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POTENTIAL THEORY ON METRIC SPACES

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Abstract

This work revolves around potential theory in metric spaces, focusing on applications of dyadic potential theory to general problems associated to functional analysis and harmonic analysis. In the first part of this work we consider the weighted dual dyadic Hardy's inequality over dyadic trees and we use the Bellman function method to characterize the weights for which the inequality holds, and find the optimal constant for which our statement holds. We also show that our Bellman function is the solution to a stochastic optimal control problem. In the second part of this work we consider the problem of quasi-additivity formulas for the Riesz capacity in metric spaces and we prove formulas of quasi-additivity in the setting of the tree boundaries and in the setting of Ahlfors-regular spaces. We also consider a proper Harmonic extension to one additional variable of Riesz potentials of functions on a compact Ahlfors-regular space and we use our quasi-additivity formula to prove a result of tangential convergence of the harmonic extension of the Riesz potential up to an exceptional set of null measure.

Introduction

Potential theory was born out of the theory of electrostatic potential, from the work of C.F. Gauss. The first notion of capacity dates back to the 1830's and it is the notion of electrostatic capacitance of a compact set $K \subseteq R^3$. We consider a distribution of charge μ over $\mathbb{R}^3$, which is a positive measure on $\mathbb{R}^3$. The amount of charge on a conductor $K \subseteq \mathbb{R}^3$ is equal to $\mu(K)$. Given a distribution of charge μ, the electrostatic potential associated to μ at the point $y \in \mathbb{R}^3$ is defined by

$$V_\mu(y) := \int_{\mathbb{R}^3} \frac{d\mu(x)}{|x-y|}, \tag{1}$$

and the energy associated to μ is defined by

$$\mathcal{E}(\mu) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} d\mu(x) d\mu(y). \tag{2}$$

A charge distribution $\tilde{\mu}$ is called an equilibrium charge distribution for a compact set $K \subseteq \mathbb{R}^3$ if $\tilde{\mu}$ is supported on K and if the potential $V_{\tilde{\mu}}(y)$ is equal to a constant value $V_{\tilde{\mu}}$ for all $y \in K$, except for a "small" set of exceptional points. An equilibrium charge distribution $\tilde{\mu}$ minimizes the value of the energy $\mathcal{E}(\mu)$ for charge distributions μ such that $\mu(K) = \tilde{\mu}(K)$. Using these notions we define the electrostatic capacitance of a compact set $K \subseteq \mathbb{R}^3$ to be ratio between the amount of charge in K and the value of the electrostatic potential associated to an equilibrium charge distribution, i.e.

$$C_{\text{E.S.}}(K) := \frac{\tilde{\mu}(K)}{V_{\tilde{\mu}}}. \tag{3}$$

Potential theory has always been strictly connected to the theory of Hilbert spaces and harmonic analysis. In the 1830's Gauss proved the existence of equilibrium potentials by minimizing a quadratic integral, the energy (see [15]). The same result was proved with modern mathematical rigor by O. Frostman in the 1930's (see [14]). This theme kept on growing during the 1940's, and it was made expecially clear in the work of H. Cartan (see [12]).
In the following decades a mathematical concept of potential theory, disconnected from the theory of the electrostatics, was developed. G. Choquet (see [13]), introduced a definition of capacity and capacitability in 1950, and in the 1970's Frostman developed the first rigorous

mathematical definition of potential theory. In the early 1960's Maz'ya developed a non linear potential theory which is connected to the theory of function spaces.
In 1990 D.R. Adams and L.I. Hedberg developed an axiomatic definition of potential theory for metric measure spaces (see [4]) which allows to prove many results in a very general setting. In this work, we refer to [4] for the definitions and main theorems about potential theory. In the 1990's J. Heinonen and P. Koskela developed a potential theory on metric spaces based on the notions of rectifiable curves and of "weak gradients" (see [18]).

General problems in potential theory

This work analyzes two problems in the field of potential theory: the weighted dual dyadic Hardy's inequality on trees and the quasi-additivity of the capacity in the setting of Ahlfors-regular spaces, with applications to the boundary behaviour of harmonic extensions of Riesz potentials.

Hardy's inequality states that

$$\int_0^{+\infty}\left(\frac{1}{x}\int_0^{+\infty}f(y)dy\right)^p dx\leq\left(\frac{p}{p-1}\right)^p\int_0^{+\infty}f(x)^p dx \tag{4}$$

for every positive measurable function f, for every $p>1$. The constant $(p/(p-1))^p$ is optimal.
In 1920 G. H. Hardy was interested in the study of the discrete version of the previous inequality

$$\sum_{j=1}^{+\infty}\left(\frac{a_1+\cdots+a_j}{j}\right)^p\leq\left(\frac{p}{p-1}\right)^p\sum_{j=1}^{+\infty}a_j^p, \tag{5}$$

where a_j are positive real numbers, motivated by the goal of giving a simpler proof of "Hilbert's inequality for double series" (see [16]). For $p=2$ the inequality was proved by Hardy in his earlier paper [17].
In the following years it became clear that Hardy's inequality and its extensions played a central role in the studies in the broad area of Harmonic analysis. A reason for the relevance of Hardy's inequality in harmonic analysis is that it is "the prototype of a (weighted) norm inequality for an integration (averaging) operator between L^p spaces", and averaging operators are among the pillars of harmonic analysis. This interpretation allows to view a vast number of theorems as Hardy's type inequalities, and lead to the construction of many inequalities of the Hardy type.
The inequality we will be considering in this work is a weighted inequality of the Hardy type

over the dyadic tree. A formulation of such inequality can be the following: given the real interval $[0,1]$ we may consider the family of dyadic subintervals

$$\mathscr{D} := \{I = [a2^{-n}, (a+1)2^{-n}] \subseteq [0,1] \mid n = 0,1,2,\dots\ ,\ a = 0,1,\dots,2^{-n}-1\}.$$

We want to find a characterization of the weights

$$\begin{aligned} \mathscr{D} \ni I &\longmapsto \alpha_I \in \mathbb{R}^+, \\ \mathscr{D} \ni I &\longmapsto \lambda_I \in \mathbb{R}^+, \end{aligned}$$

such that the inequality

$$\sum_{I \in \mathscr{D}} \alpha_I \left(\frac{1}{|I|} \sum_{\substack{J \in \mathscr{D} \\ J \subseteq I}} \phi(J) \lambda_J^{\frac{1}{p'}} \right)^p \leq C(p) \sum_{I \in \mathscr{D}} \phi(I)^p \tag{6}$$

holds for all non negative functions $\phi \in l^p(\mathscr{D})$, where $C(p)$ is an appropriate constant, and we want to find the optimal value of $C(p)$ for which the inequality holds assuming that our characterization of $\{\alpha_I\}$ and $\{\lambda_I\}$ holds.

A well-known result from potential theory (see [1]) is that the capacity $C_{K,p}(E)$ of a set $E \subseteq X$ is a sub additive map

$$C_{K,p} : \{\text{Borel subsets of } X\} \longrightarrow [0, +\infty],$$

hence we have

$$C_{K,p}\left(\bigcup_{j \in \mathbb{N}} E_j \right) \leq \sum_{j \in \mathbb{N}} C_{K,p}(E_j) \tag{7}$$

for all $E_j \subseteq X$.

In general the capacity is very far from superadditive, an example of this property coming from the theory of electrostatic potential. Suppose $B \subseteq R^3$ is a compact ball. When we consider the electrostatic capacitance $C_{\text{E.S.}}(B)$ we have

$$C_{\text{E.S.}}(B) = C_{\text{E.S.}}(\mathring{B}) = C_{\text{E.S.}}(\partial B), \tag{8}$$

hence

$$C_{\text{E.S.}}(\mathring{B}) + C_{\text{E.S.}}(\partial B) = 2 \cdot C_{\text{E.S.}}(\mathring{B} \cup \partial B), \tag{9}$$

which is a counter example to the quasi-additivity of the electrostatic capacitance.

A common problem in harmonic analysis is to find notions of "properly separated" families of sets $\{E_j\}_{j \in \mathbb{N}}$ implying that

$$\sum_{j \in \mathbb{N}} C_{K,p}(E_j) \leq C \cdot C_{K,p}\left(\bigcup_{j \in \mathbb{N}} E_j \right), \tag{10}$$

for an appropriate constant $C \geq 1$.
Results of this kind have been developed through the years, including results using a notion separation of the sets E_j based on Whitney decompositions (see [2], [4]).
In this work we consider the quasi-additivity formula in the article [3] from H.Aikawa and A.A. Borichev, and we extend it to the setting of the theory of the potential on tree boundaries. Then, we use the results from the theory of dyadic potential (see [8]) to extend the quasi-additivity formula on the setting of tree boundaries to the setting of compact Ahlfors-regular spaces. This results represents a starting point in the construction of a potential theory for the setting of Ahlfors-regular spaces analoguos to the classical potential theory in $\mathbb{R}^n$. Using the quasi-additivity formula for the compact Ahlfors-regular spaces we prove a theorem about the convergence of the values of harmonic extensions of Riesz potentials which is analoguos to a result by A. Nagel, W. Rudin and J.H. Shapiro (see [25]) in the classical setting of $\mathbb{R}^n$.

Chapter 1

The first chapter of this work will present notions of stochastic analysis needed for the understanding of the theory behind the Bellman functions, and the theory of the Bellman functions in stochastic control. The notations, definitions and theorems listed in this chapter are all thoroughly explained in the text [24] by B. Øksendal, in chapters 1, 2, 3, 4, 5, 7 and 9. The theory of the Bellman functions requires the notions of stochastic analysis necessary to define the Itô integral, which is used to define a Bellman function

$$v(x) = \sup_{\{u_t\}_{t\geq 0}} E^x \left[\int_s^{\widehat{T}} F(r, X_r, u_r)dr + K(\widehat{T}, X_{\widehat{T}})\chi_{\widehat{T}<+\infty} dB_r \right],$$

where X_t is a stochastic process solution to the stochastic differential equation

$$X_h = X_h^x = x + \int_s^h b(r, X_r, u_r)dr + \int_s^h \sigma(r, X_r, u_r)dB_r; \quad h \geq s,$$

where b and σ are proper coefficients, $\{B_t\}_{\geq 0}$ is a Brownian motion, $\widehat{T}$ is a proper stopping time and $\{u_t\}_{t\geq 0}$ is an admissible control process. Here F is a profit density and K is a "bequest" function (gain at the moment of retirement). So a Bellman function v is the solution to a stochastic optimal control problem that consists of finding the maximum average gain over a trajectory of a controlled process $\{X_t\}_{t\geq 0}$.
This chapter also includes the notions needed to prove the theorem about the Bellman function being a solution to the Hamilton-Jacobi-Bellman equation, and its converse: it contains the definitions and theorems about the strong Markov property, infinitesimal generator of a

stochastic process, Dynkin's formula and the Dirichlet-Poisson problem.

Chapter 2

In the second chapter of this work we use the method of the Bellman function to characterize the measures for which the weighted dual Hardy's inequality holds on dyadic trees. We also give an explicit interpretation of the corresponding Bellman function in terms of the theory of stochastic optimal control.

The weighted Hardy's inequality on trees was initially studied for its applications to the theory of holomorphic function spaces, but it is an interesting topic on its own. The weighted dyadic Hardy's inequality was studied (see [6] and [7]) to characterize Carleson measures for analytic Besov spaces.
In this work we study the problem and solve it for the general case $1 < p < +\infty$ and we prove that the inequality holds with constant $C(p) = \big(p/(p-1)\big)^p = \big(p'\big)^p$. See Theorem 0.0.1.

But for the best constant, our characterization of the dual dyadic Hardy's inequality is not new, see [6] and [9]. The proof we give is new and it is inspired to the linear case given in [5]. The weighted dyadic Hardy's inequality can be characterized by other equivalent, but different conditions. For instance a capacitary characterization can be given, using the Maz'ya theory, see [21].

In the past twenty years several results of this kind have been proved using the Bellman function method. The ideas behind the Bellman function technique were inspired by [11], see also [10] . The expository article [23] investigates the connection between the Bellman function technique in dyadic analysis and Bellman functions from the theory of stochastic control. The seminal article [22] presents a thorough exposition about the Bellman function technique and its applications. The article [5] solves the problem for the case $p = 2$, and the Bellman function in our article is equal to the Bellman function used in [5] when we set $p = 2$. The Bellman function we use in this article is very similar, but not completely, to the one used for the proof of the dyadic Carleson embedding theorem, see [20], and for the Carleson embedding theorem the same constant $\big(p'\big)^p$ is sharp.

We are now going to present the results in the second chapter of this work.
Given the interval $I_0 = [0, 1]$ we denote by $\mathscr{D}(I_0)$ the standard dyadic tree structure of real

intervals $I \subseteq I_0$. Let $I \in \mathscr{D}(I_0)$, let

$$\varphi : \mathscr{D}(I_0) \longrightarrow \mathbb{R}.$$

We will denote the sum of the values $\varphi(J)$ for $J \in \mathscr{D}(I)$ by

$$\sum_{J \in \mathscr{D}(I)} \varphi(J) =: \sum_{J \subseteq I} \varphi(J).$$

We consider the maps

$$I \longmapsto \alpha_I \in \mathbb{R}^+,$$
$$I \longmapsto \lambda_I \in \mathbb{R}^+,$$
$$I \longmapsto \phi(I) \in \mathbb{R}^+,$$

where we can read $\{\alpha_I\}$ as a choice of weights, $\{\lambda_I\}$ as a measure and $\{\phi(I)\}$ as a function over the dyadic tree.
The first main result of this work is the following one:

Theorem 0.0.1. *Let I_0 be a real interval. Let $\{\alpha_I\}$ and $\{\lambda_I\}$ be a choice of weights and measure. If*

$$\frac{1}{|I|} \sum_{K \subseteq I} \alpha_K \left(\frac{1}{|K|} \sum_{J \subseteq K} \lambda_J \right)^p \leq \frac{1}{|I|} \sum_{K \subseteq I} \lambda_I < +\infty \quad \forall I \in \mathscr{D}(I_0) \tag{11}$$

is satisfied, then the dual weighted dyadic Hardy's inequality holds for $\{\alpha_I\}$ and $\{\lambda_I\}$, i.e.

$$\frac{1}{|I_0|} \sum_{I \subseteq I_0} \alpha_I \left(\frac{1}{|I|} \sum_{J \subseteq I} \phi(J) \lambda_J^{\frac{1}{p'}} \right)^p \leq C(p) \frac{1}{|I_0|} \sum_{I \subseteq I_0} \phi(I)^p \quad \textit{for all } \phi \in l^p(\mathscr{D}(I_0)) \tag{12}$$

for any choice of $\{\phi(I)\}$. Here $C(p)$ is the constant

$$C(p) = \left(\frac{p}{p-1} \right)^p = (p')^p.$$

The constant $C(p)$ in the inequality (12) is sharp.
Moreover, if the inequality (12) holds with constant $C(p) = 1$ then the inequality (11) holds for any $I \in \mathscr{D}(I_0)$ by choosing $\phi(I) = \lambda_I^{\frac{1}{p}}$ and by rescaling I_0 over I.

By setting $\eta(I) = \phi(I) \lambda_I^{-\frac{1}{p}}$ and $\omega_I^{1-p} = \frac{\alpha_I}{|I|^p}$ we rewrite the inequality (12) in the form

$$\sum_{I \subseteq I_0} \omega_I^{1-p} \left(\sum_{J \subseteq I} \eta(J) \lambda_J \right)^p \leq C(p) \sum_{I \subseteq I_0} \eta(I)^p \lambda_I,$$

which, by duality, is equivalent to the weighted dyadic Hardy's inequality

$$\sum_{I\subseteq I_0}\lambda_I\Big(\sum_{J\supseteq I}\psi(J)\Big)^{p'}\leq C(p)\sum_{I\subseteq I_0}\psi(I)^{p'}\omega(I)\quad\text{for all }\psi\in l^{p'}(\mathscr{D}(I_0)).$$

To prove the results in this chapter we found and used the function

$$\mathcal{B}(F,f,A,v)=\left(\frac{p}{p-1}\right)^pF-\frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}}$$

defined over the domain

$$\mathcal{D}=\{(F,f,A,v)\in\mathbb{R}^4\mid F>0,f>0,A>0,v>0,v\geq A,f^p\leq Fv^{p-1}\}.$$

The properties of $\mathcal{B}$ we use are stated in subsection 2.1.2.

The function $\mathcal{B}$ can be interpreted as the solution to a Hamilton-Jacobi-Bellman equation associated to a stochastic problem of optimal control, which we will state in the work.

Chapter 3

In the third chapter of this work we prove formulas of quasi-additivity for the capacity associated to kernels of radial type in the setting of the boundary of a tree structure and in the setting of compact Ahlfors-regular spaces. We also define a notion of harmonic extension, to one additional variable, of a function defined over a compact Ahlfors-regular space, and we prove a result of tangential convergence of the harmonic extension to the values at the boundary.

Let $1<p<+\infty$, let $\frac{1}{p}+\frac{1}{p'}=1$, let $\frac{1}{p'}\leq s<1$. Let $f\in L^p(\mathbb{R}^n)$. Let us consider the Bessel potential of f

$$B_s*f(x):=\mathfrak{F}^{-1}\left((1+4\pi^2|\xi|^2)^{-\frac{s}{2}}\mathfrak{F}(f)(\xi)\right)(x),\tag{13}$$

where $\mathfrak{F}(f)$ denotes the Fourier transform of f, and let us consider the harmonic extension of the Bessel potential of f

$$PI(B_s*f):\mathbb{R}^n\times(0,+\infty)\longrightarrow\mathbb{R}\tag{14}$$

defined by

$$PI(B_s*f)(x_0,y_0):=\int_{\mathbb{R}^n}\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{y_0}{\left(y_0^2+|x_0-x|^2\right)^{\frac{n+1}{2}}}B_s*f(x)dx.$$

A classical result by Nagel, Rudin and Shapiro (see [25]) states that $PI(B_s * f)(x,y)$ converges to $B_s * f(x_0)$ non tangentially as $(x,y) \to (x_0, 0)$ for all $x_0 \in \mathbb{R}^n \setminus E$, where the exceptional set E is a set of Bessel capacity $C_{B_s,p}(E) = 0$, where the Bessel capacity of a set is defined by

$$C_{B_s,p}(E) := \inf\{\|g\|^p_{L^p(\mathbb{R}^n)} \mid B_s * g(x) \geq 1 \text{ for } x \in E\}. \tag{15}$$

Moreover, $PI(B_s * f)(x,y)$ converges to $B_s * f(x_0)$ as $(x,y) \to (x_0, 0)$ in a region of tangential order of contact, for all $x_0 \in \mathbb{R}^n \setminus A$, where the exceptional set A is a set of null measure.

Aikawa and Borichev (see [3]) generalized this result by Nagel, Rudin and Shapiro to a wider family of kernels, and gave a proof based on formulas of quasi-additivity of the capacity associated to kernels of radial type. In our work we generalize the theorems proved by Aikawa and Borichev to the setting of compact Ahlfors-regular spaces, and we prove a result for the tangential convergence of properly defined harmonic extensions (see section 3.3) of Riesz potentials of functions $f : X \longrightarrow \mathbb{R}$ defined on an Ahlfors-regular space X.

From the theory of the potential it is well known that the L^p capacity $C_{K,p}(E)$ of a set E is subadditive, i.e. for all countable disjoint families $\{E_j\}$ of sets such that $E = \bigcup_j E_j$ we have

$$\sum_j C_{K,p}(E_j) \leq C_{K,p}(E). \tag{16}$$

In general, the capacity is very far from superadditive. Aikawa and Borichev gave a notion of "separation" for the sets E_j that guarantees that the capacity of the sets E_j is quasi-additive, i.e. there exists a universal constant A such that

$$C_{K,p}(E) \leq \sum_j C_{K,p}(E_j) \leq A \cdot C_{K,p}(E), \tag{17}$$

for any family $\{E_j\}$ that satisfies a proper condition (see [3, Theorem 5]).

We are now going to present the results in the third chapter of this work.
We proved the following two theorems that generalize the previous result to the setting of the boundaries of tree structures and to the compact Ahlfors-regular spaces.

Theorem 0.0.2 (Quasi-additivity of the capacity for tree boundaries)**.** *Let $X = \partial T$ be the boundary of a tree T of root o, let $\rho : X \times X \longrightarrow \mathbb{R}$ be the distance defined by $\rho(x,y) = \delta^{-d(x \wedge y, o)}$ for a parameter $0 < \delta < 1$, let m be a σ-finite Borel measure on X. Suppose $K : X \times X \longrightarrow \mathbb{R}$ is a proper radial Kernel (see section 1, Theorem 3.1.2).*

Let $1 < p < +\infty$. Let $C_{K,p}(E)$ denote the L^p capacity of $E \subseteq X$ associated to the kernel K. Let $x \in X$, $r > 0$. Consider the radius

$$\eta_p(x,r) := \inf \left\{ \delta^{-n+\frac{1}{2}} \in \mathbb{R} \,\middle|\, n \in \mathbb{N},\ m(B_\rho(x, \delta^{-n+\frac{1}{2}})) \geq C_{K,p}(B_\rho(x,r)) \right\}. \tag{18}$$

and define

$$\eta_p^*(x,r) := \max\{r, \eta_p(x,r)\}. \tag{19}$$

Let J be a countable (or finite) set of indices. Let $\{B_\rho(x_j,r_j)\}_{j\in J}$ be a family of metric balls in X such that $\eta_p(x_j,r_j)$ exists for all $j \in J$. Suppose $E \subseteq X$ is a compact subset of $\bigcup_{j\in J} B_\rho(x_j,r_j)$. Suppose $\{B_\rho(x_j,\eta_p^*(x_j,r_j))\}_{j\in J}$ is disjoint.
Then

$$C_{K,p}(E) \leq \sum_{j\in J} C_{K,p}(E \cap B(x_j,r_j)) \leq A \cdot C_{K,p}(E), \tag{20}$$

where $A = A(X,K,p)$, $1 < A < +\infty$, is a constant depending only on X, K and p.

Theorem 0.0.3 (Quasi-additivity for Riesz capacity on compact Ahlfors-regular spaces)**.** *Let (X,d,m) be a compact Q-regular Ahlfors space. Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\frac{1}{p'} < s < 1$. Let $C_{K_{X,s},p}(E)$ denote the L^p Riesz capacity of $E \subseteq X$. For every $x \in X$ and $r > 0$ consider the radius*

$$\eta_{X,p}(x,r) := \inf\left\{R > 0 \mid m(B_d(x,R)) \geq C_{K_{X,s},p}(B_d(x,r))\right\}, \tag{21}$$

and define

$$\eta_{X,p}^*(x,r) := \max\{r, \eta_{X,p}(x,r)\}. \tag{22}$$

We observe that the radius $\eta_{X,p}^*(x,r)$ depends on the parameter s. Then there exists a constant $\Omega = \Omega(X,p,s) \geq 1$ such that for all $M \geq 1$ there exists a constant $1 < \tilde{A} < +\infty$ such that, for any countable family $\{B_d(x_k,r_k)\}_{k\in\mathcal{F}}$ of balls in X such that the family $\{B_d(x_k, \Omega \cdot \eta_{X,p}^*(x_k, M \cdot r_k))\}_{k\in\mathcal{F}}$ is disjoint, for any compact set $E \subseteq X$ such that $E = \bigcup_k E_k$ and $E_k \subseteq B_d(x_k,r_k)$ $\forall k$, we have

$$\sum_{k\in\mathcal{F}} C_{K_{X,s},p}(E_k) \leq \tilde{A} \cdot C_{K_{X,s},p}(E). \tag{23}$$

The constant $\tilde{A}$ depend only on the choice of the space X and the of the parameters p, s and M.

The quasi-additivity formula for compact Ahlfors-regular spaces is used to prove the following two theorems about the convergence of a "harmonic extension" of the Riesz potential of a function f defined on a compact Ahlfors-regular space X.

Theorem 0.0.4 (Non tangential convergence for the Riesz potential)**.** *Let (X,d,m) be a compact Ahlfors-regular space. Let $M \geq 1$. Let $f \in L^p(X)$. Let $K_{X,s}$ denote the Riesz kernel over X. Let $PI(g)$ denote the Poisson Integral of g over $X \times (0,+\infty)$ (see definition 3.3.1). Then $\exists E \subseteq X \times (0,+\infty)$ such that E is M-$C_{K_{X,s},p}$-thin at $X \times \{0\}$ (see definition 3.4.2) and*

$$\lim_{\substack{(x,y)=P\to(x_0,0)\\ x\in B_d(x_0,y)\\ (x,y)\notin E}} PI(K_{X,s} * f)(P) = K_{X,s} * f(x_0) \tag{24}$$

for $C_{K_{X,s},p}$-almost everywhere $x_0 \in X$, i.e. $\exists F \subset X$ such that $C_{K_{X,s},p}(F) = 0$ and (24) holds $\forall x_0 \in X \backslash F$.

Theorem 0.0.5 (Tangential convergence for the Riesz potential)**.** *Let (X,d,m) be a compact Q-regular Ahlfors space. Let $p > 1$, let $\frac{1}{p'} \leq s < 1$. Let $\Omega > 1$ be the constant defined by Theorem 3.2.4. Let $M \geq 1$.*
For every $x \in X$ and $r > 0$ consider the radius

$$\eta_{X,p}(x,r) := \inf\left\{R > 0 \mid m(B_d(x,R)) \geq C_{K_{X,s},p}(B_d(x,r))\right\}, \tag{25}$$

and define

$$\eta^*_{X,p}(x,r) := \max\{r, \eta_{X,p}(x,r)\}. \tag{26}$$

Consider the region

$$\Omega_{x_0,K_{X,s},p,\Omega,M} := \left\{(x,y) \mid x \in B_d(x_0, \Omega \cdot \eta^*_{X,p}(x_0, My))\right\} \subseteq X \times (0,+\infty). \tag{27}$$

Let $f \in L^p(X)$. Then

$$\lim_{\substack{(x,y)=P\to(x_0,0)\\ P\in\Omega_{x_0,K_{X,s},p,\Omega,M}}} PI(K_{X,s} * f)(P) = K_{X,s} * f(x_0) \tag{28}$$

for m-almost all $x_0 \in X$. The region $\Omega_{x_0,K_{X,s},p,\Omega,M}$ is tangential to the boundary $X \times \{0\}$.

These theorems generalize the results by Nagel, Rudin and Shapiro to the setting of compact Ahlfors-regular spaces, and constitute a starting point for a potential theory on Ahlfors-regular spaces analogous to the classical potential theory on $\mathbb{R}^n$. We think that Theorems 0.0.4 and 0.0.5 can be generalized to the setting of non compact Ahlfors-regular spaces.

This part of our work is heavily inspired by [3]. We reference [4] and [1] for the general notions and facts about the theory of the potential. We reference [8] for the theory of the potential on tree boundaries.

Chapter 1

Bellman functions in stochastic control

1.1 Notations and definitions

We will need the definitions of random variable and stochastic process, and we will use in most cases the same notations used in [24]. We recommend to check a text of probability and measure theory for the basic notions of probability needed in this work.

Definition 1.1.1 (Random variable)**.** We denote with $\mathscr{B}(\mathbb{R}^d)$ the σ-algebra over the set $\mathbb{R}^d$ generated by the Borel subsets of $\mathbb{R}^d$.
Given a probability space $(\Omega, \mathscr{F}, P)$, where Ω is a set, $\mathscr{F}$ is a σ-algebra over Ω and $P : \mathscr{F} \to \mathbb{R}$ is a probability measure over Ω, a random variable

$$Z : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$$

is an application

$$Z : \Omega \longrightarrow \mathbb{R}^d$$

measurable with respect to the σ-algebras $\mathscr{F}$ and $\mathscr{B}(\mathbb{R}^d)$.
Given two measurable spaces $(\Omega_1, \mathscr{F}_1)$, $(\Omega_2, \mathscr{F}_2)$ an application

$$Z : \Omega_1 \longrightarrow \Omega_2$$

is measurable if, for all $A \in \mathscr{F}_2$, $Z^{-1}(A) \in \mathscr{F}_1$.

Definition 1.1.2 (Stochastic process)**.** Given a probability space $(\Omega, \mathscr{F}, P)$, given a set of times I, given for each $t \in I$ a random variable

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)) \tag{1.1}$$

we denote by stochastic process in $\mathbb{R}^n$ over the probability space $(\Omega, \mathscr{F}, P)$ the collection of random variables $\{X_t\}_{t\in I}$.

Notation 1.1. Given a probability space $(\Omega, \mathscr{F}, P)$ and a set of times I, given a stochastic process $\{X_t\}_{t\in I}$

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)),$$

we will use the following notations to refer to $\{X_t\}_{t\in I}$:

1. $\{X_t\}_{t\in I}$ is the application X that maps each time $t \in I$ into the random variable X_t, i.e.

$$X : I \longrightarrow (\mathbb{R}^d)^\Omega \tag{1.2}$$

$$X(t)(\omega) := X_t(\omega) \quad \forall t \in I,\ \forall \omega \in \Omega.$$

2. $\{X_t\}_{t\in I}$ is the application X that maps each $\omega \in \Omega$ into the trajectory $t \mapsto X_t(\omega)$, i.e.

$$X : \Omega \longrightarrow (\mathbb{R}^d)^I \tag{1.3}$$

$$X(\omega)(t) := X_t(\omega) \quad \forall \omega \in \Omega,\ \forall t \in I.$$

3. $\{X_t\}_{t\in I}$ is the application X defined by

$$X : I \times \Omega \longrightarrow \mathbb{R}^d \tag{1.4}$$

$$X(t, \omega) := X_t(\omega) \quad \forall (t, \omega) \in I \times \Omega.$$

The notations in (1.1), (1.2), (1.3) and (1.4) are equivalent to each other, so we will use each one of them indiscriminately.
We will also denote a value $X_t(\omega) \in \mathbb{R}^d$ by

$$X_t(\omega) = X(\omega)(t) = X(t, \omega) = X(t)(\omega) \quad \forall \omega \in \Omega,\ \forall t \in I.$$

We are going to enunciate the notions needed to define a Brownian motion. We recommend to check [24, chapter 2] for a more detailed exposition.
The definition of finite-dimensional distributions is the core part in the construction of many stochastic processes, one example being the Brownian motion.

Definition 1.1.3 (Finite-dimensional distributions)**.** Given a stochastic process X=$\{X_t\}_{t\in T}$ in $\mathbb{R}^n$

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)),$$

where $T = [0, +\infty)$, we denote by finite-dimensional distributions of the proccess X the measures $\mu_{t_1,\ldots,t_k}$ defined over the Borel σ-algebra $\mathscr{B}(\mathbb{R}^{nk})$, for $k = 1, 2, \ldots,$ by

$$\mu_{t_1,\ldots,t_k}(F_1 \times F_2 \times \cdots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k]; \quad t_i \in T.$$

We recall the definition of expected value and conditional expectation from the basics of the theory of probability.

Definition 1.1.4 (Expectation, conditional expectation)**.** Given a random variable

$$X : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$$

we denote by expected value of X with respect to P the real number

$$E(X) := \int_\Omega X(\omega) dP(\omega).$$

Let $\mathscr{G}$ be a σ-algebra, $\mathscr{G} \subseteq \mathscr{F}$. Suppose that $E(|X|) < +\infty$. We denote by realization of the conditional expectation of X given $\mathscr{G}$ (with respect to P) a random variable

$$Z : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$$

such that:

1. Z is $\mathscr{G}$-measurable,
2. $$\int_G Z(\omega) dP(\omega) = \int_G X(\omega) dP(\omega) \quad \text{for all } G \in \mathscr{G}.$$

We will write $Z = E[X|\mathscr{G}]$ to denote that Z is a realization of the conditional expectation of X given $\mathscr{G}$. For all Z_1, Z_2 realizations of the conditional expectation of X given $\mathscr{G}$ we have $Z_1 = Z_2$ almost surely with respect to P, so we will sometimes just write $E[X|\mathscr{G}]$ in expressions to denote a realization Z of the conditional expectation when the expression is true for every possible realization of the conditional expectation. We recommend to check [24, appendix B] or a text of probability for an exposition over the conditional expectation.

The definition of filtration is needed for the theory of stochastic processes, and it represents the amount of "information" we know at each time t about the configuration of the stochastic process. The concept or martingale is a key element in the theory of stochastic processes, and it is also needed for the definition of the Itô integral. It represents a stochastic process $\{X_t\}_{t\geq 0}$ such that X_t can be estimated at a time $t > s$ by considering X_s.

Definition 1.1.5 (Filtration)**.** Given a measurable space $(\Omega, \mathscr{F})$, a filtration of said space is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t\geq 0}$ of σ-algebras $\mathcal{M}_t \subseteq \mathscr{F}$ such that

$$0 \leq s < t \Longrightarrow \mathcal{M}_s \subseteq \mathcal{M}_t.$$

A n-dimensional stochastic process $\{M_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathscr{F}, P)$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ (and with respect to P) if

(i) M_t is $\mathcal{M}_t$-measurable for all t,

(ii) $E[|M_t|] < +\infty$ for all t,

(iii) $E[M_s \mid \mathcal{M}_t] = M_t$ for all $s \geq t$.

Stopping times are a key element in the theory of stochastic processes. They are random times with appropriate properties that make them a usable replacement to deterministic times in most of the theorems about stochastic analysis.

Definition 1.1.6 (Stopping time, adapted process)**.** Let $(\Omega, \mathscr{F}, P)$ be a probability space, let $\{\mathcal{N}_t\}_{t\geq 0}$ be a filtration. A function

$$\tau : \Omega \longrightarrow [0, +\infty]$$

is called a (strict) stopping time with respect to $\{\mathcal{N}_t\}_{t\geq 0}$ if

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{N}_t \quad \text{for all } t \geq 0.$$

Let $\mathcal{N}_\infty$ be the smallest σ-algebra containing $\mathcal{N}_t$ for all $t \geq 0$. Then we define by $\mathcal{N}_\tau$ the σ-algebra of all sets $N \in \mathcal{N}_\infty$ such that

$$N \cap \{\tau \leq t\} \in \mathcal{N}_t \quad \text{for all } t \leq 0.$$

A stochastic process

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$$

is called adapted to the fitration $\{\mathcal{N}_t\}_{t\geq 0}$ if X_t is $\mathcal{N}_t$-measurable for all $t \geq 0$.
We denote by X_τ the random variable

$$X_\tau : (\Omega, \mathcal{N}_\tau, P) \longrightarrow (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$$

$$X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \text{if } \tau(\omega) < +\infty, \\ 0 & \text{if } \tau(\omega) = +\infty. \end{cases}$$

It can be proved that X_τ defined in this way is measurable with respect to $\mathcal{N}_\tau$ and $\mathscr{B}(\mathbb{R}^n)$.

The Brownian motion is the starting point for the theory of Itô processes. It is the most important stochastic process for this work, we recommend to check a textbook about stochastic processes to get a thorough explanation of this topic, Øksendal explains this topic in [24, chapter 2].

Definition 1.1.7 (Brownian motion)**.** Let $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$ be a fixed point and a fixed time. Define

$$p(t_1, y_1, t_2, y_2) := (2\pi(t_2 - t_1))^{-\frac{n}{2}} \cdot \exp\left(-\frac{|y_2 - y_1|^2}{2(t_2 - t_1)}\right) \qquad \text{for } y \in \mathbb{R}^n,\ t > 0.$$

For $k = 1, 2, \dots$, for $0 \le t_1 \le t_2 \le \dots \le t_k$ define a measure $\nu_{t_1,\dots,t_k}$ on $\mathscr{B}(\mathbb{R}^{nk})$ by

$$\nu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) := \tag{1.5}$$

$$= \int\limits_{F_1 \times \dots \times F_k} p(s, x, t_1, x_1) p(t_1, x_1, t_2, x_2) \dots p(t_{k-1}, x_{k-1}, t_k, x_k) dx_1 \dots dx_k,$$

where we use the convention that $p(t, y, t, z)dz = d\delta_y(z)$, the Dirac delta measure centered at y and computed at z.
We define a (version of) n-dimensional Brownian motion starting from x at the time s to be a stochastic process $B = \{B_t\}_{t \ge s}$ on a probability space $(\Omega, \mathscr{F}, P^x)$

$$B_t : (\Omega, \mathscr{F}, P^x) \longrightarrow (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$$

such that the finite-dimensional distributions of B are given by (1.5), i.e.

$$P^x(B_{t_1} \in F_1, \dots, B_{t_k} \in F_k) =$$

$$= \int\limits_{F_1 \times \dots \times F_k} p(s, x, t_1, x_1) p(t_1, x_1, t_2, x_2) \dots p(t_{k-1}, x_{k-1}, t_k, x_k) dx_1 \dots dx_k$$

for all $k \in \mathbb{N}$, $t_1, \dots, t_k \in [s, +\infty)$, for all $F_1, \dots, F_k$ Borel subsets of $\mathbb{R}^n$.
The existence of a process with such properties is guaranteed by Kolmogorov's extension theorem (see Theorem 1.2.1).

The concept of modification of a stochastic process is needed to understand Kolmogorov's continuity theorem that proves that the Brownian motion can be considered a continuous process.

Definition 1.1.8 (Modification of a stochastic process)**.** Let $X = \{X_t\}_{t \in I}$, $Y = \{Y_t\}_{t \in I}$ be stochastic processes on the same probability space $(\Omega, \mathscr{F}, P)$. We say that X is a version (or a modification) of Y if, for all $t \in I$

$$P(\{\omega \in \Omega \mid X_t(\omega) = Y_t(\omega)\}) = 1.$$

A Brownian motion $B = \{B_t\}_{t\geq 0}$ satisfies the condition (1.13) in Kolmogorov's continuity theorem (see Theorem 1.2.2) with $\alpha = 4$, $\beta = 1$, $D = n(n+2)$, so the theorem guarantees that there exists a continuous modification of B.

We need a notation for the σ-algebra generated by a Brownian motion at a time t for many propositions, expecially for the important Markov property.

Definition 1.1.9 (Filtration induced by a Brownian motion)**.** Let $\{B_t\}_{t\geq 0}$ be a n-dimensional Brownian motion. Let $t > 0$. We define $\mathcal{F}_t = \mathcal{F}_t^{(n)}$ to be the σ-algebra generated by the collection of random variables

$$\{B_s \mid 0 \leq s \leq t\},$$

i.e. the smallest σ-algebra $\mathscr{F}$ such that the random variable B_s is measurable with respect to $\mathscr{F}$ for all $0 \leq s \leq t$.

We are going to enunciate the definitions and theorems needed to define the Itô integral. We refer the reader to check a textbook about stochastic analysis for a thorough explanation of the topic. The notations and definitions are taken from the textbook by Øksendal, see [24, chapter 3].
The construction of the Itô integral begins with the construction of the Itô integral over elementary processes as a Riemann-Stieltjes integral, and then it extends the definition to a bigger class $\mathcal{V}$ of processes. We begin with the 1-dimensional case for the Itô integral.

Definition 1.1.10 (Ito integrable process)**.** Let $(\Omega, \mathscr{F}, P)$ be a probability space, let $0 \leq S < T$. Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f \in \mathcal{V}$,

$$f : [0, +\infty) \times \Omega \to \mathbb{R}$$

such that:

(i) $(t, \omega) \mapsto f(t, \omega)$ is $\mathscr{B}([0, +\infty)) \times \mathscr{F}$-measurable,

(ii) $(t, \omega) \mapsto f(t, \omega)$ is $\mathcal{F}_t$-adapted,

(iii) $E[\int_S^T f(t, \omega)^2 dt] < +\infty$.

Definition 1.1.11 (Elementary process)**.** A function $\phi \in \mathcal{V}$ is called elementary if it has the form

$$\phi(t, \omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t), \tag{1.6}$$

here e_j are functions

$$e_j : \Omega \longrightarrow \mathbb{R}.$$

The functions e_j must be $\mathcal{F}_{t_j}$-measurable since $\phi \in \mathcal{V}$.
Let $B = \{B_t\}_{t\geq 0}$ be a 1-dimensional Brownian motion over Ω. We define the Itô integral (with respect to B) for an elementary function ϕ, with the form written in (1.6), by

$$\int_S^T \phi(t,\omega)dB_t(\omega) := \sum_{j\geq 0} e_j(\omega)[B_{\tilde{t}_{j+1}} - B_{\tilde{t}_j}](\omega), \tag{1.7}$$

where $\tilde{t}_j$ are the points

$$\tilde{t}_j = \begin{cases} t_j & \text{if} \quad S \leq t_j \leq T, \\ S & \text{if} \quad t_j < S, \\ T & \text{if} \quad t_j > T. \end{cases}$$

Definition 1.1.12. (Itô integral) Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $0 \leq S < T$. Let $f \in \mathcal{V}(S,T)$. Let $B = \{B_t\}_{t\geq 0}$ be a 1-dimensional Brownian motion over Ω. Then the Itô integral of f from S to T (with respect to B) is defined by

$$\int_S^T f(t,\omega)dB_t(\omega) := \lim_{t\to+\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega) \quad (\text{limit in } L^2(P)), \tag{1.8}$$

where $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0 \quad \text{as } n \to +\infty. \tag{1.9}$$

Here the right hand side of (1.8) is defined by (1.7).
A sequence $\{\phi_n\}_{n\in\mathbb{N}}$ as such exists because of Lemma (1.2.4).

We are going to enunciate the definitions needed to define the Itô integral in the n-dimensional case.

Definition 1.1.13 (n-dimensional Itô integral)**.** Let $(\Omega, \mathscr{F}, P)$ be a probability space, let $\{B_t\}_{t\geq 0} = B = (B^1, B^2, \ldots, B^n)$ be a n-dimensional Brownian motion of components

$$B_t^k : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}, \mathscr{B}(\mathbb{R})) \quad \text{for } k = 1, 2, \ldots, n.$$

Then we denote by $\mathcal{V}_{\mathcal{H}}^{m\times n}(S,T)$ the set of matrices $v = [v_{i,j}(t,\omega)]_{i,j=1,\ldots,n}$ where each entry

$$v_{i,j} : [0,+\infty) \times \Omega \longrightarrow \mathbb{R}$$
$$(t,\omega) \longmapsto v_{i,j}(t,\omega)$$

satisfies conditions (i) and (iii) in definition (1.1.10), and it also satisfies the condition

(ii)' There exists an increasing family of σ-algebras $\{\mathcal{H}_t\}_{t\geq 0}$ such that:

a) $\{B_t\}$ is a martingale with respect to $\{\mathcal{H}_t\}$,

b) $\{v_{i,j}\}_{t\geq 0}$ is $\{\mathcal{H}_t\}$-adapted.

It is possible to construct the Itô integral for the functions $f \in \mathcal{V}_{\mathcal{H}}^{m\times n}(S,T)$ in the same way as it is done in (1.1.12).
If $v \in \mathcal{V}_{\mathcal{H}}^{m\times n}(S,T)$ we define, using matrix notation,

$$\int_S^T v dB := \int_S^T \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & & \vdots \\ v_{m,1} & \cdots & v_{m,n} \end{pmatrix} \begin{pmatrix} dB_1 \\ \vdots \\ dB_n \end{pmatrix}$$

to be the $m \times 1$ matrix whose i-th component is the following sum:

$$\sum_{j=i}^{n} \int_S^T v_{i,j}(s,\omega) dB_j(s,\omega).$$

Definition 1.1.14. Under the same notations as the previous definition, $\mathcal{W}_{\mathcal{H}}(S,T)$ denotes the class of processes

$$f : [0,+\infty) \longrightarrow \mathbb{R}$$

satisfying the conditions (i), (ii)' and the condition

(iii)' $P\left[\int\limits_S^T f(s,\omega)^2 ds < +\infty\right] = 1.$

We also define $\mathcal{W}_{\mathcal{H}} = \bigcap\limits_{T>0} \mathcal{W}_{\mathcal{H}}(0,T)$.

The Itô process is the basic example of solution to a stochastic differential equation and is the key element in the definition of the Bellman function.

Definition 1.1.15 (Itô process)**.** Let B_t be a m-dimensional Brownian motion on $(\Omega, \mathscr{F}, P)$. An Itô process (or stochastic integral) is a stochastic process $\{X_t\}_{t\geq 0}$ on $(\Omega, \mathscr{F}, P)$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dB_s. \tag{1.10}$$

Here the coefficients

$$u : [0,+\infty) \times \Omega \longrightarrow \mathbb{R}^n; \quad v : [0,+\infty) \times \Omega \longrightarrow \mathbb{R}^{n\times m}$$

satisfy proper conditions to guarantee that the object (1.10) is well defined, i.e. $v \in \mathcal{W}_{\mathcal{H}}$, so that

$$P\left[\int_0^t v(s,\omega)^2 ds < +\infty \text{ for all } t \geq 0\right] = 1.$$

We also assume that u is $\mathcal{H}_t$-adapted and

$$P\left[\int_0^t |u(s,\omega)|ds < +\infty \text{ for all } t \geq 0\right] = 1.$$

If $\{X_t\}_{t\geq 0}$ is an Itô process of the form (1.10), the equation (1.10) can be denoted by the differential expression

$$dX_t = udt + vdB_t.$$

The Itô diffusion is an example of Itô process where the coefficients of the associated stochastic differential equation do not depend on the time variable. These processes are very important for the proofs in this work, because for a process of this kind the Markov property holds.

Definition 1.1.16 (Itô diffusion)**.** A (time-homogeneous) Itô diffusion is a stochastic process

$$\begin{aligned} X : [0,+\infty) \times \Omega &\longrightarrow \mathbb{R}^n \\ (t,\omega) &\longmapsto X_t(\omega) \end{aligned}$$

satisfying a stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s;\ X_s = x, \tag{1.11}$$

where $\{B_t\}_{t\geq 0}$ is a m-dimensional Brownian motion and the coefficients

$$b : \mathbb{R}^n \longrightarrow \mathbb{R}^n; \quad \sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$$

satisfy the conditions in Theorem (1.2.6), which in this case simplify to

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x-y| \quad \text{for some } D \in \mathbb{R},\ \forall x, y \in \mathbb{R}^n.$$

We denote the unique solution to (1.11) by $X_t = X_t^{s,x}$; $t \geq s$. If $s = 0$ we write X_t^x for $X_t^{0,s}$.

We need a notation to denote the expected value of an Ito diffusion $\{X_t\}_{t\in I}$ at the time t for the theorems about Itô diffusions, like the Markov property.

Definition 1.1.17 (Expectation of an Itô diffusion)**.** Given an Itô diffusion $\{X_t\}_{t\geq 0} = \{X_t^y\}_{t\geq 0}$, for $y \in \mathbb{R}^n$, over the probability space $(\Omega, \mathscr{F}, P)$, solution to the equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t; \quad X_0 = y,$$

we denote by $\mathcal{M}_\infty$ the σ-algebra (of subsets of Ω) generated by the collection of random variables

$$\{\omega \mapsto X_t^y(\omega) \mid t > 0,\ y \in \mathbb{R}^n\}.$$

For each $x \in \mathbb{R}^n$ we define a measure Q^x over the elements of $\mathcal{M}_\infty$ by

$$Q^x[X_{t_1} \in E_1, \dots, X_{t_k} \in E_k] = P[X^x_{t_1} \in E_1, \dots, X^x_{t_k} \in E_k],$$

where $E_i \subseteq \mathbb{R}^n$ are Borel sets; $k \in \mathbb{N}$.
Q^x is the probability law of $\{X^x_t\}_{t\geq 0}$ for $x \in \mathbb{R}^n$. Q^x gives the distribution of $\{X_t\}_{t\geq 0}$ assuming that $X_0 = x$.
We denote by $E^x[X_t]$ the "expected value of X_t with respect to the measure Q^x", i.e. the expected value of the random variable $\omega \mapsto X^x_t(\omega)$ with respect to the measure P, similarly we denote by $E^x[X_t \mid \mathscr{G}]$ the conditional expectation of $\omega \mapsto X^x_t(\omega)$ with respect to the measure P given a σ-algebra $\mathscr{G} \subseteq \mathcal{M}_\infty$.

The infinitesimal generator is a key element to connect the theory of stochastic analysis with the theory of differential problems, allowing for example to solve problems like Dirichlet's problem using the tools from stochastic analysis.

Definition 1.1.18 (Infinitesimal generator)**.** Let $X = \{X_t\}_{t\geq 0}$ be an Itô diffusion in $\mathbb{R}^n$. We denote by $\mathcal{D}_A(x)$ the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that it exists the limit

$$\lim_{t\downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t}. \tag{1.12}$$

We define the infinitesimal generator of $\{X_t\}_{t\geq 0}$ in x as the operator

$$A : \mathcal{D}_A(x) \to \mathbb{R}$$

$$Af(x) = \lim_{t\downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t}.$$

We denote by D_A the set of functions for which the limit (1.12) exists for all $x \in \mathbb{R}^n$.

The exit time of a process $\{X_t\}_{t\in I}$ from a Borel set U is one of the most important examples of exit times, and it is used in important theorems about solving differential problems like the Dirichlet problem using the theory of stochastic analysis.

Definition 1.1.19 (First exit time)**.** Let $U \in \mathbb{R}^n$ be a Borel set, let $X = \{X_t\}_{t\geq 0}$ be an Itô diffusion

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)), \quad t \geq 0.$$

We denote by first exit time for X from the set U the random variable

$$\tau_U : \Omega \longrightarrow \mathbb{R}$$

$$\tau_U(\omega) = \inf\{t > 0 \mid X_t(\omega) \notin U\}.$$

We observe that τ_U is a random variable because U is a Borel set.

The definition of regular point of the boundary of a domain D is a very important definition in the theory about the Dirichlet problem, and there is the analogous version for the theory about stochastic solutions to Dirichlet problems.

Definition 1.1.20. Under the same hypotheses as the previous definition, the point $y \in \partial U$ is called regular for X if

$$Q^y[\tau_U = 0] = 1.$$

Otherwise, the point y is called irregular.
The boundary set ∂U is called regular for X if all the points $y \in \partial D$ are regular for X.

The Dirichlet-Poisson problem is used to prove the important theorem about the Bellman function being the solution to the Hamilton-Jacobi-Bellman equation.

Definition 1.1.21. Let $D \subseteq \mathbb{R}^n$ be a domain, let L denote a semi-elliptic partial differential operator on $C^2(\mathbb{R}^n)$ of the form

$$L = \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

where the functions b_i and $a_{i,j} = a_{j,i}$ are continuous functions.
Let $\phi \in C(\partial D)$ and $g \in C(D)$ be given functions. A function $w \in C^2(D)$ is called a solution to the Dirichlet-Poisson problem (over D, associated to L, ϕ, g) if

(I) $Lw = -g \quad \text{in } D,$

(II) $\lim_{\substack{x \to y \\ x \in D}} w(x) = \phi(y) \quad \text{for all } y \in \partial D.$

1.2 Theorems

The proofs of the theorems mentioned in this section can be found in the textbook [24] from Øksendal, in chapters 1,2,3,4,5,7 and 9. We will enunciate the theorems needed for the construction of the Bellman function and to prove the theorem about the Hamilton-Jacobi-Bellman equation.

Kolmogorov's extension theorem is one of the fundamental results in the theory of stochastic processes, and it allows to prove the existence of stochastic processes having given finite-dimensional distributions, like the Brownian motion.

Theorem 1.2.1 (Kolmogorov's extension theorem)**.** *Let T be a set of times. For all $k \in \mathbb{N}$, $t_1, \dots, t_k \in T$, let $\nu_{t_1,\dots,t_k}$ be probability measures on $\mathbb{R}^{nk}$ such that, for all $F_1, \dots, F_k$ Borel subsets of $\mathbb{R}^n$,*

$$\nu_{t_{\sigma(1)},\dots,t_{\sigma(k)}}(F_1 \times \cdots \times F_k) = \nu_{t_1,\dots,t_k}\left(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)}\right)$$

for all permutations σ on $\{1, 2, \dots, k\}$, and

$$\nu_{t_1,\dots,t_k}(F_1 \times \cdots \times F_k) = \nu_{t_1,\dots,t_k,t_{k+1}}(F_1 \times \cdots \times F_k \times \mathbb{R}^n).$$

Then there exists a complete probability space $(\Omega, \mathscr{F}, P)$ and a stochastic process $\{X_t\}_{t\in T}$

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$$

such that

$$\nu_{t_1,\dots,t_k}(F_1, \times, F_k) = P[X_{t_1} \in F_1, \dots, X_{t_k} \in F_k]$$

for all $t_i \in T$, $k \in \mathbb{N}$, and for all F_i Borel subsets of $\mathbb{R}^n$.

Kolmogorov's continuity theorem is another fondamental result in the theory of stochastic processes, and it is used to prove that the Brownian motion can be considered a continuous process.

Theorem 1.2.2 (Kolmogorov's continuity theorem)**.** *Let $X = \{X_t\}_{t\geq 0}$ be a stochastic process such that for all $T > 0$ there exist positive constants α, β, D such that*

$$E[|X_t - X_s|^\alpha] \leq D \cdot |t - s|^{1+\beta}; \quad \text{for } 0 \leq s, t \leq T. \tag{1.13}$$

Then there exists a continuous version of X.

The Itô isometry is one of the most important results in the theory of stochastic differential equations, and it is one of the key elements used in the construction of the Itô integral. The Itô isometry for elementary functions is used to define the Itô integral, and then using the Itô integral we can extend the Itô isometry to all the Itô integrable processes.

Lemma 1.2.3 (Itô isometry for elementary functions)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space. If*

$$\phi : [0, +\infty) \times \Omega \to \mathbb{R}^n$$

$$\phi(t, \omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j, t_j+1)}(t)$$

is a bounded elementary function, then

$$E\left[\left(\int_S^T \phi(t, \omega) dB_t(\omega)\right)^2\right] = E\left[\int_S^T \phi(t, \omega)^2 dt\right], \tag{1.14}$$

where

$$\int_S^T \phi(t,\omega)dB_t(\omega) = \sum_{j\geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega).$$

This lemma proves the three statements needed in the construction of the Itô integral.

Lemma 1.2.4. *The following three statements hold true:*

1. *Let $g \in \mathcal{V}$ be a bounded and such that $t \mapsto g(t,\omega)$ is continuous for each $\omega \in \Omega$. Then there exist a sequence of elementary functions $\phi_n \in \mathcal{V}$ such that*

$$E\Big[\int_S^T (g - \phi_n)^2 dt\Big] \to 0 \quad \textit{for } n \to +\infty.$$

2. *Let $h \in \mathcal{V}$ be bounded. Then there exist a sequence of bounded functions $g_n \in \mathcal{V}$ such that $t \mapsto g_n(t,\omega)$ is continuous for all $\omega \in \Omega$ and for all n, and*

$$E\Big[\int_S^T (h - g_n)^2 dt\Big] \to 0 \quad \textit{for } n \to +\infty.$$

3. *Let $f \in \mathcal{V}$. Then there exist a sequence of functions $h_n \in \mathcal{V}$ such that h_n is bounded for each n and*

$$E\Big[\int_S^T (f - h_n)^2 dt\Big] \to 0 \quad \textit{for } n \to +\infty.$$

Theorem 1.2.5 (Itô isometry)**.**

$$E\bigg[\Big(\int_S^T f(t,\omega)dB_t(\omega)\Big)^2\bigg] = E\Big[\int_S^T f(t,\omega)^2 dt\Big] \quad \textit{for all } f \in \mathcal{V}(S,T). \tag{1.15}$$

The following theorem allows us to prove the existence and uniqueness of solutions to stochastic differential equations, which is needed to guarantee that the Bellman function is well defined.

Theorem 1.2.6 (Existence and uniqueness of solutions to stochastic differential equations)**.** *Given $T > 0$, let*

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ \sigma &: [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m} \end{aligned}$$

be measurable functions. Suppose that there exists a constant $C > 0$ such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|); \quad \forall x \in \mathbb{R}^n,\ \forall t \in [0,T].$$

Suppose that there exists a constant $D > 0$ such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|; \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [0,T].$$

Let $\{B_t\}_{t\geq 0}$ be a m-dimensional Brownian motion, let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by the collection of random variables $\{B_s \mid s \geq 0\}$, and such that

$$E[|Z|^2] < +\infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, \ X_0 = Z \tag{1.16}$$

has a unique t-continuous solution $\{X_t\}_{0\leq t\leq T}$ such that:

1. *$\{X_t\}_{0\leq t\leq T}$ is adapted to the filtration $\{\mathcal{F}_t^Z\}_{0\leq t\leq T}$, where $\mathcal{F}_t^Z$ is the σ-algebra generated by the collection of random variables $\{Z, B_s \mid 0 \leq s \leq t\}$.*

2. $$E\left[\int_0^T |X_t|^2 dt\right] < +\infty.$$

The strong Markov property is the most important result for the theory of the Bellman functions, and it allows to prove important propositions like the Bellman principle and the theorem about the Hamilton-Bellman-Jacobi equation. The Markov property basically states that what happens to an Itô diffusion $\{X_t\}_{t\in I}$ after a time t only depends on X_t and does not depend on X_s for $s < t$.

Theorem 1.2.7 (Strong Markov property for Itô diffusions)**.** *Let $\{X_t\}_{t\geq 0}$ be a Itô diffusion in $\mathbb{R}^n$. Let f be a bounded Borel function $f : \mathbb{R}^n \to \mathbb{R}$, let $\{B_t\}_{t\geq 0}$ be a m-dimensional Brownian motion, let τ be a stopping time with respect to the σ-algebra $\mathcal{F}_t^{(m)}$ generated by $\{B_t\}_{t\geq 0}$, suppose $\tau < +\infty$ almost surely. Then*

$$E^x[f(X_{\tau+h}) \mid \mathcal{F}_\tau^{(m)}] = E^{X_\tau}[f(X_h)] \quad \forall h \geq 0.$$

The following theorem is very important for this work and for the general theory. It characterizes the infinitesimal generator of an Itô diffusion.

Theorem 1.2.8 (Characterization of the infinitesimal generator of an Itô diffusion)**.** *Let $\{X_t\}$ be the Itô diffusion*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

If $f \in C_0^2(\mathbb{R}^n)$ then $f \in \mathcal{D}_A$, and the infinitesimal generator associated to $\{X_t\}$ is

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{1.17}$$

The following lemma is used to prove Dynkin's formula, a result which is very important to understand the behaviour of the composition between a smooth function and a stochastic process.

Lemma 1.2.9. *Let* $\{X_t\}_{t\geq 0} = \{X_t^x\}_{t\geq 0}$ *be an Itô diffusion in* $\mathbb{R}^n$ *of the form*

$$X_t^x(\omega) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s(\omega),$$

where $\{B\}_{t\geq 0}$ *is a m-dimentional Brownian motion. Let* $f \in C_0^2(\mathbb{R}^n)$*, let* τ *be a stopping time with respect to the filtration* $\{\mathcal{F}_t^{(m)}\}$*, and assume that* $E^x[\tau] < +\infty$*. Assume that* $u(t,\omega)$ *and* $v(t,\omega)$ *are bounded on the set of* (t,ω) *such that* $X(t,\omega)$ *belongs to the support of* f*.Then*

$$E[f(X_t)] = f(x) + E^x\left[\int_0^\tau \left(\sum_i u_i(s,\omega)\frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2}\sum_{i,j}(vv^T)_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\right)ds\right].$$

Theorem 1.2.10 (Dynkin's formula)**.** *Under the same assumptions of Lemma (1.2.9) it follows that*

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right], \tag{1.18}$$

here A *is the infinitesimal generator of the process* X*.*

The following lemma is used in the proof of the theorem about the Hamilton-Jacobi-Bellman equation and it allows to calculate the time shift of a process stopped on an exit time from a Borel set.

Lemma 1.2.11. *Let* $H \subseteq \mathbb{R}^n$ *be measurable, let* $X = \{X_t\}_{t\geq 0}$ *be a Itô diffusion in* $\mathbb{R}^n$

$$X_t : (\Omega, \mathscr{F}, P) \longrightarrow (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)).$$

Let τ_H *be the first exit time from* H *for* X*. Let* α *be another stopping time, let* g *be a bounded continuous function on* $\mathbb{R}^n$*. Let* $\mathcal{H}$ *be the family of all* $\mathcal{M}_\infty$*-measurable functions. Let* θ_t *be the shift operator*

$$\theta_t : \mathcal{H} \longrightarrow \mathcal{H}$$

defined in the following way: given $\nu = g_1(X_{t_1})\dots g_k(X_{t_k})$*, where the functions* g_i *are Borel measurable, the shift operator is defined by*

$$\theta_t\nu := g_1(X_{t_1+t})\dots g_k(X_{t_k+t}),$$

and the definition is extended over all functions in $\mathcal{H}$ *by taking limits of sums of such functions. Consider*

$$\eta = g(X_{\tau_H}) \cdot \chi_{\{\tau_H < +\infty\}},$$

$$\tau_{_H}^{\alpha}(\omega) = \inf\{t > \alpha \mid X_t(\omega) \notin H\}, \quad \omega \in \Omega.$$

Then

$$\theta_\alpha \eta \cdot \chi_{\{\alpha<+\infty\}} = g(X_{\tau_H^\alpha}) \cdot \chi_{\{\tau_H^\alpha<+\infty\}},$$

where

$$(\theta_\alpha \eta)(\omega) = (\theta_t \eta)(\omega) \quad \textit{if } \tau(\omega) = t.$$

The Dirichlet-Poisson problem is used to prove that the Bellman function is a solution to the Hamilton-Jacobi-Bellman equation with boundary values equal to the values of the bequest function.

Theorem 1.2.12. *Let $D \subseteq \mathbb{R}^n$ be a domain. Let $X = \{X_t\}_{t\geq 0}$ be a Itô diffusion in $\mathbb{R}^n$. Let $\mathcal{A}$ be the infinitesimal generator of X. Let Q^x be the probability law of X starting at $X_0 = x$, for $x \in \mathbb{R}^n$. Let $\tau_{_D}$ be the stopping time*

$$\tau : \Omega \longrightarrow \mathbb{R}$$

$$\tau(\omega) = \inf\{t > 0 \mid X_t(\omega) \notin D\}.$$

Suppose that $\tau_{_D} < +\infty$ almost surely with respect to Q^x for all $x \in D$. Let $\phi \in C(\partial D)$ be bounded and let $g \in C(D)$ satisfy

$$E^x\left[\int_0^{\tau_D} |g(X_s)|ds\right] < +\infty \quad \textit{for all } x \in D. \tag{1.19}$$

Define

$$w(x) = E^x[\phi(X_{\tau_{_D}})] + E^x\left[\int_0^{\tau_D} g(X_s)ds\right], \quad x \in D. \tag{1.20}$$

Then the following two statements hold true

a)

$$\mathcal{A}w = -g \quad \textit{in } D \tag{1.21}$$

and

$$\lim_{t\uparrow\tau_D} w(X_t) = \phi(X_{\tau_{_D}}) \tag{1.22}$$

almost surely with respect to Q^x, for all $x \in D$.

b) If there exists a function $w_1 \in C^2(D)$ and a constant C such that

$$|w_1(x)| < C\left(1 + E^x\left[\int_0^{\tau_D} |g(X_s)|ds\right]\right) \quad \textit{for } x \in D, \tag{1.23}$$

and w_1 satisfies (1.21) and (1.22), then $w_1 = w$.

1.3 Bellman functions

Let $\{X_t\}_{t\geq 0}$ be an Itô process described by the stochastic differential equation

$$dX_t = dX_t^u = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t, \tag{1.24}$$

where $X_t(\omega) \in \mathbb{R}^n$, and the coefficients b and σ are

$$b : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n, \quad \sigma : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m},$$

and B_t is an m-dimensional Brownian motion. Here U is a given Borel set $U \subseteq \mathbb{R}^k$ and $\{u_t\}_{t\geq 0}$ is the control, i.e. a stochastic process such that $u_t(\omega) \in U$, and $\{u_t\}_{t\geq 0}$ is adapted to the filtration $\{\mathcal{F}_t^{(m)}\}_{t\geq 0}$, i.e. for all $t \geq 0$ the random variable u_t is measurable with respect to the σ-algebra $\mathcal{F}_t^{(m)}$.
Let $\{X_h^{s,x}\}_{h\geq s}$ the solution to (1.24) such that $X_s^{s,x} = x$, i.e.

$$X_h^{s,x} = x + \int_s^h b(r, X_r^{s,x}, u_r)dr + \int_s^h \sigma(r, X_r^{s,x}, u_r)dB_r; \quad h \geq s. \tag{1.25}$$

Let the probability law of X_t starting at x for $t = s$ be denoted by $Q^{s,x}$, i.e.

$$Q^{s,x}[X_{t_1} \in F_1, \dots, X_{t_k} \in F_k] = P[X_{t_1}^{s,x} \in F_1, \dots, X_{t_k}^{s,x} \in F_k] \tag{1.26}$$

for all $s \leq t_i$, F_i measurable subset of R^n; for all $1 \leq i \leq k$, $k = 1, 2, \dots$.

Let F and K be two continuos functions

$$F : \mathbb{R} \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}, \quad K : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Here F is the "utility rate" function, and K is the "bequest" function.
Let G be a fixed domain in $\mathbb{R} \times \mathbb{R}^n$ and let $\widehat{T}$ be the first exit time after s from G for the process $\{X_r^{s,x}\}_{r\geq s}$, i.e.

$$\widehat{T} = \widehat{T}^{s,x}(\omega) = \inf\{r > s \mid (r, X_r^{s,x}(\omega)) \in G\} \leq +\infty \tag{1.27}$$

Let $F^u(r, z) = F(r, z, u)$. Suppose that

$$E^{s,x}\left[\int_s^{\widehat{T}} |F^{u_r}(r, X_r)|dr + |K(\widehat{T}, X_{\widehat{T}})| \cdot \chi_{\{\widehat{T}<+\infty\}}\right] < +\infty \quad \text{for all } s, x, u. \tag{1.28}$$

We define the performance function $J^u(s, x)$ by

$$J^u(s, x) := E^{s,x}\left[\int_s^{\widehat{T}} F^{u_r}(r, X_r)dr + K(\widehat{T}, X_{\widehat{T}}) \cdot \chi_{\{\widehat{T}<+\infty\}}\right]. \tag{1.29}$$

In order to get a simpler notation we define

$$Y_t = (s+t, X^{s.x}_{s+t}) \in \mathbb{R}^{n+1} \quad \text{for } t \geq 0,\ Y_0 = (s,x), \tag{1.30}$$

and we substitute Y_t in (1.24) to get the equation

$$dY_t = dY^u_t = b(Y_t.u_t)dt + \sigma(Y_t, u_t)dB_t. \tag{1.31}$$

We denote by $Q^{s,x} = Q^y$ the probability of Y_t starting at $y = (s,x)$ for $t = 0$.
We observe that

$$\int_s^{\widehat{T}} F^{u_r}(r, X_r)dr = \int_0^{\widehat{T}-s} F^{u_{s+t}}(s+t, X_{s+t})dt = \int_s^T F^{u_{s+t}}(Y_t)dt$$

where

$$T := \inf\{t > 0 \mid Y_t \notin G\} = \widehat{T} - s \tag{1.32}$$

We also observe that

$$K(\widehat{T}, X_{\widehat{T}}) = K(Y_{\widehat{T}-s}) = K(Y_T)$$

so the performance function may be written in terms of Y as follows:

$$J^u(y) = E^y\left[\int_0^T F^{u_t}(Y_t)dt + K(Y_T)\cdot \chi_{\{T<+\infty\}}\right], \tag{1.33}$$

here $y := (s,x)$, and u_t is a time shift of the u_t in (1.31).

Definition 1.3.1 (Stochastic Bellman function)**.** Given a Borel set $U \subseteq \mathbb{R}^{n+1}$, given two continuous functions

$$F : \mathbb{R}\times\mathbb{R}^n \times U \longrightarrow \mathbb{R}, \quad K : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R},$$

and given the stochastic differential equation (1.24)

$$dX_t = dX^u_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t$$

associated to the coefficients

$$b : \mathbb{R}\times\mathbb{R}^n\times U \to \mathbb{R}^n, \quad \sigma : \mathbb{R}\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times m},$$

we denote by Bellman function associated to the equation (1.24), to the functions F and K, over a set of admissible controls $\mathcal{C}$, a function

$$\mathcal{B} : G \longrightarrow \mathbb{R}$$

$$\mathcal{B}(y) = \sup_{\{u_t\}_{t\geq 0}\in\mathcal{C}} J^u(y).$$

Here J^u is the performance function defined in (1.33), and the supremum is taken over the set $\mathcal{C}$ of admissible controls. Here $\mathcal{C}$ is a set of controls $\{u_t\}_{t\geq 0}$ that are $\mathcal{F}_t^{(m)}$-adapted, with values $u_t(\omega) \in U$.
If a control $\{u_t^*\}_{t\geq 0}$ such that

$$\mathcal{B}(y) = \sup_{\{u_t\}_{t\geq 0}\in\mathcal{C}} J^u(y) = J^{u^*}(y)$$

exists, then $\{u_t^*\}_{t\geq 0}$ is called optimal control.

We may take into consideration different types of control functions. The set of control functions that we will look into is the set of Markov controls, which is the set $\mathcal{C}$ of stochastic processes defined by

$$\mathcal{C} := \left\{ u(t,\omega) = u_0(t, X_t(\omega)) \mid \text{for } u_0 : \mathbb{R}^{n+1} \to U, \quad u_0 \text{ measurable} \right\}. \tag{1.34}$$

1.4 The Hamilton-Jacobi-Bellman Equation

Following the definitions in the previous section, we consider the set $\mathcal{C}$ of Markov controls

$$u(t,\omega) = u_0(t, X_t(\omega))$$

defined in (1.34), and, after introducing $Y_t = (s+t, X_{s+t})$ as explained in (1.30), the system equation becomes

$$dY_t = b(Y_t, u_0(Y_t))dt + \sigma(Y_t, u_0(Y_t))dB_t. \tag{1.35}$$

For every $v \in U$ and $f \in C_0^2(\mathbb{R}\times\mathbb{R}^n)$ we define the operator

$$(L^v f)(y) = \frac{\partial f}{\partial s}(y) + \sum_{i=1}^n b_i(y,v)\frac{\partial f}{\partial x_i}(y) + \sum_{i,j=1}^n a_{i,j}(y,v)\frac{\partial^2 f}{\partial x_i \partial x_j}(y), \quad \forall y \in \mathbb{R}\times\mathbb{R}^n, \tag{1.36}$$

here $a_{i,j} = \frac{1}{2}(\sigma\sigma^T)_{i,j}$, $y = (s,x)$ and $x = (x_1,\ldots,x_n)$. Then, by Theorem (1.2.8), for each choice of the function u_0 (that defines the control u), the solution $Y_t = Y_t^u$ is an Itô diffusion with infinitesimal generator A given by

$$(Af)(y) = (L^{u_0(y)} f)(y) \quad \text{for } f \in C_0^2(\mathbb{R}\times\mathbb{R}^n),\ y \in G.$$

For every $v \in U$ we define $F^v(y) = F(y,v)$. The first fundamental result in stochastic control theory is the following:

Theorem 1.4.1 (The Hamilton-Jacobi-Bellman (HJB) equation (I)). *Under the notations of the previous section, consider the Bellman function*

$$\mathcal{B}(y) = \sup\{J^u(y) \mid u = u_0(Y) \ \ \textit{Markov control} \ \}.$$

Suppose that $\mathcal{B}$ satisfies

$$E^y\left[|\mathcal{B}(Y_\alpha)| + \int_0^\alpha |L^v\mathcal{B}(Y_t)|dt\right] < +\infty$$

for all bounded stopping times $\alpha < T$, for all $y \in G$ and for all $v \in U$. Suppose that the stopping time T is $T < +\infty$ almost surely with respect to Q^y for all $y \in G$, and suppose that a optimal Markov control $u^* = u_0^*(Y)$ exists. Suppose ∂G is regular for Y^{u^*}. Then

$$\sup_{v\in U}\{F^v(y) + (L^v\mathcal{B})(y)\} = 0 \quad \textit{for all } y \in G, \tag{1.37}$$

and

$$\mathcal{B}(y) = K(y) \quad \textit{for all } y \in \partial G. \tag{1.38}$$

The supremum in (1.37) is obtained if $v = u_0^*(y)$, where $u^* = u_0^*(Y_t)$ is an optimal control. In other words

$$F(y, u_0^*(y)) + (L^{u_0^*(y)}\mathcal{B})(y) = 0 \quad \textit{for all } y \in G. \tag{1.39}$$

A converse of the previous theorem holds as well.

Theorem 1.4.2 (A converse of the HJB equation (I)). *Let ϕ be a function in $C^2(G) \cap C(\overline{G})$ such that, for all $v \in U$,*

$$F^v(y) + (L^v\phi)(y) \le 0; \quad y \in G, \tag{1.40}$$

with boundary values

$$\lim_{t\to T} \phi(Y_t) = K(Y_T) \cdot \chi_{\{T<+\infty\}} \tag{1.41}$$

almost surely with respect to Q^y, and such that

$$\{\phi(Y_\tau)\}_{\tau\le T} \quad \textit{is uniformly } Q^y\textit{-integrable} \tag{1.42}$$

for all Markov controls u and all $y \in G$. Then

$$\phi(y) \ge J^u(y) \quad \textit{for all Markov controls } u \textit{ and all } y \in G. \tag{1.43}$$

Moreover, if for each $y \in G$ we have found $u_0^*(y)$ such that

$$F^{u_0^*(y)}(y) + (L^{u_0^*(y)}\phi)(y) = 0, \tag{1.44}$$

then $u_0 = u_0^*(Y)$ is a Markov control such that

$$\phi(y) = J^{u_0}(y),$$

and hence u_0 must be a optimal control and $\phi(y) = \mathcal{B}(y)$.

The last result that we are going to mention is that, under suitable conditions on $b, \sigma, F, \partial G$ and assuming that the set of control values is compact, it is possible to show that there exists a smooth function ϕ such that

$$\sup_v \{F^v(y) + (L^v \phi)(y)\} = 0 \quad \text{for } y \in G,$$

and

$$\phi(y) = K(y) \quad y \in \partial G.$$

Moreover, by a measurable selection theorem one can find a measurable function u_0 that defines a Markov control $u_t^*(\omega) = u_0(X_t(\omega))$ such that

$$F^{u_0(y)}(y) + (L^{u_0(y)}\phi)(y) = 0$$

for almost all $y \in G$ with respect to Lebesgue measure in $\mathbb{R}^{n+1}$, and that the solution $X_t = X_t^{u^*}$ exists. For details see Øksendal [24, p. 241].
Moreover, it is always possible to get as good as a performance with Markov controls as it is with arbitrary $\mathcal{F}_t^{(m)}$-adapted controls, as long as some extra conditions are satisfied, as stated in the next theorem.

Theorem 1.4.3. *Let*

$$\Phi_M(y) = \sup\{J^u(y) \mid u = u_0(Y) \textit{ Markov control}\},$$

and

$$\Phi_a(y) = \sup\{J^u(y) \mid u = u(t,\omega)\ \mathcal{F}_t^{(m)}\textit{-adapted control}\}.$$

Suppose there exists an optimal Markov control $u^* = u_0^*(Y)$ for the Markov control problem

$$\Phi_M(y) = J^{u_0^*}(y) \quad \textit{for all } y \in G$$

such that all the boundary points of G are regular with respect to $Y_t^{u^*}$ and suppose that Φ_M is a function in $C^2(G) \cap C(\overline{G})$ satisfying

$$E^y\left[|\Phi_M(Y_\alpha)| + \int_0^\alpha |L^u \Phi_M(Y_t)|dt\right] < +\infty$$

for all bounded stopping times $\alpha \leq T$, all adapted controls u and all $y \in G$. Then

$$\Phi_M(y) = \Phi_a(y) \quad \textit{for all } y \in G.$$

For the proof of the last three theorems see Øksendal [24, chapter 11].

Chapter 2

Hardy's inequality

Introduction

In this chapter we use the method of the Bellman function to characterize the measures for which the weighted dual Hardy's inequality holds on dyadic trees. We also give an explicit interpretation of the corresponding Bellman function in terms of the theory of stochastic optimal control.

This chapter is structured as follows.
In section 2.1 we prove Theorem 0.0.1 for the optimal setting. We characterize the measures for which the weighted dyadic Hardy's inequality holds in the subsection 2.1.1, we enunciate the Bellman function $\mathcal{B}$ associated to this problem and prove its key properties in the subsection 2.1.2, and we prove the weighted dyadic Hardy's inequality using the Bellman function method in subsection 2.1.3.

In section 2.2 we prove the sharpness of the constant $C(p)$ in Theorem 0.0.1. We define a function B that satisfies the main inequality and prove the optimality of the domain $\mathcal{D}$ in the subsection 2.2.1, we improve the previous result and define a function $\hat{B}$ that satisfies the main inequality and a supplementary property in the subsection 2.2.2, and we prove that the constant $C(p)$ is sharp for dyadic Hardy's inequality in the subsection 2.2.3.

In section 2.3 we state a stochastic optimal control problem whose solution is given by the Bellman function used throughout the paper. This gives a direct probabilistic interpretation to our function. We show a natural way to transition from a dyadic inequality to a Hamilton-Jacobi-Bellman inequality in the subsection 2.3.1, we define a stochastic optimal control problem whose solution is a Bellman function that satisfies the required Hamilton-Jacobi-Bellman inequality in the subsection 2.3.2, and we prove that the Bellman function associated to the stochastic optimal control problem we defined is equal to the function $\mathcal{B}$ in subsection 2.3.3.

2.1 Hardy's inequality

2.1.1 Inequality over the dyadic tree

Let $\mathscr{D}(I_0)$ be the dyadic tree over $I_0 = [0,1]$. Let $I \in \mathscr{D}(I_0)$, let

$$\varphi : \mathscr{D}(I_0) \longrightarrow \mathbb{R}.$$

We denote the sum of the values $\varphi(J)$ for $J \in \mathscr{D}(I)$ by

$$\sum_{J\in\mathscr{D}(I)} \varphi(J) =: \sum_{J\subseteq I} \varphi(J).$$

Let Λ be a positively valued measure over the dyadic tree defined as follows: for each node $I \in \mathscr{D}(I_0)$

$$\mathscr{D}(I_0) \ni I \longmapsto \lambda_I \in \mathbb{R}^+.$$

We define the following objects as follows:

$$\begin{aligned}
\Lambda(I) &= \sum_{K\subseteq I} \lambda_K, \\
(\Lambda)_I &= \frac{1}{|I|}\sum_{K\subseteq I} \lambda_K = \frac{1}{|I|}\Lambda(I), \\
\int_I \phi \, d\Lambda &= \sum_{K\subseteq I} \phi(K)\lambda_K, \\
(\phi\Lambda)_I &= \frac{1}{|I|}\sum_{K\subseteq I} \phi(K)\lambda_K = \frac{1}{|I|}\int_I \phi \, d\Lambda.
\end{aligned}$$

Now we are going to prove Theorem (1.3) in the article [5]. in the general case $p \neq 2$.

Theorem 2.1.1 (Dual weighted Hardy's inequality for dyadic trees)**.** *Let $\mathscr{D}(I_0)$ be the dyadic tree originating at I_0, let $\{\alpha_I\}_{I\subseteq I_0}$ be a sequence of positive numbers. Let $\Lambda : \mathscr{D}(I_0) \to \mathbb{R}^+$ be a positive measure over the dyadic tree. Let $\phi : \mathscr{D}(I_0) \to \mathbb{R}^+$ be a positive function such that $\phi \in l^p(\mathscr{D}(I_0))$. Let p be a real number such that $1 < p < +\infty$. If the inequality*

$$\frac{1}{|I|}\sum_{K\subseteq I} \alpha_K (\Lambda)_K^p \leq (\Lambda)_I < +\infty \quad \forall I \in \mathscr{D}(I_0) \tag{2.1}$$

is satisfied, then

$$\frac{1}{|I_0|}\sum_{I\subseteq I_0}\alpha_I(\phi\Lambda^{\frac{1}{p'}})_I^p \le C(p)(\phi^p)_{I_0}. \tag{2.2}$$

Here $\frac{1}{p}+\frac{1}{p'}=1$, $C(p)=\big(p/(p-1)\big)^p=(p')^p$ *is a constant depending only on* p, *and*

$$(\phi\Lambda^{\frac{1}{p'}})_I=\frac{1}{|I|}\sum_{K\subseteq I}\phi(K)\lambda_K^{\frac{1}{p'}},\quad (\Lambda^p)_{I_0}=\frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I^p.$$

We will prove this theorem using the Bellman function method.

2.1.2 Bellman function for Hardy's inequality

Let $p\in\mathbb{R}$, $1<p<+\infty$. We consider the function

$$\mathcal{B}(F,f,A,v)=\left(\frac{p}{p-1}\right)^pF-\frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}} \tag{2.3}$$

defined over the domain

$$\mathcal{D}:=\left\{(F,f,A,v)\in\mathbb{R}^4 \;\middle|\; F>0,\ f>0,\ A>0,\ v>0,\ v\ge A,\ f^p\le Fv^{p-1}\right\}.$$

Let us name $C(p)=\big(p/(p-1)\big)^p$. The function $\mathcal{B}$ has the following properties:

1) $\mathcal{B}$ is a concave function defined over a convex domain.

2) $C(p)F\ge\mathcal{B}(F,f,A,v)\ge 0$.

A proof of these properties can be found in the appendix.

The next lemma is about the main inequality, which will be the key to prove the dyadic Hardy's inequality.

Lemma 2.1.2. *The function* $\mathcal{B}$ *satisfies*

$$\mathcal{B}(F,f,A,v)-\frac{1}{2}\Big[\mathcal{B}(F_-,f_-,A_-,v_-)+\mathcal{B}(F_+,f_+,A_+,v_+)\Big]\ge p^p\frac{f^p}{(A+v(p-1))^p}c, \tag{2.4}$$

which, by using the fact that $v\ge A$, *entails*

$$\mathcal{B}(F,f,A,v)-\frac{1}{2}\Big[\mathcal{B}(F_-,f_-,A_-,v_-)+\mathcal{B}(F_+,f_+,A_+,v_+)\Big]\ge\frac{f^p}{v^p}c. \tag{2.5}$$

where the inequality holds for all

$$F=\tilde F+b^p,\qquad f=\tilde f+ab,$$
$$v=\tilde v+a^{p'},\qquad A=\tilde A+c,$$

and

$$\tilde{F} = \frac{1}{2}(F_- + F_+), \qquad \tilde{f} = \frac{1}{2}(f_- + f_+),$$
$$\tilde{v} = \frac{1}{2}(v_- + v_+), \qquad \tilde{A} = \frac{1}{2}(A_- + A_+),$$

for every choice of $a \geq 0$, $b \geq 0$, $c \geq 0$. Here p' is the real number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. We start by considering the telescopic sum

$$\begin{aligned} \mathcal{B}(F, f, A, v) - \mathcal{B}(\tilde{F}, \tilde{f}, A - c, \tilde{v}) &= \mathcal{B}(F, f, A, v) - \mathcal{B}(F, f, A - c, v) + \\ &\quad \mathcal{B}(F, f, A - c, v) - \mathcal{B}(\tilde{F}, \tilde{f}, A - c, \tilde{v}). \end{aligned} \tag{2.6}$$

Since the function $\mathcal{B}$ is concave and differentiable over a convex domain, we recall that a concave differentiable function's values are lower or equal to the values of any of its tangent hyperplanes. This entails that, for every g concave and differentiable, for every choice of x, x^* in the domain of the function g

$$g(x) - g(x^*) \leq \sum_{i=1}^{4} \frac{\partial g(x^*)}{dx_i}(x_i - x_i^*). \tag{2.7}$$

By changing the sign of (2.7) we get

$$g(x^*) - g(x) \geq \sum_{i=1}^{4} \frac{\partial g(x^*)}{dx_i}(x_i^* - x_i). \tag{2.8}$$

So, when $g = \mathcal{B}$, $x = (F, f, A, v)$, $x^* = (F, f, \tilde{A}, v) = (F, f, A-c, v)$, the inequality (2.8) becomes

$$\mathcal{B}(F, f, A, v) - \mathcal{B}(F, f, A - c, v) \geq p^p \frac{f^p}{(A + (p-1)v)^p} c. \tag{2.9}$$

By combining (2.9) with (2.6) we get

$$\mathcal{B}(F, f, A, v) - \mathcal{B}(\tilde{F}, \tilde{f}, A-c, \tilde{v}) \geq \mathcal{B}(F, f, A-c, v) - \mathcal{B}(\tilde{F}, \tilde{f}, A-c, \tilde{v}) + (p-1)p^p \frac{f^p}{(A + (p-1)v)^p} c. \tag{2.10}$$

Now we consider $g = \mathcal{B}$, $x = (\tilde{F}, \tilde{f}, A - c, \tilde{v})$, $x^* = (F, f, A - c, v)$, so the inequality (2.8) becomes

$$\begin{aligned} \mathcal{B}(F, f, A - c, v) - \mathcal{B}(\tilde{F}, \tilde{f}, A - c, \tilde{v}) \;&\geq\; \left(\frac{p}{p-1}\right)^p b^p - \frac{p^{p-1}}{p-1}\left(\frac{f}{A - c + (p-1)v}\right)^{p-1} ab + \\ &\quad (p-1)p^p \left(\frac{f}{A - c + (p-1)v}\right)^p a^{p'}. \end{aligned}$$

Now let
$$y=\frac{f}{A-c+(p-1)v}.$$
We observe that $y>0$ because $f>0$, $v>0$, $A-c>0$ by definition of the domain of $\mathcal{B}$. So the last inequality can be rewritten in the form
$$\mathcal{B}(F,f,A-c,v)-\mathcal{B}(\tilde{F},\tilde{f},A-c,\tilde{v})\geq\left(\frac{p}{p-1}\right)^p b^p-\frac{p^{p+1}}{p-1}y^{p-1}ab+(p-1)p^p y^p a^{p'}=:\phi(y).$$
Now we are going to prove that $\phi(y)\geq 0$ for all $y\geq 0$.
We observe that $\phi(y)=C(p)b^p\geq 0$ when $a=0$.
Now we assume $a>0$ and we compute the derivative of the function ϕ:
$$\phi'(y)=p^{p+1}y^{p-2}\Big((p-1)a^{p'}y-ab\Big).$$
So the derivative $\phi'(y)$ is such that $\phi'(y)\leq 0$ for $0\leq y\leq\frac{b}{(p-1)a^{p'-1}}$, and $\phi'(y)\geq 0$ for $y\geq\frac{b}{(p-1)a^{p'-1}}$, so $\tilde{y}=\frac{b}{(p-1)a^{p'-1}}$ is a point of absolute minimum for ϕ, so as long as $\phi(\tilde{y})\geq 0$ the inequality holds for all $y\geq 0$. So we compute
$$\begin{aligned}\phi(\tilde{y}) &= b^p-p\tilde{y}^{p-1}ab+(p-1)\tilde{y}^p a^{p'}=\\ &\left(\frac{p}{p-1}\right)^p b^p-\frac{p^{p+1}}{p-1}\left(\frac{b}{(p-1)a^{p'-1}}\right)^{p-1}ab+(p-1)p^p\left(\frac{b}{(p-1)a^{p'-1}}\right)^p a^{p'}=\\ &\left(\frac{p}{p-1}\right)^p b^p-\frac{p^{p+1}}{p-1}b^p\frac{1}{a^{pp'-p-p'}}+\frac{p^p}{(p-1)^{p-1}}b^p\frac{1}{a^{pp'-p-p'}}.\end{aligned}$$
Now we recall that
$$\frac{1}{p}+\frac{1}{p'}=1;\quad pp'=p+p'.$$
So we get
$$\begin{aligned}\phi(\tilde{y}) &= \left(\frac{p}{p-1}\right)^p b^p-\frac{p^{p+1}}{p-1}b^p+\frac{p^p}{(p-1)^{p-1}}b^p=\\ &b^p\left(\frac{p}{p-1}\right)^p\Big[1-p+p-1\Big]=0.\end{aligned}$$
So the inequality $\phi(y)\geq 0$ holds for all $y\geq 0$, for every choice $a\geq 0$, $b\geq 0$, so the inequality (2.10) becomes
$$\mathcal{B}(F,f,A,v)-\mathcal{B}(\tilde{F},\tilde{f},A-c,\tilde{v})\geq p^p\frac{f^p}{(A+(p-1)v)^p}c. \tag{2.11}$$
Now we observe that $(\tilde{F},\tilde{f},A-c,\tilde{v})=(\tilde{F},\tilde{f},\tilde{A},\tilde{v})=\frac{1}{2}((F_+,f_+,A_+,v_+)+(F_-,f_-,A_-,v_-))$, so for the last step we use the fact that $\mathcal{B}$ is concave and we get
$$\mathcal{B}(F,f,A,v)-\frac{1}{2}\Big[\mathcal{B}(F_+,f_+,A_+,v_+)+\mathcal{B}(F_-,f_-,A_-,v_-)\Big]\geq p^p\frac{f^p}{(A+(p-1)v)^p}c.$$

Finally, using the fact that $A \leq v$, we get the weaker version of the previous inequality

$$\mathcal{B}(F, f, A, v) - \frac{1}{2}\Big[\mathcal{B}(F_+, f_+, A_+, v_+) + \mathcal{B}(F_-, f_-, A_-, v_-)\Big] \geq \frac{f^p}{v^p}c.$$

□

2.1.3 Proof of the inequality

Now we will prove Theorem 2.1.1 using the Bellman function method.

Proof. Let $I \in \mathscr{D}(I_0)$, we denote by $I_- \in \mathscr{D}(I_0)$ and $I_+ \in \mathscr{D}(I_0)$ the two children of the node I.
For every $I \in \mathscr{D}(I_0)$ we define

$$\begin{aligned} I &\longmapsto v_I \in \mathbb{R}^+, \\ I &\longmapsto F_I \in \mathbb{R}^+, \\ I &\longmapsto f_I \in \mathbb{R}^+, \\ I &\longmapsto A_I \in \mathbb{R}^+, \end{aligned}$$

as follows:

$$\begin{aligned} v_I &:= (\Lambda)_I, \\ F_I &:= (\phi^p)_I, \\ f_I &:= (\phi\Lambda^{\frac{1}{p'}})_I, \\ A_I &:= \frac{1}{|I|}\sum_{K\subseteq I}\alpha_K(\Lambda)_K^p. \end{aligned}$$

Now we define

$$\begin{aligned} a_I &:= \left(\frac{\lambda_I}{|I|}\right)^{\frac{1}{p'}}, \\ b_I &:= \frac{\phi(I)}{|I|^{\frac{1}{p}}}, \\ c_I &:= \frac{\alpha_I(\Lambda)_I^p}{|I|}, \end{aligned}$$

so we get

$$\begin{aligned}
v_I &= \frac{1}{|I|}\lambda_I + \frac{1}{2}(V_{I_-} + V_{I_+}) = a_I^{p'} + \tilde{v}_I, \\
F_I &= \frac{1}{|I|}\phi(I)^p + \frac{1}{2}(F_{I_-} + F_{I_+}) = b_I^p + \tilde{F}_I, \\
f_I &= \frac{\phi(I)\lambda_I^{\frac{1}{p'}}}{|I|} + \frac{1}{2}(f_{I_-} + f_{I_+}) = a_I b_I + \tilde{f}_I, \\
A_I &= \frac{\alpha_I(\Lambda)_I^p}{|I|} + \frac{1}{2}(A_{I_-} + A_{I_+}) = c_I + \tilde{A}_I.
\end{aligned}$$

We observe that the hypothesis (2.1) entails $A_I \leq v_I$, and we also observe that, by applying Hölder's inequality to f_I, we get

$$\begin{aligned}
f_I &= \frac{1}{|I|}\sum_{K\subseteq I}\phi_I\lambda_Y^{\frac{1}{p'}} \leq \\
& \frac{1}{|I|^{\frac{1}{p}}}\Big(\sum_{K\subseteq I}\phi_I^p\Big)^{\frac{1}{p}}\frac{1}{|I|^{\frac{1}{p'}}}\Big(\sum_{K\subseteq I}\lambda_I\Big)^{\frac{1}{p'}} = \\
& = (\phi^p)_I^{\frac{1}{p}}(\Lambda)_I^{\frac{1}{p'}} = F_I^{\frac{1}{p}}v_I^{\frac{1}{p'}}.
\end{aligned}$$

So, for all choices of $\phi : \mathscr{D}(I_0) \to \mathbb{R}^+$, $\alpha : \mathscr{D}(I_0) \to \mathbb{R}^+$, $\Lambda : \mathscr{D}(I_0) \to \mathbb{R}^+$, $I \in \mathscr{D}(I_0)$, the points

$$x_I := (F_I, f_I, A_I, v_I), \quad x_{I_-} := (F_{I_-}, f_{I_-}, A_{I_-}, v_{I_-}), \quad x_{I_+} := (F_{I_+}, f_{I_+}, A_{I_+}, v_{I_+})$$

are elements of the domain of the function $\mathcal{B}$ defined in (2.3). So we can compute the value of the function $\mathcal{B}$ at x_I, x_{I_-} and x_{I_+}, for all $I \in \mathscr{D}(I_0)$. We observe that

$$\mathcal{B}(F_I, f_I, A_I, v_I) = \mathcal{B}(\tilde{F}_I + b_I^p, \tilde{f}_I + a_I b_I, \tilde{A}_I + c_I, \tilde{v}_I + a_I^{p'}),$$

where

$$\begin{aligned}
\tilde{F} = \tfrac{1}{2}(F_- + F_+), &\qquad \tilde{f} = \frac{1}{2}(f_- + f_+), \\
\tilde{v} = \tfrac{1}{2}(v_- + v_+), &\qquad \tilde{A} = \frac{1}{2}(A_- + A_+),
\end{aligned}$$

so we can apply lemma 2.1.2 to get

$$\begin{aligned}
|I|\frac{f_I^p}{v_I^p}c_I &\leq |I|\Big[\mathcal{B}(x_I) - \frac{1}{2}\Big(\mathcal{B}(x_{I_-}) + \mathcal{B}(x_{I_+})\Big)\Big], \\
|I|\frac{f_I^p}{(\lambda)_I^p}\frac{\alpha_I(\lambda)_I^p}{|I|} &\leq |I|\mathcal{B}(x_I) - |I_-|\mathcal{B}(x_{I_-}) - |I_+|\mathcal{B}(x_{I_+}), \\
\alpha_I f_I^p &\leq |I|\mathcal{B}(x_I) - |I_-|\mathcal{B}(x_{I_-}) - |I_+|\mathcal{B}(x_{I_+}).
\end{aligned}$$

Summing over all $I \in \mathscr{D}(I_0)$ and using the telescopic nature of the sum we get

$$\sum_{I \subseteq I_0} \alpha_I f_I^p \leq |I_0| \mathcal{B}(F_{I_0}, f_{I_0}, A_{I_0}, v_{I_0}) \leq |I_0| C(p) F_{I_0}. \tag{2.12}$$

Now we recall that $F_{I_0} = (\phi^p)_{I_0}$ and $f_I = (\phi \Lambda^{\frac{1}{p'}})_I$, so we get

$$\frac{1}{|I_0|} \sum_{I \subseteq I_0} \alpha_I (\phi \Lambda^{\frac{1}{p'}})_I^p \leq \left(\frac{p}{p-1} \right)^p (\phi^p)_{I_0},$$

which is the thesis (2.2), ending the proof. □

2.2 Sharpness of the constant

In this section we prove that the constant

$$C(p) := (p')^p = \left(\frac{p}{p-1} \right)^p$$

is sharp for Theorem 2.1.1.

2.2.1 From dyadic inequality to real function

In this subsection we define a new function

$$B : \mathcal{D} \longrightarrow \mathbb{R}$$

such that B satisfies the main inequality (2.5). The tecnique used to define B is a standard technique often used in the theory of the Bellman function method.

Let $I \in \mathscr{D}(I_0)$. We consider the function

$$B(F, f, A, v) = \frac{1}{|I|} \sup \left\{ \sum_{J \subseteq I} \alpha_J \left(\frac{1}{|J|} \sum_{K \subseteq J} \phi(K) \lambda_K^{\frac{1}{p'}} \right)^p \,\middle|\, \alpha, \phi, \Lambda \text{ satisfy } (\mathrm{i})_I, (\mathrm{ii})_I, (\mathrm{iii})_I, \right. \tag{2.13}$$

$$\left. (\mathrm{iv})_I \text{ in } (F, f, A, v) \right\}, \tag{2.14}$$

where the notation α, ϕ, Λ satisfy $(\mathrm{i})_I$, $(\mathrm{ii})_I$, $(\mathrm{iii})_I$, $(\mathrm{iv})_I$ in (F, f, A, v) means that

$$\alpha : I \longmapsto \alpha_I \in \mathbb{R}^+,$$
$$\Lambda : I \longmapsto \lambda_I \in \mathbb{R}^+,$$
$$\phi : I \longmapsto \phi(I) \in \mathbb{R}^+,$$

and

(i)$_I$
$$F = \frac{1}{|I|}\sum_{J\subseteq I}\phi(J)^p,$$

(ii)$_I$
$$f = \frac{1}{|I|}\sum_{J\subseteq I}\phi(J)\lambda_J^{\frac{1}{p'}},$$

(iii)$_I$
$$A = \frac{1}{|I|}\sum_{J\subseteq I}\alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\lambda_K\right)^p,$$

(iv)$_I$
$$v = \frac{1}{|I|}\sum_{J\subseteq I}\lambda_J.$$

We observe that the function B does not depend on the choice of the interval I, and it can be proved by rescaling the weights α, the function ϕ and the measure Λ.

Lemma 2.2.1. *The function B is well defined over the domain $\mathcal{D}$, and it satisfies the main inequality (2.5).*

Proof. Given any fixed choice of $\tilde{\alpha}$, $\tilde{\phi}$, $\tilde{\Lambda}$ and $I \in \mathscr{D}(I_0)$, we consider the points

$$\begin{cases} F = \frac{1}{|I|}\sum\limits_{J\subseteq I}\tilde{\phi}(J)^p \\ f = \frac{1}{|I|}\sum\limits_{J\subseteq I}\tilde{\phi}(J)\tilde{\lambda}_J^{\frac{1}{p'}} \\ A = \frac{1}{|I|}\sum\limits_{J\subseteq I}\tilde{\alpha}_J\left(\frac{1}{|J|}\sum\limits_{K\subseteq J}\tilde{\lambda}_K\right)^p \\ v = \frac{1}{|I|}\sum\limits_{J\subseteq I}\tilde{\lambda}_J \end{cases} \tag{2.15}$$

$$\begin{cases} F_+ = \frac{1}{|I_+|}\sum\limits_{J\subseteq I_+}\tilde{\phi}(J)^p \\ f_+ = \frac{1}{|I_+|}\sum\limits_{J\subseteq I_+}\tilde{\phi}(J)\tilde{\lambda}_J^{\frac{1}{p'}} \\ A_+ = \frac{1}{|I_+|}\sum\limits_{J\subseteq I_+}\tilde{\alpha}_J\left(\frac{1}{|J|}\sum\limits_{K\subseteq J}\tilde{\lambda}_K\right)^p \\ v_+ = \frac{1}{|I_+|}\sum\limits_{J\subseteq I_+}\tilde{\lambda}_J \end{cases} \tag{2.16}$$

$$\begin{cases} F_- = \frac{1}{|I_-|}\sum\limits_{J\subseteq I_-}\tilde{\phi}(J)^p \\ f_- = \frac{1}{|I_-|}\sum\limits_{J\subseteq I_-}\tilde{\phi}(J)\tilde{\lambda}_J^{\frac{1}{p'}} \\ A_- = \frac{1}{|I_-|}\sum\limits_{J\subseteq I_-}\tilde{\alpha}_J\left(\frac{1}{|J|}\sum\limits_{K\subseteq J}\tilde{\lambda}_K\right)^p \\ v_- = \frac{1}{|I_-|}\sum\limits_{J\subseteq I_-}\tilde{\lambda}_J \end{cases} . \tag{2.17}$$

We observe that these points satisfy the following equations:

$$F = \tfrac{1}{2}(F_- + F_+) + \frac{\tilde{\phi}(I)^p}{|I|}, \qquad f = \frac{1}{2}(f_- + f_+) + \frac{\tilde{\phi}(I)\tilde{\lambda}_I^{\frac{1}{p'}}}{|I|},$$

$$A = \tfrac{1}{2}(A_- + A_+) + \frac{\tilde{\alpha}_I\left(\frac{1}{|I|}\sum\limits_{J\subseteq I}\tilde{\lambda}_J\right)^p}{|I|}, \qquad v = \frac{1}{2}(v_- + v_+) + \frac{\tilde{\lambda}_I}{|I|}.$$

By setting

$$b = \frac{\tilde{\phi}(I)}{|I|^{\frac{1}{p}}}, \quad c = \frac{\tilde{\alpha}_I\left(\frac{1}{|I|}\sum\limits_{J\subseteq I}\tilde{\lambda}_J\right)^p}{|I|}, \quad a = \frac{\tilde{\lambda}_I^{\frac{1}{p'}}}{|I|^{\frac{1}{p'}}}, \tag{2.18}$$

we get the following equations for the previous points:

$$\begin{aligned} F &= \tfrac{1}{2}(F_- + F_+) + b^p, \qquad f = \tfrac{1}{2}(f_- + f_+) + ab, \\ A &= \tfrac{1}{2}(A_- + A_+) + c, \qquad v = \tfrac{1}{2}(v_- + v_+) + a^{p'}. \end{aligned} \tag{2.19}$$

We are going to compute

$$B(F, f, A, v) \geq \frac{1}{|I|}\sum_{J\subseteq I}\tilde{\alpha}_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\tilde{\phi}(K)\tilde{\lambda}_K^{\frac{1}{p'}}\right)^p,$$

which gives us

$$\begin{aligned} B(F, f, A, v) \;\; \geq \;\; & \frac{1}{|I|}\tilde{\alpha}_I\left(\frac{1}{|I|}\sum_{J\subseteq I}\tilde{\phi}(J)\tilde{\lambda}_J^{\frac{1}{p'}}\right)^p + \\ & \frac{1}{|I|}\sum_{J\in\mathscr{D}(I_+)}\tilde{\alpha}_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\tilde{\phi}(K)\tilde{\lambda}_K^{\frac{1}{p'}}\right)^p + \\ & \frac{1}{|I|}\sum_{J\in\mathscr{D}(I_-)}\tilde{\alpha}_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\tilde{\phi}(K)\tilde{\lambda}_K^{\frac{1}{p'}}\right)^p. \end{aligned}$$

Now we observe that

$$\frac{\tilde{\alpha}_I}{|I|} = \frac{c}{v^p},$$

so the previous inequality becomes

$$B(F,f,A,v) \geq \frac{f^p}{v^p}c + \frac{1}{|I|}\sum_{J\in\mathscr{D}(I_+)} \tilde{\alpha}_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\tilde{\phi}(K)\tilde{\lambda}_K^{\frac{1}{p'}}\right)^p + \frac{1}{|I|}\sum_{J\in\mathscr{D}(I_-)} \tilde{\alpha}_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\tilde{\phi}(K)\tilde{\lambda}_K^{\frac{1}{p'}}\right)^p. \tag{2.20}$$

By construction, $\tilde{\alpha}$, $\tilde{\varphi}$ and $\tilde{\Lambda}$ satisfy the conditions $(\mathrm{i})_{I_+}$, $(\mathrm{ii})_{I_+}$, $(\mathrm{iii})_{I_+}$ and $(\mathrm{iv})_{I_+}$ in (F_+, f_+, A_+, v_+), and satisfy the conditions $(\mathrm{i})_{I_-}$, $(\mathrm{ii})_{I_-}$, $(\mathrm{iii})_{I_-}$ and $(\mathrm{iv})_{I_-}$ in (F_-, f_-, A_-, v_-).
Moreover, for any choice of α, φ and Λ such that

- α, φ and Λ satisfy the conditions $(\mathrm{i})_I$, $(\mathrm{ii})_I$, $(\mathrm{iii})_I$ and $(\mathrm{iv})_I$ in (F, f, A, v),
- α, φ and Λ satisfy the conditions $(\mathrm{i})_{I_+}$, $(\mathrm{ii})_{I_+}$, $(\mathrm{iii})_{I_+}$ and $(\mathrm{iv})_{I_+}$ in (F_+, f_+, A_+, v_+),
- α, φ and Λ satisfy the conditions $(\mathrm{i})_{I_-}$, $(\mathrm{ii})_{I_-}$, $(\mathrm{iii})_{I_-}$ and $(\mathrm{iv})_{I_-}$ in (F_-, f_-, A_-, v_-),

the following inequality holds:

$$B(F,f,A,v) \geq \frac{f^p}{v^p}c + \frac{1}{|I|}\sum_{J\in\mathscr{D}(I_+)} \alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\phi(K)\lambda_K^{\frac{1}{p'}}\right)^p + \frac{1}{|I|}\sum_{J\in\mathscr{D}(I_-)} \alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\phi(K)\lambda_K^{\frac{1}{p'}}\right)^p. \tag{2.21}$$

So, by taking the supremum over all α, φ and Λ for both the second and the third addend on the right hand side (using the fact that α, φ and Λ can be "independently" defined over $\mathscr{D}(I_+)$, $\mathscr{D}(I_-)$ and I) in the inequality (2.21), we get

$$\begin{aligned} B(F,f,A,v) &\geq \frac{f^p}{v^p}c \\ &+ \frac{1}{|I|}\sup\Bigg\{\sum_{J\subseteq I_+}\alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\phi(K)\lambda_K^{\frac{1}{p'}}\right)^p \,\Bigg|\, \alpha,\phi,\Lambda \text{ satisfy } (\mathrm{i})_{I_+}, (\mathrm{ii})_{I_+}, (\mathrm{iii})_{I_+}, \\ &\qquad\qquad (\mathrm{iv})_{I_+} \text{ in } (F_+, f_+, A_+, v_+)\Bigg\} \\ &+ \frac{1}{|I|}\sup\Bigg\{\sum_{J\subseteq I_-}\alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\phi(K)\lambda_K^{\frac{1}{p'}}\right)^p \,\Bigg|\, \alpha,\phi,\Lambda \text{ satisfy } (\mathrm{i})_{I_-}, (\mathrm{ii})_{I_-}, (\mathrm{iii})_{I_-}, \\ &\qquad\qquad (\mathrm{iv})_{I_-} \text{ in } (F_-, f_-, A_-, v_-)\Bigg\}. \end{aligned}$$

Using the definition of the function B and the fact that $|I| = 2|I_+| = 2|I_-|$ we get

$$B(F,f,A,v) \geq f^p c + \frac{1}{2}\Big[B(F_+, f_+, A_+, v_+) + B(F_-, f_-, A_-, v_-)\Big].$$

The proof is completed by showing that, for any choice of (F, f, A, v), (F_+, f_+, A_+, v_+) and (F_-, f_-, A_-, v_-) in the domain of the main inequality (2.5), there exists a choice of α, ϕ and

Λ such that the points (F, f, A, v), (F_+, f_+, A_+, v_+) and (F_-, f_-, A_-, v_-) satisfy the equations (2.15), (2.16) and (2.17).

Let (F, f, A, v), (F_+, f_+, A_+, v_+) and (F_-, f_-, A_-, v_-) be points in $\mathcal{D}$ such that they satisfy (2.19).

If we show that, for any point $(\tilde{F}, \tilde{f}, \tilde{A}, \tilde{v}) \in \mathcal{D}$ and for any interval $I \in \mathscr{D}(I_0)$, there exists a choice of α, ϕ and Λ such that

$$\begin{cases} \tilde{F} = \frac{1}{|I|} \sum\limits_{J \subseteq I} \phi(J)^p, \\ \tilde{f} = \frac{1}{|I|} \sum\limits_{J \subseteq I} \phi(J) \lambda_J^{\frac{1}{p'}}, \\ \tilde{A} = \frac{1}{|I|} \sum\limits_{J \subseteq I} \alpha_J \left(\frac{1}{|J|} \sum\limits_{K \subseteq J} \lambda_K \right)^p, \\ \tilde{v} = \frac{1}{|I|} \sum\limits_{J \subseteq I} \lambda_J, \end{cases} \tag{2.22}$$

then the proof is complete.

This is true because, in that case, given $I \in \mathscr{D}(I_0)$ we can choose α^+, ϕ^+, Λ^+ such that

$$\begin{cases} F_+ = \frac{1}{|I_+|} \sum\limits_{J \subseteq I_+} \phi^+(J)^p, \\ f_+ = \frac{1}{|I_+|} \sum\limits_{J \subseteq I_+} \phi^+(J) (\lambda_J^+)^{\frac{1}{p'}}, \\ A_+ = \frac{1}{|I_+|} \sum\limits_{J \subseteq I_+} \alpha_J^+ \left(\frac{1}{|J|} \sum\limits_{K \subseteq J} \lambda_K^+ \right)^p, \\ v_+ = \frac{1}{|I_+|} \sum\limits_{J \subseteq I_+} \lambda_J^+, \end{cases}$$

and α^-, ϕ^-, Λ^- such that

$$\begin{cases} F_- = \frac{1}{|I_-|} \sum\limits_{J \subseteq I_-} \phi^-(J)^p, \\ f_- = \frac{1}{|I_-|} \sum\limits_{J \subseteq I_-} \phi^-(J) (\lambda_J^-)^{\frac{1}{p'}}, \\ A_- = \frac{1}{|I_-|} \sum\limits_{J \subseteq I_-} \alpha_J^- \left(\frac{1}{|J|} \sum\limits_{K \subseteq J} \lambda_K^- \right)^p, \\ v_- = \frac{1}{|I_-|} \sum\limits_{J \subseteq I_-} \lambda_J^-. \end{cases}$$

We can now define

$$\lambda_J = \begin{cases} \lambda_J^+ & \text{if } J \in \mathscr{D}(I_+), \\ \lambda_J^- & \text{if } J \in \mathscr{D}(I_-), \\ |I| a^{p'} & \text{if } J = I, \\ 1 & \text{otherwise}, \end{cases}$$

$$\phi(J) = \begin{cases} \phi^+(J) & \text{if } J \in \mathscr{D}(I_+), \\ \phi^-(J) & \text{if } J \in \mathscr{D}(I_-), \\ |I|^{\frac{1}{p}} b & \text{if } J = I, \\ 1 & \text{otherwise,} \end{cases}$$

$$\alpha_J = \begin{cases} \alpha_J^+ & \text{if } J \in \mathscr{D}(I_+), \\ \alpha_J^- & \text{if } J \in \mathscr{D}(I_-), \\ \frac{|I|c}{\left(\frac{1}{|I|}\sum_{J\subseteq I}\tilde{\lambda}_J\right)^p} & \text{if } J = I, \\ 1 & \text{otherwise.} \end{cases}$$

The maps α, ϕ, Λ defined in this way are such that the points (F, f, A, v), (F_+, f_+, A_+, v_+) and (F_-, f_-, A_-, v_-) satisfy the equations (2.15), (2.16) and (2.17), which is the claim.
So, to finish the proof, all that is left to do is to prove that, for any point $(\tilde{F}, \tilde{f}, \tilde{A}, \tilde{v}) \in \mathcal{D}$ and for any interval $I \in \mathscr{D}(I_0)$, there exists a choice of α, ϕ and Λ such that (2.22) holds.
Let $(F, f, A, v) \in \mathcal{D}$ be arbitrary. We are now going to show that there exist

$$\begin{aligned} I &\longmapsto \alpha_I \in \mathbb{R}^+, \\ I &\longmapsto \lambda_I \in \mathbb{R}^+, \\ I &\longmapsto \phi(I) \in \mathbb{R}^+, \end{aligned}$$

such that the hypothesis

$$\frac{1}{|I|}\sum_{K\subseteq I}\alpha_K\left(\frac{1}{|K|}\sum_{J\subseteq K}\lambda_J\right)^p \le \frac{1}{|I|}\sum_{K\subseteq I}\lambda_I < +\infty \quad \forall I \in \mathscr{D}(I_0) \tag{2.23}$$

holds, and such that

$$\begin{aligned} F &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)^p, \qquad (2.24) \\ f &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)\lambda_I^{\frac{1}{p'}}, \\ A &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\alpha_I\left(\frac{1}{|I|}\sum_{K\subseteq I}\lambda_K\right)^p, \\ v &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I. \end{aligned}$$

Let us define the parameters

$$P_1 := \frac{f}{F^{\frac{1}{p}} v^{\frac{1}{p'}}}, \quad P_2 := \frac{A}{v}. \tag{2.25}$$

By the definition of $\mathcal{D}$ we have $v \geq A$, $f^p \leq F v^{p-1}$, so we get

$$0 < P_1 \leq 1, \quad 0 < P_2 \leq 1. \tag{2.26}$$

Let $\phi_0 > 0$, $\lambda_0 > 0$, $x_1 \in \left(0, \frac{1}{2}\right)$, $x_2 \in \left(0, \frac{1}{2^p}\right)$ to be chosen later. We define

$$\begin{aligned} \lambda_I &:= \lambda_0 \cdot |I|^{\log_{\frac{1}{2}}(x_1)}, \\ \phi(I) &:= \phi_0 \cdot |I|^{\log_{\frac{1}{2}}(x_2)}. \end{aligned} \tag{2.27}$$

Let us compute the following expressions:

$$\begin{aligned} \frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)^p &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}{\phi_0}^p \cdot |I|^{p\log_{\frac{1}{2}}(x_2)} = \\ & \frac{{\phi_0}^p}{|I_0|}\sum_{n=0}^{+\infty}\left(|I_0|\frac{1}{2^n}\right)^{p\log_{\frac{1}{2}}(x_2)} \cdot 2^n = \\ & {\phi_0}^p\,|I_0|^{p\log_{\frac{1}{2}}(x_2)-1}\sum_{n=0}^{+\infty}\left(\frac{1}{2^n}\right)^{\log_{\frac{1}{2}}({x_2}^p)} \cdot 2^n = \\ & {\phi_0}^p\,|I_0|^{\log_{\frac{1}{2}}(2{x_2}^p)}\sum_{n=0}^{+\infty}(2{x_2}^p)^n = \\ & {\phi_0}^p\,|I_0|^{\log_{\frac{1}{2}}(2{x_2}^p)}\frac{1}{1-2{x_2}^p}. \end{aligned} \tag{2.28}$$

$$\begin{aligned} \frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_0 \cdot |I|^{\log_{\frac{1}{2}}(x_1)} = \\ & \frac{\lambda_0}{|I_0|}\sum_{n=0}^{+\infty}\left(|I_0|\frac{1}{2^n}\right)^{\log_{\frac{1}{2}}(x_1)} \cdot 2^n = \\ & \lambda_0\,|I_0|^{\log_{\frac{1}{2}}(x_1)-1}\sum_{n=0}^{+\infty}\left(\frac{1}{2^n}\right)^{\log_{\frac{1}{2}}(x_1)} \cdot 2^n = \\ & \lambda_0\,|I_0|^{\log_{\frac{1}{2}}(2x_1)}\sum_{n=0}^{+\infty}(2x_1)^n = \\ & \lambda_0\,|I_0|^{\log_{\frac{1}{2}}(2x_1)}\frac{1}{1-2x_1}. \end{aligned} \tag{2.29}$$

$$\begin{aligned}\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)\lambda_I^{\frac{1}{p'}} &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi_0{\lambda_0}^{\frac{1}{p'}}\cdot|I|^{\frac{1}{p'}\log_{\frac{1}{2}}(x_1)+\log_{\frac{1}{2}}(x_2)} = \\ & \frac{\phi_0{\lambda_0}^{\frac{1}{p'}}}{|I_0|}\sum_{n=0}^{+\infty}\left(|I_0|\frac{1}{2^n}\right)^{\frac{1}{p'}\log_{\frac{1}{2}}(x_1)+\log_{\frac{1}{2}}(x_2)}\cdot 2^n = \\ & \phi_0{\lambda_0}^{\frac{1}{p'}}\,|I_0|^{\log_{\frac{1}{2}}\left(x_1{}^{\frac{1}{p'}}x_2\right)-1}\sum_{n=0}^{+\infty}\left(\frac{1}{2^n}\right)^{\log_{\frac{1}{2}}\left(x_1{}^{\frac{1}{p'}}x_2\right)}\cdot 2^n = \\ & \phi_0{\lambda_0}^{\frac{1}{p'}}\,|I_0|^{\log_{\frac{1}{2}}\left(2x_1{}^{\frac{1}{p'}}x_2\right)}\sum_{n=0}^{+\infty}\left(x_1{}^{\frac{1}{p'}}x_2\right)^n\cdot 2^n = \\ & \phi_0{\lambda_0}^{\frac{1}{p'}}\,|I_0|^{\log_{\frac{1}{2}}\left(2x_1{}^{\frac{1}{p'}}x_2\right)}\frac{1}{1-2x_1{}^{\frac{1}{p'}}x_2}.\end{aligned} \tag{2.30}$$

We observe that, since we have $I_0>0$, $x_1\in\left(0,\frac{1}{2}\right)$, $x_2\in\left(0,\frac{1}{2^p}\right)$, we get

$$0<|I_0|^{\log_{\frac{1}{2}}(2x_2{}^p)}\frac{1}{1-2x_2{}^p}<+\infty,\quad 0<|I_0|^{\log_{\frac{1}{2}}(2x_1)}\frac{1}{1-2x_1}<+\infty. \tag{2.31}$$

Now we choose

$$\phi_0:=\left(\frac{F}{|I_0|^{\log_{\frac{1}{2}}(2x_2{}^p)}\frac{1}{1-2x_2{}^p}}\right)^{\frac{1}{p}}, \tag{2.32}$$

$$\lambda_0:=\frac{v}{|I_0|^{\log_{\frac{1}{2}}(2x_1)}\frac{1}{1-2x_1}}. \tag{2.33}$$

From (2.28) and (2.32) we get

$$\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)^p=F, \tag{2.34}$$

from (2.29) and (2.33) we get

$$\frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I=v. \tag{2.35}$$

Now we are going to choose $x_1\in\left(0,\frac{1}{2}\right)$, $x_2\in\left(0,\frac{1}{2^p}\right)$ such that

$$\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)\lambda_I^{\frac{1}{p'}}=f.$$

The last equation is equivalent to

$$\frac{\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)\lambda_I^{\frac{1}{p'}}}{F^{\frac{1}{p}}v^{\frac{1}{p'}}}=\frac{f}{F^{\frac{1}{p}}v^{\frac{1}{p'}}}=P_1,$$

so, since we have (2.34) and (2.35), we are going find $x_1 \in \left(0, \frac{1}{2}\right)$, $x_2 \in \left(0, \frac{1}{2^p}\right)$ such that

$$\frac{\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)\lambda_I^{\frac{1}{p'}}}{\left(\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)^p\right)^{\frac{1}{p}}\left(\frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I\right)^{\frac{1}{p'}}} = P_1, \tag{2.36}$$

for any arbitrary value $0 < P_1 \leq 1$.
We compute (2.36) using (2.28), (2.29) and (2.30) to get

$$\begin{aligned} P_1 &= \frac{\phi_0\lambda_0{}^{\frac{1}{p'}}\left|I_0\right|^{\log_{\frac{1}{2}}\left(2x_1{}^{\frac{1}{p'}}x_2\right)}\frac{1}{1-2x_1{}^{\frac{1}{p'}}x_2}}{\left(\phi_0{}^p\left|I_0\right|^{\log_{\frac{1}{2}}(2x_2{}^p)}\frac{1}{1-2x_2{}^p}\right)^{\frac{1}{p}}\left(\lambda_0\left|I_0\right|^{\log_{\frac{1}{2}}(2x_1)}\frac{1}{1-2x_1}\right)^{\frac{1}{p'}}} = \\ & \frac{\phi_0\lambda_0{}^{\frac{1}{p'}}}{\phi_0\lambda_0{}^{\frac{1}{p'}}}\frac{\left|I_0\right|^{\log_{\frac{1}{2}}\left(2x_1{}^{\frac{1}{p'}}x_2\right)}}{\left|I_0\right|^{\frac{1}{p}\log_{\frac{1}{2}}(2x_2{}^p)}\left|I_0\right|^{\frac{1}{p'}\log_{\frac{1}{2}}(2x_1)}}\frac{(1-2x_1)^{\frac{1}{p'}}(1-2x_2{}^p)^{\frac{1}{p}}}{1-2x_1{}^{\frac{1}{p'}}x_2} = \\ & \frac{\left|I_0\right|^{\log_{\frac{1}{2}}\left(2x_1{}^{\frac{1}{p'}}x_2\right)}}{\left|I_0\right|^{\log_{\frac{1}{2}}\left(2^{\left(\frac{1}{p}+\frac{1}{p'}\right)}x_1{}^{\frac{1}{p'}}x_2\right)}}\frac{(1-2x_1)^{\frac{1}{p'}}(1-2x_2{}^p)^{\frac{1}{p}}}{1-2x_1{}^{\frac{1}{p'}}x_2} = \\ & \frac{(1-2x_1)^{\frac{1}{p'}}(1-2x_2{}^p)^{\frac{1}{p}}}{1-2x_1{}^{\frac{1}{p'}}x_2}. \end{aligned} \tag{2.37}$$

Let us define the function

$$g : \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2^p}\right) \longrightarrow \mathbb{R}, \tag{2.38}$$

$$g(x_1, x_2) := \frac{(1-2x_1)^{\frac{1}{p'}}(1-2x_2{}^p)^{\frac{1}{p}}}{1-2x_1{}^{\frac{1}{p'}}x_2}. \tag{2.39}$$

To prove that for any arbitrary $0 < P_1 \leq 1$ there exist $x_1 \in \left(0, \frac{1}{2}\right)$, $x_2 \in \left(0, \frac{1}{2^p}\right)$ such that (2.36) is satisfied, we are going to prove that

$$(0, 1] \subseteq \left\{ g(x_1, x_2) \;\middle|\; x_1 \in \left(0, \frac{1}{2}\right), x_2 \in \left(0, \frac{1}{2^p}\right) \right\}. \tag{2.40}$$

Let $x_1 \in \left(0, \frac{1}{2}\right)$, take $x_2 = x_1{}^{\frac{1}{p}}$. Then

$$g(x_1, x_2) = g(x_1, x_1{}^{\frac{1}{p}}) = \frac{(1-2x_1)^{\frac{1}{p'}}(1-2x_1)^{\frac{1}{p}}}{1-2x_1{}^{\frac{1}{p'}+\frac{1}{p}}} = 1. \tag{2.41}$$

To finish the proof we are going to show that

$$\liminf_{(x_1,x_2)\to\left(\frac{1}{2},\frac{1}{2^p}\right)} g(x_1,x_2) = 0. \tag{2.42}$$

To prove this let us consider

$$\epsilon := 1-2x_1; \quad \delta := 1-2x_2^p, \tag{2.43}$$

which gives us

$$x_1 = x_1(\epsilon) = \frac{1-\epsilon}{2}; \quad x_2 = x_2(\epsilon) = \left(\frac{1-\delta}{2}\right)^{\frac{1}{p}}. \tag{2.44}$$

So we have $\epsilon \in (0,1)$, $\delta \in (0,1)$, and (2.42) is equivalent to

$$\liminf_{(x_1,x_2)\to\left(\frac{1}{2},\frac{1}{2^p}\right)} g(x_1,x_2) = \liminf_{(\epsilon,\delta)\to(0,0)} \frac{\epsilon^{\frac{1}{p'}}\delta^{\frac{1}{p}}}{1-(1-\epsilon)^{\frac{1}{p'}}(1-\delta)^{\frac{1}{p}}} = 0. \tag{2.45}$$

Now, using big O notation, we consider the Taylor polynomial of degree 1 of the function $x \mapsto (1-x)^{\frac{1}{p}}$, i.e.

$$(1-x)^{\frac{1}{p}} = 1 - \frac{x}{p} + \underset{x\to 0}{O}(x^2). \tag{2.46}$$

So we get

$$g(x_1,x_2) = \frac{\epsilon^{\frac{1}{p'}}\delta^{\frac{1}{p}}}{1-\left(1-\frac{\epsilon}{p'}+\underset{\epsilon\to 0}{O}(\epsilon^2)\right)\left(1-\frac{\delta}{p}+\underset{\delta\to 0}{O}(\delta^2)\right)} = \tag{2.47}$$

$$\frac{\epsilon^{\frac{1}{p'}}\delta^{\frac{1}{p}}}{\frac{\epsilon}{p'}+\frac{\delta}{p}-\frac{\epsilon\delta}{pp'}-\underset{\epsilon\to 0}{O}(\epsilon^2)\left(1-\frac{\delta}{p}+\underset{\delta\to 0}{O}(\delta^2)\right)-\underset{\delta\to 0}{O}(\delta^2)\left(1-\frac{\epsilon}{p'}+\underset{\epsilon\to 0}{O}(\epsilon^2)\right)}. \tag{2.48}$$

Let $t>0$. Let us choose

$$\delta = \delta(\epsilon) := \epsilon^{1+t}. \tag{2.49}$$

Then we get

$$g(x_1,x_2) = \frac{\epsilon^{\frac{1}{p'}+\frac{1}{p}+\frac{t}{p}}}{\frac{\epsilon}{p'}+\frac{\epsilon^{1+t}}{p}-\frac{\epsilon^{1+\frac{1}{p}+\frac{1}{t}}}{pp'}-\underset{\epsilon\to 0}{O}(\epsilon^2)\left(1-\frac{\epsilon^{1+t}}{p}+\underset{\epsilon\to 0}{O}(\epsilon^{2+2t})\right)-\underset{\epsilon\to 0}{O}(\epsilon^{2+2t})\left(1-\frac{\epsilon}{p'}+\underset{\epsilon\to 0}{O}(\epsilon^2)\right)}. \tag{2.50}$$

Finally, we multiply numerator and denominator by ϵ, and we use big O notation properties, to get

$$g(x_1,x_2) = \frac{\epsilon^{\frac{t}{p}}}{\frac{1}{p'} + \frac{\epsilon^t}{p} - \frac{\epsilon^{\frac{1}{p}+\frac{1}{t}}}{pp'} - \underset{\epsilon\to 0}{O(\epsilon)}\left(1 - \frac{\epsilon^{1+t}}{p} + \underset{\epsilon\to 0}{O(\epsilon^{2+2t})}\right) - \underset{\epsilon\to 0}{O(\epsilon^{1+2t})}\left(1 - \frac{\epsilon}{p'} + \underset{\epsilon\to 0}{O(\epsilon^2)}\right)}. \tag{2.51}$$

The denominator converges to $\frac{1}{p'}$ as $\epsilon \to 0$, while the numerator converges to 0 as $\epsilon \to 0$, which proves that, under the previous choices, we have

$$g(x_1(\epsilon), x_2(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0. \tag{2.52}$$

Since $g(x_1, x_2) > 0$ by construction, this entails that

$$\liminf_{(x_1,x_2)\to\left(\frac{1}{2},\frac{1}{2^p}\right)} g(x_1,x_2) = 0. \tag{2.53}$$

The function g is a continuous function defined over the connected set $\{(x_1, x_2) \mid x_1 \in \left(0, \frac{1}{2}\right), x_2 \in \left(0, \frac{1}{2^p}\right)\}$, so, since we also proved (2.41), it follows that

$$(0,1] \subseteq \left\{ g(x_1,x_2) \,\middle|\, x_1 \in \left(0, \frac{1}{2}\right), x_2 \in \left(0, \frac{1}{2^p}\right) \right\}, \tag{2.54}$$

which is the equation we wanted to prove.
The last equation entails that (2.36) is satisfied for any $0 < P_1 \leq 1$, so we proved that for all $F > 0$, $f > 0$, $v > 0$ such that $f^p \leq F v^{p-1}$ then there exist $\lambda_0 > 0$, $\phi_0 > 0$, $0 < x_1 < \frac{1}{2}$, $0 < x_2 < \frac{1}{2^p}$ such that the maps

$$\begin{aligned} \lambda_I &:= \lambda_0 \cdot |I|^{\log_{\frac{1}{2}}(x_1)}, \\ \phi(I) &:= \phi_0 \cdot |I|^{\log_{\frac{1}{2}}(x_2)} \end{aligned} \tag{2.55}$$

satisfy the equations

$$\begin{aligned} F &= \frac{1}{|I_0|} \sum_{I \subseteq I_0} \phi(I)^p, \\ f &= \frac{1}{|I_0|} \sum_{I \subseteq I_0} \phi(I) \lambda_I^{\frac{1}{p'}}, \\ v &= \frac{1}{|I_0|} \sum_{I \subseteq I_0} \lambda_I. \end{aligned} \tag{2.56}$$

Now it is only left to prove that there exist

$$\alpha : \mathscr{D}(I_0) \longrightarrow \mathbb{R}^+ \tag{2.57}$$

such that we have

$$A = \frac{1}{|I_0|}\sum_{I\subseteq I_0}\alpha_I\left(\frac{1}{|I|}\sum_{K\subseteq I}\lambda_K\right)^p.$$

Given the previous definitions of λ_I and ϕ_I, let us consider

$$\begin{aligned} \alpha : \mathscr{D}(I_0) &\longrightarrow \mathbb{R}^+ \\ I &\longmapsto \alpha_I := P_2\cdot\lambda_I\left(\frac{1}{|I|}\sum_{K\subseteq I}\lambda_K\right)^{-p}, \end{aligned} \tag{2.58}$$

where P_2 is the parameter defined in (2.25). This expression is well defined because, by construction of λ_I, we have

$$\frac{1}{|I|}\sum_{K\subseteq I}\lambda_K > 0 \quad \forall I\in\mathscr{D}(I_0). \tag{2.59}$$

From this definition, for all $I\in\mathscr{D}(I_0)$, we get

$$\frac{1}{|I|}\sum_{J\subseteq I}\alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\lambda_K\right)^p = P_2\frac{1}{|I|}\sum_{J\subseteq I}\lambda_J \le \frac{1}{|I|}\sum_{J\subseteq I}\lambda_J < +\infty, \tag{2.60}$$

which means that the hypothesis (2.23) is satisfied. Moreover, for $I = I_0$, we get

$$\frac{1}{|I_0|}\sum_{I\subseteq I_0}\alpha_I\left(\frac{1}{|I|}\sum_{K\subseteq I}\lambda_K\right)^p = P_1\frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I = P_1\cdot v = A, \tag{2.61}$$

which is the required inequality. So we proved that for all $(F, f, A, v)\in\mathcal{D}$ there exist

$$\begin{aligned} &I\longmapsto \alpha_I\in\mathbb{R}^+,\\ &I\longmapsto \lambda_I\in\mathbb{R}^+,\\ &I\longmapsto \phi(I)\in\mathbb{R}^+, \end{aligned}$$

such that

$$\frac{1}{|I|}\sum_{K\subseteq I}\alpha_K\left(\frac{1}{|K|}\sum_{J\subseteq K}\lambda_J\right)^p \le \frac{1}{|I|}\sum_{K\subseteq I}\lambda_I < +\infty \quad \forall I\in\mathscr{D}(I_0)$$

holds, and such that

$$\begin{aligned} F &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)^p,\\ f &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)\lambda_I^{\frac{1}{p'}},\\ A &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\alpha_I\left(\frac{1}{|I|}\sum_{K\subseteq I}\lambda_K\right)^p,\\ v &= \frac{1}{|I_0|}\sum_{I\subseteq I_0}\lambda_I, \end{aligned}$$

ending the proof of the optimality of the domain $\mathcal{D}$. □

Since the function B satisfies the main inequality (2.5) it can be proved that the function B also satisfies the following infinitesimal inequalities:

3′. $d^2B \leq 0$,

3″. $\frac{\partial B}{\partial A} \geq \frac{f^p}{v^p}$.

Inequalities 3′ and 3″ use the notion of derivative from the theory of distributions. However, we will skip the technical details about the optimal regularity of the functions, and, during the remaining part of the work, we will assume that the function B has C^2 regularity.

2.2.2 Improvement of the "Bellman" type function

In this subsection we use the function B defined in (2.13) to define a new function $\hat{B}$ which satisfies the same properties, and it also satisfies an additional property which allows us to prove the sharpness of the constant $C(p)$.

In the following lemmas we are going to show that the characterization of the dual dyadic Hardy's inequality holds for a constant $K(p)$ if and only if there exists a "Bellman" type function $\hat{B}$ with the previous properties, and such that $\hat{B}$ can be written in the form

$$\hat{B}(F, f, A, v) = K(p)F - \frac{f^p}{\varphi(A, V)^{p-1}},$$

where φ is a linear function.

Lemma 2.2.2. *Let $1 < p < +\infty$ and $K(p) \in \mathbb{R}$. The following statements are equivalent:*

- *The characterization of the dual dyadic Hardy's inequality 2.1.1 holds for constant $K(p)$, i.e. if the inequality*

$$\frac{1}{|I|}\sum_{J\subseteq I}\alpha_J\left(\frac{1}{|J|}\sum_{K\subseteq J}\lambda_K\right)^p \leq \frac{1}{|I|}\sum_{J\subseteq I}\lambda_J < +\infty \quad \forall I \in \mathscr{D}(I_0) \tag{2.62}$$

  is satisfied, then

$$\frac{1}{|I_0|}\sum_{I\subseteq I_0}\alpha_I\left(\frac{1}{|I|}\sum_{J\subseteq I}\phi(J)\lambda_J^{\frac{1}{p'}}\right)^p \leq K(p)\frac{1}{|I_0|}\sum_{I\subseteq I_0}\phi(I)^p. \tag{2.63}$$

- *There exists a function*

$$\hat{B} : \mathcal{D} \longrightarrow \mathbb{R}$$

where

$$\mathcal{D} = \{(F, f, A, v) \in \mathbb{R}^4 \mid F > 0, f > 0, A > 0, v > 0, A \leq v, f^p \leq F v^{p-1}\}$$

such that

1. $\hat{B}$ *is defined over the domain* $\mathcal{D}$,
2. $K(p)F \geq \hat{B}(F, f, A, v) \geq 0$ *for all* $(F, f, A, v) \in \mathcal{D}$,

$3'$. $d^2\hat{B}(F, f, A, v) \leq 0$,

$3''$. $\frac{\partial \hat{B}}{\partial A}(F, f, A, v) \geq \frac{f^p}{v^p}$.

Moreover, the function $\hat{B}$ *may be written in the form*

$$\hat{B}(F, f, A, v) = \hat{B}^h(F, f, A, v) = K(p)F + f^p h(A, v). \tag{2.64}$$

Proof. Suppose there exists a function $\hat{B}$ that satisfies the properties 1, 2, $3'$ and $3''$.
By integration it follows that a function $\hat{B}$ that satisfies $3'$ and $3''$ also satisfies the main inequality (2.5), so the previous proof for the characterization dual dyadic Hardy's inequality holds using the Bellman function method and the function $\hat{B}$.
It is left to prove that, if the Theorem 2.1.1 holds for a constant $K(p)$, then a function $\hat{B}$ with the previous properties exists.

Let us assume that the characterization for the dual dyadic Hardy's inequality holds for constant a $K(p)$. Consider the function B defined in (2.13). The definition of B does not depend on the choice of the interval I, so we choose $I = I_0$. We observe that, by definition of the function B, we have $B(F, f, A, v) \geq 0$. We assumed that the characterization for the dual dyadic Hardy's inequality holds for the constant $K(p)$, so, since $B(F, f, A, V)$ is defined as the supremum of the left hand side of the thesis (2.63), where

$$F = \frac{1}{|I|}\sum_{J \subseteq I} \phi(J)^p, \qquad f = \frac{1}{|I|}\sum_{J \subseteq I} \phi(J)\lambda_J^{\frac{1}{p'}},$$

$$A = \frac{1}{|I|}\sum_{J \subseteq I} \alpha_J \left(\frac{1}{|J|}\sum_{K \subseteq J} \lambda_K\right)^p, \qquad v = \frac{1}{|I|}\sum_{J \subseteq I} \lambda_J,$$

we get $B(F, f, A, v) \leq K(p)F$. We also observed earlier that B satisfies $3'$ and $3''$. So we proved that B satisfies 1, 2, $3'$ and $3''$. We are now going to define the function

$$u(f, A, v) := \sup_F \left\{ B(F, f, A, v) - K(p)F \right\}.$$

Lemma 2.2.3. *Let $\phi(x,y)$ be a concave function, and let $\Phi(x) = \sup_y \phi(x,y)$. Then Φ is concave.*

The function B is concave because it satisfies $3'$, so the function u is concave. By homogeneity of B we get

$$B(a^p F, af, A, v) = a^p B(F, f, A, v) \quad \forall a > 0,$$

which entails

$$u(af, A, v) = a^p u(f, A, v) \quad \forall a > 0.$$

So u can be written in the form

$$u(f, A, v) = f^p h(A, v).$$

We are now going to consider the function

$$\hat{B}(F, f, A, v) := K(p)F + u(f, A, v),$$

which can be written in the form

$$\hat{B}(F, f, A, v) = \hat{B}^h(F, f, A, v) = K(p)F + f^p h(A, v).$$

By construction $0 \leq \hat{B}(F, f, A, v) \leq K(p)F$, $\hat{B}$ is concave and $\frac{\partial \hat{B}}{\partial A}(F, f, A, v) \geq \frac{f^p}{v^p}$. So the function $\hat{B}$ satisfies the properties 1, 2, $3'$ and $3''$ and it is written in the form (2.64), finishing the proof. $\square$

Lemma 2.2.4. *Let $1 < p < +\infty$ and $K(p) \in \mathbb{R}$. The following statements are equivalent:*

- *The characterization of the dual dyadic Hardy's inequality (2.1.1) holds for constant $K(p)$, i.e. if the inequality*

$$\frac{1}{|I|}\sum_{J \subseteq I} \alpha_J \left(\frac{1}{|J|}\sum_{K \subseteq J} \lambda_K\right)^p \leq \frac{1}{|I|}\sum_{J \subseteq I} \lambda_J < +\infty \quad \forall I \in \mathscr{D}(I_0) \tag{2.65}$$

  is satisfied, then

$$\frac{1}{|I_0|}\sum_{I \subseteq I_0} \alpha_I \left(\frac{1}{|I|}\sum_{J \subseteq I} \phi(J)\lambda_J^{\frac{1}{p'}}\right)^p \leq K(p)\frac{1}{|I_0|}\sum_{I \subseteq I_0} \phi(I)^p. \tag{2.66}$$

- *There exists a function*

$$\hat{B}_\varphi(F, f, A, v) = K(p)F - \frac{f^p}{\varphi(A, v)^{p-1}} \tag{2.67}$$

  such that

1. $\hat{B}_\varphi$ is defined over the domain

$$\mathcal{D} = \{(F, f, A, v) \in \mathbb{R}^4 \mid F > 0, f > 0, A > 0, v > 0, A \le v, f^p \le F v^{p-1}\},$$

$\mathring{2}$. $\frac{v^{p-1}}{\varphi(A,v)^{p-1}} \le K(p)$,

$\mathring{3}$. $\frac{\partial \varphi}{\partial A}(A, v) \ge \frac{1}{(p-1)v^p}\varphi(A, v)^p$,

$\mathring{4}$. $\varphi(A, v) > 0$,

$\mathring{5}$. *φ is linear.*

Proof. Let $K(p)$ be a constant. By Lemma 2.2.2 the characterization of the dual dyadic Hardy's inequality (2.1.1) holds for the constant $K(p)$ if and only if there exists a function

$$\hat{B}^h(F, f, A, v) = K(p)F + f^p h(A, v) \tag{2.68}$$

such that

1. $\hat{B}^h$ is defined over the domain

$$\mathcal{D} = \{(F, f, A, v) \in \mathbb{R}^4 \mid F > 0, f > 0, A > 0, v > 0, A \le v, f^p \le F v^{p-1}\}$$

2. $K(p)F \ge \hat{B}^h(F, f, A, v) \ge 0$ for all $(F, f, A, v) \in \mathcal{D}$

$3'$. $d^2\hat{B}^h(F, f, A, v) \le 0$

$3''$. $\frac{\partial \hat{B}^h}{\partial A}(F, f, A, v) \ge \frac{f^p}{v^p}$

The properties $3'$ and $3''$ hold for the function (2.68) if and only if

3^*. $d^2(f^p h(A, v)) \le 0$

3^{**}. $\frac{\partial h}{\partial A}(A, v) \ge \frac{1}{v^p}$

We are going to rewrite the condition 3^* in a better way.
Let us compute the charateristic polynomial of the Hessian matrix $\mathcal{H}(f^p h(A, v))$, i.e. the polynomial

$$P(\lambda)(f, A, V) = \det(\mathcal{H}(f^p h(A, v)) - \lambda I).$$

We are going to skip writing down the variables f, A and v in the following computations, to simplify the notation.
By computation, the charateristic polynomial is

$$P(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

where

$$\begin{aligned}
a_3 &= -1,\\
a_2 &= f^{p-2}\Big[f^2\Big[\frac{\partial^2 h}{\partial A^2}+\frac{\partial^2 h}{\partial v^2}\Big]+p(p-1)h\Big],\\
a_1 &= f^{2p-2}\Big[f^2\Big[p(p-1)\Big(\frac{\partial^2 h}{\partial A\partial v}\Big)^2-\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}\Big]+\\
&\quad p^2\Big[\Big(\frac{\partial h}{\partial A}\Big)^2+\Big(\frac{\partial h}{\partial v}\Big)^2\Big]-p(p-1)\Big[\frac{\partial^2 h}{\partial A^2}+\frac{\partial^2 h}{\partial v^2}\Big]h\Big],\\
a_0 &= f^{3p-2}\Big[p(p-1)\Big[\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}h-\Big(\frac{\partial^2 h}{\partial A\partial v}\Big)^2h\Big]+\\
&\quad p^2\Big[2\frac{\partial h}{\partial A}\frac{\partial h}{\partial v}\frac{\partial^2 h}{\partial A\partial v}-\Big[\Big(\frac{\partial h}{\partial v}\Big)^2\frac{\partial^2 h}{\partial A^2}+\Big(\frac{\partial h}{\partial A}\Big)^2\frac{\partial^2 h}{\partial v^2}\Big]\Big]\Big].
\end{aligned}$$

The condition 3^* holds if and only if all the roots of the polynomial $P(\lambda)$ are non-positive. Since $a_3 < 0$, a necessary condition for 3^* is

$$a_2 \le 0, \quad a_1 \le 0, \quad a_0 \le 0, \tag{2.69}$$

for all the points (f, A, V) such that $f > 0$, $0 < A < v$.
Since $f > 0$ it follows that the condition (2.69) is equivalent to

4. $f^2\Big[\frac{\partial^2 h}{\partial A^2}+\frac{\partial^2 h}{\partial v^2}\Big]+p(p-1)h \le 0,$
5. $f^2\Big[p(p-1)\Big(\frac{\partial^2 h}{\partial A\partial v}\Big)^2-\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}\Big]+p^2\Big[\Big(\frac{\partial h}{\partial A}\Big)^2+\Big(\frac{\partial h}{\partial v}\Big)^2\Big]-$
   $p(p-1)\Big[\frac{\partial^2 h}{\partial A^2}+\frac{\partial^2 h}{\partial v^2}\Big]h \le 0,$
6. $p(p-1)\Big[\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}h-\Big(\frac{\partial^2 h}{\partial A\partial v}\Big)^2h\Big]+$
   $p^2\Big[2\frac{\partial h}{\partial A}\frac{\partial h}{\partial v}\frac{\partial^2 h}{\partial A\partial v}-\Big[\Big(\frac{\partial h}{\partial v}\Big)^2\frac{\partial^2 h}{\partial A^2}+\Big(\frac{\partial h}{\partial A}\Big)^2\frac{\partial^2 h}{\partial v^2}\Big]\Big] \le 0.$

We observe that the conditions 4, 5 and 6 hold for all $f > 0$, so, by letting $f \to 0$, it follows that the conditions

$$
\begin{aligned}
&4'.\ h \le 0,\\
&5'.\ p\left[\left(\frac{\partial h}{\partial A}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2\right] - (p-1)\left[\frac{\partial^2 h}{\partial A^2} + \frac{\partial^2 h}{\partial v^2}\right]h \le 0,\\
&6.\ p(p-1)\left[\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}h - \left(\frac{\partial^2 h}{\partial A \partial v}\right)^2 h\right] +\\
&\quad p^2\left[2\frac{\partial h}{\partial A}\frac{\partial h}{\partial v}\frac{\partial^2 h}{\partial A \partial v} - \left[\left(\frac{\partial h}{\partial v}\right)^2\frac{\partial^2 h}{\partial A^2} + \left(\frac{\partial h}{\partial A}\right)^2\frac{\partial^2 h}{\partial v^2}\right]\right] \le 0.
\end{aligned}
$$

are necessary for 4, 5 and 6.
The function $\hat{B}^h$ defined in (2.68) is concave, so it follows that h is concave. Moreover, the the function h satisfies the condition 3^{**}, and $h \le 0$, so $h(A, v) < 0$ for all $0 < A < v$. So it follows that the conditions

$$
\begin{aligned}
&4''.\ h(A, v) < 0, \quad \text{for } 0 < A < v,\\
&5'.\ p\left[\left(\frac{\partial h}{\partial A}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2\right] - (p-1)\left[\frac{\partial^2 h}{\partial A^2} + \frac{\partial^2 h}{\partial v^2}\right]h \le 0,\\
&6.\ p(p-1)\left[\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}h - \left(\frac{\partial^2 h}{\partial A \partial v}\right)^2 h\right] +\\
&\quad p^2\left[2\frac{\partial h}{\partial A}\frac{\partial h}{\partial v}\frac{\partial^2 h}{\partial A \partial v} - \left[\left(\frac{\partial h}{\partial v}\right)^2\frac{\partial^2 h}{\partial A^2} + \left(\frac{\partial h}{\partial A}\right)^2\frac{\partial^2 h}{\partial v^2}\right]\right] \le 0.
\end{aligned}
$$

are necessary for $4'$, $5'$ and 6.
Since $4''$ holds, we define

$$\psi(A, v) = \left[-\frac{1}{h(A, v)}\right]^{\frac{1}{p-1}}, \quad \text{for } 0 < A < v, \tag{2.70}$$

so we get

$$h(A, v) = -\frac{1}{\psi(A, v)^{p-1}}, \quad \text{for } 0 < A < v. \tag{2.71}$$

By computation (see the Appendix), using the equation (2.71), we rewrite the conditions $4''$, $5''$ and 6 in the following way:

$$
\begin{aligned}
&\mathring{4}.\ \psi(A, v) > 0, \quad \text{for } 0 < A < v,\\
&\hat{5}.\ \Delta\psi \le 0,\\
&\hat{6}.\ \left(\frac{\partial^2 \psi}{\partial A \partial v}\right)^2 - \frac{\partial^2 \psi}{\partial A^2}\frac{\partial^2 \psi}{\partial v^2} \le 0,
\end{aligned}
$$

and we rewrite the condition 3^{**} in the following way:

$$\mathring{3}.\ \frac{\partial\psi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p}\psi(A,v)^p.$$

Conditions $\hat{5}$ and $\hat{6}$ entail that the Hessian matrix of ψ is negative semi-definite.
So we proved that, if the characterization of the dual dyadic Hardy's inequality 2.1.1 holds for a constant $K(p)$, then there exists a function

$$\hat{B}_\psi(F,f,A,v) = K(p)F - \frac{f^p}{\psi(A,v)^{p-1}}$$

such that:

1. $\hat{B}_\psi$ is defined over the domain

$$\mathcal{D} = \{(F,f,A,v) \in \mathbb{R}^4 \mid F > 0, f > 0, A > 0, v > 0, A \leq v, f^p \leq F v^{p-1}\},$$

$\mathring{2}$. $\frac{v^{p-1}}{\psi(A,v)^{p-1}} \leq K(p)$ for all $(F,f,A,v) \in \mathcal{D}$,

$\mathring{3}$. $\frac{\partial\psi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p}\psi(A,v)^p$,

$\mathring{4}$. $\psi(A,v) > 0$ for $0 < A < v$,

$\tilde{5}$. ψ is concave.

Now we are going to prove that Theorem 2.1.1 holds for the constant $K(p)$ if and only if there exists a linear function $\varphi(A,v)$ such that the function

$$\hat{B}_\varphi(F,f,A,v) = K(p)F - \frac{f^p}{\varphi(A,v)^{p-1}}$$

satisfies 1, $\mathring{2}$, $\mathring{3}$ and $\mathring{4}$.
If a function φ with such properties exists, then the function $\hat{B}_\varphi$ satisfies 1, 2, 3′ and 3″, so, by the previous lemma, Theorem 2.1.1 holds for the constant $K(p)$.
Suppose that Theorem 2.1.1 holds for the constant $K(p)$. By the previous point there exists a concave function ψ satisfying 1, $\mathring{2}$, $\mathring{3}$ and $\mathring{4}$. Let us define the function

$$t : (0,+\infty) \longrightarrow \mathbb{R}$$
$$v \longmapsto \psi(0,v).$$

The function ψ satisfies $\mathring{2}$, $\mathring{3}$ and $\mathring{4}$, so it follows that

$$\sup_{0<v}\left(\frac{v}{t(v)}\right)^{p-1} = \sup_{0<v}\left(\frac{v}{\psi(0,v)}\right)^{p-1} = \sup_{0<A\leq v}\left(\frac{v}{\psi(A,v)}\right)^{p-1} \leq K(p). \tag{2.72}$$

Moreover, the function t satisfies

$$\lim_{v\to+\infty}\left(\frac{v}{t(v)}\right) \le \sup_{0<v}\left(\frac{v}{t(v)}\right). \tag{2.73}$$

By combining (2.72) and (2.73) we get

$$\lim_{v\to+\infty}\left(\frac{v}{t(v)}\right)^{p-1} \le K(p). \tag{2.74}$$

The function t is concave and non-negative, so

$$\lim_{v\to+\infty} t'(v) \ge 0,$$

otherwise the function t would be negative as $v \to +\infty$. Moreover,

$$\lim_{v\to+\infty} t'(v) > 0.$$

Indeed, let us suppose that $t'(v) \to 0$ for $v \to +\infty$. If t is bounded by above then

$$\lim_{v\to+\infty}\left(\frac{v}{t(v)}\right)^{p-1} = +\infty \not\le K(p),$$

in contradiction with (2.74).
If $t(v) \to +\infty$ as $t \to +\infty$, then

$$\lim_{v\to+\infty}\left(\frac{v}{t(v)}\right)^{p-1} = \lim_{v\to+\infty}\left(\frac{1}{t'(v)}\right)^{p-1} = +\infty \not\le K(p),$$

in contradiction with (2.74).
Let us define a linear function

$$\varphi(A,v) = C_1 A + C_2 v \quad \text{for } 0 < A \le v, \tag{2.75}$$

and let us choose

$$C_1 > 0, \quad C_2 = \lim_{v\to+\infty} t'(v), \tag{2.76}$$

we will fix the value of C_1 later.
By construction, $C_2 > 0$, so the function φ is positive. Moreover, since $C_1 > 0$, it follows that

$$\begin{aligned}\sup_{0<A\le v}\left(\frac{v}{\varphi(A,v)}\right)^{p-1} &= \sup_{0<v}\left(\frac{v}{C_1\cdot 0 + C_2 v}\right)^{p-1} = \left(\frac{1}{C_2}\right)^{p-1} = \lim_{v\to+\infty}\left(\frac{1}{t'(v)}\right)^{p-1} = \\ &= \lim_{v\to+\infty}\left(\frac{v}{t(v)}\right)^{p-1} \le K(p),\end{aligned}$$

so we proved that

$$\frac{v^{p-1}}{\varphi(A,v)^{p-1}} \leq K(p) \quad \text{for } 0 < A \leq v. \tag{2.77}$$

It is left to prove that φ satisfies $\mathring{3}$.
We observe that the following statements are equivalent:

I)

$$\frac{\partial \varphi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p}\varphi(A,v)^p \quad \text{for } 0 < A \leq v.$$

II) There exists $v > 0$ such that

$$\frac{\partial \varphi}{\partial A}(v,v) \geq \frac{1}{(p-1)v^p}\varphi(v,v)^p,$$

where

$$\frac{\partial \varphi}{\partial A}(v,v) = \lim_{\substack{x\to v\\ x<v}} \frac{\partial \varphi}{\partial A}(x,v).$$

Indeed, *I*) trivially implies *II*). On the other hand, if φ satisfies *II*), then

$$C_1 \geq \frac{1}{(p-1)v^p}(C_1 v + C_2 v)^p.$$

However, we observe that $C_1 > 0$, and

$$\frac{\partial \varphi}{\partial A}(A,v) = \frac{\partial \varphi}{\partial A}(v,v) = C_1,$$

so it follows that

$$\frac{\partial \varphi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p}(C_1 v + C_2 v)^p \geq \frac{1}{(p-1)v^p}(C_1 A + C_2 v)^p = \frac{1}{(p-1)v^p}\varphi(A,v)^p,$$

so we proved that *I*) holds.
We are now going to prove that the function φ satisfies *II*).
Consider a fixed $\tilde{v} > 0$. Let us define the function

$$\begin{aligned} r_{\tilde{v}} : (0,\tilde{v}] &\longrightarrow \mathbb{R} \\ A &\longmapsto \psi(A,\tilde{v}). \end{aligned}$$

By construction, the function $r_{\tilde{v}}$ is concave, strictly increasing.
Moreover, we observe that $\varphi(0,v) \leq \psi(0,v)$ for all $v > 0$. Indeed, by construction, we have

$$\lim_{x\to 0} \varphi(0,x) = 0 \leq \lim_{x\to 0} \psi(0,x),$$

and, by concavity of ψ,

$$\frac{\partial}{\partial v}\varphi(0,y) = C_2 = \lim_{x\to+\infty}\frac{\partial}{\partial v}\psi(0,x) \le \frac{\partial}{\partial v}\psi(0,y) \quad \text{for all } y>0.$$

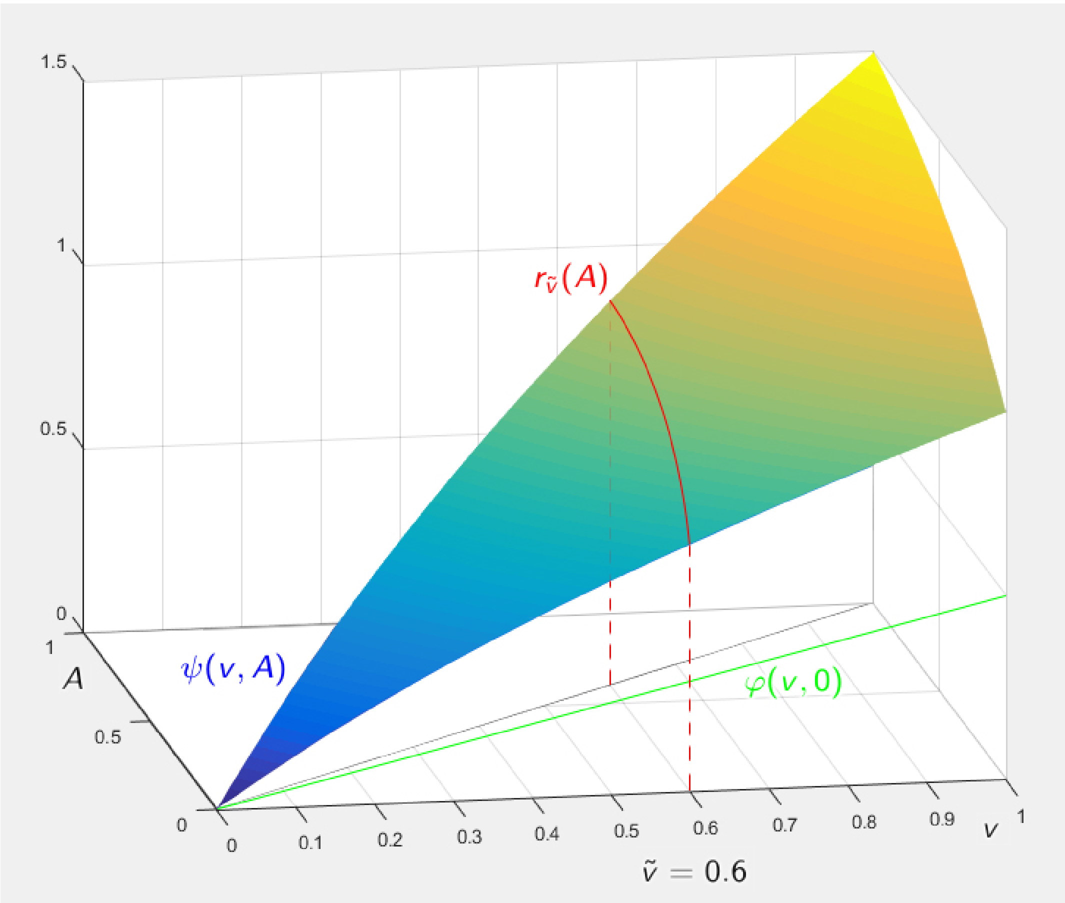
So, by integration, it follows that

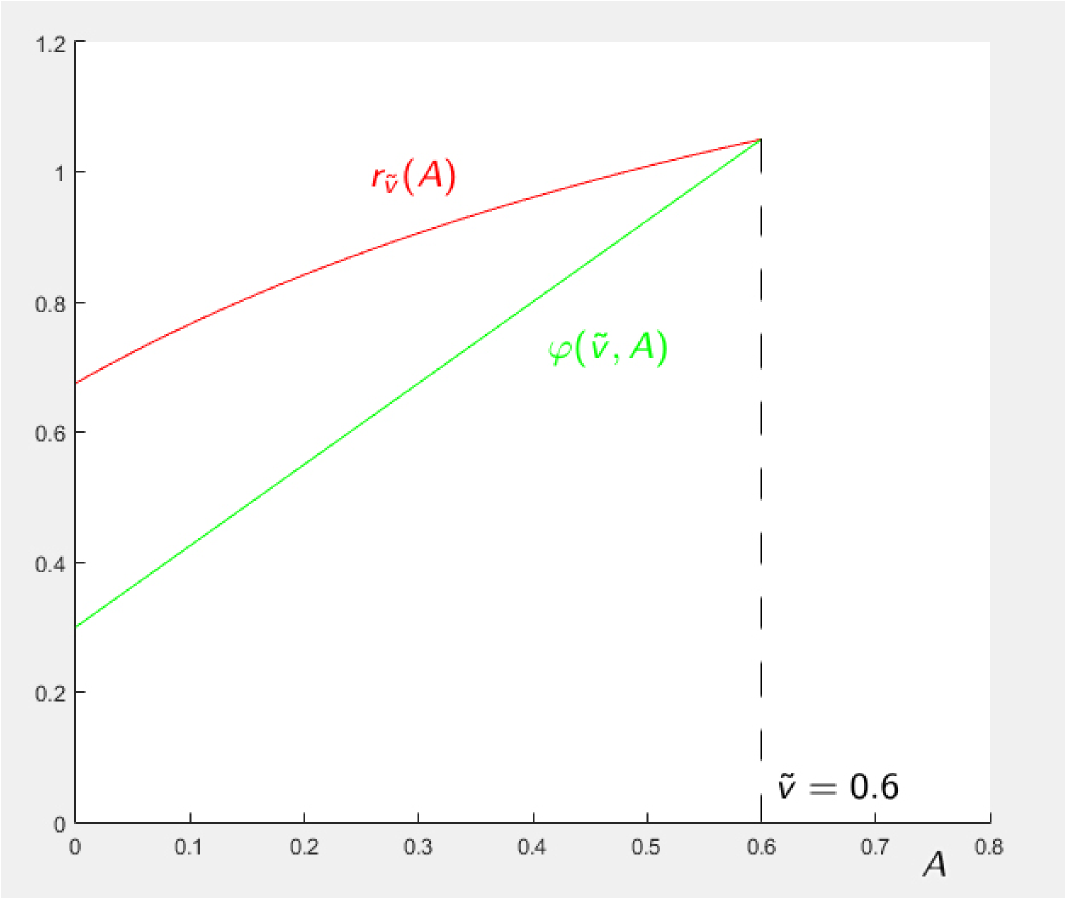
$$r_{\tilde v}(0) = \psi(0,\tilde v) \ge \varphi(0,\tilde v). \tag{2.78}$$

Now we set C_1 to be the value

$$C_1 = \frac{\psi(\tilde v,\tilde v)-\varphi(0,\tilde v)}{\tilde v} = \frac{r_{\tilde v}(\tilde v)-\varphi(0,\tilde v)}{\tilde v}. \tag{2.79}$$

The following pictures show the construction we just defined.





By construction, $C_1 > 0$, and

$$\varphi(\tilde v,\tilde v) = C_1\tilde v + C_2\tilde v = \frac{\psi(\tilde v,\tilde v)-\varphi(0,\tilde v)}{\tilde v}\tilde v + C_1\cdot 0 + C_2\tilde v = \psi(\tilde v,\tilde v) - \varphi(0,\tilde v) + \varphi(0,\tilde v) = \psi(\tilde v,\tilde v) = r_{\tilde v}(\tilde v). \tag{2.80}$$

By concavity of $r_{\tilde v}$, since $\varphi(0,\tilde v) < r_{\tilde v}(0)$, we get

$$r'_{\tilde v}(\tilde v) \le \frac{r_{\tilde v}(\tilde v) - r_{\tilde v}(0)}{\tilde v} \le \frac{r_{\tilde v}(\tilde v,\tilde v) - \varphi(0,\tilde v)}{\tilde v} = C_1. \tag{2.81}$$

However, the function $r_{\tilde v}$ satisfies equation $\mathring{3}$, i.e.

$$r'_{\tilde v}(A) \ge \frac{1}{(p-1)\tilde v^p} r_{\tilde v}(A)^p \quad \text{for } 0 < A < \tilde v, \tag{2.82}$$

so, by letting $A \to \tilde v$, since $r_{\tilde v}(\tilde v) = \varphi(\tilde v,\tilde v)$, we get

$$\frac{1}{(p-1)v^p}\varphi(\tilde v,\tilde v)^p = \frac{1}{(p-1)\tilde v^p} r_{\tilde v}(\tilde v)^p \le r'_{\tilde v}(\tilde v) \le C_1 = \frac{\partial}{\partial A}\varphi(\tilde v,\tilde v), \tag{2.83}$$

proving that II) holds.
So, by the previous remark, I) holds, i.e.

$$\frac{\partial\varphi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p}\varphi(A,v)^p \quad \text{for } 0 < A \leq v, \tag{2.84}$$

so we proved that φ satisfies $\mathring{3}$, finishing the proof. □

2.2.3 Optimality of the constant

In this section we prove the optimality of the constant $C(p)$ for Theorem 2.1.1.

Theorem 2.2.5. *The constant $C(p) = \big(p/(p-1)\big)^p$ in Theorem 2.1.1 is sharp.*

Proof. By Lemma 2.2.4, Theorem 2.1.1 holds for a constant $K(p)$ if and only if there exists a function

$$\hat{B}_\varphi(F,f,A,v) = K(p)F - \frac{f^p}{\varphi(A,v)^{p-1}}, \tag{2.85}$$

such that

1. $\hat{B}_\varphi$ is defined over the domain

$$\mathcal{D} = \{(F,f,A,v) \in \mathbb{R}^4 \mid F > 0, f > 0, A > 0, v > 0, A \leq v, f^p \leq Fv^{p-1}\},$$

$\mathring{2}$. $\frac{v^{p-1}}{\varphi(A,v)^{p-1}} \leq K(p)$,

$\mathring{3}$. $\frac{\partial\varphi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p}\varphi(A,v)^p$,

$\mathring{4}$. $\varphi(A,v) > 0$,

$\mathring{5}$. φ is linear.

We set

$$\varphi(A,v) = C_1^{-\frac{1}{p-1}}(A + C_2 v), \tag{2.86}$$

for $C_1 = C_1(p) > 0$, $C_2 = C_2(p) > 0$, getting

$$\hat{B}_\varphi(F,f,A,v) = K(p)F - C_1\frac{f^p}{(A + C_2 v)^{p-1}}. \tag{2.87}$$

This parametrization is sufficient to define all functions φ with the required properties.
By construction, $\mathring{4}$ and $\mathring{5}$ are satisfied.
Without loss of generality, we may suppose that the inequality $\mathring{2}$ is sharp, i.e.

$$\sup_{0<A<v} \frac{v^{p-1}}{\varphi(A,v)^{p-1}} = K(p). \tag{2.88}$$

Indeed, if a function φ satisfies 1, $\mathring{3}$, $\mathring{4}$, $\mathring{5}$, and it satisfies

$$\sup_{0<A<v} \frac{v^{p-1}}{\varphi(A,v)^{p-1}} = K(p) - \epsilon = \tilde{K}(p) < K(p), \tag{2.89}$$

then it also satisfies the condition (2.88) for a more optimal constant $\tilde{K}(p) < K(p)$.
By computation, since $C_1 > 0$, the condition (2.88) is equivalent to

$$\sup_{v>0} \frac{v^{p-1}}{\varphi(0,v)^{p-1}} = K(p),$$

i.e.

$$\sup_{v>0} C_1 \frac{v^{p-1}}{(0 + C_2 v)^{p-1}} = \frac{C_1}{C_2^{p-1}} = K(p).$$

So we are going to set

$$C_1 = K(p) C_2^{p-1}. \tag{2.90}$$

So, by combining (2.90) and (2.86), we get

$$\varphi(A,v) = K(p)^{-\frac{1}{p-1}} C_2^{-1} (A + C_2 v). \tag{2.91}$$

By construction, the function φ satisfies 1, $\mathring{2}$, $\mathring{4}$ and $\mathring{5}$. Now we observe that the following statements are equivalent:

I) φ satisfies $\mathring{3}$, i.e.

$$\frac{\partial \varphi}{\partial A}(A,v) \geq \frac{1}{(p-1)v^p} \varphi(A,v)^p \quad \text{for } 0 < A \leq v.$$

II) There exists $v > 0$ such that

$$\frac{\partial \varphi}{\partial A}(v,v) \geq \frac{1}{(p-1)v^p} \varphi(v,v)^p,$$

where

$$\frac{\partial \varphi}{\partial A}(v,v) = \lim_{\substack{x \to v \\ x<v}} \frac{\partial \varphi}{\partial A}(x,v).$$

III)

$$C_2 \in \mathcal{S}(K(p)) := \left\{ x > 0 \;\middle|\; (x+1)^p - (p-1)K(p)x^{p-1} \leq 0 \right\}.$$

We already proved in the previous lemma that *I*) and *II*) are equivalent.
To prove that *II*) is equivalent to *III*) we compute the condition *II*) for $v > 0$, getting

$$K(p)^{-\frac{1}{p-1}} C_2^{-1} \geq \frac{1}{(p-1)v^p} K(p)^{-\frac{p}{p-1}} C_2^{-p} (v + C_2 v)^p. \tag{2.92}$$

By computation, (2.92) is equivalent to

$$(C_2+1)^p - (p-1)K(p)C_2^{p-1} \leq 0, \tag{2.93}$$

which is equivalent to $C_2 \in \mathcal{S}(K(p))$.

So we proved that the Theorem 2.1.1 holds for a constant $K(p)$ if and only if the set $\mathcal{S}(K(p))$ is not empty.

To finish the proof that the constant

$$C(p) = (p')^p = \left(\frac{p}{p-1}\right)^p$$

is optimal for the Theorem 2.1.1, we only need to prove that

$$\mathcal{S}(K(p)) = \emptyset \quad \text{for } K(p) < C(p). \tag{2.94}$$

Consider the following function:

$$\begin{aligned} \Phi(K(p)) : (0, +\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto (x+1)^p - (p-1)K(p)x^{p-1}. \end{aligned}$$

The set $\mathcal{S}(K(p))$ is the set of points $x > 0$ such that $\Phi(K(p)) \leq 0$.

If we consider $K(p) = C(p)$ then, by computation, we get

$$\Phi(C(p))(x) = (x+1)^p - \frac{p^p}{(p-1)^{p-1}} x^{p-1}. \tag{2.95}$$

It is easy to prove that the function $\Phi(C(p))$ has a unique global minimum point at $x = p-1$, and the minimum is $\Phi(C(p))(p-1) = 0$, which entails that $\mathcal{S}(C(p)) = \{p-1\}$, and $\Phi(C(p))(x) \geq 0$ for $x > 0$.

If we consider, for any fixed $\epsilon > 0$, the constant

$$K(p) = C(p) - \epsilon < C(p),$$

then the function $\Phi(K(p))$ is the function

$$\begin{aligned} \Phi(K(p))(x) &= (x+1)^p - (p-1)(C(p)-\epsilon)x^{p-1} = \\ &= (x+1)^p - (p-1)C(p)x^{p-1} + (p-1)\epsilon x^{p-1} = \\ &= \Phi(C(p))(x) + P(\epsilon)(x), \end{aligned}$$

where $P(\epsilon)(x) = (p-1)\epsilon x^{p-1}$.

However, $P(\epsilon)(x) > 0$ for $x > 0$, and $\Phi(C(p))(x) \geq 0$ for $x > 0$, so it follows that $\Phi(K(p))(x) > 0$ for $x > 0$, which entails that the set $\mathcal{S}(K(p))$ is empty for all $K(p) < C(p)$.

So Theorem 2.1.1 does not hold for any constant $K(p)$ smaller than $C(p)$.

Moreover, we proved in section 1 that it holds for the constant $C(p) = \left(p'\right)^p$, so the constant $C(p)$ is the optimal constant for the theorem, finishing the proof. □

2.3 Stochastic approach to the problem

We will now analyze this problem from the point of view of the theory of stochastic optimal control, and we will show that the function $\mathcal{B}$ can be interpreted as the Bellman function associated to a stochastic optimal control problem naturally related to the dyadic problem. In this section we use the same notations used in [24], chapter 11. See [19] for more details about the topic.

We are going to show that

Theorem 2.3.1. *The functions $\mathcal{B}$ and g are identical.*

Here g is the Bellman function solution to the following stochastic optimal control problem associated to the inequality (2.4).
Consider $x \in \mathcal{D}$, $u = (u_1, u_2, \dots, u_5) \in \mathbb{R}^5$ such that $u_5 \geq 0$. Let us define the payoff density

$$\eta^u(x) := p^p \left(\frac{x_2}{x_3 + (p-1)x_4} \right)^p u_5.$$

Let $x \in \overline{\mathcal{D}}$. We define the bequest function

$$K(x) = \liminf_{\substack{y \to x \\ y \in \mathcal{D}}} \mathcal{B}(y).$$

We remark that, for the definition of the stochastic Bellman function, we only need to define the bequest function K on the boundary of the domain $\mathcal{D}$, however we follow the definition used in [24].
Let us define the coefficients

$$b(u,x) := (0, 0, -u_5, 0),$$

$$\sigma(u,x) := (u_1, u_2, u_3, u_4).$$

Let $\{u_t\}_{t\geq 0}$ be a control such that $u_t(\omega) \in \{u \in \mathbb{R}^5 \mid u_5 \geq 0\}$. We consider the stochastic process $\{X_t\} = \{(F_t, f_t, A_t, v_t)\}$ solution to the following stochastic differential equation

$$X_t = x_0 + \int_0^t b(u_s, X_s) ds + \int_0^t \sigma(u_s, X_s) dB_s, \tag{2.96}$$

where $x_0 \in \mathcal{D}$ is the starting point, $\{B_t\}_{t\geq 0}$ is a 1-dimensional Brownian motion and the domain of values of X_t is the set $\mathcal{D}$. Let $\tau_{\mathcal{D}}$ be the first exit time for $\{X_t\}_{t\geq 0}$ from $\mathcal{D}$, i.e.

$$\tau_{\mathcal{D}}(\omega) := \begin{cases} \inf\{s > 0 \mid X_s(\omega) \notin \mathcal{D}\} & \text{if } \{s > 0 \mid X_s(\omega) \notin \mathcal{D}\} \neq \emptyset, \\ +\infty & \text{otherwise.} \end{cases}$$

The Bellman function associated to the problem is

$$g(x) = \sup_{\{u_t\}} E^x\left[\int_0^{\tau_{\mathcal{D}}} p^p\left(\frac{f_s}{A_s+(p-1)v_s}\right)^p u_5\,ds + K(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}}\right],$$

where the supremum is taken over the set of controls $\{u_t\}_{t\geq 0}$ satisfying proper measurability conditions and whose values range in the set $\{(u_1,u_2,u_3,u_4,u_5)\in\mathbb{R}^5 \mid u_5\geq 0\}$.

We observe that, for this result, we used the stronger version of the main inequality (2.4) instead of the weaker version (2.5). By using a stronger main inequality we still get a Bellman function that can be used in the proof of Theorem 2.1.1 with the Bellman function method, however finding the solution to the problem associated to the weaker inequality (2.5) would require more work.

We are now going to show in the following subsections how we got to the stochastic optimal control problem and how we solved it.

2.3.1 From the dyadic to the stochastic problem

In this subsection we will show that the main inequality satisfied by the function $\mathcal{B}$ can be used to prove that $\mathcal{B}$ satisfies a differential inequality that will be the starting point from which we enunciate the stochastic optimal control problem having $\mathcal{B}$ as a solution.
We are going to recall the problem we are considering. Let $p\in\mathbb{R}$, $1<p<+\infty$. We consider the function

$$\mathcal{B}(F,f,A,v) = \left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}},$$

defined over the domain

$$\mathcal{D} := \left\{(F,f,A,v)\in\mathbb{R}^4 \mid F>0,\ f>0,\ A>0,\ v>0,\ v\geq A,\ f^p\leq Fv^{p-1}\right\}.$$

We proved in section 1 that $\mathcal{B}$ satisfies the inequality (2.4). We are now going to show how the inequality (2.4) entails a differential inequality for the function $\mathcal{B}$.
Let us consider a fixed point $(\tilde{F},\tilde{f},\tilde{A},\tilde{v})$ in the set of the interior points of $\mathcal{D}$. Let us consider $a\geq 0$, $b\geq 0$, $c\geq 0$. Let us consider $t>0$. We now define

$$\phi(t) = (\tilde{F}+(tb)^p, \tilde{f}+t^2ab, \tilde{A}+tc, \tilde{v}+(ta)^q)\in\mathbb{R}^4,$$

$$\psi(t) = (\tilde{F}+u_1t, \tilde{f}+u_2t, \tilde{A}+u_3t, \tilde{v}+u_4t)\in\mathbb{R}^4.$$

As long as we choose $\tilde{t} \in \mathbb{R}^+$ small enough, we have that $\phi(t) \in \mathcal{D}$ and $\psi(t) \in \mathcal{D}$ for all $0 \leq t < \tilde{t}$. So we may now compute the main inequality (2.4) in the following way

$$\mathcal{B}\Big(\tilde{F} + (t^2 b)^p, \tilde{f} + (t^2)^2 ab, \tilde{A} + t^2 c, \tilde{v} + (t^2 a)^{p'}\Big) - \mathcal{B}\Big(\tilde{F}, \tilde{f}, \tilde{A}, \tilde{v}\Big) + \mathcal{B}\Big(\tilde{F}, \tilde{f}, \tilde{A}, \tilde{v}\Big) - \frac{1}{2}\Big[\mathcal{B}\Big(\tilde{F} + u_1 t, \tilde{f} + u_2 t, \tilde{A} + u_3 t, \tilde{v} + u_4 t\Big) + \mathcal{B}\Big(\tilde{F} - u_1 t, \tilde{f} - u_2 t, \tilde{A} - u_3 t, \tilde{v} - u_4 t\Big)\Big] \geq p^p \frac{f^p}{(A + (p-1)v)^p} t^2 c,$$

which is equivalent to

$$\mathcal{B}(\phi(t^2)) - \mathcal{B}(\phi(0)) + \mathcal{B}(\psi(0)) - \frac{1}{2}\Big[\mathcal{B}(\psi(-t)) + \mathcal{B}(\psi(t))\Big] \geq p^p \frac{(\tilde{f} + (t^2)^2 ab)^p}{(\tilde{A} + t^2 c + (p-1)(\tilde{v} + (t^2 a)^{p'}))^p} t^2 c.$$

We are allowed to compute this inequality because, by setting $X = \phi(t^2)$, $\tilde{X} = \phi(0) = \psi(0)$ and $X_+ = \psi(t)$, $X_- = \psi(-t)$, we have

$$X = \tilde{X} + \Big((t^2 b)^p, (t^2 a) \cdot (t^2 b), (ta)^{p'}, c^2 t\Big),$$

$$\tilde{X} = \frac{1}{2}(X_+ + X_-),$$

and X, $\tilde{X}$, X_+, X_- are in the domain $\mathcal{D}$, so the hypotheses of the main inequality are satisfied. Dividing by t^2 and taking the limit as $t \to 0$ we get

$$\lim_{t \to 0} \frac{\mathcal{B}(\phi(t^2)) - \mathcal{B}(\phi(0)) + \mathcal{B}(\psi(0)) - \frac{1}{2}\Big[\mathcal{B}(\psi(-t)) + \mathcal{B}(\psi(t))\Big]}{t^2} \geq p^p \frac{\tilde{f}^p}{(\tilde{A} + (p-1)\tilde{v})^p} c.$$

By a change of variable we get

$$\lim_{s \to 0} \frac{\mathcal{B}(\phi(s)) - \mathcal{B}(\phi(0))}{s} + \lim_{t \to 0} \frac{\mathcal{B}(\psi(0)) - \frac{1}{2}\Big[\mathcal{B}(\psi(-t)) + \mathcal{B}(\psi(t))\Big]}{t^2} \geq p^p \frac{\tilde{f}^p}{(\tilde{A} + (p-1)\tilde{v})^p} c,$$

so we get

$$\frac{\partial}{\partial t}\mathcal{B}(\phi(t))\Big|_{t=0} - \frac{1}{2}\frac{\partial^2}{\partial t^2}\mathcal{B}(\psi(t))\Big|_{t=0} \geq p^p \frac{\tilde{f}^p}{(\tilde{A} + (p-1)\tilde{v})^p} c. \tag{2.97}$$

By computing the derivative we get

$$\langle \nabla\mathcal{B}(\phi(0)), \phi'(0)\rangle - \frac{1}{2}\Big[\langle \mathcal{H}(\mathcal{B})(\psi(0))\psi'(0), \psi'(0)\rangle + \langle \nabla\mathcal{B}(\psi(0)), \psi''(0)\rangle\Big] \geq p^p \frac{\tilde{f}^p}{(\tilde{A} + (p-1)\tilde{v})^p} c.$$

Now we observe that

$$\phi'(0) = (0,0,c,0), \quad \psi'(0) = (u_1,u_2,u_3,u_4) =: u, \quad \psi''(0) = (0,0,0,0).$$

So we get

$$\frac{\partial \mathcal{B}}{\partial x_3} \cdot c - \frac{1}{2}\langle \mathcal{H}(\mathcal{B}) \cdot u, u\rangle \geq p^p \frac{\tilde{f}^p}{(\tilde{A} + (p-1)\tilde{v})^p} c \tag{2.98}$$

for any $c \geq 0$.

We may verify that the function

$$\mathcal{B}(F,f,A,v) = \left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1} \frac{f^p}{(A + v(p-1))^{p-1}}$$

satisfies the inequality (2.98). We compute

$$\frac{\partial \mathcal{B}}{\partial x_3}(\tilde{F}, \tilde{f}, \tilde{A}, \tilde{v}) \cdot c = p^p \frac{\tilde{f}^p}{(\tilde{A} + \tilde{v}(p-1))^p} c,$$

and $\mathcal{B}$ is concave so it satisfies $-\frac{1}{2}\langle \mathcal{H}(\mathcal{B}) \cdot u, u\rangle \geq 0$, showing that (2.98) is satisfied.

It follows that the function $\mathcal{B}$ satisfies the inequality

$$-\frac{\partial \mathcal{B}(x)}{\partial x_3} u_5 + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 \mathcal{B}(x)}{\partial x_i \partial x_j} u_i u_j + p^p \left(\frac{x_2}{x_3 + (p-1)x_4}\right)^p u_5 \leq 0 \quad \forall x \in \mathcal{D},\ \forall u \in \mathbb{R}^5,\ u_5 \geq 0.$$

So the function $\mathcal{B}$ satisfies the following inequality

$$\sup_{\substack{u \in \mathbb{R}^5 \\ u_5 \geq 0}} \left\{ -\frac{\partial \mathcal{B}(x)}{\partial x_3} u_5 + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 \mathcal{B}(x)}{\partial x_i \partial x_j} u_i u_j + p^p \left(\frac{x_2}{x_3 + (p-1)x_4}\right)^p u_5 \right\} \leq 0, \tag{2.99}$$

so we will read the function $\mathcal{B}$ as a supersolution to a Hamilton-Jacobi-Bellman equation. Moreover, $\mathcal{B}$ is actually a solution to the Hamilton-Jacobi-Bellman equation by taking $u_1 = u_2 = u_3 = u_4 = 0$.

So $\mathcal{B}$ satisfies the Hamilton-Jacobi-Bellman equation

$$\sup_{\substack{u \in \mathbb{R}^5 \\ u_5 \geq 0}} \left\{ -\frac{\partial \mathcal{B}(x)}{\partial x_3} u_5 + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 \mathcal{B}(x)}{\partial x_i \partial x_j} u_i u_j + p^p \left(\frac{x_2}{x_3 + (p-1)x_4}\right)^p u_5 \right\} = 0 \quad \forall x \in \mathcal{D}. \tag{2.100}$$

So we naturally got a Hamilton-Jacobi-Bellman equation that can be interpreted as the equation associated to a stochastic optimal control problem.

2.3.2 Stochastic optimal control problem

We are now going to enunciate a stochastic optimal control which defines a Bellman function g such that $g \equiv \mathcal{B}$.
Let us consider the following extension of the function $\mathcal{B}$ to the closure $\overline{\mathcal{D}}$ of its domain:

$$\tilde{\mathcal{B}} : \overline{\mathcal{D}} \longrightarrow \mathbb{R}$$

defined in the following way:

$$\tilde{\mathcal{B}}(x) = \begin{cases} \mathcal{B}(x) & \text{if } x \in \mathcal{D}, \\ \liminf\limits_{y \to x} \mathcal{B}(y) & \text{if } x \in \overline{\mathcal{D}} \backslash \mathcal{D}. \end{cases}$$

Observation 2.3.1. For all points $x \in \overline{\mathcal{D}} \backslash \mathcal{D}$ such that $(x_3, x_4) \neq (0,0)$ the function $\mathcal{B}$ extends continuously to the value

$$\tilde{\mathcal{B}}(x_1, x_2, x_3, x_4) = \lim_{y \to x} \mathcal{B}(y) = \left(\frac{p}{p-1}\right)^p x_1 - \frac{p^p}{p-1} \frac{{x_2}^p}{(x_3 + x_4(p-1))^{p-1}}.$$

The remaining points $x \in \overline{\mathcal{D}} \backslash \mathcal{D}$ are the points $x = (F, f, 0, 0)$, however by definition of $\mathcal{D}$ we have $f^p \leq F v^{p-1}$, so $f = 0$. For the points $x = (F, 0, 0, 0)$ such that $F \geq 0$ we have

$$\tilde{\mathcal{B}}(x) = \liminf_{y \to x} \mathcal{B}(y) = 0.$$

Proof. We are going to show this fact by recalling that $\mathcal{B} \geq 0$, so $\liminf\limits_{y \to x} \mathcal{B}(y) \geq 0$, and by considering a proper sequence of points. Let $v \geq A > 0$, let $0 < t \leq 1$. Let us first assume that $F > 0$. We are going to consider the points

$$x(t) = (F, (F(tv)^{p-1})^{\frac{1}{p}}, t^2 A, tv).$$

By construction $x(t) \in \mathcal{D}$, $\lim\limits_{t \to 0} x(t) = (F, 0, 0, 0)$, and

$$\begin{aligned} \lim_{t \to 0} \mathcal{B}(x(t)) \quad &= \quad \lim_{t \to 0} \left[\left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1} \frac{F(tv)^{p-1}}{(t^2 A + (p-1)tv)^{p-1}} \right] = \\ & F \cdot \lim_{t \to 0} \left[\left(\frac{p}{p-1}\right)^p - \frac{p^p}{p-1} \frac{v^{p-1}}{(tA + (p-1)v)^{p-1}} \right] = 0. \end{aligned}$$

Let us assume $F = 0$. Let $\tilde{F} > 0$. We consider the sequence

$$x(t) = (t\tilde{F}, (t\tilde{F}(tv)^{p-1})^{\frac{1}{p}}, t^2 A, tv),$$

and the proof holds with the same argument.
So $0 \leq \liminf\limits_{y \to x} \mathcal{B}(y) \leq \lim\limits_{t \to 0} \mathcal{B}(x(t)) = 0$, which ends the proof. $\square$

Let $x \in \mathcal{D}$, $t \geq 0$, $u = (u_1, u_2, \ldots, u_5) \in \mathbb{R}^5$ such that $u_5 \geq 0$. We will now define a payoff density and a bequest function to get the stochastic optimal control problem we are looking for. These functions will not depend on the time variable, so in the notation we will skip writing it. Let us define the payoff density

$$\eta^u(x,t) \equiv \eta^u(x) := p^p \left(\frac{x_2}{x_3 + (p-1)x_4} \right)^p u_5.$$

Let $x \in \overline{\mathcal{D}}$. We define the bequest function

$$K(x_1, x_2, x_3, x_4, t) \equiv K(x) := \tilde{\mathcal{B}}(x),$$

i.e. K is the function

$$K(x_1, x_2, x_3, x_4, t) \equiv K(x) = \begin{cases} \left(\frac{p}{p-1}\right)^p x_1 - \frac{p^p}{p-1} \frac{x_2^p}{(x_3+(p-1)x_4)^{p-1}} & \text{if } x \in \mathcal{D}, \\ \liminf\limits_{y \to x} \mathcal{B}(y) & \text{if } x \in \overline{\mathcal{D}} \backslash \mathcal{D}. \end{cases}$$

To finish the formulation of the stochastic optimal control problem we define the coefficients

$$b(u, x, t) \equiv b(u, x) := (0, 0, -u_5, 0),$$

$$\sigma(u, x, t) \equiv \sigma(u, x) := (u_1, u_2, u_3, u_4).$$

Let $\{u_t\}_{t \geq 0}$ be a control such that $u_t(\omega) \in \{u \in \mathbb{R}^5 \mid u_5 \geq 0\}$. We consider the stochastic process $\{X_t\} = \{(F_t, f_t, A_t, v_t)\}$ solution to the stochastic differential equation (2.96). The Bellman function associated to the problem is

$$g(x) = \sup_{\{u_t\}} E^x \left[\int_0^{\tau_{\mathcal{D}}} p^p \left(\frac{f_s}{A_s + (p-1)v_s} \right)^p u_5 \, ds + K(X_{\tau_{\mathcal{D}}}) \chi_{\{\tau_{\mathcal{D}} < +\infty\}} \right],$$

where the supremum is taken over the set of controls $\{u_t\}_{t \geq 0}$ such that $\{u_t\}$ is measurable with respect to $\mathcal{F}_t$, where $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration generated by the variables $\{B_s \mid 0 \leq s \leq t\}$, and such that the values $u_t(\omega)$ belong to the set $\{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5 \mid u_5 \geq 0\}$.

So by Theorem 1.4.1 the function g satisfies the equation (2.100). We will also write the equation (2.100) in the following way

$$\sup_{\substack{u \in \mathbb{R}^5 \\ u_5 \geq 0}} \left\{ (\mathcal{L}^u g)(x) + p^p \left(\frac{x_2}{x_3 + (p-1)x_4} \right)^p u_5 \right\} = 0 \quad \forall x \in \mathcal{D}. \tag{2.101}$$

We recall that the operator

$$(\mathcal{L}^u \varphi)(x) = -\frac{\partial \varphi(x)}{\partial x_3} u_5 + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} u_i u_j \tag{2.102}$$

is the infinitesimal generator of the process $\{X_t\}$ solution to the equation (2.96) for the choice of the control $\{u_t\}$ such that $u_t \equiv u \in \{y \in \mathbb{R}^5 \mid y_5 \geq 0\}$. Indeed, by Theorem 1.2.8, the infinitesimal generator $\mathcal{A}$ of such process can be characterized by

$$(\mathcal{A}g)(x) = \sum_{i=1}^{4} b_i(u,x)\frac{\partial g}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{4}(\sigma\sigma^T)_{i,j}(u,x)\frac{\partial^2 g}{\partial x_i \partial x_j}(x) = (\mathcal{L}^u g)(x).$$

2.3.3 The dyadic Bellman function is a stochastic Bellman function

We are now going to give the proof of Theorem 2.3.1.

Proof. We are going to prove the stronger statement

$$g(x) = \tilde{\mathcal{B}}(x) \quad \forall x \in \overline{\mathcal{D}}. \tag{2.103}$$

By definition of $\tilde{\mathcal{B}}$ equation (2.103) entails

$$g(x) = \tilde{\mathcal{B}}(x) = \mathcal{B}(x) \quad \forall x \in \mathcal{D}, \tag{2.104}$$

which is the required statement. First we are going to prove that $g(F,f,A,v) \geq \mathcal{B}(F,f,A,v)$. We are going to compute

$$g(F,f,A,v) \geq E^{x_0}\left[\int_0^{\tau_{\mathcal{D}}} \eta^{u_s}(X_s)ds + K(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}}\right]$$

for the choice

$$u_t = (0,0,0,0,1), \quad x_0 = (F,f,A,v).$$

Let us first suppose $v > 0$. By computation we get $F_s \equiv F$, $f_s \equiv f$, $v_s \equiv v$, $A_s = A - s$, and $\tau_{\mathcal{D}} = A$, so, since the control is deterministic, we get

$$\begin{aligned}
g(F,f,A,v) \geq & \int_0^A p^p\left(\frac{f}{A-s+(p-1)v}\right)^p ds + K(F,f,A-A,v) = \\
& \left[\frac{p^p}{p-1}\frac{f^p}{(A-s+(p-1)v)^{p-1}}\right]_{s=0}^{s=A} + \tilde{\mathcal{B}}(F,f,0,v) = \\
& \left(\frac{p}{p-1}\right)^p \frac{f^p}{v^{p-1}} - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}} + \liminf_{y\to(F,f,0,v)} \mathcal{B}(y) = \\
& \left(\frac{p}{p-1}\right)^p \frac{f^p}{v^{p-1}} - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}} + \left(\frac{p}{p-1}\right)^p F - \left(\frac{p}{p-1}\right)^p \frac{f^p}{v^{p-1}} = \\
& \left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}} = \tilde{\mathcal{B}}(F,f,A,v).
\end{aligned}$$

On the other hand, if $v = 0$ then $A = 0$ and $f = 0$ by definition of the domain $\mathcal{D}$, so in this case the stopping time $\tau_{\mathcal{D}}$ is equal to 0, so the profit gain over the trajectory is 0, and we are left with the bequest gain. So the inequality becomes

$$g(F, f, A, v) \geq 0 + K(F, 0, 0, 0) = \tilde{\mathcal{B}}(F, 0, 0, 0) = 0,$$

which ends the proof that $g \geq \tilde{\mathcal{B}}$.

To prove that $g \leq \tilde{\mathcal{B}}$ we are going to first enunciate a heuristic argument to show it, using Jensen's inequality.

Let us consider a control $u = \{u_t\}_{t\geq 0}$ such that $u_t = (u_1(t), \dots, u_4(t), 0)$ for $0 \leq t < s$ and then $u_t = (0, 0, 0, 0, 1)$ for $t \geq s$. Let $\{X_t\}$ be the solution to (2.96) for this choice of the control $\{u_t\}$. The control $\{u_t\}$ lets the process $\{X_t\}$ behave like a martingale diffusion (the process has no drift) up to the time s, and on this part of the trajectory there is no profit gain (because the profit density is equal to 0 when $u_5 = 0$). Moreover, the control $\{u_t\}$ lets the process $\{X_t\}$ drift towards the boundary of the domain from the time s onwards.

Let $t \mapsto X(\omega)(t) := X_t(\omega)$ be a trajectory of the process $\{X_t\}$.

If $\tau_D(\omega) \leq s$ then, by continuity of the process $\{X_t\}$, the trajectory $X(\omega)$ lands on the point $X_{\tau_{\mathcal{D}}(\omega)}(\omega) \in \partial\mathcal{D}$ for almost all the ω with such properties. So almost all trajectories $X(\omega)$ such that $\tau_{\mathcal{D}}(\omega) \leq s$ gain an amount of profit equal to $K(X_{\tau_{\mathcal{D}}(\omega)}) = \tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}(\omega)}) = \tilde{\mathcal{B}}(X_{s\wedge\tau_{\mathcal{D}}(\omega)})$.

If $\tau_D(\omega) > s$, the trajectory $t \mapsto X_t(\omega)$ of the process $\{X_t\}$ lands on a point x in the interior of the domain $\mathcal{D}$ at the time s, without exiting the domain $\mathcal{D}$ before the time s. We observe that, in the first part of the proof, we proved that a control $\{\hat{u}_t\}$ such that $\hat{u}_t = (0, 0, 0, 0, 1)$ generates a process $\{\hat{X}_t\}$ that gains an amount of average profit equal to the value of the function $\tilde{\mathcal{B}}$ in the starting point. So, by this observation, it follows that the trajectory $X(\omega)$ gains no profit during the time $0 < t < s$, and gains an amount of profit equal to $\tilde{\mathcal{B}}(X_s) = \tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}(\omega)\wedge s})$ during the times $s \leq t \leq \tau_{\mathcal{D}}(\omega)$.

Based on these observations the average amount of profit gained by $\{X_t\}$, given a control $\{u_t\}$ of this kind, is

$$J^u(x) = E^x\left[0 + \tilde{\mathcal{B}}(X_{s\wedge\tau_{\mathcal{D}}})\right],$$

where the first addent stands for the null gain on the trajectory up to the time $s \wedge \tau_{\mathcal{D}}$, while the second addend is equal to the gain from that moment onwards (the profit gain from the bequest function at the end of times is included in the second addend).

Following this notation the Bellman function g is

$$g(x) = \sup_{u=\{u_t\}} J^u(x).$$

However, $\tilde{\mathcal{B}}$ is concave, so by Jensen's inequality we have

$$E^x\left[\tilde{\mathcal{B}}(X_{s\wedge\tau_{\mathcal{D}}})\right] \leq \tilde{\mathcal{B}}\Big(E^x[X_{s\wedge\tau_{\mathcal{D}}}]\Big).$$

Moreover, $\{X_t\}$ is a martingale up to the time s by definition, so we get

$$J^u(x) = E^x\left[\tilde{\mathcal{B}}(X_{s\wedge\tau_{\mathcal{D}}})\right] \leq \tilde{\mathcal{B}}\Big(E^x[X_{s\wedge\tau_{\mathcal{D}}}]\Big) = \tilde{\mathcal{B}}(x),$$

and the heuristic idea is that there is "independence" between letting the process drift in the third variable (which gives a non-negative gain) and letting the process be a diffusion (which gains nothing), so we can let the process be a combination of the two and the argument will still hold. So by taking the supremum over all controls $\{u_t\}$ we get

$$g(x) = \sup_{u=\{u_t\}} J^u(x) \leq \tilde{\mathcal{B}}(x).$$

We are now going to give a proof that $g \leq \tilde{\mathcal{B}}$ using Dynkin's formula 1.2.10.
We will skip some technical details in the following proof.
Let $\{u_t\}_{t\geq 0}$ be a given control. Let $\{X_t\}_{t\geq 0}$ be the process solution to (2.96) for this choice of the control $\{u_t\}_{t\geq 0}$. Let $\tau_{\mathcal{D}}$ be the first exit time for $\{X_t\}$ from $\mathcal{D}$. We will first assume that $\tau_{\mathcal{D}} < +\infty$ almost surely. We are now going to apply Dynkin's formula

$$E^x[\tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}})] = \tilde{\mathcal{B}}(x) + E^x\left[\int_0^{\tau_{\mathcal{D}}} (\mathcal{L}^{u_s}\tilde{\mathcal{B}})(X_s)dx\right] \tag{2.105}$$

to the function $\tilde{\mathcal{B}}$ and the process $\{X_t\}_{t\geq 0}$. We get

$$\tilde{\mathcal{B}}(x) = E^x[\tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}})] - E^x\int_0^{\tau_{\mathcal{D}}} (\mathcal{L}^{u_s}\tilde{\mathcal{B}})(X_s)ds.$$

Now, since $\tau_{\mathcal{D}} < +\infty$ almost surely, the event $\chi_{\{\tau_{\mathcal{D}}<+\infty\}}$ has a probability of 1, so

$$\tilde{\mathcal{B}}(x) = E^x[\tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}}] - E^x\int_0^{\tau_{\mathcal{D}}} (\mathcal{L}^{u_s}\tilde{\mathcal{B}})(X_s)ds.$$

The equation (2.101) entails that $-(\mathcal{L}^{u_s}\tilde{\mathcal{B}})(y) \geq \eta^{u_s}(y)$, so we get

$$\tilde{\mathcal{B}}(x) \geq E^x\int_0^{\tau_{\mathcal{D}}} \eta^{u_s}(X_s)ds + E^x[\tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}}].$$

However, $\tilde{\mathcal{B}}(X_{\tau_{\mathcal{D}}}) = K(X_{\tau_{\mathcal{D}}})$, so we get

$$\tilde{\mathcal{B}}(x) \geq E^x\int_0^{\tau_{\mathcal{D}}} \eta^{u_s}(X_s)ds + E^x[K(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}}] = g(x).$$

If $\tau_{\mathcal{D}}$ is not almost surely finite, we are going to show an idea of the proof. We may consider the stopping time $\tau(T) = \tau_{\mathcal{D}} \wedge T = \min\{\tau_{\mathcal{D}}, T\}$ for $T > 0$. This procedure is equivalent to considering the processes $Y_t = (t, X_t)$ in the domain $[0, T] \times \mathcal{D}$, and then defining the Bellman function $\tilde{\mathcal{B}}_T$ associated to those processes, which is a standard way to define the Bellman functions (see [24], chapter 11).
Since $\tau(T) < +\infty$ almost surely, we may apply Dynkin's formula to that stopping time and, with the same argument we used before, we get

$$\begin{aligned} \tilde{\mathcal{B}}_T(x) &\geq E^x \int_0^{\tau(T)} \eta^{u_s}(X_s)ds + E^x[K(X_{\tau(T)})\chi_{\{\tau(T)<+\infty\}}] \\ &\downarrow T \to +\infty \qquad\qquad \downarrow T \to +\infty \\ \tilde{\mathcal{B}}(x) &\geq E^x \int_0^{\tau_{\mathcal{D}}} \eta^{u_s}(X_s)ds + E^x[K(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}}]. \end{aligned}$$

So, by taking the supremum over all controls $\{u_t\}$, we get

$$\tilde{\mathcal{B}}(x) \geq \sup_{\{u_t\}} E^x \left[\int_0^{\tau_{\mathcal{D}}} \eta^{u_s}(X_s)ds + K(X_{\tau_{\mathcal{D}}})\chi_{\{\tau_{\mathcal{D}}<+\infty\}} \right] = g(x),$$

which ends the proof that $\tilde{\mathcal{B}} \equiv v$, so $\tilde{\mathcal{B}}$ is the Bellman function solution to the stochastic optimal control problem. □

2.4 Appendix

We are going to prove that the domain

$$\mathcal{D} := \left\{ (F, f, A, v) \in \mathbb{R}^4 \mid F > 0,\ f > 0,\ A > 0,\ v > 0,\ v \geq A,\ f^p \leq F v^{p-1} \right\}$$

is convex.

Proof. We write the domain $\mathcal{D}$ in the form

$$\mathcal{D} = \{v \geq A\} \cap \mathcal{A},$$

here $\mathcal{A}$ is the set

$$\mathcal{A} = \{(F, f, A, v) \in \mathbb{R}^4 \mid F > 0, f > 0, v > 0, f^p \leq F v^{p-1}\}.$$

To prove that the domain is convex we just need to prove that it is an intersection of convex sets.

The set $\{v \geq A\}$ is trivially convex because it is a half-plane. Since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{p-1}{p} = \frac{1}{p'}$, the set $\mathcal{A}$ can be written in the form

$$\mathcal{A} = \mathcal{S} \cap \{f > 0\},$$

here $\{f > 0\}$ is another half-plane (a convex set), while $\mathcal{S}$ is the set

$$\mathcal{S} = \{(F, f, A, v) \in \mathbb{R}^4 \mid F > 0, v > 0,\ f \leq F^{\frac{1}{p}} v^{\frac{1}{p'}}\}.$$

The set $\mathcal{S}$ is the subgraph of the function

$$\begin{aligned} h : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ &\longrightarrow \mathbb{R}_0^+ \\ (F, A, v) &\longmapsto F^{\frac{1}{p}} v^{\frac{1}{p'}}. \end{aligned}$$

To prove that $\mathcal{S}$ is convex, all we need to do is to prove that h is a concave function (since h is defined over a convex domain).
Since h does not depend on the variable A, we will treat it as a function over the other two variables only:

$$\begin{aligned} h : \mathbb{R}^+ \times \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ (F, v) &\longmapsto F^{\frac{1}{p}} v^{\frac{1}{p'}}. \end{aligned}$$

We compute the Hessian matrix of the function h: for all $F > 0$, $v > 0$

$$\frac{\partial h}{\partial F}(F, v) = \frac{1}{p} F^{\frac{1}{p}-1} v^{\frac{1}{p'}}, \qquad \tfrac{\partial h}{\partial v}(F, v) = \tfrac{1}{p'} F^{\frac{1}{p}} v^{\frac{1}{p'}-1}.$$

$$\begin{aligned} \frac{\partial^2 h}{\partial F^2}(F, v) &= \frac{1-p}{p^2} F^{\frac{1}{p}-2} v^{\frac{1}{p'}}, & \tfrac{\partial^2 h}{\partial v \partial F}(F, v) &= \tfrac{1}{pp'} F^{\frac{1}{p}-1} v^{\frac{1}{p'}-1}, \\ \frac{\partial^2 h}{\partial F \partial v}(F, v) & \frac{1}{pp'} F^{\frac{1}{p}-1} v^{\frac{1}{p'}-1}, & \tfrac{\partial^2 h}{\partial v^2}(F, v) &= \tfrac{1-p'}{p'^2} F^{\frac{1}{p}} v^{\frac{1}{p'}-2}. \end{aligned}$$

So the Hessian matrix is

$$\mathcal{H}(h)(F, v) = \begin{bmatrix} \frac{1-p}{p^2} F^{\frac{1}{p}-2} v^{\frac{1}{p'}} & \frac{1}{pp'} F^{\frac{1}{p}-1} v^{\frac{1}{p'}-1} \\ \\ \frac{1}{pp'} F^{\frac{1}{p}-1} v^{\frac{1}{p'}-1} & \frac{1-p'}{p'^2} F^{\frac{1}{p}} v^{\frac{1}{p'}-2} \end{bmatrix}. \tag{2.106}$$

If the Hessian matrix of h has non-positive eigenvalues then the function h is concave.

Now we compute the eigenvalues of the Hessian matrix (2.106):

$$\begin{aligned}\det(\mathcal{H}(h)(F,v)-\lambda I) &= \det\begin{bmatrix}\frac{1-p}{p^2}F^{\frac{1}{p}-2}v^{\frac{1}{p'}}-\lambda & \frac{1}{pp'}F^{\frac{1}{p}-1}v^{\frac{1}{p'}-1}\\ \\ \frac{1}{pp'}F^{\frac{1}{p}-1}v^{\frac{1}{p'}-1} & \frac{1-p'}{p'^2}F^{\frac{1}{p}}v^{\frac{1}{p'}-2}-\lambda\end{bmatrix} = \\ & \frac{(1-p)(1-p')}{(pp')^2}F^{\frac{1}{p}+\frac{1}{p}-2}v^{\frac{1}{p'}+\frac{1}{p'}-2}-\frac{1}{(pp')^2}F^{2(\frac{1}{p}-1)}v^{2(\frac{1}{p'}-1)}- \\ & \lambda\left[\frac{1-p}{p^2}F^{\frac{1}{p}-2}v^{\frac{1}{p'}}+\frac{1-p'}{p'^2}F^{\frac{1}{p}}v^{\frac{1}{p'}-2}\right]+\lambda^2.\end{aligned}$$

Now we recall that $pp'=p+p'$, so $(1-p)(1-p')=1-p-p'+pp'=1-p-p'+p+p'=1$, so we get

$$\det(\mathcal{H}(h)(F,v)-\lambda I)=\lambda^2-\lambda\left[\frac{1-p}{p^2}F^{\frac{1}{p}-2}v^{\frac{1}{p'}}+\frac{1-p'}{p'^2}F^{\frac{1}{p}}v^{\frac{1}{p'}-2}\right].$$

The eigenvalues of $\mathcal{H}(h)(F,v)$ are the solutions to the following equation equation of variable λ:

$$\det(\mathcal{H}(h)(F,v)-\lambda I)=0.$$

The solutions are the two values

$$\lambda_1=0,\quad \lambda_2=\frac{1-p}{p^2}F^{\frac{1}{p}-2}v^{\frac{1}{p'}}+\frac{1-p'}{p'^2}F^{\frac{1}{p}}v^{\frac{1}{p'}-2}.$$

Now we observe that $1-p<0$, $1-p'<0$ and $F>0$, $v>0$, so the second eigenvalue is $\lambda_2<0$, so the Hessian matrix $\mathcal{H}(h)(F,v)$ is negative semi-definite for all $F>0$ and $v>0$, so this entails that h is concave, and the subgraph $\mathcal{S}$ is a convex set. So the domain $\mathcal{D}$ of the function $\mathcal{B}$ in (2.3) is a convex set since it is an intersection of convex sets. □

We are going to prove that the function

$$\mathcal{B}(F,f,A,v)=\left(\frac{p}{p-1}\right)^pF-\frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}}$$

is concave.

Proof. We will compute the eigenvalues of the Hessian Matrix $\mathcal{H}(\mathcal{B})$. We will compute the actual 4×4 Hessian matrix (without reducing it to a 2×2 matrix), because the computation can be useful to compute the eigenvectors, which may be useful to study the properties of some of the stochastic processes associated to the problem.

In the following equations we are going to omit writing the dependence of the derivatives from the variables (F, f, A, v) to simplify the notations. The first order derivatives are

$$\frac{\partial\mathcal{B}}{\partial F} = \left(\frac{p}{p-1}\right)^p, \qquad \frac{\partial\mathcal{B}}{\partial f} = -\frac{p^{p+1}}{p-1}\frac{f^{p-1}}{(A+(p-1)v)^{p-1}},$$
$$\frac{\partial\mathcal{B}}{\partial A} = p^p\frac{f^p}{(A+(p-1)v)^p}, \qquad \frac{\partial\mathcal{B}}{\partial v} = (p-1)p^p\frac{f^p}{(A+(p-1)v)^p}.$$

The second order derivatives make up the rows of the Hessian matrix $\mathcal{HB}$.
The first row is

$$\frac{\partial^2\mathcal{B}}{\partial F^2} = 0, \qquad \frac{\partial^2\mathcal{B}}{\partial F\partial f} = 0,$$
$$\frac{\partial^2\mathcal{B}}{\partial F\partial A} = 0, \qquad \frac{\partial^2\mathcal{B}}{\partial F\partial V} = 0.$$

The second row is

$$\frac{\partial^2\mathcal{B}}{\partial f\partial F} = 0, \qquad \frac{\partial^2\mathcal{B}}{\partial f^2} = -p^{p+1}\frac{f^{p-2}}{(A+(p-1)v)^{p-1}},$$
$$\frac{\partial^2\mathcal{B}}{\partial f\partial A} = p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p}, \qquad \frac{\partial^2\mathcal{B}}{\partial f\partial V} = (p-1)p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p}.$$

The third row is

$$\frac{\partial^2\mathcal{B}}{\partial A\partial F} = 0, \qquad \frac{\partial^2\mathcal{B}}{\partial A\partial f} = p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p},$$
$$\frac{\partial^2\mathcal{B}}{\partial A^2} = -p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}}, \qquad \frac{\partial^2\mathcal{B}}{\partial A\partial V} = -(p-1)p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}}.$$

The fourth row is

$$\frac{\partial^2\mathcal{B}}{\partial V\partial F} = 0, \qquad \frac{\partial^2\mathcal{B}}{\partial V\partial f} = (p-1)p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p},$$
$$\frac{\partial^2\mathcal{B}}{\partial V\partial A} = -(p-1)p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}}, \qquad \frac{\partial^2\mathcal{B}}{\partial V^2} = -(p-1)^2p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}}.$$

So the Hessian matrix of $\mathcal{B}$ at a point (F, f, A, v) is

$$\mathcal{H}(\mathcal{B}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p^{p+1}\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} & p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p} & (p-1)p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p} \\ 0 & p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p} & -p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}} & -(p-1)p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}} \\ 0 & (p-1)p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^p} & -(p-1)p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}} & -(p-1)^2p^{p+1}\frac{f^p}{(A+(p-1)v)^{p+1}} \end{pmatrix}.$$

Let us compute the eigenvalues:

$$0 = \det(\mathcal{H}(\mathcal{B}) - \lambda I_4) =$$

$$-\lambda \det \begin{pmatrix} -p^{p+1}\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} - \lambda & p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} & (p-1)p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} \\ p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} & -p^{p+1}\frac{f^{p}}{(A+(p-1)v)^{p+1}} - \lambda & -(p-1)p^{p+1}\frac{f^{p}}{(A+(p-1)v)^{p+1}} \\ 0 & (p-1)\lambda & -\lambda \end{pmatrix} =$$

$$\lambda^2 \Bigg[(p-1)\det \begin{pmatrix} -p^{p+1}\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} - \lambda & (p-1)p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} \\ p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} & -(p-1)p^{p+1}\frac{f^{p}}{(A+(p-1)v)^{p+1}} \end{pmatrix} +$$
$$\det \begin{pmatrix} -p^{p+1}\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} - \lambda & p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} \\ p^{p+1}\frac{f^{p-1}}{(A+(p-1)v)^{p}} & -p^{p+1}\frac{f^{p}}{(A+(p-1)v)^{p+1}} - \lambda \end{pmatrix} \Bigg] =$$

$$\lambda^2 \Bigg[\lambda^2 + p^{2p+2}\Bigg[\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} + \frac{f^{p}}{(A+(p-1)v)^{p+1}} \Bigg]\lambda +$$
$$p^{2p+2}(p-1)\frac{f^{p}}{(A+(p-1)v)^{p+1}}\lambda \Bigg] =$$
$$\lambda^3 \Bigg[\lambda + p^{2p+2}\Bigg[\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} + p\frac{f^{p}}{(A+(p-1)v)^{p+1}} \Bigg]\Bigg].$$

So the eigenvalues of the Hessian matrix $\mathcal{H}(\mathcal{B})$ are 0 of algebraic multiplicity 3 and $\tilde{\lambda} = -p^{2p+2}\left[\frac{f^{p-2}}{(A+(p-1)v)^{p-1}} + p\frac{f^{p}}{(A+(p-1)v)^{p+1}}\right]$ of algebraic multiplicity 1. However, since $f > 0$, $v > 0$ and $A > 0$, then $\tilde{\lambda} > 0$, so all the eigenvalues are lower than or equal to 0. This entails that the Hessian matrix is negative semi-definite, so the function $\mathcal{B}$ is concave. □

We are going to prove that the function

$$\mathcal{B}(F, f, A, v) = \left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}}$$

satisfies $0 \le \mathcal{B}(F, f, A, v) \le \big(p/(p-1)\big)^p F$ for all $(F, f, A, v) \in \mathcal{D}$.

Proof. The thesis follows from the definition of the domain of the function. Since $p > 1$, $F > 0$,

$f > 0$, $A > 0$, $v \geq A$ and $f^p \leq Fv^{p-1}$, we get

$$\begin{aligned}
\mathcal{B}(F, f, A, v) \quad &= \quad \left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}} \geq \\
&\left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1}\frac{f^p}{(0+(p-1)v)^{p-1}} \geq \\
&\left(\frac{p}{p-1}\right)^p F - \left(\frac{p}{p-1}\right)^p \frac{Fv^{p-1}}{v^{p-1}} \geq \\
&\left(\frac{p}{p-1}\right)^p \Big[F - F\Big] = 0,
\end{aligned}$$

and

$$\mathcal{B}(F, f, A, v) = \left(\frac{p}{p-1}\right)^p F - \frac{p^p}{p-1}\frac{f^p}{(A+(p-1)v)^{p-1}} \leq \left(\frac{p}{p-1}\right)^p F.$$

□

We are going to prove that, by substitution of

$$h(A, v) = -\frac{1}{\psi(A, v)^{p-1}}, \quad \text{for } 0 < A < v, \tag{2.107}$$

in the equations

$$\begin{aligned}
&4''.\ h(A, v) < 0, \quad \text{for } 0 < A < v,\\
&5'.\ p\left[\left(\frac{\partial h}{\partial A}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2\right] - (p-1)\left[\frac{\partial^2 h}{\partial A^2} + \frac{\partial^2 h}{\partial v^2}\right] h \leq 0,\\
&6.\ p(p-1)\left[\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2} h - \left(\frac{\partial^2 h}{\partial A \partial v}\right)^2 h\right] + \\
&\quad p^2\left[2\frac{\partial h}{\partial A}\frac{\partial h}{\partial v}\frac{\partial^2 h}{\partial A \partial v} - \left[\left(\frac{\partial h}{\partial v}\right)^2 \frac{\partial^2 h}{\partial A^2} + \left(\frac{\partial h}{\partial A}\right)^2 \frac{\partial^2 h}{\partial v^2}\right]\right] \leq 0,
\end{aligned}$$

we get the conditions

$$\begin{aligned}
&\mathring{4}.\ \psi(A, v) > 0, \quad \text{for } 0 < A < v,\\
&\hat{5}.\ \Delta\psi \leq 0,\\
&\hat{6}.\ \left(\frac{\partial^2 \psi}{\partial A \partial v}\right)^2 - \frac{\partial^2 \psi}{\partial A^2}\frac{\partial^2 \psi}{\partial v^2} \leq 0.
\end{aligned}$$

Proof. Equation $\mathring{4}$ trivially follows from substition of (2.107) in $4''$.
For the other equations we compute the following derivatives:

$$\frac{\partial h}{\partial A} = \frac{\partial \psi}{\partial A}\frac{p-1}{\psi^p}, \quad \frac{\partial \psi}{\partial v} = \frac{\partial h}{\partial v}\frac{p-1}{\psi^p},$$

$$\frac{\partial^2 h}{\partial A \partial v} = \frac{\partial^2 \psi}{\partial A \partial v} \frac{p-1}{\psi^p} - \frac{\partial \psi}{\partial A} \frac{\partial \psi}{\partial v} \frac{p(p-1)}{\psi^{p+1}},$$

$$\frac{\partial^2 h}{\partial A^2} = \frac{\partial^2 \psi}{\partial A^2} \frac{p-1}{\psi^p} - \left(\frac{\partial \psi}{\partial A}\right)^2 \frac{p(p-1)}{\psi^{p+1}},$$

$$\frac{\partial^2 h}{\partial v^2} = \frac{\partial^2 \psi}{\partial v^2} \frac{p-1}{\psi^p} - \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)}{\psi^{p+1}}.$$

Now we compute the following products of derivatives:

$$\frac{\partial^2 h}{\partial v^2} \frac{\partial h^2}{\partial A^2} = \frac{\partial^2 \psi}{\partial v^2} \frac{\partial^2 \psi}{\partial v^2} \frac{(p-1)^2}{\psi^{2p}} - \frac{\partial^2 \psi}{\partial A^2} \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)^2}{\psi^{2p+1}} - \frac{\partial^2 \psi}{\partial v^2} \left(\frac{\partial \psi}{\partial A}\right)^2 \frac{p(p-1)^2}{\psi^{2p+1}} + \left(\frac{\partial \psi}{\partial A}\right)^2 \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p^2(p-1)^2}{\psi^{2p+2}}.$$

$$\frac{\partial h}{\partial A} \frac{\partial h}{\partial v} \frac{\partial^2 h}{\partial A \partial v} = \frac{\partial \psi}{\partial A} \frac{\partial \psi}{\partial v} \frac{\partial^2 \psi}{\partial A \partial v} \frac{(p-1)^3}{\psi^{3p}} - \left(\frac{\partial \psi}{\partial A}\right)^2 \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)^3}{\psi^{3p+1}},$$

$$\left(\frac{\partial h}{\partial A}\right)^2 \frac{\partial^2 h}{\partial v^2} = \left(\frac{\partial \psi}{\partial A}\right)^2 \frac{\partial^2 \psi}{\partial v^2} \frac{(p-1)^3}{\psi^{3p}} - \left(\frac{\partial \psi}{\partial A}\right)^2 \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)^3}{\psi^{3p+1}},$$

$$\left(\frac{\partial h}{\partial v}\right)^2 \frac{\partial^2 h}{\partial A^2} = \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{\partial^2 \psi}{\partial A^2} \frac{(p-1)^3}{\psi^{3p}} - \left(\frac{\partial \psi}{\partial A}\right)^2 \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)^3}{\psi^{3p+1}},$$

$$\frac{\partial^2 h}{\partial A^2} \frac{\partial^2 h}{\partial v^2} = \frac{\partial^2 \psi}{\partial A^2} \frac{\partial^2 \psi}{\partial v^2} \frac{(p-1)^2}{\psi^{2p}} - \frac{\partial^2 \psi}{\partial A^2} \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)^2}{\psi^{2p+1}} - \left(\frac{\partial \psi}{\partial A}\right)^2 \frac{\partial^2 \psi}{\partial v^2} \frac{p(p-1)^2}{\psi^{2p+1}} + \left(\frac{\partial \psi}{\partial A}\right)^2 \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p^2(p-1)^2}{\psi^{2p+2}},$$

$$\left(\frac{\partial^2 h}{\partial A \partial v}\right)^2 = \left(\frac{\partial^2 \psi}{\partial A \partial v}\right)^2 \frac{(p-1)^2}{\psi^{2p}} - 2 \frac{\partial^2 \psi}{\partial A \partial v} \frac{\partial \psi}{\partial A} \frac{\partial \psi}{\partial v} \frac{p(p-1)^2}{\psi^{2p+1}} + \left(\frac{\partial \psi}{\partial A}\right)^2 \left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p^2(p-1)^2}{\psi^{2p+2}}.$$

We substitute the previous computations in equation $5'$ and we get

$$
\begin{aligned}
0 \geq & p\left[\left(\frac{\partial h}{\partial A}\right)^2+\left(\frac{\partial h}{\partial v}\right)^2\right]-(p-1)\left[\frac{\partial^2 h}{\partial A^2}+\frac{\partial^2 h}{\partial v^2}\right] h= \\
& \left(\frac{\partial \psi}{\partial A}\right)^2 \frac{p(p-1)^2}{\psi^{2p}}+\left(\frac{\partial \psi}{\partial V}\right)^2 \frac{p(p-1)^2}{\psi^{2p}}-(p-1)\left[\frac{\partial^2 \psi}{\partial A^2} \frac{p-1}{\psi^p}-\left(\frac{\partial \psi}{\partial A}\right)^2 \frac{p(p-1)}{\psi^{p+1}}+\right. \\
& \left.\frac{\partial^2 \psi}{\partial v^2} \frac{p-1}{\psi^p}-\left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)}{\psi^{p+1}}\right] \frac{-1}{\psi(A, v)^{p-1}}= \\
& \left(\frac{\partial \psi}{\partial A}\right)^2 \frac{p(p-1)^2}{\psi^{2p}}+\left(\frac{\partial \psi}{\partial V}\right)^2 \frac{p(p-1)^2}{\psi^{2p}}+\frac{\partial^2 \psi}{\partial A^2} \frac{(p-1)^2}{\psi^{2p-1}}-\left(\frac{\partial \psi}{\partial A}\right)^2 \frac{p(p-1)^2}{\psi^{2p}}+ \\
& \frac{\partial^2 \psi}{\partial v^2} \frac{(p-1)^2}{\psi^{2p-1}}-\left(\frac{\partial \psi}{\partial v}\right)^2 \frac{p(p-1)^2}{\psi^{2p}}= \\
& \frac{(p-1)^2}{\psi^{2p-1}}\left[\frac{\partial^2 \psi}{\partial A^2}+\frac{\partial^2 \psi}{\partial v^2}\right].
\end{aligned}
$$

However, we have $\frac{(p-1)^2}{\psi^{2p-1}} \geq 0$, so we proved that equation $5'$ is equivalent to

$$\hat{5}.\ \Delta\psi \leq 0.$$

Now we substitute the previous computations in equation 6 and we get

$$
\begin{aligned}
0 \geq & p(p-1)\left[\frac{\partial^2 h}{\partial A^2}\frac{\partial^2 h}{\partial v^2}h - \left(\frac{\partial^2 h}{\partial A \partial v}\right)^2 h\right] + \\
& p^2\left[2\frac{\partial h}{\partial A}\frac{\partial h}{\partial v}\frac{\partial^2 h}{\partial A \partial v} - \left[\left(\frac{\partial h}{\partial v}\right)^2\frac{\partial^2 h}{\partial A^2} + \left(\frac{\partial h}{\partial A}\right)^2\frac{\partial^2 h}{\partial v^2}\right]\right] = \\
& - p(p-1)\left[\frac{\partial^2 \psi}{\partial A^2}\frac{\partial^2 \psi}{\partial v^2}\frac{(p-1)^2}{\psi^{3p-1}} - \frac{\partial^2 \psi}{\partial A^2}\left(\frac{\partial \psi}{\partial v}\right)^2\frac{p(p-1)^2}{\psi^{3p}} - \right. \\
& \left.\left(\frac{\partial \psi}{\partial A}\right)^2\frac{\partial^2 \psi}{\partial v^2}\frac{p(p-1)^2}{\psi^{3p}} + \left(\frac{\partial \psi}{\partial A}\right)^2\left(\frac{\partial \psi}{\partial v}\right)^2\frac{p^2(p-1)^2}{\psi^{3p+1}}\right] + \\
& p(p-1)\left[\left(\frac{\partial^2 \psi}{\partial A \partial v}\right)^2\frac{(p-1)^2}{\psi^{3p-1}} - 2\frac{\partial^2 \psi}{\partial A \partial v}\frac{\partial \psi}{\partial A}\frac{\partial \psi}{\partial v}\frac{p(p-1)^2}{\psi^{3p}} + \right. \\
& \left.\left(\frac{\partial \psi}{\partial A}\right)^2\left(\frac{\partial \psi}{\partial v}\right)^2\frac{p^2(p-1)^2}{\psi^{3p+1}}\right] + \\
& 2\frac{\partial \psi}{\partial A}\frac{\partial \psi}{\partial v}\frac{\partial^2 \psi}{\partial A \partial v}\frac{p^2(p-1)^3}{\psi^{3p}} - 2\left(\frac{\partial \psi}{\partial A}\right)^2\left(\frac{\partial \psi}{\partial v}\right)^2\frac{p^3(p-1)^3}{\psi^{3p+1}} - \\
& \left(\frac{\partial \psi}{\partial A}\right)^2\frac{\partial^2 \psi}{\partial v^2}\frac{p^2(p-1)^3}{\psi^{3p}} + 2\left(\frac{\partial \psi}{\partial A}\right)^2\left(\frac{\partial \psi}{\partial v}\right)^2\frac{p^3(p-1)^3}{\psi^{3p+1}} - \\
& \left(\frac{\partial \psi}{\partial v}\right)^2\frac{\partial^2 \psi}{\partial A^2}\frac{p^2(p-1)^3}{\psi^{3p}} = \\
& \frac{p(p-1)^3}{\psi^{3p-1}}\left[\left(\frac{\partial^2 \psi}{\partial A \partial v}\right)^2 - \frac{\partial^2 \psi}{\partial A^2}\frac{\partial^2 \psi}{\partial v^2}\right].
\end{aligned}
$$

However, we have $\frac{p(p-1)^3}{\psi^{3p-1}} \geq 0$, so we proved that equation 6 is equivalent to

$$
\hat{6}.\ \left(\frac{\partial^2 \psi}{\partial A \partial v}\right)^2 - \frac{\partial^2 \psi}{\partial A^2}\frac{\partial^2 \psi}{\partial v^2} \leq 0,
$$

finishing the proof. □

Chapter 3

Potential theory on Ahlfors-regular spaces

Introduction

In this chapter we prove formulas of quasi-additivity for the capacity associated to kernels of radial type in the setting of the boundary of a tree structure and in the setting of compact Ahlfors-regular spaces. We also define a notion of harmonic extension, to one additional variable, of a function defined over a compact Ahlfors-regular space, and we prove a result of tangential convergence of the harmonic extension to the values at the boundary.

This chapter is structured as follows.
In section 3.1 we define the capacity associated to a radial kernel on the boundary of a tree and we prove a quasi-additivity formula for the capacity.

In section 3.2 we define the capacity associated to the Riesz kernel on an Ahlfors-regular space and we prove a quasi-additivity formula for the capacity.

In section 3.3 we define the harmonic extension of a function defined over an Ahlfors-regular space and we enunciate and prove several properties of the Harmonic extension.

In section 3.4 we prove several technical lemmas and propositions and then we prove the two main results in this chapter: the non tangential convergence at the boundary of the harmonic extension of a Riesz potential up to an exceptional set of zero capacity and the tangential convergence at the boundary of the harmonic extension of a Riesz potential up to an exceptional set of null measure.

Notations

Let $a, b \in \mathbb{R}$. We write $a \lesssim b$ (respectively $a \gtrsim b$) if and only if there exists a constant $0 < C < +\infty$ such that $a \leq C \cdot b$ (respectively $a \geq C \cdot b$). Here the constant C does not depend on any of the parameters of the problem.
We write $a \lesssim_{(p_1,p_2,\ldots,p_n)} b$ (respectively $a \gtrsim_{(p_1,p_2,\ldots,p_n)} b$) if and only if there exists a constant $C = C(p_1, p_2, \ldots, p_n)$, $0 < C < +\infty$, such that $a \leq C \cdot b$ (respectively $a \geq C \cdot b$). Here the constant C depends on the paramenters $p_1, p_2, \ldots, p_n$.
We write $a \approx b$ if and only if both $a \lesssim b$ and $a \gtrsim b$ hold.
We write $a \approx_{(p_1,p_2,\ldots,p_n)} b$ if and only if both $a \lesssim_{(p_1,p_2,\ldots,p_n)} b$ and $a \gtrsim_{(p_1,p_2,\ldots,p_n)} b$ hold.
Let (X, d) be a metric space. Let $x \in X$, $r \geq 0$. We denote by $B_d(x, r)$ the metric ball of radius r and center x, i.e.

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}. \tag{3.1}$$

3.1 Quasi-additivity on tree boundaries

In this section we prove a quasi-additivity formula for capacities associated to radial kernels in the setting of the tree boundaries.

3.1.1 Setting of the problem

Let T be a tree. Suppose every node in T has at least 2 children. In this section $X := \partial T$ will denote the boundary of T. X is a metric space, where the metric on X is given by

$$\rho(x, y) = \delta^{-d(x \wedge y, o)},$$

where $\delta > 1$ is a fixed constant.
Let m be a σ-finite Borel measure on X.
Let $\mathcal{D}$ denote the set

$$\mathcal{D} := \{0\} \cup \{\delta^{-n} \mid n \in \mathbb{N}\}. \tag{3.2}$$

Let $K : \mathcal{D} \to \mathbb{R}^+$ be a function. Suppose K is lower semi-continuos in 0, i.e.

$$\liminf_{n \to +\infty} K(\delta^{-n}) \geq K(0). \tag{3.3}$$

We define, with a small abuse of notation, the kernel $K(x,y) := K(\rho(x,y))$ for $x, y \in X$. It follows that the function

$$x \longmapsto K(x, y_0)$$

is lower semi-continuous, for every choice of $y_0 \in X$.
Suppose the kernel K satisfies the following conditions:

$$\sup_{x\in X} \int_X K(x,y)dm(y) < +\infty, \qquad \sup_{y\in X} \int_X K(x,y)dm(x) < +\infty. \tag{3.4}$$

Let us denote

$$\|K\|_1 := \max\left\{\sup_{x\in X} \int_X K(x,y)dm(y), \sup_{y\in X} \int_X K(x,y)dm(x)\right\} < +\infty. \tag{3.5}$$

Definition 3.1.1. Let $1 < p < +\infty$. The capacity of a compact set $E \subseteq X$ is

$$C_{K,p}(E) := \inf\left\{\|f\|^p_{L^p(X,m)} \;\middle|\; K * f(x) \geq 1 \quad \forall x \in E\right\}, \tag{3.6}$$

where

$$K * f(x) := \int_X K(x,y)f(y)dm(y).$$

Definition 3.1.2. Let $x \in X$, $r > 0$. We define, when it exists, the radius

$$\eta_p(x,r) := \inf\left\{\delta^{-n+\frac{1}{2}} \in \mathbb{R} \;\middle|\; n \in \mathbb{N},\ m(B_\rho(x, \delta^{-n+\frac{1}{2}})) \geq C_{K,p}(B_\rho(x,r))\right\}. \tag{3.7}$$

We also define

$$\eta^*_p(x,r) := \max\{r, \eta_p(x,r)\}. \tag{3.8}$$

It follows that $B_\rho(x,r) \subseteq B_\rho(x, \eta^*_p(x,r))$.

Observation 3.1.1. The radius $\eta_p(x,r)$ does not exist when X is compact for $x \in X$ and $r > 0$ such that

$$C_{K,p}(B_\rho(x,r)) > m(X). \tag{3.9}$$

However, all the propositions and theorems using η^*_p can be proved by separating the cases where η^*_p is not defined, using other properties, like the compactness of X. This follows from the properties of the Riesz capacity of a ball in an Ahlfors-regular space (see Proposition 3.2.2). We will always assume that the radius η^*_p exists in the following proofs.

We enunciate Young's inequality for the setting of tree boundaries.

Lemma 3.1.1 (Young's inequality for tree boundaries)**.** *Let $K = K(x,y)$ be a kernel on a metric measure space (X,ρ,m), and $1 \le p \le \infty$. Then,*

$$\|K * f\|_{L^p(X,m)} := \left[\int_X \left(\int_X K(x,y)f(y)dm(y)\right)^p dm(x)\right]^{\frac{1}{p}} \le \|K\|_1 \|f\|_{L^p(X,m)},$$

where

$$\|K\|_1 := \max\left\{\sup_{x\in X}\int_X K(x,y)dm(y), \sup_{y\in X}\int_X K(x,y)dm(x)\right\}.$$

Proof. We prove this result by interpolation. For $p=\infty$ and $f \ge 0$, we have

$$\begin{aligned}\int_X K(x,y)f(y)dm(y) &\le \int_X K(x,y)dm(y)\|f\|_{L^\infty(X,m)} \le \\ &\le \left(\sup_{x\in X}\int_X K(x,y)dm(y)\right)\|f\|_{L^\infty(X,m)},\end{aligned}$$

hence

$$\|K * f\|_{L^\infty(X,m)} \le \left(\sup_{x\in X}\int_X K(x,y)dm(y)\right)\|f\|_{L^\infty(X,m)}.$$

For $p = 1$ we have

$$\begin{aligned}\int_X\int_X K(x,y)f(y)dm(y)dm(x) &= \int_X K(x,y)dm(x)\int_X f(y)dm(y) \le \\ &\le \left(\sup_{y\in X}\int_X K(x,y)dm(x)\right)\int_X f(y)dm(y),\end{aligned}$$

hence

$$\|K * f\|_{L^1(X,m)} \le \left(\sup_{y\in X}\int K(x,y)dm(x)\right)\|f\|_{L^1(X,m)}.$$

The result follows from Riesz-Thorin interpolation theorem. □

3.1.2 Quasi additivity for tree boundaries

We are now going to prove the first result in this chapter.

Theorem 3.1.2 (Quasi-additivity for tree boundaries)**.** *Let J be a countable (or finite) set of indices. Let $\{B_\rho(x_j,r_j)\}_{j\in J}$ be a family of metric balls in X such that $\eta_p(x_j,r_j)$ exists for all $j \in J$. Suppose $E \subseteq X$ is a compact subset of $\bigcup_{j\in J} B_\rho(x_j,r_j)$. Suppose $\{B_\rho(x_j,\eta_p^*(x_j,r_j))\}_{j\in J}$ is disjoint.*
Then

$$C_{K,p}(E) \le \sum_{j\in J} C_{K,p}(E\cap B_\rho(x_j,r_j)) \le A\cdot C_{K,p}(E), \tag{3.10}$$

where $A = A(X,K,p)$, $1 < A < +\infty$, is a constant depending only on X, K and p.

For the proof of Theorem 3.1.2 we recall the dual definition of capacity (see [1]).

Theorem 3.1.3 (Dual definition of capacity)**.** *Let* $\frac{1}{p}+\frac{1}{q}=1$*. Then*

$$C_{K,p}(E)=\sup\left\{\|\mu\|^p \;\middle|\; \mu \text{ is concentrated on } E;\ \|K*\mu\|_{L^q(X,m)}\leq 1\right\}, \tag{3.11}$$

where

$$\|\mu\|:=\int_X d\mu(x);\quad K*\mu(y):=\int_X K(x,y)d\mu(x)$$

Proof of Theorem 3.1.2. If $C_{K,p}(E)=0$ the proof is trivial by the monotonicity of the capacity. Suppose $C_{K,p}(E)>0$. Let $E_j:=E\cap B_\rho(x_j,r_j)$. Without loss of generality, we may assume $C_{K,p}(E_j)>0$ for all j. Indeed, let $J_0:=\{j\in J\mid C_{K,p}(E_j)>0\}$. It follows that $C_{K,p}(E_j)=0$ for all $j\in J\backslash J_0$, so

$$\sum_{j\in J}C_{K,p}(E\cap B_\rho(x_j,r_j))=\sum_{j\in J_0}C_{K,p}(E\cap B_\rho(x_j,r_j)),$$

hence we will assume $C_{K,p}(E_j)>0$ for all $j\in J$.
To prove the thesis (3.10) it is sufficient to prove that

$$\sum_{j\in J}C_{K,p}(E\cap B_\rho(x_j,r_j))\leq A\cdot C_{K,p}(E). \tag{3.12}$$

For simplicity we will assume $J\subseteq\mathbb{N}$. Let $E_j:=E\cap B_\rho(x_j,r_j)$ for $j\in J$, let $0<\epsilon<\sum_j C_{K,p}(E_j)$ be arbitrary. Let $\frac{1}{p}+\frac{1}{q}=1$. By the dual definition of capacity for every $j\in J$ there exists a measure μ_j such that

$$\begin{cases}\mu_j \text{ is concentrated on } E_j,\\ \|K*\mu_j\|_{L^q(X,m)}=1,\\ C_{K,p}(E_j)-2^{-j}\epsilon\leq\|\mu_j\|^p\leq C_{k,p}(E_j).\end{cases} \tag{3.13}$$

Let $\mu_j^*:=C_{k,p}(E_j)^{\frac{1}{q}}\mu_j$ and $\mu^*:=\sum_j\mu_j^*$.
We get

$$\|\mu_j^*\|^p=C_{k,p}(E_j)^{\frac{p}{q}}\|\mu_j\|^p. \tag{3.14}$$

From (3.13) we get

$$C_{K,p}(E_j)^{1+\frac{p}{q}}-C_{K,p}(E_j)^{\frac{p}{q}}\cdot 2^{-j}\epsilon\leq C_{K,p}(E_j)^{\frac{p}{q}}\|\mu_j\|^p\leq C_{k,p}(E_j)^{1+\frac{p}{q}}. \tag{3.15}$$

From (3.14) and (3.15) we get

$$C_{K,p}(E_j)^{1+\frac{p}{q}}-C_{K,p}(E_j)^{\frac{p}{q}}\cdot 2^{-j}\epsilon\leq\|\mu_j^*\|^p\leq C_{k,p}(E_j)^{1+\frac{p}{q}}. \tag{3.16}$$

If $2^{-j}\epsilon > C_{K,p}(E_j)$ then, from (3.16), we get

$$C_{K,p}(E_j) - 2^{-j}\epsilon < 0 < \|\mu_j^*\| \leq C_{k,p}(E_j), \tag{3.17}$$

so we proved that

$$C_{K,p}(E_j) - 2^{-j}\epsilon \leq \|\mu_j^*\| \leq C_{k,p}(E_j). \tag{3.18}$$

We are going to prove that (3.18) also holds when $2^{-j}\epsilon \leq C_{K,p}(E_j)$. To prove it we claim the following:

$$C_{K,p}(E_j) - 2^{-j}\epsilon \leq \left[C_{K,p}(E_j)^{1+\frac{p}{q}} - C_{K,p}(E_j)^{\frac{p}{q}} \cdot 2^{-j}\epsilon\right]^{\frac{1}{p}}. \tag{3.19}$$

Indeed, let $j \in J$, suppose $2^{-j}\epsilon \leq C_{K,p}(E_j)$. From (3.16) we get

$$\left[C_{K,p}(E_j)^{1+\frac{p}{q}} - C_{K,p}(E_j)^{\frac{p}{q}} \cdot 2^{-j}\epsilon\right]^{\frac{1}{p}} \leq \|\mu_j^*\| \leq C_{k,p}(E_j). \tag{3.20}$$

Let

$$\psi_j : [0, C_{K,p}(E_j)] \longrightarrow \mathbb{R},$$

$$\psi_j(x) = \left[C_{K,p}(E_j)^{1+\frac{p}{q}} - C_{K,p}(E_j)^{\frac{p}{q}} \cdot x\right]^{\frac{1}{p}}.$$

The function ψ_j is concave, and

$$\psi_j(0) = C_{K,p}(E_j); \quad \psi_j(C_{K,p}(E_j)) = 0.$$

Consider the following function:

$$\phi_j : [0, C_{K,p}(E_j)] \longrightarrow \mathbb{R},$$

$$\phi_j(x) = C_{K,p}(E_j) - x.$$

The function ψ_j is linear, and

$$\phi_j(0) = C_{K,p}(E_j); \quad \phi_j(C_{K,p}(E_j)) = 0.$$

By concavity we get $\phi_j(x) \leq \psi_j(x)$ for all $x \in [0, C_{K,p}(E_j)]$.
We observe that, by hypothesis, $0 \leq 2^{-j}\epsilon \leq C_{K,p}(E_j)$, so, using (3.20), we get

$$C_{K,p}(E_j) - 2^{-j}\epsilon = \phi_j(2^{-j}\epsilon) \leq \psi_j(2^{-j}\epsilon) = \left[C_{K,p}(E_j)^{1+\frac{p}{q}} - C_{K,p}(E_j)^{\frac{p}{q}} \cdot 2^{-j}\epsilon\right]^{\frac{1}{p}} \leq \|\mu_j^*\| \leq C_{k,p}(E_j),$$

proving the claim (3.19), and thus proving that (3.18) holds for any choice of $j \in J$ and for any ϵ.

Now we are going to prove that

$$\|K * \mu^*\|^q_{L^q(X,m)} \le C \cdot \sum_{j\in J} C_{K,p}(E_j), \tag{3.21}$$

where $C = C(X, K, p)$ is a constant depending only on X, K and p.
Let $j \in J$. We define the measure

$$d\mu'_j(y) := \frac{\|\mu^*_j\|}{m(B_\rho(x_j, \eta^*_p(x_j, r_j)))} \chi_{B_\rho(x_j,\eta^*_p(x_j,r_j))}(y) dm(y); \quad \mu' := \sum_{j\in J} \mu'_j. \tag{3.22}$$

By construction $\|\mu'_j\| = \|\mu^*_j\|$, so, using (3.18), we get

$$\|\mu'_j\| \le C_{K,p}(E_j). \tag{3.23}$$

By the definition of the function η^*_p we have

$$m(B_\rho(x_j, \eta^*_p(x_j, r_j))) \ge C_{K,p}(B_\rho(x_j, r_j)) \ge C_{K,p}(B_\rho(x_j, r_j) \cap E) = C_{K,p}(E_j), \tag{3.24}$$

so, using (3.22) and (3.24), we get

$$\begin{aligned} d\mu'_j =& \frac{\|\mu^*_j\|}{m(B_\rho(x_j, \eta^*_p(x_j, r_j)))} \chi_{B_\rho(x_j,\eta^*_p(x_j,r_j))} dm \le \\ \le& \frac{C_{K,p}(E_j)}{C_{K,p}(E_j)} \chi_{B_\rho(x_j,\eta^*_p(x_j,r_j))} dm \le \\ \le& \chi_{B_\rho(x_j,\eta^*_p(x_j,r_j))} dm. \end{aligned}$$

By construction $d\mu'$ has a density with respect to dm, so we may write $d\mu' = f \cdot dm$, where

$$f \le \sum_{j\in J} \chi_{B_\rho(x_j,\eta^*_p(x_j,r_j))}, \tag{3.25}$$

and we get

$$\|\mu'\| = \|f\|_{L^1(X,m)}. \tag{3.26}$$

By hypothesis the sets $B_\rho(x_j, \eta^*_p(x_j, r_j))$ are disjoint, so we have

$$f \le 1. \tag{3.27}$$

Using (3.4), (3.23), (3.26), (3.27) and Lemma 3.1.1 we get

$$\begin{aligned}\|K*\mu'\|^q_{L^q(X,m)} =& \int_X \left[\int_X K(\rho(x,y))d\mu'(y)\right]^q dm(x) = \\ & \int_X \left[\int_X K(\rho(x,y))f(y)dm(y)\right]^q dm(x) \leq \\ & \|K\|_1^q \|f\|^q_{L^q(X,m)} \leq \|K\|_1^q \|f\|^{q-1}_{L^\infty(X,m)} \|f\|_{L^1(X,m)} \leq \\ & \|K\|_1^q \cdot 1^{q-1} \cdot \|f\|_{L^1(X,m)} = \|K\|_1^q \|\mu'\| = \|K\|_1^q \sum_j \|\mu'_j\| \leq \\ & \|K\|_1^q \sum_j C_{K,p}(E_j).\end{aligned}$$

So we proved the following estimate for the q-norm of the potential of the measure μ':

$$\|K*\mu'\|^q_{L^q(X,m)} \leq \|K\|_1^q \sum_j C_{K,p}(E_j), \tag{3.28}$$

where $\|K\|_1^q$ is a constant depending only on X, K and p, and, by (3.4), $\|K\|_1^q < +\infty$.

We are now going to prove an estimate for $K*\mu^*(\tilde{x})$, for $\tilde{x} \in X$.
Let $j \in J$. Let $\tilde{x} \in X$ be a point such that $\tilde{x} \notin B_\rho(x_j, \eta_p^*(x_j, r_j))$. The measure μ'_j is concentrated on the set $B_\rho(x_j, \eta_p^*(x_j, r_j))$, so we get

$$\begin{aligned}K*\mu'_j(\tilde{x}) =& \int_X K(\tilde{x},y)d\mu'_j(y) = \int_{B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)d\mu'_j(y) \geq \\ & \left(\min_{y\in B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)\right)\mu'_j(B_\rho(x_j,\eta_p^*(x_j,r_j))) = \\ & \left(\min_{y\in B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)\right)\|\mu_j^*\| \frac{m(B_\rho(x_j,\eta_p^*(x_j,r_j)))}{m(B_\rho(x_j,\eta_p^*(x_j,r_j)))} = \\ & \left(\min_{y\in B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)\right)\|\mu_j^*\|.\end{aligned}$$

Moreover, μ_j^* is concentrated on $E_j \subseteq B_\rho(x_j, \eta_p^*(x_j, r_j))$, so we get

$$\begin{aligned}K*\mu_j^*(\tilde{x}) =& \int_X K(\tilde{x},y)d\mu_j^*(y) = \int_{B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)d\mu_j^*(y) \leq \\ & \left(\max_{y\in B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)\right)\mu_j^*(B_\rho(x_j,\eta_p^*(x_j,r_j))) = \\ & \left(\max_{y\in B_\rho(x_j,\eta_p^*(x_j,r_j))} K(\tilde{x},y)\right)\|\mu_j^*\|.\end{aligned}$$

So we proved that

$$K * \mu'_j(\tilde{x}) \geq \left(\min_{y \in B_\rho(x_j, \eta_p^*(x_j, r_j))} K(\tilde{x}, y) \right) \|\mu_j^*\|; \quad K * \mu_j^*(\tilde{x}) \leq \left(\max_{y \in B_\rho(x_j, \eta_p^*(x_j, r_j))} K(\tilde{x}, y) \right) \|\mu_j^*\|. \tag{3.29}$$

However, $K(\tilde{x}, y) = K(\rho(\tilde{x}, y))$, and $\tilde{x} \notin B_\rho(x_j, \eta_p^*(x_j, r_j))$, so, by definition of ρ, if follows that

$$\rho(\tilde{x}, y_1) = \rho(\tilde{x}, y_2) \quad \text{for all } y_1, y_2 \in B_\rho(x_j, \eta_p^*(x_j, r_j)). \tag{3.30}$$

So the function

$$y \mapsto \rho(\tilde{x}, y)$$

is constant when $y \in B_\rho(x_j, \eta_p^*(x_j, r_j))$, which proves that

$$\min_{y \in B_\rho(x_j, \eta_p^*(x_j, r_j))} K(\tilde{x}, y) = \max_{y \in B_\rho(x_j, \eta_p^*(x_j, r_j))} K(\tilde{x}, y). \tag{3.31}$$

So, using (3.29) and (3.31) , we get

$$K * \mu_j^*(\tilde{x}) \leq K * \mu'_j(\tilde{x}) \quad \text{for } \tilde{x} \notin B_\rho(x_j, \eta_p^*(x_j, r_j)). \tag{3.32}$$

If $\tilde{x} \notin \bigcup_j B_\rho(x_j, \eta_p^*(x_j, r_j))$ then, by applying (3.32) for all $j \in J$, we get

$$K * \mu^*(\tilde{x}) \leq K * \mu'(\tilde{x}). \tag{3.33}$$

If $\tilde{x} \in B_\rho(x_{j_0}, \eta_p^*(x_{j_0}, r_{j_0}))$ for some $j_0 \in J$, then, by the disjointness of $\{B_\rho(x_j, \eta_p^*(x_j, r_j))\}_{j \in J}$, we have

$$\tilde{x} \notin \bigcup_{j \neq j_0} B_\rho(x_j, \eta_p^*(x_j, r_j)),$$

so we get

$$\begin{aligned} K * \mu^*(\tilde{x}) =& K * \mu_{j_0}^*(\tilde{x}) + \sum_{j \neq j_0} K * \mu_j^*(\tilde{x}) \leq \\ & K * \mu_{j_0}^*(\tilde{x}) + \sum_{j \neq j_0} K * \mu'_j(\tilde{x}) \leq \\ & K * \mu_{j_0}^*(\tilde{x}) + \sum_{j} K * \mu'_j(\tilde{x}) = \\ & K * \mu_{j_0}^*(\tilde{x}) + K * \mu'(\tilde{x}). \end{aligned}$$

So we proved that, if $\tilde{x} \in B_\rho(x_{j_0}, \eta_p^*(x_{j_0}, r_{j_0}))$ for some $j_0 \in J$, then

$$K * \mu^*(\tilde{x}) \leq K * \mu_{j_0}^*(\tilde{x}) + K * \mu'(\tilde{x}). \tag{3.34}$$

Now we are going to prove that

$$\|K * \mu^*\|^q_{L^q(X,m)} \leq C \cdot \sum_j C_{k,p}(E_j), \tag{3.35}$$

where $C = C(X, K, p)$ is a constant depending only on X, K and p.
We use (3.33), (3.34) and the disjointness of the sets $\{B_\rho(x_j, \eta_p^*(x_j, r_j))\}_j$ to estimate

$$\begin{aligned}\|K * \mu^*\|^q_{L^q(X,m)} &= \int_{X\setminus\bigcup_j B_\rho(x_j,\eta_p^*(x_j,r_j))} (K * \mu^*(x))^q dm(x) + \\ &\int_{\bigcup_j B_\rho(x_j,\eta_p^*(x_j,r_j))} (K * \mu^*(x))^q dm(x) \leq \\ &\int_{X\setminus\bigcup_j B_\rho(x_j,\eta_p^*(x_j,r_j))} (K * \mu'(x))^q dm(x) + \\ &\sum_j \int_{B_\rho(x_j,\eta_p^*(x_j,r_j))} (K * \mu^*(x))^q dm(x) \leq \\ &\|K * \mu'\|^q_{L^q(X,m)} + \sum_j \int_{B_\rho(x_j,\eta_p^*(x_j,r_j))} (K * \mu_j^*(x) + K * \mu'(x))^q dm(x).\end{aligned}$$

We are going to apply Jensen's inequality for finite sums to the last inequality. Let $n \geq 2$ be a natural number, let $q > 1$, let $a_i \geq 0$ for $1 \leq i \leq n$. By Jensen's inequality we have

$$\left(\sum_{i=1}^n a_i\right)^q \leq n^{q-1} \sum_{i=1}^n a_i^q. \tag{3.36}$$

For each fixed $j \in J$ we apply (3.36) to the last estimate for the q-norm of $K * \mu^*$, where $n = 2$, $a_1 = K * \mu_j^*(x)$ and $a_2 = K * \mu'(x)$, and we get

$$\|K * \mu^*\|^q_{L^q(X,m)} \leq \|K * \mu'\|^q_{L^q(X,m)} + 2^{q-1} \sum_j \int_{B_\rho(x_j,\eta_p^*(x_j,r_j))} (K * \mu_j^*(x)^q + K * \mu'(x)^q) dm(x). \tag{3.37}$$

So we get the estimate

$$\|K * \mu^*\|^q_{L^q(X,m)} \leq \|K * \mu'\|^q_{L^q(X,m)} + 2^{q-1} \sum_j \|K * \mu_j^*\|^q_{L^q(X,m)} + 2^{q-1} \|K * \mu'\|^q_{L^q(X,m)}. \tag{3.38}$$

We observe that, by definition of μ^*, we have

$$\|K * \mu_j^*\|^q_{L^q(X,m)} = \|K * (C_{K,p}(E_j)^{\frac{1}{q}} \mu_j)\|^q_{L^q(X,m)} = C_{K,p}(E_j) \|K * \mu_j\|^q_{L^q(X,m)}, \tag{3.39}$$

but $\|K * \mu_j\|_{L^q(X,m)} = 1$ by (3.13), so we get

$$\|K * \mu_j^*\|_{L^q(X,m)}^q = C_{K,p}(E_j). \tag{3.40}$$

So we use (3.28) and (3.40) and the estimate (3.38) to get

$$\|K * \mu^*\|_{L^q(X,m)}^q \leq \|K\|_1^q \sum_j C_{K,p}(E_j) + 2^{q-1} \sum_j C_{K,p}(E_j) + 2^{q-1}\|K\|_1^q \sum_j C_{K,p}(E_j). \tag{3.41}$$

So we proved that

$$\|K * \mu^*\|_{L^q(X,m)}^q \leq \left[(2^{q-1}+1)\|K\|_1^q + 2^{q-1}\right] \sum_j C_{K,p}(E_j), \tag{3.42}$$

where $C = [(2^{q-1}+1)\|K\|_1^q + 2^{q-1}] < +\infty$ is a constant depending only on X, K and p.

Now we are going to finish the proof by defining a proper normalized measure.
We define the measure

$$\mathring{\mu} := \left[C \cdot \sum_j C_{K,p}(E_j)\right]^{-\frac{1}{q}} \mu^*. \tag{3.43}$$

From (3.42) we get

$$\|K * \mu^*\|_{L^q(X,m)} \leq \left[C \cdot \sum_j C_{K,p}(E_j)\right]^{\frac{1}{q}}. \tag{3.44}$$

By construction $\mathring{\mu}$ is concentrated on E, and, using (3.44), we get

$$\begin{aligned}\|K * \mathring{\mu}\|_{L^q(X,m)} =& \left\|K * \left[\big[C \cdot \sum_j C_{K,p}(E_j)\big]^{-\frac{1}{q}} \mu^*\right]\right\|_{L^q(X,m)} = \\ & \big[C \cdot \sum_j C_{K,p}(E_j)\big]^{-\frac{1}{q}} \|K * \mu^*\|_{L^q(X,m)} \leq \\ & \big[C \cdot \sum_j C_{K,p}(E_j)\big]^{-\frac{1}{q}} \big[C \cdot \sum_j C_{K,p}(E_j)\big]^{\frac{1}{q}} = 1.\end{aligned}$$

So we proved that the measure $\mathring{\mu}$ is a test measure for the dual definition of capacity, so we get

$$C_{K,p}(E) \geq \|\mathring{\mu}\|^p. \tag{3.45}$$

By computation, using (3.18), we get

$$\begin{aligned}\|\mathring{\mu}\| =&\left[C\cdot\sum_j C_{K,p}(E_j)\right]^{-\frac{1}{q}}\|\mu^*\| = \\ &\left[C\cdot\sum_j C_{K,p}(E_j)\right]^{-\frac{1}{q}}\sum_j\|\mu_j^*\| \geq \\ &\left[C\cdot\sum_j C_{K,p}(E_j)\right]^{-\frac{1}{q}}\sum_j(C_{K,p}(E_j)-2^{-j}\epsilon) \geq \\ &\left[C\cdot\sum_j C_{K,p}(E_j)\right]^{-\frac{1}{q}}\left[\sum_j C_{K,p}(E_j)-\epsilon\right].\end{aligned}$$

So, using the last inequality and (3.45), we get

$$C_{K,p}(E) \geq C^{-\frac{p}{q}}\left[\sum_j C_{K,p}(E_j)^{-\frac{1}{q}}\left[\sum_j C_{K,p}(E_j)-\epsilon\right]\right]^p,$$

for any arbitrary $\epsilon > 0$. We let $\epsilon \to 0$ and, since $1-\frac{1}{q}=\frac{1}{p}$, we get

$$\sum_j C_{K,p}(E_j) \leq C^{\frac{p}{q}} C_{K,p}(E). \tag{3.46}$$

So, since $\frac{p}{q}=\frac{1}{q-1}$, we proved that

$$\sum_j C_{K,p}(E_j) \leq \left[(2^{q-1}+1)\|K\|_1^q + 2^{q-1}\right]^{\frac{1}{q-1}} C_{K,p}(E), \tag{3.47}$$

where the constant $A = [(2^{q-1}+1)\|K\|_1^q + 2^{q-1}]^{\frac{1}{q-1}} < +\infty$ is a constant depending only on X, K and p, ending the proof. □

Remark 3.1.1. The previous theorem also holds (up to modifying the constant A) for a generic non-radial kernel $K = K(x,y) \geq 0$ such that:

- The function $x \mapsto K(x, y_0)$ is lower semi-continuous for any $y_0 \in X$.
- The function $y \mapsto K(x_0, y)$ is measurable for any $x_0 \in X$.
- The kernel is globally integrable, i.e.

$$\|K\|_1 := \max\left\{\sup_{x\in X}\int_X K(x,y)dm(y), \sup_{y\in X}\int_X K(x,y)dm(x)\right\} < +\infty.$$

- There exists a constant $C = C(X, K, p)$ such that at least one of the following conditions holds:

  1.
$$K * \mu_j^E(x) \leq C \cdot K * \left(\mu_j^E\right)'(x),$$
for all $x \in B_\rho(x_j, \eta_p^*(x_j, r_j))$, for all $j \in J$, where μ_j^E is the equilibrium measure for the set $E_j := E \cap B_\rho(x_j, \eta_p^*(x_j, r_j))$, and $\left(\mu_j^E\right)'$ is the measure defined by
$$d\left(\mu_j^E\right)'(y) := \frac{\|\mu_j^E\|}{m(B_\rho(x_j, \eta_p^*(x_j, r_j)))} \chi_{B_\rho(x_j, \eta_p^*(x_j, r_j))}(y) dm(y).$$

  2.
$$\sup_{y \in B_\rho(x_j, \eta_p^*(x_j, r_j))} K(\tilde{x}, y) \leq C \cdot \inf_{y \in B_\rho(x_j, \eta_p^*(x_j, r_j))} K(\tilde{x}, y),$$
for all $\tilde{x} \notin B_\rho(x_j, \eta_p^*(x_j, r_j))$, for any $j \in J$.

  We observe that the condition 2 entails the condition 1.

The following corollary allows to reformulate the previous theorem for different values of the constant A.

Corollary 3.1.4. *Let J be a countable (or finite) set of indices, and $\alpha > 0$ a real number. Consider the kernels K and αK and the associated capacities $C_{K,p}$ and $C_{\alpha K,p}$. Denote by $\eta_{\alpha K,p}(x_j, r_j)$ the radius*
$$\eta_{\alpha K,p}(x, r) := \inf\left\{\delta^{-n+\frac{1}{2}} \in \mathbb{R} \;\middle|\; n \in \mathbb{N},\ m(B_\rho(x, \delta^{-n+\frac{1}{2}})) \geq C_{\alpha K,p}(B_\rho(x, r))\right\}, \tag{3.48}$$
and define
$$\eta^*_{\alpha K,p}(x, r) := \max\{r, \eta_{\alpha K,p}(x, r)\}. \tag{3.49}$$
Let $\{B_\rho(x_j, r_j)\}_{j \in J}$ be a family of metric balls in X such that $\eta_{\alpha K,p}(x_j, r_j)$ exists for all $j \in J$. Suppose $E \subseteq X$ is a compact subset of $\bigcup_{j \in J} B_\rho(x_j, r_j)$. Suppose $\{B_\rho(x_j, \eta^*_{\alpha K,p}(x_j, r_j))\}_{j \in J}$ is disjoint.
Then
$$C_{K,p}(E) \leq \sum_{j \in J} C_{K,p}(E \cap B_\rho(x_j, r_j)) \leq A(X, K, \alpha, p) \cdot C_{K,p}(E), \tag{3.50}$$
here $A(X, K, \alpha, p) = [(2^{q-1} + 1)\alpha^q \|K\|_1^q + 2^{q-1}]^{\frac{1}{q-1}}$ is a constant depending on X, K, α and p.

3.2 Ahlfors-regular spaces

In this section we prove a quasi-additivity formula for the Riesz capacity in the setting of the Ahlfors-regular spaces.

3.2.1 Setting of the problem

Definition 3.2.1. Let (X, d, m) be a compact metric measure space. Let $Q > 0$. We say that X is a Q-regular Ahlfors space if there exist constants $0 < C_1 \leq C_2$ such that

$$C_1 r^Q \leq m(B(x, r)) \leq C_2 r^Q \tag{3.51}$$

for all $x \in X$, for all $0 < r < \mathrm{diam}(X)$.
We will say that X is an Ahlfors-regular space without mentioning the dimension Q when that paremeter is not relevant to the discussion.

Our work focuses on Q-regular Ahlfors spaces such that the measure m is the Hausdorff measure of dimensional parameter Q. Hausdorff measures are known to be regular measures, so we will assume regularity of the measure m in the following part of this work.
The following theorem and definitions allow us construct a tree structure starting from an Ahlfors-regular space (see [8] for more details).

Theorem 3.2.1 (Christ decomposition). *Let (X, d, m) be a compact Q-regular Ahlfors space. There exists a collection of qubes $\{Q_\alpha^k \subseteq X \mid \alpha \in I_k,\, k \in \mathbb{N}\}$, where I_k is a set of indices, and there exist constants $0 < \delta < 1$, $C_3 > 0$ and $C_4 > 0$ such that:*

i) $m\left(X \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k\right) = 0$ *for all* $k \in \mathbb{N}$.

ii) If $l \geq k$ then $\forall \alpha \in I_k$, $\forall \beta \in I_l$ we have either $Q_\beta^l \subseteq Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$.

iii) For all $l \in \mathbb{N}$, for all $\beta \in I_l$ and for all $k < l$ there exists a unique $\alpha \in I_k$ such that $Q_\beta^l \subseteq Q_\alpha^k$.

iv) $diam(Q_\alpha^k) \leq C_4 \cdot \delta^k$.

v) For all $k \in \mathbb{N}$, for all $\alpha \in I_k$, there exist $z_\alpha^k \in Q_\alpha^k$ such that $B(z_\alpha^k, C_3 \cdot \delta^k) \subseteq Q_\alpha^k$.

Moreover, we may assume that $I_0 = \{1\}$, and $Q_1^0 = X$.

Definition 3.2.2. Let (X, d, m) be an Ahlfors Q-regular space as above. Consider a Christ decomposition

$$\{Q_\alpha^k \subseteq X \mid \alpha \in I_k,\, k \in \mathbb{N}\}, \tag{3.52}$$

like the one in Theorem 3.2.1, such that $I_0 = \{1\}$, and $Q^0_1 = X$.
We define a tree structure $T \equiv (V(T), E(T))$, where the set of the vertices of T is

$$V(T) := \{Q^k_\alpha \subseteq X \mid \alpha \in I_k,\ k \in \mathbb{N}\}, \tag{3.53}$$

and we define the set of edges $E(T)$ in the following way: for all $Q^k_\alpha \in V(T)$ and for all $Q^l_\beta \in V(T)$ then $(Q^k_\alpha, Q^l_\beta) \in E(T)$ if and only if $k = l - 1$ and α is the unique index in I_k such that $Q^l_\beta \subseteq Q^k_\alpha$ defined by property iii) in Theorem 3.2.1.
The structure $T \equiv (V(T), E(T))$ is a tree structure such that $o := Q^0_1 = X$ is the root of the tree T.

Definition 3.2.3. (X, d, m) be a Q-regular Ahlfors space, and let $T \equiv (V(T), E(T))$ be the tree previously defined. We define the boundary ∂T of the tree T as the set of half infinite geodesics starting at the origin o, i.e.

$$\partial T := \left\{ \left(Q^0_{\alpha_0}, Q^1_{\alpha_1}, \ldots, Q^{j-1}_{\alpha_{j-1}}, Q^j_{\alpha_j}, Q^{j+1}_{\alpha_{j+1}}, \ldots \right) \,\middle|\, Q^0_{\alpha_0} = o,\ (Q^k_{\alpha_k}, Q^{k+1}_{\alpha_{k+1}}) \in E(T)\ \forall k \in N \right\}. \tag{3.54}$$

We define the map

$$\Lambda : \partial T \longrightarrow X \tag{3.55}$$
$$\Lambda((o, Q^1_{\alpha_1}, Q^2_{\alpha_2}, \ldots)) = \bigcap_{n \in N} \left(\overline{Q^n_{\alpha_n}} \right).$$

The map Λ identifies X and ∂T (see [8] for more details).
Let $x = (o, Q^1_{\alpha_1}, Q^2_{\alpha_2}, \ldots) \in \partial T$ and $y = (o, Q^1_{\beta_1}, Q^2_{\beta_2}, \ldots) \in \partial T$. We define $x \wedge y \in V(T) \cup \partial T$ in the following way: if $x = y$ then $x \wedge y := x = y$, if $x \neq y$ then $x \wedge y := Q^j_\gamma$, where $j = \max\{k \in \mathbb{N} \mid \alpha_l = \beta_l \text{ for all } l \leq k\}$, and $\gamma = \alpha_k = \beta_k$.
Let us define the distance

$$\rho : \partial T \times \partial T \longrightarrow \mathbb{R} \tag{3.56}$$
$$\rho(x, y) = \delta^{\mathrm{count}(x \wedge y, o)},$$

where $0 < \delta < 1$ is the constant defined in Theorem 3.2.1, and $\mathrm{count}(x \wedge y, o)$ is the distance that counts how many edges of the geodesic that connects $x \wedge y$ and o are in between o and $x \wedge y$. In particular, if $x \wedge y = Q^k_\alpha \in V(T)$ then $\mathrm{count}(x \wedge y, o) = k$, otherwise $x \wedge y = x = y \in \partial T$ and $\mathrm{count}(x \wedge y, o) = +\infty$ and $\rho(x, y) = 0$. We have $\mathrm{diam}(\partial T) = 1 < +\infty$.

Let $\mathcal{H}^Q_\rho$ denote the Q-dimensional Hausdorff measure on ∂T with respect to the distance ρ. The space ∂T endowed with the distance ρ and the measure $\mathcal{H}^Q_\rho$ is a compact Q-regular Ahlfors space, so there exist constants $0 < K_1 < K_2$ such that

$$K_1 \cdot r^Q \leq \mathcal{H}^Q_\rho \left(B_\rho(x, r) \right) \leq K_2 \cdot r^Q \tag{3.57}$$

for all $x \in \partial T$, for all $0 < r < \operatorname{diam}_\rho(\partial T)$. Here $(B_\rho(x,r))$ denotes the metric ball of center x and radius r in ∂T with respect to the metric ρ, and diam_ρ denotes the diameter with respect to the metric ρ.

3.2.2 Riesz potential

In this subsection we define the Riesz capacity in our setting and we enunciate some properties we will use later in this work.

Definition 3.2.4 (Riesz potential on X)**.** Let (X, d, m) be a compact Q-regular Ahlfors space. Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\frac{1}{p'} < s < 1$. We define the Riesz Kernel

$$\begin{aligned} K_{X,s} : X \times X &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \frac{1}{d(x,y)^{Q \cdot s}}. \end{aligned} \tag{3.58}$$

Let $E \subseteq X$ be a compact set. We define the L^p capacity of E associated to the kernel $K_{X,s}$:

$$C_{K_{X,s},p}(E) := \inf \left\{ \|f\|^p_{L^p(X,m)} \mid f \in L^p(X,m),\ Gf(x) \geq 1\ \forall x \in E \right\}, \tag{3.59}$$

where Gf denotes the potential of f, and it is defined by

$$Gf(x) := \int_X K_{X,s}(x,y) f(y) dm(y). \tag{3.60}$$

Definition 3.2.5 (Riesz potential on ∂T)**.** Let (X, d, m) be a compact Q-regular Ahlfors space. Let ∂T be the boundary of the tree T associated to X. Let ∂T be endowed with the distance ρ defined in 3.56 and with the Hausdorff measure $\mathcal{H}^Q_\rho$. Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\frac{1}{p'} < s < 1$. We define the Riesz Kernel

$$\begin{aligned} K_{\partial T,s} : \partial T \times \partial T &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \frac{1}{\rho(x,y)^{Q \cdot s}}. \end{aligned} \tag{3.61}$$

Let $E \subseteq \partial T$ be a compact set. We define the L^p capacity of E associated to the kernel $K_{\partial T,s}$:

$$C_{K_{\partial T,s},p}(E) := \inf \left\{ \|f\|^p_{L^p(\partial T,\rho)} \mid f \in L^p(\partial T,\rho),\ If(x) \geq 1\ \forall x \in E \right\}, \tag{3.62}$$

where If denotes the potential of f, and it is defined by

$$If(x) := \int_{\partial T} K_{\partial T,s}(x,y) f(y) d\mathcal{H}^Q_\rho(y). \tag{3.63}$$

The following estimate for the capacity of a ball in an Ahlfors-regular space is a known result from the general theory of potential on Ahlfors-regular spaces (see [8]).

Proposition 3.2.2 (Estimate for the capacity of a ball in an Ahlfors-regular space). *Let (X, d, m) be a Q-regular Ahlfors space, let $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, let $\frac{1}{p'} \leq s < 1$. Then there exist constants $0 < \tilde{C}_1 < \tilde{C}_2$, $0 < \tilde{K}_1 < \tilde{K}_2$, $r_0 > 0$ which depend only on X, Q, p and s such that for all $x \in X$, for all $r < r_0$ the following formulas hold:*

- *case $\frac{1}{p'} < s < 1$:*

$$\tilde{C}_1 \cdot r^{Qp\left(s-\frac{1}{p'}\right)} \leq C_{K_{X,s},p}(B_d(x,r)) \leq \tilde{C}_2 \cdot r^{Qp\left(s-\frac{1}{p'}\right)}, \tag{3.64}$$

- *case $s = \frac{1}{p'}$:*

$$\tilde{K}_1 \cdot \frac{1}{\log\left(\frac{1}{r}\right)} \leq C_{K_{X,s},p}(B_d(x,r)) \leq \tilde{K}_2 \cdot \frac{1}{\log\left(\frac{1}{r}\right)}. \tag{3.65}$$

The following theorem is a known result from the theory of capacity on trees and Ahlfors-regular spaces (see [8]), and it will be used in the proof of the second main result of this work.

Theorem 3.2.3 (Comparing the capacities on X and ∂T). *Let (X, d, m) be a Q-regular Ahlfors space, let $(\partial T, \rho, \mathcal{H}_\rho^Q)$ be the Q-regular Ahlfors boundary of the associated tree T. Then there exist constants A_1, A_2 such that $0 < A_1 \leq A_2$, and such that for every closed set $F \subseteq \partial T$ and for every closed set $G \subseteq X$ we have*

1.
$$A_1 \cdot C_{K_{\partial T,s},p}(F) \leq C_{K_{X,s},p}(\Lambda(F)) \leq A_2 \cdot C_{K_{\partial T,s},p}(F).$$

2.
$$A_1 \cdot C_{K_{X,s},p}(G) \leq C_{K_{\partial T,s},p}(\Lambda^{-1}(G)) \leq A_2 \cdot C_{K_{X,s},p}(G).$$

3.2.3 Quasi-additivity on compact Ahlfors-regular spaces

The following theorem is the second result in this chapter, and it will be used later in the proof of the main results in this chapter.

Theorem 3.2.4 (Quasi-additivity for Riesz capacity on compact Ahlfors-regular spaces). *Let (X, d, m) be a compact Q-regular Ahlfors space. Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\frac{1}{p'} \leq s < 1$. For every $x \in X$ and $r > 0$ let us define (when it exists) the radius*

$$\eta_{X,p}(x,r) := \inf\left\{R > 0 \mid m(B_d(x,R)) \geq C_{K_{X,s},p}(B_d(x,r))\right\}, \tag{3.66}$$

and let us define

$$\eta^*_{X,p}(x,r) := \max\{r, \eta_{X,p}(x,r)\}. \tag{3.67}$$

Then there exists a constant $\Omega = \Omega(X, p, s) \geq 1$ *such that for all* $M \geq 1$ *there exists a constant* $1 < \tilde{A} < +\infty$ *such that, for any countable family* $\{B_d(x_k, r_k)\}_{k\in\mathcal{F}}$ *of balls in* X *such that the family* $\{B_d(x_k, \Omega \cdot \eta^*_{X,p}(x_k, M \cdot r_k))\}_{k\in\mathcal{F}}$ *is disjoint, for any compact set* $E \subseteq X$ *such that* $E = \bigcup_k E_k$ *and* $E_k \subseteq B_d(x_k, r_k)\ \forall k$*, we have*

$$\sum_{k\in\mathcal{F}} C_{K_{X,s},p}(E_k) \leq \tilde{A} \cdot C_{K_{X,s},p}(E). \tag{3.68}$$

The constant $\tilde{A}$ *depend only on the choice of the space* X *and the of the parameters* p*,* s *and* M*.*

Proof. We begin the proof with the case $M > 1$. Let $M > 1$ be arbitrary. We can write M in the following way

$$M = 1 + \epsilon > 1, \quad \text{where } \epsilon > 0. \tag{3.69}$$

Let $\{B_d(x_k, r_k)\}_{k\in\mathcal{F}}$ be a family of balls in X. Let $E = \bigcup_{k\in\mathcal{F}} E_k$ be a compact set such that E_k is a compact subset of $B_d(x_k, r_k)$ for all $k \in \mathcal{F}$. Consider a Christ decomposition

$$\{Q^k_\alpha \subseteq X \mid \alpha \in I_k,\ k \in \mathbb{N}\} \tag{3.70}$$

like the one defined in Theorem 3.2.1. Consider le associated tree structure T and the boundary of the tree $(\partial T, \rho, \mathcal{H}^Q_\rho)$ defined in definition 3.2.3.
Without loss of generality we may assume $r_k \leq C_3$ for all $k \in \mathcal{F}$. Consider a fixed $k \in \mathcal{F}$. We define

$$j(k) := \max\{j \in \mathbb{N} \mid C_3 \cdot \delta^j \geq r_k\}, \tag{3.71}$$

where $0 < \delta < 1$ and $C_3 > 0$ are the constants defined in Theorem 3.2.1.
By the statement v) in Theorem 3.2.1 we have that for every $k \in \mathcal{F}$ for every $\alpha \in I_{j(k)}$ there exists $z^{j(k)}_\alpha \in Q^{j(k)}_\alpha$ such that $B(z^{j(k)}_\alpha, C_3 \cdot \delta^{j(k)}) \subseteq Q^{j(k)}_\alpha$.
Moreover, by the statement ii) in Theorem 3.2.1 we have

$$Q^{j(k)}_{\alpha_1} \cap Q^{j(k)}_{\alpha_2} = \emptyset \quad \text{for all } \alpha_1, \alpha_2 \in I_{j(k)} \text{ such that } \alpha_1 \neq \alpha_2. \tag{3.72}$$

Let us consider the family $\mathcal{G}(k) \subseteq I_{j(k)}$ of indices such that the family of qubes

$$\left\{Q^{j(k)}_\alpha \mid \alpha \in \mathcal{G}(k)\right\} \tag{3.73}$$

is the minimal covering of qubes at the level $j(k)$ of the set E_k, i.e.

$$E_k \subseteq \bigcup_{\alpha\in\mathcal{G}(k)} \overline{Q^{j(k)}_\alpha} \quad \text{and} \quad E_k \cap \overline{Q^{j(k)}_\alpha} \neq \emptyset\ \forall \alpha \in \mathcal{G}(k). \tag{3.74}$$

Consider an arbitrary $\alpha \in \mathcal{G}(k)$. Let $w(k) \in \overline{Q_\alpha^{j(k)}}$ be arbitrary. By (3.74) there exists $y(k) \in E_k \cap \overline{Q_\alpha^{j(k)}}$. By construction $E_k \subseteq B_d(x_k, r_k)$, so $d(x_k, y(k)) \leq r_k$. Moreover, by statement iv) in Theorem 3.2.1 we have $\mathrm{diam}(\overline{Q_\alpha^{j(k)}}) \leq C_4 \cdot \delta^{j(k)}$. By definition 3.71 we have $\delta^{j(k)} \leq \delta^{-1} \cdot r_k$, so we get

$$\mathrm{diam}(\overline{Q_\alpha^{j(k)}}) \leq C_4 \cdot \delta^{-1} \cdot r_k. \tag{3.75}$$

So, by triangle inequality, we get

$$\overline{Q_\alpha^{j(k)}} \subseteq B_d(x_k, r_k(1 + C_4 \cdot \delta^{-1})). \tag{3.76}$$

We are now going to repeat the previous steps, but instead of qubes at the level $j(k)$ we will use qubes at the level $j(k) + n$, where $n \in \mathbb{N}$ will be fixed later.
We define the family $\mathcal{H}(k, n) \subseteq I_{j(k)+n}$ such that

$$E_k \subseteq \bigcup_{\alpha \in \mathcal{H}(k,n)} \overline{Q_\alpha^{j(k)+n}} \quad \text{and} \quad E_k \cap \overline{Q_\alpha^{j(k)+n}} \neq \emptyset \ \forall \alpha \in \mathcal{H}(k, n). \tag{3.77}$$

For all $\alpha \in \mathcal{H}(k, n)$ we have that there exists $z_\alpha^{j(k)+n} \in Q_\alpha^{j(k)+n}$ such that

$$B(z_\alpha^{j(k)+n}, C_3 \cdot \delta^{j(k)+n}) \subseteq Q_\alpha^{j(k)+n}, \tag{3.78}$$

and such that

$$Q_{\alpha_1}^{j(k)+n} \cap Q_{\alpha_2}^{j(k)+n} = \emptyset \quad \text{for all } \alpha_1, \alpha_2 \in I_{j(k)+n} \text{ such that } \alpha_1 \neq \alpha_2. \tag{3.79}$$

We still have $\delta^{j(k)} \leq \delta^{-1} \cdot r_k$, and we also have $\mathrm{diam}(\overline{Q_\alpha^{j(k)+n}}) \leq C_4 \cdot \delta^{j(k)+n}$, so we get

$$\mathrm{diam}(\overline{Q_\alpha^{j(k)+n}}) \leq C_4 \cdot \delta^{n-1} \cdot r_k. \tag{3.80}$$

By triangle inequality we get

$$\overline{Q_\alpha^{j(k)+n}} \subseteq B_d(x_k, r_k(1 + C_4 \cdot \delta^{n-1})). \tag{3.81}$$

So, by taking

$$n := \inf\{\tilde{n} \in \mathbb{N} \mid C_4 \cdot \delta^{\tilde{n}-1} < \epsilon\} \tag{3.82}$$

and using (3.69), we have

$$\overline{Q_\alpha^{j(k)+n}} \subseteq B_d(x_k, M \cdot r_k). \tag{3.83}$$

We observe that the definition (3.82) does not depend on the choice of $k \in \mathcal{F}$.
Equation (3.83) holds for all $k \in \mathcal{F}$ and for all $\alpha \in \mathcal{H}(k, n)$, and we get

$$B_d(x_k, r_k) \subseteq \bigcup_{\alpha \in \mathcal{H}(k,n)} \overline{Q_\alpha^{j(k)+n}} \subseteq B_d(x_k, M \cdot r_k) \quad \forall k \in \mathcal{F}. \tag{3.84}$$

We observe that there exists $\tilde{N} \in \mathbb{N}$ such that

$$|\mathcal{H}(k,n)| \leq \tilde{N} \quad \forall k \in \mathcal{F}. \tag{3.85}$$

Indeed, for any $k \in \mathcal{F}$, using (3.78) and (3.79), we get that the set

$$\mathcal{S}(k) := \bigcup_{\alpha \in \mathcal{H}(k,n)} B_d(z_\alpha^{j(k)+n}, C_3 \cdot \delta^{j(k)+n}) \tag{3.86}$$

is a disjoint intersection of metric balls, and

$$\mathcal{S}(k) \subseteq \bigcup_{\alpha \in \mathcal{H}(k,n)} \overline{Q_\alpha^{j(k)+n}} \subseteq B_d(x_k, M \cdot r_k). \tag{3.87}$$

However, the space X is a Q-regular Ahlfors space, so, using (3.51) and (3.71), we get

$$\begin{aligned} m(\mathcal{S}(k)) = \sum_{\alpha \in \mathcal{H}(k,n)} m(B_d(z_\alpha^{j(k)+n}, C_3 \cdot \delta^{j(k)} \cdot \delta^n)) \geq \\ \sum_{\alpha \in \mathcal{H}(k,n)} m(B_d(z_\alpha^{j(k)+n}, r^k \cdot \delta^n)) \geq \\ \sum_{\alpha \in \mathcal{H}(k,n)} C_1 \cdot r_k^Q \cdot \delta^{nQ} = \\ |\mathcal{H}(k,n)| \cdot C_1 \cdot r_k^Q \cdot \delta^{nQ}. \end{aligned} \tag{3.88}$$

On the other hand, applying the estimate (3.51) to $B_d(x_k, M \cdot r_k)$ gives us

$$m(B_d(x_k, M \cdot r_k)) \leq C_2 \cdot M^Q \cdot r_k^Q. \tag{3.89}$$

However, $\mathcal{S}(k) \subseteq B_d(x_k, M \cdot r_k)$, so $m(\mathcal{S}(k)) \leq m(B_d(x_k, M \cdot r_k))$ and we get

$$|\mathcal{H}(k,n)| \leq \frac{C_2}{C_1} \cdot M^Q \cdot \delta^{-nQ} \quad \forall k \in \mathcal{F}, \tag{3.90}$$

proving the claim (3.85).

Consider $k \in \mathcal{F}$ and $\alpha \in I_k$. By definition 3.55 we have

$$X \supseteq Q_\alpha^k \longmapsto \Lambda^{-1}(Q_\alpha^k) \subseteq \partial T. \tag{3.91}$$

Let $S \subseteq X$ be a closed set. Let us define (with a slight abuse of notation)

$$\tilde{\Lambda}^{-1}(Q_\alpha^k \cap S) := \left\{ x = \left(Q_{\beta_0}^0, Q_{\beta_1}^1, Q_{\beta_2}^2, \dots\right) \in \partial T \;\middle|\; \Lambda(x) \in \overline{Q_\alpha^k} \cap S,\ \beta_k = \alpha \right\}, \tag{3.92}$$

and we also define $\tilde{\Lambda}^{-1}(Q^k_\alpha) := \tilde{\Lambda}^{-1}(Q^k_\alpha \cap X)$.
We have

$$\tilde{\Lambda}^{-1}(Q^k_\alpha \cap S) \subseteq \Lambda^{-1}(Q^k_\alpha \cap S), \quad \text{and} \quad \Lambda(\tilde{\Lambda}^{-1}(Q^k_\alpha \cap S)) = \overline{Q^k_\alpha} \cap S. \tag{3.93}$$

Let $k \in \mathcal{F}$. Let us denote $N(k) := |H(k,n)| < \tilde{N}$. To simplify the notation let us denote the family of indices $H(k,n)$ by

$$H(k,n) = \{\alpha_1(k), \alpha_2(k), \dots, \alpha_{N(k)}(k)\}. \tag{3.94}$$

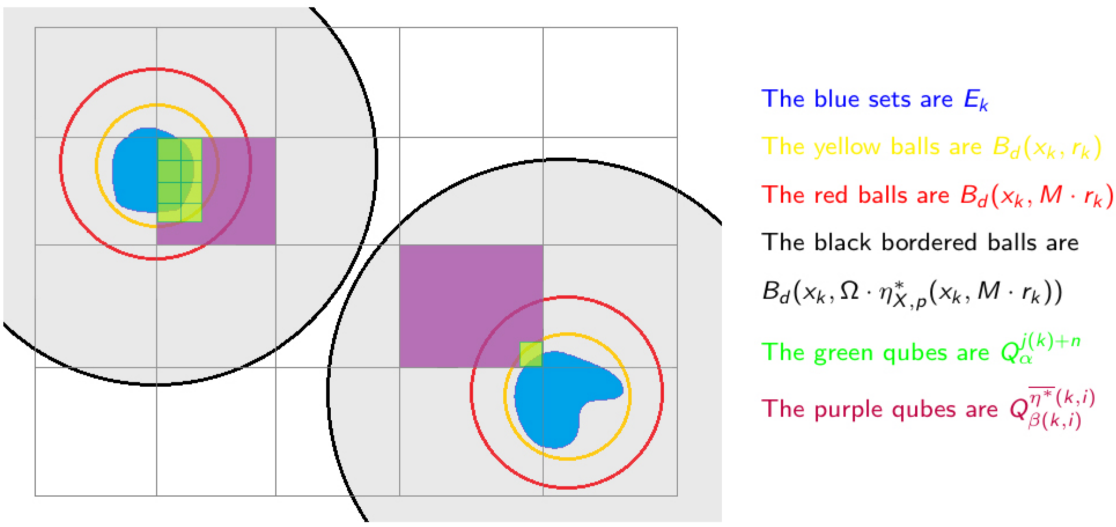
For every $k \in \mathcal{F}$, for every $i = 1, 2, \dots, N(k)$ we define, when they exist,

$$\overline{\eta^*}(k,i) \in \mathbb{N} \quad \text{and} \quad \beta(k,i) \in I_{\overline{\eta^*}(k,i)}, \tag{3.95}$$

so that the qube $Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}$ is the smallest qube such that $Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)} \supseteq Q^{j(k)+n}_{\alpha_i(k)}$ and such that

$$\mathcal{H}^Q_\rho\left(\tilde{\Lambda}^{-1}\left(Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}\right)\right) \geq C_{K_{\partial T},s,p}\left(\tilde{\Lambda}^{-1}\left(Q^{j(k)+n}_{\alpha_i(k)}\right)\right). \tag{3.96}$$

Statements ii) and iii) in Theorem 3.2.1 prove that $Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}$ is uniquely determined.
The following picture shows the construction we made, where $\Omega > 1$ is the constant mentioned in the statement of this theorem, we will fix the value of Ω later.



We claim that there exists a universal constant $\Omega_1 > 0$ such that for all $k \in \mathcal{F}$, for all $i = 1, 2, \dots, N(k)$ we have

$$\operatorname{diam}_d\left(Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}\right) \leq \Omega_1 \cdot C_{K_{\partial T},s,p}\left(\tilde{\Lambda}^{-1}\left(Q^{j(k)+n}_{\alpha_i(k)}\right)\right)^{\frac{1}{Q}}. \tag{3.97}$$

Indeed, by the Ahlfors Q-regularity of X and ∂T we can prove that there exist universal constants $0 < \Omega_2 < \Omega_3$ such that

$$\Omega_2 \cdot m\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right) \leq \mathcal{H}_\rho^Q\left(\tilde{\Lambda}^{-1}\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right)\right) \leq \Omega_3 \cdot m\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right). \tag{3.98}$$

Moreover, using the definition (3.95) and using the Ahlfors Q-regular property (3.57) we can also prove that

$$C_{K_{\partial T,s},p}\left(\tilde{\Lambda}^{-1}\left(Q_{\alpha_i(k)}^{j(k)+n}\right)\right) \leq \mathcal{H}_\rho^Q\left(\tilde{\Lambda}^{-1}\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right)\right) \leq \frac{K_2}{K_1}\delta^{-Q} \cdot C_{K_{\partial T,s},p}\left(\tilde{\Lambda}^{-1}\left(Q_{\alpha_i(k)}^{j(k)+n}\right)\right). \tag{3.99}$$

By combining (3.98) and (3.99) we get

$$m\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right) \leq \frac{1}{\Omega_2}\frac{K_2}{K_1}\delta^{-Q} \cdot C_{K_{\partial T,s},p}\left(\tilde{\Lambda}^{-1}\left(Q_{\alpha_i(k)}^{j(k)+n}\right)\right). \tag{3.100}$$

However, by statements iv) and v) in Theorem 3.2.1 we have that there exists a ball $B_d\left(z_{\beta(k,i)}^{\overline{\eta^*}(k,i)}, \tilde{r}_{k,i}\right)$ such that

$$B_d\left(z_{\beta(k,i)}^{\overline{\eta^*}(k,i)}, \tilde{r}_{k,i}\right) \subseteq Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)} \quad \text{and} \quad \mathrm{diam}_d\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right) \leq \frac{C_4}{C_3}\tilde{r}_{k,i}. \tag{3.101}$$

By the Ahlfors Q-regularity condition (3.51) we get

$$\tilde{r}_{k,i} \leq \frac{1}{C_1^{\frac{1}{Q}}} \cdot m\left(B_d\left(z_{\beta(k,i)}^{\overline{\eta^*}(k,i)}, \tilde{r}_{k,i}\right)\right)^{\frac{1}{Q}} \leq \frac{1}{C_1^{\frac{1}{Q}}} \cdot m\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right)^{\frac{1}{Q}}. \tag{3.102}$$

Finally, by combining (3.100), (3.101) and (3.102), we get

$$\mathrm{diam}_d\left(Q_{\beta(k,i)}^{\overline{\eta^*}(k,i)}\right) \leq \frac{C_4}{C_3}\left(\frac{1}{C_1\Omega_2}\frac{K_2}{K_1}\right)^{\frac{1}{Q}} \delta^{-1} \cdot C_{K_{\partial T,s},p}\left(\tilde{\Lambda}^{-1}\left(Q_{\alpha_i(k)}^{j(k)+n}\right)\right)^{\frac{1}{Q}}. \tag{3.103}$$

By choosing

$$\Omega_1 := \frac{C_4}{C_3}\left(\frac{1}{C_1\Omega_2}\frac{K_2}{K_1}\right)^{\frac{1}{Q}} \delta^{-1} \tag{3.104}$$

we prove the claim (3.97). The constant Ω_1 is universal and does not depend on the choice of $k \in \mathcal{F}$ and $i = 1, 2, \ldots, N(k)$.

Now we are going to define the value of the constant Ω defined in the hypotheses of this theorem.
Let $k \in \mathcal{F}$. We define

$$B_k^* := B_d(x_k, \eta_{X,p}^*(x_k, M \cdot r_k)). \tag{3.105}$$

By (3.83) we have $B_d(x_k, \eta^*_{X,p}(x_k, M \cdot r_k)) \supseteq \overline{Q^{j(k)+n}_{\alpha_i}}$ for all $i = 1, 2, \ldots, N(k)$. So, by the definition (3.67) we get

$$m(B^*_k) \geq C_{K_{X,s,p}} \left(B_d(x_k, \eta^*_{X,p}(x_k, M \cdot r_k)) \right) \geq C_{K_{X,s,p}} \left(\overline{Q^{j(k)+n}_{\alpha_i(k)}} \right) \qquad \forall i = 1, 2, \ldots, N(k). \tag{3.106}$$

However, by construction, we have

$$\Lambda \left(\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \right) = \overline{Q^{j(k)+n}_{\alpha_i}}, \tag{3.107}$$

and the set $\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \subseteq \partial T$ is a closed set, so we can apply Theorem 3.2.3 to get

$$m(B^*_k) \geq A_1 \cdot C_{K_{\partial T,s,p}} \left(\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \right) \quad \forall i = 1, 2, \ldots, N(k). \tag{3.108}$$

Since B^*_k is a metric ball in X we can use the Ahlfors Q-regularity condition (3.51) to get

$$\eta^*_{X,p}(x_k, M \cdot r_k) \geq \frac{1}{C_2^{\frac{1}{Q}}} \cdot m(B^*_k)^{\frac{1}{Q}} \geq \left(\frac{A_1}{C_2} \right)^{\frac{1}{Q}} \cdot C_{K_{\partial T,s,p}} \left(\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \right)^{\frac{1}{Q}} \quad \forall i = 1, 2, \ldots, N(k). \tag{3.109}$$

Now we observe that $\overline{Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}} \cap E_k \neq \emptyset$ because $Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)} \supseteq Q^{j(k)+n}_{\alpha_i(k)}$ and because of (3.77), and we also recall that $E_k \subseteq B_d(x_k, r_k)$ by construction, so, by triangle inequality and by using the estimate (3.97), we get that for all $y \in \overline{Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}}$ we have

$$d(y, x_k) \leq r_k + \operatorname{diam}_d \left(Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)} \right) \leq r_k + \Omega_1 \cdot C_{K_{\partial T,s,p}} \left(\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \right)^{\frac{1}{Q}}. \tag{3.110}$$

Finally, we are going to define the constant $\Omega > 0$ mentioned in the thesis of this theorem by

$$\Omega := 1 + \Omega_1 \cdot \left(\frac{C_2}{A_1} \right)^{\frac{1}{Q}}. \tag{3.111}$$

For all $k \in \mathcal{F}$ consider the ball $B_d(x_k, \Omega \cdot \eta^*_{X,p}(x_k, M \cdot r_k))$.
By construction $\eta^*_{X,p}(x_k, M \cdot r_k)) \geq M \cdot r_k \geq r_k$, and we proved the estimate (3.109), so we get

$$\Omega \cdot \eta^*_{X,p}(x_k, M \cdot r_k)) = \left(1 + \Omega_1 \left(\frac{C_2}{A_1} \right)^{\frac{1}{Q}} \right) \cdot \eta^*_{X,p}(x, M \cdot r_k)) \geq r_k + \Omega_1 \cdot \left(\frac{C_2}{A_1} \right)^{\frac{1}{Q}} \cdot \left(\frac{A_1}{C_2} \right)^{\frac{1}{Q}} \cdot C_{K_{\partial T,s,p}} \left(\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \right)^{\frac{1}{Q}} \geq r_k + \Omega_1 \cdot C_{K_{\partial T,s,p}} \left(\tilde{\Lambda}^{-1} \left(Q^{j(k)+n}_{\alpha_i(k)} \right) \right)^{\frac{1}{Q}}. \tag{3.112}$$

However, we proved (3.110), so we get that

$$Q_{\beta(k,i)}^{\overline{\eta^*(k,i)}} \subseteq B_d(x_k, \Omega \cdot \eta^*_{X,p}(x_k, M \cdot r_k)) \tag{3.113}$$

for every $k \in \mathcal{F}$, for every $i = 1, 2, \ldots, N(k)$.

Now we finish the proof by proving (3.68).
Let $\{B_d(x_k, r_k)\}_{k\in\mathcal{F}}$ be a family of balls in X such that $\{B_d(x_k, \Omega \cdot \eta^*_{X,p}(x_k, M \cdot r_k))\}_{k\in\mathcal{F}}$ is disjoint, Let $E \subseteq X$ be a compact set such that $E = \bigcup_k E_k$ and $E_k \subseteq B_d(x_k, r_k)$ $\forall k$.
By construction

$$E_k \subseteq \bigcup_{i=1}^{N(k)} \overline{Q_{\alpha_i(k)}^{j(k)+n}} \tag{3.114}$$

and there exists a universal $\tilde{N} \in \mathbb{N}$ such that $N(k) \leq \tilde{N}$ for all $k \in \mathcal{F}$.
We are going to define for all $k \in \mathcal{F}$ and for all $i = 1, 2, \ldots, \tilde{N}$

$$E_{k,i} := \begin{cases} E_k \cap \overline{Q_{\alpha_i(k)}^{j(k)+n}} & \text{if } i \leq N(k), \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.115}$$

It follows that

$$E_k = \bigcup_{i=1}^{\tilde{N}} E_{k,i} \quad \forall k \in \mathcal{F}. \tag{3.116}$$

Using (3.93) we get

$$E_{k,i} = \begin{cases} \Lambda\left(\tilde{\Lambda}^{-1}\left(E_k \cap Q_{\alpha_i(k)}^{j(k)+n}\right)\right) & \text{if } i \leq N(k), \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.117}$$

To simplify the notation let us denote

$$E_k \cap Q_{\alpha_i(k)}^{j(k)+n} := \emptyset, \quad \Lambda^{-1}\left(E_k \cap Q_{\alpha_i(k)}^{j(k)+n}\right) := \emptyset \tag{3.118}$$

whenever $i > N(k)$.
So we apply the subadditivity of the capacity to (3.116) to get

$$\sum_{k\in\mathcal{F}} C_{K_{X,s},p}\left(E_k\right) = \sum_{k\in\mathcal{F}} C_{K_{X,s},p}\left(\bigcup_{i=1}^{\tilde{N}} E_{k,i}\right) \leq \sum_{i=1}^{\tilde{N}} \sum_{k\in\mathcal{F}} C_{K_{X,s},p}\left(E_{k,i}\right). \tag{3.119}$$

Let $i < \tilde{N}$ be a fixed index. Let $k \in \mathcal{F}$.
If $i > N(k)$ then, by definition

$$C_{K_{\partial T,s},p}\left(\Lambda^{-1}\left(E_k \cap Q_{\alpha_i(k)}^{j(k)+n}\right)\right) = C_{K_{\partial T,s},p}(\emptyset) = 0. \tag{3.120}$$

Otherwise, we have $E_{k,i} = \Lambda\left(\tilde{\Lambda}^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)\right)$, so we can apply the estimate in Theorem 3.2.3 to get

$$\sum_{k\in\mathcal{F}} C_{K_{X,s},p}\left(E_k\right) \leq A_2 \cdot \sum_{i=1}^{\tilde{N}} \sum_{k\in\mathcal{F}} C_{K_{\partial T,s},p}\left(\tilde{\Lambda}^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)\right). \tag{3.121}$$

We observe that, for any fixed $i < \tilde{N}$, the sum

$$\sum_{k\in\mathcal{F}} C_{K_{\partial T,s},p}\left(\tilde{\Lambda}^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)\right) \tag{3.122}$$

can be estimated using Theorem 3.1.2.

Indeed, we have

$$\Lambda^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right) = \left\{x = (Q^0_{\beta_0}, Q^1_{\beta_1}, \dots) \in \partial T \;\middle|\; \Lambda(x) \in E_k \cap \overline{Q^{j(k)+n}_{\alpha_i(k)}},\ \beta_{j(k)+n} = \alpha_i(k)\right\}, \tag{3.123}$$

so it follows that, for any choice of $w_{k,i} \in \Lambda^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)$, we have

$$\Lambda^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right) \subseteq B_\rho(w_{k,i}, \delta^{j(k)+n}) \subseteq \partial T. \tag{3.124}$$

Moreover, we have

$$\Lambda^{-1}\left(Q^{j(k)+n}_{\alpha_i(k)}\right) = B_\rho(w_{k,i}, \delta^{j(k)+n}) \subseteq \partial T \quad \text{and} \quad \Lambda^{-1}\left(Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}\right) = B_\rho(w_{k,i}, \delta^{\overline{\eta^*}(k,i)}) \subseteq \partial T. \tag{3.125}$$

Using (3.96) we get that $B_\rho(w_{k,i}, \delta^{\overline{\eta^*}(k,i)})$ is the smallest possible ball such that

$$B_\rho(w_{k,i}, \delta^{\overline{\eta^*}(k,i)}) \supseteq B_\rho(w_{k,i}, \delta^{j(k)+n}), \tag{3.126}$$

and

$$\mathcal{H}^Q_\rho\left(B_\rho(w_{k,i}, \delta^{\overline{\eta^*}(k,i)})\right) \geq C_{K_{\partial T,s},p}\left(B_\rho(w_{k,i}, \delta^{j(k)+n})\right). \tag{3.127}$$

So, using definition (3.8), we have

$$\eta^*_p(w_{k,i}, \delta^{j(k)+n}) = \delta^{\overline{\eta^*}(k,i)}. \tag{3.128}$$

In (3.113) we proved

$$Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)} \subseteq B_d(x_k, \Omega \cdot \eta^*_{X,p}(x, M \cdot r_k)) \quad \forall k \in \mathcal{F},\ \forall i \leq \tilde{N}, \tag{3.129}$$

but the family $\{B_d(x_k, \Omega \cdot \eta^*_{X,p}(x, M \cdot r_k))\}_{k\in\mathcal{F}}$ is disjoint by hypothesis, so it follows that the family

$$\left\{\tilde{\Lambda}^{-1}\left(Q^{\overline{\eta^*}(k,i)}_{\beta(k,i)}\right)\right\}_{k\in\mathcal{F}} = \left\{B_\rho(w_{k,i}, \delta^{\overline{\eta^*}(k,i)})\right\}_{k\in\mathcal{F}} \tag{3.130}$$

is disjoint for every fixed $i \leq \tilde{N}$.
So the hypotheses of Theorem 3.1.2 are satisfied, and we may apply (3.10) to (3.121) to get

$$\sum_{k \in \mathcal{F}} C_{K_{X,s,p}}(E_k) \leq A \cdot A_2 \cdot \sum_{i=1}^{\tilde{N}} C_{K_{\partial T,s,p}}\left(\bigcup_{k \in \mathcal{F}} \tilde{\Lambda}^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)\right). \tag{3.131}$$

We recall the following quasi-additivity formula for the capacity of a finite union of sets:

$$\sum_{i=1}^{\tilde{N}} C_{K_{\partial T,s,p}}(S_i) \leq \tilde{N} \cdot C_{K_{\partial T,s,p}}\left(\bigcup_{i=1}^{\tilde{N}} S_i\right). \tag{3.132}$$

Applying this formula to (3.131) we get

$$\sum_{k \in \mathcal{F}} C_{K_{X,s,p}}(E_k) \leq A \cdot A_2 \cdot \tilde{N} \cdot C_{K_{\partial T,s,p}}\left(\bigcup_{i=1}^{\tilde{N}} \bigcup_{k \in \mathcal{F}} \tilde{\Lambda}^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)\right). \tag{3.133}$$

Now, to finish the proof, we observe that

$$\Lambda\left(\bigcup_{i=1}^{\tilde{N}} \bigcup_{k \in \mathcal{F}} \tilde{\Lambda}^{-1}\left(E_k \cap Q^{j(k)+n}_{\alpha_i(k)}\right)\right) = \bigcup_{i=1}^{\tilde{N}} \bigcup_{k \in \mathcal{F}} E_k \cap \overline{Q^{j(k)+n}_{\alpha_i(k)}} = \bigcup_{k \in \mathcal{F}} E_k = E. \tag{3.134}$$

So we may apply Theorem 3.2.3 to (3.133) to get

$$\sum_{k \in \mathcal{F}} C_{K_{X,s,p}}(E_k) \leq A \cdot A_2^2 \cdot \tilde{N} \cdot C_{K_{X,s,p}}(E). \tag{3.135}$$

The constant $\tilde{A} := A \cdot A_2^2 \cdot \tilde{N}$ only depends on X, s, p, M and on the chosen Christ decomposition, finishing the proof for the case $M > 1$.

The case $M = 1$ follows as a corollary from the case $M > 1$.
Indeed, choose an arbitrary $\tilde{M} > 1$. Let $\{B_d(x_k, r_k)\}_{k \in \mathcal{F}}$ be a countable family of balls in X such that the family $\{B_d(x_k, \tilde{\Omega} \cdot \eta^*_{X,p}(x_k, r_k))\}_{k \in \mathcal{F}}$ is disjoint. Here the constant $\tilde{\Omega} = \tilde{\Omega}(X, p, s, \tilde{M}) > \Omega$ will be fixed later. Then there exists $\hat{N} = \hat{N}(X, s, p, \tilde{M}) \in \mathbb{N}$ such that, up to a proper choice of the indexes of the family $\{B_d(x_k, r_k)\}_{k \in \mathcal{F}}$, we have

$$\bigcap_{k=\hat{N}}^{+\infty} B_d(x_k, \tilde{\Omega} \cdot \eta^*_{X,p}(x_k, \tilde{M} \cdot r_k)) = \emptyset. \tag{3.136}$$

This property holds if we choose $\tilde{\Omega}$ such that

$$\tilde{\Omega} \cdot \eta^*_{X,p}(x_k, r_k) \geq \Omega \cdot \eta^*_{X,p}(x_k, \tilde{M} \cdot r_k) \tag{3.137}$$

for all $k \geq \hat{N}$.

We claim that such $\hat{N}$ exists because of the properties of the Riesz capacity (see Proposition 3.2.2). Indeed, there exists $\hat{C} > 1$ and $\hat{r} > 0$ universal constants such that, for all $x \in X$, for all $0 < r < \hat{r}$ we have

$$\tilde{C} \cdot C_{K_{X,s},p}\left(B_d(x,r)\right) \geq C_{K_{X,s},p}\left(B_d(x,\tilde{M}\cdot r)\right), \tag{3.138}$$

which entails

$$\eta^*_{X,p}(x_k,r_k) \gtrsim_{X,s,p,\tilde{M}} \tilde{C}^Q \cdot \eta^*_{X,p}(x_k,\tilde{M}\cdot r_k), \tag{3.139}$$

for all $r_k < \hat{r}$. The claim follows from the compactness of X, because there exists $\hat{N} \in \mathbb{N}$ such that $r_k < \hat{r}$ for all $k \geq \hat{N}$. If such $\hat{N}$ did not exist then we would have

$$m(X) \geq m\left(\bigcup_{k\in\mathbb{N}} B_d(x_k,r_k)\right) = \sum_{k\in\mathbb{N}} m(B_d(x_k,r_k)) = +\infty, \tag{3.140}$$

which contradicts the compactness of X.

So we proved that (3.136) holds as long as we choose

$$\tilde{\Omega} := \tilde{C}\cdot\Omega. \tag{3.141}$$

Now we consider an arbitrary compact set $E \subseteq X$ such that $E = \bigcup_k E_k$ and $E_k \subseteq B_d(x_k,r_k)$ $\forall k$. We apply the quasi-additivity formula (3.135) to the family $\{E_k\}_{k\in\mathbb{N}}$ and we get

$$\sum_{k=\hat{N}}^{+\infty} C_{K_{X,s},p}\left(E_k\right) \leq \tilde{A}\cdot C_{K_{X,s},p}\left(\bigcup_{k=\hat{N}}^{+\infty} E_k\right). \tag{3.142}$$

Finally, we apply the finite quasi-additivity formula and we get

$$\begin{aligned}\sum_{k\in\mathcal{F}} C_{K_{X,s},p}\left(E_k\right) &= \sum_{k=1}^{\hat{N}-1} C_{K_{X,s},p}\left(E_k\right) + \sum_{k=\hat{N}}^{+\infty} C_{K_{X,s},p}\left(E_k\right) \leq \\ &\sum_{k=1}^{\hat{N}-1} C_{K_{X,s},p}\left(E_k\right) + \tilde{A}\cdot C_{K_{X,s},p}\left(\bigcup_{k=\hat{N}}^{+\infty} E_k\right) \leq \\ &\tilde{A}\left[\sum_{k=1}^{\hat{N}-1} C_{K_{X,s},p}\left(E_k\right) + C_{K_{X,s},p}\left(\bigcup_{k=\hat{N}}^{+\infty} E_k\right)\right] \leq \\ &\tilde{A}\cdot\hat{N}\cdot C_{K_{X,s},p}\left(\bigcup_{k=1}^{\hat{N}-1} E_k \cup \bigcup_{k=\hat{N}}^{+\infty} E_k\right),\end{aligned} \tag{3.143}$$

so we get

$$\sum_{k\in\mathcal{F}} C_{K_{X,s},p}(E_k) \lesssim_{(X,s,p)} C_{K_{X,s},p}(E), \tag{3.144}$$

so, up to choosing the new values of the constants Ω and $\tilde{A}$ for the case $M = 1$, the theorem is proved. □

3.3 Harmonic extension

In this section we define the harmonic extension of a function defined over an Ahlfors-regular space and we enunciate and prove several properties of the Harmonic extension.

3.3.1 Dyadic Poisson Integral and Riesz kernel

Let (X, d, m) be an Ahlfors Q-regular space. In the following part of this work we will be considering the space $X \times (0, +\infty)$ with the metric

$$\rho((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), |y_1 - y_2|\}. \tag{3.145}$$

We are now going to define the harmonic extension on $X \times (0, +\infty)$.

Definition 3.3.1 (Poisson Integral in $X \times (0, +\infty)$)**.** Let $f \in L^p(X)$. We define the Poisson Integral

$$PI(f)(x,y) := \int_X C(x,y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} f(z)dm(z). \tag{3.146}$$

Here $C(x, y)$ is the constant that normalizes the Poisson Kernel, i.e.

$$C(x,y) := \left[\int_X \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} dm(z)\right]^{-1}. \tag{3.147}$$

Remark 3.3.1. There exist constants $0 < C_1 \le C_2 < +\infty$ depending only on the choice of X and Q such that

$$C_1 \le C(x,y) \le C_2 \quad \forall x \in X,\ \forall y > 0, \tag{3.148}$$

i.e.

$$C(x_1, y_1) \approx_{(X,Q)} C(x_2, y_2) \quad \text{for all } (x_1, y_1), (x_2, y_2) \in X \times (0, +\infty). \tag{3.149}$$

We will use a dyadicization of the Riesz kernel to prove a property of the Poisson Integral.

Lemma 3.3.1 (Discretization of the Riesz kernel)**.** *Let $g \in L^p(X)$. Then*

$$K_{X,s} * g(x) \approx_{(Q,s)} \int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} g(z)dm(z) \tag{3.150}$$

for every $x \in X$.

Proof. Let $x \in X$. For every $z \neq x$ we have

$$\sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} = \sum_{\substack{j \text{ such that} \\ j > \log_2(d(x,z))}} \frac{1}{2^{(Qs)j}}, \tag{3.151}$$

and we have

$$\sum_{\substack{j \text{ such that} \\ j > \log_2(d(x,z))}} \frac{1}{2^{(Qs)j}} = \sum_{\substack{j \text{ such that} \\ j \geq \lceil \log_2(d(x,z)) \rceil}} \frac{1}{2^{(Qs)j}} \leq \frac{1}{1-2^{Qs}} \frac{1}{2^{(Qs)(\log_2(d(x,z)))}} = \frac{1}{1-2^{Qs}} \frac{1}{d(x,z)^{Qs}}, \tag{3.152}$$

$$\sum_{\substack{j \text{ such that} \\ j > \log_2(d(x,z))}} \frac{1}{2^{(Qs)j}} = \sum_{\substack{j \text{ such that} \\ j \geq \lceil \log_2(d(x,z)) \rceil}} \frac{1}{2^{(Qs)j}} \geq \frac{1}{1-2^{Qs}} \frac{1}{2^{(Qs)(\log_2(d(x,z))+1)}} = \frac{1}{2^{Qs}-2^{2Qs}} \frac{1}{d(x,z)^{Qs}}. \tag{3.153}$$

But $K_{X,s}(x,z) = 1/d(x,z)^{Qs}$, so we get

$$K_{X,s}(x,z) \approx_{(Q,s)} \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} \tag{3.154}$$

for all $z \neq x$. Since $m(\{x\}) = 0$ we get

$$K_{X,s} * g(x) \approx_{(Q,s)} \int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} g(z)dm(z) \tag{3.155}$$

for all $g \in L^p(X)$, for all $x \in X$, and the lemma is proved. □

3.3.2 Commutative convolution-like property

Now we will prove that the order of the Poisson Integral and the Riesz potential can be exchanged up to a universal multiplicative constant. The proof of this property in $\mathbb{R}^{n+1}$ trivially follows from the commutative property of the convolution. In the setting of compact Ahlfors-regular spaces this proof is based on a geometrical property of the dyadicization of the Riesz kernel, and on the dyadic nature of the Poisson Integral we defined.

Lemma 3.3.2 (Exchanging the order of the Poisson Integral and of the Riesz potential)**.** *Let $f \in L^p(X)$, let $y > 0$. Then*

$$K_{X,s} * (PI(f)(\cdot, y))(x) \approx_{(X,Q,s)} PI(K_{X,s} * f)(x, y) \tag{3.156}$$

for all $x \in X$.

Proof. By Lemma 3.3.1 we know that

$$K_{X,s} * (PI(f)(\cdot, y))(x) \approx_{(Q,s)} \int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} PI(f)(z, y)dm(z), \tag{3.157}$$

and

$$PI(K_{X,s} * f)(x, y) \approx_{(Q,s)} PI\left(\int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(\cdot,2^j)}(z)}{2^{(Qs)j}} f(z)dm(z)\right)(x, y), \tag{3.158}$$

so we will prove the statement by proving

$$\int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} PI(f)(z, y)dm(z) \approx_{(X,Q,s)} PI\left(\int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(\cdot,2^j)}(z)}{2^{(Qs)j}} f(z)dm(z)\right)(x, y) \tag{3.159}$$

for all $f \in L^p(X)$, for all $x, z \in X$, for all $y > 0$.
Let *L.H.S.* and *R.H.S.* denote the left hand side and the right hand side of (3.159) respectively. We compute

$$L.H.S. = \int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)}{2^{(Qs)j}} \int_X \frac{C(z,y)}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(z,2^k y)}(w)}{2^{(Q+1)k}} f(w)dm(w)dm(z) = \tag{3.160}$$
$$\int_X \int_X \frac{C(z,y)}{y^Q} \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)\chi_{B_d(z,2^k y)}(w)}{2^{[(Q+1)k+(Qs)j]}} f(w)dm(z)dm(w).$$

Remark 3.3.1 entails $C(z, y) \approx_{(X)} C(x, y)$, and we have

$$\chi_{B_d(z,2^k y)}(w) = \chi_{B_d(w,2^k y)}(z),$$

so we get

$$L.H.S. \approx_{(X)} \frac{C(x,y)}{y^Q} \int_X \left[\int_X \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)\chi_{B_d(w,2^k y)}(z)}{2^{[(Q+1)k+(Qs)j]}} dm(z)\right] f(w)dm(w). \tag{3.161}$$

Now we compute

$$R.H.S. = PI\left(\int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(\cdot,2^j)}(z)}{2^{(Qs)j}} f(z)dm(z)\right)(x,y) = \tag{3.162}$$

$$\int_X \frac{C(x,y)}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(w)}{2^{(Q+1)k}} \int_X \sum_{j=-\infty}^{+\infty} \frac{\chi_{B_d(w,2^j)}(z)}{2^{(Qs)j}} f(z)dm(z)dm(w) =$$

$$\int_X \int_X \frac{C(x,y)}{y^Q} \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(w)\chi_{B_d(w,2^j)}(z)}{2^{[(Q+1)k+(Qs)j]}} f(z)dm(z)dm(w) =$$

$$\frac{C(x,y)}{y^Q} \int_X \left[\int_X \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(w)\chi_{B_d(z,2^j)}(w)}{2^{[(Q+1)k+(Qs)j]}} dm(w)\right] f(z)dm(z).$$

We rename the bound variables in the last equation to get

$$R.H.S. = \frac{C(x,y)}{y^Q} \int_X \left[\int_X \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)\chi_{B_d(w,2^j)}(z)}{2^{[(Q+1)k+(Qs)j]}} dm(z)\right] f(w)dm(w). \tag{3.163}$$

So, to prove that $L.H.S. \approx_{(X,Q,s)} R.H.S.$ for all $f \in L^p(X)$, for all $x \in X$, for all $y > 0$, it is sufficient to prove that

$$\int_X \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^j)}(z)\chi_{B_d(w,2^k y)}(z)}{2^{[(Q+1)k+(Qs)j]}} dm(z) \approx_{(X,Q,s)} \int_X \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)\chi_{B_d(w,2^j)}(z)}{2^{[(Q+1)k+(Qs)j]}} dm(z) \tag{3.164}$$

for all $x, w \in X$, for all $y > 0$.
We will prove the stronger statement

$$\sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \approx_{(X,Q,s)} \sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(w,2^j) \cap B_d(x,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \tag{3.165}$$

for all $x, w \in X$, for all $y > 0$.
The claim is trivial when $x = w$. Let us consider $x \neq w$. So we have $d(x,w) > 0$, and we may define

$$\tilde{j}(x,w) = \lceil \log_2(d(x,w)) \rceil, \tag{3.166}$$

which is the first index j such that $w \in \overline{B_d(x,2^j)}$.
Let us consider the following equation

$$\sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} = (I) + (II) + (III), \tag{3.167}$$

where

$$(I) = \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}}, \tag{3.168}$$

$$(II) = \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}}, \tag{3.169}$$

$$(III) = \sum_{j=\tilde{j}(x,w)+1}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}}. \tag{3.170}$$

Now we are going to give estimates of the values of (I), (II) and (III).

We start from estimating (I). Consider $j \leq \tilde{j}(x,w) - 2$. We define

$$\overline{k}(j) = \max\left\{ \left\lfloor \log_2\left(\frac{d(x,w) - 2^j}{y} \right) \right\rfloor + 1, 0 \right\}, \tag{3.171}$$

which is the smallest index $k \geq 0$ such that $B_d(x,2^j) \cap B_d(w,2^k y) \neq \emptyset$.
We write

$$\begin{aligned}(I) = \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \Bigg[& \sum_{k=0}^{\overline{k}(j)-1} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ & \sum_{k=\overline{k}(j)}^{\overline{k}(j)+1} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ & \sum_{k=\overline{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \Bigg], \end{aligned} \tag{3.172}$$

with the convention $\sum_{k=a}^{b} \varphi(k) := 0$ if $b < a$.
By definition of $\overline{k}(j)$ we have $B_d(x,2^j) \cap B_d(w,2^k y) = \emptyset$ if $k < \overline{k}(j)$.
Now we consider $k \geq \overline{k}(j) + 2$. We have

$$k \geq \overline{k}(j) + 2 = \max\left\{ \left\lfloor \log_2\left(\frac{d(x,w) - 2^j}{y} \right) \right\rfloor + 1, 0 \right\} + 2 \geq \log_2\left(\frac{d(x,w) - 2^j}{y} \right) + 2, \tag{3.173}$$

hence we get

$$2^k y \geq 2^{\log_2\left(\frac{d(x,w)-2^j}{y}\right)+2} y = 4\left(d(x,w) - 2^j\right). \tag{3.174}$$

Now we consider an arbitrary point $v \in B_d(x,2^j)$. By triangle inequality we have

$$d(v,w) \leq d(v,x) + d(x,w) \leq 2^j + d(x,w). \tag{3.175}$$

However, we set $j \leq \tilde{j}(x,w) - 2$, so we get

$$j \leq \tilde{j}(x,w) - 2 = \lceil \log_2(d(x,w)) \rceil - 2 \leq \log_2(d(x,w)) - 1, \tag{3.176}$$

hence

$$2^j \leq 2^{\log_2(d(x,w))-1} \leq \frac{1}{2} d(x,w). \tag{3.177}$$

So from (3.174) and (3.177) we get

$$2^k y \geq 2d(x,w) \geq 2^j + \frac{3}{2} d(x,w). \tag{3.178}$$

From (3.175) and (3.178) we get $d(v,w) \leq 2^k y$, so we proved that $B_d(x,2^j) \subseteq B_d(w,2^k y)$ for all $x, w \in X$, for all $j \leq \tilde{j}(x,w) - 2$, for all $k \geq \overline{k}(j) + 2$.
So we proved that

$$\begin{aligned}(I) = \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \Bigg[& \sum_{k=0}^{\overline{k}(j)-1} \frac{m(\emptyset)}{2^{[(Q+1)k+(Qs)j]}} + \\ & \sum_{k=\overline{k}(j)}^{\overline{k}(j)+1} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ & \sum_{k=\overline{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}} \Bigg]. \end{aligned} \tag{3.179}$$

Now we use the equations

$$m(\emptyset) = 0, \tag{3.180}$$

$$0 \leq m(B_d(x,2^j)) \leq m(B_d(x,2^j) \cap B_d(w,2^k y)), \tag{3.181}$$

$$m(B_d(x,r)) \approx_{(X)} r^Q \quad \forall x \in X, \text{ for } 0 < r < \operatorname{diam}(X), \tag{3.182}$$

to get the following equation:

$$\begin{aligned}(I) \geq & \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \sum_{k=\overline{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}} \gtrsim_{(X)} \\ & \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \sum_{k=\overline{k}(j)+2}^{+\infty} \frac{2^{jQ}}{2^{[(Q+1)k+(Qs)j]}} = \\ & \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \frac{1}{1-\frac{1}{2^{Q+1}}} \frac{1}{2^{(Q+1)(\overline{k}(j)+2)}} \frac{2^{jQ}}{2^{(Qs)j}}. \end{aligned} \tag{3.183}$$

We remark that we may use the estimate (3.182) because for all $j \leq \tilde{j}(x,w)-1$ we have $w \notin B_d(x,2^j)$, hence $2^j < \operatorname{diam}(X)$ for all $j \leq \tilde{j}(x,w)-2$.
Now we estimate the value of $\overline{k}(j)$. Consider $j = \tilde{j}(x,w)-2$. We have

$$\overline{k}(\tilde{j}(x,w)-2) = \max\left\{\left\lfloor \log_2\left(\frac{d(x,w)-2^{\tilde{j}(x,w)-2}}{y}\right)\right\rfloor + 1, 0\right\} > \log_2\left(\frac{d(x,w)-2^{\tilde{j}(x,w)-2}}{y}\right). \tag{3.184}$$

So we get

$$2^{\overline{k}(\tilde{j}(x,w)-2)} y > d(x,w) - 2^{\tilde{j}(x,w)-2}. \tag{3.185}$$

However we use equation (3.166) to get

$$2^{\overline{k}(\tilde{j}(x,w)-2)} y > d(x,w) - 2^{\log_2(d(x,w))+1-2} = \frac{1}{2} d(x,w). \tag{3.186}$$

Now let $j < \tilde{j}(x,w)-2)$. From (3.186) we get

$$2^{\overline{k}(\tilde{j}(x,w)-2)+1} y > d(x,w), \tag{3.187}$$

hence

$$B_d(w, 2^{\overline{k}(\tilde{j}(x,w)-2)+1} y) \cap B_d(x,2^j) \neq 0 \quad \text{for all } j < \tilde{j}(x,w)-2. \tag{3.188}$$

However, from the definition of $\overline{k}(j)$, it follows that $\overline{k}(j)$ is the smallest index k such that $B_d(w,2^k y) \cap B_d(x,2^j) \neq 0$, so we proved

$$\overline{k}(j) \leq \overline{k}(\tilde{j}(x,w)-2)+1 \quad \text{for all } j \leq \tilde{j}(x,w)-2. \tag{3.189}$$

So we use equations (3.183) and (3.189) to get

$$(I) \gtrsim_{(X)} \frac{1}{1-\frac{1}{2^{Q+1}}} \frac{1}{2^{(Q+1)(\overline{k}(\tilde{j}(x,w)-2)+3)}} \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \frac{2^{jQ}}{2^{(Qs)j}} \gtrsim_{(X,Q)} \tag{3.190}$$

$$\frac{1}{2^{(Q+1)(\overline{k}(\tilde{j}(x,w)-2))}} \sum_{i=2-\tilde{j}(x,w)}^{+\infty} \frac{1}{2^{Q(1-s)i}} = \tag{3.191}$$

$$\frac{1}{2^{(Q+1)(\overline{k}(\tilde{j}(x,w)-2))}} \frac{1}{1-\frac{1}{2^{Q(1-s)}}} \frac{1}{2^{Q(1-s)(2-\tilde{j}(x,w))}} \approx_{(Q,s)} \tag{3.192}$$

$$\frac{1}{2^{(Q+1)(\overline{k}(\tilde{j}(x,w)-2))}} 2^{Q(1-s)\tilde{j}(x,w)}. \tag{3.193}$$

So we proved the lower estimate

$$(I) \gtrsim_{(X,Q,p)} \frac{1}{2^{(Q+1)(\overline{k}(\tilde{j}(x,w)-2))}} 2^{Q(1-s)\tilde{j}(x,w)}. \tag{3.194}$$

Now we prove an upper estimate for (I). We compute

$$(I) \leq \sum_{j=-\infty}^{\tilde{j}(x,w)} \sum_{k=\overline{k}(j)}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}} \lesssim_{(X)} \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \sum_{k=\overline{k}(j)}^{+\infty} \frac{2^{jQ}}{2^{[(Q+1)k+(Qs)j]}} = \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \frac{1}{1-\frac{1}{2^{Q+1}}} \frac{1}{2^{(Q+1)\overline{k}(j)}} \frac{2^{jQ}}{2^{(Qs)j}}. \tag{3.195}$$

By monotonicity in the definition of $\overline{k}(j)$ we deduce

$$\overline{k}(j) \geq \overline{k}(\tilde{j}(x,w)-2) \quad \text{for all } j \leq \tilde{j}(x,w)-2, \tag{3.196}$$

so we get

$$(I) \lesssim_{(X)} \frac{1}{1-\frac{1}{2^{Q+1}}} \frac{1}{2^{(Q+1)\overline{k}(\tilde{j}(x,w)-2)}} \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \frac{2^{jQ}}{2^{(Qs)j}} \approx_{(X,Q)} \frac{1}{2^{(Q+1)\overline{k}(\tilde{j}(x,w)-2)}} \sum_{j=-\infty}^{\tilde{j}(x,w)-2} \frac{2^{jQ}}{2^{(Qs)j}} \approx_{(Q,s)} \frac{1}{2^{(Q+1)\overline{k}(\tilde{j}(x,w)-2)}} 2^{Q(1-s)\tilde{j}(x,w)}. \tag{3.197}$$

So we proved

$$(I) \lesssim_{(X,Q,s)} \frac{1}{2^{(Q+1)\overline{k}(\tilde{j}(x,w)-2)}} 2^{Q(1-s)\tilde{j}(x,w)}, \tag{3.198}$$

which, alongside (3.194), proves

$$(I) \approx_{(X,Q,s)} \frac{1}{2^{(Q+1)\overline{k}(\tilde{j}(x,w)-2)}} 2^{Q(1-s)\tilde{j}(x,w)}. \tag{3.199}$$

Now we are going to estimate the value of (III) and we are going to compare (II) and (III).

Let us define

$$j_{diam} := \lfloor \log_2(\mathrm{diam}(X)) \rfloor + 1. \tag{3.200}$$

By computation we get

$$2^{j_{diam}} = 2^{\lfloor \log_2(\mathrm{diam}(X)) \rfloor + 1} > 2^{\log_2(\mathrm{diam}(X))} = \mathrm{diam}(X), \tag{3.201}$$

so we proved

$$X = B_d(x, 2^j) \quad \text{for all } j \geq j_{diam}. \tag{3.202}$$

Moreover we get

$$2^{j_{diam}-1} = 2^{\lfloor \log_2(\mathrm{diam}(X)) \rfloor} \leq 2^{\log_2(\mathrm{diam}(X))} = \mathrm{diam}(X). \tag{3.203}$$

We may write

$$\begin{aligned}(III) = &\sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} \sum_{k=0}^{+\infty} \frac{m(B_d(x, 2^j) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ &\sum_{j=j_{diam}}^{+\infty} \sum_{k=0}^{+\infty} \frac{B_d(w, 2^k y)}{2^{[(Q+1)k+(Qs)j]}}\end{aligned} \tag{3.204}$$

Consider $\tilde{j}(x, w) + 1 \leq j \leq j_{diam} - 1$. We define

$$\tilde{k}(j) = \max \left\{ \left\lfloor \log_2 \left(\frac{2^j - d(x, w)}{y} \right) \right\rfloor + 1, 0 \right\}, \tag{3.205}$$

which is the smallest index $k \geq 0$ such that $B_d(w, 2^k y) \not\subseteq B_d(x, 2^j)$.
Suppose $k \geq \tilde{k}(j) + 2$, suppose $v \in B_d(x, 2^j)$. By triangle inequality we get

$$d(v, w) \leq d(v, x) + d(x.w) \leq d(x, w) + 2^j. \tag{3.206}$$

However, from (3.205) we get

$$k \geq \tilde{k}(j) + 2 > \log_2 \left(\frac{2^j - d(x, w)}{y} \right) + 2, \tag{3.207}$$

hence

$$2^k y > 4(2^j - d(x, w)). \tag{3.208}$$

But from (3.166) we have

$$j \geq \tilde{j}(x, w) + 1 \geq \log_2(d(x, w)) + 1, \tag{3.209}$$

hence

$$2^j \geq 2d(x, w). \tag{3.210}$$

Now from (3.208) and (3.210) we get

$$2^k y > 2^j + (3 \cdot 2^j - 4d(x, w)) \geq 2^j + 2d(x, w). \tag{3.211}$$

Combining (3.207) and (3.211) we get

$$2^k y > d(v, w), \tag{3.212}$$

so we proved that

$$B_d(x, 2^j) \subseteq B_d(w, 2^k y) \tag{3.213}$$

for every $x, w \in X$, for every $\tilde{j}(x, w) + 1 \le j \le j_{diam} - 1$, for every $k \ge \tilde{k}(j) + 2$.
Now we use (3.213) and the definition of $\tilde{k}(j)$ to write

$$\begin{aligned}(III) = \sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} \Bigg[& \sum_{k=0}^{\tilde{k}(j)-1} \frac{m(B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ & \sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x, 2^j) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ & \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}} \Bigg] + \\ & \sum_{j=j_{diam}}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}},\end{aligned} \tag{3.214}$$

with the convention $\sum_{k=a}^{b} \varphi(k) := 0$ if $b < a$.
Let us define

$$k_{diam} := \left\lfloor \log_2 \left(\frac{\operatorname{diam}(X)}{y} \right) \right\rfloor + 1. \tag{3.215}$$

By the same argument used for j_{diam} we get

$$2^{k_{diam}} y > \operatorname{diam}(X) \quad \text{and} \quad 2^{k_{diam}-1} y \le \operatorname{diam}(X), \tag{3.216}$$

and we prove

$$X = B_d(x, 2^k y) \quad \text{for all } k \ge k_{diam}. \tag{3.217}$$

So we can write

$$
\begin{aligned}
(III) = \sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} \Bigg[& \sum_{k=0}^{\tilde{k}(j)-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\
& \sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\
& \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}} \Bigg] + \\
\sum_{j=j_{diam}}^{+\infty} \Bigg[& \sum_{k=0}^{k_{diam}-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\
& \sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \Bigg].
\end{aligned} \tag{3.218}
$$

Now we are going to prove that $(II) \lesssim_{(Q,s)} (III)$.
Suppose $\tilde{j}(x,w) > j_{diam} - 2$.
Then, the expression (3.218) reformulates to

$$
\begin{aligned}
(III) = \sum_{j=j_{diam}}^{+\infty} \Bigg[& \sum_{k=0}^{k_{diam}-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\
& \sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \Bigg].
\end{aligned} \tag{3.219}
$$

Let us denote by Φ the first addend of the first sum in (3.219) with respect to the index j, i.e

$$
\Phi = \sum_{k=0}^{k_{diam}-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j_{diam}]}} + \sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j_{diam}]}}. \tag{3.220}
$$

Now we consider the equation

$$
(II) = \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}}
$$

and we use monotonicity of the measure m to get

$$
(II) \leq \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \sum_{k=0}^{+\infty} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}}, \tag{3.221}
$$

which reformulates to

$$(II) \leq \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \left[\sum_{k=0}^{k_{diam}-1} \frac{m(B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \right]. \tag{3.222}$$

By definition we get $\tilde{j}(x,w) \leq j_{diam}$, so we have two cases:

1. Case $\tilde{j}(x,w) = j_{diam} - 1$: we get

$$(II) \leq 2^{2Qs}\Phi + 2^{Qs}\Phi. \tag{3.223}$$

2. Case $\tilde{j}(x,w) = j_{diam}$: we get

$$(II) \leq 2^{Qs}\Phi + \Phi. \tag{3.224}$$

So from (3.223) and (3.224) we get

$$(II) \lesssim_{(Q,s)} (III) \tag{3.225}$$

whenever $\tilde{j}(x,w) > j_{diam} - 2$, and the constant associated to equation (3.225) does not depend on x and w.

Now suppose $\tilde{j}(x,w) \leq j_{diam} - 2$ Let us denote by Φ first addend of the first sum in (3.218) with respect to the index j, i.e.

$$\begin{aligned} \Phi = & \sum_{k=0}^{\tilde{k}(\tilde{j}(x,w)+1)-1} \frac{m(B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)(\tilde{j}(x,w)+1)]}} + \\ & \sum_{k=\tilde{k}(\tilde{j}(x,w)+1)}^{\tilde{k}(\tilde{j}(x,w)+1)+1} \frac{m(B_d(x, 2^{\tilde{j}(x,w)+1}) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)(\tilde{j}(x,w)+1)]}} + \\ & \sum_{k=\tilde{k}(\tilde{j}(x,w)+1)+2}^{+\infty} \frac{m(B_d(x, 2^{\tilde{j}(x,w)+1}))}{2^{[(Q+1)k+(Qs)(\tilde{j}(x,w)+1)]}}. \end{aligned} \tag{3.226}$$

We may write

$$(II) = \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} = \tag{3.227}$$

$$= \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \Bigg[\sum_{k=0}^{\tilde{k}(\tilde{j}(x,w)+1)-1} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \tag{3.228}$$

$$\sum_{k=\tilde{k}(\tilde{j}(x,w)+1)}^{\tilde{k}(\tilde{j}(x,w)+1)+1} \frac{m(B_d(x,2^j)) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} +$$

$$\sum_{k=\tilde{k}(\tilde{j}(x,w)+1)+2}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \Bigg]. \tag{3.229}$$

Now we use the following facts:

1. $m(B_d(x,2^j) \cap B_d(w,2^k y)) \leq m(B_d(x,2^j))$, $m(B_d(x,2^j) \cap B_d(w,2^k y)) \leq m(B_d(w,2^k y))$, by the monotonicity of m.

2. $m(B_d(x,2^j)) \leq m(B_d(x,2^{\tilde{j}(x,w)+1}))$ for $j = \tilde{j}(x,w) - 1$ and $j = \tilde{j}(x,w)$, because $B_d(x,2^j) \subseteq B_d(x,2^{\tilde{j}(x,w)+1})$.

So we get

$$(II) \leq \sum_{j=\tilde{j}(x,w)-1}^{\tilde{j}(x,w)} \Bigg[\sum_{k=0}^{\tilde{k}(\tilde{j}(x,w)+1)-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \tag{3.230}$$

$$\sum_{k=\tilde{k}(\tilde{j}(x,w)+1)}^{\tilde{k}(\tilde{j}(x,w)+1)+1} \frac{m(B_d(x,2^{\tilde{j}(x,w)+1}) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} +$$

$$\sum_{k=\tilde{k}(\tilde{j}(x,w)+1)+2}^{+\infty} \frac{m(B_d(x,2^{\tilde{j}(x,w)+1}))}{2^{[(Q+1)k+(Qs)j]}} \Bigg]. \tag{3.231}$$

So, by computation and using the definition of Φ, we get

$$(II) \leq 2^{2Qs}\Phi + 2^{Qs}\Phi. \tag{3.232}$$

Combining (3.225) and (3.232) we get

$$(II) \lesssim_{(Q,s)} (III), \tag{3.233}$$

for all $x, w \in X$, and the constant associated to (3.233) does not depend on $x, w \in X$. Now we go back to estimating equation (III).
We consider equation (3.218):

$$(III) = \sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} \Bigg[\sum_{k=0}^{\tilde{k}(j)-1} \frac{m(B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x, 2^j) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}} \Bigg] + \sum_{j=j_{diam}}^{+\infty} \Bigg[\sum_{k=0}^{k_{diam}-1} \frac{m(B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \Bigg],$$

and we claim that

$$\sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x, 2^j) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}} \approx_{(Q,s)} \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}}. \tag{3.234}$$

Indeed, we can trivially get one part of equation (3.234) because

$$\sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}} \leq \sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x, 2^j) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}}. \tag{3.235}$$

For the other part we compute

$$\sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}} = \frac{m(B_d(x, 2^j))}{2^{(Qs)j}} \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{1}{2^{(Q+1)k}} = \frac{m(B_d(x, 2^j))}{2^{[(Qs)j+(Q+1)(\tilde{k}(j)+2)]}} \frac{1}{1 - 2^{-(Q+1)}} \tag{3.236}$$

and

$$\sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x, 2^j) \cap B_d(w, 2^k y))}{2^{[(Q+1)k+(Qs)j]}} \leq \sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x, 2^j))}{2^{[(Q+1)k+(Qs)j]}} = \frac{m(B_d(x, 2^j))}{2^{[(Qs)j+(Q+1)(\tilde{k}(j)+2)]}} \left[2^{Q+1} + 2^{2(Q+1)}\right]. \tag{3.237}$$

From (3.236) and (3.237) we get

$$\sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x,2^j)\cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \leq \frac{1}{(2^{Q+1}+2^{2(Q+1)})(1-2^{-(Q+1)})} \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}}, \tag{3.238}$$

so we proved

$$\sum_{k=\tilde{k}(j)}^{\tilde{k}(j)+1} \frac{m(B_d(x,2^j)\cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \lesssim_{(Q)} \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}}, \tag{3.239}$$

and the claim (3.234) follows.

Finally, from (3.218) and (3.234) we get

$$\begin{aligned}(III) \approx_{(Q)} \sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} \Bigg[&\sum_{k=0}^{\tilde{k}(j)-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ &\sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{m(B_d(x,2^j))}{2^{[(Q+1)k+(Qs)j]}} \Bigg] + \\ \sum_{j=j_{diam}}^{+\infty} \Bigg[&\sum_{k=0}^{k_{diam}-1} \frac{m(B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} + \\ &\sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \Bigg].\end{aligned}$$

By construction we have $2^k y \leq \operatorname{diam}(X)$ for all $\tilde{j}(x,w)+1 \leq j \leq j_{diam}-1$, for all $0 \leq k \leq \tilde{k}(j)-1$, $2^j \leq \operatorname{diam}(X)$ for all $\tilde{j}(x,w)+1 \leq j \leq j_{diam}-1$, $2^k y \leq \operatorname{diam}(X)$ for all $k \leq k_{diam}-1$, so we may apply the Ahlfors-regularity estimate (3.182) to get

$$\begin{aligned}(III) \approx_{(X,Q)} \sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} \Bigg[&\sum_{k=0}^{\tilde{k}(j)-1} \frac{2^{Qk}y^Q}{2^{[(Q+1)k+(Qs)j]}} + \\ &\sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{2^j}{2^{[(Q+1)k+(Qs)j]}} \Bigg] + \\ \sum_{j=j_{diam}}^{+\infty} \Bigg[&\sum_{k=0}^{k_{diam}-1} \frac{2^{Qk}y^Q}{2^{[(Q+1)k+(Qs)j]}} + \\ &\sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \Bigg].\end{aligned} \tag{3.240}$$

Now we recall what we proved up this point and finish the proof of the statement. We proved (3.199), (3.233) and (3.240), i.e.

$$(I) \approx_{(X,Q,s)} \varphi_1(x,w,y) := \frac{1}{2^{(Q+1)\overline{k}(\tilde{j}(x,w)-2)}} 2^{Q(1-s)\tilde{j}(x,w)},$$

$$0 \leq (II) \lesssim_{(Q,s)} (III),$$

$$\begin{aligned}(III) \approx_{(X,Q)} \varphi_2(x,w,y) := \sum_{j=\tilde{j}(x,w)+1}^{j_{diam}-1} & \left[\sum_{k=0}^{\tilde{k}(j)-1} \frac{2^{Qk} y^Q}{2^{[(Q+1)k+(Qs)j]}} + \right. \\ & \left. \sum_{k=\tilde{k}(j)+2}^{+\infty} \frac{2^j}{2^{[(Q+1)k+(Qs)j]}} \right] + \\ \sum_{j=j_{diam}}^{+\infty} & \left[\sum_{k=0}^{k_{diam}-1} \frac{2^{Qk} y^Q}{2^{[(Q+1)k+(Qs)j]}} + \right. \\ & \left. \sum_{k=k_{diam}}^{+\infty} \frac{m(X)}{2^{[(Q+1)k+(Qs)j]}} \right].\end{aligned}$$

We recall that, by definition, we have

$$\sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} = (I) + (II) + (III), \tag{3.241}$$

so we get

$$\sum_{j=-\infty}^{+\infty} \sum_{k=0}^{+\infty} \frac{m(B_d(x,2^j) \cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \approx_{(Q,s)} (I) + (III) \approx_{(X,Q,s)} \varphi_1(x,w,y) + \varphi_2(x,w,y). \tag{3.242}$$

However, $\varphi_1(x,w,y) = \varphi_1(w,x,y)$, $\varphi_2(x,w,y) = \varphi_2(w,x,y)$ for all $x,w \in X$, for all $y > 0$. Indeed, we have

$$\tilde{j}(x,w) = \lceil \log_2(d(x,w)) \rceil = \lceil \log_2(d(w,x)) \rceil = \tilde{j}(w,x), \tag{3.243}$$

$$\overline{k}(j) = \max\left\{ \left\lfloor \log_2\left(\frac{d(x,w) - 2^j}{y} \right) \right\rfloor + 1, 0 \right\} = \max\left\{ \left\lfloor \log_2\left(\frac{d(w,x) - 2^j}{y} \right) \right\rfloor + 1, 0 \right\}, \tag{3.244}$$

$$\tilde{k}(j) = \max\left\{ \left\lfloor \log_2\left(\frac{2^j - d(x,w)}{y} \right) \right\rfloor + 1, 0 \right\} = \max\left\{ \left\lfloor \log_2\left(\frac{2^j - d(w,x)}{y} \right) \right\rfloor + 1, 0 \right\}, \tag{3.245}$$

and all the other terms in the definitions of φ_1 and φ_2 do not depend on x and w, so we get that φ_1, φ_2 are symmetrical with respect to exchanging the roles of x and w.
We finish the proof by exchanging the roles of x and w in (3.242) and we get

$$\sum_{j=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{m(B_d(w,2^j)\cap B_d(x,2^k y))}{2^{[(Q+1)k+(Qs)j]}} \approx_{(X,Q,s)} \varphi_1(w,x,y)+\varphi_2(w,x,y) = \varphi_1(x,w,y)+\varphi_2(x,w,y) \approx_{(X,Q,s)} \sum_{j=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\frac{m(B_d(x,2^j)\cap B_d(w,2^k y))}{2^{[(Q+1)k+(Qs)j]}}, \tag{3.246}$$

which is the statement (3.165) we needed to prove, so the proof is finished. □

3.3.3 Other properties of the Poisson Integral

Now we will prove two more properties of the Poisson Integral which are analogous to the properties of the classical Poisson Integral in $\mathbb{R}^{n+1}$.

Definition 3.3.2. Let f be a non negative function in $L^p_+(X)$, let $M \geq 1$, $\epsilon > 0$. We define

$$E(f,\epsilon) := \{(x,y) \in X \times (0,+\infty) \mid PI(K_{X,s} * f)(x,y) > \epsilon\}, \tag{3.247}$$

$$E^{M,*}(f,\epsilon) := \bigcup_{(x,y)\in E(f,\epsilon)} B_d(x,My) \subseteq X, \tag{3.248}$$

$$E^{M'}(f,\epsilon) := \bigcup_{(x,y)\in E(f,\epsilon)} B_d(x,My)\times\{y\} \subseteq X \times (0,+\infty). \tag{3.249}$$

Lemma 3.3.3. *[Harnack-type inequality] We have*

$$PI(K_{X,s} * f)(x,y) \gtrsim_{(X,Q,M)} \epsilon \quad \forall (x,y) \in E^{M'}(f,\epsilon). \tag{3.250}$$

Proof. Let $(x,y) \in E^{M'}(f,\epsilon)$. By definition of $E^{M'}$ there exist $(\tilde{x},y) \in E(f,\epsilon)$ such that $x \in B_d(\tilde{x},My)$. By (3.148) the constant C in the Poisson integral

$$PI(K_{X,s} * f)(x,y) := \int_X C(x,y)\cdot\frac{1}{y^Q}\sum_{k=0}^{+\infty}\frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}}K_{X,s} * f(z)dm(z) \tag{3.251}$$

is bounded, so we have $C(x,y) \approx_{(X,Q)} C(\tilde{x},y)$, and we get

$$PI(K_{X,s} * f)(x,y) \gtrsim_{(X,Q)} \int_X C(\tilde{x},y)\cdot\frac{1}{y^Q}\sum_{k=0}^{+\infty}\frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}}K_{X,s} * f(z)dm(z). \tag{3.252}$$

The function f is non negative, so $K_{X,s} * f$ is also non negative, so we have

$$PI(K_{X,s} * f)(x,y) \gtrsim_{(X,Q)} \int_X C(\tilde{x},y) \cdot \frac{1}{y^Q} \sum_{k=\lfloor \log_2(M) \rfloor+1}^{+\infty} \frac{1}{2^{(Q+1)}} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)(k-1)}} K_{X,s} * f(z) dm(z). \tag{3.253}$$

Now we change the index k of summation and put $1/(2^{(Q+1)(\lfloor \log_2(M) \rfloor+1)})$ in the leading constant to get

$$PI(K_{X,s} * f)(x,y) \gtrsim_{(X,Q,M)} \int_X C(\tilde{x},y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^{k+\lfloor \log_2(M) \rfloor+1} y)}(z)}{2^{(Q+1)k}} K_{X,s} * f(z) dm(z). \tag{3.254}$$

Now let $k \in \{0,1,2,...\}$ be a fixed index. We have $d(x,\tilde{x}) < My$ by definition of $\tilde{x}$, so, for all $z \in B_d(\tilde{x}, 2^k y)$, we apply the triangle inequality to get

$$d(x,z) \leq d(\tilde{x},x) + d(\tilde{x},z) < My + 2^k y \leq (2^k + M)y \leq 2^{k+\lfloor \log_2(M) \rfloor+1} y. \tag{3.255}$$

So we proved that $B_d(\tilde{x}, 2^k y) \subseteq B_d(x, 2^{k+\lfloor \log_2(M) \rfloor+1} y)$ for all $k \in \mathbb{N}$, hence

$$\sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^{k+\lfloor \log_2(M) \rfloor+1} y)}(z)}{2^{(Q+1)k}} \geq \sum_{k=0}^{+\infty} \frac{\chi_{B_d(\tilde{x},2^k y)}(z)}{2^{(Q+1)k}}. \tag{3.256}$$

We combine (3.254) and (3.256) to get

$$PI(K_{X,s} * f)(x,y) \gtrsim_{(X,Q,M)} \int_X C(\tilde{x},y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(\tilde{x},2^k y)}(z)}{2^{(Q+1)k}} K_{X,s} * f(z) dm(z). \tag{3.257}$$

However, by the definition of Poisson Integral and of $\tilde{x}$ and of $E(f,\epsilon)$ we get

$$PI(K_{X,s} * f)(x,y) \gtrsim_{(X,Q,M)} PI(K_{X,s} * f)(\tilde{x},y) \geq \epsilon, \tag{3.258}$$

which is the required inequality. □

Observation 3.3.1. The previous Harnack-type inequality still holds if we replace $K_{X,s} * f$ in the previous definition and lemma with a generic function $g \in L^1(X)$ such that $g > 0$.

Lemma 3.3.4 (Uniform continuity at the boundary of the Poisson Integral)**.** *Let $g \in C(X)$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{x_0 \in X} \left(\sup_{P \in B_\rho((x_0,0),\delta)} |PI(g)(P) - g(x_0)| \right) \leq \epsilon \tag{3.259}$$

Proof. Let $g \in C_0(X)$. Then $g \in C(x)$. The space X is compact, so by Heine-Cantor theorem we have that for all $\epsilon_1 > 0$ there exists $\delta_1 = \delta_1(\epsilon_1) > 0$ such that for every $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta_1$ we have $|g(x_1) - g(x_2)| < \epsilon_1$.
Let $\epsilon_1 > 0$ be a number to be fixed later. Let $\delta_1(\epsilon_1) > 0$ be the number defined by the Heine-Cantor theorem. We claim that for every $0 < \epsilon_2 < 1$ there exist $\tilde{y} = \tilde{y}(\epsilon_2, \delta_1)$ such that

$$I(x, y, \delta_1) := \int_{X \setminus B_d\left(x, \frac{\delta_1}{2}\right)} C(x, y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x, 2^k y)}(z)}{2^{(Q+1)k}} dm(z) \le \epsilon_2 \tag{3.260}$$

for all $x \in X$, for all $y < \tilde{y}$.
Indeed, let $\epsilon_2 > 0$. For all $k \le \left\lfloor \log_2 \left(\frac{\delta_1}{2y} \right) \right\rfloor$ we have $B_d(x, 2^k y) \subseteq B_d(x, \frac{\delta_1}{2})$, so we can estimate

$$\int_{X \setminus B_d(x, \frac{\delta_1}{2})} C(x, y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x, 2^k y)}(z)}{2^{(Q+1)k}} dm(z) \le C_2 \sum_{k=\left\lfloor \log_2 \left(\frac{\delta_1}{2y} \right) \right\rfloor + 1}^{+\infty} \frac{m(B_d(x, 2^k y))}{y^Q 2^{(Q+1)k}}, \tag{3.261}$$

here C_2 is the uniform upper estimate of $C(x, y)$ given in (3.148).
By Ahlfors-regularity of X we get the estimate

$$I(x, y, \delta_1) \lesssim_{(X)} C_2 \sum_{k=\left\lfloor \log_2 \left(\frac{\delta_1}{2y} \right) \right\rfloor + 1}^{+\infty} \frac{y^Q 2^{kQ}}{y^Q 2^{(Q+1)k}} = C_2 \frac{2}{2^{\left(\left\lfloor \log_2 \left(\frac{\delta_1}{2y} \right) \right\rfloor + 1 \right)}} := \varphi(\delta_1, y). \tag{3.262}$$

We observe that $\varphi(\delta_1, y) \to 0$ as $y \to 0$, so there exists $\tilde{y} = \tilde{y}(\delta_1, \epsilon_2)$ such that

$$I(x, y, \delta_1) \lesssim_{(X)} \varphi(\delta_1, y) \le \epsilon_2 \tag{3.263}$$

for all $x \in X$, for all $0 < y < \tilde{y}$. Up to multiplying ϵ_2 by a constant which depends only on X we get

$$I(x, y, \delta_1) \le \epsilon_2 \tag{3.264}$$

for all $x \in X$, for all $0 < y < \tilde{y}$, proving the claim (3.260).
Now we are going to prove the statement. Let $\epsilon > 0$. Fix

$$\epsilon_1 = \epsilon_1(\epsilon) := \frac{\epsilon}{2}. \tag{3.265}$$

Let $\delta_1 = \delta_1(\epsilon_1)$ be the number defined by Heine-Cantor theorem. Let

$$\operatorname{osc}(g) := \sup(g) - \inf(g) \tag{3.266}$$

be the oscillation of the function g. If $\operatorname{osc}(g) = 0$ then the functions g and $PI(g)$ are constant, so the claim is trivial. Suppose $\operatorname{osc}(g) > 0$.
The function g is continuous over a compact set X, so $\operatorname{osc}(g) < +\infty$. Fix

$$\epsilon_2 = \epsilon_2(\epsilon) := \frac{\epsilon}{2 \cdot \operatorname{osc}(g)}. \tag{3.267}$$

Let $\tilde{y} = \tilde{y}(\epsilon_2, \delta_1)$ be the number previously defined.
Let $x_0 \in X$, let $P = (x, y) \in X \times (0, +\infty)$ such that $y < \tilde{y}$ and $d(x, x_0) < \frac{\delta_1}{2}$. We compute

$$PI(g)(P) - g(x_0) = \int_X \frac{C(x,y)}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} g(z) dm(z) - g(x_0) = \int_X \frac{C(x,y)}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} \left(g(z) - g(x_0)\right) dm(z). \tag{3.268}$$

Now we write

$$|PI(g)(P) - g(x_0)| \leq \int\limits_{B_d\left(x, \frac{\delta_1}{2}\right)} \frac{C(x,y)}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} |g(z) - g(x_0)| \, dm(z) + \int\limits_{X \setminus B_d\left(x, \frac{\delta_1}{2}\right)} \frac{C(x,y)}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} |g(z) - g(x_0)| \, dm(z). \tag{3.269}$$

If $z \in B_d\left(x, \frac{\delta_1}{2}\right)$ then, by triangle inequality, we have

$$d(x_0, z) \leq d(x_0, x) + d(x, z) \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1, \tag{3.270}$$

so by the definition of δ_1 we have $|g(z) - g(x_0)| < \epsilon_1$, so we get the estimate

$$|PI(g)(P) - g(x_0)| \leq \int\limits_{B_d\left(x, \frac{\delta_1}{2}\right)} C(x,y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} \cdot \epsilon_1 dm(z) + \int\limits_{X \setminus B_d\left(x, \frac{\delta_1}{2}\right)} C(x,y) \cdot \frac{1}{y^Q} \sum_{k=0}^{+\infty} \frac{\chi_{B_d(x,2^k y)}(z)}{2^{(Q+1)k}} \text{osc}(g) dm(z). \tag{3.271}$$

However, $y < \tilde{y}$, so we get

$$|PI(g)(P) - g(x_0)| \leq 1 \cdot \epsilon_1 + \text{osc}(g) \cdot \epsilon_2. \tag{3.272}$$

So we substitute the values of ϵ_1 and ϵ_2 and we get

$$|PI(g)(P) - g(x_0)| \leq \epsilon \tag{3.273}$$

for all $P = (x, y) \in X \times (0, +\infty)$ such that $y < \tilde{y}$, $d(x, x_0) < \frac{\delta_1}{2}$. By defining

$$\delta = \delta(\epsilon) := \min\left\{\frac{\delta_1}{2}, \tilde{y}\right\} \tag{3.274}$$

we have that if $\rho((x,y),(x_0,0)) < \delta$ then

$$d(x,x_0) < \delta \leq \frac{\delta_1}{2}, \quad \text{and} \quad |y| < \delta \leq \tilde{y}. \tag{3.275}$$

So we proved that for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that

$$|PI(g)(P) - g(x_0)| \leq \epsilon \tag{3.276}$$

for all $x_0 \in X$, for all $P \in B_\rho((x_0,0),\delta)$.
Taking the supremum over all x_0 and P gives us (3.259), ending the proof. □

3.4 Convergence at the boundary

In this section we prove several technical lemmas and propositions, and then we prove the two main results of this work: the non tangential convergence at the boundary of the harmonic extension of a Riesz potential up to an exceptional set of zero capacity and the tangential convergence at the boundary of the harmonic extension of a Riesz potential up to an exceptional set of null measure.

3.4.1 $C_{K_{X,s},p}$-thinness at the boundary

Let (X,d,m) be an Ahlfors Q-regular space. The following definitions generalize the concept of zero capacity to the space $X \times (0,+\infty)$, and allow us to formulate the main result of this work.

Definition 3.4.1. Let $M \geq 1$. Let $E \subseteq X \times (0,+\infty)$. We define

$$E_t := \{(x,y) \in E \mid 0 < y < t\}, \tag{3.277}$$

$$E^{M,*} := \bigcup_{(x,y)\in E} B_d(x, My) \subseteq X, \tag{3.278}$$

$$E_t^{M,*} := \bigcup_{\substack{(x,y)\in E \\ 0<y<t}} B_d(x, My) \subseteq X. \tag{3.279}$$

Definition 3.4.2. Let $M \geq 1$. Let $E \subseteq X \times (0,+\infty)$. E is M-$C_{K_{X,s},p}$-thin at $X \times \{0\}$ if

$$\lim_{t\to 0} C_{K_{X,s},p}(E_t^{M,*}) = 0. \tag{3.280}$$

Remark 3.4.1. If E is M-$C_{K_{X,s},p}$-thin at $X \times \{0\}$ then the essential projection of E

$$\{x \in X \mid \forall t > 0 \; \exists y < t \text{ such that } (x, y) \in E\}$$

is of $C_{K_{X,s},p}$-capacity 0, and hence of measure 0.

For every $x \in X$ and $r > 0$ let us define (when it exists) the radius

$$\eta_{X,p}(x, r) := \inf\left\{R > 0 \mid m(B_d(x, R)) \geq C_{K_{X,s},p}(B_d(x, r))\right\}, \tag{3.281}$$

and let us define

$$\eta^*_{X,p}(x, r) := \max\{r, \eta_{X,p}(x, r)\}. \tag{3.282}$$

Let $C \geq 1$, $M \geq 1$. Let $E \subseteq X$. We define

$$\tilde{E}_{K,p,C,M} := \bigcup_{x \in E} B_d(x, C \cdot \eta^*_{X,p}(x, M \cdot \delta_E(x))), \tag{3.283}$$

where

$$\delta_E(x) := d(x, E^C) = d(x, X \backslash E). \tag{3.284}$$

The following lemmas and propositions will be used to prove the main results of this work.

Proposition 3.4.1. *Let $E \subseteq X$ be a Borel set. Under the previous notations we have*

$$m(\tilde{E}_{K,p,C,M}) \lesssim_{(X,Q,s,p,C,M)} C_{K_{X,s},p}(E), \tag{3.285}$$

for all constants $C \geq \Omega$, where Ω is the constant defined by Theorem 3.2.4.

Proof. Let $\Omega \geq 1$ be the constant defined in the proof of Theorem 3.2.4. Let $C \geq \Omega$. Let F be an arbitrary compact subset of $\tilde{E}_{K,p,C,M}$. We claim that we can find a finite family of points $x_j \in E$ such that

$$\begin{aligned}
&F \subset \bigcup_j B_d(x_j, 5C \cdot \eta^*_{X,p}(x_j, M \cdot r_j)), \\
&\{B(x_j, \Omega \cdot \eta^*_{X,p}(x_j, M \cdot r_j))\} \text{ is disjoint}, \\
&r_j = \delta_E(x_j).
\end{aligned}$$

Indeed, we consider the open covering

$$\bigcup_{x \in E} B_d(x, C \cdot \eta^*_{X,p}(x, M \cdot \delta_E(x))) = \tilde{E}_{K,p,C,M} \supseteq F, \tag{3.286}$$

and by compactness of F we find a finite covering

$$\bigcup_{j=1}^{N} B_d(x_j, C \cdot \eta^*_{X,p}(x, M \cdot r_j)) \supseteq F. \tag{3.287}$$

It is not restrictive to assume that

$$r_1 \geq r_2 \geq \cdots \geq r_N. \tag{3.288}$$

We can find a finite covering

$$\bigcup_{k=1}^{\tilde{N}} B_d(x_{j_k}, 5C \cdot \eta^*_{X,p}(x, M \cdot r_{j_k})) \supseteq F, \tag{3.289}$$

such that

$$\left\{B_d(x_{j_k}, C \cdot \eta^*_{X,p}(x, M \cdot r_{j_k}))\right\}_{k=1}^{\tilde{N}} \quad \text{is disjoint.} \tag{3.290}$$

Indeed, consider r_1, which is the greatest radius r_j for $j = 1, 2, \ldots, N$. Suppose there are exactly $N(1)$ indexes $j_{k_1}, \ldots, j_{k_{N(1)}} \neq 1$ such that

$$B_d(x_{j_k}, C \cdot \eta^*_{X,p}(x, M \cdot r_{j_k})) \cap B_d(x_1, C \cdot \eta^*_{X,p}(x, M \cdot r_1)) \neq \emptyset, \tag{3.291}$$

for some $0 \leq N(1) \leq N - 1$. Then we have $r_{j_h} \leq r_1$ for $h = 1, 2, \ldots, N(1)$. By triangle inequality we get

$$B_d(x_{j_h}, C \cdot \eta^*_{X,p}(x, M \cdot r_{j_h})) \subseteq B_d(x_1, 5C \cdot \eta^*_{X,p}(x, M \cdot r_1)) \tag{3.292}$$

for all $h = 1, 2, \ldots, N(1)$. So, from (3.287) and (3.292), we get

$$F \subseteq B_d(x_1, 5C \cdot \eta^*_{X,p}(x, M \cdot r_1)) \cup \bigcup_{\substack{j=2,\ldots,N \\ j \notin \left\{j_{k_1},\ldots,j_{k_{N(1)}}\right\}}} B_d(x_j, C \cdot \eta^*_{X,p}(x, M \cdot r_j)), \tag{3.293}$$

and we have

$$B_d(x_1, C \cdot \eta^*_{X,p}(x, M \cdot r_1)) \cap B_d(x_j, C \cdot \eta^*_{X,p}(x, M \cdot r_j)) = \emptyset \tag{3.294}$$

for all $j \notin \{1, j_{k_1}, j_{k_2}, \ldots, j_{k_{(N_1)}}\}$.

So we iterate this procedure a finite amout of times, considering each time r_j the greatest radius in the family $\{r_{j_1}, \ldots, r_{j_M}\}$, and we prove that there exists a family of indexes $\{\tilde{j}_1, \ldots, \tilde{j}_{\tilde{N}}\}$ such that

$$F \subseteq \bigcup_{k=1,\ldots,\tilde{N}} B_d(x_{\tilde{j}_k}, 5C \cdot \eta^*_{X,p}(x, M \cdot r_{\tilde{j}_k})), \tag{3.295}$$

and

$$\left\{B_d(x_{\tilde{j}_k}, C \cdot \eta^*_{X,p}(x, M \cdot r_j))\right\}_{k=1,\ldots,\tilde{N}} \quad \text{is disjoint.} \tag{3.296}$$

The claim follows because $C \geq \Omega$.

Let $E' = \bigcup_j B_d(x_j, r_j)$. By definition of δ_E this is a subset of E. We apply Theorem 3.2.4 for $B(x_j, r_j)$ and E' and we get

$$\sum_j C_{K_{X,s},p}(B_d(x_j, r_j)) \lesssim_{(X,Q,s,p)} C_{K_{X,s},p}(E') \leq C_{K_{X,s},p}(E). \tag{3.297}$$

We observe that we apply Theorem 3.2.4 instead of the finite quasi-additivity formula because the number of sets in the family $\{B_d(x_j, r_j)\}_j$ depends on the choice of the sets F and E, and it can be arbitrarily large.
Now we observe that

$$m(F) \leq \sum_j m(B_d(x_j, 5C \cdot \eta^*_{X,p}(x_j, M \cdot r_j))) \approx_{(X,Q,s,p,C,M)} \sum_j m(B_d(x_j, \eta^*_{X,p}(x_j, r_j))). \quad (3.298)$$

By definition of $\eta^*_{X,p}$, using properties of the Riesz capacity (see Proposition 3.2.2) and the compactness of X, it follows that

$$m(B_d(x_j, \eta^*_{X,p}(x_j, r_j))) \approx_{(X,Q,s)} m(B_d(x_j, \eta_{X,p}(x_j, r_j))), \quad (3.299)$$

and by definition of $\eta_{X,p}$ and Ahlfors-regularity we have

$$m(B_d(x_j, \eta_{X,p}(x_j, r_j))) \approx_{(X,Q,s)} C_{K_{X,s},p}(B_d(x_j, r_j)). \quad (3.300)$$

Hence we get

$$m(F) \lesssim_{(X,Q,s,p,C,M)} C_{K_{X,s},p}(E). \quad (3.301)$$

A measure m on an Ahlfors-regular space (X, d, m) is regular, so, since F is an arbitrary compact subset of $\tilde{E}_{K,p,C,M}$, the required inequality follows and the theorem is proved. □

Lemma 3.4.2. *Let $f \in L^p(X)$, $f \geq 0$. Let $\epsilon > 0$, let $M \geq 1$. Consider*

$$E(f, \epsilon) = \{(x, y) \in X \times (0, +\infty) \mid PI(K_{X,s} * f)(x, y) > \epsilon\}, \quad (3.302)$$

$$E^{M,*}(f, \epsilon) = \bigcup_{(x,y)\in E(f,\epsilon)} B_d(x, My). \quad (3.303)$$

Then

$$C_{K_{X,s},p}(E^{M,*}(f, \epsilon)) \lesssim_{(X,Q,s,p,M)} \left(\frac{\|f\|_{L^p(X)}}{\epsilon}\right)^p. \quad (3.304)$$

Proof. Let

$$E^{M'}(f, \epsilon) = \bigcup_{(x,y)\in E(f,\epsilon)} B_d(x, My) \times \{y\} \subseteq X \times (0, +\infty). \quad (3.305)$$

Since $f \geq 0$ we have $PI(K_{X,s} * f) \geq 0$, so we may apply Lemma 3.3.3 to get

$$PI(K_{X,s} * f)(x, y) \gtrsim_{(X,Q,M)} \epsilon \quad \forall (x, y) \in E'(f, \epsilon). \quad (3.306)$$

Let us consider the maximal function

$$F(x) := \sup_{y>0} PI(f)(x, y). \quad (3.307)$$

By the maximal inequality (see [26, Theorem 3.7]) we get

$$\|F\|_{L^p(X)} \lesssim_{(X,p)} \|f\|_{L^p(X)}, \tag{3.308}$$

so $F \in L^p(X)$.
Now we compute the potential of F and we get that, for every $y > 0$, we have

$$\begin{aligned} K_{X,s} * F(x) = &\int_X K_{X,s}(x,z)F(z)dm(z) = \\ &\int_X K_{X,s}(x,z) \sup_{\tilde{y}>0} PI(f)(z,\tilde{y})dm(z) \geq \\ &\int_X K_{X,s}(x,z)PI(f)(z,y)dm(z) = \\ &K_{X,s} * (PI(f)(\cdot,y))(x). \end{aligned} \tag{3.309}$$

Now we apply Lemma 3.3.2 and we get

$$K_{X,s} * F(x) \gtrsim_{(X,Q,s)} PI(K_{X,s} * f)(x,y), \tag{3.310}$$

for every $y > 0$.
By construction $E^{M,*}(f,\epsilon)$ is the projection on the space X of the set $E^{M'}(f,\epsilon) \subseteq X \times (0,+\infty)$, so $\forall x \in E^{M,*}(f,\epsilon)$ $\exists y(x) > 0$ such that $(x,y(x)) \in E^{M'}(f,\epsilon)$.
We use (3.306) to get

$$K_{X,s} * F(x) \gtrsim_{(X,Q,s)} PI(K_{X,s} * f)(x,y(x)) \gtrsim_{(X,Q,M)} \epsilon, \quad \forall x \in E^{M,*}(f,\epsilon). \tag{3.311}$$

However, $F \in L^p(X)$, so from the definition of capacity we get that

$$C_{K_{X,s},p}(E^{M,*}(f,\epsilon)) \lesssim_{(X,Q,s,p,M)} \left(\frac{\|F\|_{L^p(X)}}{\epsilon}\right)^p. \tag{3.312}$$

Using the maximal inequality (3.308) we get

$$C_{K_{X,s},p}(E^{M,*}(f,\epsilon)) \lesssim_{(X,Q,s,p,M)} \left(\frac{\|f\|_{L^p(X)}}{\epsilon}\right)^p, \tag{3.313}$$

ending the proof. □

Lemma 3.4.3. *Let* $E \subseteq X \times (0,+\infty)$. *Let* $M \geq 1$. *Let*

$$f : X \times [0,+\infty) \longrightarrow [0,+\infty) \tag{3.314}$$

such that

1)

$$f(x, y_1) \le f(x, y_2) \quad \textit{for all } y_1 \le y_2,$$

2) There exists a constant $\alpha \ge 1$ such that for all $x_1, x_2 \in X$, for all $y \ge 0$ we have

$$f(x_1, y) \le \alpha f(x_2, y).$$

Let $\Omega_{f,x_0} := \{(x, y) \mid d(x, x_0) \le f(x_0, y)\}$. Then

$$\{x \in X \mid \Omega_{f,x} \cap E \ne \emptyset\} \subseteq \bigcup_{x \in E^{M,*}} \Omega_{\alpha f,x}(M\delta_{E^{M,*}}(x)), \tag{3.315}$$

where $\Omega_{f,x_0}(y) := \{x \in X \mid (x, y) \in \Omega_{f,x_0}\}$, and $\delta_{E^{M,*}}(x) = d\left(x, \left(E^{M,*}\right)^C\right)$.

Proof. Let $x_0 \in \{x \in X \mid \Omega_{f,x} \cap E \ne \emptyset\}$. Since $\Omega_{f,x_0} \cap E \ne \emptyset$ there exist $(\tilde{x}, \tilde{y}) \in \Omega_{f,x_0} \cap E$. We have

- $\tilde{x} \in \Omega_{f,x_0}(\tilde{y})$, hence property 2) entails $x_0 \in \Omega_{\alpha f,x}(\tilde{y})$.
- $(\tilde{x}, \tilde{y}) \in E$, hence $B_d(\tilde{x}, M\tilde{y}) \subseteq E^{M,*}$ by definition of $E^{M,*}$, so $\delta_{E^{M,*}}(\tilde{x}) \ge M\tilde{y}$.

The monotonicity of f entails the monotonicity of the regions $\Omega_{\alpha f,x}$, so we have $\Omega_{\alpha f,x}(r_1) \subseteq \Omega_{\alpha f,x}(r_2)$ $\forall r_1 < r_2$, for all $x \in X$, so we get

$$x_0 \in \Omega_{\alpha f,\tilde{x}}(\tilde{y}) \subseteq \Omega_{\alpha f,\tilde{x}}\left(\frac{1}{M}\delta_{E^{M,*}}(\tilde{x})\right) \subseteq \Omega_{\alpha f,\tilde{x}}(M\delta_{E^{M,*}}(\tilde{x})), \tag{3.316}$$

which entails the required inequality. □

Proposition 3.4.4. *Let $C \ge 1$, let $M \ge 1$. Let $p > 1$. Let $\frac{1}{p'} \le s < 1$. Consider the function*

$$f(x, y) := C \cdot \eta^*_{X,p}(x, My). \tag{3.317}$$

It can be proved that the function f satisfies the following two properties:

1)

$$f(x, y_1) \le f(x, y_2) \quad \textit{for all } y_1 \le y_2,$$

2) There exists a constant $\tilde{\alpha} = \tilde{\alpha}(X, s, p) > 1$ such that for all $x_1, x_2 \in X$, for all $y \ge 0$ we have

$$f(x_1, y) \le \tilde{\alpha} f(x_2, y).$$

The proof of the previous proposition follows from Proposition 3.2.2.

Definition 3.4.3. Let $\tilde{\alpha} = \tilde{\alpha}(X, s, p)$ denote the constant defined by property 2) in the previous proposition.

The following proposition will be used to prove the main result of this chapter.

Proposition 3.4.5. *Let $p > 1$, $\frac{1}{p'} \leq s < 1$. Let $\Omega > 1$ be the constant defined by Theorem 3.2.4. Let $M \geq 1$. If $E \subseteq X \times (0, +\infty)$ is M-$C_{K_{X,s},p}$-thin at $X \times \{0\}$ then, given*

$$\Omega_{x_0,K_{X,s},p,\Omega,M} := \left\{(x,y) \mid x \in B_d(x_0, \Omega \cdot \eta^*_{X,p}(x_0, My))\right\}, \tag{3.318}$$

for $x_0 \in X$, we have

$$m\left(\bigcap_{t>0} \left\{x \in X \mid \Omega_{x_0,K_{X,s},p,\Omega,M} \cap E_t \neq 0\right\}\right) = 0. \tag{3.319}$$

Proof. By Proposition 3.4.4 the function

$$f(x,y) := \Omega \cdot \eta^*_{X,p}(x, My). \tag{3.320}$$

satisfies the hypotheses of Lemma 3.4.3. So we apply Lemma 3.4.3 to the region

$$\Omega_{x_0,K_{X,s},p,\Omega,M} := \Omega_{f,x_0} = \{(x,y) \mid d(x,x_0) \leq \Omega \cdot \eta^*_{X,p}(x_0, My)\} \tag{3.321}$$

and we get

$$\begin{aligned}\{x \in X \mid \Omega_{x,K_{X,s},p,\Omega,M} \cap E = \emptyset\} \quad \subseteq \quad & \bigcup_{x \in E^{M,*}} \Omega_{\tilde{\alpha}f,x}(\delta_{E^{M,*}}(x)) = \\ & \bigcup_{x \in E^{M,*}} \Omega_{x,K_{X,s},p,\tilde{\alpha}\Omega,M}(\delta_{E^{M,*}}(x)).\end{aligned} \tag{3.322}$$

So by Proposition 3.4.1, with $C = \tilde{\alpha}\Omega$, we get

$$m(\{x \in X \mid \Omega_{x,K_{X,s},p,\Omega,M} \cap E \neq \emptyset\}) \lesssim_{(X,Q,s,p,\tilde{\alpha}\Omega,M)} C_{K_{X,s},p}(E^{M,*}). \tag{3.323}$$

Now we apply equation (3.323) to $E = E_t$ and we get

$$m(\{x \in X \mid \Omega_{x,K_{X,s},p,\Omega,M} \cap E_t \neq \emptyset\}) \lesssim_{(X,Q,s,p,\tilde{\alpha}\Omega,M)} C_{K_{X,s},p}(E_t^{M,*}) \to 0 \quad \text{as } t \to 0 \tag{3.324}$$

because E is M-$C_{K_{X,s},p}$-thin at $X \times \{0\}$, so the theorem follows. ∎

3.4.2 Convergence at the boundary

We will now prove one more lemma and then we will prove the main results of this chapter.

Lemma 3.4.6. *Let $f \in L^p(X)$. Let $M \geq 1$ and $\delta > 0$. Then there exist $E \subseteq X \times (0, +\infty)$ and $F \subseteq X$ such that*

1. $C_{K_{X,s},p}(E^{M,*}) < \delta$ *and* $C_{K_{X,s},p}(F) < \delta$.
2. $\forall \epsilon > 0$ *there exists* $r > 0$ *such that*

$$\sup_{x \in X \setminus F} \left(\sup_{P \in B_\rho((x,0),r) \setminus E} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x)| \right) < \epsilon. \tag{3.325}$$

Proof. Let $\delta > 0$. Let $\epsilon > 0$. Let $A = A(X, Q, s, p, M) > 0$ be the constant defined by Lemma 3.4.2 such that

$$C_{K_{X,s},p}(E^{M,*}(f, \epsilon)) \leq A \cdot \left(\frac{\|g\|_{L^p(X)}}{\epsilon} \right)^p \tag{3.326}$$

for all $g \in L^p_+(X)$, for all $\epsilon > 0$. Let $f \in L^p(x)$. Consider

$$f^+ := \max\{f, 0\}, \quad f^- := \max\{-f, 0\}. \tag{3.327}$$

Let $\epsilon_1 = \epsilon_1(j) > 0$ arbitrary to be fixed later. By Lusin's theorem and Urysohn's lemma for all $\epsilon_1 > 0$, for all $j \in \mathbb{N}$ there exist $g_j^+, g_j^- \in L^p(X) \cap C_0(X)$ and there exist sets $S_j^+, S_j^- \subseteq X$ such that $m(S_j^+) < \epsilon_1$, $m(S_j^-) < \epsilon_1$, such that $g_j^+ \equiv f^+$ in $X \setminus S_j^+$, $g_j^- \equiv f^-$ in $X \setminus S_j^-$ and such that $0 \leq g_j^+ \leq f^+$ and $0 \leq g_j^- \leq f^-$.
Indeed, let $j \in \mathbb{N}$. We apply Lusin's theorem to the function f^+ and we get that there exist an open set $S_{1,j}^+$ such that $m(S_{1,j}^+) < \frac{\epsilon_1}{2}$ and f^+ is continuous in $X \setminus S_{1,j}^+$. Let $S_{2,j}^+$ be an arbitrary closed set such that $S_{2,j}^+ \subseteq X \setminus \overline{S_{1,j}^+}$ and $m\left(\left(X \setminus \overline{S_{1,j}^+}\right) \setminus S_{2,j}^+\right) < \frac{\epsilon_1}{2}$. Such set exists because X is an Ahlfors-regular space, so the measure m is regular. We apply Urysohn's lemma and we get that there exists a function

$$h_j : X \longrightarrow [0, 1] \tag{3.328}$$

such that h_j is continuous, $h_j \equiv 1$ on $S_{2,j}^+$, $h_j \equiv 0$ on $\overline{S_{1,j}^+}$. We define

$$g_j^+ := f^+ \cdot h_j, \tag{3.329}$$

and $S_j^+ := X \setminus S_{2,j}^+$.
By construction $g_j^+ \in L^p(X) \cap C_0(X)$ (because X is compact), and we have $g_j^+ \equiv f^+$ in $X \setminus S_j^+$, and $0 \leq g_j^+ \leq f^+$. Moreover, by construction we have

$$m(S_j^+) = m(X \setminus S_{2,j}^+) \leq m\left(\left(X \setminus \overline{S_{1,j}^+}\right) \setminus S_{2,j}^+\right) + m(S_{1,j}^+) \leq \epsilon_1, \tag{3.330}$$

proving the claim for the function f^+. We repeat the same argument for f^- and the claim is proved.
We have

$$\|f^+ - g_j^+\|_{L^p(X)} = \left(\int_{S_j^+} |f^+ - g_j^+|^p dm\right)^{\frac{1}{p}} \longrightarrow 0 \quad \text{as } \epsilon_1 \to 0. \tag{3.331}$$

Now we repeat the same argument for f^- and choose $\epsilon_1 = \epsilon_1(\delta, A, j, p)$ small enough such that

$$\|f^+ - g_j^+\|_{L^p(X)} \le 2^{-j} \left(\frac{2^{-j}\delta}{2A}\right)^{\frac{1}{p}}, \tag{3.332}$$

$$\|f^- - g_j^-\|_{L^p(X)} \le 2^{-j} \left(\frac{2^{-j}\delta}{2A}\right)^{\frac{1}{p}}. \tag{3.333}$$

We define

$$E_{+,j} := E(f^+ - g_j^+, 2^{-j}) = \{(x,y) \in X \times (0,+\infty) \mid PI(K_{X,s} * (f^+ - g_j^+))(x,y) > 2^{-j}\}, \tag{3.334}$$

$$E_{-,j} := E(f^- - g_j^-, 2^{-j}) = \{(x,y) \in X \times (0,+\infty) \mid PI(K_{X,s} * (f^- - g_j^-))(x,y) > 2^{-j}\}. \tag{3.335}$$

Following definition 3.4.1 we consider

$$E_{+,j}^{M,*} := \bigcup_{(x,y)\in E_{+,j}} B_d(x, My) \subseteq X, \tag{3.336}$$

$$E_{-,j}^{M,*} := \bigcup_{(x,y)\in E_{-,j}} B_d(x, My) \subseteq X. \tag{3.337}$$

By construction we may apply Lemma 3.4.2 and we get

$$C_{K_{X,s},p}(E_{+,j}^{M,*}) \le A \left(\frac{\|f^+ - g_j^+\|_{L^p(X)}}{2^{-j}}\right)^p \le A \left(\frac{2^{-j}}{2^{-j}} \left(\frac{2^{-j}\delta}{2A}\right)^{\frac{1}{p}}\right)^p = 2^{-j}\frac{\delta}{2}, \tag{3.338}$$

$$C_{K_{X,s},p}(E_{-,j}^{M,*}) \le A \left(\frac{\|f^- - g_j^-\|_{L^p(X)}}{2^{-j}}\right)^p \le A \left(\frac{2^{-j}}{2^{-j}} \left(\frac{2^{-j}\delta}{2A}\right)^{\frac{1}{p}}\right)^p = 2^{-j}\frac{\delta}{2}. \tag{3.339}$$

Let us define

$$E := \bigcup_{j=1}^{+\infty} E_{-,j} \cup \bigcup_{j=1}^{+\infty} E_{+,j}. \tag{3.340}$$

By construction

$$E^{M,*} = \bigcup_{(x,y)\in E} B_d(x, My) = \bigcup_{j=1}^{+\infty} E_{-,j}^{M,*} \cup \bigcup_{j=1}^{+\infty} E_{+,j}^{M,*}. \tag{3.341}$$

By the subadditivity of the capacity and by equations (3.338) and (3.339) we get

$$C_{K_{X,s},p}(E^{M,*}) \leq \sum_{j=1}^{+\infty} 2^{-j}\frac{\delta}{2} + \sum_{j=1}^{+\infty} 2^{-j}\frac{\delta}{2} = \delta. \tag{3.342}$$

Now we define

$$F_{+,j} := \{x \in X \mid K_{X,s} * (f^+ - g_j^+) \geq 2^{-j}\}, \tag{3.343}$$

$$F_{-,j} := \{x \in X \mid K_{X,s} * (f^- - g_j^+) \geq 2^{-j}\}. \tag{3.344}$$

For all $x \in F_{+,j}$ we have

$$K_{X,s} * \left(\frac{f^+ - g_j^+}{2^{-j}}\right)(x) \geq 1, \tag{3.345}$$

so by the definition of capacity we get

$$C_{K_{X,s},p}(F_{+,j}) \leq \|2^j(f^+ - g_j^+)\|^p_{L^p_{(X)}} \leq \frac{2^{-j}\delta}{2A}. \tag{3.346}$$

Up to multiplying δ by the constant A we can reformulate the last equation to get

$$C_{K_{X,s},p}(F_{+,j}) \leq 2^{-j}\frac{\delta}{2}. \tag{3.347}$$

By the same argument we also get

$$C_{K_{X,s},p}(F_{-,j}) \leq 2^{-j}\frac{\delta}{2}. \tag{3.348}$$

Let us define

$$F := \bigcup_{j=1}^{+\infty} F_{-,j} \cup \bigcup_{j=1}^{+\infty} F_{+,j}. \tag{3.349}$$

By the subadditivity of the capacity and by equations (3.347) and (3.348) we get

$$C_{K_{X,s},p}(F) \leq \sum_{j=1}^{+\infty} 2^{-j}\frac{\delta}{2} + \sum_{j=1}^{+\infty} 2^{-j}\frac{\delta}{2} = \delta. \tag{3.350}$$

We define

$$g_j := g_j^+ - g_j^-. \tag{3.351}$$

By linearity we get

$$\begin{aligned} PI(K_{X,s} * f)(x,y) - PI(K_{X,s} * g_j)(x,y) &= PI(K_{X,s} * (f - g_j))(x,y) = \\ &PI(K_{X,s} * (f^+ - f^- - g_j^+ + g_j^-))(x,y) = \\ &PI(K_{X,s} * (f^+ - g_j^+))(x,y) - PI(K_{X,s} * (f^- - g_j^-))(x,y), \end{aligned} \tag{3.352}$$

so by triangle inequality we get

$$|PI(K_{X,s} * f)(x,y) - PI(K_{X,s} * g_j)(x,y)| \leq |PI(K_{X,s} * (f^+ - g_j^+))(x,y)| + |PI(K_{X,s} * (f^- - g_j^-))(x,y)|. \tag{3.353}$$

Let $j \geq 1$. Suppose $(x,y) \in X \times (0,+\infty) \backslash E$. Then $(x,y) \notin E_{+,j} \cup E_{-,j}$, so by definition of $E_{+,j}$ and $E_{-,j}$ we get

$$|PI(K_{X,s} * (f^+ - g_j^+))(x,y)| < 2^{-j}, \quad \text{and} \quad |PI(K_{X,s} * (f^- - g_j^-))(x,y)| < 2^{-j}. \tag{3.354}$$

From equations (3.353) and (3.354) we get

$$|PI(K_{X,s} * f)(x,y) - PI(K_{X,s} * g_j)(x,y)| \leq 2^{-j+1} \tag{3.355}$$

for all $(x,y) \in X \times (0,+\infty) \backslash E$. So we proved that

$$PI(K_{X,s} * g_j) \overset{j \to +\infty}{\longrightarrow} PI(K_{X,s} * f) \quad \text{uniformly on } X \times (0,+\infty) \backslash E. \tag{3.356}$$

Suppose $x \in X \backslash F$. By additivity of the potential and by triangle inequality we get that

$$|K_{X,s} * f(x) - K_{X,s} * g_j(x)| \leq |K_{X,s} * f^+(x) - K_{X,s} * g_j^+(x)| + |K_{X,s} * f^-(x) - K_{X,s} * g_j^-(x)|. \tag{3.357}$$

However, by the definition of $F_{+,j}$ and $F_{-,j}$, it follows that

$$|K_{X,s} * f(x) - K_{X,s} * g_j(x)| \leq 2^{-j+1} \tag{3.358}$$

for all $x \in X \backslash F$. So we proved that

$$K_{X,s} * g_j \overset{j \to +\infty}{\longrightarrow} K_{X,s} * f \quad \text{uniformly on } X \backslash F. \tag{3.359}$$

So from (3.356) and (3.359) we get that there exists $j_0 \in \mathbb{N}$ such that

$$\sup_{x \in X \backslash F} |K_{X,s} * f(x) - K_{X,s} * g_{j_0}(x)| \leq \frac{\epsilon}{3}, \tag{3.360}$$

$$\sup_{P \in X \times (0,+\infty) \backslash E} |PI(K_{X,s} * f)(P) - PI(K_{X,s} * g_{j_0})(P)| \leq \frac{\epsilon}{3}. \tag{3.361}$$

By construction $g_{j_0} \in C_0(X)$, so $K_{X,s} * g_{j_0} \in C(X)$ so we apply Lemma 3.3.4 to the function $K_{X,s} * g_{j_0}$ and we get that there exists $r > 0$ such that

$$\sup_{x \in X} \left(\sup_{P \in B_\rho((x,0),r)} |PI(K_{X,s} * g_{j_0})(P) - K_{X,s} * g_{j_0}(x)| \right) \leq \frac{\epsilon}{3}. \tag{3.362}$$

The statement follows from (3.360), (3.361) and (3.362) by triangle inequality. □

We are now going to prove the main results of this chapter.

Theorem 3.4.7 (Non tangential convergence for the Riesz potential)**.** *Let (X, d, m) be a compact Ahlfors-regular space. Let $f \in L^p(X)$. Let $M > 1$. Then $\exists E \subseteq X \times (0, +\infty)$ such that E is M-$C_{K_{X,s},p}$-thin at $X \times \{0\}$ and*

$$\lim_{\substack{(x,y)=P\to(x_0,0)\\ x\in B_d(x_0,My)\\ (x,y)\notin E}} PI(K_{X,s} * f)(P) = K_{X,s} * f(x_0) \tag{3.363}$$

for $C_{K_{X,s},p}$-almost everywhere $x_0 \in X$, i.e. $\exists F \subset X$ such that $C_{K_{X,s},p}(F) = 0$ and (3.363) holds $\forall x_0 \in X\backslash F$.

Proof. Let $f \in L^p(X)$. Let $\epsilon_j > 0$ be a sequence such that $\epsilon_j \downarrow 0$ as $j \to +\infty$. By Lemma 3.4.6 there exist $E_j \subseteq X \times (0, +\infty)$, $F_j \subseteq X$ and $r_j \downarrow 0$ such that

$$\sum_j C_{K_{X,s},p}(E_j^{M,*}) < +\infty, \quad \text{and} \quad \sum_j C_{K_{X,s},p}(F_j) < +\infty, \tag{3.364}$$

$$\sup_{x\in X\backslash F_j}\left(\sup_{P\in B_\rho((x,0),r_j)\backslash E_j} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x)|\right) < \epsilon_j. \tag{3.365}$$

Let us choose $t_j \downarrow 0$ such that

$$t_j < r_{j+1}, \tag{3.366}$$

$$\{(x, y) \mid x \in B_d(x_0, My)\} \cap B_\rho((x_0, 0), r_i) \subseteq \bigcup_{j=i}^{+\infty} B_\rho((x_0, 0), r_j) \cap \{(x, y) \mid y \geq t_j\}. \tag{3.367}$$

A sequence t_j with such properties exists thanks to the definition of the distance ρ.
Let us define

$$E'_j = E_j \cap \{(x, y) \mid y \geq t_j\}, \tag{3.368}$$

and

$$E = \bigcup_j E'_j. \tag{3.369}$$

Let $i \in \mathbb{N}$ be a fixed index. By construction $t_i > t_j$ for all $j > i$, and

$$E_{t_i} = \{(x, y) \in E \mid y < t_i\} = \left\{(x, y) \in \bigcup_{j=1}^{+\infty} E'_j \;\middle|\; y < t_i\right\} = \bigcup_{j=1}^{+\infty}\left\{(x, y) \in E'_j \;\middle|\; y < t_i\right\}. \tag{3.370}$$

However from (3.368) we have

$$\left\{(x, y) \in E'_j \;\middle|\; y < t_i\right\} = \emptyset \quad \forall j \leq i, \tag{3.371}$$

so we get

$$E_{t_i} = \bigcup_{j=i}^{+\infty} \left\{ (x,y) \in E'_j \;\middle|\; y < t_i \right\} \subseteq \bigcup_{j=i}^{+\infty} E_j. \tag{3.372}$$

From (3.278), (3.372) and by subadditivity of the capacity we get

$$C_{K_{X,s},p}(E^{M,*}_{t_i}) \leq C_{K_{X,s},p}\left(\bigcup_{j=i}^{+\infty} E^{M,*}_j \right) \leq \sum_{j=i}^{+\infty} C_{K_{X,s},p}\left(E^{M,*}_j\right) \tag{3.373}$$

Equation (3.364) entails

$$C_{K_{X,s},p}(E^{M,*}_{t_i}) \leq \sum_{j=i}^{+\infty} C_{K_{X,s},p}\left(E^{M,*}_j\right) \overset{i\to+\infty}{\longrightarrow} 0, \tag{3.374}$$

so, using the monotonocity of the capacity, we get

$$C_{K_{X,s},p}(E^{M,*}_{t}) \overset{t\to 0}{\longrightarrow} 0, \tag{3.375}$$

i.e. E is a M-$C_{K_{X,s},p}$-thin set at $X \times \{0\}$.
Now we define

$$F := \bigcap_{i=1}^{+\infty} \bigcup_{j=i}^{+\infty} F_j. \tag{3.376}$$

By (3.364) we have

$$C_{K_{X,s},p}(F) \leq C_{K_{X,s},p}\left(\bigcup_{j=i}^{+\infty} F_j \right) \leq \sum_{j=i}^{+\infty} C_{K_{X,s},p}(F_j) \longrightarrow 0 \quad \text{as } j \to +\infty, \tag{3.377}$$

so we have $C_{K_{X,s},p}(F) = 0$.
Let $x_0 \in X \backslash F$. By definition $\exists j_0 = j_0(x_0)$ such that $x_0 \in F_j$, $\forall j \geq j_0$. From (3.365) we get

$$\sup_{P \in B_\rho((x_0,0),r_j)\backslash E_j} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \leq \epsilon_j \quad \forall j \geq j_0. \tag{3.378}$$

Let $i \geq j_0$. Using (3.367) and the definition of E we get

$$\begin{aligned} &\sup_{(\{P=(x,y) \mid x \in B_d(x_0,My)\} \cap B_\rho((x_0,0),r_i))\backslash E} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \leq \\ &\sup_{j\geq i} \left(\sup_{P \in (B_\rho((x_0,0),r_j) \cap \{(x,y) \mid y \geq t_j\})\backslash E} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \right) \leq \\ &\sup_{j\geq i} \left(\sup_{P \in (B_\rho((x_0,0),r_j) \cap \{(x,y) \mid y \geq t_j\})\backslash E_j} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \right) \leq \\ &\sup_{j\geq i} \epsilon_j = \epsilon_i \longrightarrow 0 \quad \text{as } i \to +\infty, \end{aligned} \tag{3.379}$$

which entails the thesis, finishing the proof. □

Theorem 3.4.8 (Tangential convergence for the Riesz potential). *Let (X,d,m) be a compact Ahlfors-regular space. Let $p>1$, let $\frac{1}{p'} \le s < 1$. Let $\Omega > 1$ be the constant defined by Theorem 3.2.4. Consider the region*

$$\Omega_{x_0,K_{X,s},p,\Omega,M} := \left\{(x,y) \mid x \in B_d(x_0, \Omega \cdot \eta^*_{X,p}(x_0,My))\right\}. \tag{3.380}$$

Let $f \in L^p(X)$. Then

$$\lim_{\substack{(x,y)=P\to(x_0,0)\\ P\in\Omega_{x_0,K_{X,s},p,\Omega,M}}} PI(K_{X,s} * f)(P) = K_{X,s} * f(x_0) \tag{3.381}$$

for m-almost all $x_0 \in X$.

Proof. The proof is similar to the one of Theorem 3.4.7. Let $f \in L^p(X)$. Let $\epsilon_j > 0$ be a sequence such that $\epsilon_j \downarrow 0$ as $j \to +\infty$. By Lemma 3.4.6 there exist $E_j \subseteq X \times (0,+\infty)$, $F_j \subseteq X$ and $r_j \downarrow 0$ such that

$$\sum_j C_{K_{X,s},p}(E_j^{M,*}) < +\infty, \quad \text{and} \quad \sum_j C_{K_{X,s},p}(F_j) < +\infty, \tag{3.382}$$

$$\sup_{x\in X\setminus F_j} \left(\sup_{P\in B_\rho((x,0),r_j)\setminus E_j} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x)| \right) < \epsilon_j. \tag{3.383}$$

Let us choose $t_j \downarrow 0$ such that

$$t_j < r_{j+1}, \tag{3.384}$$

$$\{(x,y) \mid x \in B_d(x_0, \Omega \cdot \eta^*_{X,p}(x_0,My)))\} \cap B_\rho((x_0,0),r_i) \subseteq \bigcup_{j=i}^{+\infty} B_\rho((x_0,0),r_j) \cap \{(x,y) \mid y \ge t_j\}. \tag{3.385}$$

A sequence t_j with such properties exists thanks to the definition of the distance ρ.
Let us define

$$E'_j = E_j \cap \{(x,y) \mid y \ge t_j\}, \tag{3.386}$$

and

$$E = \bigcup_j E'_j. \tag{3.387}$$

By the same argument used in the proof of Theorem 3.4.7 we get

$$C_{K_{X,s},p}(E_t^{M,*}) \xrightarrow{t\to 0} 0, \tag{3.388}$$

i.e. E is a M-$C_{K_{X,s},p}$-thin set at $X \times \{0\}$.
Now we define

$$F := \bigcap_{i=1}^{+\infty} \bigcup_{j=i}^{+\infty} F_j, \tag{3.389}$$

and

$$S := F \cup \left(\bigcap_{t>0} \{x \in X \mid \Omega_{x,K_{X,s},p,\Omega,M} \cap E_t \neq \emptyset\} \right). \tag{3.390}$$

By the same argument in the proof of Theorem 3.4.7 we have $C_{K_{X,s},p}(F) = 0$, and hence $m(F) = 0$. By Proposition 3.4.5 we have

$$m\left(\bigcap_{t>0} \{x \in X \mid \Omega_{x,K_{X,s},p,\Omega,M} \cap E_t \neq \emptyset\} \right) = 0, \tag{3.391}$$

so we proved

$$m(S) = 0. \tag{3.392}$$

Let $x_0 \in X \backslash S$. By definition of S there exists $j_0 = j_0(x_0)$ such that $x_0 \in F_j$, $\forall j \geq j_0$, and such that

$$\Omega_{x_0,K_{X,s},p,\Omega,M} \cap E_t = \emptyset \quad \text{for all } t < t_{j_0}. \tag{3.393}$$

From (3.383) we get

$$\sup_{P \in B_\rho((x_0,0),r_j) \backslash E_j} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \leq \epsilon_j \quad \forall j \geq j_0. \tag{3.394}$$

Let $i \geq j_0$. Using (3.385), (3.393) and the definition of E we get

$$\sup_{\substack{P=(x,y) \in \Omega_{x_0,K_{X,s},p,\Omega,M} \\ y < t_{j_0}}} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \leq \tag{3.395}$$

$$\sup_{\substack{P=(x,y) \in \Omega_{x_0,K_{X,s},p,\Omega,M} \backslash E \\ y < t_{j_0}}} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \leq \tag{3.396}$$

$$\sup_{j \geq i} \left(\sup_{P \in (B_\rho((x_0,0),r_j) \cap \{(x,y) \mid y \geq t_j\}) \backslash E} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \right) \leq$$

$$\sup_{j \geq i} \left(\sup_{P \in (B_\rho((x_0,0),r_j) \cap \{(x,y) \mid y \geq t_j\}) \backslash E_j} |PI(K_{X,s} * f)(P) - K_{X,s} * f(x_0)| \right) \leq$$

$$\sup_{j \geq i} \epsilon_j = \epsilon_i \longrightarrow 0 \quad \text{as } i \to +\infty,$$

which entails the thesis, finishing the proof. □

Observation 3.4.1. The region $\Omega_{x_0,K_{X,s},p,\Omega,M}$ has the following properties:

- case $\frac{1}{p'} < s < 1$. There exist constants $0 < \tilde{C}_1 < \tilde{C}_2$, $y_0 > 0$ that depend only on X, s, p, M such that

$$\Omega_{x_0,K_{X,s},p,\Omega,M} \cap \{(x,y) \mid y < y_0\} \supseteq \{(x,y) \mid y < y_0,\ x \in B_d(x_0, \tilde{C}_1 \cdot y^{\frac{1}{p(s-1)+1}}\},$$

$$\Omega_{x_0,K_{X,s},p,\Omega,M} \cap \{(x,y) \mid y < y_0\} \subseteq \{(x,y) \mid y < y_0,\ x \in B_d(x_0, \tilde{C}_2 \cdot y^{\frac{1}{p(s-1)+1}}\},$$

- case $s = \frac{1}{p'}$. There exist constants $0 < \tilde{K}_1 < \tilde{K}_2$, $0 < D_1 < D_2$, $y_0 > 0$ that depend only on X, s, p, M such that

$$\begin{aligned}\Omega_{x_0,K_{X,s},p,\Omega,M} \cap \{(x,y) \mid y < y_0\} &\supseteq \{(x,y) \mid y < y_0,\ x \in B_d(x_0, \tilde{K}_1 \cdot \exp(D_1 \cdot y))\},\\ \Omega_{x_0,K_{X,s},p,\Omega,M} \cap \{(x,y) \mid y < y_0\} &\subseteq \{(x,y) \mid y < y_0,\ x \in B_d(x_0, \tilde{K}_2 \cdot \exp(D_2 \cdot y))\}.\end{aligned}$$

The proof of this observation follows from Proposition 3.2.2.

Bibliography

[1] D. R. Adams; L. I. Hedberg. *Function Spaces and Potential Theory*, Springer, 1996.

[2] H. Aikawa. *Quasiadditivity of capacity and minimal thinness.* (English summary) Ann. Acad. Sci. Fenn. Ser. A I Math. 18 (1993), no. 1, 65-75. 31B15 (31B25)

[3] H. Aikawa; A.A. Borichev. *Quasiadditivity and measure property of capacity and the tangential boundary behavior of harmonic functions.* Potential Theory - ICPT 94: Proceedings of the International Conference on Potential Theory held in Kouty, Czech Republic, August 13-20, 1994, edited by Josef Kral, Jaroslav Lukes, Ivan Netuka and Jiri Vesely, Berlin, Boston: De Gruyter, 2011, pp. 219-228. https://doi.org/10.1515/9783110818574.219

[4] H. Aikawa; M. Essen. *Potential Theory - Selected Topics*, Lecture Notes in Mathematics, 1633. Springer, 1996.

[5] N. Arcozzi; I. Holmes; P. Mozolyako; A. Volberg. *Bellman function sitting on a tree.* International Mathematics Research Notices, Volume 2021, Issue 16, August 2021, 12037-12053,

[6] N. Arcozzi; R. Rochberg; E. T. Sawyer. *Carleson measures for analytic Besov spaces.* Rev. Mat. Iberoamericana 18 (2002), no. 2, 443-510.

[7] N. Arcozzi; R. Rochberg; E. T. Sawyer. *The characterization of the Carleson measures for analytic Besov spaces: a simple proof.* Complex and harmonic analysis, 167-177, DEStech Publ., Inc., Lancaster, PA, 2007.

[8] N. Arcozzi; R. Rochberg; E. Sawyer; B. D. Wick. *Potential Theory on Trees, Graphs and Ahlfors-Regular Metric Spaces.* Potential Analysis 41, 317-366 (2014). https://doi.org/10.1007/s11118-013-9371-8

[9] N. Arcozzi; R. Rochberg; E. T. Sawyer; B. D. Wick. *The Dirichlet space and related function spaces.* Mathematical Surveys and Monographs, 239. American Mathematical Society, Providence, RI, 2019.

[10] R. Bañuelos, *The foundational inequalities of D. L. Burkholder and some of their ramifications.* Illinois J. Math. 54 (2010), no. 3, 789-868 (2012).

[11] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms.* Ann. Probab. 12 (1984), no. 3, 647-702.

[12] H. Cartan. *Theorie generale du balayage en potentiel newtonien.* Ann. Univ. Grenoble, Math. Phys. 22 (1946) 221-280.

[13] G. Choquet. *La naissance de la théorie des capacités: réflexion sur une expérience personnelle.* Comptes rendus de l'Acadmie des sciences. Srie gnrale, La Vie des sciences, vol. 3, n. 4, 1986, pp. 385-397.

[14] O. Frostman. *Potentiel d'equilibre et capacite des ensembles avec quelques applications il la theorie des fonctions.* Medd. Lunds Univ. Mat. Sem. 3 (1935) 1-118.

[15] C. F. Gauss. *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhaltnisse des Quadrats der Entfernung wirkenden Anziehungs und Abstossungs Kräfte.* Resultate aus den Beobachtungen des magnetischen Vereins im Jahre 1839, Leipzig, 1840.

[16] G. H. Hardy. *Note on a theorem of Hilbert.* Math. Z., 6(3-4), pp. 314-317, September 1920.

[17] G. H. Hardy. *Notes on some points in the integral calculus li: On hilberts double series theorem, and some connected theorems concerning the convergence of infinite series and integrals.* Messenger Math, 48: pp. 107-112, 1919.

[18] J. Heinonen; P. Koskela. *Quasiconformal maps in metric spaces with controlled geometry.* Acta Mathematica, Acta Math. 181(1), 1-61, 1998.

[19] N. V. Krylov. *Controlled diffusion processes.* Translated from the 1977 Russian original by A. B. Aries. Reprint of the 1980 edition. Stochastic Modelling and Applied Probability, 14. Springer-Verlag, Berlin.

[20] J. Lai. *The Bellman functions of the Carleson embedding theorem and the Doob's martingale inequality.* arXiv:1411.5408v3 [math.CA] 11 Feb 2015.

[21] V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations.* Second, revised and augmented edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 342. Springer, Heidelberg, 2011.

[22] F. L. Nazarov; S. R. Treil. *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz, 8:5 (1996), 32-162; St. Petersburg Math. J., 8:5 (1997), 721-824.

[23] F. L. Nazarov; S. Treil; A. Volberg. *Bellman function in stochastic control and harmonic analysis.* Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), 393-423, Oper. Theory Adv. Appl., 129, Birkhuser, Basel, 2001.

[24] B. Øksendal. *Stochastic differential equations.* An introduction with applications. Fifth edition. Universitext. Springer-Verlag, Berlin, 1998.

[25] A. Nagel; W. Rudin, J.H. Shapiro. *Tangential Boundary Behavior of Function in Dirichlet-Type Spaces.* Annals of Mathematics , Sep., 1982, Second Series, Vol. 116, No. 2 (Sep., 1982), pp. 331-360

[26] E.M. Stein; G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces.* (PMS-32). Princeton University Press, 1971. http://www.jstor.org/stable/j.ctt1bpm9w6.