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ON THE WICK PRODUCT AND WONG-ZAKAI APPROXIMATIONS.

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Dedication

To my girlfriend Simona, my parents Ana, Alberto and my brother Agustin.

Declaration

I hereby declare that I have developed and written the following PhD Thesis completely by myself. Wherever contributions of others are involved, every effort has been made to indicate this clearly through references to the Bibliography and acknowledgments.

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Ramiro Scorolli

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Abstract

This thesis is a compilation of 6 papers that the author has written together with Alberto Lanconelli[1] (chapters 3, 5 and 8) and Hyun-Jung Kim[2] (ch 7). The logic thread that link all these chapters together is the interest to analyze and approximate the solutions of certain stochastic differential equations using the so called *Wick product* as the basic tool.

In the first chapter we present arguably the most important achievement of this thesis; namely the generalization to multiple dimensions of a Wick-Wong-Zakai approximation theorem proposed by [1]. By exploiting the relationship between the *Wick product* and the *Malliavin derivative* we propose an original *reduction method* which allows us to approximate semi-linear systems of stochastic differential equations of the Itô type by solving an associated deterministic system of partial differential equations. Furthermore in chapter 4 we present a non-trivial extension of the aforementioned results to the case in which the system of stochastic differential equations are driven by a multi-dimensional fraction Brownian motion with Hurst parameter bigger than 1/2.

In chapter 5 we employ our approach and present a "short time" approximation for the solution of the *Zakai equation* from non-linear filtering theory and provide an estimation of the speed of convergence.

In chapters 6 and 7 we study some properties of the unique *mild* solution for the Stochastic Heat Equation driven by spatial white noise of the Wick-Skorohod type. In particular by means of our *reduction method* we obtain an alternative derivation of the Feynman-Kac representation for the solution, we find its optimal Hölder regularity in time and space and present a Feynman-Kac-type closed form for its spatial derivative.

Chapter 8 treats a somewhat different topic; in particular we investigate some probabilistic aspects of the unique global strong solution of a two dimensional system of semi-linear stochastic differential equations describing a predator-prey model perturbed by Gaussian noise. We obtain non-trivial estimations of upper and lower bounds for the solutions and distribution functions in a "short-time" context. It's interesting to notice that our findings are consistent with the asymptotic results present in the literature.

Key words and phrases: Wick product, Wong-Zakai approximations, stochastic differential equations, White noise, Malliavin derivative.

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Chapter 1

Introduction

The *Wick product* was originally introduced by the Italian physicist Gian Carlo Wick in the context of Quantum Fields Theory as a way to deal with certain infinite quantities. In its original formulation, the Wick product (which back then was known as *S-product*) was more of a *renormalization method* rather than an actual *product* in the usual sense of the word.

Later on Hida and Ikeda proposed to use this same technique in the context of stochastic calculus. Since then a number of researchers have actively studied properties and applications of this theoretical tool.

The *Wick product* , which is denoted by the symbol $\diamond$ could be seen as an actual product between random variables[1] satisfying the usual properties, namely *commutativity*, *associativity* and *distributivity*.

Surprisingly it is closely related to the Itô (and Skorohod) integration, and in short words one could say that *the Wick product is to Itô integration, the same as standard product is to Riemann integration.*

Motivated by this fact we decided to apply this tool to the study of the Wong-Zakai approximation method for the solutions of stochastic differential equations of the Itô type, both ordinary and partial.

The fundamental idea behind this approach is that of constructing a smooth approximation of the Brownian motion driving the equation (in the Itô sense), plug the latter into the equation and try to solve it; finally one is interested in studying the behavior of this *approximated solution* when the let the approximation of the Brownian motion become more *rough*. The fundamental result of the Wong-Zakai theory is that, due to the wild oscillations of the Brownian motion, the *approximated solution* won't converge to the solution of the original equation we were considering.

[1] As we shall see later on, we are actually able to compute the Wick product between *stochastic distributions*, i.e. *generalized* random variables.

As we shall see, this discrepancy between the starting point and the final result is a direct consequence of the use of the standard pointwise product which is not the *natural* one when dealing the integrals of the Itô type.

This thesis is devoted to partially answer a question that arise 25 years ago, namely: *what would happen if we incorporate the Wick product into the Wong-Zakai approximation? Will this scheme converge to the solution of original Itô equation?*

As we shall see this question is far from trivial and trying to provide an answer requires the use of several techniques.

Chapter 2

Preliminaries

In this chapter we will introduce some of the important mathematical concepts and tools that will be used in the following chapters. The first section deals with the basic concepts from classical stochastic calculus such as Brownian motion, Wiener space, Wiener integral, Itô integrals and a brief explanation about the Feynman-Kac representation. The second section deals with an infinite dimensional analysis i.e. Gaussian analysis and the Gaussian Hilbert space formalism which allows to treat concepts such as the Wiener chaos and stochastic integration under great generality. Furthermore this theoretical framework allows for a painless transition into another infinite dimensional calculus as the Malliavin calculus and the White noise distribution theory that we will treat latter on-

The idea is to show how this 3 different ways of looking at stochastic calculus are related.

For the sake of brevity many results are enunciated without a proof although the more important/enlightening results are treated in detail.

2.1 Brownian motion and SDE

We start by introducing arguably the most important stochastic process out there and one of the fundamental objects in this thesis, namely the Brownian motion. The following results can be found virtually in every stochastic processes book but we will attach to the following great references [2] and [3].

2.1.1 The Brownian motion

Definition 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a generic probability space. A d-dimensional stochastic process indexed by $\mathcal{T} \subset \mathbb{R}$ is a measurable map

$$X : \Omega \to (\mathbb{R}^d)^{\mathcal{T}}$$

given by

$$\Omega \ni \omega \mapsto (\mathcal{T} \ni t \mapsto X(t,\omega) \in \mathbb{R}^d).$$

Unless we state it otherwise we will consider the case in which $\mathcal{T} = [0,T]$ for some positive real constant T.

Definition 2. A d-dimensional Brownian motion $\{B(t)\}_{t\in[0,T]}$ is a stochastic process indexed by $[0,T]$ taking values in $\mathbb{R}^d$ such that

1. $B(0) = 0$ almost surely,
2. For any partition $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ for any $n \in \mathbb{N}$ it holds that $B(t_n) - B(t_{n-1}), \cdots, B(t_1) - B(t_0)$ are independent random vectors.
3. $B(t) - B(s)$ and $B(t+h) - B(s+h)$ are identically distributed for all $0 \leq s < t,\ h \geq -s$.
4. $B(t) - B(s) \sim \mathcal{N}(0,(t-s))^{\otimes d}$ where $\mathcal{N}(0,t)(dx) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx$
5. $t \mapsto B(t)$ is a.s. continuous.

Choose a finite partition $\pi := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of the interval $[0,T]$, and set $\|\pi\| := \max_{i\in\{0,1,\dots,N\}} |t_i - t_{i-1}|$. The real number $\|\pi\|$ is called *mesh* of the partition π. From now on we will assume without loss of generality that the partition is equally spaced, i.e. $t_i = \frac{iT}{N}$, for all $i \in \{0,...,N\}$; in this case we simply have $\|\pi\| := \frac{T}{N}$ but we will continue to use the notation $\pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ and $\|\pi\|$.

Proposition 3. *Let $\{B(t)\}_{t\in[0,T]}$ be a one-dimensional Brownian motion and let π be a partition of the interval $[0,T]$ as the one described above. Then*

$$\lim_{\|\pi\|\to 0} \sum_{j=1}^{N} |B(t_j) - B(t_{j-1})|^2 = T \quad \textit{convergence in } L^2(\Omega).$$

Proof. To ease the notation we set

$$S_2^N[0,T] := \sum_{j=1}^{N} |B(t_j) - B(t_{j-1})|^2.$$

Notice that

$$\begin{aligned}\mathbb{E}\left[S_2^N[0,T]\right] &= \mathbb{E}\left[\sum_{j=1}^{N}|B(t_j)-B(t_{j-1})|^2\right]\\ &= \sum_{j=1}^{N}\mathbb{E}\left[|B(t_j)-B(t_{j-1})|^2\right]\\ &= \sum_{j=1}^{N} t_j - t_{j-1} = T.\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left|S_2^N[0,T]-T\right|^2\right] = \mathbb{V}\left[S_2^N[0,T]\right],$$

where for any random variable X, $\mathbb{V}[X]$ denotes its variance.

Now notice that the independence of the Brownian increments allows us to write

$$\begin{aligned}\mathbb{V}\left[S_2^N[0,T]\right] &= \sum_{j=1}^{N}\mathbb{V}\left[|B(t_j)-B(t_{j-1})|^2\right]\\ &= 3\sum_{j=1}^{N}|t_j-t_{j-1}|^2\\ &\leq 3\max_{0\leq i\leq N}|t_i-t_{i-1}|\sum_{j=1}^{N}|t_j-t_{j-1}|\\ &= 3T\max_{0\leq i\leq N}|t_i-t_{i-1}|.\end{aligned}$$

Clearly the last term goes to zero as $N\to\infty$ proving the desired result. □

Proposition 4. *The trajectories of the Brownian motion are of unbounded variation a.s.*

Proof. We start by writing

$$\sum_{j=1}^{N}[B(t_j)-B(t_{j-1})]^2 \leq \max_{i\in\{0,1,\dots,N\}}|B(t_i)-B(t_{i-1})|\sum_{j=1}^{N}|B(t_j)-B(t_{j-1})|,$$

and notice that from the previous proposition we know that the lefter side converges in the $L^2(\Omega)$ sense (and thus almost surely for a subsequence) to T. On the other hand since the paths of the Brownian motion are a.s. uniformly continuous in $[0,T]$ we have that $\lim_{\|\pi\|\to 0}\max_{i\in\{0,1,\dots,N\}}|B(t_i)-B(t_{i-1})| = 0$. This implies that the second term on the right must diverge which proves the desired result. □

Definition 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a one-dimensional Brownian motion $\{B(t)\}_{t\in[0,T]}$. The *natural filtration* $\{\mathcal{F}_t^B\}_{t\in[0,T]} \subset \mathcal{F}$ is a family of sigma algebras defined by:

$$\mathcal{F}_t^B := \sigma(B(s); 0 \le s \le t).$$

Without loss of generality we may enrich $\mathcal{F}_0^B$ with all the $\mathbb{P}$-zero measure sets. In such a case the filtration would be called the *augmented* filtration.

2.1.2 The Wiener space

In this section we will discuss the canonical construction of the Brownian motion. For the sake of simplicity we will consider the case for $d = 1$ since the general case could be obtain under trivial modifications.

We start by introducing the following space[1]

$$\mathcal{C}_0 := \mathcal{C}_0[0,T] := \left\{f : [0,T] \to \mathbb{R} : \; f \text{ is continuous and } f(0) = 0\right\}$$

Proposition 6. *If we equip $\mathcal{C}_0$ with the metric of locally uniform convergence*

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 \wedge \sup_{0\le t\le n} |f(t) - g(t)|\right),$$

then $\mathcal{C}_0$ becomes a complete separable metric space.

Lets denote with $\mathcal{O}_\rho$ the topology induced by the metric ρ and consider the Borel sigma-algebra $\mathcal{B}(\mathcal{C}_0) := \sigma(\mathcal{O}_\rho)$ on $\mathcal{C}_0$.

Definition 7. A *cylindrical subset* A of $\mathcal{C}_0$ is a set of the form

$$A := \{f \in \mathcal{C}_0 : (f(t_1), f(t_2), ..., f(t_n)) \in U\} \tag{2.1}$$

where $0 < t_1 < \cdots < t_n < T$ and $U \in \mathcal{B}(\mathbb{R}^{nd})$ for any $n \in \mathbb{N}$.

From now on we will denote with $\mathcal{A}$ the collection of all such cylindrical subsets of $\mathcal{C}_0$.

Proposition 8. *We have the following equivalence*

$$\mathcal{B}(\mathcal{C}_0) = \sigma(\mathcal{A})$$

Proof. Under the metric ρ the *coordinate projection* map

$$\mathcal{C}_0 \ni f \mapsto \mathfrak{p}_t(f) := f(t)$$

[1] The construction of the Wiener space can be done in a much more general fashion, in particular we can costruct it over any metric space $\mathcal{M}$ not only on bounded intervals.

is continuous for every t and hence measurable with respect to the Borel sigma algebra $\mathcal{B}(\mathcal{C}_0)$. Same thing can be extended to the multi-dimensional projections and hence we conclude that since the mapping $\mathcal{C}_0 \ni f \mapsto (f(t_1), f(t_2), ..., f(t_n))$ is continuous then it's Borel measurable. This in turn implies that

$$\mathcal{A} \subset \mathcal{B}(\mathcal{C}_0) \implies \mathcal{B}(\mathcal{A}) \subset \mathcal{B}(\mathcal{C}_0).$$

We now need to show that $\mathcal{B}(\mathcal{A}) \supset \mathcal{B}(\mathcal{C}_o)$ and in order to do so it is enough to notice that the totality of the sets of the form $\{f : \sup_{0\leq t\leq n} |f(t) - f_0(t)| \leq \epsilon\}, f_0 \in \mathcal{C}_0, \epsilon > 0, n \in \mathbb{N}$ forms a basis of neighborhoods in $\mathcal{C}_0$ and

$$\left\{f : \sup_{0\leq t\leq n} |f(t) - f_0(t)| \leq \epsilon\right\} = \bigcap_{\mathbb{Q}\cap[0,n]} \{f : f(r) \in B_\epsilon(f_0(r))\}.$$

Thus such a set is representable as a countable intersection of cylindrical sets. It follows that $\mathcal{B}(\mathcal{C}_0) \subset \mathcal{B}(\mathcal{A})$. □

Now suppose that $A \in \mathcal{A}$ and define $\mathbb{P}^W(A)$ by

$$\mathbb{P}^W(A) := \int_A \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left[-\frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \right] \right) du_1 \cdots du_n.$$

Since the collection $\mathcal{A}$ of cylindrical subsets in an generator of $\mathcal{B}(\mathcal{C}_0)$ which is *stable* under countable intersections it follows from [4, theorem 5.7] that $\mathbb{P}^W$ has a unique extension to the sigma algebra $\mathcal{B}(\mathcal{C}_0)$.

The triplet $(\mathcal{C}_0, \mathcal{B}(\mathcal{C}_0), \mathbb{P}^W)$ is a probability space known as the (classical) *Wiener space*

Theorem 9. *The stochastic process* $B(t, \omega) = \omega(t)$, $\omega \in \mathcal{C}_0$, $t \geq 0$ *is a one-dimensional Brownian motion.*

We can treat each element of the Wiener space as one particular trajectory of a Brownian motion. Using this same approach we construct Brownian motion indexed by generic intervals $\mathcal{T} \subset \mathbb{R}$.

2.1.3 The Wiener integral

Let T be a positive real constant; for notational convenience, we will identify (unless stated otherwise) the triple $(\Omega, \mathcal{F}, \mathbb{P})$ with the Wiener space $(\mathcal{C}_0, \mathcal{B}(\mathcal{C}_0), \mathbb{P}^W)$ and $\{B(t)\}_{t\in[0,T]}$ will denote the one-dimensional Brownian motion introduced in theorem 9.

The aim of this section is that of introducing the concept of *integration with respect to a Brownian motion*, namely we will be concerned with objects like

$$\int_0^T f(t)dB(t)$$

where the *integrand* f is a square integrable deterministic function. This kind of integral was considered for the first time in [5]. Here we will present a simpler construction based on approximation by simple functions following [6].

The fact that the paths of a Brownian motion have almost-surely non-finite variation precludes us to define $\int_0^T f(t)dB(t)$ in the Riemann-Stieltjes sense (e.g. [7]) for a general continuous function f[2] . Nonetheless we can use an alternative approach in order to obtain a well-defined notion of integration againts a Brownian motion as follows:

<u>**Step 1:**</u> Assume that f is a *step function* given by

$$f := \sum_{i=1}^{N} c_i \ \chi_{[t_{i-1},t-i)} \tag{2.2}$$

where the c_i's are real numbers and $\{0 = t_0 < \cdots < t_N = T\}$ is a finite partition of the interval $[0, T]$. In this particular case we define the Wiener integral and denote it with $I(\bullet)$ as

$$I(f) \equiv \int_0^T f(t)dB(t) := \sum_{i=1}^{N} c_i[B(t_i) - B(t_{i-1})].$$

Notice the analogy between this construction and the usual definition of the Lebesgue integral (e.g. [4])

Proposition 10. *$I(\bullet)$ is a linear operator and for any f given by (2.2) $I(f)$ is a $\mathcal{N}\left(0, \sum_{i=1}^{N} c_i^2(t_i - t_{i-1})\right)$ random variable.*

Proof. To show that when dealing with simple functions I is indeed a linear operator is trivial from the definition.

On the other hand in this particular case it's straightforward to see that for any $i \in \{0, ..., N\}$

$$c_i(B(t_i) - B(t_{i-1})) \sim \mathcal{N}(0, c_i^2(t_i - t_{i-1})).$$

Furthermore since the intervals of the partition are disjoint we have by basic properties of the Brownian motion that

$$(B(t_i) - B(t_{i-1})) \text{ is independent of } (B(t_j) - B(t_{j-1})), \quad \text{for any } i \neq j.$$

[2] In fact the family of functions for which we would be allowed to do so is rather restrictive

Having this in mind and using the fact that linear combinations of jointly Gaussian random variables are again Gaussian random variables the conclusion follows. □

Corollary 11. *It follows from the last proposition that for any step function f we have*

$$\mathbb{E}\left[|I(f)|^2\right] = \int_0^T f(t)^2 dt,$$

where $\mathbb{E}$ denotes the expectation on $(\Omega, \mathcal{F}, \mathbb{P})$. This implies that the Wiener integral is an isometry between the subspace of step functions in $L^2([0,T])$ and $L^2(\Omega)$.

Step 2: From [4, Corollary 12.11] we have that for any function $f \in L^2([0,T])$ there exits at least one sequence of step functions $\{f_n\}_{n\in\mathbb{N}}$ such that

$$f = \lim_{n\to\infty} f_n, \quad \text{convergence in } L^2([0,T]) \ .$$

Now notice that by linearity of $I(\bullet)$ we have that

$$\mathbb{E}\left[|I(f_n) - I(f_m)|^2\right] = \mathbb{E}\left[|I(f_n - f_m)|^2\right],$$

and since the linear combination of step functions is again an step function it follows from the previous proposition that

$$\mathbb{E}\left[|I(f_n) - I(f_m)|^2\right] = \int_a^b |f_n(t) - f_m(t)|^2 dt.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is convergent in $L^2([0,T])$ then it follows that the sequence $\{I(f_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$ and thus convergent in the latter.

Definition 12. Let $f \in L^2([0,T])$. Then let us define the Wiener integral $I(f)$ as

$$I(f) = \lim_{n\to\infty} I(f_n), \quad \text{convergence in } L^2(\Omega) \tag{2.3}$$

Theorem 13. *Let $f \in L^2([0,T])$ then*

$$I(f) \sim \mathcal{N}\left(0, \|f\|_{L^2([0,T])}\right)$$

Proof. By proposition 10 the result holds true if f is a step function. Then the conclusion follows from the definition above and the fact that $L^2(\Omega)$ converges implies convergence in distribution. □

An important property of the Wiener integral is that for any $f \in L^2([0,t])$ it holds that

$$\mathbb{E}\left[I(f)|\mathcal{F}_t^B\right] = \int_0^t f(s) dB(s).$$

Another way of putting this is that if we define $X_t := I(\chi_{[0,t]}f)$ then the process $\{X_t\}_{t\in[0,T]}$ is a $\{\mathcal{F}_t^B\}_{t\in[0,T]}$ martingale.

We will now introduce a family of random variables that will be extensively used in this thesis.

Definition 14. Let $f \in L^2([0,T])$, then the stochastic exponential of f, denoted by $\mathcal{E}(f)$ is defined by

$$\mathcal{E}(f) := \exp\left\{ \int_0^T f(t)dB(t) - \frac{1}{2}\|f\|^2_{L^2([0,T])} \right\}.$$

Furthermore the family $\mathscr{E} := \{\mathcal{E}(f) : f \in L^2([0,T])\}$[3] is dense in $L^p(\Omega)$ for any $p > 1$.

Notice that this family of random variables posses the following nice properties which will become handy latter on, namely:

1. For any $f \in L^2([0,T])$ we have that $\mathbb{E}[\mathcal{E}(f)] = 1$. This result follows trivially by using the moment generating function of a Gaussian random variable.
2. For any $f \in L^2([0,T])$ define $\mathcal{E}_t(f) := \exp\left\{ \int_0^t f(s)dB(s) - \frac{1}{2}\|\chi_{[0,t]}f\|^2_{L^2([0,T])} \right\}$. Then the stochastic process $\{\mathcal{E}_t(f)\}_{t\in[0,T]}$ is a $\{\mathcal{F}_t^B\}_{t\in[0,T]}$ martingale.

2.1.4 The Itô integral

In this section we will extend the concept of "integral with respect to a Brownian motion" to random integrands (satisfying certain conditions that we will describe in the following). This construction was proposed for the first time by Itô in [8]. The strategy we will use could be summarized as follows: we start with step stochastic processes for which we are able to **define** the stochastic integral in the usual fashion. Then we will employ an approximation result which roughly states that for any *well behaved* stochastic process there exits a sequence of step stochastic processes converging to the latter in some opportune topology. Then the integral is extended to the whole family of "well behaved stochastic processes" using the isometry property.

As usual we let $\{B(t)\}_{t\in[0,T]}$ be a Brownian motion and let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be an *admissible filtration*, i.e. a sequence of sigma algebras satisfying:

(i) $\mathcal{F}_s \subseteq \mathcal{F}_t$ for any $s \leq t$, $s,t \in [0,T]$.

(ii) $B(t)$ is $\mathcal{F}_t$-measurable for any $t \in [0,T]$.

(iii) $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for any $t,s \in [0,T]$.

[3] We will always use $\mathscr{E}$ to denote the family of stochastic exponential independently of the domain of the associated functions.

Definition 15. Let $L^2_{\text{ad}}([0,T] \times \Omega)$ denote the space of all stochastic processes satisfying:

1. $f(t, \omega)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$.
2. $\int_0^T \mathbb{E}\left[|f(t)|^2\right] dt < \infty$.

In this section we will explain how we can extend the Wiener integral to integrands which belong to $L^2_{\text{ad}}([0,T] \times \Omega)$. Again we will divide our exposition in steps.

Step 1: If f is a step stochastic process in $L^2_{\text{ad}}([0,T] \times \Omega)$, i.e. a process of the form

$$f(t, \omega) = \sum_{i=1}^{N} \xi_{i-1}(\omega) \chi_{[t_{i-1}, t-i)}(t)$$

where ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$ measurable and $\mathbb{E}\left[\xi^2_{i-1}\right] < \infty$ for every $i \in \{1, ..., N\}$. In analogy with the construction presented in section 2.1.3 we define

$$\int_0^T f(t, \omega) dB(t, \omega) := \sum_{i=1}^{N} \xi_{i-1}(\omega) \cdot [B(t_i) - B(t_{i-1})](\omega). \tag{2.4}$$

Notice that a consequence of this definition is that

$$\mathbb{E}\left[\left(\int_0^T f(t) dB(t)\right)^2\right] = \mathbb{E}\left[\int_0^T |f(t)|^2 dt\right] = \sum_{i=1}^{N} \xi^2_{i-1} \|\pi\|.$$

Step 2: Now in order to extend the concept of Itô integral to more general stochastic processes we need an approximation result which roughly states the stochastic analogous of [4, Corollary 12.11] .

Lemma 16. *Suppose $f \in L^2_{\text{ad}}([0,T] \times \Omega)$. Then there exists a sequence of step stochastic processes $\{f_n\}_{n \in \mathbb{N}} \subseteq L^2_{\text{ad}}([0,T] \times \Omega)$ such that*

$$\lim_{n \to \infty} \int_0^T \mathbb{E}\left[|f(t) - f_n(t)|^2\right] dt = 0.$$

Definition 17. Let $f \in L^2_{\text{ad}}([0,T] \times \Omega)$ and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of step stochastic processes converging to f in $L^2_{\text{ad}}([0,T] \times \Omega)$. Then we defined the Itô integral of f as

$$\int_0^T f(t) dB(t) = \lim_{n \to \infty} \int_0^T f_n(t) dB(t), \quad \text{convergence in } L^2(\Omega).$$

A consequence of the definition above is the so called Itô isometry, which basically states that for any $f \in L^2_{\text{ad}}([0,T] \times \Omega)$ we have

$$\mathbb{E}\left[\left(\int_0^T f(t)dB(t)\right)^2\right] = \mathbb{E}\left[\int_0^T |f(t)|^2 dt\right],$$

i.e. the Itô integral is an isometry between the space of square integrable random variables $L^2(\Omega)$ and $L^2_{\text{ad}}([0,T] \times \Omega)$.

Remark 18. There exists various generalizations of the concept of Itô integral. In particular one could consider adapted integrands which belong to some space larger than $L^2_{\text{ad}}([0,T] \times \Omega)$ in which case the resulting integral present weaker properties. One could also consider stochastic integrals of the Itô type in which the *integrator* is not a Brownian motion, but rather a more general semi-martingale with even discontinuous trajectories (e.g. [6],[2],[9]).

Theorem 19. *Let $\{u(t)\}_{t\in[0,T]}$ be a stochastic process in $L^2_{ad}([0,T] \times \Omega)$ such that $\mathbb{E}[u(t)u(s)]$ is a continuous function in $(t,s) \in [0,T]^2$. Furthermore let $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ a uniform partition of the interval $[0,T]$ (this assumption is just for the sake of simplicity). Then it holds that*

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} u(t_i)[B(t_{i+1}) - B(t_i)] = \int_0^T u(t)dB(t), \quad \textit{convergence in } L^2([0,T]) \ . \tag{2.5}$$

Remark 20. It's important to remark the fact that unlike the deterministic Riemann sum when defining the Itô integral we must always use the left evaluation point. If instead we take the mid-point evaluation $\frac{t_i+t_{i+1}}{2}$ the limit will equal the so called Fisk-Stratonovich integral $\int_0^T u(t) \circ dB(t)$. On the other hand by taking the right evaluation point t_{i+1} the limit will be given by the Hänggi-Klimontovich integral $\int_0^T u(t) \star dB(t)$. Finally if we consider a generic evaluation point in the interval $[t_i, t_{i+1}]$, namely we evaluate the integrand at $\alpha t_i + (1-\alpha)t_{i+1}$ for some $\alpha \in [0,1]$ the limit will coincide with the *alpha* integral $\int_0^T u(t)d^\alpha B(t)$ introduced by (Da Pelo, Lanconelli, Stan)

2.1.5 The Itô rule

Theorem 21. *Let $\theta : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a functions of class $C^{1,2}([0,T] \times \mathbb{R})$ and let $\{X_t\}_{t\in[0,T]}$ be an Itô process given by*

$$X(t) = X(0) + \int_0^t h(s)dB(s) + \int_0^t g(s)ds$$

then

$$\theta(t, X(t)) = \theta(0, X(0)) + \int_0^t \frac{\partial \theta}{\partial x}(s, X(s)) f(s) dB(s) + \int_0^t \left[\frac{\partial \theta}{\partial t}(s, X(s)) + \frac{\partial \theta}{\partial x}(s, X(s)) g(s) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(s, X(s)) \right] ds,$$

or which is the same

$$d\theta(t, X(t)) = \frac{\partial \theta}{\partial x}(t, X(t)) h(t) dB(t) + \left[\frac{\partial \theta}{\partial t}(t, X(t)) + \frac{\partial \theta}{\partial x}(t, X(t)) g(t) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(t, X(t)) \right] dt$$

2.1.6 Stochastic differential equations

Consider an ordinary differential equation of the type

$$\frac{d}{dt} X(t) = f(t, X(t)), \quad X(0) = x_0;$$

then under certain conditions on f we know that a solution can be found. Now imagine that this equation describes, for instance, some *idealized* physical process. Still we know that there are sources of *noise*, and then in order to obtain a more realistic representation of the process under consideration we *perturb* the equation with the formal time derivative of a Brownian Motion $\dot{B}(t)$[4] leading to

$$\frac{d}{dt} X(t) = b(t, X(t)) + \sigma(t, X(t)) \dot{B}(t), \quad X(0) = x_0.$$

In the Itô calculus the terms $\dot{B}(t)$ and dt are combined into $dB(t)$ yielding the stochastic differential equation

$$dX(t) = b(t, X(t)) dt + \sigma(t, X(t)) dB(t), \quad X(0) = x_0,$$

which is just a shorthand for the integral equation

$$X(t) = x_0 + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dB(s)$$

where the integral on the right should be understood in the Itô sense.

Definition 22. A jointly measurable stochastic process $\{X(t)\}_{t \in [0,T]}$ is called a solution of the above stochastic integral equation if:

- The stochastic process $\sigma(t, X(t))_{t \in [0,T]}$ belongs to ... .

[4] At this point this is merely an heuristic discussion, latter on we will show how we can do this in a rigorous fashion.

- Almost all sample paths of the stochastic process $b(t, X(t))_{t\in[0,T]}$ belong to $L^1([0,T])$.
- For each $t \in [0,T]$ the expression above holds almost surely.

We now present the most important existence-uniqueness theorem for solutions of stochastic differential equations.

We must bear in mind that here we are dealing with *strong* solutions. For more information about *weak* solutions and general properties of stochastic differential equations the reader is referred to ...

2.1.7 Feynman-Kac representations

Consider the following homogeneous heat equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x), \ (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = \varphi(x) \in C^2_b(\mathbb{R}). \end{cases} \tag{2.6}$$

It's well known that

$$u(t,x) = \int_{\mathbb{R}} \varphi(y) p(t, x-y) dy,$$

where $p(t,x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$ solves the Cauchy problem 2.6. An alternative probabilistic representation of the solution is given by

$$u(t,x) = \mathbb{E}[\varphi(B(t)+x)].$$

Let us now consider a more general Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\mathscr{L}(x) u(t,x) + c(x) u(t,x), \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d \\ u(0,x) = \varphi(x), \end{cases} \tag{2.7}$$

where

$$\mathscr{L}(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i},$$

and we assume that all the coefficients of $\mathscr{L}$ and c be bounded and Lipschitz continuous and the initial condition φ is bounded and continuous.

The characteristic form of $\sum_{i,j=1}^d a_{ij}(x)$ is assumed to be positive semi-definite, i.e. $\sum_{i,j=1}^d a_{ij}(x) v_i v_j \geq 0$ for any vector $v = (v_1, ..., v_d)$, $x \in \mathbb{R}$. We also assume that the matrix $(a_{ij}(x))_{ij}$ can be represented in the form $(a_{ij}(x))_{ij} = \sigma(x)\sigma(x)^T$ where σ is a matrix whose elements are Lipschitz continuous.

Let us now consider the stochastic differential equation

$$dX^x(t) = b(X^x(t))dt + \sigma(X^x(t))dB(t),\ X^x(0) = x \in \mathbb{R}^d, \tag{2.8}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\{B(t)\}_{t\geq 0}$ is a d-dimensional Brownian motion indexed by the interval $[0, \infty)$.

It follows from [10, Theorem 7.3.3] that $\mathscr{L}$ is the infinitesimal generator of the process $\{X^x(t)\}_{t\geq 0}$ i.e.

$$\mathscr{L} f(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}[f(X^x(t))] - f(x)}{t}, \quad x \in \mathbb{R}^d,$$

(where f is such that this limit actually exists).

Theorem 23. *[11, Theorem 1.1] Let $u(t, x)$ be a solution of the Cauchy problem 2.7 which, for every $t \geq 0$, is bounded and has first and second order bounded and uniformly continuous derivatives in x and moreover has first order derivative in t which is locally uniformly continuous in t and x. Then*

$$u(t, x) = \mathbb{E}\left[\varphi(X^x(t)) \exp\left\{ \int_0^t c(X^x(s))ds \right\}\right]. \tag{2.9}$$

Notice that the theorem assumes in advance the existence of a classical solution satisfying certain conditions. Nevertheless we can *a priori* consider the expression (2.9) as a *generalized solution* of the problem 2.7. Under certain conditions regarding the coefficients of equation 2.7 the Feynman-Kac representation is a classic solution (see for instance the discussion in [11, page 122]) This is to say that even though a classical solution may not exist one could still consider the Feynman-Kac representation as a sort of *solution* for the problem in some weaker sense.

Proof. Assume that $u(t, x)$ is the solution of 2.7.

Then define

$$Z_1(s) := e^{\int_0^s c(X^x(r))dr}, \qquad Z_2(s) := u(X(s)^x, t - s).$$

Applying Itô rule we obtain the following

$$dZ_1(s) = Z_1(s)c(X(s)^x)ds$$
$$dZ_2(s) = (-\partial_t u(X^x(s), t - s) + \mathscr{L}(x)u(X^x(s), t - s))ds + \sigma(X^x(s))\nabla u(X^x(s), t - s)dB(s),$$

where ∇u denotes the spatial gradient of u.

Then by the product rule we have

$$\begin{aligned} d(Z_1(s)Z_2(s)) &= Z_1(s)dZ_2(s) + Z_2(s)dZ_1(s) \\ &= Z_1(s) \underbrace{\left[-\partial_t u + \mathscr{L}(x)u + cu\right]}_{=0} ds + Z_1(s)\sigma\nabla u B(s) \end{aligned}$$

Integrating and taking expectations

$$\mathbb{E}\left[Z_1(t)Z_2(t)\right] = \mathbb{E}\left[Z_1(0)Z_2(0)\right] = u(t,x) = \mathbb{E}\left[\varphi(X^x(t))e^{\int_0^t c(X^x(s))ds}\right].$$

□

2.1.8 Wong-Zakai approximations

The Wong-Zakai approximation theorem is a key result in the theory of stochastic calculus and it appeared for the first time in [12] and [13]. Assume we are considering a SDE of the form

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t),\ X(0) = x,\ t \in [0, T],$$

where as usual $\{B(t)\}_{t\in[0,T]}$ denotes a standard Brownian motion.

Now let's consider a smooth (differentiable) approximation $\{B^\epsilon(t)\}$ of the Brownian motion which depends on a parameter ϵ and that converges to $\{B(t)\}_{t\in[0,T]}$ in some opportune topology as $\epsilon \to 0$ (we could for instance take the convolution of B with a mollifier, a piecewise approximation, or a truncated Karuhnen-Loève series).

What if we replace the Brownian motion in the SDE above with our smooth approximation? In that case we would be left to consider the following (random) ODE

$$\dot{X}^\epsilon(t) = b(t, X^\epsilon(t)) + \sigma(t, X^\epsilon(t))\dot{B}^\epsilon(t),\ X^\epsilon(0) = x,\ t \in [0, T].$$

An interesting question then arises; will X^ϵ converge (in some opportune sense) to X as $\epsilon \to 0$?

Surprisingly (or not so much as we will see in the following sections) Wong and Zakai give us a negative answer; X^ϵ won't converge to X but rather to a close relative of it. In fact X^ϵ will converge as we let the approximation of our Brownian motion become more rough to the solution of the following SDE

$$d\mathcal{X}(t) = b(t, \mathcal{X}(t))dt + \sigma(t, \mathcal{X}(t)) \circ dB(t),\ X(0) = x,\ t \in [0, T],$$

where $\circ$ denotes the fact that the stochastic integration must be understood in the Stratonovich sense (e.g. [14]) or equivalently the following SDE of the Itô type

$$d\mathcal{X}(t) = \left[b(t, \mathcal{X}(t)) + \frac{1}{2}\sigma'\sigma(\mathcal{X}(t))\right]dt + \sigma(t, \mathcal{X}(t))dB(t),\ X(0) = x,\ t \in [0, T],$$

For a proof of this result the reader is referred to the original paper [13] and to [15].

2.2 Gaussian analysis and elements of Malliavin Calculus

2.2.1 Gaussian Hilbert spaces

In the previous section we presented the concept of Brownian motion and the stochastic integrals of Wiener and Itô type using tools from the *classical* stochastic analysis. In that setting the paths of the Brownian motion where seen as elements in the Wiener space which acted as our canonical probability space. In this section we will allow for more general probability spaces, in fact, unless stated otherwise we won't assume anything regarding the structure of the underlying probability space. This will allow to treat concepts in great generality.

The main aim of this section is that of introducing the so called Wiener chaos decomposition and explain how it relates with the concepts trated so far.

Definition 24. A Gaussian linear space is a real linear space of random variables defined on an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that each variable in the space is centered Gaussian. A *Gaussian Hilbert space* is a Gaussian linear space which is complete.

Theorem 25. *If $\mathfrak{G} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian linear space, then its closure $\overline{\mathfrak{G}}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian Hilbert space.*

A nice property of this spaces (which can be shown by linearity) is that any set of random variables in a Gaussian Hilbert space has a joint normal distribution.

Example 26. Let T be a positive real constant and let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the Wiener space[5] on the interval $[0, T]$, furthermore let $\{B(t)\}_{t\in[0,T]}$ be the canonical one-dimensional Brownian motion. An example of a Gaussian Hilbert space that will be extensively used in this thesis is given by

$$\left\{ \int_0^T f(t)dB(t) : f \in L^2([0,T]) \right\},$$

where the integral above must be understood in the Wiener sense (see section 2.1.3).

Example 27. Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$ where γ is the one-dimensional standard Gaussian measure. Let $\iota : \Omega \to \mathbb{R}$ be given by $x \mapsto \iota(x) = x$, then it's evident that $\iota \sim \mathcal{N}(0, 1)$. The space defined by

$$\{\iota t : \; t \in \mathbb{R}\},$$

is a Gaussian Hilbert space.

[5] We could actually consider any probability space rich enough to carry a Brownian motion but for the sake of conciseness we shall work with the Wiener space

Notice that in two examples above each element of the Gaussian Hilbert space could be identified with an element of a real Hilbert space. This in fact can be formalized using the following definition.

Definition 28. A *Gaussian Hilbert space indexed by a (real) Hilbert space* H is a Gaussian Hilbert space $\mathfrak{G}$ together with a specific linear isometry $h \mapsto W(h)$ of H onto $\mathfrak{G}$ (notice that the underlying probability space is not explicitated, and in fact it's irrelevant as stated in [16, page 1])

In order to emphasize the indexing space we will write $\mathfrak{G}(H)$, then the spaces in the two examples above will be denotes by $\mathfrak{G}(L^2([0,T]))$ and $\mathfrak{G}(\mathbb{R})$ respectively.

2.2.2 Wiener Chaos

This section will be devoted to introduce an extremely important result in stochastic calculus, namely the Wiener chaos decomposition. We start by presenting an abstract version of the latter and later on we will introduce the more usual one based upon the multiple Wiener integrals.

For the easiness of exposition we will focus our attention on the simpler case illustrated by assumption 1 bellow, but the reader must bear in mind that this theory could be developed under a more general setting.[6]

Assumption 1. *Let* T *be some arbitrary positive real number and let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be the Wiener space over the interval* $[0,T]$ *where we define the canonical one-dimensional Brownian motion* $\{B(t)\}_{t\in[0,T]}$. *In order to ease the notation we will denote with* $\mathfrak{H}$ *the Gaussian Hilbert space we've introduced in example 26, i.e.*

$$\mathfrak{H} := \mathfrak{G}(L^2([0,T])) = \left\{ \int_0^T f(t)dB(t) : f \in L^2([0,T]) \right\}.$$

Definition 29. Let $\{\mathcal{F}_t^B\}_{t\in[0,T]}$ be the natural filtration of $\{B(t)\}_{t\in[0,T]}$. Then a random variable F defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Brownian functional* if F is $\mathcal{F}_T^B$-measurable.

Notice that under assumption 1 it holds that $\mathcal{F}_T^B = \mathcal{F} = \sigma(\mathfrak{H})$.

From now on we will denote the *space of square integrable Brownian functionals* with $\mathcal{L}^2$ (which in our current setting equals $L^2(\Omega, \mathcal{F}, \mathbb{P})$).

It is important to keep in mind that by definition any element $\xi \in \mathfrak{H}$ can be seen as the Wiener integral $I(f)$ for some $f \in L^2([0,T])$.

[6] As a matter of fact we could assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is some generic probability space, that H is some real Hilbert space and that $\mathfrak{G}(H)$ is some Gaussian Hilbert space indexed by H. Furthermore we could let W be an isometry of H onto $\mathfrak{G}(H)$. In that case instead of working with the space of square integrable Brownian functionals $\mathcal{L}^2$ we would consider $L^2(\Omega, \sigma(\mathfrak{G}(H)), \mathbb{P})$; see for instance section 2.3.3.

Definition 30. Let, for $n \in \mathbb{N}_0$, $\overline{\mathscr{P}}_n(\mathfrak{H})$ be the closure in $\mathcal{L}^2$ of the linear space

$$\mathscr{P}_n(\mathfrak{H}) := \left\{ p(\xi_1, ..., \xi_m) : p \text{ is a polynomial of degree} \leq n \text{ on } \xi_1, ..., \xi_m \in \mathfrak{H}; m < \infty \right\},$$

and let the *n-th homogeneous chaos* be given by

$$\mathfrak{H}^{:n:} := \overline{\mathscr{P}}_n(\mathfrak{H}) \cap \overline{\mathscr{P}}_{n-1}(\mathfrak{H})^{\perp},$$

where the symbol $\perp$ denotes the orthogonal complement. For $n = 0$ we let $\mathfrak{H}^{:0:}$ be the space of constants.

We are now ready to enunciate the Wiener chaos decomposition theorem in its more abstract form.

Theorem 31. *The spaces $\mathfrak{H}^{:n:}, n \in \mathbb{N}$ are mutually orthogonal, closed subspaces of $\mathcal{L}^2$ and furthermore we have that*

$$\mathcal{L}^2 = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{:n:}.$$

Proof. The mutual orthogonality is clear from the definition so the only thing we need to show is that if F is orthogonal to $\mathfrak{H}^{:n:}$ for all $n \in \mathbb{N}_0$ then $F = 0$ a.s. .

Let $\xi \in \mathfrak{H}$ then by the triangular inequality and the Taylor expansion of the exponential function we have

$$\left| e^{i\xi} - \sum_{k=0}^{n} \frac{(i\xi)^k}{k!} \right| \leq 1 + \sum_{k=0}^{n} \frac{|\xi|^k}{k!} \leq 1 + e^{|\xi|}.$$

Taking the $\mathcal{L}^2$ norm on both sides and using the dominated convergence theorem (DCT for brevity) yields that $\sum_{k=0}^{n} \frac{(i\xi)^k}{k!}$ converges to $e^{i\xi}$ in $\mathcal{L}^2$ as $n \to \infty$. Since $\xi^k \in \mathscr{P}_k(\mathfrak{H}) \subset \bigoplus_{n=0}^{\infty} \mathfrak{H}^{:n:}$ this shows that $e^{i\xi} \in \bigoplus_{n=0}^{\infty} \mathfrak{H}^{:n:}$ whenever $\xi \in \mathfrak{H}$. Consequently if F is orthogonal to $\bigoplus_{n=0}^{\infty} \mathfrak{H}^{:n:}$ we have that $\mathbb{E}\left[Fe^{-i\xi}\right] = 0$. Finally [16, lemma 2.7] implies the desired result. □

Corollary 32. *Let $F \in \mathcal{L}^2$ then from theorem 31 it follows that*

$$F = \sum_{n=0}^{\infty} F_n \quad \textit{convergence in } \mathcal{L}^2$$

where $F_n := \pi_n(F)$ and π_n stands for the orthogonal projection of $\mathcal{L}^2$ on $\mathfrak{H}^{:n:}$.

From definition 30 it's clear that the *homogeneous chaoses* are formed by orthogonal polynomials, we will formalize this intuition in the following theorem.

Theorem 33. *Let* $\{m_j\}_{j\in\mathbb{N}}$ *be an orthonormal basis of* $L^2([0,T])$*, then it's clear that the family of random variables* $\{I(m_j)\}_{j\in\mathbb{N}}$ *forms an orthonormal basis of the Gaussian Hilbert space* $\mathfrak{H}$.

Let $\mathcal{J} := (\mathbb{N}_0^{\mathbb{N}})_c$ *denote the space of "multi-indexes", i.e. the space of integer-valued sequences with finitely many non-zero components and define*

$$\mathcal{H}_\alpha := \left(\prod_{j=1}^{\infty} \alpha_j!\right)^{-1/2} \prod_{j=1}^{\infty} H_{\alpha_j}\left(I(m_j)\right), \quad \alpha \in \mathcal{J},$$

where for any $n \in \mathbb{N}_0$, H_n *stands for the* n*-th Hermite polynomial and is given by*

$$H_n(x) := (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Then the family of random variables $\{\mathcal{H}_\alpha,\ \alpha \in \mathcal{J}_n\}$ *where* $\mathcal{J}_n := \{\alpha \in \mathcal{J} : |\alpha| = n\}$ *and for any multi-index* α *we write* $|\alpha| := \sum_{j=1}^{\infty} \alpha_j$*, forms an orthonormal basis of* $\mathfrak{H}^{:n:}$.

Proof. We start by observing that for any $\alpha, \beta \in \mathcal{J}_n$ it holds that

$$\mathbb{E}\left[\mathcal{H}_\alpha \mathcal{H}_\beta\right] = \prod_{i=1}^{\infty} \mathbb{E}\left[H_{\alpha_i}(I_n(m_i))\ H_{\beta_i}(I_n(m_i))\right] = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{otherwise.} \end{cases}$$

Thus when n varies the families $\{\mathcal{H}_\alpha, \alpha \in \mathcal{J}_n\}$ are mutually orthogonal. Furthermore it's clear that the latter belongs to $\mathscr{P}_n(\mathfrak{H})$. It is then enough to show that every polynomial random variable in $\mathscr{P}_n$ can be approximated by polynomials in $\{I(m_j)\}_{j\in\mathbb{N}}$, which is clear because the latter is a basis of $\mathfrak{H}$.

□

Corollary 34. *A corollary of this lemma due to Cameron and Martin [17] is that for any* $F \in \mathcal{L}^2$ *we have that*

$$F = \sum_{\alpha\in\mathcal{J}} F_\alpha \mathcal{H}_\alpha, \qquad \textit{convergence in } \mathcal{L}^2,$$

where $F_\alpha = \mathbb{E}\left[F\mathcal{H}_\alpha\right]$ *and*

$$\|F\|_{\mathcal{L}^2} = \sum_{\alpha\in\mathcal{J}} F_\alpha^2.$$

In the next definition we will introduce a tool that will be extensively used in this thesis, namely the *Wick product*.

Definition 35. If $\xi_1, ..., \xi_n$ is a finite sequence of elements of $\mathfrak{H}$ their *Wick product* $:\xi_1 \cdots \xi_n:$ is given by

$$:\xi_1 \cdots \xi_n: = \pi_n(\xi_1 \times \cdots \times \xi_n),$$

sometimes $:\xi_1 \cdots \xi_n:$ is called the *renormalization* of $\xi_1 \times \cdots \times \xi_n$.

Remark 36. It's important to remark the fact that although from the definition above it could seem that the Wick product is an operation exclusively defined between elements of some Gaussian Hilbert space this is not actually the case. In fact in section 2.3.6 we will present a much more general definition of the Wick product.

Proposition 37. *[16, Theorem 3.20] If $\xi_1, ..., \xi_n$ and $\eta_1, ..., \eta_m$ are centered jointly Gaussian variables, such that $\mathbb{E}[\xi_i \eta_j] = 0$ for all i and j, then*

$$:\xi_1 \cdots \xi_n \eta_1 \cdots \eta_m: = :\xi_1 \cdots \xi_n::\eta_1 \cdots \eta_m:$$

A proof of this proposition can be obtained using the tools we will develop in section 2.3, see in particular the example 103.

Proposition 38. *Let ξ be a standard Gaussian random variable, then it holds that*

$$:\xi^n: = H_n(\xi),\ n \in \mathbb{N}_0 \tag{2.10}$$

where H_n denotes the n-th Hermite polynomial and we impose $H_0(x) \equiv 1$.

Proof. Since the Wick product $:\xi_1 \cdots \xi_n:$ coincides on any Gaussian Hilbert space containing $\xi_1, \dots, \xi_n$, it is enough to consider the Wick power $:\xi^n:$ on the one dimensional Gaussian Hilbert space $\Xi \subset \mathfrak{H}$ given by the span of some arbitrary $\xi \in \mathfrak{H}$.

In particular by theorem 33, $\Xi^{:n:}$ is one-dimensional and is spanned by $H_n(\xi)$. Therefore

$$:\xi^n: = \pi_n(\xi^n) = \lambda H_n(\xi)$$

a.s. for some $\lambda \in \mathbb{R}$.

To identify λ, notice that by standard properties of orthogonal projections

$$\mathbb{E}[:\xi^n: H_n(\xi)] = \mathbb{E}[\xi^n H_n(\xi)]$$

Now since $\xi^n \in \bigoplus_{j=0}^n \Xi^{:j:}$, we have the decomposition $\xi^n = \sum_{j=0}^n \mathbb{E}[\xi^n H_n(\xi)] H_j(\xi)$. Since the leading coefficient of H_n is equal to 1 and n-th order coefficients must match, we see that $\mathbb{E}[\xi^n H_n(\xi)] = 1$. Therefore

$$\lambda = \mathbb{E}[:\xi^n: H_n(\xi)] = \mathbb{E}[\xi^n H_n(\xi)] = 1.$$

□

Corollary 39. *Let $n \in \mathbb{N}_0$ be fixed, and let H_n denote the n-th Hermite polynomial. Then we have the following equivalence*

$$\mathfrak{H}^{:n:} \equiv \operatorname{span}\left\{H_n(\xi); \quad \xi \in \mathfrak{H}\right\} \tag{2.11}$$

Remark 40. The connection between Wick powers and Hermite polynomials was already present in the Physics' literature, specially when computing powers of the position operator in Quantum Field Theory (e.g. [18],[19] or [20])

If $\xi_1, ..., \xi_n$ and $\eta_1, ..., \eta_m$ are centered jointly normal variables, then

$$\mathbb{E}\left[: \xi_1 \cdots \xi_n : : \eta_1 \cdots \eta_m :\right] = \begin{cases} \sum_{\sigma \in \mathcal{P}_n} \prod_{i=1}^n \mathbb{E}\left[\xi_i \eta_{\sigma(i)}\right], & m = n, \\ 0, & m \neq n. \end{cases}$$

Following [21] we have that since the left hand side above is symmetric, it suffices to prove it for $\xi_1 = \eta_1, ..., \xi_n = \eta_n$. Furthermore using the polarization identity it suffices to consider the case in which $\xi_1 = \cdots = \xi_n = \xi$ where we assume as well that ξ is a standard Gaussian random variable. Then under this assumptions the formula above reads

$$\mathbb{E}\left[| : \xi^n : |^2\right] = \mathbb{E}\left[H_n(\xi)^2\right] = n!,$$

which is the well known normalization of Hermite polynomials.

Theorem 41. *If $\xi_1, ..., \xi_n \in \mathfrak{H}$ then the map*

$$\xi_1 \odot \cdots \odot \xi_n \mapsto : \xi_1 \cdots \xi_n :$$

defines a Hilbert space isometry of $\mathfrak{H}^{\odot n}$ onto $\mathfrak{H}^{:n:}$.

Proof. From the definition of symmetric tensor product in the Hilbert space context we know that

$$(\xi_1 \odot \cdots \odot \xi_n \, , \, \eta_1 \odot \cdots \odot \eta_n)_{\mathfrak{H}^{\odot n}} = \sum_{\sigma \in \mathcal{P}_n} \left(\xi_1, \eta_{\sigma(1)}\right)_{\mathfrak{H}} \cdots \left(\xi_n, \eta_{\sigma(n)}\right)_{\mathfrak{H}}.$$

This together with the fact that for any $\xi \in \mathfrak{H}$ it holds that $(\xi, 1)_{\mathfrak{H}} = \mathbb{E}[\xi] = 0$ implies the result stated above. □

The previous result states that from an abstract point of view the Wick product, understood as an operation between Gaussian random variables is the *same* as the symmetric tensor product if we consider those same random variables as elements of a (Gaussian) Hilbert space.

2.2.3 Multiple stochastic integration and Wiener Chaos.

In section 2.1.3 we've introduced the concept of Wiener integral in a way that resembles the construction of the Lebesgue integral, namely we've started from step functions and then extended the definition by a density argument. In this section instead, we will work under a more abstract framework . For the sake of simplicity all throughout this section we will work under assumption 1 (the reader is referred to [16] and [22] for a more general treatment).

From example 26 we have that the Wiener integral I is an isometry of $L^2([0,T])$ onto the Gaussian Hilbert space $\mathfrak{H}$; then the tensor power $I^{\odot n}$ is an isometry of $L^2([0,T])^{\odot n}$ onto $\mathfrak{H}^{\odot n}$ which as we stated in theorem 41 can be identified with $\mathfrak{H}^{:n:}$.

In accordance with assumption 1 we will consider only the case of functions defined on a bounded interval $[0,T]$ for some positive real constant T, but the results can nonetheless be extended to the case of a generic interval $\mathcal{T} \subset \mathbb{R}$ and more generally to the case of a metric space $\mathcal{M}$ (see [16, chapter 7.2])

Proposition 42. *Let T be some positive real constant and define the simplex*

$$\mathbb{T}^n_{[0,T]} := \{(t_1, ..., t_n) : 0 < t_1 < \cdots < t_n < T\}. \tag{2.12}$$

We have the following identifications

$$L^2([0,T])^{\odot n} \cong L^2_{\text{sym}}([0,T]^n) \cong L^2(\mathbb{T}^n_{[0,T]}),$$

where as usual the symbol $\cong$ denotes the fact that the spaces are isomorphism and $L^2_{\text{sym}}([0,T]^n)$ denotes the subspace of symmetric functions in $L^2([0,T])$ where for "symmetric function" we refer to any function $f \in L^2([0,T]^n)$ such that

$$f(t_1, ..., t_n) = f\left(t_{\sigma(1)}, ..., t_{\sigma(n)}\right), \quad (t_1, ..., t_n) \in \mathbb{R}^n$$

for any permutation $\sigma \in \mathcal{P}_n$ where $\mathcal{P}_n$ denotes the permutation group of $\{1, 2, ..., n\}$.

With all this in hand we are now ready to introduce the multiple Wiener integral.

Theorem 43. *For each $n \in \mathbb{N}_0$ there exists a map*

$$I_n : L^2_{\text{sym}}([0,T]^n) \to \mathfrak{H}^{:n:},$$

such that

$$I_n(f_1 \odot \cdots \odot f_n) = :I(f_1) \cdots I(f_n): \tag{2.13}$$

and

$$\mathbb{E}\left[|I_n(f_1 \odot \cdots \odot f_n)|^2\right] = n! \|f_1 \odot \cdots \odot f_n\|_{L^2([0,T]^n)}. \tag{2.14}$$

The idea is now to show that the map I_n defines a n-fold multiple Wiener integral.

Let $\mathcal{X}(t_1, ..., t_n) := \chi_{(a_1,b_1]} \odot \cdots \odot \chi_{(a_n,b_n]}(t_1, ..., t_n)$ where $0 \leq a_1 < b_1 \leq a_2 < \cdots < b_{n-1} \leq a_n < b_n \leq T$.

Then by the definition above, the independence of the increments of the Brownian motion over disjoint time intervals and proposition 37 we have that

$$I_n(\mathcal{X}) =: \prod_{i=1}^{n} [B(b_i) - B(a_i)] := \prod_{i=1}^{n} [B(b_i) - B(a_i)],$$

and

$$I_{n-1}(\mathcal{X}(\bullet, t_n)) = \prod_{i=1}^{n-1} [B(b_i) - B(a_i)] \chi_{(a_n,b_n]}(t_n).$$

The latter is clearly Itô-integrable and it's clear that

$$I_n(\mathcal{X}) = \int_0^T I_{n-1}(\mathcal{X}(\bullet, t_n)) dB(t_n);$$

where the integral above must be understood in the Itô sense. Proceeding by induction we obtain

$$I_n(\mathcal{X}) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} \mathcal{X}(t_1, ..., t_n) dB(t_1) \cdots dB(t_n),$$

or using a more compact notation

$$I_n(\mathcal{X}) = n! \int_{\mathbb{T}^n_{[0,T]}} \mathcal{X}(t_1, ..., t_n) dB^{\otimes n}(t_1, ..., t_n) = n! J_n(\mathcal{X}),$$

where J_n denotes the n-fold iterated Wiener integral. The result holds for any function in $L^2_{\text{sym}}([0,T]^n)$ since functions like $\mathcal{X}$ are dense in the latter.

This implies that if $f \in L^2_{\text{sym}}([0,T]^n)$

$$I_n(f) := n! \int_{\mathbb{T}^n_{[0,T]}} f(t_1, ..., t_n) dB^{\otimes n}(t_1, ..., t_n),$$

since the simplex $\mathbb{T}^n_{[0,T]}$ occupies a fraction of the box $[0,T]^n$ equal to $1/n!$ is clear from the symmetric nature of f that we can think of I_n as an n-fold multiple Wiener integral over $[0,T]^n$.

In order to show (2.14) we iteratively apply Itô's isometry to get

$$\begin{aligned}\mathbb{E}\left[J_n^2(f)\right] &= \mathbb{E}\left[\left(\int_{\mathbb{T}^n_{[0,T]}} f(t_1,...,t_n)dB^{\otimes n}(t_1,...,t_n)\right)^2\right] \\ &= \mathbb{E}\left[\int_0^T \left(\int_{\mathbb{T}^{n-1}_{[0,t_n]}} f(t_1,...,t_n)dB^{\otimes(n-1)}(t_1,...,t_{n-1})\right)^2 dt_n\right] \\ &\vdots \\ &= \int_{\mathbb{T}^n_{[0,T]}} f(t_1,...,t_n)^2 dt_1\cdots dt_n = \|f\|^2_{L^2\left(\mathbb{T}^n_{[0,T]}\right)}.\end{aligned}$$

Then it follows that

$$\mathbb{E}\left[|I_n(f)|^2\right] = (n!)^2\mathbb{E}\left[|J_n(f)|^2\right] = (n!)^2\|f\|^2_{L^2\left(\mathbb{T}^n_{[0,T]}\right)} = n!\|f\|^2_{L^2([0,T])},$$

where in the last equality we used the symmetry of f.

The n-fold Wiener integral can be extended to a non-symmetric function $h \in L^2([0,T]^n)$ by letting

$$I_n(h) := I_n(\operatorname{Sym} h), \tag{2.15}$$

where

$$\operatorname{Sym} h(t_1,...,t_n) := \frac{1}{n!}\sum_{\rho\in\mathcal{P}_n} h\left(t_{\rho(1)},...,t_{\rho(n)}\right).$$

is called the *symmetrization* of h.

Corollary 44. *From theorem 43 and proposition 38 it follows that if $f \in L^2_{\mathrm{sym}}([0,T]^n)$ then*

$$I_n(f) = H_n(I(f);\sigma) = \sigma^n H_n(I(f)/\sigma)$$

with

$$\sigma = \|f\|_{L^2([0,T]^n)}.$$

Now we are ready to present the connection between Wiener chaos expansion and multiple Wiener integrals.

Lets start by considering any square integrable Brownian functional $F \in \mathcal{L}^2$. Theorem 31 states that such a random variable can be written as an infinite series of the form $\sum_{n=0}^{\infty}\pi_n(F)$ where $\pi_n(F)$ denotes the orthogonal projection of the random variable F on the n-th homogeneous chaos. Furthermore from theorem 33 we know that the family $\{\mathcal{H}_\alpha : \alpha \in \mathcal{J}_n\}$ forms an orthonormal basis of $\mathfrak{H}^{:n:}$. Then we must have that

$$\mathfrak{H}^{:n:} \ni \pi_n(F) = \sum_{\alpha\in\mathcal{J}_n} \mathbb{E}\left[F\mathcal{H}_\alpha\right]\mathcal{H}_\alpha \tag{2.16}$$

In the following we will need the next result which shows the connection between multiple Wiener integrals and the generalized Hermite polynomials.

Lemma 45. *Let $\{m_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2([0,T])$ and let $\alpha \in \mathcal{J}_n$, $n \in \mathbb{N}$ then it holds that*

$$\mathcal{H}_\alpha = \left(\prod_{j=1}^{\infty} \alpha_j!\right)^{-1/2} I_n\left(\bigodot_{j=1}^{\infty} m_j^{\odot\alpha_j}\right). \tag{2.17}$$

Proof. Let us write $\boldsymbol{\alpha}! := \left(\prod_{j=1}^{\infty} \alpha_j!\right)$ then by definition (see theorem 33) we have that

$$\mathcal{H}_\alpha := (\boldsymbol{\alpha}!)^{-1/2} \prod_{j=0}^{\infty} H_{\alpha_j}(I(m_j)),$$

and since the m's have unitary norm we can use theorem 43 and proposition 38 to write the latter as

$$(\boldsymbol{\alpha}!)^{-1/2} \prod_{j=0}^{\infty} I_{\alpha_j}(m_j).$$

The orthogonality of the elements of $\{m_j\}_{j\in\mathbb{N}}$ together with Itô's isometry implies that $\mathbb{E}[I(m_k)I(m_j)] = 0$, $\forall j \neq k$ and then by proposition 37 we can see that the product above could as well be interpreted as a Wick product, i.e. we can write the last expression as

$$(\boldsymbol{\alpha}!)^{-1/2} : \prod_{j=0}^{\infty} I_{\alpha_j}(m_j) \ : .$$

Finally the conclusion follows from theorem 43. □

Plugging (2.17) into (2.16) we obtain

$$\pi_n(F) = I_n(f_n),$$

where

$$L^2_{\text{sym}}([0,T]^n) \ni f_n := \sum_{\alpha\in\mathcal{J}_n} \mathbb{E}\left[F\mathcal{H}_\alpha\right](\alpha!)^{-1/2} \bigodot_{j=1}^{\infty} m_j^{\odot\alpha_j}. \tag{2.18}$$

The discussion above allows us to state the most well known form of Wiener chaos decomposition in terms on multiple Wiener integrals.

Theorem 46. *Let $F \in \mathcal{L}^2$ then*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \textit{convergence in } \mathcal{L}^2,$$

where $f_n \in L^2_{\text{sym}}([0,T]^n),\ n \in \mathbb{N}$.

From this theorem we are able to see that each square integrable Brownian functional $F \in \mathcal{L}^2$ is in a one-to-one relationship with a sequence of `kernels`. This last consequence could be summarized as (see [16] and [23] for in-detailed explanations)

Theorem 47. *Let $\mathcal{L}^2$ denote the space of square integrable Brownian functionals then the Itô-Wiener-Segal isomorphism tells us that*

$$\mathcal{L}^2 \cong \Gamma(L^2([0,T]))$$

where the symmetric Fock space $\Gamma(L^2([0,T]))$ over $L^2([0,T])$ is defined by

$$\Gamma(L^2([0,T])) := \left\{ (f_n)_{n=0}^{\infty} : f_n \in L^2_{\text{sym}}([0,T]^n), \quad \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2_{\text{sym}}([0,T]^n)} < \infty \right\}.$$

Remark 48. One could adapt all the discussions we've presented in this chapter to the case in which the Gaussian Hilbert space is indexed by $L^2(M, \mathcal{M}, m)$ where $(M, \mathcal{M}, m)$ is some arbitrary measure space (see [16, chapter 7.2]).

An alternative (and less technical) derivation of Wiener chaos decomposition.

An alternative derivation of the Wiener chaos decomposition in terms of multiple Wiener integrals can be obtained by an application of the Itô representation theorem. This proof based upon the set of notes [24].

Let $F \in \mathcal{L}^2$, the Itô representation theorem tells us that there exist a $\{\mathcal{F}_t^B\}_{t\in[0,T]}$-adapted (where as usual $\{\mathcal{F}_t^B\}_{t\in[0,T]}$ denotes the natural filtration) process $\{\varphi_1(t)\}_{t\in[0,T]}$ such that

$$F = \mathbb{E}[F] + \int_0^T \varphi_1(t_1) dB(t_1) \tag{2.19}$$

with

$$\mathbb{E}\left[\int_0^T |\varphi_1(t_1)|^2 dt_1\right] \leq \|F\|^2_{\mathcal{L}^2}.$$

Now for any fixed $t_1 \in [0,T]$ we can apply the Itô representation theorem to the random variable $\varphi_1(t_1)$ and find a $\{\mathcal{F}_t^B\}_{t\in[0,T]}$ adapted process $\{\varphi_2(t_1,t)\}_{t\in[0,t_1]}$ such that

$$\varphi_1(t_1) = \mathbb{E}[\varphi_1(t_1)] + \int_0^{t_1} \varphi_2(t_1,t_2)dB(t_2), \tag{2.20}$$

and

$$\mathbb{E}\left[\int_0^T |\varphi_2(t_1,t_2)|^2 dt_2\right] \leq \|\varphi_1(t_1)\|^2_{\mathcal{L}^2}.$$

Plugging (2.20) into (2.19) we obtain

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[\varphi_1(t_1)]dB(t_1) + \int_0^T \int_0^{t_1} \varphi_2(t_1,t_2,\omega)dB(t_2)dB(t_1). \tag{2.21}$$

We can continue iteratively with this procedure and we would arrive to the following equality

$$\begin{aligned} F = \mathbb{E}[F] &+ \sum_{n=1}^{N} \int_{\mathbb{T}^n_{[0,T]}} \mathbb{E}[\varphi_n(t_1,...,t_n)]dB^{\otimes n}(t_1,...,t_n) \\ &+ \int_{\mathbb{T}^{N+1}_{[0,T]}} \varphi_{N+1}(t_1,...,t_{N+1})dB^{\otimes(N+1)}(t_1,...,t_{N+1}). \end{aligned}$$

We can ease the notation by letting $g_0 := \mathbb{E}[F]$, $\;g_n(t_1,...,t_n) := \mathbb{E}[\varphi_n(t_1,...,t_n)], n \in \{1,...,N\}$ and using J_n to denote the n-fold **iterated** Wiener integral. By doing so we obtain the following expression

$$F = \sum_{n=0}^{N} J_n(g_n) + \int_{\mathbb{T}^{N+1}_{[0,T]}} \varphi_{N+1}(t_1,...,t_{N+1})dB^{\otimes(N+1)}(t_1,...,t_{N+1})$$

An iterative application of Itô's isometry shows that

$$\mathbb{E}\left[\left(\int_{\mathbb{T}^{N+1}_{[0,T]}} \varphi_{N+1}dB^{\otimes(N+1)}\right) J_n(f_n)\right] = 0, \text{ for any } f_n \in L^2(\mathbb{T}^n_{[0,T]}) \text{ with } n \leq N,$$

and hence

$$\|F\|^2_{\mathcal{L}^2} = \sum_{n=0}^{N} \|J_n(g_n)\|^2_{\mathcal{L}^2} + \left\|\int_{\mathbb{T}^{N+1}_{[0,T]}} \varphi_{N+1}dB^{\otimes(N+1)}\right\|^2_{\mathcal{L}^2}, \quad \text{for any interger } N.$$

This in turn implies that $\sum_{n=0}^{\infty} J_n(g_n)$ is convergent in $\mathcal{L}^2$ and thus

$$\Psi := \mathcal{L}^2 - \lim_{N\to\infty} \int_{\mathbb{T}^{N+1}_{[0,T]}} \varphi_{N+1} dB^{\otimes(N+1)},$$

must exits. By the argument above we must have that

$$\mathbb{E}[\Psi J_n(f_n)] = 0, \quad \text{for any } f_n \in L^2(\mathbb{T}^n_{[0,T]}),\ n \in \mathbb{N},$$

and thus it follows from and the density of Hermite polynomials that $\Psi = 0$. This implies that

$$F = \sum_{n=0}^{\infty} J_n(g_n), \quad \text{convergence in } \mathcal{L}^2,$$

but we would like to express the random variable as a series involving I_n not J_n. In order to do that we can extend the domain of g_n from the simplex to the whole box by imposing its value on $[0,T]^n \backslash \mathbb{T}^n_{[0,T]}$ to be simply 0; let us call this modified function $\widetilde{g_n}$. Then take the symmetrization of $\widetilde{g_n}$ and notice that then

$$I_n\left(\mathrm{Sym}\,\widetilde{g_n}\right) = J_n(g_n),$$

which finally allow us to write

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

and since $\|g_n\|_{L^2(\mathbb{T}^n_{[0,T]})} = n!\|f_n\|_{L^2([0,T]^n)}$ it follows that

$$\|F\|_{\mathcal{L}^2} = \sum_{n=0}^{\infty} n!\|f_n\|_{L^2([0,T]^n)}.$$

2.2.4 Elements of Malliavin Calculus

The Malliavin calculus (also known as the stochastic calculus of variations) is an infinite dimensional on the Wiener space. The fundations of this theory were set by Paul Malliavin in [25] in which he proposed a probabilistic proof of Hörmander's theorem. Later on this a very rich theoretical apparatus was build upon this original idea.

In the following we will introduce some of basic ideas regarding the Malliavin calculus, in particular we will be concerned with the computation of derivatives of random variables with respect to the chance parameter ω.

Once again for the sake of uniformity and easiness of exposition in this section we will work under the setting described in assumption 1 (we should bear in mind the fact that we could consider much more general settings, see for instance [22],[16],[26])and in correspondence with the notation used so far we will denote with $\mathcal{L}^p$ the space of p-integrable Brownian functionals for any $p \geq 1$.

The Malliavin derivative

In the following we will be interested in calculating the derivative of a square integrable random variable $F \in \mathcal{L}^2$ with respect to the chance parameter $\omega \in \Omega$.

Let $\mathbb{S}$ denote the class of smooth random variables F having the form

$$F = f(I(h_1), ..., I(h_n)), \quad n \in \mathbb{N},$$

where $h_1, ..., h_n \in L^2([0,T])$ and $f \in C_p^\infty(\mathbb{R}^n)$ which stands for the set of all infinitely continuously differentiable functions such that each function together with all its derivatives has a polynomial growth. We will refer to $\mathbb{S}$ as the family of *smooth Brownian functionals.*

Proposition 49. *The space $\mathbb{S}$ is dense in $\mathcal{L}^2$. This is easily seen by noticing that $\mathbb{S}_p \subset \mathbb{S}$ where $\mathbb{S}$ is the class of random variables of the form $F = f(I(h_1), ..., I(h_n))$ where f is a polynomial, and that $\mathbb{S}_p$ is dense in $\mathcal{L}^2$.*

Definition 50. The Malliavin derivative of a smooth random variable F is the $L^2([0,T])$-valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f(I(h_1), \dots, I(h_n))h_i,$$

where ∂_i denotes the partial derivative with respect to the i-th variable. In the same way, we can define the k-th derivative of F for any $k \in \mathbb{N}$, which will be a $L^2([0,T])^{\otimes k}$-valued random variable.

Definition 51. Let $\mathbb{S}$ be the space of smooth random variables, and define the following semi-norm on $\mathbb{S}$ for $k \in \mathbb{N}$ and $p \geq 1$,

$$\|F\|_{k,p} := \left[\mathbb{E}\left(|F|^p\right) + \sum_{j=1}^{k} \mathbb{E}\left(\|D^j F\|^p_{L^2([0,T])^{\otimes j}}\right)\right]^{1/p}.$$

We will denote by $\mathbb{D}^{k,p}$ the completion of the family $\mathbb{S}$ with respect to the norm $\| \bullet \|_{k,p}$ and for any $F \in \mathbb{D}^{k,p}$, we will let

$$D^k F = \lim_{n\to\infty} D^k F_n \quad \text{in } \mathcal{L}^p\left(L^2([0,T])^{\otimes k}\right),$$

where $(F_n)_{n\in\mathbb{N}} \subset \mathbb{S}$ is any sequence converging to F in $\mathcal{L}^p$. The spaces $\mathbb{D}^{k,p}$ are referred to as *Sobolev-Malliavin* spaces in analogy with the usual Sobolev spaces (e.g. [27]).

Proposition 52. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F_1, ..., F_m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$ for some $p \geq 1$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \partial_i \varphi(F) DF_i \tag{2.22}$$

Notice that the condition regarding the boundedness of the derivatives of φ is a somewhat restrictive condition and it is by no means optimal (see [28, Proposition 2.3.7]). In fact we have the following two well known examples

Example 53. Let $f \in L^2([0,T])$ then

$$DI(f)^n = mfI(f)^{(m-1)}, \quad m \in \mathbb{N}.$$

Example 54. Let $f \in L^2([0,T])$ then

$$D\mathcal{E}(f) = f\mathcal{E}(f),$$

and since $\mathcal{E}(f) \in \mathcal{L}^p$ for any $p \geq 1$ we conclude that the family of stochastic exponentials $\mathscr{E}$ is contained in $\mathbb{D}^{k,p}$ for any $k \in \mathbb{N}, p \geq 1$.

Proposition 55. *The space* $\mathbb{D}^{1,2}$ *is dense in* $\mathcal{L}^2$*.*

Proof. The proof follows from the fact that the family of stochatic exponentials $\mathscr{E}$ is contained in $\mathbb{D}^{1,2}$. □

Fix $h \in L^2([0,T])$ then we can define the *directional Malliavin derivatite of F in the direction of h* by

$$D_h F := (DF\ ,\ h)_{L^2([0,T])}$$

Definition 56. The space of *smooth random variables* (in the sense of Malliavin) will be given by

$$\mathbb{D}^{\infty,2} := \bigcap_{k\geq 1} \mathbb{D}^{k,2}.$$

Proposition 57.

Let $F = I_n(f_n)$ for some $f_n \in L^2_{\text{sym}}([0,T]^n)$. Then

$$D_t F = nI_{n-1}(f_n(\bullet;t)), \quad \textit{for any } t \in [0,T]. \tag{2.23}$$

Proof. We consider a simple symmetric function of the form

$$f_n(t_1,...,t_n) = \sum_{i_1,...,i_n=1}^{N} a_{i_1,...,i_n}\chi_{A_{i_1}}(t_1) \times \cdots \times \chi_{A_{i_n}}(t_n)$$

where $a_{i_1,\dots,i_n} = a_{i_{\rho(1)},\dots,i_{\rho(n)}}$ for any permutation $\rho \in \mathcal{P}_n$ and $a_{i_1,\dots,i_n} = 0$ if $i_k = i_l$ for some $i, l \in \{1, ..., n\}$, $k \neq l$ and the elements of $\{A_i\}_{i\in\{1,\dots,n\}}$ are disjoint subsets of $\mathcal{B}([0, T])$. In that case

$$I_n(f_n) = \sum_{i_1,\dots,i_n=1}^{N} a_{i_1,\dots,i_n} I(\chi_{A_{i_1}}) \times \cdots \times I(\chi_{A_{i_n}}),$$

and by definition 50 we obtain

$$D_t I_n(f_n) = \sum_{j=1}^{n} \sum_{i_1,\dots,i_n=1}^{N} a_{i_1,\dots,i_n} I(\chi_{A_{i_1}}) \times \cdots \chi_{A_{i_j}}(t) \times \cdots \times I(\chi_{A_{i_n}}) = nI_{n-1}(f_n(\bullet; t)).$$

The result follows easily by a limit argument. □

Proposition 58. *Let $F \in \mathcal{L}^2$ have the following Wiener chaos decomposition*

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

Then

$$F \in \mathbb{D}^{1,2} \iff \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L^2([0,T]^n)} < \infty,$$

and in this case

$$D_t F = \sum_{n=1}^{\infty} n I_n(f_n(\bullet; t)), \quad \text{convergence in } \mathcal{L}^2.$$

Lemma 59. *Let $F \in \mathbb{D}^{\infty,2}$ have the following chaos decomposition*

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

Then it holds that $f_n = \frac{1}{n!}\mathbb{E}[D^n F]$. This is sometimes referred to as the Strook-Taylor formula (see for instance [29]).

Example 60. From example 54 we know that for any $f \in L^2([0, T])$ the random variable $\mathcal{E}(f)$ belongs to the space $\mathbb{D}^{\infty,2}$. Furthermore we also showed that $\mathbb{E}[\mathcal{E}(f)] = 1$ for any $f \in L^2([0, T])$, thus from the lemma above we have the following Wiener chaos decomposition of the stochastic exponential

$$\mathcal{E}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes n}).$$

The divergence operator and Skorohod integral

The Malliavin derivative D introduced in the previous section is an unbounded operator from $\mathcal{L}^2$ into $\mathcal{L}^2(L^2([0,T]))$ (space of square integrable $L^2([0,T])$-valued Brownian functionals). Nonetheless it's densely defined (remember that the Malliavin-Sobolev space $\mathbb{D}^{1,2}$ is dense in $\mathcal{L}^2$) and that means that we can define its formal adjoint δ. The domain of this operator denote by $\mathrm{Dom}(\delta)$, is the set of random variables $v \in \mathcal{L}^2(L^2([0,T]))$ such that for any $F \in \mathbb{D}^{1,2}$,

$$\left|\mathbb{E}\left[(DF\ ,\ v)_{L^2([0,T])}\right]\right| \leq c\|F\|_{\mathcal{L}^2},$$

where c is a constant depending on v. If $v \in \mathrm{Dom}(\delta)$ then $\delta(v)$ is a random variable in $\mathcal{L}^2$ characterized by

$$\mathbb{E}\left[\delta(v)F\right] = \mathbb{E}\left[(DF\ ,\ v)_{L^2([0,T])}\right], \quad \text{for any } F \in \mathbb{D}^{1,2}. \tag{2.24}$$

Equation (2.24) a generalization of *integration by parts* formula

Proposition 61. *Let $F \in \mathbb{D}^{1,2}$ and let $u \in \mathrm{Dom}(\delta)$ such that $Fu \in \mathcal{L}^2(L^2([0,T]))$. Then $Fu \in \mathrm{Dom}(\delta)$ and*

$$\delta(Fu) = F\delta(u) - (DF, u)_{L^2([0,T])}\,. \tag{2.25}$$

Theorem 62. *The class of Itô-integrable stochastic processes is contained in $\mathrm{Dom}(\delta)$. Morover if $\{u(t)\}_{t\in[0,T]}$ is an Itô-integrable stochastic process, then*

$$\delta(u) = \int_0^T u(t)dB(t),$$

where the right-hand side denotes an Itô integral.

Proof. Let $\{u(t)\}_{t\in[0,T]}$ be a Itô integrable stochastic process such that $\mathbb{E}[u(t)u(s)]$ is a continuous function of $t \in [0,T]$ and $s \in [0,T]$[7]. Then lets consider the simple process

$$u^N(t) := \sum_{j=1}^{N} u(t_j)\chi_{[t_{j-1},t_j)}(t), \quad t \in [0,T], N \geq 1,$$

where $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ is an arbitrary partition of the interval $[0,T]$. It's clear that u^N converges to u in $\mathcal{L}^2(L^2([0,T]))$ and from the discussion present in section 2.1.4 we have that

$$\int_0^T u(t)dB(t) = \lim_{N\to\infty} \int_0^T u^N(t)dB(t), \quad \text{convergence in } \mathcal{L}^2.$$

[7] This assumption is not needed for the validity of the theorem but it eases the presentation of the proof

Assume for a moment that $u(t_j) \in \mathbb{S}$ for any $j \in \{0, 1, ..., N\}$ then we have that

$$\begin{aligned}
\delta\left(u^N\right) &= \delta\left(\sum_{j=0}^{N} u(t_j)\chi_{[t_j,t_{j+1})}\right) = \sum_{j=0}^{N} \delta(\chi_{[t_j,t_{j+1})}) \\
&= \sum_{j=0}^{N} u(t_j) I(\chi_{[t_j,t_{j+1})}) - \sum_{j=0}^{N} \left(Du(t_j)\ ,\ \chi_{[t_j,t_{j+1})}\right)_{L^2([0,T])} \\
&= \sum_{j=0}^{N} u(t_j)[B(t_{j+1}) - B(t_j)] \\
&= \int_0^T u^N(t) dB(t),
\end{aligned}$$

since $D_t u(t_j) = 0$ for $t > t_j$. This holds in general if we approximate $u(t_j), j = 0, 1, ..., N$ with a sequence of smooth random variables in $\mathbb{S}$ (such a sequence exists due to the density of $\mathbb{S}$ in $\mathcal{L}^2$) This shows that δ and the Itô integral coincides on the class of adapted simple processes. A limit argument shows that if $u \in \mathrm{Dom}(\delta)$ then $\delta(u) = \int_0^T u(t)dB(t)$. □

If the integrand is non-adapted then the operator δ coincides with an anticipating stochastic integral introduced by Skorohod in [30]. In that case we will write instead

$$\delta(u) = \int_0^T u(t)\delta B(t),$$

where the right-hand-side will be referred to as a Skorohod integral.

2.3 White noise distribution theory.

The basis for white noise calculus were presented by Hida in his work [31] and was subsequently developed by several authors (see for instance [32],[23],[33] and references therein). This theoretical framework allows for a rigorous construction of the *white noise functionals* (in analogy with the concept of *Brownian functionals* we've introduced in the previous sections), where the *white noise process* is identified with the derivative of a Brownian motion.

Although it's well known that the paths of Brownian motions are not smooth enough to posses an *actual* derivative we are able to compute its derivative in a distributional sense.

The mathematical framework is based upon an infinite dimensional analogue of the Schwartz distribution theory where the role of the Lebesgue measure on $\mathbb{R}$ is played by an infinite dimensional Gaussian measure [23].

So far we have always consider the case in which the indexing interval was given by $[0,T]$ for some positive real constant T. This is due to the fact that in general we interpret the interval $[0,T]$ as a *time* interval which for the great majority of applications can be assumed to be bounded.

Under risk of slight confusion in what follows we shall, instead, use the whole real line as the indexing space, intuitively this means we will have to do with the *derivative* of a Brownian motion indexed by $\mathbb{R}$ [8]. This is particularly useful when dealing with applications in which the indexing space represents (one of the dimensions of ...) the physical space as we will do in chapters 6 and 7. For a construction of the white noise distribution theory based upon bounded intervals the reader is referred to [34].

2.3.1 The Schwartz Space

Definition 63. We start by introducing the so called Schwartz space of test functions, which, together with its dual will be two key elements in the development of the white noise theory.

The *Schwartz space of rapidly decreasing functions* $\mathscr{S}(\mathbb{R})$ is the linear space of all functions $f:\mathbb{R}\to\mathbb{C}$ satisfying

$$\sup_{x\in\mathbb{R}}\left|x^a\frac{d^b f(x)}{dx^b}\right|<\infty$$

for all $a,b\in\mathbb{N}_0$ where $\frac{d^0}{dx^0}$ equals the identity operator.

The finiteness condition implies that $f\in\mathscr{S}(\mathbb{R})$ together with all its derivatives goes to zero *faster* than any polynomial as $|x|\to\infty$. From now on we will refer to any element of $\mathscr{S}(\mathbb{R})$ as a *Schwartz function*. Obviously we can extend the definition above to functions defined on $\mathbb{R}^d,\ d\in\mathbb{N}$ this is, let $\alpha,\beta\in(\mathbb{N}_0)^d$ be two *multi-indices* then the multi-dimensional Schwartz space is given by the functions $f:\mathbb{R}^d\to\mathbb{C}$ satisfying

$$\sup_{\mathbf{x}\in\mathbb{R}^d}|\mathbf{x}^\alpha\mathcal{D}^\beta f(\mathbf{x})|<\infty$$

where we use the following notation

$$|\gamma|=\gamma_1+\cdots+\gamma_d,\ \text{for all}\ \gamma=(\gamma_1,...,\gamma_d)\in(\mathbb{N}_0)^d$$

$$x^\alpha:=\prod_{i=1}^d x_i^{\alpha_i},\quad\text{and}\quad\mathcal{D}^\beta:=\frac{\partial^{|\beta|}}{\partial_{x_1}^{\beta_1}\cdots\partial_{x_d}^{\beta_d}}.$$

[8] More generally one could use $\mathbb{R}^d$ as the indexing space, but we attach to $d=1$ case for the sake of simplicity

Definition 64. If we write

$$[[\bullet]]_{a,b} := \sup_{x\in\mathbb{R}} \left| x^a \frac{d^b}{dx^b} \bullet \right|$$

then $\{[[\bullet]]_{a,b}, a, b \in \mathbb{N}_0\}$ constitutes a family of *semi-norms* in $\mathscr{S}(\mathbb{R})$.

Reconstruction of the Schwartz space.

Following [35] and [33] we present a construction of the *Schwartz space* which will be extremely useful in the following.

Lets consider the operator

$$A = -\frac{d^2}{dx^2} + x^2 + 1, \tag{2.26}$$

which coincides with the Hamiltonian of the *harmonic oscillator* (after some opportune normalization, e.g. [19]). This operator plays a very important role in the construction of the Schwartz space as we shall see.

Let $n \in \mathbb{N}$ and define

$$e_n(x) := \frac{1}{\sqrt{\sqrt{\pi}2^{n-1}(n-1)!}} H_{n-1}(\sqrt{2}x)e^{-x^2/2} \tag{2.27}$$

where $H_n(x) := (-1)^n e^{-x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$, $n \in \mathbb{N}_0$ is the n-th *Hermite polynomial.* The family $\{e_n\}_{n\in\mathbb{N}}$ are the so called Hermite functions[9] and will be used several times throughout this thesis.

In fact it is well known that the family $\{e_n\}_{n\in\mathbb{N}}$ are solutions of the time-independent Schrödinger equation (e.g. [36]); in particular they satisfy the following eigenvalue problem

$$Ae_n = 2ne_n, \quad n \in \mathbb{N},$$

and forms an orthonormal basis of the space $L^2(\mathbb{R})$.

An important property that follows easily from the definition (notice the presence of the Gaussian kernel) is that $e_n \in \mathscr{S}(\mathbb{R})$ for any $n \in \mathbb{N}$ and from here we can immediately see that $\mathscr{S}(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$.

Now for any $p \geq 0$ define

$$|f|_p = \|A^p f\|_{L^2(\mathbb{R})}.$$

[9] Notice that we use a different numbering of the Hermite functions in order to avoid having a 0 order term.

Equivalently $|f|_p$ is given by

$$|f|_p = \left(\sum_{n=1}^{\infty} (2n)^{2p} (f \ , \ e_n)^2_{L^2(\mathbb{R})} \right)^{1/2} .$$

Let

$$\mathscr{S}_p(\mathbb{R}) := \big\{ f \in L^2(\mathbb{R}) : |f|_p < \infty \big\}. \tag{2.28}$$

Notice that the space $\mathscr{S}_p(\mathbb{R})$ is a Hilbert space with norm $|\bullet|_p$, furthermore we have the following facts:

1. $\mathscr{S}(\mathbb{R}) = \bigcap_{p \geq 0} \mathscr{S}_p(\mathbb{R})$.
2. The families $\{[[\bullet]]_{a,b} \ ; \ a, b \in \mathbb{N}\}$ and $\{|\bullet|_p \ ; \ p \geq 0\}$ are equivalent in the sense that they generate the same topology on $\mathscr{S}(\mathbb{R})$.

2.3.2 Space of Tempered distributions

The dual space of $\mathscr{S}(\mathbb{R})$ is known as the space of *Tempered distributions* and it's denoted by $\mathscr{S}'(\mathbb{R})$. One of the most important examples of a tempered distribution is the so called Dirac's delta *function* δ_0.

Following the reconstruction of the Schwartz space we've presented in the last section we can conclude that formally

$$\mathscr{S}'(\mathbb{R}) = \bigcup_{p \geq 0} \mathscr{S}_{-p}(\mathbb{R}),$$

where $\mathscr{S}_{-p}(\mathbb{R})$ is the dual space of $\mathscr{S}_p(\mathbb{R})$ for $p \geq 0$.

Finally we arrive to the following chain of continuous inclusions maps

$$\mathscr{S}(\mathbb{R}) \subseteq \mathscr{S}_p(\mathbb{R}) \subseteq \mathbb{L}^2(\mathbb{R}) \subseteq \mathscr{S}_{-p}(\mathbb{R}) \subseteq \mathscr{S}'(\mathbb{R}).$$

Proposition 65. *The following properties will be used extensively in this thesis:*

1. *The Schwartz space $\mathscr{S}(\mathbb{R})$ is separable.*
2. *$\mathscr{S}(\mathbb{R})$ is dense in $\mathscr{S}'(\mathbb{R})$ in the weak topology on $\mathscr{S}'(\mathbb{R})$.*

It follows that if $\phi \in \mathscr{S}'(\mathbb{R})$ we have the following series expansion

$$\phi = \sum_{n=1}^{\infty} \langle \phi \ , \ e_n \rangle e_n, \qquad \text{weak convergence on } \mathscr{S}'(\mathbb{R}),$$

where $\langle \bullet, \bullet \rangle$ denotes the bilinear product between $\mathscr{S}'(\mathbb{R})$ and $\mathscr{S}(\mathbb{R})$ and $\{e_n\}_{n \in \mathbb{N}}$ denotes as always the family of Hermite functions. The latter will be referred to as the *Hermite expansion*.

2.3.3 The White noise probability space.

The aim of this section is that of introducing a particular probability space will play a role analogous to that of the Wiener space we've defined in section 2.1.2, i.e. that will constitute the basis for the canonical construction of the so called *white noise functionals.*

Definition 66. The *white noise probability space* will be given by the triplet $(\mathscr{S}'(\mathbb{R}), \mathscr{B}, \mu)$ where $\mathscr{B} := \mathcal{B}(\mathscr{S}'(\mathbb{R}))$ denotes the Borel sigma algebra on $\mathscr{S}'(\mathbb{R})$ which can be identified with the sigma algebra generated by the family $\mathcal{A}$ of cylindrical subsets

$$A := \left\{\omega \in \mathscr{S}'(\mathbb{R}) : (\langle\omega, \varphi_1\rangle, ..., \langle\omega, \varphi_n\rangle) \in U\right\}$$

where $\varphi_1, ..., \varphi_n \in \mathscr{S}(\mathbb{R})$ and $U \in \mathcal{B}(\mathbb{R}^n)$ for any $n \in \mathbb{N}$. The measure μ which will be called from now one the *hite noise measure* is an infinite dimensional analog of the Gaussian measure whose existence is guaranteed by the following theorem.

Theorem 67. *There exists a unique probability measure μ on $\mathscr{S}'(\mathbb{R})$ with the following property:*

$$\int_{\mathscr{S}'(\mathbb{R})} e^{i\langle\omega,\varphi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}}, \quad \forall\varphi \in \mathscr{S}(\mathbb{R}).$$

Remark 68. Interestingly if instead of $\mathscr{S}'(\mathbb{R})$ we consider any infinite dimensional Hilbert space the theorem above is no longer valid. In particular if H is an infinite dimensional Hilbert space with norm $\| \bullet \|_H$ there is no Borel measure μ in H such that

$$\widehat{\mu}(x) = e^{-\frac{1}{2}\|x\|^2_H},$$

where $\hat{\bullet}$ denotes the Fourier transform (see for instance [33] or [37]).

Corollary 69. *It's easy to see that for any $\varphi \in \mathscr{S}(\mathbb{R})$ the random variable $\omega \mapsto \langle\omega, \varphi\rangle$ is well defined for any $\omega \in \mathscr{S}'(\mathbb{R})$. From theorem 67 it follows that $\mu \circ \langle\bullet, \varphi\rangle^{-1}$ equals the centered Gaussian measure with variance equal to $\|\varphi\|^2_{L^2(\mathbb{R})}$ or which is equivalent*

$$\langle\bullet, \varphi\rangle \sim \mathcal{N}\left(0, \|\varphi\|^2_{L^2(\mathbb{R})}\right), \quad \forall\varphi \in \mathscr{S}(\mathbb{R}). \tag{2.29}$$

So far we've denoted with $\mathcal{L}^p$ the space of p-integrable Brownian functionals, i.e. the space of p-integrable random variables defined on the Wiener space $(\mathcal{C}_0, \mathcal{B}, \mathbb{P}^W)$. In analogy with this notation we will write

$$(L^p) := L^p(\mathscr{S}'(\mathbb{R}), \mathscr{B}, \mu), \quad p \geq 1. \tag{2.30}$$

in particular when $p = 2$, (L^2) denote the space of *square integrable white noise functionals.*

We can extend the dual paring to elements in $L^2(\mathbb{R})$ by a density argument, let $\{f_n\}_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathbb{R})$ be a sequence of Schwartz functions converging to f in $L^2(\mathbb{R})$ (the existence of such a sequence of functions is ensured by the density of $\mathscr{S}(\mathbb{R})$ in $L^2(\mathbb{R})$.) Then

$$\|\langle \bullet, f_n\rangle - \langle \bullet, f_m\rangle\|_{(L^2)} = \|\langle \bullet, f_n - f_m\rangle\|_{(L^2)} = \|f_n - f_m\|_{L^2(\mathbb{R})},$$

where in the last equality we used the fact that $\langle \bullet, f_n - f_m\rangle \sim \mathcal{N}\left(0, \|f_n - f_m\|^2_{L^2(\mathbb{R})}\right)$. Remembering that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is convergent, and thus Cauchy we can see that $\{\langle \bullet, f_n\rangle\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (L^2) and we can define

$$\langle \bullet, f\rangle := \lim_{n\to\infty} \langle \bullet, f_n\rangle \quad \text{convergence in } (L^2).$$

Proposition 70. *Let $\mathfrak{G}(\texttt{WN})$ be defined by*

$$\mathfrak{G}(\texttt{WN}) := \left\{ \langle \bullet, f\rangle, \quad f \in L^2(\mathbb{R}) \right\}.$$

The the latter is Gaussian Hilbert space (see [16, example 1.16]) and by definition $\sigma(\mathfrak{G}(\texttt{WN})) = \mathcal{B}(\mathscr{S}'(\mathbb{R}))$.

If $\langle \bullet, f\rangle$ is defined as a (L^2) limit then it must be the case that $\langle \bullet, f\rangle$ is $\mathscr{B}$ measurable, and thus

$$\sigma(\mathfrak{G}(\texttt{WN})) \subseteq \mathcal{B}(\mathscr{S}'(\mathbb{R})).$$

On the other hand since by definition $\langle \bullet, f\rangle$ is $\sigma(\mathfrak{G}(\texttt{WN}))$-measurable for any $f \in L^2(\mathbb{R})$ then it must be the case that $\langle \bullet, \varphi\rangle$ is $\sigma(\mathfrak{G}(\texttt{WN}))$-measurable for any $\varphi \in \mathscr{S}(\mathbb{R})$.

If we define

$$B(t,\omega) := \begin{cases} \langle \omega, \chi_{[0,t]}\rangle, & \text{if } t \geq 0; \\ -\langle \omega, \chi_{[t,0]}\rangle, & \text{if } t < 0, \end{cases} \tag{2.31}$$

for any $\omega \in \mathscr{S}'(\mathbb{R})$ it is easy to check that $\{B(t)\}_{t\in\mathbb{R}}$ is a Brownian motion.

Remark 71. We will now provide an heuristic explanation on why $(\mathscr{S}'(\mathbb{R}), \mathcal{B}, \mu)$ is called the *white noise* probability space.

If for $\omega \in \mathscr{S}'(\mathbb{R})$ we formally write [10]

$$\dot{B}(t,\omega) = \langle \omega, \delta_t\rangle = \text{“}\omega(t)\text{”}$$

[10]It is clear that the expression above is just symbolic, and it does not have a proper meaning since, in general, one cannot define the dual product between two tempered distributions. In the next section we will explain how to make sense of this expression by means of an approximation result in which the convergence must be understood in some weaker topology.

then $\dot{B}(t,\omega)$ could be regarded as the *distributional derivative* of the Brownian motion defined in (2.31).

In fact for any fixed $\omega \in \mathscr{S}'(\mathbb{R})$ and $\varphi \in \mathscr{S}(\mathbb{R})$ we have that by definition of the distributional derivative

$$\begin{aligned}\left\langle \frac{d}{dt}B(\omega),\varphi \right\rangle &= -\int_{\mathbb{R}} \varphi'(t)B(t,\omega)dt = \int_{\mathbb{R}} \varphi(t)dB(t) \\ &= \lim_{\Delta t_j \to 0} \sum_j \varphi(t_j)[B(t_{j+1}) - B(t_j)](\omega) = \lim_{\Delta t_j \to 0} \sum_j \varphi(t_j)\langle \omega, \chi_{(t_j,t_{j+1}]}\rangle \\ &= \lim_{\Delta t_j \to 0} \left\langle \omega\ ,\ \sum_j \varphi(t_j)\chi_{(t_j,t_{j+1}]} \right\rangle = \langle \omega, \varphi\rangle,\end{aligned}$$

where the limits must be understood in the (L^2) sense. Formally placing δ_t in the place of φ we obtain

$$\dot{B}(t,\omega) = \langle \omega, \delta_t\rangle = \left(\frac{d}{dt}B\right)(t,\omega).$$

By this point we are able to see the some similarities between the classical stochastic calculus, the Malliavin calculus and the White noise analysis.

In the classical stochastic calculus we've introduced in the first few sections the basic probability space was given by the Wiener space $(\mathcal{C}_0, \mathcal{B}, \mathbb{P}^W)$. In that setting the Brownian motion the *sample-paths* were taken to be functions in the space $\mathcal{C}_0(\mathbb{R})$[11] of continuous functions defined over $\mathbb{R}$ starting from the origin. Over the aforementioned space we defined random variables which we referred to as *Brownian functionals* and considered in particular the subset $\mathcal{L}^2$ of square integrable Brownian functionals.

In the white noise calculus setting, instead, the underlying probability space is given by the triplet $(\mathscr{S}'(\mathbb{R}), \mathscr{B}, \mu)$, where intuitively the space of tempered distributions $\mathscr{S}'(\mathbb{R})$ could be seen as the space of *sample-paths* of the distributional derivative of a Brownian motion.

Notice that although the Malliavin calculus was originally introduced as an analog of the calculus of variations on the Wiener space we may extend it as well to the case in which the underlying probability space is given by $(\mathscr{S}'(\mathbb{R}), \mathscr{B}, \mu)$.

So far we haven't really introduced anything *new*, in fact we have only considered random variables defined on a particular probability space. In the next section we will be introducing the theoretical framework that allow us to give a rigorous sense to *strange* objects such as the *derivative* of a Brownian motion among several other things.

[11] Here we changed the indexing interval to be consistent with the setting used in this section.

2.3.4 Stochastic distributions and test-functions.

In this section we will mimic the construction of the Schwartz space and the space of tempered distributions in such a way to obtain stochastic analogs of the latter two. In simple words the idea will be that of obtaining a pair of spaces, one of which will contain random variables that are very *regular* and the other will contain *ill-behaved* objects that cannot be regarded as random variables, but rather, will be referred to as *stochastic distributions.*

$$\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}),$$

For a more detailed treatment the reader is referred to [32].

Just as in section 2.3.1 we start by consider the operator A given in (2.26). The second thing we must do is to remember the Itô-Wiener-Segal isomorphism between a space of square integrable *Brownian functionals* and the symmetric Fock-Boson space that we described in ... .

In our current setting the latter could be summarized by

$$(L)^2 \cong \Gamma(L^2(\mathbb{R})) := \bigoplus_{n=0}^{\infty} L^2(\mathbb{R})^{\odot n};$$

this implies that every random variable $F \in (L^2)$ is uniquely determined by a sequence of symmetric square integrable `kernels` $f_n \in L^2_{\text{sym}}(\mathbb{R}^n),\ n \in \mathbb{N}$ via the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad \text{convergence in } (L^2),$$

where $I_n(\bullet)$ denotes the n-fold Wiener integral with respect to the Brownian motion given in (2.31).

Definition 72. Let $(L^2) \ni F = \sum_{n=0}^{\infty} I_n(f_n)$ and let A be the operator given by (2.26). Assume that

$$\sum_{n=0}^{\infty} n! \| A^{\otimes n} f_n \|^2_{L^2(\mathbb{R}^n)} < \infty,$$

we then define $\Gamma(A)F \in (L^2)$ to be

$$\Gamma(A)F := \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n).$$

The operator $\Gamma(A)$ is known as the *second quantization* of the operator A.

With this in mind we proceed as in section 2.3.1; first thing for $p \geq 0$ we define the norm

$$\|F\|_p := \|\Gamma(A^p)F\|_{(L^2)},$$

and we let

$$(S_p) := \{F \in (L^2) : \|F\|_p < \infty\},$$

i.e. the closure of (L^2) with respect to the norm $\| \bullet \|_p$; then (S_p), $p \geq 0$ is a Hilbert space with respect to the latter.

Definition 73. The *Hida test-function space* denoted by (S) is given by

$$(S) := \bigcap_{p \geq 0} (S_p).$$

Its dual will be referred to as the *Hida distribution space* and is formally given by

$$(S)^* := \bigcup_{p \leq 0} (S_p).$$

Remark 74. This definition implies that any element of $\varphi \in (S)$ has a Wiener chaos decomposition given by

$$\varphi = \sum_{n=0}^{\infty} I_n(\varphi_n), \quad \varphi_n \in \mathscr{S}_{\text{sym}}(\mathbb{R}^n),$$

where $\mathscr{S}_{\text{sym}}(\mathbb{R}^n)$ denotes the symmetric part of $\mathscr{S}(\mathbb{R}^n)$. In fact notice that if φ belongs to (S) then

$$\|\varphi\|_p = \sum_{n=0}^{\infty} n! \|(A^{\otimes n})^p \varphi_n\|_{L^2(\mathbb{R}^n)} = \sum_{n=0}^{\infty} n! |\varphi_n|_p < \infty, \quad \text{for any } p \geq 0.$$

From there it follows that all the kernels $\varphi_n, n \in \mathbb{N}_0$ belong to $\mathscr{S}_{\text{sym}}(\mathbb{R}^n)$.

We thus obtain the following triplet of spaces

$$(S) \subset (L^2) \subset (S)^*,$$

and the bilinear pairing between (S) and $(S)^*$ will be denoted by $\langle\langle \bullet, \bullet \rangle\rangle$.

Proposition 75. *Any element $\Phi \in (S)^*$ can be represented by a formal series as*

$$\Phi = \sum_{n=0}^{\infty} I_n(\Phi_n), \quad \Phi_n \in \mathscr{S}'_{\text{sym}}(\mathbb{R}^n),$$

where $\mathscr{S}'_{\text{sym}}(\mathbb{R}^n)$ denotes the symmetric part of $\mathscr{S}'(\mathbb{R}^n)$.

Proof. We must bear in mind that this representation is just formal, in particular notice that for $n = 1$ we can't define the Wiener integral of a generic tempered distribution, because that would be the same as taking the dual product between two tempered distributions which is not defined in general, but still we can make sense of it if we consider the duality product. Let $\Phi \in (S)^*$, by construction we know that there exits $p \in \mathbb{N}_0$ such that $\Phi \in (S)_{-p}$ therefore for every $\varphi \in (S)$ it holds that

$$|\langle\!\langle \Phi, \varphi \rangle\!\rangle| \leq \|\Phi\|_{-p}\|\varphi\|_p.$$

Consider for instance the case in which $\varphi := I_n(f_n),\ f_n \in \mathscr{S}_{\text{sym}}(\mathbb{R}^n)$. In this case we would have

$$|\langle\!\langle \Phi, \varphi \rangle\!\rangle| \leq \sqrt{n!}\|\Phi\|_{-p}|f_n|_p.$$

This implies that if we treat Φ as a functional defined on (S) then its restriction to $(S) \cap \mathfrak{G}^{:n:}(\mathtt{WN})$ is in one-to-one correspondence with an element $\Phi_n \in \mathscr{S}'_{\text{sym}}(\mathbb{R}^n)$ such that

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = n!\langle \Phi_n, f_n \rangle.$$

Since a general $\varphi \in (S)$ is a sum of such elements we can write

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\langle \Phi_n, \varphi_n \rangle,$$

and thus we identify Φ with the formal series $\sum_{n=0}^{\infty} I_n(\Phi_n)$. □

Example 76. The distributional derivative $\{\dot{B}(x)\}_{x\in\mathbb{R}}$ of the Brownian motion $\{B(x)\}_{x\in\mathbb{R}}$ given in (2.31) (also known as a *singular white noise process*, e.g. [38]) is an $(S)^*$-valued stochastic process.

In fact as we've anticipated one can formally write

$$\dot{B}(x;\omega) = \langle \omega, \delta_x \rangle, \quad \text{for any } x \in \mathbb{R}.$$

It follows that $\|\dot{B}(x)\|_{-p} = \|A^{-p}\delta_x\|_{L^2(\mathbb{R})}$ and using the series expansion

$$\delta_x(\bullet) = \sum_{n=1}^{\infty} e_n(\bullet)e_n(x),$$

we obtain

$$\|A^{-p}\delta_x\|^2_{L^2(\mathbb{R})} = \sum_{n=1}^{\infty} (2n)^{-2p} e_n(x)^2.$$

It's well known that $\sup_{t\in\mathbb{R}} |e_n(t)| = O(n^{-1/12})$. Hence for any $p > 5/12$ the series above converges which implies that for any $t \in [0, T]$, $\dot{B}(t) \in (S_{-p})$ for such p.

Remember the definition of *stochastic exponential* we gave in the section 2.1.3, in our current setting they are given by

$$\mathcal{E}(f) := \exp\left\{ I(f) - \frac{1}{2}\|f\|^2_{L^2(\mathbb{R})} \right\}, \quad f \in L^2(\mathbb{R}). \tag{2.32}$$

Just as in example ... by applying the Strook-Taylor formula (lemma 59) and noticing that[12]

$$D^k\mathcal{E}(f) = f^{\otimes k}\mathcal{E}(f) \quad \text{and} \quad \mathbb{E}[\mathcal{E}(f)] = 1,$$

we obtain the following Wiener chaos decomposition (c.f. ...)

$$\mathcal{E}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes n}). \tag{2.33}$$

Proposition 77. *For any $\eta \in \mathscr{S}(\mathbb{R})$ the stochastic exponential $\mathcal{E}(\eta)$ belongs to the space (S).*

The proof for this can be done by direct calculations,

$$\|\mathcal{E}(\eta)\|_p = \sum_{n=0}^{\infty} \frac{1}{n!}|\eta^{\otimes n}|_p^2 = \sum_{n=0}^{\infty} \frac{1}{n!}|\eta|_p^{2n} = e^{|\eta|_p^2} < \infty, \ \forall p \geq 0.$$

Now imagine that $\phi \in \mathscr{S}'(\mathbb{R})\backslash L^2(\mathbb{R})$, i.e. ϕ is a non-square integrable temepered distribution. Then by formally using the formula in equation (2.33) we can define its stochastic exponential of a tempered distribution, i.e.

$$\mathcal{E}(\phi) := \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\phi^{\otimes n}) \quad \phi \in \mathscr{S}'(\mathbb{R}^n),$$

where this series must be understood in the sense of proposition 75. Notice however that in this case we cannot in general write an analog for (2.32) since some of the components wouldn't be defined.

Still we can make the following interesting heuristic observation. We know that $I(\phi)$ is not a random variable but rather a stochastic distribution and for this reason the expression $\exp\{I(\phi)\}$ has no actual meaning (just as we cannot define the exponential of a Schwartz distribution). In analogy to (2.32) we can write the following informal expression

$$\mathcal{E}(\phi) \text{``}=\text{''} \exp\left\{ I(\phi) - \frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})} \right\} = \exp\{I(\phi)\} \times \exp\left\{ \frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})} \right\}$$

[12]Notice that here we are using a generalization of the concept of Malliavin derivative we've introduced in section 2.2.4 where the indexing space is $L^2(\mathbb{R})$.

At this point we notice that in general we have $\|\phi\|^2_{L^2(\mathbb{R})} = +\infty$ so in the expression above we are subtracting a divergent constant to $I(\phi)$ in the argument of the exponential (or alternatively multiplying "$\exp\{I(\phi)\}$" by "0")) and we obtain as a result a well defined object. This is why the stochastic exponential $\mathcal{E}(\phi)$ is called the *renormalization* of the exponential $\exp\{I(\phi)\}$.

2.3.5 S-transform and general Wick products

Definition 78. Let $\Phi \in (S)^*$ be given by the formal series

$$\Phi = \sum_{n=0}^{\infty} I_n(\Phi_n).$$

Then its S-transform is given by

$$\mathcal{S}(\Phi)(\eta) := \langle\!\langle \Phi, \mathcal{E}(\eta) \rangle\!\rangle = \sum_{n=0}^{\infty} \langle \Phi_n, \eta^{\otimes n} \rangle, \quad \eta \in \mathscr{S}(\mathbb{R}). \tag{2.34}$$

It's worth to notice that if we are dealing with a square integrable random variable $F \in (L^2)$, then we can extend the domain of $\mathcal{S}(F)$ to the whole $L^2(\mathbb{R})$ in which case we can write

$$\mathcal{S}(F)(h) = \langle\!\langle F, \mathcal{E}(h) \rangle\!\rangle \equiv \mathbb{E}\left[F\mathcal{E}(h)\right].$$

Corollary 79. *Let Φ be a Hida distribution in $(S)^*$. We know that since Φ is not an actual random variable but rather a stochastic distribution it does not posses a mean. Nonetheless we can compute a generalized mean which is given by*

$$\mathcal{S}(\Phi)(0) \tag{2.35}$$

Definition 80. A function $F : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ is called a U-functional if

1. For every $\phi, \varphi \in \mathscr{S}(\mathbb{R})$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda\phi + \varphi) \in \mathbb{C}$ has an entire extension to $z \in \mathbb{C}$.

2. There are constants $0 < \mathtt{K}_1, \mathtt{K}_2, p < \infty$ such that
$$|F(\varphi)| \leq \mathtt{K}_1 \exp\left(\mathtt{K}_2 |\varphi|_p^2\right), \quad \varphi \in \mathscr{S}(\mathbb{R}).$$

We are now ready to introduce a characterization result for the space $(S)^*$ due to Potthoff and Streit [39].

Theorem 81. *The S-transform defines a bijection between the space $(S)^*$ and the space of U-functionals.*

This result is extremely useful since it tells us that if two Hida distributions Φ and Ψ have the same S-transform then they should be equal almost surely, i.e.

$$\mathcal{S}(\Phi)(\eta) = \mathcal{S}(\Psi)(\eta),\ \forall \eta \in \mathscr{S}(\mathbb{R}) \iff \Phi = \Psi \ a.s.$$

2.3.6 The general Wick product

In the previous chapter we briefly introduced the concept of *Wick product* as an operation between elements of a Gaussian Hilbert space. This first definition is closer to the original idea that Wick introduced in [40] as a method for *renormalizing* infinite quantities in Quantum field theory.

In fact this is a particular case of a more general definition. In this section we shall introduce the concept of *Wick product* in a more general setting and we will explain the reason why this operation lies in the very core of the theory of Stochastic integration theory and Gaussian analysis.

It's important to remark the fact that even though in this section we will work on the White noise probability space the Wick product can be defined on much more general probability spaces (see for instance [16, Chapter 3]).

The *Wick product* was originally introduced by Wick in [40] (he originally called it *S-product*) as a way of renormalizing certain infinite quantities in the Quantum Field theory (see also [18]). Latter on this concept was brought to the theory of Stochastic analysis by the pioneer work of Hida and Ikeda (...). Subsequently several authors have used this powerful tool in many areas of stochastic calculus and applications (see [33][38][41] and references therein). The reason is that the Wick product lies in the very core of the theory of stochastic calculus and, as we will see in the following, is closely connected with the Itô-Skorohod integration.

Unfortunately many probabilists and mathematicians ignore the relevance of this tool and for this reason its use is not widely spread across the discipline.

Definition 82. Let $\Phi, \Psi \in (S)^*$ be two Hida distributions. We define the *Wick product* between the two elements, denoted by $\Phi \diamond \Psi$, to be the unique Hida distribution satisfying

$$\mathcal{S}(\Phi \diamond \Psi)(\eta) = \mathcal{S}(\Phi)(\eta) \cdot \mathcal{S}(\Psi)(\eta), \quad \text{for every } \eta \in \mathscr{S}(\mathbb{R}).$$

Remark 83. Notice the similarity between this definition and the well known property that relates the Fourier transform with the convolution operator, namely

$$\widehat{g \star h} = \widehat{g} \cdot \widehat{h}, \quad g, h \in \mathscr{S}'(\mathbb{R}).$$

Proposition 84. *Let $F_n \in \mathscr{S}'_{\mathrm{sym}}(\mathbb{R}^n)$ and $G_m \in \mathscr{S}'_{\mathrm{sym}}(\mathbb{R}^m)$ then it hols that*

$$I_n(F_n) \diamond I_m(G_m) = I_{n+m}(F_n \odot G_m)$$

Proof. The proof is an straightforward application of the S-transform,

$$\begin{aligned}\mathcal{S}(I_n(F_n)\diamond I_m(G_m))(\eta) &= \mathcal{S}(I_n(F_n))(\eta)\cdot\mathcal{S}(I_m(G_m))(\eta)\\ &= \langle\!\langle I_n(F_n),\mathcal{E}(\eta)\rangle\!\rangle\,\langle\!\langle I_m(G_m),\mathcal{E}(\eta)\rangle\!\rangle\\ &= \langle F_n,\eta^{\otimes n}\rangle\langle G_m,\eta^{\otimes m}\rangle\\ &= \langle F_n\odot G_m,\eta^{\otimes(n+m)}\rangle = \mathcal{S}(I_{n+m}(F_n\odot G_m))(\eta),\end{aligned}$$

the uniqueness of the S-transform implies the desired result. □

Remark 85. A consequence of the latter is that if $\Phi,\Psi\in(S)^*$ are given by the formal series

$$\Phi=\sum_{n=0}^{\infty}I_n(F_n),\qquad \Psi=\sum_{n=0}^{\infty}I_n(G_n),$$

then the series representation for the Wick product between $\Phi,\Psi\in(S)^*$ can be obtained by computing the Cauchy product between the series and using the proposition above, i.e.

$$\left(\sum_{n=0}^{\infty}I_n(F_n)\right)\diamond\left(\sum_{n=0}^{\infty}I_n(G_n)\right)=\sum_{p=0}^{\infty}\sum_{l=0}^{p}I_l(F_l)\diamond I_{p-l}(G_{p-l})=\sum_{p=0}^{\infty}I_p(K_p),$$

where

$$K_p:=\sum_{l=0}^{p}F_l\odot G_{p-l}.$$

Remark 86. The concept of *Wick product* we just introduced is strictly related to the one we gave in defintion 35 which was confined to a Gaussian setting. In particular if $(\xi_1,...,\xi_n)$ belong to some Gaussian Hilbert space $\mathfrak{G}$ then it holds that

$$\xi_1\diamond\cdots\diamond\xi_n=\,:\xi_1\cdots\xi_n:$$

The following basic algebraic properties of the Wick product could easily be verified using the definition;

Lemma 87.

1. *(Commutative law)* $\Phi,\Psi\in(S)^*\implies\Phi\diamond\Psi=\Psi\diamond\Phi.$
2. *(Associative law)*

$$\Phi,\Psi,\Xi\in(S)^*\implies\Phi\diamond(\Psi\diamond\Xi)=(\Phi\diamond\Psi)\diamond\Xi$$

3. (Distributive law)

$$\Phi, \Psi, \Xi \in (S)^* \implies \Phi \diamond (\Psi + \Xi) = \Phi \diamond \Psi + \Phi \diamond \Xi$$

The Wick powers are defined inductively as follows

$$\Phi \in (S)^*, \quad \begin{cases} \Phi^{\diamond 0} = 1 \\ \Phi^{\diamond k} = \Phi \diamond \Phi^{\diamond (k-1)} \quad \textit{for } k = 1, 2, \dots \end{cases}$$

Corollary 88. *The Hida distribution space $(S)^*$ is an algebra with respect to the Wick product $\diamond$.*

Definition 89. Let $\Phi \in (S)^*$ the *Wick exponential* is defined by

$$\exp^{\diamond}(\Phi) := \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\diamond n},$$

i.e. in the Taylor expansion of the standard exponential function we replace the regular powers with Wick-powers. In particular we have that

$$\mathcal{S}\left(\exp^{\diamond}(\Phi)\right)(\phi) = \exp\left(\mathcal{S}(\Phi)(\phi)\right).$$

Remark 90. Using the same reasoning one could construct *Wick versions* or *Wick composition* of entire functions. For instance let h be a entire function and let $\Phi \in (S)^*$. Then the *Wick composition* or *Wick version* $h^{\diamond}(\Phi)$ is defined by

$$\mathcal{S}(h^{\diamond}(\Phi))(\eta) = h(\mathcal{S}(\Phi)(\eta)), \quad \eta \in \mathscr{S}(\mathbb{R}).$$

Proposition 91. *Let $f \in L^2(\mathbb{R})$ then it holds that*

$$\mathcal{E}(f) = \exp^{\diamond}\left\{ \int_{\mathbb{R}} f(x) dB(x) \right\},$$

i.e. the stochastic exponential of f equals the Wick exponential of the Wiener integral $I(f)$.

Proof. From lemma 59 , remark 86 and the definition above we have that

$$\mathcal{E}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes n}) = \sum_{n=0}^{\infty} \frac{1}{n!} I(f)^{\diamond n} = \exp^{\diamond}\left\{ I(f) \right\}.$$

□

Proposition 92. *Let $\alpha \in \mathcal{J}$ be a multi-index and let $\mathcal{H}_\alpha$ denote the generalized Hermite polynomial, which in our current setting will be given by (c.f. ...)*

$$\mathcal{H}_\alpha := (\boldsymbol{\alpha}!)^{-1/2} \prod_{j=1}^{\infty} H_{\alpha_j}(I(e_j)).$$

Using ... we can rewrite the latter as

$$\mathcal{H}_\alpha = (\boldsymbol{\alpha}!)^{-1/2} \prod_{j=1}^{\infty} I(e_j)^{\diamond \alpha_j}.$$

Furthermore by ... and the observation above we can write this as

$$\mathcal{H}_\alpha = (\boldsymbol{\alpha}!)^{-1/2} \bigdiamond_{j=1}^{\infty} I(e_j)^{\diamond \alpha_j}$$

where $\bigdiamond$ denotes the productory in the Wick sense. By the associativity of the Wick product we can see that for $\alpha, \beta \in \mathcal{J}$ it holds that

$$\mathcal{H}_\alpha \diamond \mathcal{H}_\beta = (\boldsymbol{\alpha}!)^{-1/2} (\boldsymbol{\beta}!)^{-1/2} \left(\bigdiamond_{j=1}^{\infty} I(e_j)^{\diamond \alpha_j} \right) \diamond \left(\bigdiamond_{j=1}^{\infty} I(e_j)^{\diamond \beta_j} \right) = \sqrt{\frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!}{\boldsymbol{\alpha}!\boldsymbol{\beta}!}} \mathcal{H}_{\alpha+\beta}. \tag{2.36}$$

For example we have

$$\mathcal{H}_\alpha \diamond I(e_j) = \mathcal{H}_\alpha \diamond \mathcal{H}_{\epsilon_j} = \sqrt{\alpha_j + 1} \mathcal{H}_{\alpha+\epsilon_j} =: \sqrt{\alpha_j + 1} \mathcal{H}_{\alpha^+_{(j)}}$$

2.3.7 Wick multiplication and stochastic integration.

It's a well known fact that if $f : \mathbb{R} \to \mathbb{R}$ is a good enough function and $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function then

$$\int_a^b f(x) dg(x) = \int_a^b f(x) g'(x) dx$$

where the lefter integral is of Riemann-Stieltjes type and as usual the prime $'$ denotes the derivative. The aim of this section is that of showing an analog property for stochastic integrals of the Itô-Skorohod type. One issue that we may face is the fact that if we formally let g be Brownian motion then $g' = \dot{B}$ and we know that in general we are not able to compute the point-wise product between Hida distributions. This is precisely where the Wick product comes into play.

Remember the definition of the Skorohod integral we gave in section 2.2.4, and assume that $\{Y(t)\}_{t\in\mathbb{R}}$ is a Skorohod-integrable stochastic process then it holds that

$$\int_{\mathbb{R}} Y(t)\delta B(t) = \int_{\mathbb{R}} Y(t) \diamond \dot{B}(t)dt \tag{2.37}$$

where the left hand integral must be understood as Skorohod integal and right hand one as an $(S)^*$-valued Pettis (or Bochner) integral (see for instance [42] and [43] for the theory of vector valued integrals in a deterministic setting).

The integral on the right must be understood as a vector valued integral since we have that

$$\mathbb{R} \ni t \mapsto (\mathscr{S}'(\mathbb{R}) \ni \omega \mapsto \dot{B}(t;\omega))$$

i.e. $\mathbb{R} \ni t \mapsto \dot{B}(t)$ is an $(S)^*$-valued map.

This construction allows us to extend the definition of Skorohod integral to stochastic processes that do not belong to $\mathrm{Dom}(\delta)$ (sometimes this generalization is referred to as Hitsuda-Skorohod integral, see for instance [44]), in particular we are able to deal with $(S)^*$-valued processes that satisfy certain conditions.

We start this section giving an heuristic and somewhat informal justification for the equality above. For the sake of simplicity we will consider the case in which the underlying probability space is given by the Wiener space over of a finite interval $[0, T]$, that $\{B(t)\}_{t\in[0,T]}$ is the canonical Brownian motion defined on the latter and we assume that $\{Y(t)\}_{t\in[0,T]}$ is an Itô-integrable stochastic process adapted to the natural filtration $\{\mathcal{F}_t^B\}_{t\in[0,T]}$.

By definition of the S-transform we have that for any $h \in L^2([0, T])$

$$\mathcal{S}\left(\int_0^T Y(t)dB(t)\right)(h) = \mathbb{E}\left[\int_0^T Y(t)dB(t)\ \mathcal{E}(h)\right].$$

Remember that from ... we know that if we define $\mathcal{E}_t(h) := \mathcal{E}(\chi_{[0,t]}h)$ then the the stochastic process $\{\mathcal{E}_t(h)\}_{t\in[0,T]}$ is a $\{\mathcal{F}_t^B\}_{t\in[0,T]}$-martingale. An application of the Itô formula yields

$$\mathcal{E}(h) = \mathcal{E}_T(h) = \left(1 + \int_0^T h(t)\mathbb{E}[\mathcal{E}(h)|\mathcal{F}_t^B]dB(t)\right).$$

Plugging this in the expression above

$$\begin{aligned}\mathcal{S}\left(\int_0^T Y(t)dB(t)\right)(h) &= \mathbb{E}\left[\int_0^T Y(t)dB(t)\ \left(1 + \int_0^T h(t)\mathbb{E}[\mathcal{E}(h)|\mathcal{F}_t^B]dB(t)\right)\right] \\ &= \mathbb{E}\left[\int_0^T Y(t)dB(t)\ \left(\int_0^T h(t)\mathbb{E}[\mathcal{E}(h)|\mathcal{F}_t^B]dB(t)\right)\right],\end{aligned}$$

where in the last equality we've used the fact that by assumption Y is an Itô-integrable process which implies that the mean of its Itô integral equals 0. An application of Itô isometry, using the standard properties of the conditional expectation and Fubini-Tonelli lemma we obtain that

$$\mathcal{S}\left(\int_0^T Y(t)dB(t)\right)(h) = \int_0^T \mathbb{E}\left[Y(t)\mathcal{E}(h)\right]h(t)dt.$$

Finally noticing that formally $\mathcal{S}(\dot{B}(t))(h) := \langle\!\langle I(\delta_t), \mathcal{E}(h)\rangle\!\rangle = \langle \delta_t, h\rangle = h(t)$ the latter implies that

$$\mathcal{S}\left(\int_0^T Y(t)dB(t)\right)(h) = \int_0^T \mathcal{S}(Y(t))(h)\cdot\mathcal{S}(\dot{B}(t))(h)dt = \int_0^T \mathcal{S}(Y(t)\diamond\dot{B}(t))(h)dt.$$

Formally interchaning the S-transform and the time integral (we will latter see that indeed we are actually allowed to do so, e.g. equation (2.38)) we obtain

$$\mathcal{S}\left(\int_0^T Y(t)dB(t)\right)(h) = \mathcal{S}\left(\int_0^T Y(t)\diamond\dot{B}(t)dt\right)(h)$$

which by the uniqueness of the S-transform implies the desired result.

Definition 93. A function $\Phi : \mathbb{R} \to (S)^*$ (also called an $(S)^*$-valued process) is called $(S)^*$-integrable if

$$\langle\!\langle \Phi(t), \varphi\rangle\!\rangle \in L^1(\mathbb{R}) \quad \text{for all } \varphi \in (S).$$

Then the $(S)^*$-integral of Y denoted by $\int_{\mathbb{R}} \Phi(t)dt$, is the unique Hida distribution satisfying

$$\left\langle\!\!\left\langle \int_{\mathbb{R}} \Phi(t)dt\ ,\ \varphi \right\rangle\!\!\right\rangle = \int_{\mathbb{R}} \langle\!\langle \Phi(t), \varphi\rangle\!\rangle dt.$$

Technically speaking this integral is a $(S)^*$-valued Pettis[13] integral and in a sense allows us to extend the concept of Skorohod integral to distributional stochastic processes.

An immediate consequence of the definition above is that the $(S)^*$-integral commutes with the S-transform, i.e.

$$\mathcal{S}\left(\int_{\mathbb{R}} \Phi(t)dt\right)(\eta) = \int_{\mathbb{R}} \mathcal{S}\left(\Phi(t)\right)(\eta)dt, \quad \eta \in \mathscr{S}(\mathbb{R}). \tag{2.38}$$

In fact both expressions are equivalent due to the density of the stochastic exponentials in (S).

[13] In certain cases we are also able to define the integral in the Bochner sense, see for instance [33, Chapter 13]

Theorem 94. *Assume $\{\Phi(t)\}_{t\in\mathbb{R}}$ is a Skorohod-integrable stochastic process. Then $\Phi \diamond \dot{B}$ is $(S)^*$-integrable and we have*

$$\int_{\mathbb{R}} \Phi(t)\delta B(t) = \int_{\mathbb{R}} \Phi(t) \diamond \dot{B}(t)dt. \tag{2.39}$$

Proof. Assume that for any fixed $t \in [0,T]$ the random variable $\Phi(t)$ has the representation (e.g. Cameron Martin)

$$\Phi(t) = \sum_{\alpha\in\mathcal{J}} \Phi_\alpha(t)\mathcal{H}_\alpha,$$

and remembering that $\dot{B}(t) = I(\delta_t) = I\left(\sum_{k=1}^{\infty} e_k(t)e_k\right) = \sum_{k\in\mathbb{N}} e_k(t)\mathcal{H}_{\epsilon_k}$ where $\epsilon_k := (\underbrace{0,0,...,0}_{k-1},1,0,0,...)$ we have that

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(t) \diamond \dot{B}(t)dt &= \int_{\mathbb{R}} \sum_{\alpha\in\mathcal{J}} \Phi_\alpha(t)\mathcal{H}_\alpha \diamond \sum_{k\in\mathbb{N}} e_k(t)\mathcal{H}_{\epsilon^{(k)}}dt \\
&= \sum_{\alpha\in\mathcal{J},\, k\in\mathbb{N}} \left(\int_{\mathbb{R}} \Phi_\alpha(t)e_k(t)dt\right) \mathcal{H}_\alpha \diamond \mathcal{H}_{\epsilon^{(k)}} \\
&= \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} (\Phi_\alpha\ ,\ e_k)_{L^2(\mathbb{R})} \sqrt{\alpha_k+1}\mathcal{H}_{\alpha^+_{(k)}},
\end{aligned}$$

where $\alpha^+_{(k)} := \alpha + \epsilon_k$. For the left-hand side we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(t)\delta B(t) &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} I_n(\Phi_n(\bullet,t))\delta B(t) \\
&= \int_{\mathbb{R}} \sum_{n=0}^{\infty} I_n\left(\sum_{\alpha\in\mathcal{J}_n^K} \Phi_\alpha(\alpha!)^{-1/2} \bigodot_{i=1}^{\infty} e_i^{\odot\alpha_i}\right)\delta B(t) \\
&= \int_{\mathbb{R}} \sum_{n=0}^{\infty} I_n\left(\sum_{\alpha\in\mathcal{J}_n^K} \sum_{k=1}^{\infty} (\Phi_\alpha\ ,\ e_k)_{L^2(\mathbb{R})}(\alpha!)^{-1/2} \bigodot_{i=1}^{\infty} e_i^{\odot\alpha_i} \odot e_k\right)\delta B(t) \\
&= \sum_{n=0}^{\infty} \sum_{\alpha\in\mathcal{J}_n^K} \sum_{k=1}^{\infty} (\Phi_\alpha\ ,\ e_k)_{L^2(\mathbb{R})}(\alpha!)^{-1/2} I_{n+1}\left(\bigodot_{i=1}^{\infty} e_i^{\odot\alpha_i} \odot e_k\right) \\
&= \sum_{\alpha\in\mathcal{J},k\in\mathbb{N}} (\Phi_\alpha\ ,\ e_k)_{L^2(\mathbb{R})} \sqrt{\alpha_k+1}\mathcal{H}_{\alpha^+_{(k)}}.
\end{aligned}$$

which proves the result. An alternative proof makes use of the integration by parts formula,

$$\begin{aligned}\mathcal{S}(\delta(\Phi))(\eta) = \mathbb{E}\left[\delta(\Phi)\mathcal{E}(\eta)\right] &= \mathbb{E}\left[(\Phi, D\mathcal{E}(\eta))_{L^2(\mathbb{R})}\right] \\ &= \mathbb{E}\left[(\Phi, \eta)_{L^2(\mathbb{R})}\mathcal{E}(\eta)\right] \\ &= \int_{\mathbb{R}} \mathcal{S}(\Phi(t))(\eta)\eta(t)dt \\ &= \int_{\mathbb{R}} \mathcal{S}\left(\Phi(t) \diamond \dot{B}(t)\right)(\eta)dt \\ &= \mathcal{S}\left(\int_{\mathbb{R}} \Phi(t) \diamond \dot{B}(t)dt\right)(\eta),\end{aligned}$$

and the conclusion follows from the uniqueness of the S-transform □

Remark 95. An interesting property that follows from the latter is that the Wick product satisfies the usual rules of calculus, in fact we can see that for any $a, b \in \mathbb{R}$ with $a < b$ it holds that

$$\begin{aligned}\int_a^b B(t)^{\diamond n} dB(t) = \int_a^b B(t)^{\diamond n} \diamond \dot{B}(t)dt &= \int_a^b \frac{d}{dt}\left(\frac{B(t)^{\diamond(n+1)}}{n+1}\right)dt \\ &= \left(\frac{B(b)^{\diamond(n+1)} - B(a)^{\diamond(n+1)}}{n+1}\right), \quad n \in \mathbb{N}_0,\end{aligned}$$

where by convention $B(t)^{\diamond 1} \equiv B(t)$.

Lemma 96. *Assume that $\{Y(t)\}_{t\in[0,T]}$ is Skorohod integrable, that $X \in (S)^*$ does not depend on t and that $\{X \diamond Y(t)\}_{t\in[0,T]}$ is Hitsuda-Skorohod integrable. Then*

$$X \diamond \int_0^T Y(t)\delta B(t) = \int_0^T X \diamond Y(t)\delta B(t). \tag{2.40}$$

Corollary 97. *Using this result we can see theorem 43 under a new light. Let $f_1, ..., f_n \in L^2(\mathbb{R})$ then remark 86 and the lemma above we can formally write*

$$\begin{aligned}I_n(f_1 \odot \cdots \odot f_n) &= I(f_1) \diamond \cdots \diamond I(f_n) \\ &= \int_{\mathbb{R}} f_1(t_1)dB(t_n) \diamond \cdots \diamond \int_{\mathbb{R}} f_n(t_n)dB(t_n) \\ &= \int_{\mathbb{R}^n} f_1(t_1)\cdots f_n(t_n)dB(t_1)\cdots dB(t_n) \\ &= \int_{\mathbb{R}^n} f_1 \odot \cdots \odot f_n(t_1, ..., t_n)dB^{\otimes n}(t_1, ..., t_n)\end{aligned}$$

In the following we will discuss an interesting and useful connection between the Wick product and the so called *translation operator*.

Definition 98. Fix $\omega_0 \in \mathscr{S}'(\mathbb{R})$.

1. The map $\mathtt{T}_{\omega_0} : (S) \to (S)$ is called the *translation operator* and is given by

$$\mathtt{T}_{\omega_0} X(\omega) = X(\omega + \omega_0), \ \omega \in \mathscr{S}'(\mathbb{R}),$$

for any $X \in (S)$. Notice that if we assume instead that $\omega_0 \in L^2(\mathbb{R})$ then we can extend the domain of the translation operator to (L^p) for any $p > 1$.

2. The adjoint operator is the map

$$\mathtt{T}^*_{\omega_0} : (S)^* \to (S)^*$$

defined by

$$\langle\langle \mathtt{T}^*_{\omega_0} \Phi, \varphi \rangle\rangle = \langle\langle \Phi, \mathtt{T}_{\omega_0} \varphi \rangle\rangle.$$

Theorem 99. *Let $\omega_0 \in \mathscr{S}'(\mathbb{R})$ and $\Phi \in (S)^*$, then*

$$\mathtt{T}^*_{\omega_0} \Phi = \Phi \diamond \exp^\diamond \left(\langle \bullet, \omega_0 \rangle \right).$$

Proof. By direct computations we have that

$$\begin{aligned}
\mathcal{S}\left(\mathtt{T}^*_{\omega_0}\Phi\right)(\phi) &= \left\langle\!\!\left\langle \mathtt{T}^*_{\omega_0}\Phi, \exp^\diamond\left(\langle \bullet, \phi\rangle\right)\right\rangle\!\!\right\rangle \\
&= \langle\langle \Phi, \mathtt{T}_{\omega_0} \exp^\diamond\left(\langle \bullet, \phi\rangle\right)\rangle\rangle \\
&= \langle\langle \Phi, \exp^\diamond\left(\langle \bullet + \omega_0, \phi\rangle\right)\rangle\rangle \\
&= \langle\langle \Phi, \exp^\diamond\left(\langle \bullet, \phi\rangle\right) \diamond \exp^\diamond\left(\langle \omega_0, \phi\rangle\right)\rangle\rangle \\
&= \langle\langle \Phi, \exp^\diamond\left(\langle \bullet, \phi\rangle\right)\rangle\rangle \cdot \exp\left(\langle \omega_0, \phi\rangle\right) \quad \text{(because } \langle \omega_0, \phi\rangle \text{ is a constant)} \\
&= \mathcal{S}(\Phi)(\phi) \cdot \mathcal{S}\left(\exp^\diamond\left(\langle \bullet, \omega_0\rangle\right)\right)(\phi) \\
&= \mathcal{S}\left(\Phi \diamond \exp^\diamond\left(\langle \bullet, \omega_0\rangle\right)\right)(\phi),
\end{aligned}$$

and the conclusion follows from the uniqueness of the S-transform. We used the fact that $\langle \omega_0, \phi\rangle$ is a constant, and then the Wick exponential of a constant equals the standard exponential. □

We are now ready to enunciate a key result that will be used in chapter 3 and 4 due to Gjessing (e.g. [45], [38])

Theorem 100. *Let $\phi \in L^2(\mathbb{R})$ and $X \in (L^p)$ for some $p > 1$. Then $X \diamond \mathcal{E}(\langle \bullet, \phi\rangle) \in (L^q)$ for all $q < p$ and almost surely we have*

$$(X \diamond \mathcal{E}(\langle \bullet, \phi\rangle))(\omega) = \mathtt{T}_{-\phi} X \cdot \mathcal{E}(\langle \omega, \phi\rangle).$$

It's important to remark the fact that this hold under more general assumptions regarding our underlying probability space, see for instance chapter 3 where we employed Gjessing's formula in the Wiener space and chapter 4 for a fractional version.

2.3.8 Relationship between the Wick product and the Malliavin derivative

The aim of this section is that of introducing a property of the Wick product that represent a key tool in this thesis. In simple words it states a relationship between the Wick product and the dot product of a random variable with an element of the first Wiener chaos, i.e. a centered Gaussian random variable.

In chapters 3, 4, 5, 6 this feature is the key ingredient to construct candidate approximation for the solution of certain stochastic differential equations. At the end of this section we will present some interesting results that we've obtain by making use of this property.

We now present the result in the simpler setting

Proposition 101. *Let $\mathcal{T} \subset \mathbb{R}$ be some arbitrary interval, $f \in L^2(\mathcal{T})$ and $F \in \mathbb{D}^{1,2}$ then*

$$F \diamond \int_{\mathcal{T}} f(t)dB(t) = F \cdot \int_{\mathcal{T}} f(t)dB(t) - (DF\ ,\ f)_{L^2(\mathcal{T})} \tag{2.41}$$

Proof. We will start by showing the result for stochastic exponentials. By definition it holds that

$$\mathcal{E}(f) \diamond \mathcal{E}(\epsilon g) = \mathcal{E}(f + \epsilon g),\ \epsilon \in \mathbb{R}.$$

If we differentiate the expression above with respect to ϵ and evaluate the result at $\epsilon = 0$ we obtain for the left hand side

$$\frac{d}{d\epsilon}\mathcal{E}(f) \diamond \mathcal{E}(\epsilon g)\Big|_{\epsilon=0} = \mathcal{E}(f) \diamond \int_{\mathcal{T}} g(t)dB(t). \tag{2.42}$$

For the right hand side we have instead

$$\begin{aligned} \frac{d}{d\epsilon}\mathcal{E}(f + \epsilon g)\Big|_{\epsilon=0} &= \mathcal{E}(f)\left[\int_{\mathcal{T}} g(t)dB(t) - \int_{\mathcal{T}} f(t)g(t)dt\right] \\ &= \mathcal{E}(f) \cdot \int_{\mathcal{T}} g(t)dB(t) - \mathcal{E}(f) \cdot \int_{\mathcal{T}} f(t)g(t)dt. \end{aligned} \tag{2.43}$$

Notice that $\mathcal{E}(f)$ does not depend on t and thus we can bring it inside the integral in the last term above. Now using the fact that $f\mathcal{E}(f) = D\mathcal{E}(f)$ (this follows directly from definition 50) we can combine (2.42) and (2.43) to obtain

$$\mathcal{E}(f) \diamond \int_{\mathcal{T}} g(t)dB(t) = \mathcal{E}(f) \cdot \int_{\mathcal{T}} g(t)dB(t) - (D\mathcal{E}(f)\ ,\ g)_{L^2(\mathcal{T})}.$$

Since the set of finite linear combinations of stochastic exponentials is dense in $\mathbb{D}^{1,2}$ by using a limit argument we obtain the desired result. □

In the following we will present a toy example that would allow for a clearer exposition of the connection between the Wick product and the Malliavin derivative.

Example 102. Consider the one-dimensional probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$ where γ denotes the standard Gaussian measure on $\mathbb{R}$, i.e.

$$\gamma(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{\frac{-x^2}{2}} dx, \quad \text{for any Borel set } B \in \mathcal{B}(\mathbb{R}). \tag{2.44}$$

The random variable $\iota : \mathbb{R} \to \mathbb{R}$ given by the map $x \mapsto \iota(x) = x$ is $\mathcal{N}(0,1)$ as shown by the following simple calculation,

$$\gamma\left(\{x \in \mathbb{R}\} : \iota(x) \in B\right) = \gamma(B), \quad \text{for any Borel set } B \in \mathcal{B}(\mathbb{R}).$$

From this it follows that

$$\{\iota_t : t \in \mathbb{R}\} := \{t\iota : t \in \mathbb{R}\}$$

forms a one dimensional Gaussian Hilbert space indexed by $\mathbb{R}$. (e.g. [16, example 1.15])

Let $L^2(\gamma)$ denote the space of square integrable random variables defined on this probability space and let $\{H_n\}_{n\in\mathbb{N}_0}$ denote the family of Hermite polynomials. It's well know that the family of Hermite polynomials forms an orthogonal basis of $L^2(\gamma)$ and thus any random variable $F \in L^2(\gamma)$ has the following series representation

$$F(x) = \sum_{n=0}^{\infty} f_n H_n(x), \quad \text{for a.e. } x \in \mathbb{R},$$

and

$$\|F\|_{L^2(\gamma)} = \sum_{n=0}^{\infty} n!|f_n|^2.$$

This is a one dimensional version of the Wiener chaos decomposition, in fact remember that from theorem 43 and remark 86 it holds that

$$I_n(f^{\odot n}) = :I(f)\cdots I(f): = I(f)^{\diamond n}, \quad \text{for } f \in L^2([0,T]).$$

In this very simple framework f would be a scalar and as we've mentioned before the isometry between the indexing space (in this case $\mathbb{R}$) and the Gaussian Hilbert space is given by the multiplication with ι, hence in a slight abuse of notation we have that

$$\text{“}I_n(f^{\odot n})\text{”} = f^n \iota^{\diamond n}(x) = f^n H_n(\iota(x)) = f^n H_n(x),$$

where we used the relationship between Wick powers and Hermite polynomials (which holds in any arbitrary Gaussian Hilbert space).

It follows that the Wick product between F and ι can be simply written as

$$(F \diamond \iota)(x) = \sum_{n=0}^{\infty} f_n H_{n+1}(x).$$

Using the well known recursion formula for Hermite polynomials that tells us that

$$H_{n+1}(x) = xH_n(x) - \frac{d}{dx} H_n(x)$$

we can write the latter as

$$(F \diamond \iota)(x) = \sum_{n=0}^{\infty} f_n \left(xH_n(x) - \frac{d}{dx} H_n(x) \right).$$

Given that the series converges we can write this as

$$(F \diamond \iota)(x) = \iota(x) \cdot F(x) - \frac{d}{dx} F(x) = \left(x - \frac{d}{dx} \right) F(x). \tag{2.45}$$

From here we can see that on the Sobolev space $\mathcal{W}^{1,2}(\gamma)$ (which is the same as $\mathbb{D}^{1,2}$ in this one-dimension probability space) the Wick product acts as a differential operator. Notice furthermore that $\frac{d}{dx}$ coincides with the Malliavin derivative D.

It is important to remark the fact that this property is just a particular case of a much more general property connecting the Wick product with the Malliavin derivative which states that for $X, Y \in (S)$ (see for instance [46] for the proof of a more general result) it holds that

$$X \diamond Y = \sum_{n \in \mathbb{N}_0} \frac{(-1)^n}{n!} \int_{[0,T]^n} D^n_{t_1,\ldots,t_n} X \cdot D^n_{t_1,\ldots,t_n} Y dt_1 \cdots dt_n$$

Example 103. Let $\xi_1, ..., \xi_n$ and $\eta_1, ..., \eta_m$ be centered jointly Gaussian random variables such that $\mathbb{E}[\xi_i \eta_j] = 0$ for all i and j. In particular we can think that $\xi_i = \int_0^T f_i(t) dB(t)$ and $\eta_i = \int_0^T g_i(t) dB(t)$ where B is a one dimensional Brownian motion and the kernels are deterministic square integrable functions. Using the isometry property of the Wiener integral we have that by assumption $\int_0^T f_i(t) g_j(t) dt = 0$ for all i and j.

Using the polarization identity it suffices to show this result in case in which $\xi_1 = \cdots = \xi_n = \xi$ and $\eta_1 = \cdots = \eta_m = \eta$.

In that case we have

$$\begin{aligned} :\underbrace{\xi \cdots \xi}_{n} \quad \underbrace{\eta \cdots \eta}_{m} : &= \xi^{\diamond n} \diamond \eta^{\diamond m} \\ &= \sum_{k=0}^{n \wedge m} \frac{(-1)^k}{k!} \int_{[0,T]^k} D^k_{t_1,\ldots,t_k} \xi^{\diamond n} \cdot D^k_{t_1,\ldots,t_k} \eta^{\diamond m} dt_1 \cdots dt_k \end{aligned}$$

Notice that $D^i_{t_1,...,t_k}(\xi^{\diamond n}) = D^i_{t_1,...,t_k} I_n(f^{\odot n}) = f^{\otimes i}(t_1, ..., t_k) I_{n-k}(f^{\odot(n-k)})$ and analogously for $\eta^{\diamond m}$. The result follows invoking the orthogonality between f and g.

Some applications

The following theorem was obtained by the the author and Alberto Lanconelli and as far as we know is not present in the literature. It basically states that if in the construction of the Itô integral (see equation ...) one replaces the pointwise product with the Wick product then it doesn't matter whether we select the right or left point evaluation.

Theorem 104. *Let $f \in C^1_b(\mathbb{R})$ i.e. a bounded, continuously differentiable function with bounded derivative. Then for any finite partition $\pi = \{0 = t_0 < t_1 < \cdots t_N = T\}$ with mesh $\|\pi\|$ we have that*

$$\lim_{\|\pi\|\to 0} \sum_{j=1}^{N} f(B(t_j)) \diamond [B(t_j) - B(t_{j-1})] = \int_0^T f(B(t))dB(t), \quad \textit{convergence in } L^2(\Omega). \tag{2.46}$$

Proof. For the sake of simplicity we will assume as usual that the partition π is uniform, i.e. that $t_k = \frac{kT}{N}$ for any $k \in \{0, 1, ..., N\}$.

Adding and subtracting $f(B(t_{j-1}))\diamond[B(t_j)-B(t_{j-1})]$ to each term in the summation above we obtain

$$\begin{aligned}\sum_{j=1}^{N} f(B(t_j)) \diamond [B(t_j) - B(t_{j-1})] &= \sum_{j=1}^{N} [f(B(t_j)) - f(B(t_{j-1}))] \diamond [B(t_j) - B(t_{j-1})] \\ &\quad \cdots + \sum_{j=1}^{N} f(B(t_{j-1})) \diamond [B(t_j) - B(t_{j-1})] \\ &= \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

An important property that follows from the disjointness of the increments of the Brownian motion is that

$$\begin{aligned}f(B(t_{j-1})) \diamond [B(t_j) - B(t_{j-1})] &= f(B(t_{j-1})) \cdot [B(t_j) - B(t_{j-1})] - \left(Df(B(t_{j-1})), \chi_{[t_{j-1},t_j)}\right)_{L^2([0,T])} \\ &= f(B(t_{j-1})) \cdot [B(t_j) - B(t_{j-1})] \\ &\quad - f'(B(t_{j-1})) \left(\chi_{[0,t_{j-1})}, \chi_{[t_{j-1},t_j)}\right)_{L^2([0,T])} \\ &= f(B(t_{j-1})) \cdot [B(t_j) - B(t_{j-1})],\end{aligned}$$

thus in $\mathcal{I}_2$ we can replace the Wick product with the point-wise product.

The boundedness of f allows us to use the DCT to show that $\mathbb{E}[f(B(s))f(B(t))]$ is continuous function on $[0,T]^2$ and thus (e.g.[6])

$$\mathcal{I}_2 = \sum_{j=1}^{N} f(B(t_{j-1})) \cdot [B(t_j) - B(t_{j-1})] \xrightarrow{\|\pi\|\to 0} \int_0^T f(B(s))dB(s), \quad \text{convergence in } L^2(\Omega).$$

On the other hand by denoting $\Delta_j f := [f(B(t_j)) - f(B(t_{j-1}))]$ and $\Delta_j B := [B(t_j) - B(t_{j-1})]$ we can write

$$\mathcal{I}_1 = \sum_{j=1}^{N} \Delta_j f \diamond \Delta_j B = \sum_{j=1}^{N} \Big[\Delta_j f \cdot \Delta_j B - \big(D\Delta_j f, \chi_{[t_{j-1},t_j]}\big)_{L^2([0,T])}\Big] = \mathcal{I}_3 - \mathcal{I}_4.$$

Now we have that by definition

$$\mathcal{I}_3 = \sum_{j=1}^{N} \Delta_j f \cdot \Delta_j B \xrightarrow{\|\pi\|\to 0} [f(B), B]_T = \int_0^T f'(B(s))ds, \quad \text{convergence in } L^2(\Omega)$$

where $[\bullet,\bullet]_T$ denotes the quadratic covariation evaluated at time T.

By using the chain rule for the Malliavin derivative we obtain

$$D\Delta_j f = f'(B(t_j))\chi_{[0,t_j]} - f'(B(t_{j-1}))\chi_{[0,t_{j-1}]},$$

and thus (thanks to f''s boundedness)

$$\mathcal{I}_4 = \sum_{j=1}^{N} f'(B(t_j))\|\pi\| \xrightarrow{\|\pi\|\to 0} \int_0^T f'(B(s))ds \ a.s.$$

which by means of DCT (again by virtue of f' being bounded) implies the $L^2(\Omega)$ convergence proving the desired result. □

Theorem 105. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a one dimensional Brownian motion $\{B(t)\}_{t\in[0,T]}$. Let $\{X(t)\}_{t\in[0,T]}$ be an Itô integrable stochastic process satisfying:*

1. *$X(t) \in \mathbb{D}^{1,2}$ for a.a. $t \in [0,T]$;*
2. *$\{X(t)\}_{t\in[0,T]}$ is (uniformly) square-mean continuous;*
3. *$\operatorname{ess\,sup}_{(t,s)\in[0,T]^2} \mathbb{E}\left[|D_s X(t)|^2\right] < \infty$.*

Then for any finite partition $\pi := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ with mesh $\|\pi\|$, it holds that

$$\sum_{j=1}^{N} X(t_j) \diamond [B(t_j) - B(t_{j-1})] \xrightarrow{\|\pi\|\to 0} \int_0^T X(t)dB(t), \quad in\ L^2(\Omega). \tag{2.47}$$

Proof. As usual we will assume that the partition π is uniform and just as before we start by adding and subtracting to each term in the summation the quantity $X(t_{j-1}) \diamond [B(t_j) - B(t_{j-1})]$. Under our assumptions we can use proposition 101 to write

$$X(t_{j-1}) \diamond [B(t_j) - B(t_{j-1})] = X(t_{j-1}) \cdot [B(t_j) - B(t_{j-1})] - \left(DX_{t_{j-1}}\ ,\ \chi_{[t_{j-1},t_j)}\right)_{L^2([0,T])},$$

notice that since by assumption X is an adapted process then $D_s X_t = 0, \forall s > t$ and thus the second term above vanishes, i.e. $X(t_{j-1}) \diamond [B(t_j) - B(t_{j-1})] = X(t_{j-1}) \cdot [B(t_j) - B(t_{j-1})]$ for any $j \in \{1, ..., N\}$.

Then

$$\begin{aligned}\sum_{j=1}^{N} X(t_j) \diamond [B(t_j) - B(t_{j-1})] &= \sum_{j=1}^{N} [X(t_j) - X(t_{j-1})] \diamond [B(t_j) - B(t_{j-1})] \\ &\quad + \sum_{j=1}^{N} X(t_{j-1}) \cdot [B(t_j) - B(t_{j-1})] \\ &= \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

By our assumptions we have that $\mathcal{I}_2$ converges in $L^2(\Omega)$ to the Itô integral and thus the result would follow if we manage to show that $\mathcal{I}_1 \to 0$ in the $L^2(\Omega)$ norm. Notice that each Brownian increment on the summation in $\mathcal{I}_1$ equals $\int_0^T \chi_{[t_{j-1},t_j)}(s) dB(s)$ and using theorem ... we can rewrite each summand as

$$[X(t_j) - X(t_{j-1})] \diamond [B(t_j) - B(t_{j-1})] = \int_0^T [X(t_j) - X(t_{j-1})] \chi_{[t_{j-1},t_j)}(s) \delta B(s), \tag{2.48}$$

where the stochastic integral must be understood as a Skorohod integral since the integrand is not adapted. Thus it follows that

$$\mathcal{I}_1 = \int_0^T \left[\sum_{j=1}^{N} [X(t_j) - X(t_{j-1})] \chi_{[t_{j-1},t_j)}(s)\right] \delta B(s). \tag{2.49}$$

Then by letting $u(t) := \left[\sum_{j=1}^{N} [X(t_j) - X(t_{j-1})] \chi_{[t_{j-1},t_j)}(t)\right]$ we obtain (e.g. [22, eq. 1.60])

$$\mathbb{E}[|\mathcal{I}_1|^2] = \mathbb{E}\left[\int_0^T |u(t)|^2 dt\right] + \mathbb{E}\left[\int_{[0,T]^2} D_s u(t) D_t u(s) ds dt\right] = \mathcal{I}_3 + \mathcal{I}_4.$$

Now by definition we have that

$$\mathcal{I}_3 = \mathbb{E}\left[\int_0^T \left(\sum_{j=1}^N [X(t_j) - X(t_{j-1})]\chi_{[t_{j-1},t_j)}(t)\right)^2 dt\right] = \|\pi\| \cdot \sum_{j=1}^N \mathbb{E}\left[(X(t_j) - X(t_{j-1}))^2\right]$$
$$\leq \|\pi\| N \sup_{i\in\{1,\dots,N\}} \mathbb{E}\left[(X(t_i) - X(t_{i-1}))^2\right]$$
$$= T \sup_{i\in\{1,\dots,N\}} \mathbb{E}\left[(X(t_i) - X(t_{i-1}))^2\right] \to 0,$$

as $\|\pi\|$ goes to 0, where the convergence follows from the (uniform) square-mean continuity of $\{X(t)\}_{t\in[0,T]}$.

Using the definition of u we have that $\mathcal{I}_4$ can be treated as follows,

$$\begin{aligned}
\mathcal{I}_4 &= \sum_{j=1}^N \mathbb{E}\left[\left(\int_{t_{j-1}}^{t_j} D_s X(t_j) ds\right)^2\right] \\
&\leq \|\pi\| \sum_{j=1}^N \mathbb{E}\left[\int_{t_{j-1}}^{t_j} |D_s X(t_j)|^2 ds\right] \\
&= \|\pi\| \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \mathbb{E}\left[|D_s X(t_j)|^2\right] ds \\
&\leq \|\pi\| \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \text{ess sup}_{(t,u)\in[0,T]^2} \mathbb{E}\left[|D_u X(t)|^2\right] ds \\
&= \|\pi\| T \,\text{ess sup}_{(t,u)\in[0,T]^2} \mathbb{E}\left[|D_u X(t)|^2\right] \to 0,
\end{aligned}$$

as $\|\pi\|$ goes to 0, this implies the desired result.

□

We will now present a result which is the analogous of .... . This is yet another proof that Wick product and Itô integration are deeply connected

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space carrying a one-dimensional Brownian motion $\{B(t)\}_{t\in[0,T]}$, where as usual T is some arbitrary positive real constant.

As in theorem ... we denote with $\pi := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ a finite partition of the interval $[0,T]$; for the sake of simplicity we shall assume that the partition is uniform, namely that $t_k = \frac{kT}{N}$ for any $k \in \{1, 2, ..., N\}$.

We define the polygonal approximation of $\{B(t)\}_{t\in[0,T]}$ as

$$B^\pi(t) := \int_0^T K_t^\pi(u) dB(u), \quad \text{for any } t \in [0,T] \tag{2.50}$$

with

$$K_t^\pi(u) := \sum_{k=0}^{N-1} \left(\chi_{[0,t_k)}(u) + \frac{t - t_k}{t_{k+1} - t_k} \chi_{[t_k,t_{k+1})}(u) \right) \chi_{[t_k,t_{k+1})}(t). \tag{2.51}$$

From here it's straightforward to see that

$$\dot{B}^\pi(t) := \int_0^T \partial_t K_t^\pi(u) dB(u) \quad \text{with} \quad \partial_t K_t^\pi(u) = \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \chi_{[t_k,t_{k+1})}(u) \chi_{[t_k,t_{k+1})}(t). \tag{2.52}$$

With this in hand we are ready to introduce the main theorem

Theorem 106. *Let $f \in C_b^2(\mathbb{R})$ i.e. a bounded, twice continuously differentiable function with bounded derivatives. Then it holds that*

$$\int_0^T f(B^\pi(t)) \diamond \dot{B}^\pi(t) dt \longrightarrow \int_0^T f(B(t)) dB(t) \quad \text{in } L^2(\Omega) \tag{2.53}$$

as $\|\pi\| \to 0$.

Proof. We start by defining $g(x) := \int_0^x f(y)dy$ such that $g'(x) = f(x)$ and this together with our assumptions imply that g is a Lipschitz continuous function.

By means of the usual chain rule we obtain

$$\frac{d}{dt} g(B_t^\pi) = g'(B_t^\pi) \cdot \dot{B}_t^\pi = f(B_t^\pi) \cdot \dot{B}_t^\pi. \tag{2.54}$$

From equation (2.54) and proposition ... we have that

$$\frac{d}{dt} g(B^\pi(t)) = f(B^\pi(t)) \diamond \dot{B}^\pi(t) + f'(B^\pi(t)) \left(K_t^\pi \ , \ \partial_t K_t^\pi \right)_{L^2([0,T])} .$$

Integrating both sides over the interval $[0, T]$ we obtain

$$g(B^\pi(T)) - g(B^\pi(0)) = \int_0^T f(B^\pi(t)) \diamond \dot{B}^\pi(t) dt + \int_0^T f'(B^\pi(t)) \left(K_t^\pi \ , \ \partial_t K_t^\pi \right)_{L^2([0,T])} dt.$$

Notice that the lefter side converges in the $L^2(\Omega)$-norm to the Stratonovich integral $\int_0^T f(B(t)) \circ dB(t) = g(B(T))$, in fact

$$\begin{aligned} \|g(B^\pi(T)) - g(B^\pi(0)) - g(B(T)) + g(B(0))\| &\leq \|g(B^\pi(T)) - g(B(T))\| + \|g(B^\pi(0)) - g(B(0))\| \\ &\leq L\big(\|B^\pi(T) - B(T)\| + \|B^\pi(0) - B(0)\|\big) \xrightarrow{\|\pi\| \to 0} 0. \end{aligned}$$

where in the last inequality we've used the Lipschitz continuity of g.

Then using the equivalency

$$\int_0^T f(B(t)) \circ dB(t) = \int_0^T f(B(t))dB(t) + \frac{1}{2}\int_0^T f'(B(t))dt,$$

it would suffice to show that

$$\int_0^T f'(B^\pi(t)) \left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} dt \longrightarrow \frac{1}{2}\int_0^T f'(B(t))dt$$

in the $L^2(\Omega)$ norm as $\|\pi\| \to 0$.

$$\begin{aligned}
&\left\| \int_0^T f'(B^\pi(t)) \left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} dt - \frac{1}{2}\int_0^T f'(B(t))dt \right\| \\
&\leq \left\| \int_0^T f'(B^\pi(t)) \left[\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} - \frac{1}{2}\right] dt \right\| + \left\| \frac{1}{2}\int_0^T f'(B^\pi(t)) - f'(B(t))dt \right\| \\
&\leq \left\| \int_0^T f'(B^\pi(t)) \left[\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} - \frac{1}{2}\right] dt \right\| + \frac{1}{2}\int_0^T \left\| f'(B^\pi(t)) - f'(B(t)) \right\| dt = \mathcal{I}_1 + \mathcal{I}_2,
\end{aligned}$$

where in the last inequality we've employed the Minkowski inequality for integrals.

The boundedness of f' allows us to employ the DCT which together with the Lipchitz continuity of f' (which is implied by the boundedness of f'') implies that

$$\mathcal{I}_2 \to 0 \quad \text{as } \|\pi\| \to 0.$$

Let's now consider $\mathcal{I}_1$; we start by adding and subtracting $f'(B(t)) \left[\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} - \frac{1}{2}\right]$ to the integrand, then by means of the triangular inequality we have that

$$\begin{aligned}
\mathcal{I}_1 \leq &\left\| \int_0^T \left[f'(B^\pi(t)) - f'(B(t))\right] \left[\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} - \frac{1}{2}\right] dt \right\| \\
&+ \left\| \int_0^T f'(B(t)) \left[\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} - \frac{1}{2}\right] dt \right\| = \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned}$$

Notice that the function $\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} = \sum_{i=0}^{n-1} \frac{t-t_i}{\|\pi\|}\chi_{(t_i,t_{i+1}]}(t)$ converges to the constant $\frac{1}{2}$ in the weak topology of $L^2([0,T])$ (as $\|\pi\| \to 0$), this implies that

$$\int_0^T f'(B(t)) \left[\left(K_t^\pi \ , \ \partial_t K_t^\pi\right)_{L^2([0,T])} - \frac{1}{2}\right] dt \to 0 \, a.s. \quad \text{when } \|\pi\| \to 0, \tag{2.55}$$

then noticing that for any $t \in [0,T]$, $|\,(K_t^\pi\ ,\ \partial_t K_t^\pi)_{L^2([0,T])}\,| \leq 1$ an application of the DCT implies that $\mathcal{I}_4 \to 0$ as $\|\pi\| \to 0$.

On the other hand applying Mikowski inequality for integrals, using the bound on $|\,(K_t^\pi\ ,\ \partial_t K_t^\pi)_{L^2([0,T])}\,|$ we can use the same reasoning as for $\mathcal{I}_2$ to show that $\mathcal{I}_3 \to 0$ as $\|\pi\| \to 0$. This completes the proof. $\square$

Chapter 3

Wong-Zakai approximations for quasilinear systems of Itô's type stochastic differential equations

Based on: Lanconelli, A., & Scorolli, R. (2021). Wong–Zakai approximations for quasilinear systems of Itô's type stochastic differential equations. Stochastic Processes and their Applications, 141, 57-78.

Abstract

We extend to the multidimensional case a Wong-Zakai-type theorem proved by Hu and Øksendal in [1] for scalar quasi-linear Itô stochastic differential equations (SDEs). More precisely, with the aim of approximating the solution of a quasilinear system of Itô's SDEs, we consider for any finite partition of the time interval $[0, T]$ *a system of differential equations, where the multidimensional Brownian motion is replaced by its polygonal approximation and the product between diffusion coefficients and smoothed white noise is interpreted as a Wick product. We remark that in the one dimensional case this type of equations can be reduced, by means of a transformation related to the method of characteristics, to the study of a random ordinary differential equation. Here, instead, one is naturally led to the investigation of a semilinear hyperbolic system of partial differential equations that we utilize for constructing a solution of the Wong-Zakai approximated systems. We show that the law of each element of the approximating sequence solves in the sense of distribution a Fokker-Planck equation and that the sequence converges to the solution of the Itô equation, as the mesh of the partition tends to zero.*

3.1 Introduction and statement of the main results

Let $\{B(t)\}_{t\in[0,T]}$ be a standard one dimensional Brownian motion and, for a given finite partition π of the interval $[0,T]$, denote by $\{B^\pi(t)\}_{t\in[0,T]}$ its polygonal approximation. Then, under suitable conditions on the coefficients $b:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $\sigma:[0,T]\times\mathbb{R}\to\mathbb{R}$, the solution $\{Y^\pi(t)\}_{t\in[0,T]}$ of the random ordinary differential equation

$$\frac{dY^\pi(t)}{dt} = b(t,Y^\pi(t)) + \sigma(t,Y^\pi(t))\cdot\frac{dB^\pi(t)}{dt}, \tag{3.1}$$

converges, as the mesh of π tends to zero, to the strong solution $\{Y(t)\}_{t\in[0,T]}$ of the Stratonovich stochastic differential equation (SDE, for short)

$$dY(t) = b(t,Y(t))dt + \sigma(t,Y(t))\circ dB(t), \tag{3.2}$$

or equivalently (see [2]) of the Itô SDE

$$dY(t) = \left[b(t,Y(t)) + \frac{1}{2}\sigma(t,Y(t))\partial_y\sigma(t,Y(t))\right]dt + \sigma(t,Y(t))dB(t).$$

This is the famous Wong-Zakai theorem [12],[13] whose extension to the multidimensional case can be found in [47].
In [1] the authors suggested how to modify equation (3.1) to get in the limit the Itô's interpretation of (3.2): they considered the case with $\sigma(t,x)=\sigma(t)x$, where $\sigma:[0,T]\to\mathbb{R}$ is a deterministic function, and proved that the solution $\{X^\pi(t)\}_{t\in[0,T]}$ of the differential equation

$$\frac{dX^\pi(t)}{dt} = b(t,X^\pi(t)) + \sigma(t)X^\pi(t)\diamond\frac{dB^\pi(t)}{dt}, \tag{3.3}$$

converges, as the mesh of π tends to zero, to the strong solution $\{X(t)\}_{t\in[0,T]}$ of the Itô SDE

$$dX(t) = b(t,X(t))dt + \sigma(t)X(t)dB(t). \tag{3.4}$$

Observe that the achievement of [1] is twofold: existence of a solution for (3.3) and its convergence towards the solution of (3.4) (see also the related works [48] and [49]). As far as the existence is concerned, equation (3.3) is not a standard random ordinary differential equation but instead an infinite dimensional partial differential equation. In fact, by proposition 101 we have that

$$X^\pi(t)\diamond\frac{dB^\pi(t)}{dt} = X^\pi(t)\frac{dB^\pi(t)}{dt} - D_{\partial_t K^\pi(t,\cdot)}X^\pi(t), \tag{3.5}$$

where $K^\pi(t,\cdot)$ is a deterministic function that verifies the identity

$$B^\pi(t) = \int_0^T K^\pi(t,s)dB(s),$$

while $D_{\partial_t K^\pi(t,\cdot)}$ stands for the directional Malliavin derivative along the function $s \mapsto \partial_t K^\pi(t,s)$, one recognizes equation (3.3) as a nonlinear evolution equation driven by an infinite dimensional gradient. Nevertheless, the particular form of $\sigma(t,x)$ considered in [1] allows for a reduction method which transforms that into a random ordinary differential equation. We now briefly describe such method: we Wick-multiply both sides of (3.3) by

$$\mathtt{E}^\pi(0,t) := e^{-\int_0^t \sigma(s)\frac{dB^\pi(s)}{ds}ds - \frac{1}{2}\mathbb{E}\left[\left(\int_0^t \sigma(s)\frac{dB^\pi(s)}{ds}ds\right)^2\right]}, \quad t \in [0,T],$$

to obtain

$$\frac{dX^\pi(t)}{dt} \diamond \mathtt{E}^\pi(0,t) = b(t, X^\pi(t)) \diamond \mathtt{E}^\pi(0,t) + \sigma(t)X^\pi(t) \diamond \frac{dB^\pi(t)}{dt} \diamond \mathtt{E}^\pi(0,t),$$

or equivalently,

$$\frac{dX^\pi(t)}{dt} \diamond \mathtt{E}^\pi(0,t) = b(t, X^\pi(t)) \diamond \mathtt{E}^\pi(0,t) - X^\pi(t) \diamond \frac{d\mathtt{E}^\pi(0,t)}{dt}.$$

Here, we utilized the identity

$$\frac{d\mathtt{E}^\pi(0,t)}{dt} = \sigma(t)\frac{dB^\pi(t)}{dt} \diamond \mathtt{E}^\pi(0,t).$$

Rearranging the terms and exploiting the Leibniz rule for the Wick product we can write

$$\frac{d}{dt}\left(X^\pi(t) \diamond \mathtt{E}^\pi(0,t)\right) = b(t, X^\pi(t)) \diamond \mathtt{E}^\pi(0,t). \tag{3.6}$$

Now, if we set

$$\mathcal{X}^\pi(t) := X^\pi(t) \diamond \mathtt{E}^\pi(0,t), \quad t \in [0,T],$$

and recall that

$$\mathtt{E}^\pi(0,t) \diamond \mathcal{E}^\pi(0,t) = 1, \quad \text{for all } t \in [0,T],$$

where

$$\mathcal{E}^\pi(0,t) := e^{\int_0^t \sigma(s)\frac{dB^\pi(s)}{ds}ds - \frac{1}{2}\mathbb{E}\left[\left(\int_0^t \sigma(s)\frac{dB^\pi(s)}{ds}ds\right)^2\right]}, \quad t \in [0,T],$$

we can reduce (3.6) to

$$\frac{d\mathcal{X}^\pi(t)}{dt} = b(t, \mathcal{X}^\pi(t) \diamond \mathcal{E}^\pi(0,t)) \diamond \mathtt{E}^\pi(0,t). \tag{3.7}$$

Equation (3.7) doesn't look simpler than (3.3); however, in (3.7) one can apply theorem 100 which produces a Wick product-free expression. First, we observe that resorting to the definition of $\{B^{\pi}(t)\}_{t\in[0,T]}$ (see equation (3.16) below) one gets the representation

$$\int_0^t \sigma(s)\frac{dB^{\pi}(s)}{ds}ds = \int_0^T \sigma^{\pi}(t,s)dB(s),$$

for a suitable $\sigma^{\pi}:[0,T]\times[0,T]\to\mathbb{R}$. With this notation at hand, Gjessing's formula (theorem 100) can be simply stated as

$$\mathcal{Z}\diamond\mathcal{E}^{\pi}(0,t) = \mathtt{T}_{-\sigma^{\pi}(t,\cdot)}\mathcal{Z}\cdot\mathcal{E}^{\pi}(0,t), \tag{3.8}$$

and

$$\mathcal{Z}\diamond\mathtt{E}^{\pi}(0,t) = \mathtt{T}_{\sigma^{\pi}(t,\cdot)}\mathcal{Z}\cdot\mathtt{E}^{\pi}(0,t), \tag{3.9}$$

for a general random variable $\mathcal{Z}$ belonging to $L^p(\Omega)$, for some $p>1$. It's important to notice that here $\mathtt{T}_f$ denotes the operator that translates the Brownian path by the function $\int_0^{\cdot} f(s)ds$ (see formula (3.22) below), the difference between this formulation and the one introduced in theorem 100 is due to the fact that in the latter we were working with the white noise probability space while here we deal with the Wiener space.

An application to equation (3.7) of Gjessing's formula yields the following random ordinary differential equation

$$\frac{d\mathcal{X}^{\pi}(t)}{dt} = b\left(t,\mathcal{X}^{\pi}(t)\cdot(\mathtt{E}^{\pi}(0,t))^{-1}\right)\cdot\mathtt{E}^{\pi}(0,t). \tag{3.10}$$

Standard assumptions on the coefficients ensure the existence of a unique solution $\{\mathcal{X}^{\pi}(t)\}_{t\in[0,T]}$ which, together with equality $X^{\pi}(t)=\mathcal{X}^{\pi}(t)\diamond\mathcal{E}^{\pi}(0,t)$, provides a unique solution also for (3.3). It is important to remark that the success of this reduction method is due to the opposite signs appearing in front of $\sigma^{\pi}(t,\cdot)$ in equations (3.8) and (3.9); this results in the disappearance of the translation operator, and hence of the Wick product, from equation (3.7).

Aim of the present paper is the extension to the multidimensional case of the existence theorem for (3.3) and its convergence to (3.4) proven in [1]. More precisely, for each finite partition π of the interval $[0,T]$ we introduce the Cauchy problem

$$\begin{cases} \frac{dX_i^{\pi}(t)}{dt} = b_i(t,X^{\pi}(t)) + \sigma_i(t)X_i^{\pi}(t)\diamond\frac{dB_i^{\pi}(t)}{dt}, \\ \qquad \text{for } t\in]0,T] \text{ and } i=1,...,d; \\ X_i^{\pi}(0)=c_i\in\mathbb{R}, \quad \text{for } i=1,...,d, \end{cases} \tag{3.11}$$

where $\{B^{\pi}(t)=(B_1^{\pi}(t),...,B_d^{\pi}(t))^*\}_{t\in[0,T]}$ stands for the polygonal approximation, relative to the partition π, of the standard d-dimensional Brownian motion $\{B(t)=(B_1(t),...,B_d(t))^*\}_{t\in[0,T]}$; the functions $b_1,...,b_d:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ and $\sigma_1,...,\sigma_d$:

$[0,T] \to \mathbb{R}$ are measurable while $c \in \mathbb{R}^d$ is a deterministic initial condition. System (3.11) should be thought as a Wong-Zakai-type approximation for the system of Itô's SDEs

$$\begin{cases} dX_i(t) = b_i(t, X(t))dt + \sigma_i(t)X_i(t)dB_i(t), \\ \qquad \text{for } t \in]0,T] \text{ and } i = 1, ..., d; \\ X_i(0) = c_i \in \mathbb{R}, \quad \text{for } i = 1, ..., d. \end{cases} \tag{3.12}$$

We will assume throughout the paper the following regularity properties for the coefficients: they guarantee the existence of a unique strong solution for (3.12).

Assumption 2.

- *The functions $b(t,x)$, $\partial_{x_1} b(t,x)$,..., $\partial_{x_d} b(t,x)$ are bounded and continuous;*
- *the functions $\sigma_1(t), ..., \sigma_d(t)$ are bounded and continuous.*

Our first main theorem concerns the existence of a solution for (3.11). It is worth mentioning that the reduction method described above doesn't apply to such systems, unless very particular cases are considered. In fact, the disappearance of the translation operator mentioned before takes place only when the same one dimensional Brownian motion drives all the equations in (3.11) and moreover $\sigma_1(t) = \cdots = \sigma_d(t)$, for all $t \in [0,T]$. Therefore, to prove the existence of a solution for (3.11) we have to employ a different approach which can be summarized as follows.
Using identity (3.5) we rewrite (3.11) as

$$\begin{cases} \frac{dX_i^\pi(t)}{dt} = b_i(t, X^\pi(t)) + \sigma_i(t)X_i^\pi(t)\frac{dB_i^\pi(t)}{dt} - \sigma_i(t)D^{(i)}_{K^\pi(t,\cdot)}X_i^\pi(t), \\ \qquad \text{for } t \in]0,T] \text{ and } i = 1, ..., d; \\ X_i^\pi(0) = c_i \in \mathbb{R}, \quad \text{for } i = 1, ..., d. \end{cases} \tag{3.13}$$

(Here, $D^{(i)}$ stands for the Mallivian derivative with respect to the i-th component of the multidimensional Brownian motion $\{B(t)\}_{t\geq 0}$). If we now divide the interval $[0,T]$ according to the partition $\pi = \{t_0, ..., t_N\}$ and search on any subinterval $]t_k, t_{k+1}]$ for a solution to (3.13) of the form

$$X_i^\pi(t) := u_i(t, B^\pi(t_{k+1}) - B^\pi(t_k)), \quad t \in]t_k, t_{k+1}], i = 1, ..., d$$

where $u_i : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ are deterministic functions, we see that $u = (u_1, ..., u_d)$ has to solve a semilinear hyperbolic system of partial differential equations of the type

$$\begin{cases} \partial_t u_i(t,x) = -\sigma_i(t)\partial_{x_i} u_i(t,x) + \sigma_i(t)\frac{x_i}{h}u_i(t,x) + b_i(t, u(t,x)), \\ \qquad \text{for } t \in]t_k, t_{k+1}], x \in \mathbb{R}^d \text{ and } i = 1, ..., d; \\ u_i(r,x) = \alpha_i, \quad \text{for } x \in \mathbb{R}^d \text{ and } i = 1, ..., d. \end{cases} \tag{3.14}$$

Here, h denotes the mesh of the partition π while $\alpha_1, ..., \alpha_d$ are suitable deterministic initial conditions. This link to the theory of partial differential equations allows us to state our first main result whose proof can be found in Section 3.3. We will deal with a weak notion of solution, see Definition 111 below, that doesn't require any Malliavin differentiability property of the solution (as it should be implied by the last term in (3.13)).

Theorem 107 (Existence). *Under Assumption equation (3.11) possesses a mild solution $\{X^\pi(t)\}_{t\in[0,T]}$ in the sense of Definition 111 below.*

Our second main result shows that system (3.11) is naturally connected with a Fokker-Planck-type equation which is solved in the sense of distributions by the law of the mild solution $\{X^\pi(t)\}_{t\in[0,T]}$. This establishes a further similarity between the Wong-Zakai approximating equation (3.11) and its exact counterpart (3.12). This theorem generalizes the one obtained in [50] for the scalar problem (3.3). The proof is postponed to Section 3.4.

Theorem 108 (Fokker-Planck equation). *The law*

$$\mu^\pi(t, A) := P(X^\pi(t) \in A), \quad t \in [0, T], A \in \mathcal{B}(\mathbb{R}^d)$$

of the random vector $X^\pi(t)$ solves in the sense of distributions the Fokker-Planck equation

$$\left(\partial_t + \sum_{i,j=1}^{d} \sigma_i(t) x_i g_{ij}(t, x_i) \partial^2_{x_i x_j} + \sum_{i=1}^{d} b_i(t, x) \partial_{x_i}\right)^* \mathtt{u}(t, x) = 0, \quad t \in [0, T], x \in \mathbb{R}^d. \tag{3.15}$$

Here, $g_{ij} : [0, T] \times \mathbb{R} \to \mathbb{R}$ is the measurable function defined in (3.37) and (3.38) below.

Lastly, we present the convergence of $\{X^\pi(t)\}_{t\in[0,T]}$ towards the solution of the Itô equation (3.12), as the mesh $\|\pi\|$ of the partition π tends to zero. For the proof the reader is referred to Section 3.5.

Theorem 109 (Convergence). *The mild solution $\{X^\pi(t)\}_{t\in[0,T]}$ converges, as the mesh of π tends to zero, to the unique strong solution $\{X(t)\}_{\in[0,T]}$ of the Itô SDE (3.12). More precisely,*

$$\lim_{\|\pi\|\to 0} \sum_{i=1}^{d} \mathbb{E}\left[|X_i^\pi(s) - X_i(s)|\right] = 0, \quad \text{for all } t \in [0, T].$$

Remark 110. In the next chapter we will a similar idea and approach to Itô-type stochastic differential equations driven by fractional Brownian motions (see Section 6.1 in [51]).

The paper is organized as follows: in Section 2 we describe our framework and formalize all the mathematical concepts utilized in the introduction to present the problem; Section 3 contains the most novel part of our paper that consists in the link between the Wong-Zakai equation (3.11) and the semilinear hyperbolic system of partial differential equations (3.14); here, we describe in details the construction of the mild solution $\{X^\pi(t)\}_{t\in[0,T]}$; in Section 4 the proof of Theorem 108 on the Fokker-Planck equation passes through a careful interplay between the Gaussian nature of the noise and structure of the hyperbolic system; Section 5 concludes the manuscript with the proof of Theorem 109 which greatly benefits from the notion of mild solution introduced in Section 2.

3.2 Notation and preliminary results

In this section we set the notation and prepare the ground for proving our main theorems. We fix a positive time horizon T and a dimension $d \in \mathbb{N}$. Let $(\Omega, \mathcal{F}, P)$ be the classical Wiener space over the time interval $[0,T]$ with values on $\mathbb{R}^d$ (see section 2.1.2); we denote by $\{B(t) = (B_1(t), ..., B_d(t))^*\}_{t\in[0,T]}$ the coordinate process, i.e.

$$\begin{aligned} B(t) : \Omega &\to \mathbb{R}^d \\ \omega &\mapsto B(t)(\omega) := \omega(t); \end{aligned}$$

by construction, $\{B(t)\}_{t\in[0,T]}$ is a standard d-dimensional Brownian motion.
We choose a finite partition $\pi := \{t_0, ..., t_N\}$ of the interval $[0,T]$, i.e.

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

and set $\|\pi\| := \max_{k\in\{0,1,...,N\}} |t_k - t_{k-1}|$. The real number $\|\pi\|$ is called *mesh* of the partition π. We will assume without loss of generality that the partition is equally spaced, i.e. $t_k = \frac{kT}{N}$, for all $k \in \{0, ..., N\}$; in this case we simply have $\|\pi\| := \frac{T}{N}$ but we will continue to use the notation $\pi = \{t_0, ..., t_N\}$ and $\|\pi\|$.
We associate to the partition π the *polygonal* approximation of the Brownian motion $\{B(t)\}_{t\in[0,T]}$:

$$B^\pi(t) := \left(1 - \frac{t - t_k}{t_{k+1} - t_k}\right) B(t_k) + \frac{t - t_k}{t_{k+1} - t_k} B(t_{k+1}), \quad \text{if } t \in [t_k, t_{k+1}[\tag{3.16}$$

and $B^\pi(T) := B(T)$. It is well known that for any $\varepsilon > 0$ and $p \geq 1$ there exists a positive constant $C_{p,T,\varepsilon}$ such that

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} |B(t)^\pi - B(t)|^p\right]\right)^{1/p} \leq C_{p,T,\varepsilon} \|\pi\|^{1/2-\varepsilon}.$$

We refer the reader to [15, Lemma 11.8] for a sharper estimate. For $i = 1, ..., d$, we set

$$\Sigma_i(s,t) := \int_s^t \sigma_i(r)dr, \quad 0 \leq s \leq t \leq T, \tag{3.17}$$

and observe that

$$\begin{aligned}\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr &= \int_s^{t_j} \sigma_i(r)\dot{B}_i^\pi(r)dr + \int_{t_j}^{t_{j+1}} \sigma_i(r)\dot{B}_i^\pi(r)dr + \cdots + \int_{t_k}^t \sigma_i(r)\dot{B}_i^\pi(r)dr \\ &= \Sigma_i(s,t_j)\frac{B_i(t_j) - B_i(t_{j-1})}{h} + \Sigma_i(t_j,t_{j+1})\frac{B_i(t_{j+1}) - B_i(t_j)}{h} \\ &\cdots + \Sigma_i(t_k,t)\frac{B_i(t) - B_i(t_k)}{h},\end{aligned} \tag{3.18}$$

when $t_{j-1} \leq s < t_j < \cdots < t_k \leq t$, for some $j \leq k$ in $\{1, ..., N-1\}$. In particular, if $s, t \in [t_k, t_{k+1}]$ for some $k \in \{0, ..., N\}$, the last expression simplifies to

$$\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr = \Sigma_i(s,t)\frac{B_i(t_{k+1}) - B_i(t_k)}{h}.$$

It is important to remark that according to (3.18) the quantity $\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr$ is a linear combination of independent Gaussian random variables with

$$\mathbb{E}\left[\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr\right] = 0$$

and

$$\mathbb{E}\left[\left(\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr\right)^2\right] = \frac{1}{h}\left(\Sigma_i(s,t_j)^2 + \Sigma_i(t_j,t_{j+1})^2 + \cdots + \Sigma_i(t_k,t)^2\right)$$

We now set

$$\mathcal{E}_i^\pi(s,t) := e^{\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr - \frac{1}{2}\mathbb{E}\left[\left(\int_s^t \sigma_i(r)\dot{B}_i^\pi(r)dr\right)^2\right]}$$

and observe that if $s, t \in [t_k, t_{k+1}]$, for some $k \in \{0, ..., N\}$, we get

$$\mathcal{E}_i^\pi(s,t) = e^{\Sigma_i(s,t)\frac{B_i(t_{k+1}) - B_i(t_k)}{h} - \frac{1}{2h}\Sigma_i(s,t)^2}.$$

It is easy to verify, using the independence of Brownian increments on disjoint subintervals $[t_k, t_{k+1}]$, that

$$\mathcal{E}_i^\pi(s,t_k)\mathcal{E}_i^\pi(t_k,t) = \mathcal{E}_i^\pi(s,t) \tag{3.19}$$

when $s \leq t_k \leq t$ for some $k \in \{1, ..., N-1\}$ and $s, t \in [0, T]$.
A key role in the following will be played by the notion of *Wick product* (see section 2.3.6). For the sake of completeness we briefly remind two particular cases of proposition 101 and theorem 100

- if X belongs to the Sobolev-Malliavin space $\mathbb{D}^{1,p}$, for some $p > 1$, and $f \in L^2([0,T])$ is a deterministic function, then

$$X \diamond \int_0^T f(t)dB_i(t) := X \cdot \int_0^T f(t)dB_i(t) - D_f^{(i)}X, \tag{3.20}$$

with $D_f^{(i)}$ being the directional Mallivian derivative with respect to the i-th component of the multidimensional Brownian motion $\{B(t)\}_{t\geq 0}$ in the direction f;

- if $X \in L^p(\Omega)$, for some $p > 1$, and $s,t \in [t_k, t_{k+1}]$, for some $k \in \{1, ..., N-1\}$, we set

$$X \diamond \mathcal{E}_i^\pi(s,t) := \mathtt{T}_{-\sigma_i,k}X \cdot \mathcal{E}_i^\pi(s,t), \tag{3.21}$$

where $\mathtt{T}_{-\sigma_i,k}$ stands for the *translation* operator

$$(\mathtt{T}_{-\sigma_i,k}X)(\omega) := X\left(\omega - \epsilon_i \frac{\Sigma_i(s,t)}{h} \int_0^\cdot \mathbf{1}_{[t_k,t_{k+1}]}(r)dr\right). \tag{3.22}$$

Here, $\{\epsilon_1, ..., \epsilon_d\}$ denotes the canonical basis of $\mathbb{R}^d$ (recall that we are working with a d-dimensional Brownian motion and hence $\mathtt{T}_{-\sigma_i,k}$ acts only on the i-th component of $\{B(t)\}_{t\in[0,T]}$).

We observe that both definitions (3.20) and (3.21) are actually consequences of the general definition of Wick product: the first one being related to the interplay between Wick product and Skorohod integral (e.g. theorem 94)and the latter being nothing else than Gjessing's Lemma (recall the use we made of that in the introduction).
It is known (see for instance [16, Theorem 14.1 (vi)]) that the translation operator maps $L^p(\Omega)$ into $L^q(\Omega)$, for all $q < p$; therefore, since $\mathcal{E}_i^\pi(s,t) \in L^p(\Omega)$ for any $p \geq 1$, we conclude that $X \diamond \mathcal{E}_i^\pi(s,t)$ belongs to $L^q(\Omega)$, for all $q < p$. It is immediate to verify using definition (3.21) that

$$\mathcal{E}_i^\pi(s,t) \diamond \mathcal{E}_j^\pi(s,t) = \mathcal{E}_i^\pi(s,t) \cdot \mathcal{E}_j^\pi(s,t), \quad \text{if } i \neq j,$$

and

$$\mathcal{E}_i^\pi(s,t_k) \diamond \mathcal{E}_i^\pi(t_k,t) = \mathcal{E}_i^\pi(s,t_k) \cdot \mathcal{E}_i^\pi(t_k,t) = \mathcal{E}_i^\pi(s,t), \quad \text{if } s \leq t_k \leq t \leq t_{k+1}.$$

By means of the last identity we can extend definition (3.21) to the case where s and t do not necessarily belong to the same subinterval $[t_k, t_{k+1}]$. In fact, assume that $t_{k-1} \leq s \leq t_k \leq t \leq t_{k+1}$: then,

$$\begin{aligned} X \diamond \mathcal{E}_i^\pi(s,t) :=& (X \diamond \mathcal{E}_i^\pi(s,t_k)) \diamond \mathcal{E}_i^\pi(t_k,t) \\ =& (\mathtt{T}_{-\sigma_i,k-1}X \cdot \mathcal{E}_i^\pi(s,t_k)) \diamond \mathcal{E}_i^\pi(t_k,t) \\ =& \mathtt{T}_{-\sigma_i,k}(\mathtt{T}_{-\sigma_i,k-1}X \cdot \mathcal{E}_i^\pi(s,t_k)) \cdot \mathcal{E}_i^\pi(t_k,t) \\ =& \mathtt{T}_{-\sigma_i,k}\mathtt{T}_{-\sigma_i,k-1}X \cdot \mathcal{E}_i^\pi(s,t_k) \cdot \mathcal{E}_i^\pi(t_k,t) \\ =& \mathtt{T}_{-\sigma_i,k}\mathtt{T}_{\sigma_i,k-1}X \cdot \mathcal{E}_i^\pi(s,t). \end{aligned}$$

The transformation (3.21) inherits from the translation operator a monotonicity property:

$$\text{if } X \leq Y, \text{ then } X \diamond \mathcal{E}_i^{\pi}(s,t) \leq Y \diamond \mathcal{E}_i^{\pi}(s,t).$$

In particular,

$$|X \diamond \mathcal{E}_i^{\pi}(s,t)| \leq |X| \diamond \mathcal{E}_i^{\pi}(s,t). \tag{3.23}$$

We are now able to formalize the solution concept that we utilize for solving (3.11).

Definition 111. A d-dimensional stochastic process $\{X^{\pi}(t)\}_{t\in[0,T]}$ is said to be a *mild* solution of equation (3.11) if:

1. the function $t \mapsto X^{\pi}(t)$ is almost surely continuous;
2. for $i = 1, ..., d$ and $t \in [0,T]$, the random variable $X_i(t)$ belongs to $L^p(\Omega)$ for some $p > 1$;
3. for $i = 1, ..., d$, the identity

   $$X_i^{\pi}(t) = c_i \mathcal{E}_i^{\pi}(0,t) + \int_0^t b_i(s, X^{\pi}(s)) \diamond \mathcal{E}_i^{\pi}(s,t) ds, \quad t \in [0,T], \tag{3.24}$$

   holds almost surely.

Remark 112. The way one can go from (3.11) to (3.24) is pretty similar to the reduction method described in the introduction for the scalar case. Namely, if we Wick-multiply by $\mathtt{E}_i^{\pi}(0,t)$ both sides of

$$\frac{dX_i^{\pi}(t)}{dt} = b_i(t, X^{\pi}(t)) + \sigma_i(t) X_i^{\pi}(t) \diamond \frac{dB_i^{\pi}(t)}{dt},$$

and employ the properties of Wick product mentioned there, we will end up with the corresponding multidimensional analogue of (3.7), i.e.

$$\frac{d\mathcal{X}_i^{\pi}(t)}{dt} = b_i(t, \mathcal{X}^{\pi}(t) \diamond \mathcal{E}_i^{\pi}(0,t)) \diamond \mathtt{E}_i^{\pi}(0,t), \tag{3.25}$$

where

$$\mathcal{X}_i^{\pi}(t) := X_i^{\pi}(t) \diamond \mathtt{E}_i^{\pi}(0,t), \quad t \in [0,T].$$

We now write (3.25) in the integral form

$$\mathcal{X}_i^{\pi}(t) = c_i + \int_0^t b_i(s, \mathcal{X}^{\pi}(s) \diamond \mathcal{E}_i^{\pi}(0,s)) \diamond \mathtt{E}_i^{\pi}(0,s) ds;$$

this identity together with

$$X_i^\pi(t) = \mathcal{X}_i^\pi(t) \diamond \mathcal{E}_i^\pi(0,t),$$

gives

$$X_i^\pi(t) \diamond \mathtt{E}_i^\pi(0,t) = c_i + \int_0^t b_i(s, X_i^\pi(t)) \diamond \mathtt{E}_i^\pi(0,s)ds.$$

If we now Wick-multiply both sides above by $\mathcal{E}_i^\pi(0,t)$, we obtain (3.24). We recall that the application of Gjessing's Lemma here doesn't reduce the previous equation to a random ordinary differential equation and hence to prove the existence of a solution for (3.24) we have to resort to the technique described in the next section.

3.3 Proof of Theorem 107

3.3.1 An auxiliary semilinear hyperbolic system of PDEs

To prove the existence of a mild solution for equation (3.11) we introduce the following auxiliary semilinear hyperbolic system of partial differential equations

$$\begin{cases} \partial_t u_i(t,x) = -\sigma_i(t)\partial_{x_i} u_i(t,x) + \sigma_i(t)\frac{x_i}{h} u_i(t,x) + b_i(t, u(t,x)), \\ \qquad \text{for } t \in]r,R], x \in \mathbb{R}^d \text{ and } i = 1, ..., d; \\ u_i(r,x) = \alpha_i, \quad \text{for } x \in \mathbb{R}^d \text{ and } i = 1, ..., d, \end{cases} \tag{3.26}$$

where $\alpha_1, ..., \alpha_d$ are constant initial conditions and h denotes the mesh of the partition under consideration. The validity of Assumption 2 implies the existence of a unique classical solution for the Cauchy problem (3.26) (see for instance [52] and [53]).
Now, if u solves (3.26), then from the trivial identity

$$\partial_{x_i}\left(u_i(t,x)e^{-\frac{|x|^2}{2h}}\right) = \partial_{x_i} u_i(t,x)e^{-\frac{|x|^2}{2h}} - \frac{x_i}{h} u_i(t,x)e^{-\frac{|x|^2}{2h}},$$

we can argue that the function

$$v(t,x) := u(t,x)e^{-\frac{|x|^2}{2h}}, \quad t \in [r,R], x \in \mathbb{R}^d, \tag{3.27}$$

is a classical solution of

$$\begin{cases} \partial_t v_i(t,x) = -\sigma_i(t)\partial_{x_i} v_i(t,x) + b_i\left(t, v(t,x)e^{\frac{|x|^2}{2h}}\right)e^{-\frac{|x|^2}{2h}}, \\ \qquad \text{for } t \in]r,R], x \in \mathbb{R}^d \text{ and } i = 1, ..., d; \\ v_i(r,x) = \alpha_i e^{-\frac{|x|^2}{2h}}, \quad \text{for } x \in \mathbb{R}^d \text{ and } i = 1, ..., d. \end{cases} \tag{3.28}$$

Rewriting system (3.28) in the mild form

$$\begin{cases} v_i(t,x) = \alpha_i e^{-\frac{|x-\Sigma_i(r,t)\epsilon_i|^2}{2h}} \\ \qquad\qquad + \int_r^t b_i\left(t, v(s, x-\Sigma_i(s,t)\epsilon_i) e^{\frac{|x-\Sigma_i(s,t)\epsilon_i|^2}{2h}}\right) e^{-\frac{|x-\Sigma_i(s,t)\epsilon_i|^2}{2h}} ds \\ \text{for } t \in [r,R], x \in \mathbb{R}^d \text{ and } i = 1, ..., d, \end{cases}$$

(recall the definition of $\Sigma_i(s,t)$ in (3.17)) and using identity (3.27), we obtain that u solves

$$\begin{cases} u_i(t,x) e^{-\frac{|x|^2}{2h}} = \alpha_i e^{-\frac{|x-\Sigma_i(r,t)\epsilon_i|^2}{2h}} \\ \qquad\qquad + \int_r^t b_i\left(t, u(s, x-\Sigma_i(s,t)\epsilon_i)\right) e^{-\frac{|x-\Sigma_i(s,t)\epsilon_i|^2}{2h}} ds \\ \text{for } t \in [r,R], x \in \mathbb{R}^d \text{ and } i = 1, ..., d, \end{cases}$$

or equivalently,

$$\begin{cases} u_i(t,x) = \alpha_i e^{\frac{x_i}{h}\Sigma_i(r,t) - \frac{1}{2h}\Sigma_i(r,t)^2} \\ \qquad\qquad + \int_r^t b_i\left(t, u(s, x-\Sigma_i(s,t)\epsilon_i)\right) e^{\frac{x_i}{h}\Sigma_i(s,t) - \frac{1}{2h}\Sigma_i(s,t)^2} ds \\ \text{for } t \in [r,R], x \in \mathbb{R}^d \text{ and } i = 1, ..., d. \end{cases} \tag{3.29}$$

Note that from the previous identity we get the estimate

$$\begin{aligned} |u_i(t,x)| \le & |\alpha_i| e^{\frac{x_i}{h}\Sigma_i(r,t) - \frac{1}{2h}\Sigma_i(r,t)^2} \\ & + \int_r^t \left|b_i\left(t, u(s, x-\Sigma_i(s,t)\epsilon_i)\right)\right| e^{\frac{x_i}{h}\Sigma_i(s,t) - \frac{1}{2h}\Sigma_i(s,t)^2} ds \\ \le & |\alpha_i| e^{\frac{x_i}{h}\Sigma_i(r,t) - \frac{1}{2h}\Sigma_i(r,t)^2} + M \int_r^t e^{\frac{x_i}{h}\Sigma_i(s,t) - \frac{1}{2h}(\Sigma_i(s,t)^2} ds \\ \le & |\alpha_i| e^{\frac{x_i}{h}\Sigma_i(r,t)} + M \int_r^t e^{\frac{x_i}{h}\Sigma_i(s,t)} ds. \end{aligned} \tag{3.30}$$

Here, M denotes a positive constant satisfying $|b_i(t,x)| \le M$, for all $t \in [0,T]$, $x \in \mathbb{R}^d$ and $i = 1, ..., d$.

3.3.2 Construction of a mild solution for 3.11

In the sequel, in order to stress the dependence on specific initial conditions, we will write

$$u(t,x;r,\alpha) = (u_1(t,x;r,\alpha), ..., u_d(t,x;r,\alpha))^*, \quad t \in [r,R], x \in \mathbb{R}^d$$

to denote the unique classical solution of (3.26). We define the process $\{X^\pi(t)\}_{t\in[0,T]}$ inductively:

$$X^\pi(t) := \begin{cases} u(t, B(t_1); 0, c), & \text{if } t \in [0, t_1]; \\ u(t, B(t_2) - B(t_1); t_1, X^\pi(t_1)), & \text{if } t \in]t_1, t_2]; \\ \quad \cdots & \quad \cdots \\ u(t, B(T) - B(t_{N-1}); t_{N-1}, X^\pi(t_{N-1})), & \text{if } t \in]t_{N-1}, T]. \end{cases} \tag{3.31}$$

We now verify that $X(t)$ is a mild solution of (3.11), that is we check the conditions of Definition 111.

The almost sure continuity of $t \in [0, T] \mapsto X(t)$ follows immediately from the continuity of $t \in [r, T] \mapsto u(t, x; r, \alpha)$, for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$ (u is a classical solution of (3.26)) and the fact that for all $k \in \{1, ..., N-1\}$ we have by construction

$$\lim_{t\to t_k^-} X^\pi(t) = \lim_{t\to t_k^+} X^\pi(t).$$

We now verify that $X^\pi(t) \in L^p(\Omega)$, for some $p > 1$ and all $t \in [0, T]$. If $t \in [0, t_1]$, then by the definition of $X^\pi(t)$ and estimate (3.30) we can write

$$\begin{aligned} |X_i^\pi(t)| =& |u(t, B(t_1); 0, c)| \\ \leq& |c_i| e^{\frac{B_i(t_1)}{h}\Sigma_i(0,t)} + M \int_0^t e^{\frac{B_i(t_1)}{h}\Sigma_i(s,t)} ds, \end{aligned}$$

and hence

$$\begin{aligned} \|X_i^\pi(t)\|_{L^p(\Omega)} \leq& |c_i| \|e^{\frac{B_i(t_1)}{h}\Sigma_i(0,t)}\|_{L^p(\Omega)} + M \int_0^t \|e^{\frac{B_i(t_1)}{h}\Sigma_i(s,t)}\|_{L^p(\Omega)} ds \\ =& |c_i| e^{p\frac{\Sigma_i(0,t)^2}{2h}} + M \int_0^t e^{p\frac{\Sigma_i(s,t)^2}{2h}} ds \\ =& |c_i| e^{p\frac{\Sigma_i(0,t)^2}{2h}} + Mte^{\frac{p}{2h}\sup_{s\in[0,t]}\Sigma_i(s,t)^2}. \end{aligned} \tag{3.32}$$

This proves the membership of $X_i^\pi(t)$ to $L^p(\Omega)$, for all $i = 1, ..., d$, $t \in [0, t_1]$ and $p \geq 1$. Let us now take $t \in]t_1, t_2]$; again, by the definition of $X^\pi(t)$ and estimate (3.30) we can write

$$\begin{aligned} |X_i^\pi(t)| =& |u(t, B(t_2) - B(t_1); t_1, X^\pi(t_1))| \\ \leq& |X_i^\pi(t_1)| e^{\frac{B_i(t_2)-B_i(t_1)}{h}\Sigma_i(t_1,t)} + M \int_{t_1}^t e^{\frac{B_i(t_2)-B_i(t_1)}{h}\Sigma_i(s,t)} ds, \end{aligned}$$

and hence, using Hölder inequality,

$$\begin{aligned}\|X_i^\pi(t)\|_{L^p(\Omega)} \leq & \||X_i^\pi(t_1)|e^{\frac{B_i(t_2)-B_i(t_1)}{h}\Sigma_i(t_1,t)}\|_{L^p(\Omega)} + M\int_{t_1}^t \|e^{\frac{B_i(t_2)-B_i(t_1)}{h}\Sigma_i(s,t)}\|_{L^p(\Omega)}ds \\ \leq & \|X_i^\pi(t_1)\|_{L^q(\Omega)}\|e^{\frac{B_i(t_2)-B_i(t_1)}{h}\Sigma_i(t_1,t)}\|_{q'} + M\int_{t_1}^t \|e^{\frac{B_i(t_2)-B_i(t_1)}{h}\Sigma_i(s,t)}\|_{L^p(\Omega)}ds \\ \leq & \|X_i^\pi(t_1)\|_{L^q(\Omega)}e^{q'\frac{\Sigma_i(t_1,t)^2}{2h}} + M\int_{t_1}^t e^{p\frac{\Sigma_i(s,t)^2}{2h}}ds \\ \leq & \|X_i^\pi(t_1)\|_{L^q(\Omega)}e^{q'\frac{\Sigma_i(t_1,t)^2}{2h}} + M(t-t_1)e^{\frac{p}{2h}\sup_{s\in[t_1,t]}\Sigma_i(s,t)^2}.\end{aligned}$$

This last estimate combined with (3.32) provides the desired upper bound for $\|X_i^\pi(t)\|_{L^p(\Omega)}$ on the interval $]t_1, t_2]$. It also clear that in a similar manner one obtains analogous estimates for the $L^p(\Omega)$-norm of $X_i(t)$ on any subinterval $]t_k, t_{k+1}]$ for $k = 2, ..., N-1$.

We are left with the verification that $\{X^\pi(t)\}_{t\in[0,T]}$ as defined in (3.31) satisfies identity (3.24). To this aim we prove the following auxiliary result.

Proposition 113. *Identity* (3.24) *is equivalent to*

$$X_i^\pi(t) = X_i^\pi(t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1}, t) + \int_{t_{k-1}}^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t)ds, \quad t \in [t_{k-1}, t_k], \tag{3.33}$$

for all $k \in \{1, ..., N\}$.

Proof. Assume identity (3.24) to be true; then, for $t \in]t_{k-1}, t_k]$ we can write

$$\begin{aligned}
X_i^\pi(t) =& c_i \mathcal{E}_i^\pi(0,t) + \int_0^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds \\
=& c_i \mathcal{E}_i^\pi(0,t) + \int_0^{t_{k-1}} b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds + \int_{t_{k-1}}^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds \\
=& c_i \mathcal{E}_i^\pi(0,t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1},t) + \int_0^{t_{k-1}} b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1},t) ds \\
&+ \int_{t_{k-1}}^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds \\
=& c_i \mathcal{E}_i^\pi(0,t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1},t) + \left(\int_0^{t_{k-1}} b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t_{k-1}) ds \right) \diamond \mathcal{E}_i^\pi(t_{k-1},t) \\
&+ \int_{t_{k-1}}^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds \\
=& \left(c_i \mathcal{E}_i^\pi(0,t_{k-1}) + \int_0^{t_{k-1}} b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t_{k-1}) ds \right) \diamond \mathcal{E}_i^\pi(t_{k-1},t) \\
&+ \int_{t_{k-1}}^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds \\
=& X_i^\pi(t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1},t) + \int_{t_{k-1}}^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds.
\end{aligned}$$

This proves (3.33). If we now start from (3.33) and replace iteratively $X_i^\pi(t_{k-1})$ with

$$X_i^\pi(t_{k-2}) \diamond \mathcal{E}_i^\pi(t_{k-2}, t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds,$$

and then replace $X_i^\pi(t_{k-2})$ with

$$X_i^\pi(t_{k-3}) \diamond \mathcal{E}_i^\pi(t_{k-3}, t_{k-2}) + \int_{t_{k-3}}^{t_{k-2}} b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) ds,$$

and so on, we will end up with (3.24). □

Remark 114. We observe that according to the definition of $X^\pi(t)$ in (3.31), for any $k \in \{1, ..., N\}$ and $t \leq t_{k-1}$ the random vector $X^\pi(t)$ depends only on the Brownian increments on the intervals $[0, t_1]$,...,$[t_{k-2}, t_{k-1}]$. Therefore, the term

$$X_i^\pi(t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1}, t), \quad t \in]t_{k-1}, t_k]$$

in (3.33) can be rewritten for our particular mild solution as

$$X_i^\pi(t_{k-1}) \mathcal{E}_i^\pi(t_{k-1}, t), \quad t \in]t_{k-1}, t_k].$$

In fact, according to (3.21) one has

$$\begin{aligned} X_i^\pi(t_{k-1}) \diamond \mathcal{E}_i^\pi(t_{k-1}, t) &= \mathtt{T}_{-\sigma_i,k-1} X_i^\pi(t_{k-1}) \mathcal{E}_i^\pi(t_{k-1}, t) \\ &= X_i^\pi(t_{k-1}) \mathcal{E}_i^\pi(t_{k-1}, t). \end{aligned}$$

(The translation acts on a part of Brownian path which is disjoint from the increments on which $X_i^\pi(t_{k-1})$ depends).

We are now ready to prove that $X^\pi(t)$ defined in (3.31) verifies identity (3.24) through the equivalent equalities (3.33). Let $t \in [0, t_1]$; then, identity (3.29) and definition (3.31) give

$$\begin{aligned} X_i^\pi(t) =& u_i(t, B(t_1); 0, c) \\ =& c_i e^{\frac{B_i(t_1)}{h}\Sigma_i(0,t) - \frac{1}{2h}\Sigma_i(0,t)^2} \\ &+ \int_0^t b_i\left(t, u(s, B(t_1) - \Sigma_i(s,t)e_i; 0, c)\right) e^{\frac{B_i(t_1)}{h}\Sigma_i(s,t) - \frac{1}{2h}\Sigma_i(s,t)^2} ds \\ =& c_i \mathcal{E}_i^\pi(0,t) + \int_0^t b_i\left(t, u(s, B(t_1) - \Sigma_i(s,t)e_i; 0, c)\right) \mathcal{E}_i^\pi(s,t) ds \\ =& c_i \mathcal{E}_i^\pi(0,t) + \int_0^t \mathtt{T}_{-\sigma_i,0} b_i\left(t, u(s, B(t_1); 0, c)\right) \mathcal{E}_i^\pi(s,t) ds \\ =& c_i \mathcal{E}_i^\pi(0,t) + \int_0^t b_i\left(t, u(s, B(t_1); 0, c)\right) \diamond \mathcal{E}_i^\pi(s,t) ds \\ =& c_i \mathcal{E}_i^\pi(0,t) + \int_0^t b_i\left(t, X^\pi(s)\right) \diamond \mathcal{E}_i^\pi(s,t) ds. \end{aligned}$$

This corresponds to (3.33) for $t \in [0, t_1]$. Let us now consider the general subinterval

$]t_k, t_{k+1}]$, with $k \in \{1, ..., N-1\}$; identity (3.29) and definition (3.31) give

$$\begin{aligned}
X_i^\pi(t) =& u_i(t, B(t_{k+1}) - B(t_k); t_k, X^\pi(t_k)) \\
=& X_i^\pi(t_k) e^{\frac{B_i(t_{k+1}) - B_i(t_k)}{h}\Sigma_i(t_k,t) - \frac{1}{2h}\Sigma_i(t_k,t)^2} \\
& + \int_{t_k}^t b_i\left(t, u(s, B(t_{k+1}) - B(t_k) - \Sigma_i(s,t)\epsilon_i; t_k, X^\pi(t_k))\right) \times \\
& \times e^{\frac{B(t_{k+1}) - B(t_k)}{h}\Sigma_i(s,t) - \frac{1}{2h}\Sigma_i(s,t)^2} ds \\
=& X_i^\pi(t_k)\mathcal{E}_i^\pi(t_k, t) \\
& + \int_{t_k}^t b_i\left(t, u(s, B(t_{k+1}) - B(t_k) - \Sigma_i(s,t)\epsilon_i; t_k, X^\pi(t_k))\right)\mathcal{E}_i^\pi(s,t) ds \\
=& X_i^\pi(t_k)\mathcal{E}_i^\pi(t_k, t) + \int_{t_k}^t \mathtt{T}_{-\sigma_i,k} b_i\left(t, u(s, B(t_{k+1}) - B(t_k); t_k, X^\pi(t_k))\right)\mathcal{E}_i^\pi(s,t) ds \\
=& X_i^\pi(t_k)\mathcal{E}_i^\pi(t_k, t) + \int_{t_k}^t b_i\left(t, u(s, B(t_{k+1}) - B(t_k); t_k, X^\pi(t_k))\right) \diamond \mathcal{E}_i^\pi(s,t) ds \\
=& X_i^\pi(t_k)\mathcal{E}_i^\pi(t_k, t) + \int_{t_k}^t b_i\left(t, X^\pi(s)\right) \diamond \mathcal{E}_i^\pi(s,t) ds.
\end{aligned}$$

This corresponds to (3.33) and the proof is complete.

3.4 Proof of Theorem 108

Let $\varphi \in C_0^2([0,T] \times \mathbb{R}^d)$; then,

$$\begin{aligned}
0 =& \varphi(T, X^\pi(T)) - \varphi(0, c) \\
=& \sum_{k=1}^N \varphi(t_k, X^\pi(t_k)) - \varphi(t_{k-1}, X^\pi(t_{k-1})) \\
=& \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left[\partial_t \varphi(t, X^\pi(t)) + \sum_{i=1}^d \partial_{x_i}\varphi(t, X^\pi(t)) \frac{d}{dt} X_i^\pi(t)\right] dt \\
=& \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \partial_t \varphi(t, u(t, B(t_k) - B(t_{k-1}); t_{k-1}, X^\pi(t_{k-1}))) dt \\
& + \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \sum_{i=1}^d \partial_{x_i}\varphi(t, u(t, B(t_k) - B(t_{k-1}); t_{k-1}, X^\pi(t_{k-1}))) \\
& \qquad \cdots \times \partial_t u_i(t, B(t_k) - B(t_{k-1}); t_{k-1}, X^\pi(t_{k-1})) dt.
\end{aligned}$$

To ease the notation, we now suppress the explicit dependence on the initial conditions in the function u and set $Z(k) := B(t_k) - B(t_{k-1})$; therefore, the previous identity reads

$$0 = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \partial_t \varphi(t, u(t, Z(k)))dt + \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k))) \partial_t u_i(t, Z(k))dt. \tag{3.34}$$

We recall that u_i is a classical solution of (3.26) and hence we get

$$\partial_t u_i(t, Z(k)) = -\sigma_i(t)\partial_{x_i} u_i(t, Z(k)) + \sigma_i(t)\frac{Z_i(k)}{h} u_i(t, Z(k)) + b_i(t, u(t, Z(k))).$$

Substituting this identity into (3.34) yields

$$\begin{aligned}
0 =& \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \partial_t \varphi(t, u(t, Z(k)))dt \\
&- \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k)))\sigma_i(t)\partial_{x_i} u_i(t, Z(k))dt \\
&+ \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k)))\sigma_i(t)\frac{Z_i(k)}{h} u_i(t, Z(k))dt \\
&+ \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k)))b_i(t, u(t, Z(k)))dt \\
=& \mathcal{A} - \mathcal{B} + \mathcal{C} + \mathcal{D},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A} &:= \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \partial_t \varphi(t, u(t, Z(k)))dt, \\
\mathcal{B} &:= \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k)))\sigma_i(t)\partial_{x_i} u_i(t, Z(k))dt, \\
\mathcal{C} &:= \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k)))\sigma_i(t)\frac{Z_i(k)}{h} u_i(t, Z(k))dt, \\
\mathcal{D} &:= \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \partial_{x_i} \varphi(t, u(t, Z(k)))b_i(t, u(t, Z(k)))dt.
\end{aligned}$$

We now take the expectation of the first and last members above and get

$$0 = \mathbb{E}[\mathcal{A}] - \mathbb{E}[\mathcal{B}] + \mathbb{E}[\mathcal{C}] + \mathbb{E}[\mathcal{D}]. \tag{3.35}$$

Let us analyse $\mathbb{E}[\mathcal{C}]$:

$$\begin{aligned}\mathbb{E}[\mathcal{C}] &= \sum_{k=1}^{N}\sum_{i=1}^{d}\int_{t_{k-1}}^{t_k} \mathbb{E}\left[\partial_{x_i}\varphi(t,u(t,Z(k)))\sigma_i(t)\frac{Z_i(k)}{h}u_i(t,Z(k))\right]dt \\ &= \sum_{k=1}^{N}\sum_{i=1}^{d}\int_{t_{k-1}}^{t_k} \mathbb{E}\left[\mathbb{E}\left[\partial_{x_i}\varphi(t,u(t,Z(k)))\sigma_i(t)\frac{Z_i(k)}{h}u_i(t,Z(k))\Big|\mathcal{F}^B_{t_{k-1}}\right]\right]dt;\end{aligned} \tag{3.36}$$

here $\{\mathcal{F}^B_t\}_{t\in[0,T]}$ stands for the natural filtration of the Brownian motion $\{B(t)\}_{t\in[0,T]}$. We remark that $u(t,Z(k))$ depends implicitly also on the increments $Z(1),....,Z(k-1)$ through the initial condition; however, these increments are measurable with respect to the sigma-algebra $\mathcal{F}^B_{t_{k-1}}$. Therefore, the conditional expectation can be computed as follows

$$\begin{aligned}&\mathbb{E}\left[\partial_{x_i}\varphi(t,u(t,Z(k)))\sigma_i(t)\frac{Z_i(k)}{h}u_i(t,Z(k))\Big|\mathcal{F}^B_{t_{k-1}}\right] \\ &= \int_{\mathbb{R}^d}\partial_{x_i}\varphi(t,u(t,x))\sigma_i(t)\frac{x_i}{h}u_i(t,x)\frac{e^{-|x|^2/2h}}{(2\pi h)^{d/2}}dx \\ &= -\int_{\mathbb{R}^d}\partial_{x_i}\varphi(t,u(t,x))\sigma_i(t)u_i(t,x)\partial_{x_i}\left(\frac{e^{-|x|^2/2h}}{(2\pi h)^{d/2}}\right)dx \\ &= \int_{\mathbb{R}^d}\sigma_i(t)\partial_{x_i}\left(\partial_{x_i}\varphi(t,u(t,x))u_i(t,x)\right)\frac{e^{-|x|^2/2h}}{(2\pi h)^{d/2}}dx \\ &= \sum_{j=1}^{d}\int_{\mathbb{R}^d}\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,x))\partial_{x_i}u_j(t,x)u_i(t,x)\frac{e^{-|x|^2/2h}}{(2\pi h)^{d/2}}dx \\ &\quad + \int_{\mathbb{R}^d}\sigma_i(t)\partial_{x_i}\varphi(t,u(t,x))\partial_{x_i}u_i(t,x)\frac{e^{-|x|^2/2h}}{(2\pi h)^{d/2}}dx \\ &= \mathbb{E}\left[\sum_{j=1}^{d}\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_j(t,Z(k))u_i(t,Z(k))\Big|\mathcal{F}^B_{t_{k-1}}\right] \\ &\quad + \mathbb{E}\left[\sigma_i(t)\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_i(t,Z(k))\Big|\mathcal{F}^B_{t_{k-1}}\right];\end{aligned}$$

in the third equality we performed an integration by parts. Inserting the last expression in (3.36) gives

$$\begin{aligned}\mathbb{E}[\mathcal{C}] &= \sum_{k=1}^{N}\sum_{i=1}^{d}\int_{t_{k-1}}^{t_k}\mathbb{E}\left[\mathbb{E}\left[\partial_{x_i}\varphi(t,u(t,Z(k)))\sigma_i(t)\frac{Z_i(k)}{h}u_i(t,Z(k))\middle|\mathcal{F}^B_{t_{k-1}}\right]\right]dt\\ &= \sum_{k=1}^{N}\sum_{i=1}^{d}\int_{t_{k-1}}^{t_k}\mathbb{E}\left[\sum_{j=1}^{d}\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_j(t,Z(k))u_i(t,Z(k))\right]dt\\ &\quad+\sum_{k=1}^{N}\sum_{i=1}^{d}\int_{t_{k-1}}^{t_k}\mathbb{E}\left[\sigma_i(t)\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_i(t,Z(k))\right]dt.\end{aligned}$$

Note that last term above coincides with $\mathbb{E}[\mathcal{B}]$ which appear with a negative sign in (3.35); hence,

$$\begin{aligned}&-\mathbb{E}[\mathcal{B}]+\mathbb{E}[\mathcal{C}]\\ &\quad=\sum_{k=1}^{N}\sum_{i,j=1}^{d}\int_{t_{k-1}}^{t_k}\mathbb{E}\left[\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_j(t,Z(k))u_i(t,Z(k))\right]dt.\end{aligned}$$

Before recollecting all the parts of our computation, we make a further step; if we denote by $\mathcal{G}^{(k)}_{i,t}$ the sigma algebra generated by the random variable $u_i(t,Z(k))$, for $t\in[t_{k-1},t_k]$, $k=1,...,N$ and $i=1,...,d$, we can rewrite the expectation inside the integral above as

$$\begin{aligned}&\mathbb{E}\left[\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_j(t,Z(k))u_i(t,Z(k))\right]\\ &\quad=\mathbb{E}\left[\mathbb{E}\left[\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))\partial_{x_i}u_j(t,Z(k))u_i(t,Z(k))\middle|\mathcal{G}^{(k)}_{i,t}\right]\right]\\ &\quad=\mathbb{E}\left[\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))u_i(t,Z(k))\mathbb{E}\left[\partial_{x_i}u_j(t,Z(k))\middle|\mathcal{G}^{(k)}_{i,t}\right]\right]\\ &\quad=\mathbb{E}\left[\sigma_i(t)\partial_{x_j}\partial_{x_i}\varphi(t,u(t,Z(k)))u_i(t,Z(k))g^{(k)}_{ij}(t,u_i(t,Z(k)))\right],\end{aligned}$$

where $g^{(k)}_{ij}:[t_{k-1},t_k]\times\mathbb{R}\to\mathbb{R}$ is a measurable function, whose existence is guaranteed by Doob's Lemma, chosen to satisfy

$$g^{(k)}_{ij}(t,u_i(t,Z(k)))=\mathbb{E}\left[\partial_{x_i}u_j(t,Z(k))\middle|\mathcal{G}^{(k)}_{i,t}\right]. \tag{3.37}$$

Now, starting from (3.35) and using the last two identities we obtain

$$
\begin{aligned}
0 =&\mathbb{E}[\mathcal{A}] - \mathbb{E}[\mathcal{B}] + \mathbb{E}[\mathcal{C}] + \mathbb{E}[\mathcal{D}] \\
=&\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \mathbb{E}[\partial_t \varphi(t, u(t, Z(k)))] dt \\
&+ \sum_{k=1}^{N} \sum_{i,j=1}^{d} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[\sigma_i(t) \partial_{x_j} \partial_{x_i} \varphi(t, u(t, Z(k))) u_i(t, Z(k)) g_{ij}^{(k)}(t, u_i(t, Z(k)))\right] dt \\
&+ \sum_{k=1}^{N} \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \mathbb{E}[\partial_{x_i} \varphi(t, u(t, Z(k))) b_i(t, u(t, Z(k)))] dt \\
=&\int_0^T \mathbb{E}[\partial_t \varphi(t, X^\pi(t))] dt \\
&+ \sum_{k=1}^{N} \sum_{i,j=1}^{d} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[\sigma_i(t) \partial_{x_j} \partial_{x_i} \varphi(t, X^\pi(t)) X_i^\pi(t) g_{ij}^{(k)}(t, X_i^\pi(t))\right] dt \\
&+ \int_0^T \sum_{i=1}^{d} \mathbb{E}[\partial_{x_i} \varphi(t, X^\pi(t)) b_i(t, X^\pi(t))] dt \\
=&\int_0^T \mathbb{E}[\partial_t \varphi(t, X^\pi(t))] dt \\
&+ \sum_{i,j=1}^{d} \int_0^T \mathbb{E}\left[\sigma_i(t) \partial_{x_j} \partial_{x_i} \varphi(t, X^\pi(t)) X_i^\pi(t) g_{ij}(t, X_i^\pi(t))\right] dt \\
&+ \int_0^T \sum_{i=1}^{d} \mathbb{E}[\partial_{x_i} \varphi(t, X^\pi(t)) b_i(t, X^\pi(t))] dt,
\end{aligned}
$$

where $g_{ij} : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$
g_{ij}(t, y) := g_{ij}^{(k)}(t, y), \qquad \text{if } t \in [t_{k-1}, t_k]. \tag{3.38}
$$

Observe that the last member above contains expectations of functions of the random vector $X^\pi(t)$, for $t \in [0, T]$; therefore, writing the law of this random vector as

$$
\mu^\pi(t, A) := P(X^\pi(t) \in A), \quad A \in \mathcal{B}(\mathbb{R}^d),
$$

we can write

$$
\begin{aligned}
0 =& \int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(t,x) d\mu^\pi(t,x)dt + \sum_{i,j=1}^d \int_0^T \int_{\mathbb{R}^d} \sigma_i(t) \partial_{x_j} \partial_{x_i} \varphi(t,x) x_i g_{ij}(t,x_i) d\mu^\pi(t,x)dt \\
&+ \sum_{i=1}^d \int_0^T \int_{\mathbb{R}^d} \partial_{x_i} \varphi(t,x) b_i(t,x) d\mu^\pi(t,x)dt \\
=& \int_0^T \int_{\mathbb{R}^d} \left[\partial_t \varphi(t,x) + \sum_{i,j=1}^d \sigma_i(t) \partial^2_{x_i x_j} \varphi(t,x) x_i g_{ij}(t,x_i) + \langle b(t,x), \nabla \varphi(t,x) \rangle \right] d\mu^\pi(t,x)dt.
\end{aligned}
$$

The last equalities hold for any test function $\varphi \in C_0^2([0,T] \times \mathbb{R}^d)$ and this completes the proof of Theorem 108.

3.5 Proof of Theorem 109

The aim of this section is to prove that the mild solution of

$$
\begin{cases}
\frac{dX_i^\pi(t)}{dt} = b_i(t, X^\pi(t)) + \sigma_i(t) X_i^\pi(t) \diamond \frac{dB_i^\pi(t)}{dt}, \\
\qquad \text{for } t \in]0,T] \text{ and } i = 1, ..., d; \\
X_i^\pi(0) = c_i \in \mathbb{R}, \quad \text{for } i = 1, ..., d,
\end{cases}
\tag{3.39}
$$

as defined in (3.31), converges in $L^1(\Omega)$ to the unique strong solution of the Itô SDE

$$
\begin{cases}
dX_i(t) = b_i(t, X(t))dt + \sigma_i(t) X_i(t) dB_i(t), \\
\qquad \text{for } t \in]0,T] \text{ and } i = 1, ..., d; \\
X_i(0) = c_i \in \mathbb{R}, \quad \text{for } i = 1, ..., d.
\end{cases}
\tag{3.40}
$$

First of all, by means of the Itô formula we rewrite equation (3.40) in a form that resembles identity (3.24). In fact, setting

$$
\mathtt{E}_i(s,t) := e^{-\int_s^t \sigma_i(r) dB_i(r) - \frac{1}{2} \int_s^t \sigma_i(r)^2 dr}, \quad 0 \le s \le t \le T,
$$

and

$$
\mathcal{E}_i(s,t) := e^{\int_s^t \sigma_i(r) dB_i(r) - \frac{1}{2} \int_s^t \sigma_i(r)^2 dr}, \quad 0 \le s \le t \le T,
$$

we write

$$
\begin{aligned}
d\left(X_i(t) \diamond \mathtt{E}_i(0,t) \right) =& d(\mathtt{T}_{\sigma_i} X_i(t) \cdot \mathtt{E}_i(0,t)) \\
=& \mathtt{E}_i(0,t) \cdot d\mathtt{T}_{\sigma_i} X_i(t) + \mathtt{T}_{\sigma_i} X_i(t) \cdot d\mathtt{E}_i(0,t) \\
&+ d\mathtt{T}_{\sigma_i} X_i(t) \cdot d\mathtt{E}_i(0,t).
\end{aligned}
$$

Now,

$$
\begin{aligned}
d\mathtt{T}_{\sigma_i}X_i(t) &= [b_i(t,\mathtt{T}_{\sigma_i}X(t)) + \sigma_i(t)^2\mathtt{T}_{\sigma_i}X_i(t)]dt + \sigma_i(t)\mathtt{T}_{\sigma_i}X_i(t)dB_i(t),\\
d\mathtt{E}_i(0,t) &= -\sigma_i(t)\mathtt{E}_i(0,t)dB_i(t),
\end{aligned}
$$

and hence

$$
\begin{aligned}
&d\left(X_i(t)\diamond \mathtt{E}_i(0,t)\right)\\
&\quad= [b_i(t,\mathtt{T}_{\sigma_i}X(t))\mathtt{E}_i(0,t) + \sigma_i(t)^2\mathtt{T}_{\sigma_i}X_i(t)\mathtt{E}_i(0,t)]dt + \sigma_i(t)\mathtt{T}_{\sigma_i}X_i(t)\mathtt{E}_i(0,t)dB_i(t)\\
&\qquad - \sigma_i(t)\mathtt{T}_{\sigma_i}X_i(t)\mathtt{E}_i(0,t)dB_i(t) - \sigma_i(t)^2\mathtt{T}_{\sigma_i}X_i(t)\mathtt{E}_i(0,t)dt\\
&\quad= b_i(t,\mathtt{T}_{\sigma_i}X(t))\mathtt{E}_i(0,t)dt\\
&\quad= b_i(t,X(t))\diamond \mathtt{E}_i(0,t)dt.
\end{aligned}
$$

This is equivalent to

$$
X_i(t)\diamond \mathtt{E}_i(0,t) = c_i + \int_0^t b_i(s,X(s))\diamond \mathtt{E}_i(0,s)ds,
$$

or

$$
\begin{aligned}
X_i(t) &= c_i\mathcal{E}_i(0,t) + \int_0^t b_i(s,X(s))\diamond \mathtt{E}_i(0,s)\diamond \mathcal{E}_i(0,t)ds\\
&= c_i\mathcal{E}_i(0,t) + \int_0^t b_i(s,X(s))\diamond \mathcal{E}_i(s,t)ds.
\end{aligned}
$$

Here, we utilized the equality

$$
\mathtt{E}_i(0,t)\diamond \mathcal{E}_i(0,t) = 1, \qquad \text{for all } t\in[0,T].
$$

Therefore, the solution of the Itô SDE (3.40) verifies the integral identity

$$
X_i(t) = c_i\mathcal{E}_i(0,t) + \int_0^t b_i(s,X(s))\diamond \mathcal{E}_i(s,t)ds, \tag{3.41}
$$

for all $t \in [0,T]$ and $i = 1, ..., d$. We are now ready to prove the convergence:

$$
\begin{aligned}
&|X_i^\pi(t) - X_i(t)| \\
&\quad = \left| c_i \left(\mathcal{E}_i^\pi(t,0) - \mathcal{E}_i(0,t) \right) + \int_0^t b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i\left(s, X(s)\right) \diamond \mathcal{E}_i(s,t) ds \right| \\
&\quad \leq |c_i| \left| \mathcal{E}_i^\pi(0,t) - \mathcal{E}_i(0,t) \right| + \int_0^t \left| b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i\left(s, X(s)\right) \diamond \mathcal{E}_i(s,t) \right| ds \\
&\quad \leq |c_i| \left| \mathcal{E}_i^\pi(0,t) - \mathcal{E}_i(0,t) \right| + \int_0^t \left| b_i(s, X^\pi(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t) \right| ds \\
&\qquad + \int_0^t \left| b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i\left(s, X(s)\right) \diamond \mathcal{E}_i(s,t) \right| ds \\
&\quad \leq |c_i| \left| \mathcal{E}_i^\pi(0,t) - \mathcal{E}_i(0,t) \right| + \int_0^t \left| b_i(s, X^\pi(s)) - b_i(s, X(s)) \right| \diamond \mathcal{E}_i^\pi(s,t) ds \\
&\qquad + \int_0^t \left| b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i\left(s, X(s)\right) \diamond \mathcal{E}_i(s,t) \right| ds \\
&\quad \leq |c_i| \left| \mathcal{E}_i^\pi(0,t) - \mathcal{E}_i(0,t) \right| + L \int_0^t \sum_{j=1}^d \left| X_j^\pi(s) - X_j(s) \right| \diamond \mathcal{E}_i^\pi(s,t) ds \\
&\qquad + \int_0^t \left| b_i(s, X(s)) \diamond \left(\mathcal{E}_i^\pi(s,t) - \mathcal{E}_i(s,t) \right) \right| ds;
\end{aligned}
$$

In the last two estimates we utilized inequality (3.23) together with the Lipschitz continuity of b, which is implied by Assumption 2. We now take the expectation of the first and last members above to get

$$
\begin{aligned}
&\mathbb{E}[|X_i^\pi(t) - X_i(t)|] \\
&\quad \leq |c_i| \mathbb{E}\left[\left| \mathcal{E}_i^\pi(0,t) - \mathcal{E}_i(0,t) \right| \right] + L \int_0^t \sum_{j=1}^d \mathbb{E}\left[\left| X_j^\pi(s) - X_j(s) \right| \right] ds \\
&\qquad + \int_0^t \mathbb{E}\left[\left| b_i(s, X(s)) \diamond \left(\mathcal{E}_i^\pi(s,t) - \mathcal{E}_i(s,t) \right) \right| \right] ds.
\end{aligned}
$$

The previous inequality is valid for all $i = 1, ..., d$ and $t \in [0,T]$; therefore, summing over i and setting

$$
\mathtt{X}^\pi(t) := \sum_{i=1}^d \mathbb{E}\left[\left| X_i^\pi(s) - X_i(s) \right| \right],
$$

we obtain

$$\begin{aligned}\mathtt{X}^{\pi}(t) \leq &\sum_{i=1}^{d} |c_i| \mathbb{E}\left[|\mathcal{E}_i^{\pi}(0,t) - \mathcal{E}_i(0,t)|\right] + Ld \int_0^t \mathtt{X}^{\pi}(s)ds \\ &+ \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[|b_i(s,X(s)) \diamond (\mathcal{E}_i^{\pi}(s,t) - \mathcal{E}_i(s,t))|\right] ds \\ =&\mathcal{M}^{\pi}(t) + Ld \int_0^t \mathtt{X}^{\pi}(s)ds,\end{aligned}$$

with

$$\begin{aligned}\mathcal{M}^{\pi}(t) := &\sum_{i=1}^{d} |c_i| \mathbb{E}\left[|\mathcal{E}_i^{\pi}(0,t) - \mathcal{E}_i(0,t)|\right] \\ &+ \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[|b_i(s,X(s)) \diamond (\mathcal{E}_i^{\pi}(s,t) - \mathcal{E}_i(s,t))|\right] ds.\end{aligned}$$

According to Gronwall's inequality the previous estimate yields

$$\mathtt{X}^{\pi}(t) \leq \mathcal{M}^{\pi}(t) + Ld \int_0^t \mathcal{M}^{\pi}(s) e^{Ld(t-s)} ds; \tag{3.42}$$

the proof will be complete if we show that $\mathcal{M}^{\pi}(t)$ is bounded for all $t \in [0,T]$ and any finite partition π and that

$$\lim_{\|\pi\| \to 0} \mathcal{M}^{\pi}(t) = 0, \qquad \text{for all } t \in [0,T];$$

this will allow us to use dominated convergence for the Lebesgue integral appearing in (3.42) and conclude that

$$\lim_{\|\pi\| \to 0} \mathtt{X}^{\pi}(t) = \lim_{\|\pi\| \to 0} \sum_{i=1}^{d} \mathbb{E}\left[|X_i^{\pi}(s) - X_i(s)|\right] = 0.$$

We start with the boundedness:

$$\begin{aligned}
\mathcal{M}^\pi(t) \leq & \sum_{i=1}^{d} |c_i| \left(\mathbb{E}\left[|\mathcal{E}_i^\pi(0,t)| \right] + \mathbb{E}\left[|\mathcal{E}_i(0,t)| \right] \right) \\
& + \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)| \right] ds \\
\leq & 2 \sum_{i=1}^{d} |c_i| + \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t)| \right] + \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)| \right] ds \\
\leq & 2 \sum_{i=1}^{d} |c_i| + \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[|b_i(s, X(s))| \diamond \mathcal{E}_i^\pi(s,t) \right] + \mathbb{E}\left[|b_i(s, X(s))| \diamond \mathcal{E}_i(s,t) \right] ds \\
\leq & 2 \sum_{i=1}^{d} |c_i| + 2dMt.
\end{aligned}$$

We now check the convergence:

$$\begin{aligned}
\lim_{\|\pi\| \to 0} \mathcal{M}^\pi(t) = & \lim_{\|\pi\| \to 0} \sum_{i=1}^{d} |c_i| \mathbb{E}\left[|\mathcal{E}_i^\pi(0,t) - \mathcal{E}_i(0,t)| \right] \\
& + \lim_{\|\pi\| \to 0} \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[|b_i(s, X(s)) \diamond (\mathcal{E}_i^\pi(s,t) - \mathcal{E}_i(s,t))| \right] ds \\
= & \sum_{i=1}^{d} \lim_{\|\pi\| \to 0} \int_0^t \mathbb{E}\left[|b_i(s, X(s)) \diamond (\mathcal{E}_i^\pi(s,t) - \mathcal{E}_i(s,t))| \right] ds.
\end{aligned}$$

We now prove that we can take the last limit inside the integral; first of all, note that the integrand is bounded: in fact,

$$\begin{aligned}
\mathbb{E}\left[|b_i(s, X(s)) \diamond (\mathcal{E}_i^\pi(s,t) - \mathcal{E}_i(s,t))| \right] = & \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t) - b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)| \right] \\
\leq & \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i^\pi(s,t)| \right] + \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)| \right] \\
\leq & \mathbb{E}\left[|b_i(s, X(s))| \diamond \mathcal{E}_i^\pi(s,t) \right] + \mathbb{E}\left[|b_i(s, X(s))| \diamond \mathcal{E}_i(s,t) \right] \\
= & \mathbb{E}\left[|b_i(s, X(s))| \right] + \mathbb{E}\left[|b_i(s, X(s))| \right] \\
\leq & 2M.
\end{aligned}$$

We proceed by proving that

$$\lim_{\|\pi\| \to 0} \mathbb{E}\left[|b_i(s, X(s)) \diamond (\mathcal{E}_i^\pi(s,t) - \mathcal{E}_i(s,t))| \right] = 0.$$

Let us rewrite the expected value as follows:

$$\begin{aligned}
&\mathbb{E}\left[\left|b_i(s,X(s))\diamond(\mathcal{E}_i^\pi(s,t)-\mathcal{E}_i(s,t))\right|\right]\\
&\quad=\mathbb{E}\left[\left|b_i(s,X(s))\diamond\mathcal{E}_i^\pi(s,t)-b_i(s,X(s))\diamond\mathcal{E}_i(s,t)\right|\right]\\
&\quad=\mathbb{E}\left[\left|\mathtt{T}_{\sigma_i,\pi}b_i(s,X(s))\mathcal{E}_i^\pi(s,t)-\mathtt{T}_{\sigma_i}b_i(s,X(s))\mathcal{E}_i(s,t)\right|\right]\\
&\quad\le\mathbb{E}\left[\left|\mathtt{T}_{\sigma_i,\pi}b_i(s,X(s))\mathcal{E}_i^\pi(s,t)-\mathtt{T}_{\sigma_i}b_i(s,X(s))\mathcal{E}_i^\pi(s,t)\right|\right]\\
&\qquad+\mathbb{E}\left[\left|\mathtt{T}_{\sigma_i}b_i(s,X(s))\mathcal{E}_i^\pi(s,t)-\mathtt{T}_{\sigma_i}b_i(s,X(s))\mathcal{E}_i(s,t)\right|\right]\\
&\quad=\mathbb{E}\left[\left|b_i(s,\mathtt{T}_{\sigma_i,\pi}X(s))-b_i(s,\mathtt{T}_{\sigma_i}X(s))\right|\mathcal{E}_i^\pi(s,t)\right]\\
&\qquad+\mathbb{E}\left[\left|b_i(s,\mathtt{T}_{\sigma_i}X(s))\right|\left|\mathcal{E}_i^\pi(s,t)-\mathcal{E}_i(s,t)\right|\right]\\
&\quad\le L\mathbb{E}\left[\left|\mathtt{T}_{\sigma_i,\pi}X(s)-\mathtt{T}_{\sigma_i}X(s)\right|\mathcal{E}_i^\pi(s,t)\right]\\
&\qquad+M\mathbb{E}\left[\left|\mathcal{E}_i^\pi(s,t)-\mathcal{E}_i(s,t)\right|\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\|\pi\|\to 0}\mathbb{E}\left[\left|b_i(s,X(s))\diamond(\mathcal{E}_i^\pi(s,t)-\mathcal{E}_i(s,t))\right|\right]\\
&\quad\le\lim_{\|\pi\|\to 0}L\mathbb{E}\left[\left|\mathtt{T}_{\sigma_i,\pi}X(s)-\mathtt{T}_{\sigma_i}X(s)\right|\mathcal{E}_i^\pi(s,t)\right]\\
&\qquad+\lim_{\|\pi\|\to 0}M\mathbb{E}\left[\left|\mathcal{E}_i^\pi(s,t)-\mathcal{E}_i(s,t)\right|\right].
\end{aligned}$$

By the properties of the translation operator,

$$\lim_{\|\pi\|\to 0}\mathtt{T}_{\sigma_i,\pi}X(s)=\mathtt{T}_{\sigma_i}X(s),\qquad\text{in } L^p(\Omega)\text{ for all } p\ge 1;$$

on the other hand

$$\lim_{\|\pi\|\to 0}\mathcal{E}_i^\pi(s,t)=\mathcal{E}_i(s,t),\qquad\text{in } L^p(\Omega)\text{ for all } p\ge 1.$$

These two facts imply

$$\lim_{\|\pi\|\to 0}L\mathbb{E}\left[\left|\mathtt{T}_{\sigma_i,\pi}X(s)-\mathtt{T}_{\sigma_i}X(s)\right|\mathcal{E}_i^\pi(s,t)\right]=0,$$

completing the proof.

Chapter 4

Wong-Zakai approximations for quasilinear systems of Itô's type fractional stochastic differential equations driven by fBm

Based on : Scorolli, R. (2021). Wong-Zakai approximations for quasilinear systems of Itô's type stochastic differential equations driven by fBm with $H > 1/2$. To appear in Infinite Dimensional Analysis, Quantum Probability and Related Topics.

Abstract

In a recent article Lanconelli and Scorolli (2021) extended to the multidimensional case a Wong-Zakai-type approximation for Itô stochastic differential equations proposed by Øksendal and Hu (1996). The aim of the current paper is to extend the latter result to system of stochastic differential equations of Itô type driven by fractional Brownian (fBm) motion like those considered by Hu (2018). This extension is not trivial since the covariance structure of the fBm precludes us from using the same approach as that used by Lanconelli and Scorolli. Instead we employ a truncated Cameron-Martin expansion as the approximation for the fBm. We are naturally led to the investigation of a semilinear hyperbolic system of evolution equations in several space variables that we utilize for constructing a solution of the Wong–Zakai approximated systems. We show that the law of each element of the approximating sequence solves in the sense of distribution a Fokker-Planck equation and that the sequence converges to the solution of the Itô equation, as the number of terms in the expansion goes to infinite.

4.1 Introduction and statements of the main results.

Our aim is to extend the results introduced in the last chapter to the case in which the equations are driven by a fractional Brownian motion (fBm for short) with Hurst paramenter $H > 1/2$ as those considered by [51]. We stress the fact that the approach proposed in the previous chapter is not entirely suitable for this particular situation; in specific the independence of the Brownian increments was a key part in the proof we proposed, and it's well known that the fBm does not posses this desirable property.

To overcome this difficulty we introduce the following Cauchy problem

$$\begin{cases} \frac{dX_i^K(t)}{dt} = b_i(t, X^K(t)) + \sigma_i(t) X_i^K(t) \diamond \frac{dB_i^{H,K}(t)}{dt}, \ t \in]0,T] \\ X_i(0) = c_i \in \mathbb{R}, \quad \text{for} \ \ i \in \{1, ..., d\}, \end{cases} \tag{4.1}$$

where $\{B^{H,K}(t)\}_{t\in[0,T]}$ stands for the *truncated* (up to the K-th term) Cameron-Martin expansion of the d-dimensional fractional Brownian motion $\{B^H(t)\}_{t\in[0,T]}$(see equation (4.6) for the definition), the coefficients $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \to \mathbb{R}^d$ satisfy certain condition which will be specified later on, while $c \in \mathbb{R}^d$ is a deterministic initial condition.

We must interpret (4.1) as Wong-Zakai approximation of the stochastic Cauchy problem of the Itô type:

$$\begin{cases} dX_i(t) = b_i(t, X(t))dt + \sigma_i(t) X_i(t) dB_i^H(t), \ t \in]0,T] \\ X_i(0) = c_i \in \mathbb{R}, \ \text{for} \ i \in \{1, ..., d\}. \end{cases} \tag{4.2}$$

It's important to stress the fact that (4.1) is not a system of random ordinary differential equations, but rather an evolution equation involving an infinite dimensional gradient (see for instance [51, equation 1.5]).

Throughout this chapter we will assume that the coefficients b and σ posses enough regularity to ensure that the Cauchy problem (4.2) has a unique strong solution (e.g. [51]). This conditions could be summarized as

Assumption 3.

- *The functions $b(t, x)$, $\partial_{x_1} b(t, x)$,..., $\partial_{x_d} b(t, x)$ are bounded and continuous;*
- *the functions $\sigma_1(t), ..., \sigma_d(t)$ are bounded and continuous.*

We are now ready to state our main results; the first of which ensures the existence of a *solution* for the approximating equation.

Theorem 115 (Existence)**.** *Let Assumption 3 be in force. Then (4.1) has a mild solution in the sense of definition 118.*

Our second theorem states that the law of the *approximation* solves in a distributional sense a Fokker-Planck-like equation.

Theorem 116 (Fokker-Planck equation)**.** *The law*

$$\mu^K(t,A) := \mathbb{P}(\{\omega \in \Omega : X^K(t,\omega) \in A\}), \quad t \in [0,T], A \in \mathcal{B}(\mathbb{R}^d)$$

of the random vector $X^K(t)$ solves in the sense of distributions the Fokker-Planck equation

$$\left(\partial_t + \sum_{i,j=1}^{d}\sum_{k=1}^{K} \sigma_i(t)\zeta_k(t)x_i g_{ik}^{(j)}(t,x)\partial^2_{x_i x_j} + \sum_{i=1} b_i(t,x)\partial_{x_i}\right)^* \mathtt{u}(t,x) = 0, \quad (t,x) \in [0,T]\times\mathbb{R}^d \tag{4.3}$$

where $g_{ik}^{(j)} : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a measurable function defined in (4.23) and $\zeta_k : [0,T] \to \mathbb{R}$ is introduced in (4.7)

The third and last theorem states that the *approximation* indeed converges to the strong solution of the Itô SDE;

Theorem 117 (Convergence)**.** *The mild solution $\{X^K(t)\}_{t\in[0,T]}$ converges as K tends to infinite, to the unique strong solution $\{X(t)\}_{\in[0,T]}$ of the Itô SDE (4.2). More precisely,*

$$\lim_{K\to\infty}\sum_{i=1}^{d}\mathbb{E}\left[\left|X_i^K(t) - X_i(t)\right|\right] = 0, \qquad \textit{for all } t \in [0,T].$$

4.2 Preliminaries

4.2.1 Elements on fractional Brownian motion.

In this section we will introduce the basic concepts that will be needed in order to prove our results. For further details the interested reader is referred to the excellent references [15][54][55].

Start by fixing $H \in \left(\frac{1}{2},1\right)$, and let $\Omega := C_0([0,T],\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions endowed with the topology of uniform convergence. There is a probability measure P^H on $(\Omega,\mathcal{B}(\Omega))$, such that on $(\Omega,\mathcal{B}(\Omega),P^H)$ the coordinate process $B^H : \Omega \to \mathbb{R}^d$ defined as

$$B^H(t,\omega) = \omega(t), \quad \omega \in \Omega$$

is a d-dimensional fBm (c.f. section 2.1.2), i.e. a d-dimensional centered Gaussian stochastic process in which for each $i \in \{1,...,d\}$ it holds that

$$\mathbb{E}\left[B_i^H(t)B_i^H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),\ s,t \in [0,T],$$

where from now on $\mathbb{E}$ denotes the expectation in the aforementioned probability space.

Let

$$\phi(s,t) := H(2H-1)|s-t|^{2H-2},\ s,t \in [0,T], \tag{4.4}$$

and define

$$\mathscr{H}_\phi := \left\{ f : [0,T] \to \mathbb{R} : |f|_\phi^2 := \int_0^T \int_0^T f(s)f(t)\phi(s,t)dsdt < \infty \right\}.$$

If $\mathscr{H}_\phi$ is equipped with the inner product

$$(f,g)_\phi = \int_0^T \int_0^T f(s)g(t)\phi(s,t)dsdt,$$

then it becomes a separable Hilbert space, moreover we can see that $\mathscr{H}_\phi$ equals the closure of $L^2([0,T])$ with respect to the inner product $(\cdot,\cdot)_\phi$. For $f \in \mathscr{H}_\phi$ we denote with $\Phi[f] : [0,T] \to \mathbb{R}$ the following continuous map

$$\begin{aligned} &[0,T] \ni t \to \mathbb{R}, \\ &t \mapsto \int_0^T f(s)\phi(t,s)ds. \end{aligned}$$

For a deterministic function $f \in \mathscr{H}_\phi$ we can define in the usual manner a *fractional* Wiener integral satisfying the following isometry property

$$\mathbb{E}\left[\left(\int_0^T f(s)dB_i^H(s) \right)^2 \right] = |f|_\phi^2. \tag{4.5}$$

For $f \in \mathscr{H}_\phi$ and $i \in \{1,...,d\}$ define the (fractional) *stochastic exponential of* f by

$$\mathcal{E}_i(f) := \exp\left\{ \int_0^T f(s)B_i^H(s) - \frac{1}{2}|f|_\phi^2 \right\}.$$

It can be shown that the linear span of $\mathscr{E} := \{\mathcal{E}_i(f); f \in \mathscr{H}_\phi,\ i \in \{1,...,d\}\}$ is dense in $L^p(\Omega)$ for any $p \in (1,\infty)$ (e.g. [55]).

4.2.2 Approximating equation.

The first step when constructing a Wong-Zakai approximation is to choose a sequence of smooth stochastic processes converging to the Brownian motion that drives the original equation.

The key feature that the approximation must have in order to employ the approach proposed in the previous chapter is that of *separating* the random coefficient from the time parameter. In the previous chapter we have employed the *polygonal* approximation however the independence of the Brownian increments was a key feature when proving the result (see for instance the discussion following) and this precludes from using the same approach.

Instead let's assume that $\{e_k\}_{k\geq 1}$ is a complete orthonormal system (CONS) of the Hilbert space $\mathscr{H}_\phi$, then (e.g. [56, equation 3.21]) the d-dimensional fractional Brownian motion has the following Cameron-Martin expansion

$$B_i^H(t) = \sum_{k=1}^{\infty} \left[\int_0^t \left(\int_0^T e_k(r)\phi(v,r)dr \right) dv \right] \int_0^T e_k(s)dB_i^H(s),\ t \in [0,T],\ \text{for } i \in \{1,...,d\}.$$

From this expression it's then straightforward to see that a natural approximation for the fractional white noise is given by

$$\frac{dB_i^{H,K}(t)}{dt} := \sum_{k=1}^{K} \left(\int_0^T e_k(r)\phi(t,r)dr \right) \int_0^T e_k(s)dB_i^H(s),\ t \in [0,T],\ \text{for } i \in \{1,...,d\}; \tag{4.6}$$

i.e. the time derivative of the *truncated* Karuhnen-Loève expansion. The convergence of this object to the "singular fractional white noise" as K goes to infinity must be understood in a space of generalized random variables (see [56] for further details).

Notice that due to (4.5) and the orthonormality of $\{e_k\}_{k\geq 1}$ if we let $Z_k^{(i)} := \int_0^T e_k(s)dB_i^H(s)$, then $\left(Z_k^{(i)}\right)_{(k,i)\in\{1,...,K\}\times\{1,...,d\}}$ is a family of i.i.d. standard Gaussian random variables.

For the ease of notation let

$$\zeta_k(\cdot) = \Phi[e_k](\cdot), \qquad \text{for all } k \in \{1,...,K \tag{4.7}$$

and hence our approximation for the fractional white noise can be written as

$$\frac{dB_i^{H,K}(t)}{dt} = \sum_{k=1}^{K} \zeta_k(t)Z_k^{(i)},\ t \in [0,T],\ \text{for } i \in \{1,...,d\}. \tag{4.8}$$

Just as in the previous chapter we remind some properties of the Wick product that will be used latter on:

- if $X \in L^p(\Omega)$ for some $p > 1$ and $f \in \mathscr{H}_\phi$ we set

$$X \diamond \mathcal{E}_i(f) := \mathtt{T}_{-\Phi[f]_i} X \cdot \mathcal{E}_i(f) \tag{4.9}$$

where $\mathtt{T}_{-\Phi[f]_i}$ stands for the *translation* operator

$$(\mathtt{T}_{-\Phi[f]_i}X)(\omega) := X\left(\omega - \epsilon_i \int_0^{\cdot} \Phi[f](r)dr\right). \tag{4.10}$$

Here $\{\epsilon_1, ..., \epsilon_d\}$ denotes the canonical basis of $\mathbb{R}^d$. This is the fractional analog of the Gjessing's formula (theorem 100) From the latter we are able to see that the *Wick product* with a *stochastic exponential* preserves the monotonicity, i.e.

$$\text{if } X \leq Y \text{ then } X \diamond \mathcal{E}_i(f) \leq Y \diamond \mathcal{E}_i(f).$$

- A consequence of the latter and the density of the linear span of $\mathscr{E}$ is that if $g \in \mathscr{H}_\phi$, $F \in L^p(\Omega)$ and $\left(D_\phi^{(i)} F \,,\, g\right)_{L^2(\mathbb{R})} \in L^p(\Omega)$ for some $p > 1$ then

$$F \diamond \int_0^T g(s)dB_i^H(s) = F \int_0^T g(s)dB_i^H(s) - \left(D_\phi^{(i)} F \,,\, g\right)_{L^2([0,T])} \tag{4.11}$$

  where $D_\phi^{(i)}$ denotes the ϕ-derivative (e.g. [56][55]) with respect to the i-th fBm (c.f. proposition 101).

With all this in hand we are able to provide a solution concept for (4.1):

Definition 118. A d-dimensional stochastic process $\{X^K(t)\}_{t\in[0,T]}$ is said to be a *mild* solution of equation (4.1) if:

1. the function $t \mapsto X^K(t)$ is almost surely continuous;
2. for all $i \in \{1, ..., d\}$ and $t \in [0, T]$, the random variable $X_i^K(t)$ belongs to $L^p(\Omega)$ for some $p > 1$;
3. for all $i \in \{1, ..., d\}$, the identity

$$X_i^K(t) = c_i \mathcal{E}_i^K(0, t) + \int_0^t b_i(s, X^K(s)) \diamond \mathcal{E}_i^K(s, t)ds, \quad t \in [0, T], \tag{4.12}$$

   holds almost surely, where for any $t, r \in [0, T], r \leq t$, $\mathcal{E}_i^K(r, t)$ is a shorthand for $\mathcal{E}_i(\sigma_i^K(r, t))$ where $\sigma_i^K(r, t; \cdot)$ denotes the orthogonal projection of $\chi_{[r,t]}\sigma_i(\cdot)$ on $\text{span}\{e_1, ..., e_K\} \subset \mathscr{H}_\phi$.

4.3 Proof of theorem 115

In order to prove the existence of a mild solution for (4.1) we will introduce a system of partial differential equations which is related to (4.1) by the following heuristic considerations.

Remark 119. Using (4.8) and formally applying identity (4.11) we can rewrite (4.1) as

$$\begin{cases} \frac{dX_i^K(t)}{d} = b_i(t, X^K(t)) + \sigma_i(t)X_i(t) \cdot \left(\sum_{k=1}^K \zeta_k(t) Z_k^{(i)}\right) - \sum_{k=1}^K \sigma_i(t)\zeta_k(t) \left(D_\phi^{(i)} X_i(t), e_k\right)_{L^2([0,T])}, \\ t \in]0, T] \\ X_i(0) = c_i \in \mathbb{R}, \text{ for } i \in \{1, ..., d\}. \end{cases}$$

If we now search for a solution of the form

$$X_i^K(t, \omega) = u_i(t, \mathbf{z}(\omega)),$$

for $u_i : [0, T] \times \mathbb{R}^{K \times d} \to \mathbb{R}$ where we identify $\mathbb{R}^{K \times d}$ with the space of $(K \times d)$ matrices and $\mathbf{z}_{ki}(\omega) = Z_k^{(i)}(\omega)$ then by a simple application of the chain rule for the ϕ-derivative we see that $u = (u_1, ..., u_d)$ has to solve the following semilinear hyperbolic system of partial differential equations

$$\begin{cases} \partial_t u_i = b_i(t, u) + \sigma_i(t) \sum_{k=1}^K \zeta_k(t) \left[x_{ki} u_i - \partial_{x_{ki}} u_i\right], \\ (t, \mathbf{x}) \in]0, T] \times \mathbb{R}^{K \times d}, \\ u_i(0, \mathbf{x}) = c_i \in \mathbb{R}, \text{ for } i \in \{1, ..., d\}. \end{cases} \tag{4.13}$$

Unfortunately to the best of our knowledge the latter does not satisfy the basic assumption of the main existence-uniqueness theorems present in the literature. For that reason we will introduce the following auxiliary Cauchy problem

$$\begin{cases} \partial_t v_i = b_i\left(t, v(t) \exp\left\{\frac{1}{2}\|\mathbf{x}\|_F^2\right\}\right) \exp\left\{-\frac{1}{2}\|\mathbf{x}\|_F^2\right\} - \sigma_i(t) \sum_{k=1}^K \zeta_k(t) \partial_{x_{ki}} v_i, \\ (t, \mathbf{x}) \in]0, T] \times \mathbb{R}^{K \times d}, \\ v_i(0, \mathbf{x}) = c_i \exp\left\{-\frac{1}{2}\|\mathbf{x}\|_F^2\right\}, \text{ for } i \in \{1, ..., d\}, \end{cases} \tag{4.14}$$

where $\|\cdot\|_F$ denotes the Frobenious norm, i.e. $\|\mathbf{x}\|_F^2 := \sum_{i=1}^d \sum_{k=1}^K |x_{ki}|^2$. Our motivation for doing so will be clear in a moment.

A closer inspection would allow the reader to see that the latter is a d-dimensional semilinear symmetric hyperbolic system of evolution equations in $(K \times d)$ spatial variables.

Remark 120. Just as in the previous chapter using the properties of the Wick product we are led to consider a system of evolution equations. The difference relies in the fact that in the latter we were able to partition the problem and we considered multiple Cauchy problems with a single spatial variable. The difficulty in that case was that of "glueing" the pieces together. In current setting we must study a single Cauchy problem with multiple space variables.

The validity of assumption 3 implies the existence of a unique classical solution of (4.14) for any arbitrary time interval $[0, T]$ (see for instance [53],[52] and [57]).

If we let

$$\Sigma_{i,k}(r,t) := \int_r^t \sigma_i(s)\zeta_k(s)ds = \left(\chi_{[r,t]}\sigma_i\ ,\ e_k\right)_\phi , \tag{4.15}$$

then we can write down a *mild solution* for (4.14) as

$$\begin{cases} v_i(t,\mathbf{x}) = c_i \exp\left\{ -\frac{1}{2}\|\mathbf{x}-\mathbf{\Sigma}^{(i)}(t)\|_F^2\right\} \\ \qquad + \int_0^t b_i\left(s, v(s,\mathbf{x}-\mathbf{\Sigma}^{(i)}(s,t)\right) \exp\left\{\frac{1}{2}\|\mathbf{x}-\mathbf{\Sigma}^{(i)}(s,t)\|_F^2\right\}\Big) \exp\left\{ -\frac{1}{2}\|\mathbf{x}-\mathbf{\Sigma}^{(i)}(s,t)\|_F^2\right\} ds, \\ \text{for } t\in[0,T], \mathbf{x}\in\mathbb{R}^{K\times d},\ i\in\{1,...,d\} \end{cases} \tag{4.16}$$

where for any pair $r,t\in[0,T], t\geq r$, we denote with $\mathbf{\Sigma}^{(i)}(r,t)$ the $(K\times d)$-matrix where the i-th column is given by $[\Sigma_{i,1}(r,t),...,\Sigma_{i,K}(r,t)]^T$ and all the remaining components are equal 0.

Now if we let $u_i(t,\mathbf{x}) := v_i(t,\mathbf{x})\exp\left\{\frac{1}{2}\|\mathbf{x}\|_F^2\right\}$ for all $i\in\{1,...,d\}$ a simple application of the chain rule shows that u solves (4.13), furthermore using (4.16) we have that the following mild representation holds

$$\begin{cases} u_i(t,\mathbf{x}) = c_i \exp\left\{ \sum_{k=1}^K \left[x_{ik}\Sigma_{i,k}(0,t) - \frac{1}{2}|\Sigma_{i,k}(0,t)|^2\right] \right\} \\ \qquad + \int_0^t b_i\big(s, u(s,\mathbf{x}-\mathbf{\Sigma}^{(i)}(s,t))\big) \exp\left\{ \sum_{k=1}^K \left[x_{ik}\Sigma_{i,k}(s,t) - \frac{1}{2}|\Sigma_{i,k}(s,t)|^2\right] \right\} ds, \\ \text{for } t\in[0,T], \mathbf{x}\in\mathbb{R}^{K\times d}, i\in\{1,...,d\}. \end{cases} \tag{4.17}$$

Remark 121. This equation is the analog of [58, equation 3.4], where instead of shifting the i-th component of the vector of spatial variables we shift the i-th column of the matrix of spatial variables.

At this point we define the candidate solution $\{X^K(t)\}_{t\in[0,T]}$ as

$$X_i^K(t,\omega) = u_i(t,\mathbf{z}(\omega)),\ t\in[0,T], \omega\in\Omega,\ \text{for } i\in\{1,...,d\} \tag{4.18}$$

where again $\mathbf{z}(\omega)$ is the $K\times d$-matrix in which the (k,i)-th component is given by $Z_k^{(i)}(\omega)$.

Next we must verify that $\{X^K(t)\}_{t\in[0,T]}$ is indeed a mild solution of the system (4.1), i.e. that satisfies the conditions imposed by definition 118.

The almost surely continuity of the path is given by the continuity of $[0,T]\ni t\mapsto u(t,\mathbf{x}) := v(t,\mathbf{x})e^{\frac{\|\mathbf{x}\|_F^2}{2}}$ for all $\mathbf{x}\in\mathbb{R}^{K\times d}$ (remember that v is a classical solution).

Noticing that

$$
\begin{aligned}
&\exp\left\{\sum_{k=1}^{K}\left[x_{ik}\Sigma_{i,k}(r,t)-\frac{1}{2}|\Sigma_{i,k}(r,t)|^2\right]\right\}\Bigg|_{x_{ki}=Z_k^{(i)}(\omega)}\\
&=\exp\left\{\int_0^T\sum_{k=1}^{K}\left(\int_r^t\sigma_i(s)\zeta_k(s)ds\right)e_k(q)dB_i^H(q)-\frac{1}{2}\sum_{k=1}^{K}\left(\int_r^t\sigma_i(s)\zeta_k(s)ds\right)^2\right\}\\
&=\exp\left\{\int_0^T\sum_{k=1}^{K}\left(\chi_{[r,t]}\sigma_i,e_k\right)_\phi e_k(q)dB_i^H(q)-\frac{1}{2}\sum_{k=1}^{K}\left(\chi_{[r,t]}\sigma_i,e_k\right)_\phi^2\right\}\\
&=\exp\left\{\int_0^T\sigma_i^K(r,t;q)dB_i^H(q)-\frac{1}{2}|\sigma_i^K(r,t;\cdot)|_\phi^2\right\}=:\mathcal{E}_i^K(r,t),
\end{aligned}
\tag{4.19}
$$

and using assumption 3 we have that

$$
|X_i^K(t)|\leq|c_i|\exp\left\{\int_0^T\sigma_i^K(0,t;q)dB_i^H(q)\right\}+M\int_0^t\exp\left\{\int_0^T\sigma_i^K(s,t;q)dB_i^H(q)\right\}ds,
$$

where $M>0$ is a constant such that $|b(t,\mathbf{x})|\leq M$. Taking the $L^p(\Omega)$-norm on both sides above and using the triangular inequality we obtain

$$
\begin{aligned}
\|X_i^K(t)\|_{L^p(\Omega)}\leq{}&|c_i|\left\|\exp\left\{\int_0^T\sigma_i^K(0,t;q)dB_i^H(q)\right\}\right\|_{L^p(\Omega)}\\
&+M\int_0^t\left\|\exp\left\{\int_0^T\sigma_i^K(s,t;q)dB_i^H(q)\right\}\right\|_{L^p(\Omega)}ds\\
\leq{}&|c_i|\exp\left\{\frac{p}{2}|\sigma_i^K(0,t;\cdot)|_\phi^2\right\}+M\int_0^t\exp\left\{\frac{p}{2}|\sigma_i^K(s,t;\cdot)|_\phi^2\right\}ds\\
\leq{}&|c_i|\exp\left\{\frac{p}{2}|\sigma_i^K(0,t;\cdot)|_\phi^2\right\}+Mt\sup_{s\in[0,t]}\exp\left\{\frac{p}{2}|\sigma_i^K(s,t;\cdot)|_\phi^2\right\}<\infty,
\end{aligned}
$$

where we used the fact that the fractional Wiener integral of $\sigma_i^K(r,t;\cdot)$ is a Gaussian random variable. This proves the membership of $X_i^K(t)$ to $L^p(\Omega), p\geq 1$ for all $i\in\{1,2,...,d\}$ and $t\in[0,T]$

Last thing we need to do is to prove that the process $X^K(t)$ satisfies the representation (4.12). First we notice that for any $l\in\{1,...,K\}$ and $i\in\{1,...,d\}$

$$
Z_l^{(i)}-\Sigma_{i,l}(s,t)=Z_l^{(i)}-\sum_{k=1}^{K}\left(\chi_{[s,t]}\sigma_i,e_k\right)_\phi\int_0^T\int_0^T e_k(r)e_l(q)\phi(r,q)drdq,=\mathtt{T}_{-\Phi[\sigma_i^K(s,t)]}Z_l^{(i)}
$$

where $=\mathtt{T}_{-\Phi[\sigma_i^K(s,t)]}$ is a shorthand for $=\mathtt{T}_{-\Phi[\sigma_i^K(s,t)]_i}$. Then it follows from (4.17) and (4.19) that

$$
X_i^K(t)=c_i\mathcal{E}_i^K(0,t)+\int_0^t\mathtt{T}_{-\Phi[\sigma_i^K(s,t)]}b_i\left(s,X^K(s)\right)\cdot\mathcal{E}_i^K(s,t)\,ds.
$$

Using identity (4.9) we have that for all $t \in [0,T]$ and $i \in \{1,...,d\}$ the following holds a.s.

$$X_i^K(t) = c_i \mathcal{E}_i^K(0,t) + \int_0^t b_i\big(s, X^K(s)\big) \diamond \mathcal{E}_i^K(s,t)\, ds;$$

completing the proof.

4.4 Proof of theorem 117

The aim of this section is to show that the *mild solution* of

$$\begin{cases} \frac{dX_i^K(t)}{dt} = b_i(t, X^K(t)) + \sigma_i(t) X_i^K(t) \diamond \frac{dB_i^{H,K}(t)}{dt}, \ t \in]0,T] \\ X_i(0) = c_i \in \mathbb{R}, \quad \text{for } i \in \{1,...,d\}; \end{cases}$$

converges in $L^1(\Omega)$ to the unique strong solution of

$$\begin{cases} dX_i(t) = b_i(t, X(t))dt + \sigma_i(t) X_i(t) dB_i^H(t), \ t \in]0,T] \\ X_i(0) = c_i \in \mathbb{R}, \quad \text{for } i \in \{1,...,d\}. \end{cases}$$

Let

$$\mathtt{E}_i(0,t) := \exp\left\{ -\int_0^t \sigma_i(s) dB_i^H(s) - \frac{1}{2} |\chi_{[0,t]}\sigma_i|_\phi^2 \right\},$$

and

$$\mathcal{E}_i(0,t) := \exp\left\{ \int_0^t \sigma_i(s) dB_i^H(s) - \frac{1}{2} |\chi_{[0,t]}\sigma_i|_\phi^2 \right\}.$$

Using equation (3.41) of we can formally write (4.2) as

$$\begin{cases} \frac{dX_i(t)}{dt} = b_i(t, X(t)) + \sigma_i(t) X_i(t) \diamond \frac{dB_i^H(t)}{dt}, \ t \in]0,T] \\ X_i(0) = c_i, \ \text{for } i \in \{1,...,d\}, \end{cases}$$

where we must bare in mind that the time derivative of the fBm is not well defined as a random variable, so in order to make sense of the expression above we must interpret it as a differential equation in some space of generalized random variables (like the fractional Hida space e.g. [54][56]). Then we can Wick-multiply both sides of the equality above by $\mathtt{E}_i(0,t)$ which gives, after rearranging

$$\frac{dX_i(t)}{dt} \diamond \mathtt{E}_i(0,t) - \sigma_i(t) X_i(t) \diamond \frac{dB_i^H(t)}{dt} \diamond \mathtt{E}_i(0,t) = b_i(t, X(t)) \diamond \mathtt{E}_i(0,t).$$

By means of the identity

$$\frac{d\mathtt{E}_i(0,t)}{dt} = \sigma_i(t)\mathtt{E}_i(0,t) \diamond \frac{dB_i^H(t)}{dt}.$$

and the Leibniz rule for the *Wick product* we obtain

$$\frac{d\mathcal{X}_i(t)}{dt} = b_i(t, X(t)) \diamond \mathtt{E}_i(0,t), \tag{4.20}$$

where

$$\mathcal{X}_i(t) := X_i(t) \diamond \mathtt{E}_i(0,t).$$

It follows that

$$\mathcal{X}_i(t) = X_i(t) \diamond \mathtt{E}_i(0,t) = c_i + \int_0^t b_i(s, X(s)) \diamond \mathtt{E}_i(0,s)ds$$

or which is equivalent

$$X_i(t) = c_i\mathcal{E}_i(0,t) + \int_0^t b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)ds$$

where we used the identity

$$\mathcal{E}_i(0,t) \diamond \mathtt{E}_i(0,t) = 1, \ a.s. \text{ for all } t \in [0,T].$$

We conclude that the unique strong solution of (4.2) satisfies the following integral equation

$$X_i(t) = c_i\mathcal{E}_i(0,t) + \int_0^t b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)ds \tag{4.21}$$

for all $t \in [0,T]$ and $i \in \{1,...,d\}$.

Remark 122. Under assumption 3 it follows that for any $t \in [0,T]$ the strong solution of (4.2) belongs to $L^p(\Omega)$ for any $p \geq 1$.

Now we are ready to prove the convergence;

$$\begin{aligned}
|X_i(t) - X_i^K(t)| &\leq |c_i||\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)| \\
&+ \int_0^t |b_i(s, X(s)) \diamond \mathcal{E}_i(s,t) - b_i(s, X^K(s)) \diamond \mathcal{E}_i^K(s,t)| ds \\
&= |c_i||\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)| \\
&+ \int_0^t \Big| b_i(s, X(s)) \diamond \mathcal{E}_i(s,t) - b_i(s, X(s)) \diamond \mathcal{E}_i^K(s,t) \\
&+ b_i(s, X(s)) \diamond \mathcal{E}_i^K(s,t) - b_i(s, X^K(s)) \diamond \mathcal{E}_i^K(s,t) \Big| ds \\
&\leq |c_i||\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)| \\
&+ \int_0^t \left| b_i(s, X(s)) \diamond \big(\mathcal{E}_i(s,t) - \mathcal{E}_i^K(s,t)\big) \right| ds \\
&+ \int_0^t \left| b_i(s, X(s)) - b_i(s, X^K(s)) \right| \diamond \mathcal{E}_i^K(s,t) ds.
\end{aligned}$$

Using the Lipschitz continuity of b_i and the fact that the *Wick product* with a *stochastic exponential* preserves the monotonicity we have that

$$\begin{aligned}
|X_i(t) - X_i^K(t)| \leq |c_i||\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)| + \int_0^t \left| b_i(s, X(s)) \diamond \big(\mathcal{E}_i(s,t) - \mathcal{E}_i^K(s,t)\big) \right| ds \\
+ L \int_0^t \sum_{j=1}^d |X_j(s) - X_j^K(s)| \diamond \mathcal{E}_i^K(s,t) ds
\end{aligned}$$

where L is a positive constant such that for all $t \in [0,T]$ it holds $|b_i(t,X) - b_i(t,Y)| \leq L|X - Y|_1$; here $|\cdot|_1$ denotes the ℓ^1 norm. Now we take expectation yielding

$$\begin{aligned}
\mathbb{E}\left[|X_i(t) - X_i^K(t)|\right] \leq |c_i| \mathbb{E}\left[|\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)|\right] + \int_0^t \mathbb{E}\left[\left| b_i(s, X(s)) \diamond \big(\mathcal{E}_i(s,t) - \mathcal{E}_i^K(s,t)\big) \right|\right] ds \\
+ L \int_0^t \sum_{j=1}^d \mathbb{E}\left[|X_j(s) - X_j^K(s)|\right] ds.
\end{aligned}$$

The previous inequality is valid for all $i = 1, ..., d$ and $t \in [0,T]$; therefore, summing over i and setting

$$\mathtt{X}^K(t) := \sum_{i=1}^d \mathbb{E}\left[|X_j(t) - X_j^K(t)|\right]$$

we obtain

$$\mathtt{X}^K(t) \le \sum_{i=1}^d |c_i| \mathbb{E}\left[|\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)| \right] + \sum_{i=1}^d \int_0^t \mathbb{E}\left[\left| b_i(s, X(s)) \diamond \left(\mathcal{E}_i(s,t) - \mathcal{E}_i^K(s,t) \right) \right| \right] ds$$
$$+ Ld \int_0^t \mathtt{X}^K(s) ds$$
$$= \mathcal{M}^K(t) + Ld \int_0^t \mathtt{X}^K(s) ds,$$

where

$$\mathcal{M}^K(t) := \sum_{i=1}^d |c_i| \mathbb{E}\left[|\mathcal{E}_i(0,t) - \mathcal{E}_i^K(0,t)| \right] + \sum_{i=1}^d \int_0^t \mathbb{E}\left[\left| b_i(s, X(s)) \diamond \left(\mathcal{E}_i(s,t) - \mathcal{E}_i^K(s,t) \right) \right| \right] ds.$$

According to Gronwall's inequality the previous estimate yields

$$\mathtt{X}^K(t) \le \mathcal{M}^K(t) + Ld \int_0^t \mathcal{M}^K(s) e^{Ld(t-s)} ds; \tag{4.22}$$

and hence the proof will be complete if we show that $\mathcal{M}^K(t)$ is bounded for all $t \in [0,T]$ and it holds that

$$\lim_{K\to\infty} \mathcal{M}^K(t) = 0, \quad \text{for all } t \in [0,T];$$

this will allow us to use dominated convergence for the Lebesgue integral appearing in (4.22) and conclude that

$$\lim_{K\to\infty} \mathtt{X}^K(t) = 0.$$

In order to prove the boundedness we write

$$\mathcal{M}^K(t) \le \sum_{i=1}^d |c_i| \left(\mathbb{E}\left[|\mathcal{E}_i(0,t)\right] + \mathbb{E}\left[\mathcal{E}_i^K(0,t)| \right] \right)$$
$$+ \int_0^t \mathbb{E}\left[\left| b_i(s, X(s)) \diamond \mathcal{E}_i(s,t) - b_i(s, X(s)) \diamond \mathcal{E}_i^K(s,t) \right| \right] ds$$
$$\le 2 \sum_{i=1}^d |c_i| + \sum_{i=1}^d \int_0^t \mathbb{E}\left[|b_i(s, X(s)) \diamond \mathcal{E}_i(s,t)| \right] + \mathbb{E}\left[\left| b_i(s, X(s)) \diamond \mathcal{E}_i^K(s,t) \right| \right] ds$$
$$\le 2 \sum_{i=1}^d |c_i| + 2dMt;$$

the boundedness also follows from the continuity of $t \mapsto \mathcal{M}^K(t)$ and the compactness of $[0,T]$.

Thus it suffices to prove that

$$\lim_{K\to\infty} \mathcal{M}^K(t) = 0, \quad \text{for all } t \in [0,T].$$

Using the fact that $\mathcal{E}_i^K(0,t)$ converges in $L^p(\Omega), p \geq 1$ to $\mathcal{E}_i(0,t)$ (see Appendix A) it follows that

$$\begin{aligned}
\lim_{K\to\infty} \mathcal{M}^K(t) = &\lim_{K\to\infty} \sum_{i=1}^{d} |c_i| \mathbb{E}\left[\left|\mathcal{E}_i^K(0,t) - \mathcal{E}_i(0,t)\right|\right] \\
&+ \lim_{K\to\infty} \sum_{i=1}^{d} \int_0^t \mathbb{E}\left[\left|b_i(s,X(s)) \diamond \left(\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right)\right|\right] ds \\
= &\sum_{i=1}^{d} \lim_{K\to\infty} \int_0^t \mathbb{E}\left[\left|b_i(s,X(s)) \diamond \left(\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right)\right|\right] ds.
\end{aligned}$$

We now prove that we can take the last limit inside the integral; first of all, note that the integrand is bounded: in fact,

$$\begin{aligned}
\mathbb{E}\left[\left|b_i(s,X(s)) \diamond \left(\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right)\right|\right] = &\mathbb{E}\left[\left|b_i(s,X(s)) \diamond \mathcal{E}_i^K(s,t) - b_i(s,X(s)) \diamond \mathcal{E}_i(s,t)\right|\right] \\
\leq &\mathbb{E}\left[\left|b_i(s,X(s)) \diamond \mathcal{E}_i^K(s,t)\right|\right] + \mathbb{E}\left[\left|b_i(s,X(s)) \diamond \mathcal{E}_i(s,t)\right|\right] \\
\leq &\mathbb{E}\left[\left|b_i(s,X(s))\right| \diamond \mathcal{E}_i^K(s,t)\right] + \mathbb{E}\left[\left|b_i(s,X(s))\right| \diamond \mathcal{E}_i(s,t)\right] \\
= &\mathbb{E}\left[\left|b_i(s,X(s))\right|\right] + \mathbb{E}\left[\left|b_i(s,X(s))\right|\right] \\
\leq &2M.
\end{aligned}$$

We proceed by proving that

$$\lim_{K\to\infty} \mathbb{E}\left[\left|b_i(s,X(s)) \diamond \left(\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right)\right|\right] = 0.$$

Let us rewrite the expected value as follows:

$$\mathbb{E}\left[\left|b_i(s,X(s)) \diamond \left(\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right)\right|\right] = \mathbb{E}\left[\left|b_i(s,X(s)) \diamond \mathcal{E}_i^K(s,t) - b_i(s,X(s)) \diamond \mathcal{E}_i(s,t)\right|\right]$$

Using (4.9) we get rid of the *Wick product* and write

$$= \mathbb{E}\left[\left|\mathtt{T}_{-\Phi[\sigma^K(s,t)]} b_i(s,X(s)) \mathcal{E}_i^K(s,t) - \mathtt{T}_{-\Phi[\sigma_i(s,t)]} b_i(s,X(s)) \mathcal{E}_i(s,t)\right|\right].$$

Adding and subtracting $\mathtt{T}_{-\Phi[\sigma_i(s,t)]} b_i(s,X(s)) \mathcal{E}_i^K(s,t)$ inside the absolute value and then using the triangular inequality yields

$$\begin{aligned}
\leq\, &\mathbb{E}\left[\left|\mathtt{T}_{-\Phi[\sigma^K(s,t)]} b_i(s,X(s)) \mathcal{E}_i^K(s,t) - \mathtt{T}_{-\Phi[\sigma_i(s,t)]} b_i(s,X(s)) \mathcal{E}_i^K(s,t)\right|\right] \\
&+ \mathbb{E}\left[\left|\mathtt{T}_{-\Phi[\sigma_i(s,t)]} b_i(s,X(s)) \mathcal{E}_i^K(s,t) - \mathtt{T}_{-\Phi[\sigma_i(s,t)]} b_i(s,X(s)) \mathcal{E}_i(s,t)\right|\right] \\
=\, &\mathbb{E}\left[\left|b_i(s, \mathtt{T}_{-\Phi[\sigma_i^K(s,t)]} X(s)) - b_i(s, \mathtt{T}_{-\Phi[\sigma_i(s,t)]} X(s))\right| \mathcal{E}_i^K(s,t)\right] \\
&+ \mathbb{E}\left[\left|b_i(s, \mathtt{T}_{-\Phi[\sigma_i(s,t)]} X(s))\right| \left|\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right|\right] \\
\leq\, &L\mathbb{E}\left[\left|\mathtt{T}_{-\Phi[\sigma_i^K(s,t)]} X(s) - \mathtt{T}_{-\Phi[\sigma_i(s,t)]} X(s)\right| \mathcal{E}_i^K(s,t)\right] \\
&+ M\mathbb{E}\left[\left|\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)\right|\right].
\end{aligned}$$

Hence,

$$\lim_{K\to\infty}\mathbb{E}\left[\left|b_i(s,X(s))\diamond\left(\mathcal{E}_i^K(s,t)-\mathcal{E}_i(s,t)\right)\right|\right]\leq\lim_{K\to\infty}L\mathbb{E}\left[\left|\mathtt{T}_{-\Phi[\sigma_i^K(s,t)]}X(s)-\mathtt{T}_{-\Phi[\sigma_i(s,t)]}X(s)\right|\mathcal{E}_i^K(s,t)\right]$$
$$+\lim_{K\to\infty}M\mathbb{E}\left[\left|\mathcal{E}_i^K(s,t)-\mathcal{E}_i(s,t)\right|\right].$$

The second term above converges to zero by the discussion on Appendix A.

At this point we will need the following lemma;

Lemma 123. *Let $Y\in L^q(\Omega)$ for some $q\in(0,\infty)$ and let $\{f_n\}_{n\geq1}$ be a sequence converging to f in $\mathscr{H}_\phi$, then it holds that*

$$\lim_{n\to\infty}\mathtt{T}_{\Phi f_n}Y=\mathtt{T}_{\Phi f}Y,\qquad \textit{in } L^p(\Omega) \textit{ for all } 0<p<q<\infty$$

Proof. For simplicity we will consider the case of random variable Y depending only on a one dimensional fBm that can be seen as one of the components of our d-dimensional fBm, the general case does not present further difficulties. Notice that

$$\begin{aligned}\mathtt{T}_{\Phi f_n}B^H(t)&=B^H(t)+\int_0^t\int_0^T f_n(s)\phi(s,r)dsdr\\&=B^H(t)+\int_0^T\int_0^T\chi_{[0,t]}(r)f_n(s)\phi(s,r)dsdr\\&=B^H(t)+\left(f_n,\chi_{[0,t]}\right)_\phi,\end{aligned}$$

at this point we use the fact that convergence in norm implies the weak convergence, and hence if f_n converges in $\mathscr{H}_\phi$ to f as $n\to\infty$ we have that $\left(f_n,\chi_{[0,t]}\right)_\phi$ converges to $\left(f,\chi_{[0,t]}\right)_\phi$.

This implies that

$$\mathbb{E}[|\mathtt{T}_{\Phi f}B^H(t)-\mathtt{T}_{\Phi f}B^H(t)|^p]=|\left(f_n,\chi_{[0,t]}\right)_\phi-\left(f,\chi_{[0,t]}\right)_\phi|^p\to0,\text{ as }n\to\infty.$$

Furthermore notice that this holds for any random variable in the Gaussian Hilbert space

$$\mathfrak{G}(\mathscr{H}_\phi):=\left\{\int_0^T g(s)dB^H(s);\ g\in\mathscr{H}_\phi\right\}.$$

At this point if $Y\in L^q(\Omega)$ we have that for any $\epsilon>0$ there's a polynomial random variable P (which is a polynomial in some random variables in $\mathfrak{G}(\mathscr{H}_\phi)$) such that $\|Y-P\|_{L^q(\Omega)}<\epsilon$ (the existence of such a random variable is guaranteed by [56, Theorem 3.2] together with [16, Theorem 2.11]). By the triangle inequality we have that for $0<p<q$

$$\|\mathtt{T}_{\Phi f_n}X-\mathtt{T}_{\Phi f}X\|_{L^p(\Omega)}\leq\|\mathtt{T}_{\Phi f_n}X-\mathtt{T}_{\Phi f_n}P\|_{L^p(\Omega)}+\|\mathtt{T}_{\Phi f_n}P-\mathtt{T}_{\Phi f}P\|_{L^p(\Omega)}+\|\mathtt{T}_{\Phi f}P-\mathtt{T}_{\Phi f}Y\|_{L^p(\Omega)}$$

Now using the fractional Girsanov's theorem we have

$$\begin{aligned}\|\mathtt{T}_{\Phi f}P - \mathtt{T}_{\Phi f}Y\|_{L^p(\Omega)} &= \mathbb{E}\left[|P-Y|^p\mathcal{E}(f)\right]^{1/p}\\ &\le \mathbb{E}\left[|P-Y|^{pp_1}\right]^{1/(pp_1)}\mathbb{E}\left[\mathcal{E}(f)^{p_2/p}\right]^{1/(pp_2)}\\ &= \|P-Y\|_{L^q(\Omega)}\|\|\mathcal{E}(f)^{1/p^2}\|_{L^r(\Omega)}\end{aligned}$$

where $q := p_1 p$, $r := p_2 p$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Same happens with $\|\mathtt{T}_{\Phi f_n}P - \mathtt{T}_{\Phi f_n}Y\|_{L^p(\Omega)}$. At this point we notice that

$$\begin{aligned}\|\mathcal{E}(f_n)^{1/p^2}\|_{L^r(\Omega)} &\le \mathbb{E}\left[\exp\left\{r/p^2\int_0^T f_n(s)dB_s^H\right\}\right]^{1/r}\\ &\le \sup_n \exp\left\{\frac{r}{2p^4}|f_n|_\phi^2\right\} =: C.\end{aligned}$$

It follows that

$$\begin{aligned}\|\mathtt{T}_{\Phi f_n}X - \mathtt{T}_{\Phi f}X\|_{L^p(\Omega)} &\le 2C\|P-Y\|_{L^q(\Omega)} + \|\mathtt{T}_{f_n}P - \mathtt{T}_{\Phi f}P\|_{L^p(\Omega)}\\ &\le (2C+1)\epsilon\end{aligned}$$

provided n is large enough; since ϵ was arbitrary the proof is complete. □

From remark 122, lemma 128 and the fact that

$$\lim_{K\to\infty}\mathcal{E}_i^K(s,t) = \mathcal{E}_i(s,t), \qquad \text{in } L^p(\Omega) \text{ for all } p \ge 1$$

it follows that

$$\lim_{K\to\infty} L\mathbb{E}\left[|\mathtt{T}_{-\Phi\sigma_i^K(s,t)}X(s) - \mathtt{T}_{-\Phi\sigma_i(s,t)}X(s)|\mathcal{E}_i^K(s,t)\right] = 0,$$

completing the proof.

4.5 Proof theorem 116

Let $\varphi \in C_0^2([0,T]\times\mathbb{R}^d)$, i.e. a two times continuously differentiable function on $[0,T]\times\mathbb{R}^d$ with compact support, and in order to ease the notation we set $\mathbf{z} := \mathbf{x}|_{x_{ki}=Z_k^{(i)}}$.

Then by 4.18 we have

$$
\begin{aligned}
0 &= \varphi(T, X^K(T)) - \varphi(0, c) \\
&= \int_0^T \left[\partial_t \varphi(r, u(r, \mathbf{z})) + \sum_{i=1}^d \partial_i \varphi(r, u(r, \mathbf{z})) \partial_t u_i(r, \mathbf{z}) \right] dr \\
&= \int_0^T \partial_t \varphi(r, u(r, \mathbf{z})) dr \\
&+ \sum_{i=1}^d \int_0^T \partial_i \varphi(r, u(r, \mathbf{z})) \left(b_i(t, u(r, \mathbf{z})) + \sigma_i(t) \sum_{k=1}^K \zeta_k(t) \left[x_{ki} u_i(r, \mathbf{z}) - \partial_{x_{ki}} u_i(r, \mathbf{z}) \right] \right) dr \\
&= \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A} &= \int_0^T \partial_t \varphi(r, u(r, \mathbf{z})) dr, \\
\mathcal{B} &= \int_0^T \nabla \varphi(r, u(r, \mathbf{z})) * b(s, u(r, \mathbf{z})) dr, \\
\mathcal{C} &= \sum_{i=1}^d \sum_{k=1}^K \int_0^T \partial_i \varphi(r, u(r, \mathbf{z})) \sigma_i(r) \zeta_k(r) [x_{ki} u_i(r, \mathbf{z})] dr, \\
\mathcal{D} &= - \sum_{i=1}^d \sum_{k=1}^K \int_0^T \partial_i \varphi(r, u(r, \mathbf{z})) \sigma_i(r) \zeta_k(r) \partial_{ik} u_i(r, \mathbf{z}) dr,
\end{aligned}
$$

where $*$ denotes the inner product in $\mathbb{R}^d$. Taking expectation to the first and last term above we obtain

$$
0 = \mathbb{E}[\mathcal{A}] + \mathbb{E}[\mathcal{B}] + \mathbb{E}[\mathcal{C}] + \mathbb{E}[\mathcal{D}].
$$

Now using the fact that $\mathbf{z}$ is a standard Gaussian matrix where the components are mutually independent, we have

$$
\mathbb{E}[\mathcal{C}] = \sum_{i=1}^d \sum_{k=1}^K \int_0^T \sigma_i(r) \zeta_k(r) \int_{\mathbb{R}^{K \times d}} \partial_i \varphi(r, u(r, \mathbf{x})) u_i(r, \mathbf{x}) x_{ki} (2\pi)^{-K \times d/2} e^{-\frac{1}{2} \|\mathbf{x}\|_F^2} d\mathbf{x} dr,
$$

integration by parts yields

$$= \sum_{i=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\int_{\mathbb{R}^{K\times d}} \partial_i\varphi(r,u(r,\mathbf{x}))u_i(r,\mathbf{x})(2\pi)^{-K\times d/2}\partial_{x_{ki}}e^{-\frac{1}{2}\|\mathbf{x}\|_F^2}d\mathbf{x}dr$$

$$= \sum_{i,j=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\int_{\mathbb{R}^{K\times d}} \partial_j\partial_i\varphi(r,u(r,\mathbf{x}))\partial_{x_{ki}}u_j(r,\mathbf{x})u_i(r,\mathbf{x})(2\pi)^{-K\times d/2}\partial_{x_{ki}}e^{-\frac{1}{2}\|\mathbf{x}\|_F^2}d\mathbf{x}dr$$

$$+ \sum_{i=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\int_{\mathbb{R}^{K\times d}} \partial_i\varphi(r,u(r,\mathbf{x}))\partial_{x_{ki}}u_i(r,\mathbf{x})(2\pi)^{-K\times d/2}\partial_{x_{ki}}e^{-\frac{1}{2}\|\mathbf{x}\|_F^2}d\mathbf{x}dr$$

$$= \mathbb{E}\left[\sum_{i,j=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\partial_j\partial_i\varphi(r,u(r,\mathbf{z}))\partial_{x_{ki}}u_j(r,\mathbf{z})u_i(r,\mathbf{z})dr\right]$$

$$+ \mathbb{E}\left[\sum_{i=1}^{d}\sum_{k=1}^{K}\int_0^T \partial_i\varphi(r,u(r,\mathbf{z}))\sigma_i(r)\zeta_k(r)\partial_{x_{ki}}u_i(r,\mathbf{z})dr\right]$$

and now notice that the last term above equals $-\mathbb{E}[\mathcal{D}]$.

At this point we have that

$$0 = \mathbb{E}[\mathcal{A}] + \mathbb{E}[\mathcal{B}] + \mathbb{E}\left[\sum_{i,j=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\partial_j\partial_i\varphi(r,u(r,\mathbf{z}))\partial_{x_{ki}}u_j(r,\mathbf{z})u_i(r,\mathbf{z})dr\right].$$

Using Tower's property yields

$$0 = \mathbb{E}[\mathcal{A}] + \mathbb{E}[\mathcal{B}] + \mathbb{E}\left[\sum_{i,j=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\partial_j\partial_i\varphi(r,u(r,\mathbf{z}))u_i(r,\mathbf{z})\mathbb{E}[\partial_{x_{ki}}u_j(r,\mathbf{z})|\mathcal{G}_i(r)]dr\right]$$

$$= \mathbb{E}[\mathcal{A}] + \mathbb{E}[\mathcal{B}] + \mathbb{E}\left[\sum_{i,j=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\partial_j\partial_i\varphi(r,u(r,\mathbf{z}))u_i(r,\mathbf{z})g_{ki}^{(j)}(r,u_i(r,\mathbf{z}))dr\right]$$

where $\mathcal{G}_i(r)$ is the sigma algebra generated by the random variable $u_i(r,\mathbf{z})$, and the function $g_{ki}^{(j)} : [0,T]\times\mathbb{R}^{K\times d}$ is a measurable function, whose existence is guaranteed by the Doob's lemma chosen to satisfy

$$g_{ki}^{(j)}(r,u_i(r,\mathbf{z})) = \mathbb{E}[\partial_{x_{ki}}u_j(r,\mathbf{z})|\mathcal{G}_i(r)]. \tag{4.23}$$

Putting everything together and using (4.18) we obtain

$$0 = \mathbb{E}\Big[\int_0^T \partial_t\varphi(r,X^K(r))dr + \int_0^T \nabla\varphi(r,X^K(r)) * b(s,X^K(r))dr$$
$$+ \sum_{i,j=1}^{d}\sum_{k=1}^{K}\int_0^T \sigma_i(r)\zeta_k(r)\partial_j\partial_i\varphi(r,X^K(r))X_i^k(r)g_{ki}^{(j)}(r,X_i^K(r))dr\Big]$$

Observe that the last member above contains expectations of functions of the random vector $X^K(r)$, for $r \in [0,T]$; therefore, writing the law of this random vector as

$$\mu^K(r,A) := \mathbb{P}(\{\omega \in \Omega : X^K(r,\omega) \in A\}), \quad r \in [0,T], A \in \mathcal{B}(\mathbb{R}^d)$$

$$\begin{aligned}0 = &\int_0^T \int_{\mathbb{R}^d} \Bigg[\partial_t \varphi(r,x) + \nabla\varphi(r,x) * b(s,x) \\ &+ \sum_{i,j=1}^{d} \sum_{k=1}^{K} \int_0^T \sigma_i(r)\zeta_k(r)\partial_j\partial_i\varphi(r,x)x_i g_{ki}^{(j)}(r,x_i)dr\Bigg] d\mu^K(r,x)dr.\end{aligned}$$

The last equalities hold for any test function $\varphi \in C_0^2([0,T]\times\mathbb{R}^d)$ and this completes the proof.

Appendix A

Fix $s,t \in [0,T]$ and without loss of generality assume that $t \geq s$, then using the basic inequality $|e^X - e^Y| \leq |e^X + e^Y| \cdot |X - Y|$ it holds that

$$\begin{aligned}|\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)| &\leq |\mathcal{E}_i^K(s,t) + \mathcal{E}_i(s,t)| \\ &\quad \times \left| \int_0^T \sigma_i^K(t,s;q)dB_i^H(q) - \frac{1}{2}|\sigma_i^K(t,s;\cdot)|_\phi^2 - \int_0^t \sigma_i(s)dB_i^H(s) + \frac{1}{2}|\chi_{[0,t]}\sigma_i|_\phi^2 \right| \\ &\leq |\mathcal{E}_i^K(s,t) + \mathcal{E}_i(s,t)| \\ &\quad \times \left(\left| \int_0^T [\sigma_i^K(t,s;q) - \sigma_i(q)]dB_i^H(q) \right| + \frac{1}{2}\left| |\sigma_i^K(t,s;\cdot)|_\phi^2 - |\chi_{[s,t]}\sigma_i|_\phi^2 \right| \right).\end{aligned}$$

Now let's write

$$\begin{aligned}\left| |\sigma_i^K(t,s;\cdot)|_\phi^2 - |\chi_{[s,t]}\sigma_i|_\phi^2 \right| &\leq \left| (|\sigma_i^K(t,s;\cdot)|_\phi + |\chi_{[s,t]}\sigma_i|_\phi)(|\sigma_i^K(t,s;\cdot)|_\phi - |\chi_{[s,t]}\sigma_i|_\phi) \right| \\ &\leq 2S^2T^{2H}|\sigma_i^K(t,s) - \chi_{[s,t]}\sigma_i|_\phi\end{aligned}$$

where we used the triangular inequality and S is a constant such that $|\sigma_i(t)| \leq S$ for all $t \in [0,T]$.

At this point we raise both sides to the $p \geq 1$ and take expectation yielding

$$\begin{aligned}\mathbb{E}\left[|\mathcal{E}_i^K(s,t) - \mathcal{E}_i(s,t)|^p\right] \leq 2^{p-1}\mathbb{E}\Bigg[&|\mathcal{E}_i^K(s,t) + \mathcal{E}_i(s,t)|^p \Bigg(|I(\sigma_i^K(t,s) - \chi_{[s,t]}\sigma_i)|^p \\ &+ S^{2p}T^{2Hp}|\sigma_i^K(t,s) - \chi_{[s,t]}\sigma_i|_\phi^p \Bigg)\Bigg]\end{aligned}$$

where $I(\cdot)$ denotes the fractional Wiener integral. Using Hölder's inequality where $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ we have

$$\leq 2^{p-1}\Big\{\|\mathcal{E}_i^K(s,t)+\mathcal{E}_i(s,t)\|_{p_1}^p\|I(\sigma_i^K(t,s)-\chi_{[s,t]}\sigma_i)\|_{p_2}^p$$

$$+S^{2p}T^{2Hp}\|\mathcal{E}_i^K(s,t)+\mathcal{E}_i(s,t)\|_p^p|\sigma_i^K(t,s)-\chi_{[s,t]}\sigma_i|_\phi^p\Big\}$$

$$\leq 2^{p-1}\Big\{\left(2^p e^{p(p_1-1)/2|\chi_{[s,t]}\sigma_i|_\phi^2}\right)2^{p/2}\Gamma(p_2+1)^{p/p_2}/\sqrt{\pi}|\sigma_i^K(t,s)-\chi_{[s,t]}\sigma_i|_\phi^p$$

$$+S^{2p}T^{2Hp}\left(2^p e^{p(p-1)/2|\chi_{[s,t]}\sigma_i|_\phi^2}\right)|\sigma_i^K(t,s)-\chi_{[s,t]}\sigma_i|_\phi^p\Big\}$$

and hence

$$\mathbb{E}\left[|\mathcal{E}_i^K(s,t)-\mathcal{E}_i(s,t)|^p\right]\leq \mathtt{C}|\sigma_i^K(t,s)-\chi_{[s,t]}\sigma_i|_\phi^p\to 0,$$

as $K\to\infty$ where

$$\mathtt{C}=2^{3p/2-1}e^{p(p_1-1)/2|\chi_{[s,t]}\sigma_i|_\phi^2}\Gamma(p_2+1)^{p/p_2}/\sqrt{\pi}+2^{2p-1}S^{2p}T^{2Hp}e^{p(p-1)/2|\chi_{[s,t]}\sigma_i|_\phi^2}.$$

Chapter 5

A small time approximation for the solution to the Zakai Equation

Based on: Lanconelli, A., & Scorolli, R. (2021). A small time approximation for the solution to the Zakai Equation. Potential Analysis, 1-11.

Abstract

We propose a novel small time approximation for the solution to the Zakai equation from nonlinear filtering theory. We prove that the unnormalized filtering density is well described over short time intervals by the solution of a deterministic partial differential equation of Kolmogorov type; the observation process appears in a pathwise manner through the degenerate component of the Kolmogorov's type operator. The rate of convergence of the approximation is of order one in the lenght of the interval. Our approach combines ideas from Wong-Zakai-type results and Wiener chaos approximations for the solution to the Zakai equation. The proof of our main theorem relies on the well-known Feynman-Kac representation for the unnormalized filtering density and careful estimates which lead to completely explicit bounds.

5.1 Introduction and statement of the main result

In this chapter we derive an original small time approximation for the solution to the so called *Zakai equation*

$$u(t,x) = u_0(x) + \int_0^t \mathscr{L}^\star(x)u(s,x)ds + \int_0^t h(x)u(s,x)dY(s), \quad (t,x) \in [0,1] \times \mathbb{R}^d \tag{5.1}$$

Here:

- $\mathscr{L}^\star(x)$ is the formal adjoint of $\mathscr{L}(x)$, infinitesimal generator of the d-dimensional *signal* process $\{X(t)\}_{t\in[0,1]}$ which is assumed to solve the stochastic differential equation

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s), \quad t \in [0,1]; \tag{5.2}$$

  the process $\{B(t)\}_{t\in[0,1]}$ is a standard d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;

- $\{Y(t)\}_{t\in[0,1]}$ is the one-dimensional *observation* process described by

$$Y(t) = y_0 + \int_0^t h(X(s))ds + W(t), \quad t \in [0,1], \tag{5.3}$$

  with $\{W(t)\}_{t\in[0,1]}$ being a standard one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of $\{B(t)\}_{t\in[0,1]}$.

The solution $\{u(t,x)\}_{t\in[0,1],x\in\mathbb{R}^d}$ to the Zakai equation (5.1), usually called *unnormalized filtering density*, plays a crucial role in the nonlinear filtering problem since it identifies uniquely the conditional distribution of $X(t)$ given $\mathcal{F}^Y(t) := \sigma(Y(s), 0 \leq s \leq t)$. The reader is referred to the original paper [59] and the references quoted there; for an exhaustive treatment of the subject we suggest the excellent review [60], as well as the books [61] and [62].
Existence, uniqueness and regularity properties for the solution to (5.1) can be found for instance, under different sets of assumptions and solution concepts, in the classic works [63],[64],[65],[66] and the more recent paper [67]. We also mention a useful Feynman-Kac representation for the solution $\{u(t,x)\}_{t\in[0,1],x\in\mathbb{R}^d}$ obtained in [65] and, in a slightly different form, in [67]. This representation will play a crucial role in our investigation.
From the applications point of view, closed form expressions for the solution to the Zakai equation are certainly desirable; however, as pointed in [68] only few particular cases of (5.1) allow for explicit computations. The important issue of deriving simple approximation schemes for the solution to (5.1) have been considered in [69] and [70] which employ splitting up methods and time discretization, respectively; Wong-Zakai-type results were investigated in [71] and [72] while [63] and [73] proposed a Wiener chaos approach. We also mention the so called *pathwise filtering* that steams from the problem of having a robust, with respect to the observation process, filter; this has been discussed in [74] and [75].
The approach proposed in the current paper combines ideas from the Wong-Zakai approximation proposed in[72], where the signal process is smoothed through a polygonal approximation, and the Wiener chaos approach presented in [63] and [73], where one relates equation (5.1) to a system of nested deterministic partial differential equations solved by the kernels of the Cameron-Martin decomposition of the solution $\{u(t,x)\}_{t\in[0,1],x\in\mathbb{R}^d}$. We refer the reader to Remark 126 below for the

heuristic idea supporting our analysis and its link to the aforementioned approaches. The main novelty of our result is the connection between equation (5.1) and a deterministic partial differential equation of Kolmogorov type (see e.g. [76]), where the observation process enters as a degenerate component of the second order differential operator $\mathscr{L}^\star(x)$. We prove that the solution $\{u(t,x)\}_{t\in[0,1],x\in\mathbb{R}^d}$ to the Zakai equation (5.1) can be approximated over small intervals of time by the solution of the aforementioned degenerate partial differential equation, with the observation process having a pathwise role. This approximation has the same rate of convergence of one obtained in [73] and is described by completely explicit constants.
To be more specific, we now introduce some notation and state our main result. In the sequel the following regularity conditions will be in force.

Assumption 4.

1. *For $1 \le i,j \le d$, the functions $b_i : \mathbb{R}^d \to \mathbb{R}^d$ and $a_{ij} : \mathbb{R}^d \to \mathbb{R}^d$, where*

$$a_{ij}(x) := \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x), \quad x \in \mathbb{R}^d, \tag{5.4}$$

   are bounded with bounded partial derivatives up to the third order. Moreover, the matrix $\{a_{ij}(x)\}_{1\le i,j\le d}$ is uniformly elliptic, i.e. there exists two positive constants $\mu_1 < \mu_2$ such that

$$\mu_1|z|^2 \le \sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \le \mu_2 |z|^2, \qquad \textit{for all } z \in \mathbb{R}^d,$$

   with $|z|^2 := z_1^2 + \cdots + z_d^2$.

2. *The initial data X_0 in (5.2) is random, independent of $\{B(t)\}_{t\in[0,1]}$ and its distribution is absolutely continuous with respect to the d-dimensional Lebesgue measure; its density $u_0 : \mathbb{R}^d \to \mathbb{R}$ is bounded and acts as initial data in (5.1).*

3. *The function $h : \mathbb{R}^d \to \mathbb{R}$ is bounded and globally Lipschitz continuous.*

Remark 124. We observe that, according to Assumption 4, there exists a positive constant L such that

$$|h(x_1) - h(x_2)| \le L|x_1 - x_2|, \qquad \text{for all } x_1, x_2 \in \mathbb{R}^d. \tag{5.5}$$

Moreover, there exists a positive constant M such that

$$\max\{|a(x)|^2, |b^\star(x)|\} \le M, \qquad \text{for all } x \in \mathbb{R}^d, \tag{5.6}$$

where $b_i^\star(x) := \sum_{j=1}^d \partial_{x_j} a_{ij}(x) - b_i(x)$, $i = 1, ..., d,$. We will need these two constants in the statement of our main theorem.

According to the Girsanov theorem and thanks to the assumption of boundedness on h, the prescription

$$\widetilde{\mathbb{P}}(A) := \int_A e^{-\int_0^1 h(X(s)(\omega))dW_s(\omega) - \frac{1}{2}\int_0^1 h(X(s)(\omega))^2 ds} d\mathbb{P}(\omega), \quad A \in \mathcal{F},$$

defines a probability measure on $(\Omega, \mathcal{F})$; moreover, the stochastic process $\{Y(t) - y_0\}_{t\in[0,1]}$ in (5.3) becomes on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ a one dimensional Brownian motion independent of $\{B(t)\}_{t\geq 0}$. In the sequel we will write $\widetilde{\mathbb{E}}$ to denote the expectation under the probability measure $\widetilde{\mathbb{P}}$.

We are now ready to state our main result.

Theorem 125. *Let Assumption 4 be in force and, for $0 < T < 1$, let*

$$[0, T] \times \mathbb{R}^d \times \mathbb{R} \ni (t, x, y) \mapsto v(t, x, y)$$

be a classical solution of the Cauchy problem

$$\begin{cases} \partial_t v(t, x, y) = \mathscr{L}^\star(x)v(t, x, y) - h(x)\partial_y v(t, x, y), & (t, x, y) \in]0, T] \times \mathbb{R}^d \times \mathbb{R}; \\ v(0, x, y) = u_0(x)e^{-\frac{y^2}{2T}}, & (x, y) \in \mathbb{R}^d \times \mathbb{R}. \end{cases} \tag{5.7}$$

Then, for any $q \geq 1$ and $K > 0$, we have

$$\sup_{|x|\leq K} \widetilde{\mathbb{E}}\left[\left|u(T, x) - e^{\frac{(Y_T - y_0)^2}{2T}} v(T, x, Y_T - y_0)\right|^q\right]^{1/q} \leq \mathtt{C}T,$$

with

$$\mathtt{C} := \frac{2}{\sqrt{3}}\|u_0\|_\infty e^{T\left(\|c\|_\infty + \frac{q_1 - 1}{2}\|h\|_\infty^2 + \sqrt{M} + M/2\right)}\left(\kappa(q_2) + \sqrt{T}\|h\|_\infty\right) L\sqrt{2(1 + K^2)(1 + T)}.$$

Here L and M are defined in (5.5) and (5.6), respectively; the constants $q_1, q_2 \geq 1$ verify the identity $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$; $\kappa(q_2)$ is given by $\sqrt{2}\left(\Gamma(\frac{q_2+1}{2})/\sqrt{\pi}\right)^{1/q_2}$; $\|u_0\|_\infty$ and $\|h\|_\infty$ denotes the $L^\infty(\mathbb{R}^d)$-norms of u_0 and h, respectively.

Remark 126. The heuristic idea that links equation (5.1) to equation (5.7) is as follows. Write (5.1) in the differential form

$$\partial_t u(t, x) = \mathscr{L}^\star(x)u(t, x) + h(x)u(t, x) \diamond \frac{dY(t)}{dt}, \quad u(0, x) = u_0(x), \tag{5.8}$$

where $\diamond$ denotes the Wick product associated to the Brownian motion $\{Y(t) - y_0\}_{t\in[0,1]}$ on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$. The use of the Wick product is dictated by the Itô's interpretation of (5.1) (see [38], [16] or section 2.3.7 for a discussion on this issue and detailed analysis of the Wick product). If equation (5.8) is considered on a small time interval $[0,T]$, one may replace $\frac{dY(t)}{dt}$ with $\frac{Y(T)-y_0}{T}$ (this amounts at considering a Wong-Zakai approximation with the rudest possible partition of the interval $[0,T]$); this gives

$$\partial_t u(t,x) = \mathscr{L}^\star(x)u(t,x) + \frac{h(x)}{T}u(t,x)\diamond(Y(T)-y_0), \quad u(0,x)=u_0(x). \tag{5.9}$$

Since, $Y(T)-y_0=\int_0^1 \chi_{[0,T]}(s)dY(s)$ is an element in the first Wiener chaos associated with the Brownian motion $\{Y(t)-y_0\}_{t\in[0,1]}$ and probability space $(\Omega,\mathcal{F},\widetilde{\mathbb{P}})$, we can formally apply proposition 101 to transform equation (5.9) into

$$\partial_t u(t,x) = \mathscr{L}^\star(x)u(t,x) + \frac{h(x)}{T}\left[u(t,x)\cdot(Y(T)-y_0) - D_{\chi_{[0,T]}}u(t,x)\right]. \tag{5.10}$$

We now search for a solution $u(t,x)$ to equation (5.10) of the form

$$u(t,x,\omega) = \tilde{u}(t,x,Y(T)(\omega)-y_0), \tag{5.11}$$

for some $\tilde{u}:[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ to be determined. A substitution of (5.11) in (5.10) yields, together with the chain rule for the Malliavin derivative,

$$\begin{aligned}\partial_t\tilde{u}(t,x,Y(T)-y_0) =& \mathscr{L}^\star(x)\tilde{u}(t,x,Y(T)-y_0) + \frac{h(x)}{T}\tilde{u}(t,x,Y(T)-y_0)(Y(T)-y_0)\\ &- h(x)\partial_y\tilde{u}(t,x,Y(T)-y_0);\end{aligned}$$

note that here the term $Y(T)-y_0$ can be tackled at a path-wise level. Equation (5.7) is now obtained via the simple transformation

$$v(t,x,y) := \tilde{u}(t,x,y)e^{-\frac{y^2}{2T}}, \quad t\in[0,T], x\in\mathbb{R}^d, y\in\mathbb{R}.$$

It is not difficult to see, using Theorem 4.12 in [16] and the Feynman-Kac representation for $\{u(t,x)\}_{t\in[0,1],x\in\mathbb{R}^d}$ in [67], that we also have

$$\widetilde{\mathbb{E}}[u(T,x)|Y_T-y_0] = e^{\frac{(Y_T-y_0)^2}{2T}}v(T,x,Y_T-y_0);$$

this spots the analogy between our approach and the one in [73] where projections of $u(T,x)$ on suitable families of elements from the Wiener chaos were utilized to propose approximation schemes for the solution to (5.1).

Remark 127. The existence of a classical solution for the Cauchy problem (5.7) is actually not needed for the validity of Theorem 125 (the statement is presented this way for easiness of exposition). In fact, in the proof of our main result we deal with the Feynman-Kac representation for the solution to (5.7) (see formula (5.13) below) without using its differentiability properties with respect to t and x. The right hand side of (5.13) is well defined under mild conditions on the coefficients of equation (5.7) (largely covered by Assumption 4) and this makes our proof consistent. It is worth mentioning that the right hand side of (5.13) becomes a classical solution if suitable regularity assumptions on the coefficients of (5.12) are in force. For more details on this issue we refer the reader to [77] and [11, page 122] .

5.2 Proof of Theorem 125

We start with some notation. The infinitesimal generator $\mathscr{L}(x)$ of the signal process $\{X(t)\}_{t\in[0,1]}$ in (5.2) is

$$\mathscr{L}(x)f(x)=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(x)\partial^2_{x_ix_j}f(x)+\sum_{i=1}^{d}b_i(x)\partial_{x_i}f(x),$$

where the $a_{ij}(x)$'s are defined in (5.4). The adjoint operator $\mathscr{L}^\star(x)$ is given by

$$\mathscr{L}^\star(x)f(x)=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(x)\partial^2_{x_ix_j}f(x)+\sum_{i=1}^{d}b_i^\star(x)\partial_{x_i}f(x)+c(x)f(x),$$

with

$$b_i^\star(x):=\sum_{j=1}^{d}\partial_{x_j}a_{ij}(x)-b_i(x),\quad i=1,...,d,$$

(see Remark 124) and

$$c(x):=\sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j,k=1}^{d}\partial^2_{x_jx_k}a_{ij}(x)-\partial_{x_k}b_i(x)\right).$$

It is convenient to split the operator $\mathscr{L}^\star(x)$ as

$$\mathscr{L}^\star(x)f(x)=\mathtt{L}^\star(x)f(x)+c(x)f(x)$$

where we set

$$\mathtt{L}^\star(x)f(x):=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(x)\partial^2_{x_ix_j}f(x)+\sum_{i=1}^{d}b_i^\star(x)\partial_{x_i}f(x).$$

With this notation at hand, the Cauchy problem (5.7) takes the form

$$\begin{cases}\partial_t v(t,x,y)=\mathtt{L}^\star(x)v(t,x,y)+c(x)v(t,x,y)-h(x)\partial_y v(t,x,y)\\ \quad (t,x,y)\in]0,T]\times\mathbb{R}^d\times\mathbb{R};\\ v(0,x,y)=u_0(x)e^{-\frac{y^2}{2T}}.\end{cases}\tag{5.12}$$

Now, assume

$$[0,T]\times\mathbb{R}^d\times\mathbb{R}\ni(t,x,y)\mapsto v(t,x,y)$$

to be a classical solution of (3.26). According to the Feynman-Kac formula (see, for instance, Theorem 1.1, page 120 in [11]) we can write

$$\begin{aligned} v(T,x,y) &= \widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{-\frac{\left(y-\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}e^{\int_0^T c(\widehat{\xi}_s^x)ds}\right] \\ &= e^{-\frac{y^2}{2T}}\widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{\int_0^T c(\widehat{\xi}_s^x)ds}e^{\frac{y\int_0^T h(\widehat{\xi}_s^x)ds}{T}-\frac{\left(\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}\right], \end{aligned} \tag{5.13}$$

where $\{\widehat{\xi}_s^x\}_{s\in[0,1]}$ solves the SDE

$$d\widehat{\xi}_s^x = b^\star(\widehat{\xi}_s^x) + \sigma(\widehat{\xi}_s^x)d\widehat{B}(s), \quad \widehat{\xi}_0^x = x, \tag{5.14}$$

on the auxiliary probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ with d-dimensional Brownian motion $\{\widehat{B}(s)\}_{s\in[0,1]}$ and where $\widehat{\mathbb{E}}$ denotes the expectation on that space. This gives

$$e^{\frac{(Y(T)-y_0)^2}{2T}}v(T,x,Y(T)-y_0) = \widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{\int_0^T c(\widehat{\xi}_s^x)ds}e^{\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}_s^x)ds}{T}-\frac{\left(\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}\right]. \tag{5.15}$$

It is well known that the solution $u(t,x)$ to the Zakai equation (5.1) also possesses a Feynman-Kac representation: see formula (1.4) page 132 in [65]. Here, we use instead an equivalent formulation due to [67] (see formula (2.9) there), namely

$$u(T,x) = \widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{\int_0^T c(\widehat{\xi}_s^x)ds}e^{\int_0^T h(\widehat{\xi}_{T-s}^x)dY(s)-\frac{1}{2}\int_0^T h^2(\widehat{\xi}_s^x)ds}\right], \tag{5.16}$$

where $\{\widehat{\xi}_s^x\}_{s\in[0,1]}$ is defined in (5.14). A comparison between (5.15)and (5.16)gives

$$\begin{aligned} &u(T,x) - e^{\frac{(Y(T)-y_0)^2}{2T}}v(T,x,Y(T)-y_0) \\ &\quad= \widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{\int_0^T c(\widehat{\xi}_s^x)ds}e^{\int_0^T h(\widehat{\xi}_{T-s}^x)dY(s)-\frac{1}{2}\int_0^T h^2(\widehat{\xi}_s^x)ds}\right] \\ &\qquad- \widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{\int_0^T c(\widehat{\xi}_s^x)ds}e^{\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}_s^x)ds}{T}-\frac{\left(\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}\right] \\ &\quad= \widehat{\mathbb{E}}\left[u_0(\widehat{\xi}_T^x)e^{\int_0^T c(\widehat{\xi}_s^x)ds}\left(e^{\int_0^T h(\widehat{\xi}_{T-s}^x)dY(s)-\frac{1}{2}\int_0^T h^2(\widehat{\xi}_s^x)ds} - e^{\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}_s^x)ds}{T}-\frac{\left(\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}\right)\right], \end{aligned}$$

and hence

$$\begin{aligned} &\left|u(T,x) - e^{\frac{(Y(T)-y_0)^2}{2T}}v(T,x,Y(T)-y_0)\right| \\ &\quad\leq \widehat{\mathbb{E}}\left[|u_0(\widehat{\xi}_T^x)|e^{\int_0^T c(\widehat{\xi}_s^x)ds}\left|e^{\int_0^T h(\widehat{\xi}_{T-s}^x)dY(s)-\frac{1}{2}\int_0^T h^2(\widehat{\xi}_s^x)ds} - e^{\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}_s^x)ds}{T}-\frac{\left(\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}\right|\right] \\ &\quad\leq \|u_0\|_\infty e^{T\|c\|_\infty}\widehat{\mathbb{E}}\left[\left|e^{\int_0^T h(\widehat{\xi}_{T-s}^x)dY(s)-\frac{1}{2}\int_0^T h^2(\widehat{\xi}_s^x)ds} - e^{\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}_s^x)ds}{T}-\frac{\left(\int_0^T h(\widehat{\xi}_s^x)ds\right)^2}{2T}}\right|\right]. \end{aligned}$$

We now take $q \geq 1$ and compute the $L^q(\widetilde{\mathbb{P}})$-norm of the first and last members above; an application of Minkowsky's inequality gives

$$\begin{aligned} &\left\| u(T,x) - e^{\frac{(Y(T)-y_0)^2}{2T}} v(T,x,Y(T)-y_0) \right\|_{L^q(\widetilde{\mathbb{P}})} \\ &\quad \leq \|u_0\|_\infty e^{T\|c\|_\infty} \widehat{\mathbb{E}} \left[\left\| e^{\int_0^T h(\widehat{\xi}^x_{T-s})dY(s) - \frac{1}{2}\int_0^T h^2(\widehat{\xi}^x_s)ds} - e^{\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}^x_s)ds}{T} - \frac{\left(\int_0^T h(\widehat{\xi}^x_s)ds\right)^2}{2T}} \right\|_{L^q(\widetilde{\mathbb{P}})} \right]. \end{aligned} \tag{5.17}$$

We need the following result.

Lemma 128. *Let $f, g : [0,T] \to \mathbb{R}$ be bounded measurable deterministic functions. Then, for any $q \geq 1$ we have*

$$\begin{aligned} &\widetilde{\mathbb{E}}\left[\left| e^{\int_0^T f(s)dY(s) - \frac{1}{2}|f|_2^2} - e^{\int_0^T g(s)dY(s) - \frac{1}{2}|g|_2^2} \right|^q \right]^{\frac{1}{q}} \\ &\quad \leq \left(e^{\frac{q_1-1}{2}T\|f\|_\infty^2} + e^{\frac{q_1-1}{2}T\|g\|_\infty^2} \right) \left(\kappa(q_2) + \frac{\sqrt{T}}{2}(\|f\|_\infty + \|g\|_\infty) \right) |f-g|_2, \end{aligned}$$

where $q_1, q_2 \geq 1$ satisfy $1/q_1 + 1/q_2 = 1/q$ while $\kappa(q_2) := \sqrt{2}\left(\Gamma(\frac{q_2+1}{2})/\sqrt{\pi}\right)^{1/q_2}$. Moreover, $|\bullet|_2$ and $\|\bullet\|_\infty$ stand for the $L^2([0,T])$ and $L^\infty([0,T])$ norms respectively (not to be confused with the p-seminorm defined in section 2.3.1)

Proof. By means of the elementary inequality $|e^a - e^b| \leq (e^a + e^b)|a-b|$ we can write

$$\begin{aligned} &\left| e^{\int_0^T f(s)dY(s) - \frac{1}{2}|f|_2^2} - e^{\int_0^T g(s)dY(s) - \frac{1}{2}|g|_2^2} \right| \\ &\quad \leq \left(e^{\int_0^T f(s)dY(s) - \frac{1}{2}|f|_2^2} + e^{\int_0^T g(s)dY(s) - \frac{1}{2}|g|_2^2} \right) \\ &\quad\quad \cdots \times \left| \int_0^T [f(s)-g(s)]dY(s) - \frac{1}{2}\left(|f|_2^2 - |g|_2^2\right) \right| \\ &\quad \leq \left(e^{\int_0^T f(s)dY(s) - \frac{1}{2}|f|_2^2} + e^{\int_0^T g(s)dY(s) - \frac{1}{2}|g|_2^2} \right) \\ &\quad\quad \cdots \times \left(\left| \int_0^T [f(s)-g(s)]dY(s) \right| + \frac{1}{2}\left| |f|_2^2 - |g|_2^2 \right| \right). \end{aligned}$$

Now, for $q \geq 1$ we take the $L^q(\widetilde{\mathbb{P}})$-norm of the first and last members above and apply Hölder's inequality with exponents $q_1, q_2 \geq 1$ satisfying $1/q_1 + 1/q_2 = 1/q$.

This gives

$$\begin{aligned}
&\left\| e^{\int_0^T f(s)dY(s)-\frac{1}{2}|f|_2^2} - e^{\int_0^T g(s)dY(s)-\frac{1}{2}|g|_2^2} \right\|_{L^q(\widetilde{\mathbb{P}})} \\
&\quad \le \left\| e^{\int_0^T f(s)dY(s)-\frac{1}{2}|f|_2^2} + e^{\int_0^T g(s)dY(s)-\frac{1}{2}|g|_2^2} \right\|_{L^{q_1}(\widetilde{\mathbb{P}})} \\
&\qquad \cdots \times \left(\left\| \int_0^T [f(s)-g(s)]dY(s) \right\|_{L^{q_2}(\widetilde{\mathbb{P}})} + \frac{1}{2}\left| |f|_2^2 - |g|_2^2 \right| \right). \qquad (5.18)
\end{aligned}$$

Under the measure $\widetilde{\mathbb{P}}$, the random variables $\int_0^T f(s)dY(s)$ and $\int_0^T g(s)dY(s)$ are Gaussian with mean zero and variances $|f|_2^2$ and $|g|_2^2$, respectively. Hence,

$$\begin{aligned}
&\left\| e^{\int_0^T f(s)dY(s)-\frac{1}{2}|f|_2^2} + e^{\int_0^T g(s)dY(s)-\frac{1}{2}|g|_2^2} \right\|_{L^{q_1}(\widetilde{\mathbb{P}})} \\
&\quad \le \left\| e^{\int_0^T f(s)dY(s)-\frac{1}{2}|f|_2^2} \right\|_{L^{q_1}(\widetilde{\mathbb{P}})} + \left\| e^{\int_0^T g(s)dY(s)-\frac{1}{2}|g|_2^2} \right\|_{L^{q_1}(\widetilde{\mathbb{P}})} \\
&\quad = e^{\frac{q_1-1}{2}|f|_2^2} + e^{\frac{q_1-1}{2}|g|_2^2} \\
&\quad \le e^{\frac{q_1-1}{2}T\|f\|_\infty^2} + e^{\frac{q_1-1}{2}T\|g\|_\infty^2}. \qquad (5.19)
\end{aligned}$$

Moreover, using once more the normality, under the measure $\widetilde{\mathbb{P}}$, of the random variable $\int_0^T [f(s)-g(s)]dY(s)$ we get

$$\left\| \int_0^T [f(s)-g(s)]dY(s) \right\|_{L^{q_2}(\widetilde{\mathbb{P}})} = \kappa(q_2)|f-g|_2, \qquad (5.20)$$

where $\kappa(q_2) := \sqrt{2}\left(\Gamma(\frac{q_2+1}{2})/\sqrt{\pi}\right)^{1/q_2}$ (see, for instance, Formula (1.1) in [16]). Furthermore,

$$\begin{aligned}
\left| |f|_2^2 - |g|_2^2 \right| &= (|f|_2 + |g|_2)\left| |f|_2 - |g|_2 \right| \\
&\le (|f|_2 + |g|_2)\,|f-g|_2 \\
&\le \sqrt{T}(\|f\|_\infty + \|g\|_\infty)|f-g|_2. \qquad (5.21)
\end{aligned}$$

Therefore, combining (5.18) with (5.19),(5.20) and (5.21) we get

$$\begin{aligned}
&\left\| e^{\int_0^T f(s)dY(s)-\frac{1}{2}|f|_2^2} - e^{\int_0^T g(s)dY(s)-\frac{1}{2}|g|_2^2} \right\|_{L^q(\widetilde{\mathbb{P}})} \\
&\le \left(e^{\frac{q_1-1}{2}T\|f\|_\infty^2} + e^{\frac{q_1-1}{2}T\|g\|_\infty^2} \right)\left(\kappa(q_2) + \frac{\sqrt{T}}{2}(\|f\|_\infty + \|g\|_\infty) \right)|f-g|_2.
\end{aligned}$$

The proof is complete. □

Thanks to the identities

$$\frac{(Y(T)-y_0)\int_0^T h(\widehat{\xi}_s^x)ds}{T} = \int_0^T \left(\frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr \right) dY(s),$$

and

$$\int_0^T \left(\frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr\right)^2 ds = \frac{\left(\int_0^T h(\widehat{\xi}_r^x)dr\right)^2}{T},$$

we are in a position to apply Lemma 128 to the last term in (5.17)with

$$f(s) := h(\widehat{\xi}_{T-s}^x) \quad \text{and} \quad g(s) := \frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr;$$

note that such choices imply $\|f\|_\infty \le \|h\|_\infty$ and $\|g\|_\infty \le \|h\|_\infty$ (here, the norms are on the corresponding domains). Therefore,

$$\begin{aligned}
&\left\| u(T,x) - e^{\frac{(Y(T)-y_0)^2}{2T}} v(T,x,Y(T)-y_0) \right\|_{L^q(\widetilde{\mathbb{P}})} \\
&\le 2\|u_0\|_\infty e^{T\left(\|c\|_\infty + \frac{q_1-1}{2}\|h\|_\infty^2\right)} \left(\kappa(q_2) + \sqrt{T}\|h\|_\infty\right) \\
&\quad \cdots \times \widehat{\mathbb{E}}\left[\left(\int_0^T \left| h(\widehat{\xi}_{T-s}^x) - \frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr\right|^2 ds\right)^{1/2}\right].
\end{aligned}$$

We now focus on the last expectation; using a combination of Jensen's inequalities and Tonelli's theorem we get

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[\left(\int_0^T \left| h(\widehat{\xi}_{T-s}^x) - \frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr\right|^2 ds\right)^{1/2}\right] \\
&\le \left(\widehat{\mathbb{E}}\left[\int_0^T \left| h(\widehat{\xi}_{T-s}^x) - \frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr\right|^2 ds\right]\right)^{1/2} \\
&= \left(\int_0^T \widehat{\mathbb{E}}\left[\left| h(\widehat{\xi}_{T-s}^x) - \frac{1}{T}\int_0^T h(\widehat{\xi}_r^x)dr\right|^2\right] ds\right)^{1/2} \\
&= \left(\int_0^T \widehat{\mathbb{E}}\left[\left| \frac{1}{T}\int_0^T (h(\widehat{\xi}_{T-s}^x) - h(\widehat{\xi}_r^x))dr\right|^2\right] ds\right)^{1/2} \\
&\le \left(\int_0^T \widehat{\mathbb{E}}\left[\frac{1}{T}\int_0^T |h(\widehat{\xi}_{T-s}^x) - h(\widehat{\xi}_r^x)|^2 dr\right] ds\right)^{1/2} \\
&= \left(\int_0^T \left(\frac{1}{T}\int_0^T \widehat{\mathbb{E}}\left[|h(\widehat{\xi}_{T-s}^x) - h(\widehat{\xi}_r^x)|^2\right] dr\right) ds\right)^{1/2}.
\end{aligned}$$

The Lipschitz continuity of h and Theorem 4.3, Chapter 2 in [78] yield

$$\begin{aligned}
\widehat{\mathbb{E}}\left[|h(\widehat{\xi}_{T-s}^x) - h(\widehat{\xi}_r^x)|^2\right] &\le L^2 \widehat{\mathbb{E}}\left[|\widehat{\xi}_{T-s}^x - \widehat{\xi}_r^x|^2\right] \\
&\le 2L^2(1+|x|^2)(1+T)e^{2(\sqrt{M}+M/2)T}|T-s-r|,
\end{aligned}$$

where L and M come from (5.5) and (5.6). Moreover,

$$\left(\int_0^T \left(\frac{1}{T}\int_0^T |T-s-r|dr\right) ds\right)^{1/2} = \frac{T}{\sqrt{3}}.$$

Combining all our estimates we obtain

$$\begin{aligned} &\left\| u(T,x) - e^{\frac{(Y(T)-y_0)^2}{2T}} v(T,x,Y(T)-y_0) \right\|_{L^q(\widetilde{\mathbb{P}})} \\ &\quad \leq \frac{2}{\sqrt{3}} \|u_0\|_\infty e^{T\left(\|c\|_\infty + \frac{q_1-1}{2}\|h\|_\infty^2 + \sqrt{M} + M/2\right)} \left(\kappa(q_2) + \sqrt{T}\|h\|_\infty\right) \\ &\qquad \cdots \times L\sqrt{2(1+|x|^2)(1+T)}T, \end{aligned}$$

as desired.

Chapter 6

An alternative derivation of the Feynman-Kac formula for the heat equation driven by spatial white noise potential

Abstract

We present an alternative derivation of the Feynman-Kac formula for the 1*-dimensional stochastic heat equation (SHE) driven by a space-only Gaussian white noise potential, where the noise is interpreted in the Wick-Itô-Skorokhod sense. Our approach consists in constructing a Wong-Zakai-type approximation for the SHE from which we are able to obtain an "approximating Feynman-Kac" representation via the reduction of the approximated SHE to a deterministic partial differential equation (PDE). Then we will show that those "approximating Feynman-Kac" converge to a well defined object we will call "formal Feynman-Kac" representation which happens to coincide with the unique solution of the SHE.*

6.1 Introduction.

In this work we will deal with the 1-dimensional stochastic heat equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\partial^2_{xx} u(t,x) + u(t,x) \diamond \dot{W}(x), & (t,x) \in]0,T] \times \mathbb{R} \\ u(0,x) = u_0(x), \end{cases} \tag{6.1}$$

driven by the (distributional) derivative of the Brownian motion $\{W(x)\}_{x\in\mathbb{R}}$ aka a (spatial) *spatial white noise process.*

The initial condition $u_0 : \mathbb{R} \to \mathbb{R}$ is assumed to be a bounded, deterministic Borel-measurable function. In light of what we discussed in section 2.3.7 the use of the Wick product in equation (6.1) implies that the noise should be interpreted in the Itô-Skorohod sense.

In the case of space-time white noise potential, equation (6.1) has been extensively studied (e.g. [79],[80],[81],[82] and references therein). On the other hand if the noise is assumed to be only time-dependent equation (6.1) is just a particular case of the well known *Zakai equation* that we treated in chapter 5.

Nevertheless the case of spatial white noise hasn't received the same attention. In [83] the author has showed that in this one-dimensional setting, equation (6.1) admits a unique *mild* solution which is square integrable for any $(t, x) \in [0, T] \times \mathbb{R}$; such a solution is constructed employing the Wiener Chaos expansion (see also Theorem 3.9 of [84]).

In [85] and [86] the author treated the SHE with spatial noise in the d-dimensional case, showing existence, and providing numerous estimations of the Lyapunov exponent of the solutions. The former treats the case of fractional Brownian motion, while in the latter the solution is showed to exist in a *flat* Hilbert space similar to those introduced by Kondratiev (see for instance [38]).

In [84] the authors studied, among other things, the existence and regularity of the multidimensional version of (6.1) when the covariance structure of $\{W(x)\}_{x\in\mathbb{R}^d}$ satisfies certain conditions. They also propose some formal Feynman-Kac representations for the solutions of the SHE in various settings. Although they hinted a way for constructing the Feynman-Kac representation for the solution of equation (6.1) the details were omitted.

In [87] the authors studied the SHE with space-only white noise potential in a bounded interval using the concept of Wiener chaos solution and *propagator* introduced in [88] (see also [73]). They obtain original estimates of the Hölder regularity of the solutions (in particular in space).

The aim of this chapter is to construct a Feynman-Kac representation for the unique mild solution of (6.1), using an alternative approach to that in [84], the main difference between the two being the way in which the noise is regularized.

As in the previous chapters we introduce an "approximating equation" which is reduced to a deterministic partial differential equation by means of the connection between the Wick product and the Malliavin derivative. Applying the classical Feynman-Kac formula to the latter we are able to derive a probabilistic representation for the Wong-Zakai approximation of the SHE. Then we show that as the regularization of the noise vanishes this representation converges to a well-defined random variable (for fixed t and x) and that this limit-object coincides with the unique mild solution of (6.1) present in the literature. It's worth noticing that due to the structure of the approximated noise the sequence of "approximated solutions" converges not only in the $L^p(\Omega)$ norm for any $p \in [1, \infty)$, but also almost surely.

Thus our achievement is two-fold, first we propose an alternative and original derivation of the Feynman-Kac formula for the solution of the SHE with purely spatial noise, and on the other hand it spots an interesting interconnection between the SHE and an approximating deterministic PDE.

6.2 Construction of the approximating equation.

For technical and notational convenience we will denote with $(\Omega, \mathcal{F}, \mathbb{P})$ the *White noise probability space* which was defined in section 2.3.3. Furthermore let $\{W(x)\}_{x\in\mathbb{R}}$ denote the Brownian motion defined in 2.31 and let $\mathbb{E}$ be the expectation operator in this space.

The aim of this section is that of introducing a Wong-Zakai-type approximation for (6.1), and thus the first thing to do is to construct an opportune smooth approximation of the singular White noise process $\{\dot{W}(x)\}_{x\in\mathbb{R}}$. As we already mentioned (see section 2.3.3) the point evaluation of singular white noise at $x \in \mathbb{R}$ could formally be seen as

$$\dot{W}(x) = \text{“}\int_{\mathbb{R}} \delta_x(y) dW(y)\text{”},$$

i.e. the stochastic integral of a Dirac delta function with mass at $x \in \mathbb{R}$.

One possible approximation could be obtained by truncating the Hermite expansion of δ_x (see section 2.3.1) up to a certain finite value K yielding:

$$\dot{W}^K(x) := \sum_{j=1}^{K} e_j(x) \int_{\mathbb{R}} e_j(y) dW(y), \tag{6.2}$$

(notice that the latter is nothing more than the derivative of a truncated Karuhnen-Loève expansion of the Brownian motion W) clearly $\dot{W}^K$ converges to $\dot{W}$ in $(S)^*$ (e.g. [38]). We can also write

$$\dot{W}^K(x) = \Gamma(P_K)\dot{W}(x),$$

where P_K is the projection operator on $\text{span}(e_1, ..., e_K)$ (see definition 139) and $\Gamma(P_K)$ denotes its second quantization.

If we substitute the singular white noise in (6.1) with (6.2) we obtain the following "*ala* Wong-Zakai approximating equation":

$$\begin{cases} \partial_t u^K_{t,x} = \frac{1}{2}\partial^2_{xx} u^K_{t,x} + u^K_{t,x} \diamond \dot{W}^K_x, & (t,x) \in]0,T] \times \mathbb{R} \\ u(0,x) = u_0(x), \end{cases} \tag{6.3}$$

where in order to simplify the notation we write W_x^K for $W^K(x)$.

Since the equation above involves non-trivial operations, such as taking the Wick product between the solution and a random potential we should state what a "solution" of the latter actually is. Following [80] we have:

Definition 129. Let K be any arbitrary positive integer, then random field u^K : $[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ is said to be a *mild solution* of (6.3) if for any fixed $(t,x)\in[0,T]\times\mathbb{R}$ we have that $u_{t,x}^K\in L^2(\mathbb{P})$ and for any random variable $F\in\mathbb{D}^{1,2}$ it holds that:

$$\begin{aligned}\mathbb{E}\left[u_{t,x}^K F\right] &= (P_t u_0)(x)\mathbb{E}[F] \\ &\quad + \mathbb{E}\left[\left(\left(\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\left(\sum_{j=1}^K e_j(y)e_j(\bullet)\right)u_{s,y}^K dy\, ds \quad , \quad D_{(\bullet)}F\right)_{L^2(\mathbb{R})}\right],\end{aligned} \tag{6.4}$$

where D denotes the Malliavin derivative and $\mathbb{D}^{1,2}$ the Malliavin-Sobolev space.

Remark 130. This definition is essentially equivalent to the concept of *soft solution* (e.g. [89][90]).

6.3 Reduction method

From now on we will write $Z_j := \int_{\mathbb{R}} e_j(y)dW(y),\ \ j\in\{1,...,K\}$, then using the linearity of the Wick product we can rewrite equation (6.3) as

$$\begin{cases}\partial_t u_{t,x}^K = \frac{1}{2}\partial_{xx}^2 u_{t,x}^K + \sum_{j=1}^K e_j(x)u_{t,x}^K\diamond Z_j, & (t,x)\in\,]0,T]\times\mathbb{R}\\ u(0,x)=u_0(x).\end{cases}$$

We can formally use proposition 101 and write

$$u_{t,x}^K\diamond Z_j = u_{t,x}^K\cdot Z_j - \left(Du_{t,x}^K, e_j\right)_{L^2(\mathbb{R})}.$$

Thus replacing this into the equation above yields

$$\begin{cases}\partial_t u_{t,x}^K = \frac{1}{2}\partial_{xx}^2 u_{t,x}^K + \sum_{j=1}^K\left[u_{t,x}^K\cdot Z_j - \left(Du_{t,x}^K, e_j\right)_{L^2(\mathbb{R})}\right], & (t,x)\in\,]0,T]\times\mathbb{R}\\ u(0,x)=u_0(x).\end{cases}$$

If we assume that the solution of this equation is of the form

$$u_{t,x}^K(\omega) = \mathfrak{u}^K(t,x,z_1,...,z_K)\big|_{z_k=Z_k(\omega),\ k=1,...,K}$$

for some (well behaved) function $\mathfrak{u}^K : [0,T] \times \mathbb{R} \times \mathbb{R}^K \to \mathbb{R}$ that must be determined then an application of the chain rule implies that

$$\left(Du^K_{t,x}, e_j\right)_{L^2(\mathbb{R})} = \sum_{i=1}^{K} \partial_{z_i} \mathfrak{u}^K_{t,x,z} \left(DZ_i, e_j\right)_{L^2(\mathbb{R})} = \partial_{z_j} \mathfrak{u}^K(t,x,z_1,...,z_K)\big|_{z_k = Z_k(\omega),\, k=1,...,K} \quad .$$

It follows that $\mathfrak{u}^K(t,x,z)$ must solve the deterministic Cauchy problem

$$\begin{cases} \partial_t \mathfrak{u}^K_{t,x,z} = \frac{1}{2}\partial^2_{xx}\mathfrak{u}^K_{t,x,z} + \sum_{j=1}^K e_j(x)[\mathfrak{u}^K_{t,x,z} \cdot z_j - \partial_{z_j}\mathfrak{u}^K_{t,x,z}], \\ (t,x,z) \in]0,T] \times \mathbb{R} \times \mathbb{R}^K \\ \mathfrak{u}^K(0,x) = u_0(x). \end{cases} \tag{6.5}$$

Notice that by doing so we've managed to reduce the *approximating* Wong-Zakai equation (6.3) to a deterministic partial differential equation. This in turn implies that we can obtain an approximation for the solution of (6.1) by solving a deterministic PDE. This by itself is an interesting result since it's generally true that dealing with PDEs is considerably easier than solving SPDEs.

Upon multiplying both sides of the equation by $\exp\left(-\frac{1}{2}\sum_{j=1}^K z_j^2\right)$ and defining

$$\mathfrak{v}^K_{t,x,z} := \mathfrak{u}^K_{t,x,z} \exp\left(-\frac{1}{2}\sum_{j=1}^K z_j^2\right),$$

we are able to get rid of the zero-order term and obtain the following:

$$\begin{cases} \partial_t \mathfrak{v}^K = \frac{1}{2}\partial^2_{xx}\mathfrak{v}^K - \sum_{j=1}^K e_j(x)\partial_{z_j}\mathfrak{v}^K, \\ \mathfrak{v}^K(0,x) = u_0(x) \times \exp\left(-\frac{1}{2}\sum_{j=1}^K z_j^2\right). \end{cases} \tag{6.6}$$

Proposition 131. *Straightforward computations show that the differential operator*

$$\mathscr{L} = \frac{1}{2}\partial^2_{xx} - \sum_{j=1}^K e_j(x)\partial_{z_j}$$

is the infinitesimal generator of a $(K+1)$*-dimensional stochastic process given by*

$$\left(B^x(t), z_1 - \int_0^t e_1(B^x(s))ds, ..., z_K - \int_0^t e_K(B^x(s))ds\right)_{t\in[0,T]},$$

where $\{B^x(t)\}_{t\in[0,T]}$ *is a* 1*-dimensional Brownian motion defined on the auxiliary probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *starting at* $x \in \mathbb{R}$*.*

Proof. Let

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad t \in]0,T], \quad X(0) = (x, z_1, ..., z_K)$$

where $b = (0, -e_1(X_1(t)), ..., -e_K(X_1(t)))^T$ and $\sigma = (1, 0, ..., 0)^T$. Then by [14, theorem 7.3.3] the infinitesimal generator of X is given by $\mathscr{L}$. □

The last proposition and a formal application of the classical Feynman-Kac formula yields

$$\mathfrak{v}^K(t,x,z) = \widetilde{\mathbb{E}}\left[u_0(B^x(t)) \exp\left\{ -\frac{1}{2}\sum_{j=1}^{K}\left(z_j - \int_0^t e_j(B^x(s))ds \right)^2 \right\} \right], \tag{6.7}$$

where $\widetilde{\mathbb{E}}$ denotes the expectation with respect to $\widetilde{\mathbb{P}}$.

The expression given by (6.7) is sometimes referred to as a "generalized solution" of (6.6). It is worth mentioning that the latter becomes a classical solution if suitable regularity assumptions on the coefficients of (6.6) are in force. For more details see the discussion on [11, page 122] . By definition the latter implies that

$$\mathfrak{u}^K(t,x,z) = \widetilde{\mathbb{E}}\left[u_0(B^x(t)) \exp\left\{ \sum_{j=1}^{K} z_j \left(\int_0^t e_j(B^x(s))ds \right) - \frac{1}{2}\sum_{j=1}^{K}\left(\int_0^t e_j(B^x(s))ds \right)^2 \right\} \right]. \tag{6.8}$$

Letting $u^K_{t,x}(\omega) := \mathfrak{u}^K(t, x, Z_1(\omega), ..., Z_K(\omega))$ we obtain the formula in theorem 133.

6.3.1 Wong-Zakai and propagator.

There exists an interesting connection between our approach and the concept of Chaos solution (e.g. [88][73]). For fixed (t,x), it is easy to shown that the *generalized solution* (6.8) belongs to $L^2\left(\mathbb{R}^K, \gamma^{\otimes K}\right)$ where $\gamma^{\otimes K}$ stands for the standard K-dimensional Gaussian measure where the matrix of covariance equals the identity. Then, $\mathfrak{u}^K(t,x,z)$ can be written as a series involving the Hermite polynomials. Thus formally replacing $\mathfrak{u}^K(t,x,z)$ by $\sum_{\alpha \in \mathcal{J}^K} u_\alpha(t,x) \left(\prod_{j=1}^{\infty} \frac{H_{\alpha_j}(z_j)}{\sqrt{\alpha_j!}} \right) =: \sum_{\alpha \in \mathcal{J}^K} u_\alpha(t,x) \mathtt{H}_\alpha(z)$, where $\mathcal{J}^K := \{\alpha \in \mathcal{J} : \alpha_i = 0, \forall i > K\}$ and $\mathtt{H}_\alpha(z) := \left(\prod_{j=1}^{\infty} \frac{H_{\alpha_j}(z_j)}{\sqrt{\alpha_j!}} \right)$, we obtain after reordering the terms and formally bringing the differential operators inside the series:

$$\begin{cases} \left(\sum_{\alpha\in\mathcal{J}^K} \partial_t u_\alpha(t,x)\mathtt{H}_\alpha(z)\right) = \left(\sum_{\alpha\in\mathcal{J}^K} \frac{1}{2}\partial^2_{xx}u_\alpha(t,x)\mathtt{H}_\alpha(z)\right) \\ \qquad\qquad + \sum_{\alpha\in\mathcal{J}^K}\sum_{j=1}^K e_j(x)u_\alpha(t,x)\Big[\mathtt{H}_\alpha(z)z_j - \partial_{z_j}\mathtt{H}_\alpha(z)\Big], \\ (t,x,z)\in]0,T]\times\mathbb{R}\times\mathbb{R}^K \\ \sum_{\alpha\in\mathcal{J}^K} u_\alpha(0,x)\mathtt{H}_\alpha(z) = u_0(x). \end{cases}$$

Note that the term inside the square brackets above is equal to

$$\begin{aligned} \mathtt{H}_\alpha(z)z_j - \partial_{z_j}\mathtt{H}_\alpha(z) &= \frac{1}{\sqrt{\alpha!}}\left(\prod_{\substack{k=1\\k\neq j}}^{\infty} H_{\alpha_k}(z_k)\right)\left[H_{\alpha_j}(z_j)z_j - H'_{\alpha_j}(z_j)\right] \\ &= \frac{1}{\sqrt{\alpha!}}\left(\prod_{\substack{k=1\\k\neq j}}^{\infty} H_{\alpha_k}(z_k)\right) H_{\alpha_j+1}(z_j) = \sqrt{\alpha_j+1}\,\mathtt{H}_{\alpha^+_{(j)}}(z), \end{aligned}$$

where $\alpha^+_{(j)} := \alpha + \epsilon_j$ (c.f. ...) and thus we can rewrite the system of equations as

$$\begin{cases} \left(\sum_{\alpha\in\mathcal{J}^K} \partial_t u_\alpha(t,x)\mathtt{H}_\alpha(z)\right) = \left(\sum_{\alpha\in\mathcal{J}^K} \frac{1}{2}\partial^2_{xx}u_\alpha(t,x)\mathtt{H}_\alpha(z)\right) \\ \qquad\qquad + \sum_{\alpha\in\mathcal{J}^K}\sum_{j=1}^K e_j(x)u_\alpha(t,x)\sqrt{\alpha_j+1}\,\mathtt{H}_{\alpha^+_{(j)}}(z), \\ (t,x,z)\in]0,T]\times\mathbb{R}\times\mathbb{R}^K \\ \sum_{\alpha\in\mathcal{J}^K} u_\alpha(0,x)\mathtt{H}_\alpha(z) = u_0(x). \end{cases}$$

By comparing the coefficients of $\mathtt{H}_\alpha$, we can obtain the following triangular system of equations which is equivalent to the *propagator* presented in [87]

$$\begin{cases} \partial_t u_{(\mathbf{0})}(t,x) = \frac{1}{2}\partial^2_{xx}u_{(\mathbf{0})}(t,x), \quad u_{(\mathbf{0})}(0,x) = u_0(x), \\ \partial_t u_\alpha(t,x) = \frac{1}{2}\partial^2_{xx}u_\alpha(t,x) + \sum_{j=1}^K e_j(x)u_{\alpha^-_{(j)}}(t,x)\sqrt{\alpha_j}, \quad u_\alpha(0,x) = 0,\ \alpha\in\mathcal{J}^K, \end{cases} \tag{6.9}$$

where $\alpha^-_{(j)} = \alpha - \epsilon_j := (\alpha_1, \alpha_2, ..., \max\{\alpha_j - 1, 0\}, ...)$.

Then, solving this propagator (6.9) gives us $\mathtt{u}^K(t,x,z)$ and by evaluating it at $(t, x, Z_1, \ldots, Z_K)$, we can obtain the solution of (6.3).

Remark 132. This same technique should work for any linear equation in which the noise enters in a multiplicative fashion like this one.

6.4 Statements of the theorems.

Theorem 133. *The random field $u^K : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ defined by:*

$$u^K_{t,x} = \widetilde{\mathbb{E}}\left[u_0(B^x(t)) \times \exp\left\{ \int_{\mathbb{R}} \sum_{j=1}^K \left(\int_0^t e_j(B^x(s))ds\right) e_j(y)dW(y) - \frac{1}{2}\sum_{j=1}^K \left(\int_0^t e_j(B^x(s))ds\right)^2 \right\}\right], \tag{6.10}$$

is a mild solution of (6.3) (in the sense of definition 129)

Theorem 134. *The family of random variables $\{\Psi^K_{t,x};\ K \in \mathbb{N}\}$ given by:*

$$\Psi^K_{t,x} := \int_{\mathbb{R}} \sum_{j=1}^K \left(\int_0^t e_j(B^x(s))ds\right) e_j(y)dW(y) - \frac{1}{2}\sum_{j=1}^K \left(\int_0^t e_j(B^x(s))ds\right)^2,$$

converges in $L^2(\mathbb{P} \otimes \widetilde{\mathbb{P}})$ to a well defined random variable given by

$$\Psi_{t,x} = \int_{\mathbb{R}} L^x_y(t)\ dW(y)\ - \frac{1}{2}\int_{\mathbb{R}} L_a(t)^2 da,$$

where $\{L^x_y(t)\}$ denotes the local time of the auxiliary Brownian motion $\{B^x(t)\}_{t\in[0,T]}$ defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

Furthermore, conditional on $\widetilde{\mathcal{F}}^B_T$ it holds that $\Psi_{t,x} \sim \mathcal{N}\left(-\frac{1}{2}\int_{\mathbb{R}} |L_a(t)|^2 da, \int_{\mathbb{R}} |L_a(t)|^2 da\right)$, where $\{L_a(t); (t,a) \in [0,T] \times \mathbb{R}\}$ is the local time of $\{B(t)\}_{t\in[0,T]}$.

Theorem 135. *For fixed $(x,t) \in [0,T] \times \mathbb{R}$, $p \in [1,\infty)$, let $u^K_{t,x} := \widetilde{\mathbb{E}}\left[u_0(B^x(t)) \exp\{\Psi^K_{t,x}\}\right]$ and denote $u_{t,x} := \widetilde{\mathbb{E}}\left[u_0(B^x(t)) \exp\{\Psi_{t,x}\}\right]$, then it holds that:*

$$\lim_{K\to\infty} \|u^K_{t,x} - u_{t,x}\|_{L^p(\mathbb{P})} = 0, \tag{6.11}$$

and

$$\lim_{K\to\infty} u^K_{t,x} = u_{t,x}, \quad \mathbb{P} - a.s. \tag{6.12}$$

Furthermore we have that

$$[0,T] \times \mathbb{R} \times \Omega \ni (t,x,\omega) \mapsto u_{t,x}(\omega)$$

is the unique solution for (6.1) (in the sense of theorem 3.1 of [83]).

Using our Feynman-Kac representation we are able to derive the following formulae for the moments of the solution (this formula has also been obtained in [84] but the proof was ommited).

Theorem 136. *Let $q \geq 2$ then the q-th moment of the unique solution of (6.1) is given by*

$$\mathbb{E}\left[(u_{t,x})^q\right] = \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{q} u_0(B(t)^{(i)}+x)\right)\exp\left\{\sum_{i<j}^{q}\int_0^t\int_0^t \delta_0(B(s)^{(i)}-B(r)^{(j)})dsdr\right\}\right], \tag{6.13}$$

$(B^{(1)},...,B^{(q)})$ are q independent 1-dimensional Brownian motions defined on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ and $\int_0^t\int_0^t \delta_0(B(s)-B(r)')dsdr$ denotes the intersection local time of the Brownian motions B and B'(e.g. [91]).

6.5 Proof of theorem 133

We will only consider the case in which $F=\mathcal{E}(h),\ h\in L^2(\mathbb{R})$ (which amounts to deal with the S-transform) because the general case could be obtained by a density argument. We then have

$$\begin{aligned}\mathbb{E}\left[u^K_{t,x}\mathcal{E}(h)\right] &= \mathbb{E}\left[\widetilde{\mathbb{E}}\left[u_0(B^x(t))\times\exp\{\Psi^K_{t,x}\}\right]\times\mathcal{E}(h)\right]\\ &= \widetilde{\mathbb{E}}\left[u_0(B^x(t))\mathbb{E}\left[\exp\{\Psi^K_{t,x}\}\times\mathcal{E}(h)\right]\right]\\ &= \widetilde{\mathbb{E}}\left[u_0(B^x(t))\exp\left\{\int_0^t\sum_{j=1}^{K}e_j(B^x(s))\int_{\mathbb{R}}e_j(y)h(y)dy\ ds\right\}\right],\end{aligned}$$

hence by the classical Feynman-Kac formula we can see that the latter is the solution of

$$\begin{cases}\partial_t\mathcal{S}\left(u^K_{t,x}\right)(h)=\frac{1}{2}\partial^2_{xx}\mathcal{S}\left(u^K_{t,x}\right)(h)+\mathcal{S}\left(u^K_{t,x}\right)(h)\cdot\left(\sum_{j=1}^K e_j(x)\int_{\mathbb{R}}e_j(y)h(y)dy\right),\ (t,x)\in]0,T]\times\mathbb{R}\\ \mathcal{S}_{0,x}(h)=u_0(x),\end{cases}$$

for any $h\in L^2(\mathbb{R})$, where as usual $\mathcal{S}\left(u^K_{t,x}\right)(h)=\mathbb{E}\left[u^K_{t,x}F\right]$ for any fixed K.

The solution of this equation can be written in mild form as

$$\begin{aligned}\mathcal{S}\left(u^K_{t,x}\right)(h) &= (P_tu_0)(x)+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\mathcal{S}\left(u^K_{s,x}\right)(h)\sum_{j=1}^{K}e_j(y)\int_{\mathbb{R}}e_j(z)h(z)dz\ dy,\\ &= (P_tu_0)(x)+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\mathbb{E}\left[u^K_{s,x}\mathcal{E}(h)\right]\sum_{j=1}^{K}e_j(y)\int_{\mathbb{R}}e_j(z)h(z)dz\ dy,\end{aligned}$$

or which is equivalent,

$$\mathbb{E}\left[u^K_{t,x}\mathcal{E}(h)\right]=(P_tu_0)(x)+\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\left(\sum_{j=1}^{K}e_j(y)e_j(\bullet)\right)u^K_{s,y}dyds\ ,\ h(\bullet)\mathcal{E}(h)\right)_{L^2(\mathbb{R})}\right],$$

which implies (6.4) since $\mathbb{E}[\mathcal{E}(h)]=1$ and $D\mathcal{E}(h)=h\mathcal{E}(h)$ and the fact that the stochastic exponentials are a dense family in $\mathbb{D}^{1,2}$.

6.6 Proof of theorem 134

We start by showing that $\{\Psi^K_{t,x}\}_{K\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{P}\otimes\widetilde{\mathbb{P}})$. .

Without lost of generality assume that $N > M+1$ and let $\||\bullet\||_p := \mathbb{E}\widetilde{\mathbb{E}}\left[|\bullet|^p\right]^{1/p}$ for any $p \geq 1$, then

$$\begin{aligned}
\||\Psi^N_{t,x} - \Psi^M_{t,x}\||_2^2 &= \left\|\left|\sum_{j=M+1}^{N} Z_j \times \left(\int_0^t e_j(B^x(s))ds\right) - \frac{1}{2}\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2\right\|\right|_2^2 \\
&= \widetilde{\mathbb{E}}\mathbb{E}\left[\left|\sum_{j=M+1}^{N} Z_j \times \left(\int_0^t e_j(B^x(s))ds\right) - \frac{1}{2}\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2\right|^2\right] \\
&= \widetilde{\mathbb{E}}\mathbb{E}\left[\left(\sum_{j=M+1}^{N} Z_j \times \left(\int_0^t e_j(B^x(s))ds\right)\right)^2\right. \\
&\quad - \left(\sum_{j=M+1}^{N} Z_j \times \left(\int_0^t e_j(B^x(s))ds\right)\right)\left(\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2\right) \\
&\quad \left. + \frac{1}{4}\left(\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2\right)^2\right] \\
&= \widetilde{\mathbb{E}}\left[\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2 + \frac{1}{4}\left(\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2\right)^2\right].
\end{aligned}$$

It would suffice to show that

$$\lim_{N,M\to\infty}\widetilde{\mathbb{E}}\left[\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2 + \frac{1}{4}\left(\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2\right)^2\right] = 0.$$

Using (6.21) we have that

$$\sum_{j=M+1}^{N}\left(\int_0^t e_j(B^x(s))ds\right)^2 \leq \int_{\mathbb{R}} |L_a(t)|^2 da =: \alpha_t,$$

for any positive integers $N > M$. This together with the fact that the random variable α_t is exponentially integrable (e.g. [91, page 178]) allows us to use the Dominated Convergence Theorem to bring the limit inside the expectation.

Finally from (6.21) we know that the sequence

$$S_n = \sum_{j=1}^{n}\left(\int_0^t e_j(B^x(s))ds\right)^2,$$

is $\widetilde{\mathbb{P}}$-a.s. convergent and thus we have that

$$|S_N - S_M| = \sum_{j=M+1}^{N} \left(\int_0^t e_j(B^x(s))ds \right)^2 \to 0, \quad \widetilde{\mathbb{P}} - a.s.$$

when $N, M \to \infty$.

This implies that

$$\lim_{N,M\to\infty} \|\Psi_{t,x}^N - \Psi_{t,x}^M\|_2^2 = 0$$

which shows that $\{\Psi_{t,x}^K\}_{K\in\mathbb{N}}$ is a Cauchy sequence.

Furthermore notice that

$$\left\| \int_{\mathbb{R}} \sum_{j=1}^{K} \left(\int_0^t e_j(B^x(s))ds \right) e_j(y)dW(y) - \frac{1}{2} \sum_{j=1}^{K} \left(\int_0^t e_j(B^x(s))ds \right)^2 - \int_{\mathbb{R}} L_y^x(t)\, dW(y) + \frac{1}{2} \int_{\mathbb{R}} L_a(t)^2 da \right\|_2^2$$

$$\leq 2 \left\| \int_{\mathbb{R}} \sum_{j=1}^{K} \left(\int_0^t e_j(B^x(s))ds \right) e_j(y)dW(y) - \int_{\mathbb{R}} L_y^x(t)\, dW(y) \right\|_2^2$$

$$+ \left\| \sum_{j=1}^{K} \left(\int_0^t e_j(B^x(s))ds \right)^2 - \int_{\mathbb{R}} L_a(t)^2 da \right\|_2^2$$

$$= 2\widetilde{\mathbb{E}} \left[\int_{\mathbb{R}} \left| \sum_{j=1}^{K} \left(\int_0^t e_j(B^x(s))ds \right) e_j(y) - L_y^x(t) \right|^2 dy \right]$$

$$+ \widetilde{\mathbb{E}} \left[\left| \sum_{j=1}^{K} \left(\int_0^t e_j(B^x(s))ds \right)^2 - \int_{\mathbb{R}} L_a(t)^2 da \right|^2 \right] \longrightarrow 0,$$

where in the last equality we've employed fact that conditional on $\widetilde{\mathcal{F}}_T^B$ the stochastic integrals are Wiener integrals and the convergence to 0 follows from an application of Dominated Convergence Theorem.

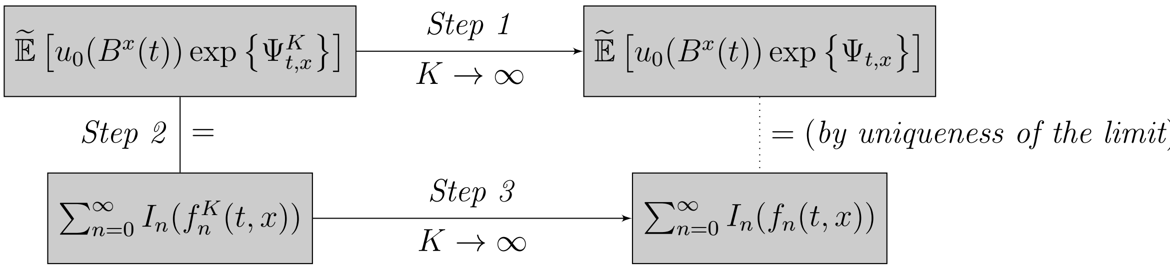
6.7 Proof of theorem 135

The proof of our main theorem will be done in several steps:

1. *Show that the "approximated Feynman-Kac" formula converges in $L^2(\mathbb{P})$ to the "formal Feynman-Kac".*
2. *Obtain the Wiener Chaos expansion of the "approximated Feynman-Kac".*
3. *Show that the latter converges in $L^2(\mathbb{P})$ (as $K \to \infty$) to the solution of (6.1) represented by the formal series given in [83] and [86].*

Then since the limit in $L^2(\mathbb{P})$ is $\mathbb{P}$-a.s. unique, we conclude that the solution given by the formal Wiener chaos series in [83] and [86] coincides with the "formal Feynman-Kac" formula.

The previous can be summarized by the following diagram,



Step 1:

We start by showing the convergence of the "approximated Feynman-Kac" formula:

$$\begin{aligned}\mathbb{E}\left[|u^K_{t,x} - u_{t,x}|^p\right] &= \mathbb{E}\left[\left|\widetilde{\mathbb{E}}\left[u_0(B^x(t))\left(\exp\{\Psi^K_{t,x}\} - \exp\{\Psi_{t,x}\}\right)\right]\right|^p\right] \\ &\leq \|u_0\|^p_\infty \mathbb{E}\widetilde{\mathbb{E}}\left[\left|\exp\{\Psi^K_{t,x}\} - \exp\{\Psi_{t,x}\}\right|^p\right].\end{aligned}$$

Since $\Psi^K_{t,x} \to \Psi_{t,x}$ in $L^2(\mathbb{P}\otimes\widetilde{\mathbb{P}})$, then $\exp\{\Psi^K_{t,x}\} \to \exp\{\Psi_{t,x}\}$ in probability, and in order to show the desired result we just need to prove that $\|\!|\exp\{\Psi^K_{t,x}\}|\!\|_p \to \|\!|\exp\{\Psi_{t,x}\}|\!\|_p$.

Using the Tower rule and the fact that conditional on $\widetilde{\mathcal{F}}^B_T$ the random variables $\Psi_{t,x}$ and $\Psi^K_{t,x}$ are Gaussian we have that

$$\begin{aligned}\|\!|\exp\{\Psi_{t,x}\}|\!\|^p_p &= \mathbb{E}\widetilde{\mathbb{E}}|\exp\{p\Psi_{t,x}\}| \\ &= \widetilde{\mathbb{E}}\left[\mathbb{E}\left[\exp\{p\Psi_{t,x}\}|\widetilde{\mathcal{F}}^B_T\right]\right] \\ &= \widetilde{\mathbb{E}}\left[\exp\left\{\frac{p(p-1)}{2}\int_{\mathbb{R}}|L_a(t)|^2 da\right\}\right] < \infty,\end{aligned}$$

and

$$\begin{aligned}\|\!|\exp\{\Psi^K_{t,x}\}|\!\|^p_p &= \mathbb{E}\widetilde{\mathbb{E}}|\exp\{p\Psi^K_{t,x}\}| \\ &= \widetilde{\mathbb{E}}\left[\mathbb{E}\left[\exp\{p\Psi_{t,x}\}|\mathcal{F}^B_T\right]\right] \\ &= \widetilde{\mathbb{E}}\left[\exp\left\{\frac{p(p-1)}{2}\sum_{j=1}^{K}\left(\int_0^t e_j(B^x(s))ds\right)^2\right\}\right].\end{aligned}$$

At this point we can use Monotone convergence theorem to bring the limit inside the expectation, the continuity of the exponential function and (6.21) implies the desired result.

Step 2:

Now we need to obtain the Wiener chaos decomposition of (6.10) and start by noticing that conditional on $\widetilde{\mathcal{F}}_T^B$ we can write (e.g. lemma 59)

$$\exp\{\Psi_{t,x}^K\} = \sum_{n=0}^{\infty} \frac{1}{n!} I_n\left(g_n^K(t,x,\bullet)\right), \quad \text{convergence in } L^2(\mathbb{P})$$

where the n-th kernel is given by the following element of $L^2_{\text{sym}}(\mathbb{R}^n)$:

$$g_n^K(t,x,\bullet) = \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \frac{1}{n!} \left(\int_{[0,t]^n} e_{i_1}(B^x(s_1)) \times \cdots \times e_{i_n}(B^x(s_n)) d\mathbf{s} \right) (e_{i_1} \otimes \cdots \otimes e_{i_n})(\bullet),$$

and where $d\mathbf{s} = ds_1...ds_n$. From the latter it follows that

$$u_{t,x}^K = \sum_{n=0}^{\infty} I_n \left(\sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \frac{1}{n!} \widetilde{\mathbb{E}} \left[u_0(B^x(t)) \int_{[0,t]^n} e_{i_1}\left(B^x(s_1)\right) \times \cdots \times e_{i_n}\left(B^x(s_n)\right) d\mathbf{s} \right] e_{i_1} \otimes \cdots \otimes e_{i_n} \right);$$

then the n-th kernel of the Chaos decomposition of $u_{t,x}^K$ is given by

$$\sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \frac{1}{n!} \widetilde{\mathbb{E}} \left[u_0(B^x(t)) \int_{[0,t]^n} e_{i_1}\left(B^x(s_1)\right) \times \cdots \times e_{i_n}\left(B^x(s_n)\right) d\mathbf{s} \right] e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

Using the multi-index notation we can rewrite the series above as

$$\frac{1}{n!} \sum_{\alpha \in \mathcal{J}_n^K} \frac{1}{\alpha!} \sum_{\sigma \in \mathcal{P}_n} \widetilde{\mathbb{E}} \left[u_0(B^x(t)) \int_{[0,t]^n} e_{k_{\sigma(1)}}\left(B^x(s_1)\right) \times \cdots \times e_{k_{\sigma(n)}}\left(B^x(s_n)\right) d\mathbf{s} \right] e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(1)}}$$

where for any $\alpha \in \mathcal{J}_n$ we denote its characteristic vector by $k_\alpha = (k_1, \ldots, k_n)$ (e.g. [87, Section 2]).

Using the symmetry of the term inside the expectation we see that this equals

$$\frac{1}{n!} \sum_{\alpha \in \mathcal{J}_n^K} \widetilde{\mathbb{E}} \left[u_0(B^x(t)) \int_{[0,t]^n} e_{k_1}\left(B^x(s_1)\right) \times \cdots \times e_{k_n}\left(B^x(s_n)\right) d\mathbf{s} \right] \frac{1}{\alpha!} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(1)}}$$

Definition 137. Let $\alpha \in \mathcal{J}$ and as usual let $\boldsymbol{\alpha}! = \prod_{j=1}^{\infty} \alpha_j!$. We define

$$\mathfrak{e}_\alpha := \sqrt{\frac{n!}{\boldsymbol{\alpha}!}} \bigodot_{j=1}^{\infty} e_j^{\odot \alpha_j}. \tag{6.14}$$

Then $\{\mathfrak{e}_\alpha : \alpha \in \mathcal{J}_n\}$ is an orthonormal basis for $L^2_{\text{sym}}(\mathbb{R}^n)$ (e.g. [23, proposition 2.3.7])

Using the basis we've introduced above we can rewrite the latter expression as

$$\begin{aligned}
&\frac{1}{n!}\sum_{\alpha\in\mathcal{J}_n^K}\sqrt{\frac{n!}{\alpha!}}\widetilde{\mathbb{E}}\left[u_0(B^x(t))\int_{[0,t]^n} e_{k_1}\left(B^x(s_1)\right)\times\cdots\times e_{k_n}\left(B^x(s_n)\right)d\mathbf{s}\right]\mathfrak{e}_\alpha\\
&=\frac{1}{n!}\sum_{\alpha\in\mathcal{J}_n^K}\sqrt{\frac{n!}{\alpha!}}\widetilde{\mathbb{E}}\left[u_0(B^x(t))\left(L^x(t)^{\otimes n}, e_{k_1}\otimes\cdots\otimes e_{k_n}\right)_{L^2(\mathbb{R}^n)}\right]\mathfrak{e}_\alpha\\
&=\frac{1}{n!}\sum_{\alpha\in\mathcal{J}_n^K}\sqrt{\frac{n!}{\alpha!}}\widetilde{\mathbb{E}}\left[u_0(B^x(t))\left(L^x(t)^{\otimes n},\frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} e_{k_{\sigma(1)}}\otimes\cdots\otimes e_{k_{\sigma(n)}}\right)_{L^2(\mathbb{R}^n)}\right]\mathfrak{e}_\alpha\\
&=\frac{1}{n!}\sum_{\alpha\in\mathcal{J}_n^K}\widetilde{\mathbb{E}}\left[u_0(B^x(t))\left(L^x(t)^{\otimes n},\mathfrak{e}_\alpha\right)_{L^2(\mathbb{R}^n)}\right]\mathfrak{e}_\alpha.
\end{aligned}$$

Using the time occupation formula again we obtain

$$\frac{1}{n!}\sum_{\alpha\in\mathcal{J}_n^K}\widetilde{\mathbb{E}}\left[\int_{[0,t]^n} u_0(B^x(t))\mathfrak{e}_\alpha(B^x(s_1),...,B^x(s_n))d\mathbf{s}\right]\mathfrak{e}_\alpha.$$

Now let $\sigma\in\mathcal{P}_n$ be the permutation of $\{1,...,n\}$ such that $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}$ then by the symmetry of $\mathfrak{e}_\alpha$ we can rewrite the last expression as

$$\frac{1}{n!}\sum_{\alpha\in\mathcal{J}_n^K}\widetilde{\mathbb{E}}\left[\int_{[0,t]^n} u_0(B^x(t))\mathfrak{e}_\alpha\left(B^x_{s_{\sigma(1)}},...,B^x_{s_{\sigma(n)}}\right)d\mathbf{s}\right]\mathfrak{e}_\alpha,$$

at this point we can simply compute the expectation and obtain the following

Proposition 138. *Let $u^K_{t,x}$ be given by (6.10) then it holds that:*

$$u^K_{t,x}=\sum_{n=0}^{\infty} I_n(f_n^K(t,x)),\tag{6.15}$$

where

$$\begin{cases} f_0^K(t,x) &= u_{(0)}(t,x),\\ f_n^K(t,x,\bullet) &= \sum_{\alpha\in\mathcal{J}_n^K}\mathfrak{e}_\alpha(\bullet)\\ & \frac{1}{n!}\int_{[0,t]^n}\int_{\mathbb{R}^n} u_{(0)}(t-s_{\sigma(n)},x_{\sigma(n)})p_{s_\sigma(n)-s_\sigma(n)}(x_{\sigma(n)}-x_{\sigma(n-1)})\\ & \qquad\cdots\times p_{s_{\sigma(1)}}(x_{\sigma(1)}-x)\mathfrak{e}_\alpha(x_{\sigma(1)},...,x_{\sigma(n)})d\mathbf{x}\, d\mathbf{s}\end{cases}\tag{6.16}$$

where $d\mathbf{x}:=dx_1\cdots dx_n$, and σ denotes the permutation of $\{1,...,n\}$ such that $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$.

In [84] the authors have shown that if the initial condition is deterministic and bounded, then equation (6.1) has a unique mild solution given by the Wiener Chaos expansion:

$$u(t,x)=\sum_{n=0}^{\infty} I_n(f_n(t,x)),\tag{6.17}$$

where

$$\begin{cases} f_0(t,x) = u_{(0)}(t,x) & := (P_t u_0)(x), \\ f_n(t,x;x_1,...,x_n) & = \frac{1}{n!}\int_{[0,t]^n} p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)}) \\ & \quad \cdots \times p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})u_{(0)}(x_{\sigma(1)},s_{\sigma(1)})d\mathbf{s}, \end{cases} \tag{6.18}$$

where σ denotes the permutation of $\{1,2,...,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ (see also equations (4.2) and (4.3) of [86]).

Now the idea is to show that (6.15) converges in $L^2(\mathbb{P})$ to (6.17) as $K \to \infty$. Comparing (6.16) and (6.18) it is evident, after a simple reordering of the terms, that f_n^K equals the orthogonal projection of f_n on the span of $\{\mathfrak{e}_\alpha : \alpha \in \mathcal{J}_n^K\}$, i.e. the symmetric part of $\text{span}\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{i_1,...,i_n=1}^K$.

Definition 139. Let K be some fixed positive integer. Then $P_K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a self-adjoint projection operator defined by the action

$$P_K f = P_K\left(\sum_{j=1}^{\infty} (f,e_j)_{L^2(\mathbb{R})}\ e_j\right) = \sum_{j=1}^{K} (f,e_j)_{L^2(\mathbb{R})}\ e_j, \tag{6.19}$$

for any $f \in L^2(\mathbb{R})$, i.e. the orthogonal projection on $\text{span}\{e_1,...,e_K\}$.

Straightforward calculations, similar to those in Appendix B show that the following holds

Proposition 140. *Let $u(t,x), (t,x) \in [0,T]\times\mathbb{R}$ denote the mild solution of (6.1) given by (6.17) . Then for any $(t,x) \in [0,T]\times\mathbb{R}$ it holds that:*

$$u_{t,x}^K = \Gamma(P_K)u(t,x), \tag{6.20}$$

where $\Gamma(P_K)$ stands for the second quantization of the projection operator P_K.

Step 3:

It's straightforward to see that

$$\begin{aligned} \left\|\sum_{n=0}^{\infty} I_n(f_n^K(t,x)) - \sum_{n=0}^{\infty} I_n(f_n(t,x))\right\|_{L^2(\mathbb{P})}^2 &= \left\|\sum_{n=0}^{\infty} I_n(f_n(t,x) - P_K^{\otimes n} f_n(t,x))\right\|_{L^2(\mathbb{P})}^2 \\ &= \sum_{n=0}^{\infty} n! \|f_n(t,x) - P_K^{\otimes n} f_n(t,x)\|_{L^2(\mathbb{R}^n)}^2 \to 0 \end{aligned}$$

as $K \to \infty$. This together with the results obtained in *Step 1* and the unicity of the $L^2(\mathbb{P})$ limit we conclude that

$$\widetilde{\mathbb{E}}\left[u_0(B^x(t)) \exp\left\{\Psi_{t,x}\right\}\right] = \sum_{n=0}^{\infty} I_n(f_n(t,x)), \quad \mathbb{P}-a.s.$$

On the other hand from the propositions 140 and 141 we see that

$$u^K_{t,x} = \mathbb{E}\left[u(t,x)|\mathcal{K}\right], \quad \mathcal{K} := \sigma(Z_1, ..., Z_K)$$

and we also notice that $\mathcal{K} \uparrow \mathcal{F} = \sigma(\mathfrak{H}(W))$. Then the martingale convergence theorem (e.g. theorem 35.6 of [92]) gives us the $\mathbb{P}$-a.s. convergence

6.8 Proof of theorem 136

The convergence in $L^p(\mathbb{P})$, $p \in [1, \infty)$ of $u^K_{t,x}$ to the solution of $u_{t,x}$ implies that for any $q \in \mathbb{N}$ the q-th moment of $u^K_{t,x}$ converges to that of the solution.

$$\begin{aligned}
\mathbb{E}\left[\left(u^K_{t,x}\right)^q\right] &= \mathbb{E}\left[\prod_{i=1}^q \widetilde{\mathbb{E}}\left[u_0(B(t)^{(i)} + x)\exp\left\{\Psi^{K,(i)}_{t,x}\right\}\right]\right] \\
&= \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^q u_0(B(t)^{(i)} + x)\right)\mathbb{E}\left[\exp\left\{\sum_{i=1}^q \Psi^{K,(i)}_{t,x}\right\}\middle|\mathcal{F}^B_T\right]\right] \\
&= \widetilde{\mathbb{E}}\Bigg\{\left(\prod_{i=1}^q u_0(B(t)^{(i)} + x)\right) \\
&\quad \times \mathbb{E}\left[\exp\left\{\int_{\mathbb{R}}\sum_{i=1}^q\sum_{k=1}^K\left(\int_0^t e_k(B(s)^{(i)} + x)ds\right)e_k(y)dW(y)\right\}\middle|\mathcal{F}^B_T\right] \\
&\quad \times \exp\left\{-\frac{1}{2}\sum_{i=1}^q\left(\int_0^t e_k(B(s)^{(i)} + x)ds\right)^2\right\}\Bigg\},
\end{aligned}$$

using the fact that conditional on $\widetilde{\mathcal{F}}^B_T$ the stochastic integral appearing in the exponential is a centered Gaussian random variable we can see that the latter equals

$$\begin{aligned}
&= \widetilde{\mathbb{E}}\Bigg[\left(\prod_{i=1}^q u_0(B(t)^{(i)} + x)\right)\exp\left\{\left\|\sum_{i=1}^q\sum_{k=1}^K\left(\int_0^t e_k(B(s)^{(i)} + x)ds\right)e_k(\bullet)\right\|^2_{L^2(\mathbb{R})}\right\} \\
&\times \exp\left\{-\frac{1}{2}\sum_{i=1}^q\left(\int_0^t e_k(B(s)^{(i)} + x)ds\right)^2\right\}\Bigg] \\
&= \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^q u_0(B(t)^{(i)} + x)\right)\exp\left\{\sum_{i<j}\sum_{k=1}^K\left(\int_0^t e_k(B(s)^{(i)} + x)ds\right)\left(\int_0^t e_k(B(r)^{(j)} + x)dr\right)\right\}\right].
\end{aligned}$$

Now we must take the limit for $K \to \infty$ (we can see that the exponential function is dominated by $\exp\{q \max_{1\leq i\leq q}\int_{\mathbb{R}}|L^{(i)}_a(t)|^2 da\}$ which is integrable) yielding

$$\mathbb{E}\left[(u_{t,x})^q\right] = \lim_{K\to\infty} \mathbb{E}\left[\left(u_{t,x}^K\right)^q\right]$$
$$= \widetilde{\mathbb{E}}\left[\left(\prod_{i=1}^{q} u_0(B(t)^{(i)}+x)\right)\exp\left\{\sum_{i<j}^{q}\int_0^t\int_0^t \delta_0(B(s)^{(i)}-B(r)^{(j)})dsdr\right\}\right].$$

6.9 Appendix A: Local time

Consider the Brownian local time of a one dimensional Brownian motion $\{B^x(t)\}_{t\in[0,T]}$ starting at $x\in\mathbb{R}$, at level $a\in\mathbb{R}$ and time $t\in[0,T]$:

$$L_a^x(t) = \int_0^t \delta_a(B^x(s))ds,$$

and notice that the latter can be seen as the usual Brownian local time $L_{a-x}(t)$.

It's known (e.g. the proof of proposition XIII-2.1. of [93]) that for a fixed t the map $\mathbb{R}\ni a\mapsto L_a(t)$ is a.s. continuous and has compact support, hence it follows that

$$\alpha_t := \int_{\mathbb{R}} |L_a(t)|^2 da < \infty,\ a.s.,$$

this together with the invariance of Lebesgue measure implies that $a\mapsto L_a^x(t)$ belongs to $L^2(\mathbb{R})$ almost surely.

Then the following Fourier-like series expansion holds a.s.

$$L_a^x(t) = \sum_{j=1}^{\infty}\left(\int_{\mathbb{R}} L_y^x(t)e_j(y)dy\right)e_j(a)$$
$$= \sum_{j=1}^{\infty}\left(\int_0^t e_j(B^x(s))ds\right)e_j(a),$$

where in the last equality we've used the occupation time formula.

By the Parseval's identity we have:

$$\sum_{j=1}^{\infty}\left(\int_0^t e_j(B^x(s))ds\right)^2 = \int_{-\infty}^{\infty}|L_a^x(t)|^2 da = \int_{-\infty}^{\infty}|L_a(t)|^2 da < \infty\ a.s. \tag{6.21}$$

6.10 Appendix B: Second quantization and Conditional expectation

Let $(\Omega,\mathcal{A},P)$ be a probability space. Then it's well know that if $X\in L^2(\Omega,\mathcal{A},P)$ and $\mathcal{B}\subset\mathcal{A}$ is a sub-sigma-algebra, the conditional expectation $\mathbb{E}[X|\mathcal{B}]$ can be seen as the orthogonal projection of X on $L^2(\Omega,\mathcal{B},P)$. In this appendix we will show an analogous property of the second quantization operator.

Proposition 141. *Let P_K be the projection operator of definition 139. Then the second quantization operator $\Gamma(P_K)$ coincides with the conditional expectation $\mathbb{E}[\bullet|\mathcal{K}]$[1] on $L^2(\mathbb{P})$.*

Proof. We consider again the complete probability space $(\Omega, \mathfrak{B}, P)$ treated in the introduction. Let $X \in L^2(\mathbb{P})$ then by theorem 33 X has a series expansion of the form

$$X = \sum_{\alpha \in \mathcal{J}} x_\alpha \mathcal{H}_\alpha, \quad \text{convergence in } L^2(\mathbb{P}).$$

Now let us take the conditional expectation of X given the sigma algebra $\mathcal{K}$. It's well known that we are allowed to interchange conditional expectation with an L^2 convergent series, yielding

$$\mathbb{E}\left[X|\mathcal{K}\right] = \sum_{\alpha \in \mathcal{J}} x_\alpha E\left[\mathcal{H}_\alpha|\mathcal{K}\right] = \sum_{\alpha \in \mathcal{J}} x_\alpha \mathbb{E}\left[\prod_{j=1}^{\infty} \frac{H_{\alpha_j}(Z_j)}{\sqrt{\alpha_j!}}\middle|\mathcal{K}\right],$$

and at this point we notice that the terms of the product involving Z_j for $j \in \{1, 2, ..., N\}$ are $\mathcal{K}$-measurable and hence can be pulled outside the conditional expectation, furthermore the mutual independence of the Z's yield

$$\mathbb{E}[X|\mathcal{K}] = \sum_{\alpha \in \mathcal{J}} x_\alpha \prod_{j=1}^{K} \frac{H_{\alpha_j}(Z_j)}{\sqrt{\alpha_j!}} \prod_{j=K+1}^{\infty} \mathbb{E}\left[\frac{H_{\alpha_j}(Z_j)}{\sqrt{\alpha_j!}}\right].$$

Furthermore since the Hermite polynomials of a standard Gaussian random variables can be seen as its Wick power i.e. $H_n(Z_j) = Z_j^{\diamond n}$ (proposition 38), and since $\mathbb{E}(Z_j^{\diamond n}) = \mathbb{E}(Z_j)^n = 0$ we see that the only non-vanishing terms are those corresponding to the α's containing only positive values in the first K entries (remember that $H_0(\cdot) \equiv 1$). This allows us to conclude that

$$\mathbb{E}[X|\mathcal{K}] = \sum_{\alpha \in \mathcal{J}^K} x_\alpha \mathcal{H}_\alpha \tag{6.22}$$

where $\mathcal{J}^K := \{\alpha \in \mathcal{J} : \alpha_i = 0, \forall i > K\}$.

On the other hand we could write the Chaos decomposition in terms of multiple Wiener integrals, i.e.

$$X = \sum_{n=0}^{\infty} I_n(f_n),$$

where the kernel f_n is a symmetric function in $L^2(\mathbb{R}^n)$. Then by definition of the second quantization operator we have

$$\begin{aligned}\Gamma(P_K)X &= \sum_{n=0}^{\infty} I_n((P_K)^{\otimes n} f_n) \\ &= \sum_{n=0}^{\infty} I_n\left(\sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} (f_n, e_{i_1} \otimes \cdots \otimes e_{i_n})_{L^2(\mathbb{R}^n)} e_{i_1} \otimes \cdots \otimes e_{i_n}\right),\end{aligned}$$

[1] Remember that $\mathcal{K} := \sigma(Z_1, ..., Z_K)$

using the multi-index notation this could be written as

$$= \sum_{n=0}^{\infty} I_n \left(\sum_{\alpha \in \mathcal{J}_n^K} \frac{1}{\alpha!} \sum_{\sigma \in \mathcal{P}_n} \left(f_n, e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right)_{L^2(\mathbb{R}^n)} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right).$$

where as usual $k_\alpha = (k_1, ..., k_n)$ denotes the characteristic vector for $\alpha \in \mathcal{J}_n$. Since f_n is symmetric then it follows that the latter equals

$$\sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{J}_n^K} \frac{n!}{\alpha!} (f_n, e_{k_1} \otimes \cdots \otimes e_{k_n})_{L^2(\mathbb{R}^n)} I_n \left(\frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right)$$
$$= \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{J}_n^K} \sqrt{n!} \left(f_n, \frac{1}{\sqrt{\alpha! n!}} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right)_{L^2(\mathbb{R}^n)} \frac{1}{\sqrt{\alpha!}} I_n \left(\frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right).$$

Using the identity presented in equation (2.17)

$$\mathcal{H}_\alpha = \frac{1}{\sqrt{\alpha!}} I_n \left(\bigodot_{j=1}^{\infty} e_j^{\odot \alpha_j} \right), \quad \text{for } \alpha \in \mathcal{J}_n$$

and letting (c.f.(2.18))

$$x_\alpha = \sqrt{n} \, (f_n, \mathfrak{e}_\alpha)_{L^2(\mathbb{R}^n)},$$

we can write the latter as

$$\sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{J}_n^K} x_\alpha \mathcal{H}_\alpha = \sum_{\alpha \in \mathcal{J}^K} x_\alpha \mathcal{H}_\alpha = \mathbb{E}[X|\mathcal{K}],$$

and this completes the proof. □

Chapter 7

A Feynman-Kac approach for the spatial derivative of the SHE driven by spatial white noise potential.

Based on: Kim, H. J., & Scorolli, R. (2021). A Feynman-Kac approach for the spatial derivative of the solution to the Wick stochastic heat equation driven by time homogeneous white noise. To appear in Infinite Dimensional Analysis, Quantum Probability and Related Topics.

Abstract

We consider the (unique) mild solution $u(t,x)$ of a 1-dimensional stochastic heat equation on $[0,T]\times\mathbb{R}$ driven by time-homogeneous white noise in the Wick-Skorokhod sense. The main result of this paper is the computation of the spatial derivative of $u(t,x)$, denoted by $\partial_x u(t,x)$, and its representation as a Feynman-Kac type closed form. The chaos expansion of $\partial_x u(t,x)$ makes it possible to find its (optimal) Hölder regularity especially in space.

7.1 Introduction

As an extension of the results obtained in the previous chapter, we will further investigate the (unique) mild solution of the 1-dimensional stochastic heat equation (SHE):

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\partial^2_{xx}u(t,x) + u(t,x)\diamond \dot{W}(x), & (t,x)\in(0,T]\times\mathbb{R}, \\ u(0,x) = u_0(x) \end{cases} \tag{7.1}$$

where $T>0$, u_0 is a function satisfying certain conditions, $\{\dot{W}(x)\}_{x\in\mathbb{R}}$ is a spatial Gaussian *white noise* i.e. the distributional derivative of the Brownian motion $\{W(x)\}_{x\in\mathbb{R}}$ defined on the white noise probability space[1] $(\mathscr{S}'(\mathbb{R}), \mathscr{B}, \mu)$, and $\diamond$ stands for the *Wick product* which by proposition 101 implies that the stochastic integration must be understood in the Itô-Skorohod sense.

Definition 142. We say $u : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be a *mild solution* of (7.1) if for any fixed $(t,x) \in [0,T] \times \mathbb{R}$, $u(t,x) \in (L^2)$ and it satisfies

$$u(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)u(s,y) \diamond \dot{W}(y)dyds, \ \mu\text{-almost surely,} \tag{7.2}$$

where as usual $p(t,x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ is the Gaussian heat kernel.

Notice that from the discussion in section 2.3.7 we can rewrite equation (142) using the Skorohod integral as

$$u(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)u(s,y)\delta W(y)ds, \ \mu\text{-almost surely.} \tag{7.3}$$

Let $\{u(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ be a mild solution of (7.1). Then for any fixed (t,x), the random variable $u(t,x)$ admits the following multiple Wiener chaos expansion (e.g. [86], [84] or [83]):

$$u(t,x) = \sum_{n=0}^{\infty} I_n \left(F_n^{\mathtt{MW}}(t,x) \right), \tag{7.4}$$

where $I_n(\bullet)$ is the n-th multiple Wiener integral with respect to the Brownian motion $\{W(x)\}_{x\in\mathbb{R}}$,

$$\begin{cases} F_0^{\mathtt{MW}}(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)\, dy; \\ F_n^{\mathtt{MW}}(t,x;y_1,\ldots,y_n) \\ = \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} \int_{[0,t]^n} p\left(t - r_{\rho(n)}, x - y_{\rho(n)}\right) \\ \qquad \cdots \times p\left(r_{\rho(2)} - r_{\rho(1)}, y_{\rho(2)} - y_{\rho(1)}\right) F_0^{\mathtt{MW}}(r_{\rho(1)}, x_{\rho(1)})\, d\mathbf{r},\ n \geq 1, \end{cases}$$

and ρ denotes the permutation of $\{1,\ldots,n\}$ such that $0 < r_{\rho(1)} < \cdots < r_{\rho(n)} < t$. For simplicity, we have denoted $d\mathbf{r} := dr_1 dr_2 \cdots dr_n$.

To distinguish among different representations of the mild solution, let us call (7.4) the ***multiple Wiener solution*** $u^{\mathtt{MW}}(t,x)$ of (7.1). There are a few papers considering this representation:

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[1]Even though the choice of the underlying probability space is immaterial, we will employ this in order to ease the presentation of the results

(i) The paper [83, Theorem 3.1] shows that $u^{\mathtt{MW}}$ is the unique mild solution in $C\left([0,T];L^2(\mathbb{R});(L^2)\right)$ if $u_0 \in L^2(\mathbb{R})$ by showing,

$$\sup_{t\in[0,T]} \sum_{n=0}^{\infty} n! \int_{\mathbb{R}} \left\| F_n^{\mathtt{MW}}(t,x;\bullet)\right\|^2_{L^2(\mathbb{R}^n)} dx \le \mathtt{C}\|u_0\|^2_{L^2(\mathbb{R})} < \infty.$$

Note that $u_0 \in L^2(\mathbb{R})$ does not cover $u_0 \equiv 1$.

(ii) When $u_0 \in L^\infty(\mathbb{R})$, [86, Section 4] shows that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \sum_{n=0}^{\infty} n! \left\| F_n^{\mathtt{MW}}(t,x;\bullet)\right\|^2_{L^2(\mathbb{R}^n)} \le \mathtt{C}\|u_0\|^2_{L^\infty(\mathbb{R})} < \infty.$$

Hence, we can say that $u^{\mathtt{MW}}$ is the unique mild solution in $C\left([0,T]\times\mathbb{R};(L^2)\right)$ if $u_0 \in L^\infty(\mathbb{R})$.

There is an alternative chaos expansion of the mild solution u (e.g. [94, Theorem 3.11]):

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} u_\alpha^{\mathtt{CS}}(t,x)\mathcal{H}_\alpha. \tag{7.5}$$

As usual we denote with $\mathbb{T}^n_{[0,t]}$ the simplex $\{0 \le s_1 \le \cdots \le s_n \le t\}$, and then we can write $u^{\mathtt{CS}}_{(\mathbf{0})}(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)dy$, and for $|\alpha| = n \ge 1$,

$$u_\alpha^{\mathtt{CS}}(t,x) = \sqrt{n!}\int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^n} p(t-s_n,x-y_n)\times\cdots\times p(s_2-s_1,y_2-y_1)u^{\mathtt{CS}}_{(\mathbf{0})}(s_1,y_1)\ \mathfrak{e}_\alpha(y_1,\ldots,y_n)\ d\mathbf{s}d\mathbf{y},$$

for $\alpha \in \mathcal{J}_n := \{\alpha \in \mathcal{J} : |\alpha| = n\}$, and $\left\{\mathfrak{e}_\alpha := \sqrt{\frac{n!}{\alpha!}} \bigodot_{j=1}^\infty e_j^{\odot\alpha_j}, \alpha \in \mathcal{J}_n\right\}$ forms an orthonormal basis of $L^2_{\mathrm{sym}}(\mathbb{R}^n)$. We will call (7.5) the ***chaos solution*** $u^{\mathtt{CS}}(t,x)$ of (7.1). The existence and uniqueness of this representation can be proved by showing the following:

(i') We can prove that $u^{\mathtt{CS}}$ is the unique mild solution in $C\left([0,T];L^2(\mathbb{R});(L^2)\right)$ when $u_0 \in L^2(\mathbb{R})$ by showing (c.f. [87, Theorem 4.1])

$$\sup_{t\in[0,T]} \sum_{n=0}^{\infty}\sum_{\alpha\in\mathcal{J}_n} \left\| u_\alpha^{\mathtt{CS}}(t,\bullet)\right\|^2_{L^2(\mathbb{R})} \le \mathtt{C}\|u_0\|^2_{L^2(\mathbb{R})} < \infty. \tag{7.6}$$

(ii') We can also show that $u^{\mathtt{CS}}$ is the unique mild solution in $C\left([0,T]\times\mathbb{R};(L^2)\right)$ if $u_0 \in L^\infty(\mathbb{R})$ (c.f. [87, Theorem 4.3]) by achieving

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \sum_{n=0}^{\infty}\sum_{\alpha\in\mathcal{J}_n} \left| u_\alpha^{\mathtt{CS}}(t,x)\right|^2 \le \mathtt{C}\|u_0\|^2_{L^\infty(\mathbb{R})} < \infty. \tag{7.7}$$

Indeed, (7.6) and (7.7) can be easily obtained as follows:

(7.6) To use the same argument as [87, Theorem 4.1], it is enough to show

$$U_0 := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} p(s, y - z_1) u_0(z_1) dz_1 \right) \cdot \left(\int_{\mathbb{R}} p(s, y - z_2) u_0(z_2) dz_2 \right) dy \le \|u_0\|^2_{L^2(\mathbb{R})},$$

and it is clear by semigroup property and Hölder inequality,

$$\begin{aligned} U_0 &= \int_{\mathbb{R}} \int_{\mathbb{R}} p(s + r, z_1 - z_2) u_0(z_1) u_0(z_2) dz_1 dz_2 \\ &= \int_{\mathbb{R}} p(s + r, z_1) \int_{\mathbb{R}} u_0(z_1 + z_2) u_0(z_2) dz_2 dz_1 \le \|u_0\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

(7.7) To use the same argument as [87, Theorem 4.3], it is enough to show $\left| \int_{\mathbb{R}} p(t, x - y) u_0(y) dy \right| \le \|u_0\|_{L^\infty(\mathbb{R})}$, and it automatically follows from the fact $\int_{\mathbb{R}} p(t, x) dx = 1$.

Then, it is not surprising that $u^{\mathtt{MW}} = u^{\mathtt{CS}}$ if $u_0 \in L^\infty(\mathbb{R})$ since the mild solution is unique in $C\left([0, T] \times \mathbb{R}; (L^2)\right)$.

We now discuss one more possible representation of the mild solution. In fact, the condition (7.2) is equivalent to definition 129 In the previous chapter we obtained a Feynman-Kac representation of the unique mild solution of (7.1) when $u_0 \in L^\infty(\mathbb{R})$. This is given by

$$u(t, x) = \widetilde{\mathbb{E}} \left[u_0(B^x(t)) \exp\{\Psi_{t,x}\} \right],$$

where $\{B^x(t)\}_{t \ge 0}$ is a one-dimensional Brownian motion starting at x, and for fixed $(t, x) \in [0, T] \times \mathbb{R}$, the random variable $\Psi_{t,x}$ is given by

$$\Psi_{t,x} := \int_{\mathbb{R}} L^x_y(t) dW(y) - \frac{1}{2} \int_{\mathbb{R}} |L^x_y(t)|^2 dy. \tag{7.8}$$

Here $L^x_a(t)$ denotes the local time of $\{B^x(s)\}_{s \ge 0}$ at level a and time t. Let us call the Feynman-Kac representation the *Feynman-Kac solution* $u^{\mathtt{FK}}(t, x)$ of (7.1).

Combining all, as long as $u_0 \in L^\infty(\mathbb{R})$, we can say

$$u := u^{\mathtt{FK}} = u^{\mathtt{MW}} = u^{\mathtt{CS}} \in C\left([0, T] \times \mathbb{R}; (L^2)\right). \tag{7.9}$$

In this paper, we will provide an alternative proof for the equivalence (7.9) using a more direct approach.

The main motivation for the current article is as follows: As we stated above, the equation (7.1) may have three possible representations for the unique mild solution u, namely (I) Feynman-Kac solution $u^{\mathtt{FK}}$, (II) multiple Wiener-Itô integral solution $u^{\mathtt{MW}}$, and (III) chaos solution $u^{\mathtt{CS}}$. Unfortunately, there is no enough discussion on Hölder regularity of the mild solution. In particular,

(I) There is no Hölder regularity result for $u^{\texttt{FK}}$ in the existing literature.

(II) For $u^{\texttt{MW}}$, [83, Theorem 4.1] proves that $u^{\texttt{MW}} \in \mathcal{C}^{1/2-\varepsilon,1/2-\varepsilon}([0,T],\mathbb{R})$ for any small $\varepsilon > 0$ if $u_0 \in C_b^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Here, $C_b^1(\mathbb{R})$ (for the definition of this space see the next section) denotes the space of all bounded differentiable functions on $\mathbb{R}$ with bounded continuous derivatives.

(III) On the one hand, no one discusses the regularity of $u^{\texttt{CS}}$ on the whole line. On the other hand, the paper [87] discuss the same equation as (7.1), but on a bounded domain, say $[0,\pi]$; the authors show that there exists a unique mild solution (using chaos expansion) $u_b^{\texttt{CS}} \in C([0,T]; L^2([0,\pi]); (L^2))$ if $u_0 \in L^2([0,\pi])$, and moreover $u_b^{\texttt{CS}} \in \mathcal{C}^{3/4-\varepsilon,3/2-\varepsilon}([0,T]\times[0,\pi])$ for any small $\varepsilon > 0$ if $u_0 \in \mathcal{C}^{3/2}([0,\pi])$.

Since Hölder continuity is a local property, it is natural to expect that $u^{\texttt{CS}} \in \mathcal{C}^{3/4-\varepsilon,3/2-\varepsilon}([0,T]\times\mathbb{R})$ for any small $\varepsilon > 0$ (under a suitable initial condition on u_0) like the bounded case $u_b^{\texttt{CS}}$. Furthermore, it is impossible that the other representations $u^{\texttt{FK}}$ and $u^{\texttt{MW}}$ have a different regularity from the one of $u^{\texttt{CS}}$ (by uniqueness). In this sense, we would say that the existing Hölder regularity results of the mild solution on $\mathbb{R}$ should be improved, and in this paper, we will suggest an idea of how to get the desired result. We emphasize that the regularity almost 3/4 in time and almost 3/2 in space is optimal in the classical PDE sense, since $\dot{W}$ is understood to have regularity $-1/2-\varepsilon$ for any small $\varepsilon > 0$ (c.f. [95, Lemma 1.1]).

The main aim of this chapter is to find the optimal spatial regularity of the unique mild solution u. The first task is to find $\partial_x u$ and check if it is well-defined. One could formally compute $\partial_x u$ from $u^{\texttt{CS}}$ using the same argument as in [87], but we will focus on the Feynman-Kac representation and compute the spatial derivative of u using $u^{\texttt{FK}}$ instead. This approach allows us to obtain a Feynman-Kac-type closed formula for $\partial_x u$. We remark that we can also derive the chaos decomposition of $\partial_x u$ using the Taylor-Strook formula, and it is exactly the same as the one obtained after formally differentiating $u^{\texttt{CS}}$ with respect to x directly. With this in hand, we can achieve the optimal Hölder regularity of $\partial_x u$ that is almost 1/4 in time and almost 1/2 in space.

7.1.1 Hölder spaces and classical Hölder regularity results

In this subsection, we first give a definition of Hölder spaces on $\mathcal{T} \subseteq \mathbb{R}$. For $0 < \gamma < 1$, we let

$$]f[_\gamma := \sup_{z_1 \neq z_2 \in \mathcal{T}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\gamma}.$$

We say that f is Hölder continuous with Hölder exponent γ (or Hölder γ continuous) on $\mathcal{T}$ if

$$\sup_{z\in\mathcal{T}} |f(z)| +]f[_\gamma < \infty.$$

The collection of Hölder γ continuous functions on $\mathcal{T}$ is denoted by $\mathcal{C}^\gamma(\mathcal{T})$ with the norm

$$]]f[[_\gamma := \sup_{z\in\mathcal{T}} |f(z)| +]f[_\gamma.$$

For $k \in \mathbb{N}$, we say that f is a k times continuously differentiable function on $\mathcal{T}$ if the m-th derivative of f, denoted by $\partial^m f$, exists and is continuous for all $m \leq k$. The collection of k times continuously differentiable functions on $\mathcal{T}$ such that $\partial^k f \in \mathcal{C}^\gamma(\mathcal{T})$ with $0 < \gamma < 1$, is denoted by $\mathcal{C}^{k+\gamma}(G)$ with the norm

$$]]f[[_{k+\gamma} := \sum_{1\leq m\leq k} \sup_{z\in\mathcal{T}} |\partial^m f(z)| +]\partial^k f[_\gamma < \infty.$$

In a similar manner, we can define the Hölder spaces on $[0,T]\times\mathbb{R}$ for $T > 0$ as follows. For $0 < \gamma_1, \gamma_2 < 1$, we define

$$]f[_{\gamma_1,\gamma_2} := \sup_{(t,x)\neq(s,x)\in[0,T]\times\mathbb{R}} \frac{|f(t,x)-f(s,x)|}{|t-s|^{\gamma_1}} + \sup_{(t,x)\neq(t,y)\in[0,T]\times\mathbb{R}} \frac{|f(t,x)-f(t,y)|}{|x-y|^{\gamma_2}}.$$

Then, f is said to be Hölder (γ_1,γ_2) continuous on $[0,T]\times\mathbb{R}$ if

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |f(t,x)| +]f[_{\gamma_1,\gamma_2} < \infty,$$

and the collection of Hölder (γ_1,γ_2) continuous functions on $[0,T]\times\mathbb{R}$ is denoted by $\mathcal{C}^{\gamma_1,\gamma_2}([0,T]\times\mathbb{R})$ with the norm

$$]]f[[_{\gamma_1,\gamma_2} := \sup_{(t,x)\in[0,T]\times\mathbb{R}} |f(t,x)| +]f[_{\gamma_1,\gamma_2}.$$

Let $k_1, k_2 \in \mathbb{N}$ and $0 < \gamma_1, \gamma_2 < 1$. The Hölder space, denoted by $\mathcal{C}^{k_1+\gamma_1,k_2+\gamma_2}([0,T]\times\mathbb{R})$, is defined by the collection of all functions on $([0,T]\times\mathbb{R})$ such that f is k_1 times continuously differentiable in t and k_2 times continuously differentiable in x and the norm

$$]]f[[_{k_1+\gamma_1,k_2+\gamma_2} := \sum_{0\leq i\leq k_1,\ 0\leq j\leq k_2} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\partial_t^i \partial_x^j f(t,x)\right| + \Big]\partial_t^{k_1}\partial_x^{k_2} f\Big[_{\gamma_1,\gamma_2} < \infty.$$

Here, $\partial_t := \dfrac{\partial}{\partial t}$ $\left(\partial_x := \dfrac{\partial}{\partial x}\right)$ represents the differentiation operator with respect to t (resp. x).

7.2 Basic regularity of u

We again assume that $u_0 \in L^\infty(\mathbb{R})$ so that $u = u^{\texttt{FK}} = u^{\texttt{MW}} = u^{\texttt{CS}}$ and we will denote $\|\bullet\|_\infty := \|\bullet\|_{L^\infty(\mathbb{R})}$ In this section, we will provide a few basic regularity of u using the Feynman-Kac representation.

Definition 143. [96] For any $\lambda \in \mathbb{R}$, let $\mathcal{G}_\lambda$ be the closure of (L^2) with respect to the norm $\left\|\Gamma(e^\lambda \mathcal{I})\bullet\right\|_2$, where $\mathcal{I}$ stands for the identity operator. More explicitly,

$$\mathcal{G}_\lambda := \left\{F = \sum_{n=0}^\infty I_n(f_n) \in (L^2) : \sum_{n=0}^\infty n! e^{2\lambda n} \|f_n\|^2_{L^2(\mathbb{R}^n)} < \infty\right\},$$

and now we set

$$\mathcal{G} := \bigcap_{\lambda\in\mathbb{R}} \mathcal{G}_\lambda.$$

In particular, it is not hard to see the following inclusions:

$$(S) \subset \mathcal{G} \subset (L^2)$$

We note that if $F \in (L^2)$ can be written as $\sum_{\alpha\in\mathcal{J}} F_\alpha \mathcal{H}_\alpha$, then one can show that $F \in \mathcal{G}_\lambda$ if

$$\sum_{n=0}^{\infty} e^{2\lambda n} \sum_{\alpha\in\mathcal{J}_n} |F_\alpha|^2 < \infty.$$

Theorem 144. *For every $(t,x) \in [0,T] \times \mathbb{R}$,*

$$u(t,x) \in \mathcal{G}.$$

Proof. We have for $\phi \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned}\mathcal{S}(u(t,x))(\phi) &:= \mathbb{E}\left[u(t,x)\mathcal{E}(\phi)\right] = \mathbb{E}\widetilde{\mathbb{E}}\left[u_0(B^x(t))\mathcal{E}(L^x(t))\mathcal{E}(\phi)\right] \\ &= \mathbb{E}^B\left[u_0(B^x(t)) \exp\left\{\int_{\mathbb{R}} L^x_y(t)\phi(y)dy\right\}\right],\end{aligned} \tag{7.10}$$

where the last equality comes from Fubini Lemma and [33, Theorem 5.13].

Let $P_m : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ be the orthogonal projection of $\mathscr{S}'(\mathbb{R})$ on $\text{span}\{e_1,\dots,e_m\}$, $m \geq 1$. Then for $\eta \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R}) := \mathscr{S}'(\mathbb{R}) \oplus i\mathscr{S}'(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we have

$$\mathcal{S}(u(t,x))(\lambda P_m\eta) = \mathbb{E}^B\left[u_0(B^x(t)) \exp\left\{\int_{\mathbb{R}} \lambda P_m\eta(y) L^x_y(t)dy\right\}\right].$$

Using the fact that for $\phi, \eta \in \mathscr{S}'_{\mathbb{C}}(\mathbb{R})$ it holds that $\langle \eta, P_m\phi\rangle = \langle \phi, P_m\eta\rangle$, we can write

$$\mathcal{S}(u(t,x))(\lambda P_m\eta) = \mathbb{E}^B\left[u_0(B^x(t)) \exp\left\{\lambda \int_{\mathbb{R}} \eta(y)(P_m L^x(t))(y)dy\right\}\right]$$

and

$$|\mathcal{S}(u(t,x))(\lambda P_m\eta)|^2 = \left|\mathbb{E}^B\left[u_0(B^x(t)) \exp\left\{\lambda \int_{\mathbb{R}} (\eta_1(y) + i\eta_2(y))(P_m L^x(t))(y)dy\right\}\right]\right|^2.$$

By Jensen's inequality, we have

$$|\mathcal{S}(u(t,x))(\lambda P_m\eta)|^2 \leq \|u_0\|_\infty^2 \mathbb{E}^B\left[\left|\exp\left\{\lambda \int_{\mathbb{R}} (\eta_1(y) + i\eta_2(y))(P_m L^x(t))(y)dy\right\}\right|^2\right].$$

Since $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ for any $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}
&|\mathcal{S}(u(t,x))(\lambda P_m \eta)|^2 \\
&\leq \|u_0\|_\infty^2 \mathbb{E}^B \left[\left| \exp\left\{ \lambda \int_{\mathbb{R}} \eta_1(y)(P_m L^x(t))(y) dy \right\} \right|^2 \left| \exp\left\{ i\lambda \int_{\mathbb{R}} \eta_2(y)(P_m L^x(t))(y) dy \right\} \right|^2 \right] \\
&= \|u_0\|_\infty^2 \widetilde{\mathbb{E}} \left[\exp\left\{ 2\lambda \left(\eta_1 \,,\ P_m L^x(t)\right)_{L^2(\mathbb{R})} \right\} \right].
\end{aligned}$$

Thus,

$$\int_{\mathscr{S}'_{\mathbb{C}}(\mathbb{R})} |\mathcal{S}(u(t,x))(\lambda P_m \eta)|^2 \nu(d\eta) \leq \|u_0\|_\infty^2 \mathbb{E}^B \left[\int_{\mathscr{S}'_{\mathbb{C}}(\mathbb{R})} \exp\left\{ 2\lambda \left(\eta_1 \,,\ P_m L^x(t)\right)_{L^2(\mathbb{R})} \right\} \nu(d\eta) \right],$$

where the measure ν is given by the product measure $\mu_{\frac{1}{2}} \otimes \mu_{\frac{1}{2}}$, where $\mu_{\frac{1}{2}}$ is the measure on $(\mathscr{S}'(\mathbb{R}), \mathscr{B})$ with the characteristic function given by:

$$\int_{\mathscr{S}'(\mathbb{R})} e^{i\langle \omega, \varphi \rangle} d\mu_{\frac{1}{2}}(\omega) = e^{-\frac{1}{4}\|\varphi\|_{L^2(\mathbb{R})}^2}, \qquad \varphi \in \mathscr{S}(\mathbb{R}).$$

It is clear that for any $\varphi \in \mathscr{S}(\mathbb{R})$, $\mu_{\frac{1}{2}} \circ \left\langle \bullet, \frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R})}} \right\rangle^{-1}$ is a centered Gaussian measure with variance 1/2 as in [38, Lemma 2.1.2]. Therefore,

$$\begin{aligned}
\int_{\mathscr{S}'_{\mathbb{C}}(\mathbb{R})} |\mathcal{S}(u(t,x))(\lambda P_m \eta)|^2 \nu(d\eta) &\leq \|u_0\|_\infty^2 \widetilde{\mathbb{E}} \left[\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2\lambda y \|P_m L^x(t)\|_{L^2(\mathbb{R})}} e^{-y^2} dy \right] \\
&= \|u_0\|_\infty^2 \widetilde{\mathbb{E}} \left[e^{\lambda^2 \|P_m L^x(t)\|_{L^2(\mathbb{R})}^2} \right].
\end{aligned}$$

Finally, we obtain

$$\lim_{m\to\infty} \int_{\mathscr{S}'_{\mathbb{C}}(\mathbb{R})} |\mathcal{S}(u(t,x))(\lambda P_m \eta)|^2 \nu(d\eta) \leq \|u_0\|_\infty^2 \widetilde{\mathbb{E}} \left[e^{\lambda^2 \int_{\mathbb{R}} |L_y^x(t)|^2 dy} \right] < \infty, \ \forall \lambda \in \mathbb{R},$$

which implies by [97, Corollary 5.1], $u(t,x)$ belongs to $\mathcal{G}$. □

Next, we state the basic Hölder regularity of u both in time and space.

Theorem 145. *Let $0 < \varepsilon < 1/2$ be arbitrary and C be a constant.*

(i) Assume that $u_0 \equiv C$. Then,

$$u \in \mathcal{C}^{3/4-\varepsilon, 1/2-\varepsilon}([0,T] \times \mathbb{R}).$$

(ii) Assume that $u_0 \not\equiv C$ and $u_0 \in L^\infty(\mathbb{R})$ is (globally) Lipschitz continuous on $\mathbb{R}$. Then,

$$u \in \mathcal{C}^{1/2-\varepsilon, 1/2-\varepsilon}([0,T] \times \mathbb{R}).$$

Proof. Let $\|\!| \bullet \|\!|_p := \left(\mathbb{E}\widetilde{\mathbb{E}}|\bullet|^p\right)^{1/p}$ be the norm on the Banach space $L^p(\mu \otimes \widetilde{\mathbb{P}})$ for $p \geq 1$. From the calculations we've presented in the last chapter, we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \|\!| \exp\{\Psi_{t,x}\} \|\!|_p < \infty. \tag{7.11}$$

Also, (7.8) conditional on B, becomes

$$\Psi_{t,x} \sim \mathcal{N}\left(-\frac{1}{2}\int_{\mathbb{R}} |L_a(t)|^2 da, \int_{\mathbb{R}} |L_a(t)|^2 da\right), \tag{7.12}$$

where $\mathcal{N}\left(\mu, \sigma^2\right)$ denotes the Gaussian random variable with mean μ and variance σ^2.

(i) Let u_0 be a constant. Then, using the fact $|e^x - e^y| \leq (e^x + e^y)|x-y|$ for all $x, y \in \mathbb{R}$, Cauchy-Schwarz inequality, Minkowski inequality, (7.11), and (7.12), we can obtain for $p \geq 2$.

$$\mathbb{E}\left[|u(t,x) - u(s,y)|^p\right] \leq c_p \left\{\widetilde{\mathbb{E}}\mathbb{E}\left[|\Psi_{t,x} - \Psi_{s,y}|^2\right]^{1/2}\right\}^p = c_p \|\!|\Psi_{t,x} - \Psi_{s,y}\|\!|_2^p,$$

for some $c_p > 0$. Also, the triangular inequality implies

$$\mathbb{E}\left[|u(t,x) - u(s,y)|^p\right] \leq c_p \left\{\|\!|\Psi_{t,x} - \Psi_{t,y}\|\!|_2 + \|\!|\Psi_{t,y} - \Psi_{s,y}\|\!|_2\right\}^p =: c_p\left(A_1^{1/2} + A_2^{1/2}\right)^p.$$

Let us now work with

$$A_1 = \|\!|\Psi_{t,x} - \Psi_{t,y}\|\!|_2^2 = \widetilde{\mathbb{E}}\mathbb{E}\left[\Psi_{t,x}^2 - 2\Psi_{t,x}\Psi_{t,y} + \Psi_{t,y}^2\right].$$

By (7.12), we have

$$A_1 = \widetilde{\mathbb{E}}\left[2\int_{\mathbb{R}} |L_a(t)|^2 da + \frac{1}{2}\left(\int_{\mathbb{R}} |L_a(t)|^2 da\right)^2 - 2\mathbb{E}[\Psi_{t,x}\Psi_{t,y}]\right].$$

Recall the Dirac-delta function $\delta_x(y) = \lim_{\varepsilon\to 0}(\pi\varepsilon)^{-1/2} e^{-|x-y|^2/\varepsilon}$, $x, y \in \mathbb{R}$. Since

$$\widetilde{\mathbb{E}}\mathbb{E}\left[\Psi_{t,x}\Psi_{t,y}\right] = \widetilde{\mathbb{E}}\left[\int_0^t \int_0^t \delta_0(B_u - B(r) - (x-y))dudr + \frac{1}{4}\left(\int_{\mathbb{R}} |L_a(t)|^2 da\right)^2\right],$$

we get

$$A_1 = \widetilde{\mathbb{E}}\left[2\int_{\mathbb{R}} |L_a(t)|^2 da - 2\int_0^t \int_0^t \delta_0(B_u - B(r) - (x-y))dudr\right].$$

The next step is done rigorously (See [15] for instance) by the translation invariant property of the Lebesgue measure:

$$A_1 = \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}} |L_{a-x}(t)|^2 da - 2\int_0^t \int_0^t \delta_0(B_u - B(r) - (x-y))dudr + \int_{\mathbb{R}} |L_{a-y}(t)|^2 da\right],$$

and this yields, by [15, Proposition 9.2],

$$A_1 = \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}} |L_{a-x}(t) - L_{a-y}(t)|^2 \, da\right] = 4t|x-y| + \mathcal{O}(|x-y|^2),$$

which implies $u(t, \bullet)$ is almost Hölder 1/2 continuous uniformly for all $t \in [0, T]$.

On the other hand, let us compute, for $0 \le s \le t \le T$,

$$\begin{aligned} A_2 &= \widetilde{\mathbb{E}}\mathbb{E}\left[\Psi_{t,y}^2 - 2\Psi_{t,y}\Psi_{s,y} + \Psi_{s,y}^2\right] \\ &= \widetilde{\mathbb{E}}\Bigg[\int_{\mathbb{R}} |L_a(t)|^2 da + \frac{1}{4}\left(\int_{\mathbb{R}} |L_a(t)|^2 da\right)^2 + \int_{\mathbb{R}} |L_a(s)|^2 da + \frac{1}{4}\left(\int_{\mathbb{R}} |L_a(s)|^2 da\right)^2 \\ &\quad - 2\mathbb{E}\left[\Psi_{t,y}\Psi_{s,y}\right]\Bigg]. \end{aligned}$$

Since

$$\widetilde{\mathbb{E}}\mathbb{E}\left[\Psi_{t,y}\Psi_{s,y}\right] = \widetilde{\mathbb{E}}\left[\int_0^t \int_0^s \delta_0(B_u - B(r)) du dr + \frac{1}{4}\left(\int_{\mathbb{R}} |L_a(t)|^2 da\right)\left(\int_{\mathbb{R}} |L_a(s)|^2 da\right)\right],$$

we have

$$\begin{aligned} A_2 &= \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}} |L_a(t)|^2 da - 2\int_0^t \int_0^s \delta_0(B(r) - B(z)) dr dz + \int_{\mathbb{R}} |L_a(s)|^2 da\right] \\ &\quad + \frac{1}{4}\widetilde{\mathbb{E}}\left[\left(\int_{\mathbb{R}} |L_a(t)|^2 da\right)^2 - 2\left(\int_{\mathbb{R}} |L_a(t)|^2 da\right)\left(\int_{\mathbb{R}} |L_a(s)|^2 da\right) + \left(\int_{\mathbb{R}} |L_a(s)|^2 da\right)^2\right] \\ &= \widetilde{\mathbb{E}}\left[\int_{\mathbb{R}} |L_a(t)|^2 da - 2\int_0^t \int_0^s \delta_0(B(r) - B(z)) dr dz + \int_{\mathbb{R}} |L_a(s)|^2 da\right] \\ &\quad + \frac{1}{4}\widetilde{\mathbb{E}}\left[\left(\int_{\mathbb{R}} \left(|L_a(t)|^2 - |L_a(s)|^2\right) da\right)^2\right]. \end{aligned}$$

We note that

$$\widetilde{\mathbb{E}}\left[\int_{\mathbb{R}} |L_a(t)|^2 da\right] = \widetilde{\mathbb{E}}\left[\int_0^t \int_0^t \delta_0(B(r) - B(z)) dr dz\right],$$

which implies

$$\begin{aligned} A_2 &= \widetilde{\mathbb{E}}\left[\int_0^t \int_0^t \delta_0(B(r) - B(z)) dr dz - 2\int_0^t \int_0^s \delta_0(B(r) - B(z)) dr dz + \int_0^s \int_0^s \delta_0(B(r) - B(z)) dr dz\right] \\ &\quad + \frac{1}{4}\widetilde{\mathbb{E}}\left[\left(\int_0^t \int_0^t \delta_0(B(r) - B(z)) dr dz - \int_0^s \int_0^s \delta_0(B(r) - B(z)) dr dz\right)^2\right] \\ &= \widetilde{\mathbb{E}}\left[\int_s^t \int_s^t \delta_0(B(r) - B(z)) dr dz\right] \\ &\quad + \frac{1}{4}\widetilde{\mathbb{E}}\left[\left(\int_0^t \int_0^t \delta_0(B(r) - B(z)) dr dz - \int_0^s \int_0^s \delta_0(B(r) - B(z)) dr dz\right)^2\right] \\ &=: A_3 + A_4. \end{aligned}$$

We can easily compute A_3:

$$A_3 = \int_s^t \int_s^t (2\pi|r-z|)^{-1/2} drdz = \mathtt{C}(t-s)^{3/2} \quad \text{for some } \mathtt{C} > 0 \text{ independent of } x.$$

For A_4, we have

$$\begin{aligned}
4A_4 &= \int_s^t \int_s^t \int_s^t \int_s^t \widetilde{\mathbb{E}}\left[\delta_0(B(z)-B(r))\delta_0(B(q)-B(p))\right] dpdqdrdz \\
&\overset{\text{symmetry}}{=} 4! \int_s^t \int_s^z \int_s^r \int_s^q \widetilde{\mathbb{E}}\left[\delta_0(B(z)-B(r))\right] \widetilde{\mathbb{E}}\left[\delta_0(B(q)-B(p))\right] dpdqdrdz \\
&= 4! \int_s^t \int_s^z \int_s^r \int_s^q \frac{1}{\sqrt{2\pi(z-r)}} \frac{1}{\sqrt{2\pi(q-p)}} dpdqdrdz \\
&\leq 4! \int_s^t \int_s^z \frac{1}{\sqrt{2\pi(z-r)}} \left(\int_s^t \int_s^q \frac{1}{\sqrt{2\pi(q-p)}} \right) dpdqdrdz \\
&= \mathtt{C}(t-s)^3 \quad \text{for some } \mathtt{C} > 0 \text{ independent of } x.
\end{aligned}$$

Combining all together, we obtain

$$A_2 \leq \mathtt{C}|t-s|^{3/2},$$

which implies $u(\bullet, x)$ is almost Hölder 3/4 continuous uniformly for all $x \in \mathbb{R}$.

(ii) If u_0 is not a constant function on $\mathbb{R}$, then we have

$$\begin{aligned}
\mathbb{E}\left[|u(t,x)-u(s,y)|^p\right] &= \mathbb{E}\left[\left|\widetilde{\mathbb{E}}\left(u_0(x+B(t))\exp\{\Psi_{t,x}\}\right) - \widetilde{\mathbb{E}}\left(u_0\left(y+B(s)\right)\exp(\Psi_{s,y})\right)\right|^p\right] \\
&= \mathbb{E}\Bigg[\Bigg|\widetilde{\mathbb{E}}\left((u_0(x+B(t)) - u_0(y+B(s)))\exp\{\Psi_{t,x}\}\right) \\
&\qquad + \widetilde{\mathbb{E}}\left(u_0(y+B(s))\left(\exp\{\Psi_{t,x}\} - \exp(\Psi_{s,y})\right)\right)\Bigg|^p\Bigg].
\end{aligned}$$

Since $|f+g|^p \leq 2^{p-1}\left(|f|^p + |g|^p\right)$ for $p \geq 1$,

$$\begin{aligned}
\mathbb{E}\left[|u(t,x)-u(s,y)|^p\right] &\leq 2^{p-1}\Bigg(\mathbb{E}\left[\left|\widetilde{\mathbb{E}}\left(\left(u_0\left(x+B(t)\right) - u_0\left(y+B(s)\right)\right)\exp\{\Psi_{t,x}\}\right)\right|^p\right] \\
&\qquad + \mathbb{E}\left[\left|\widetilde{\mathbb{E}}\left(u_0\left(y+B(s)\right)\left(\exp\{\Psi_{t,x}\} - \exp(\Psi_{s,y})\right)\right)\right|^p\right]\Bigg) \\
&=: 2^{p-1}\left(\bar{A}_1 + \bar{A}_2\right).
\end{aligned}$$

For $\overline{\mathcal{A}}_1$, by Cauchy-Schwarz inequality,

$$\bar{A}_1 \leq \left(\widetilde{\mathbb{E}}\left[|u_0(x+B(t)) - u_0(y+B(s))|^2\right]\right)^{p/2} \cdot \mathbb{E}\left(\widetilde{\mathbb{E}}\left[\exp\{2\Psi_{t,x}\}\right]\right)^{p/2}.$$

Since u_0 is Lipschitz continuous on $\mathbb{R}$, we have

$$\begin{aligned}\bar{A}_1 &\leq \left(|t-s| + (x-y)^2\right)^{p/2} \cdot \mathbb{E}\left(\widetilde{\mathbb{E}}\left[\exp\{2\Psi_{t,x}\}\right]\right)^{p/2} \\ &\leq \left(|t-s|^{1/2} + |x-y|\right)^{p} \cdot \mathbb{E}\left(\widetilde{\mathbb{E}}\left[\exp\{2\Psi_{t,x}\}\right]\right)^{p/2}.\end{aligned}$$

By Minkowski inequality, Hölder inequality for $p \geq 2$, and (7.11), we also have

$$\bar{A}_1 \leq \left(|t-s|^{1/2} + |x-y|\right)^{p} \cdot \mathbb{E}\widetilde{\mathbb{E}}\left[\exp\{p\Psi_{t,x}\}\right] < \infty.$$

For $\bar{A}_2$, since $u_0 \in L^\infty(\mathbb{R})$, we can apply the same argument in (i). As a result, we can say that u is Hölder continuous almost $1/2$ both in time and space. □

As we argued in the introduction, one expects that we can still improve the spatial regularity of u. We will derive our desired result in Section 7.3.

7.3 The spatial derivative of u

As we anticipated in the introduction, we expect that $u(t,\bullet) \in \mathcal{C}^{3/2-\varepsilon}(\mathbb{R})$ for any small $\varepsilon > 0$. To verify this assertion, we first compute the spatial derivative of u using the Feynman-Kac representation and then find its chaos expansion to see if it is well-defined and to get the optimal spatial regularity of u.

Let us start with a useful Lemma. The following result will serve as a key idea for finding $\partial_x u(t,x)$.

Lemma 146. *For fixed (t,x) the map $\widetilde{\omega} \ni \widetilde{\Omega} \mapsto \widetilde{\Phi}_{t,x}(\widetilde{\omega}) \in (S)^*$ given by*

$$\widetilde{\Phi}_{t,x}(\widetilde{\omega}) = \mathcal{E}(L^x(t;\widetilde{\omega})) \diamond \left[u_0'(B^x(t)(\widetilde{\omega})) + u_0(B^x(t)) I\left(\partial_x L^x(t;\widetilde{\omega})\right)\right],$$

is Bochner integrable in $(S)^*$. Here, $\partial_x L^x(t;\widetilde{\omega}) \in \mathscr{S}'(\mathbb{R}), \widetilde{\omega} \in \widetilde{\Omega}$ denotes the distributional derivative of $L^x(t,\widetilde{\omega})$.

Proof. This immediately follows from [32, Theorem 4.51] and the facts

1. $\mathcal{S}\left(\widetilde{\Phi}_{t,x}(\bullet)\right)(\phi) = \left(\int_0^t \phi(B^x(s)(\bullet))ds\right) \times \left[u_0'(B^x(t)(\bullet)) + u_0(B^x(t)) \int_0^t \phi'(B^x(s)(\bullet))ds\right]$ is measurable for any $\phi \in \mathscr{S}(\mathbb{R})$.
2. $\left|\mathcal{S}\left(\widetilde{\Phi}_{t,x}(\widetilde{\omega})\right)(\phi)\right| \leq \left(\|u_0'\|_\infty + t\|u_0\|_\infty\|\phi'\|_\infty\right) e^{t\|\phi\|_\infty} \leq e^{\|u_0'\|_\infty + t(\|u_0\|_\infty\|\phi'\|_\infty + \|\phi\|_\infty)} \leq \mathtt{K}_1 e^{\mathtt{K}_2|\phi|_p^2}$ for some $p, \mathtt{K}_1, \mathtt{K}_2 \geq 0$. The last inequality comes from the equivalence between standard seminorms in the Schwartz space and the system of *p-seminorms.*

□

Recall the Feynman-Kac representation of u:

$$u(t,x) = \widetilde{\mathbb{E}}\left[u_0(B^x(t))\exp\{\Psi_{t,x}\}\right] = \widetilde{\mathbb{E}}\left[u_0(B^x(t))\mathcal{E}(L^x(t))\right]$$

and recall $C_b^1(\mathbb{R})$ is the space of all bounded differentiable functions on $\mathbb{R}$ with bounded continuous derivative.

Theorem 147. *Assume that $u_0 \in C_b^1(\mathbb{R})$. Then, for each $t > 0$, $u(t,\bullet)$ is weakly continuously differentiable in $(S)^*$, and*

$$\partial_x u(t,x) = \widetilde{\mathbb{E}}\left[\mathcal{E}(L^x(t)) \diamond \left\{u_0'(B^x(t)) + u_0(B^x(t)) I\left(\partial_x L^x(t)\right)\right\}\right],$$

where $\partial_x L^x(t;\widetilde{\omega}) \in \mathscr{S}'(\mathbb{R}), \widetilde{\omega} \in \widetilde{\Omega}$ is the distributional derivative of $L^x(t;\widetilde{\omega})$, and $\widetilde{\mathbb{E}}$ must be understood as a Bochner integral in $(S)^*$.

Proof. To find $\partial_x u(t,x)$, we start by computing the S-transform of u. From (7.10), for $\phi \in \mathscr{S}(\mathbb{R})$, we have

$$\mathcal{S}\left(u(t,x)\right)(\phi) = \mathbb{E}^B\left[u_0(B^x(t))\exp\left\{\int_{\mathbb{R}} L_y^x(t)\phi(y)dy\right\}\right] = \widetilde{\mathbb{E}}\left[u_0(B^x(t))\exp\left\{\int_0^t \phi(B^x(s))ds\right\}\right], \tag{7.13}$$

where the last equality follows by the occupation time formula.

It is clear that $x \in \mathbb{R} \mapsto \mathcal{S}(u(t,x))(\phi)$ is continuous for all $\phi \in \mathscr{S}(\mathbb{R})$, and $|S\left(u(t,x)\right)(\phi)| \le \mathtt{K}_1 e^{\mathtt{K}_2|\phi|_p^2}$ for some $\mathtt{K}_1, \mathtt{K}_2, p > 0$. Then, by [98, Lemma A.1.2], we can see that $u(t,x) = \mathcal{S}^{-1}(\mathcal{S}(u(t,x)))$ is *weakly* continuous in $(S)^*$ (with respect to the x variable).

In order to prove that $u(t,\bullet)$ is *weakly* continuously differentiable in $(S)^*$ we must first take the spatial derivative on both sides of (7.13). Using the fact $\phi, \phi' \in \mathscr{S}(\mathbb{R}) \subset L^\infty(\mathbb{R})$, by dominated convergence theorem (DCT), we obtain

$$\begin{aligned}\partial_x \mathcal{S}(u(t,x))(\phi) = \widetilde{\mathbb{E}}\Big[&u_0'(B^x(t))\exp\left\{\int_0^t \phi(B^x(s))ds\right\} \\ &+ u_0(B^x(t))\exp\left\{\int_0^t \phi(B^x(s))ds\right\} \times \int_0^t \phi'(B^x(s))ds\Big],\end{aligned}$$

and it is clear that the map $x \mapsto \partial_x \mathcal{S}(u(t,x))(\phi)$ is continuous for all $\phi \in \mathscr{S}(\mathbb{R})$.

We also need to show $\partial_x \mathcal{S}\left(u(t,x)\right)$ is a U-functional (see Definition 80). By direct computation, we can verify that, as in the proof of Lemma 146,

$$|\partial_x \mathcal{S}\left(u(t,x)\right)(\phi)| \le \mathtt{K}_1 e^{\mathtt{K}_2|\phi|_p^2},$$

where $\mathtt{K}_1, K_2, p$ are positive real constants. Also, it is clear that the map $z \mapsto \partial_x \mathcal{S}(u)(z\phi+\eta)$ is entire for any $\phi, \eta \in \mathscr{S}(\mathbb{R})$ and $z \in \mathbb{C}$. Hence, $\partial_x \mathcal{S}\left(u(t,x)\right)$ is indeed a U-functional, and thus there exists a unique $\Phi \in (S)^*$ such that $\partial_x \mathcal{S}(u(t,x)) = S(\Phi)$; then from [98, Lemma A.3], we can conclude that $u(t,\bullet)$ is *weakly* continuously differentiable in the Hida distribution space $(S)^*$. Following the aforementioned reference, the *weak* spatial derivative $\partial_x u$ of u is defined as the unique element in $(S)^*$ such that

$$\mathcal{S}(\partial_x u(t,x))(\phi) = \partial_x \mathcal{S}(u(t,x))(\phi).$$

Using Lemma 146, we can see that $\partial_x \mathcal{S}\left(u(t,x)\right) = \widetilde{\mathbb{E}}\left(\mathcal{S}\left(\widetilde{\Phi}_{t,x}\right)\right)$, and furthermore we have that $\widetilde{\mathbb{E}}\left(\mathcal{S}\left(\widetilde{\Phi}_{t,x}\right)\right) = \mathcal{S}\left(\widetilde{\mathbb{E}}\left(\widetilde{\Phi}_{t,x}\right)\right)$ since the Bochner integral $\widetilde{\mathbb{E}}$ and the S-transform can be interchanged (e.g. [32, Theorem 4.51]). Finally, by definition 82, we can conclude

$$\partial_x u(t,x) = \widetilde{\mathbb{E}}\left[\mathcal{E}(L^x(t)) \diamond \left\{ u_0'(B^x(t)) + u_0(B^x(t)) I\left(\partial_x L^x(t)\right) \right\}\right],$$

where $\widetilde{\mathbb{E}}$ must be understood as a Bochner integral in $(S)^*$. □

From this result, we can only say that $\partial_x u(t,x) \in (S)^*$ for each $(t,x) \in [0,T] \times \mathbb{R}$. But, in the following subsection, we will show that $\partial_x u(t,x) \in \mathcal{G}$ using its chaos decomposition, and furthermore, we will investigate its Hölder regularity.

7.3.1 Chaos decomposition for $\partial_x u$

Let us find the chaos expansion of

$$\partial_x u(t,x) = \widetilde{\mathbb{E}}\left[\mathcal{E}(L^x(t)) \diamond \left\{ I_0(u_0'(B^x(t))) + u_0(B^x(t)) I\left(\partial_x L^x(t)\right) \right\}\right],$$

and notice that as usual the Wiener integral of order 0 equals the identity operator, but nonetheless we explicitly write I_0 for notational convenience.

By Lemma 59, we have

$$\mathcal{E}(L^x(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n\left(L^x(t)^{\otimes n}\right), \quad \text{convergent in } (L^2),$$

and by the definition of *Wick product*, we see that

$$\mathcal{E}(L^x(t)) \diamond \left\{ I_0(u_0'(B^x(t))) + u_0(B^x(t)) I\left(\partial_x L^x(t)\right) \right\} = \sum_{n=0}^{\infty} I_n(h_n(t,x)), \quad \text{convergent in } (S)^*,$$

where $h_0(t,x) = u_0'\left(B^x(t)\right)$, and

$$\begin{aligned} \mathscr{S}'(\mathbb{R}^n) \ni h_n(t,x;\bullet) =& u_0(B^x(t)) \operatorname{Sym}\left[\left(\frac{L^x(t)^{\otimes(n-1)}}{(n-1)!}\right) \otimes \partial_x L^x(t)\right](\bullet) \\ &+ u_0'(B^x(t)) \left(\frac{L^x(t)^{\otimes n}}{n!}\right)(\bullet), \qquad n \geq 1. \end{aligned}$$

It is known that (e.g. [33, Chapter 13.3]) if $\Psi(u) = \sum_{n=0}^{\infty} I_n(F_n(u))$ is Bochner integrable on the measure space $(M, \sigma(M), m)$, then for any $n \in \mathbb{N}_0$, F_n is Bochner integrable on $(M, \sigma(M), m)$, and it holds that

$$\int_M \Psi(u) m(du) = \sum_{n=0}^{\infty} I_n \left(\int_M F_n(u) m(du) \right).$$

In our case, letting $(M, \sigma(M), m) = \left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$, this would imply that

$$\partial_x u(t,x) = \sum_{n=0}^{\infty} I_n \left(\widetilde{\mathbb{E}} \left[h_n(t,x) \right] \right),$$

where $\widetilde{\mathbb{E}}$ should be understood as a Bochner integral in $\mathscr{S}'(\mathbb{R}^n)$.

We can easily check that the first term of $\partial_x u(t,x)$ is $\widetilde{\mathbb{E}}\left[u_0'\left(B^x(t)\right)\right] = \partial_x u_{(\mathbf{0})}(t,x)$. Let's check the general n-th terms of $\partial_x u(t,x)$ for $n \geq 1$. Since $h_n(t,x;\bullet)$ is a symmetric element of $\mathscr{S}'(\mathbb{R}^n)$, we can expand it with respect to $\{\mathfrak{e}_\alpha : \alpha \in \mathcal{J}_n\}$ as

$$h_n(t,x;\bullet) = \sum_{\alpha \in \mathcal{J}_n} \langle h_n(t,x), \mathfrak{e}_\alpha \rangle \, \mathfrak{e}_\alpha(\bullet), \quad \text{convergence in } \mathscr{S}'(\mathbb{R}^n), \tag{7.14}$$

where $\langle \bullet, \bullet \rangle$ is the bilinear product between $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$, and

$$\mathfrak{e}_\alpha(y_1, ..., y_n) = \frac{1}{\sqrt{n!\alpha!}} \sum_{\sigma \in \mathcal{P}_n} e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n).$$

It is clear by direct calculations that $\langle \operatorname{Sym} f, \operatorname{Sym} g \rangle = \langle f, \operatorname{Sym}^2 g \rangle = \langle f, \operatorname{Sym} g \rangle$. Then, we have

$$\begin{aligned}
\langle h_n(t,x), \mathfrak{e}_\alpha \rangle &= \frac{u_0(B^x(t))}{\sqrt{n!\alpha!}(n-1)!} \left\langle \left[L^x(t)^{\otimes(n-1)} \otimes \ \partial_x L^x(t) \right], \sum_{\sigma \in \mathcal{P}_n} \left[e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right] \right\rangle \\
&\quad + \frac{u_0'(B^x(t))}{\sqrt{n!\alpha!}n!} \int_{\mathbb{R}^n} L^x(t)^{\otimes(n)}(y_1, ..., y_n) \times \sum_{\sigma \in \mathcal{P}_n} \left[e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} \right] (y_1, ..., y_n) d\mathbf{y} \\
&= \frac{u_0(B^x(t))}{\sqrt{n!\alpha!}(n-1)!} \sum_{\sigma \in \mathcal{P}_n} \int_{[0,t]^n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e'_{k_{\sigma(n)}} (B^x(s_1), \ldots, B^x(s_n)) \, d\mathbf{s} \\
&\quad + \frac{u_0'(B^x(t))}{\sqrt{n!\alpha!}n!} \sum_{\sigma \in \mathcal{P}_n} \int_{[0,t]^n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}} (B^x(s_1), \ldots, B^x(s_n)) \, d\mathbf{s},
\end{aligned} \tag{7.15}$$

where in the last expression we used the occupation time formula and the fact that $-\partial_x L^x$ equals distributional derivative of the Brownian local time.

Also, taking $\widetilde{\mathbb{E}}$ on both sides of (7.14), we have

$$\widetilde{\mathbb{E}}[h_n(t,x;\bullet)] = \widetilde{\mathbb{E}} \left[\sum_{\alpha \in \mathcal{J}_n} \langle h_n(t,x), \mathfrak{e}_\alpha \rangle \, \mathfrak{e}_\alpha(\bullet) \right]. \tag{7.16}$$

At this point, we need the following Lemma to compute (7.16).

Lemma 148. *[42, Lemma 11.45] Let $f : M \to X$ be Bochner integrable on $(M, \sigma(M), m)$ in X and let Y be a Banach space. If $T : X \to Y$ is a bounded operator, then $Tf : M \to Y$ is Bochner integrable on $(M, \sigma(M), m)$ in Y and it holds that*

$$\int_M Tf dm = T\left(\int_M f dm\right).$$

In our case, we set $\mathscr{S}'_{\text{sym}}(\mathbb{R}^n)$:= the symmetric part of $\mathscr{S}'(\mathbb{R}^n)$ and $T : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'_{\text{sym}}(\mathbb{R}^n)$ equals the orthogonal projection on $\mathscr{S}'_{\text{sym}}(\mathbb{R}^n)$, which is clearly a bounded linear operator. Even though if it is well-known that the space of tempered distributions is not a Banach space, we can think of $\mathscr{S}'(\mathbb{R})$ as the inductive limit of a family of Hilbert spaces (e.g [33, Section 3.2] or [99]), and an analogous reasoning extends to the multi-dimensional case; thus the lemma above holds true by letting Y be some of those Hilbert spaces.

Then, (7.16) becomes

$$\widetilde{\mathbb{E}}[h_n(t,x;\bullet)] = \widetilde{\mathbb{E}}\left[\sum_{\alpha\in\mathcal{J}_n} \langle h_n(t,x), \mathfrak{e}_\alpha\rangle \mathfrak{e}_\alpha(\bullet)\right] = \sum_{\alpha\in\mathcal{J}_n} \left\langle \widetilde{\mathbb{E}}\left[h_n(t,x)\right], \mathfrak{e}_\alpha \right\rangle \mathfrak{e}_\alpha(\bullet).$$

It is known that if a function is Bochner integrable, then its Pettis and Bochner integrals coincide (see for instance the discussion on page 80 of [43]). Therefore, by definition of the Pettis integral, we have

$$\left\langle \widetilde{\mathbb{E}}\left[h_n(t,x)\right], \mathfrak{e}_\alpha \right\rangle = \widetilde{\mathbb{E}}\left[\langle h_n(t,x), \mathfrak{e}_\alpha\rangle\right].$$

Hence, we have

$$\widetilde{\mathbb{E}}[h_n(t,x;\bullet)] = \sum_{\alpha\in\mathcal{J}_n} \widetilde{\mathbb{E}}\left[\langle h_n(t,x), \mathfrak{e}_\alpha\rangle\right] \mathfrak{e}_\alpha(\bullet).$$

Next we compute $\widetilde{\mathbb{E}}\left[\langle h_n(t,x), \mathfrak{e}_\alpha\rangle\right]$, and so far from (7.15), we have

$$\begin{aligned}\widetilde{\mathbb{E}}\left[\langle h_n(t,x), \mathfrak{e}_\alpha\rangle\right] = \widetilde{\mathbb{E}}\Bigg[&\frac{u_0(B^x(t))}{\sqrt{n!\alpha!}(n-1)!}\int_{[0,t]^n}\sum_{\sigma\in\mathcal{P}_n} e_{k_{\sigma(1)}}\otimes\cdots\otimes e'_{k_{\sigma(n)}}(B^x(s_1),\ldots,B^x(s_n))\, d\mathbf{s} \\ &+ \frac{u'_0(B^x(t))}{\sqrt{n!\alpha!}n!}\sum_{\sigma\in\mathcal{P}_n}\int_{[0,t]^n} e_{k_{\sigma(1)}}\otimes\cdots\otimes e_{k_{\sigma(n)}}(B^x(s_1),\ldots,B^x(s_n))\, d\mathbf{s}\Bigg].\end{aligned} \tag{7.17}$$

To simplify the expression in (7.17), we first observe

$$\begin{aligned}\partial_x\left[e_{k_1}(B^x(s_1))\cdots e_{k_n}(B^x(s_n))\right] = &\left[e'_{k_1}(B^x(s_1))\times\cdots\times e_{k_n}(B^x(s_n))\right] \\ &\cdots + \left[e_{k_1}(B^x(s_1))\times\cdots\times e'_{k_n}(B^x(s_n))\right].\end{aligned}$$

Since the Lebesgue measure is invariant under rotations, we see that for any $f : [0,t]^n \to \mathbb{R}$, it holds that

$$\int_{[0,t]^n} f(s_1,\dots,s_n)\, d\mathbf{s} = \int_{[0,t]^n} \operatorname{Sym} f(s_1,\dots,s_n)\, d\mathbf{s}.$$

Thus, for any permutation σ of $\{1,\dots,n\}$, we have

$$\begin{aligned}\int_{[0,t]^n} \frac{u_0(B^x(t))}{(n-1)!} \sum_{\sigma\in\mathcal{P}_n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e'_{k_{\sigma(n)}}(B^x(s_1),\dots,B^x(s_n))\, d\mathbf{s} \\ = \int_{[0,t]^n} u_0(B^x(t))\partial_x \left[e_{k_1}(B^x(s_1)) \times \cdots \times e_{k_n}(B^x(s_n))\right]\, d\mathbf{s},\end{aligned}$$

and

$$\begin{aligned}\int_{[0,t]^n} \frac{u_0'(B^x(t))}{n!} \sum_{\sigma\in\mathcal{P}_n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}}(B^x(s_1),\dots,B^x(s_n))\, d\mathbf{s} \\ = \int_{[0,t]^n} u_0'(B^x(t)) e_{k_1}(B^x(s_1)) \times \cdots \times e_{k_n}(B^x(s_n))\, d\mathbf{s},\end{aligned}$$

which implies

$$\widetilde{\mathbb{E}}\left[\langle h_n(t,x), \mathfrak{e}_\alpha\rangle\right] = \frac{1}{\sqrt{n!\alpha!}} \widetilde{\mathbb{E}}\left[\int_{\mathbb{T}^n_{[0,t]}} \partial_x \left[u_0(B^x(t)) e_{k_1}(B^x(s_1)) \times \cdots \times e_{k_n}(B^x(s_n))\right]\, d\mathbf{s}\right], \tag{7.18}$$

where again $\mathbb{T}^n_{[0,t]}$ denotes the simplex $\{0 \le s_1 \le \cdots \le s_n \le t\}$.

Furthermore, we notice that

$$\begin{aligned}&\int_{[0,t]^n} \left[e_{k_1}(B^x(s_1)) \times \cdots \times e_{k_n}(B^x(s_n))\right]\, d\mathbf{s} = \int_{[0,t]^n} \operatorname{Sym}\left[e_{k_1}(B^x(s_1)) \times \cdots \times e_{k_n}(B^x(s_n))\right]\, d\mathbf{s} \\ &= \int_{[0,t]^n} \frac{1}{n!} \sum_{\sigma\in\mathcal{P}_p} \left[e_{k_1}(B^x_{s_{\sigma(1)}}) \times \cdots \times e_{k_p}(B^x_{s_{\sigma(n)}})\right]\, d\mathbf{s} = \sqrt{\frac{\alpha!}{n!}} \int_{[0,t]^n} \partial_x \mathfrak{e}_\alpha(B^x(s_1),\dots,B^x(s_n))\, d\mathbf{s} \\ &= \sqrt{\alpha! n!} \int_{\mathbb{T}^n_{[0,t]}} \partial_x \mathfrak{e}_\alpha(B^x(s_1),\dots,B^x(s_n))\, d\mathbf{s},\end{aligned}$$

and similarly,

$$\int_{[0,t]^n} \partial_x \left[e_{k_1}(B^x(s_1)) \times \cdots \times e_{k_n}(B^x(s_n))\right]\, d\mathbf{s} = \sqrt{\alpha! n!} \int_{\mathbb{T}^n_{[0,t]}} \partial_x \mathfrak{e}_\alpha(B^x(s_1),\dots,B^x(s_n))\, d\mathbf{s}.$$

Therefore, (7.18) becomes

$$\widetilde{\mathbb{E}}\left[\langle h_n(t,x),\mathfrak{e}_\alpha\rangle\right] = \widetilde{\mathbb{E}}\left[\int_{\mathbb{T}^n_{[0,t]}} \partial_x\left[u_0(B^x(t))\mathfrak{e}_\alpha(B^x(s_1),\dots,B^x(s_n))\right]\, d\mathbf{s}\right]$$
$$= \int_{\mathbb{T}^n_{[0,t]}} \widetilde{\mathbb{E}}\left[\partial_x\left[u_0(B^x(t))\mathfrak{e}_\alpha(B^x(s_1),\dots,B^x(s_n))\right]\right]\, d\mathbf{s}, \qquad (7.19)$$

where the second equality holds true by Fubini lemma.

By conditioning iteratively on the filtration of B at the sites $s_n, s_{n-1}, \dots, s_1$, we can rewrite (7.19) as

$$\widetilde{\mathbb{E}}\left[\langle h_n(t,x),\mathfrak{e}_\alpha\rangle\right] = \int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^n} \partial_x\mathfrak{e}_\alpha(y_1+x,y_2,\dots,y_n)p(s_n-s_{n-1},y_n-y_{n-1})$$
$$\cdots\times p(s_1,y_1)u_{(\mathbf{0})}(t-s_n,y_n)d\mathbf{y}\, d\mathbf{s}$$
$$= \int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^p} \partial_{y_1}\mathfrak{e}_\alpha(y_1+x,y_2,\dots,y_n)p(s_n-s_{n-1},y_n-y_{n-1})$$
$$\cdots\times p(s_1,y_1)u_{(\mathbf{0})}(t-s_n,y_n)d\mathbf{y}\, d\mathbf{s}$$
$$= -\int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^n} \mathfrak{e}_\alpha(y_1,y_2,\dots,y_n)p(s_n-s_{n-1},y_n-y_{n-1})$$
$$\cdots\times \partial_{y_1}p(s_1,y_1-x)u_{(\mathbf{0})}(t-s_n,y_n)d\mathbf{y}\, d\mathbf{s}. \qquad (7.20)$$

Noticing that $-\partial_y p(s,y-x) = \partial_x p(s,x-y)$ and letting $r_i = t - s_{n+1-i}$ for $i \in \{1,\dots,n\}$, the equation (7.20) becomes

$$\int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^n} \mathfrak{e}_\alpha(y_1,y_2,\dots,y_n)p(r_2-r_1,y_n-y_{n-1})\times\cdots\times\partial_x p(t-r_n,x-y_1)u_{(\mathbf{0})}(r_1,y_n)d\mathbf{y}\, d\mathbf{r},$$

which yields, after relabeling the y's,

$$\widetilde{\mathbb{E}}\left[\langle h_n(t,x),\mathfrak{e}_\alpha\rangle\right] = \int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^n} \partial_x p(t-r_n,x-y_n)p(r_n-r_{n-1},y_n-y_{n-1})\times\cdots\times p(r_2-r_1,y_2-y_1)$$
$$\cdots\times\mathfrak{e}_\alpha(y_1,y_2,\dots,y_n)u_{(\mathbf{0})}(r_1,y_1)d\mathbf{y}\, d\mathbf{r}.$$

Using the definition of $\mathfrak{e}_\alpha$, it equals

$$\frac{1}{\sqrt{\alpha!}}\frac{1}{\sqrt{n!}}\sum_{\sigma\in\mathcal{P}_n}\int_{\mathbb{T}^n_{[0,t]}}\int_{\mathbb{R}^n} \partial_x p(t-r_n,x-y_n)p(r_n-r_{n-1},y_n-y_{n-1})\times\cdots\times p(r_2-r_1,y_2-y_1)$$
$$\cdots\times e_{k_{\sigma(1)}}(y_1)\times\cdots\times e_{k_{\sigma(n)}}(y_n)u_{(\mathbf{0})}(r_1,y_1)d\mathbf{y}\, d\mathbf{r} =: \frac{1}{\sqrt{n!}}\mathfrak{K}_\alpha(t,x), \quad |\alpha| = n \geq 1.$$

Putting all together, we obtain

$$\widetilde{\mathbb{E}}\left[h_n(t,x;\bullet)\right] = \sum_{\alpha\in\mathcal{J}_n} \frac{1}{\sqrt{n!}} \mathfrak{K}_\alpha(t,x)\mathfrak{e}_\alpha(\bullet)$$

and

$$I_n\left(\widetilde{\mathbb{E}}\left[h_n(t,x)\right]\right) = \sum_{\alpha\in\mathcal{J}_n} \mathfrak{K}_\alpha(t,x) I_n\left(\frac{\mathfrak{e}_\alpha}{\sqrt{n!}}\right) = \sum_{\alpha\in\mathcal{J}_n} \mathfrak{K}_\alpha(t,x)\mathcal{H}_\alpha \quad \text{by (??)}.$$

Finally, we have the chaos expansion of $\partial_x u(t,x)$ as follows:

$$\partial_x u(t,x) = \sum_{n=0}^{\infty} I_n\left(\widetilde{\mathbb{E}}\left[h_n(t,x)\right]\right) = \sum_{n=0}^{\infty}\sum_{\alpha\in\mathcal{J}_n} \mathfrak{K}_\alpha(t,x)\mathcal{H}_\alpha = \sum_{\alpha\in\mathcal{J}} \mathfrak{K}_\alpha(t,x)\mathcal{H}_\alpha, \quad (7.21)$$

where $\mathfrak{K}_{(\mathbf{0})}(t,x) = \partial_x u_{(\mathbf{0})}(t,x)$, and for $|\alpha| = n \geq 1$,

$$\mathfrak{K}_\alpha(t,x) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)$$
$$\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r}.$$

Remark 149. Using the Feynman-Kac representation of $u(t,x)$, it was possible to compute the weak derivative of $u(t,x)$ with respect to x in $(S)^*$ as follows:

$$\partial_x u(t,x) = \sum_{\alpha\in\mathcal{J}} \mathfrak{K}_\alpha(t,x)\mathcal{H}_\alpha,$$

where $\mathfrak{K}_{(\mathbf{0})}(t,x) = \partial_x u_{(\mathbf{0})}(t,x)$, and for $|\alpha| = n \geq 1$,

$$\mathfrak{K}_\alpha(t,x) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)$$
$$\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r}.$$

In fact, the weak spatial derivative is the usual spatial derivative in (L^2) sense (and hence P almost sure sense). One can verify this assertion by (formally) computing $\partial_x u(t,x)$ using the chaos expansion of $u(t,x)$ directly and by the uniqueness of mild solution.

Before we show $\partial_x u(t,x) \in \mathcal{G}$ for each $t > 0$ and $x \in \mathbb{R}$, we first need the following lemma:

Lemma 150. *Assume that $u_0 \in C^1_b(\mathbb{R})$. Then, for each $\alpha \in \mathcal{J}$, $t > 0$ and $x \in \mathbb{R}$, $\mathfrak{K}_\alpha(t,x)$ is well-defined, and moreover, for $|\alpha| \geq 1$,*

$$\lim_{\varepsilon\to 0^+} \mathfrak{K}^\epsilon_\alpha(t,x) = \mathfrak{K}_\alpha(t,x),$$

where

$$\mathfrak{K}^{\epsilon}_{\alpha}(t,x) := \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)$$
$$\cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \qquad \textit{for } \epsilon > 0,$$

with $\mathbb{T}^n_{[0,t-\epsilon]} := \{0 \leq s_1 \leq \cdots \leq s_n \leq t-\epsilon\}$.

Proof. We will decompose $\mathfrak{K}_{\alpha}(t,x)$ as a finite sum of well-defined terms. Without loss of generality, we let $|\alpha| = n \geq 1$. Notice that

$$\begin{aligned}
\mathfrak{K}_{\alpha}(t,x) &= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} \partial_x p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) \\
&\qquad \cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&= -\frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} \partial_{y_n} p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) \\
&\qquad \cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) \partial_{y_n} \Big(p(r_n - r_{n-1}, y_n - y_{n-1}) e_{k_{\sigma(n)}}(y_n) \Big) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-1)}}(y_{n-1}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) \partial_{y_n} p(r_n - r_{n-1}, y_n - y_{n-1}) e_{k_{\sigma(n)}}(y_n) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-1)}}(y_{n-1}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&\quad + \frac{1}{\sqrt{\alpha!}} \sum_{\sigma\in\mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e'_{k_{\sigma(n)}}(y_n) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-1)}}(y_{n-1}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&=: a_1 + b_1.
\end{aligned}$$

We note that b_1 is well-defined since $u_0 \in L^{\infty}(\mathbb{R})$, and for a_1, we do a similar step as above:

$$
\begin{aligned}
a_1 &= -\frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) \partial_{y_{n-1}} p(r_n - r_{n-1}, y_n - y_{n-1}) e_{k_{\sigma(n)}}(y_n) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-1)}}(y_{n-1}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e_{k_{\sigma(n)}}(y_n) \\
&\qquad \cdots \times \partial_{y_{n-1}} \Big(p(r_{n-1} - r_{n-2}, y_{n-1} - y_{n-2}) e_{k_{\sigma(n-1)}}(y_{n-1}) \Big) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-2)}}(y_{n-2}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e_{k_{\sigma(n)}}(y_n) \\
&\qquad \cdots \times \partial_{y_{n-1}} p(r_{n-1} - r_{n-2}, y_{n-1} - y_{n-2}) e_{k_{\sigma(n-1)}}(y_{n-1}) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-2)}}(y_{n-2}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&\quad + \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) e_{k_{\sigma(n)}}(y_n) \\
&\qquad \cdots \times p(r_{n-1} - r_{n-2}, y_{n-1} - y_{n-2}) e'_{k_{\sigma(n-1)}}(y_{n-1}) \\
&\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n-2)}}(y_{n-2}) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r} \\
&=: a_2 + b_2.
\end{aligned}
$$

Again, b_2 is well-defined. By iterating this process, we can get

$$
\mathfrak{K}_\alpha(t,x) = a_{n-1} + \sum_{i=1}^{n-1} b_i,
$$

where $\sum_{i=1}^{n-1} b_i$ is well-defined and

$$
\begin{aligned}
a_{n-1} = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^n} & p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) \\
& \cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) \partial_{y_1} u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y}\, d\mathbf{r}.
\end{aligned}
$$

Since $u_{(\mathbf{0})}(r_1, y_1) d\mathbf{y} = \int_{\mathbb{R}} p(r_1, y_1 - y_0) u_0(y_0) dy_0$, a_{n-1} becomes

$$
\begin{aligned}
a_{n-1} = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{\mathbb{T}^n_{[0,t]}} \int_{\mathbb{R}^{n+1}} & p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1) \\
& \cdots \times e_{k_{\sigma(1)}}(y_1) \times \cdots \times e_{k_{\sigma(n)}}(y_n) p(r_1, y_1 - y_0) u_0'(y_0) dy_0 d\mathbf{y}\, d\mathbf{r},
\end{aligned}
$$

which is clearly well-defined since $u_0' \in L^\infty(\mathbb{R})$.

Moreover, one can show that $\lim_{\varepsilon \to 0^+} \mathfrak{K}^\epsilon_\alpha(t,x) = \mathfrak{K}_\alpha(t,x)$ easily by considering the same argument as $\mathfrak{K}_\alpha(t,x)$ for $\mathfrak{K}^\epsilon_\alpha(t,x)$. □

Theorem 151. *If $u_0 \in C^1_b(\mathbb{R})$, for each $t > 0$ and $x \in \mathbb{R}$,*

$$\partial_x u(t,x) \in \mathcal{G}.$$

Proof. From (7.21), we have

$$\mathbb{E}|\partial_x u(t,x)|^2 = |\mathfrak{K}_{(\mathbf{0})}(t,x)|^2 + \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{J}_n} |\mathfrak{K}_\alpha(t,x)|^2.$$

We note that $\mathfrak{K}_{(\mathbf{0})}(t,x) = \partial_x u_{(\mathbf{0})}(t,x) < \infty$ for each $(t,x) \in [0,T] \times \mathbb{R}$. For $|\alpha| = n \geq 1$, we use $\mathfrak{K}^\epsilon_\alpha$: For $\epsilon > 0$, we notice that (by Fubini lemma)

$$\mathfrak{K}^\epsilon_\alpha(t,x) = \sqrt{n!} \int_{\mathbb{R}^n} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \partial_x p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1}) \times \cdots \times p(r_2 - r_1, y_2 - y_1)$$
$$\cdots \times \mathfrak{e}_\alpha(y_1, \ldots, y_n) u_{(\mathbf{0})}(r_1, y_1)\, d\mathbf{r} d\mathbf{y}.$$

Using Bessel's inequality, we have that

$$\sum_{\alpha \in \mathcal{J}_n} |\mathfrak{K}^\epsilon_\alpha(t,x)|^2 \leq n! \int_{\mathbb{R}^n} \Bigg(\int_{\mathbb{T}^n_{[0,t-\epsilon]}} \partial_x p(t-r_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1})$$
$$\cdots \times p(r_2 - r_1, y_2 - y_1) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{r} \Bigg)^2 d\mathbf{y}$$
$$\leq n! \|u_0\|^2_\infty \int_{\mathbb{R}^n} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \partial_x p(t-r_n, x-y_n) \partial_x p(t-s_n, x-y_n) p(r_n - r_{n-1}, y_n - y_{n-1})$$
$$\cdots \times p(s_n - s_{n-1}, y_n - y_{n-1}) \cdots p(r_2 - r_1, y_2 - y_1) p(s_2 - s_1, y_2 - y_1)\, d\mathbf{r} d\mathbf{s} d\mathbf{y}.$$

Again using Fubini lemma and the semigroup property of p, we can write the last expression as

$$n! \|u_0\|^2_\infty (2\pi)^{-(n-1)/2} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \prod_{k=1}^{n-1} (s_{k+1} + r_{k+1} - s_k - r_k)^{-1/2}$$
$$\cdots \times \int_{\mathbb{R}} \partial_x p(t-r_n, x-y_n) \partial_x p(t-s_n, x-y_n) dy_n\, d\mathbf{r} d\mathbf{s}.$$

Since

$$\int_{\mathbb{R}} \partial_x p(t-r_n, x-y_n) \partial_x p(t-s_n, x-y_n) dy_n = (2\pi)^{-1/2} (2t - s_n - r_n)^{-3/2},$$

and $(a+b)^{-1/2} \le 2^{-1/2}a^{-1/4}b^{-1/4}$ for $a, b > 0$, we can finally see that

$$\begin{aligned}\sum_{\alpha\in\mathcal{J}_n} |\mathfrak{K}^{\epsilon}_{\alpha}(t,x)|^2 &\le n!\|u_0\|^2_{\infty}(2\pi)^{-n/2}\left(\int_{\mathbb{T}^n_{[0,t-\epsilon]}} (t-s_n)^{-3/4}\prod_{k=1}^{n-1}(s_{k+1}-s_k)^{-1/4}d\mathbf{s}\right)^2\\ &\le n!\|u_0\|^2_{\infty}(2\pi)^{-n/2}\left(\int_{\mathbb{T}^n_{[0,t]}} (t-s_n)^{-3/4}\prod_{k=1}^{n-1}(s_{k+1}-s_k)^{-1/4}d\mathbf{s}\right)^2\\ &\le \|u_0\|^2_{\infty}\mathtt{C}^n t^{3n-4}n^{-n/2} \quad \text{for some } \mathtt{C} > 0,\end{aligned}$$

where the last inequality follows from [87, equation (4.10)].

For each $t > 0$ and $x \in \mathbb{R}$, the convergence is uniform in ϵ and $\mathfrak{K}^{\epsilon}_{\alpha}(t,x) \to \mathfrak{K}_{\alpha}(t,x)$ as $\epsilon \to 0$ by Lemma 150,

$$\sum_{\alpha\in\mathcal{J}_n} |\mathfrak{K}_{\alpha}(t,x)|^2 \le \|u_0\|^2_{\infty}\mathtt{C}^n t^{\frac{3n}{4}-1}n^{-n/2} \quad \text{for some constant } \mathtt{C} > 0.$$

Since $\mathtt{C}^n t^{\frac{3n}{4}-1}n^{-n/2}e^{2\lambda n}$ is summable in n for any $\lambda \in \mathbb{R}$, the conclusion follows from Definition 143. □

Remark 152. We have the following Feynman-Kac type formula for the spatial derivative of u:

$$\partial_x u(t,x) = \widetilde{\mathbb{E}}\left[\mathcal{E}(L^x(t)) \diamond \left\{u_0'(B^x(t)) + u_0(B^x(t))I\left(\partial_x L^x(t)\right)\right\}\right],$$

where $\widetilde{\mathbb{E}}$ must be understood as a Bochner integral in $(S)^*$. Notice that the integrand of $\widetilde{\mathbb{E}}$ on the right-hand side is in fact a Hida distribution. However, after taking the expectation $\widetilde{\mathbb{E}}$, which is interpreted as a Bochner integral in $(S)^*$, we end up with a regular random variable in $\mathcal{G}$. This means that the *white noise integral* (or the Bochner integral in $(S)^*$) has a regularizing effect.

Theorem 153. *Let $0 < \varepsilon < 1/2$ be arbitrary and assume that $u_0 \in \mathcal{C}^{3/2}(\mathbb{R})$. Then, for each $t > 0$,*

$$\partial_x u(t,\bullet) \in \mathcal{C}^{1/2-\varepsilon}(\mathbb{R}).$$

Proof. Let $p > 1$, $t > 0$ and $x \in \mathbb{R}$. By [87, Proposition 2.1], we have

$$(\mathbb{E}\,|\partial_x u(t,x+h) - \partial_x u(t,x)|^p)^{1/p} = \sum_{n=0}^{\infty}(p-1)^{n/2}\left(\sum_{\alpha\in\mathcal{J}_n}|\mathfrak{K}_{\alpha}(t,x+h) - \mathfrak{K}_{\alpha}(t,x)|^2\right)^{1/2}. \tag{7.22}$$

For $|\alpha| = 0$, we have

$$\mathfrak{K}_{(\mathbf{0})}(t,\bullet) = \partial_x u_{(\mathbf{0})}(t,\bullet) \in \mathcal{C}^{1/2}(\mathbb{R}).$$

For $|\alpha| = n \geq 1$, similarly to the proof of Theorem 151, we can get

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{J}_n} |\mathfrak{K}^{\epsilon}_{\alpha}(t,x+h) - \mathfrak{K}^{\epsilon}_{\alpha}(t,x)|^2 \\
&\leq n! \int_{\mathbb{R}^n} \bigg(\int_{\mathbb{T}^n_{[0,t-\epsilon]}} (\partial_x p(t-r_n, x+h-y_n) - \partial_x p(t-r_n, x-y_n))\, p(r_n - r_{n-1}, y_n - y_{n-1}) \\
&\qquad\qquad \cdots \times p(r_2 - r_1, y_2 - y_1) u_{(\mathbf{0})}(r_1, y_1) d\mathbf{r} \bigg)^2 d\mathbf{y} \\
&\leq n! \|u_0\|^2_\infty (2\pi)^{-(n-1)/2} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \int_{\mathbb{T}^n_{[0,t-\epsilon]}} \prod_{k=1}^{n-1} (s_{k+1} + r_{k+1} - s_k - r_k)^{-1/2} \\
&\quad \cdots \times \int_{\mathbb{R}} (\partial_x p(t-r_n, x+h-y_n) - \partial_x p(t-r_n, x-y_n)) \\
&\quad \cdots \times (\partial_x p(t-s_n, x+h-y_n) - \partial_x p(t-s_n, x-y_n))\, dy_n \, d\mathbf{r} d\mathbf{s}.
\end{aligned}$$

We next compute

$$\begin{aligned}
\int_{\mathbb{R}} \partial_x p(t_1, x_1 - z) \partial_x p(t_2, x_1 - z) dz &= \frac{1}{2\pi t_1^{3/2} t_2^{3/2}} \int_{\mathbb{R}} (x_1 - z)(x_2 - z) e^{-\frac{(x_1-z)^2}{2t_1} - \frac{(x_2-z)^2}{2t_2}} dz \\
&= \frac{1}{2\pi t_1^{3/2} t_2^{3/2}} \int_{\mathbb{R}} z\,(z - (x_1 - x_2))\, e^{-\frac{z^2}{2t_1} - \frac{(z-(x_1-x_2))^2}{2t_2}} dz \\
&= \frac{e^{-\frac{(x_1-x_2)^2}{2(t_1+t_2)}}}{2\pi t_1^{3/2} t_2^{3/2}} \int_{\mathbb{R}} (z^2 - (x_1 - x_2) z) e^{-\frac{(t_1+t_2)}{2t_1 t_2}\left(z - \frac{(x_1-x_2)t_1}{t_1+t_2}\right)^2} dz \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-x_2)^2}{2(t_1+t_2)}} (t_1+t_2)^{-3/2} \left(1 - \frac{(x_1-x_2)^2}{(t_1+t_2)}\right).
\end{aligned}$$

The last equality can be verified using the mean and variance of a normal distribution $\mathcal{N}\left(\frac{(x-y)t_1}{t_1+t_2}, \frac{t_1 t_2}{t_1+t_2}\right)$. Then, we can easily check, using the fact $1 - e^{-z} \leq z^{\gamma}$ for any $z \geq 0$ and $0 < \gamma \leq 1$,

$$\begin{aligned}
&\int_{\mathbb{R}} (\partial_x p(t-r_n, x+h-y_n) - \partial_x p(t-r_n, x-y_n)) \\
&\qquad\qquad \cdots \times (\partial_x p(t-s_n, x+h-y_n) - \partial_x p(t-s_n, x-y_n))\, dy_n \\
&= \sqrt{\frac{2}{\pi}} (2t-s-r)^{-3/2} \left(1 - e^{-\frac{h^2}{2(2t-s-r)}} + \frac{h^2 e^{-\frac{h^2}{2(2t-s-r)}}}{2t-s-r}\right) \leq \mathtt{C} h^{2\gamma} (2t-s-r)^{-3/2-\gamma}.
\end{aligned} \tag{7.23}$$

At this point, we restrict $0 < \gamma < 1/2$ so that (7.23) is integrable both in s and r variables near t.

This leads to

$$\sum_{\alpha\in\mathcal{J}_n} |\mathfrak{K}_\alpha^\epsilon(t,x+h) - \mathfrak{K}_\alpha^\epsilon(t,x)|^2 \le h^{2\gamma}\|u_0\|_\infty^2 \mathtt{C}^n(\gamma) t^{\frac{3n}{4}-1-\frac{\gamma}{2}} n^{-n/2},$$

with $0 < \gamma < 1/2$ for some constant $\mathtt{C}(\gamma) > 0$ depending only on γ.

After taking $\epsilon \to 0$, the desired result follows from (7.22) and the Kolmogorov continuity theorem. □

Remark 154. Under the same initial condition in Theorem 153, we can achieve the optimal temporal regularity of $\partial_x u$ in a similar manner, i.e., $\partial_x u(\bullet, x) \in C^{1/4-\varepsilon}([\varepsilon_0, T])$ for every $x \in \mathbb{R}$, $0 < \varepsilon_0 < T$, and $0 < \varepsilon < 1/4$.

Chapter 8

Upper and lower bounds for the solution of a stochastic prey-predator system with foraging arena scheme

Based on: Lanconelli, A., & Scorolli, R. (2020). Upper and lower bounds for the solution of a stochastic prey-predator system with foraging arena scheme. arXiv preprint arXiv:2009.14516.

Abstract

In this chapter we investigate some probabilistic aspects of the unique global strong solution of a two dimensional system of stochastic differential equations describing a preypredator model perturbed by Gaussian noise. We first establish, for any fixed $t > 0$, almost sure upper and lower bounds for the components $X(t)$ and $Y(t)$ of the solution vector: these explicit estimates emphasize the interplay between the various parameters of the model and agree with the asymptotic results found in the literature. Then, standing on the aforementioned bounds, we derive upper and lower estimates for the joint moments and distribution function of $(X(t); Y(t))$. Our analysis is based on a careful use of comparison theorems for stochastic differential equations and exploits several peculiar features of the noise driving the equation.

8.1 Introduction

In theoretical ecology the system of equations

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(a_1 - b_1 x(t)) - c_1 h(x(t), y(t))y(t), & x(0) = x; \\ \frac{dy(t)}{dt} = y(t)(-a_2 - b_2 y(t)) + c_2 h(x(t), y(t))y(t), & y(0) = y, \end{cases} \tag{8.1}$$

constitutes a fundamental class of models for predator-prey interaction. Here, $x(t)$ and $y(t)$ represent the population densities of prey and predator at time $t \geq 0$, respectively; a_1 the prey intrinsic growth rate; a_2 the predator intrinsic death rate; a_1/b_1 the carrying capacity of the ecosystem; b_2 the predator intraspecies competition; $h(x(t), y(t))$ the intake rate of predator; c_2/c_1 the trophic efficiency. We observe that equation (8.1) encompasses the classic Lotka-Volterra model [100][101] which is obtained setting $b_1 = b_2 = 0$ and $h(x, y) = x$.
To catch the different features of specific environments, several choices for the so-called functional response $h(x, y)$ have been suggested in the literature; we mention, among others,

- Holling II function [102]: $h(x, y) = \frac{x}{\beta+x}$;
- ratio dependent functional responses [103],[104]: $h(x, y) = \widetilde{h}(x/y)$;
- foraging arena models [105],[106]: $h(x, y) = \frac{x}{\beta+\alpha_2 y}$;
- Beddington-DeAngelis model [107],[108]: $h(x, y) = \frac{x}{\beta+\alpha_1 x+\alpha_2 y}$;
- Crowley-Martin model [109]: $h(x, y) = \frac{x}{\beta+\alpha_1 x+\alpha_2 y+\alpha_3 xy}$;
- Hassell-Varley model [110]: $h(x, y) = \frac{x}{\alpha_1 x+\alpha_2 y^m}$.

($\beta, \alpha_1, \alpha_2, \alpha_3$, are positive real numbers, $m \in \mathbb{N}$ and $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ a suitable regular function). What distinguishes the Holling II function from other models is the absence of y; on this issue the paper [111] presents statistical evidence from 19 predator-prey systems that the Beddington-DeAngelis, Crowley-Martin and Hassell-Varley models (whose functional responses depend on both prey and predator abundances) can provide better descriptions compared to those with Holling-type functions (see also [112]). Moreover, as remarked in [113], models based on ratio-dependent functional responses exhibit singular behaviours.

With the aim of introducing environmental noise in the model, different types of stochastic perturbation for the system (8.1) have been considered and studied. Among the most common, we find the Itô-type stochastic differential equation

$$\begin{cases} dX(t) = [X(t)(a_1 - b_1 X(t)) - c_1 h(X(t), Y(t))Y(t)]\, dt + \sigma_1 X(t) dB_1(t), & X(0) = x; \\ dY(t) = [Y(t)(-a_2 - b_2 Y(t)) + c_2 h(X(t), Y(t))Y(t)]\, dt + \sigma_2 Y(t) dB_2(t), & Y(0) = y, \end{cases} \tag{8.2}$$

where $\{(B_1(t), B_2(t))\}_{t\geq 0}$ is a standard two dimensional Brownian motion and σ_1, σ_2 positive real numbers. System (8.2) tries to catch random fluctuations in the growth rate a_1 and death rate a_2. Some references in this stream of research are [114], in the case of foraging arena schemes, [115], [116], [117] treating the case of Beddington-DeAngelis functional response, and [118] dealing with Hassell-Varley model. It is worth mentioning that all these papers are devoted to the study of global existence, uniqueness, positivity and asymptotic properties for the specific model of type (8.2) considered.

Our investigation is focused on the system

$$\begin{cases} dX(t) = \left[X(t)(a_1 - b_1X(t)) - c_1\frac{X(t)Y(t)}{\beta+Y(t)}\right]dt + \sigma_1X(t)dB_1(t), & X(0) = x; \\ dY(t) = \left[Y(t)(-a_2 - b_2Y(t)) + c_2\frac{X(t)Y(t)}{\beta+Y(t)}\right]dt + \sigma_2Y(t)dB_2(t), & Y(0) = y, \end{cases} \tag{8.3}$$

which is proposed and analysed in [114]. It corresponds to equation (8.2) with a foraging arena functional response. It is proved in [114] that system (8.3) possesses a unique global strong solution $\{(X(t), Y(t))\}_{t\geq 0}$ fulfilling the condition

$$\mathbb{P}(X(t) > 0 \text{ and } Y(t) > 0, \text{ for all } t \geq 0) = 1.$$

Moreover, the authors investigate the asymptotic behaviours of $X(t)$ and $Y(t)$, as t tends to infinity, and identify three different regimes:

- if $a_1 < \frac{\sigma_1^2}{2}$, then

$$\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty} Y(t) = 0, \tag{8.4}$$

  almost surely and exponentially fast;

- if $\frac{\sigma_1^2}{2} < a_1 < \frac{\sigma_1^2}{2} + \frac{b_1\beta a_2}{c_2} + \frac{b_1\beta\sigma_2^2}{2c_2} =: \phi$, then almost surely

$$\lim_{t\to+\infty} Y(t) = 0, \qquad \text{exponentially fast,} \tag{8.5}$$

  and

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t X(r)dr = \frac{a_1 - \sigma_1^2/2}{b_1}; \tag{8.6}$$

- if $a_1 > \frac{\phi}{1-\sigma_2^2/2c_2 - a_2/c_2}$ and $a_2 + \frac{\sigma_2^2}{2} < c_2$, then system (8.3) has a unique stationary distribution.

The case

$$\phi < a_1 < \frac{\phi}{1 - \sigma_2^2/2c_2 - a_2/c_2},$$

with $a_2 + \frac{\sigma_2^2}{2} < c_2$, is not investigated but the authors mention that computer simulations indicate the existence of stationary distributions for both $X(t)$ and $Y(t)$ also in that regime (see for this case the general results proved in [119]).

The goal of our work is to present a novel analysis for systems of the type (8.2), which in the current study take the form (8.3). We derive explicit upper and lower

bounds for the components $X(t)$ and $Y(t)$ of the solution of equation (8.3) at any fixed time $t \geq 0$. Such almost sure estimates depend solely on the parameters describing the model under investigation and the noise driving the equation. Their derivation is based on a careful use of comparison theorems for stochastic differential equations and standard stochastic calculus' tools. The estimates we obtain reflect the intrinsic interplay between the parameters of the model and enlighten the probabilistic dependence structure of $X(t)$ and $Y(t)$. We also remark that our bounds, which are valid for any fixed time $t \geq 0$, agree in the limit as t tends to infinity with the asymptotic results proven in [114] and summarized above. We then utilize the previously mentioned bounds to get upper and lower estimates for the joint moments and distribution function of $(X(t), Y(t))$. We propose closed form expressions which rely on new estimates for a logistic-type stochastic differential equation.
It is important to remark that, while systems of the type (8.2) with Beddington-DeAngelis or Crowley-Martin or Hassell-Varley functional responses can be treated, as far as finite time analysis is concerned, with a change of measure approach, the unboundedness of $h(x,y) = \frac{x}{\beta+\alpha_2 y}$, as a function of x, prevents from the use of a similar approach for (8.3). We will in fact prove in Section 3.1 below the failure of the Novikov condition for the corresponding change of measure.

The paper is organized as follows: Section 2 collects some auxiliary results on the solution of a logistic stochastic differential equation that plays a major role in our analysis; in Section 3 we state and prove our first main theorem: almost sure upper and lower bounds for $X(t)$ and $Y(t)$, for any $t \geq 0$. Here, we also comment on the impossibility of a change of measure approach and compare our findings with the asymptotic results from [114]; Section 4 contains our second main result, which proposes upper and lower estimates for the joint moments of $(X(t), Y(t))$; in Section 5 upper and lower bounds for the joint probability function of $(X(t), Y(t))$ constitutes our third and last main theorem; the last section contains a discussion of the result obtained in the paper and some numerical simulations of the proposed bounds.

8.2 Preliminary results

In this section we will prove some auxiliary results concerning the solution of the logistic stochastic differential equation

$$dL(t) = L(t)(a - bL(t))dt + \sigma L(t)dB(t), \quad L(0) = \lambda. \tag{8.7}$$

Here a, b, σ and λ are positive real numbers and $\{B(t)\}_{t\geq 0}$ is a standard one dimensional Brownian motion. It is well known (see for instance formula (4.51) in [120] or formula (2.1) in [121] for the case of time-dependent parameters) that equation (8.7) possesses a unique global positive strong solution which can be represented as

$$L(t) = \frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1 + b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma B(r)}dr}, \quad t \geq 0. \tag{8.8}$$

We start focusing on the asymptotic behaviour of the solution of equation (8.7). We also refer the reader to the paper [122] for a small time analysis of $\{L(t)\}_{t\geq 0}$.

Proposition 155. *Let $\{L(t)\}_{t\geq 0}$ be the unique global strong solution of (8.7). Then,*

- *if $a < \sigma^2/2$,*

$$\lim_{t\to+\infty} L(t) = 0 \quad \textit{almost surely}; \tag{8.9}$$

- *if $a \geq \sigma^2/2$, then $L(t)$ is recurrent on $]0,+\infty[$;*
- *if $a > \sigma^2/2$, then $L(t)$ converges in distribution, as t tends to infinity, to the unique stationary distribution $\texttt{Gamma}(\frac{2a}{\sigma^2}-1, \frac{2b}{\sigma^2})$.*

Proof. See Proposition 3.3 in [123]. □

From formula (8.8) we see that, for any $t > 0$, the random variable $L(t)$ is a function of the Geometric Brownian motion $e^{(a-\sigma^2/2)t+\sigma B(t)}$ and its integral $\int_0^t e^{(a-\sigma^2/2)r+\sigma B(r)}dr$. Using the joint probability density function of the random vector

$$\left(e^{(a-\sigma^2/2)t+\sigma B(t)}, \int_0^t e^{(a-\sigma^2/2)r+\sigma B(r)}dr\right),$$

which can be found in [124], the authors of [125] write down an expression for the probability density function of $L(t)$: see formula (40) there. However, the authors mention that, due to the presence of oscillating integrals, the numerical treatment of such expression is rather tricky.
In the next two results, instead of insisting with exact formulas, we propose upper and lower estimates for the moments $\mathbb{E}[L(t)^p]$ and distribution function $\mathbb{P}(L(t) \leq z)$; the bounds we obtain involve integrals whose numerical approximations do not present the aforementioned difficulties. We also mention the paper [126] which uses an approach based on power series to approximate the moments of $L(t)$.
In the sequel, we will write for $t > 0$

$$\mathcal{N}_{0,t}(r) := \frac{1}{\sqrt{2t}} e^{-\frac{r^2}{2t}}, \quad r \in \mathbb{R},$$

and

$$\mathcal{N}'_{0,t}(r) := \frac{d}{dr}\mathcal{N}_{0,t}(r) = -\frac{r}{t}\frac{1}{\sqrt{2t}}e^{-\frac{r^2}{2t}}, \quad r \in \mathbb{R}.$$

For notational convenience we also set

$$m(t) := \inf_{r\in[0,t]} B(r) \quad \text{and} \quad M(t) := \sup_{r\in[0,t]} B(r). \tag{8.10}$$

Proposition 156. *Let* $\{L(t)\}_{t\geq 0}$ *be the unique global strong solution of (8.7). Then, for any* $p \geq 0$*, we have*

$$\mathbb{E}[L(t)^p] \leq 2k_p(t)\int_0^{+\infty}\left(1+b\lambda e^{-\sigma z}K_p(t)\right)^{-p}\mathcal{N}_{0,t}(z)dz, \tag{8.11}$$

and

$$\mathbb{E}[L(t)^p] \geq 2k_p(t)\int_0^{+\infty}\left(1+b\lambda e^{\sigma z}K_p(t)\right)^{-p}\mathcal{N}_{0,t}(z)dz, \tag{8.12}$$

where

$$k_p(t) := \lambda^p e^{p(a-\sigma^2/2)t+p^2\sigma^2 t/2} \qquad \text{and} \qquad K_p(t) := \lambda\frac{e^{(a-\sigma^2/2+p\sigma^2)t}-1}{a-\sigma^2/2+p\sigma^2}.$$

Proof. Fix $p \geq 0$; then,

$$\begin{aligned}
\mathbb{E}[L(t)^p] &= \mathbb{E}\left[\frac{\lambda^p e^{p(a-\sigma^2/2)t+p\sigma B(t)}}{\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma B(r)}dr\right)^p}\right]\\
&= \mathbb{E}\left[\frac{\lambda^p e^{p(a-\sigma^2/2)t+p^2\sigma^2 t/2}e^{p\sigma B(t)-p^2\sigma^2 t/2}}{\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma B(r)}dr\right)^p}\right]\\
&= \lambda^p e^{p(a-\sigma^2/2)t+p^2\sigma^2 t/2}\mathbb{E}\left[\frac{e^{p\sigma B(t)-p^2\sigma^2 t/2}}{\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma B(r)}dr\right)^p}\right]\\
&= k_p(t)\mathbb{E}\left[\frac{e^{p\sigma B(t)-p^2\sigma^2 t/2}}{\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma B(r)}dr\right)^p}\right].
\end{aligned}$$

We now observe that, according to the Girsanov's theorem, for any $T > 0$ the law of $\{B(t)\}_{t\in[0,T]}$ under the equivalent probability measure

$$d\mathbb{Q} := e^{p\sigma B(t)-p^2\sigma^2 t/2}d\mathbb{P} \quad \text{on } \mathcal{F}_T^B$$

coincides with the one of $\{B(t)+p\sigma t\}_{t\in[0,T]}$ under the measure $\mathbb{P}$. Therefore,

$$\begin{aligned}
\mathbb{E}[L(t)^p] &= k_p(t)\mathbb{E}\left[\frac{e^{p\sigma B(t)-p^2\sigma^2 t/2}}{\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma B(r)}dr\right)^p}\right]\\
&= k_p(t)\mathbb{E}\left[\frac{1}{\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma(B(r)+p\sigma r)}dr\right)^p}\right]\\
&= k_p(t)\mathbb{E}\left[\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma(B(r)+p\sigma r)}dr\right)^{-p}\right].
\end{aligned}$$

Now, adopting the notation (8.10), we can estimate as

$$\begin{aligned}\mathbb{E}[L(t)^p] &= k_p(t)\mathbb{E}\left[\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma(B(r)+p\sigma r)}dr\right)^{-p}\right]\\ &\geq k_p(t)\mathbb{E}\left[\left(1+be^{\sigma M(t)}\int_0^t \lambda e^{(a-\sigma^2/2+p\sigma^2)r}dr\right)^{-p}\right]\\ &= k_p(t)\mathbb{E}\left[\left(1+be^{\sigma M(t)}K_p(t)\right)^{-p}\right],\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{E}[L(t)^p] &= k_p(t)\mathbb{E}\left[\left(1+b\int_0^t \lambda e^{(a-\sigma^2/2)r+\sigma(B(r)+p\sigma r)}dr\right)^{-p}\right]\\ &\leq k_p(t)\mathbb{E}\left[\left(1+be^{\sigma m(t)}\int_0^t \lambda e^{(a-\sigma^2/2+p\sigma^2)r}dr\right)^{-p}\right]\\ &= k_p(t)\mathbb{E}\left[\left(1+be^{\sigma m(t)}K_p(t)\right)^{-p}\right].\end{aligned}$$

Moreover, recalling that, for $A \in \mathcal{B}(\mathbb{R})$ and $t > 0$, we have

$$\mathbb{P}(m(t)\in A) = 2\int_A \mathcal{N}_{0,t}(z)\mathbf{1}_{]-\infty,0]}(z)dz \quad \text{and} \quad \mathbb{P}(M(t)\in A) = 2\int_A \mathcal{N}_{0,t}(z)\mathbf{1}_{[0,+\infty[}(z)dz,$$

(see formula (8.2) in Chapter 2 from [2]) we can conclude that

$$\begin{aligned}\mathbb{E}[L(t)^p] &\geq k_p(t)\mathbb{E}\left[\left(1+be^{\sigma M(t)}K_p(t)\right)^{-p}\right]\\ &= 2k_p(t)\int_0^{+\infty}\left(1+be^{\sigma z}K_p(t)\right)^{-p}\mathcal{N}_{0,t}(z)dz,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[L(t)^p] &\leq k_p(t)\mathbb{E}\left[\left(1+be^{\sigma m(t)}K_p(t)\right)^{-p}\right]\\ &= 2k_p(t)\int_0^{+\infty}\left(1+be^{-\sigma z}K_p(t)\right)^{-p}\mathcal{N}_{0,t}(z)dz.\end{aligned}$$

□

Proposition 157. *Let $\{L(t)\}_{t\geq 0}$ be the unique global strong solution of (8.7). Then, for any $z > 0$ and $t > 0$, we have the bounds*

$$\mathbb{P}(L(t)\leq z) \leq -2\int_{\left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}}\leq z\right\}\cap\{v>0\}\cap\{u<v\}} \mathcal{N}'_{0,t}(2v-u)dudv, \qquad (8.13)$$

and

$$\mathbb{P}(L(t) \le z) \ge -2 \int_{\left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}} \le z\right\} \cap \{v<0\} \cap \{u>v\}} \mathcal{N}'_{0,t}(u-2v)dudv, \tag{8.14}$$

with

$$k(t) := \lambda e^{(a-\sigma^2/2)t} \quad \text{and} \quad K(t) := \lambda \frac{e^{(a-\sigma^2/2)t}-1}{a-\sigma^2/2}.$$

Proof. We first prove (8.14): from (8.8) we have

$$L(t) \ge \frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1+be^{\sigma M(t)}\int_0^t \lambda e^{(a-\sigma^2/2)r}dr} = \frac{k(t)e^{\sigma B(t)}}{1+bK(t)e^{\sigma M(t)}}.$$

The last member above is a function of the two dimensional random vector $(B(t), M(t))$, whose joint probability density function is given by the expression

$$f_{B(t),M(t)}(u,v) = \begin{cases} -2\mathcal{N}'_{0,t}(2v-u), & \text{if } v>0 \text{ and } u<v, \\ 0, & \text{otherwise} \end{cases}$$

(see formula (8.2) in Chapter 2 from [2]) Therefore, for any $z > 0$, we obtain

$$\begin{aligned} \mathbb{P}(L(t) \le z) &\le \mathbb{P}\left(\frac{k(t)e^{\sigma B(t)}}{1+bK(t)e^{\sigma M(t)}} \le z\right) \\ &= -2\int_{\left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}} \le z\right\} \cap \{v>0\} \cap \{u<v\}} \mathcal{N}'_{0,t}(2v-u)dudv, \end{aligned}$$

completing the proof of (8.14). Similarly,

$$L(t) \le \frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1+be^{\sigma m(t)}\int_0^t \lambda e^{(a-\sigma^2/2)r}dr} = \frac{k(t)e^{\sigma B(t)}}{1+bK(t)e^{\sigma m(t)}}.$$

The last member above is a function of the two dimensional random vector $(B(t), m(t))$, whose joint probability density function is given by the expression

$$f_{B(t),m(t)}(u,v) = \begin{cases} -2\mathcal{N}'_{0,t}(u-2v), & \text{if } v<0 \text{ and } u>v, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for any $z > 0$, we obtain

$$\begin{aligned} \mathbb{P}(L(t) \le z) &\ge \mathbb{P}\left(\frac{k(t)e^{\sigma B(t)}}{1+bK(t)e^{\sigma m(t)}} \le z\right) \\ &= -2\int_{\left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}} \le z\right\} \cap \{v<0\} \cap \{u>v\}} \mathcal{N}'_{0,t}(u-2v)dudv. \end{aligned}$$

The proof is complete. □

Remark 158. We observe that the inequality $u < v$ implies

$$\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}} \leq \frac{k(t)e^{\sigma v}}{1+bK(t)e^{\sigma v}} \leq \frac{k(t)}{bK(t)}.$$

Therefore, the upper bound (8.13) becomes trivial for $z \geq \frac{k(t)}{bK(t)}$; in fact, in that case

$$\{u < v\} \Rightarrow \left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}} \leq \frac{k(t)}{bK(t)}\right\} \Rightarrow \left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}} \leq z\right\}$$

which yields

$$\begin{aligned}&\int_{\left\{\frac{k(t)e^{\sigma u}}{1+bK(t)e^{\sigma v}}\leq z\right\}\cap\{u>0\}\cap\{u<v\}} -2\mathcal{N}'_{0,t}(2v-u)dudv \\ &\quad = \int_{\{v>0\}\cap\{u<v\}} -2\mathcal{N}'_{0,t}(2v-u)dudv = 1.\end{aligned}$$

8.3 First main theorem: almost sure bounds

Our first main theorem provides explicit almost sure upper and lower bounds for the solution of (8.3) at any given time t. It is useful to introduce the following notation: let

$$L_1(t) := \frac{G_1(t)}{1+b_1\int_0^t G_1(r)dr}, \quad t \geq 0, \tag{8.15}$$

and

$$L_2(t) := \frac{G_2(t)}{1+b_2\int_0^t G_2(r)dr}, \quad t \geq 0, \tag{8.16}$$

where for $t \geq 0$ we set

$$G_1(t) := xe^{(a_1-\sigma_1^2/2)t+\sigma_1 B_1(t)} \quad \text{and} \quad G_2(t) := ye^{-(a_2+\sigma_2^2/2)t+\sigma_2 B_2(t)};$$

the parameters $a_1, a_2, b_1, b_2, \sigma_1, \sigma_2, x, y$ are those appearing in equation (8.3). According to the previous section, the stochastic processes $\{L_1(t)\}_{t\geq 0}$ and $\{L_2(t)\}_{t\geq 0}$ satisfy the equations

$$dL_1(t) = L_1(t)(a_1 - b_1L_1(t))dt + \sigma_1 L_1(t)dB_1(t), \quad L_1(0) = x, \tag{8.17}$$

and

$$dL_2(t) = L_2(t)(-a_2 - b_2L_2(t))dt + \sigma_2 L_2(t)dB_2(t), \quad L_2(0) = y, \tag{8.18}$$

respectively. Therefore, the two dimensional process $\{(L_1(t), L_2(t))\}_{t\geq 0}$ is the unique strong solution of system (8.3) when $c_1 = c_2 = 0$, i.e. when the interaction term $\frac{X(t)Y(t)}{\beta+Y(t)}$ is not present.

8.3.1 Comments on the use of Girsanov theorem

We have just mentioned that, by removing the ratio $\frac{X(t)Y(t)}{\beta+Y(t)}$ from its drift, equation (8.3) reduces to the uncoupled system

$$\begin{cases} dL_1(t) = L_1(t)(a_1 - b_1 L_1(t))dt + \sigma_1 L_1(t) dB_1(t), & L_1(0) = x; \\ dL_2(t) = L_2(t)(-a_2 - b_2 L_2(t))dt + \sigma_2 L_2(t) dB_2(t), & L_2(0) = y, \end{cases} \tag{8.19}$$

whose solution is explicitly represented via formulas (8.15) and (8.16). Since drift removals can in general be performed with the use of Girsanov theorem, one may wonder whether the almost sure properties of (8.3) can be deduced from those of (8.19) under a suitable equivalent probability measure. Aim of the present subsection is to show that this not case: we are in fact going to prove that the Novikov condition corresponding to the just mentioned drift removal is not fulfilled.

First of all, we notice that system (8.19) can be rewritten as

$$\begin{cases} dL_1(t) = L_1(t)(a_1 - b_1 L_1(t))dt + \sigma_1 L_1(t) \left(dB_1(t) + \frac{c_1 L_2(t)}{\sigma_1(\beta + L_2(t))} dt - \frac{c_1 L_2(t)}{\sigma_1(\beta + L_2(t))} dt \right); \\ L_1(0) = x; \\ dL_2(t) = L_2(t)(-a_2 - b_2 L_2(t))dt + \sigma_2 L_2(t) \left(dB_2(t) - \frac{c_2 L_1(t)}{\sigma_2(\beta + L_2(t))} dt + \frac{c_2 L_1(t)}{\sigma_2(\beta + L_2(t))} dt \right); \\ L_2(0) = y, \end{cases}$$

or equivalently

$$\begin{cases} dL_1(t) = L_1(t)(a_1 - b_1 L_1(t))dt - c_1 \frac{L_1(t) L_2(t)}{\beta + L_2(t)} dt + \sigma_1 L_1(t) d\widetilde{B_1}(t), & L_1(0) = x; \\ dL_2(t) = L_2(t)(-a_2 - b_2 L_2(t))dt + c_2 \frac{L_1(t) L_2(t)}{\beta + L_2(t)} dt + \sigma_2 L_2(t) d\widetilde{B_2}(t), & L_2(0) = y, \end{cases} \tag{8.20}$$

where we set

$$\widetilde{B_1}(t) := B_1(t) + \int_0^t \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} dr, \quad t \geq 0,$$

and

$$\widetilde{B_2}(t) := B_2(t) - \int_0^t \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} dr, \quad t \geq 0.$$

Now, if the Novikov condition

$$\mathbb{E}\left[\exp\left\{ \frac{1}{2} \int_0^T \left(\frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} \right)^2 + \left(\frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} \right)^2 dr \right\} \right] < +\infty \tag{8.21}$$

is satisfied for some $T > 0$, then the stochastic process $\{(\widetilde{B_1}(t), \widetilde{B_1}(t))\}_{t \in [0,T]}$ is according to the Girsanov theorem a standard two dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$ (here $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the augmented Brownian filtration) with

$$\begin{aligned} d\mathbb{Q} := \exp\left\{ -\int_0^T \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} dB_1(r) - \frac{1}{2}\int_0^T \left(\frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))}\right)^2 dr \right\} \\ \times \exp\left\{ \int_0^T \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} dB_2(r) - \frac{1}{2}\int_0^T \left(\frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))}\right)^2 dr \right\} d\mathbb{P}. \end{aligned}$$

Moreover, in this case equation (8.20) implies that the two dimensional process $\{(L_1(t), L_2(t))\}_{t\in[0,T]}$ is a weak solution of (8.3) with respect to $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{Q}, \{(\widetilde{B_1}(t), \widetilde{B_1}(t))\}_{t\in[0,T]})$. We now prove that condition (8.21) cannot be true without additional assumptions on the parameters of our model. In fact,

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))}\right)^2 + \left(\frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))}\right)^2 dr\right\}\right] \\ &\quad \geq \mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))}\right)^2 dr\right\}\right] \\ &\quad = \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2}\int_0^T \frac{L_1^2(r)}{(\beta + L_2(r))^2} dr\right\}\right] \\ &\quad \geq \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2 \mathcal{M}_2}\int_0^T L_1^2(r) dr\right\}\right] \end{aligned}$$

where we introduced the notation

$$\mathcal{M}_2 := \sup_{r\in[0,T]} (\beta + L_2(r))^2.$$

We now apply Jensen's inequality to the Lebesgue integral and use the identity

$$\int_0^T L_1(r) dr = \frac{1}{b_1} \ln\left(1 + b_1 \int_0^T G_1(r) dr\right)$$

to get

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left(\frac{c_1L_2(r)}{\sigma_1(\beta+L_2(r))}\right)^2+\left(\frac{c_2L_1(r)}{\sigma_2(\beta+L_2(r))}\right)^2dr\right\}\right]\\
&\quad\geq \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2}\int_0^T L_1^2(r)dr\right\}\right]\\
&\quad= \mathbb{E}\left[\exp\left\{\frac{c_2^2T}{2\sigma_2^2\mathcal{M}_2T}\int_0^T L_1^2(r)dr\right\}\right]\\
&\quad\geq \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2T}\left(\int_0^T L_1(r)dr\right)^2\right\}\right]\\
&\quad= \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2Tb_1^2}\left(\ln\left(1+b_1\int_0^T G_1(r)dr\right)\right)^2\right\}\right]\\
&\quad\geq \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2Tb_1^2}\left(\ln\left(1+b_1K_1(T)e^{\sigma_1m_1(T)}\right)\right)^2\right\}\right]\\
&\quad\geq \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2Tb_1^2}\left(\ln\left(b_1K_1(T)e^{\sigma_1m_1(T)}\right)\right)^2\right\}\right]\\
&\quad= \mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2Tb_1^2}\left(\sigma_1m_1(T)+\ln(b_1K_1(T))\right)^2\right\}\right].
\end{aligned}
$$

Here, we set

$$
K_1(T)=\frac{e^{(a_1-\sigma_1^2/2)T}-1}{a_1-\sigma_1^2/2}\qquad\text{and}\qquad m_1(T):=\min_{t\in[0,T]}B_1(t).
$$

Using the independence between B_1 and B_2, we can write the last expectation as

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{\frac{c_2^2}{2\sigma_2^2\mathcal{M}_2Tb_1^2}\left(\sigma_1m_1(T)+\ln(b_1K_1(T))\right)^2\right\}\right]\\
&\qquad=\int_{\beta^2}^{+\infty}\left(\int_{-\infty}^0 e^{\frac{C}{2Tz}(\sigma_1u+D)^2}\frac{2}{\sqrt{2\pi T}}e^{-\frac{u^2}{2T}}du\right)d\mu(z),
\end{aligned}
$$

where μ stands for the law of $\mathcal{M}_2$, $C:=\frac{c_2^2}{\sigma_2^2b_1^2}$ and $D:=\ln(b_1K_1(T))$. It is now clear that the inner integral above is finite if and only if $z\geq C\sigma_1^2$. Since z ranges in the interval $]\beta^2,\infty[$, we deduce that the last condition is verified for all $z\in]\beta^2,+\infty[$ only when $\beta^2\geq C\sigma_1^2$, which in our notation means

$$
\beta\geq\frac{c_2\sigma_1}{b_1\sigma_2}. \tag{8.22}
$$

Therefore, if the parameters describing system (8.3) do not respect the bound (8.22), then inequality

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left(\frac{c_1L_2(r)}{\sigma_1(\beta+L_2(r))}\right)^2+\left(\frac{c_2L_1(r)}{\sigma_2(\beta+L_2(r))}\right)^2dr\right\}\right]\\&\quad\geq 2\int_{\beta^2}^{+\infty}\left(\int_{-\infty}^0 e^{\frac{C}{2Tz}(\sigma_1u+D)^2}\frac{1}{\sqrt{2\pi T}}e^{-\frac{u^2}{2T}}du\right)d\mu(z)=+\infty,\end{aligned}$$

which is valid for all $T>0$, implies the failure of Novikov condition (8.21). From this point of view the almost sure properties of the solution of (8.3) cannot be deduced from those of the uncoupled system (8.19).

Remark 159. The functional response in the foraging arena model formally appears to be a particular case of the one that characterizes the Beddington-DeAngelis model (take $\alpha_1=0$). However, referring to the change of measure technique mentioned above, we see that the Novikov condition corresponding to the Beddington-DeAngelis model would amount at the finiteness of

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T\left(\frac{c_1L_2(r)}{\sigma_1(\beta+\alpha_1L_1(r)+\alpha_2L_2(r))}\right)^2+\left(\frac{c_2L_1(r)}{\sigma_2(\beta+\alpha_1L_1(r)+\alpha_2L_2(r))}\right)^2dr\right\}\right].$$

Since the two ratios in the Lebesgue integral are upper bounded almost surely by $\frac{c_1}{\sigma_1\alpha_2}$ and $\frac{c_2}{\sigma_2\alpha_1}$, respectively, we get immediately the finiteness, for all $T>0$, of the expectation above. Therefore, in the Beddington-DeAngelis model one may utilize the change of measure approach to study almost sure properties of the solution on any finite interval of time $[0,T]$. The same reasoning applies also to the Crowley-Martin and Hassell-Varley functional responses.

8.3.2 Statement and proof of the first main theorem

Recall that, according to the discussion in Section 1, the quantity

$$\phi:=\frac{\sigma_1^2}{2}+\frac{b_1\beta a_2}{c_2}+\frac{b_1\beta\sigma_2^2}{2c_2}$$

is a threshold determining the asymptotic behaviour of $X(t)$ and $Y(t)$.

Theorem 160. *Let $\{(X(t),Y(t))\}_{t\geq 0}$ be the unique global strong solution of (8.3). Then, for all $t\geq 0$ the following bounds hold almost surely:*

$$L_2(t)\leq Y(t)\leq L_2(t)\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}}; \tag{8.23}$$

if $a_1<\phi$, then

$$L_1(t)e^{-\frac{c_1}{\beta b_2}\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}}\ln\left(1+b_2\int_0^t G_2(r)dr\right)}\leq X(t)\leq L_1(t); \tag{8.24}$$

if $a_1 > \phi$, *then*

$$L_1(t)e^{-c_1 t} \leq X(t) \leq L_1(t). \tag{8.25}$$

Remark 161. We assumed at the beginning of this manuscript that the Brownian motions $\{B_1(t)\}_{t\geq 0}$ and $\{B_2(t)\}_{t\geq 0}$, driving the two dimensional system (8.3), are independent. However, this assumption is not needed in the derivation of the almost sure bounds stated above, as long as system (8.3) possesses a positive global strong solution. Therefore, the estimates (8.23), (8.24) and (8.25) remain true in the case of correlated Brownian motions as well.

Remark 162. The bounds in Theorem 160 are consistent with the asymptotic results obtained in [114]. In fact:

- $a_1 < \frac{\sigma_1^2}{2}$: taking the limit as t tends to infinity in the second inequality of (8.24) we get

  $$0 \leq \lim_{t\to+\infty} X(t) \leq \lim_{t\to+\infty} L_1(t),$$

  which, in combination with (8.9) for L_1, gives

  $$\lim_{t\to+\infty} X(t) = 0.$$

  On the other hand, if we take the limit in (8.23) we obtain

  $$0 \leq \lim_{t\to+\infty} Y(t) \leq \lim_{t\to+\infty} L_2(t)\left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}}.$$

  According to formula 1.8.4 page 612 in [127] the random variable $\int_0^{+\infty} G_1(r)dr$ is finite almost surely; this fact and (8.9) for L_2 yield

  $$\lim_{t\to+\infty} Y(t) = 0,$$

  completing the proof of (8.4);

- $\frac{\sigma_1^2}{2} < a_1 < \phi = \frac{\sigma_1^2}{2} + \frac{b_1\beta a_2}{c_2} + \frac{b_1\beta\sigma_2^2}{2c_2}$: first of all, we write

  $$L_2(t) \leq G_2(t) = e^{-(a_2+\sigma_2^2/2)t+\sigma_2 B_2(t)};$$

  moreover, since

  $$\int_0^t G_1(r)ds \leq e^{\sigma_1 M_1(t)}K_1(t),$$

where $M_1(t) := \max_{t\in[0,t]} B_1(r)$ and

$$K_1(t) := x\frac{e^{(a_1-\sigma_1^2/2)t}-1}{a_1-\sigma_1^2/2},$$

we get

$$\begin{aligned}\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}} &\leq \left(1+b_1 e^{\sigma_1 M_1(t)}K_1(t)\right)^{\frac{c_2}{\beta b_1}}\\ &\leq \left(1+Ce^{\sigma_1 M_1(t)}e^{(a_1-\sigma_1^2/2)t}\right)^{\frac{c_2}{\beta b_1}},\end{aligned}$$

for a suitable positive constant C. Therefore,

$$\begin{aligned}L_2(t)\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}} &\leq e^{-(a_2+\sigma_2^2/2)t+\sigma_2 B_2(t)}\left(1+Ce^{\sigma_1 M_1(t)}e^{(a_1-\sigma_1^2/2)t}\right)^{\frac{c_2}{\beta b_1}}\\ &= \left(e^{-\frac{(a_2+\sigma_2^2/2)\beta b_1}{c_2}t+\frac{\sigma_2\beta b_1}{c_2}B_2(t)}+Ce^{-\frac{(a_2+\sigma_2^2/2)\beta b_1}{c_2}t+\frac{\sigma_2\beta b_1}{c_2}B_2(t)}e^{\sigma_1 M_1(t)}e^{(a_1-\sigma_1^2/2)t}\right)^{\frac{c_2}{\beta b_1}}\\ &= \left(e^{-\frac{(a_2+\sigma_2^2/2)\beta b_1}{c_2}t+\frac{\sigma_2\beta b_1}{c_2}B_2(t)}+Ce^{\left(a_1-\sigma_1^2/2-\frac{(a_2+\sigma_2^2/2)\beta b_1}{c_2}\right)t+\frac{\sigma_2\beta b_1}{c_2}B_2(t)}e^{\sigma_1 M_1(t)}\right)^{\frac{c_2}{\beta b_1}}.\end{aligned} \tag{8.26}$$

Recalling that

$$\mathbb{P}\left(\lim_{t\to+\infty}\frac{B(t)}{t}=0\right)=\mathbb{P}\left(\lim_{t\to+\infty}\frac{M_1(t)}{t}=0\right)=1,$$

(see for instance [78]), we can say that both terms inside the parenthesis in (8.26) will tend to zero as t tends to infinity if the constants multiplying t in the exponentials are negative. While this is obvious for the first exponential, the negativity of the constant

$$a_1-\sigma_1^2/2-\frac{(a_2+\sigma_2^2/2)\beta b_1}{c_2}$$

is equivalent to the condition $a_1 < \phi$, i.e. the regime under consideration. Hence, passing to the limit in (8.23), we conclude that

$$\lim_{t\to+\infty} Y(t) = 0;$$

this corresponds to (8.5). In addition, from (8.24) we obtain

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t X(r)dr \leq \lim_{t\to+\infty}\frac{1}{t}\int_0^t L_1(r)dr = \frac{a_1-\sigma_1^2/2}{b_1}.$$

Here, we utilized Proposition 155 for L_1 with $a_1 > \sigma_1^2/2$, in particular the ergodic property

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t L_1(r)dr = \mathbb{E}[L_\infty],$$

with $\mathbb{E}[L_\infty]$ being the expectation of the unique stationary distribution. This partially proves (8.6).

Proof. We start finding the Itô's differential of the stochastic process $\frac{1}{L_1(t)}$:

$$\begin{aligned} d\frac{1}{L_1(t)} &= -\frac{1}{L_1^2(t)}dL_1(t) + \frac{1}{L_1^3(t)}d\langle L_1\rangle_t \\ &= -\frac{a_1 - b_1L_1(t)}{L_1(t)}dt - \frac{\sigma_1}{L_1(t)}dB_1(t) + \frac{\sigma_1^2}{L_1(t)}dt \\ &= \frac{\sigma_1^2 - a_1 + b_1L_1(t)}{L_1(t)}dt - \frac{\sigma_1}{L_1(t)}dB_1(t). \end{aligned}$$

Combining this expression with the first equation in (8.3) we get

$$\begin{aligned} d\frac{X(t)}{L_1(t)} =& X(t)d\frac{1}{L_1(t)} + \frac{1}{L_1(t)}dX(t) + d\left\langle X, 1/L_1\right\rangle(t) \\ =& X(t)\left(\frac{\sigma_1^2 - a_1 + b_1L_1(t)}{L_1(t)}dt - \frac{\sigma_1}{L_1(t)}dB_1(t)\right) \\ &+ \frac{1}{L_1(t)}\left[X(t)\left(a_1 - b_1X(t) - \frac{c_1Y(t)}{\beta + Y(t)}\right)dt + \sigma_1X(t)dB_1(t)\right] \\ &- \sigma_1^2\frac{X(t)}{L_1(t)}dt \\ =& \frac{X(t)}{L_1(t)}\left[\sigma_1^2 - a_1 + b_1L_1(t) + a_1 - b_1X(t) - \frac{c_1Y(t)}{\beta + Y(t)} - \sigma_1^2\right]dt \\ =& \frac{X(t)}{L_1(t)}\left[b_1(L_1(t) - X(t)) - \frac{c_1Y(t)}{\beta + Y(t)}\right]dt. \end{aligned}$$

Since $\frac{X(0)}{L_1(0)} = 1$, the last chain of equalities implies

$$\frac{X(t)}{L_1(t)} = \exp\left\{b_1\int_0^t (L_1(r) - X(r))dr - c_1\int_0^t \frac{Y(r)}{\beta + Y(r)}dr\right\}. \tag{8.27}$$

Following the previous reasoning we also find that

$$\begin{aligned} d\frac{1}{L_2(t)} &= -\frac{1}{L_2^2(t)}dL_2(t) + \frac{1}{L_2^3(t)}d\langle L_2\rangle_t \\ &= -\frac{-a_2 - b_2L_2(t)}{L_2(t)}dt - \frac{\sigma_2}{L_2(t)}dB_2(t) + \frac{\sigma_2^2}{L_2(t)}dt \\ &= \frac{\sigma_2^2 + a_2 + b_2L_2(t)}{L_2(t)}dt - \frac{\sigma_2}{L_2(t)}dB_2(t). \end{aligned}$$

Combining this expression with the second equation in (8.3) we get

$$\begin{aligned}
d\frac{Y(t)}{L_2(t)} =& Y(t)d\frac{1}{L_2(t)} + \frac{1}{L_2(t)}dY(t) + d\langle Y, 1/L_2\rangle(t) \\
=& Y(t)\left(\frac{\sigma_2^2 + a_2 + b_2 L_2(t)}{L_2(t)}dt - \frac{\sigma_2}{L_2(t)}dB_2(t)\right) \\
&+ \frac{1}{L_2(t)}\left[Y(t)\left(-a_2 - b_2 X(t) + \frac{c_2 X(t)}{\beta + Y(t)}\right)dt + \sigma_2 Y(t) dB_2(t)\right] \\
&- \sigma_2^2 \frac{Y(t)}{L_2(t)}dt \\
=& \frac{Y(t)}{L_2(t)}\left[\sigma_2^2 + a_2 + b_2 L_2(t) - a_2 - b_2 Y(t) + \frac{c_2 X(t)}{\beta + Y(t)} - \sigma_2^2\right]dt \\
=& \frac{Y(t)}{L_2(t)}\left[b_2(L_2(t) - Y(t)) + \frac{c_2 X(t)}{\beta + Y(t)}\right]dt.
\end{aligned}$$

Since $\frac{Y(0)}{L_2(0)} = 1$, the last chain of equalities implies

$$\frac{Y(t)}{L_2(t)} = \exp\left\{b_2 \int_0^t (L_2(r) - Y(r))dr + c_2 \int_0^t \frac{X(r)}{\beta + Y(r)}dr\right\}. \tag{8.28}$$

We now observe that

$$\mathbb{P}\left(\frac{X(t)Y(t)}{\beta + Y(t)} > 0\right) = 1, \qquad \text{for any } t \geq 0$$

(remember that $X(t)$ and $Y(t)$ are positive for all $t \geq 0$); therefore, by means of standard comparison theorems for SDEs (see for instance Theorem 1.1 in Chapter VI from [9]) applied to (8.3) we deduce that

$$X(t) \leq L_1(t), \quad \text{for all } t \geq 0, \tag{8.29}$$

and

$$Y(t) \geq L_2(t), \quad \text{for all } t \geq 0, \tag{8.30}$$

where $\{L_1(t)\}_{t\geq 0}$ and $\{L_2(t)\}_{t\geq 0}$ solve (8.17) and (8.18), respectively. Therefore, equation (8.27) leads to

$$\exp\left\{-c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)}dr\right\} \leq \frac{X(t)}{L_1(t)} \leq 1,$$

or equivalently,

$$L_1(t) \exp\left\{-c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)}dr\right\} \leq X(t) \leq L_1(t), \tag{8.31}$$

while equation (8.28) leads to

$$1 \leq \frac{Y(t)}{L_2(t)} \leq \exp\left\{c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr\right\},$$

or equivalently,

$$L_2(t) \leq Y(t) \leq L_2(t) \exp\left\{c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr\right\}. \tag{8.32}$$

The lower bound in (8.31) and upper bound in (8.32) are not explicit yet since they depend on the solution itself. To solve this problem we first recall that the process $\{L_2(t)\}_{t\geq 0}$ is positive and converges almost surely to zero exponentially fast, as t tends to infinity. Now, by virtue of (8.29), (8.30) and the infinitesimal behaviour of L_2, we can upper bound the right hand side in (8.32) as

$$\begin{aligned} L_2(t) \exp\left\{c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr\right\} &\leq L_2(t) \exp\left\{c_2 \int_0^t \frac{L_1(r)}{\beta + L_2(r)} dr\right\} \\ &\leq L_2(t) \exp\left\{\frac{c_2}{\beta} \int_0^t L_1(r) dr\right\}, \end{aligned}$$

In addition, since

$$L_1(t) = \frac{1}{b_1} \frac{d}{dt} \ln\left(1 + b_1 \int_0^t G_1(r) dr\right),$$

the last member above can be rewritten as

$$\begin{aligned} L_2(t) \exp\left\{\frac{c_2}{\beta} \int_0^t L_1(r) dr\right\} &= L_2(t) \exp\left\{\frac{c_2}{\beta b_1} \ln\left(1 + b_1 \int_0^t G_1(r) dr\right)\right\} \\ &= L_2(t) \left(1 + b_1 \int_0^t G_1(r) dr\right)^{\frac{c_2}{\beta b_1}}. \end{aligned}$$

Combining this estimate with (8.32) we obtain (8.23).

For the lower bound in (8.31), we observe that the function $y \mapsto \frac{y}{\beta + y}$, for $y > 0$, can be sharply upper bounded by affine functions in two different ways: the upper bound $y \mapsto 1$ is sharp at infinity but not accurate at zero while the upper bound $y \mapsto \frac{y}{\beta}$ is sharp at zero but very bad at infinity. Therefore, according to the asymptotic results proved in [114] and mentioned in the Introduction, we now proceed distinguishing two different regimes:

- when $a_1 < \phi$, the process $\{Y_t\}_{t\geq 0}$ tends to zero exponentially fast and hence we utilize the process $\frac{Y_r}{\beta}$ to upper bound $\frac{Y_r}{\beta + Y_r}$. The left hand side of (8.31) is

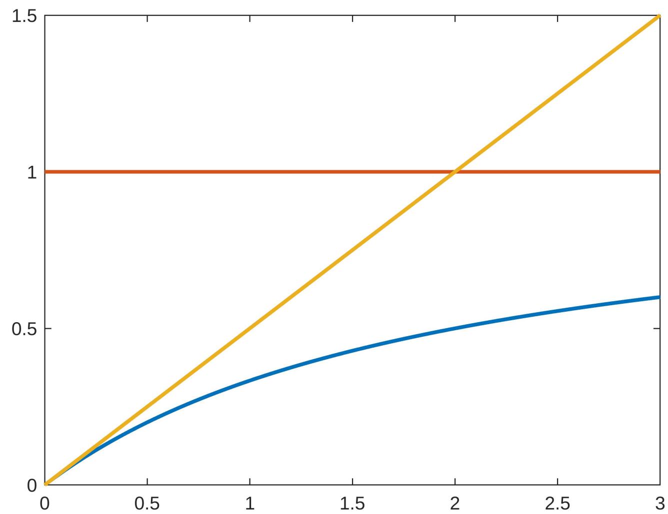

Figure 8.1: Upper bounds for the function $y \mapsto \frac{y}{2+y}$ (blue line) with the function $y \mapsto \frac{y}{2}$ (yellow line) and the function $y \mapsto 1$ (red line).

then simplified to

$$\begin{aligned} L_1(t)\exp\left\{-c_1\int_0^t \frac{Y(r)}{\beta+Y(r)}dr\right\} &\geq L_1(t)\exp\left\{-\frac{c_1}{\beta}\int_0^t Y(r)dr\right\} \\ &\geq L_1(t)\exp\left\{-\frac{c_1}{\beta}\int_0^t L_2(r)\left(1+b_1\int_0^r G_1(u)du\right)^{\frac{c_2}{\beta b_1}} dr\right\} \\ &\geq L_1(t)\exp\left\{-\frac{c_1}{\beta}\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}}\int_0^t L_2(r)dr\right\} \\ &= L_1(t)\exp\left\{-\frac{c_1}{\beta b_2}\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}}\ln\left(1+b_2\int_0^t G_2(r)dr\right)\right\}. \end{aligned} \tag{8.33}$$

Here, in the second inequality we utilized the upper bound in (8.23) while in the last equality we employed the identity

$$L_2(t) = \frac{1}{b_2}\frac{d}{dt}\ln\left(1+b_2\int_0^t G_2(r)dr\right).$$

Inserting (8.33) in the left hand side of (8.31), one gets (8.24);

- when $a_1 > \phi$, the process $\{Y_t\}_{t\geq 0}$ has a more oscillatory behaviour; therefore, we prefer to upper bound the ratio $\frac{Y_r}{\beta+Y_r}$ with one. This gives

$$L_1(t)\exp\left\{-c_1\int_0^t \frac{Y(r)}{\beta+Y(r)}dr\right\} \geq L_1(t)e^{-c_1 t},$$

and (8.31) reduces to (8.25).

□

Remark 163. It is important to emphasize that both the lower bounds in (8.24) and (8.25) remain valid without restrictions on the parameters: this is clear from the proof of Theorem 160 and in particular from the use of the comparison principle we made. In fact, one may combine the two lower estimates as

$$L_1(t)\max\left\{e^{-\frac{c_1}{\beta b_2}\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}}\ln\left(1+b_2\int_0^t G_2(r)dr\right)}, e^{-c_1 t}\right\}\leq X(t)\leq L_1(t),$$

and argue on the different values attained by the maximum above. However, such analysis would necessarily involve the non directly observable quantities $\int_0^t G_1(r)dr$, $\int_0^t G_2(r)dr$ and their probabilities. That is why we preferred to suggest which lower bound is better suited for the given set of parameters.

8.4 Second main theorem: bounds for the moments

The next theorem presents upper and lower estimates for the joint moments of $X(t)$ and $Y(t)$ at any given time t. These bounds, which rely on the almost sure inequalities (8.23), (8.24) and (8.25) are represented through closed form expressions involving Lebesgue integrals; such integrals can be evaluated via numerical approximations or Monte Carlo simulations.
We also mention that in [114] the authors prove an asymptotic upper bound for the moments $\mathbb{E}\left[(X(t)^2+Y(t)^2)^{\theta/2}\right]$ with θ being a positive real number.

Theorem 164. *Let $\{(X(t),Y(t))\}_{t\geq 0}$ be the unique global strong solution of (8.3). For all $t\geq 0$ we have the following estimates:*

1. *if $p,q\geq 0$ with $\frac{qc_2}{\beta b_1}-p\geq 1$, then*

$$\begin{aligned}\mathbb{E}\left[X(t)^pY(t)^q\right]\leq & 2k_{1,p}(t)k_{2,q}(t)\left(1+b_1x\frac{e^{\left(a_1+\left(\frac{qc_2}{\beta b_1}+p-1\right)\frac{\sigma_1^2}{2}\right)t}-1}{a_1+\left(\frac{qc_2}{\beta b_1}+p-1\right)\frac{\sigma_1^2}{2}}\right)^{\frac{qc_2}{\beta b_1}-p}\\ &\times\int_0^{+\infty}\left(1+b_2ye^{-\sigma_2 z}K_{2,q}(t)\right)^{-q}\mathcal{N}_{0,t}(z)dz. \qquad (8.34)\end{aligned}$$

2. *if $p,q\geq 0$ and $a_1>\phi$, then*

$$\begin{aligned}\mathbb{E}\left[X(t)^pY(t)^q\right]\geq & 4e^{-pc_1t}k_{1,p}(t)k_{2,q}(t)\int_0^{+\infty}\left(1+b_1xe^{\sigma_1 z}K_{1,p}(t)\right)^{-p}\mathcal{N}_{0,t}(z)dz\\ &\times\int_0^{+\infty}\left(1+b_2ye^{\sigma_2 z}K_{2,q}(t)\right)^{-q}\mathcal{N}_{0,t}(z)dz. \qquad (8.35)\end{aligned}$$

3. if $p, q \geq 0$ and $a_1 < \phi$, then

$$\mathbb{E}[X(t)^p] \geq -4k_1(t)^p \int_A \frac{e^{p\sigma_1 u_1 - \frac{pc_1}{\beta b_2}(1+b_1K_1(t)e^{\sigma_1 v_1})^{\frac{c_2}{\beta b_1}} \ln(1+b_2K_2(t)e^{\sigma_2 v_2})}}{(1+b_1K_1(t)e^{\sigma_1 v_1})^p} \times \mathcal{N}'_{0,t}(2v_1 - u_1)\mathcal{N}_{0,t}(v_2)du_1dv_1dv_2, \tag{8.36}$$

where

$$A := \{(u_1, v_1, v_2) \in \mathbb{R}^3 : v_1 > 0, u_1 < v_1, v_2 > 0\},$$

while

$$\mathbb{E}[Y(t)^q] \geq 2k_{2,q}(t) \int_0^{+\infty} (1 + b_2 y e^{\sigma_2 z} K_{2,q}(t))^{-q} \mathcal{N}_{0,t}(z)dz. \tag{8.37}$$

Here,

$$k_1(t) := xe^{(a_1-\sigma_1^2/2)t}, \quad K_1(t) := x\frac{e^{(a_1-\sigma_1^2/2)t}-1}{a_1-\sigma_1^2/2}, \quad K_2(t) := y\frac{e^{(a_2-\sigma_2^2/2)t}-1}{a_2-\sigma_2^2/2},$$

$$k_{1,p}(t) := x^p e^{p(a_1-\sigma_1^2/2)t+p^2\sigma_1^2 t/2}, \qquad K_{1,p}(t) := x\frac{e^{(a_1-\sigma_1^2/2+p\sigma_1^2)t}-1}{a_1-\sigma_1^2/2+p\sigma_1^2},$$

$$k_{2,p}(t) := y^p e^{p(a_2-\sigma_2^2/2)t+p^2\sigma_2^2 t/2} \qquad K_{2,p}(t) := y\frac{e^{(a_2-\sigma_2^2/2+p\sigma_2^2)t}-1}{a_2-\sigma_2^2/2+p\sigma_2^2}.$$

Proof. 1. Using (8.23) and (8.24) (or (8.25)), we can write

$$\begin{aligned}
\mathbb{E}\left[X(t)^p Y(t)^q\right] \leq & \mathbb{E}\left[L_1(t)^p L_2(t)^q \left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{qc_2}{\beta b_1}}\right] \\
= & \mathbb{E}\left[L_1(t)^p \left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{qc_2}{\beta b_1}}\right]\mathbb{E}\left[L_2(t)^q\right] \\
= & \mathbb{E}\left[\frac{G_1(t)^p}{\left(1+b_1\int_0^t G_1(r)dr\right)^p}\left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{qc_2}{\beta b_1}}\right]\mathbb{E}\left[L_2(t)^q\right] \\
= & \mathbb{E}\left[G_1(t)^p \left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{qc_2}{\beta b_1}-p}\right]\mathbb{E}\left[L_2(t)^q\right] \\
= & \mathcal{I}_1\mathcal{I}_2,
\end{aligned}$$

where we set

$$\mathcal{I}_1 := \mathbb{E}\left[G_1(t)^p \left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{qc_2}{\beta b_1}-p}\right] \quad \text{and} \quad \mathcal{I}_2 := \mathbb{E}\left[L_2(t)^q\right].$$

From (8.11) we get immediately that

$$\mathcal{I}_2 \leq 2k_{2,q}(t) \int_0^{+\infty} \left(1 + b_2 y e^{-\sigma_2 z} K_{2,q}(t)\right)^{-q} \mathcal{N}_{0,t}(z) dz.$$

Now, mimicking the proof of Proposition 156 we can write

$$\begin{aligned}\mathcal{I}_1 &= k_{1,p}(t)\mathbb{E}\left[e^{p\sigma_1 B_1(t) - p^2\sigma_1^2 t/2}\left(1 + b_1\int_0^t G_1(r)dr\right)^{\frac{qc_2}{\beta b_1}-p}\right] \\ &= k_{1,p}(t)\mathbb{E}\left[\left(1 + b_1\int_0^t G_1(r)e^{p\sigma_1^2 r}dr\right)^{\frac{qc_2}{\beta b_1}-p}\right] \\ &= k_{1,p}(t)\left\|1 + b_1\int_0^t G_1(r)e^{p\sigma_1^2 r}dr\right\|_{\mathbb{L}^{\frac{qc_2}{\beta b_1}-p}(\Omega)}^{\frac{qc_2}{\beta b_1}-p}.\end{aligned}$$

Observe that the condition $\frac{qc_2}{\beta b_1} - p \geq 1$ allows for the use of triangle and Minkowski's inequalities for the norm of the space $\mathbb{L}^{\frac{qc_2}{\beta b_1}-p}(\Omega)$; therefore, we obtain

$$\begin{aligned}\mathcal{I}_1 &= k_{1,p}(t)\left\|1 + b_1\int_0^t G_1(r)e^{p\sigma_1^2 r}dr\right\|_{\mathbb{L}^{\frac{qc_2}{\beta b_1}-p}(\Omega)}^{\frac{qc_2}{\beta b_1}-p} \\ &\leq k_{1,p}(t)\left(1 + b_1\left\|\int_0^t G_1(r)e^{p\sigma_1^2 r}dr\right\|_{\mathbb{L}^{\frac{qc_2}{\beta b_1}-p}(\Omega)}\right)^{\frac{qc_2}{\beta b_1}-p} \\ &\leq k_{1,p}(t)\left(1 + b_1\int_0^t \|G_1(r)\|_{\mathbb{L}^{\frac{qc_2}{\beta b_1}-p}(\Omega)} e^{p\sigma_1^2 r}dr\right)^{\frac{qc_2}{\beta b_1}-p} \\ &= k_{1,p}(t)\left(1 + b_1 x\frac{e^{\left(a_1 + \left(\frac{qc_2}{\beta b_1}+p-1\right)\frac{\sigma_1^2}{2}\right)t} - 1}{a_1 + \left(\frac{qc_2}{\beta b_1}+p-1\right)\frac{\sigma_1^2}{2}}\right)^{\frac{qc_2}{\beta b_1}-p}.\end{aligned}$$

Combining the estimates for $\mathcal{I}_1$ and $\mathcal{I}_2$ we obtain

$$\begin{aligned}\mathbb{E}\left[X(t)^p Y(t)^q\right] \leq & 2k_{1,p}(t)k_{2,q}(t)\left(1 + b_1 x\frac{e^{\left(a_1 + \left(\frac{qc_2}{\beta b_1}+p-1\right)\frac{\sigma_1^2}{2}\right)t} - 1}{a_1 + \left(\frac{qc_2}{\beta b_1}+p-1\right)\frac{\sigma_1^2}{2}}\right)^{\frac{qc_2}{\beta b_1}-p} \\ & \times \int_0^{+\infty} \left(1 + b_2 y e^{-\sigma_2 z} K_{2,q}(t)\right)^{-q} \mathcal{N}_{0,t}(z) dz.\end{aligned}$$

2. From (8.23) and (8.25) we can write

$$\begin{aligned}\mathbb{E}\left[X(t)^p Y(t)^q\right] &\geq e^{-pc_1 t}\mathbb{E}\left[L_1(t)^p L_2(t)^q\right] \\ &= e^{-pc_1 t}\mathbb{E}\left[L_1(t)^p\right]\mathbb{E}\left[L_2(t)^q\right].\end{aligned}$$

Inequality (8.12) completes the proof of (8.35).

3. The lower bound (8.37) is obtained setting $p = 0$ in (8.35); to prove the lower bound (8.36) we observe that

$$\begin{aligned}
X(t) &\geq L_1(t) e^{-\frac{c_1}{\beta b_2}\left(1+b_1\int_0^t G_1(v)dv\right)^{\frac{c_2}{\beta b_1}} \ln\left(1+b_2\int_0^t G_2(r)dr\right)} \\
&= \frac{G_1(t) e^{-\frac{c_1}{\beta b_2}\left(1+b_1\int_0^t G_1(v)dv\right)^{\frac{c_2}{\beta b_1}} \ln\left(1+b_2\int_0^t G_2(r)dr\right)}}{1+b_1\int_0^t G_1(r)dr} \\
&\geq \frac{G_1(t) e^{-\frac{c_1}{\beta b_2}\left(1+b_1 K_1(t) e^{\sigma_1 M_1(t)}\right)^{\frac{c_2}{\beta b_1}} \ln\left(1+b_2 K_2(t) e^{\sigma_2 M_2(t)}\right)}}{1+b_1 K_1(t) e^{\sigma_1 M_1(t)}} \\
&= \frac{k_1(t) e^{\sigma_1 B_1(t)-\frac{c_1}{\beta b_2}\left(1+b_1 K_1(t) e^{\sigma_1 M_1(t)}\right)^{\frac{c_2}{\beta b_1}} \ln\left(1+b_2 K_2(t) e^{\sigma_2 M_2(t)}\right)}}{1+b_1 K_1(t) e^{\sigma_1 M_1(t)}}. \qquad (8.38)
\end{aligned}$$

The last member above is a function of the three dimensional random vector $(B_1(t), M_1(t), M_2(t))$ whose joint probability density function is given by

$$\begin{aligned}
&f_{B_1(t),M_1(t),M_2(t)}(u_1, v_1, v_2) \\
&\quad = \begin{cases} -4\mathcal{N}'_{0,t}(2v_1-u_1)\mathcal{N}_{0,t}(v_2), & \text{if } v_1 > 0,\ u_1 < v_1 \text{ and } v_2 > 0, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, for any $p \geq 0$ we get

$$\begin{aligned}
\mathbb{E}[X(t)^p] &\geq \mathbb{E}\left[\left|\frac{k_1(t) e^{\sigma_1 B_1(t)-\frac{c_1}{\beta b_2}\left(1+b_1 K_1(t) e^{\sigma_1 M_1(t)}\right)^{\frac{c_2}{\beta b_1}} \ln\left(1+b_2 K_2(t) e^{\sigma_2 M_2(t)}\right)}}{1+b_1 K_1(t) e^{\sigma_1 M_1(t)}}\right|^p\right] \\
&= -4k_1(t)^p \int_A \frac{e^{p\sigma_1 u_1-\frac{pc_1}{\beta b_2}(1+b_1 K_1(t) e^{\sigma_1 v_1})^{\frac{c_2}{\beta b_1}} \ln(1+b_2 K_2(t) e^{\sigma_2 v_2})}}{(1+b_1 K_1(t) e^{\sigma_1 v_1})^p} \\
&\qquad \times \mathcal{N}'_{0,t}(2v_1-u_1)\mathcal{N}_{0,t}(v_2) du_1 dv_1 dv_2,
\end{aligned}$$

where

$$A := \{(u_1, v_1, v_2) \in \mathbb{R}^3 : v_1 > 0, u_1 < v_1, v_2 > 0\}.$$

This proves (8.36).

□

Remark 165. Due to the complexity of the left hand side in (8.24) we were not able to obtain a lower bound for the joint moments $\mathbb{E}\left[X(t)^p Y(t)^q\right]$ in the regime $a_1 < \phi$. However, according to the argument of Remark 163, inequality (8.35) can be utilize also in that regime.

8.5 Third main theorem: bounds for the distribution functions

The last main theorem of this paper concerns with upper and lower estimates for the distribution functions of $X(t)$ and $Y(t)$.

Theorem 166. *Let $\{(X(t), Y(t))\}_{t\geq 0}$ be the unique global strong solution of (8.3). Then, for all $t \geq 0$ and $z_1, z_2 > 0$ we have the following bounds:*

1.
$$\mathbb{P}(X(t) \leq z_1) \geq -2 \int_{\left\{\frac{k_1(t)e^{\sigma u}}{1+b_1K_1(t)e^{\sigma_1 v}} \leq z_1\right\} \cap \{v>0\} \cap \{u<v\}} \mathcal{N}'_{0,t}(2v-u)dudv, \quad (8.39)$$

and

$$\mathbb{P}(Y(t) \leq z_2) \geq -\frac{4\beta b_1}{\sigma_1 c_2} \int_0^{z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}} \left(\int_{\left\{\frac{k_2(t)e^{\sigma_2 u}}{1+b_2K_2(t)e^{\sigma_2 v}} \leq \zeta\right\} \cap \{v<0\} \cap \{u>v\}} \mathcal{N}'_{0,t}(u-2v)dudv \right)$$
$$\times \mathcal{N}_{0,t}\left(\frac{1}{\sigma_1} \ln \left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}} - 1}{b_1 K_1(t)} \right) \right) \frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}}{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}} - 1} \frac{1}{\zeta} d\zeta; \quad (8.40)$$

2. *if $a_1 > \phi$, then*
$$\mathbb{P}\left(X(t) \leq z_1, Y(t) \leq z_2\right) \leq 4 \int_{\left\{\frac{k_1(t)e^{\sigma_1 u}}{1+b_1K_1(t)e^{\sigma_1 v}} \leq z_1 e^{c_1 t}\right\} \cap \{v>0\} \cap \{u<v\}} \mathcal{N}'_{0,t}(2v-u)dudv$$
$$\times \int_{\left\{\frac{k_2(t)e^{\sigma_2 u}}{1+b_2K_2(t)e^{\sigma_2 v}} \leq z_2\right\} \cap \{v>0\} \cap \{u<v\}} \mathcal{N}'_{0,t}(2v-u)dudv; \quad (8.41)$$

3. *if $a_1 < \phi$, then*
$$\mathbb{P}(X(t) \leq z_1) \leq -4 \int_{A_{z_1} \cap \{v_1>0, u_1<v_1, v_2>0\}} \mathcal{N}'_{0,t}(2v_1-u_1)\mathcal{N}_{0,t}(v_2)du_1dv_1dv_2, \quad (8.42)$$

where

$$A_{z_1} := \left\{ (u_1, v_1, v_2) \in \mathbb{R}^3 : \frac{k_1(t)e^{\sigma_1 u_1 - \frac{c_1}{\beta b_2}(1+b_1K_1(t)e^{\sigma_1 v_1})^{\frac{c_2}{\beta b_1}} \ln(1+b_2K_2(t)e^{\sigma_2 v_2})}}{1+b_1K_1(t)e^{\sigma_1 v_1}} \leq z_1 \right\},$$

and

$$\mathbb{P}(Y(t) \leq z_2) \leq -2 \int_{\left\{\frac{k_2(t)e^{\sigma u}}{1+b_2K(t)e^{\sigma_2 v}} \leq z_2\right\} \cap \{v>0\} \cap \{u<v\}} \mathcal{N}'_{0,t}(2v-u)dudv. \quad (8.43)$$

Here,

$$k_1(t) := xe^{(a_1-\sigma_1^2/2)t} \quad \text{and} \quad K_1(t) := x\frac{e^{(a_1-\sigma_1^2/2)t}-1}{a_1-\sigma_1^2/2},$$

while

$$k_2(t) := ye^{(a_2-\sigma_2^2/2)t} \quad \text{and} \quad K_2(t) := y\frac{e^{(a_2-\sigma_2^2/2)t}-1}{a_2-\sigma_2^2/2}.$$

Proof. 1. The upper bound in (8.24) (or (8.25)) yields

$$\mathbb{P}(X(t) \le z_1) \ge \mathbb{P}(L_1(t) \le z_1)$$

which in combination with (8.14) gives (8.39). We now prove (8.40); the estimate

$$\int_0^t G_1(r)dr \le K_1(t)e^{\sigma_1 M_1(t)},$$

together with the upper estimate in (8.23), entails

$$\begin{aligned}
\mathbb{P}(Y(t) \le z_2) &\ge \mathbb{P}\left(L_2(t)\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}} \le z_2\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(L_2(t)\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}} \le z_2 \middle| \mathcal{F}_t^2\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\left(1+b_1\int_0^t G_1(r)dr\right)^{\frac{c_2}{\beta b_1}} \le \frac{z_2}{L_2(t)} \middle| \mathcal{F}_t^2\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\int_0^t G_1(r)dr \le \left(\left(\frac{z_2}{L_2(t)}\right)^{\frac{\beta b_1}{c_2}}-1\right)/b_1 \middle| \mathcal{F}_t^2\right)\right] \\
&\ge \mathbb{E}\left[\mathbb{P}\left(K_1(t)e^{\sigma_1 M_1(t)} \le \left(\left(\frac{z_2}{L_2(t)}\right)^{\frac{\beta b_1}{c_2}}-1\right)/b_1 \middle| \mathcal{F}_t^2\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(M_1(t) \le \frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z_2}{L_2(t)}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right) \middle| \mathcal{F}_t^2\right)\right].
\end{aligned}$$

Here $\{\mathcal{F}_t^2\}_{t\ge 0}$ denotes the natural augmented filtration of the Brownian motion $\{B_2(t)\}_{t\ge 0}$. Note that the almost sure positivity of the random variable $M_1(t)$ implies that the probability in the last member above is different from zero if and only if

$$\frac{\left(\frac{z_2}{L_2(t)}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)} > 1$$

which is equivalent to say that

$$L_2(t) \leq \frac{z_2}{(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}}.$$

Therefore,

$$\mathbb{P}(Y(t) \leq z_2) \geq \mathbb{E}\left[\mathbb{P}\left(M_1(t) \leq \frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z_2}{L_2(t)}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\middle|\mathcal{F}_t^2\right)\right]$$
$$= \int_0^{z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}} \mathbb{P}\left(M_1(t) \leq \frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\right) dF_2(\zeta),$$

where F_2 denotes the distribution function of the random variable $L_2(t)$. We now integrate by parts and notice that $\mathbb{P}\left(M_1(t) \leq \frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\right) = 0$ if $\zeta = z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}$ while $F_2(\zeta) = 0$ when $\zeta = 0$. This gives

$$\mathbb{P}(Y(t) \leq z_2) \geq \int_0^{z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}} \mathbb{P}\left(M_1(t) \leq \frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\right) dF_2(\zeta)$$
$$= \frac{2\beta b_1}{\sigma_1 c_2}\int_0^{z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}} F_2(\zeta)\mathcal{N}_{0,t}\left(\frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\right)\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}}{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}\frac{1}{\zeta}d\zeta.$$

Moreover, since from (8.14) we know that

$$F_2(\zeta) = \mathbb{P}(L_2(t) \leq \zeta) \geq -2\int_{\left\{\frac{k_2(t)e^{\sigma_2 u}}{1+b_2K_2(t)e^{\sigma_2 v}} \leq \zeta\right\}\cap\{v<0\}\cap\{u>v\}} \mathcal{N}'_{0,t}(u-2v)dudv,$$

we can conclude that

$$\mathbb{P}(Y(t)\le z_2)\ge \frac{2\beta b_1}{\sigma_1 c_2}\int_0^{z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}} F_2(\zeta)\mathcal{N}_{0,t}\left(\frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\right)\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}}{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}\frac{1}{\zeta}d\zeta$$

$$\ge -\frac{4\beta b_1}{\sigma_1 c_2}\int_0^{z_2/(1+b_1K_1(t))^{\frac{c_2}{\beta b_1}}}\left(\int_{\left\{\frac{k_2(t)e^{\sigma_2 u}}{1+b_2K_2(t)e^{\sigma_2 v}}\le\zeta\right\}\cap\{v<0\}\cap\{u>v\}}\mathcal{N}'_{0,t}(u-2v)dudv\right)$$
$$\times\mathcal{N}_{0,t}\left(\frac{1}{\sigma_1}\ln\left(\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}{b_1K_1(t)}\right)\right)\frac{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}}{\left(\frac{z}{\zeta}\right)^{\frac{\beta b_1}{c_2}}-1}\frac{1}{\zeta}d\zeta.$$

2. Using the lower bounds in (8.23) and (8.25) we obtain

$$\begin{aligned}\mathbb{P}\left(X(t)\le z_1, Y(t)\le z_2\right)&\le\mathbb{P}\left(L_1(t)e^{-c_1t}\le z_1, L_2(t)\le z_2\right)\\&=\mathbb{P}\left(L_1(t)e^{-c_1t}\le z_1\right)\mathbb{P}\left(L_2(t)\le z_2\right)\\&=\mathbb{P}\left(L_1(t)\le z_1e^{c_1t}\right)\mathbb{P}\left(L_2(t)\le z_2\right).\end{aligned}$$

With the help of (8.13) we conclude that

$$\mathbb{P}\left(X(t)\le z_1, Y(t)\le z_2\right)\le 4\int_{\left\{\frac{k_1(t)e^{\sigma_1 u}}{1+b_1K_1(t)e^{\sigma_1 v}}\le z_1e^{c_1t}\right\}\cap\{v>0\}\cap\{u<v\}}\mathcal{N}'_{0,t}(2v-u)dudv$$
$$\times\int_{\left\{\frac{k_2(t)e^{\sigma_2 u}}{1+b_2K_2(t)e^{\sigma_2 v}}\le z_2\right\}\cap\{v>0\}\cap\{u<v\}}\mathcal{N}'_{0,t}(2v-u)dudv$$

3. We now prove (8.42); we know from (8.24) and (8.38) that

$$\begin{aligned}X(t)&\ge L_1(t)e^{-\frac{c_1}{\beta b_2}\left(1+b_1\int_0^t G_1(v)dv\right)^{\frac{c_2}{\beta b_1}}\ln\left(1+b_2\int_0^t G_2(r)dr\right)}\\&\ge\frac{k_1(t)e^{\sigma_1B_1(t)-\frac{c_1}{\beta b_2}\left(1+b_1K_1(t)e^{\sigma_1M_1(t)}\right)^{\frac{c_2}{\beta b_1}}\ln\left(1+b_2K_2(t)e^{\sigma_2M_2(t)}\right)}}{1+b_1K_1(t)e^{\sigma_1M_1(t)}}.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\mathbb{P}(X(t)\le z_1)&\le\mathbb{P}\left(\frac{k_1(t)e^{\sigma_1B_1(t)-\frac{c_1}{\beta b_2}\left(1+b_1K_1(t)e^{\sigma_1M_1(t)}\right)^{\frac{c_2}{\beta b_1}}\ln\left(1+b_2K_2(t)e^{\sigma_2M_2(t)}\right)}}{1+b_1K_1(t)e^{\sigma_1M_1(t)}}\le z_1\right)\\&=-4\int_{A_{z_1}\cap\{v_1>0,u_1<v_1,v_2>0\}}\mathcal{N}'_{0,t}(2v_1-u_1)\mathcal{N}_{0,t}(v_2)du_1dv_1dv_2,\end{aligned}$$

where

$$A_{z_1} := \left\{ (u_1, v_1, v_2) \in \mathbb{R}^3 : \frac{k_1(t) e^{\sigma_1 u_1 - \frac{c_1}{\beta b_2}(1+b_1 K_1(t) e^{\sigma_1 v_1})^{\frac{c_2}{\beta b_1}} \ln(1+b_2 K_2(t) e^{\sigma_2 v_2})}}{1 + b_1 K_1(t) e^{\sigma_1 v_1}} \leq z_1 \right\}.$$

This coincides with (8.42). Moreover, from the lower estimate in (8.23) we get

$$\mathbb{P}(Y(t) \leq z_2) \leq \mathbb{P}(L_2(t) \leq z_2);$$

inequality (8.13) completes the proof of (8.43).

□

8.6 Discussion

In this paper, we propose a finite-time analysis for the solution of the two dimensional system (8.3) which describes a foraging arena model in presence of environmental noise. We derive in Theorem 160 almost sure upper and lower bounds for the components on the solution vector; these bounds emphasis the interplay between the parameters describing the model and different sources of randomness involved in the system. While such relationship is hardly visible in the description of the asymptotic behaviour of the solution, our estimates agree, if let the time tend to infinity, with the classification in asymptotic regimes obtained by [114]: this is shown in details in Remark 162. The accuracy of our bounds, which are obtained via a careful use of comparison theorems for stochastic differential equations, is evident in the simulations below (see Figures 2-6 below). There we plot for a given set of parameters the solution of the deterministic version of (8.3), i.e. with $\sigma_1 = \sigma_2 = 0$, a computer simulation of the solution of the stochastic equation (8.3) for different noise intensities and the corresponding upper and lower bounds from Theorem 160. Then, we utilize the bounds for the solution from Theorem 160 to derive two sided estimates for some statistical aspects of the solution. More precisely, in Theorem 164 and Theorem 166 we propose upper and lower bounds for the joint moments and distribution function of the components of the solution vector, respectively. These estimates are expressed via integrals whose numerical approximation is pretty standard. Again, the roles of the parameters describing our model are explicitly described in the proposed estimates. Figure 7 and Figure 8 below provide a numerical implementation for the bounds obtained in Theorem 164 for $p = q = 1$ and a given set of parameters.

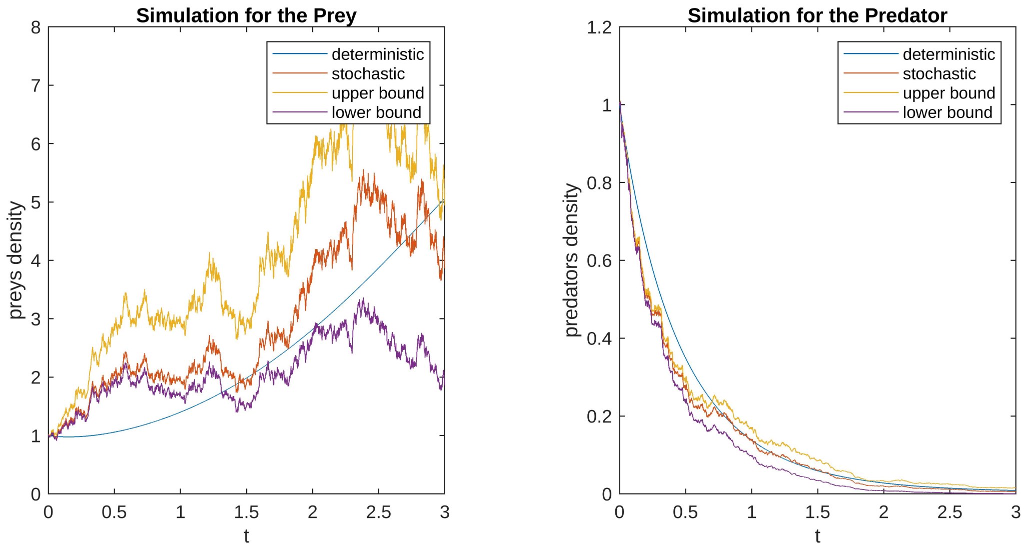

Figure 8.2: $a_1 = 1$, $a_2 = 2$, $b_1 = 0.1$, $b_2 = 0.5$, $c_1 = 6$, $c_2 = 0.9$, $\beta = 4$ $\sigma_1 = 0.5$,$\sigma_2 = 0.3$

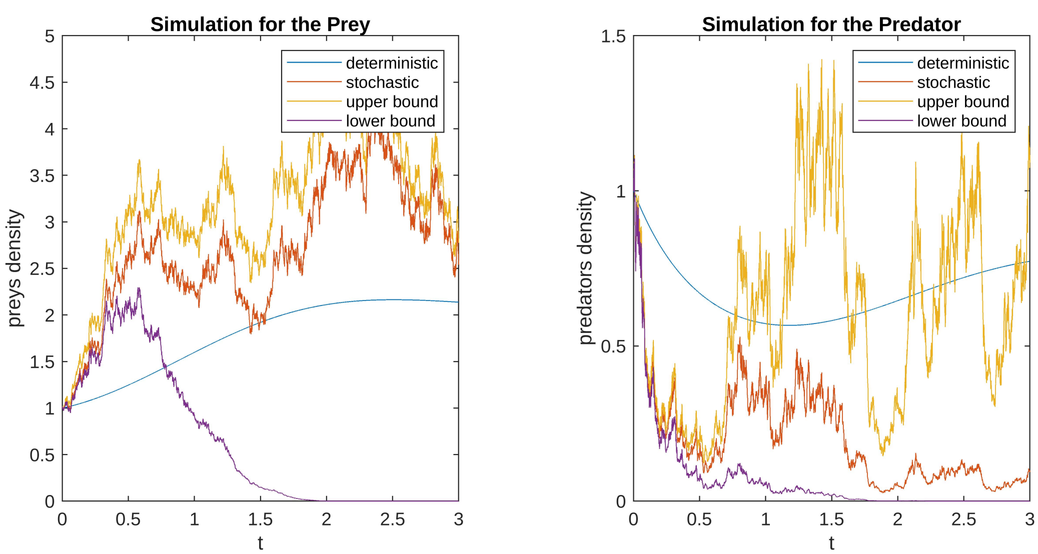

Figure 8.3: $a_1 = 2$, $a_2 = 1$, $b_1 = 0.5$, $b_2 = 0.9$, $c_1 = 6$, $c_2 = 4$, $\beta = 4$ $\sigma_1 = 0.5$,$\sigma_2 = 1.3$

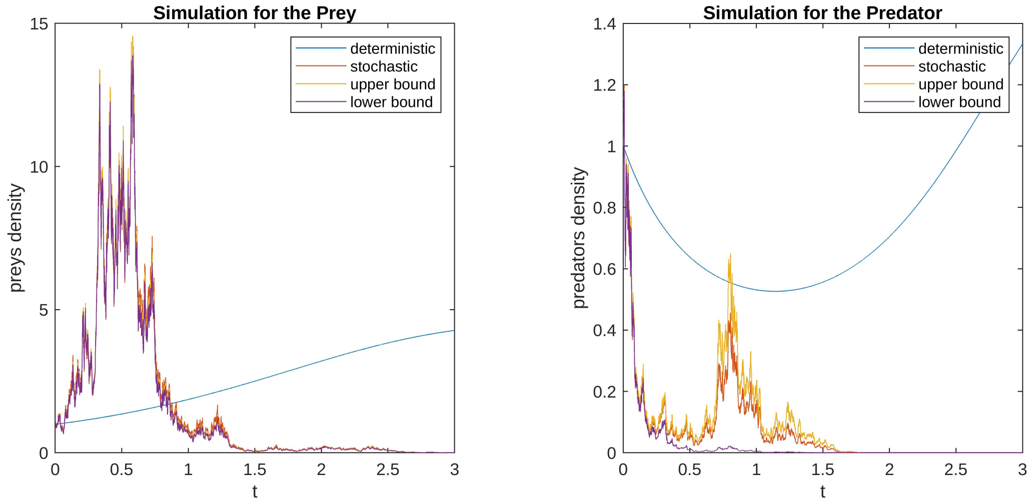

Figure 8.4: $a_1 = 2$, $a_2 = 1$, $b_1 = 0.1$, $b_2 = 0.9$, $c_1 = 2$, $c_2 = 4$, $\beta = 5$ $\sigma_1 = 2.5$,$\sigma_2 = 2.5$

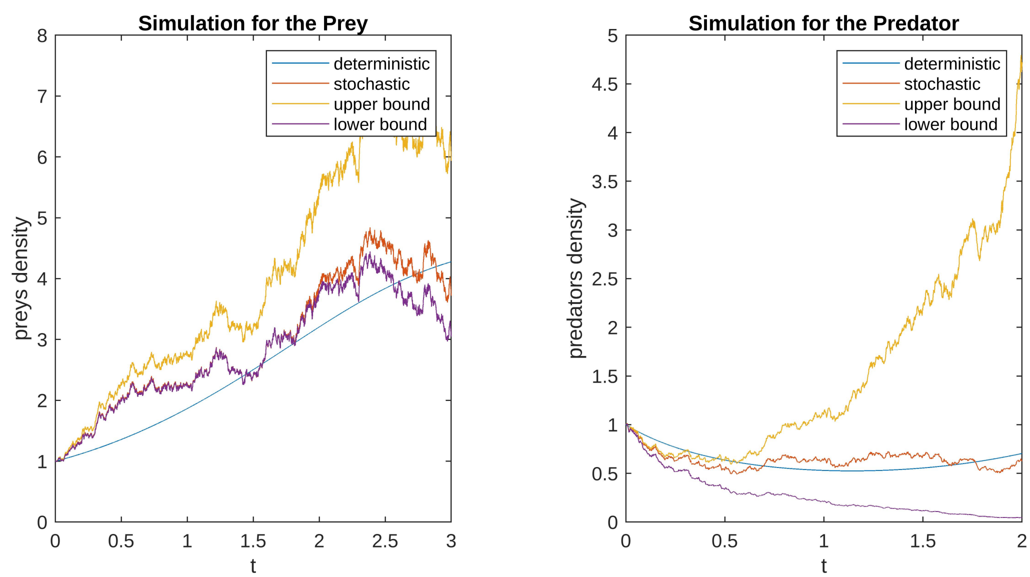

Figure 8.5: $a_1 = 1$, $a_2 = 1$, $b_1 = 0.1$, $b_2 = 0.9$, $c_1 = 2$, $c_2 = 4$, $\beta = 5$ $\sigma_1 = 0.3$,$\sigma_2 = 0.3$

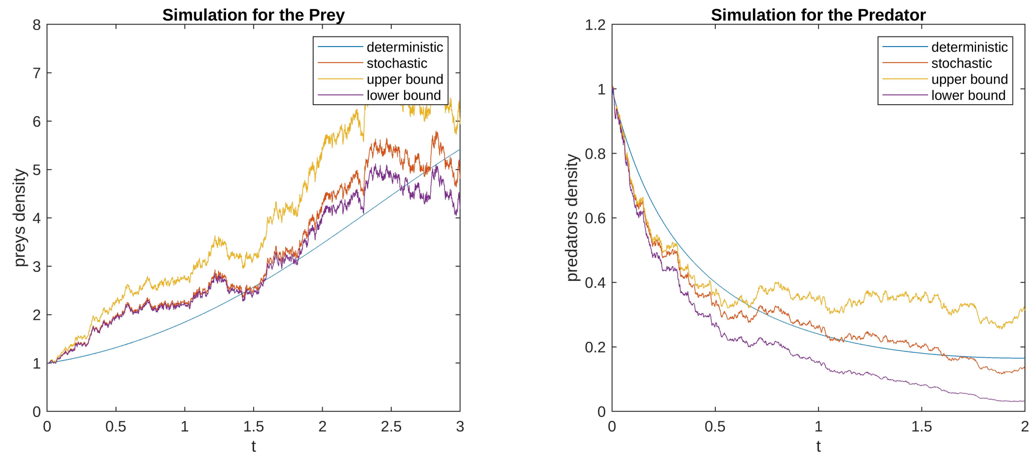

Figure 8.6: $a_1 = 1$, $a_2 = 1$, $b_1 = 0.1$, $b_2 = 2$, $c_1 = 3$, $c_2 = 2$, $\beta = 5$ $\sigma_1 = 0.3$,$\sigma_2 = 0.3$

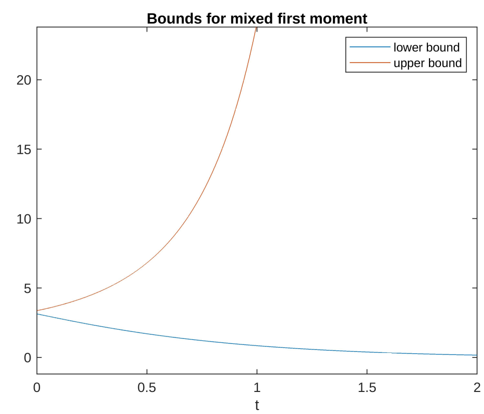

Figure 8.7: $a_1 = 1$, $a_2 = 1$, $b_1 = 0.1$, $b_2 = 0.9$, $c_1 = 2$, $c_2 = 4$, $\beta = 5$ $\sigma_1 = 0.3$,$\sigma_2 = 0.3$

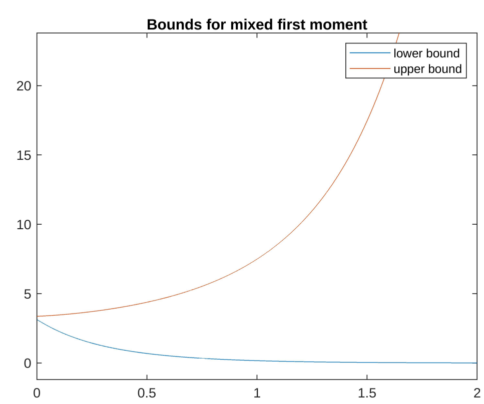

Figure 8.8: $a_1 = 1$, $a_2 = 1$, $b_1 = 0.1$, $b_2 = 0.9$, $c_1 = 2$, $c_2 = 4$, $\beta = 5$ $\sigma_1 = 0.3$,$\sigma_2 = 0.3$

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