ \tilde{\alpha}_0|_{t=0} = I_N. \end{cases}$$

Since the characteristics are straight lines, the solution exists for all times, the matrix-valued term $\tilde{b}_0$ being, as already mentioned, harmless. One proceeds similarly for the other transport equations (which have the same structure, with initial condition the zero-matrix and source terms depending on the $\tilde{\alpha}_{-j}$s already constructed, as usual). Observe that $\tilde{b}_0 - \frac{1}{2}\{\tilde{p}_1, \tilde{\phi}_1\} I_N$ is homogeneous of degree 0 and that the higher transport equations for $\tilde{\alpha}_{-j}$ preserve homogeneity ($\tilde{\alpha}_{-j}$ is homogeneous of degree $-j$). The characteristics being straight lines, the $\tilde{\alpha}_{-j}(t)$ exist for all times. Taking $\tilde{\alpha} \sim \sum_{j\geq 0} \tilde{\alpha}_{-j}$ concludes the proof. ☐

Next, we need a composition result for quadratic phase functions (analogous to Proposition 4.2 in [12]).

Proposition 7.1.4. *Let $A \in \mathsf{M}_{2n}$ be a real symmetric matrix and $a, b \in \mathscr{S}(\mathbb{R}^{2n})$. We have*

$$(e^{i\langle A\cdot,\cdot\rangle}a\#b)(X) = \pi^{-4n}e^{i\langle AX,X\rangle}\int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle}a(X+Y_1)b(X+JAX+Y_2)\,dY_1\,dY_2,$$

where $Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{R}^{4n}$, $X \in \mathbb{R}^{2n}$, the $4n \times 4n$ matrix Q is given by

$$Q := \left[\begin{array}{c|c} -A & -J \\ \hline J & 0 \end{array}\right], \tag{7.1.7}$$

and where $J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$ is the standard $2n \times 2n$ symplectic matrix, $\#$ being the composition operator in the Weyl calculus.

Proof. The proof follows by the integral representation for the composition of Schwartz symbols (see Zworski [61]) and a change of coordinates in the integral. In fact, by [61] Theorem 4.11,

$$\begin{aligned}(e^{i\langle A\cdot,\cdot\rangle}&a\#b)(X)\\ &=\pi^{-4n}e^{i\langle AX,X\rangle}\int_{\mathbb{R}^{4n}} e^{-2i\sigma(Y_1,Y_2)+i\langle A(X+Y_1),X+Y_1\rangle}a(X+Y_1)b(X+Y_2)\,dY_1\,dY_2.\end{aligned} \tag{7.1.8}$$

where

$$\sigma(Y_1,Y_2) := \frac{1}{2}\left\langle \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\right\rangle.$$

Now, the change of coordinates in (7.1.8)

$$Y_1 = \tilde{Y}_1, \quad Y_2 = \tilde{Y}_2 + JAX,$$

leads to (using Y_1, Y_2 again)

$$(e^{i\langle A\cdot,\cdot\rangle}a\#b)(X)$$
$$=\pi^{-4n}e^{i\langle AX,X\rangle}\int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle}a(X+Y_1)b(X+JAX+Y_2)\,dY_1\,dY_2.$$

In fact,

$$2\sigma(Y_1,Y_2)=\left\langle \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}\tilde{Y},\tilde{Y}\right\rangle+\left\langle AX,\tilde{Y}_1\right\rangle+\left\langle A\tilde{Y}_1,X\right\rangle,$$

where $\tilde{Y}:=\begin{bmatrix}\tilde{Y}_1\\ \tilde{Y}_2\end{bmatrix}\in\mathbb{R}^{4n}$. Hence, $-2\sigma(Y_1,Y_2)+\langle A(X+Y_1),X+Y_1\rangle=-\langle QY,Y\rangle$ and the proof is complete. □

By Proposition 7.1.4 we may compute how quadratic exponentials act on oscillating functions.

Proposition 7.1.5. *Let ϕ_1 be real, homogeneous of degree 1 and smooth on $\mathbb{R}^{2n}\setminus\{0\}$. Let $a\in S(m^{\mu_1},g;\mathsf{M}_N)$, and $b\in S(m^{\mu_2},g;\mathsf{M}_N)$. For any given real symmetric and positive-definite (resp. negative-definite) matrix $A\in\mathsf{M}_{2n}$ we have*

$$(e^{i\langle A\cdot,\cdot\rangle}a\#e^{i\phi_1}b)(X)=e^{i\langle AX,X\rangle+i\phi_1(X+JAX)}c,$$

where $X\in\mathbb{R}^{2n}$ and $c\in S(m^{\mu_1+\mu_2},g;\mathsf{M}_N)$.

Proof. Since the linear map defined by Q (see (7.1.7)) is injective we may use the usual approximation argument and a non-stationary phase argument to extend the previous approach to semiregular symbols. In fact, let $\chi\in\mathscr{S}(\mathbb{R}^{4n})$ with $\chi(0)=1$ and let χ_ε be the function $\mathbb{R}^{4n}\ni Y\longmapsto\chi(\varepsilon Y)$ with $\varepsilon>0$. Thus, $\chi_\varepsilon a$, $\chi_\varepsilon e^{i\phi_1}b\in\mathscr{S}$ and we can apply Proposition 7.1.4. We may hence consider

$$(e^{i\langle A\cdot,\cdot\rangle}a\sharp e^{i\phi_1}b)(X)=\pi^{-4n}e^{i\langle AX,X\rangle+i\phi_1(X+JAX)}\times$$
$$\times\int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle}a(X+Y_1)e^{i(\phi_1(X+JAX+Y_2)-\phi_1(X+JAX))}b(X+JAX+Y_2)dY_1dY_2. \tag{7.1.9}$$

Now we follow an argument similar to the one of the proof of Lemma 4.2 in [11] to show that the integral in the right-hand side of (7.1.9) is a symbol $c \in S(m^{\mu_1+\mu_2}, g; \mathsf{M}_N)$. Note that, when $A > 0$ (resp. $A < 0$), then $I_{2n} + JA$ is invertible. We next define, for $\lambda > 0$,

$$c_\lambda(X) := \pi^{-4n} \int_{\mathbb{R}^{4n}} e^{-i\langle QY,Y\rangle} a(\sqrt{\lambda}X+Y_1) e^{i(\phi_1(\sqrt{\lambda}(I_{2n}+JA)X+Y_2)-\phi_1(\sqrt{\lambda}(I_{2n}+JA)X))} \times$$

$$\times b(\sqrt{\lambda}(I_{2n} + JA)X + Y_2)dY_1dY_2,$$

$X \in \mathbb{R}^{2n}$. In order to prove that c is a symbol it suffices to show that there is $\lambda_0 \geq 1$ such that, for all $\lambda \geq \lambda_0$ and all $1 \leq |X| \leq 2$, we have

$$|\partial_X^\alpha c_\lambda(X)| \leq C_\alpha \lambda^{(\mu_1+\mu_2)/2}, \tag{7.1.10}$$

where C_α is independent of λ (and X). In fact, if (7.1.10) holds, then, for any given $X \in \mathbb{R}^{2n}$ with $|X| \geq \lambda_0$, we may find $\lambda \geq \lambda_0$ for which $\sqrt{\lambda} \leq |X| \leq 2\sqrt{\lambda}$ and hence may find a unique $\tilde{X}$ with $1 \leq |\tilde{X}| \leq 2$ such that $X = \sqrt{\lambda}\,\tilde{X}$. Therefore,

$$|\partial_X^\alpha c(X)| = \lambda^{-|\alpha|/2} |\partial_{\tilde{X}} c_\lambda(\tilde{X})| \leq C'_\alpha \lambda^{(\mu_1+\mu_2-|\alpha|)/2},$$

for $\lambda \geq \lambda_0$ and with $C'_\alpha \geq C_\alpha$. Since $\frac{1}{2}|X| \leq \sqrt{\lambda} \leq |X|$, we hence have that

$$|\partial_X^\alpha c(X)| \leq C''_\alpha \langle X\rangle^{\mu_1+\mu_2-|\alpha|},$$

for $|X| \geq \lambda_0$ and with $C''_\alpha \geq C'_\alpha$, that is, $c \in S(m^{\mu_1+\mu_2}, g; \mathsf{M}_N)$.

Now, for $\lambda \geq \lambda_0$, let

$$f_\lambda \colon \mathbb{R}^{2n}_X \times \mathbb{R}^{4n}_Y \ni (X, Y) \longmapsto a(\sqrt{\lambda}(X + Y_1)) b(\sqrt{\lambda}((I_{2n} + JA)X + Y_2)).$$

Note that, for any fixed constant C with $0 < C < C_{\min} := \min_{1\leq|X|\leq 2} |(I_{2n} + JA)X|$, one has

$$|\partial_X^\alpha f_\lambda(X, Y)| \leq C_\alpha \lambda^{(\mu_1+\mu_2)/2}, \tag{7.1.11}$$

uniformly in $1 \leq |X| \leq 2$ and $|Y| \leq C$, because a and b are symbols.

For $\mu \in \mathbb{R}$, define

$$\Phi_\mu(X,Y) := -\langle QY, Y\rangle + \mu\phi_1((I_{2n}+JA)X + Y_2) - \mu\phi_1((I_{2n}+JA)X),$$

and

$$c_{\lambda,\mu}(X) = \pi^{-4n}\lambda^{2n}\int e^{i\lambda\Phi_\mu(X,Y)} f_\lambda(X,Y)dY. \tag{7.1.12}$$

By using the homogeneity of the phase and the dilation $Y \mapsto Y/\sqrt{\lambda}$, we have that

$$c_\lambda = c_{\lambda,\lambda^{-1/2}}.$$

We next study $c_{\lambda,\mu}$ as $\lambda \to +\infty$ and μ lies in a neighborhood of zero ($1 \le |X| \le 2$ and $|Y| \le C$).
Let $C_\mu = \{Y \neq 0;\ d_Y\Phi_\mu = 0\}$ denote the set of stationary points. Thus, $Y \in C_\mu$ iff

$$\begin{cases} -AY_1 - JY_2 = 0, \\ JY_1 + \mu\nabla\phi_1((I_{2n}+JA)X + Y_2) = 0. \end{cases}$$

By the Implicit Function Theorem, we may parametrize Y by (μ, X) near any fixed X_0 with $1 \le |X_0| \le 2$ for $|\mu|$ sufficiently small. In fact, the Jacobian of $d_Y\Phi_\mu$with respect to Y is

$$\begin{bmatrix} -A & -J \\ J & \mu(\partial_{Y_2}\nabla\phi_1((I_{2n}+JA)X + Y_2)) \end{bmatrix},$$

which is invertible when $\mu = 0$. Hence, we obtain that $|Y(\mu, X)| \le C'|\mu|$. In particular, $|Y(\mu, X)| \le C < C_{\min}$ for $|\mu|$ sufficiently small, and $1 \le |X| \le 2$, whence the bounds (7.1.11) for f_λ. Moreover, note that $(I_{2n}+JA)X+Y_2 \neq 0$ by taking C small enough, since $|Y(\mu, X)| \le C$ if $1 \le |X| \le 2$ when μ is sufficiently small.

Next, without loss of generality we may assume that f_λ vanishes on the complement of $\{(X,Y);\ 1 \le |X| \le 2,\ |Y| \le C/2\}$. In fact, we wish to prove (7.1.10) for $1 \le |X| \le 2$ and Φ_μ is stationary only if $|Y| \le C/2$ (by taking $|\mu|$ even smaller), so that the contribution to the integral $c_{\lambda,\mu}(X)$ when $|Y| > C/2$ (and $1 \le |X| \le 2$) is $O(\lambda^{-\infty})$, by a non-stationary phase argument.

We may now estimate (7.1.12) and its derivatives, by initially thinking of μ as a parameter. Consider $\partial_X^\gamma c_{\lambda,\mu}$. It is a sum of terms, where those with $\ell \leq |\gamma|$ derivatives landing on the exponential factor can be written as

$$\pi^{-4n}\lambda^{2n}(\lambda\mu)^\ell \int e^{i\lambda\Phi_\mu(X,Y)} \left(\partial_z^{\gamma'} f_\lambda(X,Y)\right) \sum_{|\beta|=\ell} Y_2^\beta h_\beta(X,Y,\mu)dY, \quad (7.1.13)$$

for some smooth functions h_β and $|\gamma'| = |\gamma| - \ell$. In fact, expanding ϕ_1 at $(I_{2n} + JA)X$, with $|Y| \leq C/2$ and $1 \leq |X| \leq 2$, we have

$$\begin{aligned}\phi_1((I_{2n} + JA)X + Y_2) =&\phi((I_{2n} + JA)X) + \langle Y_2, \nabla\phi_1((I_{2n} + JA)X)\rangle \\ &+ \sum_{|\alpha|=2} Y_2^\alpha \psi_\alpha(X, Y_2),\end{aligned}$$

for some smooth functions ψ_α. Hence, for any given $\mu \in \mathbb{R}$,

$$\begin{aligned}\Phi_\mu(X,Y) :=& - \langle QY, Y\rangle + \mu\phi_1((I_{2n} + JA)X + Y_2) - \mu\phi_1((I_{2n} + JA)X) \\ =& - \langle QY, Y\rangle + \mu(\langle Y_2, \nabla\phi_1((I_{2n} + JA)X)\rangle + \mu \sum_{|\alpha|=2} Y_2^\alpha \psi_\alpha(X, Y_2)).\end{aligned}$$

Now, by the stationary-phase method, recalling the bounds (7.1.11), we have that at the critical set C_μ each term $Y_2^\beta h_\beta(X,Y,\mu)$ in (7.1.13) gives an additional factor of order $O(|\mu|^\ell)$, since $Y(\mu, X) = O(|\mu|)$. Hence, this cancels the factor of $\lambda^{\ell/2}$ in front of the integral in (7.1.13). The stationary-phase formula eliminates the prefactor λ^{2n} and setting $\mu = \lambda^{-1/2}$ gives

$$|\partial_X^\alpha c_\lambda(X)| \leq C_\alpha \lambda^{(\mu_1+\mu_2)/2},$$

in a neighborhood of X_0 (for λ large). Since $\{X \in \mathbb{R}^{2n};\ 1 \leq |X| \leq 2\}$ is compact, this implies the symbol estimates (7.1.10), and the proof is complete. □

We are now ready to use the previous results to obtain a parametrix of the unitary group of B^{w} (still in the case where the principal and semiprincipal parts are scalar) by composing the parametrix of the unitary group of the harmonic oscillator obtained by Hörmander with that of the reduced propagator.

Lemma 7.1.6. *Let $B = B^*$ be as in Lemma 7.1.3. Then, for all $k \in \mathbb{Z}$ and for $\varepsilon > 0$ sufficiently small, setting $I_\varepsilon(k) := (2k\pi - \varepsilon, 2k\pi + \varepsilon) \subset \mathbb{R}_t$, there are functions*

$$\phi_j \in C^\infty(I_\varepsilon(k) \times \dot{\mathbb{R}}^{2n}; \mathbb{R}),\ j = 1, 2,$$

homogeneous of degree j in $X \neq 0$ and

$$\alpha \in C^\infty(\mathbb{R}_t; S(1, g; \mathsf{M}_N)),$$

such that

$$U - \tilde{U} \in C^\infty(I_\varepsilon(k); \mathscr{L}(\mathscr{S}', \mathscr{S}) \otimes \mathsf{M}_N),$$

where U is the unitary group of B^{w} and $\tilde{U} := (e^{i(\phi_2+\phi_1)}\alpha)^{\mathrm{w}} = U_0\tilde{F}$, with

$$U_0(t) := \cos(t/2)^{-n}(e^{i\phi_2(t)})^{\mathrm{w}}$$

the unitary group of the harmonic oscillator with $t \notin \pi + 2\pi\mathbb{Z}$, $\phi_2(t) := -2\tan(t/2)p_2$, and $\tilde{F} = (e^{i\tilde{\phi}_1}\tilde{\alpha})^{\mathrm{w}}$ the reduced parametrix obtained in Lemma 7.1.3. In addition,

$$\phi_2 \colon I_\varepsilon(k) \times \mathbb{R}^{2n}_X \ni (t, X) \longmapsto -2\tan(t/2)p_2(X),$$

$$\phi_1 \colon I_\varepsilon(k) \times \mathbb{R}^{2n}_X \ni (t, X) \longmapsto \tilde{\phi}_1((I_{2n} - 2\tan(t/2)J)(X)).$$

Proof. The main idea of the proof is to compose by Proposition 7.1.5 the parametrix computed in Lemma 7.1.3 with the one obtained by Hörmander (see p. 427 of [28]) for the unitary group of the harmonic oscillator i.e.

$$U_0(t) := \cos(t/2)^{-n}(e^{i\phi_2(t)})^{\mathrm{w}},$$

with $t \notin \pi + 2\pi\mathbb{Z}$ and $\phi_2(t) := -2\tan(t/2)p_2$. Hence, one has that $\tilde{U} := U_0\tilde{F}$

is a parametrix on $I_\varepsilon(k)$ because

$$(i\partial_t - B^{\mathrm{w}})U_0\tilde{F} = \underbrace{((i\partial_t - p_2^{\mathrm{w}})U_0)}_{\in C^\infty(I_\varepsilon(k);\mathscr{L}(\mathscr{S}',\mathscr{S}))}\tilde{F} + U_0(i\partial_t\tilde{F}) - (B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F},$$

and

$$U_0(i\partial_t\tilde{F}) - (B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F} = U_0(i\partial_t\tilde{F} - U_0^{-1}(B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F}).$$

With P the reduced propagator, as

$$P - U_0^{-1}(B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0 \in C^\infty(I_\varepsilon(k);\mathscr{L}(\mathscr{S}',\mathscr{S})\otimes \mathsf{M}_N),$$

we have

$$i\partial_t\tilde{F} - U_0^{-1}(B^{\mathrm{w}} - p_2^{\mathrm{w}})U_0\tilde{F} \in C^\infty(I_\varepsilon(k);\mathscr{L}(\mathscr{S}',\mathscr{S})\otimes \mathsf{M}_N),$$

and $(U_0\tilde{F})|_{t=0} = I_N + R$, which shows that $\tilde{U}$ is a parametrix of $e^{-itB^{\mathrm{w}}}$ on $I_\varepsilon(k)$. By Proposition 7.1.5, we finally have $\phi_1(t) = \tilde{\phi}_1 \circ (I_{2n} - 2\tan(t/2)J)$. □

We next consider a general ψdo system A^{w} whose symbol belongs to the class SMGES (see Definition 1.2.6). As already anticipated, we determine an asymptotic expansion of $\mathsf{N} * \rho$ with a suitable $\rho \in \mathscr{S}(\mathbb{R})$, which leads immediately to the Weyl law (see (7.1.15) below). We exploit the construction of the parametrix in the blockwise diagonal case to obtain a parametrix of the Schrödinger group $e^{-itA^{\mathrm{w}}}$.

Theorem 7.1.7 (Weyl law). *Let $A = A^*$, with $A \sim \sum_{j\geq 0} a_{2-j} \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, be a second-order SMGES, with principal symbol $p_2 I_N$, p_2 being the harmonic oscillator. Adopting the notation used in Definition 1.2.6, we hence denote by $\lambda_{1,j}$, (with multiplicity N_j), $1 \leq j \leq r$, the eigenvalues of the semiprincipal part. Then, if $\rho \in \mathscr{S}(\mathbb{R})$ is chosen such that $\hat{\rho}$ has compact support in $(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon > 0$ and $\hat{\rho} = 1$ on a neighborhood of 0,*

$$(\mathsf{N}*\rho)(\lambda)=\left(\sum_{j=1}^{r}\left(\frac{N_j}{(2\pi)^n}\int_{p_2+\lambda_{1,j}\le\lambda}dX\right)-(2\pi)^{-n}\int_{p_2=\lambda}\mathsf{Tr}\,(a_0)\,\frac{ds}{|\nabla p_2|}\right) \tag{7.1.14}$$
$$+\,O(\lambda^{n-3/2}),$$

as $\lambda\to+\infty$ *(recall that* Tr *is the matrix trace).*
Therefore

$$\mathsf{N}(\lambda)=\left(\frac{N}{(2\pi)^n}\int_{p_2\le 1}dX\right)\lambda^n-\left((2\pi)^{-n}\int_{p_2=1}\mathsf{Tr}\,(a_1)\,\frac{ds}{|\nabla p_2|}\right)\lambda^{n-1/2} \tag{7.1.15}$$
$$+\,O(\lambda^{n-1}),$$

as $\lambda\to+\infty$*.*

Proof. In the first place we obtain a parametrix $U_A(t)$ of the unitary group $t\mapsto e^{-itA^{\mathrm{w}}}$ of A^{w} by a parametrix of the unitary group of its diagonalization B^{w}. Then we study the distribution $\hat\rho\mathrm{Tr}(U_A)$ where $\mathrm{Tr}(U_A)=\mathrm{Tr}_\Delta\mathsf{Tr}(U_A)$ denotes the trace of the Schwartz kernel of U_A (where Tr_Δ denotes the restriction to the diagonal). Since $N'*\rho=\mathcal{F}^{-1}\{\hat\rho\mathrm{Tr}(U_A)\}$, modulo a rapidly decreasing term, we finally get the result.

• **The parametrix** U_A**.** Recall that the decoupling Theorem 3.1.3 of Section 3 diagonalizes A^{w} (modulo smoothing operators), so that the principal symbol b_2 of the blockwise diagonal operator B^{w} is p_2 while the semiprincipal symbol $b_1=\mathrm{diag}(\lambda_{1,j}I_{N_j};\,1\le j\le r)$ is blockwise scalar. Hence, there is an operator S with Schwartz kernel $\mathsf{K}_S\in C^\infty(\mathbb{R}_t;\mathscr{S}(\mathbb{R}^{2n}_{x,y}))$ such that

$$e^{-itA^{\mathrm{w}}}=E^{\mathrm{w}}e^{-itB^{\mathrm{w}}}(E^{\mathrm{w}})^*+S(t),\quad\forall t\in\mathbb{R}$$

(see, for instance, Lemma 5.2 of [41]).
For notational simplicity, we suppose that the number r of blocks is 2 (the proof extends to the case $r\ge 3$ with no difficulties). Hence, consider the

symbols in blockwise form

$$B =: \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right],$$

where B_j is an $N_j \times N_j$ block ($j = 1, 2$), and

$$E =: \left[\begin{array}{c|c} E_{11} & E_{12} \\ \hline E_{21} & E_{22} \end{array}\right],$$

where E_{kj} is an $N_k \times N_j$ block ($j, k = 1, 2$).
Since for $j = 1, 2$ the semiprincipal term $\lambda_{1,j}$ of B_j is scalar, we obtain a parametrix U_{B_j} of the unitary group of B_j^{w} by Lemma 7.1.6. Thus,

$$U_A(t) := E^{\mathrm{w}} \left[\begin{array}{c|c} U_{B_1}(t) & 0 \\ \hline 0 & U_{B_2}(t) \end{array}\right] (E^{\mathrm{w}})^*$$

is a parametrix of the unitary group U_A, and its entries on the principal diagonal are given by

$$E_{11}^{\mathrm{w}} U_{B_1}(t)(E_{11}^{\mathrm{w}})^* + E_{12}^{\mathrm{w}} U_{B_2}(t)(E_{12}^{\mathrm{w}})^* \quad \text{and} \quad E_{21}^{\mathrm{w}} U_{B_1}(t)(E_{21}^{\mathrm{w}})^* + E_{22}^{\mathrm{w}} U_{B_2}(t)(E_{22}^{\mathrm{w}})^*.$$

• **Use of the parametrix.** Recall that

$$(\mathcal{F}_{\lambda\to t}\mathsf{N}')(t) = \mathrm{Tr}(e^{-itA^{\mathrm{w}}}),$$

where $\mathrm{Tr}(e^{-itA^{\mathrm{w}}})$ is well defined as a tempered distribution. Hence,

$$\mathcal{F}_{\lambda\to t}\{\mathsf{N}' * \rho\}(t) = \hat{\rho}(t)\mathrm{Tr}(e^{-itA^{\mathrm{w}}}),$$

and we may consider the distribution

$$K(t) = \hat{\rho}(t)\mathrm{Tr}(U_A)(t)$$

$$= \hat{\rho}(t)\mathrm{Tr}\Big(E_{11}^{\mathrm{w}} U_{B_1}(t)(E_{11}^{\mathrm{w}})^* + E_{12}^{\mathrm{w}} U_{B_2}(t)(E_{12}^{\mathrm{w}})^* + E_{21}^{\mathrm{w}} U_{B_1}(t)(E_{21}^{\mathrm{w}})^* + E_{22}^{\mathrm{w}} U_{B_2}(t)(E_{22}^{\mathrm{w}})^*\Big),$$

for $t \in (-\varepsilon, \varepsilon)$. Next, for $j, k = 1, 2$ let

$$K_{kj}(t) := \hat{\rho}(t)\mathrm{Tr}\left(E^{\mathrm{w}}_{kj} U_{B_j}(t) (E^{\mathrm{w}}_{kj})^*\right) = \hat{\rho}(t)\mathrm{Tr}_{\Delta}\mathsf{Tr}\left(E^{\mathrm{w}}_{kj} U_{B_j}(t) (E^{\mathrm{w}}_{kj})^*\right).$$

Denote by $\tilde{\phi}_{1,j}$, $\tilde{\alpha}_j$, α_j and $\phi_{1,j}$, $j = 1, 2$, respectively, the $\tilde{\phi}_1$, $\tilde{\alpha}$, α and ϕ_1 constructed in Lemmas 7.1.3 and 7.1.6 when $B = B_j$. Now,

$$E^{\mathrm{w}}_{kj} U_{B_j}(t) (E^{\mathrm{w}}_{kj})^* := E^{\mathrm{w}}_{kj} U_0(t) F_j(t) (E^{\mathrm{w}}_{kj})^*,$$

where $F_j(t)$ is the parametrix of the reduced propagator $e^{itp_2^{\mathrm{w}}}(B_j^{\mathrm{w}} - p_2^{\mathrm{w}})e^{-itp_2^{\mathrm{w}}}$. Hence,

$$E^{\mathrm{w}}_{kj} U_{B_j}(t) (E^{\mathrm{w}}_{kj})^* = U_0(t) (E_{kj} \circ \exp(tH_{p_2}))^{\mathrm{w}} \left(E^{\mathrm{w}}_{kj} F_j(t)^*\right)^*,$$

where $F_j(t)^* = (e^{-i\tilde{\phi}_{1,j}(t)}\tilde{\alpha}_j(t)^*)^{\mathrm{w}}$. By Proposition 4.1 of [12] and Lemma 7.1.6, we have

$$K_{kj}(t) := (2\pi)^{-n}\hat{\rho}(t) \int e^{i(\phi_2(t,X)+\phi_{1,j}(t,X))} c_{kj}(t, X)\, dX$$

(which makes sense since $\hat{\rho}$ has support on the interval where U_{B_j} is well defined). Now, by construction of ϕ_2 and $\phi_{1,j}$, we have $\phi_2(0, X)+\phi_{1,j}(0, X) = 0$, which yields by a Taylor's expansion

$$\phi_2(t, \cdot) + \phi_{1,j}(t, \cdot) = t\psi_j(t, \cdot),$$

where ψ_j is given to leading order in t by

$$\psi_j(t, \cdot) = -(p_2 + \lambda_{1,j}) + \frac{t}{2}(-H_{p_2}\lambda_{1,j}) + t^2 r_j(t, \cdot).$$

Following Hörmander [27], Lemma 29.1.3, we define

$$Q_{kj}(t, \lambda) := (2\pi)^{-n} \int_{\{-\psi_j(t,\cdot)\leq\lambda\}} c_{kj}(t, X)\hat{\rho}(t)\ dX.$$

For sufficiently small $|t|$, the function $\psi_j(t, \cdot)$ is elliptic in $S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$, and by the above mentioned lemma by Hörmander, Q_{kj} is a Kohn-Nirenberg symbol in $S^n(\mathbb{R}_\lambda)$ for $|t|$ sufficiently small. Furthermore,

$$K_{kj}(t) = \int_{\mathbb{R}} e^{-it\lambda} \partial_\lambda Q_{kj}(t, \lambda)\ d\lambda.$$

Thus, $K_{kj}(t)$ is a conormal distribution, which can be written as the Fourier transform of a symbol independent of t (see [26], Lemma 18.2.1). Defining

$$\tilde{Q}_{kj}(\lambda) := e^{iD_t D_\lambda} Q_{kj}(0, \lambda) \tag{7.1.16}$$

and recalling the definition of $K_{kj}(t)$,

$$\mathcal{F}^{-1}_{t\to\lambda}\Big(\hat{\rho}(t)\mathrm{Tr}(E^{\mathrm{w}}_{kj} U_{B_j}(t)(E^{\mathrm{w}}_{kj})^*)\Big)(\lambda) = \partial_\lambda \tilde{Q}_{kj}(\lambda).$$

From (7.1.16) we have

$$\tilde{Q}_{kj}(\lambda) = Q_{kj}(0, \lambda) - i\partial_t \partial_\lambda Q_{kj}(0, \lambda) + R_{kj}(\lambda),\ \ R_{kj} \in S^{n-2}(\mathbb{R}_\lambda). \tag{7.1.17}$$

• **The expansion $\tilde{Q}_{kj}$.** For the first term in (7.1.17) we have

$$Q_{kj}(0, \lambda) = (2\pi)^{-n} \int_{\{p_2 + \lambda_{1,j} \le \lambda\}} c_{kj}(0, X)\ dX.$$

Now,

$$c_{kj}(t, X) = \mathsf{Tr}\left(e^{-i(\phi_2+\phi_{1,j})}(E_{kj}\#(e^{i(\phi_2+\phi_{1,j})}\alpha_j)\#E^*_{kj})\right)(t, X),\ t \in (-\varepsilon, \varepsilon),$$

whence

$$c_{kj}(0, X) = \mathsf{Tr}\left(E_{kj}\#E^*_{kj}\right)(X), \tag{7.1.18}$$

since $U_0(0) = I$ and $F_j(0) = I_{N_j}$ by construction.
As for the next term in the expansion (7.1.17), with $\langle\cdot|\cdot\rangle$ denoting the distributional duality in the X variables, and recalling that

$$Q_{kj}(t,\lambda) = (2\pi)^{-n}\langle H(\psi_j(t,\cdot)+\lambda)|c_{jk}(t,\cdot)\hat{\rho}(t)\rangle,$$

we have

$$\begin{aligned}-i(\partial_t Q_{kj})(0,\lambda) =&(2\pi)^{-n}\left.\langle H(\psi_j+\lambda)|-i\partial_t c_{kj}\rangle\right|_{t=0}\\ &-i(2\pi)^{-n}\left.\langle \delta(\psi_j+\lambda)|c_{kj}\partial_t\psi_j\rangle\right|_{t=0}\\ =&-(2\pi)^{-n}\left.\langle H(\lambda-p_2)|i\partial_t c_{kj}\rangle\right|_{t=0}+\tilde{r}_{kj}(\lambda),\end{aligned} \tag{7.1.19}$$

where $\tilde{r}_{kj}\in S^{n-1/2}(\mathbb{R}_\lambda)$, and H and δ are the Heaviside and Delta distributions. Therefore, we need to compute $\partial_t c_{kj}(0,\cdot)$. Put $h_0(t,\cdot)$ for the (Weyl) symbol of $U_0(t)$, and for $j=1,2$ denote by $h_j(t,\cdot)$ and by $f_j(t,\cdot)$ those of $U_{B_j}(t)$ and of $F_j(t)$, respectively. We then have

$$\begin{aligned}\partial_t c_{kj}(0,\cdot) =& \partial_t \mathsf{Tr}\left(e^{-i(\phi_2(t)+\phi_{1,j}(t))}E_{kj}\#h_j\#E^*_{kj}\right)\Big|_{t=0}\\ =&\mathsf{Tr}\left((\partial_t e^{-i(\phi_2(t)+\phi_{1,j}(t))})E_{kj}\#h_j\#E^*_{kj}\right)\Big|_{t=0}\\ &+\mathsf{Tr}\left(e^{-i(\phi_2(t)+\phi_{1,j}(t))}E_{kj}\#\partial_t h_0\#f_j\#E^*_{kj}\right)\Big|_{t=0}\\ &+\mathsf{Tr}\left(e^{-i(\phi_2(t)+\phi_{1,j}(t))}E_{kj}\#h_0\#\partial_t f_j\#E^*_{kj}\right)\Big|_{t=0}\\ =&\mathsf{Tr}\left(ip_2E_{kj}\#E^*_{kj}\right)+\mathsf{Tr}\left(i\lambda_{1,j}E_{kj}\#E^*_{kj}\right)+\mathsf{Tr}\left(-iE_{kj}\#p_2\#E^*_{kj}\right)\\ &+\mathsf{Tr}\left(-iE_{kj}\#\lambda_{1,j}\#E^*_{kj}\right)+\mathsf{Tr}\left(-iE_{kj}\#b_{0,j}\#E^*_{kj}\right).\end{aligned}$$

Recalling that $b_{0,j}$ is the subprincipal term of B_j^{w} and denoting by $e_{0,kj}$ the principal symbol of E^{w}_{kj}, we therefore have

$$\partial_t c_{kj}(0,\cdot) = -\frac{1}{2}\mathsf{Tr}\Big(e_{0,kj}\{p_2,e^*_{0,kj}\}+\{e_{0,kj},p_2\}e^*_{0,kj}\Big)-i\mathsf{Tr}\left(e_{0,kj}b_{0,j}e^*_{0,kj}\right)+s_{kj}, \tag{7.1.20}$$

where $s_{kj}\in S(m^{-1},g)$. By taking ∂_λ of $(\partial_t Q_{kj})(0,\cdot)$ in (7.1.19) we hence have

$$-i(\partial_\lambda\partial_t Q_{kj})(0,\lambda) = -i(2\pi)^{-n}\left.\langle\delta(\lambda-p_2)|\partial_t c_{kj}\rangle\right|_{t=0}+O(\lambda^{n-3/2}),\ \lambda\to+\infty. \tag{7.1.21}$$

• **The asymptotics of $\mathsf{N}' * \rho$.** To obtain the result we have to integrate the following equation, which holds for any given real exponent $\gamma > 0$ (see [18], Lemma IV.1):

$$
\begin{aligned}
&(\mathsf{N}' * \rho)(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\Big(\hat{\rho}\mathrm{Tr}(e^{-itA^{\mathrm{w}}})\Big)(\lambda) \\
&= \mathcal{F}^{-1}_{t\to\lambda}\Big(\hat{\rho}\mathrm{Tr}\left(E^{\mathrm{w}}_{11}U_{B_1}(E^{\mathrm{w}}_{11})^* + E^{\mathrm{w}}_{12}U_{B_2}(E^{\mathrm{w}}_{12})^* + E^{\mathrm{w}}_{21}U_{B_1}(E^{\mathrm{w}}_{21})^* + E^{\mathrm{w}}_{22}U_{B_2}(E^{\mathrm{w}}_{22})^*\right)\Big)(\lambda) \\
&+O(\lambda^{-\gamma}) = \sum_{k,j=1}^{2} \partial_\lambda \tilde{Q}_{kj}(\lambda) + O(\lambda^{-\gamma}).
\end{aligned}
$$

Hence, to obtain (7.1.14) we need to compute $\sum_{k,j=1}^{2} \partial_\lambda \tilde{Q}_{kj}(\lambda)$. In the first place we note that, by (7.1.18), one has

$$
c_{1j}(0,\cdot) + c_{2j}(0,\cdot) = \mathsf{Tr}\left(E_{1j}\# E^*_{1j} + E_{2j}\# E^*_{2j}\right), \quad j = 1, 2.
$$

Hence, with $e_{-1,kj}$ denoting the semiprincipal symbol of E^{w}_{kj}, for $\tilde{r}$ a suitable symbol in $S_{\mathrm{sreg}}(m^{-2}, g)$, we have

$$
\begin{aligned}
(c_{1j} + c_{2j})(0,\cdot) =& \sum_{k=1}^{2} \mathsf{Tr}\left(e_{0,kj}e^*_{0,kj} + e_{-1,kj}e^*_{0,kj} + e_{0,kj}e^*_{-1,kj}\right) + \tilde{r} \\
=& \sum_{k=1}^{2} \mathsf{Tr}\left(e^*_{0,kj}e_{0,kj} + e^*_{0,kj}e_{-1,kj} + e^*_{-1,kj}e_{0,kj}\right) + \tilde{r} \qquad (7.1.22) \\
=& \mathsf{Tr}\left(I_{N_j}\right) + \tilde{r} = N_j + \tilde{r},
\end{aligned}
$$

where the third equality follows from the symbolic identity $E^*\# E = I_N$. Hence, by (7.1.21) we get

$$
\sum_{k=1}^{2} -i(\partial_\lambda\partial_t Q_{kj})(0,\lambda) = -i(2\pi)^{-n}\Big\langle \delta(\lambda - p_2)\Big|\sum_{k=1}^{2}\partial_t c_{kj}\Big\rangle\Big|_{t=0} + O(\lambda^{n-3/2}), \qquad (7.1.23)
$$

as $\lambda \to +\infty$. By (7.1.22) and (7.1.23), we have

$$\sum_{j=1}^{2}\sum_{k=1}^{2}\tilde{Q}_{kj}(\lambda) = \sum_{j=1}^{2}\sum_{k=1}^{2}\partial_\lambda Q_{kj}(0,\lambda) - i\partial_t\partial_\lambda Q_{kj}(0,\lambda) + R_{kj}(\lambda)$$

$$= (2\pi)^{-n}\sum_{j=1}^{2}N_j\int_{\{p_2+\lambda_{1,j}\leq\lambda\}}dX - i(2\pi)^{-n}\Big\langle\delta(\lambda-p_2)\Big|\sum_{k,j=1}^{2}\partial_t c_{kj}\Big\rangle\Big|_{t=0} + O(\lambda^{n-3/2}), \tag{7.1.24}$$

as $\lambda \to +\infty$, and, by (7.1.20),

$$-i\sum_{k,j=1}^{2}\partial_t c_{kj}(0) = -\,\mathsf{Tr}\Big(e_0 b_0 e_0^* - \frac{i}{2}(e_0\{p_2, e_0^*\} + \{e_0, p_2\}e_0^*)\Big) + \sum_{k,j=1}^{2}s_{kj} \tag{7.1.25}$$

$$= -\,\mathsf{Tr}\Big(b_0 + \frac{i}{2}(e_0^*\{p_2, e_0\} + \{e_0^*, p_2\}e_0)\Big) + \sum_{k,j=1}^{2}s_{kj}$$

$$= -\,\mathsf{Tr}(a_0) + \sum_{k,j=1}^{2}s_{kj},$$

where the third equality follows from Corollary 4.2.1.
Hence, (7.1.14) is obtained by substituting (7.1.25) into (7.1.24) and by recalling that $\delta(\lambda - p_2) = ds/|\nabla p_2|\big|_{p_2=\lambda}$.

From (7.1.14) one immediately gets the asymptotics (7.1.15), using the well-known polynomial growth of N, by writing the volume of $\{p_2 + \lambda_{1,j} \leq \lambda\}$ as λ^n times the volume of $\{p_2 + \lambda^{-1/2}\lambda_{1,j} \leq 1\}$ and expanding the latter in powers of $\lambda^{-1/2}$. □

We finally prove the refined asymptotics of $\mathsf{N}(\lambda)$ for a positive ψdo system A^{w} satisfying the hypotheses of Theorem 7.1.7 and **Condition DGW** (7.1.1). The proof goes through comparing N with $\mathsf{N} * \rho$ by a Tauberian argument whose hypotheses will be verified thanks to (7.1.14) and **Condition DGW** (7.1.1).

Theorem 7.1.8 (Refined Weyl law)**.** *Let $A = A^* \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$ be a second-order SMGES satisfying the hypotheses of Theorem 7.1.7. If* ***Condition DGW*** *(7.1.1) is satisfied, then*

$$\mathsf{N}(\lambda) = (2\pi)^{-n}\left(\sum_{j=1}^{r}\left(N_j\int_{p_2+\lambda_{1,j}\le\lambda} dX\right) - \int_{p_2=\lambda}\mathsf{Tr}\,(a_0)\,\frac{ds}{|\nabla p_2|}\right) + o(\lambda^{n-1}),\ \lambda\to+\infty. \tag{7.1.26}$$

In particular, as $\lambda \to +\infty$,

$$\begin{aligned}\mathsf{N}(\lambda) = (2\pi)^{-n}\Big(N\lambda^n\int_{p_2\le 1} dX - \lambda^{n-1/2}\int_{p_2=1}\mathsf{Tr}(a_1)\frac{ds}{|\nabla p_2|}\\ +\lambda^{n-1}\int_{p_2=1}\Big(\frac{n}{2}\mathsf{Tr}(a_1^2) - \mathsf{Tr}(a_0)\Big)\frac{ds}{|\nabla p_2|}\Big) + o(\lambda^{n-1}).\end{aligned} \tag{7.1.27}$$

Proof. Fix an even and positive cutoff function $\rho \in \mathscr{S}(\mathbb{R}^{2n})$ in the time variable such that $\hat\rho = 1$ on $(-\varepsilon, \varepsilon)$ for some $\varepsilon \in (0, \pi/2)$ and $\operatorname{supp}\hat\rho \subset (-\pi/2, \pi/2)$.
We have to show that, under our assumptions, $(\mathsf{N} * \rho)(\lambda) = \mathsf{N}(\lambda)$ modulo an error which is $o(\lambda^{n-1})$, so that the result follows from the asymptotics (7.1.14) by using the following Tauberian theorem (see [55], Theorem B.5.1) which allows the required comparison between N and $\mathsf{N} * \rho$.

Lemma 7.1.9. *Let ρ be fixed as above. If there is a real number γ such that $(\mathsf{N}' * \rho)(\lambda) = O(\lambda^\gamma)$ and $(\mathsf{N}' * \chi)(\lambda) = o(\lambda^\gamma)$ for all χ satisfying $\hat\chi \in C_c^\infty(\mathbb{R})$,* $\operatorname{supp}\hat\chi \subset (0, +\infty)$*, then $\mathsf{N}(\lambda) = (\mathsf{N} * \rho)(\lambda) + o(\lambda^\gamma)$ as $\lambda \to +\infty$.*

We have therefore to prove that

$$\mathcal{F}^{-1}_{t\to\lambda}\big(\chi(t)\mathrm{Tr}\; e^{-itA^{\mathrm{w}}}\big)(\lambda) = o(\lambda^{n-1}), \tag{7.1.28}$$

for any given $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (0, \infty)$ (here χ is playing the role of $\hat\chi$ in Lemma 7.1.9). Since Theorem 7.1.7, in particular, shows that

$$(\mathsf{N}' * \rho)(\lambda) = O(\lambda^{n-1}),$$

it follows that if we have (7.1.28) then the hypotheses of Lemma 7.1.9 are fulfilled.

Now, by Proposition 1.1 and Section 3 in [11] we have that sing supp Tr $U(t) \subset 2\pi\mathbb{Z}$, whence we need to check (7.1.28) only for $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (2\pi k - \varepsilon, 2\pi k + \varepsilon)$ where $k \in \mathbb{Z} \setminus \{0\}$ and $\varepsilon \in (0, \pi/2)$. Now, for all real γ (again, we suppose without loss of generality $r = 2$), we have

$$\begin{aligned}\mathcal{F}^{-1}_{t\to\lambda}\big(\chi(t)\mathrm{Tr}\; e^{-itA^{\mathrm{w}}}\big)(\lambda) =&\mathcal{F}^{-1}_{t\to\lambda}\Big(\chi\mathrm{Tr}\,(E^{\mathrm{w}}_{11}U_{B_1}(E^{\mathrm{w}}_{11})^* + E^{\mathrm{w}}_{12}U_{B_2}(E^{\mathrm{w}}_{12})^*\\ &+E^{\mathrm{w}}_{21}U_{B_1}(E^{\mathrm{w}}_{21})^* + E^{\mathrm{w}}_{22}B_2(E^{\mathrm{w}}_{22})^*)\Big)(\lambda) + O(\lambda^{-\gamma}),\end{aligned}$$

(see Lemma 4.7 in [11]) and for all j, k

$$\mathcal{F}^{-1}_{t\to\lambda}\left(\chi\mathrm{Tr}\left(E^{\mathrm{w}}_{kj}U_{B_j}(E^{\mathrm{w}}_{kj})^*\right)\right)(\lambda) = \int e^{it\lambda}e^{i(\phi_2(t,X)+\phi_{1,j}(t,X))}\chi(t)c(t,X)\,dt\,dX,$$

where c is a suitable amplitude and ϕ_2, $\phi_{1,j}$ are given as in the proof of Theorem 7.1.7. Hence, we are in a position to use Proposition 5.1 of [11], with $\psi_2 := \phi_2$ and $\psi_1 := \phi_{1,j}$. Since $\phi_2 := -2\tan(t/2)p_2$ and χ is supported close to $2\pi k$, the hypotheses of that proposition for the phases ψ_2, ψ_1 and amplitude c are satisfied (in the notation of that proposition, we take $t_0 = 2\pi k$ and $r_0 = \sqrt{2}$).

Now, since $\phi_{1,j}(2k\pi, X) = -k\mathsf{R}(\lambda_{1,j})(X)$, **Condition DGW** (7.1.1) yields that the set of the $\omega \in \mathbb{S}^{2n-1}$ at which $\partial^\alpha_\omega\phi_{1,j}(2k\pi,\omega)$, $|\alpha| = 1$, vanish to infinite order ($j = 1, \ldots, r$) has measure zero for all $k \in \mathbb{Z} \setminus \{0\}$. Thus, Proposition 5.1 in [11] shows that

$$\mathcal{F}^{-1}_{t\to\lambda}\left(\chi\mathrm{Tr}\left(E^{\mathrm{w}}_{kj}U_{B_j}(E^{\mathrm{w}}_{kj})^*\right)\right) = o(\lambda^{n-1}),$$

for all $j, k = 1$, ..., r.for all $j, k = 1$, ..., r.

The final formula (7.1.27) is obtained by Taylor-expanding the volume term in (7.1.26).

The proof is complete. □

7.2 Some examples.

In this section we will provide the calculation of the Weyl-asymptotics in the case of the JC-model and of Refined Weyl-asymptotics in the case of bigger size systems.

7.2.1 Refined Weyl-asymptotics for the JC-model 2.1.1 ($n = 1,\ N = 2$).

In this case we have that $\lambda_{\pm}(X) = \pm|\alpha||\psi(X)|$, and $a_0(X) = \gamma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $\alpha \neq 0$ and γ are real numbers. Since the eigenvalues of a_1 are constant on the level sets of $p_2(X) = |X|^2/2$, **Condition DGW** does not hold and we may only have the classical Weyl law

$$\mathsf{N}(\lambda) = (2\pi)^{-1} 2\lambda \int_{p_2 \leq 1} dX + O(1) = 2\lambda + O(1),\ \lambda \to +\infty.$$

7.2.2 Refined Weyl-asymptotics for bigger size systems.

Note that in the JC-models with $n = N - 1$ atom levels (and their geometric generalizations) we have that an eigenvalue of the semiprincipal term is 0 with a fixed multiplicity. In this case $\Pi_{2\pi} = \mathbb{S}^{2n-1}$for the eigenvalue 0 and we cannot achieve a refined Weyl-law. However, let us consider the following deformation of the JC-model in the Ξ-configuration (2.1.3). Let $n = 2$ and $N = 3$. Recall that $\psi_j(X) = (x_j + i\xi_j)/\sqrt{2}$ is the symbol of the annihilation operator in the x_j variable, $j = 1, 2$. For $\alpha_1, \alpha_2 \neq 0$ real, we put $\alpha\psi := (\alpha_1\psi_1, \alpha_2\psi_2)$ and consider the functions $f_j(X) = \alpha_j\psi_j(X)/|\alpha\psi(X)|$, $X \neq 0$, which are homogeneous of degree 0, $j = 1, 2$. Let

$$\lambda_+,\ \lambda_-,\ \mu \in C^\infty(\dot{\mathbb{R}}^{2n}; \mathbb{R})$$

be homogeneous of degree 1, such that

$$\lambda_-(X) < \lambda_+(X),\ \lambda_+(X) - \lambda_-(X) \approx |X|,\ |\lambda_\pm(X) - \mu(X)| \approx |X|,\ X \neq 0,$$

with

$$\textit{either } \mu(X) \in (\lambda_-(X), \lambda_+(X)), \;\; \textit{or} \;\; \mu(X) \notin (\lambda_-(X), \lambda_+(X)), \;\; \forall X \neq 0.$$

We consider then

$$\begin{aligned} A_{1,\mu} &= \begin{bmatrix} \mu|f_2|^2 + \frac{\lambda_+ + \lambda_-}{2}|f_1|^2 & \frac{\lambda_+ - \lambda_-}{2}\bar{f}_1 & (-\mu + \frac{\lambda_+ + \lambda_-}{2})\bar{f}_1\bar{f}_2 \\ \frac{\lambda_+ - \lambda_-}{2} f_1 & \frac{\lambda_+ + \lambda_-}{2} & \frac{\lambda_+ - \lambda_-}{2}\bar{f}_2 \\ (-\mu + \frac{\lambda_+ + \lambda_-}{2}) f_1 f_2 & \frac{\lambda_+ - \lambda_-}{2} f_2 & \mu|f_1|^2 + \frac{\lambda_+ + \lambda_-}{2}|f_2|^2 \end{bmatrix} \\ &= e_0 \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda_+ & 0 \\ 0 & 0 & \lambda_- \end{bmatrix} e_0^*, \end{aligned}$$

where

$$e_0(X) = \begin{bmatrix} -\overline{f_2(X)} & \overline{f_1(X)}/\sqrt{2} & \overline{f_1(X)}/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ f_1(X) & f_2(X)/\sqrt{2} & f_2(X)/\sqrt{2} \end{bmatrix},$$

is smooth and unitary for $X \in \mathbb{R}^4$, $X \neq 0$, and homogeneous of degree 0. Then, if we require that the sets

$$\begin{aligned} &\{\omega \in \mathbb{S}^3;\; \partial_\omega^\alpha \mu(\omega) = 0, \; \forall \alpha \in \mathbb{N}^{2n-1} \setminus \{0\}\}, \\ &\{\omega \in \mathbb{S}^3;\; \partial_\omega^\alpha \lambda_\pm(\omega) = 0, \; \forall \alpha \in \mathbb{N}^{2n-1} \setminus \{0\}\} \end{aligned}$$

have measure zero, we have a refined Weyl-law

$$\begin{aligned} \mathsf{N}(\lambda) = (2\pi)^{-2}\Big(3|\mathbb{S}^3|\lambda^2 - \lambda^{3/2}\int_{p_2=1} (\lambda_+ + \lambda_- + \mu)\frac{ds}{|\nabla p_2|} \\ + \lambda \int_{p_2=1} \Big(\lambda_+^2 + \lambda_-^2 + \mu^2 - (\gamma_1 + \gamma_2)\Big)\frac{ds}{|\nabla p_2|} + o(1)\Big), \;\; \lambda \to +\infty. \end{aligned}$$

In particular, in the case of Subsection 2.1.3 we have $\lambda_\pm = \pm|\alpha\psi|$, and a computation shows that in coordinates

$$\omega = (\sin\theta_3 \sin\theta_2 \cos\theta_1, \sin\theta_3 \sin\theta_2 \sin\theta_1, \sin\theta_3 \cos\theta_2, \cos\theta_3),$$

with $\theta_1 \in [0, 2\pi]$ and $\theta_2, \theta_3 \in [0, \pi]$,

$$\lambda_+(\omega)^2 = \frac{1}{2}\Big(\alpha_2^2 + \sin^2\theta_3 \sin^2\theta_2(\alpha_1^2 - \alpha_2^2)\Big).$$

Therefore, when $\alpha_1^2 \neq \alpha_2^2$, the sets $\Pi_{2\pi,\pm}$ have measure zero. Hence, considering

$$\mu(X) = \kappa\lambda_+(X) + (1-\kappa)\lambda_-(X), \text{ for some } \ \kappa \in (0,1),$$

yields that for $\alpha_1^2 \neq \alpha_2^2$ and $\kappa \neq 1/2$, the system with the semiprincipal part

$$A_{1,\mu} = \begin{bmatrix} \mu|f_2|^2 & \frac{\lambda_+-\lambda_-}{2}\bar{f}_1 & -\mu\bar{f}_1\bar{f}_2 \\ \frac{\lambda_+-\lambda_-}{2}f_1 & 0 & \frac{\lambda_+-\lambda_-}{2}\bar{f}_2 \\ -\mu f_1 f_2 & \frac{\lambda_+-\lambda_-}{2}f_2 & \mu|f_1|^2 \end{bmatrix}$$

satisfies the hypoteses of the refined Weyl-law and we therefore have

$$\mathsf{N}(\lambda) = (2\pi)^{-2}\Big(3|\mathbb{S}^3|\lambda^2 - \lambda^{3/2}\int_{p_2=1}\mu\frac{ds}{|\nabla p_2|}$$

$$+\lambda\int_{p_2=1}\Big(\lambda_+^2+\lambda_-^2+\mu^2-(\gamma_1+\gamma_2)\Big)\frac{ds}{|\nabla p_2|}+o(1)\Big), \ \lambda\to+\infty.$$

By tensorizing the symbols with I_2, one readily obtains, in the same hypotheses on μ, the refined Weyl-law for the 6×6 Laplacian $\Box_1^{(3)}$ (see Subsection 2.2.1) with semiprincipal term $A_{1,\mu}\otimes I_2$.

Chapter 8

The spectral zeta function: meromophic continuation

In this chapter we give a meromorphic continuation of the spectral zeta function associated with a class of elliptic *semiregular* differential systems, including models of semiregular NCHOs in the class SMGES, relevant to Quantum Optics, as those introduced in Section 2.1. As an application of our results, we first compute the meromorphic continuation of the JC-model spectral zeta function. Then we compute the spectral zeta function of the JC-model generalization to a 3-level atom in a cavity in the Ξ-configuration. For both of them we show that it has only one pole in $s = 1$.

8.1 Statement of the problem

One of the most important observables of the spectrum of an elliptic operator is the spectral zeta function. For a complex Hilbert space H and a densely defined linear operator $P : D(P) \subset H \to H$, we denote the set of the eigenvalues (repeated by multiplicity) of P by $\operatorname{Spec} P$. When $\operatorname{Spec} P$ is discrete we can define the spectral zeta function of P as

$$\zeta_P(s) := \sum_{\lambda \in \operatorname{Spec} P} \lambda^{-s},$$

for any given complex number s for which it makes sense. In particular, if P is an elliptic, self-adjoint and positive global pseudodifferential operator of order $\mu > 0$ on $\mathbb{R}^n$, then $s \mapsto \zeta_P(s)$ is holomorphic for $\mathrm{Re} s > 2n/\mu$ since the defining series is absolutely convergent there (see Corollary 4.4.4. in [45]). For instance, if we denote by $P = \frac{x^2-\partial_x^2}{2}$ the harmonic oscillator defined as the maximal operator in $L^2(\mathbb{R})$, then $\mathrm{Spec}\, P = \{k + 1/2;\ k \in \mathbb{Z}_+\}$ with multiplicity 1, and

$$\zeta_P(s) = \sum_{k\geq 0}(k+1/2)^{-s} = (2^s - 1)\zeta(s),$$

where $\zeta(s)$ denotes the Riemann zeta function. Note that ζ_P is holomorphic for $\mathrm{Re}(s) > 1$, and has a meromorphic continuation to the whole complex plane. Furthermore, ζ_P has the only pole at $s = 1$, and we have $\zeta_P(s) = 0$ for $s = -2k$, $k \in \mathbb{Z}_+$ which are, thus, called trivial zeros. Moreover, the spectral zeta function entangles information about the spectrum of P in its analytical properties. For instance, the residues of the zeta function at its poles gives the coefficients of the Weyl law for P by the Ikehara Tauberian theorem (see Section 14 of Shubin [58]. See also Proposition (IV.6) in [18] and the references in Ivrii [31]).

The notion of spectral zeta function was introduced for the first time for the Laplacian on a two-dimensional Euclidean domains Ω by Carleman [8], who studied the Dirichlet-type series

$$\sum_{\lambda_j\in\mathrm{Spec}\,\Delta}\frac{\phi_{\lambda_j}(x_1)\phi_{\lambda_j}(x_2)}{\lambda_j^s},\ x_1, x_2 \in \Omega, \tag{8.1.1}$$

where ϕ_{λ_j} is the eigenfunction of Δ associated with the eigenvalue λ_j. Later, in the case of a bounded Euclidean domain V of arbitrary dimension N, Minakshisundaram [36] showed, through a method different from Carleman's, that (8.1.1) is an entire function of s with zeros at negative integers and that

$$\sum_{\lambda_j\in\mathrm{Spec}\,\Delta}\frac{\phi_{\lambda_j}(x_1)^2}{\lambda_j^s}$$

can be continued as a meromorphic function of s with a unique simple pole at $N/2$ and negative integer zeros. Next, the analytic continuation of the spectral zeta function was studied by Minakshisundaram and Pleijel [37] for the Laplacian on a general compact manifold by a method that is a generalization of Carleman's. Seeley [56] studied the spectral zeta function of an elliptic ψdo on a compact manifold without boundary through the trace of complex powers of ψdos, furthermore giving the value of the zeta function at 0.

Many different techniques have been used to obtain properties of the spectral zeta function. Duistermaat and Guillemin [13] (see also [16] and the references in Hormander [28]) studied systematically the spectral zeta function of ψdos on compact boundaryless manifolds basing their approach on the construction of a parametrix for the wave equation. Robert [51] (see also Aramaki [2]) extended meromorphically the spectral zeta function of an elliptic ψdo on $\mathbb{R}^n$ to the whole complex plane with simple poles that he computed along with the corresponding residues. He generalized to the global setting the techniques by Seeley to construct the parametrix of the resolvent by complex powers.

Moreover, we recall that a second order regular Non-Commutative Harmonic Oscillators (NCHOs) is the class of the *regular* global partial differential systems of second order with polynomial coefficients. From now on we will omit the expression "second order" since all the NCHOs considered will be of second order.

Ichinose and Wakayama [30] obtained a meromorphic continuation of the spectral zeta function of a subclass of regular NCHOs and determined some of its special values. In addition, they showed that such a spectral zeta function has only a simple pole at 1 and that the sequence of its trivial zeros coincides with the one of the Riemann zeta function, the non-positive even integers. Their approach is based on the Mellin transform of the heat-semigroup of the operator in the approximation given by a parametrix which they computed *directly*, without using the one for the resolvent, obtaining its asymptotic expansion (see (15) and (16) in their paper). Later, Parmeggiani [45] generalized that approach to obtain the meromorphic continuation of the spectral zeta function of all the regular NCHOs. Nevertheless, while gaining in generality, unfortunately his result did not explicitly locate the trivial zeros of the continuation of the spectral zeta function as could Ichinose and Wakayama.

Ichinose and Wakayama's and Parmeggiani's papers deal with *regular* systems. Regarding the *semiregular* systems, Sugiyama explored in [59] the Hurwitz-type spectral zeta function for the quantum Rabi model which describes the interaction of light and matter of a two-level atom coupled with a single quantized photon of the electromagnetic field even when the field is not near resonance with the atomic transition and the coupling strength is not weak. This model will be treated in Chapter 10.

In this chapter we study the properties of the spectral zeta function associated with a positive elliptic *semiregular positive partial differential systems* with polynomial coefficients, including *also* models of semiregular NCHOs in the class SMGES. This class contains models relevant to Quantum Optics, such as the Jaynes-Cummings model. Here we follow the construction of the zeta function provided by Ichinose and Wakayama, in analogy to the approach by Parmeggiani in Theorem 7.2.1 of [45].

We will prove a result about the continuation of the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ which turns out to be a meromorphic function whose poles are real and accumulate at $-\infty$. Namely, we will give the continuation as a linear combination of the meromorphic functions $s \mapsto \frac{1}{s-(n-j)+h/2}$, $j \geq 0$ and $h = 0$, 1, modulo a function that is holomorphic on a complex half-plane. Notice that even if in principle our extension can have poles in all the negative semi-integers, unlike the results in [30], [45] and [59] (where the poles are all positive), we prove that only the positive integers are poles for the spectral zeta function of a differential operator with polynomial coefficients. The meromorphic continuation is obtained by following the approach of Theorem 7.2.1 in [45] where the parametrix approximation $U_A(t)$ of the heat-semigroup $e^{-tA^{\mathrm{w}}}$is used. More precisely, by the Mellin transform we can write $\zeta_{A^{\mathrm{w}}}$ as $s \mapsto \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt$ for $\mathrm{Re} s > 2n/2 = n$ and, at this point, the asymptotic expansion $\sum\limits_{j \geq 0} b_{-j}(t)$ (in the sense of Remark 6.1.5 at p. 83 of [45]) of $U_A(t)$ with $t \in \overline{\mathbb{R}}_+$ becomes crucial. In fact, the approximation of

$s \mapsto \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt$ by $s \mapsto \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{Tr}\, U_A(t)\, dt$ leads to the study of integrals of the form

$$(2\pi)^{-n}\int_{\mathbb{R}^{2n}} \chi(X)\mathsf{Tr}\,(b_{-2j-h}(t,X))\, dX,\ j\in\mathbb{N},\ h=0,1, \tag{8.1.2}$$

where χ is a chosen excision function and Tr is the classical matrix trace. In fact, the computation of (8.1.2) will give the coefficients of the linear combination of the aforementioned meromorphic functions. These coefficients will contribute to determine the residues and zeros of the spectral zeta function. Now one needs to go through a Taylor expansion argument as the time variable $t \to 0+$ of the terms arising from the study of $\mathrm{Tr}\, e^{-tA^{\mathrm{w}}} - \sum_{j=0}^{\nu}\sum_{h=0}^{1}\mathrm{Tr}\, B_{-2j-h}(t)$, where B_{-k} has principal symbol b_{-k} (the behavior of $e^{-tA^{\mathrm{w}}}$ as $t \to +\infty$ does not affect the result).

This is a delicate argument since the behaviour of the coefficients of the linear combination of the above meromorphic functions must be controlled as $t \to 0+$.

The plan of this chapter is the following. First of all, the notation adopted will be introduced in Section 8.2, along with the parabolic ψdifferential calculus needed to define the heat-semigroup parametrix which will be constructed directly in Section 8.3 by computing the terms of its asymptotic expansion through the solution of eikonal and transport equations. After that, in Section 8.4, we will control the behaviour of the coefficients. We will give the proof of our theorem in Section 8.5. Actually, in Section 8.5 we will also obtain a meromorphic continuation for the Hurwitz spectral zeta function $\zeta_{A^{\mathrm{w}}+\tau I}$ for all $\tau \geq 0$. Next, in Section 8.6, by using our results in the previous sections, we will compute the meromorphic continuation of the spectral zeta function for the Hamiltonians of Jaynes-Cummings and its generalization to a 3-level atom in one cavity. For these Hamiltonians we will show that the meromorphic continuation has only a simple pole at $s = 1$ and no other (even if, recall, the general formula allows all the negative semi-integer as poles). Finally, in Section 8.7 we prove that the spectral zeta function of a differential operator with polynomial coefficients does not have semi-integer poles as a corollary of the previous results of this chapter.

8.2 Parabolic calculus

In this section, similarly to what is done by Parenti and Parmeggiani in [40] (see also Section 6.1 of [45]), we will introduce a class of symbols suitable for the construction of a pseudodifferential approximation of $e^{-tA^{\mathrm{w}}}$. Let us recall the notation $\overline{\mathbb{R}}_+ = [0, +\infty)$. We will be using the following notation for the Hörmander metric and admissible weight (see Hörmander [26]): with $X = (x, \xi)$, $Y = (y, \eta)$, etc., belonging to the phase-space $\mathbb{R}^n \times \mathbb{R}^n$, and $m(X) := \langle X \rangle = (1 + |X|^2)^{1/2}$ the usual "Japanese bracket", we consider the Hörmander metric $g_X = |dX|^2/m(X)^2$. Then, m is an admissible function (and so is m^μ for any given $\mu \in \mathbb{R}$), and we may exploit the full power of the Weyl-Hörmander pseudodifferential calculus.

Definition 8.2.1. *Let M_N denote the algebra of $N \times N$ complex-values matrices. A symbol $a \in S(m^\mu, g; \mathsf{M}_N)$ is said to be* ***semiregular*** *(see Remark 3.2.4 of [45]), and we write $a \in S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$ if it possesses an asymptotic expansion $\sum_{j \geq 0} a_{\mu - j}$ in isotropic (i.e. positively homogenous and smooth outside the origin) terms $a_{\mu-j}$ positively homogeneous of degree $\mu - j$.*

Moreover, we define also a class of symbols depending on the time variable t.

Definition 8.2.2. *Let $r \in \mathbb{R}$. By $S(\mu, r)$ we denote the set of all smooth maps $b : \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathsf{M}_N$ satisfying the following estimates: for any given $\alpha \in \mathbb{Z}_+^{2n}$ and any given p, $j \in \mathbb{Z}_+$ there exists $C > 0$ such that*

$$\sup \left| t^p \left(\frac{d}{dt} \right)^j \partial_X^\alpha b(t, X) \right| \leq C m(X)^{r - |a| + (j-p)\mu}. \tag{8.2.1}$$

For $b \in S(\mu, r)$ we then consider the pseudodifferential operator

$$b^{\mathrm{w}}(t, x, D) u(x) = (2\pi)^{-n} \iint e^{i(x-y, \xi)} b\left(t, \frac{x+y}{2}, \xi\right) u(y) dy d\xi, \; u \in \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N),$$

and we shall say that $B \in \mathrm{OP}S(\mu, r)$ *if* $B = b^{\mathrm{w}}(t, x, D) + R$*, where* R *is smoothing. In this setting, a smoothing operator* R *is any* ***continuous*** *map*

$$R\colon \mathscr{S}'(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathscr{S}(\overline{\mathbb{R}}_+; \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)).$$

Then we introduce the "classical operators". In this case, the key is to take into account the correct homogeneity properties. The basic example to keep in mind is the matrix $e^{-ta_\mu(x,\xi)}$.

Definition 8.2.3. *We say that the operator* $B \in \mathrm{OP}S(\mu, r)$*,* $B = b^{\mathrm{w}} + R$ *is* ***classical****, and write* $B \in \mathrm{OP}S_{\mathrm{cl}}(\mu, r)$*, if there exists a sequence of functions* $b_{r-2j} = b_{r-2j}(t, X), j \geq 0, t \geq 0$ *and* $X \neq 0$*, such that:*

1. *One has the homogeneity*

$$b_{r-2j}(t, \tau X) = \tau^{r-2j} b_{r-2j}(\tau^\mu t, X),\ \forall \tau > 0,\ \forall j \geq 0; \tag{8.2.2}$$

2. *The function*

$$\mathbb{R}^{2n} \setminus \{0\} \ni X \longmapsto b_{r-2j}(\cdot, X) \in \mathscr{S}(\overline{\mathbb{R}}_+; \mathsf{M}_N),$$

is smooth for all $j \geq 0$*;*

3. *For all* $\nu \geq 1$

$$b(t, X) - \sum_{j=0}^{\nu=1} \chi(X) b_{r-2j}(t, X) \in S(\mu, r - 2\nu), \tag{8.2.3}$$

where χ *is an excision function.*

Remark 8.2.4. *We call* $b_r = \sigma_r(B)$ *the* ***principal symbol*** *of* B*.*

Remark 8.2.5. ***Semiregular classical*** *symbols are defined accordingly, considering also terms with odd degree of homogeneity in the expansion formula (8.2.3), and the class of pseudodifferential operators associated with them is denoted by* $\mathrm{OP}S_{\mathrm{sreg}}(\mu, r)$*.*

8.3 Parametrix of the heat-semigroup

In this section we will construct the parametrix of the heat-semigroup of a semiregular positive elliptic pseudodifferential operator.

Lemma 8.3.1. *Let* $A = A^*$*, with* $A \sim \sum_{j\geq 0} a_{\mu-j} \in S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$*, be an elliptic system such that* $A^{\mathrm{w}} > 0$*. Then, there exists* $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, 0)$ *such that*

$$\frac{d}{dt}U_A + A^{\mathrm{w}}U_A : \mathscr{S}'(\mathbb{R}^n;\mathbb{C}^N) \to \mathscr{S}(\overline{\mathbb{R}}_+;\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N))$$

is smoothing, and

$$U_A|_{t=0} - I_N : \mathscr{S}'(\mathbb{R}^n;\mathbb{C}^N) \to \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$$

is smoothing. Moreover, the principal symbol of U_A *is*

$$\overline{\mathbb{R}}_+ \times \big(\mathbb{R}^{2n} \setminus \{0\}\big) \ni (t, X) \mapsto e^{-ta_\mu(X)}.$$

Proof. We will prove the lemma by constructing the terms of the expansion of the symbol of U_A. In fact, we determine those terms by solving a sequence of transport equations.

Let

$$\overline{\mathbb{R}}_+ \times \big(\mathbb{R}^{2n} \setminus \{0\}\big) \ni (t, X) \mapsto b_0(t, X) := e^{-ta_\mu(X)},$$

and let $B_0 \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, 0)$ with principal symbol given by b_0. Hence, by Lemma 6.1.3 at p. 81 of [45], we have that $\frac{d}{dt}B_0 + A^{\mathrm{w}}B_0 \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, \mu-1)$ with principal symbol $r_{\mu-1} := a_{\mu-1}b_0$. Moreover, $B_0|_{t=0} - I_N$ is a pseudodifferential system with symbol in $S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$ and we denote its principal symbol by p_{-1}.

Next, we look for a symbol $b_{-1}(t, X)$, positively homogeneous of degree -1 (in the sense of (8.2.2)), such that

$$\begin{cases} \frac{d}{dt}b_{-1} + a_\mu b_{-1} = -r_{\mu-1}, \\ b_{-1}|_{t=0} = -p_{-1}. \end{cases} \tag{8.3.1}$$

The solution of (8.3.1),

$$b_{-1}(t, X) := -e^{-ta_\mu(X)} p_{-1}(X) - \int_0^t e^{-(t-t')a_\mu(X)} r_{\mu-1}(t', X)\, dt',$$

is easily seen to be smooth and have the required homogeneity properties since

$$\begin{aligned} b_{-1}(t, \tau X) &= - e^{-ta_\mu(\tau X)} p_{-1}(\tau X) - \int_0^t e^{-(t-t')a_\mu(\tau X)} r_{\mu-1}(t', \tau X)\, dt' \\ &= - e^{-\tau^\mu t a_\mu(X)} \tau^{-1} p_{-1}(X) - \int_0^t e^{-\tau^\mu(t-t')a_\mu(X)} \tau^{\mu-1} r_{\mu-1}(\tau^\mu t', X)\, dt' \\ &= \tau^{-1} \left(-e^{-\tau^\mu t a_\mu(X)} p_{-1}(X) - \int_0^{\tau^\mu t} e^{-(\tau^\mu t - t')a_\mu(X)} r_{\mu-1}(t', X)\, dt' \right) \\ &= \tau^{-1} b_{-1}(\tau^\mu t, X), \end{aligned}$$

where the last equality follows from the change of variable $t \to \tau^{-\mu} t$ in the integral. Taking $B_{-1} \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, -1)$ with principal symbol given by b_{-1} gives

$$\frac{d}{dt}(B_0 + B_{-1}) + A^{\mathrm{w}}(B_0 + B_{-1}) \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, \mu - 2).$$

Moreover, $(B_0 + B_{-1})|_{t=0} - I_N$ is a pseudodifferential system with symbol in $S_{\mathrm{sreg}}(m^{-2}, g; \mathsf{M}_N)$ and we denote its principal symbol by p_{-2}.

Iterating the above procedure gives a formal series

$$\sum_{k \geq 0} B_{-k}, \; B_{-k} \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, -k).$$

Hence, there exists an operator $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(\mu, 0)$ for which

$$U_A - \sum_{k=0}^{\nu-1} B_{-k} \in \mathrm{OP}S(\mu, -\nu), \; \forall \nu \geq 1,$$

by an adaptation of Proposition 3.2.15 at p. 32 of [45], and therefore we obtain the required parametrix.

□

Remark 8.3.2. *In the applications of Lemma 8.3.1 we shall always consider a parametrix approximation of* $e^{-tA^{\mathrm{w}}}$ *where* $b_{-j}|_{t=0} = 0$ *for* $j \geq 1$,

$$B_{-j} := (\chi b_{-j})^{\mathrm{w}}(t, x, D_x),$$

for all $t \in \overline{\mathbb{R}}_+$, *where* χ *is a chosen excision function. Hence, consider the symbol* $c_A(t, X)$ *of* $U_A(t)$, *i.e.* $U_A(t) = c_A^{\mathrm{w}}(t, x, D)$, *given by*

$$c_A(t, X) = \sum_{j \geq 0} \chi_j(X) b_{-j}(t, X), \tag{8.3.2}$$

where $\chi_0(X) := \chi(X)$ *and* $\chi_j(X) := \chi(X/R_j)$, $j \geq 1$, *with* $R_j \nearrow +\infty$, *as* $j \to +\infty$, *sufficiently fast (for instance, see the proof of Proposition 3.2.15 at p. 32 of [45]). Thus, the series (8.3.2) is locally finite in* X *and, hence,* $c_A(t, \cdot) \in C^\infty$ *for all* $t \in \overline{\mathbb{R}}_+$.

From now on we will write $U_A \sim \sum_{j \geq 0} B_{-j}$.

8.4 Vanishing property

Let A^{w} be as in the previous section. In this section we prove the technical proposition that we need to control the behavior of the terms b_{-j} constructed in Lemma 8.3.1 as $t \to 0+$, that is, its vanishing property, for a class of positive and self-adjoint elliptic *differential* systems with symbol in $S_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$. Hence, we will suppose the symbol of A^{w} to be $a_2 + a_1 + a_0$ where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$.

Proposition 8.4.1. *Let* $A = a_2 + a_1 + a_0$ *be an elliptic symbol of second order where* a_j *is an* $N \times N$ *matrix-valued function on* $\mathbb{R}^{2n}$ *with homogeneous polynomial of degree* j *entries for all* $j = 0, 1, 2$. *Let* $A^{\mathrm{w}} > 0$, *and let* U_A *be the heat-semigroup* $e^{-tA^{\mathrm{w}}}$ *parametrix constructed by Lemma 8.3.1. Then, denoting again by* $\sum_{j \geq 0} B_{-j}$ *the expansion of* U_A *constructed in the proof of Lemma 8.3.1 and by* b_{-j} *the principal symbol of* B_{-j}, *we have, for all* $j \geq 0$, *and* $h = 0, 1$

$$b_{-2j-h}(t,\omega) = O(t^{j+h}),\ t \to 0+,$$

and for all $\alpha,\ \beta \in \mathbb{Z}_+^n$, *with* $|\alpha| = 2k+1,\ k \geq 0$ *and* $|\beta| \leq 1$ *we have:*

$$\partial_X^{\alpha+\beta} b_{-2j-h}(t,\omega) = O(t^{j+k+h|\beta|+1}),\ t \to 0+,$$

where the constants in $O(\cdot)$ *do not depend on* $\omega \in \mathbb{S}^{2n-1}$.

Proof. We prove this theorem by induction taking into account the definition of the terms b_{-j}.

We won't be writing the dependence on ω, and we will write $b_{-j}^{(\ell)}$ for a generic $\partial_X^\alpha b_{-j}$ with $|\alpha| = \ell$.

First of all, we remind that, given two pseudodifferential operators with symbol a and b, then by the composition law for pseudodifferential operators (see, for instance, formula (3.3) at p. 19 of [45]), $a^{\mathrm{w}} b^{\mathrm{w}}$ has symbol

$$a \# b \sim ab + \sum_{j \geq 1} \frac{1}{j!} \left(\frac{-i}{2}\right)^j \{a, b\}_{(j)},$$

where $\{\cdot,\cdot\}_{(1)} = \{\cdot,\cdot\}$ is the Poisson bracket.

The terms r_{2-j}, $j \geq 1$ obtained in the proof of Lemma 8.3.1 is

$$\begin{aligned} r_{2-j} =& a_0 b_{-(j-2)} + a_1 b_{-(j-1)} + \frac{1}{2}\left(\frac{-i}{2}\right)^2 \{a_2, b_{-(j-4)}\}_{(2)} \\ &- \frac{i}{2}\{a_2, b_{-(j-2)}\} - \frac{i}{2}\{a_1, b_{-(j-3)}\},\ j \geq 0, \end{aligned} \tag{8.4.1}$$

where we set $b_k \equiv 0$ for all $k = 1, \dots, 4$ and we recall that a_0 is a *constant* $N \times N$ Hermitian matrix. Therefore, by the construction in the proof of Lemma 8.3.1,

$$\begin{cases} b_0(t,X) = e^{-ta_2(X)}, \\ b_{-j}(t,X) = -\int_0^t e^{-(t-t')a_2} r_{2-j}(t',X)\,dt',\ j \geq 1. \end{cases} \tag{8.4.2}$$

In fact, $p_{-j} = 0$ for any $j \geq 1$ under our hypotheses.

Denote by $E(a_2^{(2)}, b_{-j}^{(2)})$, resp. $E(a_2^{(1)}, b_{-j}^{(1)})$, a generic expression obtained by taking the (matrix) product of derivatives of order 2, resp. order 1, of a_2 with derivatives of order 2, resp. order 1, of b_{-j}. Hence, for all $j \geq 0$,

$$\{a_2, b_{-j}\} = E(a_2^{(1)}, b_{-j}^{(1)}), \text{ and } \{a_2, b_{-j}\}_{(2)} = E(a_2^{(2)}, b_{-j}^{(2)}).$$

Therefore, in $\{a_2, b_{-j}\}_{(2)} = E(a_2^{(2)}, b_{-j}^{(2)})$ we have a *constant* coefficient matrix (given by partial derivatives of order 2 of a_2) times partial derivatives of order 2 of b_{-j}.

We proceed by induction. We start with the case $j = 0$ and $h = 0$. In this case b_0 is the solution of

$$\begin{cases} \partial_t b_0 + a_2 b_0 = 0, \\ b_0|_{t=0} = I_N, \end{cases} \tag{8.4.3}$$

whence $b_0(t) = O(1)$ as $t \to 0+$.

Next, by induction on ℓ we show that $b_0^{(\ell)}$ has the claimed property.

For $\ell = 1$ we take a 1st-order partial derivative with respect to X of (8.4.3) and find

$$\begin{cases} \partial_t b_0^{(1)} + a_2 b_0^{(1)} = -a_2^{(1)} b_0, \\ b_0^{(1)}|_{t=0} = 0, \end{cases}$$

whence

$$b_0^{(1)}(t) = -\int_0^t e^{-(t-t')a_2} a_2^{(1)} b_0(t')\, dt' = O(t),\ t \to 0+\,. \tag{8.4.4}$$

For $\ell = 2$ we take a 1st-order partial derivative with respect to X of (8.4.4) and find

$$\begin{aligned} b_0^{(2)}(t) = &-\int_0^t (e^{-(t-t')a_2})^{(1)} a_2^{(1)} b_0(t')\, dt' - \int_0^t e^{-(t-t')a_2} a_2^{(2)} b_0(t')\, dt' \\ &- \int_0^t e^{-(t-t')a_2} a_2^{(1)} b_0(t')^{(1)}\, dt' \\ =&O(t^2) + O(t) + O(t^2) = O(t),\ t \to 0+\,. \end{aligned}$$

Next, suppose $b_0^{(2k-1+\ell)}(t) = O(t^k)$ as $t \to 0+$, for $\ell = 0, 1$ and $k \geq 0$. We want to prove that $b_0^{(2k+1+\ell)}(t) = O(t^{k+1})$, as $t \to 0+$, for $\ell = 0, 1$. Using (8.4.3) and taking a $2k+1$-st partial derivative with respect to X we obtain (recall that $a_2^{(p)} = 0$ for all $p \geq 3$ since a_2 has polynomial of degree 2 entries)

$$\begin{cases} \partial_t b_0^{(2k+1)} + a_2 b_0^{(2k+1)} = -a_2^{(1)} b_0^{(2k)} - a_2^{(2)} b_0^{(2k-1)} = O(t^k) + O(t^k) = O(t^k), \\ b_0^{(2k+1)}|_{t=0} = 0, \end{cases}$$

whence $b_0^{(2k+1)}(t) = O(t^{k+1})$ as $t \to 0+$. Then, as before,

$$\begin{aligned} b_0^{(2k+2)}(t) = & -\int_0^t (e^{-(t-t^{'})a_2})^{(1)} (a_2^{(1)} b_0^{(2k)}(t^{'})) + a_2^{(2)} b_0^{(2k-1)}(t^{'}))\, dt^{'} \\ & - \int_0^t e^{-(t-t^{'})a_2} \partial_X (a_2^{(1)} b_0^{(2k)}(t^{'})) + a_2^{(2)} b_0^{(2k-1)}(t^{'}))\, dt^{'} \\ = & O(t^{k+2}) + O(t^{k+1}) = O(t^{k+1}),\ t \to 0+. \end{aligned}$$

Hence, the result is proved for b_0.

Next, we prove the result for the case $j = 0$ and $h = 1$. In this case by (8.4.2)

$$\begin{aligned} b_{-1}(t) = & -\int_0^t e^{-(t-t^{'})a_2} r_{2-1}(t^{'})\, dt^{'} \\ = & -\int_0^t e^{-(t-t^{'})a_2} a_1 \underbrace{b_0(t^{'})}_{=O(1),\, t^{'} \to 0+} dt^{'} \\ = & O(t),\ t \to 0+. \end{aligned} \tag{8.4.5}$$

By taking the derivative in X of (8.4.5)

$$
\begin{aligned}
b_{-1}^{(1)}(t) = & -\int_0^t (e^{-(t-t^{'})a_2})^{(1)} a_1 b_0(t^{'})\, dt^{'} \\
& -\int_0^t e^{-(t-t^{'})a_2} a_1^{(1)} b_0(t^{'})\, dt^{'} \\
& -\int_0^t e^{-(t-t^{'})a_2} a_1 \underbrace{b_0^{(1)}(t^{'})}_{=O(t^{'}),\, t^{'}\to 0+} dt^{'} \\
=& O(t^2) + O(t) + O(t^2) = O(t),\ t \to 0+ .
\end{aligned}
$$

By taking another derivative in X we obtain that $b_{-1}^{(2)}(t) = O(t^2)$ (recall that $a_1^{(p)} = 0$ for all $p \geq 2$ since a_1 has polynomial of degree 1 entries).

Next, suppose $b_{-1}^{(2k-1+\ell)}(t) = O(t^{k+\ell})$, as $t \to 0+$, for $\ell = 0, 1$ and $k \geq 0$. We want to prove that $b_{-1}^{(2k+1+\ell)}(t) = O(t^{k+\ell+1})$, as $t \to 0+$, for $\ell = 0$, 1. First of all, we notice that, by (8.4.2), b_{-1} is the solution of the Cauchy problem

$$
\begin{cases}
\partial_t b_{-1} + a_2 b_{-1} = -r_1 = -a_1 b_0, \\
b_{-1}|_{t=0} = 0,
\end{cases}
\tag{8.4.6}
$$

By using (8.4.6) and taking a $2k+1$-st partial derivative with respect to X

$$
\begin{cases}
\begin{aligned}
\partial_t b_{-1}^{(2k+1)} + a_2 b_{-1}^{(2k+1)} &= -a_2^{(1)} b_{-1}^{(2k)} - a_2^{(2)} b_{-1}^{(2k-1)} - a_1^{(1)} b_0^{(2k)}, \\
&= O(t^{k+1}) + O(t^k) + O(t^k)
\end{aligned} \\
b_{-1}^{(2k+1)}|_{t=0} = 0,
\end{cases}
$$

whence $b_{-1}^{(2k+1)}(t) = O(t^{k+1})$ as $t \to 0+$. Then, as before,

$$
\begin{aligned}
b_{-1}^{(2k+2)}(t) = & -\int_0^t (e^{-(t-t^{'})a_2})^{(1)} (a_2^{(1)} b_{-1}^{(2k)}(t^{'}) + a_2^{(2)} b_{-1}^{(2k-1)}(t^{'}) + a_1^{(1)} b_0^{(2k)}(t^{'}))\, dt^{'} \\
& -\int_0^t e^{-(t-t^{'})a_2} \partial_X (a_2^{(1)} b_{-1}^{(2k)}(t^{'}) + a_2^{(2)} b_{-1}^{(2k-1)}(t^{'}) + a_1^{(1)} b_0^{(2k)}(t^{'}))\, dt^{'} \\
=& O(t^{k+2}) + O(t^{k+2}) = O(t^{k+2}),\ t \to 0+ .
\end{aligned}
$$

Hence, the result has been proved for b_{-1}.

Next, suppose, by induction, that for all $\ell = 0, 1$, all $h = 0, 1$ and all $j^{'} \leq j$

$$b_{-2j'-h} = O(t^{j^{'}+h}),\ b^{(2k+1+\ell)}_{-2j'-h} = O(t^{j^{'}+k+h\ell+1}), \quad t \to 0+ .$$

We want to prove $b_{-2(j+1)} = O(t^{j+1})$ and $b^{(2k+1+\ell)}_{-2(j+1)} = O(t^{j+1+k+1})$ for $\ell = 0, 1$, as $t \to 0+$, that is, the case $h = 0$ (after that, we will prove that $b_{-2(j+1)-1} = O(t^{j+1+1})$ and $b^{(2k+1+\ell)}_{-2(j+1)-1} = O(t^{j+1+k+\ell+1})$ for $t \to 0+$, i.e. the case $h = 1$). To do it, we have to examine $r_{2-2(j+1)}$ (see (8.4.2)). In the first place we have, from (8.4.1),

$$\begin{aligned} r_{2-2(j+1)} =& a_0 b_{-2j} + a_1 b_{-2j-1} + \frac{1}{2}\left(\frac{-i}{2}\right)^2 \{a_2, b_{-2(j-1)}\}_{(2)} - \frac{i}{2}\{a_2, b_{-2j}\} \\ &- \frac{i}{2}\{a_1, b_{-(2j-1)}\} \\ =& O(t^j) + O(t^{j+1}) + O(t^{j-1+1}) + O(t^{j+1}) + O(t^{j-1+1}) \\ =& O(t^j), \quad t \to 0+ . \end{aligned}$$

Consider next, keeping into account that $a^{(p)}_q = 0$ for all $p \geq q+1$, since a_q has polynomial of degree $q = 1, 2$ entries,

$$\begin{aligned} r^{(2k+1)}_{2-2(j+1)} =& a_0 b^{(2k+1)}_{-2j} + a_1 b^{(2k+1)}_{-2j-1} + E(a^{(1)}_1, b^{(2k)}_{-2j-1}) + E(a^{(2)}_2, b^{(2k+3)}_{-2(j-1)}) \\ &+ E(a^{(1)}_2, b^{(2k+2)}_{-2j}) + E(a^{(2)}_2, b^{(2k+1)}_{-2j}) + E(a^{(1)}_1, b^{(2k+2)}_{-(2j-1)}) \\ =& O(t^{j+k+1}) + O(t^{j+k+1}) + O(t^{j+k-1+1+1}) + O(t^{j-1+k+1+1}) \\ &+ O(t^{j+k+1}) + O(t^{j+k+1}) + O(t^{j-1+k+1+1}) \\ =& O(t^{j+k+1}), \quad t \to 0+ . \end{aligned}$$

Taking an extra derivative, one immediately sees also that

$$r^{(2k+2)}_{2-2(j+1)} = O(t^{j+k+1}), \quad t \to 0+ .$$

Hence, for all $\ell = 0, 1$ and for $k \geq -1$

$$r^{(2k+1+\ell)}_{2-2(j+1)} = O(t^{j+k+1}), \quad t \to 0+$$

(when $k = -1$ we take $\ell = 1$). Since $b_{-2(j+1)}$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t b_{-2(j+1)} + a_2 b_{-2(j+1)} = -r_{2-2(j+1)}, \\ b_{-2(j+1)}|_{t=0} = 0, \end{cases} \tag{8.4.7}$$

we obtain $b_{-2(j+1)}(t) = O(t^{j+1})$ as $t \to 0+$. As before, taking one partial derivative with respect to X yields

$$\begin{cases} \partial_t b^{(1)}_{-2(j+1)} + a_2 b^{(1)}_{-2(j+1)} = -a_2^{(1)} b_{-2(j+1)} - r^{(1)}_{2-2(j+1)} = O(t^{j+1}) + O(t^{j+1}), \\ b^{(1)}_{-2(j+1)}|_{t=0} = 0, \end{cases}$$

whence it follows that $b^{(1)}_{-2(j+1)}(t) = O(t^{j+2})$, and, taking an extra derivative, we also see that, as $t \to 0+$,

$$b^{(2)}_{-2(j+1)}(t) = -\partial_X \left(\int_0^t e^{-(t-t')a_2} \left(a_2^{(1)} b_{-2(j+1)} + r^{(1)}_{2-2(j+1)} \right) dt' \right) = O(t^{j+2}).$$

Supposing then by induction the estimates up to order $2k-1$ proved and using

$$\begin{cases} \partial_t b^{(2k+1)}_{-2(j+1)} + a_2 b^{(2k+1)}_{-2(j+1)} = -E(a_2^{(1)}, b^{(2k)}_{-2(j+1)}) - E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)}) - r^{(2k+1)}_{2-2(j+1)}, \\ \qquad\qquad = O(t^{j+1+k-1+1}) + O(t^{j+1+k-1+1}) + O(t^{j+k+1}), \\ b^{(2k+1)}_{-1}|_{t=0} = 0, \end{cases}$$

we obtain $b^{(2k+1)}_{-2(j+1)}(t) = O(t^{j+1+k+1})$, as $t \to 0+$. By using

$$b^{(2k+2)}_{-2(j+1)}(t) = -\partial_X \left(\int_0^t e^{-(t-t')a_2} \left(E(a_2^{(1)}, b^{(2k)}_{-2(j+1)}) + E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)}) + r^{(2k+1)}_{-2j} \right) dt' \right),$$

we also see that

$$b^{(2k+2)}_{-2(j+1)}(t) = O(t^{j+1+k+1}),\ t \to 0+,$$

which proves the result for the case $h = 0$.

Now, to complete the proof of this proposition, we need to prove the result for the case $h = 1$, that is, for all $\ell = 0,\, 1$

$$b_{-2(j+1)-1} = O(t^{j+2}),\ b^{(2k+1+\ell)}_{-2(j+1)-1} = O(t^{j+1+k+\ell+1}),\, t \to 0+ .$$

To do it, we have to examine $r_{2-2(j+1)-1}$ and its derivatives, that is,

$$\begin{aligned} r_{2-2(j+1)-1} =& a_0 b_{-2j-1} + a_1 b_{-2(j+1)} + \frac{1}{2}\left(\frac{-i}{2}\right)^2 \{a_2, b_{-2(j-1)-1}\}_{(2)} - \frac{i}{2}\{a_2, b_{-2j-1}\} \\ &- \frac{i}{2}\{a_1, b_{-2j}\} \\ =& O(t^{j+1}) + O(t^{j+1}) + O(t^{j-1+1+1}) + O(t^{j+1}) + O(t^{j+1}) \\ =& O(t^{j+1}),\ t \to 0+, \end{aligned}$$

and

$$\begin{aligned} r^{(2k+1)}_{2-2(j+1)-1} =& a_0 b^{(2k+1)}_{-2j-1} + a_1 b^{(2k+1)}_{-2(j+1)} + E(a^{(1)}_1, b^{(2k)}_{-2(j+1)}) + E(a^{(2)}_2, b^{(2k+3)}_{-2(j-1)-1}) \\ &+ E(a^{(1)}_2, b^{(2k+2)}_{-2j-1}) + E(a^{(2)}_2, b^{(2k+1)}_{-2j-1}) + E(a^{(1)}_1, b^{(2k+2)}_{-2j}) \\ =& O(t^{j+k+1}) + O(t^{j+1+k+1}) + O(t^{j+1+k-1+1}) + O(t^{j-1+k+1+1}) \\ &+ O(t^{j+k+1+1}) + O(t^{j+k+1}) + O(t^{j+k+1}) \\ =& O(t^{j+k+1}),\ t \to 0+ . \end{aligned}$$

Taking an extra derivative, one immediately sees also that

$$r^{(2k+2)}_{2-2(j+1)-1} = O(t^{j+k+2}),\ t \to 0+ .$$

Hence, for all $\ell = 0, 1$ and for all $k \geq -1$

$$r^{(2k+1+\ell)}_{2-2(j+1)-1} = O(t^{j+k+\ell+1}),\ t \to 0+ .$$

(again, when $k = -1$ we take $\ell = 1$). Since $b_{-2(j+1)-1}$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t b_{-2(j+1)-1} + a_2 b_{-2(j+1)-1} = -r_{2-2(j+1)-1}, \\ b_{-2(j+1)-1}|_{t=0} = 0, \end{cases} \tag{8.4.8}$$

we obtain $b_{-2(j+1)-1}(t) = O(t^{j+1+1})$ as $t \to 0+$. As before, taking one partial derivative with respect to X yields

$$\begin{cases} \partial_t b^{(1)}_{-2(j+1)-1} + a_2 b^{(1)}_{-2(j+1)-1} = -a_2^{(1)} b_{-2(j+1)-1} - r^{(1)}_{2-2(j+1)-1} = O(t^{j+2}) + O(t^{j+1}), \\ b^{(1)}_{-2(j+1)-1}|_{t=0} = 0, \end{cases}$$

whence it follows $b^{(1)}_{-2(j+1)-1}(t) = O(t^{j+2})$, and, taking an extra derivative, we also see that, as $t \to 0+$,

$$b^{(2)}_{-2(j+1)-1}(t) = -\partial_X \left(\int_0^t e^{-(t-t^{'})a_2} \left(a_2^{(1)} b_{-2(j+1)-1} + r^{(1)}_{2-2(j+1)} \right) dt^{'} \right) = O(t^{j+3}).$$

Supposing then by induction the estimates up to order $2k$ proved and making use of

$$\begin{cases} \partial_t b^{(2k+1)}_{-2(j+1)-1} + a_2 b^{(2k+1)}_{-2(j+1)-1} = & -E(a_2^{(1)}, b^{(2k)}_{-2(j+1)-1}) - E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)-1}) - r^{(2k+1)}_{2-2(j+1)-1}, \\ \qquad = & O(t^{j+1+k-1+1+1}) + O(t^{j+1+k-1+1+1}) + O(t^{j+1+k+1}) \\ b^{(2k+1)}_{-1}|_{t=0} = 0, \end{cases}$$

we obtain $b^{(2k+1)}_{-2(j+1)-1}(t) = O(t^{j+k+2})$, as $t \to 0+$. Finally, using

$$\begin{aligned} b^{(2k+2)}_{-2(j+1)-1}(t) = & -\partial_X \left(\int_0^t e^{-(t-t^{'})a_2} \left(E(a_2^{(1)}, b^{(2k)}_{-2(j+1)-1}) + E(a_2^{(2)}, b^{(2k-1)}_{-2(j+1)-1}) \right) dt^{'} \right) \\ & -\partial_X \left(\int_0^t e^{-(t-t^{'})a_2} \left(r^{(2k+1)}_{2-2(j+1)-1} \right) dt^{'} \right), \end{aligned}$$

we also see that

$$b^{(2k+2)}_{-2(j+1)-1}(t) = O(t^{j+1+k+1+1}),\ t \to 0+,$$

which proves the proposition.

□

8.5 Meromorphic continuation of $\zeta_{A^{\mathrm{w}}}$

Let A^{w} be as in the Section 8.3. In this section we will use the parametrix approximation of the heat-semigroup constructed in Lemma 8.3.1 to prove the result about the continuation of the spectral zeta function of the class of positive and self-adjoint elliptic operators A^{w} satisfying the hypotheses of Proposition 8.4.1. Namely, $\zeta_{A^{\mathrm{w}}}$ can be rewritten modulo a term holomorphic on a half plane of $\mathbb{C}$ as a linear complex combination of meromorphic functions. Moreover, we will give explicit formulas for the coefficients of this linear combination.

Theorem 8.5.1. *Let $A = a_2 + a_1 + a_0$ be an elliptic system of second order where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$. Moreover, suppose $A^{\mathrm{w}} > 0$.*

Then, there exist constants $c_{-2j-h,n}$ with $0 \le j \le n-1$, $h = 0, 1$, and constants $c_{-2j-1,n}$, C_{-2j} with $j \ge n$, such that, for any given integer $\nu \in \mathbb{Z}_+$ with $\nu \ge n$,

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)} \Bigg[\left(\sum_{h=0}^{1} \sum_{j=0}^{n-1} \frac{c_{-2j-h,n}}{s-(n-j)+h/2} \right) + \left(\sum_{j=n}^{\nu} \frac{c_{-2j-1,n}}{s-(n-j)+1/2} \right) \tag{8.5.1}$$
$$+ \left(\sum_{j=n}^{\nu} \frac{C_{-2j}}{s-(n-j)} \right) + H_\nu(s) \Bigg],$$

where $\Gamma(s)$ is the Euler gamma function, and H_ν is holomorphic in the region $\mathrm{Re}s > (n-\nu)-1$. Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ is meromorphic in the whole complex plane $\mathbb{C}$ with at most simple poles at $s = n$, $n-1$, $n-2$, ..., 1 and $s = n-\frac{1}{2}$, $n-\frac{3}{2}$, $n-\frac{5}{2}$, ... One has

$$c_{-2j-h,n} = (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\left(b_{-2j-h}(\rho^2,\omega)\right) \rho^{2(n-j)-1-h}\, d\omega\, d\rho, \tag{8.5.2}$$

where $0 \le j \le n-1$, $h = 0, 1$ or $j \ge n$, $h = 1$. In (8.5.2) the b_{-2j-h} are the terms in the symbol of the parametrix $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(2,0)$ constructed in the *proof of Lemma 8.3.1 and Remark 8.3.2,*

$$U_A \sim \sum_{j\geq 0} B_{-j}.$$

Proof. The proof follows the idea to make use of the asymptotic expansion given by Lemma 8.3.1 to obtain an asymptotic expansion for the continuation of $\zeta_{A^{\mathrm{w}}}$.To do that, we write $\zeta_{A^{\mathrm{w}}}$ by the Mellin transform, which gives it in terms of the heat-semigroup of A^{w}. Hence, via Lemma 8.3.1 we compute an approximation of $\zeta_{A^{\mathrm{w}}}$ whose asymptotic terms, given by integrals, are the $\frac{c_{-2j-h,n}}{s-(n-j)+h/2}$ in (8.5.1), obtained by a Taylor expansion argument. Actually, we need the integrals defining the $c_{-2j-h,n}$ to converge. That is why we use Proposition 8.4.1 to have a control on the vanishing of the asymptotic terms of the parametrix of the heat-semigroup as $t \to 0+$. Finally, we take into account the residuals given by the approximations made and we sum their contributes. Namely, we notice that they do not affect the values of the $c_{-2j-h,n}$ for $j \leq n-1$ and those for $h = 1$ if $j \geq n$.

By the properties of the heat semigroup $0 \leq t \to e^{-tA^{\mathrm{w}}}$, we may use the Mellin transform and write

$$(A^{\mathrm{w}})^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} e^{-tA^{\mathrm{w}}}\, dt,\ \mathrm{Re}s > 2n/2 = n,$$

so that

$$s \mapsto \zeta_{A^{\mathrm{w}}}(s) = \mathrm{Tr}(A^{\mathrm{w}})^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt.$$

Let hence $U_A \sim \sum_{j\geq 0} B_{-j} \in \mathrm{OP}S_{\mathrm{sreg}}(2,0)$ be the parametrix approximation of $e^{-tA^{\mathrm{w}}}$ constructed in Lemma 8.3.1. We write

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 + \int_1^{+\infty} \right) t^{s-1} \mathrm{Tr}\, e^{-tA^{\mathrm{w}}}\, dt =: Z_0(s) + Z_\infty(s).$$

In the first place, we claim that $Z_\infty(s)$ is holomorphic in $\mathbb{C}$. In fact, notice first that $t \mapsto \mathrm{Tr}R(t)$ is rapidly decreasing for $t \to +\infty$ (where $R(t) :=$

$e^{-tA^{\mathrm{w}}} - U_A(t))$, so we have that, for all $p \in \mathbb{N}$ and for all $t \geq 1$,

$$|\mathrm{Tr}\, R(t)| \lesssim t^{-p}.$$

Second, given any $\nu \geq 0$ and any symbol $b \in S(2, -2\nu)$, we have (by definition of the class $S(\mu, \nu)$ at p. 79 of [45]) that for all $t \geq 1$ and all $p \in \mathbb{N}$

$$\begin{aligned}\left|(2\pi)^{-n}\int_{\mathbb{R}^{2n}} \mathsf{Tr}\, b(t,X)\, dX\right| &= \left|(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\, b(t,\rho\omega)\rho^{2n-1}\, d\omega\, d\rho\right| \\ &= t^{-p}\left|(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}} t^p \mathsf{Tr}\, b(t,\rho\omega)\rho^{2n-1}\, d\omega\, d\rho\right| \\ &\lesssim t^{-p}\int_0^{+\infty} \frac{\rho^{2n+1}}{(1+\rho)^{2\nu+2p}}\, d\rho \\ &\lesssim t^{-p}.\end{aligned}$$

(Here, we uses the polar coordinates $0 \neq X = |X|\dfrac{X}{|X|}$ with $\rho \in \mathbb{R}_+$, $\omega \in \mathbb{S}^{2n-1}$, and $d\omega$ is the induced Riemann measure on $\mathbb{S}^{2n-1}$.) It thus follows that for all $p \in \mathbb{N}$ and for all $t \geq 1$

$$|\mathrm{Tr}\, U_A(t)| \lesssim t^{-p}.$$

In conclusion, since

$$\mathrm{Tr}\, e^{-tA^{\mathrm{w}}} = \mathrm{Tr}\, U_A(t) + \mathrm{Tr}\, R(t),$$

for every $p \geq 1$ there exists $C_p > 0$ such that

$$|\mathrm{Tr}\, e^{-tA^{\mathrm{w}}}| \leq C_p t^{-p}, \ \forall t \geq 1,$$

which proves the claim, since the term $1/\Gamma(s)$ is already holomorphic in $\mathbb{C}$. Therefore, the crucial point is the study of the function $Z_0(s)$. To this aim we need a better understanding of the terms $\mathrm{Tr}\, B_{-2j-h}$, $j \geq 0$, $h = 0, 1$. Hence, we recall that, by the homogeneity of the b_{-2j-h}, for $t > 0$, $j \geq 0$, and $h = 0, 1$,

$$\begin{aligned}\operatorname{Tr} B_{-2j-h}(t) =&(2\pi)^{-n}\int_{\mathbb{R}^{2n}}\chi(X)\mathsf{Tr}\left(b_{-2j-h}(t,X)\right)\,dX\\ =&(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}}\chi(\rho\omega)\mathsf{Tr}\left(b_{-2j-h}(t,\rho\omega)\right)\rho^{2n-1}\,d\omega\,d\rho\\ =&(2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}}\chi(\rho\omega)\mathsf{Tr}\left(b_{-2j-h}(\rho^2 t,\omega)\right)\rho^{2(n-j)-1-h}\,d\omega\,d\rho.\end{aligned}$$

We consider

$$c_{-2j-h,n} := (2\pi)^{-n}\int_0^{+\infty}\int_{\mathbb{S}^{2n-1}}\mathsf{Tr}\left(b_{-2j-h}(\rho^2,\omega)\right)\rho^{2(n-j)-1-h}\,d\omega\,d\rho,$$

and claim that

$$|c_{-2j-h,n}| < +\infty,\ \forall j\in\mathbb{Z}_+,\ h=0,1.$$

In fact, the integral is convergent at $\rho=+\infty$ for all j since $\mathsf{Tr}\left(b_{-2j-h}(\cdot,\omega)\right)$ is a Schwartz function. It is clearly convergent at $\rho=0$ for $0\le 2j+h\le 2n-1$. Finally, it is convergent at $\rho=0$ also when $2j+h\ge 2n$, for the singularity at 0 of the factor $\rho^{2(n-j)-1-h}$ is compensated by $\mathsf{Tr}\left(b_{-2j-h}(t,\omega)\right)=O(t^{j+h})$ as $t\to 0+$.

We define now the function

$$f_{-2j-h}(t) := -(2\pi)^{-n}\int_0^1\int_{\mathbb{S}^{2n-1}}(1-\chi(\rho\omega))\mathsf{Tr}\left(b_{-2j-h}(t,\rho\omega)\right)\rho^{2n-1}\,d\omega\,d\rho,\ j\in\mathbb{Z}_+.$$

Then, $f_{-j'}\in C^\infty([0,+\infty);\mathbb{C})$, for all $j'\in\mathbb{Z}_+$, and by Proposition 8.4.1

$$f_{-2j-h}(t) = O(t^{j+h}),\ t\to 0+\,. \tag{8.5.3}$$

It follows that

$$\operatorname{Tr} B_{-2j-h}(t) = c_{-2j-h,n}t^{-(n-j)+h/2} + f_{-2j-h}(t) = c_{-2j-h,n}t^{-(n-j)+h/2} + O(t^{j+h}), \tag{8.5.4}$$

as $t\to 0+$, for all $j\ge 0$, $h=0,1$, and that (by the proof of Proposition

3.2.15 at p. 32 of [45] adapted to the present setting),

$$\mathrm{Tr}\, U_A(t) - \sum_{h=0}^{1}\sum_{j=0}^{\nu} \mathrm{Tr}\, B_{-2j-h}(t) =: \mathrm{Tr}\, R_{2\nu+2}(t) = O(t^{\nu+1}),\ t \to 0+,$$

$\forall \nu \in \mathbb{Z}_+$, $h = 0, 1$.

However, the information contained in (8.5.4) alone is not yet sufficient to obtain the continuation of $\zeta_{A^{\mathrm{w}}}$, and we need a better control of f_{-2j-h} . Notice that for all $j, k \in \mathbb{Z}_+$, denoting $\partial_t^k f_{-2j-h}(t)$ by $f_{-2j-h}^{(k)}(t)$,

$$f_{-2j-h}^{(k)}(t) = -(2\pi)^{-n} \int_0^1 \int_{\mathbb{S}^{2n-1}} (1-\chi(\rho\omega))\mathsf{Tr}\left(\partial_t^k b_{-2j-h}(t,\rho\omega)\right) \rho^{2n-1}\, d\omega\, d\rho,$$

so that $f_{-2j-h}^{(k)}(0)$ is finite and can be computed through (8.4.1), and, through the differential equations (8.4.3), (8.4.6), (8.4.7), and (8.4.8), used to construct the b_{-2j-h}. Note, in particular, that

$$f_0^{(k)}(0) = (-1)^{k+1}(2\pi)^{-n} \int_0^1 \int_{\mathbb{S}^{2n-1}} (1-\chi(\rho\omega))\mathsf{Tr}\left(a_2(\rho\omega)^k\right) \rho^{2n-1}\, d\omega\, d\rho.$$

We next apply Lemma 7.2.3 at p. 99 of [45] to the functions f_{-2j-h}, so that for any given $\nu \in \mathbb{Z}_+$ we may write, by (8.5.3),

$$F_{-2j-h}(s) := \int_0^1 t^{s-1} f_{-2j-h}(t)\, dt = \sum_{k=0}^{\nu} \frac{f_{-2j-h}^{(j+h+k)}(0)}{(j+h+k)!} \frac{1}{s+j+h+k} + F_{-2j-h,\nu}(s),$$

where $F_{-2j-h,\nu}$ is holomorphic for $\mathrm{Re}s > -j-h-\nu-1$.

Using this in (8.5.4) we have that for each $j \geq 0$, $h = 0, 1$, for any given $\nu \in \mathbb{Z}_+$,

$$\begin{aligned} s \mapsto \int_0^1 t^{s-1} \mathrm{Tr}\, B_{-2j-h}(t)\, dt = & \frac{c_{-2j-h,n}}{s-(n-j)+h/2} \\ & + \left(\sum_{k=0}^{\nu} \frac{f_{-2j-h}^{(j+h+k)}(0)}{(j+h+k)!} \frac{1}{s+j+h+k} \right) + F_{-2j-h,\nu}(s), \end{aligned}$$

where $F_{-2j-h,\nu}$ is holomorphic for $\mathrm{Re}s > -j-h-\nu-1$.

Analogously, since $0 \leq t \mapsto \mathrm{Tr}\, R(t) \in \mathscr{S}(\overline{\mathbb{R}}_+;\mathbb{C})$ and by Lemma 7.2.3 at p. 99 of [45] we also have, with $f_R(t) := \mathrm{Tr}\, R(t)$, that for any given $\nu \in \mathbb{Z}_+$

$$\int_0^1 t^{s-1} f_R(t)\, dt = \sum_{k=0}^{\nu} \frac{f_R^{(k)}(0)}{k!} \frac{1}{s+k} + F_{R,\nu}(s),$$

where $F_{R,\nu}$ is holomorphic for $\mathrm{Re} s > -\nu - 1$.

We therefore obtain that for any given $\nu \in \mathbb{Z}_+$.

$$\begin{aligned} Z_0(s) =& \frac{1}{\Gamma(s)} \left[\left(\sum_{h=0}^{1} \sum_{j=0}^{\nu} \int_0^1 t^{s-1} \mathrm{Tr}\, B_{-2j-h}(t)\, dt \right) \right. \\ & \left. + \int_0^1 t^{s-1} \mathrm{Tr}\, R_{2\nu+2}(t)\, dt + \int_0^1 t^{s-1} \mathrm{Tr}\, R(t)\, dt \right]. \end{aligned}$$

Since the function $s \mapsto \int_0^1 t^{s-1} \mathrm{Tr} R_{2\nu+2}(t)\, dt =: F_{2\nu+2}(s)$ is holomorphic for $\mathrm{Re} s > -\nu - 1$, we thus obtain that, for any given $\nu \in \mathbb{Z}_+$ with $\nu \geq n$,

$$\begin{aligned} Z_0(s) &= \frac{1}{\Gamma(s)} \left[\sum_{h=0}^{1} \sum_{j=0}^{\nu} \frac{c_{-2j-h,n}}{s-(n-j)+h/2} + \left(\sum_{h=0}^{1} \sum_{j,k=0}^{\nu} \frac{f_{-2j-h}^{(j+h+k)}(0)}{(j+h+k)!} \frac{1}{s+j+h+k} \right) \right. \\ &\qquad \left. + \sum_{k=0}^{\nu} \frac{f_R^{(k)}(0)}{k!} \frac{1}{s+k} + \left(\sum_{h=0}^{1} \sum_{j=0}^{\nu} F_{-2j-h,\nu}(s) \right) + F_{R,\nu}(s) + F_{2\nu+2}(s) \right] \\ &= \frac{1}{\Gamma(s)} \left[\left(\sum_{h=0}^{1} \sum_{j=0}^{n-1} \frac{c_{-2j-h,n}}{s-(n-j)+h/2} \right) + \left(\sum_{j=n}^{\nu} \frac{c_{-2j-1,n}}{s-(n-j)+1/2} \right) \right. \\ &\qquad \left. + \left(\sum_{j=n}^{\nu} \frac{C_{-2j}}{s-(n-j)} \right) + \tilde{H}_\nu(s) \right], \end{aligned}$$

with $s \mapsto \tilde{H}_\nu(s)$ holomorphic for $\mathrm{Re} s > (n-\nu)-1$. Since the function $1/\Gamma(s)$ is holomorphic in $\mathbb{C}$ and has zeros at the non-positive integers $-k$, $k \in \mathbb{Z}_+$, this proves the theorem.

□

Remark 8.5.2. *An interesting problem can be to use in our setting the asymptotics for resolvent expansions and trace regularizations by [21] and [22] to try to get an improvement of the result.*

Theorem 8.5.1 has the following corollary for the Hurwitz-type spectral zeta function of A^{w}.

Corollary 8.5.3. *Let $A = a_2 + a_1 + a_0$ be an elliptic system of second order where a_j is an $N \times N$ matrix-valued function on $\mathbb{R}^{2n}$ with homogeneous polynomial of degree j entries for all $j = 0, 1, 2$. Moreover, suppose $A^{\mathrm{w}} > 0$.*

For all $\tau > 0$ there exist constants $c_{-2j-h,n}$ with $0 \leq j \leq n-1$, $h = 0, 1$, and constants $c_{-2j-1,n}$, C_{-2j} with $j \geq n$, such that, for any given integer $\nu \in \mathbb{Z}_+$ with $\nu \geq n$,

$$\zeta_{A^{\mathrm{w}}+\tau I}(s) = \frac{1}{\Gamma(s)} \left[\left(\sum_{h=0}^{1} \sum_{j=0}^{n-1} \frac{c_{-2j-h,n}}{s-(n-j)+h/2} \right) + \left(\sum_{j=n}^{\nu} \frac{c_{-2j-1,n}}{s-(n-j)+1/2} \right) \right. \tag{8.5.5}$$

$$\left. + \left(\sum_{j=n}^{\nu} \frac{C_{-2j}}{s-(n-j)} \right) + H_\nu(s) \right],$$

where $\Gamma(s)$ is the Euler gamma function, and H_ν is holomorphic in the region $\mathrm{Re}s > (n-\nu) - 1$. *Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}+\tau I}$ is meromorphic in the whole complex plane $\mathbb{C}$ with at most simple poles at $s = n$, $n-1$, $n-2$, …, 1 and $s = n - \frac{1}{2}$, $n - \frac{3}{2}$, $n - \frac{5}{2}$, … . One has*

$$\begin{aligned} &c_{-2j-h,n} \\ &= (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\left(b_{-2j-h}(\rho^2, \omega)\right) \rho^{2(n-j)-1-h} \, d\omega \, d\rho \\ &- \tau (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \int_0^{\rho^2} e^{-(\rho^2 - t')a_2} \mathsf{Tr}(b_{2-2j-h}(t', \omega)) \rho^{2(n-j)-1-h} \, dt' \, d\omega \, d\rho, \end{aligned} \tag{8.5.6}$$

where $0 \leq j \leq n-1$, $h = 0, 1$ or $j \geq n$, $h = 1$. In (8.5.6) the b_{-2j-h} are the terms in the symbol of the parametrix $U_A \in \mathrm{OP}S_{\mathrm{sreg}}(2, 0)$ constructed in the proof of Lemma 8.3.1 and Remark 8.3.2,

$$U_A \sim \sum_{j \geq 0} B_{-j},$$

where we set $b_k \equiv 0$ for all $k = 1, 2$.

Proof. The proof follows from the one of Theorem 8.5.1. In fact, we use of the equations in the proof of Lemma 8.3.1 and (8.4.2) to link the asymptotic expansion of the parametrix of the heat semigroup of $A^{\mathrm{w}} + \tau I$ to the one of A^{w}. Let b_j, r_{2-j} be the terms constructed in the proof of Lemma 8.3.1 (see also (8.4.1) and (8.4.2)) for A^{w} and $b_{-j}(\tau)$, $r_{2-j}(\tau)$ those for $A^{\mathrm{w}} + \tau I$. Then,

$$\begin{cases} b_0(\tau)(t,X) = e^{-ta_2(X)}, \\ b_{-1}(\tau)(t,X) = \int_0^t e^{-(t-t')a_2} r_{2-j}(t',X)\,dt' \\ b_{-j}(\tau)(t,X) = -\int_0^t e^{-(t-t')a_2} r_{2-j}(t',X)\,dt' - \tau \int_0^t e^{-(t-t')a_2} b_{2-j}(t',X)\,dt',\ j \geq 2, \end{cases} \tag{8.5.7}$$

since for all $j \geq 2$

$$\begin{cases} \frac{d}{dt} b_{-j}(\tau) + a_2 b_{-j}(\tau) = -r_{2-j}(\tau) = -r_{2-j} - \tau b_{2-j}, \\ b_{-j}(\tau)|_{t=0} = 0. \end{cases}$$

Now, we apply Theorem 8.5.1 to $\zeta_{A^{\mathrm{w}}+\tau I}$, obtaining (8.5.5) with coefficients

$$c_{-2j-h,n}(\tau) = (2\pi)^{-n} \int_0^{+\infty} \int_{\mathbb{S}^{2n-1}} \mathsf{Tr}\left(b_{-2j-h}(\tau)(\rho^2,\omega)\right) \rho^{2(n-j)-1-h}\, d\omega\, d\rho. \tag{8.5.8}$$

Actually, substituting in (8.5.8) the expressions for $\tilde{b}_{-j}$ given by (8.5.7), we obtain (8.5.6) which completes the proof.

□

8.6 Examples

8.6.1 The meromorphic continuation of Jaynes-Cummings model spectral zeta function ($n = 1,\ N = 2$).

We recall from Section 2.1.1 that the JC-model is the model of a two-level atom in one cavity, given by the 2×2 system in one real variable $x \in \mathbb{R}$

$$A^{\mathrm{w}}(x,D) = p_2^{\mathrm{w}}(x,D) I_2 + \alpha\Big(\boldsymbol{\sigma}_+ \psi^{\mathrm{w}}(x,D)^* + \boldsymbol{\sigma}_- \psi^{\mathrm{w}}(x,D)\Big) + \gamma \boldsymbol{\sigma}_3, \quad \gamma > 0, \alpha \in \mathbb{R},$$

where $\psi(x,D) := \frac{x+\partial_x}{\sqrt{2}}$, $\boldsymbol{\sigma}_\pm := \frac{1}{2}(\boldsymbol{\sigma}_1 \pm i\boldsymbol{\sigma}_2)$ with $\boldsymbol{\sigma}_j$, $j = 0, \ldots, 3$, the Pauli-matrices, that is

$$\boldsymbol{\sigma}_0 := I_2, \quad \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and the atom levels are given by $\pm\gamma$.

To apply Theorem 8.5.1 we need to compute the terms b_{-j} of the asymptotic expansion of the semigroup parametrix constructed in Lemma 8.3.1. First of all, if A is a the Hamiltonian of the JC-model and, in the notation of the previous sections, $A = a_2 + a_1 + a_0$, then

$$a_1 a_0 = -a_0 a_1,\ a_0^2 = I_2,\ \text{and } a_1^2 = p_2, \tag{8.6.1}$$

where p_2 is the harmonic oscillator symbol. Hence, the product of any number of factors equal to a_1 or a_0 can be rewritten as the multiple (by a function in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$) of a_1, a_0,a_0a_1 or I_2 by using iteratively the identities (8.6.1). This fact motivates the following definition.

Definition 8.6.1. *Given a linear combinations of products of any number of a_0 and a_1, we say that it is written in irreducible form if it is a linear combination of a_1, a_0, a_0a_1 and I_2 with coefficients in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$.*

We are going to prove a lemma determining the structure of the b_j as linear combination with coefficients in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$ of a_1, a_0, a_0a_1 and I_2.

Lemma 8.6.2. *Let $A = a_2 + a_1 + a_0$ be the Hamiltonian of the JC-model with a_j homogeneous of degree j. Then, the b_{-j} can be written in irreducible form. Moreover,*

$$j \text{ odd} \;\Rightarrow\; \text{the coefficients of } a_0,\ I_2 \text{ in the irreducible form of } b_{-j} \text{ are } 0, \tag{8.6.2}$$

$$j \text{ even} \;\Rightarrow\; \text{the coefficients of } a_1,\ a_0a_1 \text{ in the irreducible form of } b_{-j} \text{ are } 0. \tag{8.6.3}$$

Proof. The proof is by induction. It follows the construction of the parametrix in Lemma 8.3.1 and here we will use the same notations employed there. First of all,

$$b_0(t,X) = e^{-tp_2(X)}I_2, \quad b_{-1}(t,X) = -te^{-tp_2(X)}a_1$$

(see also (8.4.2)). Hence, b_0 and b_{-1} are already written in irreducible form and satisfy (8.6.2) and (8.6.3).

Now, we suppose that for all $j' \leq 2j-1$ ($j \geq 2$) the claim holds true and we want to prove the result for b_{2j} and b_{2j+1}. By the construction in Lemma 8.3.1 and since A is a differential operator (that is, its expansion contains only terms with degree of homogeneity ≥ 0),

$$\begin{cases} \frac{d}{dt}b_{-2j} + p_2 b_{-2j} = -a_0 b_{2-2j} - a_1 b_{1-2j}, \\ b_{-2j}|_{t=0} = 0. \end{cases}$$

Hence, by inductive hypothesis

$$\begin{aligned} a_0 b_{2-2j} + a_1 b_{1-2j} =& a_0(f_1 a_0 + f_2 I_2) + a_1(g_1 a_1 + g_2 a_0 a_1) \\ =& f_1 a_0^2 + f_2 a_0 + g_1 a_1^2 + g_2 a_1 a_0 a_1 \\ =& f_1 I_2 + f_2 a_0 - g_1 p_2 I_2 - g_2 p_2 a_0, \end{aligned}$$

where the third equality follows from (8.6.1) and where the f_j and g_j are functions in $C^\infty(\mathbb{R}_t; C^\infty(\mathbb{R}^{2n}))$. Hence, the claim is true for b_{-2j}. Repeating the argument for b_{-2j-1}, we have

$$\begin{cases} \frac{d}{dt}b_{-2j-1} + p_2 b_{-2j-1} = -a_0 b_{2-2j-1} - a_1 b_{1-2j-1}, \\ b_{-2j-1}|_{t=0} = 0, \end{cases}$$

which, since

$$\begin{aligned} a_0 b_{1-2j} + a_1 b_{-2j} =& a_0(\tilde{f}_1 a_1 + \tilde{f}_2 a_0 a_1) + a_1(\tilde{g}_1 a_0 + \tilde{g}_2 I_2) \\ =& \tilde{f}_1 a_0 a_1 + \tilde{f}_2 a_0^2 a_1 + \tilde{g}_1 a_1 a_0 + \tilde{g}_2 a_1 \\ =& \tilde{f}_1 a_0 a_1 + \tilde{f}_2 a_1 - \tilde{g}_1 a_0 a_1 + \tilde{g}_2 a_1, \end{aligned}$$

shows that the claim is true also for b_{-2j-1} and completes the proof.

□

Remark 8.6.3. *By Lemma 8.6.2 we have that* $\mathsf{Tr}\,(b_{-2j-1}) = 0$ *since it is a linear combination of matrices with zeros on the principal diagonal. Hence, by (8.6.2) and (8.5.2) we have that*

$$c_{-2j-1,1} = (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \mathsf{Tr}\left(b_{-2j-1}(\rho^2, \omega)\right) \rho^{-2j}\, d\omega\, d\rho = 0,\ j \geq 0,$$

and

$$\begin{aligned} c_{0,1} =& (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \mathsf{Tr}\left(b_0(\rho^2, \omega)\right) \rho\, d\omega\, d\rho \\ =& 2(2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} e^{-\rho^2/2} \rho\, d\omega\, d\rho \\ =& 2 \int_0^{+\infty} e^{-\rho^2/2} \rho\, d\rho = 2. \end{aligned}$$

Therefore, if A is the JC-model Hamiltonian, by (8.5.1) the spectral zeta function associated with A^{w} is

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)} \left[\frac{2}{s-1} + \left(\sum_{j=1}^{\nu} \frac{C_{-2j}}{s-(1-j)} \right) + H_\nu(s) \right],$$

where $\nu \geq 1$, H_ν is holomorphic in the region $\mathrm{Re}s > -\nu$ *and the $c_{-2,1}$, C_{-2j} has been defined in Theorem 8.5.1. Consequently, the spectral zeta function $\zeta_{A^{\mathrm{w}}}$ is meromorphic in the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$. Thus, $\zeta_{A^{\mathrm{w}}}$ has a meromorphic continuation to $\mathbb{C}$.*

8.6.2 The JC-model for one atom with 3-level and one cavity-mode in the so called Ξ-*configuration*.

We recall from Section 2.1.3 (or, equivalently, see Section 3.2 in [35])) that the generalization of the JC-model for one atom with 3-level and one cavity-mode in the so called Ξ-*configuration* (that we will denote by JC-3-Ξ in this section) describes a 3-level atom in one cavity, given by the 3×3 system in one real variable $x \in \mathbb{R}^2$. In this configuration every level of energy can interact only with the ones near to it, that is the electron can absorb (or emit) a photon moving from the j-th level of energy to the $j+1$-st (or from the $j+1$-st level of energy to the j-th) for $j = 1, 2$. That is mathematically represented by the following Hamiltonian operator. For $\alpha > 0$, $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ with $\gamma_1 < \gamma_2 < \gamma_3$,

$$\begin{aligned} A^{\mathrm{w}}(x,D) =& p_2^{\mathrm{w}}(x,D) I_3 + \frac{1}{2} \sum_{k=1}^{2} \alpha_k \Big(\psi^{\mathrm{w}}(x,D)^* E_{k,k+1} + \psi^{\mathrm{w}}(x,D) E_{k+1,k} \Big) \\ & + \sum_{k=1}^{3} \gamma_k E_{kk}, \end{aligned}$$

with

$$E_{jk} := e_k^* \otimes e_j, \quad 1 \leq j,k \leq 3$$

forming the basis of the 3×3 complex matrices, where E_{jk} acts on $\mathbb{C}^3$ as

$$E_{jk} w = \langle w, e_k \rangle e_j, \quad w \in \mathbb{C}^3,$$

and $\psi(x,D) := \frac{x+\partial_x}{\sqrt{2}}$.

Lemma 8.6.4. *Let $A = a_2 + a_1 + a_0$ be the Hamiltonian of the JC-3-Ξ with a_j homogeneous of degree j. Then,*

$$j \text{ odd } \Rightarrow \text{the principal and secondary diagonal entries of } b_{-j} \text{ are } 0, \quad (8.6.4)$$

$$j \text{ even } \Rightarrow \text{the subdiagonal and superdiagonal entries of } b_{-j} \text{ are } 0. \quad (8.6.5)$$

Proof. Again, the proof is by induction, and follows the construction of the parametrix in Lemma 8.3.1. Here we will use the same notation employed there. First of all,

$$b_0(t,X) = e^{-tp_2(X)} I_2, \quad b_{-1}(t,X) = -t e^{-tp_2(X)} a_1.$$

Hence, b_0 and b_{-1} satisfy (8.6.4) and (8.6.5). Now, we suppose that, for all

$j^{'} \leq 2j-1$ ($j \geq 2$), the claim is verified and we want to prove the result for b_{2j} and b_{2j+1}. By the construction in Lemma 8.3.1 and since A is a differential operator

$$\begin{cases} \frac{d}{dt}b_{-2j} + p_2 b_{-2j} = -a_0 b_{2-2j} - a_1 b_{1-2j}, \\ b_{-2j}|_{t=0} = 0. \end{cases}$$

Therefore, by the inductive hypothesis, $a_0 b_{2-2j}$ off-diagonal entries are 0 since a_0 is a diagonal matrix. Moreover, $a_1 b_{1-2j}$ off-diagonal entries are 0 since the principal and secondary diagonal entries of b_{1-2j} are 0. Hence, the claim is verified for b_{-2j}. Repeating the argument for b_{-2j-1}, we have

$$\begin{cases} \frac{d}{dt}b_{-2j-1} + p_2 b_{-2j-1} = -a_0 b_{2-2j-1} - a_1 b_{1-2j-1}, \\ b_{-2j-1}|_{t=0} = 0. \end{cases}$$

Thus, by the inductive hypothesis, $a_0 b_{1-2j}$ has principal and secondary diagonal entries that are 0 since a_0 is diagonal. Moreover, $a_1 b_{-2j}$ principal and secondary diagonal entries are 0 since b_{-2j} diagonal entries are 0. Hence, the claim is verified also for b_{-2j-1}.

□

Remark 8.6.5. *By Lemma 8.6.4 we have that* $\mathsf{Tr}\left(b_{-2j-1}\right) = 0$ *since* b_{-2j-1} *principal diagonal entries are* 0. *Hence, by (8.6.4) and (8.6.5) we have that*

$$c_{-2j-1,2} = (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \mathsf{Tr}\left(b_{-2j-1}(\rho^2, \omega)\right) \rho^{-2j}\, d\omega\, d\rho = 0,\ j \geq 0,$$

and

$$\begin{aligned} c_{0,2} =& (2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} \mathsf{Tr}\left(b_0(\rho^2, \omega)\right) \rho\, d\omega\, d\rho \\ =& 3(2\pi)^{-1} \int_0^{+\infty} \int_0^{2\pi} e^{-\rho^2/2} \rho\, d\omega\, d\rho \\ =& 3 \int_0^{+\infty} e^{-\rho^2/2} \rho\, d\rho = 3. \end{aligned}$$

Therefore, if A is the JC-3-Ξ Hamiltonian, by (8.5.1) the spectral zeta func-

tion associated with A^{w} *is*

$$\zeta_{A^{\mathrm{w}}}(s) = \frac{1}{\Gamma(s)}\left[\frac{3}{s-1} + \left(\sum_{j=1}^{\nu} \frac{C_{-2j}}{s-(1-j)}\right) + H_\nu(s)\right],$$

where $\nu \geq 1$*,* H_ν *is holomorphic in the region* $\operatorname{Re} s > -\nu$ *and the* $c_{-2,1}$*,* C_{-2j} *has been defined in Theorem 8.5.1. Consequently, the spectral zeta function* $\zeta_{A^{\mathrm{w}}}$ *is meromorphic in the whole complex plane* $\mathbb{C}$ *with a simple pole at* $s = 1$*. Thus,* $\zeta_{A^{\mathrm{w}}}$ *has a meromorphic continuation to* $\mathbb{C}$*.*

8.7 Non-existence of semi-integer poles

In this section we prove, as a corollary of Theorem 8.5.1, that for a differential operator of order 2 with polynomial coefficients there are no semi-integer poles.

Corollary 8.7.1. *Let* $A = a_2 + a_1 + a_0$ *be an elliptic system of second order where* a_j *is an* $N \times N$ *matrix-valued function on* $\mathbb{R}^{2n}$ *with homogeneous polynomial of degree* j *entries for all* $j = 0, 1, 2$*. Moreover, suppose* $A^{\mathrm{w}} > 0$*. Then,* $c_{-2j-1,n} = 0$ *in (8.5.1) for all* $j \geq 0$*.*

Proof. We prove this corollary by showing that for $j \geq 0$ the b_{-2j-1} (respectively, b_{-2j}) in the construction of the parametrix in Lemma 8.3.1 are odd (respectively, even) functions in the phase variable X, by an induction argument on j. In fact, this completes the proof since in (8.5.1) $\mathsf{Tr}\,(b_{-2j-1})$ is again an odd function of X and in the integral over $\mathbb{S}^{2n-1}$ is 0. First of all,

$$b_0(t, X) = e^{-tp_2(X)} I_2, \quad b_{-1}(t, X) = -te^{-tp_2(X)} a_1$$

(see also (8.4.2)). Hence, b_0 and b_{-1} are, respectively, an even and odd function of X since a_1 is an homogeneous polynomial of degree 1. Now, we suppose that for all $j' \leq 2j-1$ ($j \geq 1$) the claim holds true, and we want to prove the result for b_{-2j} and b_{-2j-1}. By the construction in Lemma 8.3.1, and since A is a differential operator (that is, its expansion contains only terms with degree of homogeneity ≥ 0),

$$\begin{cases} \frac{d}{dt}b_{-2j} + p_2 b_{-2j} = -a_0 b_{2-2j} - a_1 b_{1-2j}, \\ b_{-2j}|_{t=0} = 0. \end{cases}$$

Hence, by the inductive hypothesis, b_{2-2j} and b_{1-2j} are, respectively, an even and odd function of X. Therefore, the claim holds true for b_{-2j}. Repeating the argument for b_{-2j-1}, we have

$$\begin{cases} \frac{d}{dt}b_{-2j-1} + p_2 b_{-2j-1} = -a_0 b_{2-2j-1} - a_1 b_{1-2j-1}, \\ b_{-2j-1}|_{t=0} = 0, \end{cases}$$

which completes the proof since, by the inductive hypothesis, b_{1-2j-1} and b_{2-2j-1} are, respectively, an even and odd function of X.

□

Chapter 9

Spectral quasi-clustering estimates

In this chapter we prove a spectral quasi-clustering estimate for large eigenvalues of a class of systems in the SMGES class. At first, we consider systems with principal symbol given by the harmonic oscillator p_2, semiprincipal symbol with matrix invariants that are functions of the harmonic oscillator and subprincipal of its diagonalized which is constant on the bicharacteristics of p_2. This is a relevant case since the Jaynes-Cummings model and its generalizations in Chapter 2.1 with $\alpha_k = \alpha$ for all k satisfy this property for all N. Then, we extend the result to be able to include also the case with $\alpha_k \neq \alpha_{k'}$ for some $k \neq k'$.

Namely, by spectral quasi-clustering estimate we mean the concentration of the spectrum of a positive self-adjoint ψdo within the union of certain intervals with centers on a sequence determined through invariants of the symbol and width decreasing as the centers go to infinity. Moreover, the determination of a clustering like that is interesting since it actually completes the spectral asymptotic information given by the Weyl law asymptotics. In fact, it gives a precise location of the spectrum for high eigenvalues in case the intervals are disjoint when the centers are in an neighbourhood of $+\infty$ on the real line.

Duistermaat and Guillemin [13] gave a clustering result for the μ-th root of a scalar positive elliptic self-adjoint ψdo P of order $\mu > 0$ on a compact smooth boundaryless manifold under the hypothesis that the bicharacteristics of $\sqrt[\mu]{p}$ are periodic, all with the same period, where p denotes the principal symbol of P. Conversely, they showed that if that clustering occurs, then the flow of $\sqrt[\mu]{p}$ is periodic. Next, Weinstein [60] proved also a eigenvectors clustering result for the reduced Schrödinger operator (on a compact Riemannian manifold) deepening the description of the asymptotic structure of the clusters. We will later recall the arguments used in that paper since they are relevant in our work. Later, Colin de Verdière [9] gave an even more precise result in the case of the square of a first order ψdo with zero subprincipal symbol and 2π-periodic flux on a compact smooth manifold. In fact, he also recovered the multiplicities of the eigenvalues in the disjoint intervals. All these works treated scalar operators. Regarding systems, Helffer [20] obtained a clustering result for the case of a second order global elliptic *regular* positive self-adjoint pseudodifferential operator under the hypothesis that the X-ray transform of the subprincipal symbol is identically a scalar constant.

We generalize to semiregular systems an idea proposed by Weinstein [60]. Namely, he studied ψdos on a compact Riemannian manifold of the form $A^2 + B$ with A a 1st-order, self-adjoint, positive, elliptic ψdo, B a self-adjoint ψdo of order 0, and $e^{2\pi i A} = cI$ for some constant c. His approach is based on an averaging technique: the subprincipal symbol is X-ray transformed on the bicharacteristics of the principal symbol by a unitary operator conjugation and the new subprincipal part commutes with the principal one. Thus, the spectrum of the sum of the principal and subprincipal term can be obtained by studying the one of the two terms individually and it gives the sequence at which the intervals are centered. The remainder, that is, the difference between the conjugated operator and the operator itself, is a compact operator and gives the diameter of the intervals. In fact, the compactness of the remainder leads to an operator inequality and the minimax principle completes the analysis.

In Section 9.2 we prove that the blockwise diagonalization with scalar semiprincipal blocks of a system of our class is equal, modulo a system of order -1, to a system whose principal, semiprincipal, and subprincipal terms commute. Here, we take inspiration from the work by Weinstein. In fact, we study the non-compact part of the operator obtaining an explicit expression for its spectrum. Next, we recapture the spectrum of the whole operator thanks to an operator inequality which leads to our estimate by using the minimax principle.

Actually, the minimax principle alone is not sufficient to obtain the result. In fact, we need to link the spectrum of an operator with the one of its conjugation by the diagonalizing operator E that appears in the inequality of operators mentioned before. This is achieved by, at first, supposing that E^* can be made into an isometry, that is $EE^* = I$, by adding a smoothing term. Then, we prove the result without hypotheses on E by “doubling” our $N \times N$ system, that is by studying a system $2N \times 2N$ having decoupling operator $\tilde{E}$ such that $\tilde{E}^*$ can be made into an isometry. Hence, in Section 9.1 we carry out the task of proving that it is possible for a Fredholm operator (with non positive index and parametrix given by its adjoint) to be made into an isometry by adding a compact operator.

9.1 The “isometrization”

In this section we consider a *quasi-unitary* pseudodifferential system U.

Definition 9.1.1. *We call quasi-unitary a system such that* $U^*U = I + F_1$ *and* $UU^* = I + F_2$ *where the* F_j *are compact and* I *is the identity operator.*

We show by Lemma 9.1.2 that we can perturb a quasi-unitary pseudodifferential system U by an operator of the same order of F_1 to make it into an isometry. This is fundamental for Section 9.2 since in Theorem 9.2.3 and Theorem 9.2.5, as we stated, we will need to relate the spectrum of the diagonalization of the SMGES under study with the spectrum of the SMGES itself. In fact, in the proofs of Theorem 9.2.3 and Theorem 9.2.5 we will see that the conjugation by an isometry change the point spectrum of a positive ψdo by adding, at most, the eigenvalue 0. Hence, the conjugation by an isometry conserve the spectrum asymptotic property of a positive SMGES.

Lemma 9.1.2. *Let* $u \in S_{\mathrm{sreg}}(1, g; \mathsf{M}_N)$ *such that* $U^*U = I + F_1$, $UU^* = I + F_2$ *and* $\operatorname{ind} U \le 0$ *where the* F_j *are* ψ*dos with negative order and* $U := u^{\mathrm{w}}$*. Then, there is a* ψ*do* K *such that* $U + K$ *is an isometry (that is* $(U+K)^*(U+K) = I$*). Moreover,* K *has the same order as* F_1*.*

Proof. First of all, we consider the case of a ψdo U such that $U^*U = I + F$ where F has negative order and U is one-to-one (that is, $-1 \notin \mathrm{Spec}(F)$). We want to construct a ψdo R such that UR is an isometry where $R = I + K'$ with K' a ψdo with the same order as F. More precisely, R would (formally) be the inverse of the squared root of $I + F$. To give a precise meaning to R as a ψdo we follow the approach by Helffer in [20]. It is based on the construction of an abstract linear operator (seen as a $L^2 \to L^2$ bounded operator) commuting with F . Next, one shows that it can be approximated by a ψdo H by using the inverse squared root series of F_0 where F_0 is the projection of F on a finite codimension vector subspace of L^2 such that $\|F_0\|_{L^2 \to L^2} \le 1/2$.

After that, our aim is to show that U can be modified by a smoothing operator to become one-to-one. Here we make use of the hypothesis $\operatorname{ind} U \le 0$. In fact, we construct a bijection $Q : \ker U \to S \subset \ker U^*$ which, then, we extend to L^2 by imposing $Q|_{(\ker U)^\perp} = 0$. Finally, the claim is true since $R = I + K'$ where K' has the same order as F_1 and, hence, $R(U+Q) = U + K$ is an isometry with $K := Q + K'(U+Q)$ having the same order as F_1.

First of all, we consider the orthonormal basis $(\phi_j)_{j \ge 1}$ of L^2 given by the eigenfunctions of F (which is self-adjoint and compact, its order denoted by $-\ell$ for $\ell > 0$). Hence, we can abstractly define the bounded operator $R : L^2 \to L^2$ by

$$R\phi_j := (1 + \nu_j)^{-1/2}\phi_j, \ \forall j \ge 1,$$

where ν_j is the eigenvalue of F associated with the eigenfunction ϕ_j. R is well defined since we supposed $-1 \notin \mathrm{Spec}(F)$. Now, we consider the ψdo H such that

$$H - \sum_{j=0}^{k} c_j F^j \text{ is a } \psi\text{do of order } -\ell(k+1), \quad \forall k \ge 0, \tag{9.1.1}$$

where c_j is the j-th Taylor coefficient of the expansion of $t \mapsto (1+t)^{-1/2}$ at the origin. We want to prove that H approximates R, that is, $H - R$ is smoothing. To do that, we would like to write the inverse squared root series $\sum_{j\geq 0} c_j F^j$. However, this series is only formal since there could be ν_j such that $|\nu_j| \geq 1$. Hence, to give meaning to it, we introduce the linear operator F_0 defined on the basis $(\phi_j)_{j\geq 1}$ by

$$F_0\phi_j := \begin{cases} 0, & \text{if } |\nu_j| > 1/2, \\ F\phi_j, & \text{if } |\nu_j| \leq 1/2. \end{cases}$$

Since F is a compact operator, $\nu_j \to 0$ as $j \to \infty$ and, hence, $|\nu_j| > 1/2$ are finitely many. Moreover, F_0 is a ψdo because

$$F^j - F_0^j \text{ is smoothing, } \forall j \geq 0 \tag{9.1.2}$$

since

$$(F^j - F_0^j)\phi = \sum_{k\colon |\nu_k|>1/2} \nu_k^j (\phi, \phi_k)_{L^2}\phi_k, \quad \forall \phi \in L^2(\mathbb{R}^n),$$

Thus, it is a $L^2 \to L^2$ bounded operator with Schwartz kernel

$$\mathbb{R}^{2n} \ni (x,y) \mapsto \mathsf{K}_{F^j - F_0^j}(x,y) := \sum_{k\colon |\nu_k|>1/2} \nu_k^j \overline{\phi_k(y)}\phi_k(x) \in \mathscr{S}(\mathbb{R}^{2n}).$$

In fact, $\phi_k \in \mathscr{S}$ when $|\nu_k| > 1/2$ by ellipticity of the 0-th order operator $F - \nu_k I$. Therefore, $R_0 := \sum_{j\geq 0} c_j F_0^j$ is well-defined as bounded operator on L^2 since $\|F_0\|_{L^2\to L^2} \leq 1/2$. In addition,

$$R - R_0 \text{ is smoothing.} \tag{9.1.3}$$

In fact,

$$(R - R_0)\phi = \sum_{j\colon |\nu_j|>1/2} (1+\nu_j)^{-1/2}(\phi, \phi_j)_{L^2}\phi_j, \ \forall \phi \in L^2(\mathbb{R}^n),$$

and, hence, it is a $L^2 \to L^2$ bounded operator with Schwartz kernel

$$\mathbb{R}^{2n} \ni (x,y) \mapsto \mathsf{K}_{R-R_0}(x,y) := \sum_{j\colon |\nu_j|>1/2} (1+\nu_j)^{-1/2}\overline{\phi_j(y)}\phi_j(x) \in \mathscr{S}(\mathbb{R}^{2n}).$$

Now, we prove that $H-R$ is smoothing by a telescoping sum argument. Actually, by (9.1.3) it is sufficient to show that $H-R_0$ is smoothing. We start by writing

$$H - R_0 = A_k + B_k - C_k,$$

where for all $k \geq 0$

$$A_k := H - \sum_{j=0}^{2k} c_j F_0^j,\ B_k := \sum_{j=0}^{2k} c_j (F^j - F_0^j),\ C_k := R_0 - \sum_{j=0}^{2k} c_j F_0^j$$

are $L^2 \to L^2$ bounded operators. Now, A_k is a ψdo of order $-\ell(2k+1)$ by (9.1.1) and B_k is smoothing by (9.1.2) for all the k. We only need to study C_k, that is

$$C_k = F_0^{2k+1} \sum_{j\geq 2k+1} c_j F_0^{j-(2k+1)} = F_0^{k+1}\left(\sum_{j\geq k+1} c_j F_0^{j-(k+1)}\right) F_0^k,$$

which means that C_k is a $B^{-\ell k} \to B^{\ell(k+1)}$ bounded operator for all k. In fact, $\sum_{j\geq k+1} c_j F_0^{j-(2k+1)}$ is a bounded operator on L^2 and F_0^j is a ψdo of order $-\ell j$ for all j. Hence, $H - R_0$ is smoothing, too.

Therefore, we complete the first part of the proof by denoting UR by $\tilde{U}$ and we have that $\tilde{U}^*\tilde{U} = R^*U^*UR = (I+F)R^2 = I$ since R and $I+F$ commute because their eigenspaces coincide. We notice that $R = I + K'$ with K' a ψdo of order $-\ell$ by the definition of H. Hence, $\tilde{U} = U(I+K') = U + \tilde{K}$ with $\tilde{K} := UK'$ a ψdo of order $-\ell$. We highlight that R is invertible since R^2 is, by construction, the inverse of $I + F = U^*U > 0$.

Thus, we proved the theorem under the hypothesis $-1 \notin \operatorname{Spec}(F)$.

Now we show that we can modify U by a smoothing operator Q such that $\tilde{U} := U + Q$ is one-to-one.

We consider the orthonormal basis $(\phi_j)_{j\geq 1}$ of L^2 given by the eigenfunctions of F_1 and we call ν_j the eigenvalue of F_1 corresponding to ϕ_j. We denote by Z_1 the set of all j such that $\nu_j = -1$. Z_1 is a finite set since F_1 is compact. Then, we consider the orthonormal basis $(\psi_j)_{j\geq 1}$ of L^2 given by the eigenfunctions of F_2 and we call $\tilde{\nu}_j$ the eigenvalue of F_2 corresponding to ψ_j. We denote by Z_2 the set of all j such that $\tilde{\nu}_j = -1$. Z_2 is a finite set since F_2 is compact. Next,

$$\begin{aligned}\ker U &= \{\phi \in L^2;\, U\phi = 0\} = \{\phi \in L^2;\, U^*U\phi = 0\} \\ &= \operatorname{Span}\{\phi_j;\, U^*U\phi_j = 0\} = \operatorname{Span}\{\phi_j;\, j \in Z_1\},\end{aligned} \tag{9.1.4}$$

and

$$\begin{aligned}\ker U^* &= \{\psi \in L^2;\, U^*\psi = 0\} = \{\psi \in L^2;\, UU^*\psi = 0\} \\ &= \operatorname{Span}\{\psi_j;\, UU^*\psi_j = 0\} = \operatorname{Span}\{\psi_j;\, j \in Z_2\}.\end{aligned} \tag{9.1.5}$$

Now we construct the one-to-one linear operator $Q : L^2 \to L^2$ being non-zero on $\ker U \setminus \{0\}$ and ranging on $\ker U^*$. First, there is a one-to-one function $\rho : Z_1 \to Z_2$ since $\operatorname{ind} U \leq 0$ implies $\operatorname{card} Z_1 \leq \operatorname{card} Z_2$ by (9.1.4) and (9.1.5). Now, we define

$$Q\phi := \sum_{j\in Z_1} (\phi, \phi_j)_{L^2} \psi_{\rho(j)},\ \forall \phi \in L^2(\mathbb{R}^n),$$

which is smoothing since it is a $L^2 \to L^2$ bounded operator with Schwartz kernel

$$\mathbb{R}^{2n} \ni (x, y) \mapsto \mathsf{K}_Q(x, y) := \sum_{j\in Z_1} \overline{\phi_j(y)} \psi_{\rho(j)}(x) \in \mathscr{S}(\mathbb{R}^{2n}).$$

Hence, $U + Q$ is one-to-one. In fact, by denoting the range of U by $\operatorname{Ran} U$

$$(U + Q)\phi = 0 \ \Leftrightarrow\ \underbrace{U\phi}_{\in \operatorname{Ran} U} = \underbrace{-Q\phi}_{\in \ker U^*},$$

and, thus, $\operatorname{Ran} U = (\ker U^*)^\perp$ implies

$$U\phi = -Q\phi \in (\ker U^*)^\perp \cap \ker U^* = \{0\}.$$

Therefore, $\phi = 0$ since $\ker Q \cap \ker U = \{0\}$ by definition of Q. This concludes the proof.

□

Remark 9.1.3. *If in Lemma 9.1.2* $\operatorname{ind} U = 0$ *then* ρ *can be constructed to be bijective and, hence,* $U + Q$ *is surjective, too. In fact,* $(U+Q)^*\phi = 0$ *implies*

$$\underbrace{U^*\phi}_{\in \operatorname{Ran} U^*} = \underbrace{-Q^*\phi}_{\in \ker U}.$$

Hence,

$$U^*\phi = -Q^*\phi \in (\ker U)^\perp \cap \ker U = \{0\},$$

which means $\phi = 0$ *since* $\ker Q^* \cap \ker U^* = \{0\}$ *by bijectivity of* ρ. *Since* $R^*\tilde{U}^*\tilde{U}R = I$ *with* R *invertible,* $\tilde{U}R$ *is unitary. In fact, it is invertible (*$\tilde{U}$ *and* R *are invertible) and the left inverse is unique.*

In addition, if $\operatorname{ind} U > 0$ *then* $\operatorname{ind} U^* < 0$ *and we can repeat the proof of Lemma 9.1.2 with* U^* *in place of* U. *Thus, there is* K *such that* $U^* + K$ *is an isometry and* K *has the same order as* F_2.

Moreover, if $\operatorname{ind} U \leq 0$ *and* F_1 *is smoothing (or* $\operatorname{ind} U \geq 0$ *and* F_2 *is smoothing), then* K *is smoothing, too.*

9.2 Spectral quasi-clustering theorems

In this section we are going to determine a quasi-clustering estimate for a class of SMGES. First of all, we state and prove Theorem 9.2.3 where we consider the class of SMGES having the semiprincipal matrix eigenvalues that are function of the scalar harmonic oscillator and the subprincipal of the diagonalized system that are constant on the bicharacteristics of the harmonic oscillator. This is a relevant class since the Jaynes-Cummings Hamiltonian operator and its generalizations in Chapter 2.1 with $\alpha_k = \alpha$ for all k satisfy this property for all N.

Actually, we look for a result for a more general class of SMGES which includes at least the models with $\alpha_k \neq \alpha_{k'}$ for some $k \neq k'$. Therefore, we state and prove Theorem 9.2.5 where we consider the class of SMGES which have the semiprincipal matrix eigenvalues that are function of the polynomial $p_{2,\alpha} := \sum_{k=1}^{n} \alpha_k^2 p_{2,k}$ with $p_{2,k}(X) := \frac{x_k^2+\xi_k^2}{2}$ for all $X \in \mathbb{R}^{2n}$, $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, and $\alpha_j \neq 0$ for all j.

We recall that we use Lemma 9.1.2 and Remark 9.1.3 in the following theorems to relate the spectrum of the diagonalization of a positive SMGES with the SMGES itself.

In this section (and in the following chapter) we use this notation: letting a be a semiregular symbol in $S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$, we denote by A (that is, by using the corresponding capital letter) the unbounded densely defined and closed operator $D(A) \subset L^2 \to L^2$ which is the realization of the ψdo a^{w}. If a is elliptic and $\mu \geq 0$, $D(A) = B^\mu$.

First of all, we prove a lemma that, actually, shows our result under the hypothesis that the decoupling operator has non-negative index.

Lemma 9.2.1. *Let $a = a^* \sim \sum_{j\geq 0} a_{2-j} \in S^2_{\mathrm{sreg}}(m^2, g; \mathsf{M}_N)$ be a 2nd-order SMGES with principal symbol $a_2 = p_2 I_N$, such that the corresponding unbounded operator satisfies $A > 0$. Suppose that the coefficients of the characteristic polynomial $\det(\lambda - a_1(X))$ of the semiprincipal term a_1 are functions of p_2 and that there is a unitary diagonalizer e_0 for the semiprincipal symbol such that, denoting by $b_0 := \mathrm{diag}(b_{0,h};\ 1 \leq h \leq r)$ the subprincipal symbol of the resulting blockwise diagonalization of A, we have that*

$$b_0 \circ \exp(tH_{p_2}) = b_0, \quad b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0, \quad \forall t \in \mathbb{R}.$$

In addition, suppose that $\operatorname{ind} E_0 \geq 0$.

Then, the eigenspaces of P_2 are invariant for $B_{0,h}$, for all $h = 1, \dots, r$. Moreover, for each $h = 1, \dots, r$, there is an orthonormal basis of $L^2(\mathbb{R}^n; \mathbb{C}^{N_h})$

$$\{\phi_{h,k,j}\}_{k\in\mathbb{Z}_+, 1\leq j\leq N_h} \subset \mathscr{S}(\mathbb{R}^n; \mathbb{C}^{N_h})$$

such that

$$\mathrm{Ker}\big(P_2 \otimes I_{N_h} - (k+\frac{n}{2})\big) = \mathrm{Span}\{\phi_{h,k,j};\ 1 \le j \le N_h\},$$

$$B_{0,h}\phi_{h,k,j} = \mu_{h,k,j}\phi_{h,k,j}, \quad \textit{with } |\mu_{h,k,j}| \le \|B_{0,h}\|_{L^2 \to L^2}, \quad \forall k,\ \forall j = 1,\dots,N_h,$$

and a smooth function $p_{1/2}\colon \mathbb{R}_+ \longrightarrow \mathbb{R}^r$*, positively homogeneous of degree* $1/2$*, such that* $\lambda_{1,h} = p_{1/2,h}(p_{2,\alpha})$*,* $1 \le h \le r$*, and* $M > 0$ *such that*

$$\mathrm{Spec}(A) \subset \bigcup_{h=1}^{r} \bigcup_{k\ge 0} \bigcup_{j=1}^{N_h} S_{h,k,j}(A), \tag{9.2.1}$$

where, for each $h = 1,\dots,r$*,*

$$S_{h,k,j}(A) := \big(k + \frac{n}{2} + p_{1/2,h}(k+n/2) + \mu_{h,k,j}\big) + \Big[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\Big]. \tag{9.2.2}$$

Remark 9.2.2. *To assume that* $A > 0$ *in Lemma 9.2.1 (and in the theorems and lemma that will follow in this section) is not a loss of generality since this is true for a SMGES of second order modulo an additive positive scalar constant. In fact, the principal symbol of a SMGES is always elliptic and positive and, thus, there is* $C > 0$ *such that* $(A\phi,\phi)_{L^2} > -C$ *for all* $\phi \in B^2\backslash\{0\}$ *by the Sharp-Gårding inequality (see [26] and its form as in Theorem. 3.3.22 of [45]).*

Proof. The proof takes inspiration from the approach by Weinstein [60] adapted to semiregular ψdos. The main idea is to investigate the spectrum of A by studying the spectrum of the part of its blockwise diagonalization B which has positive order as a ψdo (the difference being a compact operator). Of course, it will suffice to work for a single block of B, which is parametrized by $h = 1,\dots,r$. Hence, we may suppose that $r = 1$ and that b_2, b_1 are scalar operators.

Let P be the self-adjoint L^2 realization of $p^{\mathrm{w}} := p_2^{\mathrm{w}} + p_{1/2}(p_2^{\mathrm{w}}) + b_0^{\mathrm{w}}$ with $D(P) = D(P_2) = B^2(\mathbb{R}^n;\mathbb{C}^N)$. Recall that, for the semiprincipal term b_1 of B, we have that $b_1(X) = (p_{1/2} \circ p_2)(X)$ for $X \neq 0$, with $p_{1/2}$ smooth and positively homogeneous of degree 1/2, by virtue of the hypothesis (namely, that the characteristic polynomial of a_1 have coefficients which are (smooth) functions of p_2).

The first step in the proof is to show that $b^{\mathrm{w}} - p^{\mathrm{w}} =: k_1^{\mathrm{w}}$ has order -1. Since $b^{\mathrm{w}} = p_2^{\mathrm{w}} + (p_{1/2} \circ p_2)^{\mathrm{w}} + b_0^{\mathrm{w}}$ modulo a term of order -1 and since, by Theorem 1.11.2 in [20], $(p_{1/2} \circ p_2)^{\mathrm{w}} - p_{1/2}(p_2^{\mathrm{w}})$ has order -1, we obtain indeed that $k_1 \in S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$.

We next show that the commutator $[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}] = 0$.
Since $[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}]\big|_{\mathscr{S}} = [P_2, B_0]\big|_{\mathscr{S}}$ and $[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}]$ has order 0, it follows that we may extend $[P_2, B_0]\big|_{\mathscr{S}}$ as a bounded linear operator $[P_2, B_0] \colon L^2 \longrightarrow L^2$. Hence, if we show that $[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}] = 0$, then also $[P_2, B_0] = 0$. Now, $b_0 \circ \exp(tH_{p_2}) = b_0$ for all t by hypoyhesis. Hence $b_0 = \mathsf{R}(b_0)$ and (on $\mathscr{S}$)

$$[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}] = [p_2^{\mathrm{w}}, \mathsf{R}(b_0)^{\mathrm{w}}] = \frac{-i}{2\pi} \int_0^{2\pi} \partial_t (e^{itP_2} b_0^{\mathrm{w}} e^{-itP_2}) \, dt = \frac{-i}{2\pi} [e^{itP_2} b_0^{\mathrm{w}} e^{-itP_2}]_0^{2\pi} = 0.$$

Therefore, the eigenspaces of P_2 are invariant under B_0. We may hence choose an orthonormal system $\{\phi_{k,j};\ k \in \mathbb{Z}_+,\ 1 \le j \le N\} \subset \mathscr{S}(\mathbb{R}^n; \mathbb{C}^N)$ of $L^2(\mathbb{R}^n; \mathbb{C}^N)$, made of eigenfunctions of both P and P_2, that also diagonalizes $B_0\big|_{W_k}$ on each space $W_k := \mathrm{Span}_{\mathbb{C}}\{\phi_{k,j};\ 1 \le j \le N\}$, $k \in \mathbb{Z}_+$. It follows that the eigenvalue of P associated with the eigenfunctions $\phi_{k,j}$, for $1 \le j \le N$ is

$$k + \frac{n}{2} + p_{1/2}(k + n/2) + \mu_{k,j}, \quad 1 \le j \le N,$$

where $\mu_{k,j} \in \mathbb{R}$ is such that $B_0 \phi_{k,j} = \mu_{k,j} \phi_{k,j}$.

Hence,

$$\mathrm{Spec}(P) = \bigcup_{k \ge 0} \bigcup_{j=1}^{N} C_{k,j},$$

where

$$C_{k,j} := k + \frac{n}{2} + p_{1/2}(k + n/2) + \mu_{k,j}.$$

We next wish to show that there is $M > 0$ such that

$$\mathrm{Spec}(A) \subset \bigcup_{k \ge 0} \bigcup_{j=1}^{N} \left(C_{k,j} + \Big[-\frac{M}{\sqrt{k + n/2}}, \frac{M}{\sqrt{k + n/2}} \Big] \right). \tag{9.2.3}$$

For that, we have to consider the chosen ψdo diagonalizer e^{w} (and hence its L^2 bounded extension E) for a^{w} (see Theorem 3.1.3 in [37]). Then, $\operatorname{ind} E = \operatorname{ind} E_0 \geq 0$ by hypothesis since the index of an operator is invariant under compact perturbations. Thus, by the quasi-isometrization Theorem 9.1.2, we may assume that $E^*\colon L^2 \to L^2$ is an isometry (i.e. $EE^* = I$). Letting

$$\tilde{r}^{\mathrm{w}} := (e^{\mathrm{w}})^* a^{\mathrm{w}} e^{\mathrm{w}} - p^{\mathrm{w}} = ((e^{\mathrm{w}})^* a^{\mathrm{w}} e^{\mathrm{w}} - b^{\mathrm{w}}) + (b^{\mathrm{w}} - p^{\mathrm{w}}),$$

which has order -1, and noting that $(p_2^{\mathrm{w}})^{1/4}$ is a ψdo of order $1/2$ with principal symbol $p_2^{1/4}$, we have that $(p_2^{\mathrm{w}})^{1/4}\tilde{r}^{\mathrm{w}}(p_2^{\mathrm{w}})^{1/4}$ can be extended to a bounded operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$. Hence, there is $M > 0$ such that for all $\psi \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$

$$-M\|\psi\|_{L^2}^2 \leq \big((p_2^{\mathrm{w}})^{1/4}\tilde{r}^{\mathrm{w}}(p_2^{\mathrm{w}})^{1/4}\psi, \psi\big)_{L^2} \leq M\|\psi\|_{L^2}^2, \tag{9.2.4}$$

that we rewrite in terms of the L^2 realizations of the ψdos involved

$$-M\|\psi\|_{L^2}^2 \leq \left(P_2^{1/4}\tilde{R}P_2^{1/4}\psi, \psi\right)_{L^2} \leq M\|\psi\|_{L^2}^2. \tag{9.2.5}$$

Now, recalling that $P_2^{1/4}\colon D(P_2^{1/4}) \subset L^2 \longrightarrow L^2$ is the self-adjoint unbounded L^2 realization of $(p_2^{\mathrm{w}})^{1/4}$ which is elliptic, we have that $D(P_2^{1/4}) = B^{1/2}(\mathbb{R}^n;\mathbb{C}^N)$ (which is dense in L^2) and $P_2^{1/4}$ is invertible with bounded inverse $P_2^{-1/4} : L^2 \to B^{1/2} \subset L^2$. Therefore, by substituting $P_2^{-1/4}E^*\phi$ for ψ in (9.2.5), we get that for all $\phi \in \mathscr{S}$

$$-M\left(P_2^{-1/2}\phi, \phi\right)_{L^2} \leq \left(\tilde{R}\phi, \phi\right)_{L^2} \leq M\left(P_2^{-1/2}\phi, \phi\right)_{L^2}.$$

Hence, for all $\phi \in B^2$,

$$\begin{aligned}\left(\left(P - MP_2^{-1/2}\right)\phi, \phi\right)_{L^2} &\leq (\underbrace{P + \tilde{R}}_{=E^*AE}\phi, \phi)_{L^2} \\ &\leq \left(\left(P + MP_2^{-1/2}\right)\phi, \phi\right)_{L^2},\end{aligned}$$

which leads to (9.2.3) by minimax principle. In fact,

$$\mathrm{Spec}\,(A) \subset \mathrm{Spec}\,(E^*AE)\,,$$

since

$$A\phi_\lambda = \lambda\phi_\lambda, \quad \phi_\lambda \neq 0$$

implies

$$(E^*AE)\,E^*\phi_\lambda = \lambda E^*\phi_\lambda,\ E^*\phi_\lambda \neq 0.$$

Moreover,

$$\mathrm{Spec}\,(E^*AE) \setminus \{0\} \subset \mathrm{Spec}\,(A)\,,$$

since

$$E^*AE\psi_\eta = \eta\psi_\eta, \quad \psi_\eta \neq 0 \tag{9.2.6}$$

implies

$$AE\psi_\eta = \eta E\psi_\eta.$$

On the one hand, if $\psi_\eta \notin \ker\, E$ then $\eta \in \mathrm{Spec}\,(A)$. On the other hand, if $\psi_\eta \in \ker\, E$ then $\eta = 0$ by (9.2.6) and $0 \notin \mathrm{Spec}\,(A)$ since $A > 0$. Therefore

$$\mathrm{Spec}\,(E^*AE) \setminus \{0\} = \mathrm{Spec}\,(A)\,,$$

which concludes the proof of the lemma.

□

We want to generalize Lemma 9.2.1 by removing the hypothesis on the non-negativity of the decoupling operator index.

Theorem 9.2.3. *Let $a = a^* \sim \sum_{j\geq 0} a_{2-j} \in S^2_{\mathrm{sreg}}$ be a 2nd-order SMGES with principal symbol $a_2 = p_2 I_N$, such that the corresponding unbounded operator $A > 0$ (this is no restriction, in view of the Sharp-Gårding inequality). Suppose that the coefficients of the characteristic polynomial* $\det(\lambda - a_1(X))$ *of the semiprincipal term a_1 are functions of p_2 and that there exist smooth functions $\mathbb{R} \times \dot{\mathbb{R}}^{2n}_X \ni (t, X) \mapsto f_t(X) \in \mathsf{M_N}$ such that*

$$\{p_2, f_t\} = 0, \quad e_0 \circ \exp(t\mathsf{H}_{p_2}) = f_t e_0, \quad a_0 = f_t^*\,(a_0 \circ \exp(t\mathsf{H}_{p_2}))\,f_t. \tag{9.2.7}$$

Then, with $b_0 = \mathrm{diag}(b_{0,h};\ 1 \le h \le r)$ *denoting the subprincipal symbol of the blockwise diagonalization of* a^{w}*, the eigenspaces of* P_2 *are invariant for* $B_{0,h}$*, for all* $h = 1, \ldots, r$*. Moreover, for each* $h = 1, \ldots, r$*, there is an orthonormal basis* $\{\phi_{h,k,j}\}_{k\in\mathbb{Z}_+, 1\le j\le N_h} \subset \mathscr{S}(\mathbb{R}^n;\mathbb{C}^{N_h})$ *of* $L^2(\mathbb{R}^n;\mathbb{C}^{N_h})$ *such that* $\mathrm{Ker}\big(P_2 \otimes I_{N_h} - (k+\frac{n}{2})\big) = \mathrm{Span}\{\phi_{h,k,j};\ 1 \le j \le N_h\}$*, and*

$$B_{0,h}\phi_{h,k,j} = \mu_{h,k,j}\phi_{h,k,j}, \quad \textit{with } |\mu_{h,k,j}| \le \|B_0\|_{L^2\to L^2}, \quad \forall k,\ \forall j = 1, \ldots, N_h,$$

a smooth function $p_{1/2}\colon \mathbb{R}_+ \longrightarrow \mathbb{R}^r$*, positively homogeneous of degree* $1/2$*, such that* $\lambda_{1,h} = p_{1/2,h}(p_2)$*,* $1 \le h \le r$*, and* $M, c > 0$ *such that*

$$\mathrm{Spec}(A) \subset \bigcup_{h=1}^{r+1} \bigcup_{k\ge 0} \bigcup_{j=1}^{N_h} S_{h,k,j}(A), \tag{9.2.8}$$

where, for each $h = 1, \ldots, r$*,*

$$S_{h,k,j}(A) := \big(k + \frac{n}{2} + p_{1/2,h}(k+n/2) + \mu_{h,k,j}\big) + \Big[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\Big], \tag{9.2.9}$$

where $N_{r+1} = N$*,* $p_{1/2,r+1} = 0$ *and* $\mu_{r+1,k,j} = c$ *for all* j*.*

Proof. The proof follows from an argument based on the construction of a system $\tilde{A}$ associated with A having a decoupling operator $\tilde{e}_0$ with non-negative index. Namely, $\tilde{A}$ is a block-diagonal system with two $N \times N$ blocks, the first being A and the second being $P_2 I_N + P_0$,

$$\begin{aligned} p_0 := &- e_0^*(e_{-2}e_0^* p_2 + p_2 e_0 e_{-2}^* - \frac{i}{2}\left(e_0\left\{a_\mu, e_0^*\right\} + \left\{e_0, a_\mu e_0^*\right\}\right) \\ &+ e_{-1}p_2 e_{-1}^*)e_0 + cI_N, \end{aligned}$$

with $c > 0$ a real constant such that $P_0 > 0$, and

$$\tilde{A} := \begin{bmatrix} A & 0_N \\ 0_N & P_2 I_N + P_0 \end{bmatrix}.$$

Hence,

$$\tilde{E}^*\tilde{A}\tilde{E} = \begin{bmatrix} B+R' & 0_N \\ 0_N & (P_2+c)I_N+R \end{bmatrix},$$

where $\tilde{e} := \begin{bmatrix} e & 0_N \\ 0_N & e^* \end{bmatrix}$, R' is smoothing and $r \in S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$ since, by Proposition 4.0.1 (or by a straightforward computation, using the composition formula for ψdos), the subprincipal symbol of $E(P_2 I_N + P_0)E^*$ is cI_N by definition of p_0. Actually, $\tilde{A}$ verifies the hypothesis of Lemma 9.2.1. In fact, $\tilde{A} > 0$, $\mathrm{ind}\,(\tilde{E}) = \mathrm{ind}\,(E) + \mathrm{ind}\,(E^*) = 0$ and, by Corollary 4.2.1,

$$b_{0,h} = \pi_h(e_0^* a_{\mu-2} e_0 - i e_0^* \{a_\mu, e_0\})\pi_h^*.$$

Therefore $b_0 \circ \exp(tH_{p_2}) = b_0$ for all t since

$$e_0 \circ \exp(tH_{p_2}) = f_t e_0, \quad a_0 = f_t^* \left(a_0 \circ \exp(tH_{p_2})\right) f_t, \ \forall t,$$

by hypothesis. Hence, by Lemma 9.2.1, we have (9.2.1) for $\tilde{A}$, that is

$$\mathrm{Spec}(\tilde{A}) \subset \bigcup_{h=1}^{r+1} \bigcup_{k\geq 0} \bigcup_{j=1}^{N_h} S_{h,k,j}(A),$$

where, for each $h = 1, \ldots, r+1$,

$$S_{h,k,j}(A) := \left(k + \frac{n}{2} + p_{1/2,h}(k+n/2) + \mu_{h,k,j}\right) + \left[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\right]$$

with $N_{r+1} = N$, $p_{1/2,r+1} = 0$ and $\mu_{r+1,k,j} = c$ for all j. Moreover, $\mathrm{Spec}\left(\tilde{A}\right) = \mathrm{Spec}\,(A) \sqcup \mathrm{Spec}\,(P_2 I_N + P_0)$ since $\tilde{A}$ is blockwise diagonal. (The disjoint union symbol $\sqcup$ means that we are counting the eigenvalues with their multiplicities and summing them up.) Hence,

$$\mathrm{Spec}(A) \subset \bigcup_{h=1}^{r+1} \bigcup_{k\geq 0} \bigcup_{j=1}^{N_h} S_{h,k,j}(A),$$

which completes the proof.

□

Now we want to extend Theorem 9.2.3 to the case in which $\alpha_k \neq \alpha_{k'}$ for some $k \neq k'$. Hence, we define $p_{2,k}(X) := \frac{x_k^2+\xi_k^2}{2}$ and $p_{2,\alpha}(X) := \sum_{k=1}^{n} \alpha_k \frac{x_k^2+\xi_k^2}{2}$ with $X \in \mathbb{R}^{2n}$ and $\alpha \in \mathbb{R}^n$ with $\alpha_k \neq 0$ for all k and state a generalization of Lemma 9.2.1 to the case in which the coefficients of the characteristic polynomial of the semiprincipal symbol are functions of $p_{2,\alpha}$.

Lemma 9.2.4. *Let $a = a^* \sim \sum_{j\geq 0} a_{2-j} \in S^2_{\mathrm{sreg}}$ be a 2nd-order SMGES with principal symbol $a_2 = p_2 I_N$, such that the corresponding unbounded operator $A > 0$ (this is no restriction, in view of the Sharp-Gårding inequality). Suppose that the coefficients of the characteristic polynomial* $\det(\lambda - a_1(X))$ *of the semiprincipal term a_1 are functions of $p_{2,\alpha}$ and that there is a unitary diagonalizer e_0 for the semiprincipal symbol such that, denoting by $b_0 :=$* $\mathrm{diag}(b_{0,h};\ 1 \leq h \leq r)$ *the subprincipal symbol of the resulting blockwise diagonalization of A, we have that*

$$b_0 \circ \exp(tH_{p_2}) = b_0, \quad b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0, \quad \forall t \in \mathbb{R}.$$

In addition, suppose that $\mathrm{ind}\, E_0 \geq 0$ *and that there are $m_1, \dots, m_n \in \mathbb{Z}_+ \setminus \{0\}$ coprime such that*

$$\frac{m_1}{\alpha_1} = \dots = \frac{m_n}{\alpha_n} =: L. \tag{9.2.10}$$

Then, the eigenspaces of P_2 are invariant for $P_{2,\alpha}$ and the eigenspaces of $(P_2 + P_{2,\alpha}) \otimes I_{N_h}$ are invariant for $B_{0,h}$, for all $h = 1, \dots, r$. Moreover, for each $h = 1, \dots, r$, there is an orthonormal basis $\{\phi_{h,\gamma,j}\}_{\gamma\in\mathbb{Z}^n_+, 1\leq j\leq N_h} \subset \mathscr{S}(\mathbb{R}^n; \mathbb{C}^{N_h})$ of $L^2(\mathbb{R}^n; \mathbb{C}^{N_h})$ such that $\mathrm{Ker}\big(P_2 \otimes I_{N_h} - (k + \frac{n}{2})\big) = \mathrm{Span}\{\phi_{h,\gamma,j};\ |\gamma| = k,\ 1 \leq j \leq N_h\}$*, and*

$$B_{0,h}\phi_{h,\gamma,j} = \mu_{h,\gamma,j}\phi_{h,\gamma,j}, \ \ \textit{with}\ |\mu_{h,\gamma,j}| \leq \|B_{0,h}\|_{L^2\to L^2}, \quad \forall\gamma,\ \forall j = 1, \dots, N_h,$$

and a smooth function $p_{1/2} \colon \mathbb{R}_+ \longrightarrow \mathbb{R}^r$, positively homogeneous of degree

$1/2$, *such that* $\lambda_{1,h} = p_{1/2,h}(p_{2,\alpha})$, $1 \le h \le r$, *and* $M > 0$ *such that*

$$\operatorname{Spec}(A) \subset \bigcup_{h=1}^{r} \bigcup_{k\ge 0} \bigcup_{\substack{\gamma\in\mathbb{Z}_+^n\\ |\gamma|=k}} \bigcup_{j=1}^{N_h} S_{h,k,\gamma,j}(A), \tag{9.2.11}$$

where, for each $h = 1, \dots, r$,

$$S_{h,k,\gamma,j}(A) := \big(k+\frac{n}{2}+p_{1/2,h}(\alpha(\gamma+1/2))+\mu_{h,\gamma,j}\big)+\Big[-\frac{M}{\sqrt{k+n/2}}, \frac{M}{\sqrt{k+n/2}}\Big], \tag{9.2.12}$$

where $\alpha(\gamma + 1/2) := \sum_{j=1}^{n} \alpha_j(\gamma_j + 1/2)$.

Proof. The proof takes inspiration from the approach by Weinstein [60] and adapted to semiregular ψdos. The main idea is to investigate the spectrum of A by studying the spectrum of the part of its blockwise diagonalization B which has positive order as a ψdo (the difference being a compact operator). Of course, it will suffice to work for a single block of B, which is parametrized by $h = 1, \dots, r$. Hence, we may suppose that $r = 1$ and that b_2, b_1 are scalar operators.

Let P be the self-adjoint L^2 realization of $p^{\mathrm{w}} := p_2^{\mathrm{w}} + p_{1/2}(p_{2,\alpha}^{\mathrm{w}}) + b_0^{\mathrm{w}}$ with $D(P) = D(P_2) = B^2(\mathbb{R}^n;\mathbb{C}^N)$. Recall that for the semiprincipal term b_1 of B we have that $b_1(X) = (p_{1/2} \circ p_{2,\alpha})(X)$ for $X \neq 0$ with $p_{1/2}$ smooth and positively homogeneous of degree $1/2$, by virtue of the hypothesis that the characteristic polynomial of a_1 have coefficients which are (smooth) functions of $p_{2,\alpha}$.

The first step in the proof is to show that $b^{\mathrm{w}} - p^{\mathrm{w}} =: k_1^{\mathrm{w}}$ has order -1. Since $b^{\mathrm{w}} = p_2^{\mathrm{w}} + (p_{1/2} \circ p_{2,\alpha})^{\mathrm{w}} + b_0^{\mathrm{w}}$ modulo a term of order -1 and since, by Theorem 1.11.2 in [20], $(p_{1/2} \circ p_{2,\alpha})^{\mathrm{w}} - p_{1/2}(p_{2,\alpha}^{\mathrm{w}})$ has order -1, we obtain indeed that $k_1 \in S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$.

For a later purpose, it is now convenient to prove that that $e^{\pm i2\pi L P_{2,\alpha}} = \mathrm{id}$ where $2\pi L$ is the period of the bicharacteristics of $p_{2,\alpha}$. In fact, for $\phi \in \mathscr{S}$,

$$e^{\pm i2\pi L P_{2,\alpha}}\phi = \bigotimes_{k=1}^{n} e^{\pm i2\pi L\alpha_k P_{2,k}}\phi = \bigotimes_{k=1}^{n} \underbrace{e^{\pm i2\pi m_k P_{2,k}}}_{=\mathrm{id}}\phi = \phi,$$

since the $P_{2,k}$ commute over $\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$. The fact that $2\pi L$ is an integer multiple of the period of the bicharacteristics of $p_{2,\alpha}$ follows form the fact that, as

$$H_{p_{2,\alpha}} = \sum_{j=1}^{n} \alpha_j \Big(\xi_j \partial_{x_j} - x_j \partial_{\xi_j}\Big),$$

the bicharacteristic flow is for all $t \in \mathbb{R}$ and $X \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ given by

$$\exp(tH_{p_{2,\alpha}})(X) = \left(\sum_{j=1}^{n} (\cos(\alpha_j t)x_j + \sin(\alpha_j t)\xi_j), \sum_{j=1}^{n} (-\sin(\alpha_j t)x_j + \cos(\alpha_j t)\xi_j) \right).$$

We now want to show that $2\pi L$ is indeed *the* period of the bicharacteristics of $P_{2,\alpha}$. Suppose, by contradiction, that there is $0 < L' < L$ such that $2\pi L = 2\pi L' m'$ with $0 < m' \in \mathbb{Z}_+$ and $\exp(\pm 2\pi L' H_{p_{2,\alpha}}) = \mathrm{id}$. Then, we must have $\exp(\pm 2\pi L' \alpha_k H_{p_{2,k}}) = \mathrm{id}$ for all $k = 1, \ldots, n$. Therefore, $2\pi L' \alpha_k \in 2\pi \mathbb{Z}$, which implies that m' divides m_k for all k, which is impossible. Hence, $2\pi L$ is the period of the bicharacteristics of $p_{2,\alpha}$.

We next show that the commutator $[p^{\mathrm{w}}_{2,\alpha}, b^{\mathrm{w}}_0] = 0$.

Since $[p^{\mathrm{w}}_{2,\alpha}, b^{\mathrm{w}}_0]\big|_{\mathscr{S}} = [P_{2,\alpha}, B_0]\big|_{\mathscr{S}}$ and since $[p^{\mathrm{w}}_{2,\alpha}, b^{\mathrm{w}}_0]$ is a ψdo of order 0, it follows that we may extend $[P_{2,\alpha}, B_0]\big|_{\mathscr{S}}$ as a bounded linear operator $[P_{2,\alpha}, B_0]\colon L^2 \longrightarrow L^2$. Hence, if we show that $[p^{\mathrm{w}}_{2,\alpha}, b^{\mathrm{w}}_0] = 0$ then also $[P_{2,\alpha}, B_0] = 0$. Now, $b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0$ for all t by hypoyhesis. Hence $b_0 = \mathsf{R}_\alpha(b_0)$ (R_α being the X-ray transform with respect to the bicharacteristics of $p_{2,\alpha}$) and (on $\mathscr{S}$)

$$\begin{aligned}
[p^{\mathrm{w}}_{2,\alpha}, b^{\mathrm{w}}_0] = [p^{\mathrm{w}}_{2,\alpha}, \mathsf{R}_\alpha(b_0)^{\mathrm{w}}] &= \frac{-i}{2\pi L} \int_0^{2\pi L} \partial_t (e^{itP_{2,\alpha}} b^{\mathrm{w}}_0 e^{-itP_{2,\alpha}})\, dt \\
&= \frac{-i}{2\pi L} [e^{itP_{2,\alpha}} b^{\mathrm{w}}_0 e^{-itP_{2,\alpha}}]_0^{2\pi L} = 0.
\end{aligned}$$

In addition, $b_0 \circ \exp(tH_{p_2}) = b_0$ for all t by hypothesis. Hence, we have also that (on $\mathscr{S}$)

$$
\begin{aligned}
[p_2^{\mathrm{w}}, b_0^{\mathrm{w}}] = [p_2^{\mathrm{w}}, \mathsf{R}(b_0)^{\mathrm{w}}] &= \frac{-i}{2\pi L}\int_0^{2\pi L} \partial_t (e^{itP_2} b_0^{\mathrm{w}} e^{-itP_2})\, dt \\
&= \frac{-i}{2\pi L}[e^{itP_2} b_0^{\mathrm{w}} e^{-itP_2}]_0^{2\pi} = 0.
\end{aligned}
$$

Recall that the eigenspaces, made of Hermite functions, of P_2 are invariant under $P_{2,\alpha}$ and viceversa. Therefore, the eigenspaces of $P_2+P_{2,\alpha}$ are invariant under B_0. We may hence choose an orthonormal system $\{\phi_{\gamma,j};\ \gamma\in\mathbb{Z}_+^n,\ 1\le j\le N\}\subset\mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$ of $L^2(\mathbb{R}^n;\mathbb{C}^N)$, made of eigenfunctions of both P and P_2, that also diagonalizes $B_0\big|_{W_k}$ on each space $W_k := \mathrm{Span}_{\mathbb{C}}\{\phi_{\gamma,j};\ |\gamma| = k,\ 1\le j\le N\}$, $k\in\mathbb{Z}_+$. It follows that the eigenvalue of P associated with the eigenfunctions $\phi_{\gamma,j}$, for $|\gamma|=k$ and $1\le j\le N$, are

$$
k+\frac{n}{2}+p_{1/2}(\alpha(\gamma+1/2))+\mu_{\gamma,j},\quad 1\le j\le N,
$$

where $\mu_{\gamma,j}\in\mathbb{R}$ is such that $B_0\phi_{\gamma,j}=\mu_{\gamma,j}\phi_{\gamma,j}$.

Hence,

$$
\mathrm{Spec}(P) = \bigcup_{k\ge 0}\ \bigcup_{\substack{\gamma\in\mathbb{Z}_+^n\\ |\gamma|=k}}\ \bigcup_{j=1}^{N} C_{k,\gamma,j},
$$

where

$$
C_{k,\gamma,j} := k+\frac{n}{2}+p_{1/2}(\alpha(\gamma+1/2))+\mu_{\gamma,j}.
$$

We next wish to show that there is $M>0$ such that

$$
\mathrm{Spec}(A)\subset \bigcup_{k\ge 0}\ \bigcup_{\substack{\gamma\in\mathbb{Z}_+^n\\ |\gamma|=k}}\ \bigcup_{j=1}^{N}\left(C_{k,\gamma,j}+\Big[-\frac{M}{\sqrt{k+n/2}},\frac{M}{\sqrt{k+n/2}}\Big]\right). \tag{9.2.13}
$$

For that, we have to consider the chosen ψdo diagonalizer e^{w} (and hence its L^2 bounded extension E) for a^{w} (see Theorem 3.1.3 in [37]). Then, $\mathrm{ind}\,E=\mathrm{ind}\,E_0\ge 0$ by hypothesis since the index of an operator is invariant under compact perturbations. Thus, by the quasi-isometrization Theorem 9.1.2, we may assume that $E^*\colon L^2\to L^2$ is an isometry (i.e., recall, $EE^*=I$).

Letting

$$\tilde{r}^{\mathrm{w}} := (e^{\mathrm{w}})^* a^{\mathrm{w}} e^{\mathrm{w}} - p^{\mathrm{w}} = ((e^{\mathrm{w}})^* a^{\mathrm{w}} e^{\mathrm{w}} - b^{\mathrm{w}}) + (b^{\mathrm{w}} - p^{\mathrm{w}}),$$

which is a ψdo of order -1, and noting that $(p_2^{\mathrm{w}})^{1/4}$ has order $1/2$ with principal symbol $p_2^{1/4}$, we have that $(p_2^{\mathrm{w}})^{1/4}\tilde{r}^{\mathrm{w}}(p_2^{\mathrm{w}})^{1/4}$ can be extended to a bounded operator in $L^2(\mathbb{R}^n;\mathbb{C}^N)$. Hence, there is $M > 0$ such that for all $\psi \in \mathscr{S}(\mathbb{R}^n;\mathbb{C}^N)$

$$-M\|\psi\|^2_{L^2} \leq \left((p_2^{\mathrm{w}})^{1/4}\tilde{r}^{\mathrm{w}}(p_2^{\mathrm{w}})^{1/4}\psi,\psi\right)_{L^2} \leq M\|\psi\|^2_{L^2}, \tag{9.2.14}$$

that we rewrite in terms of the L^2 realizations of the ψdos involved:

$$-M\|\psi\|^2_{L^2} \leq \left(P_2^{1/4}\tilde{R}P_2^{1/4}\psi,\psi\right)_{L^2} \leq M\|\psi\|^2_{L^2}, \tag{9.2.15}$$

Now, recalling that $P_2^{1/4}\colon D(P_2^{1/4}) \subset L^2 \longrightarrow L^2$ is the self-adjoint unbounded L^2 realization of $(p_2^{\mathrm{w}})^{1/4}$, which is elliptic, we have that $D(P_2^{1/4}) = B^{1/2}(\mathbb{R}^n;\mathbb{C}^N)$ (which is dense in L^2) and $P_2^{1/4}$ is invertible with bounded inverse $P_2^{-1/4} : L^2 \to B^{1/2} \subset L^2$. Therefore, by substituting $P_2^{-1/4}E^*\phi$ for ψ in (9.2.15), we get that for all $\phi \in \mathscr{S}$

$$-M\left(P_2^{-1/2}\phi,\phi\right)_{L^2} \leq \left(\tilde{R}\phi,\phi\right)_{L^2} \leq M\left(P_2^{-1/2}\phi,\phi\right)_{L^2}.$$

Hence, for all $\phi \in B^2$,

$$\begin{aligned}\left(\left(P - MP_2^{-1/2}\right)\phi,\phi\right)_{L^2} &\leq (\underbrace{P+\tilde{R}}_{=E^*AE}\phi,\phi)_{L^2}\\ &\leq \left(\left(P + MP_2^{-1/2}\right)\phi,\phi\right)_{L^2},\end{aligned}$$

which leads to (9.2.13) by minimax principle. In fact,

$$\operatorname{Spec}(A) \subset \operatorname{Spec}(E^*AE),$$

since

$$A\phi_\lambda = \lambda\phi_\lambda, \quad \phi_\lambda \neq 0$$

implies

$$(E^*AE)\,E^*\phi_\lambda = \lambda E^*\phi_\lambda,\ E^*\phi_\lambda \neq 0.$$

Moreover,

$$\mathrm{Spec}\,(E^*AE) \setminus \{0\} \subset \mathrm{Spec}\,(A)\,,$$

since

$$E^*AE\psi_\eta = \eta\psi_\eta, \quad \psi_\eta \neq 0 \tag{9.2.16}$$

implies

$$AE\psi_\eta = \eta E\psi_\eta.$$

On the one hand, if $\psi_\eta \notin \ker\, E$ then $\eta \in \mathrm{Spec}\,(A)$. On the other hand, if $\psi_\eta \in \ker\, E$ then $\eta = 0$ by (9.2.16), and $0 \notin \mathrm{Spec}\,(A)$ since $A > 0$. Therefore

$$\mathrm{Spec}\,(E^*AE) \setminus \{0\} = \mathrm{Spec}\,(A)\,,$$

which concludes the proof of the lemma.

□

Now we want to generalize Lemma 9.2.4 by removing the hypothesis on the non-negativity of the decoupling operator index.

Theorem 9.2.5. *Let $a = a^* \sim \sum_{j\geq 0} a_{2-j} \in S^2_{\mathrm{sreg}}$ be a 2nd-order SMGES with principal symbol $a_2 = p_2 I_N$, such that the corresponding unbounded operator $A > 0$ (this is no restriction, in view of the Sharp-Gårding inequality). Suppose that the coefficients of the characteristic polynomial $\det(\lambda - a_1(X))$ of the semiprincipal term a_1 are functions of $p_{2,\alpha}$ and that there is a unitary diagonalizer e_0 for the semiprincipal symbol such that, denoting by $b_0 := \mathrm{diag}(b_{0,h};\ 1 \leq h \leq r)$ the subprincipal symbol of the resulting blockwise diagonalization of A, we have that*

$$b_0 \circ \exp(tH_{p_2}) = b_0, \quad b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0, \quad \forall t \in \mathbb{R}. \tag{9.2.17}$$

In addition, suppose there are $m_1, \ldots, m_n \in \mathbb{Z}_+ \setminus \{0\}$ coprime such that

$$\frac{m_1}{\alpha_1} = \ldots = \frac{m_n}{\alpha_n} =: L. \tag{9.2.18}$$

Then, the eigenspaces of P_2 *are invariant for* $P_{2,\alpha}$ *and the eigenspaces of* $(P_2+P_{2,\alpha})\otimes I_{N_h}$ *are invariant for* $B_{0,h}$*, for all* $h=1,\dots,r$*. Moreover, for each* $h=1,\dots,r$*, there is an orthonormal basis* $\{\phi_{h,\gamma,j}\}_{\gamma\in\mathbb{Z}^n_+,1\le j\le N_h}\subset\mathscr{S}(\mathbb{R}^n;\mathbb{C}^{N_h})$ *of* $L^2(\mathbb{R}^n;\mathbb{C}^{N_h})$ *such that* $\mathrm{Ker}\big(P_2\otimes I_{N_h}-(k+\frac{n}{2})\big)=\mathrm{Span}\{\phi_{h,\gamma,j};\ |\gamma|=k,\ 1\le j\le N_h\}$*, and*

$$B_{0,h}\phi_{h,\gamma,j}=\mu_{h,\gamma,j}\phi_{h,\gamma,j},\ \ \textit{with}\ |\mu_{h,\gamma,j}|\le\|B_0\|_{L^2\to L^2},\quad\forall\gamma,\ \forall j=1,\dots,N_h,$$

a smooth function $p_{1/2}\colon\mathbb{R}_+\longrightarrow\mathbb{R}^r$*, positively homogeneous of degree* $1/2$*, such that* $\lambda_{1,h}=p_{1/2,h}(p_{2,\alpha})$*,* $1\le h\le r$*, and* $M,c>0$ *such that*

$$\mathrm{Spec}(A)\subset\bigcup_{h=1}^{r+1}\bigcup_{k\ge0}\bigcup_{\substack{\gamma\in\mathbb{Z}^n_+\\|\gamma|=k}}\bigcup_{j=1}^{N_h}S_{h,k,\gamma,j}(A),\tag{9.2.19}$$

where, for each $h=1,\dots,r$*,*

$$S_{h,k,\gamma,j}(A):=\big(k+\frac{n}{2}+p_{1/2,h}(\alpha(\gamma+1/2))+\mu_{h,\gamma,j}\big)+\Big[-\frac{M}{\sqrt{k+n/2}},\frac{M}{\sqrt{k+n/2}}\Big],\tag{9.2.20}$$

where $\alpha(\gamma+1/2):=\sum_{j=1}^{n}\alpha_j(\gamma_j+1/2)$*,* $N_{r+1}=N$*,* $p_{1/2,r+1}=0$ *and* $\mu_{r+1,\gamma,j}=c$ *for all* γ *and* j*.*

Proof. The proof follows from an argument based on the construction of a system $\tilde{A}$ associated with A having a decoupling operator $\tilde{e}_0$ with non-negative index. Namely, $\tilde{A}$ is a block-diagonal system with two $N\times N$ blocks, the first being A and the second being $P_2I_N+P_0$ where p_2 is the harmonic oscillator,

$$\begin{aligned}p_0:=&-e_0^*(e_{-2}e_0^*p_2+p_2e_0e_{-2}^*-\frac{i}{2}\left(e_0\left\{a_\mu,e_0^*\right\}+\left\{e_0,a_\mu e_0^*\right\}\right)\\&+e_{-1}p_2e_{-1}^*)e_0+cI_N,\end{aligned}$$

with $c > 0$ a real constant such that $P_0 > 0$, and

$$\tilde{A} := \begin{bmatrix} A & 0_N \\ 0_N & P_2 I_N + P_0 \end{bmatrix}.$$

Hence,

$$\tilde{E}^* \tilde{A} \tilde{E} = \begin{bmatrix} B & 0_N \\ 0_N & (P_2 + c) I_N + R \end{bmatrix},$$

where $\tilde{e} := \begin{bmatrix} e & 0_N \\ 0_N & e^* \end{bmatrix}$ and $r \in S_{\mathrm{sreg}}(m^{-1}, g; \mathsf{M}_N)$ since, by Proposition 4.0.1 (or by a straightforward computation, using the composition formula for ψdos), the subprincipal symbol of $E(P_2 I_N + P_0)E^*$ is cI_N by definition of p_0. Actually, $\tilde{A}$ verifies the hypothesis of Lemma 9.2.4. In fact, $\tilde{A} > 0$, $\mathrm{ind}\,(\tilde{E}) = \mathrm{ind}\,(E) + \mathrm{ind}\,(E^*) = 0$ by hypothesis. Hence, by Lemma 9.2.4, we have (9.2.11) for $\tilde{A}$, that is

$$\mathrm{Spec}(\tilde{A}) \subset \bigcup_{h=1}^{r+1} \bigcup_{k \geq 0} \bigcup_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = k}} \bigcup_{j=1}^{N_h} S_{h,k,\gamma,j}(A),$$

where, for each $h = 1, \ldots, r+1$,

$$S_{h,k,\gamma,j}(A) := \big(k + \frac{n}{2} + p_{1/2,h}(\alpha(\gamma + 1/2)) + \mu_{h,\gamma,j}\big) + \Big[-\frac{M}{\sqrt{k + n/2}}, \frac{M}{\sqrt{k + n/2}}\Big]$$

with $N_{r+1} = N$, $p_{1/2,r+1} = 0$ and $\mu_{r+1,\gamma,j} = c$ for all γ and j. Moreover, $\mathrm{Spec}(\tilde{A}) = \mathrm{Spec}\,(A) \cup \mathrm{Spec}\,(P_2 I_N + P_0)$ since $\tilde{A}$ is blockwise diagonal. Hence,

$$\mathrm{Spec}(A) \subset \bigcup_{h=1}^{r+1} \bigcup_{k \geq 0} \bigcup_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = k}} \bigcup_{j=1}^{N_h} S_{h,k,\gamma,j}(A),$$

which completes the proof.

□

9.3 Remarks on Theorem 9.2.3 and Theorem 9.2.5

We notice that condition (9.2.7) in Theorem 9.2.3 is used as a sufficient condition for the subprincipal symbol to be constant on the bicharacteristics of p_2. Moreover, it is satisfied by the JC-model and many of its generalizations in Chapter 2.1 with $\alpha_k = \alpha$ for all k. Namely, we have that for the JC-model in Subsection 2.1.1 (where we take, for simplicity's sake, $\alpha = 1$) a diagonalizer for the semiprincipal symbol is

$$\mathbb{R}^2 \setminus \{0\} \ni (x,\xi) \mapsto e_0(x,\xi) := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{x+i\xi}{\sqrt{2(x^2+\xi^2)}} & \frac{-(x+i\xi)}{\sqrt{2(x^2+\xi^2)}} \end{pmatrix},$$

and, hence, for $(x,\xi) \in \mathbb{R}^2 \setminus \{0\}$

$$e_0(\exp(t\mathsf{H}_{p_2})(x,\xi)) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{e^{-it}(x+i\xi)}{\sqrt{2(x^2+\xi^2)}} & \frac{-e^{-it}(x+i\xi)}{\sqrt{2(x^2+\xi^2)}} \end{pmatrix} = (f_t e_0)(x,\xi),$$

where $f_t := \begin{pmatrix} 1 & 0 \\ 0 & e^{-it} \end{pmatrix}$. Moreover, $f_t^*(\gamma\boldsymbol{\sigma}_3)f_t = \gamma\boldsymbol{\sigma}_3$ (where $\gamma > 0$) and $\{p_2, f_t\} = 0$. Hence, condition (9.2.7) is verified the Jayne-Cummings model.

Furthermore, also generalizations of the JC-model satisfy the condition (9.2.7). In fact, if we consider, for example, the JC-model for a 3-level atom and 2 cavity-modes in the Ξ-*configuration* in Subsection 2.1.3 with $\alpha_1 = \alpha_2$, we have that a diagonalizer for the semiprincipal symbol is

$$\mathbb{R}^4 \setminus \{0\} \ni (x,\xi) \mapsto e_0(x,\xi) := \begin{pmatrix} \frac{\overline{\psi_2}}{|\psi|} & \frac{\overline{\psi_1}}{\sqrt{2}|\psi|} & \frac{\overline{\psi_1}}{\sqrt{2}|\psi|} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\psi_1}{|\psi|} & \frac{\psi_2}{\sqrt{2}|\psi|} & \frac{\psi_2}{\sqrt{2}|\psi|} \end{pmatrix},$$

where $\psi_j(X) := \frac{x_j+i\xi_j}{\sqrt{2}}$, $j = 1, 2$ with $X = (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and, hence, for $(x,\xi)\in\mathbb{R}^4\setminus\{0\}$

$$e_0(\exp(t\mathsf{H}_{p_2})(x,\xi))=\begin{pmatrix}\frac{e^{it}\overline{\psi_2}}{|\psi|} & \frac{e^{it}\overline{\psi_1}}{\sqrt{2}|\psi|} & \frac{e^{it}\overline{\psi_1}}{\sqrt{2}|\psi|}\\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{e^{-it}\psi_1}{|\psi|} & \frac{e^{-it}\psi_2}{\sqrt{2}|\psi|} & \frac{e^{-it}\psi_2}{\sqrt{2}|\psi|}\end{pmatrix}=(f_te_0)(x,\xi),$$

where $f_t:=\begin{pmatrix}e^{it} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-it}\end{pmatrix}$. Moreover,

$$f_t^*\left(\sum_{k=1}^{N-1}\gamma_k E_{k+1,k+1}\right)f_t=\sum_{k=1}^{N-1}\gamma_k E_{k+1,k+1}$$

(where $\gamma_1,\dots\gamma_N\in\mathbb{R}$ with $\gamma_1<\gamma_2<\dots<\gamma_N$) and $\{p_2,f_t\}=0$.

In the statement of Theorem 9.2.5 we can replace the condition (9.2.7) by requiring that there exist smooth functions $\mathbb{R}\times\dot{\mathbb{R}}^{2n}_X\ni(t,X)\mapsto f_t(X),\ g_t(X)\in \mathsf{M}_\mathsf{N}$ such that

$$\{p_2,f_t\}=0,\quad e_0\circ\exp(tH_{p_{2,\alpha}})=f_te_0,\quad a_0=f_t^*\Big(a_0\circ\exp(tH_{p_{2,\alpha}})\Big)f_t,\tag{9.3.1}$$

$$\{p_2,g_t\}=0,\quad e_0\circ\exp(tH_{p_2})=g_te_0,\quad a_0=g_t^*\Big(a_0\circ\exp(tH_{p_2})\Big)g_t.\tag{9.3.2}$$

(The two conditions (9.3.1) and (9.3.2) are equivalent to (9.2.7) in the case $\alpha_k=\alpha_{k'}$ for all $k,\ k'$ by taking $g_t=f_t$.) Actually, even if (9.3.1) and (9.3.2) are explicit conditions on e_0, they are sufficient and non-necessary condition for having (9.2.17) satisfied. On the one hand, they are sufficient since $b_0=e_0^*a_{\mu-2}e_0-i\{e_0^*,a_\mu\}e_0$ by Corollary 4.2.1 and, therefore, $b_0\circ\exp(tH_{p_{2,\alpha}})=b_0$ by (9.3.1) and (9.3.2). On the other hand, they are non-necessary because of the following example. We consider the JC-model for a 3-level atom and 2 cavity-modes in the Ξ-*configuration* in Subsection 2.1.3 (with 1 as coefficient of the principal term $p_2^{\mathrm{w}}(x,D)I_N$) with $\alpha_1\neq\alpha_2$ and $\psi_j(X):=\alpha_j\frac{x_j+i\xi_j}{\sqrt{2}},\quad j=1,2$ with $X=(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n=\mathbb{R}^{2n}$. Then, again, a diagonalizer for the semiprincipal symbol is

$$\mathbb{R}^4 \setminus \{0\} \ni (x,\xi) \mapsto e_0(x,\xi) := \begin{pmatrix} \frac{\overline{\psi_2}}{|\psi|} & \frac{\overline{\psi_1}}{\sqrt{2}|\psi|} & \frac{\overline{\psi_1}}{\sqrt{2}|\psi|} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\psi_1}{|\psi|} & \frac{\psi_2}{\sqrt{2}|\psi|} & \frac{\psi_2}{\sqrt{2}|\psi|} \end{pmatrix},$$

and, hence, for $(x,\xi) \in \mathbb{R}^4 \setminus \{0\}$ with $\alpha := (\alpha_1, \alpha_2)$

$$e_0(\exp(t\mathsf{H}_{p_2,\alpha})(x,\xi)) = \begin{pmatrix} \frac{e^{i\alpha_2 t}\overline{\psi_2}}{|\psi|} & \frac{e^{i\alpha_1 t}\overline{\psi_1}}{\sqrt{2}|\psi|} & \frac{e^{i\alpha_1 t}\overline{\psi_1}}{\sqrt{2}|\psi|} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{e^{-i\alpha_1 t}\psi_1}{|\psi|} & \frac{e^{-i\alpha_2 t}\psi_2}{\sqrt{2}|\psi|} & \frac{e^{-i\alpha_2 t}\psi_2}{\sqrt{2}|\psi|} \end{pmatrix} = (f_t e_0)(x,\xi),$$

where

$$f_t := \begin{pmatrix} \frac{e^{i\alpha_2 t}|\psi_2|^2 + e^{it\alpha_1}|\psi_1|^2}{|\psi|^2} & 0 & (e^{it\alpha_1} - e^{i\alpha_2 t})\frac{\overline{\psi_2\psi_1}}{|\psi|^2} \\ 0 & 1 & 0 \\ (e^{-it\alpha_1} - e^{-i\alpha_2 t})\frac{\psi_2\psi_1}{|\psi|^2} & 0 & \frac{e^{-i\alpha_2 t}|\psi_2|^2 + e^{-it\alpha_1}|\psi_1|^2}{|\psi|^2} \end{pmatrix}.$$

Therefore, $f_t^* \left(\sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1} \right) f_t \neq \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}$ (where $\gamma_1, \dots \gamma_N \in \mathbb{R}$ with $\gamma_1 < \gamma_2 < \dots < \gamma_N$). However, recalling Corollary 4.2.1

$$b_0(X) = \mathsf{diag}(\pi_j e_0^* a_{\mu-2} e_0 \pi_j^* - i\pi_j e_0^* \{p_\mu, e_0\} \pi_j^*; 1 \le j \le 3)(X)$$

$$= \begin{pmatrix} \frac{(\gamma_1+1)|\psi_2|^2 + (\gamma_3+1)|\psi_1|^2}{|\psi|^2} & 0 & 0 \\ 0 & \frac{(\gamma_1+1)|\psi_1|^2 + \gamma_2|\psi|^2 + (\gamma_3-1)|\psi_2|^2}{2|\psi|^2} & 0 \\ 0 & 0 & \frac{(\gamma_1+1)|\psi_1|^2 + \gamma_2|\psi|^2 + (\gamma_3-1)|\psi_2|^2}{2|\psi|^2} \end{pmatrix},$$

which means that $b_0 \circ \exp(tH_{p_{2,\alpha}}) = b_0 \circ \exp(tH_{p_2}) = b_0$. Hence, we have just shown that there is a e_0 such that (9.3.1) is not satisfied, but (9.2.17) is.

Chapter 10

The Rabi model

In this chapter we determine a refined Weyl law result for the Rabi model. The Rabi model is the model which describes the interaction of a 2-level atom and one cavity-mode electromagnetic field even when the field is not near resonance with the atomic transition and the coupling strength is not weak. In fact, it can be seen as the model leading to the Jaynes-Cummings model by rotating waves approximation, which is valid if the field is near resonance with the atomic transition and the coupling strength is weak. For a physical description of the model see the seminal papers [53] and [54] by Rabi and also [6] by Braak.

The Hamiltonian operator describing this model is the Weyl-quantization of the symbol

$$\mathbb{R}^2 \ni (x,\xi) \longmapsto a_{\mathrm{Rabi}}(x,\xi) := \frac{x^2+\xi^2}{2} I_2 + C \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} + \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad (10.0.1)$$

where C and $\gamma_1 < \gamma_2$ are real numbers. We are going to give a spectral asymptotics result for the counting function of the Rabi Hamiltonian.

The main problem that we face in the study of the asymptotics of the counting function for (10.0.1) is the non-ellipticity of the semiprincipal term and the non-separation of the eigenvalues of the semiprincipal symbol. In fact, for a general model with non-elliptic semiprincipal symbol, one would not be able to decouple the system as done in Chapter 3, since there would be no symbol $e_0 \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}; \mathsf{M}_2)$ diagonalizing the semiprincipal symbol or there would crossing of the eigenvalues of the semiprincipal symbol. Hence, it would not be an SMGES. Actually, regarding the non-smooth diagonalizability of the semiprincipal, this is not the case of a_{Rabi} since

$$e_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

diagonalizes $\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} = x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, leading to the matrix $x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is smooth as a function of $X \in \mathbb{R}^{2n}$, but it does not satisfies the separation of the eigenvalues property. Hence, Theorem 3.1.3 does not give a decoupling of the Hamiltonian operator describing the Rabi model. Therefore, Theorem 7.1.8 does not lead to a Weyl Law result.

In this section, as in Section 9.2, we use the following notation: letting a be a semiregular symbol in $S_{\mathrm{sreg}}(m^\mu, g; \mathsf{M}_N)$, we denote by A (that is, by using the corresponding capital letter) the unbounded, densely defined and closed operator $D(A) \subset L^2 \to L^2$ which is the realization of the ψdo a^{w}. If a is elliptic and $\mu \geq 0$, $D(A) = B^\mu$.

10.1 The Rabi model generalizations by NCHOs

In this section we introduce classes of NCHOs which generalize the Rabi model. For these classes is stated and proved a non-refined Weyl law result in Section 10.2.

To formalize these generalized models mathematically we use the same notations adopted in Section 2.1.

10.1.1 The Rabi model for an N-level atom and $n = N-1$ cavity-modes in the Ξ-*configuration*

In this case, for $\alpha > 0$, $\alpha, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$a^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_N + \sum_{k=1}^{N-1} \alpha_k x_k \Big(E_{k,k+1} + E_{k+1,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

In this case, the levels of the atom are given by 0 and the γ_k.

10.1.2 The Rabi model for an N-level atom and $n = N-1$ cavity-modes in the $\bigwedge$-*configuration*

In this case, for $\alpha > 0$, $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$a^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_N + \sum_{k=1}^{N-1} \alpha_k x_k \Big(E_{k,N} + E_{N,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

In this case, the levels of the atom are given by 0 and the γ_k.

10.1.3 The Rabi model for an N-level atom and $n = N-1$ cavity-modes in the so-called $\bigvee$-*configuration*

In this case, for $\alpha > 0$, $\alpha_1, \dots \alpha_{N-1} \in \mathbb{R} \setminus \{0\}$, $\gamma_1, \dots \gamma_{N-1} \in \mathbb{R}$ with $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{N-1}$, we consider the $N \times N$ system in $\mathbb{R}^n$, $n = N-1$, given by

$$a^{\mathrm{w}}(x, D) = p_2^{\mathrm{w}}(x, D) I_N + \sum_{k=1}^{N-1} \alpha_k x_k \Big(E_{1,k+1} + E_{k+1,1} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}.$$

In this case, the levels of the atom are given by 0 and the γ_k.

10.2 Non-refined Weyl law for the generalized Rabi models

For the classes introduced in Section 10.1 we can state the following non-refined Weyl law.

Theorem 10.2.1. *Let* $a = a^* = \tilde{a}_2 + \tilde{a}_1$ *with*

$$\tilde{a}_2 \sim \sum_{j\geq 0} a_{2-2j} \in S_{\mathrm{cl}}(m^2, g; \mathsf{M}_N),$$

$$\tilde{a}_1 \sim \sum_{j\geq 0} a_{1-2j} \in S_{\mathrm{cl}}(m, g; \mathsf{M}_N),$$

defined on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ $(n := N-1)$ *where* $a_k = a_k^*$ *is positively homogeneous of degree* k *and* $A > 0$. *Moreover, suppose that* $a_2 = p_2 I_2$ *with* p_2 *the scalar harmonic oscillator and that there is a smooth on* $\mathbb{R}^{2n} \setminus \{0\}$ *and positively homogeneous of order* 1 *matrix-valued function* b *on* $\mathbb{R}^{2n} \setminus \{0\}$ *such that* $a_\epsilon := a + \varepsilon b$ *is an SMGES for all* $\varepsilon \in (0,1)$. *Then, if* $\mathbb{R} \ni \lambda \longmapsto \mathsf{N}(\lambda)$ *denotes the spectral counting function associated with* a^{w},

$$\mathsf{N}(\lambda) = (2\pi)^{-n} \left(N\lambda^n \int_{p_2 \leq 1} dX - \lambda^{n-1/2} \int_{p_2=1} \mathsf{Tr}(a_1) \frac{ds}{|\nabla p_2|} \right) + o(\lambda^{n-1/2}),$$

as $\lambda \to +\infty$.

Proof. To prove the theorem we use a perturbation argument. Namely, we obtain an operator inequality between A_ε and A which, by minimax principle, leads to a spectral inequality between A_ε and A. Next, we use Theorem 7.1.7 to have a Weyl law for A_ε. Finally, we can obtain a Weyl law for A by the spectral inequality just proven. Actually, the Weyl law for the second order operator A is not refined since the perturbation has order 1 and, hence, only the first term after the leading term of the asymptotics can be determined precisely.

Since $A > 0$, we can define $A^{\pm 1/4}$ as the unbounded realization of the ψdo

$(a^{\mathrm{w}})^{\pm 1/4}$ and for all $\phi \in \mathscr{S}$

$$\begin{aligned} |(A^{-1/4}(A_\varepsilon - A)A^{-1/4}\phi, \phi)_{L^2}| =& \varepsilon |(A^{-1/4}BA^{-1/4}\phi, \phi)_{L^2}| \\ \leq& \varepsilon \left\| A^{-1/4}BA^{-1/4} \right\|_{L^2 \to L^2} \|\phi\|_{L^2} = \varepsilon C \|\phi\|_{L^2}, \end{aligned} \tag{10.2.1}$$

where $C := \left\| A^{-1/4}BA^{-1/4} \right\|_{L^2 \to L^2}$. Now, by density of $\mathscr{S}$ in L^2 and $L^2 \to L^2$ boundedness of $A^{-1/4}(A_\varepsilon - A)A^{-1/4}$, (10.2.1) holds for all $\phi \in L^2$. Since $A^{1/4} : D(A^{1/4}) \subset L^2 \to L^2$ is an elliptic ψdo, $D(A^{1/4}) = B^{1/2}$. Hence, since $A^{1/4}$ is surjective, we can substitute $\phi \in L^2$ with $A^{1/4}\psi$ ($\psi \in B^{1/2}$) in (10.2.1) and we obtain for all $\psi \in B^2 \subset B^{1/2}$

$$(A\psi, \psi)_{L^2} - \varepsilon C(\sqrt{A}\psi, \psi)_{L^2} \leq (A_\varepsilon \psi, \psi)_{L^2} \leq (A\psi, \psi)_{L^2} + \varepsilon C(\sqrt{A}\psi, \psi)_{L^2}. \tag{10.2.2}$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ (respectively, $\lambda_{1,\varepsilon} \leq \lambda_{2,\varepsilon} \leq \ldots$) be the eigenvalues of A (respectively A_ε), repeated according to multiplicities. By minimax principle and (10.2.2)

$$\lambda_j - \varepsilon C\sqrt{\lambda_j} \leq \lambda_{j,\varepsilon} \leq \lambda_j + \varepsilon C\sqrt{\lambda_j}, \quad \forall j \geq 1,$$

which leads to an estimate for the counting function N_A of A and $\mathsf{N}_{A_\varepsilon}$ of A_ε. In fact, for ε small enough

$$\begin{aligned} \mathsf{N}_{A_\varepsilon}(\lambda) :=& \#\{j;\ \lambda_{j,\varepsilon} \leq \lambda\} \\ \geq& \#\{j;\ \lambda_j + \varepsilon C\sqrt{\lambda_j} \leq \lambda\} \\ =& \#\{j;\ \nu_{\varepsilon,+}^{-1}(\lambda_j) \leq \lambda\} = \mathsf{N}_A(\nu_{\varepsilon,+}(\lambda)), \end{aligned}$$

and

$$\begin{aligned} \mathsf{N}_{A_\varepsilon}(\lambda) :=& \#\{j;\ \lambda_{j,\varepsilon} \leq \lambda\} \\ \leq& \#\{j;\ \lambda_j - \varepsilon C\sqrt{\lambda_j} \leq \lambda\} \\ =& \#\{j;\ \nu_{\varepsilon,-}^{-1}(\lambda_j) \leq \lambda\} = \mathsf{N}_A(\nu_{\varepsilon,-}(\lambda)), \end{aligned}$$

where $\nu_{\varepsilon,\pm} : \lambda \mapsto \frac{\varepsilon^2}{2}C^2 + \lambda \mp \varepsilon C\sqrt{\frac{\varepsilon^2}{4}C^2 + \lambda}$ is a $(c_0, +\infty) \to (\nu_{\varepsilon,\pm}(c_0), +\infty)$ smooth function with $c_0 > 0$ a lower bound of Spec A which is increasing and invertible for $\varepsilon < \varepsilon_0 := \frac{2\sqrt{c_0}}{C}$ and, namely, $\nu_{\varepsilon,\pm}^{-1} : \lambda \mapsto \lambda \pm \varepsilon C\sqrt{\lambda}$. Therefore, for $\varepsilon < \varepsilon_0$

$$\mathsf{N}_{A_\varepsilon}(\nu_{\varepsilon,-}^{-1}(\lambda)) \leq \mathsf{N}_A(\lambda) \leq \mathsf{N}_{A_\varepsilon}(\nu_{\varepsilon,+}^{-1}(\lambda)),$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_{A_\varepsilon}(\nu_{\varepsilon,-}^{-1}(\lambda)) - (2\pi)^{-n}\lambda^n N\int_{p_2\leq 1} dX\right)\\
\leq&\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_A(\lambda) - (2\pi)^{-n}\lambda^n N\int_{p_2\leq 1} dX\right)\\
\leq&\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_{A_\varepsilon}(\nu_{\varepsilon,+}^{-1}(\lambda)) - (2\pi)^{-n}\lambda^n N\int_{p_2\leq 1} dX\right). \qquad (10.2.3)
\end{aligned}$$

Now, we study the behavior of $\mathsf{N}_{A_\varepsilon} \circ \nu_{\varepsilon,+}^{-1}$ when $\lambda \to +\infty$. By Theorem 7.1.7 and since the semiprincipal symbol of A_ε is $a_1 + \varepsilon b$,

$$\begin{aligned}
\mathsf{N}_{A_\varepsilon}(\nu_{\varepsilon,+}^{-1}(\lambda)) =&(2\pi)^{-n}\Bigg(N(\nu_{\varepsilon,+}^{-1}(\lambda))^n\int_{p_2\leq 1} dX\\
&- (\nu_{\varepsilon,+}^{-1}(\lambda))^{n-1/2}\int_{p_2=1}\mathsf{Tr}(a_1+\varepsilon b)\frac{ds}{|\nabla p_2|}\Bigg) + O(\lambda^{n-1})\\
=&(2\pi)^{-n}\left(\lambda^n N\left(\int_{p_2\leq 1} dX\right)\right.\\
&\left.- \lambda^{n-1/2}\left(\int_{p_2=1}\mathsf{Tr}(a_1+\varepsilon b)\frac{ds}{|\nabla p_2|} - \varepsilon nCN\int_{p_2\leq 1} dX\right)\right) + O(\lambda^{n-1}),
\end{aligned}$$

as $\lambda \to +\infty$, where the second equality follows from Newton's generalized binomial theorem. In a similar way, for $\mathsf{N}_{A_\varepsilon} \circ \nu_{\varepsilon,-}^{-1}$,

$$\begin{aligned}
\mathsf{N}_{A_\varepsilon}(\nu_{\varepsilon,-}^{-1}(\lambda)) =&(2\pi)^{-n}\left(\lambda^n N\left(\int_{p_2\leq 1} dX\right)\right.\\
&\left.- \lambda^{n-1/2}\left(\int_{p_2=1}\mathsf{Tr}(a_1+\varepsilon b)\frac{ds}{|\nabla p_2|} + \varepsilon nCN\int_{p_2\leq 1} dX\right)\right) + O(\lambda^{n-1}),
\end{aligned}$$

as $\lambda \to +\infty$.

Hence, since we are looking for the behavior of N_A as $\lambda \to +\infty$, by (10.2.3)

$$\begin{aligned}
&-(2\pi)^{-n}\left(\int_{p_2=1}\mathsf{Tr}(a_1+\varepsilon b)\frac{ds}{|\nabla p_2|}+\varepsilon nCN\int_{p_2\leq 1}dX\right)\\
\leq&\liminf_{\lambda\to+\infty}\left(\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_A(\lambda)-(2\pi)^{-n}\lambda^n N\int_{p_2\leq 1}dX\right)\right)\\
\leq&\limsup_{\lambda\to+\infty}\left(\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_A(\lambda)-(2\pi)^{-n}\lambda^n N\int_{p_2\leq 1}dX\right)\right)\\
\leq&-(2\pi)^{-n}\left(\int_{p_2=1}\mathsf{Tr}(a_1+\varepsilon b)\frac{ds}{|\nabla p_2|}-\varepsilon nCN\int_{p_2\leq 1}dX\right),
\end{aligned}$$

for all $\varepsilon < \varepsilon_0$ which implies, by computing the limit as $\varepsilon \to 0+$,

$$\begin{aligned}
-(2\pi)^{-n}\int_{p_2=1}\mathsf{Tr}(a_1)\frac{ds}{|\nabla p_2|}&\leq\liminf_{\lambda\to+\infty}\left(\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_A(\lambda)-(2\pi)^{-n}\lambda^n N\int_{p_2\leq 1}dX\right)\right)\\
&\leq\limsup_{\lambda\to+\infty}\left(\frac{1}{\lambda^{n-1/2}}\left(\mathsf{N}_A(\lambda)-(2\pi)^{-n}\lambda^n N\int_{p_2\leq 1}dX\right)\right)\\
&\leq-(2\pi)^{-n}\int_{p_2=1}\mathsf{Tr}(a_1)\frac{ds}{|\nabla p_2|},
\end{aligned}$$

and completes the proof.

□

Remark 10.2.2. *Theorem 10.2.1 can be used to obtain a non-refined Weyl law for the models of Section 10.1. Namely, in the notation of Theorem 10.2.1, for the models of Subsection 10.1.1 we have that*

$$a = p_2 I_N + \sum_{k=1}^{N-1}\alpha_k x_k\Big(E_{k,k+1}+E_{k+1,k}\Big)+\sum_{k=1}^{N-1}\gamma_k E_{k+1,k+1},$$

$$b = \sum_{k=1}^{N-1}\alpha_k\Big((i\xi_k)^* E_{k,k+1}+i\xi_k E_{k+1,k}\Big).$$

In fact,

$$\begin{aligned} a + \varepsilon b =& p_2 I_N + \sum_{k=1}^{N-1} \alpha_k \Big(\psi_{k,\varepsilon}^* E_{k,k+1} + \psi_{k,\varepsilon} E_{k+1,k} \Big) \\ &+ \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1}, \end{aligned}$$

where $\psi_{k,\varepsilon} := \frac{x_j + i\varepsilon \xi_j}{\sqrt{2}}, \quad 1 \leq j \leq n,$ *is an SMGES for all* $\varepsilon \in (0,1)$.

In a similar way, for the models of Subsection 10.1.2, Theorem 10.2.1 can be used to obtain a non-refined Weyl law with

$$a = p_2 I_N + \sum_{k=1}^{N-1} \alpha_k x_k \Big(E_{k,N} + E_{N,k} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1},$$

$$b = \sum_{k=1}^{N-1} \alpha_k \Big((i\xi_k)^* E_{k,N}^* + i\xi_k E_{N,k} \Big),$$

while for the models of Subsection 10.1.2, Theorem 10.2.1 can be used with

$$a = p_2 I_N + \sum_{k=1}^{N-1} \alpha_k x_k \Big(E_{1,k+1} + E_{k+1,1} \Big) + \sum_{k=1}^{N-1} \gamma_k E_{k+1,k+1},$$

$$b = \sum_{k=1}^{N-1} \alpha_k \Big((i\xi_k)^* E_{k,k+1}^* + i\xi_k E_{k+1,k} \Big).$$

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Bibliography

[1] A.O. Barut, J.F.V. *Huele._Quantum electrodynamics based on self-energy: Lamb shift and spontaneous emission without field quantization.* Phys. Rev. A 53: 2816 (1996)

[2] J. Aramaki. *Complex powers of vector valued operators and their application to asymptotic behavior of eigenvalues.* J. Funct. Anal. 87 (1989), 294–320.

[3] J.E. Bayeld. *Quantum evolution.* ohn-Weily and Sons (1999)

[4] G. Barton._*Introduction to Advanced Field Theory.* New York: Interscience (1963)

[5] L. Boutet de Monvel. *Hypoelliptic operators with double characteristics and related pseudo-differential operators.* Comm. Pure Appl. Math. 27 (1974), 585–639. doi.org/10.1002/cpa.3160270502.

[6] D. Braak. *Analytic solutions of basic models in quantum optics.* Applications + Practical Conceptualization + Mathematics = fruitful Innovation - Proceedings of the Forum of Mathematics for Industry 2014, Mathematics for Industry 11 (eds. R. Anderssen et al.) Springer, 2015, 75-92.

[7] T. Carleman. *Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes.* Skand. Matent. Kongress (1934).

[8] J. Chazarain. *Spectre d'un Hamiltonien quantique et Mécanique Classique.* Comm. P.D.E. 5, no.6, 595-644 (1980)

[9] Y.C. de Verdière. Sur le spectre des opérateurs elliptiques a bicaractéristiques toutes periodiques. Commentarii Mathematici Helvetici 54, 508–522 (1979). https://doi.org/10.1007/BF02566290

[10] E. del Valle Reboul. *Quantum Electrodynamics with Quantum Dots in Microcavities.* Thesis submitted for the degree of Doctor of Philosophy, Universidad Autónoma de Madrid, p. 27-39 (February 2009)

[11] M. Doll,O. Gannot, J. Wunsch. *Refined Weyl law for homogeneous perturbations of the harmonic oscillator.* Comm. Math. Phys. **362** (2018), no. 1, 269–294. doi: 10.1007/s00220-018-3100-5

[12] M. Doll, S. Zelditch. *Schrödinger trace invariants for homogeneous perturbations of the harmonic oscillator.* J. Spectr. Theory 10 (2020), 1303–1332 DOI 10.4171/JST/328

[13] J.J. Duistermaat, V.W. Guillemin. *The spectrum of positive elliptic operators and periodic bicharacteristics.* Invent Math 29, 39–79 (1975). https://doi-org.ezproxy.unibo.it/10.1007/BF01405172

[14] R.J. Glauber. *The quantum theory of optical coherence.* Phys. Rev. 130: 2529 (1963)

[15] R.J. Glauber. *Coherent and incoherent states of the radiation field.* Phys. Rev. 131: 2766 (1963)

[16] V.W. Guillemin. *25 Years of Fourier Integral Operators.* Brüning, J., Guillemin, V.W. (eds) Mathematics Past and Present Fourier Integral Operators. Springer, Berlin, Heidelberg (1994). https://doi.org/10.1007/978-3-662-03030-1_1

[17] R. Hanbury Brown, R.Q. Twiss. *A test of a new type of stellar interferometer on Sirius.* Nature 178: 1046 (1956)

[18] B. Helffer, D. Robert. *Comportement asymptotique précisé du spectre d'opérateurs globalement elliptiques dans* $\mathbb{R}^n$. Séminaire Équations aux dérivées partielles (Polytechnique) (1980-1981), exp. no. 2, p. 1-22

[19] B. Helffer, D. Robert. *Propriétés asymptotiques du spectre d'opérateurs pseudodifférentiels sur* $\mathbb{R}^n$. Comm. Partial Differential Equations **7** (1982), 795–882.

[20] B. Helffer. *Théorie spectrale pour des opérateurs globalement elliptiques.* Astérisque, tome 112 (1984)

[21] M. Hitrik and I. Polterovich. *Regularized traces and Taylor expansions for the heat semigroup.* J. London Math. Soc. (2) 68 (2003), 402–418.

[22] M. Hitrik and I. Polterovich. *Resolvent expansions and trace regularizations for Schrödinger operators.* Advances in differential equations and mathematical physics (Birmingham, AL, 2002), 161–173, Contemp. Math. 327, Amer. Math. Soc., Providence, RI, 2003.

[23] J.J. Hopfield. *Theory of the contribution of excitons to the complex dielectric constant of crystals*. hys. Rev. 112: 1555 (1958)

[24] R. Howe, E.C. Tan. *Non-Abelian Harmonic Analysis. Applications of SL(2,*$\mathbb{R}$*).* Springer-Verlag, 1992

[25] L. Hörmander. *On the asymptotic distribution of the eigenvalues of pseudodifferential operators in* $\mathbb{R}^n$. Ark. Mat. **17**, 297–313 (1979).

[26] L. Hörmander. *The Analysis of Linear Partial Differential Operators III.* Classics in Mathematics. Springer, Berlin (2007)

[27] L. Hörmander. *The Analysis of Linear Partial Differential Operators IV.* Classics in Mathematics. Springer, Berlin (2009). Fourier integral operators; Reprint of the 1994 edition

[28] L. Hörmander. *Symplectic classification of quadratic forms, and general Mehler formulas.* Math. Z. 219 (1995), no. 3, 413–449. MR 1339714 Zbl 0829.35150

[29] L. Hormander. *The spectral function of an elliptic operator.* Acta Math. 121: 193-218 (1968). DOI: 10.1007/BF02391913

[30] T. Ichinose and M. Wakayama. *On the spectral zeta function for the non-commutative harmonic oscillator.* Rep. Math. Phys. 59 (2007), 421–432.

[31] V. Ivrii. . *100 years of Weyl's law.* Bull. Math. Sci. (2016) 6:379–452 DOI 10.1007/s13373-016-0089-y

[32] V. Ivrii. . *100 years of Weyl's law.* Bull. Math. Sci. (2016) 6:379–452 DOI 10.1007/s13373-016-0089-y

[33] E. T. Jaynes and F. W. Cummings. *Comparison of quantum and semi-classical radiation theories with application to the beam maser.* Proc. IEEE 51 (1963), 89-109.

[34] J. Larson. *Extended Jaynes-Cummings models in cavity QED.* Thesis submitted for the degree of Doctor of Philosophy, Department of Physics, The Royal Institute of Technology, Albanova, 9-12 (2005)

[35] M. Malagutti, A. Parmeggiani. *Spectral Asymptotic Properties of Semiregular Non-Commutative Harmonic Oscillators.* Submitted preprint (2022)

[36] S. Minakshisundaram. *A Generalization of Epstein Zeta Functions. Canadian Journal of Mathematics.* 1(4), 320-327. doi:10.4153/CJM-1949-029-3 (1949)

[37] S. Minakshisundaram, A. Pleijel. *Some properties of eigenfunctions of the Laplace operator on Riemannian manifolds.* Canadian J. Math. 1 (1949) 242–256

[38] M.A. Nielsen, I. L. Chuang. *Quantum Computation and Quantum Information.* Cambridge (2000)

[39] C. Parenti, A. Parmeggiani. _*Lower bounds for systems with double characteristics.* J. Anal. Math. 86 (2002), 49–91

[40] C. Parenti, A. Parmeggiani. *A Lyapunov Lemma for elliptic systems.* Ann. Glob. Anal. Geom. 25 (2004), 27–41

[41] A. Parmeggiani. *On the spectrum of certain noncommutative harmonic oscillators.* Ann. Univ. Ferrara Sez. VII Sci. Mat. **52** (2006), no. 2, 431–456. doi: 10.1007/s11565-006-0030-5

[42] A. Parmeggiani. *On the spectrum of certain non-commutative harmonic oscillators and semiclassical analysis.* Comm. Math. Phys. **279** (2008), no. 2, 285–308. doi: 10.1007/s00220-008-0436-2

[43] A. Parmeggiani. *On the Fefferman-Phong inequality for systems of PDEs.* In "Phase Space Analysis of Partial Differential Equations". Birkhäuser Verlag, Progress in Nonlinear Differential Equations and their Applications 69 (2006), 247–266

[44] A. Parmeggiani. *On positivity of certain systems of partial differential equations.* Proc. Natl. Acad. Sci. USA 104 (2007), 723–726

[45] A. Parmeggiani. *Spectral theory of Non-Commutative Harmonic Oscillators: An Introduction.* Lecture Notes in Mathematics, 1992. Springer-Verlag, Berlin, 2010. xii+254 pp. doi: 10.1007/978-3-642-11922-4

[46] A. Parmeggiani. *Non-commutative harmonic oscillators and related problems.* Milan J. Math. **82** (2014), no. 2, 343–387. doi: 10.1007/s00032-014-0220-z

[47] A. Parmeggiani, M. Wakayama. *Oscillator representations and systems of ordinary differential equations.* Proceedings of the National Academy of Sciences U.S.A. **98** (2001), 2630. doi: 10.1073/pnas.98.1.26

[48] A. Parmeggiani, M. Wakayama. *Non-commutative harmonic oscillators-I, -II.* Forum Mathematicum **14** (2002), 539–604 ibid. 669–690. doi: 10.1515/form.2002.025 and 10.1515/form.2002.029

[49] R. Paschotta. *Encyclopedia of Laser Physics and Technology*. Vol. 1: A-M. Wiley-VCH. p. 580 (2008)

[50] M.I. Petrov, D.F. Kornovan, I.V. Toftul. *Introduction to Quantum Optics: an amateur's view.* Lecture notes, ITMO University, Department of Physics and Mathematics, p. 21-24 (2019)

[51] D. Robert. *Propriétés spectrales d'opérateurs pseudo-différentiels.* Communications in Partial Differential Equations (1978) , 3:9, 755-826, doi: 10.1080/03605307808820077

[52] D. Robert. *Propriétés spectrales d'opérateurs pseudodifférentiels.* Séminaire Équations aux dérivées partielles (Polytechnique) (1977-1978), exp. no 5, p. 1-10

[53] I. I. Rabi. *On the process of space quantization.* Phys. Rev. 49 (1936), 324-328.

[54] I. I. Rabi. *Space quantization in a gyrating magnetic field.* Phys. Rev. 51 (1937), 652-65

[55] Y. Safarov, D. Vassiliev. *The asymptotic distribution of eigenvalues of partial differential operators.* Translations of Mathematical Monographs, vol. 155. American Mathematical Society, Providence (1997)

[56] R.T. Seeley. *Complex powers of an elliptic operator.* Proc. Symp. Pure Math. 10, pp. 288–307, Amer. Math. Society, Providence (1968)

[57] B. W. Shore, P. L. Knight. *The Jaynes-Cummings model.* J. Modern Opt. **40** (1993), no. 7, 1195–1238. doi: 10.1080/09500349314551321

[58] M. A. Shubin. *Pseudodifferential operators and spectral theory* (second edition). Springer-Verlag, Berlin, 2001. xii+288 pp.

[59] S. Sugiyama. *Spectral zeta functions for the quantum Rabi models.* Nagoya Math. J. 229 (2018), 52–98.

[60] A. Weinstein. A symptotics of eigenvalue clusters for the Laplacian plus a potential. Duke Math. J. 44(4): 883-892 (December 1977). DOI: 10.1215/S0012-7094-77-04442-8

[61] M. Zworski. *Semiclassical analysis.* Graduate Studies in Mathematics, **138**. American Mathematical Society, Providence (2012)