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QUANTUM INTEGRABILITY AS A NEW METHOD FOR EXACT RESULTS ON N=2 SUPERSYMMETRIC GAUGE THEORIES AND BLACK HOLES

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Quantum Integrability as a New Method for Exact Results on $\mathcal{N} = 2$ Supersymmetric Gauge Theories and Black Holes

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Abstract

In this thesis I show a triple new connection we found between quantum integrability, $\mathcal{N} = 2$ supersymmetric gauge theories and black holes perturbation theory. I use the approach of the ODE/IM correspondence between Ordinary Differential Equations (ODE) and Integrable Models (IM), first to connect basic integrability functions - the Baxter's Q, T and Y functions - to the gauge theory periods. This fundamental identification allows several new results for both theories, for example: an exact non linear integral equation (Thermodynamic Bethe Ansatz, TBA) for the gauge periods; an interpretation of the integrability functional relations as new exact R-symmetry relations for the periods; new formulas for the local integrals of motion in terms of gauge periods. This I develop in all details at least for the $SU(2)$ gauge theory with $N_f = 0, 1, 2$ matter flavours. Still through to the ODE/IM correspondence, I connect the mathematically precise definition of quasinormal modes of black holes (having an important role in gravitational waves' obervations) with quantization conditions on the Q, Y functions. In this way I also give a mathematical explanation of the recently found connection between quasinormal modes and $\mathcal{N} = 2$ supersymmetric gauge theories. Moreover, it follows a new simple and effective method to numerically compute the quasinormal modes - the TBA - which I compare with other standard methods. The spacetimes for which I show these in all details are in the simplest $N_f = 0$ case the D3 brane in the $N_f = 1, 2$ case a generalization of extremal Reissner-Nordström (charged) black holes. Then I begin treating also the $N_f = 3, 4$ theories and argue on how our integrability-gauge-gravity correspondence can generalize to other types of black holes in either asymptotically flat ($N_f = 3$) or Anti-de-Sitter ($N_f = 4$) spacetime. Finally I begin to show the extension to a 4-fold correspondence with also Conformal Field Theory (CFT), through the renowned AdS/CFT correspondence.

تقدیم به مهسا

“Questions posed by nature are vastly deeper and more fruitful than ones we humans would tend to pose for ourselves.”

نیما ارکانی حامد

(Strings 2021)

“La storia della scienza può servire a renderci consapevoli del fatto che la razionalità, il rigore logico, la controllabilità delle asserzioni, la pubblicità dei risultati e dei metodi, la stessa struttura del sapere scientifico come qualcosa che è capace di crescere su se stesso, non sono categorie perenni dello spirito né dati eterni della storia umana, ma conquiste storiche, che, come tutte le conquiste, sono, per definizione, suscettibili di andare perdute.”

Paolo Rossi

(La nascita della scienza moderna in Europa)

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1. Summary

In our work [1], I and my supervisor Prof. D. Fioravanti found a novel kind of correspondence between $\mathcal{N} = 2$ supersymmetric deformed gauge theory (or super Yang-Mills, SYM) and integrable models (IM). Our basic result was that the gauge periods a, a_D (from which one computes the prepotential) are directly connected to the Baxter's Q and T functions. Such functions can be expanded in the integrals of motion of some two dimensional integrable model and satisfy certain exact functional relations among them. This connection allowed several new results for both theories, for example: an exact non linear integral equation (Thermodynamic Bethe Ansatz, TBA) for the gauge periods; an interpretation of the integrability functional relations as new exact R symmetry relations for the periods; new formulas for the local integrals of motion in terms of gauge periods. The general method we used is the ODE/IM correspondence [2, 3, 4] between Ordinary Differential Equations (ODEs) and Integrable Models. It allows to derive the characteristic structures of integrable models by studying the connection coefficients of the solutions of ordinary differential equations.

All this we showed to hold for pure ($N_f = 0$) $SU(2)$ SYM in the Nekrasov-Shatashvili (NS) limit of the Ω-background (a deformation of spacetime used to compute instanton contributions to the partition function) and self-dual Liouville integrable model. These may seem a very particular choice of SYM and IM, but already back then it was intuitively clear to us that our correspondence should hold much more generally. Thus about two years ago we begun a long and meticulous generalization and extension work, with the new collaborator Dr. Hongfei Shu, to the $N_f = 1$ and $N_f = 2$ $SU(2)$ NS-deformed gauge theories, in correspondence with more general IMs, which ended up in the paper [5].

As interesting as this new kind of gauge-integrability correspondence may be regarded, arguably much more interesting developments followed. In fact, the very same NS deformed $\mathcal{N} = 2$ $SU(2)$ gauge theories were found to be useful to compute quasinormal modes (QNMs) of black holes (BHs) and black branes [6, 7, 8, 9]. This constitutes an unexpected application of supersymmetric gauge theory, specifically to already experimentally observable/testable physics in the form of astrophysical black holes as modelled by either General Relativity (GR) or String Theory (ST) or modified theories of gravity, which seem to vastly increase the general interest and trust to the whole subject [10].

To our wonder, as soon as I began doing research on this new line, under inspiration from also my PhD abroad visit's supervisor Prof. Konstantin Zarembo, I immediately found a new fundamental connection between QNMs and other BH observables to also the integrable models we were involved connecting to $\mathcal{N} = 2$ NS-deformed $SU(2)$ gauge theories. Our other work [11] rapidly followed, were we showed that QNMs are nothing but the zeros (Bethe roots) of the Baxter's Q function - as defined in the ODE/IM correspondence approach - and can be computed very efficiently with a new method tipical of integrability: the TBA. This there we sketched for the $N_f = 0$ and $N_f = 2$ $SU(2)$ gauge theories, in correspondence with the D3 brane and the intersection of four stacks of D3 branes, respectively. The latter can be regarded as a mathematical generalization of the extremal (maximally charged) Reissner-Nordström (RN) BH.

In the subsequent work [5], beyond showing the extension of the integrability-gauge correspondence to the $SU(2)$ $N_f = 1, 2$ theory, we have shown the generalization of the integrability-gravity correspondence to also $N_f = 1$ theory. It corresponds physically to just the null entropy limit in the system of intersection of four stacks of D3 branes.

Moreover here I begin setting up the same triple gauge-integrability-gravity correspondence for the $N_f = (0, 2)$ and $N_f = 3$ $SU(2)$ theory. In this case the gravity counterpart are asymptotically flat (non-extremal) general relatity (GR) black holes or various string theory black holes (for instance, fuzzballs). The integrability counterpart is less clear for the moment, though.

Finally, I begin extending the triple correspondence to 4-fold correspondence, by studying asymptotically AdS black holes (exploiting AdS/CFT correspondence). In particular I connect BTZ (AdS_3) black hole and its CFT_2 counterpart, to class $\mathcal{S}$ gauge theory and the integrable XXZ spin chain at the supersymmetric point, gaining among other things a new understanding of the poles skipping phenomenon for the retarded correlator in the CFT_2.

1.1. Acknowledgments

Among the various collaborators I have been fortunate enough to have, I especially thank my supervisor at INFN and Bologna U. Prof. Davide Fioravanti, as well as my hosts in NORDITA Prof. Konstantin Zarembo and Dr. Hongfei Shu (now BIMSA). An exceptional thank I would like to give to also Prof. Alba Grassi (CERN, Genève U.), for connecting for the first time our theoretical research field to astrophysical black holes [6].

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2. Introduction to quantum Seiberg-Witten theory

2.1. General $\mathcal{N} = 2$ Supersymmetry

2.1.1. General Supersymmetry algebra

Supersymmetry (SUSY) is broadly speaking a conjectured *symmetry between matter and radiation* [12]. It is thought to be a correspondence between *fermion* (matter) and *boson* (force or radiation) particle, by which for every known elementary particle of one kind there exists another particle of the other kind and viceversa. Supersymmetry helps explaining several theoretical short- comings of the Standard Model, but observations demand that supersymmetry, if present at all, be broken at the observed energy scales [13].

Mathematically, such correspondence can be expressed in terms of the action *supersymmetric charges* $Q_{i\,\alpha}$, with $i = 1, 2, ..., N$, which indeed exchange boson and fermion particles. $Q_{i\,\alpha}$ ($i = 1, 2, ..., \mathcal{N}$) are Majorana fermions. The (graded) algebra they satisfy is

$$\begin{aligned}
\{Q_{i\,\alpha}, \bar{Q}^j_{\dot{\beta}}\} &= 2\delta^j_i \sigma^\mu_{\alpha\dot{\beta}} P_\mu \\
[Q_{i\alpha}, P_\mu] &= 0 \\
[\bar{Q}_{i\dot{\alpha}}, P_\mu] &= 0 \\
[Q_{i\alpha}, M^{\mu\nu}] &= (\sigma^{\mu\nu})^\beta_\alpha Q_{i\beta} \\
[\bar{Q}^{\dot{\alpha}}_i, M^{\mu\nu}] &= (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} Q^{\dot{\beta}}_i
\end{aligned} \tag{2.1}$$

and it is by construction an extension of Poincaré algebra. Different pairing in the anti-commutators gives the *central charges* Z_{ij}

$$\begin{aligned}
\{Q_{i\alpha}, Q_{j\beta}\} &= \epsilon_{\alpha\beta} Z_{ij} \\
\{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} (Z_{ij})^* \,.
\end{aligned} \tag{2.2}$$

It holds $Z_{ij} = -Z_{ji}$. On the SUSY generators $Q_{i\alpha}$ it can act at most an $U(N)$ internal symmetry (called *R-symmetry*) group, with generators B_r:

$$\begin{aligned}
[Q_{i\alpha}, B_r] &= (b_r)^j_i Q_{j\alpha} \\
[\bar{Q}^i_{\dot{\alpha}}, B_r] &= -(b^*_r)^i_j \bar{Q}^j_{\dot{\alpha}} \,.
\end{aligned} \tag{2.3}$$

2.1.2. Superfields

An $\mathcal{N} = 1$ *chiral superfield* ϕ $(0, \frac{1}{2})$ is made of a scalar z, fermion ψ and auxiliary field f. It is denoted in terms of spin components $(0, \frac{1}{2})$. Using the superspace coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ it can be expressed as

$$\phi(y, \theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y) \,. \tag{2.4}$$

Under SUSY transormations the $\mathcal{N} = 1$ chiral superfield's components vary as

$$\begin{aligned}
\delta z &= \sqrt{2}\epsilon\psi \\
\delta\psi &= \sqrt{2}i\partial_\mu z\sigma^\mu\bar{\epsilon} - \sqrt{2}f\epsilon \\
\delta f &= \sqrt{2}i\partial_\mu\psi\sigma^\mu\bar{\epsilon} \,.
\end{aligned} \tag{2.5}$$

A $\mathcal{N}=1$ *vector superfield* V is defined to have spin components $(\frac{1}{2},1)$ and it is endowed of a gauge symmetry

$$V \to V + \phi + \phi^\dagger\,. \tag{2.6}$$

In the so-called Wess-Zumino gauge it can be expanded in terms of components fields as

$$V = \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\,. \tag{2.7}$$

The $\mathcal{N}=2$ *vector superfield* is defined to have spin components $(0,\frac{1}{2},\frac{1}{2},1)$ and thus it is the sum of $N=1$ chiral and $N=1$ vector superfield. All fields are in the adjoint representation of the gauge group. $N=2$ hypermultiplet has spin components $(-\frac{1}{2},0,0,\frac{1}{2})$ is composed of 2 complex scalar fields and 1 Dirac fermion, with 2 complex auxiliary fields and thus it describes matter. In terms of $\mathcal{N}=1$ superfields it is the composition of a chiral and antichiral $\mathcal{N}=1$ superfields.

2.1.3. $\mathcal{N}=2$ supersymmetric microscopic Lagrangian

The *exact* or so-called *microscopic Lagrangian* for $\mathcal{N}=2$ SUSY is

$$\mathcal{L}_{YM}^{N=2} = \frac{1}{32\pi}\Im\left(\tau\int d^2\theta \mathrm{Tr}\, W_\alpha W^\alpha\right) + \int d^2\theta d^2\bar{\theta}\mathrm{Tr}\,\phi^\dagger e^{2gV}\phi \tag{2.8}$$

$$= \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} - i\psi\sigma^\mu D_\mu\bar{\psi} + (D_\mu z)^\dagger D^\mu z + \frac{\theta}{32\pi^2}g^2F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}D^2 + f^\dagger f\right.$$

$$\left. + i\sqrt{2}gz^\dagger\{\lambda,\psi\} - i\sqrt{2}g\{\bar{\psi},\bar{\lambda}\}z + gD[z,z^\dagger]\right)\,. \tag{2.9}$$

The kinetic term for the vector field

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V\,, \tag{2.10}$$

is constructed through the covariant derivative

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{2.11}$$

and it reads explicitly

$$W_\alpha(y) = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta(\sigma^\mu D_\mu\bar{\lambda}(y))_\alpha\,, \tag{2.12}$$

with $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu]$.

The microscopic action is invariant under the $U(1)$ R-symmetry acting as

$$\phi \to e^{2i\alpha}\phi \qquad W_\alpha \to e^{i\alpha}W_\alpha \qquad \theta \to e^{i\alpha}\theta\,. \tag{2.13}$$

Instanton corrections break the continuous $U(1)$ symmetry group to the discrete symmetry group $\mathbb{Z}_8$.

The auxiliary fields equations of motion are

$$f^a = 0 \tag{2.14}$$
$$D^a = -g[z, z^\dagger]^a \tag{2.15}$$

and when inserted in $\mathcal{L}_{YM}^{N=2}$ produce a term called *scalar potential*

$$V(z, z^\dagger) = \frac{1}{2} g^2 \mathrm{Tr}\,([z, z^\dagger])^2 \,. \tag{2.16}$$

2.1.4. Supersymmetry breaking

Unbroken SUSY requires $V = 0$. For $SU(2)$ gauge group $V(z) = 0$ requires classically that $z = \frac{1}{2} a^{(0)} \sigma_3$, or including quantum fluctuations

$$\langle z \rangle = \frac{1}{2} a^{(0)} \sigma_3 \,. \tag{2.17}$$

Gauge transformations can take $a^{(0)} \to -a^{(0)}$ and therefore

$$u = \langle \mathrm{tr}\, z^2 \rangle \tag{2.18}$$

labels gauge inequivalent vacua: the *moduli space* $\mathcal{M}$. u and $-u$ correspond to *physically equivalent vacua* related by the $\mathbb{Z}_8$ R-symmetry. The gauge symmetry is broken as $SU(2) \to U(1)$.

2.2. Classical pure Seiberg-Witten theory

The *Wilsonian effective action* S_W is defined as the generating function of the vertex functions Γ except that all loop momenta are integrated down to an infrared cut-off μ. In particular, the low energy Wilsonian effective action S_W for $U(1)$ is

$$\frac{1}{16\pi} \Im \int d^4x \left[\frac{1}{2} \int d^2\theta \mathcal{F}''(\phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} \phi^\dagger \mathcal{F}'(\phi) \right] , \tag{2.19}$$

where $\mathcal{F}$ is the *prepotential*, a holomorphic function.

One can define the *dual field* and *dual prepotential* as

$$\phi_D = \frac{\partial \mathcal{F}(\phi)}{\partial \phi} \qquad \frac{\partial \mathcal{F}_D(\phi_D)}{\partial \phi_D} = -\phi \,, \tag{2.20}$$

or

$$a_D^{(0)} = \frac{\partial \mathcal{F}}{\partial a^{(0)}} \qquad \frac{\partial \mathcal{F}_D}{\partial a_D^{(0)}} = -a^{(0)} \,. \tag{2.21}$$

Similarly for vectors, W_D^α is defined relative to V_D, which is a lagrange multiplier in the functional integral for the Bianchi identity $\Im(D_\alpha W^\alpha) = 0$. The effective action is duality invariant.

The *coupling constant*

$$\tau(a^{(0)}) = \mathcal{F}''(a^{(0)}) = \frac{\theta(a^{(0)})}{2\pi} + \frac{4\pi i}{g^2(a^{(0)})} \tag{2.22}$$

enjoys a *weak-strong coupling duality*

$$\tau_D(a_D^{(0)}) = -\frac{1}{\tau(a^{(0)})} \, . \tag{2.23}$$

The duality symmetry

$$\begin{pmatrix} \phi \\ \phi_D \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi_D \end{pmatrix} \tag{2.24}$$

and the symmetry

$$\begin{pmatrix} \phi \\ \phi_D \end{pmatrix} \to \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \phi_D \end{pmatrix} \tag{2.25}$$

generate $Sl(2,\mathbb{Z})$ group of duality symmetries.

All fields (n_e, n_m) satisfy the *BPS condition*

$$m^2 = 2|Z|^2 \, , \qquad Z = n_m a_D^{(0)}(u) + n_e a^{(0)}(u) \, . \tag{2.26}$$

These states are *collective excitations - solitons*. For example, the *magnetic monopole* $(0,1)$ is described by a $N=2$ hypermultiplet H which couples locally to the dual fields ϕ_D and W_D. Also, the *electron* $(1,0)$ is described by H which couples locally to ϕ and W. Roughly in the circle $|u| < \Lambda^2$ we have only the *monopole* $\pm(0,1)$ and the *dyon* $\pm(\pm 1, 1)$, while outside the circle we have also all other *dyons* $\pm(n,1)$ and the W *bosons* $(0, \pm 1)$.

We remark that we are considering here only the *pure* (with zero number of fundamental matter flavours $N_f = 0$) Seiberg-Witten (SW) theory. For this theory, the *Seiberg-Witten cycles* or *Seiberg-Witten periods* are

$$a^{(0)}(u,\Lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{2u - 2\Lambda^2 \cos z} \, dz \tag{2.27}$$

$$= \Lambda \sqrt{2(u/\Lambda^2 + 1)} \; {}_2F_1(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{1 + u/\Lambda^2}) \; , \tag{2.28}$$

$$a_D^{(0)}(u,\Lambda) = \frac{1}{2\pi} \int_{-\arccos(u/\Lambda^2) - i0}^{\arccos(u/\Lambda^2) - i0} \sqrt{2u - 2\Lambda^2 \cos z} \, dz \tag{2.29}$$

$$= i\Lambda \frac{(1 - u/\Lambda^2)}{2} \; {}_2F_1(\frac{1}{2}, \frac{1}{2}, 2; \frac{1 - u/\Lambda^2}{2}) \, , \tag{2.30}$$

We notice that they are given by the well-known *Gauss Hypergeometric function* ${}_2F_1$. They are integrals of the *SW differential*

$$\lambda = \sqrt{2u - 2\Lambda^2 \cos z} \, . \tag{2.31}$$

Inverting $a^{(0)}(u)$ as $u(a^{(0)})$, substituting $a_D^{(0)}(a^{(0)}) = \frac{\partial \mathcal{F}}{\partial a^{(0)}}$ and integrating one can obtain prepotential $\mathcal{F}(a^{(0)})$. This is, in a nutshell, *classical Seiberg-Witten (SW) theory* [14, 15, 16].

2.3. Ω background

Quantum Seiberg-Witten (qSW) theory is essentially the effective $\mathcal{N} = 2$ supersymmetric gauge theory in the spacetime deformation called Ω *background*. The latter deformation is a mere an artifact, but it is very useful to compute instanton corrections to the partition function, called *Nekrasov partition function*.

The Ω background is introduced formally as follows. Some differential geometry (or topology)'s preliminaries to understand it are given in appendix A. Given the 4D $\mathcal{N} = 2$ theory T_4, one can find a 6D $\mathcal{N} = 1$ theory T_6 whose dimensional reduction gives T_4. Then one should compactify T_6 on a manifold X^6 which is an $\mathbb{R}^4$ vector bundle over the two-torus $\mathbb{T}^2$ of area r^2, with a flat $Spin(4) = SU(2)_+ \times SU(2)_-$ connection, whose holonomies around the two non-contractible cycles are

$$\left(e^{\frac{ir}{2}\Re(\epsilon_1+\epsilon_2)\sigma_3}, e^{\frac{ir}{2}\Re(\epsilon_1-\epsilon_2)\sigma_3}\right), \left(e^{\frac{ir}{2}\Im(\epsilon_1+\epsilon_2)\sigma_3}, e^{\frac{ir}{2}\Im(\epsilon_1-\epsilon_2)\sigma_3}\right). \tag{2.32}$$

Then one should embed also the $SU(2)_+$ part of the flat connection into the R-symmetry $SU(2)$ of T_6. Finally, one should take the limit $r \to 0$ while keeping the complex numbers ϵ_1, ϵ_2 finite and obtain thus the Ω background. We remark that the Ω background is needed to compute *instanton contributions* to the partition function [17].

2.4. Nekrasov-Shatashvili limit and pure quantum Seiberg-Witten theory

In this work, we will need to consider only the *Nekrasov-Shatashvili (NS) limit* of the Ω background $\epsilon_2 \to 0$, $\epsilon_1 \neq 0$ [18].

For the *pure qSW theory in the NS limit*, the expectation value for scalar field is

$$\langle \tilde{z} \rangle = \frac{1}{2} a(\epsilon_1, u, \Lambda)\sigma_3 \tag{2.33}$$

where a is the *quantum SW period*. This latter corresponds also to the *Floquet exponent* for the Mathieu equation:

$$-\frac{\epsilon_1^2}{2}\frac{d^2}{dz^2}\psi(z) + [\Lambda^2 \cos z - u]\psi(z) = 0\,. \tag{2.34}$$

that is the quasi-periodicity index of the quasiperiodic solution (or *wave function*)

$$\psi(z + 2\pi) = e^{\frac{2\pi i}{\hbar}a}\psi(z) \tag{2.35}$$

One can obtain in a similar way from the quasiperiodic wave function the *quantum SW dual period* [19]

$$\psi(\arccos\frac{u}{\Lambda^2}) = e^{\frac{2\pi i}{\hbar}a_D}\psi(-\arccos\frac{u}{\Lambda^2})\,. \tag{2.36}$$

It is common to call the differential equation whose The logarithmic derivative of the wave function $\mathcal{P}(z) = -i\frac{d}{dz}\psi(z)$ is called also *quantum SW differential*.

2.5. Quantum Seiberg-Witten theory with fundamental matter

The Seiberg-Witten (SW) curve for $\mathcal{N} = 2$ $SU(2)$ with N_f fundamental matter flavour hypermultiplets is given by

$$K(p) - \frac{\bar{\Lambda}}{2}(K_+(p)e^{ix} + K_-(p)e^{-ix}) = 0 \tag{2.37}$$

where

$$\bar{\Lambda} = \begin{cases} \Lambda_0^2 & N_f = 0 \\ \Lambda_1^{3/2} & N_f = 1 \\ \Lambda_2^1 & N_f = 2 \\ \Lambda_3^{1/2} & N_f = 3 \\ \sqrt{q} & N_f = 4\,. \end{cases} \tag{2.38}$$

$$K(p) = \begin{cases} p^2 - u & N_f = 0 \\ p^2 - u & N_f = 1 \\ p^2 - u + \frac{\Lambda_2^2}{8} & N_f = 2 \\ p^2 - u + \frac{\Lambda_3}{4}(p + \frac{m_1+m_2+m_3}{2}) & N_f = 3 \\ (1+\frac{q}{2})p^2 - u + \frac{q}{4}p\sum_{i=1}^4 m_i + \frac{q}{8}\sum_{i<j} m_i m_j & N_f = 4\,. \end{cases} \tag{2.39}$$

$$K_+(p) = \prod_{j=1}^{N_+}(p+m_j)\,, \quad K_-(p) = \prod_{j=N_++1}^{N_f}(p+m_j)\,. \tag{2.40}$$

u is the Coulomb moduli parameter and m_i are the masses $1 \leq N_+ \leq N_f$. By introducing $y_{SW} = \bar{\Lambda}K_+(p)e^{ix} - K(p)$ we get the SW curve in standard form

$$y_{SW}^2 = K(p)^2 - \bar{\Lambda}^2 K_+(p)K_-(p) \tag{2.41}$$

The SW differential is then defined to be

$$\lambda = p d\ln\frac{K_-}{K_+} - 2\pi i p\,dx \tag{2.42}$$

and defines a symplectic form $d\lambda = dp \wedge dx$, which doubly integrated gives the SW periods [20]

$$a = \oint_A p(x)\,dx \qquad a_D = \oint_B p(x)\,dx\,. \tag{2.43}$$

The quantum SW curve is obtained by letting p become the differential operator $-i\hbar\frac{d}{dx}$[20]:

$$\left(K(-i\hbar\partial_x)) - \frac{\bar{\Lambda}}{2}(e^{ix/2}K_+(-i\hbar\partial_x)e^{ix/2} + e^{-ix/2}K_-(-i\hbar\partial_x)e^{-ix/2}\right)\psi(x) = 0\,. \tag{2.44}$$

Let $N_f = 0$ and $x = -iy$. We get

$$-\hbar^2\frac{d^2}{dy^2}\psi + (\Lambda_0^2\cosh y + u)\psi = 0 \tag{2.45}$$

Let $N_f = 1$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \left[\frac{1}{16}\Lambda_1^3 e^{2y} + \frac{1}{2}\Lambda_1^{3/2} e^{-y} + \frac{1}{2}\Lambda_1^{3/2} m_1 e^y + u\right]\psi = 0 \tag{2.46}$$

Let $N_f = 1$ and $x = -iy$, $y \to y - \frac{1}{2}\ln\Lambda_1 + \ln 2$. We get

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \left[\frac{1}{4}\Lambda_1^2 (e^{2y} + e^{-y}) + \Lambda_1 m_1 e^y + u\right]\psi = 0 \tag{2.47}$$

Let $N_f = 2$, $N_+ = 1$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \left[\frac{1}{16}\Lambda_2^2 (e^{2y} + e^{-2y}) + \frac{1}{2}\Lambda_2 m_1 e^y + \frac{1}{2}\Lambda_2 m_2 e^{-y} + u\right]\psi = 0 \tag{2.48}$$

Let $N_f = 2$, $N_+ = 2$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \frac{e^{2y}\Lambda_2^2 (m_1 - m_2)^2 + e^y \left(\Lambda_2^3 - 2\Lambda_2\hbar^2 + 8\Lambda_2 m_1 m_2 - 8\Lambda_2 u\right) + 16u - 6\Lambda_2^2 + 8\Lambda_2 e^{-y}}{4(\Lambda_2 e^y - 2)^2}\psi = 0 \tag{2.49}$$

Let $N_f = 3$, $N_+ = 2$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \frac{4e^{2y}\Lambda_3 (m_1 - m_3)^2 + 4e^y\sqrt{\Lambda_3}\left(-2\hbar^2 + 8m_1 m_3 + \Lambda_3 m_2 - 8u\right)}{16\left(\sqrt{\Lambda_3}e^y - 2\right)^2}\psi \tag{2.50}$$

$$+ \frac{\Lambda_3^2 + 64u - 24\Lambda_3 m_2 + 4e^{-y}\sqrt{\Lambda_3}(8m_2 - \Lambda_3) + 4\Lambda_3 e^{-2y}}{16\left(\sqrt{\Lambda_3}e^y - 2\right)^2}\psi = 0 \tag{2.51}$$

Let $N_f = 4$, $N_+ = 2$ and $x = -iy$.

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \frac{1}{16\left(-2\sqrt{q}\cosh(y) + q + 2\right)^2}\Big[4e^{2y}q(m_1 - m_2) + 4e^{-2y}q(m_3 - m_4)^2 \tag{2.52}$$

$$+ e^y\left(-4m_1^2 q^{3/2} + 12m_1 m_2 q^{3/2} + 32m_1 m_2\sqrt{q} - 4m_2^2 q^{3/2} + 4m_3 m_4 q^{3/2} - 4q^{3/2}\hbar^2 - 32\sqrt{q}u - 8\sqrt{q}\hbar^2\right) \tag{2.53}$$

$$+ e^{-y}\left(4m_1 m_2 q^{3/2} - 4m_3^2 q^{3/2} + 12m_3 m_4 q^{3/2} + 32m_3 m_4\sqrt{q} - 4m_4^2 q^{3/2} - 4q^{3/2}\hbar^2 - 32\sqrt{q}u - 8\sqrt{q}\hbar^2\right) \tag{2.54}$$

$$+ 32qu + 16q\hbar^2 + 64u + m_1^2 q^2 - 2m_1 m_2 q^2 - 24m_1 m_2 q - 2m_1 m_3 q^2 - 2m_1 m_4 q^2 + m_2^2 q^2 \tag{2.55}$$

$$- 2m_2 m_3 q^2 - 2m_2 m_4 q^2 + m_3^2 q^2 - 2m_3 m_4 q^2 - 24m_3 m_4 q + m_4^2 q^2\Big]\psi = 0 \tag{2.56}$$

2.6. A comment on phenomenology

The missed discovery of supersymmetry at the electro-weak scale after the first run of LHC in 2013 generated a lot of skepticism towards such paradigm, especially among experimentalists. However, testing SUSY in general, independently from each particular model, “is an extremely challenging task”, so the LHC results do not exclude some particular realizations of supersymmetry which remain untested [21]. Still, the observed Higgs mass is compatible with supersymmetry only if the superpartners are quite heavy (tens of TeV) and beyond the current reach of LHC. Moreover, extended supersymmetry models (that is, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry) are very interesting for their mathematical richness which allow to apply them beyond particle physics [22, 23].

3. Introduction to quasinormal modes of black holes

3.1. General context

Very recently, on 14th September 2015, the LIGO experiment has revealed the first gravitation wave signal of the history of humankind [24]. The enormous gravitational field variations which produce such *oscillations of spacetime itself* are most tipically realized in events of *black holes merging*. Thus these revelations (which now counts in more than one-hundred [25]) are first of all very effective means for studying black holes which, in the words of the great astrophysicist S. Chandrasekhar:

> “are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well.” [26]

Black holes merging events can be naturally divided in three phases:

1. *inspiral*, in which the two original black holes approach each other increasingly closely;

2. *merging*, in which the chaotic behaviour is not easy to understand by analytical means;

3. *ringdown*, when the final merged black hole is formed and spacetime oscillates in a damped way swiflty down to zero.

Perturbation theory can be used to study this third and last ringdown phase. The characteristic frequencies of oscillations of spacetime in it are called Quasinormal Modes (QNMs). The term ”quasinormal”, rather than ”normal”, is used because perturbed BH spacetimes are instrinsically dissipative due to the presence of an event horizon (the system is not time-symmetric). Indeed, in general, the QNMs ω_n have an imaginary part $\Im\omega_n < 0$, so that the perturbation they describe is damped to zero as $t \to \infty$ [27].

QNMs provide informations on the manner in which gravitational waves, incident on the black-hole, are scattered and absorbed. Thus on the astrophysical side, they can be used to prove that the compact objects observed are indeed rotating BHs, that is, QNMs can be used to infer mass and angular momentum of BHs and to test the no-hair theorem of general relativity [27]. Also on the theoretical side, such information from QNMs

> “has a more trascendent interest: it provides insight, in its simplest and purest context, into the deeper aspects of space time as conceived in general relativity; and it reveals the analytical richness of the theory.” [26]

Moreover, nowadays the importance of QNMs is not confined to a better understanding of General Relativity. Indeed, since many decades, a pressing issue in theoretical physics is to reconcile this theory with the other pillar of modern physics, namely Quantum Mechanics. Hence the astrophysical dark compact objects we name “black holes” could be far better modeled by means of, for instance String Theory, rather than General Relativity. Thus argues for instance S. D. Mathur:

> “The black hole information paradox is probably the most important issue for fundamental physics today. If we cannot understand its resolution, then we cannot understand how quantum theory and gravity work together. [...] I conclude with a brief outline of how the paradox is resolved in string theory: quantum gravity effects are not confined to a bounded length [...], and the information of the hole is spread thorughout its interior, creating a ’fuzzball’.” [28]

Thus with gravitational wave astronomy it has become possible to make progress in also fundamental physics, testing General Relativity (GR) in extreme regimes and in particular to discriminate between GR Black Holes (BHs) and Exotic Compact Objects (ECOs) or Fuzzballs appearing in Modified Theories of Gravity or String Theory. This is possible importantly again by analysing the Quasinormal Modes. At later ringdown stages, ECOs and fuzzballs produce a peculiar train of echoes, probing their internal cavity and not only their external walls, with significant deviations from GR. The crucial role played by QNMs in discriminating BHs from fuzzballs or other ECOs motivated renewed effort in their determination with higher and higher accuracy [7, 9, 29].

3.2. Black hole perturbation theory

The Einstein-Hilbert action for a d-dimensional spacetime with cosmological constant Λ is

$$S = \frac{1}{16\pi G}\int d^d x\,\sqrt{-g}(R - 2\Lambda + \mathcal{L}_m) \tag{3.1}$$

where $g = \det g_{\mu\nu}$, R is the Ricci scalar and $\mathcal{L}_m$ is the Lagrangian for the matter fields coupled to gravity. It gives rise to the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.2}$$

and must be supplemented by the equation of motion for the matter fields Φ. In general these equations form a complicated system of non-linear partial differential equations (PDEs). However, they can be greatly simplified into linear equations in the approximation of small perturbations $h_{\mu\nu}$, ϕ of the background fields $g^{BG}_{\mu\nu}$, Φ^{BG}: $g_{\mu\nu} = g^{BG}_{\mu\nu} + h_{\mu\nu}$, $\Phi = \Phi^{BG} + \phi$ [27].

Maximally symmetric vacuum solutions are Minkovski, de Sitter (dS) and anti-de Sitter (AdS) spacetimes. AdS spacetimes arise also as natural groundstates of supergravity theories and as near-horizon geometry of extremal BHs and p-branes in string theory. The non-rotating, uncharged, Schwarschild AdS (SAdS) BH has the line element [27]

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2_{d-2} \tag{3.3}$$

where $d\Omega^2_{d-2}$ is the metric of the $(d-2)$ sphere, $f(r)$ is

$$f(r) = 1 + \frac{r^2}{L^2} - \frac{r_0^{d-3}}{r^{d-3}} \tag{3.4}$$

where L is the AdS curvature radius related to the cosmological constant as

$$L^2 = -\frac{(d-2)(d-1)}{2\Lambda} \tag{3.5}$$

and the parameter r_0 is related to the mass M as

$$M = \frac{(d-2)A_{d-2}r_0^{d-3}}{16\pi}\,, \quad A_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \tag{3.6}$$

3.2.1. Scalar perturbations

To obtain the physical quasinormal associated to gravitational waves, it is necessary to study tensor perturbations of Einstein's field equations. In that way, after linearization and choice of gauge, one reduces Einstein's 10 coupled non-linear PDEs to just 2 linear ODEs: the Regge-Wheeler and Zerilli equation [30]. However, for more purely theoretical investigations it is often considered the simplified though completely analogous problem with scalar perturbations, in which the physical equation of interest is the Klein-Gordon equation in curved spacetime [7].

The Lagrangian for a complex scalar field with conformal comping γ is

$$\mathcal{L}_m = -(\partial_\mu \Phi)^\dagger \partial \mu \Phi - \frac{d-2}{4(d-1)}\gamma R \Phi^\dagger \Phi - m^2 \Phi^\dagger \Phi \tag{3.7}$$

For $\gamma = 1, m = 0$ the action is invariant under the conformal transformations $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $\Phi \to \Omega^{1-d/2}\Phi$. Consider a massless scalar $m = 0$. The equations of motion are

$$\nabla_\mu \nabla^\mu \Phi = \frac{d-2}{4(d-1)}\gamma R \Phi\,, \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.8}$$

The equations for the linear perturbations $h_{\mu\nu}$ and ϕ decouple. The scalar fluctuation satisfies

$$\frac{1}{\sqrt{-g_{BG}}}\partial_\mu \left(\sqrt{-g_{BG}} g_{BG}^{\mu\nu} \partial_\nu \phi\right) = \frac{d(d-2)\gamma}{4L^2}\phi \tag{3.9}$$

With a stationary and spherically symmetric metric the perturbation decomposes in spherical harmonics Y_{lm}

$$\phi(t, r, \theta) = \sum_{l,m} e^{-i\omega t} \frac{\Psi_{s=0}(r)}{r^{(d-2)/2}} Y_{lm}(\theta) \tag{3.10}$$

where we have omited the integral over frequency in the Fourier transform. The equation for the radial part $\Psi_{s=0}(r)$ is

$$f^2 \frac{d^2\Psi_{s=0}}{dr^2} + f f' \frac{d\Psi_{s=0}}{dr} + (\omega^2 - V_{s=0})\Psi_{s=0} = 0 \tag{3.11}$$

with potential

$$V_{s=0} = f\left[\frac{l(l+d-3)}{r^2} + \frac{d-2}{4}\left(\frac{(d-4)f}{r^2} + \frac{2f'}{r} + \frac{d\gamma}{L^2}\right)\right]. \tag{3.12}$$

By introducing the "tortoise" coordinate r_*

$$dr_* = \frac{1}{f} dr \tag{3.13}$$

such that the horizon $r \to r_+$ is at $r_* \to -\infty$ and infinity at $r^* \to \infty$ or $r^* \to$ const. for respectively flat or SAdS spacetime, we can reduce the equation to canonical form [27]

$$\frac{d^2\Psi_{s=0}}{dr_*^2} + (\omega^2 - V_{s=0})\Psi_{s=0} = 0 \tag{3.14}$$

3.3. Mathematical definition of quasinormal modes

We recall the definition of quasinormal modes following [30]. A linear perturbation of a BH is a solution $\Phi(t,x)$ of some linear PDE derived from the equations for the fields and metric. It has the form

$$\left\{+\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial w^2} + U(w)\right\}\Phi(t,w) = 0\,, \tag{3.15}$$

where w here is a coordinate (the "tortoise" coordinate) such that the BH horizon is put at $w \to -\infty$ and spacetime infinity at $w \to +\infty$. If we take the Laplace transform of Φ

$$\hat{\Psi}(s,w) = \int_0^\infty e^{-st}\Phi(t,w)\,dt\,, \tag{3.16}$$

then $\hat{f}$ satisfies the non-homogeneous ODE:

$$\left\{-\frac{\partial^2}{\partial w^2} + U(w) + s^2\right\}\hat{\Psi}(s,w) = -\mathcal{I}(s,w)\,, \tag{3.17}$$

with the non-homogeneous term given by the initial time values of the perturbation as

$$\mathcal{I}(s,w) = -s\Psi(t,w)\bigg|_{t=0} - \frac{\partial\Psi(t,w)}{\partial t}\bigg|_{t=0}\,. \tag{3.18}$$

The corresponding homogeneous equation is exactly the ODE we are going to study in the next sections

$$\left\{-\frac{\partial^2}{\partial w^2} + U(w) + s^2\right\}\Psi(s,w) = 0\,. \tag{3.19}$$

Its solutions bounded at $w \to \pm\infty$, for $\Re s > 0$, are

$$\begin{aligned}\Psi_+(s,w) &\sim e^{-sw}\,, \qquad w \to +\infty\\ \Psi_-(s,w) &\sim e^{sw}\,, \qquad w \to -\infty\,.\end{aligned} \tag{3.20}$$

The solution of the homogenous equation is then found to be given by the Green function G as

$$\hat{\Psi}(s,w) = \int_{-\infty}^\infty G(s,w,w')\mathcal{I}(s,w')dw'\,, \qquad G(s,w,w') = \frac{1}{W[\Psi_-,\Psi_+]}\Psi_-(s,w_<)\Psi_+(s,w_>)\,, \tag{3.21}$$

with

$$w_< = \text{min}(w',w)\,, \quad w_< = \text{max}(w',w)\,. \tag{3.22}$$

Then taking the antiplace transform of $\hat{\Psi}$

$$\Phi(t,w) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st}\hat{\Psi}(s,w)\,ds\,, \tag{3.23}$$

we get the original perturbation as

$$\begin{aligned}
\Phi(t,w) &= \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st}\int_{-\infty}^{\infty} G(s,w,w')\mathcal{I}(s,w')\,dw'\,ds \\
&= \frac{1}{2\pi i}\oint e^{st}\frac{1}{W(s)}\int_{-\infty}^{\infty}\Psi_-(s,w_<)\Psi_+(s,w_>)\mathcal{I}(s,w')\,dw'\,ds \\
&= \sum_q e^{s_q t}\text{Res}\left(\frac{1}{W(s)}\right)\bigg|_{s_q}\int_{-\infty}^{\infty}\Psi_-(s_q,w_<)\Psi_+(s_q,w_>)\mathcal{I}(s_q,w')\,dw'\,.
\end{aligned} \tag{3.24}$$

The crucial point for us is that the perturbation is a sum over the residues of the inverse wronskian of the regular solutions (3.20):

$$W(s_n) = W[\Psi_+,\Psi_-] = 0\,. \tag{3.25}$$

Besides, condition (3.25) means that at these special points the two solutions (3.20) (in general independent) become linearly dependent. By setting $s = i\omega$ we recover the usual intuitive definition of QNMs as the frequencies of plane wave solutions both incoming at the horizon and outgoing at infinity. However, as well explained in [30], this last definition is not mathematically rigorous, since it would lead to diverging boundary conditions. Instead, the QNMs ω_n have an imaginary part $\Im\omega_n < 0$, so that the perturbation they describe is damped to zero as $t \to \infty$.

3.4. Methods of computation of quasinormal modes

We report here two of the main methods of computation of QNMs. The first is approximate, the second is exact.

3.4.1. The WKB approximation

As is typical of normal modes of vibration of any object, also quasinormal modes of black holes can be thought of as waves travelling around the BH. More precisely, QNMs can be interpreted as waves rapped at the unstable null circular geodesic (called ”light-ring”) and slowly leaking out [27].

This intuitive idea is related to the more rigorous WKB approximation procedure for computing QNMs. Indeed in that approach one expands the coefficient $Q = \omega^2 - V$ of the ODE for the perturbation

$$\frac{d^2\Psi}{dr_*^2} + Q\Psi = 0 \tag{3.26}$$

around the extremum of the potential r_0, which also defines the light ring. In this approximation one gets the ODE

$$\frac{d^2\Psi}{dr_*^2} + \left[Q_0 + \frac{1}{2}Q_0''(r_* - r_0)^2\right]\Psi = 0 \tag{3.27}$$

where Q_0 stands for $Q(r_0)$. (3.27) has the parabolic cylinder form

$$\frac{d^2w}{dz^2} + (\nu + \frac{1}{2} - \frac{1}{4}z^2)w = 0 \tag{3.28}$$

So the exact solution of (3.27) is in terms of the parabolic cylinder functions D_ν, $D_{-\nu-1}$

$$\Psi = AD_\nu(z) + BD_{-\nu-1}(iz) \tag{3.29}$$

with

$$z = \sqrt[4]{-2Q_0''}(r_* - r_0) \qquad \nu = -i\frac{Q_0}{\sqrt{2Q_0''}} - \frac{1}{2} \tag{3.30}$$

Asymptotically expanding for $z \to \infty$ we get

$$\Psi \sim Ae^{-i\pi\nu}z^\nu e^{-\frac{z^2}{2}} - i\sqrt{2\pi}A[\Gamma(-\nu)]^{-1}e^{5i\pi/4}z^{-\nu-1}e^{\frac{z^2}{2}} \tag{3.31}$$

QNMs boundary conditions imply that the term proportional to $e^{\frac{z^2}{2}}$, corresponding to outgoing waves at infinity, should be absent. So

$$\frac{1}{\Gamma(-\nu)} = 0 \tag{3.32}$$

that is

$$\frac{Q_0}{\sqrt{2Q_0''}} = i\left(n + \frac{1}{2}\right) \qquad n \in \mathbb{N} \tag{3.33}$$

This relation defines the QNMs in the WKB approximation. We notice that it appears like a "Bohr-Sommerfeld quantization rule" in the old quantum theory [27]. In particular the QNMs turn out to be given by

$$\omega \simeq \omega_0 - i(2n+1)\lambda_0 \tag{3.34}$$

where ω_0 is the root together with r_0 of the system

$$Q(\omega_0, r_0) = 0 \qquad \partial_r Q(\omega_0, r_0) = 0 \tag{3.35}$$

and λ_0 is the term corresponding to the quantization condition (3.33)

$$\lambda_0 = \frac{Q_0}{\sqrt{Q_0''}}\,. \tag{3.36}$$

It turns out λ_0 is the Lyapunov exponent governing the chaotic behaviour of nearly critical geodesics around it [7].

The WKB approximation works best for low overtones n (with small imaginary part ω_I) and in the limit of large l (which corresponds to large ω_R/ω_I). This method also assumes that the potential has a single extremum [27].

3.4.2. The continued fraction (Leaver) method

The continued fraction method by Leaver has been regarded the most successful algorithm to compute QNMs. It is based on the observation that the Teukolsky equation for the perturbation of Kerr BHs is a special case of the spheroidal wave equations that appear in the calculation of the electronic spectra of the hydrogen molecule ion. These equations are characterized by the fact that their solution near the horizon can be expanded in power series with coefficients that satisfy a three-term recursion relation. The boundary condition at infinity which defines QNMs is also satisfied when the series is absolutely convergent and that imposes a particular continued fraction condition on the terms of the recursion which gives the QNMs [27].

Let us consider for illustration the Schwarshild BH (with $2M = 1$) . The perturbation equation is the Regge-Wheeler equation

$$r(r-1)\frac{d^2}{dr^2} + \frac{d}{dr}\psi + \left[\frac{\omega^2 r^3}{r-1} - l(l+1) + \frac{s}{r}\right] \tag{3.37}$$

The boundary conditions for QNMs are

$$\psi \to (r-1)^{-i\omega} \qquad \psi \to r^{i\omega} e^{i\omega r} \tag{3.38}$$

The solution which has the desired behaviour at the event horizon can be expanded in power series as

$$\psi = (r-1)^{-i\omega} r^{2i\omega} e^{i\omega(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r}\right)^n \tag{3.39}$$

The coefficients a_n satisfy the three term recursion relation

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \qquad n = 1, 2, \ldots \tag{3.40}$$

with initial condition

$$\alpha_0 a_1 + \beta_0 a_0 = 0\,. \tag{3.41}$$

and where we defined

$$\begin{aligned} \alpha_n &= n^2 + (-2i\omega + 2)n - 2i\omega + 1 \\ \beta_n &= -[2n^2 + (-8i\omega + 2)n - 8\omega^2 - 4i\omega + l(l+1) - s] \\ \gamma_n &= n^2 - 4i\omega n - 4\omega^2 - s - 1 \end{aligned} \tag{3.42}$$

The boundary condition at spatial infinity will be satisfied by those values of ω for which the series for the solution is absolutely convergent. It can be proven that happens if the ration of successive a_n is given by the infinite continued fraction

$$\frac{a_{n+1}}{a_n} = -\frac{\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} - \ldots}}} \tag{3.43}$$

This equation can be though as an "$n = \infty$ boundary condition" for the sequence a_n and we obtain a characteristic equation for QNMs by evaluating it also at $n = 0$ (so an "$n = 0$ boundary condition"). In particular for $n = 0$ we get by (3.41)

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 - \ldots}}} \tag{3.44}$$

which determines the basic overtone ω_0. Higher overtones are obtained by inversion of (3.44) n times

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-\ldots-\frac{\alpha_0\gamma_1}{\beta_0}}} = \frac{\alpha_n\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-\frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-\ldots}}} \tag{3.45}$$

For every $n > 0$ are equivalent in the sense that every solution of (3.44) is also a solution of (3.45) and viceversa. So either one may be taken as defining ω_n. The problem is reduced to solving albegraic equation and ssually ω_n is found to be the most stable root of the n-th inversion [31].

We notice that this method assumes that the ODE for the perturbation has two regular and one irregular singularities. This happens in particular for the Confluent Heun equation (CHE) (see appendix D). If the ODE has two irregular singularities and no regular singularities as the Doubly Confluent Heun equation (DCHE) which will be the case for most of the model we are going to study, then it should be first mapped in the CHE as explained in [7, 9].

3.5. Quasinormal modes in AdS/CFT and holography

The AdS/CFT correspondence was originally formulated between type IIB string theory on the product space $AdS_5 \times S_5$ and $\mathcal{N} = 4$ supersymmetric gauge theory (which is a Conformal Field Theory, CFT) [32]. Later it has been extended much further, so that is called more generically *holographic correspondence*. In particular, it provides a method for an effective description of a non-perturbative, strongly coupled regime of certain gauge theories in terms of higher dimensional classical gravity.

Quasinormal spectra of the dual gravitational backgrounds give the location (in momentum space) of the poles of the retarded correlators in the gauge theory. This is a standard tool to study the near-equilibrium behavior of gauge theory plasmas with a dual gravity description [27].

3.6. From gauge to gravity and back

In the last two years, a surprising connection between $\mathcal{N} = 2$ $SU(2)$ gauge theories NS deformed and black holes (BHs) perturbation theory has emerged [6]. It was found first that (Bohr-Sommerfeld like) quantisations conditions on quantum gauge periods a_D, a provide a new analytic exact characterisation of quasinormal modes (QNMs)[1] and could be practically used to also compute them [6]. Thank to this and exploting the AGT duality [33, 34] between four dimensional $\mathcal{N} = 2$ gauge theories and two dimensional Conformal Field Theories (CFTs), also the latter kind of theories found applications to BHs [8][2]. For instance thus were made new computations of other BHs observables such the greybody

[1] QNMs are the characteristic frequencies of the gravitational wave signal in ringdown (after merging) phase.

[2] These CFTs are different from ours. In fact, we relate to $N_f = 0$ gauge theory the $c = 25$ self-dual Liouville, rather then the $c \to +\infty$ Liouville as AGT does for the NS limit [33]. Further investigations on the relation between such two Liouville models would be interesting.

factor and Love numbers[3], sometimes also more accurate [8, 35, 36]. From these many other applications and new results followed, like for instance

- an isospectral simpler equation to the perturbation ODE [37];
- improved theoretical proofs of BHs stability [38];
- a simpler interpretation of Chandrasekhar transformation as exchange of gauge mass parameters [39];
- precise determination of the conditions of invariance under (Couch-Torrence) transformations which exchange inner horizon and null infinity [40];
- an exact formula for the thermal scalar two-point function in four-dimensional holographic conformal field theories [41].

Moreover, we emphasise that the BHs which can be studied through this approaches are also very 'real' (for instance, the Schwarschild and Kerr BHs) and enter astrophysics and gravitation phenomenology [6, 29]. For instance, if real BHs possesed horizon-scale structure, forbidden by General Relativity (GR) but allowed by modified theories of gravity or String Theory, it would manifest itself as echoes in the gravitational wave signal in the later ringdown phase and would be accessible to future higher precision detectors [42, 7]. An explanation of this correspondence has been constructed in a rather general case [8] by exploiting another correspondence between $\mathcal{N} = 2$ gauge theory and Conformal Field Theory [33]. However, we are going to show that it is possible to explain this so-called *SW-QNM correspondence* [9] by analysing closely the Ordinary Differential Equations (ODEs) describing the perturbations in gravitational physics. We are able to do this on the basis of our previous works [1, 43], where we have connected the $\mathcal{N} = 2$ gauge theories to quantum integrable theories, in particular the gauge periods to the Baxter's Q and T functions. To this aim we have started from the ODEs characterizing the periods and developed further the elegant ODE/IM correspondence between ODEs and Integrable Models (IM) [2, 3, 4, 44].

[3]The greybody factor, or absorption coefficient, is associated to Hawking radiation, while Love numbers describe tidal deformations of BHs.

4. $SU(2)\ N_f = 0$ gauge theory, Liouville model, D3 brane

4.1. Liouville ODE/IM

The original ODE/IM correspondence establishes an exact parallel between a particular Schrödinger equation (ODE) and ground state of the Minimal Conformal models (IM) [2, 3], without masses (cf. also [45], and see [46, 47] for the correspondence with the excited states). In particular, the equation used is the following:

$$\left\{ -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + x^{2M} - E \right\} \phi(x) = 0 \,, \tag{4.1}$$

where $M > 0$ is related to the central charge c as

$$M = \beta^{-2} - 1 \qquad c = 1 - 6(\beta - \beta^{-1})^2 \tag{4.2}$$

and l is related to the conformal dimension Δ as

$$p = \frac{2l+1}{4M+4} \qquad \Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24} \,. \tag{4.3}$$

Let ϕ_∞ denote the solution which has the subdominant asymptotic at the irregular singular point $x \to \infty$

$$\phi_\infty(x) \simeq x^{-M/2} \exp\left(-\frac{1}{M+1} x^{M+1} \right) \qquad x \to \infty \tag{4.4}$$

and let ϕ_0 denote the solution which has the power law behaviour at the regular singular point $x \to 0$

$$\phi_0(x) \simeq x^{-l} \qquad x \to 0 \,. \tag{4.5}$$

Then, the Baxter's Q function for the minimal models can be defined as the wronksian

$$Q_- = W[\phi_\infty, \phi_0] \,, \tag{4.6}$$

or, alternatively, as the limit [2]

$$Q_- = \lim_{x \to 0} \left[(2l+1) x^l \phi_\infty(x) \right] \,. \tag{4.7}$$

Later, the ODE/IM correspondence was extended to the massive ground state [48].

Although there was already a bold suggestion already in [4], the conjecture for the (conformal) Liouville field theory ($M < -1$) came only in a brilliant draft paper [49] by the late scholar Al. B. Zamolodchikov and takes the form of the *Generalized Mathieu equation* (GME):

$$\left\{ -\frac{d^2}{dy^2} + e^{\frac{\alpha+y}{b}} + e^{b(\alpha-y)} + P^2 \right\} \psi(y) = 0 \,. \tag{4.8}$$

The parameters b and P are respectively the *Liouville coupling* and *momentum* and express the central charge

$$c = \ 1 + 6(b + b^{-1})^2 \tag{4.9}$$

and the conformal weight

$$\Delta = (c-1)/24 - P^2\,. \tag{4.10}$$

Liouville field theory enjoys a duality symmetry for $b \to 1/b$ (*self-dual* point $b = 1$). Following [4], we may immagine that this equation could be obtained *heuristically* from the ODE/IM equation (4.1) for the minimal models, through some continuation in $\beta = ib$ and some transformation on the independent variable (crucially, the Langer transform).

However, we found the form (4.8) not adequate for the large rapidity expansion, as e^α appears with two different powers. We have solved this problem by the shift $y \to y + \alpha\frac{b-1/b}{b+1/b}$, after which the GME acquires the modified Schrödinger form:

$$\left\{-\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2\right\}\psi(y) = 0 \tag{4.11}$$

with the rapidity θ defined as $\theta = \alpha/(b + b^{-1})$.

4.1.1. Heuristic derivation of Generalized Mathieu equation

In particular, we apply a succession of transformations, the first of which is the Langer transform: $x = e^{\hat{y}}$, $\phi(x) = e^{\hat{y}/2}\psi(\hat{y})$, so that the equation becomes

$$\left\{-\frac{d^2}{d\hat{y}^2} + e^{2\hat{y}/\beta^2} - Ee^{2\hat{y}} + \left(l + \frac{1}{2}\right)^2\right\}\psi(\hat{y}) = 0\,. \tag{4.12}$$

Now, we continue to transform, passing to the variables $\hat{y} = \frac{\beta}{2}\bar{y}$

$$\left\{-\frac{d^2}{d\bar{y}^2} + \frac{\beta^2}{4}e^{\bar{y}/\beta} - \frac{\beta^2}{4}Ee^{\beta\bar{y}} + \frac{\beta^2}{4}\left(l + \frac{1}{2}\right)^2\right\}\psi(\bar{y}) = 0 \tag{4.13}$$

and also $\bar{y} = i\tilde{y}$

$$\left\{+\frac{d^2}{d\tilde{y}^2} + \frac{\beta^2}{4}e^{i\tilde{y}/\beta} - \frac{\beta^2}{4}Ee^{i\beta\tilde{y}} + \frac{\beta^2}{4}\left(l + \frac{1}{2}\right)^2\right\}\psi(\tilde{y}) = 0\,. \tag{4.14}$$

Now we send $\beta = ib$, with $b > 0$, that is, give imaginary values to β. The ODE/IM equation for the minimal models is thus transformed into the equation for the Liouville model. We also define a new parameter

$$P^2 = \frac{p^2}{b^2} = \frac{(l+\frac{1}{2})^2 b^2}{4}\,. \tag{4.15}$$

Then the equation becomes

$$\left\{-\frac{d^2}{d\tilde{y}^2} + \frac{b^2}{4}e^{\tilde{y}/b} - \frac{b^2}{4}Ee^{-b\tilde{y}} + P^2\right\}\psi(\tilde{y}) = 0\,. \tag{4.16}$$

A final change of variable $y = \tilde{y} - \alpha + b\ln\frac{b^2}{4}$ and of parametrization $\alpha = \frac{\ln(-E)}{2b} + \frac{b^2+1}{2b}\ln\frac{b^2}{4}$ delivers now the Generalized Mathieu Equation (4.8). The parameter α is related to the

TBA rapidity, as we explain below. We observe also that the initial variable x can be conveniently expressed in terms of the final variable y as

$$x = \exp\left[-\frac{yb}{2} - \frac{\alpha b}{2} + \frac{b^2}{2}\ln\frac{b^2}{4}\right] \tag{4.17}$$
$$\frac{\phi(x)}{\sqrt{x}} = \psi(y) \tag{4.18}$$

This shows that $x \to 0$ corresponds to $\Re y \to +\infty$, while $x \to +\infty$ corresponds to $\Re y \to -\infty$.

4.1.2. Functional relations

In the rest of this Section we will summarise our understanding of draft paper [49] by using the GME (4.11). It has the subdominant asymptotic solutions: for $\Re y \to +\infty$, within $|\Im(\theta + \frac{y}{2b})| < \frac{3}{2}\pi$ and for $\Re y \to -\infty$, within $|\Im(\theta - \frac{by}{2})| < \frac{3}{2}\pi$, respectively

$$U_0(y) \simeq \frac{1}{\sqrt{2}}\exp\Big\{-\theta/2 - y/4b\Big\}\exp\Big\{-2be^{\theta+y/2b}\Big\} \qquad \Re y \to +\infty\ ; \tag{4.19}$$
$$V_0(y) \simeq \frac{1}{\sqrt{2}}\exp\Big\{-\theta/2 + yb/4\Big\}\exp\Big\{-\frac{2}{b}e^{\theta-yb/2}\Big\} \qquad \Re y \to -\infty\ . \tag{4.20}$$

Other solutions can be generated applying on these the following discrete symmetries of the GME (4.11)

$$\Lambda_b : \theta \to \theta + i\pi\frac{b}{q} \qquad y \to y + \frac{2\pi i}{q} \quad , \quad \Omega_b : \theta \to \theta + i\pi\frac{1}{bq} \qquad y \to y - \frac{2\pi i}{q} \tag{4.21}$$

where $q = b + 1/b$: concisely $U_k = \Lambda_b^k U_0$ and $V_k = \Omega_b^k V_0$, with U_k invariant under Ω_b and V_k under Λ_b. We may interpret this phenomenon as a spontaneous symmetry breaking for the differential equation (vacua are the solutions). Now we apply these (broken) symmetries to derive interesting functional and integral equations for the gauge theory. On the other hand, the symmetry $\Pi : \theta \to \theta + i\pi$ would not do the same job in the present case with two irregular singularities as it transforms simultaneously $U_0 \to U_1$ and $V_0 \to V_1$ (differently from [50] and [51] with only one irregular singularity, see also [52] for a detailed examination of the two kinds of symmetries).

In fact, we will prove correct (as conjectured by [49]) to define the Baxter's Q function as the wronskian

$$Q(\theta, P^2) = W[U_0, V_0]\ . \tag{4.22}$$

We can say that the dependence of Q is on the *square* of P, because equation (4.11) is invariant inverting the sign of P and also the boundary conditions (4.19) and (4.20) are invariant. Notice, however, that in all the functional relations below P^2 is fixed. Definition (4.22) gives rise to $Q(\theta + i\pi p) = W[U_1, V_0](\theta)$ upon action of Λ_b: these are equivalent to the linear dependence

$$iV_0(y) = Q(\theta + i\pi p)U_0(y) - Q(\theta)U_1(y)\,, \tag{4.23}$$

where $p = b/q$ (from the asymptotic calculation $W[U_1, U_0] = i$). Which is transformed by Ω_b into

$$iV_1(y) = Q(\theta + i\pi)U_0(y) - Q(\theta + i\pi(1-p))U_1(y)\,, \tag{4.24}$$

namely $Q(\theta + i\pi(1-p)) = W[U_0, V_1](\theta)$ and $Q(\theta + i\pi) = W[U_1, V_1](\theta)$. The basilar functional relation (anticipated for the massive theory by other means in [53]), the QQ relation is obtained by taking the wronskian $W[V_0, V_1]$ ($= i$ from asymptotics) between the right hand sides

$$1 + Q(\theta + i\pi(1-p))Q(\theta + i\pi p) = Q(\theta + i\pi)Q(\theta)\,. \tag{4.25}$$

If we define the two (dual) T functions as

$$T(\theta) = Q(\theta - i\pi p)Q(\theta + i\pi) - Q(\theta + i\pi p)Q(\theta + i\pi(1-2p))\,, \quad \tilde{T}(\theta) = T(\theta)\Big|_{b\to 1/b}\,, \tag{4.26}$$

(also $T = i\,W[U_{-1}, U_1]$ and $\tilde{T} = -i\,W[V_{-1}, V_1]$) by using the QQ relation (4.25), these two Baxter's TQ relations follow

$$T(\theta)Q(\theta) = Q(\theta + i\pi p) + Q(\theta - i\pi p) \qquad \tilde{T}(\theta)Q(\theta) = Q(\theta + i\pi(1-p)) + Q(\theta - i\pi(1-p))\,, \tag{4.27}$$

as well as the periodicity of T [49]

$$T(\theta + i\pi(1-p)) = T(\theta) \qquad \tilde{T}(\theta + i\pi p) = \tilde{T}(\theta)\,. \tag{4.28}$$

We make now some comparison between the functional relations of the Liouville model and those of the minimal models. For the minimal models Q_- is the wronskian between the eigenfunctions defined by the asymptotic at 0 and $+\infty$ in x, as in (4.6). This property is kept for the Liouville model, in (4.22), since by (4.17) $x = 0$ corresponds to $y = +\infty$ and $x = +\infty$ corresponds to $y = -\infty$. Besides, for the minimal models there is only one TQ system, while for the Liouville model there are two different TQ systems. This is because, essentially, in the Langer variable y, $+\infty$ and $-\infty$ are symmetrical, that is, the eigenfunctions have analogous form. Accordingly, for the minimal models there is only one T function, while for the Liouville model there are two T functions. However, for the Liouville model there only one Q function, while for the minimal models there are actually two $Q_\pm$ functions, which are obtained through the action of the symmetry Λ_{MM} [3] which sends $p \to -p$. The two symmetries used in the ODE/IM construction for the range $\beta^2 > 0$ (minimal models) are very different: Ω_{MM} [3] acts on the solutions at $x \to +\infty$ only through x; while Λ_{MM} [3] acts on the solutions at $x \to 0$ only through l. Now, the the two symmetries used for the range $\beta^2 < 0$ (Liouville model) are very similar: both Λ_b and Ω_b act on the solutions at $y \to \pm\infty$ through y and θ. In the Liouville model, $P^2 \to P^2$ under the minimal models symmetry Λ_{MM} (cf. (4.15)).

4.1.3. Perturbative limit

In the limits $y \to +\infty$ and $y \to -\infty$, the GME (4.11) reduces to the approximate equations, respectively:

$$\left\{-\frac{d^2}{dy^2} + e^{2\theta + y/b} + P^2\right\} U_0(y) \simeq 0 \qquad y \to +\infty\,, \tag{4.29}$$

$$\left\{-\frac{d^2}{dy^2}+e^{2\theta-by}+P^2\right\}V_0(y)\simeq 0 \qquad y\to-\infty\,. \tag{4.30}$$

By the changes of variables $u=2be^{\theta+y/2b}$ and $v=2/be^{\theta-by/2}$, we see that these equations are Modified Bessel equations:

$$\left\{u^2\frac{d^2}{du^2}+u\frac{d}{du}-(2bP)^2-u^2\right\}U_0(u)\simeq 0 \qquad u\to+\infty\,, \tag{4.31}$$

$$\left\{v^2\frac{d^2}{dv^2}+v\frac{d}{dv}-(2P/b)^2-v^2\right\}V_0(v)\simeq 0 \qquad v\to+\infty\,. \tag{4.32}$$

From the asymptotics (4.19) and (4.20), it follows that the basic solutions U_0 and V_0 correspond the modified bessel functions as

$$U_0(u)\simeq\sqrt{\frac{2b}{\pi}}K_{2bP}(u) \qquad u\to+\infty\,, \tag{4.33}$$

$$V_0(v)\simeq\sqrt{\frac{2}{\pi b}}K_{2P/b}(v) \qquad v\to+\infty\,. \tag{4.34}$$

In the perturbative limit $\theta\to-\infty$, the approximate equations become the same:

$$\left\{-\frac{d^2}{dy^2}+P^2\right\}U_0(y)\simeq 0 \qquad \theta\to-\infty\,, \tag{4.35}$$

$$\left\{-\frac{d^2}{dy^2}+P^2\right\}V_0(y)\simeq 0 \qquad \theta\to-\infty\,. \tag{4.36}$$

and then we are justified in combining the solutions of both equations for each $y\in\mathbb{R}$. The modified-Bessel function $K_\nu(x)$ behaves, as $x\to 0$ as

$$K_\nu\left(x\right)=\frac{\Gamma(\nu)}{2^{1-\nu}}x^{-\nu}+\frac{\Gamma(-\nu)}{2^{1+\nu}}x^{\nu}+O\left(x^2\right) \tag{4.37}$$

therefore the U_0 and V_0 solutions are approximately equal to

$$U_0\simeq\frac{\sqrt{b}}{\sqrt{2\pi}}\left[b^{2bP}\Gamma(2bP)e^{-2b\theta P}e^{-Py}+b^{-2bP}\Gamma(-2bP)e^{2b\theta P}e^{Py}\right] \tag{4.38}$$

$$V_0\simeq\frac{1}{\sqrt{2\pi b}}\left[b^{-2P/b}\Gamma(2P/b)e^{-2\theta P/b}e^{Py}+b^{2P/b}\Gamma(-2P/b)e^{2\theta P/b}e^{-Py}\right] \tag{4.39}$$

and their wronskian (4.22), the Q function, is approximately equal to

$$Q(\theta,b,P^2)\simeq\frac{1}{2\pi}\left\{\frac{\Gamma(1+2P/b)\Gamma(2Pb)}{b^{-1+2bP-2P/b}}e^{-2qP\theta}+\frac{\Gamma(1-2bP)\Gamma(-2P/b)}{b^{1-2bP+2P/b}}e^{2qP\theta}\right\} \tag{4.40}$$

4.1.4. Thermodynamic Bethe Ansatz

Also the Liouville Y-system can be obtained from the QQ-system, by defining $Y(\theta) = Q(\theta + i\pi a/2)Q(\theta - i\pi a/2)$, where $a = 1 - 2p$

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \Big(1 + Y(\theta + ia\pi/2)\Big)\Big(1 + Y(\theta - ia\pi/2)\Big). \tag{4.41}$$

This functional equation can be inverted into the Thermodynamic Bethe Ansatz (TBA) equation for the logarithm $\varepsilon(\theta) = -\ln Y(\theta)$, the pseudoenergy, in the integral form

$$\varepsilon(\theta) = \frac{8\sqrt{\pi^3}\,q}{\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})}e^{\theta} - \int_{-\infty}^{\infty}\left[\frac{1}{\cosh(\theta - \theta' + ia\pi/2)} + \frac{1}{\cosh(\theta - \theta' - ia\pi/2)}\right]\ln\left[1 + \exp\{-\varepsilon(\theta')\}\right]\frac{d\theta'}{2\pi}\,, \tag{4.42}$$

where the coefficient of the forcing term (zero-mode) is fixed by the leading order of Q below, (4.96). This TBA equation goes into that in [49, 53, 54] upon a real shift on θ:

$$\theta \to \theta + \ln\frac{8\sqrt{\pi}q}{\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})}\,. \tag{4.43}$$

The Liouville TBA can be derived as a massless limit of the one concerning Sinh-Gordon [53]:

$$\varepsilon(\theta) = mR\cosh\theta - \int_{-\infty}^{\infty} d\theta'\,\varphi(\theta - \theta')\ln\left[1 + e^{-\varepsilon(\theta')}\right], \tag{4.44}$$

where

$$\varphi(\theta) = \frac{1}{2\pi}\left[\frac{1}{\cosh(\theta + i\pi a/2)} + \frac{1}{\cosh(\theta - i\pi a/2)}\right]. \tag{4.45}$$

Here, though, we wish to show its arising from the Stokes relations of the Schrödinger equation: the Q system (4.25) or the equivalent Y system (4.41). Boundary conditions must also be fixed in order for the TBA to be uniquely determined. We begin by making a shift of $-i\pi/2$ on the Q system (4.25).

$$Q(\theta - i\pi/2)Q(\theta + i\pi/2) = 1 + Q(\theta + i\pi a/2)Q(\theta - i\pi a/2)\,. \tag{4.46}$$

Now define $Y(\theta)$ as

$$Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Y(\theta) \tag{4.47}$$

and note that such a definition of $Y(\theta)$ implies the relation

$$Y(\theta) = Q(\theta + ia\pi/2)Q(\theta - ia\pi/2)\,. \tag{4.48}$$

We now use a theorem of [55], which we report here. Let ξ be a function such that its Fourier transform $\hat{\xi}$ belongs to L^1. If we define another function χ as

$$\chi(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi(\theta')}{\cosh(\theta - \theta')}d\theta'\,, \tag{4.49}$$

then χ is bounded, analytic in the strip $|\Im\theta| < \frac{\pi}{2}$ and satisfies

$$\chi(\theta + i\pi/2) + \chi(\theta - i\pi/2) = \xi(\theta)\,, \tag{4.50}$$

for real θ. Conversely if ξ is bounded and analytic in the strip $|\Im\theta| < \frac{\pi}{2}$ and if (4.50) holds, then so does (4.49) [55]. We observe that (4.50) leaves the freedom to add to ξ a "zero-mode function" ϕ, solution of the homogeneous equation

$$\phi(\theta + i\pi/2) + \phi(\theta - i\pi/2) = 0\,. \tag{4.51}$$

A possible zero mode function is $\cosh\theta$ or $\exp\theta$. Thus, the most general expression for ξ is

$$\chi(\theta) = \phi(\theta) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi(\theta')}{\cosh(\theta-\theta')}d\theta' \tag{4.52}$$

Using this theorem, we can write the expression for $Q(\theta)$ in terms of $Y(\theta)$, starting from the definition (4.47), whose logarithm reads

$$\ln Q(\theta + i\pi/2) + \ln Q(\theta - i\pi/2) = \ln\left[1 + Y(\theta)\right], \tag{4.53}$$

which is an example of relation (4.50), if we set $\chi(\theta) = \ln Q(\theta)$ and $\xi(\theta) = \ln[1 + Y(\theta)]$. Taking $\phi(\theta) = -ce^{\theta}$ as zero mode, or boundary condition for $\theta \to \infty$, by (4.52) we get

$$\ln Q(\theta) = -ce^{\theta} + \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\ln\left[1 + Y(\theta')\right] \tag{4.54}$$

Considering $\varepsilon(\theta) = -\ln Y(\theta)$ and applying formula (4.48) we get

$$\begin{aligned}-\varepsilon(\theta) &= -ce^{\theta}[e^{i\pi a/2} + e^{-i\pi a/2}]\\ &+ \int_{-\infty}^{\infty}\frac{1}{2\pi}\left[\frac{1}{\cosh(\theta + i\pi a/2 - \theta')} + \frac{1}{\cosh(\theta - i\pi a/2 - \theta')}\right]\ln\left[1 + Y(\theta')\right]d\theta'\,.\end{aligned} \tag{4.55}$$

Choosing c such that the coefficient of the forcing term is $c_0 = 8\sqrt{\pi^3}\,q/\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})$ and recognizing the Sinh-Gordon kernel (4.45) we can write finally the Liouville TBA:

$$\varepsilon(\theta) = \frac{8\sqrt{\pi^3}\,q}{\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})}e^{\theta} - \int_{-\infty}^{\infty}\varphi(\theta-\theta')\ln\left[1 + Y(\theta')\right]d\theta'\,. \tag{4.56}$$

Notice also that Q can be written as

$$\ln Q(\theta) = -\frac{8\sqrt{\pi^3}\,q}{2\sin\pi p\,\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})}e^{\theta} + \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\ln\left[1 + Y(\theta')\right]. \tag{4.57}$$

Q is an entire function, free of zeroes inside the strip $|\Im\theta| < \pi/2 + \epsilon$, for some finite $\epsilon > 0$ [53], [56]. Zamolodchikov conjectured that the same expression could be obtained by taking the wronskian (4.22) of the solutions of the ODE. We have already proven his conjecture partially, the remaining part of the proof will be given below by fixing the boundary coefficient c directly from the equation (4.8).

In the Liouville TBA, P does not appear explicitly, but (numerically) in the asymptotic linear behaviour of $\varepsilon(\theta, P^2)$ at $\theta \to -\infty$[49], which matches the analytic computation of the wronskian (4.22) via $1 + Y(\theta) = Q(\theta + i\pi/2)Q(\theta - i\pi/2)$. In fact, from expression (4.40), we derive the asymptotic $\theta \to -\infty$ behaviour of the pseudoenergy:

$$\varepsilon(\theta, b, P) \simeq 4qP\theta - 2C(b, P) \qquad \theta \to -\infty \,, \tag{4.58}$$

with the constant

$$C(b, P) = \ln \frac{\Gamma(1 + 2P/b)\Gamma(2Pb)}{2\pi b^{-1+2bP-2P/b}} \,. \tag{4.59}$$

On the other hand from (4.41) we only know that Y must diverge.

$$Y(\theta) \simeq \left(\frac{1}{2\pi} \frac{\Gamma(1 + 2P/b)\Gamma(2Pb)}{b^{-1+2bP-2P/b}} \right)^2 e^{-4qP\theta} \,. \tag{4.60}$$

As a consequence $L = \ln[1 + e^{-\varepsilon}]$ tends to

$$L(\theta, P) \simeq -4qP\theta + 2C(b, P) \qquad \theta \to -\infty \,. \tag{4.61}$$

Since the Liouville TBA (4.42) does not depend explicitly on P, in order to solve it numerically, we must add the P dependent boundary condition (4.58) in the forcing term and subtract it in the convolution. To this end, define the functions L_0 and L_1 which reproduce for $\theta \in \mathbb{R}$ the asymptotic (4.61) as:

$$L_0 = 2qP \ln\left[1 + e^{-2\theta}\right] \,, \tag{4.62}$$

$$L_1 = C(b, P)(1 - \tanh\theta) \,. \tag{4.63}$$

In order to compute the convolutions $f_k = \varphi * L_k$of this terms with the kernel, set

$$L_k(\theta) = l_k(\theta + i\pi/2) + l_k(\theta - i\pi/2) \tag{4.64}$$

which specifically means

$$l_0(\theta) = 2qP \ln\left[1 + e^{-\theta}\right] \tag{4.65}$$

$$l_1(\theta) = \frac{C}{2}\left(1 - \tanh\frac{\theta}{2}\right) \,. \tag{4.66}$$

The convolutions are in general

$$\varphi * L_k = l_k(\theta + ia\pi/2) + l_k(\theta - ia\pi/2) \tag{4.67}$$

and in particular

$$f_0 = \varphi * L_0 = 2qP\left\{\ln\left[1 + e^{-(\theta + ia\pi/2)}\right] + \ln\left[1 + e^{-(\theta - ia\pi/2)}\right]\right\} \,, \tag{4.68}$$

$$f_1 = \varphi * L_1 = C\left[1 - \frac{1}{2}\tanh\left(\frac{\theta}{2} + \frac{i\pi a}{4}\right) - \frac{1}{2}\tanh\left(\frac{\theta}{2} - \frac{i\pi a}{4}\right)\right] \,. \tag{4.69}$$

Then the numerically solvable TBA reads

$$\varepsilon(\theta, P) = c_0 e^{\theta} - f_0 - f_1 - \varphi * (L - L_0 - L_1)\,. \tag{4.70}$$

Until now we considered real positive $P > 0$. We observe that for imaginary or complex P the asymptotic of the pseudoenergy is much more complex (cf. (4.71)). In fact letting $P^2 \to -P^2$ in (4.40) we get

$$Q(\theta, b, -P^2) \simeq K_1 e^{2iqP\theta} + K_2 e^{-2iqP\theta}\,. \tag{4.71}$$

where we defined the complex constants

$$K_1 = \frac{\Gamma(1-2ibP)\Gamma(-2iP/b)}{2\pi b^{1-2ibP+2iP/b}}\,, \qquad K_2 = \frac{\Gamma(1+2iP/b)\Gamma(2ibP)}{2\pi b^{-1+2ibP-2iP/b}}\,. \tag{4.72}$$

However, it is easy to verify the Y system QQ-system even in the case of imaginary P or complex P. The LHS of Q system reads

$$1 + Q(\theta + ia\pi/2, b, P)Q(\theta - ia\pi/2, b, P) = 1 + K_1^2 e^{4iqP\theta} + 2K_1K_2\cosh(2\pi qPa) + K_2^2 e^{-4iqP\theta}\,, \tag{4.73}$$

while the RHS reads

$$Q(\theta + i\pi/2, b, P)Q(\theta - i\pi/2, b, P) = K_1^2 e^{4iqP\theta} + K_2^2 e^{-4iqP\theta} + 2K_1K_2\cosh(2\pi qP)\,. \tag{4.74}$$

In order for the Q system to hold, it must hold that

$$\cosh 2\pi qPa + \frac{1}{2K_1K_2} = \cosh 2\pi qP\,, \tag{4.75}$$

which is in fact true

$$\begin{aligned} 2K_1K_2 &= \frac{\Gamma(1-2ibP)\Gamma(2ibP)\Gamma(1+2iP/b)\Gamma(-2iP/b)}{2\pi^2} \\ &= \frac{1}{2\sinh(2\pi bP)\sinh(2\pi P/b)} = \frac{1}{\cosh 2\pi(b+1/b)P - \cosh 2\pi(b-1/b)P} \\ &= \frac{1}{\cosh 2\pi qP - \cosh 2\pi qPa}\,. \end{aligned} \tag{4.76}$$

For complex P it is not clear how to do the procedure to set up the TBA as in (4.70). However, the gauge/integrability correspondence we are going to state permits to overcome this difficulty (see below (4.307)).

4.1.5. Self-dual case

The self-dual GME ($b = 1$ in (4.11)) is known in literature as *m*odified Mathieu equation:

$$\left\{-\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2\right\}\psi(y) = 0\,, \tag{4.77}$$

and is the non-compact version of equation (2.34), so establishing a contact with gauge theory (which importantly exhibits two irregular singularities). In particular, the discrete symmetry (4.21) is an enhanced (by the covering $y = \ln x$) version of the original $\mathbb{Z}_2$ spontaneously broken symmetry (in the x variable) of SW [57]. Because $a = 0$, the Q system simplifies into

$$Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q^2(\theta)\,, \tag{4.78}$$

while, since $T(\theta) = \tilde{T}(\theta)$, the two TQ systems reduce to a single one

$$T(\theta)Q(\theta) = Q(\theta + i\pi/2) + Q(\theta - i\pi/2) \tag{4.79}$$

and the T periodicity reads

$$T(\theta + i\pi/2) = T(\theta)\,. \tag{4.80}$$

In the self dual case (4.40) becomes

$$Q(\theta, P^2) \simeq \frac{1}{2\pi}\left[\Gamma(1+2P)\Gamma(2P)e^{-4P\theta} + \Gamma(1-2P)\Gamma(-2P)e^{4P\theta}\right] \tag{4.81}$$

and the T function is approximately equal to

$$T(\theta, P^2) \simeq 2\cos 2\pi P\,. \tag{4.82}$$

In the gauge variables (see below (4.223)) these expressions become

$$Q(\theta, u) \simeq \frac{1}{2\pi}\left[\Gamma(1+2\frac{\sqrt{2u}}{\hbar})\Gamma(2\frac{\sqrt{2u}}{\hbar})\left(\frac{\hbar}{\Lambda}\right)^{4\frac{\sqrt{2u}}{\hbar}} + \Gamma(1-2\frac{\sqrt{2u}}{\hbar})\Gamma(-2\frac{\sqrt{2u}}{\hbar})\left(\frac{\hbar}{\Lambda}\right)^{-4\frac{\sqrt{2u}}{\hbar}}\right] \tag{4.83}$$

and

$$T(\theta, u) \simeq 2\cos 2\pi\frac{\sqrt{2u}}{\hbar}\,. \tag{4.84}$$

We observe that the limit $\theta \to -\infty$ with P finite in the integrability variables corresponds to the limit $\Lambda \to 0$ with $\hbar$ and u finite in the gauge variables. Since $b = 1$, the Y function is just the square of the Q function

$$Y(\theta, P^2) = Q^2(\theta, P^2) \tag{4.85}$$

and the Y system reads

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = \Big(1 + Y(\theta)\Big)^2\,. \tag{4.86}$$

Its inversion, the TBA equation simplifies as

$$\varepsilon = \frac{16\sqrt{\pi^3}}{\Gamma^2(\frac{1}{4})}e^\theta - 2\hat{\varphi} * \ln\left(1 + e^{-\varepsilon}\right), \tag{4.87}$$

with a new simplified kernel which (because $a = 0$ at $b = 1$) is half of the former kernel (4.45)

$$\hat{\varphi}(\theta) = \frac{1}{2\pi}\frac{1}{\cosh\theta}\,. \tag{4.88}$$

Since $a = 0$ then $Q^2 = Y = \exp[-\varepsilon]$ and the TBA becomes an integral equation for the Baxter's Q function [49]

$$\ln Q(\theta) = -\frac{8\sqrt{\pi^3}}{\Gamma^2(\frac{1}{4})} e^{\theta} + \int_{-\infty}^{\infty} \frac{\ln\left[1+Q^2(\theta')\right]}{\cosh(\theta-\theta')} \frac{d\theta'}{2\pi} \,. \tag{4.89}$$

4.2. Local integrals of motion

We wish here to compute the Baxter's Q function and then the Liouville Local Integrals of Motion (LIM). About Q, (4.23) says that it can regarded as the regularised value of the solution V_0 (4.20) at $y \to +\infty$:

$$Q(\theta) = -i \lim_{y\to+\infty} \frac{V_0(y;\theta)}{U_1(y;\theta)} = \sqrt{2} e^{\frac{\theta}{2}} \lim_{y\to+\infty} e^{\frac{y}{4b} - 2be^{\theta+\frac{y}{2b}}} V_0(y;\theta) \,. \tag{4.90}$$

We can write V_0 (4.20) in terms of $\Pi(w) = -i\, d\ln(\sqrt[4]{c_b(y)} V_0(w))/dw$ in a convergent form of (B.5)

$$V_0(y;\theta) = \frac{e^{-\frac{\theta}{2}}}{\sqrt{2}\sqrt[4]{c_b(y)}} \exp\left\{ -\frac{2}{b} e^{\theta - \frac{by}{2}} + 2be^{\theta+\frac{y}{2b}} + \int_{-\infty}^{y} \left[\sqrt{c_b(y')}\Pi(y';\theta) - e^{\theta}(e^{-\frac{by'}{2}} + e^{\frac{y'}{2b}}) \right] dy' \right\} \tag{4.91}$$

where $c_b(y) = -\phi(y) = e^{y/b} + e^{-yb}$ and $dw = \sqrt{\phi(y)}dy = -i\sqrt{c_b(y)}dy$. Hence, we write an integral expression for the Q function (4.90) and its asymptotic series (denoted again by $\doteq$) for $\theta \to +\infty$, by using formula (B.26) (integrating on $\mathbb{R}$, the decaying derivatives do not contribute):

$$\ln Q(\theta) = \int_{-\infty}^{+\infty} \left[\sqrt{c_b(y)}\Pi(y) - e^{\theta}(e^{-\frac{by}{2}} + e^{\frac{y}{2b}}) \right] dy \tag{4.92}$$

$$\doteq e^{\theta} \int_{-\infty}^{\infty} \left[\sqrt{c_b(y)} - e^{\frac{y}{2b}} - e^{-\frac{by}{2}} \right] dy - \sum_{n=1}^{\infty} \frac{e^{\theta(1-2n)}}{2n-1} \int_{-\infty}^{\infty} \sqrt{c_b(y)} R_n(y)\, dy \,. \tag{4.93}$$

Notice for the future developments that $\ln Q$ is given i times the integral of the regularised momentum

$$\mathcal{P}_{reg}(y) = -i\sqrt{c_b(y)}\Pi(y) + ie^{\theta}(e^{-\frac{by}{2}} + e^{\frac{y}{2b}}) = \mathcal{P}(y) + ie^{\theta}(e^{-\frac{by}{2}} + e^{\frac{y}{2b}}) - \frac{i}{4}\frac{c_b'}{c_b} \tag{4.94}$$

thanks to (B.6): this fact is valid for any b and connects Q to SW-NS periods (*cf.* below the development for the pure gauge case $b = 1$). Moreover, upon identification of the n-th local integral of motion I_{2n-1} up to an arbitrary normalisation B_n

$$B_n I_{2n-1} = \frac{1}{2n-1} \int_{-\infty}^{\infty} \sqrt{c_b(y)} R_n(y)\, dy \,, \tag{4.95}$$

they are given by the large θ asymptotic expansion of the Baxter's Q function (4.93):

$$Q(\theta, P^2, b) \doteq \exp\left\{ -e^{\theta} \frac{4\sqrt{\pi^3} q}{\sin(\pi b/q)\Gamma(\frac{b}{2q})\Gamma(\frac{1}{2bq})} - \sum_{n=1}^{\infty} e^{\theta(1-2n)} B_n(b) I_{2n-1}(b, P^2) \right\} . \tag{4.96}$$

For instance, we can use the normalisation constants:

$$B_n(b) = \frac{\Gamma(\frac{(2n-1)b}{2q})\Gamma(\frac{(2n-1)}{2bq})}{2\sqrt{\pi} n! q} . \tag{4.97}$$

This expansion matches the numerical results from TBA (4.42), by the formula

$$B_n I_{2n-1} = (-1)^n \int_{-\infty}^{\infty} \frac{d\theta}{\pi} e^{(2n-1)\theta} \ln\left[1 + e^{-\varepsilon(\theta)}\right] \tag{4.98}$$

Now, we can make explicit the one-step recusion procedure (B.16) for the R_n in this particular case (4.11). We will give the details elsewhere and just give the final formula for the LIMs

$$I_{2n-1}(b, P^2) = \frac{(2n)!!}{(2n-1)!!} \sum_{m=n}^{3n} \frac{\Gamma(n-1/2)}{\Gamma(n-1/2+m-n)} \frac{\Gamma(\frac{n-1/2}{bq} + m - n)}{\Gamma(\frac{n-1/2}{bq})} a_{n,m}(b, P^2) , \tag{4.99}$$

with the recursion for the coefficients $a_{n,m}$

$$a_{n+1,m+1} = -\sum_{k=m+1}^{3n+3} \frac{m!}{k! q} \frac{\Gamma(-\frac{(n+1)b}{q} + k + 1)}{\Gamma(-\frac{(n+1)b}{q} + m + 2)} \sum_{l=0}^{3} F_l(n, k-l)\, a_{n,k-l} , \tag{4.100}$$

from the initial condition $a_{0,0} = 1$ and where the F_l functions are defined as

$$F_0(n,m) = \frac{1}{4}(m+\frac{1}{2})^3 q^3 - \frac{3}{4}(n+\frac{1}{2})(m+\frac{1}{2})^2 q^2 b + \frac{3}{4}(n+\frac{1}{2})^2(m+\frac{1}{2}) q b^2 - \frac{1}{4}(n+\frac{1}{2})^3 b^3 - P^2\left[(m+\frac{1}{2})q - (n+\frac{1}{2})b\right] \tag{4.101}$$

$$F_1(n,m) = -\frac{3}{4}(m+\frac{1}{2})(m^2+2m+\frac{13}{12})q^3 + \frac{3}{2}(n+\frac{1}{2})(m+\frac{1}{2})(m+1)q^2 b - \frac{3}{4}(m+\frac{1}{2})(n+\frac{1}{2})^2 q b^2 + P^2(m+\frac{1}{2})q \tag{4.102}$$

$$F_2(n,m) = \frac{3}{4}(m+\frac{1}{2})(m+\frac{3}{2})^2 q^3 - \frac{3}{4}(n+\frac{1}{2})(m+\frac{1}{2})(m+\frac{3}{2}) b q^2 \tag{4.103}$$

$$F_3(n,m) = -\frac{1}{4}(m+\frac{1}{2})(m+\frac{3}{2})(m+\frac{5}{2})q^3 . \tag{4.104}$$

Since the recursion for the Gelfand-Dikii coefficients is *one-step*, using formula (4.99) and (4.100) is a very efficient way of computing the I_{2n-1}, which have also been checked numerically by exploiting TBA equation (4.42). Besides, we have repeated the calculations in the case of the minimal models and have found the same formulæ in terms of c and Δ (as expected).

4.3. Deformed SW cycles

According to Seiberg-Witten theory [57], the low energy effective Lagrangian of 4d $\mathcal{N}=2$ SUSY $SU(2)$ pure gauge theory is expressed through an holomorphic function $\mathcal{F}_{\text{SW}}(a^{(0)})$ called *prepotential*. It may be thought of as constructed from the Seiberg-Witten one-cycle period $a^{(0)}$, such that the v.e.v. of the scalar field $\langle\Phi\rangle = a^{(0)}\sigma_3$, and its *(Legendre) dual* $a_D^{(0)} = \partial\mathcal{F}_{SW}/\partial a^{(0)}$, as expressed by (2.28),(2.30): which are functions of the modulus $u = \langle \text{tr}\,\Phi^2\rangle$ (for fixed parameter Λ[4]) upon eliminating u to obtain $a_D^{(0)}(a^{(0)})$ (and finally integrating). The $\mathcal{N}=2$ SYM classical action enjoys a $U(1)_{\mathcal{R}}$ $\mathcal{R}$-symmetry, which is broken to $\mathbb{Z}_8$ by one-loop anomaly and instanton contributions. Eventually it is broken down to $\mathbb{Z}_4$ by the vacuum, so that the (spontaneously) broken part, which is a $\mathbb{Z}_2$, *i.*e. $u \to -u$, connects two equivalent vacua [57]: we will see that somehow *t*his broken symmetry plays an important rôle also in the deformed theory.

The exact partition function for $\mathcal{N}=2$ SYM theories, with all instanton corrections, has been obtained through equivariant localisation techniques in [58, 59]: two super-gravity parameters, ϵ_1 and ϵ_2, *the Omega background* deform space-time. When both $\epsilon_1\,,\epsilon_2 \to 0$, the logarithm of the partition function reproduces the Seiberg-Witten prepotential $\mathcal{F}_{\text{SW}}$ [59]. The latter can also be thought of as a successive limit of the Nekrasov-Shatashvili (NS) limiting theory [60], defined by the quantisation/deformation (of SW) $\epsilon_1 = \hbar$, $\epsilon_2 \to 0$.

More specifically, having in mind the AGT corresponding Liouville field theory [33, 34] and precisely its level 2 degenerate field equation [61], we may think of it as a quantisation/deformation[5] of the quadratic SW differential which takes up the form of the Mathieu equation (2.34)[6] The Seiberg-Witten cycles (2.28)-(2.30) are the leading order asymptotic representations, as $\hbar \to 0$, of the two exact deformed cycle period

$$a(\hbar,u,\Lambda) = \frac{\hbar}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}(z;\hbar,u,\Lambda)\,dz \tag{4.105}$$

(in gauge theory $a = 2\langle\tilde{\Phi}\rangle$), as well as the exact deformed dual cycle period

$$a_D(\hbar,u,\Lambda) = i\hbar\sum_{z_n\in B}\text{Res}\,\mathcal{P}(z_n;\hbar,u,\Lambda)\,dz \tag{4.106}$$

(the set of poles B will be shown below, *cf.* figure 4.7.4) of the quantum SW differential $\mathcal{P}(z) = -i\frac{d}{dz}\ln\psi(z)$. Also, we may expand asymptotically, around $\hbar=0$, $\mathcal{P}(z) \doteq \sum_{n=-1}^{\infty}\hbar^n\mathcal{P}_n(z)$, and then the NS-deformed periods (modes) are

$$a^{(n)}(u,\Lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}_{2n-1}(z;u,\Lambda)\,dz \qquad a_D^{(n)}(u,\Lambda) = \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0}\mathcal{P}_{2n-1}(z;u,\Lambda)\,dz\,. \tag{4.107}$$

[4]We may calculate the first integral for $u > \Lambda^2$ while the second one for $u < \Lambda^2$ along a *c*ontinuous (without jumps, and hence changing sheet) path in z and then analytically continue in u; we will analyse better the complex structure below, in Section 4.7.

[5]We shall prefer this latter denotation as the former generates sometimes confusion with gauge theory quantisation.

[6]In this section on the $SU(2)$ $N_f=0$ gauge theory we use a different convention on $\hbar$. To get the conventions we use for the higher N_f theories we need to let $\hbar \to \sqrt{2}\hbar$.

The asymptotic expansion of the deformed prepotential $\mathcal{F}_{\text{NS}}$ (logarithm of the partition function) may be derived as above by eliminating u between the two deformed cycle periods [19]. Alternatively, we can use Matone's formula connecting $\mathcal{F}_{\text{NS}}$, a, and u [62], still valid upon deformation [63] asymptotically. However, we have found that the exact dual deformed cycle period a_D differs by the a-derivative A_D of the deformed prepotential $\mathcal{F}_{\text{NS}}$ by $\hbar$ non-perturbative terms.

$$a_D \neq \frac{\partial \mathcal{F}_{\text{NS}}}{\partial a} = A_D \tag{4.108}$$

Similarly, a_D is not connected to the Matone's formula, exactly in $\hbar$. The precise relation between a_D and A_D is given below in (4.321).

4.3.1. Gelfand-Dikii recursion

Exploiting the mathematical result of appendix B, we proceed now to systematically calculate the Nekrasov-Shatashvili deformed integrals (4.107). The equation to be considered is the Mathieu equation (2.34), the asymptotic expansion is for small $\hbar$ as in subsection B.2. Hence we can apply formula (B.45) with

$$\phi(z) = 2u - 2\Lambda^2 \cos z\,. \tag{4.109}$$

By direct inspection of the first Gelfand-Dikii polynomials (cf. (B.18)-(B.21)), we see that they can be expanded in the basis of the inverse powers of $\phi(z)^{-m}$ and conjecture the general form

$$R_n(z; u, \Lambda) = \sum_{m=n}^{3n} \frac{a_{n,m}(u,\Lambda)}{\phi^m(z; u, \Lambda)}\,, \tag{4.110}$$

which will be proved by the structure of the recursion.

The coefficients $a_{n,m}(u,\Lambda)$ will clearly satisfy some one-step recursion relation, which we now find by using the Gelfand Dikii recursion equation (B.45). Inserting the ansatz (4.110) in this recursion, on the $n+1$ side we find

$$\sum_{m=n+1}^{3n+3} a_{n+1,m}(u)\left[-m\frac{\phi'(z)}{\phi^{m+1}(z)}\right], \tag{4.111}$$

while on the n side we find

$$\sum_{m=n}^{3n} a_{n,m}\frac{1}{\phi^{m+1}}\left\{-\left[\frac{1}{4}m+\frac{1}{8}\right]\frac{\phi'''}{\phi}\right.$$
$$\left.+\left[\frac{3}{4}m(m+1)+\frac{3}{4}m+\frac{9}{16}\right]\frac{\phi'\phi''}{\phi^2}-\left[\frac{1}{4}m(m+1)(m+2)+\frac{3}{8}m(m+1)+\frac{9}{16}m+\frac{15}{32}\right]\frac{\phi'^3}{\phi^3}\right\}. \tag{4.112}$$

We collect useful expressions for the derivatives of ϕ with respect to z:

$$\frac{\partial^2\phi}{\partial z^2} = -\phi + 2u \qquad \frac{\partial^3\phi}{\partial z^3} = -\phi' \qquad \phi'^2 = 4\left(\Lambda^4 - u^2 + \phi u - \frac{\phi^2}{4}\right). \tag{4.113}$$

Using these expressions, (4.112) becomes

$$\sum_{m=n+1}^{3n+3} m a_{n+1,m}\frac{1}{\phi^{m+1}} = \sum_{m=n}^{3n} a_{n,m}\Bigg\{\Big[+\frac{1}{4}(m+\frac{1}{2})^3\Big]\frac{1}{\phi^{m+2}} - \Big[u(m+\frac{1}{2})(m+1)(m+\frac{3}{2})\Big]\frac{1}{\phi^{m+3}} + (u^2-\Lambda^4)\Big[(m+\frac{1}{2})(m+\frac{3}{2})(m+\frac{5}{2})\Big]\frac{1}{\phi^{m+4}}\Bigg\} \tag{4.114}$$

We finally find the one-step recursion for the Gelfand Dikii coefficients $a_{n,m}$ of the small $\hbar$ expansion of the Mathieu equation (2.34)

$$a_{n+1,m+1} = \frac{1}{4}\frac{(m+\frac{1}{2})^3}{m+1}a_{n,m} - u\frac{(m-\frac{1}{2})m(m+\frac{1}{2})}{m+1}a_{n,m-1} + (u^2-\Lambda^4)\frac{(m-\frac{3}{2})(m-\frac{1}{2})(m+\frac{1}{2})}{m+1}a_{n,m-2} \tag{4.115}$$

with the initial condition $a_{0,0}=1$. We verified the correctness of this recursion by direct computation of R_1, R_2 and R_3. We report here the coefficients of these tested first polynomials: for R_1

$$a_{1,1} = \frac{1}{32} \qquad a_{1,2} = -\frac{3}{8}u \qquad a_{1,3} = \frac{5}{8}(u^2-\Lambda^4)\,, \tag{4.116}$$

for R_2

$$\begin{aligned} a_{2,2} &= \frac{27}{2048} \qquad a_{2,3} = -\frac{145}{256}u \qquad a_{2,4} = \frac{1085u^2}{256} - \frac{455\Lambda^4}{256} \\ a_{2,5} &= -\frac{693}{64}u\left(u^2-\Lambda^4\right) \qquad a_{2,6} = \frac{1155}{128}\left(u^2-\Lambda^4\right)^2\,, \end{aligned} \tag{4.117}$$

for R_3

$$\begin{aligned} a_{3,3} &= \frac{1125}{65536} \qquad a_{3,4} = -\frac{26285u}{16384} \qquad a_{3,5} = \frac{435015u^2}{16384} - \frac{134379\Lambda^4}{16384} \\ a_{3,6} &= \frac{245553\Lambda^4 u}{2048} - \frac{349503u^3}{2048} \qquad a_{3,7} = -\frac{429\left(u^2-\Lambda^4\right)\left(1235\Lambda^4-4943u^2\right)}{4096} \\ a_{3,8} &= -\frac{765765u\left(u^2-\Lambda^4\right)^2}{1024} \qquad a_{3,9} = \frac{425425\left(u^2-\Lambda^4\right)^3}{1024}\,. \end{aligned} \tag{4.118}$$

4.3.2. Cycles integrals

Considering the ansatz (4.110) and equivalence formula (B.46), we have the basic integrals

$$A_m = \int_{-\pi}^{\pi} \left[2u - 2\Lambda^2 \cos z\right]^{-m+1/2} dz \tag{4.119}$$

$$= 2\int_0^1 (2u + 2\Lambda^2 - 4\Lambda^2 t)^{-m+1/2} t^{-1/2}(1-t)^{-1/2}\, dt \tag{4.120}$$

$$= \frac{\Lambda^{1-2m} 2^{3/2-m}\pi}{[(u/\Lambda^2+1)]^{m-1/2}}\, {}_2F_1(m-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u/\Lambda^2+1}) \tag{4.121}$$

$$B_m = \int_{-\arccos u/\Lambda^2 - i0}^{\arccos u/\Lambda^2 - i0} \left[2u - 2\Lambda^2 \cos z\right]^{-m+1/2} dz \tag{4.122}$$

$$= 2i(-1)^m \int_0^{\frac{1-u/\Lambda^2}{2}} (-2u + 2\Lambda^2 - 4\Lambda^2 s)^{-m+1/2} s^{-1/2}(1-s)^{-1/2}\, ds \tag{4.123}$$

$$= \frac{2^{1-m} i(-1)^m \Lambda^{1-2m}}{(1-u/\Lambda^2)^{-m+1}} \int_0^1 r^{-1/2}(1-r)^{-m+1/2}\left(1 - \frac{1-u/\Lambda^2}{2} r\right)^{-1/2} dr \tag{4.124}$$

$$= \frac{i(-1)^m \Lambda^{1-2m}}{2^{m-1}(1-u/\Lambda^2)^{m-1}} \frac{\sqrt{\pi}\Gamma(\frac{3}{2}-m)}{\Gamma(2-m)}\, {}_2F_1(\frac{1}{2}, \frac{1}{2}, 2-m; \frac{1-u/\Lambda^2}{2}) \tag{4.125}$$

$$= \frac{-i\Lambda^{1-2m}}{2^{2m-2}} \frac{\sqrt{\pi}\Gamma(m-\frac{1}{2})}{\Gamma(m)}\, {}_2F_1(m-\frac{1}{2}, m-\frac{1}{2}, m; \frac{1-u/\Lambda^2}{2}) \tag{4.126}$$

Finally, the deformed cycles (4.107) can be expressed as

$$a^{(n)} = -\frac{1}{2\pi(2n-1)} \sum_{m=n}^{3n} a_{nm} A_m \tag{4.127}$$

$$a_D^{(n)} = -\frac{1}{2\pi(2n-1)} \sum_{m=n}^{3n} a_{nm} B_m \tag{4.128}$$

The basic integrals have been regularized by the use of the exponential parameter m. Yet, another way to regularize the integral is to define them through some differential operators which act on the SW (regular) cycles:

$$\begin{pmatrix} a^{(n)}(u,\Lambda) \\ a_D^{(n)}(u,\Lambda) \end{pmatrix} = \sum_{m=n}^{3n} \alpha_{n,m}(u,\Lambda) \frac{\partial^m}{\partial u^m} \begin{pmatrix} a^{(0)}(u,\Lambda) \\ a_D^{(0)}(u,\Lambda) \end{pmatrix}. \tag{4.129}$$

with new coefficients

$$\alpha_{n,m}(u,\Lambda) = \frac{(-1)^m a_{n,m}(u,\Lambda)}{(2n-1)(2m-3)!!}. \tag{4.130}$$

Formula (4.129) can be immediately proven observing that

$$\frac{1}{\phi^{m-1/2}} = \frac{(-1)^{m+1}}{(2m-3)!!} \frac{\partial^m}{\partial u^m} \phi^{1/2}. \tag{4.131}$$

The coefficients (4.130) satisfy the one-step recursion (simply obtained from (4.115) and (4.130)).

$$\alpha_{n+1,m+1}=-\frac{n-\frac{1}{2}}{n+\frac{1}{2}}\left\{\frac{\frac{1}{8}(m+\frac{1}{2})^3}{(m+1)(m-\frac{1}{2})}\alpha_{n,m}+\frac{\frac{1}{4}m(m+\frac{1}{2})u}{(m+1)(m-\frac{3}{2})}\alpha_{n,m-1}+\frac{\frac{1}{8}(m+\frac{1}{2})(u^2-\Lambda^4)}{(m+1)(m-\frac{5}{2})}\alpha_{n,m-2}\right\} \tag{4.132}$$

with initial condition $\alpha_{0,0}=1$. For example, the first two differential operators are

$$\hat{\mathcal{H}}^{(1)}_{red}(u,\Lambda)=-\frac{1}{32}\frac{\partial}{\partial u}-\frac{3u}{8}\frac{\partial^2}{\partial u^2}-\frac{5(u^2-\Lambda^4)}{24}\frac{\partial^3}{\partial u^3} \tag{4.133}$$

and

$$\hat{\mathcal{H}}^{(2)}_{red}(u,\Lambda)=\frac{9}{2048}\frac{\partial^2}{\partial u^2}+\frac{145u}{2304}\frac{\partial^3}{\partial u^3}+\frac{7\,(31u^2-13\Lambda^4)}{2304}\frac{\partial^4}{\partial u^4}+\frac{11}{320}u(u^2-\Lambda^4)\frac{\partial^5}{\partial u^5}+\frac{11\,(u^2-\Lambda^4)^2}{3456}\frac{\partial^6}{\partial u^6} \tag{4.134}$$

He and Miao [64] conjectured the existence of slightly simpler operators. In the next subsection, we will derive those operators from ours and rigorously proven their conjecture. However, we have found no simple recursion formulas for such operators. Our operators have instead the advantage of being given by the very efficient one-step recursion (4.132).

We have now two methods to compute the deformed cycles, both exploiting the efficiency of one-step recursions. Using the software Wolfram Mathematica we find that the most efficient formulæ are (4.127)-(4.128), since the computation of high order derivatives in (4.129) is rather slow.

4.3.3. Homogeneous operators

He and Miao [65] conjectured the existence of simple differential operators in u which give the Seiberg-Witten deformed cycles:

$$a^{(n)}(u,\Lambda)=\hat{\mathcal{H}}^{(n)}(u)a^{(0)}(u,\Lambda)=\sum_{k=0}^{n}h_{n,k}u^k\frac{\partial^{n+k}}{\partial u^{n+k}}a^{(0)}(u,\Lambda) \tag{4.135}$$

where $n=0,1,2...$ and the $h_{n,k}$ ($k=0,1,...,n$) are numerical coefficients (rational numbers). For example, the first homogeneous operator acting on the SW cycles is

$$\hat{\mathcal{H}}^{(1)}(u)=\frac{1}{48}\frac{\partial}{\partial u}+\frac{u}{24}\frac{\partial^2}{\partial u^2}\,, \tag{4.136}$$

while the second homogeneous operator is

$$\hat{\mathcal{H}}^{(2)}(u)=\frac{5}{1536}\frac{\partial^2}{\partial u^2}+\frac{u}{192}\frac{\partial^3}{\partial u^3}+\frac{7u^2}{5760}\frac{\partial^4}{\partial u^4}\,. \tag{4.137}$$

In this section, we give a rigorous proof of the existence and uniqueness of the homogeneous differential operators by giving a general algorithm for calculating them. A first step for the proof was already made in the previous subsection, with formula (4.131). However the "redundant operator" (4.129) was not exactly that homogeneous of He-Miao [65], but more complex. In fact, the number of derivatives involved was double and they multiplied polynomials of u and Λ, rather than simple powers of u. The coefficients of the first 10 of the these operators are given in appendix B.3.

4.4. Quantum Picard Fuchs in moduli parameter

The Seiberg-Witten cycles (2.28) and (2.30) are both constrained by the Picard-Fuchs equation [66]

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+\frac{1}{4}\right\}a^{(0)}(u,\Lambda)=0\,. \tag{4.138}$$

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+\frac{1}{4}\right\}a_D^{(0)}(u,\Lambda)=0\,. \tag{4.139}$$

In this section we derive an explicit formula for computing the coefficients of all the *q*uantum Picard-Fuchs equations (constraining both periods $a_D^{(n)}(u,\Lambda)$ and $a^{(n)}(u,\Lambda)$), *e.g.*:

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+4u\frac{\partial}{\partial u}+\frac{5}{4}\right\}a_D^{(1)}(u,\Lambda)=0 \tag{4.140}$$

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+6u\frac{\frac{u^2}{\Lambda^4}+\frac{111}{8}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\frac{\partial}{\partial u}+\frac{21}{4}\frac{\frac{u^2}{\Lambda^4}+\frac{689}{32}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\right\}a_D^{(2)}(u,\Lambda)=0\,. \tag{4.141}$$

In the last equation, as in higher order equations, these coefficients show additional singularities which have been checked also numerically to be apparent ones (not of the solution). Eventually, from the knowledge of the periods we can determine the partition function by different means as explained in Section 4.3.

4.4.1. General derivation of quantum Picard-Fuchs

The action of the classical Picard-Fuchs operator on the n-th cycle can be expressed through a commutator with the homogeneous operator as

$$\hat{\mathcal{F}}_0 a^{(n)}=(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}a^{(n)}+\frac{1}{4}a^{(n)}=\left[\hat{\mathcal{F}}_0,\sum_{k=0}^{n}h_{n,k}u^k\frac{\partial^{n+k}}{\partial u^{n+k}}\right]a^{(0)}\,. \tag{4.142}$$

Using the basic commutators

$$\left[\frac{\partial^2}{\partial u^2},u^k\frac{\partial^{k+n}}{\partial u^{k+n}}\right]=k(k-1)u^{k-2}\frac{\partial^{k+n}}{\partial u^{k+n}}+2ku^{k-1}\frac{\partial^{k+n+1}}{\partial u^{k+n+1}} \tag{4.143}$$

$$\left[u^2\frac{\partial^2}{\partial u^2},u^k\frac{\partial^{k+n}}{\partial u^{k+n}}\right]=\left[-n^2-n(2k-1)\right]u^k\frac{\partial^{k+n}}{\partial u^{k+n}}-2nu^{k+1}\frac{\partial^{k+n+1}}{\partial u^{k+n+1}} \tag{4.144}$$

we obtain

$$\hat{\mathcal{F}}_0 a^{(n)} = \sum_{k=0}^{n+1} C_{n,k}(u,\Lambda) \frac{\partial^{k+n}}{\partial u^{k+n}} a^{(0)} \,, \tag{4.145}$$

with

$$C_{n,k}(u,\Lambda) = h_{n,k}\left[(-n^2-2nk+n)u^k - \Lambda^4 k(k-1)u^{k-2}\right] + h_{n,k-1}\left[-2nu^k - 2\Lambda^4(k-1)u^{k-2}\right] . \tag{4.146}$$

Differentiating the classical Picard-Fuchs (4.139) we get the formula

$$\frac{\partial^N}{\partial u^N} a^{(0)} = -2(N-2)\frac{u}{u^2-\Lambda^4}\frac{\partial^{N-1}}{\partial u^{N-1}} a^{(0)} - \left[(N-2)(N-3)+\frac{1}{4}\right]\frac{1}{u^2-\Lambda^4}\frac{\partial^{N-2}}{\partial u^{N-2}} a^{(0)} \,, \tag{4.147}$$

which if used repeatedly, allows to reduce the number of derivatives in expression (4.145) to only two:

$$\hat{\mathcal{F}}_0 a^{(n)} = \chi_{n,0}(u,\Lambda)\frac{\partial^n}{\partial u^n} a^{(0)} + \chi_{n,1}(u,\Lambda)\frac{\partial^{n+1}}{\partial u^{n+1}} a^{(0)} \,, \tag{4.148}$$

where $\chi_{n,0}$ and $\chi_{n,1}$ are rational expressions of u and Λ. If $n \geq 2$ we can write another expression:

$$\left[\hat{\mathcal{F}}_0 + (n^2-n)\right] a^{(n)} = \sum_{k=1}^{n+1}\left[C_{n,k} + (n^2-n)u^k h_{n,k}\right]\frac{\partial^{n+k}}{\partial u^{n+k}} a^{(0)} \,, \tag{4.149}$$

which similarly can be simplified with the aid of rational functions $\xi_{n,0}$ and $\xi_{n,1}$

$$\left[\hat{\mathcal{F}}_0 + (n^2-n)\right] a^{(n)} = \xi_{n,0}(u,\Lambda)\frac{\partial^n}{\partial u^n} a^{(0)} + \xi_{n,1}(u,\Lambda)\frac{\partial^{n+1}}{\partial u^{n+1}} a^{(0)} \,. \tag{4.150}$$

(4.148) and (4.150) constitute a system of equations for the $n-1$-th and n-th derivative of $a^{(0)}$, which we can solve as:

$$\frac{\partial^n}{\partial u^n} a^{(0)} = -\frac{1}{\chi_{n,0}\xi_{n,1} - \chi_{n,1}\xi_{n,0}}\left\{(\chi_{n,1}-\xi_{n,1})\left[(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} a^{(n)} + \frac{1}{4} a^{(n)}\right] + (n^2-n)\chi_{n,1} a^{(n)}\right\} \tag{4.151}$$

$$\frac{\partial^{n+1}}{\partial u^{n+1}} a^{(0)} = \frac{1}{\chi_{n,0}\xi_{n,1} - \chi_{n,1}\xi_{n,0}}\left\{(\chi_{n,0}-\xi_{n,0})\left[(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} a^{(n)} + \frac{1}{4} a^{(n)}\right] + (n^2-n)\chi_{n,0} a^{(n)}\right\} . \tag{4.152}$$

Now, differentiating the expression for the n-th derivative and subtracting to the expression for the $n+1$-th we obtain a third order equation for only $a^{(n)}$

$$\begin{aligned}
0 = {} & (u^2-\Lambda^4)\frac{\partial^3}{\partial u^3} a^{(n)} + \left\{(u^2-\Lambda^4)\left[-\frac{(\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0})'}{\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0}} + \frac{\chi'_{n,1}-\xi'_{n,1}+\chi_{n,0}-\xi_{n,0}}{\chi_{n,1}-\xi_{n,1}}\right] + 2u\right\}\frac{\partial^2}{\partial u^2} a^{(n)} \\
& + \left\{+\frac{(n^2-n)\chi_{n,1}}{\chi_{n,1}-\xi_{n,1}} + \frac{1}{4}\right\}\frac{\partial}{\partial u} a^{(n)} + \left\{\frac{1}{4}\left[-\frac{(\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0})'}{\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0}} + \frac{\chi'_{n,1}-\xi'_{n,1}+\chi_{n,0}-\xi_{n,0}}{\chi_{n,1}-\xi_{n,1}}\right]\right. \\
& \left. + \frac{n^2-n}{\chi_{n,1}-\xi_{n,1}}\left[-\frac{(\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0})'}{\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0}}\chi_{n,1} + \chi'_{n,1} + \chi_{n,0}\right]\right\} a^{(n)} \,,
\end{aligned} \tag{4.153}$$

where $' = \frac{\partial}{\partial u}$. Since for each n there are only two quantum cycles, we expect the quantum Picard-Fuchs equation to be of second order, hence we need to eliminate the third derivative. In order to do this, we do for the derivative Picard Fuchs equation the same passages as before

$$\frac{\partial}{\partial u}\left(\hat{\mathcal{F}}_0 a^{(n)}\right) = (u^2-\Lambda^4)\frac{\partial^3}{\partial u^3}a^{(n)} + 2u\frac{\partial^2}{\partial u^2}a^{(n)} + \frac{1}{4}\frac{\partial}{\partial u}a^{(n)} = \tag{4.154}$$

$$= \sum_{k=0}^{n+2}\left\{C_{n,k-1} + \frac{\partial}{\partial u}C_{n,k}\right\}\frac{\partial^{k+n}}{\partial u^{k+n}}a^{(0)} \tag{4.155}$$

$$= \gamma_{n,0}(u,\Lambda)\frac{\partial^n}{\partial u^n}a^{(0)} + \gamma_{n,1}(u,\Lambda)\frac{\partial^{n+1}}{\partial u^{n+1}}a^{(0)} \tag{4.156}$$

with rational functions $\gamma_{n,0}(u,\Lambda)$ and $\gamma_{n,1}(u,\Lambda)$. We obtain the expression

$$(u^2-\Lambda^4)\frac{\partial^3}{\partial u^3}a^{(n)} = -\frac{1}{4}\frac{\partial}{\partial u}a^{(n)} - 2u\frac{\partial^2}{\partial u^2}a^{(n)} + \gamma_{n,0}(u)\frac{\partial^n}{\partial u^n}a^{(0)} + \gamma_{n,1}(u)\frac{\partial^{n+1}}{\partial u^{n+1}}a^{(0)}\,, \tag{4.157}$$

which inserted in (4.153) give the general second order Picard-Fuchs equation for the n-th deformed cycle:

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + \alpha_n(u,\Lambda)\frac{\partial}{\partial u} + \beta_n(u,\Lambda)\right\}a^{(n)} = 0\,, \tag{4.158}$$

with coefficients

$$\alpha_n = \frac{(n^2-n)\chi_{n,1}}{\chi_{n,1}-\xi_{n,1}}\Delta_n^{-1} \tag{4.159}$$

$$\beta_n = \frac{1}{4} + (n^2-n)\left[\frac{\chi_{n,0}\gamma_{n,1} - \chi_{n,1}\gamma_{n,0} - (\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0})'\frac{\chi_{n,1}}{\chi_{n,1}-\xi_{n,1}}}{\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0}} + \frac{\chi'_{n,1}+\chi_{n,0}}{\chi_{n,1}-\xi_{n,1}}\right]\Delta_n^{-1}\,, \tag{4.160}$$

where

$$\Delta_n = \frac{(\chi_{n,0}-\xi_{n,0})\gamma_{n,1} - (\chi_{n,1}-\xi_{n,1})\gamma_{n,0} - (\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0})'}{\chi_{n,0}\xi_{n,1}-\chi_{n,1}\xi_{n,0}} + \frac{\chi'_{n,1}-\xi'_{n,1}+\chi_{n,0}-\xi_{n,0}}{\chi_{n,1}-\xi_{n,1}}\,. \tag{4.161}$$

4.4.2. Examples

$n=1$ For $n=1$ the passage (4.150) fails and the coefficients (4.159) and (4.160) are singular. However, the general procedure of section 4.4.1 can be slightly modified and we can still obtain a quantum Picard-Fuchs equation.

The Picard Fuchs operator for the Seiberg-Witten order commuted with the first homogeneous operator gives

$$\hat{\mathcal{F}}^{(0)}(u)\hat{\mathcal{H}}^{(1)}(u)a^{(0)} = [\hat{\mathcal{F}}^{(0)}(u),\hat{\mathcal{H}}^{(1)}(u)]a^{(0)} = \left[\frac{-u^2-\Lambda^4}{12}\frac{\partial^3}{\partial u^3} - \frac{u}{8}\frac{\partial^2}{\partial u^2}\right]a^{(0)}\,, \tag{4.162}$$

which using (4.147) becomes

$$\hat{\mathcal{F}}_0 a^{(1)} = \chi_{1,0}(u)\frac{\partial}{\partial u}a^{(0)} + \chi_{1,1}(u)\frac{\partial^2}{\partial u^2}a^{(0)}\,, \tag{4.163}$$

with

$$\chi_{1,0} = -\frac{-\Lambda^4 - u^2}{48\,(u^2-\Lambda^4)} \qquad \chi_{1,1} = \frac{u\,(7\Lambda^4+u^2)}{24\,(u^2-\Lambda^4)} \tag{4.164}$$

Using now the homogeneous operator identities

$$\frac{\partial}{\partial u}a^{(0)} = 48a^{(1)} - 2u\frac{\partial^2}{\partial u^2}a^{(0)} \tag{4.165}$$

$$\frac{\partial^2}{\partial u^2}a^{(0)} = \frac{24}{u}a^{(1)} - \frac{1}{2u}\frac{\partial}{\partial u}a^{(0)} \tag{4.166}$$

we can write two equations for the first quantum cycle $a^{(1)}$, but still involving also $a^{(0)}$:

$$\left\{(u^2-\Lambda^4)^2\frac{\partial^2}{\partial u^2} - \frac{3u^2+5\Lambda^4}{4}\right\}a^{(1)}(u) = \frac{u\Lambda^4}{4}\frac{\partial^2}{\partial u^2}a^{(0)}(u) \tag{4.167}$$

$$\left\{(u^2-\Lambda^4)^2\frac{\partial^2}{\partial u^2} - \frac{3u^2+29\Lambda^4}{4}\right\}a^{(1)}(u) = -\frac{\Lambda^4}{8}\frac{\partial}{\partial u}a^{(0)}(u)\,. \tag{4.168}$$

Such equations are the analogue of (4.151) and (4.152). In fact, multiplying on the left the second equation by $2u\frac{\partial}{\partial u}$ and adding it to the first we get a third order equation for only the first quantum cycle:

$$\left\{u(u^2-\Lambda^4)^2\frac{\partial^3}{\partial u^3} + \frac{(9u^2-\Lambda^4)(u^2-\Lambda^4)}{2}\frac{\partial^2}{\partial u^2} - \frac{u(3u^2+29\Lambda^4)}{4}\frac{\partial}{\partial u} - \frac{5(\Lambda^4+3u^2)}{8}\right\}a^{(1)}(u) = 0\,. \tag{4.169}$$

We need to eliminate the third derivative, because even a quantum Picard-Fuchs has only two independent solutions: $a^{(1)}$ and $a_D^{(1)}$. Hence, expliciting $a^{(0)}$ and using (4.147) we obtain

$$u(u^2-\Lambda^4)^2\frac{\partial^3}{\partial u^3}\left[\frac{1}{48}\frac{\partial}{\partial u} + \frac{u}{24}\frac{\partial^2}{\partial u^2}\right]a^{(0)} = \left[\frac{7u(u^2-\Lambda^4)^2}{48}\frac{\partial^4}{\partial u^4} + \frac{u^2(u^2-\Lambda^4)^2}{24}\frac{\partial^5}{\partial u^5}\right]a^{(0)} \tag{4.170}$$

$$= \left[-\frac{5u^4+102u^2\Lambda^4+21\Lambda^8}{64(u^2-\Lambda^4)}u\frac{\partial^2}{\partial u^2} - \frac{u^2\,(5u^2+27\Lambda^4)}{128\,(u^2-\Lambda^4)}\frac{\partial}{\partial u}\right]a^{(0)}\,. \tag{4.171}$$

Using equations (4.168) and (4.167) we finally get the Picard-Fuchs equation for first quantum Seiberg-Witten cycle

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + 4u\frac{\partial}{\partial u} + \frac{5}{4}\right\}a^{(1)}(u,\Lambda) = 0\,. \tag{4.172}$$

$n \geq 2$ For $n \geq 2$ we can apply the general procedure of section 4.4.1.

We begin with $n = 2$. The commutator of the second homogeneous operator with the classical Picard-Fuchs equation gives

$$\mathcal{F}_0 a^{(2)} = \left\{ -\frac{7u\left(\Lambda^4+u^2\right)}{1440}\frac{\partial^5}{\partial u^5} - \frac{\left(37\Lambda^4+95u^2\right)}{2880}\frac{\partial^4}{\partial u^4} - \frac{17u}{384}\frac{\partial^3}{\partial u^3} - \frac{5}{768}\frac{\partial^2}{\partial u^2} \right\} a^{(0)} \tag{4.173}$$

and can be simplified to

$$\mathcal{F}_0 a^{(2)} = \chi_{2,0}\frac{\partial^2}{\partial u^2}a^{(0)} + \chi_{2,1}\frac{\partial^3}{\partial u^3}a^{(0)}\,, \tag{4.174}$$

with

$$\chi_{2,0} = \frac{-17\Lambda^8 - 47\Lambda^4u^2 + u^4}{480\left(\Lambda^4-u^2\right)^2} \tag{4.175}$$

$$\chi_{2,1} = \frac{u\left(-363\Lambda^8 - 313\Lambda^4u^2 + 4u^4\right)}{2880\left(\Lambda^4-u^2\right)^2}\,. \tag{4.176}$$

The other auxiliary functions in (4.150) and (4.156) are

$$\xi_{2,0} = \frac{-37\Lambda^8 - 135\Lambda^4u^2 + 4u^4}{1280\left(\Lambda^4-u^2\right)^2} \tag{4.177}$$

$$\xi_{2,1} = \frac{u\left(-111\Lambda^8 - 115\Lambda^4u^2 + 2u^4\right)}{960\left(\Lambda^4-u^2\right)^2} \tag{4.178}$$

$$\gamma_{2,0} = \frac{u\left(-795\Lambda^8 - 553\Lambda^4u^2 + 4u^4\right)}{1280\left(\Lambda^4-u^2\right)^3} \tag{4.179}$$

$$\gamma_{2,1} = -\frac{155\Lambda^{12} + 1220\Lambda^8u^2 + 419\Lambda^4u^4 - 2u^6}{960\left(\Lambda^4-u^2\right)^3} \tag{4.180}$$

Putting these expressions in (4.159) and (4.160) we get the second quantum Picard-Fuchs equation

$$\left\{ (u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + 6u\frac{\frac{u^2}{\Lambda^4}+\frac{111}{8}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}}\frac{\partial}{\partial u} + \frac{21}{4}\frac{\frac{u^2}{\Lambda^4}+\frac{689}{32}}{\frac{u^2}{\Lambda^4}+\frac{325}{32}} \right\} a^{(2)}(u,\Lambda) = 0\,. \tag{4.181}$$

We now consider $n = 3$. The auxiliary rational functions of (4.148), (4.150) and (4.156) are

$$\chi_{3,0} = -\frac{11469\Lambda^{12} + 72268\Lambda^8u^2 + 15367\Lambda^4u^4 + 96u^6}{516096\left(\Lambda^4-u^2\right)^3} \tag{4.182}$$

$$\chi_{3,1} = \frac{u\left(70535\Lambda^{12} + 148050\Lambda^8u^2 + 19399\Lambda^4u^4 + 96u^6\right)}{1290240\left(u^2-\Lambda^4\right)^3} \tag{4.183}$$

$$\xi_{3,0} = -\frac{5\left(1851\Lambda^{12} + 14360\Lambda^8u^2 + 3597\Lambda^4u^4 + 32u^6\right)}{516096\left(\Lambda^4-u^2\right)^3} \tag{4.184}$$

$$\xi_{3,1} = \frac{u\left(62273\Lambda^{12} + 152734\Lambda^8u^2 + 22913\Lambda^4u^4 + 160u^6\right)}{1290240\left(u^2-\Lambda^4\right)^3} \tag{4.185}$$

$$\gamma_{3,0} = -\frac{5u\left(155875\Lambda^{12} + 288266\Lambda^8u^2 + 31923\Lambda^4u^4 + 96u^6\right)}{1032192\left(\Lambda^4 - u^2\right)^4} \tag{4.186}$$

$$\gamma_{3,1} = -\frac{39683\Lambda^{16} + 548813\Lambda^{12}u^2 + 514877\Lambda^8u^4 + 39315\Lambda^4u^6 + 96u^8}{516096\left(\Lambda^4 - u^2\right)^4} \tag{4.187}$$

and put in (4.159) and (4.160) produce the third quantum Picard-Fuchs equation

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + 8u\frac{\frac{u^4}{\Lambda^8} + \frac{617u^2}{64\Lambda^4} + \frac{731043}{2048}}{\frac{u^4}{\Lambda^8} + \frac{117u^2}{16\Lambda^4} + \frac{242433}{1024}}\frac{\partial}{\partial u} + \frac{45}{4}\frac{\frac{u^4}{\Lambda^8} + \frac{251u^2}{48\Lambda^4} + \frac{675177}{1024}}{\frac{u^4}{\Lambda^8} + \frac{117u^2}{16\Lambda^4} + \frac{242433}{1024}}\right\} a^{(3)}(u,\Lambda) = 0\,. \tag{4.188}$$

We report also the quantum Picard-Fuchs equation for $n = 4$

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + 16u\frac{40960\frac{u^6}{\Lambda^{12}} + 7291392\frac{u^4}{\Lambda^8} - 61637640\frac{u^2}{\Lambda^4} + 721916729}{65536\frac{u^6}{\Lambda^{12}} + 9689088\frac{u^4}{\Lambda^8} - 73211328\frac{u^2}{\Lambda^4} + 731068145}\frac{\partial}{\partial u}\right. \tag{4.189}$$

$$\left. + \frac{77}{4}\frac{65536\frac{u^6}{\Lambda^{12}} + 13694976\frac{u^4}{\Lambda^8} - 192704448\frac{u^2}{\Lambda^4} + 2338851605}{65536\frac{u^6}{\Lambda^{12}} + 9689088\frac{u^4}{\Lambda^8} - 73211328\frac{u^2}{\Lambda^4} + 731068145}\right\} a^{(4)} = 0 \tag{4.190}$$

and write the coefficients of the quantum Picard-Fuchs equation for $n = 5$

$$\alpha_5 = 4u\frac{62914560\frac{u^8}{\Lambda^{16}} + 35696148480\frac{u^6}{\Lambda^{12}} + 2044215361536\frac{u^4}{\Lambda^8} - 20435136246144\frac{u^2}{\Lambda^4} + 93217274165643}{20971520\frac{u^8}{\Lambda^{16}} + 10186915840\frac{u^6}{\Lambda^{12}} + 507233746944\frac{u^4}{\Lambda^8} - 4653859998464\frac{u^2}{\Lambda^4} + 19108840832975} \tag{4.191}$$

$$\beta_5 = \frac{117\left(20971520\frac{u^8}{\Lambda^{16}} + 13311672320\frac{u^6}{\Lambda^{12}} + 912813637632\frac{u^4}{\Lambda^8} - 12830541348608\frac{u^2}{\Lambda^4} + 66392092574911\right)}{4\left(20971520\frac{u^8}{\Lambda^{16}} + 10186915840\frac{u^6}{\Lambda^{12}} + 507233746944\frac{u^4}{\Lambda^8} - 4653859998464\frac{u^2}{\Lambda^4} + 19108840832975\right)}\,. \tag{4.192}$$

We expect, in general perturbation theory

$$\left\{(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2} + \frac{\sum_{j=0}^{n-1} p_j^{(n)}\left(\frac{u}{\Lambda^2}\right)^{2j}}{\sum_{k=0}^{n-1} q_k^{(n)}\left(\frac{u}{\Lambda^2}\right)^{2k}}u\frac{\partial}{\partial u} + \frac{\sum_{l=0}^{n-1} r_l^{(n)}\left(\frac{u}{\Lambda^2}\right)^{2l}}{\sum_{m=0}^{n-1} s_m^{(n)}\left(\frac{u}{\Lambda^2}\right)^{2m}}\right\} a^{(n)}(u,\Lambda) = 0 \tag{4.193}$$

where $p_j^{(n)}$, $q_k^{(n)}$, $r_l^{(n)}$ and $s_m^{(n)}$ are rational numbers.

4.4.3. Alternative derivation

The Picard-Fuchs may be found also from the series which are resummation of the LIMs, as explained below in section 4.7.3. We report here the first two of such series:

$$2\pi i a_D^{(1)}(u,\Lambda) = -\Lambda^{-1}\sum_{n=0}^{\infty}\left[(-1)^n 2^n \frac{(n+\frac{1}{2})\Gamma^2(\frac{n}{2}+\frac{1}{4})}{48\sqrt{\pi}n!}\right]\left(\frac{u}{\Lambda^2}\right)^n \tag{4.194}$$

$$2\pi i a_D^{(2)}(u,\Lambda) = +\Lambda^{-3}\sum_{n=0}^{\infty}\left[(-1)^n 2^n \frac{(n+\frac{3}{2})(7n+\frac{25}{2})\Gamma^2(\frac{n}{2}+\frac{3}{4})}{5760\sqrt{\pi}n!}\right]\left(\frac{u}{\Lambda^2}\right)^n. \tag{4.195}$$

These formulæ are valid for $|u| < \Lambda^2$.

From (4.193), we get the following ansatz for the first quantum Picard-Fuchs equation

$$\left[(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+f_0 u\frac{\partial}{\partial u}+g_0\right]a_D^{(1)}(u)=0\,. \tag{4.196}$$

Applying this ansatz to the series (4.194) we get the following relation for $n \in \mathbb{N}$

$$g_0+f_0 n-4n-\frac{5}{4}=0\,, \tag{4.197}$$

which is solved immediately as

$$f_0=4 \qquad g_0=\frac{5}{4}\,. \tag{4.198}$$

Again from the general form (4.193), we get the following ansatz for the second quantum Picard-Fuchs equation

$$\left[(u^2-\Lambda^4)\frac{\partial^2}{\partial u^2}+f_0\frac{f_1\Lambda^4+u^2}{f_2\Lambda^4+u^2}u\frac{\partial}{\partial u}+g_0\frac{g_1\Lambda^4+u^2}{g_2\Lambda^4+u^2}\right]a_D^{(2)}(u)=0 \tag{4.199}$$

Applying this ansatz to the series (4.195) we get the following relations, for the first powers $(u/\Lambda^2)^n$:

$$n=0\;: \qquad \frac{g_0 g_1}{g_2}=\frac{1113}{100} \tag{4.200}$$

$$n=1\;: \qquad \frac{f_0 f_1}{f_2}=\frac{2664}{325} \tag{4.201}$$

$$n=2\;: \qquad \frac{g_0 g_1}{g_2}\left[\frac{1}{g_1}-\frac{1}{g_2}\right]=-\frac{4704}{8125} \tag{4.202}$$

$$n=3\;: \qquad \frac{f_0 f_1}{f_2}\left[\frac{1}{f_1}-\frac{1}{f_2}\right]=-\frac{22848}{105625} \tag{4.203}$$

$$n=4\;: \qquad \frac{g_0 g_1}{g_2}\left[\frac{1}{g_1}-\frac{1}{g_2}\right]\left(-\frac{1}{g_2}\right)=\frac{150528}{2640625} \tag{4.204}$$

$$n=5\;: \qquad \frac{f_0 f_1}{f_2}\left[\frac{1}{f_1}-\frac{1}{f_2}\right]\left(-\frac{1}{f_2}\right)=\frac{731136}{34328125}\,. \tag{4.205}$$

These equations can be solved without any algebraic problem to give the already known coefficients

$$f_0=6 \qquad f_1=\frac{111}{8} \qquad f_2=\frac{325}{32} \tag{4.206}$$

$$g_0=\frac{21}{4} \qquad g_1=\frac{689}{32} \qquad g_2=\frac{325}{32}\,. \tag{4.207}$$

4.5. Quantum Picard-Fuchs in the cut-off scale

4.5.1. SW order

Set $\gamma = \Lambda^2$. By combining the already known Picard-Fuchs equation [66] [19]

$$\left[(u^2-\gamma^2)\frac{\partial^2}{\partial u^2}+\frac{1}{4}\right]a_D^{(0)}=0 \tag{4.208}$$

$$\left[\gamma\frac{\partial^2}{\partial u^2}+\gamma\frac{\partial^2}{\partial \gamma^2}+2u\frac{\partial^2}{\partial u\partial\gamma}\right]a_D^{(0)}=0 \tag{4.209}$$

and the relation (which we have derived from the formula below (4.282))

$$\frac{\partial}{\partial u}a_D^{(0)}=\frac{1}{2u}a_D^{(0)}-\frac{\gamma}{u}\frac{\partial}{\partial\gamma}a_D^{(0)} \tag{4.210}$$

We find another Picard-Fuchs equation

$$\left[\frac{\partial^2}{\partial\gamma^2}+\frac{1}{\gamma}\frac{\partial}{\partial\gamma}+\frac{1}{4(u^2-\gamma^2)}\right]a_D^{(0)}=\hat{\mathcal{P}}_\Lambda^{(0)}a_D^{(0)}=0 \tag{4.211}$$

4.5.2. Quantum orders

We find as differential operator in $\gamma=\Lambda^2$ which gives the first cycle

$$a^{(1)}=\hat{\mathcal{H}}_\Lambda^{(1)}a^{(0)}=\left[\frac{\gamma^2}{24u}\frac{\partial^2}{\partial\gamma^2}+\frac{\gamma}{48u}\frac{\partial}{\partial\gamma}\right]a^{(0)} \tag{4.212}$$

For the second

$$a^{(2)}=\left[+\frac{7\gamma^4}{5760u^2}\frac{\partial^4}{\partial\gamma^4}+\frac{\gamma^3}{144u^2}\frac{\partial^3}{\partial\gamma^3}+\frac{11\gamma^2}{1536u^2}\frac{\partial^2}{\partial\gamma^2}+\frac{\gamma}{1536u^2}\frac{\partial}{\partial\gamma}\right]a^{(0)} \tag{4.213}$$

(and of course the same operators for the dual cycles). We have derived these expression by setting an ansatz for the differential operator in γ and finding its coefficients by comparison with the higher cycles calculated through differential operators in u, simplified to the elliptic integrals of the first and second kind.

We find a Picard-Fuchs equation in Λ for the first quantum cycle (and dual cycle):

$$\left[\frac{\partial^2}{\partial\gamma^2}+\frac{(3\gamma^2+u^2)}{\gamma^3-\gamma u^2}\frac{\partial}{\partial\gamma}+\frac{3}{4\left(\gamma^2-u^2\right)}\right]a_D^{(1)}=0 \tag{4.214}$$

The details of the derivation are as follows. We calculate the commutating of the operator $\hat{\mathcal{H}}_\Lambda^{(1)}$ (4.212) with the operator $\hat{\mathcal{P}}_\Lambda^{(0)}$ associated to equation (4.211) and simplify the result by using equation (4.211)

$$[\hat{\mathcal{P}}_\Lambda^{(0)},\hat{\mathcal{H}}_\Lambda^{(1)}]a^{(0)}=\frac{(3\gamma^4+5\gamma^2u^2-u^4)}{24\gamma u\left(u^2-\gamma^2\right)^2}\frac{\partial}{\partial\gamma}a^{(0)}+\frac{(12\gamma^4-11\gamma^2u^2+7u^4)}{24u\left(u^2-\gamma^2\right)^2}\frac{\partial^2}{\partial\gamma^2}a^{(0)}+\frac{\gamma}{6u}\frac{\partial^3}{\partial\gamma^3}a^{(0)} \tag{4.215}$$

Then by using the expression for $a^{(1)}$ we write this result in two ways in a mixed form with $a^{(0)}$

$$[\hat{\mathcal{P}}^{(0)}_\Lambda, \hat{\mathcal{H}}^{(1)}_\Lambda]a^{(0)} = \frac{\gamma\left(3\gamma^2 u - u^3\right)}{16\left(\gamma^3-\gamma u^2\right)^2}\frac{\partial}{\partial\gamma}a^{(0)} + \frac{4\gamma\left(u^2-\gamma^2\right)^2}{\left(\gamma^3-\gamma u^2\right)^2}\frac{\partial}{\partial\gamma}a^{(1)} - \frac{\left(-2\gamma^4-9\gamma^2u^2+3u^4\right)}{\left(\gamma^3-\gamma u^2\right)^2}a^{(1)} \tag{4.216}$$

$$[\hat{\mathcal{P}}^{(0)}_\Lambda, \hat{\mathcal{H}}^{(1)}_\Lambda]a^{(0)} = \frac{\gamma^2 u\left(u^2-3\gamma^2\right)\mathsf{a0}''(\gamma)}{8\left(\gamma^3-\gamma u^2\right)^2}\frac{\partial^2}{\partial\gamma^2} + \frac{4\gamma\left(u^2-\gamma^2\right)^2}{\left(\gamma^3-\gamma u^2\right)^2}\frac{\partial}{\partial\gamma}a^{(1)} - \frac{2\left(-\gamma^4-9\gamma^2u^2+3u^4\right)}{\left(\gamma^3-\gamma u^2\right)^2}a^{(1)} \tag{4.217}$$

Exploiting the fact that

$$[\hat{\mathcal{P}}^{(0)}_\Lambda, \hat{\mathcal{H}}^{(1)}_\Lambda]a^{(0)} = \hat{\mathcal{P}}^{(0)}_\Lambda\hat{\mathcal{H}}^{(1)}_\Lambda a^{(0)} = \left[\frac{\partial^2}{\partial\gamma^2} + \frac{1}{\gamma}\frac{\partial}{\partial\gamma} + \frac{1}{4(u^2-\gamma^2)}\right]a^{(1)} \tag{4.218}$$

and differentiating on of the two mixed equations (4.216) (4.217), we arrive at a third order equation in only $a^{(1)}$

$$\frac{\partial^3}{\partial\gamma^3}a^{(1)} + \frac{\left(3\gamma^4+16\gamma^2u^2-3u^4\right)}{2\gamma\left(3\gamma^4-4\gamma^2u^2+u^4\right)}\frac{\partial^2}{\partial\gamma^2}a^{(1)} + \frac{\left(117\gamma^6-65\gamma^2u^4+6\gamma^4u^2+6u^6\right)}{4\gamma^2\left(u^2-\gamma^2\right)\left(3\gamma^4-4\gamma^2u^2+u^4\right)}\frac{\partial}{\partial\gamma}a^{(1)} \tag{4.219}$$

$$+ \frac{3\left(27\gamma^4-20\gamma^2u^2+u^4\right)}{8\gamma\left(u^2-\gamma^2\right)\left(3\gamma^4-4\gamma^2u^2+u^4\right)}a^{(1)} = 0 \tag{4.220}$$

We can simplify the third derivative of $a^{(1)}$ by writing the derivatives of $a^{(1)}$ in terms of derivatives of $a^{(0)}$ through (4.212) and simplifying higher order derivatives using the differentiation of (4.211). We end up with the second order equation (4.214).

4.6. Baxter's T function at self-dual point as Seiberg-Witten period

This section is devoted to the $b = 1$ case, where we first analyse an important connexion between thr unique Baxter's T function $T(\theta) = \tilde{T}(\theta)$ and the Floquet exponent, as proven numerically by [49]. Then, we give both T and Q two peculiar SW theory interpretations. As anticipated, in the self-dual GME (4.77), we shall rotate the real into the imaginary axis, $z = -iy - \pi$, and obtain the Mathieu equation

$$-\frac{d^2}{dz^2}\psi(z,\theta) + \left[2e^{2\theta}\cos z - P^2\right]\psi(z,\theta) = 0\,. \tag{4.221}$$

According to Floquet theorem, there exist two linearly independent (quasi-periodic) solutions of the Mathieu equation (4.221) of the form $\psi_+(z) = e^{\nu z}p(z)$ and $\psi_-(z) = e^{-\nu z}p(-z)$, with periodic $p(z) = p(z + 2\pi)$ and monodromy exponent $\nu = \nu(\theta, P)$, *the Floquet index*. As anticipated, already [49] conjectures this identification

$$T(\theta, P^2) = 2\cosh\left\{2\pi\nu(\theta, P^2)\right\}\,. \tag{4.222}$$

We will prove this formula in the next subsection. This identity has a very relevant interpretation in gauge theory once we add the other important ingredient, namely the coincidence of the quantum SW period (4.105) $a/\hbar = -i\nu$ with the Floquet exponent[7]. More precisely, the Mathieu ODE/IM equation (4.221) coincides with the Seiberg-Witten one (2.34), provided we set the change of variables

$$\frac{\hbar}{\Lambda} = e^{-\theta} \,, \qquad \frac{u}{\Lambda^2} = \frac{P^2}{2e^{2\theta}}. \tag{4.223}$$

Thus, the above (4.222) can be interpreted as a direct connexion between the Baxter's T function and the quantum SW period (4.105):

$$T(\hbar, u, \Lambda) \equiv T(\theta, P^2) = 2\cos\left\{\frac{2\pi}{\hbar} a(\hbar, u, \Lambda)\right\} \,. \tag{4.224}$$

4.6.1. Exact analytic proof

Define the periodicity operator

$$M\psi(y) = \psi(y + 2\pi i) \tag{4.225}$$

We can express it in terms of the Λ_1, Ω_1 symmetry operators

$$M = \Lambda_1 \Omega_1^{-1} \tag{4.226}$$

Then we write

$$\begin{aligned} \psi_{+,-1}(y + 2\pi i) &= \psi_{+,0} \\ \psi_{+,0}(y + 2\pi i) &= \psi_{+,1} = -\psi_{+,-1} + T(\theta)\psi_{+,0} \end{aligned} \tag{4.227}$$

Or in matrix form, defining $\psi = (\psi_{+,-1}, \psi_{+,0})^T$

$$M\psi_+ = \Upsilon_+ \psi_+ \tag{4.228}$$

with

$$\Upsilon_+ = \begin{pmatrix} 0 & 1 \\ -1 & T(\theta) \end{pmatrix} \tag{4.229}$$

Now we can say that ν is a characteristic exponent of the Doubly confluent Heun equation (4.77) if and only if $e^{\pm 2\pi i\nu}$ are eigenvalues of Υ_+. It then follows that ν is determined from

$$2\cos 2\pi\nu = \mathrm{tr}\,\Upsilon_+ \tag{4.230}$$

or more explicitly

$$2\cos 2\pi\nu = T(\theta) \tag{4.231}$$

[7] We have carried preliminary successful comparisons with the few instanton Nekrasov partition function in terms of Young diagrams upon using Matone's relation, as in [43].

4.6.2. Numerical exact proof

In order to compute numerically $T(\theta, P^2)$, we use the TQ relation (4.27)

$$T(\theta, P^2) = \frac{Q(\theta + i\pi/2, P^2)}{Q(\theta, P^2)} + \frac{Q(\theta - i\pi/2, P^2)}{Q(\theta, P^2)} \tag{4.232}$$

The analytic continuation of the TBA (4.70) gives $\varepsilon(\theta + i\pi/2, P^2) = -2\ln Q(\theta + i\pi/2, P^2)$ with the real part given by the contribution of (half) the residue at $\theta' = \theta + i\pi/2$ of the integrand

$$\Re\varepsilon(\theta + i\pi/2, P^2) = -\ln\left[1 + Q^2(\theta, P^2)\right] \tag{4.233}$$

and the imaginary part [49]

$$\Im\varepsilon(\theta+i\pi/2, P^2) = c_0 e^{\theta} + 8P\arctan[e^{-\theta}] + \frac{C(P)}{\cosh\theta} - 2\,\text{v.p.}\int_{-\infty}^{\infty} \frac{L(\theta') - 4P\ln[1+e^{-2\theta'}] - C[1-\tanh(\theta')]}{\sinh(\theta-\theta')}\frac{d\theta'}{2\pi}\,. \tag{4.234}$$

Considering that for real θ and P^2 we have $Q(\theta+i\pi/2, P^2) = Q(\theta-i\pi/2, P^2)^*$ the TQ simplifies as

$$T(\theta, P^2) = 2\frac{\sqrt{1+Q^2(\theta, P^2)}}{Q(\theta, P^2)}\cos\left\{\frac{1}{2}\Im\varepsilon(\theta + i\pi/2, P^2)\right\}\,, \tag{4.235}$$

Asymptotically for $\theta \to -\infty$, we find easily

$$T(\theta, P) \simeq 2\cos 2\pi P \qquad \theta \to -\infty\,, \tag{4.236}$$

which is consistent with $T(\theta, P) \simeq 2\cosh 2\pi\nu(\theta, P)$ since $\nu \simeq iP$. At finite θ we must compute ν through the Hill determinant (see below and for instance [67]) and find confirmed Zamolodchikov's conjecture [49]:

$$T(\theta, P) = 2\cosh 2\pi\nu(\theta, P)\,. \tag{4.237}$$

Another check of this relation was given by H. Poghosyan in [68].

Hill determinant Here we give the details of the computation of the Hill determinant. Let us consider the modified Mathieu ODE (4.77) written as:

$$-\frac{d^2}{dy^2}\psi(y) + [e^{2\theta}e^{y} + e^{2\theta}e^{-y} + P^2]\psi(y) = 0\,, \tag{4.238}$$

Noticing that the potential is periodic under $y \to y + 2\pi i$, the Floquet index is defined through

$$\psi_+(y) = e^{-i\nu y}\hat{p}(y)\,, \qquad \psi_+(y + 2\pi i) = e^{2\pi\nu}\psi_+(y)\,, \tag{4.239}$$

where $\hat{p}(y)$ is a $2\pi i$ periodic function, which expanded in Fourier modes gives

$$\psi_+(y) = e^{-i\nu y}\sum_{n=-\infty}^{\infty} b_n e^{ny}\,. \tag{4.240}$$

θ	$T(\theta, P)$ TBA	$2\cosh 2\pi\nu_M$ Mathematica	$2\cosh 2\pi\nu_H$ Hill
$-10.$	0.617594	0.618034	0.618034
$-8.$	0.61583	0.618034	0.618034
$-6.$	0.598208	0.618034	0.618034
$-4.$	0.479008	0.618026	0.618026
$-2.$	-0.176943	0.594172	0.594172
$0.$	-50.9945	-47.0357	-47.0357
$1.$	-16061.2	-18715.7	-18715.7
$2.$	$1.4194 \cdot 10^{11}$	$1.46531 \cdot 10^{11}$	$1.46531 \cdot 10^{11}$
$3.$	$3.19213 \cdot 10^{29}$	$3.67017 \cdot 10^{29}$	$3.65387 \cdot 10^{29}$
$4.$	$-4.23969 \cdot 10^{80}$	N.R.	$-4.2823 \cdot 10^{80}$
$5.$	$5.14167 \cdot 10^{218}$	N.C.	$5.13 \cdot 10^{218}$

Table 4.1: Here we make a table, with $P = 0.2$ and several θ in the lines, of three quantities: $T(\theta, P)$ from the TBA, $2\cosh 2\pi\nu_M$, were ν_M is Mathematica's Floquet and $2\cosh 2\pi\nu_H$ were ν_M is Hill's Floquet. N.C. stays for not computable, N.R. for not reliable (because a little beyond it becomes uncomputable).

Substituting this expression in the Modified Mathieu ODE, we obtain

$$\sum_{n=-\infty}^{\infty} \left\{-(n-i\nu)^2 b_n + \left[e^{2\theta} b_{n-1} + e^{2\theta} b_{n+1} + P^2 b_n\right]\right\} \left(e^{(n-i\nu)y}\right) = 0 \tag{4.241}$$

In order to have a nontrivial solution, we need to impose the following condition on the Fourier modes b_n:

$$b_{n-1} + \left[\frac{-(n-i\nu)^2 + P^2}{e^{2\theta}}\right] b_n + b_{n+1} = 0 \tag{4.242}$$

or

$$\xi_n b_{n-1} + b_n + \xi_n b_{n+1} = 0 \qquad \text{with } \xi_n = \frac{e^{2\theta}}{P^2 - (n-i\nu)^2} \tag{4.243}$$

In the matrix form, we have

$$\begin{pmatrix} \vdots & & & & & \vdots \\ \cdots & \xi_n & 1 & \xi_n & 0 & \cdots \\ \cdots & 0 & \xi_{n+1} & 1 & \xi_{n+1} & \cdots \\ \cdots & 0 & 0 & \xi_{n+2} & 1 & \cdots \\ \cdots & 0 & 0 & 0 & \xi_{n+3} & \cdots \\ \vdots & & & & & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ b_{n-1} \\ b_n \\ b_{n+1} \\ b_{n+2} \\ \vdots \end{pmatrix} = 0. \tag{4.244}$$

Let the determinant of the matrix at the left hand side be $\Delta(\nu)$, we thus have

$$\Delta(\nu) = 0. \tag{4.245}$$

We also introduce a $(2n+1)\times(2n+1)$ matrix

$$\mathcal{A}_n(\nu) = \begin{pmatrix} \begin{smallmatrix} 1 & \xi_{-n} & 0 & 0 \\ \xi_{-n+1} & 1 & \xi_{-n+1} & 0 \\ 0 & \xi_{-n+2} & 1 & \xi_{-n+2} \end{smallmatrix} & & & & \\ & \vdots & \ddots & & \\ & & \begin{smallmatrix} \xi_{-1} & 1 & \xi_{-1} & 0 & 0 \\ 0 & \xi_0 & 1 & \xi_0 & 0 \\ 0 & 0 & \xi_1 & 1 & \xi_1 \end{smallmatrix} & & \\ & & & \ddots & \vdots \\ & & & & \begin{smallmatrix} \xi_{n-2} & 1 & \xi_{n-2} & 0 \\ 0 & \xi_{n-1} & 1 & \xi_{n-1} \\ 0 & 0 & \xi_n & 1 \end{smallmatrix} \end{pmatrix} \tag{4.246}$$

Let $\Delta(\nu) = 0$, we find by the ordinary Floquet theory

$$2\cosh 2\pi\nu = 2\left[1 - 2\Delta(0)\sin^2 \pi P\right] \tag{4.247}$$

where

$$\Delta(0) = \lim_{n\to\infty} \det \mathcal{A}_n(0) \tag{4.248}$$

In this way, we can check (4.237) for all θ, as Zamolodchikov did [49].

4.6.3. Identification with instanton period

The gauge a period is defined through the relation

$$2u = a^2 - \frac{\Lambda_0}{4}\frac{\partial \mathcal{F}_{NS}^{inst}}{\partial \Lambda_0} = a^2 + \frac{\Lambda_0^4}{2(4a^2 - \hbar^2)} + O(\Lambda_0^8) \tag{4.249}$$

where the instanton prepotential $\mathcal{F}$ is given by

$$\mathcal{F}_{NS}^{inst} = \sum_{n=0}^{\infty} \Lambda^{4n} \mathcal{F}_{NS}^{(n)} \tag{4.250}$$

with

$$\begin{aligned} \mathcal{F}_{NS}^{(1)} &= -\frac{2}{4a^2 - \hbar^2} \\ \mathcal{F}_{NS}^{(2)} &= -\frac{20a^2 + 7\hbar^2}{4(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \\ \mathcal{F}_{NS}^{(3)} &= -\frac{4(144a^4 + 232a^2\hbar^2 + 29\hbar^4)}{3(4a^2 - \hbar^2)^5(4a^4 - 13a^2\hbar^2 + 9\hbar^4)} \end{aligned} \tag{4.251}$$

In table 4.2 we check the equality to this order of approximation

$$\frac{a}{\hbar} = \nu \tag{4.252}$$

Λ_0	u	a	ν
$\frac{1}{10}$	$\frac{1}{5}$	0.6321918255	0.6321918255
$\frac{1}{5}$	$\frac{1}{5}$	0.6281906897	0.6281906283
$\frac{3}{10}$	$\frac{1}{5}$	0.6096988347	0.6096495640
$\frac{2}{5}$	$\frac{1}{5}$	0.5463504374	0.5303270313

Λ_0	u	a	ν
$\frac{1}{10}$	$\frac{1}{5}$	0.8164231121	0.8164231121
$\frac{1}{5}$	$\frac{1}{5}$	0.8153248650	0.8153248652
$\frac{3}{10}$	$\frac{1}{5}$	0.8106453170	0.8106454455
$\frac{2}{5}$	$\frac{1}{5}$	0.7986254589	0.7986371992

Table 4.2: Comparison of a for $N_f = 0$ as computed by instanton series and the built-in Mathematica Floquet exponent ν (with $\hbar = 1$).

4.7. Baxter's Q function at self-dual point as Seiberg-Witten dual period

Now we find an analogous link for the Q-function, $Q(\theta, P^2)$, upon writing (4.77) in the gauge variables (4.223)

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi(y) + [\Lambda^2 \cosh y + u]\psi(y) = 0 \ , \tag{4.253}$$

which is the same as equation (2.34) upon substitution $\psi(y) = \psi(z)$ with $y = iz + i\pi$. Equation (4.253) gives rise for $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ to the Riccati equation

$$\mathcal{P}^2(y,\hbar,u) - i\frac{d\mathcal{P}(y,\hbar,u)}{dy} = -(\frac{2u}{\hbar^2} + \frac{2\Lambda^2}{\hbar^2}\cosh y), \tag{4.254}$$

while $\mathcal{P}(z) = -i\frac{d}{dz}\ln\psi(z)$ (so that $\mathcal{P}(y)dy = \mathcal{P}(z)dz$) verifies

$$\mathcal{P}^2(z,\hbar,u) - i\frac{d\mathcal{P}(z,\hbar,u)}{dz} = \frac{2u}{\hbar^2} - \frac{2\Lambda^2}{\hbar^2}\cos z. \tag{4.255}$$

4.7.1. Seiberg-Witten order proof

Let us consider the integral for $\ln Q$ at the leading $\hbar$ (Seiberg-Witten) order. For the modified Mathieu equation (4.253), $\phi = -2\Lambda^2\cosh y - 2u$ (cf. (B.43)). Then, the leading order of the quantum momentum is

$$\mathcal{P}_{-1} = -i\Lambda\sqrt{2\cosh y' + 2\frac{u}{\Lambda^2}}\,. \tag{4.256}$$

Since, in the limits $y \to \pm\infty$, we have $\mathcal{P}_{-1} = -i\frac{\Lambda}{\hbar}e^{\pm y/2} + O(e^{\mp y/2})$, it follows that the Seiberg-Witten regularized momentum is

$$\mathcal{P}_{reg,-1}(y) = \mathcal{P}_{-1}(y) + 2i\Lambda\cosh\frac{y}{2} = -i\Lambda\Big[\sqrt{2\cosh y' + 2\frac{u}{\Lambda^2}} - 2\cosh\frac{y'}{2}\Big]\,. \tag{4.257}$$

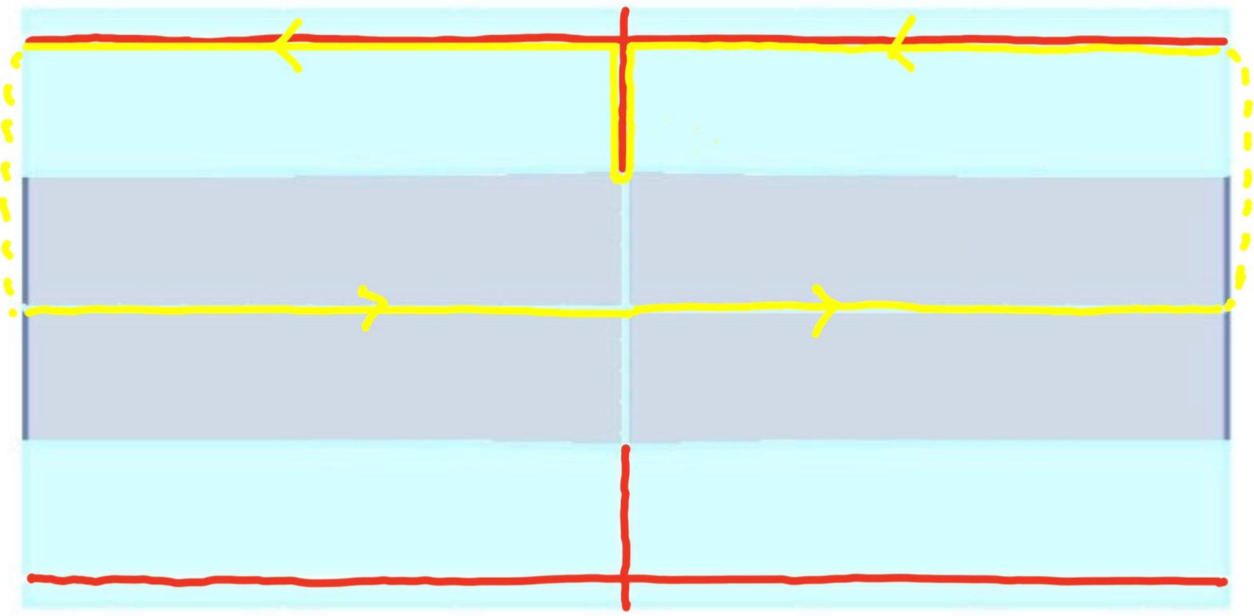

Figure 4.1: A region of the y complex plane, where in yellow we show the contour of integration of SW differential for the $SU(2)$ $N_f = 0$ theory we use for the proof equality of the dual SW period $a_D^{(0)}$ and the leading $\hbar \to 0$ order of the logarithm of the Baxter's Q function $\ln Q^{(0)}$. In red are shown the branch cuts of the SW differential.

From (4.94) and (4.92), the leading order of $\ln Q$ is

$$\ln Q^{(0)}(u,\Lambda) = \int_{-\infty}^{\infty} i\mathcal{P}_{reg,-1}(y)\,dy = \Lambda \int_{-\infty}^{\infty} \Big[\sqrt{2\cosh y + 2\frac{u}{\Lambda^2}} - 2\cosh\frac{y}{2}\Big]\,dy\,. \tag{4.258}$$

We assume $u < \Lambda^2$. Let us consider the integral of $i\mathcal{P}_{reg,-1}(y)$ on the (oriented) closed curve which runs along the real axis, slightly below the cut and closes laterally. Mathematically, it is $\gamma = \gamma_1 \cup \gamma_{lat,R} \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5 \cup \gamma_{lat,L}$, with $\gamma_1 = (-\infty, +\infty)$, $\gamma_2 = (+\infty + i\pi - i0, 0^+ + i\pi - i0)$, $\gamma_3 = (0^+ + i\pi - i0, 0^+ + i\pi - i\arccos(u/\Lambda^2)$, $\gamma_4 = (0^- + i\pi - i\arccos(u/\Lambda^2), 0^- + i\pi - i0)$, $\gamma_5 = (0^- + i\pi - i0, -\infty + i\pi - i0)$, and $\gamma_{lat,L}$ $\gamma_{lat,R}$ are the lateral contours which close the curve (see figure 4.1).

We expect the integral of $\mathcal{P}_{reg,-1}(y)$ on γ to be zero, since the branch cuts are avoided and no singularities are inside the curve. By expanding the square root for $\Re y \to \pm\infty$, $|\Im y| < \pi$, we get the asymptotic behaviour:

$$\frac{\hbar}{\Lambda} i\mathcal{P}_{reg,-1}(y) = -(\frac{u}{\Lambda^2} + 1)e^{-y/2} + o(e^{-y/2}) \qquad \Re y \to +\infty \tag{4.259}$$

$$\frac{\hbar}{\Lambda} i\mathcal{P}_{reg,-1}(y) = -(\frac{u}{\Lambda^2} + 1)e^{y/2} + o(e^{y/2}) \qquad \Re y \to -\infty\,, \tag{4.260}$$

from which, we deduce that the integrals on the lateral contours $\gamma_{lat,L/R}$ are exponentially suppressed. For γ_2 and γ_5, we consider $\mathcal{P}_{reg,-1}(t + i\pi - i0)$ for $t \in \mathbb{R}$:

$$\frac{\hbar}{\Lambda} i\mathcal{P}_{reg,-1}(t + i\pi - i0) = \sqrt{-2\cosh t + 2\frac{u}{\Lambda^2}} - 2i\sinh\frac{t}{2}\,. \tag{4.261}$$

Since for $t = 0$ it is necessary to cross a cut, we find the oddness property $\mathcal{P}_{-1}(t + i\pi - i0) = -\mathcal{P}_{-1}(-t + i\pi - i0)$. Besides also the regularizing part is odd and therefore, for $t \in \mathbb{R}$ we have

$$\mathcal{P}_{reg,-1}(t + i\pi - i0) = -\mathcal{P}_{reg,-1}(-t + i\pi - i0) \tag{4.262}$$

As a consequence, the integrals on γ_2 and γ_5 cancel each other. The integrals on γ_3 and γ_4, around the cut, can be better taken into account in the variable $z = -iy - \pi$. There is no contribution from the regularizing part, which has no cut. Instead $\mathcal{P}_{-1}$, which is

$$\mathcal{P}_{-1}(z - i0) = \Lambda\sqrt{-2\cos(z - i0) + 2\frac{u}{\Lambda^2}}\,, \tag{4.263}$$

has the oddness property

$$\mathcal{P}_{-1}(-z + i0) = -\mathcal{P}_{-1}(z - i0) \qquad z \in \mathbb{R} \tag{4.264}$$

It follows that the integrals on γ_3 and γ_4 add to each other

$$\int_{-\arccos(u/\Lambda^2)}^{0} \mathcal{P}_{-1}(z - i0)\,dz + \int_{0}^{-\arccos(u/\Lambda^2)} \mathcal{P}_{-1}(z + i0)\,dz = \int_{-\arccos(u/\Lambda^2)-i0}^{+\arccos(u/\Lambda^2)-i0} \mathcal{P}_{-1}(z)\,dz\,. \tag{4.265}$$

In conclusion, we find a relation between the integrals on γ_1 and on γ_3 and γ_4:

$$\int_{-\infty}^{+\infty} i\mathcal{P}_{reg,-1}(y)\,dy = \int_{-\arccos(u/\Lambda^2)-i0}^{+\arccos(u/\Lambda^2)-i0} i\mathcal{P}_{-1}(z)\,dz\,, \tag{4.266}$$

which in terms of physical quantities is

$$\ln Q^{(0)}(u,\Lambda) = 2\pi i a_D^{(0)}(u,\Lambda)\,. \tag{4.267}$$

4.7.2. Higher orders asymptotic proof

We now give an asymptotic proof for also all higher orders in the $\hbar \to 0$ expansion of Q:

$$\ln Q \doteq \sum_{n=0}^{\infty} \hbar^{2n-1} \ln Q^{(n)} \qquad \hbar \to 0\,. \tag{4.268}$$

The small $\hbar$ asymptotic expansion of (4.92) is analogous but different from the large θ expansion (4.93), since for the former u is finite while for the latter P is finite. If we expand the Gelfand-Dikii polynomials in the basis $(2u + 2\Lambda^2\cosh y)^m$, we obtain the same Gelfand-Dikii coefficients $a_{n,m}$ of (4.115) and basic integrals Z_m given by

$$Z_m = \int_{-\infty}^{\infty} \left[2\Lambda^2\cosh y + 2u\right]^{-m+1/2} dy\,, \tag{4.269}$$

(regular for $m \geq 1$). Then $\ln Q^{(n)}$ is given by

$$\ln Q^{(n)} = -\frac{1}{2n-1}\sum_{m=n}^{3n} a_{nm} Z_m\,. \tag{4.270}$$

We compute the basic integral as follows

$$Z_m = \int_{-\infty}^{\infty} \left[2\Lambda^2 \cosh y + 2u\right]^{-m+1/2} dy \tag{4.271}$$

$$= 2(2\Lambda)^{-2m+1} \int_0^{\infty} \left(\cosh^2 \frac{y}{2} - \frac{1-u/\Lambda^2}{2}\right)^{-m+1/2} dy \tag{4.272}$$

$$= 2(2\Lambda)^{-2m+1} \int_1^{\infty} \left(t - \frac{1-u/\Lambda^2}{2}\right)^{-m+1/2} (t-1)^{-1/2} t^{-1/2}\, dt \tag{4.273}$$

$$= 2(2\Lambda)^{-2m+1} \int_0^{1} \left(1 - \frac{1-u/\Lambda^2}{2} s\right)^{-m+1/2} (1-s)^{-1/2} s^{m-3/2}\, ds \tag{4.274}$$

$$= 2^{-2m+2}\Lambda^{-2m+1} \frac{\Gamma(m-\frac{1}{2})\sqrt{\pi}}{\Gamma(m)} {}_2F_1(m-\frac{1}{2}, m-\frac{1}{2}, m, \frac{1-u/\Lambda^2}{2})\,. \tag{4.275}$$

Comparing with formula (4.126) for the deformed dual cycles, we get $Z_m = iB_m$ and therefore

$$\ln Q^{(n)}(u,\Lambda) = 2\pi i a_D^{(n)}(u,\Lambda)\,. \tag{4.276}$$

The full asymptotic expansion of $\ln Q$ reads:

$$\ln Q(\hbar, u, \Lambda) \doteq \sum_{n=0}^{\infty} \hbar^{2n-1} 2\pi i a_D^{(n)}(u,\Lambda) \qquad \hbar \to 0\,, \tag{4.277}$$

by which we prove asymptotically the equality

$$\ln Q(\hbar, u, \Lambda) \doteq \frac{2\pi i}{\hbar} a_D(\hbar, u, \Lambda) \qquad \hbar \to 0\,. \tag{4.278}$$

4.7.3. Resummed formulæ for the cycles

In consideration of the one to one relation between θ and $\hbar$ (4.223) we can use the first in place of the latter. Thus, these two asymptotic expansions hold in the strip $|\Im\theta| < \frac{\pi}{2} + \epsilon$, $\epsilon > 0$ for $\Re\theta \to +\infty$ (small $\hbar$)

$$T(\theta, P^2) = T(\theta, u) \doteq 2\cos\left\{2\pi \sum_{n=0}^{\infty} e^{\theta(1-2n)} \Lambda^{2n-1} a^{(n)}(u,\Lambda)\right\} \tag{4.279}$$

$$Q(\theta, P^2) = Q(\theta, u) \doteq \exp\left\{2\pi i \sum_{n=0}^{\infty} e^{\theta(1-2n)} \Lambda^{2n-1} a_D^{(n)}(u,\Lambda)\right\} \qquad . \tag{4.280}$$

We now find a new way to compute the NS-deformed Seiberg Witten periods modes, which will also reveal itself to be an asymptotic check of the identification (4.280). Considering the large energy asymptotic expansion (4.96) of Q in terms of the LIM, we observe that, since in Seiberg-Witten theory u is finite as $\theta \to +\infty$, it is necessary that also $P^2(\theta) = 2\frac{u}{\Lambda^2} e^{2\theta} \to +\infty$. In this double limit, an infinite number of LIMs $I_{2n-1}(b=1)$ are re-summed into an NS-deformed dual period mode (a sort of charge in its turn). Then the n-th mode of the Q function in the small $\hbar$ expansion (4.280) is a series which gives the n-th dual period

$$2\pi i a_D^{(n)}(u,\Lambda) = -\Lambda^{1-2n}\sum_{k=0}^{\infty} 2^k C_{n+k}\Upsilon_{n+k,k}\left(\frac{u}{\Lambda^2}\right)^k . \tag{4.281}$$

From here, closed formulæ can be obtained through the previous powerful method for determining the LIM through the one-step Gelfand-Dikii recursion explained in [1]; they are very simple series (cf. (4.129)) convergent in the circle $|u| < \Lambda^2$:

$$2\pi i a_D^{(0)}(u,\Lambda) = -\Lambda\sum_{n=0}^{\infty}\left[(-1)^n 2^n \frac{\Gamma^2(\frac{n}{2}-\frac{1}{4})}{4\sqrt{\pi}n!}\right]\left(\frac{u}{\Lambda^2}\right)^n \tag{4.282}$$

$$2\pi i a_D^{(1)}(u,\Lambda) = \Lambda^{-1}\sum_{n=0}^{\infty}\left[(-1)^n 2^n \frac{(n+\frac{1}{2})\Gamma^2(\frac{n}{2}+\frac{1}{4})}{48\sqrt{\pi}n!}\right]\left(\frac{u}{\Lambda^2}\right)^n \tag{4.283}$$

$$2\pi i a_D^{(2)}(u,\Lambda) = -\Lambda^{-3}\sum_{n=0}^{\infty}\left[(-1)^n 2^n \frac{(n+\frac{3}{2})(7n+\frac{25}{2})\Gamma^2(\frac{n}{2}+\frac{3}{4})}{5760\sqrt{\pi}n!}\right]\left(\frac{u}{\Lambda^2}\right)^n \tag{4.284}$$

$$2\pi i a_D^{(3)}(u,\Lambda) = \Lambda^{-5}\sum_{n=0}^{\infty}(-1)^n 2^n \frac{\left(n+\frac{5}{2}\right)\left(124n^2+740n+1107\right)\Gamma^2\left(\frac{n}{2}+\frac{5}{4}\right)}{1935360\sqrt{\pi}n!}\left(\frac{u}{\Lambda^2}\right)^n \tag{4.285}$$

$$2\pi i a_D^{(4)}(u,\Lambda) = -\Lambda^{-7}\sum_{n=0}^{\infty}(-1)^n 2^n \frac{\left(n+\frac{7}{2}\right)\left[n\left(508n^2+6406n+27021\right)+\frac{76145}{2}\right]\Gamma^2\left(\frac{n}{2}+\frac{7}{4}\right)}{154828800\sqrt{\pi}n!}\left(\frac{u}{\Lambda^2}\right)^n \tag{4.286}$$

We obtained (4.282)-(4.284) directly from the resummation of the LIMs, as in [1]. For higher orders, however, we found easier to use homogeneous operators. In general if

$$2\pi i a_D^{(n)}(u) = \sum_{m=0}^{n} h_{n,m}u^m \frac{\partial^{n+m}}{\partial u^{n+m}} 2\pi i a_D^{(0)}(u) \tag{4.287}$$

then

$$2\pi i a_D^{(n)}(u) = \sum_{k=0}^{\infty}\left\{\sum_{m=0}^{n} h_{n,m}\frac{(k+n)!}{(k-m)!}\right\}(-1)^{k+n}2^{k+n}C_{n+k}\left(\frac{u}{\Lambda^2}\right)^k \tag{4.288}$$

and thus

$$\Upsilon_{n,n-k} = (-1)^n 2^k \sum_{m=0}^{k} h_{k,m}\frac{n!}{(n-k-m)!} . \tag{4.289}$$

The first leading terms are, for the *natural number* n

$$\Upsilon_{n,n} = (-1)^n \qquad \Upsilon_{n,n-1} = \frac{(-1)^n}{12} n(n-\frac{1}{2}) \qquad \Upsilon_{n,n-2} = (-1)^n \frac{14n-3}{2880}(n-1)n(n-\frac{1}{2}) \tag{4.290}$$

$$\Upsilon_{n,n-3} = (-1)^n \frac{124n^2-4n+3}{483840}(n-2)(n-1)n(n-\frac{1}{2}) \tag{4.291}$$

$$\Upsilon_{n,n-4} = (-1)^n \frac{1016n^3+620n^2+314n-55}{77414400}(n-3)(n-2)(n-1)n(n-\frac{1}{2}) \tag{4.292}$$

$$\Upsilon_{n,n-5} = (-1)^n \frac{40880n^4+71136n^3+71656n^2+18648n-7965}{61312204800}(n-4)(n-3)(n-2)(n-1)n(n-\frac{1}{2}) \tag{4.293}$$

We have found explicit formulæ until $\Upsilon_{n,n-9}$ and all has been tested with the already known charges $I_1 \to I_{17}$.

From the alternative derivation of the quantum Picard-Fuchs presented in section 4.4.3 we learn how to interpret in integrability such equations. Since the analytic series (4.281) are essentially the P^2 coefficients of the LIMs, we can interpret in integrability the quantum Picard-Fuchs as fixing the LIMs for $b = 1$. Therefore, thanks to the quantum Picard-Fuchs equations (4.139-4.141), we can express explicitly the LIM themselves at all orders.

Conversely, we can invert (4.281) and expresses the LIMs in terms of the the deformed periods.

$$\Upsilon_{n,n-k} = \frac{(-1)^n 2^k n!}{(n-k)!} \frac{1}{\frac{\partial^n}{\partial u^n} a_D^{(0)}(0,\Lambda)} \frac{\partial^{n-k}}{\partial u^{n-k}} a_D^{(k)}(0,\Lambda) \tag{4.294}$$

We emphasize that formulæ similar to (4.281) hold also for the $a^{(n)}$ cycles, by taking linear combinations as follows trivially from the formula (4.311) below.

4.7.4. Exact analytic proof

We can also imagine here an $\hbar$-exact analytic proof of the relation between the Baxter's Q function and a_D period.

$$Q(\theta, P) = \exp \frac{2\pi i \, a_D(\hbar, u, \Lambda_0)}{\hbar} \tag{4.295}$$

following on the lines of the $\hbar \to 0$ (classical SW) proof, by using Cauchy theorem to relate the exact integral for the Baxter's Q function and a_D period. Since $\ln Q$ is i times the integral over $(-\infty, +\infty)$ of the regularised NS momentum (as $b = 1$) (as in (5.49), but see also [1])

$$\mathcal{P}_{reg}(y) = \mathcal{P}(y) + 2ie^{\theta} \cosh \frac{y}{2} - \frac{i}{4} \tanh y \ , \tag{4.296}$$

let us consider the integral of $i\mathcal{P}_{reg}(y)$ on the (oriented) closed curve with the actual numerically computed poles in figure 4.7.4. We can define the exact dual periods as the

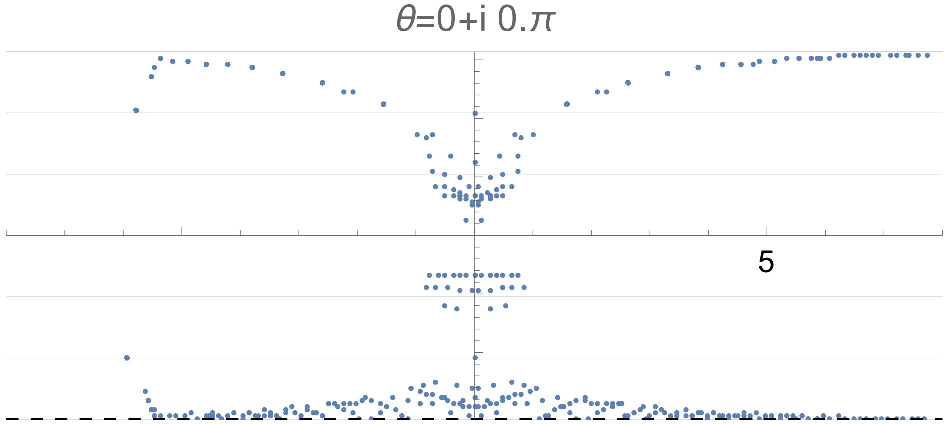

Figure 4.2: Poles for the quantum SW differental $\mathcal{P}(\hbar, u, \Lambda_0)$ for the $SU(2)$ $N_f = 0$ theory. The set of poles in the periodicity strip $|\Im y| < \pi$ we denote by B.

exact integrals of $\mathcal{P}(y) = -i\frac{d}{dy}\ln\psi(y)$ written as sum over residues at the poles which as $\hbar \to 0$ reduce to the classical cycles (branch cuts), as shown in figure 4.7.4.

$$\frac{1}{\hbar}a_D(\hbar, u, \Lambda_0) \doteq \oint_B \mathcal{P}(y, \hbar, u, \Lambda_0)\, dy = 2\pi i \sum_n \mathrm{Res}\mathcal{P}(y)\Big|_{y_n^B} \tag{4.297}$$

One may argue that the choice of poles for the two cycles is not well defined. However, on one hand we numerically find that the period a is given precisely as the integral from $-i\pi$ to $i\pi$ as required by the equality $a = \nu$. On the other hand the choice of poles for the period a_D is unambiguous because it includes all of them. Along this lines we should be able to prove analytically precisely (4.295). However, another exact, precise and unambiguous proof , though perhaps less illuminating since numerical, will be given in the next subsection.

4.7.5. Gauge TBA

As we have a gauge interpretation (4.224) and (4.295) of the self-dual Liouville integrability Baxter's T and Q functions, respectively, we can search for a gauge interpretation of the integrability functional relations (the QQ system, the TQ relation, the periodicity relation, *cf.* Section 4.1 with $b = 1$). First, we write the QQ relation (4.25) at $b = 1$, and then the same in the gauge variables (4.223)

$$1+Q^2(\theta, P^2) = Q(\theta-i\pi/2, P^2)Q(\theta+i\pi/2, P^2)\,, \qquad 1+Q^2(\theta, u) = Q(\theta-i\pi/2, -u)Q(\theta+i\pi/2, -u)\,, \tag{4.298}$$

where we have considered that $\theta \to \theta \mp i\pi/2$ means $u \to -u$ (as P^2 is fixed). The latter equation, the gauge QQ system, has been verified by using the expansion (4.280) in several complex regions of u, in particular in the circle $|u| < \Lambda^2$. In the present case it is a 'square root' of the Y system and then gives us the gauge TBA equations. In fact, we can take the logarithm of both members and invert to obtain an explicit expression for $\ln Q(\theta, u)$. As usual, this inversion possesses zero-modes and so does not fix completely the forcing term. For it we need to consider the asymptotic expansion (4.280) as $\Re\theta \to +\infty$, $\ln Q(\theta,u) \simeq 2\pi i a_D^{(0)}(u,\Lambda)e^{\theta}/\Lambda$. In this way we find a TBA integral equation for the deformed dual period $-2\ln Q(\theta,u) = \varepsilon(\theta,u) = -4\pi i a_D(\hbar(\theta),u)$ and then we close the system by writing the same for modulus $u \to -u$

$$\begin{aligned} \varepsilon(\theta,u,\Lambda) &= -4\pi i a_D^{(0)}(u,\Lambda)\frac{e^{\theta}}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta',-u,\Lambda)\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi} \\ \varepsilon(\theta,-u,\Lambda) &= -4\pi i a_D^{(0)}(-u,\Lambda)\frac{e^{\theta}}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta',u,\Lambda)\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}\,. \end{aligned} \tag{4.299}$$

In contrast with Liouville TBA (where was no P), the forcing terms have non-trivial u-dependences, the SW periods indeed, which can be interpreted (as in [50]) as the mass of a BPS state of a monopole and dyon (via Bilal-Ferrari [69] formulæ, *i.e.* (4.310) for $n=0$), respectively. Actually, the quantum period

$$2\pi i a_D(\hbar(\theta),-u,\Lambda) = 2\pi i a_D^{(0)}(-u,\Lambda)\frac{e^{\theta}}{\Lambda} + \int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\left\{4\pi i a_D(\hbar(\theta'),u,\Lambda)\right\}\right]}{\cosh(\theta-\theta')}\frac{d\theta'}{2\pi}\,. \tag{4.300}$$

can take the place of the first period $a(\hbar,u)$ (linked to T in any case) as the latter can be expressed in terms of the former two via (4.308). From the large θ asymptotic expansion of the integral part, we find all the quantum dual periods modes ($m \geq 1$), as well

$$2\pi i\, a_D^{(m)}(u,\Lambda) = -\Lambda^{1-2m}(-1)^m \int_{-\infty}^{\infty} e^{\theta'(2m-1)} \ln\Big[1+\exp\{-\varepsilon(\theta',-u,\Lambda))\}\Big]\,\frac{d\theta'}{\pi}\,. \tag{4.301}$$

By solving with numerical iterations the two coupled equations of gauge TBA (4.299), we tested these expressions with the analytic WKB recursive periods (4.129, 4.132) for a region of the complex plane slightly larger than $|u| < \Lambda^2$(see for example table 4.3). The $u=0$ unique equation from (4.299) was conjectured numerically in [70]. In order to get more precise result, it is convenient to add the boundary condition and subtract it within the convolution, as done in [71]:

$$\varepsilon(\theta,u) \simeq -2\ln\left(-\frac{2\theta}{\pi}\right) \simeq -2\ln\left[1+\frac{2}{\pi}\ln(1+e^{-\theta})\right] \qquad \theta \to -\infty \tag{4.302}$$

However, even if the procedure is the same that leads to the integrability TBA (4.70), it differs by it because in that case the boundary condition is strictly necessary to solve the TBA (not just to improve the precision), because only in the boundary condition is present the parameter P (while in the gauge TBA is is present also in the leading order).

When u is complex, an alternative way to write the TBA can be given. Adding to the real θ the phase

$$\phi(u) = -\arg\{-i a_D^{(0)}(u)\} \tag{4.303}$$

	$a_D^{(1)}$	$a_D^{(2)}$	$a_D^{(3)}$	$a_D^{(4)}$	$a_D^{(5)}$	$a_D^{(6)}$
WKB	$-0.0445523\,\mathrm{i}$	$+0.0141647\,\mathrm{i}$	$-0.0272573i$	$+0.132656i$	$-1.23083i$	$+18.6813i$
TBA	$-0.0445535\,\mathrm{i}$	$+0.0141649\,\mathrm{i}$	$-0.0272576i$	$+0.132657i$	$-1.23084i$	$+18.6814i$

Table 4.3: A table of comparison between the WKB (4.129) and TBA (4.301) results for the higher cycle modes. Here $u = 1/40$, $\Lambda = 1/4$. They match rather well, at around 1 part in 10^6 for higher cycles beyond the first, slightly less precise at 1 part in 10^5 for the first higher cycle. Here we use about 2000 iterations of the successive approximations method for solving the TBA (in details, within the interval $\theta \in (-200, 200)$ divided in 2^{12} discrete parts). Of course, for other values of the parameters, similar matches hold.

we get the TBA

$$\begin{aligned}\varepsilon(\theta + i\phi(u), u) &= 4\pi| - ia_D^{(0)}(u)|\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\left\{-\varepsilon(\theta', -u)\right\}\right]}{\cosh(\theta + i\phi(u) - \theta')}\frac{d\theta'}{2\pi}\,, \\ \varepsilon(\theta + i\phi(-u), -u) &= 4\pi| - ia_D^{(0)}(-u)|\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\left\{-\varepsilon(\theta', u)\right\}\right]}{\cosh(\theta + i\phi(-u) - \theta')}\frac{d\theta'}{2\pi}\,.\end{aligned} \tag{4.304}$$

or (defining $\Delta\phi(u) = \phi(u) - \phi(-u)$)

$$\begin{aligned}\varepsilon(\theta + i\phi(u), u) &= 4\pi| - ia_D^{(0)}(u)|\frac{e^\theta}{\Lambda} - 2\int_{-\infty - i\phi(-u)}^{\infty - i\phi(-u)} \frac{\ln\left[1 + \exp\left\{-\varepsilon(\theta' + i\phi(-u), -u)\right\}\right]}{\cosh(\theta + i\Delta\phi(u) - \theta')}\frac{d\theta'}{2\pi}\,, \\ \varepsilon(\theta + i\phi(-u), -u) &= 4\pi| - ia_D^{(0)}(-u)|\frac{e^\theta}{\Lambda} - 2\int_{-\infty - i\phi(u)}^{\infty - i\phi(u)} \frac{\ln\left[1 + \exp\left\{-\varepsilon(\theta' + i\phi(u), u)\right\}\right]}{\cosh(\theta + i\Delta\phi(-u) - \theta')}\frac{d\theta'}{2\pi}\,.\end{aligned} \tag{4.305}$$

Applying Cauchy theorem, we can relate the integral on the shifted real axis to the integral on the real axis. If $\Delta\phi(u) < \pi/2$, we have no poles inside the contour and find that the two integrals are equal. Therefore, the analytic continuation in θ of the TBA is:

$$\begin{aligned}\varepsilon(\theta + i\phi(u), u) &= 4\pi| - ia_D^{(0)}(u)|\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\left\{-\varepsilon(\theta' + i\phi(-u), -u)\right\}\right]}{\cosh(\theta + i\Delta\phi(u) - \theta')}\frac{d\theta'}{2\pi}\,, \\ \varepsilon(\theta + i\phi(-u), -u) &= 4\pi| - ia_D^{(0)}(-u)|\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\left\{-\varepsilon(\theta' + i\phi(u), u)\right\}\right]}{\cosh(\theta + i\Delta\phi(-u) - \theta')}\frac{d\theta'}{2\pi}\,.\end{aligned} \tag{4.306}$$

The range $\Delta\phi(u) < \pi/2$ corresponds to the strong coupling region of Seiberg-Witten spectrum. The formulation (4.306) can be useful as a starting point for the extension of the TBA to the weak coupling region. Since different particles are present in the two regions[8], we do expect some fundamental change in our relations to take place. Also, the Q, Y function in integrability are defined to be entire in θ, while the gauge periods not.

[8] The particles at strong coupling being only the magnetic monopole associated to $a_D(u)$ and the dyon associated to $a_D(-u)$ [69, 1]. At weak coupling there are infinite dyonic BPS particles differing by units of electric charge, associated to a [69, 71].

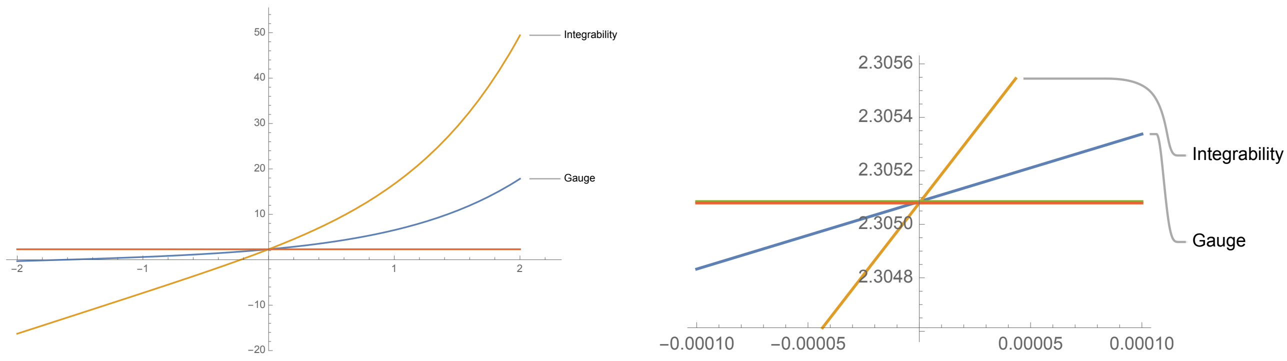

Figure 4.3: Comparison of gauge (blue) $\varepsilon(\theta, u)$ (4.299) and integrability (light orange) $\varepsilon(\theta, P^2)$ (4.70) pseudoenergies for $u = 1/40$, $\Lambda = 1/4$ and $P = \sqrt{2u}e^{\theta_0}/\Lambda$, with $\theta_0 = 0$. At $\theta = \theta_0$ they match very well, up to 1 part in 10^6, with about 2000 iterations of the successive approximations method for solving the TBA (in details, within the interval $\theta \in (-200, 200)$ divided in 2^{12} discrete parts). The horizontal lines (dark orange visible, superimposed to an another green not visible) corresponds to the values of $\varepsilon(\theta_0, u) = 2.30509$ and $\varepsilon(\theta_0, P^2) = 2.30508$. Of course, for other values of the parameters, similar matches hold.

We compared numerically the solution $\varepsilon(\theta, u)$ of the gauge TBA (4.299) and that $\varepsilon(\theta, P^2)$ of the integrability TBA (4.70) and we found that

$$\varepsilon(\theta, u) = \varepsilon(\theta, P^2) \qquad \text{when} \quad P^2 = 2ue^{2\theta}/\Lambda^2 \,, \tag{4.307}$$

that is, we verified numerically the relation (4.295) between the Baxter function and the dual cycle. See for example the plots in figure 4.7.5.

4.7.6. Functional relations and $\mathbb{Z}_2$ symmetry

Consider now the TQ relation (4.27) at $b = 1$, which we also write in the gauge variables (4.223)

$$T(\theta, P^2) = \frac{Q(\theta - i\pi/2, P^2) + Q(\theta + i\pi/2, P^2)}{Q(\theta, P^2)} \,, \qquad T(\theta, u) = \frac{Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u)}{Q(\theta, u)} \tag{4.308}$$

For the asymptotic $\hbar \to 0$ analysis of the latter relation, we keep only the dominant exponents (fixed by SW order (4.282))

$$\exp\Big\{ -\operatorname{sgn}(\Im u) 2\pi i \sum_{n=0}^{\infty} e^{\theta(1-2n)} a^{(n)}(+u) \Big\} \doteq \exp\Big\{ -2\pi \sum_{n=0}^{\infty} e^{\theta(1-2n)} \Big[\operatorname{sgn}(\Im u)(-1)^n a_D^{(n)}(-u) + i a_D^{(n)}(u) \Big] \Big\} . \tag{4.309}$$

Thus, the TQ relation entails

$$a_D^{(n)}(-u) = i(-1)^n \Big[-\operatorname{sgn}(\Im u)\, a_D^{(n)}(u) + a^{(n)}(u) \Big] . \tag{4.310}$$

These relations are, in fact, the extension of the $\mathbb{Z}_2$ symmetry relation in SW ($n = 0$) [69] to the NS-deformed theory [72]. In a nutshell, the TQ relation encodes these $\mathbb{Z}_2$ relations among the asymptotic modes as a unique exact equation. Besides, relation (4.310) – as well as the TQ relation – allows one to express the NS-periods completely in terms of the NS-dual periods in the form:

$$a^{(n)}(u) = \text{sgn}\,(\Im u)\, a_D^{(n)}(u) - i(-1)^n a_D^{(n)}(-u)\,, \tag{4.311}$$

into which we can use the new formulas (4.281) (4.301) for $a_D^{(n)}(u)$.

We finally consider the (integrability)T periodicity relation at $b = 1$ (4.28):

$$T(\theta, P^2) = T(\theta - i\pi/2, P^2) \qquad T(\theta, u) = T(\theta - i\pi/2, -u) \tag{4.312}$$

To interpret this relation through the asymptotic identification (4.279). Thus, the (4.312) relation truncates to

$$\exp\left\{-\,\text{sgn}\,(\Im u) 2\pi i \sum_{n=0}^{\infty} e^{\theta(1-2n)} a^{(n)}(u)\right\} \doteq \exp\left\{+2\pi \sum_{n=0}^{\infty} e^{\theta(1-2n)} (-1)^n a^{(n)}(-u)\right\} \tag{4.313}$$

from which, we deduce the $\mathbb{Z}_2$ symmetry relation for the other period [69] extended to the NS-deformed theory [72]

$$a^{(n)}(-u) = -i(-1)^n\, \text{sgn}\,(\Im u)\, a^{(n)}(u)\,. \tag{4.314}$$

We conclude that, thanks to the identifications (4.224) (4.295) between the integrability and gauge quantities, we can interpret the Baxter's TQ relation (4.308) and T periodicity relation (4.312) as *non-perturbative* $\mathbb{Z}_2$ symmetry relations.

4.7.7. Relation with other gauge period

It was found in [71] a relation between the Q function and the gauge periods A_D, a (in our conventions)

$$Q(\hbar, a, \Lambda_0) = i \frac{\sinh \frac{1}{\hbar} A_D(\hbar, a, \Lambda_0)}{\sinh \frac{2\pi i}{\hbar} a} \tag{4.315}$$

Actually, we could easily check numerically this relation by computing the l.h.s. by the Liouville TBA (4.87) for $b = 1$ and the r.h.s. relies on the expansion of the prepotential $\mathcal{F}$ in Λ_0 (number of instantons) [73, 74]: the period a is related to the moduli parameter u (or P) through the Matone's relation [62, 63] and the dual one is given by $A_D = \partial \mathcal{F}/\partial a$. In this respect we noticed that only the first instanton contributions are easily accessible and summing them up (naively) is accurate as long as $|\Lambda_0|/\hbar \ll 1$. The gauge period is defined as[9]

$$\frac{A_D}{\hbar} = \frac{4a}{\hbar} \ln \frac{i\hbar}{\Lambda_0} + \ln \frac{\Gamma(1 + \frac{2a}{\hbar})}{\Gamma(1 - \frac{2a}{\hbar})} + \frac{1}{\hbar} \frac{8a}{(4a^2 - \hbar^2)^2} \Lambda_0^4 + O(\Lambda_0^8) \tag{4.316}$$

[9] Beware that for the $N_f = 0$ theory with respect to the $N_f = 1, 2$ theories we rescale $\hbar \to \hbar/\sqrt{2}$. This explains the differences with the formulas in subsection 5.3.2.

$\{\Lambda_0, p, \hbar\}$	$-\frac{1}{2}\epsilon(\theta, p)$	$\ln i \sinh A_D / \sinh(2\pi i a/\hbar)$
$\{\frac{\Gamma^2(\frac{1}{4})}{16\sqrt{\pi}}, 2, -i\}$	9.27325	9.273204
$\{\frac{\Gamma^2(\frac{1}{4})}{16\sqrt{\pi}}, 3, -i\}$	18.7522	18.752173
$\{e^{-1}\frac{\Gamma^2(\frac{1}{4})}{16\sqrt{\pi}}, 2, -i\}$	17.2829	17.282910
$\{e^{1}\frac{\Gamma^2(\frac{1}{4})}{16\sqrt{\pi}}, 2, -i\}$	1.04849	1.04235

Table 4.4: Numerical check of formula (4.315). We used only two instanton contribution and so to have a good match we have to restrict to small Λ_0.

$$2u = a^2 - \frac{\Lambda_0}{4}\frac{\partial \mathcal{F}}{\partial \Lambda_0} = a^2 + \frac{\Lambda_0^4}{2(4a^2 - \hbar^2)} + O(\Lambda_0^8) \tag{4.317}$$

The instanton prepotential is given by

$$\mathcal{F}_{NS}^{inst} = \sum_{n=0}^{\infty} \Lambda_0^{4n} \mathcal{F}_{NS}^{(n)} \tag{4.318}$$

with

$$\begin{aligned}
\mathcal{F}_{NS}^{(1)} &= -\frac{2}{4a^2 - \hbar^2} \\
\mathcal{F}_{NS}^{(2)} &= -\frac{20a^2 + 7\hbar^2}{4(a^2 - \hbar^2)(4a^2 - \hbar^2)^3} \\
\mathcal{F}_{NS}^{(3)} &= -\frac{4(144a^4 + 232a^2\hbar^2 + 29\hbar^4)}{3(4a^2 - \hbar^2)^5(4a^4 - 13a^2\hbar^2 + 9\hbar^4)}
\end{aligned} \tag{4.319}$$

$A_D(\hbar, u)$ is very different from our dual cycle $a_D(\hbar, u)$: it is not a cycle integral at all and is defined as the derivative of the prepotential (logarithm of the partition function) coming from instanton counting:

$$A_D(\hbar, u) = \frac{\partial \mathcal{F}_{NS}}{\partial a} \tag{4.320}$$

Thus, thanks to (4.295), relation (4.315) of Grassi, Gu and Marino becomes a relation between the two definition of dual cycles

$$i\frac{\sinh \frac{1}{\hbar}A_D(\hbar, a, \Lambda_0)}{\sinh \frac{2\pi i}{\hbar}a} = \exp \frac{2\pi i\, a_D(\hbar, u)}{\hbar} \,. \tag{4.321}$$

This relation means that the two cycles a_D and A_D differ by non-perturbative terms in $\hbar$. From the gauge theory point of view, they are precisely respectively the dyon and monopole period in the strong coupling region [71].

4.8. D3 brane's quasinormal modes

The D3 brane is described by the line element

$$ds^2 = H(r)^{-\frac{1}{2}}(-dt^2 + d\mathrm{x}^2) + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2)\,, \tag{4.322}$$

where x are the longitudinal coordinates, $H(r) = 1 + L^4/r^4$and $d\Omega_5^2$ denotes the metric of the transverse round S^5-sphere [7]. The ODE which describes the scalar field perturbation of the D3 brane is [7, 75]

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi = 0\,. \tag{4.323}$$

Upon the change of variables

$$r = Le^{\frac{y}{2}} \quad \omega L = -2ie^{\theta} \quad P = \frac{1}{2}(l+2)\,, \tag{4.324}$$

the equation reduces to the generalized Mathieu equation

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0\,. \tag{4.325}$$

Crucially, the QNMs condition (3.25) translates into

$$Q(\theta_n) = 0\,, \tag{4.326}$$

namely the zeros of the Baxter's Q function which are the Bethe roots [76].

We prove now that the Bethe roots condition (4.326) recovers the QNMs characterisation of [6], namely as quantization condition on the gauge period a_D. Indeed, it was found in [71] relation (4.315) between the Q function, as obtained from TBA (4.87), and the gauge periods a_D, a.

Now, (4.326) is the same as the quantization of the A_D period, as originally stated in [6]

$$\frac{1}{\hbar}A_D(a, \Lambda_{0,n}, \hbar) = i\pi n\,, \qquad n \in \mathbb{Z}\,. \tag{4.327}$$

Nevertheless, we found it very difficult to reach, by summing instantons, the QNMs values $|\Lambda_{0,n}|/\hbar \gg 1$.

On the contrary we found very easy using Thermodynamic Bethe Ansatz (TBA) integral equation for the pseudoenergy $\varepsilon(\theta) = -\ln Y(\theta)$. Eventually, the QQ system (4.78) characterizes the QNMs as $Y(\theta_n - i\pi/2) = -1$, i.e. the *TBA quantization condition*

$$\varepsilon(\theta_{n'} - i\pi/2) = -i\pi(2n'+1)\,, \qquad n' \in \mathbb{Z} \tag{4.328}$$

which can be easily implemented by using the TBA (4.87) as table 4.5 shows. These values match very well with those of obtained by the standard method of continued fractions by Leaver [31, 7] and is consistent with the ($l \to \infty$) WKB approximation (geodetic method).

We note that the physical condition $\Im\omega < 0$ becomes by (4.324) $-\pi/2 + 2\pi n < \Im\theta < \pi/2 + 2\pi n$, for $n \in \mathbb{Z}$. However, the TBA (4.87) is valid only for the fundamental strip $|\Im\theta| < \pi/2$. In fact, in this region we find directly the QNMs for overtone number $n = 0 = n'$. We expect that analytically continuing the TBA by using the Y-system (4.86) in the other strips $|\Im(\theta - 2\pi i n)| < \pi/2$, we would obtain the other overtone numbers. We leave more details on this for future work.

Within our set-up of functional and integral equations for entire functions in θ (integrability), we can find other quantization conditions on the roots θ_n (QNMs). For instance, the TQ relation [1]

$$T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)\,. \tag{4.329}$$

n	l	TBA	Leaver	WKB
0	0	$1.36912 - 0.504048i$	$1.36972 - 0.504311i$	$1.41421 - 0.5i$
0	1	$2.09118 - 0.501788i$	$2.09176 - 0.501811i$	$2.12132 - 0.5i$
0	2	$2.8057 - 0.501009i$	$2.80629 - 0.501000i$	$2.82843 - 0.5i$
0	3	$3.51723 - 0.500649i$	$3.51783 - 0.500634i$	$3.53553 - 0.5i$
0	4	$4.22728 - 0.500453i$	$4.22790 - 0.500438i$	$4.24264 - 0.5i$

Table 4.5: Comparison of QNMs of the D3 brane from TBA (4.87) (through (4.328) with $n' = 0$), Leaver (continued fractions) method and WKB (geodetic) approximation ($L = 1$).

means $Q(\theta_n - i\pi/2) + Q(\theta_n + i\pi/2) = 0$. This and the QQ relation (4.78) actually fixes $Q(\theta_n + i\pi/2)Q(\theta_n - i\pi/2) = 1$ and then

$$Q(\theta_n \pm i\pi/2) = \pm i \tag{4.330}$$

are fixed, too. Again (4.78) around θ_n forces $Q(\theta + i\pi/2) = i \pm Q(\theta) + \dots$ and $Q(\theta - i\pi/2) = -i \pm Q(\theta) + \dots$ up to smaller corrections (dots). Therefore, (4.329) imposes

$$T(\theta_n) = \pm 2\,. \tag{4.331}$$

Now, in [1] we have identified the T function through the a period (or Floquet index ν) as

$$T(\theta) = 2\cos\left(\frac{2\pi}{\hbar}a\right)\,. \tag{4.332}$$

In conclusion, condition (4.326) means that also the period a is quantized

$$\frac{1}{\hbar}a(\theta_n) = \frac{n}{2}\,, \qquad n \in \mathbb{Z}\,. \tag{4.333}$$

This is exactly the condition used by [7]. Yet, here we have fixed the general limits of its validity as relying on specific forms of the TQ and QQ systems (4.329) and (4.78) respectively: it does not work in general, but we will see in the next section the specific conditions for its validity.

4.9. On D3 brane's greybody factor

Eventually, we note that much of the BH theory seems to go in parallel to the ODE/IM correspondence construction and its 2D statistical field theory interpretation, beyond the determination of QNMs: as an example, the *absorption coefficient* or *greybody factor* seems a ration of Qs. We aim to give more details about this statement in the future.

4.10. General conclusions

We have shown how quantum integrability, in the approach of the ODE/IM correspondence, can be applied to the $SU(2)$ $N_f = 0$ NS-deformed SW theory, as well as to the D3 brane gravitational perturbation theory, to obtain both new mathematical physics results and improve the general understanding of such theories and their interrelation.

In the next section, we will show a direct albeit technical complex generalization of the same triple correspondence to the $SU(2)$ $N_f = 1$ and $N_f = 2 = (1,1)$ theories. In the subsequent sections, we will begin showing something similar for also the $N_f = 2 = (0,2)$ and $N_f = 3$, though in a much less complete way. In the final section, we will continue doing so for the $N_f = 4$ theory and its simplified version (a certain class $\mathcal{S}$ gauge theory) and further extend our triple correspondence to a 4-fold correspondence, thanks to AdS/CFT.

5. $SU(2)$ $N_f = 1, 2$ gauge theory, Hairpin model, extremal black holes

5.1. ODE/IM correspondence for gauge theory

5.1.1. Gauge/Integrability dictionary

The quantum Seiberg-Witten curves for $SU(2)$ $N_f = 1, 2$ $\mathcal{N} = 2$ gauge theory, deformed in the Nekrasov Shatashvili limit $\epsilon_2 \to 0$, $\epsilon = \hbar \neq 0$ can be derived from the classical one as explained in section 2.5 and they are the following ODEs. For $N_f = 1$

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \left[\frac{\Lambda_1^2}{4}(e^{2y} + e^{-y}) + \Lambda_1 m e^y + u\right]\psi(y) = 0\,, \tag{5.1}$$

for $N_f = 2$ (with the first realization $N_+ = 1$, see section 2.5):

$$-\hbar^2 \frac{d^2}{dy^2}\psi(y) + \left[\frac{\Lambda_2^2}{8}\cosh(2y) + \frac{1}{2}\Lambda_2 m_1 e^y + \frac{1}{2}\Lambda_2 m_2 e^{-y} + u\right]\psi(y) = 0\,, \tag{5.2}$$

where u is the moduli parameter, Λ_1, Λ_2 are the instanton coupling parameters, m, m_1, m_2 are masses of the flavour hypermultiplets [20]. We notice that both equations are of the Doubly Confluent Heun equations [77], with two irregular singularities at $y \to \pm\infty$, as shown in appendix D.

The first physical observation we can make is that they can be mapped into the ODEs for the Integrable Perturbed Hairpin model (IPHM) in the ODE/IM correspondence approach [78] and its generalization:

$$-\frac{d^2}{dy^2}\psi(y) + [e^{2\theta}(e^{2y} + e^{-y}) + 2e^{\theta} q e^y + p^2]\psi(y) = 0\,, \tag{5.3}$$

$$-\frac{d^2}{dy^2}\psi(y) + [2e^{2\theta}\cosh(2y) + 2e^{\theta} q_1 e^y + 2e^{\theta} q_2 e^{-y} + p^2]\psi(y) = 0\,, \tag{5.4}$$

where θ is the TBA rapidity, p, q parametrizes the Fock vacuum of the IPHM and q_1, q_2 their generalization. For $q = 0$, equation (5.3) can be related to the ODE (Generalized Mathieu equation) associated to the Integrable Liouville model with Liouville coupling $b = \sqrt{2}$ [79, 1, 49]. In particular, the gauge/integrability parameter dictionary is the following

$$\frac{\hbar}{\Lambda_1} = \frac{1}{2}e^{-\theta} \qquad \frac{u}{\Lambda_1^2} = \frac{1}{4}p^2 e^{-2\theta} \qquad \frac{m}{\Lambda_1} = \frac{1}{2}q e^{-\theta}\,, \tag{5.5}$$

$$\frac{\hbar}{\Lambda_2} = \frac{1}{4}e^{-\theta} \qquad \frac{u}{\Lambda_2^2} = \frac{1}{16}p^2 e^{-2\theta} \qquad \frac{m_{1,2}}{\Lambda_2} = \frac{1}{4}q_{1,2} e^{-\theta}\,, \tag{5.6}$$

or also

$$\frac{u}{\hbar^2} = p^2 \qquad \frac{m}{\hbar} = q \tag{5.7}$$

$$\frac{u}{\hbar^2} = p^2 \qquad \frac{m_1}{\hbar} = q_1 \qquad \frac{m_2}{\hbar} = q_2\,. \tag{5.8}$$

In [78], P and q were considered fixed, on the other hand, in the gauge theory, it is natural to keep Λ_1, u and m fixed. The mixed dependence on θ gives then a nontrivial map, producing for instance different integrable structures in different parameters.

5.1.2. Integrability functional relations

The integrability equations are invariant under the following discrete symmetries. For $N_f = 1$

$$\begin{aligned} \Omega_+ &: y \to y + 2\pi i/3 \qquad \theta \to \theta + i\pi/3 \qquad q \to -q\,, \\ \Omega_- &: y \to y - 2\pi i/3 \qquad \theta \to \theta + 2\pi i/3 \qquad q \to q\,, \end{aligned} \tag{5.9}$$

for $N_f = 2$

$$\begin{aligned} \Omega_+ &: \; y \to y + i\pi/2\,,\; \theta \to \theta + i\pi/2\,,\; q_1 \to -q_1\,,\; q_2 \to +q_2\,, \\ \Omega_- &: \; y \to y - i\pi/2\,,\; \theta \to \theta + i\pi/2\,,\; q_1 \to q_1\,,\; q_2 \to -q_2\,. \end{aligned} \tag{5.10}$$

This symmetry is spontaneously broken by the regular solutions for $\Re y \to \pm\infty$, defined by the asymptotics, for $N_f = 1$:

$$\begin{aligned} \psi_{+,0}(y) &\simeq 2^{-\frac{1}{2}-q} e^{-(\frac{1}{2}+q)\theta - \left(\frac{1}{2}+q\right)y - e^{\theta+y}} \qquad y \to +\infty\,, \\ \psi_{-,0}(y) &\simeq 2^{-\frac{1}{2}} e^{-\frac{1}{2}\theta + \frac{1}{4}y - 2e^{\theta - y/2}} \qquad y \to -\infty \end{aligned} \tag{5.11}$$

and for $N_f = 2$:

$$\begin{aligned} \psi_{+,0}(y) &\simeq 2^{-\frac{1}{2}-q_1} e^{-(\frac{1}{2}+q_1)\theta - (\frac{1}{2}+q_1)y} e^{-e^{\theta+y}} \qquad \Re y \to +\infty \\ \psi_{-,0}(y) &\simeq 2^{-\frac{1}{2}-q_2} e^{-(\frac{1}{2}+q_2)\theta + (\frac{1}{2}+q_2)y} e^{-e^{\theta-y}} \qquad \Re y \to -\infty\,. \end{aligned} \tag{5.12}$$

The solutions $(\psi_{+,0}, \psi_{-,0})$ of course form a basis. However, we can generate other independent solutions by using the symmetries as

$$\psi_{+,k} = \Omega_+^k \psi_+\,, \qquad \psi_{-,k} = \Omega_-^k \psi_- \qquad k \in \mathbb{Z}\,. \tag{5.13}$$

For $k \neq 0$ such solutions are in general diverging for $y \to \pm\infty$. A basis of solutinos is then given also, for instance, by $(\psi_{+,0}, \psi_{+,1})$. Importantly, the solutions $\psi_\pm$ are invariant under the symmetry $\Omega_\mp$ respectively:

$$\Omega_+ \psi_{-,k} = \psi_{-,k} \qquad \Omega_- \psi_{+,k} = \psi_{+,k}\,. \tag{5.14}$$

The normalization so that we have the following wronskians for next neighbour k-$k+1$ solutions. For $N_f = 1$

$$W[\psi_{+,k+1}, \psi_{+,k}] = i e^{(-1)^k i\pi q} \qquad W[\psi_{-,k+1}, \psi_{-,k}] = -i \tag{5.15}$$

for $N_f = 2$

$$W[\psi_{+,k+1}, \psi_{+,k}] = i e^{(-1)^k i\pi q_1} \qquad W[\psi_{-,k+1}, \psi_{-,k}] = -i e^{(-1)^k i\pi q_2} \tag{5.16}$$

As is usual in ODE/IM correspondence, we can define the integrability Baxter's Q function as the wronskian of the regular solutions at different singular points $y \to \pm\infty$

$$Q = W[\psi_{+,0}, \psi_{-,0}] \tag{5.17}$$

Mathematically this quantity is called also the central connection coefficient, since it appears in the connection relations for solutions at different singular points $y \to \pm\infty$. To write such relations it is convenient to introduce the notation, for $N_f = 1$:

$$Q_\pm(\theta) = W[\psi_{+,0}, \psi_{-,0}](\theta, p, \pm q) \tag{5.18}$$

and for $N_f = 2$:

$$Q_{\pm,\pm}(\theta) = W[\psi_{+,0}, \psi_{-,0}](\theta, p, \pm q_1, \pm q_2) \qquad Q_{\pm,\mp}(\theta) = W[\psi_{+,0}, \psi_{-,0}](\theta, p, \pm q_1, \mp q_2) \tag{5.19}$$

We have to expand the solutions $(\psi_{-,0}, \psi_{-,1})$ in terms of $(\psi_{+,0}, \psi_{+,1})$ with coefficients obtained very simply by taking the wronskians of both sides of the relations and using the symmetries $\Omega_\pm$ to change the parameters of Q. Thus we obtain, for $N_f = 1$

$$ie^{i\pi q}\psi_{-,0} = Q_-(\theta + i\frac{\pi}{3})\psi_{+,0} - Q_+(\theta)\psi_{+,1} \tag{5.20}$$

$$ie^{i\pi q}\psi_{-,1} = Q_-(\theta + i\pi)\psi_{+,0} - Q_+(\theta + i\frac{2\pi}{3})\psi_{+,1} \tag{5.21}$$

and for $N_f = 2$

$$\begin{aligned} ie^{i\pi q_1}\psi_{-,0} &= Q_{-,+}(\theta + i\frac{\pi}{2})\psi_{+,0} - Q_{+,+}(\theta)\psi_{+,1} \\ ie^{i\pi q_1}\psi_{-,1} &= Q_{-,-}(\theta + i\pi)\psi_{+,0} - Q_{+,-}(\theta + i\frac{\pi}{2})\psi_{+,1} \,. \end{aligned} \tag{5.22}$$

By taking the wronskian of the first line with the second line (and also shifting θ and flipping the sign of q), we obtain the first integrability structure, that is the QQ system. For $N_f = 1$

$$Q_+(\theta + i\frac{\pi}{2})Q_-(\theta - i\frac{\pi}{2}) = e^{-i\pi q} + Q_+(\theta - i\frac{\pi}{6})Q_-(\theta + i\frac{\pi}{6}) \,. \tag{5.23}$$

and for $N_f = 2$

$$Q_{+,-}(\theta + \frac{i\pi}{2})Q_{-,+}(\theta - \frac{i\pi}{2}) = e^{-i\pi(q_1 - q_2)} + Q_{-,-}(\theta)Q_{+,+}(\theta) \,. \tag{5.24}$$

For this particular ODEs with two irregular singularities it is possible to define also an integrability Y function and obtain a Y system relation starting directly from the Q function and QQ system relation, rather than from the T functions and T system. So we define a function as, for $N_f = 1$

$$Y_\pm(\theta) = e^{\pm i\pi q}Q_\pm(\theta - i\frac{\pi}{6})Q_\mp(\theta + i\frac{\pi}{6}) \,, \tag{5.25}$$

and for $N_f = 2$

$$Y_{+,\pm}(\theta) = e^{i\pi(q_1 \mp q_2)}Q_{+,\pm}(\theta)Q_{-,\mp}(\theta) \qquad Y_{-,\pm}(\theta) = e^{i\pi(-q_1 \mp q_2)}Q_{-,\pm}(\theta)Q_{+,\mp}(\theta) \,. \tag{5.26}$$

We notice that for $N_f = 1$ in the Y function the Q functions appear with different θ arguments and this will lead to several technical complications for this case, albeit corresponding to one hypermultiplet less. Equivalent definitions are obtained by the QQ systems as, for $N_f = 1$:

$$e^{\pm i\pi q}Q_\pm(\theta + i\frac{\pi}{2})Q_\mp(\theta - i\frac{\pi}{2}) = 1 + Y_\pm(\theta) \tag{5.27}$$

and for $N_f = 2$:

$$e^{i\pi(q_1 - q_2)}Q_{+,-}(\theta + \frac{i\pi}{2})Q_{-,+}(\theta - \frac{i\pi}{2}) = 1 + Y_{+,+}(\theta) \,. \tag{5.28}$$

The Y systems can be now obtained by taking a product of the QQ system with itself with suitable parameters so to obtain a close relation in terms of Y functions. For $N_f = 1$

$$Y_\pm(\theta + i\frac{\pi}{2})Y_\mp(\theta - i\frac{\pi}{2}) = \left[1 + Y_\mp(\theta + i\frac{\pi}{6})\right]\left[1 + Y_\pm(\theta - i\frac{\pi}{6})\right] \tag{5.29}$$

and for $N_f = 2$

$$Y_{+,-}(\theta + \frac{i\pi}{2})Y_{-,+}(\theta - \frac{i\pi}{2}) = [1 + Y_{+,+}(\theta)][1 + Y_{-,-}(\theta)]\,. \tag{5.30}$$

Now, the presence of the irregular singularities of ODEs (5.3)-(5.4) at $y \to +\infty$ (Stokes phenomenon) plays a rôle for defining the T functions, for $N_f = 1$

$$T_+(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}]\,, \qquad \tilde{T}_+(\theta) = iW[\psi_{+,-1}, \psi_{+,1}]\,. \tag{5.31}$$

and for $N_f = 2$

$$T_{+,+}(\theta) = -iW[\psi_{-,-1}, \psi_{-,1}]\,, \qquad \tilde{T}_{+,+}(\theta) = iW[\psi_{+,-1}, \psi_{+,1}]\,. \tag{5.32}$$

(with of course T_- $T_{\mp,\pm}$ defined with the flipped masses as in (5.18) (5.19).) By expanding $\psi_{\pm,1}$ in terms of $\psi_{\pm,0}$, $\psi_{\pm,-1}$, for $N_f = 1$

$$\psi_{+,1} = -e^{2i\pi q}\psi_{+,-1} + e^{i\pi q}\tilde{T}_{+,+}(\theta)\psi_{+,0} \qquad \psi_{-,1} = -\psi_{-,-1} + T_{+,+}(\theta)\psi_{-,0} \tag{5.33}$$

or for $N_f = 2$

$$\psi_{+,1} = -e^{2i\pi q_1}\psi_{+,-1} + e^{i\pi q_1}\tilde{T}_{+,+}(\theta)\psi_{+,0} \qquad \psi_{-,1} = -e^{2i\pi q_2}\psi_{-,-1} + T_{+,+}(\theta)e^{i\pi q_2}\psi_{-,0} \tag{5.34}$$

we obtain the TQ relations, for $N_f = 1$

$$\begin{aligned} T_\pm(\theta)Q_\pm(\theta) &= Q_\pm(\theta - i\frac{2\pi}{3}) + Q_\pm(\theta + i\frac{2\pi}{3}) \\ \tilde{T}_\pm(\theta)Q_\pm(\theta) &= e^{\pm i\pi q_1}Q_\mp(\theta - \frac{i\pi}{3}) + e^{\mp i\pi q_1}Q_\mp(\theta + \frac{i\pi}{3}) \end{aligned} \tag{5.35}$$

or for $N_f = 2$

$$\begin{aligned} T_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi q_2}Q_{+,-}(\theta - \frac{i\pi}{2}) + e^{-i\pi q_2}Q_{+,-}(\theta + \frac{i\pi}{2}) \\ \tilde{T}_{+,+}(\theta)Q_{+,+}(\theta) &= e^{i\pi q_1}Q_{-,+}(\theta - \frac{i\pi}{2}) + e^{-i\pi q_1}Q_{-,+}(\theta + \frac{i\pi}{2})\,. \end{aligned} \tag{5.36}$$

By applying the Ω_+ and Ω_- symmetries to the T and $\tilde{T}$ functions it is immediate to obtain also the periodicity relations, for $N_f = 1$

$$T_\pm(\theta + i\frac{\pi}{3}) = T_\mp(\theta) \qquad \tilde{T}_\pm(\theta + i\frac{2\pi}{3}) = \tilde{T}_\pm(\theta) \tag{5.37}$$

and for $N_f = 2$

$$T_{-,+}(\theta + i\frac{\pi}{2}) = T_{+,+}(\theta) \qquad \tilde{T}_{+,-}(\theta + i\frac{\pi}{2}) = \tilde{T}_{+,+}(\theta)\,. \tag{5.38}$$

5.1.3. Q function's exact expressions and asymptotic expansion

From the ODE/IM analysis, cf. equations (5.20)-(5.22), we find a limit formula Baxter's Q function as, for $N_f = 1$

$$Q_+(\theta) = -ie^{i\pi q} \lim_{y\to+\infty} \frac{\psi_{-,0}(y,\theta)}{\psi_{+,1}(y,\theta)}\,, \tag{5.39}$$

or for $N_f = 2$

$$Q_{+,+}(\theta) = -ie^{i\pi q_1} \lim_{y\to+\infty} \frac{\psi_{-,0}(y,\theta)}{\psi_{+,1}(y,\theta)}\,. \tag{5.40}$$

From this formula we can obtain another which concretely allows to compute Q as an integral. However, to do that, it is convenient first to transform the second order linear ODEs (5.3)-(5.4) for ψ into their equivalent first order nonlinear Riccati equations for the logarithmic derivative of ψ. Besides, since we will need later to asymptotically expand the solution for $y \to \pm\infty$ and $\theta \to \infty$, it is convenient to change variable so to single out the leading order behaviour in y, θ and simplify higher orders calculations. So we change variable as

$$dw = \sqrt{\phi}\, dy \qquad \phi = \begin{cases} -e^{2y} - e^{-y} & N_f = 1 \\ -2\cosh(2y) & N_f = 2 \end{cases}\,. \tag{5.41}$$

To keep the ODE in normal form we have to let $\psi \to \sqrt[4]{\phi}\psi$. Then we take the logarithmic derivative of ψ in the new variable w

$$\Pi = -i\frac{d}{dw}\ln(\sqrt[4]{\phi}\psi) \tag{5.42}$$

and we get for it the Riccati equation

$$\Pi(y)^2 - i\frac{1}{\sqrt{\phi}}\frac{d}{dy}\Pi(y) = e^{2\theta} - e^{\theta}V(y) - U(y)\,, \tag{5.43}$$

with

$$\begin{aligned} V(y) &= \begin{cases} -\frac{2qe^y}{e^{-y}+e^{2y}} & N_f = 1 \\ -\frac{q_1e^y+q_2e^{-y}}{\cosh(2y)} & N_f = 2\,, \end{cases} \\ U(y) &= \begin{cases} -\frac{p^2}{e^{-y}+e^{2y}} + \frac{e^y-40e^{4y}+4e^{7y}}{16(e^{3y}+1)^3} & N_f = 1 \\ \frac{1}{2\cosh(2y)}\left[-p^2 - 1 + \frac{5}{4}\tanh^2(2y)\right] & N_f = 2\,. \end{cases} \end{aligned} \tag{5.44}$$

The first asymptotic expansion we make is the one for $y \to \pm\infty$, in the formal parameter $e^{\mp y}$. The Riccati equation gets approximated, at the leading and subleading order as

$$\Pi(y)^2 - i\frac{1}{\sqrt{\phi}}\frac{d}{dy}\Pi(y) \simeq \begin{cases} e^{2\theta} + 2e^{\theta}\delta_+ qe^{-y} & N_f = 1 \\ e^{2\theta} + 2e^{\theta}q_{1,2}e^{\mp y} & N_f = 2 \end{cases} \qquad y \to \pm\infty \tag{5.45}$$

where for $N_f = 1$ $\delta_+ = 1$ for $y \to +\infty$, $\delta_+ = 0$ for $y \to -\infty$. Then the solution is asymptotic to

$$\Pi(y) \simeq \begin{cases} e^{\theta} + \delta_+ qe^{-y} & N_f = 1 \\ e^{\theta} + q_{1,2}e^{\mp y} & N_f = 2 \end{cases} \qquad y \to \pm\infty \tag{5.46}$$

This leading expansion for $y \to \pm\infty$ allows us to fix the regularization in the integrals formulas we now write for the (logarithm) of $\psi_{-,0}$, for $N_f = 1$

$$\psi_{-,0}(y) = \frac{2^{-\frac{1}{2}} e^{-\frac{1}{2}\theta}}{\sqrt[4]{e^{2y} + e^{-y}}} \exp\left\{-e^{\theta}(2e^{-y/2} - e^{y}) + 2q\ln(1 + e^{y/2})\right\} \times \\ \exp\left\{\int_{-\infty}^{y} dy' \left[\sqrt{e^{2y'} + e^{-y'}}\Pi(y', \theta, p, q) - e^{\theta}\left(e^{y'} + e^{-y'/2}\right) - q\frac{1}{1 + e^{-y'/2}}\right]\right\} \tag{5.47}$$

and for $N_f = 2$

$$\psi_{-,0}(y) = \frac{2^{-\frac{1}{2} - q_2} e^{-(\frac{1}{2} + q_2)\theta}}{\sqrt[4]{e^{2y} + e^{-2y}}} \exp\left\{-e^{\theta}(e^{-y} - e^{y}) + 2q_1\ln(1 + e^{y/2}) - 2q_2\ln(1 + e^{-y/2})]\right\} \times \\ \exp\left\{\int_{-\infty}^{y} dy' \left[\sqrt{e^{2y'} + e^{-2y'}}\Pi(y', \theta, p, q_1, q_2) - e^{\theta}(e^{y'} + e^{-y'}) - q_1\frac{1}{1 + e^{-y'/2}} - q_2\frac{1}{1 + e^{y'/2}}\right]\right\} . \tag{5.48}$$

Then from the limit formula for Q we get also an integral expression for it, for $N_f = 1$

$$\ln Q_+(\theta) = \int_{-\infty}^{\infty} dy \left[\sqrt{e^{2y} + e^{-y}}\Pi(y, \theta, q, p) - e^{\theta}e^{y} - e^{\theta}e^{-y/2} - q\frac{1}{1 + e^{-y/2}}\right] - (\theta + \ln 2)\, q\, . \tag{5.49}$$

and for $N_f = 2$

$$\ln Q_{+,+}(\theta) = \int_{-\infty}^{\infty} dy \left[\sqrt{2\cosh(2y)}\Pi(y, \theta, q_1, q_2, p) - 2e^{\theta}\cosh y - \left(\frac{q_1}{1 + e^{-y/2}} + \frac{q_2}{1 + e^{y/2}}\right)\right] - (\theta + \ln 2)\,(q_1 + q_2) \tag{5.50}$$

To get the vacuum eigenvalues of the local integrals of motion (LIMs) we make instead the $\theta \to +\infty$ asymptotic expansion, at all orders

$$\Pi(y, \theta) \doteq e^{\theta} + \sum_{n=0}^{\infty} \Pi_n(y) e^{-n\theta} \qquad \theta \to +\infty\, . \tag{5.51}$$

Its coefficients Π_n satisfy the recursion relation

$$\Pi_{n+1} = \frac{1}{2}\left(\frac{i}{\sqrt{\phi}}\frac{d}{dy}\Pi_n - \sum_{m=0}^{n}\Pi_m\Pi_{n-m}\right) \qquad n \geq 1 \tag{5.52}$$

with initial conditions

$$\Pi_0 = -\frac{1}{2}V \\ \Pi_1 = \frac{1}{2}\left(\frac{i}{\sqrt{\phi}}\frac{d}{dy}\Pi_0 - \Pi_0^2 - U\right) \tag{5.53}$$

The expansion of $\ln Q$ in terms of the LIMs is, for $N_f = 1$

$$\ln Q_+(\theta) \doteq -\frac{4\sqrt{3\pi^3}}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}e^{\theta} - (\theta + \frac{1}{3}\ln 2)q - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbb{I}_n \qquad \theta \to +\infty \tag{5.54}$$

and for $N_f = 2$

$$\ln Q_{+,+}(\theta) \doteq -\frac{4\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^\theta - (\theta + \frac{1}{2}\ln 2)(q_1+q_2) - \sum_{n=1}^{\infty} e^{-n\theta} C_n \mathbb{I}_n \qquad \theta \to +\infty\,, \tag{5.55}$$

with the local integrals of motion $\mathbb{I}_n$ times some normalization constants C_n given by the integrals

$$C_n \mathbb{I}_n(p,q) = -i\int_{-\infty}^{\infty} dy\,\sqrt{\phi(y)}\Pi_n(y,p,q) \qquad n \geq 1\,. \tag{5.56}$$

$\mathbb{I}_n(p,q)$ are in geneal polynomials in p,q, where q of course here stands for either q for $N_f = 1$ or (q_1, q_2) for $N_f = 2$. We have checked the first ones for $N_f = 1$ to match with those of IPHM given in [78].

$$\begin{aligned} \mathbb{I}_1(p,q) &= \frac{1}{12}\left(4q^2 - 12p^2 - 1\right) \\ \mathbb{I}_2(p,q) &= \frac{1}{6\sqrt{3}} q\left(\frac{20}{3}q^2 - 12p^2 - 3\right) \end{aligned} \tag{5.57}$$

For $N_f = 2$ they were never given in the literature to our knowledge and they have the peculiar feature that the mixed q_1, q_2 terms have trascendental coefficients (Gamma functions). We notice also that the one step recursion very effective method of computation of LIMs explained in [1] does not directly generalize to this case where all $e^{-n\theta}$ are present in the asymptotic expansion. Further investigations on such LIMs issues could be pursued.

5.1.4. Integrability TBA

Define as usual the pseudoenergy $\varepsilon(\theta) = -\ln Y(\theta)$ and $L = \ln[1+\exp(-\varepsilon)]$ (with suitable subscripts omitted of course). Using the analytic properties of pseudoenergy ϵ, we can transform the Y system (5.29) into the following 'integrability TBAs'. For $N_f = 1$ [78]

$$\begin{aligned} \varepsilon_+(\theta) &= \frac{12\sqrt{\pi^3}}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}e^\theta - \frac{4}{3}i\pi q - (\varphi_{++} * L_+)(\theta) - (\varphi_{+-} * L_-)(\theta) \\ \varepsilon_-(\theta) &= \frac{12\sqrt{\pi^3}}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}e^\theta + \frac{4}{3}i\pi q - (\varphi_{++} * L_-)(\theta) - (\varphi_{+-} * L_+)(\theta)\,, \end{aligned} \tag{5.58}$$

and for $N_f = 2$

$$\begin{aligned} \varepsilon_{+,+}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^\theta - i\pi(q_1 - q_2) - \varphi * (L_{+-} + L_{-+}) \\ \varepsilon_{+,-}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^\theta - i\pi(q_1 + q_2) - \varphi * (L_{++} + L_{--}) \\ \varepsilon_{-,+}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^\theta + i\pi(q_1 + q_2) - \varphi * (L_{--} + L_{++}) \\ \varepsilon_{-,-}(\theta) &= \frac{8\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^\theta + i\pi(q_1 - q_2) - \varphi * (L_{-+} + L_{+-})\,. \end{aligned} \tag{5.59}$$

The leading (driving) term follows directly from the expansions (5.54)-(5.55) under the definitions for $Y = \exp -\varepsilon$ (5.25)-(5.26). The symbol $*$ stands for the $(-\infty, +\infty)$ convolution, which for general functions f, g

$$(f * g)(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} f(\theta - \theta') g(\theta') \tag{5.60}$$

The kernel for $N_f = 2$ is the simple usual hyperbolic secant [80]

$$\varphi(\theta) = \frac{1}{\cosh\theta}, \tag{5.61}$$

while the one for $N_f = 1$ is slightly more involved because of the shifts in θ also on the RHS of the Y system (5.29) but can be obtained by taking Fourier transform as explained in [81]

$$\varphi_{+\pm}(\theta) = \frac{\sqrt{3}}{2\cosh\theta \pm 1}. \tag{5.62}$$

We notice that q, q_1, q_2 enter the integrability TBAs as chemical potentials [82]. In these TBAs the parameter p does not appear, but it enters in the boundary condition for the solution ε at $\theta \to -\infty$, for $N_f = 1$

$$\varepsilon_\pm(\theta, p) \simeq 6p\theta \mp i\pi q + 2C(p, q) \qquad \theta \to -\infty, \tag{5.63}$$

and for $N_f = 2$

$$\varepsilon_{+,+}(\theta, p) \simeq 4p\theta - i\pi(q_1 - q_2) + 2C(p, q_1, q_2) \qquad \theta \to -\infty \tag{5.64}$$

with

$$C(p) = \begin{cases} \ln\left[\frac{2^{-p}\Gamma(2p)\Gamma(1+2p)}{\sqrt{2\pi}\sqrt{\Gamma(\frac{1}{2}+p+q)\Gamma(\frac{1}{2}+p-q)}}\right] & N_f = 1 \\ \ln\left[\frac{2^{1-2p}p\,\Gamma(2p)^2}{\sqrt{\Gamma(p+\frac{1}{2}-q_1)\Gamma(p+\frac{1}{2}+q_1)\Gamma(p+\frac{1}{2}-q_2)\Gamma(p+\frac{1}{2}-q_2)}}\right] & N_f = 2\,. \end{cases} \tag{5.65}$$

This asymptotic behaviour follows of course from the $\theta \to -\infty$ perturbative expansion of the ODE (shifting y by $\pm\theta$ in the ODE so to eliminate the leading terms at $y \to \mp\infty$ and get the solution as confluent hypergeometric function, expanding it in e^θ and taking the wronskian, see also [78] for $N_f = 1$). We can solve therefore this TBA by adding and subtracting outside and inside the convolutions the boundary condition for $\theta \to -\infty$ which depends on p. For example, for $N_f = 1$ the numerically solvable integrability TBA reads

$$\begin{aligned} \varepsilon_+(\theta) &= \frac{12\sqrt{\pi^3}}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)} e^\theta - \frac{4}{3} i\pi q - f_0(\theta) - f_1(\theta) - (\varphi_{++} * (L_+ - L_0 - L_1))(\theta) - (\varphi_{+-} * (L_- - L_0 - L_1))(\theta) \\ \varepsilon_-(\theta) &= \frac{12\sqrt{\pi^3}}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)} e^\theta + \frac{4}{3} i\pi q - f_0(\theta) - f_1(\theta) - (\varphi_{++} * (L_- - L_0 - L_1))(\theta) - (\varphi_{+-} * (L_+ - L_0 - L_1))(\theta), \end{aligned} \tag{5.66}$$

where the explicit terms can be derived in analogue way to [49] as

$$\begin{aligned} L_0(\theta) &= 3p\ln\left[1+e^{-2\theta}\right] , \\ L_1(\theta) &= C(p)(1-\tanh\theta) , \\ f_0(\theta) &= \varphi * L_0 = 3p\left\{\ln\left[1+e^{-(\theta+i\pi/6)}\right] + \ln\left[1+e^{-(\theta-i\pi/6)}\right]\right\} , \\ f_1(\theta) &= \varphi * L_1 = C(p)\left[1-\frac{1}{2}\tanh\left(\frac{\theta}{2}+\frac{i\pi}{12}\right)-\frac{1}{2}\tanh\left(\frac{\theta}{2}-\frac{i\pi}{12}\right)\right] . \end{aligned} \tag{5.67}$$

We notice that the constant term $i\pi q$ in (5.63) is automatically produced by the contribution of the the complex convolution. We notice also that boundary condition (5.63) requires strictly $p > 0$, which in gauge theory will correspond to $u/\Lambda^2_{1,2} > 0$ by (5.5). However, we shall see that we can solve the TBA in gauge variables for $u/\Lambda^2_{1,2} \in \mathbb{C}$ (small), thus providing an analytic continuation of the integrability TBA. For $N_f = 2$ instead the corresponding auxiliary functions are

$$\begin{aligned} L_0(\theta) &= 2p\ln\left[1+e^{-2\theta}\right] , \\ L_1(\theta) &= C(p)(1-\tanh\theta) , \\ f_0(\theta) &= \varphi * L_0 = 4p\ln\left[1+e^{-\theta}\right] , \\ f_1(\theta) &= \varphi * L_1 = C(p)\left[1-\tanh\left(\frac{\theta}{2}\right)\right] . \end{aligned} \tag{5.68}$$

We notice that (5.59) generalizes the TBA found in [78] for the Perturbed Hairpin IM and therefore we call the IM involved (with no much creativity admittedly) Generalized Perturbed Hairpin IM.

Now from the TBA solution we can obtain also Q as follows. Writing from the QQ system for $N_f = 1$ (5.27)

$$\begin{gathered} [Q_+(\theta+i\pi/2)Q_-(\theta+i\pi/2)][Q_+(\theta-i\pi/2)Q_-(\theta-i\pi/2)] = [1+Y_+(\theta)][1+Y_-(\theta)] \\ \left[\frac{Q_+(\theta+i\pi/2)}{Q_-(\theta+i\pi/2)}\right]\left[\frac{Q_+(\theta-i\pi/2)}{Q_-(\theta-i\pi/2)}\right]^{-1} = e^{-2\pi i q}\frac{1+Y_+(\theta)}{1+Y_-(\theta)} \end{gathered} \tag{5.69}$$

we easily deduce the following integral expression for Q for $N_f = 1$

$$\begin{aligned} \ln Q_\pm(\theta) = &-\frac{4\sqrt{3\pi^3}}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}e^\theta \mp (\theta+\frac{1}{3}\ln 2)q \\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\left\{\frac{\ln[1+\exp\{-\varepsilon_+(\theta')\}][1+\exp\{-\varepsilon_-(\theta')\}]}{\cosh(\theta-\theta')} \mp i\frac{e^{\theta'-\theta}}{\cosh(\theta-\theta')}\ln\left[\frac{1+\exp\{-\varepsilon_+(\theta')\}}{1+\exp\{-\varepsilon_-(\theta')\}}\right]\right\} . \end{aligned} \tag{5.70}$$

Similarly for $N_f = 2$ it follows

$$\begin{aligned} \ln Q_{\pm,\mp}(\theta) = &-\frac{4\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}e^\theta \mp (\theta+\frac{1}{2}\ln 2)(q_1-q_2) \\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\left\{\frac{\ln[1+\exp\{-\varepsilon_{+,+}(\theta')\}][1+\exp\{-\varepsilon_{-,-}(\theta')\}]}{\cosh(\theta-\theta')} \mp i\frac{e^{\theta'-\theta}}{\cosh(\theta-\theta')}\ln\left[\frac{1+\exp\{-\varepsilon_{+,+}(\theta')\}}{1+\exp\{-\varepsilon_{-,-}(\theta')\}}\right]\right\} . \end{aligned} \tag{5.71}$$

5.2. Integrability Y function and dual gauge period

5.2.1. Gauge TBA

To establish a connection between integrability and gauge theory, we need first of all to express all integrability definitions and relations in gauge variables through the parameter dictionaries (5.5)-(5.6). Thus for $N_f = 1$ we can introduce 6 gauge Q and Y functions defined, for $k = 0, 1, 2$, as

$$Q_{\pm,k}(\theta) = Q(\theta, -u_k, \pm m_k, \Lambda_1)\,, \quad Y_{\pm,k}(\theta) = Y(\theta, u_k, \pm i m_k, \Lambda_1)\,. \tag{5.72}$$

where for simplicity we denote

$$u_k = e^{2\pi i k/3} u \qquad m_k = e^{-2\pi i k/3} m \qquad k = 0, 1, 2 \tag{5.73}$$

The explicit relation between Q and Y is for example, for $k = 0$ (from (5.25) and (5.5))

$$Y_{\pm,0}(\theta) = Y(\theta, u, \pm i m, \Lambda_1) = e^{\mp 2\pi \frac{m}{\Lambda_1} e^{\theta}} Q_{\pm,2}(\theta - i\pi/6) Q_{\pm,1}(\theta + i\pi/6)\,. \tag{5.74}$$

For $N_f = 2$ instead we have 8 Q and Y functions

$$Y_{\pm,\pm}(\theta) = Y(\theta, u, \pm m_1, \pm m_2, \Lambda_2) \qquad \bar{Y}_{\pm,\pm}(\theta) = Y(\theta, -u, \mp i m_1, \pm i m_2, \Lambda_2) \tag{5.75}$$

It is convenient to write the gauge Y system (5.29) explicitly as, for $N_f = 1$

$$\begin{aligned}
Y_{\pm,0}(\theta + i\pi/2) Y_{\pm,0}(\theta - i\pi/2) &= [1 + Y_{\pm,1}(\theta + i\pi/6)]\,[1 + Y_{\pm,2}(\theta - i\pi/6)] \\
Y_{\pm,1}(\theta + i\pi/2) Y_{\pm,1}(\theta - i\pi/2) &= [1 + Y_{\pm,2}(\theta + i\pi/6)]\,[1 + Y_{\pm,0}(\theta - i\pi/6)] \\
Y_{\pm,2}(\theta + i\pi/2) Y_{\pm,2}(\theta - i\pi/2) &= [1 + Y_{\pm,0}(\theta + i\pi/6)]\,[1 + Y_{\pm,1}(\theta - i\pi/6)]
\end{aligned} \tag{5.76}$$

and for $N_f = 2$

$$\begin{aligned}
\bar{Y}_{\pm,\pm}(\theta + i\pi/2) \bar{Y}_{\pm,\pm}(\theta - i\pi/2) &= [1 + Y_{\pm,\pm}(\theta)][1 + Y_{\mp,\mp}(\theta)] \\
Y_{\pm,\pm}(\theta + i\pi/2) Y_{\pm,\pm}(\theta - i\pi/2) &= [1 + \bar{Y}_{\pm,\pm}(\theta)][1 + \bar{Y}_{\mp,\mp}(\theta)]
\end{aligned} \tag{5.77}$$

Notice that with respect to what happens in the integrability variables, in the gauge variables the number of Q, Y functions increases (triples for $N_f = 1$, doubles for $N_f = 2$), as it happens for the $SU(2)$ $N_f = 0$ theory (where it doubles) [1]. Besides the Q and Y systems in gauge variables simplify their dependence on the flipped masses.

Again, as explained in [81], it straightforward to invert the Y-systems into the following 'gauge TBAs'. For $N_f = 1$:

$$\begin{aligned}
\varepsilon_{\pm,0}(\theta) &= \varepsilon^{(0)}_{\pm,0} e^{\theta} - (\varphi_+ * L_{\pm,1})(\theta) - (\varphi_- * L_{\pm,2})(\theta) \\
\varepsilon_{\pm,1}(\theta) &= \varepsilon^{(0)}_{\pm,1} e^{\theta} - (\varphi_+ * L_{\pm,2})(\theta) - (\varphi_- * L_{\pm,0})(\theta) \\
\varepsilon_{\pm,2}(\theta) &= \varepsilon^{(0)}_{\pm,2} e^{\theta} - (\varphi_+ * L_{\pm,0})(\theta) - (\varphi_- * L_{\pm,1})(\theta)\,,
\end{aligned} \tag{5.78}$$

and for $N_f = 2$

$$\begin{aligned}
\varepsilon_{\pm,\pm}(\theta) &= \varepsilon^{(0)}_{\pm,\pm} e^{\theta} - \varphi * (\bar{L}_{\pm\pm} + \bar{L}_{\mp\mp})(\theta) \\
\bar{\varepsilon}_{\pm,\pm}(\theta) &= \bar{\varepsilon}^{(0)}_{\pm,\pm} e^{\theta} - \varphi * (L_{\pm\pm} + L_{\mp\mp})(\theta)\,.
\end{aligned} \tag{5.79}$$

The new kernels for $N_f = 1$ $\varphi_\pm$ are defined as

$$\varphi_\pm(\theta) = \frac{1}{\cosh(\theta \pm i\pi/6)} \,. \tag{5.80}$$

The leading order coeffient for $N_f = 1$, for example for $k = 0$ writes explicitly as

$$\varepsilon^{(0)}_{\pm,k} = -e^{-i\pi/6} \ln Q^{(0)}(-e^{-2\pi i/3}u_k, \pm e^{2\pi i/3}m_k, \Lambda_1) - e^{i\pi/6} \ln Q^{(0)}(-e^{2\pi i/3}u_k, \pm e^{-2\pi i/3}m_k, \Lambda_1) \pm \frac{8}{3}\pi \frac{m_k}{\Lambda_1} \,, \tag{5.81}$$

where $\ln Q^{(0)}(u, m, \Lambda_1)$ is given by the integral

$$\ln Q^{(0)}(u, m, \Lambda_1) = \int_{-\infty}^{\infty} \left[\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} e^y + \frac{4u}{\Lambda_1^2}} - e^y - e^{-y/2} - 2\frac{m}{\Lambda_1} \frac{1}{1 + e^{-y/2}} \right] dy \,. \tag{5.82}$$

For $N_f = 2$ also

$$\begin{aligned} \varepsilon^{(0)}_{\pm,\pm} &= -\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) - \ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) \mp \frac{4\pi i}{\Lambda_2}(m_1 - m_2) \\ \bar{\varepsilon}^{(0)}_{\pm,\pm} &= -\ln Q^{(0)}(-u, -im_1, im_2, \Lambda_2) - \ln Q^{(0)}(-u, im_1, -im_2, \Lambda_2) \mp \frac{4\pi}{\Lambda_2}(m_1 + m_2) \end{aligned} \tag{5.83}$$

and

$$\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) \tag{5.84}$$

$$= \int_{-\infty}^{\infty} \left[\sqrt{2\cosh(2y) + \frac{8m_1}{\Lambda_2} e^y + \frac{8m_2}{\Lambda_2} e^{-y} + \frac{16u}{\Lambda_2^2}} - 2\cosh y - \frac{4m_1}{\Lambda_2} \frac{1}{1 + e^{-y/2}} - \frac{4m_2}{\Lambda_2} \frac{1}{1 + e^{y/2}} \right] dy \,. \tag{5.85}$$

We can simply compute concretely $\ln Q^{(0)}$ by expanding the square root integrand in multiple binomial series for small parameters and then getting simple Beta function integrals. In particular, for $N_f = 1$ we get

$$\ln Q^{(0)}(u, m, \Lambda_1) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \binom{1/2}{n} \binom{1/2 - n}{l} B_1(n, l) \left(\frac{4m}{\Lambda_1} \right)^n \left(\frac{4u}{\Lambda_1^2} \right)^l \tag{5.86}$$

with

$$\begin{aligned} B_1(n, l) &= \frac{1}{3} B\left(\frac{1}{6}(2l + 4n - 1), \frac{1}{3}(2l + n - 1) \right) \qquad (n, l) \neq (1, 0) \\ B_1(1, 0) &= \frac{2\ln(2)}{3} \end{aligned} \tag{5.87}$$

and for $N_f = 2$ we obtain

$$\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) = \sum_{l,m,n=0}^{\infty} \binom{\frac{1}{2}}{l} \binom{\frac{1}{2} - l}{m} \binom{-l - m + \frac{1}{2}}{n} B_2(l, m, n) \left(\frac{8m_1}{\Lambda_2} \right)^n \left(\frac{16u}{\Lambda_2^2} \right)^m \left(\frac{8m_2}{\Lambda_2} \right)^l \tag{5.88}$$

with

$$
\begin{aligned}
B_2(l,m,n) &= \frac{\Gamma\left(\frac{1}{4}(3l+2m+n-1)\right)\Gamma\left(\frac{1}{4}(l+2m+3n-1)\right)}{4\Gamma\left(l+m+n-\frac{1}{2}\right)} \\
B_2(1,0,0) &= \frac{1}{2}(\ln 2 - 1) \quad B_2(0,0,1) = \frac{1}{2}\ln 2\,.
\end{aligned}
\tag{5.89}
$$

Of course, when u, m, Λ_1 (u, m_1, m_2, Λ_2) are such that the leading order (5.81) computed through (5.82) has a negative real part, the TBA (5.78) no longer converges. In general, we find the convergence region to correspond to u, m (u, m_1, m_2) finite but small with respect to Λ_1 (Λ_2). For instance in the $N_f = 1$ massless case, this region corresponds on the real axis of u precisely to the strong coupling region $-3\Lambda_1^2/2^{8/3} < u < 3\Lambda_1^2/2^{8/3}$. For $N_f = 2$ instead it corresponds to the region $-3\Lambda_2^2/8 < u < 3\Lambda_2^2/8$ [83].

Following [49, 71], it is easy to find the boundary condition at $\theta \to -\infty$ for the gauge TBA

$$
\varepsilon_{\pm,k}(\theta) \simeq -2\ln\left(-\frac{2}{\pi}\theta\right) \simeq \hat{f}(\theta), \quad \theta \to -\infty, \tag{5.90}
$$

with

$$
\hat{f}(\theta) = -\ln\left(1 + \frac{2}{\pi}\ln\left(1 + e^{-\theta - \frac{\pi i}{6}}\right)\right) - \ln\left(1 + \frac{2}{\pi}\ln\left(1 + e^{-\theta + \frac{\pi i}{6}}\right)\right). \tag{5.91}
$$

Numerically, this condition is imposed by modified the TBA equations to be

$$
\varepsilon_{\pm,k}(\theta) = \varepsilon_{\pm,k}^{(0)} e^{\theta} + \hat{f}(\theta) - \left(\varphi_+ * \left(L_{\pm,(k+1)\,\mathrm{mod}\,3} + \hat{L}\right)\right)(\theta) - \left(\varphi_- * \left(L_{\pm,(k+2)\,\mathrm{mod}\,3} + \hat{L}\right)\right)(\theta), \tag{5.92}
$$

where $\hat{L}$ is fixed by $\hat{f} = (\varphi_+ + \varphi_-) * \hat{L}$. Under this boundary condition (5.90), the dilogarithm trick leads to the "effective central charge" associated with the TBA equations (5.78)

$$
c_{eff} = \frac{6}{\pi^2}\int d\theta e^{\theta} \sum_{j=0}^{2} \varepsilon_{\pm,j}^{(0)} L_{\pm,j}(\theta) = 3, \tag{5.93}
$$

which coincides the numeric test and thus tests the validity of our boundary condition. We notice that even if we had not added the boundary condition at $\theta \to -\infty$, the solution of the gauge TBA (5.78) would have been fixed anyway, just giving a less precise numerical solution. We remark that the same thing would have not been true for the integrability TBA (5.58), since the boundary condition is strictly necessary to fix p, which does not enter the forcing term [49].

Similarly for $N_f = 2$ the effective central charge of $N_f = 2$ case is found to be

$$
c_{eff} = \frac{6}{\pi^2}\int d\theta e^{\theta} \sum_{\pm} \left(\varepsilon_{\pm,\pm}^{(0)} L_{\pm,\pm}(\theta) + \bar{\varepsilon}_{\pm,\pm}^{(0)} \bar{L}_{\pm,\pm}(\theta)\right) = 4, \tag{5.94}
$$

and we find the consistent boundary condition at $\theta \to -\infty$:

$$
\varepsilon_{\pm,\pm}(\theta) \simeq -2\ln\left(-\frac{2\theta}{\pi}\right) \simeq -2\ln\left[1 + \frac{2}{\pi}\ln(1 + e^{-\theta})\right] \qquad \theta \to -\infty \tag{5.95}
$$

and so

$$
\hat{f}(\theta) = -2\ln\left(1 + \frac{2}{\pi}\ln\left(1 + e^{-\theta}\right)\right). \tag{5.96}
$$

5.2.2. Seiberg-Witten gauge/integrability identification

We can now begin to find, first at the leading $\hbar \to 0$ ($\theta \to +\infty$) order, a relation between the integrability quantity $\varepsilon^{(0)}$ and the gauge periods. It is just a more complex version of the proof reported in section 4.7.1 for the simpler $SU(2)$ $N_f = 0$ gauge theory. We do now the proof for $N_f = 1$. In (5.81) for $k = 0$ we have the following integral contributions

$$e^{-i\pi/6} \ln Q^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) = \tag{5.97}$$

$$= \int_{-\infty-2\pi i/3}^{\infty-2\pi i/3} \left[\sqrt{-e^{2y} - \frac{4m}{\Lambda_1}e^{+y} + \frac{4u}{\Lambda_1^2} - e^{-y}} - ie^{y} + ie^{-y/2} - i2\frac{m}{\Lambda_1}\frac{1}{1+e^{-y/2-\pi i/3}} \right] dy \tag{5.98}$$

and

$$e^{i\pi/6} \ln Q^{(0)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m) = \tag{5.99}$$

$$= \int_{-\infty+2\pi i/3}^{\infty+2\pi i/3} \left[-\sqrt{-e^{2y} - \frac{4m}{\Lambda_1}e^{+y} + \frac{4u}{\Lambda_1^2} - e^{-y}} + ie^{y} - ie^{-y/2} + i2\frac{m}{\Lambda_1}\frac{1}{1+e^{-y/2+\pi i/3}} \right] dy\,. \tag{5.100}$$

We notice that the integrands in (5.97) and (5.99) are equal except for the mass regularizing term, which gives an integral difference[10]

$$2\frac{im}{\Lambda_1} \int_{-\infty+2\pi i/3}^{\infty+2\pi i/3} \left[\frac{1}{1+e^{-y/2+\pi i/3}} - \frac{1}{1+e^{-y/2-\pi i/3}} \right] dy = \frac{2}{\Lambda_1}\frac{4\pi m}{3}\,. \tag{5.101}$$

We can then consider only the integrand of $\ln Q^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m)$. We observe that such integrand is nothing but the Seiberg-Witten differential λ, up to a total derivative

$$e^{-i\pi/6} \ln Q^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) \tag{5.102}$$

$$= i \int_{-\infty-2\pi i/3}^{+\infty-2\pi i/3} dy \left[\frac{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1}e^{y} - \frac{4u}{\Lambda_1^2}}{\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1}e^{y} - \frac{4u}{\Lambda_1^2}}} - \frac{d}{dy}\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1}e^{y} - \frac{4u}{\Lambda_1^2}} - \text{reg.} \right] \tag{5.103}$$

$$= 4i \int_{-\infty-2\pi i/3}^{+\infty-2\pi i/3} dy \left[\frac{\frac{3}{8}e^{-y} + \frac{1}{2}\frac{m}{\Lambda_1}e^{y} - \frac{u}{\Lambda_1^2}}{\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1}e^{y} - \frac{4u}{\Lambda_1^2}}} - \text{reg.} \right] = -\frac{4\sqrt{2}\pi}{\Lambda_1} \int_{-\infty-2\pi i/3}^{+\infty-2\pi i/3} \lambda(y, -u, m, \Lambda_1)\, dy \tag{5.104}$$

where the SW differential λ [84] is defined as usual in the variable $x = -\frac{\Lambda_1^2}{4}e^{-y}$ as

$$\lambda(x,-u)dx = \frac{1}{2\pi\sqrt{2}} \frac{-u - \frac{3}{2}x - \frac{\Lambda_1^3}{8}\frac{m}{x}}{\sqrt{x^3 + ux^2 + \frac{\Lambda_1^3 m}{4}x - \frac{\Lambda_1^6}{64}}} dx = -\frac{i\Lambda_1}{2\pi\sqrt{2}} 2 \frac{\frac{3}{8}e^{-y} + \frac{1}{2}\frac{m}{\Lambda_1}e^{y} - \frac{u}{\Lambda_1^2}}{\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1}e^{y} - \frac{4u}{\Lambda_1^2}}} dy = \lambda(y,-u)\, dy \tag{5.105}$$

Now we consider for $-i\lambda(y)$ the countour of integration as in figure 5.2.2. We have horizontal branch cuts for $\Im y = \pm\pi$, $\Re y < \Re y_1$ and other two curved branch cuts $b_\pm$ from the

[10] We notice that the integrand of (5.101) has poles only at $y = \pm 4\pi i/3$ with periodicity of $4\pi i$.

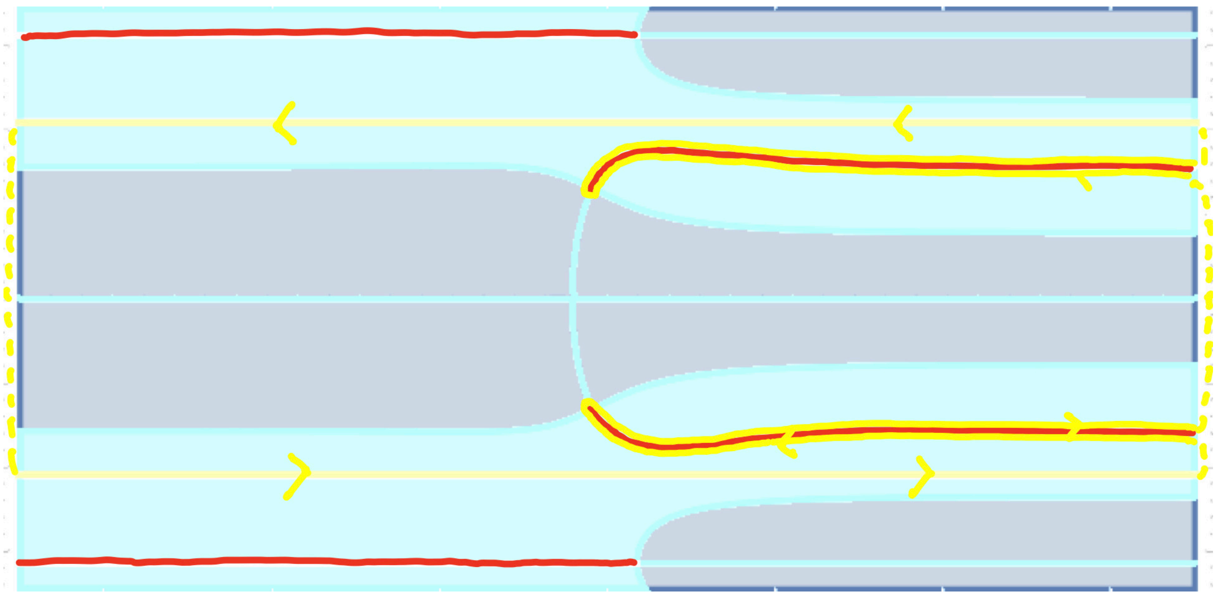

Figure 5.1: A strip of the y complex plane, where in yellow we show the contour of integration of SW differential for the $SU(2)$ $N_f = 1$ theory we use for the proof equality of the (alternative) SW period $a_1^{(0)}$ and the leading $\hbar \to 0$ order of the (minus the logarithm of the) integrability Y function $\varepsilon^{(0)} = -\ln Y^{(0)}$. In red are shown the branch cuts of the SW differential.

branch points y_2, y_3 to their asymptotics at $\Im y = \pm\frac{\pi}{2}$ for $\Re y \to +\infty$. (This can be shown easily by considering the asymptotics of $e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} - \frac{4u}{\Lambda_1}^2$ at $\Re y \to \pm\infty$ and $\Im y = \pm\frac{\pi}{2}, \pm\pi$, which are negative real). Now, the integral from the complex-conjugate branch points y_2 and y_3 is defined as the alternative gauge period $a_1^{(0)}$ (see for the definition appendix C)

$$a_1^{(0)}(-u, m, \Lambda_1) = 2\int_{y_3}^{y_2} \lambda(y, -u, m, \Lambda_1)\, dy \tag{5.106}$$

We now find some symmetry properties of $\lambda(y)$ for $y \in \mathbb{C}$. Since for $y \in \mathbb{R}$ and $m, \Lambda > 0$ and $u > 0$ not large we have

$$i\lambda(y) \in \mathbb{R} \qquad y \in \mathbb{R} \tag{5.107}$$

the analytic continuation is such that

$$i\lambda(y^*) = (i\lambda(y))^* \qquad y \in \mathbb{C} \tag{5.108}$$

From this it follows that along the branch cuts upper $b_\pm^+$ and lower $b_\pm^-$ edge of the curved branch cuts $b_\pm$, where $i\lambda \in i\mathbb{R}$ we have the properties

$$i\lambda(y)\Big|_{b_+^+} = -i\lambda(y)\Big|_{b_-^-} = -i\lambda(y)\Big|_{b_+^-} = +i\lambda(y)\Big|_{b_-^+} \qquad \in i\mathbb{R} \tag{5.109}$$

where of course the change of sign between $b_\pm^+$ and $b_\pm^-$ is due to the fact these are branch cuts for a square root. Thus by considering the integration contour $C_2 = (y_2, y_3) \cup b_+^- \cup b_-^+$

closed also at infinity (where thanks to the regularization there is no contribution) we have

$$0 = \oint_{C_2} i\lambda(y)\,dy = -\frac{i}{2}a_1^{(0)} + 2\int_{b_+^-} i\lambda(y)\,dy\,, \tag{5.110}$$

that means we can express the gauge period also as an integral along the branch cut b_+^-

$$a_1^{(0)} = 4\int_{b_+^-} \lambda(y)\,dy\,. \tag{5.111}$$

On the other hand by considering the integration contour $C_1 = (-\infty + 2\pi i/3, +\infty + 2\pi i/3) \cup b_+^+ \cup b_+^- \cup b_-^+ \cup b_-^- \cup (\infty - 2\pi i/3, -\infty - 2\pi i/3)$ closed also at infinity we have

$$0 = \oint_{C_1} i\lambda(y)\,dy = +\frac{1}{8\sqrt{2}\pi\Lambda_1}\varepsilon^{(0)}(u, im) - 4\int_{b_+^-} i\lambda(y)\,dy \tag{5.112}$$

Hence

$$\varepsilon^{(0)}(u, im, \Lambda_1) = \frac{4\sqrt{2}\pi}{\Lambda_1}a_1^{(0)}(-u, m, \Lambda_1) \tag{5.113}$$

This result is also confirmed numerically. The change of basis of the periods is, at least for $u > 0$ (see for the derivation appendix C)

$$\begin{aligned} a^{(0)}(-u,m) &= -a_1^{(0)}(-u,m) + a_2^{(0)}(-u,m) + \frac{m}{\sqrt{2}} \\ a_D^{(0)}(-u,m) &= -2a_1^{(0)}(-u,m) + a_2^{(0)}(-u,m) + \frac{3}{2}\frac{m}{\sqrt{2}} \end{aligned} \tag{5.114}$$

Hence, we can write the gauge-integrability relation for all 3 gauge TBA's forcing terms as [11]

$$\begin{aligned} \varepsilon^{(0)}(u, im) &= 2\pi\sqrt{2}\left[a^{(0)}(-u,m) - a_D^{(0)}(-u,m) + \frac{1}{2}\frac{m}{\sqrt{2}}\right]\frac{2\sqrt{2}}{\Lambda_1} \\ \varepsilon^{(0)}(e^{2\pi i/3}u, ie^{-2\pi i/3}m) &= 2\pi\sqrt{2}\left[a^{(0)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m) - a_D^{(0)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m) + \frac{e^{-2\pi i/3}m}{\sqrt{2}}\right]\frac{2\sqrt{2}}{\Lambda_1} \\ \varepsilon^{(0)}(e^{-2\pi i/3}u, ie^{2\pi i/3}m) &= 2\pi\sqrt{2}\left[-2a^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) + a_D^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) + \frac{1}{2}\frac{e^{2\pi i/3}m}{\sqrt{2}}\right]\frac{2\sqrt{2}}{\Lambda_1}\,. \end{aligned} \tag{5.115}$$

We notice that for all three pseudoenergies of the gauge TBA we find that the forcing term (leading order) is of the form of a central charge for the SW theory for $SU(2)$ with $N_f = 1$ [15]:

$$Z = n_m a_D^{(0)} - n_e a^{(0)} + s\frac{m}{\sqrt{2}}\,, \tag{5.116}$$

so that the mass of the BPS state is $M_{BPS} = \sqrt{2}|Z|$ [84]. We find a perfect match between the expected electric and magnetic charges n_e, n_m which multiply the periods $a^{(0)}$ and $a_D^{(0)}$

[11] We have checked this expression also numerically through the use of elliptic integrals of appendix C [84, 85] for the periods and the hypergeometric integral (5.86) to calculate $\varepsilon_{\pm,k}^{(0)}$.

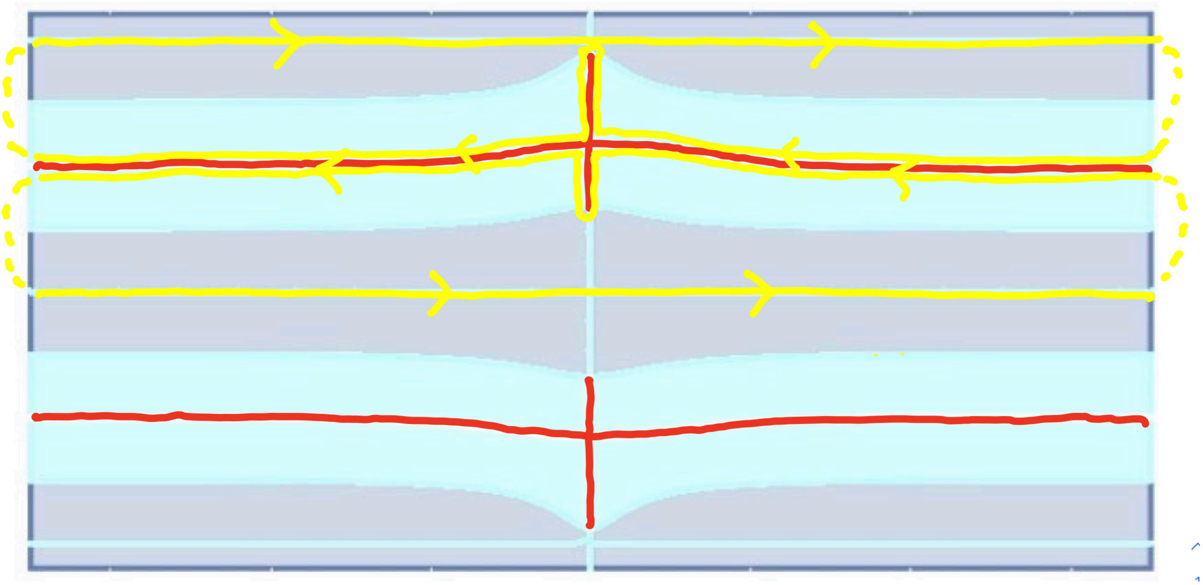

Figure 5.2: A strip of the y complex plane, where in yellow we show the contour of integration of SW differential for the $SU(2)$ $N_f = 2 = (1,1)$ theory we use for the proof equality of the (alternative) SW period $a_2^{(0)}$ and the leading $\hbar \to 0$ order of the (minus the logarithm of the) integrability Y function $\varepsilon^{(0)} = -\ln Y^{(0)}$. In red are shown the branch cuts of the SW differential.

respectively (precisely, $(-1,0)$, $(1,-1)$ and $(0,1)$ [83])[12][13]. We notice that with this observation on the spectrum the same TBA equations can be derived formally by taking the conformal limit of the integral equations in the framework of Gaiotto, Moore and Neitzke in[87, 50, 88, 70, 71]. However, we remark that though their framework is used very generally, it is for that very reason arguably more conjectural than our bottom-up approach from the precise four dimensional gauge theory ODEs.

Similarly for $N_f = 2$ we find At the $\hbar \to 0$ leading SW order we have the relations, for

[12] The mass constant term (physical flavour charge [86]) is ambiguous, but that it is just because the periods themselves are defined up to the well-known SW monodromy of exactly an integer multiple of $\frac{1}{2}\frac{m}{\sqrt{2}}$ [15, 85, 84]. We emphasize that that the central charge and mass of BPS states have no ambiguity. We notice also that, in integrability, there is no ambiguity since the wave functions and therefore the Q function in (5.39) cannot change. In other words, we are fixing through integrability what is in gauge theory is in general ambiguous.

[13] The periods $(a^{(0)}, a_D^{(0)})$ are discontinuous on the moduli space, due to the singularities, and can be analytically continued to $(\tilde{a}^{(0)}, \tilde{a}_D^{(0)})$ by using the monodromy matrix around the singularity on the moduli space. Correspondingly, the charges (n_e, n_m) will also be transformed by the inverse of the monodromy matrix, because one needs to keep the physical mass and central charge invariant. Since the driving terms of TBA equations are given by the central charge, more precisely the BPS mass, the TBA equations are invariant under the monodromy transformation.

$u, m_1, m_2 > 0$

$$\begin{aligned}
\varepsilon^{(0)}(-u, im_1, -im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(0)}(u, m_1, m_2, \Lambda_2) \\
\varepsilon^{(0)}(-u, -im_1, im_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(0)}(u, m_1, m_2, \Lambda_2) + \frac{8\pi}{\Lambda_2}(m_1 + m_2) \\
\varepsilon^{(0)}(u, m_1, m_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(0)}(-u, -im_1, im_2, \Lambda_2) - \frac{4\pi i}{\Lambda_2}(m_1 - m_2) \\
\varepsilon^{(0)}(u, -m_1, -m_2, \Lambda_2) &= \frac{8\sqrt{2}\pi}{\Lambda_2} a_D^{(0)}(-u, -im_1, im_2, \Lambda_2) + \frac{4\pi i}{\Lambda_2}(m_1 - m_2)
\end{aligned} \tag{5.117}$$

We give an analytic proof also of this result. The leading order of ε as $\hbar \to 0$ (that is, $\theta \to \infty$) is

$$\begin{aligned}
\varepsilon(\theta, u, m_1, m_2, \Lambda_2) &\simeq e^{\theta} \varepsilon^{(0)}(u, m_1, m_2, \Lambda_2) \\
&= e^{\theta}\left[-\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) - \ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) + \frac{4\pi i}{\Lambda_2}(m_1 - m_2)\right]
\end{aligned} \tag{5.118}$$

with

$$\begin{aligned}
\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) &= \int_{-\infty}^{\infty} \Bigg[\sqrt{e^{2y} + e^{-2y} + \frac{8m_1}{\Lambda_2}e^{y} + \frac{8m_2}{\Lambda_2}e^{-y} + \frac{16u}{\Lambda^2}} \\
&\qquad - 2\cosh y - \frac{4m_1}{\Lambda_2}\frac{1}{1 + e^{-y/2}} - \frac{4m_2}{\Lambda_2}\frac{1}{1 + e^{y/2}}\Bigg] dy \\
\ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) &= \int_{-\infty}^{\infty} \Bigg[\sqrt{e^{2y} + e^{-2y} - \frac{8m_1}{\Lambda_2}e^{y} - \frac{8m_2}{\Lambda_2}e^{-y} + \frac{16u}{\Lambda^2}} \\
&\qquad - 2\cosh y + \frac{4m_1}{\Lambda_2}\frac{1}{1 + e^{-y/2}} + \frac{4m_2}{\Lambda_2}\frac{1}{1 + e^{y/2}}\Bigg] dy
\end{aligned} \tag{5.119}$$

Now we can trade the change of sign in the masses as a shift in y by $i\pi$

$$\begin{aligned}
\ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) &= \int_{-\infty + i\pi}^{\infty + i\pi} \Bigg[\sqrt{e^{2y} + e^{-2y} + \frac{8m_1}{\Lambda_2}e^{y} + \frac{8m_2}{\Lambda_2}e^{-y} + \frac{16u}{\Lambda^2}} \\
&\qquad + 2\cosh y + \frac{4m_1}{\Lambda_2}\frac{1}{1 + ie^{-y/2}} + \frac{4m_2}{\Lambda_2}\frac{1}{1 - ie^{y/2}}\Bigg] dy
\end{aligned} \tag{5.120}$$

We can use the same integrand and integrate it in the countour of figure 5.2 if we separate and add outside the term coming from the regularizing part

$$\int_{-\infty}^{\infty}\left[\frac{4\left(m_1 e^{y/2} + m_2\right)}{\Lambda_2\left(e^{y/2} + 1\right)} - \frac{4\left(m_1 e^{y/2} + im_2\right)}{\Lambda_2\left(e^{y/2} + i\right)}\right] dy = \frac{4i\pi(m_1 - m_2)}{\Lambda_2} \tag{5.121}$$

Now the SW differential as defined in section 2.5 from the *quartic* SW curve (2.41) gives

$$\begin{aligned}
\int \lambda(x, -u, -im_1, im_2, \Lambda_2)\, dx &= \int \Bigg[\sqrt{e^{2y} + e^{-2y} + \frac{8m_1}{\Lambda_2}e^{y} + \frac{8m_2}{\Lambda_2}e^{-y} + \frac{16u}{\Lambda^2}} \\
&\qquad - 2\cosh y - \frac{4m_1}{\Lambda_2}\frac{1}{1 + e^{-y/2}} - \frac{4m_2}{\Lambda_2}\frac{1}{1 + e^{y/2}} + \frac{d}{dy}(\dots)\Bigg] dy
\end{aligned} \tag{5.122}$$

Therefore

$$\varepsilon^{(0)}(u,m_1,m_2,\Lambda_2) = \oint \lambda(y)\,dy - \frac{4\pi i}{\Lambda_2}(m_1-m_2) \tag{5.123}$$

We notice that for $y=t+is$ along the (almost) horizontal branch cuts we have

$$\begin{aligned} \Re\mathcal{P}^{(0)}(y) &= 0 \\ \Im\mathcal{P}^{(0)}(t+is) &= -\Im\mathcal{P}^{(0)}(-t+is) \end{aligned} \tag{5.124}$$

so that the only contribution is from the vertical branch cuts. That is

$$\varepsilon^{(0)}(u,m_1,m_2,\Lambda_2) = \frac{8\sqrt{2}\pi}{\Lambda_2} a_2^{(0)}(-u,-im_1,im_2,\Lambda_2) - \frac{4\pi i}{\Lambda_2}(m_1-m_2) \tag{5.125}$$

For $u\to\infty,\Lambda_2$ we have $a_2^{(0)}(-u,m_1,m_2,\Lambda_2)\sim a_D(-u,m_1,m_2,\Lambda_2)\sim\frac{i}{2\pi}\sqrt{2u}\ln\frac{u}{\Lambda_2^2}$ and then (5.117) follows. In this way TBA (5.79) constitute a generalization of that found in [89] $N_f=2$ gauge theory with equal masses $m_1=m_2$ respectively (see a numerical test for different masses below in table 5.2).

5.2.3. Exact quantum gauge/integrability identification for Y

We can use the following differential operators [20] to get higher $\hbar\to 0$ ($\theta\to+\infty$) orders of either the periods a_k or $\ln Q$

$$\begin{aligned} a_k(\theta,u,m,\Lambda_1) &\doteq \sum_{n=0}^{\infty} e^{-2n\theta} a_k^{(n)}(u,m,\Lambda_1) \qquad \theta\to+\infty \\ \ln Q(\theta,u,m,\Lambda_1) &\doteq \sum_{n=0}^{\infty} e^{\theta(1-2n)} \ln Q^{(n)}(u,m,\Lambda_1) \qquad \theta\to+\infty \end{aligned} \tag{5.126}$$

For $N_f=1$ they are

$$\begin{aligned} a_k^{(1)}(u,m,\Lambda_1) &= \left(\frac{\Lambda_1}{2}\right)^2 \frac{1}{12}\left[\frac{\partial}{\partial u} + 2m\frac{\partial}{\partial m}\frac{\partial}{\partial u} + 2u\frac{\partial^2}{\partial u^2}\right] a_k^{(0)}(u,m,\Lambda_1) \\ a_k^{(2)}(u,m,\Lambda_1) &= \left(\frac{\Lambda_1}{2}\right)^4 \frac{1}{1440}\left[28m^2\frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial m^2} + 28u^2\frac{\partial^4}{\partial u^4} + 132m\frac{\partial^2}{\partial u^2}\frac{\partial}{\partial m} + 56mu\frac{\partial^3}{\partial u^3}\frac{\partial}{\partial m}\right. \\ &\qquad \left. + 81\frac{\partial^2}{\partial u^2} + 124u\frac{\partial^3}{\partial u^3}\right] a_k^{(0)}(u,m,\Lambda_1) \end{aligned} \tag{5.127}$$

and for $N_f = 2$

$$
\begin{aligned}
a_k^{(1)}(u,m_1,m_2,\Lambda_2) &= \left(\frac{\Lambda_2}{4}\right)^2 \frac{1}{6}\left[2u\frac{\partial^2}{\partial u^2} + \frac{3}{2}\left(m_1\frac{\partial}{\partial m_1}\frac{\partial}{\partial u} + m_2\frac{\partial}{\partial m_2}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial u}\right] a_k^{(0)}(u,m_1,m_2,\Lambda_2) \\
a_k^{(2)}(u,m_1,m_2,\Lambda_2) &= \left(\frac{\Lambda_2}{4}\right)^4 \frac{1}{360}\Bigg[28u^2\frac{\partial^4}{\partial u^4} + 120u\frac{\partial^3}{\partial u^3} + 75\frac{\partial^2}{\partial u^2} + 42\left(um_1\frac{\partial}{\partial m_1}\frac{\partial^3}{\partial u^3} + um_2\frac{\partial}{\partial m_2}\frac{\partial^3}{\partial u^3}\right) \\
&\quad + \frac{345}{4}\left(m_1\frac{\partial}{\partial m_1}\frac{\partial^2}{\partial u^2} + m_2\frac{\partial}{\partial m_2}\frac{\partial^2}{\partial u^2}\right) + \frac{63}{4}\left(m_1^2\frac{\partial^2}{\partial m_1^2}\frac{\partial^2}{\partial u^2} + m_2^2\frac{\partial^2}{\partial m_2^2}\frac{\partial^2}{\partial u^2}\right) \\
&\quad + \frac{126}{4}m_1m_2\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2}\frac{\partial^2}{\partial u^2}\Bigg] a_k^{(0)}(u,m_1,m_2,\Lambda_2)\,.
\end{aligned}
\tag{5.128}
$$

The same operators can be used also to obtain $\ln Q^{(n)}$ of course.

Remarkably, we find the same higher orders of a_k to be given by the asymptotic expansion of the gauge TBA. For $N_f = 1$

$$
\begin{aligned}
\varepsilon_{+,0}^{(1)} &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\, e^{\theta}\left\{-2(-1)^{5/6}L_{+,1}(\theta') + 2(-1)^{1/6}L_{+,2}(\theta')\right\} \\
&= -e^{i\pi/6}\ln Q^{(1)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m, \Lambda_1) - e^{-i\pi/6}\ln Q^{(1)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m, \Lambda_1) \\
&= -\frac{4\pi\sqrt{2}}{\Lambda_1}\, a_1^{(1)}(-u,m,\Lambda_1) \\
\varepsilon_{+,0}^{(2)} &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\, e^{3\theta}\left\{2iL_{+,1}(\theta') - 2iL_{+,2}(\theta')\right\} \\
&= -e^{i\pi/2}\ln Q^{(2)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m, \Lambda_1) - e^{-i\pi/2}\ln Q^{(2)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m, \Lambda_1) \\
&= \frac{4\pi\sqrt{2}}{\Lambda_1}a_1^{(2)}(-u,m,\Lambda_1)\,.
\end{aligned}
\tag{5.129}
$$

The numerical check is shown in table 5.1. Thus we have the asymptotic expansion, for $N_f = 1$

$$
\varepsilon(\theta,u,im,\Lambda_1) \doteq \sum_{n=0}^{\infty} e^{\theta(1-2n)}\varepsilon^{(n)}(u,im,\Lambda_1) = \frac{4\sqrt{2}\pi}{\Lambda_1}\sum_{n=0}^{\infty} e^{\theta(1-2n)}(-1)^n a_1^{(n)}(-u,m,\Lambda_1) \qquad \theta \to +\infty
\tag{5.130}
$$

	k=1	k=2
$\varepsilon^{(k)}_{+,0}$	-0.143902	0.00285479
$d_k\Delta[\ln Q^{(k)}]_{+,0}$	-0.143905	0.00285481
$(-1)^k c_k a_1^{(k)}(-u_0, m_0)$	-0.143905	0.00285481
$\varepsilon^{(k)}_{+,1}$	$-0.140549 + 0.00193600i$	$-0.00142739 + 0.00311155i$
$d_k\Delta[\ln Q^{(k)}]_{+,1}$	$-0.140552 + 0.00193603i$	$-0.00142740 + 0.00311157i$
$(-1)^k c_k a_1^{(k)}(-u_1, m_2)$	$-0.140552 + 0.00193603i$	$-0.00142740 + 0.00311157i$
$\varepsilon^{(k)}_{+,2}$	$-0.140549 - 0.00193600i$	$-0.00142739 - 0.00311155i$
$d_k\Delta[\ln Q^{(k)}]_{+,2}$	$-0.140552 - 0.00193603i$	$-0.00142740 - 0.00311157i$
$(-1)^k c_k a_1^{(k)}(-u_2, m_1)$	$-0.140552 - 0.00193603i$	$-0.00142740 - 0.00311157i$

Table 5.1: Comparison between the higher $\hbar \to 0$ asymptotic expansion modes for the $N_f = 1$ gauge theory and Perturbed Hairpin IM. The first line is the result from the $\theta \to \infty$ expansion of the gauge TBA (5.78). The second line is the result from the differential operators (5.127) acting on the leading order $\ln Q^{(0)}$ computed through hypergeometric functions (5.81), (5.86). The third line are the higher periods computed through the same differential operators acting on the elliptic integral of the SW order, as in appendix C, with $c_k = \frac{4\pi\sqrt{2}}{\Lambda_1}\left(\frac{\Lambda_1^2}{8}\right)^k$ and $d_k = \left(\frac{\Lambda_1^2}{8}\right)^k$. Here the parameters are $u = 0.1$, $\Lambda_1 = 1.$, $m = \frac{1}{20\sqrt{2}}$ and of course $u_k = e^{2\pi i k/3}u$, $m_k = e^{-2\pi i k/3}m$.

	$k=1, m_1 = m_2$	$k=2, m_1 = m_2$	$k=1, m_1 \neq m_2$	$k=2, m_1 \neq m_2$
$\varepsilon^{(k)}(u, m_1, m_2, \Lambda_2)$	-0.2395247	0.0158881	-0.2379413	0.01513637
$(-1)^k c_k a_2^{(k)}(-u, -im_1, im_2, \Lambda_2)$	-0.2395130	0.0158902	-0.2379297	0.01513580
$\varepsilon^{(k)}(u, im_1, -im_2, \Lambda_2)$	-0.5025004	0.3120101	-0.5000211	0.29418949
$c_k a_2^{(k)}(u, m_1, m_2, \Lambda_2)$	-0.5024841	0.3120003	-0.5000048	0.29418016

Table 5.2: Comparison of higher orders $\varepsilon^{(k)}$ from gauge TBA (5.79) and $a_2^{(k)}$ from elliptic integrals (through differential operators (5.128), with $c_k = \frac{8\pi\sqrt{2}}{2^{4k}}\Lambda_2^{2k-1}$). In the second and third column $m_1 = m_2 = \frac{1}{8}$, $\Lambda_2 = 4$, $u = 1$. In the fourth and fifth column $m_1 = \frac{1}{16}$, $m_2 = \frac{1}{8}$, $\Lambda_2 = 4$, $u = 1$.

For $N_f = 2$ we have similarly

$$\begin{aligned}
\varepsilon^{(n)}_{+,+} &= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\theta\, e^{\theta(2n-1)}\left[\bar{L}_{+,+}(\theta') + \bar{L}_{-,-}(\theta')\right] \\
&= -\ln Q^{(n)}(u, m_1, m_2, \Lambda_2) - \ln Q^{(n)}(u, -m_1, -m_2, \Lambda_2) \\
&= (-1)^n \frac{8\pi\sqrt{2}}{\Lambda_2} a_2^{(n)}(-u, -im_1, im_2, \Lambda_2) \\
\bar{\varepsilon}^{(n)}_{+,+} &= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\theta\, e^{\theta(2n-1)}\left[L_{+,+}(\theta') + L_{-,-}(\theta')\right] \\
&= -\ln Q^{(n)}(-u, -im_1, im_2, \Lambda_2) - \ln Q^{(n)}(-u, im_1, -im_2, \Lambda_2) \\
&= (-1)^n \frac{8\pi\sqrt{2}}{\Lambda_2} a_2^{(n)}(u, m_1, m_2, \Lambda_2)\,.
\end{aligned} \tag{5.131}$$

The numerical check is shown in table 5.2. Thus we have the asymptotic expansion, for $N_f = 2$

$$\begin{aligned}
&\varepsilon(\theta, u, \pm m_1, \pm m_2, \Lambda_2) \doteq \sum_{n=0}^{\infty} e^{\theta(1-2n)} \varepsilon^{(n)}(u, \pm m_1, \pm m_2, \Lambda_2) \\
&= \frac{8\sqrt{2}\pi}{\Lambda_2}\left[e^{\theta} a_2^{(0)}(-u, -im, +im_2, \Lambda_2) \mp \frac{1}{2\sqrt{2}}(im_1 - im_2) + \sum_{n=1}^{\infty} e^{\theta(1-2n)} a_2^{(n)}(-u, -im, +im_2, \Lambda_2)\right] \qquad \theta \to +\infty
\end{aligned} \tag{5.132}$$

Therefore we can identify the exact gauge pseudoenergy ε as defining the exact periods a_k. Moreover, we can numerically prove that the exact gauge pseudoenergy is equivalent, under change of variable, to the exact integrability pseudoenergy.

$$\begin{aligned}
\varepsilon(\theta, p, q) &= \varepsilon(\theta, u, m, \Lambda) & \frac{u}{\Lambda_1^2} &= \frac{1}{4} p^2 e^{-2\theta}\,, & \frac{m}{\Lambda_1} &= \frac{1}{2}\, q\, e^{-\theta_0} \\
\varepsilon(\theta, p, q_1, q_2) &= \varepsilon(\theta, u, m_1, m_2, \Lambda_2) & \frac{u}{\Lambda_2^2} &= \frac{1}{16} p^2 e^{-2\theta} & \frac{m_i}{\Lambda_2} &= \frac{1}{4} q_i e^{-\theta}
\end{aligned} \tag{5.133}$$

This check is shown in tables 5.3-5.4 and figure 5.3.

We have defined the exact gauge periods as cycle integrals of the solution of the Riccati equation $\mathcal{P}(y)$, the Seiberg-Witten quantum differential (see section 4.7.4). However, in gauge theory they are properly defined from the instanton expansion (around $\Lambda_1 = 0$), which is, for also small $\hbar$, for $N_f = 1$

$$\begin{aligned}
a(\theta, u, m, \Lambda_1) &= \sqrt{\frac{u}{2}} - \frac{\Lambda_1^3 m \left(\frac{1}{u}\right)^{3/2}}{2^4\sqrt{2}} + \frac{3\Lambda_1^6 \left(\frac{1}{u}\right)^{5/2}}{2^{10}\sqrt{2}} + \ldots \\
&+ \hbar(\theta)^2\left(-\frac{\Lambda_1^3 m \left(\frac{1}{u}\right)^{5/2}}{2^6\sqrt{2}} + \frac{15\Lambda_1^6 \left(\frac{1}{u}\right)^{7/2}}{2^{12}\sqrt{2}} - \frac{35\Lambda_1^6 m^2 \left(\frac{1}{u}\right)^{9/2}}{2^{11}\sqrt{2}} + \ldots\right) \\
&+ \hbar(\theta)^4\left(-\frac{\Lambda_1^3 m \left(\frac{1}{u}\right)^{7/2}}{2^8\sqrt{2}} + \frac{63\Lambda_1^6 \left(\frac{1}{u}\right)^{9/2}}{2^{14}\sqrt{2}} - \frac{273\Lambda_1^6 m^2 \left(\frac{1}{u}\right)^{11/2}}{2^{14}\sqrt{2}} + \ldots\right) + \ldots\,,
\end{aligned} \tag{5.134}$$

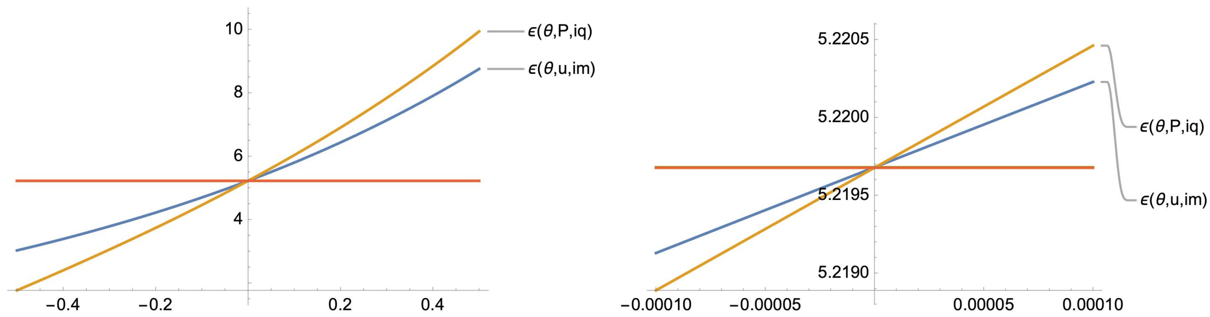

Figure 5.3: Plots (at low and high magnification) of the matching between the $N_f = 1$ gauge and Perturbed Hairpin IM pseudoenergies $\varepsilon(\theta, u, im)$ and $\varepsilon(\theta, P, iq)$ for $u = 0.1$, $q = 0.1$, $\Lambda = 1$, for $\theta_0 = 0$.

$(\theta_0, u, m, \Lambda_1)$	$(0, \frac{1}{10}, \frac{1}{20}, 1)$
$\varepsilon(\theta_0, u, im)$	5.21968
$\varepsilon(\theta_0, P, iq)$	5.21968
$\varepsilon(\theta_0, u, im)_{\text{Riccati}}$	5.21933

Table 5.3: Table which shows the very good match between $N_f = 1$ gauge and integrability pseudoenergies at $\theta = \theta_0 = 0$ with parameters $u = 0.1$, $m = \frac{1}{20}$, $\Lambda_1 = 1$. In the third line we show also a match with the result from direct numerical integration of the Riccati equation (5.43).

(θ_0, q_1, q_2, P)	$(0, \frac{1}{8}, \frac{1}{8}, 1)$	$(0, \frac{1}{16}, \frac{1}{8}, 1)$
$\varepsilon^{INT}_{\pm,\pm}$	1.428378	$1.416945047 \pm 0.196349544i$
$\varepsilon^{GAUGE}_{\pm,\pm}$	1.428383	$1.416939137 \pm 0.196349544i$
$\varepsilon^{INT}_{\pm,\mp}$	$1.4133849 \mp 0.78539816i$	$1.40946127 \mp 0.58904862i$
$\varepsilon^{GAUGE}_{\pm,\mp}$	$1.4133714 \mp 0.78539816i$	$1.40944721 \mp 0.5890486i$

Table 5.4: Comparison of $N_f = 2$ gauge and generalized Perturbed Hairpin IM TBA for different values of parameters.

$$a_D(\theta, u, m, \Lambda_1) = \frac{i}{2\sqrt{2}\pi}\left[\sqrt{2}\left[\sum_{k=0}^{2}\hbar(\theta)^{2k}a^{(k)}(u,m,\Lambda_1)\right]\left(i\pi - 3\ln\frac{16u}{\Lambda_1^2}\right) + \left(6\sqrt{u} + \frac{m^2}{\sqrt{u}} + \frac{\frac{m^4}{6} - \frac{1}{4}\Lambda_1^3 m}{u^{3/2}} + \dots\right)\right.$$
$$+ \hbar(\theta)^2\left(-\frac{1}{4\sqrt{u}} - \frac{m^2}{12u^{3/2}} + \frac{-\frac{9}{64}\Lambda_1^3 m - \frac{m^4}{12}}{u^{5/2}} + \dots\right)$$
$$\left. + \hbar(\theta)^4\left(\frac{1}{160u^{3/2}} + \frac{7m^2}{240u^{5/2}} + \frac{\frac{7m^4}{96} - \frac{127\Lambda_1^3 m}{2560}}{u^{7/2}} + \dots\right) + \dots\right]. \tag{5.135}$$

The results are shown in table 5.5. Notice that through formulas (5.134) and (5.135) we can reach even the non-perturbative (non-WKB) large $\hbar$ regime, the important thing to be necessarily small being the ratio Λ_1^2/u.

Hence we find a first identification between an integrability quantity, the Y function,

$\{\theta,\Lambda_1,p,q\}$	$\{-5,0.1,5,0.1\}$	$\{-2.5,0.1,10,0.1\}$	$\{0,0.1,5,0.1\}$
$\{\hbar,u,m,\frac{\Lambda_1^2}{u}\}$	$\sim\{5,10^3,0.5,10^{-6}\}$	$\sim\{0.5,10^2,10^{-1},10^{-4}\}$	$\sim\{10^{-2},10^{-1},10^{-3},10^{-1}\}$
$\varepsilon(\theta,p,iq)$	-267.1186026	-381.1795517	-54.9818090
$2\pi\sqrt{2}\frac{a-a_D+\frac{1}{2}\frac{m}{\sqrt{2}}}{\hbar(\theta)}$	-267.1186297	-381.1797573	-54.9949700

Table 5.5: A table which shows the match between the integrability pseudoenergy for positive mass iq in the θ-non-perturbative region and the instanton expansion for the right combination of the $N_f=1$ gauge periods which we have analytically proven to be equal to it.

and the exact gauge periods. For $N_f=1$, for $u,m,\Lambda_1>0$

$$\varepsilon(\theta,p,iq)=\frac{2\pi\sqrt{2}}{\hbar(\theta)}\left[a(\theta-i\pi/2,-u,m)-a_D(\theta-i\pi/2,-u,m)+\frac{1}{2}\frac{m}{\sqrt{2}}\right]\qquad u,m,\Lambda_1>0 \tag{5.136}$$

or more generally for $u,m\in\mathbb{C}$, with $\arg u=-\arg m$

$$\varepsilon(\theta,p,iq)=\frac{2\pi\sqrt{2}}{\hbar(\theta)}a_1(\theta-i\pi/2,-u,m)=\frac{2\pi\sqrt{2}\,i}{\hbar(\theta-i\pi/2)}a_1(\theta-i\pi/2,-u,m)\qquad \arg u=-\arg m\quad \Lambda_1>0\,. \tag{5.137}$$

Similarly for $N_f=2$ and $u,m,\Lambda_2>0$[14].

$$\begin{aligned}\varepsilon(\theta,ip,iq_1,-iq_2)&=\frac{2\sqrt{2}\pi}{\hbar(\theta)}a_D(\theta,u,m_1,m_2,\Lambda_2)\\ \varepsilon(\theta,ip,-iq_1,iq_2)&=\frac{2\sqrt{2}\pi}{\hbar(\theta)}\left[a_D(\theta,u,m_1,m_2,\Lambda_2)+\frac{1}{\sqrt{2}}(m_1+m_2)\right]\\ \varepsilon(\theta,p,q_1,q_2)&=\frac{2\sqrt{2}\pi}{\hbar(\theta)}\left[a_D(\theta,-u,-im_1,im_2,\Lambda_2)-\frac{i}{2\sqrt{2}}(m_1-m_2)\right]\\ \varepsilon(\theta,p,-q_1,-q_2)&=\frac{2\sqrt{2}\pi}{\hbar(\theta)}\left[a_D(\theta,-u,-im_1,im_2,\Lambda_2)+\frac{i}{2\sqrt{2}}(m_1-m_2)\right]\end{aligned} \tag{5.138}$$

Relations (5.136)-(5.138) show a new connection between the $SU(2)$ $N_f=1,2$ gauge periods and the Y function (Generalized) Perturbed Hairpin integrable model. This generalizes to the case of massive hypermultiplets matter the integrability-gauge correspondence already developed for the $SU(2)$ $N_f=0$ and the self-dual Liouville model (*cf.* (4.295), with $Q=\sqrt{Y}$) [1]. (5.136) and (5.138) are in some sense expressions for a $N_f=1,2$ SW exact central charge. As explained in section 4.7.5 by considering different particles in the spectrum or definition of gauge periods other than the integral one, different relations could be found like those for the $N_f=0$ and $N_f=1$ theory in [71, 79].

Besides, we remark that these gauge-integrability identifications holds as they are written only in a restricted strip of of the complex θ plane: $\Im\theta<\pi/3$ and $\Im\theta<\pi/2$ for the

[14] We remark that the first two relation with imaginary p parameters are not directly implemented in the integrability variables (since the integrability TBA does not converge), but they will in the gravity variables in section 5.6 (in (5.220) precisely this range of parameters is involved).

$N_f = 1$ and $N_f = 2$ theory. Beyond such strips the gauge TBAs (5.78) (5.79) needs analytic continuation (of its solution) since poles of the kernels are found on the θ' integrating axis. A modification of TBAs equation as usually done in integrability by adding the residue is possible, but then the Ys no longer identifies with the gauge periods: in fact the former are entire functions while the latter are not [90, 91, 51]. This is a manifestation of the so-called wall-crossing phenomenon, whereby the spectrum of SW theory changes and therefore a fundamental change in its relation to integrability is to be expected. We hope to investigate further and write more on this issue in the future.

5.3. Integrability T function and gauge period

5.3.1. T function and Floquet exponent

In this subsection we follow and adapt the monograph on Doubly Confluent Heun equation in [77]. Define the periodicity operator

$$\Upsilon\psi(y) = \psi(y + 2\pi i) \tag{5.139}$$

We can express Υ in terms of the $\Omega_\pm$ symmetry operators, for $N_f = 1$ as

$$\Upsilon = \Omega_+^2 \Omega_-^{-1} \tag{5.140}$$

and for $N_f = 2$ as

$$\Upsilon = \Omega_+^2 \Omega_-^{-2} \tag{5.141}$$

Then we write, for $N_f = 1$

$$\begin{aligned} \psi_{+,-1}(y + 2\pi i) &= \psi_{+,1} = -e^{2\pi i q}\psi_{+,-1} + i e^{i\pi q}\tilde{T}_+(\theta)\psi_{+,0} \\ \psi_{+,0}(y + 2\pi i) &= \psi_{+,2} = -e^{i\pi q}\tilde{T}_-(\theta + i\pi/3)\psi_{+,-1} + [-e^{-2\pi i q} + \tilde{T}_-(\theta + i\pi/3)\tilde{T}_+(\theta)]\psi_{+,1} \end{aligned} \tag{5.142}$$

and for $N_f = 2$

$$\begin{aligned} \psi_{+,-1}(y + 2\pi i) &= \psi_{+,1} = -e^{2\pi i q_1}\psi_{+,-1} + i e^{i\pi q_1}\tilde{T}_{+,+}(\theta)\psi_{+,0} \\ \psi_{+,0}(y + 2\pi i) &= \psi_{+,2} = -e^{i\pi q_1}\tilde{T}_{-,+}(\theta + i\pi/2)\psi_{+,-1} + [-e^{-2\pi i q_1} + \tilde{T}_{-,+}(\theta + i\pi/2)\tilde{T}_{+,+}(\theta)]\psi_{+,1}\,. \end{aligned} \tag{5.143}$$

We can write these relations also in matrix form

$$\Upsilon\psi_+ = \mathcal{T}_+\psi_+ \tag{5.144}$$

where we defined $\psi = (\psi_{+,-1}, \psi_{+,0})$ and, for $N_f = 1$

$$\mathcal{T}_+ = \begin{pmatrix} -e^{2\pi i q} & e^{i\pi q}\tilde{T}_{+,+}(\theta) \\ e^{i\pi q}\tilde{T}_-(\theta + i\pi/3) & [-e^{-2\pi i q} + \tilde{T}_-(\theta + i\pi/3)\tilde{T}_+(\theta)] \end{pmatrix} \tag{5.145}$$

and for $N_f = 2$

$$\mathcal{T}_+ = \begin{pmatrix} -e^{2\pi i q_1} & e^{i\pi q_1}\tilde{T}_{+,+}(\theta) \\ e^{i\pi q_1}\tilde{T}_{-,+}(\theta + i\pi/2) & [-e^{-2\pi i q_1} + \tilde{T}_{-,+}(\theta + i\pi/2)\tilde{T}_{+,+}(\theta)] \end{pmatrix}\,. \tag{5.146}$$

θ	$T(\theta,p,0)$ TBA, TQ	$\exp\left[-2\pi\nu(\theta+i\pi/3,p,0)\right]+\exp\left[2\pi\nu(\theta-i\pi/3,p,0)\right]$ Hill
$-10.$	-0.409791	-0.409791
$-8.$	-0.409791	-0.409791
$-6.$	-0.40979	-0.409791
$-4.$	-0.409786	-0.409791
$-2.$	-0.412355	-0.412353
$-1.$	-1.44334	-1.44332
$0.$	-371.911	-371.912
$1.$	$-3.99263\cdot 10^{6}$	$-3.99263\cdot 10^{6}$
$2.$	$-1.02835\cdot 10^{17}$	$-1.02835\cdot 10^{17}$
$3.$	$1.00886\cdot 10^{48}$	$1.00886\cdot 10^{48}$
$4.$	$-2.63656\cdot 10^{130}$	$-2.63656\cdot 10^{130}$
$5.$	$6.00739\cdot 10^{353}$	$6.00739\cdot 10^{353}$

Table 5.6: Here we make a table, with $p=0.2$ and several θ in the lines, of three quantities: $T(\theta,p,q=0)$ from the TBA and TQ system (Q function), $\exp\left[-2\pi\nu(\theta+i\pi/3,p,0)\right]+\exp\left[2\pi\nu(\theta-i\pi/3,p,0)\right]$, were ν is Hill's Floquet (see appendix F). (Here in θ we discretize the interval $(-50,50)$ in 2^8 parts, which is no big effort, but we go up to 2^{13} iterations for the TBA or 2^{14} as the Hill matrix's width.)

Now we can say that ν is a characteristic exponent of the Doubly confluent Heun equation (5.4) if and only if $e^{\pm 2\pi i\nu}$ are eigenvalues of Υ_+. It then follows that ν is determined from

$$2\cos 2\pi\nu = \mathrm{tr}\,\mathcal{T}_+ \tag{5.147}$$

or more explicitly, for $N_f=1$

$$2\cos 2\pi\nu + 2\cos 2\pi q = 4\cos\pi(q+\nu)\cos\pi(q-\nu) = \tilde{T}_+(\theta)\tilde{T}_-(\theta+i\frac{\pi}{3}) \tag{5.148}$$

and for $N_f=2$

$$2\cos 2\pi\nu + 2\cos 2\pi q_1 = 4\cos\pi(q_1+\nu)\cos\pi(q_1-\nu) = \tilde{T}_{+,+}(\theta)\tilde{T}_{-,+}(\theta+i\frac{\pi}{2}) \tag{5.149}$$

Similarly we can prove relations for T, for $N_f=1$

$$2\cos 2\pi\nu = 4\cos^2\pi\nu = T_+(\theta)T_+(\theta+i\frac{2\pi}{3}) = T_+^2(\theta) \tag{5.150}$$

and for $N_f=2$

$$2\cos 2\pi\nu + 2\cos 2\pi q_2 = 4\cos\pi(q_2+\nu)\cos\pi(q_2-\nu) = T_{+,+}(\theta)T_{+,-}(\theta+i\frac{\pi}{2}) \tag{5.151}$$

These relations between T and ν generalize both what found numerically by Zamolodchikov and us [49, 1] for the self-dual Liouville model ($N_f=0$) and also that found by D.F. and R. Poghossian and H. Poghosyan for $SU(3)$ $N_f=0$ [43].

For $N_f = 1$, from the T periodicity $T_+(\theta + i\pi/3) = T_-(\theta)$ it follows the Floquet (anti)-periodicity

$$\nu(\theta + i\frac{\pi}{3}, -q) = \nu(\theta, q) = \pm\nu(\theta - i\frac{\pi}{3}, -q) \qquad \text{mod}(n) \in \mathbb{Z} \tag{5.152}$$

Thus for $N_f = 1$ we prove the following conjecture by Fateev and Lukyanov [78].

$$T_+(\theta) = T(\theta, p, q) = \exp\{-i\pi\nu(\theta + i\pi/3, p, -q)\} + \exp\{i\pi\nu(\theta - i\pi/3, p, -q)\}\,, \tag{5.153}$$

which follows immediately from (5.150) and (5.152). We show also its numerical proof in the massless case in table 5.6, where ν is computed in practice through the well-known method of the Hill determinant [92] (see appendix F).

5.3.2. Exact quantum gauge/integrability identification for T

The gauge a period is defined from the Λ_1 (Λ_2) derivative of the instanton part of the gauge prepotential $\mathcal{F}_{NS}$ through the Matone's relation, for $N_f = 1$

$$2u = a^2 - \frac{\Lambda_1}{3}\frac{\partial \mathcal{F}_{NS}^{inst}}{\partial \Lambda_1} \tag{5.154}$$

and for $N_f = 2$

$$2u = a^2 - \frac{\Lambda_2}{2}\frac{\partial \mathcal{F}_{NS}^{inst}}{\partial \Lambda_2}\,. \tag{5.155}$$

where the instanton prepotential $\mathcal{F}_{NS}^{inst}$ is given by, for $N_f = 1$

$$\mathcal{F}_{NS}^{inst} = \sum_{n=0}^{\infty} \Lambda_1^{3n} \mathcal{F}_{NS}^{(n)} \tag{5.156}$$

with first terms

$$\begin{aligned} \mathcal{F}_{NS}^{(1)} &= -\frac{2m_1}{4(4a^2 - 2\hbar^2)} \\ \mathcal{F}_{NS}^{(2)} &= -\frac{4m_1^2\left(20a^2 + 14\hbar^2\right) - 3\left(4a^2 - 2\hbar^2\right)^2}{256\left(a^2 - 2\hbar^2\right)\left(4a^2 - 2\hbar^2\right)^3} \\ \mathcal{F}_{NS}^{(3)} &= -\frac{4m_1^3\left(144a^4 + 464a^2\hbar^2 + 116\hbar^4\right) - m_1\left(28a^2 + 34\hbar^2\right)\left(4a^2 - 2\hbar^2\right)^2}{192\left(4a^2 - 2\hbar^2\right)^5\left(4a^4 - 26a^2\hbar^2 + 36\hbar^4\right)} \end{aligned} \tag{5.157}$$

and for $N_f = 2$

$$\mathcal{F}_{NS}^{inst} = \sum_{n=0}^{\infty} \Lambda_2^{2n} \mathcal{F}_{NS}^{(n)} \tag{5.158}$$

with

$$\begin{aligned} \mathcal{F}_{NS}^{(1)} &= -\frac{1}{8} + \left[\frac{1}{8} - \frac{4m_1m_2}{8(4a^2 - 2\hbar^2)}\right] \\ \mathcal{F}_{NS}^{(2)} &= -\frac{64a^2(a^4 + 3a^2(m_1^2 + m_2^2) + 5m_1^2m_2^2) - 8\hbar^6 + 48\hbar^2(a^2 + m_1^2 + m_2^2) - 32\hbar^2[3a^4 + 6a^2(m_1^2 + m_2^2) - 7m_1^2m_2^2]}{1024\left(a^2 - 2\hbar^2\right)\left(4a^2 - 2\hbar^2\right)^3}\,. \end{aligned} \tag{5.159}$$

Λ_1	u	m	ν	$\frac{a}{\hbar}$	Λ_1	u	m	ν	$\frac{a}{\hbar}$
0.04	1.1	0	0.0488088	$1+0.0488088$	0.04	1.1	0.3	0.0488075	$1+0.0488085$
0.08	1.1	0	0.0488089	$1+0.0488088$	0.08	1.1	0.3	0.0487981	$1+0.0488062$
0.12	1.1	0	0.0488089	$1+0.0488089$	0.12	1.1	0.3	0.047726	$1+0.0487998$
0.16	1.1	0	0.0488094	$1+0.0488089$	0.16	1.1	0.3	0.0487231	$1+0.0487874$

Table 5.7: Comparison of ν as computed by the Hill determinant and a for $N_f = 1$ as computed from the instanton series (with $\hbar = 1$).

Λ_2	u	m_1	m_2	ν	$\frac{a}{\hbar}$
0.04	1.1	0	0	0.0488088	$1+0.0488088$
0.08	1.1	0	0	0.0488085	$1+0.0488088$
0.12	1.1	0	0	0.0488069	$1+0.0488084$
0.16	1.1	0	0	0.0488027	$1+0.0488073$
Λ_2	u	m_1	m_2	ν	$\frac{a}{\hbar}$
0.04	1.1	0.2	0.2	0.0488043	$1+0.0488077$
0.08	1.1	0.2	0.2	0.0487906	$1+0.0488043$
0.12	1.1	0.2	0.2	0.048767	$1+0.0487982$
0.16	1.1	0.2	0.2	0.0487325	$1+0.0487892$

Table 5.8: Comparison of ν as computed by the Hill determinant and a for $N_f = 2$ as computed from the instanton series (with $\hbar = 1$).

In tables 5.7 and 5.8 we check the equality to this order of approximation

$$\nu = \frac{1}{\sqrt{2}}\frac{a}{\hbar} \qquad \mathrm{mod}(n)\,, \quad n \in \mathbb{Z} \tag{5.160}$$

We notice that (for $N_f = 2$) the first instanton series coeffient match the general mathematical analytical result (from continued fractions tecnique) for the expansion of the eigenvalue of Doubly Confluent Heun equation in Λ given in [77] in terms of $\mu = \nu\,\mathrm{mod}(n)$ with the identification (5.160), as shown in (D.24) of appendix D. This provides a very strong analytical check of our gauge period-Floquet identification.

In conclusion, from the a period-Floquet identification (5.160) and the Floquet-T function identifications (5.148)-(5.151) follow new gauge-integrability basic connection formulas for the T function and a period. For $N_f = 1$

$$\begin{aligned} T_+^2(\theta) &= 2\cos\frac{\sqrt{2}\pi a}{\hbar} \\ \tilde{T}_+(\theta)\tilde{T}_-(\theta + i\frac{\pi}{3}) &= 2\cos\frac{\sqrt{2}\pi a}{\hbar} + 2\cos\frac{2\pi m}{\hbar} \end{aligned} \tag{5.161}$$

and for $N_f = 2$

$$\begin{aligned} T_{+,+}(\theta)T_{+,-}(\theta + i\frac{\pi}{2}) &= 2\cos\frac{\sqrt{2}\pi a}{\hbar} + 2\cos\frac{2\pi m_2}{\hbar} \\ \tilde{T}_{+,+}(\theta)\tilde{T}_{-,+}(\theta + i\frac{\pi}{2}) &= 2\cos\frac{\sqrt{2}\pi a}{\hbar} + 2\cos\frac{2\pi m_1}{\hbar} \end{aligned} \tag{5.162}$$

5.4. Applications of gauge-integrability correspondence

We now show some applications of the gauge-integrability correspondence as new results on both sides. In particular, for gauge theory we find a gauge interpretation of integrability's functional relations, namely as exact R-symmetry relations never found before to our knowledge. For integrability instead we find new formulas for the local integrals of motions in terms of the asymptotic gauge periods, which may sometimes be convenient.

5.4.1. Applications to gauge theory

Consider first $N_f = 2$. We have the relation (5.151) which considering that $a = \nu$ (cf. (5.160)) becomes

$$T_{++}(\theta)T_{+-}(\theta + i\pi/2) = 4\cos(a - q_2)\cos(a + q_2) \tag{5.163}$$

Now using the T periodicity relation (5.38) and the TQ relation (5.36) becomes

$$\begin{aligned} T_{++}(\theta)T_{--}(\theta) = \frac{1}{Q_{++}(\theta)Q_{--}(\theta)}\Big[&Q_{+-}(\theta + i\pi/2)Q_{-+}(\theta + i\pi/2) + Q_{+-}(\theta - i\pi/2)Q_{-+}(\theta - i\pi/2) \\ &+ e^{2i\pi q_2}Q_{+-}(\theta + i\pi/2)Q_{-+}(\theta - i\pi/2) + e^{-2i\pi q_2}Q_{+-}(\theta - i\pi/2)Q_{-+}(\theta + i\pi/2)\Big] \end{aligned} \tag{5.164}$$

Now we claim that thanks to our connection of T function and Q/Y function to gauge periods a and a_D, this TQ relation becomes an $\mathbb{Z}_2$ R-symmetry relation for the *exact* gauge periods a, a_D. Indeed, such relations where already known in the $SU(2)$ $N_f = 0$ case for the $\hbar \to 0$ *asymptotic* expansion modes $a^{(n)}$, $a_D^{(n)}$ [72]. For the massless $SU(2)$ $N_f = 2$ case the periods are the same, up to a factor 2 [83]. If $u > 0$ they are

$$\begin{aligned} a^{(0)}(-u, 0, 0) &= -ia^{(0)}(u, 0, 0) \\ a_D^{(0)}(-u, 0, 0) &= -i[a_D^{(0)}(u, 0, 0) - a^{(0)}(u, 0, 0)] \end{aligned} \tag{5.165}$$

Indeed, expressing (5.164) in terms of gauge periods through (5.163) and (5.117) we get

$$a^{(0)}(-u, 0, 0) = -a_D^{(0)}(-u) - ia_D^{(0)}(u) \tag{5.166}$$

which is consistent with the same relations (5.165). Actually, relations (5.165) can be considered to be *derived* from the TQ relation when coupled with the T periodicity relation (5.38)

$$T_{-+}(\theta + i\pi/2) = T_{++}(\theta) \tag{5.167}$$

which inside (5.163) reads

$$T_{++}(\theta)T_{--}(\theta) = T_{-+}(\theta + i\pi/2)T_{+-}(\theta + i\pi/2) \tag{5.168}$$

and is then another $\mathbb{Z}_2$ R-symmetry relation for the *exact* gauge periods a. Indeed, in the massless $N_f = 2$ case reduces precisely to the first of (5.165). Thus we conclude that $\mathbb{Z}_2$ R-symmetry for exact gauge theory periods is encoded in the integrability TQ and T periodicity functional relations.

Similarly for $N_f = 1$ case the T periodicity is easily shown to be interpreted in gauge theory in the same way. If $u > 0$ and $m = 0$ the other exact relation from the T periodicity (5.37) reduces to the $\mathbb{Z}_3$ symmetry in the asymptotic $\hbar \to 0$ (cf. (C.9))

$$a^{(0)}(e^{-2\pi i/3}u, 0) = -e^{2\pi i/3}a^{(0)}(u, 0) \tag{5.169}$$

$$a^{(n)}(e^{-2\pi i/3}u, 0) = -e^{2\pi i/3(1-n)}a^{(n)}(u, 0) \tag{5.170}$$

We avoid though for the moment considering the $N_f = 1$ TQ relation since it requires some non-trivial analytic continuation of gauge-integrabiliy relations beyond the complex strip $\Im\theta < \pi/3$ in which the TBA holds without analytic continuation.

We see that the new exact relations following from the integrability functional relations are a $\mathbb{Z}_2$, $\mathbb{Z}_3$ $N_f = 2, 1$ R-symmetry relations. They were never found previously in the literature, to our knowledge. We knew only the $\hbar \to 0$ perturbative relations, also in the massless case in [83].

5.4.2. Applications to integrability

We now find a new ways to compute either the local integrals of motions for the Perturbed Hairpin IM or the asymptotic expansion modes of the $N_f = 1$ quantum gauge periods.

Consider the large energy asymptotic expansion (5.54) of Q in terms of the LIMs. We set first $q = 0$ so to recover the LIMs of Liouville $b = \sqrt{2}$. For this particular case the expansion simplifies as

$$\ln Q(\theta, p) \doteq -C_0 e^{\theta} - \sum_{n=1}^{\infty} e^{\theta(1-2n)} C_n \mathbb{I}_{2n-1}\,, \qquad \theta \to +\infty\,, \qquad p \quad \text{finite}\,. \tag{5.171}$$

The normalization constants are given (*cf.* [1] with $b = \sqrt{2}$)

$$C_n = \frac{\Gamma\left(\frac{2n}{3} - \frac{1}{3}\right)\Gamma\left(\frac{n}{3} - \frac{1}{6}\right)}{3\sqrt{2\pi}n!}\,. \tag{5.172}$$

We can also expand the LIMs $\mathbb{I}_{2n-1}$, as polynomials in p^2 with coefficients $\Upsilon_{n,k}$

$$\mathbb{I}_{2n-1} = \sum_{k=0}^{n} \Upsilon_{n,k} p^{2k}\,. \tag{5.173}$$

The leading and subleading coefficients are found to be [1]

$$\Upsilon_{n,n} = (-1)^n\,, \qquad \Upsilon_{n,n-1} = \frac{1}{24}(-1)^n n(2n-1)\,. \tag{5.174}$$

Now, since in Seiberg-Witten theory u is finite as $\theta \to +\infty$, to connect the IM $\theta \to +\infty$ *asymptotic* expansion, it is necessary to take the further limit

$$p^2(\theta) = 4\frac{u}{\Lambda_1^2}e^{2\theta} \to +\infty\,. \tag{5.175}$$

In this double limit, an infinite number of LIMs $\mathbb{I}_{2n-1}(b=\sqrt{2})$, through their coefficients $\Upsilon_{n,k}$, are re-summed into a an quantum gauge period asymptotic mode (a sort of LIM on its way). For instance the leading order is obtained from the resummation of all $\Upsilon_{n,n}=(-1)^n$ terms as

$$\ln Q^{(0)}(u,0,\Lambda_1) = -\sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n}{3}-\frac{1}{3}\right)\Gamma\left(\frac{n}{3}-\frac{1}{6}\right)}{3\sqrt{2\pi}n!}\left(-\frac{4u}{\Lambda_1^2}\right)^n \tag{5.176}$$

and from it we can derive the higher orders as usual through differential operators (5.127). In particular, in the massless case the first simplify as

$$\begin{aligned}
\ln Q^{(1)}(u,0,\Lambda_1) &= \left(\frac{\Lambda_1}{2}\right)^2\left[\frac{u}{6}\frac{\partial^2}{\partial u^2}+\frac{1}{12}\frac{\partial}{\partial u}\right]\ln Q^{(0)}(u,0,\Lambda_1)\\
\ln Q^{(2)}(u,0,\Lambda_1) &= \left(\frac{\Lambda_1}{2}\right)^4\left[\frac{7}{360}u^2\frac{\partial^4}{\partial u^4}+\frac{31}{360}u\frac{\partial^3}{\partial u^3}+\frac{9}{160}\frac{\partial^2}{\partial u^2}\right]\ln Q^{(0)}(u,0,\Lambda_1)\\
\ln Q^{(3)}(u,0,\Lambda_1) &= \left(\frac{\Lambda_1}{2}\right)^6\left[\frac{31u^3}{15120}\frac{\partial^6}{\partial u^6}+\frac{443u^2}{18144}\frac{\partial^5}{\partial u^5}+\frac{43u}{576}\frac{\partial^4}{\partial u^4}+\frac{557}{10368}\frac{\partial^3}{\partial u^3}\right]\ln Q^{(0)}(u,0,\Lambda_1)\,.
\end{aligned} \tag{5.177}$$

Indeed these expression match with the resummation of LIMs at higher orders:

$$\ln Q^{(1)}(u,0,\Lambda_1) = \left(\frac{\Lambda_1}{2}\right)^2\sum_{n=0}^{\infty}\left[\frac{n}{12}+\frac{1}{24}\right]\frac{\Gamma\left(\frac{2n}{3}+\frac{1}{3}\right)\Gamma\left(\frac{n}{3}+\frac{1}{6}\right)}{3\sqrt{2\pi}n!}\left(-\frac{4u}{\Lambda_1^2}\right)^n \tag{5.178}$$

$$\ln Q^{(2)}(u,0,\Lambda_1) = -\left(\frac{\Lambda_1}{2}\right)^4\sum_{n=0}^{\infty}\left[\frac{(14n+27)(2n+3)}{5760}\right]\frac{\Gamma\left(\frac{2n}{3}+1\right)\Gamma\left(\frac{n}{3}+\frac{1}{2}\right)}{3\sqrt{2\pi}n!}\left(-\frac{4u}{\Lambda_1^2}\right)^n \tag{5.179}$$

$$\ln Q^{(3)}(u,0,\Lambda_1) = \left(\frac{\Lambda_1}{2}\right)^6\sum_{n=0}^{\infty}\left[\frac{1}{8}\frac{[4n(93n+596)+3899](2n+5)}{362880}\right]\frac{\Gamma\left(\frac{2n}{3}+\frac{5}{3}\right)\Gamma\left(\frac{n}{3}+\frac{5}{6}\right)}{3\sqrt{2\pi}n!}\left(-\frac{4u}{\Lambda_1^2}\right)^n \tag{5.180}$$

So in general we find the relation

$$\ln Q^{(k)}(u,0,\Lambda_1) = (-1)^{k+1}\left(\frac{\Lambda_1}{2}\right)^{2k}\sum_{n=0}^{\infty}\Upsilon_{n+k,n}\frac{\Gamma\left(\frac{k+n}{3}-\frac{1}{6}\right)\Gamma\left(\frac{2(k+n)}{3}-\frac{1}{3}\right)}{3\sqrt{2\pi}(k+n)!}\left(\frac{4u}{\Lambda_1^2}\right)^n\,. \tag{5.181}$$

Thus this procedure can actually be a convenient way to compute the LIMs coefficients $\Upsilon_{n+k,n}$ for general n at each successive k order. Alternatively and equivalently, we can use it to compute the k-th mode of the (alternative dual) quantum period a_1

$$\frac{4\sqrt{2}\pi}{\Lambda_1}a_1^{(k)}(u,0,\Lambda_1) = -\sum_{k=0}^{\infty}\Upsilon_{n+k,n}\frac{\Gamma\left(\frac{k+n}{3}-\frac{1}{6}\right)\Gamma\left(\frac{2(k+n)}{3}-\frac{1}{3}\right)}{3\sqrt{2\pi}(k+n)!}2\sin\left(\frac{1}{3}\pi(k+n+1)\right)\left(\frac{4u}{\Lambda_1^2}\right)^n. \tag{5.182}$$

5.5. Limit to lower flavours gauge theories

5.5.1. Limit from $N_f = 1$ to $N_f = 0$

The Seiberg-Witten curve for $N_f = 1$

$$y_{SW,1}^2 = x^2(x-u) + \frac{\Lambda_1}{4} m_1 x - \frac{\Lambda_1^6}{64} \tag{5.183}$$

in the limit

$$\Lambda_1 \to 0\,, \quad m_1 \to \infty\,, \quad \text{with } \Lambda_1^3 m_1 = \Lambda_0^4\,. \tag{5.184}$$

flows to the Seiberg-Witten curve for $N_f = 0$

$$y_{SW,0}^2 = x^2(x-u) + \frac{\Lambda_0^4}{4} x\,. \tag{5.185}$$

Similarly the $N_f = 1$ quantum Seiberg-Witten curve:

$$-\hbar^2 \frac{d^2}{dy_1^2}\psi + \left[\frac{1}{16}\Lambda_1^3 e^{2y_1} + \frac{1}{2}\Lambda_1^{3/2} e^{-y_1} + \frac{1}{2}\Lambda_1^{3/2} m_1 e^{y_1} + u\right]\psi = 0\,. \tag{5.186}$$

if we let

$$y_1 = y_0 - \frac{1}{2}\ln m_1 \to -\infty \tag{5.187}$$

becomes

$$-\hbar^2 \frac{d^2}{dy_0^2}\psi + \left[\frac{1}{16}\frac{\Lambda_1^3}{m_1} e^{2y_0} + \frac{1}{2}\Lambda_1^{3/2} m_1^{1/2} e^{-y_0} + \frac{1}{2}\Lambda_1^{3/2} m_1^{1/2} e^{y_0} + u\right]\psi = 0 \tag{5.188}$$

that is precisely reduce to the $N_f = 0$ equation:

$$-\hbar^2 \frac{d^2}{dy_0^2}\psi + (\Lambda_0^2 \cosh y_0 + u)\psi = 0\,. \tag{5.189}$$

We can also consider the limit on the integrability equation as follows. The Perturbed Hairpin IM ODE/IM equation is

$$-\frac{d^2}{d{y_1}^2}\psi(y_1) + [e^{2\theta_1}(e^{2y_1} + e^{-y_1}) + 2q e^{\theta_1} e^{y_1} + p_1^2]\psi(y_1) = 0\,. \tag{5.190}$$

and it must reduce to the ODE/IM equation for the Liouville model studied in [1]

$$-\frac{d^2}{d{y_0}^2}\psi(y_0) + \{e^{2\theta_0}[e^{y_0} + e^{-y_0}] + p_0^2\}\psi(y_0) = 0\,, \tag{5.191}$$

In order for (5.190) to go into (5.191) we need to impose

$$\begin{gathered} e^{2\theta_1 + 2y_1} \to 0 \\ e^{2\theta_1 - y_1} = e^{2\theta_0 - y_0} \qquad 2q e^{\theta_1 + y_1} = e^{2\theta_0 + y_0} \qquad p_1 = p_0 \end{gathered} \tag{5.192}$$

or

$$q = \frac{1}{2}\frac{e^{4\theta_0}}{e^{3\theta_1}} \tag{5.193}$$
$$y_1 = y_0 - 2\theta_0 + 2\theta_1$$

Now the limit requires $\theta_1 + y_1 \to -\infty$, that is

$$\theta_1 \to -\infty \tag{5.194}$$

and as a consequence

$$q \sim e^{-3\theta_1} \to \infty \qquad \theta_1 \to -\infty \tag{5.195}$$

We now consider also the limit on gauge periods. We numerically find, for $u, m_1, \Lambda_1 > 0$, $\Lambda_1 \to 0, m_1 \to \infty, \Lambda_1^3 m_1 = \Lambda_0^4$

$$a^{(0)}_{1,1}(u, m_1, \Lambda_1) \to -a^{(0)}_{0,D}(u, \Lambda_0) \tag{5.196}$$

$$a^{(0)}_{1,1}(-u, m_1, \Lambda_1) \to -a^{(0)}_{0,D}(-u, \Lambda_0) + a^{(0)}_0(-u + i0, \Lambda_0) \tag{5.197}$$

$$= -ia^{(0)}_{0,D}(u, \Lambda_0) \tag{5.198}$$

$$a^{(0)}_{1,2}(\pm u, m_1, \Lambda_1) + \frac{m_1}{\sqrt{2}} \to \frac{1}{2}a^{(0)}_0(\pm u, \Lambda_0) \tag{5.199}$$

$$a^{(0)}_{1,1}(e^{\pm 2\pi i/3}u, e^{\mp 2\pi i/3}m_1, \Lambda_1) - \frac{e^{\mp 2\pi i/3}m_1}{\sqrt{2}} \to \frac{1}{2}a^{(0)}_0(u, e^{\mp i\pi/6}\Lambda_0) \tag{5.200}$$

$$a^{(0)}_{1,1}(-e^{+2\pi i/3}u, e^{-2\pi i/3}m_1, \Lambda_1) - \frac{e^{-2\pi i/3}m_1}{\sqrt{2}} \to e^{-2\pi i/3}[a^{(0)}_{0,D}(-u, \Lambda_0) - \frac{1}{2}a^{(0)}_0(-u + i0, \Lambda_0)] \tag{5.201}$$

$$a^{(0)}_{1,1}(-e^{-2\pi i/3}u, e^{2\pi i/3}m_1, \Lambda_1) - \frac{e^{2\pi i/3}m_1}{\sqrt{2}} \to e^{2\pi i/3}[-\frac{1}{2}a^{(0)}_0(-u + i0, \Lambda_0)] \tag{5.202}$$

5.5.2. Limit from $N_f = 2$ to $N_f = 1$

Staring from the $N_f = 2$ quantum Seiberg Witten curve

$$-\hbar^2 \frac{d^2}{dy_2^2}\psi + \left[\frac{1}{16}\Lambda_2^2(e^{2y_2} + e^{-2y_2}) + \frac{1}{2}\Lambda_2 m_1 e^{y_2} + \frac{1}{2}\Lambda_2 m_2 e^{-y_2} + u\right]\psi = 0\,, \tag{5.203}$$

since we have

$$\Lambda_2^2 m_2 = \Lambda_1^3 \qquad m_2 \to \infty \qquad \Lambda_2 \to 0 \tag{5.204}$$

we can set

$$y_2 = y_1 + \frac{1}{2}\ln m_2 \to +\infty \tag{5.205}$$

so the equation becomes

$$-\hbar^2 \frac{d^2}{dy_1^2}\psi + \left[\frac{1}{16}\Lambda_2^2\left(m_2 e^{2y_1} + \frac{1}{m_2}e^{-2y_1}\right) + \frac{1}{2}\Lambda_2\sqrt{m_2}m_1 e^{y_2} + \frac{1}{2}\Lambda_2\sqrt{m_2}e^{-y_2} + u\right]\psi = 0 \tag{5.206}$$

which in the limit reduces to the $N_f = 1$ quantum Seiberg-Witten curve equation:

$$-\hbar^2 \frac{d^2}{dy_1^2}\psi + \left[\frac{1}{16}\Lambda_1^3 e^{2y_1} + \frac{1}{2}\Lambda_1^{3/2} e^{-y_1} + \frac{1}{2}\Lambda_1^{3/2} m_1 e^{y_1} + u\right]\psi = 0\,. \tag{5.207}$$

In integrability variables, we impose the conditions that allow the limit of the differential equations

$$e^{2\theta+2y_2} = e^{2\theta_1+2y_1}\,, \quad e^{\theta_2+y_2}q_1 = e^{\theta_1+y_1}q_1\,, \quad 2e^{\theta_2-y_2}q_2 = e^{2\theta_1-y_1}\,, \quad e^{2\theta_2-2y_2} \to 0\,, \quad p_2^2 = p_1^2\,. \tag{5.208}$$

from which we deduce that we have to take the limit

$$y_2 = -\theta_2 + \theta_1 + y_1 \qquad \theta_2 \to -\infty \qquad q_2 = \frac{1}{2}e^{3\theta_1 - 2\theta_2} \to \infty \tag{5.209}$$

5.6. Gravitational correspondence and applications

5.6.1. Gravitational correspondence $N_f = 2$

Our two-fold integrability-gauge correspondence actually is three-fold method as black hole's perturbation theory involves the same ODEs we use. In particular the Doubly Confluent Heun equation (see appendix D) we have for the $SU(2)$ $N_f = 0, 1, 2$ gauge theory and Generalized Perturbed Hairpin integrable model is typically associated to *extremal black holes*. In particular, for the $N_f = 2$ we consider now the gravitational background given by the intersection of four stacks of D3-branes in type IIB supergravity. This geometry is characterised by four different charges $\mathcal{Q}_i$ which, if all equal, lead to an extremal RN BH, that is maximally charged. In isotropic coordinates the line element writes [93, 9]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\,, \tag{5.210}$$

with $f(r) = \prod_{i=1}^4 (1 + \mathcal{Q}_i/r)^{-\frac{1}{2}}$. The ODE describing the scalar perturbation is, with $\Sigma_k = \sum_{i_1<\ldots<i_k}^4 \mathcal{Q}_{i_1}\cdots\mathcal{Q}_{i_k}$

$$\frac{d^2\phi}{dr^2} + \left[-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \omega^2 \sum_{k=0}^4 \frac{\Sigma_k}{r^k}\right]\phi = 0\,. \tag{5.211}$$

Changing variables as $r = \sqrt[4]{\Sigma_4}e^y$ and

$$\omega\sqrt[4]{\Sigma_4} = -ie^\theta \qquad q_j = \frac{1}{2}\frac{\Sigma_{2j-1}}{\sqrt[4]{\Sigma_4}^{2j-1}}e^\theta \qquad p^2 = (l+\frac{1}{2})^2 - \omega^2\Sigma_2\,, \tag{5.212}$$

($j = 1, 2$) the ODE takes precisely the form of the Generalized Perturbed Hairpin IM (5.4).

Setting up ODE/IM in gravity variables (5.212), we notice that the discrete symmetries (5.10) are consistent with the brane dictionary (5.212), as the brane parameters vary as $\Sigma_1 \to \pm i\Sigma_1\,, \Sigma_2 \to -\Sigma_2\,, \Sigma_3 \to \mp i\Sigma_3\,, \Sigma_4 \to \Sigma_4$[15]. So in gravity variables the Y system reads

$$Y(\theta + \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3)Y(\theta - \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3) = [1 + Y(\theta, \Sigma_1, \Sigma_2, \Sigma_3)][1 + Y(\theta, -\Sigma_1, \Sigma_2, -\Sigma_3)]\,, \tag{5.213}$$

[15] This observation does not mean that a dictionary not consistent with the discrete symmetry would imply ODE/IM cannot be used: in that case we should just do ODE/IM in the suitable variables and then *afterwards* change to the variables of interest.

(with Σ_4 omitted since it is fixed). We remark we shall pay particular attention to the change of variables from gravity or gauge to integrability: this results in different TBA equations as first noted in [1]. Indeed, Y system (5.213) can be inverted into the TBA in gravitational variables

$$\begin{aligned}\varepsilon_{\pm,\pm}(\theta) &= [f_{0,+} \mp \frac{i\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}} - \frac{\Sigma_3}{\Sigma_4^{3/4}})]e^{\theta} - \varphi * (\bar{L}_{\pm\pm} + \bar{L}_{\mp\mp})(\theta) \\ \bar{\varepsilon}_{\pm,\pm}(\theta) &= [\bar{f}_{0,+} \pm \frac{\pi}{2}(\frac{\Sigma_1}{\Sigma_4^{1/4}} + \frac{\Sigma_3}{\Sigma_4^{3/4}})]e^{\theta} - \varphi * (L_{\pm\pm} + L_{\mp\mp})(\theta)\end{aligned} \tag{5.214}$$

where we defined $\varepsilon(\theta) = -\ln Y(\theta, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, $\bar{\varepsilon}(\theta) = \varepsilon(\theta, i\Sigma_1, -\Sigma_2, -i\Sigma_3, \Sigma_4)$, $L = \ln[1 + \exp\{-\varepsilon\}]$, $\varphi(\theta) = (\cosh(\theta))^{-1}$and

$$f_{0,\pm} = c_{0,+,\pm} + c_{0,-,\mp} \qquad c_{0,+,\pm} = c_0(\Sigma_1, \Sigma_2, \pm\Sigma_3, \Sigma_4) \tag{5.215}$$

with

$$c_0(\,) = \int_{-\infty}^{\infty} \left[\sqrt{2\cosh(2y) + \frac{\Sigma_1}{\sqrt[4]{\Sigma_4}}e^{y} + \frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}}e^{-y} + \frac{\Sigma_2}{\sqrt{\Sigma_4}}} - 2\cosh y - \frac{1}{2}\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}}\frac{1}{1+e^{-y/2}} - \frac{1}{2}\frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}}\frac{1}{1+e^{y/2}}\right] dy \tag{5.216}$$

which in turn can be expressed either through a triple power series for small parameters or as an elliptic integral as

$$\begin{aligned}c_0(\,) &= \sum_{l,m,n=0}^{\infty} \binom{\frac{1}{2}}{l}\binom{\frac{1}{2}-l}{m}\binom{-l-m+\frac{1}{2}}{n}\mathcal{B}_2(l,m,n)\left(\frac{\Sigma_1}{\sqrt[4]{\Sigma_4}}\right)^n\left(\frac{\Sigma_2}{\sqrt{\Sigma_4}}\right)^m\left(\frac{\Sigma_3}{\sqrt[4]{\Sigma_4^3}}\right)^l \\ \mathcal{B}_2(l,m,n) &= \frac{\Gamma\left(\frac{1}{4}(3l+2m+n-1)\right)\Gamma\left(\frac{1}{4}(l+2m+3n-1)\right)}{4\Gamma\left(l+m+n-\frac{1}{2}\right)} \\ \mathcal{B}_2(1,0,0) &= \frac{1}{2}(\ln 2 - 1)\,, \quad \mathcal{B}_2(0,0,1) = \frac{1}{2}\ln 2\end{aligned} \tag{5.217}$$

We have to numerically input l in the TBA with the boundary condition at $\theta \to -\infty$:

$$\varepsilon_{\pm,\pm}(\theta) \simeq 4P\theta \simeq 4(l+1/2) + 2C(p)\theta \qquad \theta \to -\infty\,, \tag{5.218}$$

$$C(p) = \ln\left(\frac{2^{1-2p}p\Gamma(2p)^2}{\Gamma\left(p+\frac{1}{2}\right)^2}\right) \tag{5.219}$$

also following from the asymptotic of the ODE (5.4) (the precision improves by adding also the constant at the subleading order, as explained in [78]). Through this TBA we find again the QNMs to be given by the Bethe roots condition

$$\bar{\varepsilon}_{+,+}(\theta_{n'} - i\pi/2) = -i\pi(2n'+1)\,, \qquad Q_{+,+}(\theta_n) = 0 \qquad n' \in \mathbb{Z} \tag{5.220}$$

and we show in tables 5.10 their agreement with continued fraction (Leaver) method and WKB approximation ($l \to \infty$) [31]. We notice that for $\Sigma_1 \neq \Sigma_3$ and $\Sigma_4 \neq 1$ the Leaver method

n	l	TBA	Leaver	WKB
0	1	$0.869623 - 0.372022i$	$0.868932 - 0.372859i$	$0.89642 - 0.36596i$
0	2	$1.477990 - 0.368144i$	$1.477888 - 0.368240i$	$1.4940 - 0.36596i$
0	3	$2.080200 - 0.367076i$	$2.080168 - 0.367097i$	$2.0916 - 0.36596i$
0	4	$2.680363 - 0.366637i$	$2.680350 - 0.366642i$	$2.6893 - 0.36596i$

Table 5.9: Comparison of QNMs obtained from TBA (5.214), Leaver method (through (5.220) with $n' = 0$) and WKB approximation ($\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$.)

n	l	TBA	Leaver	WKB
0	1	$0.896681 - 0.40069i$	N.A.	$0.93069 - 0.39458i$
0	2	$1.5308 - 0.39676i$	N.A.	$1.5511 - 0.39458i$
0	3	$2.15708 - 0.395689i$	N.A.	$2.1716 - 0.39458i$
0	4	$2.78077 - 0.39525i$	N.A.	$2.7921 - 0.39458i$

Table 5.10: Comparison of QNMs obtained from TBA (5.214), (through (5.220) with $n' = 0$) and WKB approximation ($\Sigma_1 = 0.1$, $\Sigma_2 = 0.2$, $\Sigma_3 = 0.3$, $\Sigma_4 = 1$). Since $\Sigma_1 \neq \Sigma_3$ the Leaver method seems not applicable, at least in its original version (N.A.).

is not applicable, at least in its original version since the recursion produced by the ODE involves more than 3 terms (compare [7, 31]) and thus also for this reason the TBA method may be regarded as convenient. However, we point out that there exists a development of the Leaver method, the so-called matrix Leaver method which is still applicable [94, 95]. Thinking to gauge theory, it is clear from the black hole physical requirement in ODE/IM language (5.220) and our gauge-integrability identication (5.138), it follows that the integral gauge period a_D must be quantized.

$$\frac{2\sqrt{2}\pi}{\hbar(\theta)} a_D(\theta, u, m_1, m_2, \Lambda_2) = -i\pi(2n' + 1) \tag{5.221}$$

This constitutes a (mathematical?) proof of the essential finding of [6] and the following literature (see the introduction).

A note of caution, though. Literature following [6] uses another definition of gauge period which we denote by A_D which derives from the instanton expansion of the prepotential. As we explain in section 4.8 the two definitions can be actually related by formulas like (4.315) for the $N_f = 0$ theory. Generalizations of formula (4.315), already exist for the subcase of the $N_f = 1$ gauge theory [79] (see next subsection) and so we expect them to exist also for the whole $N_f = 2$ theory and even more generally. In this way we expect that in general the integrable Bethe roots condition, which we have shown to follow straightforwardly from BHs physics, in gauge theory indeed corresponds to the quantization of the gauge A_D period as stated in [6].

By making considerations on these TQ systems and the QQ system (5.24) like done in [11] and reported in section 4.8, we are not in general able to conclude any quantization condition on the T function, except in the case of equal masses $q_1 = q_2 \equiv q$ where we find

$$T_{+,+}(\theta_n) T_{-,-}(\theta_n) = 4\,. \tag{5.222}$$

that generalizes (4.331) for $N_f = 0$. We now prove (5.222). From the QQ system (5.24) we can write, for general q_1, q_2

$$e^{i\pi q_1} Q_{-+}(\theta - i\pi/2) = c_0 \left[1 \pm i e^{i\pi \frac{q_1 - q_2}{2}} \sqrt{Q_{+,+}(\theta) Q_{-,-}(\theta)}\right] \tag{5.223}$$

$$e^{-i\pi q_2} Q_{+-}(\theta + i\pi/2) = \frac{1}{c_0} \left[1 \mp i e^{i\pi \frac{q_1 - q_2}{2}} \sqrt{Q_{+,+}(\theta) Q_{-,-}(\theta)}\right] \tag{5.224}$$

$$e^{i\pi q_2} Q_{+-}(\theta - i\pi/2) = \frac{1}{c_0'} \left[1 \pm i e^{-i\pi \frac{q_1 - q_2}{2}} \sqrt{Q_{+,+}(\theta) Q_{-,-}(\theta)}\right] \tag{5.225}$$

$$e^{-i\pi q_1} Q_{-+}(\theta + i\pi/2) = c_0' \left[1 \mp i e^{-i\pi \frac{q_1 - q_2}{2}} \sqrt{Q_{+,+}(\theta) Q_{-,-}(\theta)}\right] . \tag{5.226}$$

From the 2 TQ system (5.36) at the Bethe roots we get the same relation

$$c_0(-q_1, q_2) = -c_0'(-q_1, q_2) . \tag{5.227}$$

We can also exchange the masses in (5.223) and (5.225) to obtain the relation

$$c_0(-q_1, q_2) c_0(-q_2, q_1) = -1 . \tag{5.228}$$

In addition, considering real parameters, we have

$$c_0 = -c_0^* . \tag{5.229}$$

However, *we cannot fix c_0 completely in general*, only when $q_1 = q_2 = q$ we can say

$$c_0(q_1, q_2 = q_1) = \pm i . \tag{5.230}$$

We notice also that

$$Q_{+,-} = Q_{-,+} \qquad q_1 = q_2 = q . \tag{5.231}$$

We can generalize the $N_f = 0$ procedure by considering the Y system instead of the Q system.

$$\begin{aligned} T_{+,+}(\theta) T_{-,-}(\theta) Y_{+,+}(\theta) &= [e^{i\pi q} Q_{+,-}(\theta - i\pi/2) + e^{-i\pi q} Q_{+,-}(\theta + i\pi/2)][e^{i\pi q} Q_{-,+}(\theta - i\pi/2) + e^{-i\pi q} Q_{-,+}(\theta + i\pi \\ &= Y_{+,-}(\theta - i\pi/2) + Y_{-,+}(\theta + i\pi/2) + 2 + 2 Y_{+,+}(\theta) . \end{aligned} \tag{5.232}$$

Notice that we can write shifted Y as

$$\begin{aligned} Y_{+,-}(\theta - i\pi/2) &= e^{2\pi i q} Q_{+,-}(\theta - i\pi/2) Q_{-,+}(\theta - i\pi/2) \\ &= -1 \mp 2i \sqrt{Q_{+,+}(\theta) Q_{-,-}(\theta)} + Q_{+,+}(\theta) Q_{-,-}(\theta) \\ &= -1 \mp 2i \sqrt{Y_{+,+}(\theta)} + Y_{+,+}(\theta) \end{aligned} \tag{5.233}$$

and

$$Y_{-,+}(\theta + i\pi/2) = -1 \pm 2i \sqrt{Y_{+,+}(\theta)} + Y_{+,+}(\theta) . \tag{5.234}$$

Inserting these shifted-Y expressions in what we could call the TY relation (5.232) we find

$$T_{+,+}(\theta)T_{-,-}(\theta)Y_{+,+}(\theta) = +4Y_{+,+}(\theta)\,, \tag{5.235}$$

that is nothing but quantization relation on T (5.222).

Now, on plugging the T periodicity relations (5.38) $T_{+,-}(\theta + i\frac{\pi}{2}) = T_{+,+}(\theta)$, $\tilde{T}_{-,+}(\theta + i\frac{\pi}{2}) = \tilde{T}_{+,+}(\theta)$ in the relations between T, $\tilde{T}$ and ν (5.151), (5.149) we get the simplification to only one T

$$\begin{aligned} \pm\sqrt{2\cos 2\pi\nu + 2\cos 2\pi q_2} &= T_{+,+}(\theta) \\ \pm\sqrt{2\cos 2\pi\nu + 2\cos 2\pi q_1} &= \tilde{T}_{+,+}(\theta) \end{aligned} \tag{5.236}$$

Now we notice from the $\nu = a$ instanton series terms (5.159) that

$$\nu(q_1, q_2) = \nu(-q_1, -q_2) \tag{5.237}$$

so we can write the same relations for also opposite masses

$$\begin{aligned} \pm\sqrt{2\cos 2\pi\nu + 2\cos 2\pi q_2} &= T_{-,-}(\theta) \\ \pm\sqrt{2\cos 2\pi\nu + 2\cos 2\pi q_1} &= \tilde{T}_{-,-}(\theta)\,. \end{aligned} \tag{5.238}$$

Now from T quantization for $q_1 = q_2 = q$ (5.222)

$$T_{+,+}(\theta)T_{-,-}(\theta) = \pm[2\cos 2\pi\nu + 2\cos 2\pi q] = 4 \tag{5.239}$$

it follows a quantization condition on the *combination* of ν and q

$$[\cos 2\pi\nu + \cos 2\pi q]_{\theta=\theta_n} = \pm 2\,. \tag{5.240}$$

In conclusion, from this derivation we do not expect that the alternative QNMs quantization condition on the gauge a period found in [7] for $N_f = 0$ generalizes to other gauge theories, both because the integrabilty T function is not quantized generally (for different masses m_1, m_2 q_1, q_2) and because even when it is, it implies a quantization on only the combination of a ν period and masses.

Now we can find also an integrability interpretation of the symmetry under Couch-Torrence transformation found for this gravitational background in [96], thanks to identifications of certain scattering angles with the SW a period. It refers to the symmetry that exchange infinity ($y \to +\infty$) and the (analogue) horizon ($y \to -\infty$), leaving the photon sphere ($y = 0$) fixed. In our ODE approach, it correspondence to the following wave function properties

$$\psi_{+,0}(y) = \psi_{-,0}(-y)\,, \qquad (q_1 = q_2) \tag{5.241}$$

which we notice holds only for equal masses. In this respect, under (5.241) we have the T and $\tilde{T}$ identity

$$\tilde{T}_{+,+}(\theta) = T_{+,+}(\theta) \qquad (q_1 = q_2)\,, \tag{5.242}$$

as can be understood by looking to their very definitions (5.32).

All the considerations of this subsection show how integrability structures give valuable insights in several gauge-gravity correspondence mathematical physics issues.

5.6.2. Gravitational correspondence $N_f = 1$

Now, to get a gravitation counterpart of the $N_f = 1$ gauge theory, we can simply take the limit from the $N_f = 2$ theory, as explained in section 5.5. In gravity variables such limit corresponds to

$$\Sigma_4 \to 0 \tag{5.243}$$

and in terms of charges can be realised for instance with $\mathcal{Q}_4 \to 0$. Upon this limit, get the following gravity-integrability parameters dictionary

$$\omega\sqrt[3]{\Sigma_3} = -ie^{\theta}\,, \qquad \frac{\Sigma_1}{\sqrt[3]{\Sigma_3}} = 2q_1 e^{-\theta}\,, \qquad p^2 = (l+\frac{1}{2})^2 - \omega^2\Sigma_2 \tag{5.244}$$

The $N_f = 1$ Y system in gravitational variables reads

$$\begin{aligned} &Y(\theta + i\pi/2, -i\Sigma_1, -\Sigma_2)Y(\theta - i\pi/2, -i\Sigma_1, -\Sigma_2) \\ &= [1 + Y(\theta + i\pi/6, -ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2)][1 + Y(\theta - i\pi/6, -ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2)]\,, \end{aligned} \tag{5.245}$$

from which it appears convenient to define

$$\begin{aligned} &Y_{0,+}(\theta) = Y(\theta, i\Sigma_1, -\Sigma_2) \quad Y_{1,+}(\theta) = Y(\theta, ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2) \quad Y_{2,+}(\theta) = Y(\theta, ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2) \\ &Y_{0,-}(\theta) = Y(\theta, -i\Sigma_1, -\Sigma_2) \quad Y_{1,-}(\theta) = Y(\theta, -ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2) \quad Y_{2,-}(\theta) = Y(\theta, -ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2)\,. \end{aligned} \tag{5.246}$$

The Y system can be inverted in a TBA made of 6 coupled equations as

$$\varepsilon_{0,\pm}(\theta) = (f_{0,\pm} \pm \frac{4}{3}\pi\Sigma_1)e^{\theta} - (\varphi_- * L_{1,\pm})(\theta) - (\varphi_+ * L_{2,\pm})(\theta) \tag{5.247}$$

$$\varepsilon_{1,\pm}(\theta) = (f_{1,\pm} \pm \frac{4}{3}\pi e^{2\pi i/3}\Sigma_1)e^{\theta} - (\varphi_- * L_{2,\pm})(\theta) - (\varphi_+ * L_{0,\pm})(\theta) \tag{5.248}$$

$$\varepsilon_{2,\pm}(\theta) = (f_{2,\pm} \pm \frac{4}{3}\pi e^{-2\pi i/3}\Sigma_1)e^{\theta} - (\varphi_- * L_{0,\pm})(\theta) - (\varphi_+ * L_{1,\pm})(\theta) \tag{5.249}$$

with of course $L_{k,\pm} = \ln[1 + \exp\{-\varepsilon_{k,\pm}\}]$ and the kernels

$$\varphi_\pm(\theta) = \frac{1}{2\pi\cosh(\theta \pm i\pi/6)} \tag{5.250}$$

Under change to gravity variables $q(\theta) = \frac{1}{2}\frac{\Sigma_1}{\sqrt[3]{\Sigma_3}}e^{\theta}$ and so the leading order is given by

$$f_{k,\pm} = -e^{-i\pi/6}c_0(\mp ie^{\frac{2\pi i(1+k)}{3}}\Sigma_1, -e^{-\frac{2\pi i(1+k)}{3}}\Sigma_2) - e^{i\pi/6}c_0(\mp ie^{\frac{2\pi i(-1+k)}{3}}\Sigma_1, -e^{-\frac{2\pi i(-1+k)}{3}}\Sigma_2) \tag{5.251}$$

$$c_0(\Sigma_1;\Sigma_2,\Sigma_3) = \int_{-\infty}^{\infty}\left[\sqrt{e^{2y} + e^{-y} + \frac{\Sigma_1}{\sqrt[3]{\Sigma_3}}e^{y} + \frac{\Sigma_2}{\sqrt[3]{\Sigma_3^2}}} - e^{y} - e^{-y/2} - \frac{1}{2}\frac{\Sigma_1}{\sqrt[3]{\Sigma_3}}\frac{1}{1+e^{-y/2}}\right]dy\,. \tag{5.252}$$

We can compute this integral analytically as usual by expanding it in double binomial series for small Σ_1, Σ_2

$$c_0(\Sigma_1;\Sigma_2,\Sigma_3) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{\Sigma_1}{\sqrt[3]{\Sigma_3}}\right)^n\left(\frac{\Sigma_2}{\sqrt[3]{\Sigma_3^2}}\right)^l\binom{1/2}{l}\binom{1/2-l}{n}\mathcal{B}(n,l) \tag{5.253}$$

$\{n,l,\Sigma_1,\Sigma_2,\Sigma_3\}$	TBA	WKB
$\{0,1,0.1,0.2,1.\}$	$0.996031 - 0.308972i$	$1.018635 - 0.317055i$
$\{0,2,0.1,0.2,1.\}$	$1.6945 - 0.301444i$	$1.69772 - 0.31706i$
$\{0,3,0.1,0.2,1.\}$	$2.39612 - 0.294969i$	$2.37681 - 0.31706i$
$\{0,1,0.2,0.4,1.\}$	$0.943852 - 0.263758i$	$0.959219 - 0.281322i$
$\{0,2,0.2,0.4,1.\}$	$1.59951 - 0.250208i$	$1.59870 - 0.28132i$
$\{0,3,0.2,0.4,1.\}$	$2.25939 - 0.237859i$	$2.23818 - 0.28132i$
$\{0,1,0.4,0.1,1.\}$	$0.966828 - 0.337457i$	$0.990202 - 0.300483i$
$\{0,2,0.4,0.1,1.\}$	$1.64269 - 0.357236i$	$1.65034 - 0.30048i$
$\{0,3,0.4,0.1,1.\}$	$2.32242 - 0.37745i$	$2.31047 - 0.30048i$

Table 5.11: QNMs for $N_f = 1$. Since the Leaver method is not applicable to this case, at least in its original version, we were able to compare only with the WKB approximation, by which however the match is necessarily very rough.

$$\mathcal{B}(n,l) = \frac{1}{3}B\left(\frac{1}{6}(2l+4n-1), \frac{1}{3}(2l+n-1)\right) \qquad (n,l) \neq (1,0)$$
$$\mathcal{B}(1,0) = \frac{2\log(2)}{3} \tag{5.254}$$

As in the Liouville model, also in the Hairpin model the TBA does not contain explicitly p, so that is has to be solved through the boundary condition

$$\varepsilon_\pm(\theta) \simeq 6p\theta \simeq 6(l+\frac{1}{2})\theta_1 + 2C(p)\,, \quad \theta \to -\infty \tag{5.255}$$

$$C(p) = \log\left(\frac{2^{-\frac{p}{\sqrt{2}}}p\Gamma\left(\sqrt{2}p\right)\Gamma\left(2\sqrt{2}p\right)}{\pi}\right) \tag{5.256}$$

From the general analysis of [11] we can safely affirm that the QNMs are given by zeros of Q_+

$$Q_+(\theta_n) = 0\,. \tag{5.257}$$

or the equivalent condition on Y

$$Y_{0,+}(\theta_n - i\pi/2) = -1. \tag{5.258}$$

or ε

$$\varepsilon_{0,+}(\theta_n - i\pi/2) = -i\pi(2n'+1) \qquad n' \in \mathbb{Z} \tag{5.259}$$

With the last relation we can actually compute the QNMs as usual[16]. We report their values obtained in table 5.6.2. Again, we find the Leaver method is not applicable to this case, at least in its original version [31], so we are able to compare only with the WKB approximation, which gives however necessarily a very rough match. Now from our

[16] We notice that to implement this condition through TBA it is NOT necessary to analytically continue (since Y functions are analytic) beyond the poles of the kernels (5.250) at the points $\theta - \theta' = i\frac{\pi}{3}$ by adding their residue.

gauge-integrability identification (5.137) we can prove a quantization on the (alternative) gauge period a_1

$$\frac{2\pi\sqrt{2}}{\hbar}a_1(\theta - i\pi/2, u, m) = -i\pi(2n' + 1) \qquad n' \in \mathbb{Z} \tag{5.260}$$

and as discussed in the previous subsection we surely expect a similar quantization condition on the other differently defined A_D period actually used in the literature on the new gauge-gravity correspondence following [6]. In particular, we can now compare directly with the work [79] in which eq. 8.12 (in the first arXiv version) shows that zeros of Q correspond to quantization conditions on the gauge periods, thus again recovering the characterization of QNMs of [6].

Applying the $N_f = 1$ TQ system (5.35) to also this background, we find the same limitations as for $N_f = 2$ in finding quantization conditions for T and a as in (5.222) and (5.240).

6. $SU(2)$ $N_f = (0,2), 3$ gauge theory, asymptotically flat black holes and fuzzballs

6.1. $N_f = (0,2)$ ODE/IM

The quantum Seiberg-Witten curve for $N_f = (0,2)$ $SU(2)$ gauge theory is, shifting $y \to y - \ln \Lambda_2$ in (2.49).

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \frac{e^{2y}(m_1-m_2)^2 + e^y\left(\Lambda_2^2 - 2\hbar^2 + 8m_1m_2 - 8u\right) + 16u - 6\Lambda_2^2 + 8\Lambda_2^2 e^{-y}}{4\left(e^y-2\right)^2}\psi = 0 \tag{6.1}$$

Defining the integrability variables as

$$\frac{\Lambda_2}{\hbar} = \sqrt{2}e^\theta \qquad \frac{m_i}{\hbar} = q_i \qquad \frac{u}{\hbar^2} = p^2 \tag{6.2}$$

we transform the equation as

$$-\frac{d^2}{dy^2}\psi + \frac{e^{2y}(q_1-q_2)^2 + e^y\left(2e^{2\theta} - 2 + 8q_1q_2 - 8p^2\right) + 16p^2 - 12e^{2\theta} + 16e^{2\theta}e^{-y}}{4\left(e^y-2\right)^2}\psi = 0 \tag{6.3}$$

Equation (6.3) enjoys the symmetries:

$$\begin{aligned}
\Omega_+ &: \quad y \to y\,, && \theta \to \theta\,, && q_1 \to -q_1\,, && q_2 \to -q_2 \\
\Omega_- &: \quad y \to y\,, && \theta \to \theta + i\pi\,, && q_1 \to q_1\,, && q_2 \to q_2\,, \\
\Upsilon &: \quad y \to y + 2\pi i\,, && \theta \to \theta\,, && q_1 \to q_1\,, && q_2 \to q_2\,, \\
E &: \quad y \to y\,, && \theta \to \theta\,, && q_1 \to q_2\,, && q_2 \to q_1\,.
\end{aligned} \tag{6.4}$$

The fundamental regular solutions are given by the asymptotics:

$$\begin{aligned}
\psi_{-,0} &\simeq e^{-\theta/2+y/4}\exp\left\{-e^{\theta-\frac{y}{2}}\right\} && y \to -\infty \\
\psi_{+,0} &\simeq \frac{1}{\sqrt{q_1-q_2}}\exp\left\{-\frac{q_1-q_2}{2}y\right\} && y \to +\infty \\
\psi_{0,0} &\simeq \frac{1}{\sqrt{2(q_1+q_2)}}\left(e^y-2\right)^{\frac{1+q_1+q_2}{2}} && y \to \ln 2\,.
\end{aligned} \tag{6.5}$$

We can generate new solutions by action of the symmetries as

$$\begin{aligned}
\psi_{-,k} &= \Omega_-^k \psi_{-,0}\,, \\
\psi_{+,1} &= \Omega_+ \psi_{+,0} \\
\psi_{0,1} &= \Omega_+ \psi_{0,0} \qquad \psi_{0,2} = \psi_{0,0} \\
\tilde{\psi}_{+,1} &= E\psi_{+,0}
\end{aligned} \tag{6.6}$$

and have the invariance properties

$$\begin{aligned}
&\Omega_+\psi_{-,0} = \psi_{-,0}\,, \\
&\Omega_-^k\psi_{+,0} = \psi_{+,0}\,, \qquad \Omega_-^k\psi_{0,0} = \psi_{0,0}\,, \\
&E\psi_{-,0} = \psi_{-,0}\,, \qquad E\psi_{0,0}\,, \qquad E\Omega_+\psi_{+,0} = \psi_{+,0}
\end{aligned} \tag{6.7}$$

The solutions are normalized so that the wronskians among nearby $k, k+1$ solutions around the same singular point are

$$\begin{aligned} W[\psi_{-,k+1}, \psi_{-,k}] &= -i \\ W[\psi_{+,1}, \psi_{+,0}] &= i \\ W[\psi_{0,1}, \psi_{0,0}] &= i \\ W[\tilde{\psi}_{+,1}, \psi_{+,0}] &= -i\,. \end{aligned} \tag{6.8}$$

We can define apparently 3 Q functions

$$Q_+^{(1)} = W[\psi_{+,0}, \psi_{-,0}]\,, \qquad Q_+^{(2)} = W[\psi_{-,0}, \psi_{0,0}]\,, \qquad Q_+^{(3)} = W[\psi_{+,0}, \psi_{0,0}] \tag{6.9}$$

but that they are indeed different Q functions and not merely differently defined wronskians should be shown. Define also the shorthand notation

$$\begin{aligned} Q_\pm^{(1)}(\theta) &= Q^{(1)}(\theta, \pm q_1, \pm q_2, p) \\ Q_\pm^{(2)}(\theta) &= Q^{(2)}(\theta, \pm q_1, \pm q_2, p) \\ Q_\pm^{(3)}(\theta) &= Q^{(3)}(\theta, \pm q_1, \pm q_2, p) \\ \tilde{Q}_\pm^{(3)}(\theta) &= Q^{(3)}(\theta, \pm q_2, \pm q_1, p) \end{aligned} \tag{6.10}$$

We have the connection relations between $y \to +\infty$ and $y \to -\infty$

$$\begin{aligned} \psi_{+,0} &= -iQ_+^{(1)}(\theta + i\pi)\psi_{-,0} + iQ_+^{(1)}(\theta)\psi_{-,1} \\ \psi_{+,1} &= -iQ_-^{(1)}(\theta + i\pi)\psi_{-,0} + iQ_-^{(1)}(\theta)\psi_{-,1}\,. \end{aligned} \tag{6.11}$$

As usual, taking the wronskian of the first with the second line we get the QQ system

$$1 = Q_+^{(1)}(\theta)Q_-^{(1)}(\theta + i\pi) - Q_+^{(1)}(\theta + i\pi)Q_-^{(1)}(\theta) \tag{6.12}$$

or, shifting θ

$$Q_+^{(1)}(\theta - i\pi/2)Q_-^{(1)}(\theta + i\pi/2) - Q_-^{(1)}(\theta - i\pi/2)Q_+^{(1)}(\theta + i\pi/2) = 1 \tag{6.13}$$

Similarly

$$\begin{aligned} \psi_{-,0} &= -iQ_-^{(2)}(\theta)\psi_{0,0} + iQ_+^{(2)}(\theta)\psi_{0,1} \\ \psi_{-,1} &= -iQ_-^{(2)}(\theta + i\pi)\psi_{0,0} + iQ_+^{(2)}(\theta + i\pi)\psi_{0,1} \end{aligned} \tag{6.14}$$

$$Q_-^{(2)}(\theta - i\pi/2)Q_+^{(2)}(\theta + i\pi/2) - Q_+^{(2)}(\theta - i\pi/2)Q_-^{(2)}(\theta + i\pi/2) = 1 \tag{6.15}$$

We notice the QQ relations for Q and $\tilde{Q}$ have the same form of those of the minimal models [2]. Also

$$\begin{aligned} \psi_{+,0} &= -iQ_-^{(3)}(\theta)\psi_{0,0} + iQ_+^{(3)}(\theta)\psi_{0,1} \\ \tilde{\psi}_{+,1} &= -i\tilde{Q}_-^{(3)}(\theta)\psi_{0,0} + i\tilde{Q}_+^{(3)}(\theta)\psi_{0,1} \end{aligned} \tag{6.16}$$

from this we obtain the constraint

$$Q_+^{(3)}(\theta)\tilde{Q}_-^{(3)}(\theta) - \tilde{Q}_+^{(3)}(\theta)Q_-^{(3)}(\theta) = 1 \tag{6.17}$$

Let's build then a Y system from the QQ system (6.12).

$$\begin{aligned} Q_+^{(1)}(\theta)Q_-^{(1)}(\theta+i\pi) &= 1+Q_+^{(1)}(\theta+i\pi)Q_-^{(1)}(\theta) \\ Q_-^{(1)}(\theta)Q_+^{(1)}(\theta-i\pi) &= 1+Q_+^{(1)}(\theta)Q_-^{(1)}(\theta-i\pi) \\ Q_-^{(1)}(\theta+i\pi)Q_+^{(1)}(\theta)Q_-^{(1)}(\theta)Q_+^{(1)}(\theta-i\pi) &= [1+Q_+^{(1)}(\theta+i\pi)Q_-^{(1)}(\theta)][1+Q_+^{(1)}(\theta)Q_-^{(1)}(\theta-i\pi)] \end{aligned} \tag{6.18}$$

Define a Y function as

$$Y_+^{(1)}(\theta) = Q_+^{(1)}(\theta-i\pi/2)Q_-^{(1)}(\theta+i\pi/2) \tag{6.19}$$

We get a possible Y system as

$$Y_+^{(1)}(\theta+i\pi/2)Y_+^{(1)}(\theta-i\pi/2) = [1+Y_-^{(1)}(\theta+i\pi/2)][1+Y_-^{(1)}(\theta-i\pi/2)] \tag{6.20}$$

6.2. Gravity dictionary of $N_f=(2,0)$ to D1D5 circular fuzzball

We report here the dictionary with gravity given in [9]. Le'ts consider a D1D5 circular fuzzball with radius a_f and equal charges $\mathcal{Q}_1=\mathcal{Q}_5=L^2$. The smooth horizonless metric is given by [97]

$$\begin{aligned} ds^2 &= H_f^{-1}\left[(dv+\omega_\psi\, d\psi)^2-(dt+\omega_\phi\, d\phi)^2\right]+ \\ &+H_f\left[d\phi^2\sin^2\theta(\rho^2+a_f^2)+\frac{\Sigma_f}{\rho^2+a_f^2}\left[d\rho^2+(\rho^2+a_f^2)d\theta^2\right]+\rho^2 d\psi^2\cos^2\theta\right] \end{aligned} \tag{6.21}$$

with

$$\omega_\phi=\frac{L^2 a_f\sin^2\theta}{\Sigma_f}\,,\quad \omega_\psi=\frac{L^2 a_f\cos^2\theta}{\Sigma_f}\,,\quad H_f=1+\frac{L^2}{\Sigma_f}\,,\quad \Sigma_f=\rho^2+a_f^2\cos^2\theta \tag{6.22}$$

Setting

$$\Phi=e^{-i\omega t+iP_v v+im_\phi\phi+im_\psi\psi}R(\rho)S(\chi) \tag{6.23}$$

the wave equation can be separated into two ODEs, which can be matched to that of $SU(2)$ gauge theory with $N_f=(0,2)$ fundamentals.

$$\frac{d^2}{d\rho^2}R_0(\rho)+\left\{\frac{\left(a_f^2-\rho^2\right)^2+4\left[\rho^2\mathcal{L}_\phi^2-\left(a_f^2+\rho^2\right)\left(\mathcal{L}_\psi^2+\rho^2\left(1+K^2-\left(2L^2+\rho^2\right)\tilde{\omega}^2\right)\right)\right]}{4\rho^2\left(a_f^2+\rho^2\right)^2}\right\}R_0(\rho)=0 \tag{6.24}$$

$$\frac{d^2}{d\chi^2}S_0(\chi)+\left\{\frac{\left(\chi^2+1\right)^2-4\left[\chi^2 m_\phi^2+\left(1-\chi^2\right)\left(m_\psi^2-\chi^2\left(1+K^2+\tilde{\omega}^2 a_f^2\chi^2\right)\right)\right]}{4\chi^2\left(1-\chi^2\right)^2}\right\}S_0(\chi)=0 \tag{6.25}$$

where $\chi=\cos\theta$ and we defined

$$\mathcal{L}_\phi=a_f\, m_\phi-L^2\omega \quad,\quad \mathcal{L}_\psi=a_f\, m_\psi-L^2P_v \quad,\quad \tilde{\omega}^2=\omega^2-P_v^2 \tag{6.26}$$

The gauge/gravity dictionary for the radial equation reads

$$\frac{q}{\hbar^2}=\frac{a_f^2\tilde{\omega}^2}{4}\,,\quad \frac{u}{\hbar^2}=\frac{1+K^2+\tilde{\omega}^2\left(a_f^2-2L^2\right)}{4}\,,\quad \frac{m_{1,2}}{\hbar}=\frac{\mathcal{L}_\phi\mp\mathcal{L}_\psi}{2a_f}\,;\quad y=\frac{\rho^2}{a_f^2} \tag{6.27}$$

while for the angular equation one finds

$$\frac{q^\chi}{\hbar^2}=\frac{a_f^2\tilde{\omega}^2}{4}\quad,\quad \frac{u^\chi}{\hbar^2}=\frac{1+K^2+\tilde{\omega}^2a_f^2}{4}\quad,\quad \frac{m^\chi_{1,2}}{\hbar}=\frac{m_\phi\pm m_\psi}{2}\quad;\quad y^\chi=-\chi^2 \tag{6.28}$$

In the BH limit $a_f=0$ the gauge coupling goes to zero while both masses diverge ($q_{\rm BH}=m_1m_2\,q$ is finite), the resulting theory is $N_f=(0,0)$ with radial dictionary

$$\frac{q_{\rm BH}}{\hbar^4}=\left(\frac{L\tilde{\omega}}{2}\right)^4\quad,\quad \frac{u}{\hbar^2}=\frac{1+K^2-2\tilde{\omega}^2L^2}{4}\quad;\quad \hbar^2\,y_{\rm BH}=\frac{4\rho^2}{\tilde{\omega}^2L^4} \tag{6.29}$$

The $\Omega_\pm$ symmetries are compatible with the dictionary of the D1D5 fuzzball's angular equation:

$$\frac{\Lambda_2^2}{\hbar^2}=\frac{a_f^2\tilde{\omega}^2}{4}\,,\quad \frac{u}{\hbar^2}=\frac{1+K^2+\tilde{\omega}^2a_f^2}{4}\,,\quad \frac{m_{1,2}}{\hbar}=\frac{m_\phi\pm m_\psi}{2}\,, \tag{6.30}$$

where K is the original PDE separation constant, m_ϕ and m_ψ are the projections of the total angular momentum along two orthogonal 2-planes. Indeed:

$$\begin{aligned}\Omega_+ :&\quad \omega\to\omega\,, &&a_f\to -a_f\,, &&m_\phi\to -m_\phi\,, &&m_\psi\to -m_\psi\\ \Omega_- :&\quad \omega\to -\omega\,, &&a_f\to a_f\,, &&m_\phi\to m_\phi\,, &&m_\psi\to m_\psi\,.\end{aligned} \tag{6.31}$$

However, we expect that also for the radial problem where the symmetries do not directly apply, ODE/IM still can be applied, by changing to radial gravity parameters after the ODE/IM derivation.

The authors [9] give dictionaries for both angular and radial problems of other geometries, namely CCLP five-dimensional BHs, JMaRT and GMS geometries.

6.3. $N_f=3$ ODE/IM

Now shift $y\to y-\frac{1}{2}\ln\Lambda_3$.

$$\begin{aligned}&-\frac{d^2}{dy^2}\psi+\left\{\frac{e^{2y}\left(4(m_1-m_2)^2\right)}{\hbar^2}+\frac{e^{y}\left(-8\hbar^2+32m_1m_2+4\Lambda_3m_3-32u\right)}{\hbar^2}\right.\\ &\left.+\frac{\left(\Lambda_3^2-24\Lambda_3m_3+64u\right)}{\hbar^2}+\frac{e^{-y}\left(32\Lambda_3m_3-4\Lambda_3^2\right)}{\hbar^2}+\frac{4\Lambda_3^2e^{-2y}}{\hbar^2}\right\}\frac{1}{16\left(e^y-2\right)^2}\psi=0\end{aligned} \tag{6.32}$$

Change the variables as

$$\frac{\Lambda_3}{\hbar}=4e^\theta\qquad \frac{m_i}{\hbar}=q_i\qquad \frac{u}{\hbar^2}=p^2 \tag{6.33}$$

$$
\begin{aligned}
&-\frac{d^2}{dy^2}\psi + \left\{ \frac{1}{4} e^{2y}(q_1 - q_2)^2 + e^y \left(-\frac{1}{2} + 2q_1 q_2 + e^\theta q_3 - 2p^2 \right) \right. \\
&\left. + \left(e^{2\theta} - 6e^\theta q_3 + 4p^2 \right) + e^{-y} \left(8e^\theta q_3 - 4e^{2\theta} \right) + 4e^{2\theta} e^{-2y} \right\} \frac{1}{(e^y - 2)^2} \psi = 0
\end{aligned}
\tag{6.34}
$$

We have the discrete symmetries

$$
\begin{array}{rlllll}
\Omega_- : & \theta \to \theta + i\pi \,, & y \to y \,, & q_1 \to q_1 \,, & q_2 \to q_2 \,, & q_3 \to -q_3 \\
\Omega_+ : & \theta \to \theta \,, & y \to y \,, & q_1 \to -q_1 \,, & q_2 \to -q_2 \,, & q_3 \to q_3 \\
\Upsilon : & \theta \to \theta \,, & y \to y + 2\pi i \,, & q_1 \to q_1 \,, & q_2 \to q_2 \,, & q_3 \to q_3 \\
E : & \theta \to \theta \,, & y \to y \,, & q_1 \to q_2 \,, & q_2 \to q_1 \,, & q_3 \to q_3
\end{array}
\tag{6.35}
$$

$$
\begin{aligned}
\psi_{-,0} &\simeq e^{-\theta/2 + y/2 + yq_3} \exp\left\{ -e^{\theta - y} \right\} && \Re y \to -\infty \\
\psi_{+,0} &\simeq \frac{1}{\sqrt{q_1 - q_2}} \exp\left\{ -\frac{q_1 - q_2}{2} y \right\} && \Re y \to +\infty \,, \quad (\Re q_1 > \Re q_2) \\
\psi_{0,0} &\simeq \frac{1}{\sqrt{2(q_1 + q_2)}} \left(e^y - 2 \right)^{\frac{1}{2}(1 + q_1 + q_2)} && y \to \ln 2
\end{aligned}
\tag{6.36}
$$

We can generate new solutions by

$$
\begin{aligned}
\psi_{-,k} &= \Omega_-^k \psi_{-,0} \,, \\
\psi_{+,1} &= \Omega_+ \psi_{+,0} \\
\psi_{0,1} &= \Omega_+ \psi_{0,0} \qquad \psi_{0,2} = \psi_{0,0} \\
\tilde{\psi}_{+,1} &= E \psi_{+,0}
\end{aligned}
\tag{6.37}
$$

and have the invariance properties

$$
\begin{aligned}
\Omega_+ \psi_{-,0} &= \psi_{-,0} \,, \\
\Omega_-^k \psi_{+,0} &= \psi_{+,0} \,, \qquad \Omega_-^k \psi_{0,0} = \psi_{0,0} \,, \\
E \psi_{-,0} &= \psi_{-,0} \,, \qquad E \psi_{0,0} = \psi_{0,0} \,, \qquad E \Omega_+ \psi_{+,0} = \psi_{+,0}
\end{aligned}
\tag{6.38}
$$

The wronskians are

$$
\begin{aligned}
W[\psi_{-,k+1}, \psi_{-,k}] &= -2i \\
W[\psi_{+,1}, \psi_{+,0}] &= i \\
W[\psi_{0,1}, \psi_{0,0}] &= i \\
W[\tilde{\psi}_{+,1}, \psi_{+,0}] &= -i
\end{aligned}
\tag{6.39}
$$

Define

$$
Q^{(1)}_{+,+}(\theta) = W[\psi_{+,0}, \psi_{-,0}] \qquad Q^{(2)}_{+,+}(\theta) = W[\psi_{0,0}, \psi_{-,0}] \qquad Q^{(3)}_{+,+}(\theta) = W[\psi_{+,0}, \psi_{0,0}]
\tag{6.40}
$$

with

$$
\begin{aligned}
Q^{(1)}_{\pm,\pm}(\theta) &= Q^{(1)}(\theta, \pm q_1, \pm q_2, \pm q_3) \qquad Q^{(1)}_{\pm,\mp}(\theta) = Q^{(1)}(\theta, \pm q_1, \pm q_2, \mp q_3) \\
Q^{(2)}_{\pm,\pm}(\theta) &= Q^{(2)}(\theta, \pm q_1, \pm q_2, \pm q_3) \qquad Q^{(2)}_{\pm,\mp}(\theta) = Q^{(2)}(\theta, \pm q_1, \pm q_2, \mp q_3)
\end{aligned}
\tag{6.41}
$$

and

$$
\begin{aligned}
Q^{(3)}_{\pm,\pm}(\theta) &= Q^{(3)}(\theta, \pm q_1, \pm q_2, \pm q_3) \qquad & Q^{(3)}_{\pm,\mp}(\theta) &= Q^{(3)}(\theta, \pm q_1, \pm q_2, \mp q_3) \\
\tilde{Q}^{(3)}_{\pm,\pm}(\theta) &= Q^{(3)}(\theta, \pm q_2, \pm q_1, \pm q_3) \qquad & \tilde{Q}^{(3)}_{\pm,\mp}(\theta) &= Q^{(3)}(\theta, \pm q_2, \pm q_1, \mp q_3)\,.
\end{aligned} \tag{6.42}
$$

We have the linear relations

$$
\begin{aligned}
\psi_{-,0} &= iQ^{(1)}_{-,+}(\theta)\psi_{+,0} - iQ^{(1)}_{+,+}(\theta)\psi_{+,1} \\
\psi_{-,1} &= iQ^{(1)}_{-,-}(\theta + i\pi)\psi_{+,0} - iQ^{(1)}_{+,-}(\theta + i\pi)\psi_{+,1}
\end{aligned} \tag{6.43}
$$

and the QQ system is then (taking wronskians of both sides and shifting θ)

$$
Q^{(1)}_{+,+}(\theta - i\pi/2)Q^{(1)}_{-,-}(\theta + i\pi/2) - Q^{(1)}_{-,+}(\theta - i\pi/2)Q^{(1)}_{+,-}(\theta + i\pi/2) = 2\,. \tag{6.44}
$$

Similarly for $Q^{(2)}$

$$
\begin{aligned}
\psi_{-,0} &= iQ^{(2)}_{-,+}(\theta)\psi_{0,0} - iQ^{(2)}_{+,+}(\theta)\psi_{0,1} \\
\psi_{-,1} &= iQ^{(2)}_{-,-}(\theta + i\pi)\psi_{0,0} - iQ^{(2)}_{+,-}(\theta + i\pi)\psi_{0,1}
\end{aligned} \tag{6.45}
$$

$$
Q^{(2)}_{+,+}(\theta - i\pi/2)Q^{(2)}_{-,-}(\theta + i\pi/2) - Q^{(2)}_{-,+}(\theta - i\pi/2)Q^{(2)}_{+,-}(\theta + i\pi/2) = 2\,. \tag{6.46}
$$

Also for $Q^{(3)}$

$$
\begin{aligned}
\psi_{+,0} &= -iQ^{(3)}_{-,+}(\theta)\psi_{0,0} + iQ^{(3)}_{+,+}(\theta)\psi_{0,1} \\
\tilde{\psi}_{+,1} &= -i\tilde{Q}^{(3)}_{-,+}(\theta)\psi_{0,0} + i\tilde{Q}^{(3)}_{+,+}(\theta)\psi_{0,1}
\end{aligned} \tag{6.47}
$$

$$
Q^{(3)}_{+,+}(\theta)\tilde{Q}^{(3)}_{-,+}(\theta) - \tilde{Q}^{(3)}_{+,+}(\theta)Q^{(3)}_{-,+}(\theta) = 1\,. \tag{6.48}
$$

6.4. Gravity dictionaries for $N_f = 3$

6.4.1. Schwarshild black holes

The ODE which governs the perturbation of Schwarschild BHs is the Regge-Wheeler equation

$$
f(r)\frac{d}{dr}f(r)\frac{d}{dr}\phi(r) + [\omega^2 - V(r)]\phi(r) = 0 \tag{6.49}
$$

with

$$
V(r) = f(r)\left[\frac{l(l+1)}{r^2} + (1 - s^2)\frac{2M}{r^3}\right] \qquad l \in \mathbb{N} \qquad l \geq |s| \tag{6.50}
$$

Changing variable as

$$
r = \frac{4Me^{-y}}{\sqrt{\Lambda_3}} \tag{6.51}
$$

the Regge-Wheeler equation (6.49) becomes the $SU(2)$ $N_f = 3$ quantum SW curve.

The dictionary of parameters to Schwarzschild asymptotically flat BHs is [6]

$$
\begin{aligned}
&\hbar = 1 \qquad \Lambda_3 = -16iM\omega \\
&u = -l(l+1) + 8M^2\omega^2 - \frac{1}{4} \\
&m_1 = s - 2iM\omega\,, \quad m_2 = -s - 2iM\omega\,, \quad m_3 = -2iM\omega
\end{aligned} \tag{6.52}
$$

The condition $\Re m_1 > \Re m_2$ means just $\Re s > 0$.

6.4.2. Kerr black holes

The angular Teukolsy equation for Kerr BHs reads

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + (cx)^2 - 2csx + {}_sA_{lm} + s - \frac{(m+sx)^2}{1-x^2}\right]{}_sS_{lm}(x) = 0, \tag{6.53}$$

where $x = \cos\theta$ and s is the (minus of) spin of a perturbing field. Moreover

$$l = 0, 1, 2\cdots, \quad \text{with} \quad |m| \leq l, \tag{6.54}$$

where $m \in \mathbb{Z}$ for integer spins and $m \in \frac{1}{2} + \mathbb{Z}$ for half integer spins. In the black hole perturbation, the parameter c is related to the angular momentum α and the frequency ω by

$$c = \alpha\omega.$$

The eigenfunction ${}_sS_{lm}(x)$ is called the spin-weighted spheroidal harmonics in the literature. Its eigenvalue ${}_sA_{lm}$ is determined by the regularity condition of ${}_sS_{lm}(x)$ at $x = \pm 1$. For general s, l, m and c, no closed form of ${}_sA_{lm}$ is known so far. However, for $c = 0$ the spheroidal harmonics ${}_sS_{lm}(x)$ reduces to the spin-weighted spherical harmonics ${}_sY_{lm}$ and one has

$${}_sA_{lm}(c=0) = l(l+1) - s(s+1). \tag{6.55}$$

The radial Teukolsky equation for Kerr BHs is

$$\Delta(r)R''(r) + (s+1)\Delta'(r)R'(r) + V_T(r)R(r) = 0, \tag{6.56}$$

where $\Delta(r) = r^2 - 2Mr + \alpha^2$. The potential is

$$V_T(r) = \frac{K(r)^2 - 2is(r-M)K(r)}{\Delta(r)} - {}_sA_{lm} + 4is\omega r + 2\alpha m\omega - \alpha^2\omega^2, \tag{6.57}$$

where $K(r) = (r^2+\alpha^2)\omega - \alpha m$. The radial differential equation (6.56) has (regular) singular points at $r = r_\pm := M \pm \sqrt{M^2-\alpha^2}$ corresponding to the Cauchy and event horizons. In addition, (6.56) is supplied by the following boundary conditions

$$R(r) \sim \begin{cases} (r_+ - r_-)^{-1-s+i\omega+i\sigma_+} e^{i\omega r_+} (r-r_+)^{-s-i\sigma_+} & \text{if} \quad r \to r_+\,, \\ A(\omega) r^{-1-2s+i\omega} e^{i\omega r} & \text{if} \quad r \to \infty\,, \end{cases} \tag{6.58}$$

where

$$\sigma_+ = \frac{\omega r_+ - \alpha\frac{m}{2M}}{\sqrt{1-\frac{\alpha^2}{M^2}}}. \tag{6.59}$$

Both the angular and the radial parts of the Teukolsky equation have the same singularity structure as the confluent Heun equation.

For the angular part, we change the variable $z = (1+x)/2$, and define $y(z) := \sqrt{1-x^2}\,{}_sS_{lm}(x)/2$. Then we obtain

$$y''(z) + Q(z)y(z) = 0, \tag{6.60}$$

where $Q(z)$ takes the form

$$Q(z) = \frac{1}{z^2(z-1)^2} \sum_{i=0}^{4} A_i z^i \,. \tag{6.61}$$

The coefficients in $Q(z)$ are computed straightforwardly. Similarly, defining $z = (r - r_-)/(r_+ - r_-)$ and $y(z) := \Delta(r)^{(s+1)/2} R(r)$ for the radial part, we obtain the same form as (6.60) and (6.61) with different coefficients.

For the *angular* part, we find [6]

$$\begin{aligned} &\Lambda_3 = 16c, \quad u = -{}_sA_{\ell m} - s(s+1) - c^2 - \frac{1}{4}, \\ &m_1 = -m, \quad m_2 = m_3 = -s. \end{aligned} \tag{6.62}$$

For the *radial* part of asymptotically flat Kerr BHs, we have [6]

$$\begin{aligned} \Lambda_3 &= -16i\omega\sqrt{M^2 - \alpha^2}, \\ u &= -{}_sA_{lm} - s(s+1) + (8M^2 - \alpha^2)\omega^2 - \frac{1}{4}, \\ m_1 &= s - 2iM\omega, \qquad m_3 = -s - 2iM\omega, \\ m_2 &= \frac{i(-2M^2\omega - \alpha m)}{\sqrt{M^2 - \alpha^2}} = -2iM\omega\frac{M}{\sqrt{M^2 - \alpha^2}} - i\alpha m\frac{1}{\sqrt{M^2 - \alpha^2}}. \end{aligned} \tag{6.63}$$

When $\alpha = 0$, it reproduces the identification in the Schwarzschild case by exchanging $m_2 \leftrightarrow m_3$. This relabelling comes from the fact that the Teukolsky equation at $\alpha = 0$ does not take the form of the Regge-Wheeler equation.

7. $SU(2)\ N_f = 4$ or class $\mathcal{S}$ gauge theory, spin chains and asymptotically AdS black holes

7.1. BTZ black hole

The rotating BTZ (Bañados-Teitelboim-Zanelli) is the black hole of standard 2+1 Einstein-Maxwell theory with a negative cosmological constant Λ [98]. In particular, the rotating BTZ BH has line element: [99]

$$ds^2 = -F dt^2 + \frac{dr^2}{F} + r^2 \left(dx - \frac{r_+ r_-}{r^2} dt \right)^2 \tag{7.1}$$

with

$$F = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} \,. \tag{7.2}$$

For simplicity we have set the AdS radius to $L = 1$. For the BTZ BH x is angular compact coordinate such that $x \simeq x + 2\pi$. However, following [99] we will consider it as a noncompact coordinate. The BTZ BH temperature T, mass M, angular momentum J and angular momentum potential Ω are given by

$$\begin{aligned} 2\pi T &= \frac{r_+^2 - r_-^2}{r_+} \,, \quad M = r_+^2 + r_-^2 \,, \\ J &= 2 r_+ r_- \,, \quad \Omega = \frac{r_-}{r_+} \,. \end{aligned} \tag{7.3}$$

(We use the standard convention $8G = 1$). A two-dimensional CFT has two independent modes, left-movers and right-movers, and one can introduce temperatures for each sectors:

$$\begin{aligned} 2\pi T_L &= \frac{2\pi T}{1+\Omega} = r_+ - r_- \\ 2\pi T_R &= \frac{2\pi T}{1-\Omega} = r_+ + r_- \\ \frac{2}{T} &= \frac{1}{T_L} + \frac{1}{T_R} \,, \\ \Omega &= \frac{T_R - T_L}{T_R + T_L} \,. \end{aligned} \tag{7.4}$$

In the standard convention, $T_L \to 0$ corresponds to the extreme limit in the BTZ BH.

The authors [99] consider a scalar field perturbation of the form $\phi(r) e^{-i\omega t + iqx}$. The perturbation behaves as

$$\phi \sim A \left(\frac{1}{r} \right)^{\Delta_-} + B \left(\frac{1}{r} \right)^{\Delta_+} \qquad (r \to \infty) \tag{7.5}$$

with

$$\Delta_\pm = 1 \pm \nu \qquad \nu = \sqrt{1 + m^2} \tag{7.6}$$

where m is the mass of the scalar field ϕ. According to the AdS/CFT dictionary the retarded Green function is given by

$$G^R = -(2\nu) \frac{B}{A} \,. \tag{7.7}$$

7.2. Naive ODE/IM construction

The equation for a minimally coupled scalar field ϕ in the background is

$$\left[(1-z)\partial_z((1-z)\partial_z)+\frac{(\omega-\Omega q)^2-(\Omega\omega-q)^2(1-z)}{(4\pi T)^2 z}-\frac{(\nu^2-1)(1-z)}{4z^2}\right]\phi=0 \tag{7.8}$$

where we introduced the variable

$$z=\frac{r_+^2-r_-^2}{r^2-r_-^2}\,. \tag{7.9}$$

for which asymptotic infinity is located at $z=0$ and the outer horizon is located at $z=1$.
Asymptotically for $z\to 0$ (spacetime infinity)

$$\phi\sim z^{\frac{\Delta_\pm}{2}}\,. \tag{7.10}$$

The solution with $\Delta_+=1+\nu$ is regular.

$$\phi_{-,0}\sim z^{\frac{\Delta_+}{2}} \tag{7.11}$$

Near the outer horizon for $z\to 1$

$$\phi\sim(1-z)^{\pm i\lambda}\,,\qquad \lambda=\frac{\omega-\Omega q}{4\pi T}\,. \tag{7.12}$$

The solution with $+i\lambda$ is regular, since $\Im\omega<0$.

$$\phi_{1,0}\sim(1-z)^{+i\lambda} \tag{7.13}$$

Near $z\to\infty$

$$\phi\sim z^{\pm i\mu}\qquad \mu=\frac{q-\omega\Omega}{4\pi T} \tag{7.14}$$

the regular solution is

$$\phi_{+,0}\sim z^{i\mu} \tag{7.15}$$

The ODE (7.8) has symmetries

$$\begin{aligned}
\Omega_-:&\quad z\to z\,,\quad \omega\to\omega\,,\quad \Omega\to\Omega\,,\quad q\to q\,,\quad T\to T\,,\quad \nu\to-\nu\,,\quad \lambda\to\lambda\,,\quad \mu\to\mu\\
\Omega_1:&\quad z\to z\,,\quad \omega\to e^{i\pi}\omega\,,\quad \Omega\to e^{-i\pi}\Omega\,,\quad q\to q\,,\quad T\to T\,,\quad \nu\to\nu\quad \lambda\to-\lambda\,,\quad \mu\to\mu\\
\Omega_+:&\quad z\to z\,,\quad \omega\to e^{i\pi}\omega\,,\quad \Omega\to e^{-i\pi}\Omega\,,\quad q\to q\,,\quad T\to -T\,,\quad \nu\to\nu\quad \lambda\to\lambda\,,\quad \mu\to-\mu
\end{aligned} \tag{7.16}$$

We have

$$\begin{aligned}
\Omega_-\phi_{-,0}&=\phi_{-,1}\sim z^{\frac{\Delta_-}{2}}\\
\Omega_-\phi_{1,0}&=\phi_{1,0}\\
\Omega_-\phi_{+,0}&=\phi_{+,0}
\end{aligned} \tag{7.17}$$

$$\begin{aligned}
\Omega_1\phi_{-,0}&=\phi_{-,0}\\
\Omega_1\phi_{1,0}&=\phi_{1,1}\sim(1-z)^{-i\lambda}\\
\Omega_1\phi_{+,0}&=\phi_{+,0}
\end{aligned} \tag{7.18}$$

$$\begin{aligned} \Omega_+\phi_{-,0} &= \phi_{-,0} \\ \Omega_+\phi_{1,0} &= \phi_{1,0} \\ \Omega_+\phi_{+,0} &= \phi_{+,1} \sim z^{-i\mu} \end{aligned} \tag{7.19}$$

By transforming the dependent variable as $\psi = \sqrt{z-1}\phi$ we get the canonical equation

$$\partial_z^2\psi(z) + \frac{4\pi^2 m^2 T^2(z-1) + z\left[q^2\left(\Omega^2 + z - 1\right) - 2q\omega\Omega z + 4\pi^2 T^2 z + \omega^2 + \omega^2\Omega^2(z-1)\right]}{16\pi^2 T^2 (z-1)^2 z^2}\psi(z) = 0 \tag{7.20}$$

Define the normalized solutions

$$\begin{aligned} \psi_{-,0} &\simeq \frac{1}{\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}}\sqrt{z-1}z^{\frac{\Delta_+}{2}} & \psi_{-,1} &\simeq \frac{1}{\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}}\sqrt{z-1}z^{\frac{\Delta_-}{2}} & z &\to 0 \\ \psi_{1,0} &\simeq \frac{1}{\sqrt{2i\lambda}}\sqrt{z-1}(1-z)^{i\lambda} & \psi_{1,1} &\simeq \frac{1}{\sqrt{-2i\lambda}}\sqrt{z-1}(1-z)^{-i\lambda} & z &\to 1 \\ \psi_{+,0} &\simeq \frac{1}{\sqrt{2i\mu}}\sqrt{z-1}z^{i\mu} & \psi_{+,1} &\simeq \frac{1}{\sqrt{-2i\mu}}\sqrt{z-1}z^{-i\mu} & z &\to \infty \end{aligned} \tag{7.21}$$

$$\begin{aligned} W[\psi_{-,1}, \psi_{-,0}] &= i \\ W[\psi_{1,1}, \psi_{1,0}] &= i\frac{\lambda}{\sqrt{\lambda^2}} = i \quad \text{(conventionally)} \\ W[\psi_{+,1}, \psi_{+,0}] &= i\frac{\mu}{\sqrt{\mu^2}} = -i \quad \text{(conventionally)} \end{aligned} \tag{7.22}$$

We can define the wronskians of the regular solutions as

$$\begin{aligned} Q_{-,1}(\omega, q, \Omega, T, \nu) &= W[\psi_{-,0}, \psi_{1,0}] \\ Q_{-,+}(\omega, q, \Omega, T, \nu) &= W[\psi_{-,0}, \psi_{+,0}] \\ Q_{1,+}(\omega, q, \Omega, T, \nu) &= W[\psi_{1,0}, \psi_{+,0}] \end{aligned} \tag{7.23}$$

By expanding $\psi_{-,0}$ and $\psi_{-,1}$ in terms of $\psi_{1,0}$ and $\psi_{1,1}$

$$\begin{aligned} \psi_{-,0} &= iQ_{-,1}(e^{i\pi}\omega, e^{i\pi}\Omega, T, \nu)\psi_{1,0} - iQ_{-,1}(\omega, \Omega, T, \nu)\psi_{1,1} \\ \psi_{-,1} &= iQ_{-,1}(e^{i\pi}\omega, e^{i\pi}\Omega, T, -\nu)\psi_{1,0} - iQ_{-,1}(\omega, \Omega, T, -\nu)\psi_{1,1} \end{aligned} \tag{7.24}$$

and taking wronskians we obtain the QQ relation with the $Q_{-,1}$ (by also suitably normalizing the wave functions):

$$Q_{-,1}(\omega, \Omega, T, \nu)Q_{-,1}(e^{\pi i}\omega, e^{i\pi}\Omega, T, -\nu) = 1 + Q_{-,1}(e^{i\pi}\omega, e^{i\pi}\Omega, T, \nu)Q_{-,1}(\omega, \Omega, T, -\nu) \tag{7.25}$$

By expanding $\psi_{-,0}$ and $\psi_{-,1}$ in terms of $\psi_{+,0}$ and $\psi_{+,1}$

$$\begin{aligned} \psi_{-,0} &= -iQ_{-,+}(e^{i\pi}\omega, e^{i\pi}\Omega, -T, \nu)\psi_{+,0} + iQ_{-,+}(\omega, \Omega, T, \nu)\psi_{+,1} \\ \psi_{-,1} &= -iQ_{-,+}(e^{i\pi}\omega, e^{i\pi}\Omega, -T, -\nu)\psi_{+,0} + iQ_{-,+}(\omega, \Omega, T, -\nu)\psi_{+,1} \end{aligned} \tag{7.26}$$

and taking wronskians we obtain the QQ relation with the $Q_{-,+}$ (by also suitably normalizing the wave functions):

$$Q_{-,+}(\omega,\Omega,T,\nu)Q_{-,+}(e^{\pi i}\omega,e^{i\pi}\Omega,-T,-\nu)=1+Q_{-,+}(e^{i\pi}\omega,e^{i\pi}\Omega,-T,\nu)Q_{-,+}(\omega,\Omega,T,-\nu) \quad (7.27)$$

By expanding $\psi_{1,0}$ and $\psi_{1,1}$ in terms of $\psi_{+,0}$ and $\psi_{+,1}$

$$\begin{aligned}\psi_{1,0}&=-iQ_{1,+}(e^{i\pi}\omega,e^{i\pi}\Omega,-T,\nu)\psi_{+,0}+iQ_{1,+}(\omega,\Omega,T,\nu)\psi_{+,1}\\ \psi_{1,1}&=-iQ_{1,+}(e^{2\pi i}\omega,e^{2\pi i}\Omega,-T,\nu)\psi_{+,0}+iQ_{1,+}(e^{i\pi}\omega,e^{i\pi}\Omega,T,\nu)\psi_{+,1}\end{aligned} \quad (7.28)$$

and taking wronskians we obtain the QQ relation with the $Q_{1,+}$ (by also suitably normalizing the wave functions):

$$Q_{1,+}(\omega,\Omega,T,\nu)Q_{1,+}(e^{2\pi i}\omega,e^{2\pi i}\Omega,-T,\nu)=1+Q_{1,+}(e^{i\pi}\omega,e^{i\pi}\Omega,-T,\nu)Q_{1,+}(e^{i\pi}\omega,e^{i\pi}\Omega,T,\nu) \quad (7.29)$$

7.3. Exact expressions for Q functions

For this simple theory we can actually get exact expressions for the Q functions as the ODE in other variables reduces to a hypergeometric ODE, for which connection coefficients are known to be rational combinations of Euler Gamma functions.

To see that, set the ansatz

$$\phi=z^{(1+\nu)/2}(1-z)^{i\lambda}f(z)\,. \quad (7.30)$$

Then the field equation becomes the hypergeometric differential equation

$$z(1-z)\partial_z^2 f+[c-(1+a+b)z]\partial_z f-abf=0 \quad (7.31)$$

with

$$\begin{aligned}c(m)&=1+\nu=1+\sqrt{1+m^2}\\ a(\omega,q,\Omega,T,m)&=\frac{c(m)}{2}+\frac{i(1+\Omega)}{4\pi T}(\omega-q)=\frac{\Delta_+}{2}+i\frac{\omega-q}{4\pi T_L}\\ b(\omega,q,\Omega,T,m)&=\frac{c(m)}{2}+\frac{i(1-\Omega)}{4\pi T}(\omega+q)=\frac{\Delta_-}{2}+i\frac{\omega+q}{4\pi T_R}\end{aligned} \quad (7.32)$$

with $a+b=c+2i\lambda$.

We consider now the standard Kummer solutions w_n. Around $z=0$ we have the regular and irregular fundamental solutions given respectively by

$$\begin{aligned}w_1(z)&={}_2F_1(a,b,c;z)\,,\\ w_2(z)&=z^{1-c}{}_2F_1(1+a-c,1+b-c,2-c;z)\,.\end{aligned} \quad (7.33)$$

Around $z=1$ we have the regular and irregular fundamental solutions given respectively by

$$\begin{aligned}w_3(z)&={}_2F_1(a,b,1+a+b-c;1-z)\,,\\ w_4(z)&=(1-z)^{c-a-b}{}_2F_1(c-a,c-b,c-a-b+1;1-z)\,.\end{aligned} \quad (7.34)$$

We assume

$$\Re(iq)<0 \quad (7.35)$$

Around $z=\infty$ we have the regular and irregular fundamental solutions given respectively by

$$\begin{aligned} w_5(z) &= e^{a\pi i} z^{-a} {}_2F_1(a, 1+a-c, 1+a-b; \frac{1}{z}), \\ w_6(z) &= e^{b\pi i} z^{-b} {}_2F_1(b, 1+b-c, 1+b-a; \frac{1}{z}). \end{aligned} \tag{7.36}$$

It is easy to see that the normalized solutions we used for the ODE/IM construction of the previous section are related to the Kummer solutions w_1, w_3, w_5 as follows

$$\begin{aligned} \psi_{-,0} &= \frac{1}{\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}} \sqrt{z-1} z^{(1+\nu)/2} (1-z)^{i\lambda} w_1(z) = \frac{1}{\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}} p(z) w_1(z) \\ \psi_{1,0} &= \frac{1}{\sqrt{2i\lambda}} \sqrt{z-1} z^{(1+\nu)/2} (1-z)^{i\lambda} w_3(z) = \frac{1}{\sqrt{2i\lambda}} p(z) w_3(z) \\ \psi_{+,0} &= \frac{1}{\sqrt{2i\mu}} e^{-i\pi\frac{1+\nu}{2} - \frac{q-\Omega\omega}{4T}} \sqrt{z-1} z^{(1+\nu)/2} (1-z)^{i\lambda} w_5(z) = \frac{1}{\sqrt{2i\mu}} e^{-i\pi\frac{1+\nu}{2} - \frac{q-\Omega\omega}{4T}} p(z) w_5(z) \end{aligned} \tag{7.37}$$

with

$$p(z) = z^{(1+\nu)/2} (1-z)^{i\lambda+1/2} = \sqrt{\frac{1-c}{W[w_1, w_2]}} = \sqrt{\frac{a+b-c}{W[w_3, w_4]}} \tag{7.38}$$

The Q functions are then defined as

$$\begin{aligned} Q_{-,1}(\omega, q, \Omega, T, m) &= W[\psi_{-,0}, \psi_{1,0}] = \frac{1}{\sqrt{2i\lambda}\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}} p^2(z) W[w_1, w_3](z) \\ Q_{-,+}(\omega, q, \Omega, T, m) &= W[\psi_{-,0}, \psi_{+,0}] = \frac{e^{-i\pi\frac{1+\nu}{2} - \frac{q-\Omega\omega}{4T}}}{\sqrt{2i\mu}\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}} p^2(z) W[w_1, w_5](z) \\ Q_{1,+}(\omega, q, \Omega, T, m) &= W[\psi_{1,0}, \psi_{+,0}] = \frac{e^{-i\pi\frac{1+\nu}{2} - \frac{q-\Omega\omega}{4T}}}{\sqrt{2i\mu}\sqrt{2i\lambda}} p^2(z) W[w_3, w_5](z) \end{aligned} \tag{7.39}$$

By Kummer's connection relations we can eventually get the following exact expression for the Baxter's Q functions:

$$\begin{aligned} Q_{-,1}(\omega, q, \Omega, T, m) &= \frac{1}{\sqrt{2i\lambda}\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}} \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} (1-c) \\ Q_{-,+}(\omega, q, \Omega, T, m) &= -\frac{e^{-i\pi\frac{1+\nu}{2} - \frac{q-\Omega\omega}{4T}}}{\sqrt{2i\mu}\sqrt[4]{(1-\Delta_-)(1-\Delta_+)}} e^{(c-1)i\pi} \frac{\Gamma(c-1)\Gamma(a-b+1)}{\Gamma(a)\Gamma(c-b)} (1-c) \\ Q_{1,+}(\omega, q, \Omega, T, m) &= \frac{e^{-i\pi\frac{1+\nu}{2} - \frac{q-\Omega\omega}{4T}}}{\sqrt{2i\mu}\sqrt{2i\lambda}} e^{(c-b)\pi i} \frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)} (a+b-c) \end{aligned} \tag{7.40}$$

7.4. Poles skipping

The so called "poles skipping" phenomenon can be summarized as follows [100, 101, 102]. At certain imaginary values of the frequency and momentum there is no unique ingoing solution at the BH horizon. As a consequence, near these points in Fourier space the holographic retarded Green's function is no longer uniquely defined. Its values depend on the direction in which we approach the special point. Such behaviour has been dubbed "pole-skipping" as a line of poles intersects a line of zeros for the Green's function.

The incoming wave is written as

$$\begin{aligned} w_3 \propto w_1 + \frac{\Gamma(c-1)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(-c+1)\Gamma(a)\Gamma(b)} w_2 = w_1 + \frac{Q_{-,1}}{Q_{1,+}} \frac{e^{(b-c)\pi i}\Gamma(b-c+1)}{\Gamma(a)} \frac{\Gamma(a-b+1)}{\Gamma(c)} w_2 \\ = w_1 + \frac{\Gamma(\nu)\Gamma(a-\nu)\Gamma(b-\nu)}{\Gamma(-\nu)\Gamma(a)\Gamma(b)} w_2 \end{aligned} \tag{7.41}$$

The asymptotic behaviour at infinity is

$$\phi \sim A \left(\frac{z}{r_+^2 - r_-^2} \right)^{\frac{\Delta_-}{2}} + B \left(\frac{z}{r_+^2 - r_-^2} \right)^{\frac{\Delta_+}{2}}, \qquad z \to 0 \tag{7.42}$$

The Green's function is then given by

$$G^R = -2\nu(4\pi^2 T_L T_R)^\nu \frac{\Gamma(\nu)\Gamma(a-\nu)\Gamma(b-\nu)}{\Gamma(-\nu)\Gamma(a)\Gamma(b)} \tag{7.43}$$

We have the following poles and zeroes for the Green function. Left poles:

$$a - \nu = a - c + 1 = \frac{\Delta_-}{2} + i\frac{\omega - q}{4\pi T_L} = -n_L^p \tag{7.44}$$

$$\omega_L = q - i(2\pi T_L)(\Delta_- 2n_L^p) \tag{7.45}$$

Right poles:

$$b - \nu = b - c + 1 = \frac{\Delta_-}{2} + i\frac{\omega + q}{4\pi T_R} = -n_R^p \tag{7.46}$$

$$\omega_R = -q - i(2\pi T_R)(\Delta_- 2n_R^p) \tag{7.47}$$

Left zeros:

$$a = \frac{\Delta_+}{2} + i\frac{\omega - q}{4\pi T_L} = -n_L^z \tag{7.48}$$

$$\omega_L = q - i(2\pi T_L)(\Delta_+ + 2n_L^z) \tag{7.49}$$

Right zeros:

$$b = \frac{\Delta_+}{2} + i\frac{\omega + q}{4\pi T_R} = -n_R^z \tag{7.50}$$

$$\omega_R = -q - i(2\pi T_R)(\Delta_+ + 2n_R^z) \tag{7.51}$$

$\Gamma(a)$ and $\Gamma(a-\nu)$ do not diverge simultaneously. Similarly also $\Gamma(b)$ and $\Gamma(b-\nu)$ do not diverge simultaneously. Therefore poles skipping is given by combination of left poles and right zeros

$$\begin{aligned} i\omega = 2\pi T_R \left(\frac{\Delta_+}{2} + n_R^z \right) + 2\pi T_L \left(\frac{\Delta_-}{2} + n_L^p \right) \\ iq = 2\pi T_R \left(\frac{\Delta_+}{2} + n_R^z \right) - 2\pi T_L \left(\frac{\Delta_-}{2} + n_L^p \right) \end{aligned} \tag{7.52}$$

or right poles and left zeros

$$
\begin{aligned}
i\omega &= 2\pi T_R \left(\frac{\Delta_-}{2} + n_R^p\right) + 2\pi T_L \left(\frac{\Delta_+}{2} + n_L^z\right) \\
iq &= 2\pi T_R \left(\frac{\Delta_-}{2} + n_R^p\right) - 2\pi T_L \left(\frac{\Delta_+}{2} + n_L^z\right)
\end{aligned}
\tag{7.53}
$$

This is exactly the result of [99], except that poles and zeros are exchanged.

Through the exact expressions for the Q functions, we can give an interpretation of poles skipping in integrability as follows:

$$
\Gamma(b) \sim \infty\,, \qquad \text{and} \qquad \Gamma(a-c+1) \sim \infty \tag{7.54}
$$

mean

$$
Q_{-,1} = 0\,, \qquad \text{and} \qquad Q_{1,+} = 0\,. \tag{7.55}
$$

Alternatively:

$$
\Gamma(a) \sim \infty\,, \qquad \text{and} \qquad \Gamma(b-c+1) \sim \infty\,, \quad (\Gamma(c-b) = 0) \tag{7.56}
$$

mean again

$$
Q_{-,1} = 0\,, \qquad \text{and} \qquad Q_{1,+} = 0 \tag{7.57}
$$

We see then that poles skipping in integrability corresponds to simultaneous zeros for two Q functions. This is to compare with the general characterization of quasinormal modes as zeros of a single Q function.

We seek now an alternative characterization of poles skipping which might be used even when there is no exact analytic expression for the Q function. In particular in that case poles skipping is expected to corresponds to simultaneous zeros of two Q functions, but with different arguments. We check now the consistency of such characterizations.

The action of the symmetries on the hypergeometric parameters is

$$
\begin{aligned}
&\Omega_+ a = b\,, \quad \Omega_+ b = a\,, \quad \Omega_+ c = c \\
&\Omega_- a = a - c + 1\,, \quad \Omega_- b = b - c + 1\,, \quad \Omega_- c = 2 - c \\
&\Omega_1 a = c - b\,, \quad \Omega_1 b = c - a\,, \quad \Omega_1 c = c
\end{aligned}
\tag{7.58}
$$

We notice that the poles skipping condition is invariant under the action of the symmetries Ω_+ and Ω_-, but the action of the symmetry Ω_1 makes poles of Gamma functions become zero.

We have also

$$
w_3 \sim \Omega_- Q_{-,1} w_1 + Q_{-,1} w_2 \tag{7.59}
$$

and we find indeed

$$
\Omega_- Q_{-1} \propto -\frac{\Gamma(2-c)\Gamma(3-3c+a+b)}{\Gamma(a+1-c)\Gamma(b+1-c)} \sim \frac{1}{\Gamma(a+1-c)\Gamma(b+1-c)} \tag{7.60}
$$

from which we notice that $\Omega_- Q_{-1}$ has the same poles-skipping zeros as $Q_{1,+}$ (poles of $\Gamma(a+1-c)$). We notice also that simultaneous zeros of $Q_{-,1}$ and $\Omega_- Q_{-,1}$ seem at first sight incompatible with the QQ system. However, in the QQ system we have actually also $\Omega_1 Q_{-,1}$ which multiplies $\Omega_- Q_{-1}$ and cancels its poles-skipping zeros

$$\Omega_1 Q_{-,1} \propto -\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} \sim \Gamma(a+1-c)\Gamma(b+1-c) \tag{7.61}$$

while

$$\Omega_1 \Omega_- Q_{-,1} \propto -\frac{\Gamma(2-c)\Gamma(3-c-a-b)}{\Gamma(1-b)\Gamma(1-a)} \sim \frac{1}{Q_{-,1}} \tag{7.62}$$

7.5. Relation to gauge theory

We can relate (7.20) to Gaiotto's opers for the three-punctured sphere [70]

$$-\epsilon^2 \partial_z^2 \psi + \left[\frac{-c_0^2 - c_1^2 + c_\infty^2 + \epsilon^2/4}{(z-1)z} + \frac{c_0^2 - \epsilon^2/4}{z^2} + \frac{c_1^2 - \epsilon^2/4}{(z-1)^2} \right] \psi = 0 \tag{7.63}$$

by

$$c_0 = \pm \frac{1}{2}\sqrt{m^2+1}\epsilon, \qquad c_1 = \pm \frac{i\epsilon(\omega - q\Omega)}{4\pi T}, \qquad c_\infty = \pm \frac{i\epsilon(q - \omega\Omega)}{4\pi T} \tag{7.64}$$

However, such gauge theory is a non-Lagrangian...

By the change of variable $z \to 1/x$ we get the equation

$$\epsilon^2 \frac{d^2}{dx^2}\psi(x) + \frac{-4c_0^2(x-1)x - 4c_1^2 x - 4c_\infty^2(x-1) + x^2\epsilon^2 - x\epsilon^2 + \epsilon^2}{4(x-1)^2 x^2}\psi(x) = 0 \tag{7.65}$$

and make contact with the SW curve in the conventions of [103]

$$y_{SW}^2(x) = -\frac{m_1^2}{(x-1)x^2} + \frac{m_2^2}{(x-1)^2 x} + \frac{m_3^2}{(x-1)x} \tag{7.66}$$

with

$$m_1 = \pm i c_\infty \qquad m_2 = \pm c_1 \qquad m_3 = \pm c_0 \tag{7.67}$$

The gauge periods are defined as integrals of the SW differential

$$\lambda = \sqrt{y_{SW}^2} \tag{7.68}$$

among the branch points given by the roots

$$x^2(x-1)^2 y_{SW}^2(x) = 0 \tag{7.69}$$

which are ∞ and

$$\begin{aligned} x_1 &= \frac{-\sqrt{(-m_1^2+m_2^2-m_3^2)^2 - 4m_1^2 m_3^2} + m_1^2 - m_2^2 + m_3^2}{2m_3^2} \\ x_2 &= \frac{\sqrt{(-m_1^2+m_2^2-m_3^2)^2 - 4m_1^2 m_3^2} + m_1^2 - m_2^2 + m_3^2}{2m_3^2} \end{aligned} \tag{7.70}$$

so we can write

$$\lambda = m_3 \frac{\sqrt{(x-x_1)(x-x_2)}}{x(x-1)} \tag{7.71}$$

For example

$$\int_{x_1}^{x_2} dx\, \lambda = \pi m_3 \left(-\sqrt{x_1-1}\sqrt{1-x_2} - i\sqrt{x_1}\sqrt{x_2} + \frac{\sqrt{x_2-x_1}}{\sqrt{x_1-x_2}} \right) \tag{7.72}$$

Eventually, one expects to be able relate the SW periods of this differential to the leading order of the Q function

$$\ln Q^{(0)}_{-+} = \int_0^\infty \left[\frac{\sqrt{(x-x_1)(x-x_2)}}{x(x-1)} - \frac{\sqrt{x_1 x_2}}{x^2} - \frac{\sqrt{(1-x_1)(1-x_2)}}{x-1} \right] dx \tag{7.73}$$

We not yet completed developing the details, though.

7.6. XXZ spin chain at the supersymmetric point and poles skipping

Following [44] we can construct a more proper ODE/IM construction by mapping the equation with only regular singularities to one with an irregular singularity, so that there is indeed the Stokes phenomenon and we also can define a T function. For that ODE the authors have a proper QQ and TQ system, from which they actually derive the ODE:

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w', \bar{w}'|\lambda) + \lambda^2 \psi_\pm(w', \bar{w}'|\lambda) = U_\pm(w'; \bar{w}')\psi_\pm(w', \bar{w}'|\lambda) \tag{7.74}$$

Then they transform its energy parameter λ into the independent variable u of an another ODE with only regular singularities in that variable.

$$\lambda = e^{\frac{3iu}{2}} \tag{7.75}$$

They end up with the following ODEs (one for each independent solution $\psi_\pm$ of the original ODE)

$$\begin{aligned} \frac{d^2 f_+}{du^2} - 6n\cot(3u+2\phi)\frac{df_+}{du} + (1-9n^2)f_+ = 0 \\ \frac{d^2 f_-}{du^2} - 6n\cot(3u+2\phi)\frac{df_-}{du} + (4-9n^2)f_- = 0 \end{aligned} \tag{7.76}$$

where $\phi = \arg w' = -\arg \bar{w}'$. If n is an integer and $\phi = 0$, these equations coincide with those found by Stroganov for the XXZ spin chain with an odd $2n+1$ number of sites at the supersymmetric point (anisotropy $\Delta = -\frac{1}{2}$) [104]. Then by the procedure of [44] they can associate with their solutions $f_\pm$ some proper $Q_\pm(u) = Q_\pm(\lambda)$ functions for the ground state of such spin chain.

Now, we can transform easily ODEs (7.76) to the hypergeometric equation. Set

$$u = -\frac{i}{6}\ln(-z) \tag{7.77}$$

k	a	b	c	$a-c+1$	$b-c+1$
1	$\frac{2}{3}+n$	$1+n$	$2+2n$	$-\frac{1}{3}-n$	$-n$
1	$-\frac{1}{3}-n$	$-n$	$-2n$	$\frac{2}{3}+n$	$1+n$
1	$-n$	$\frac{1}{3}-n$	$-2n$	$1+n$	$\frac{4}{3}+n$
1	$1+n$	$\frac{4}{3}+n$	$2+2n$	$-n$	$\frac{1}{3}-n$
1	$\frac{4}{3}+n$	$1+n$	$2+2n$	$\frac{1}{3}-n$	$-n$
1	$\frac{1}{3}-n$	$-n$	$-2n$	$\frac{4}{3}+n$	$1+n$
1	$-n$	$-\frac{1}{3}-n$	$-2n$	$1+n$	$\frac{2}{3}+n$
1	$1+n$	$\frac{2}{3}+n$	$2+2n$	$-n$	$-\frac{1}{3}-n$
k	a	b	c	$a-c+1$	$b-c+1$
4	$\frac{1}{3}+n$	$1+n$	$2+2n$	$-\frac{2}{3}-n$	$-n$
4	$-\frac{2}{3}-n$	$-n$	$-2n$	$\frac{1}{3}+n$	$1+n$
4	$-n$	$\frac{2}{3}-n$	$-2n$	$1+n$	$\frac{5}{3}+n$
4	$1+n$	$\frac{5}{3}+n$	$2+2n$	$-n$	$\frac{2}{3}-n$
4	$\frac{5}{3}+n$	$1+n$	$2+2n$	$\frac{2}{3}-n$	$-n$
4	$\frac{2}{3}-n$	$-n$	$-2n$	$\frac{5}{3}+n$	$1+n$
4	$-n$	$-\frac{2}{3}-n$	$-2n$	$1+n$	$\frac{1}{3}+n$
4	$1+n$	$\frac{1}{3}+n$	$2+2n$	$-n$	$-\frac{2}{3}-n$

Table 7.1: Values of the hypergeometric parameters for the XXZ spin chain with $2n+1$ number of sites at the supersymmetric point. We notice that for $k=1,4$ there is no poles skipping.

and $k = 1, 4$. We get

$$\frac{d^2}{dz^2} g - \frac{kz^2 + 2kz + k - 36n^2 z - 36nz - 9z^2 - 18z - 9}{36z^2(z+1)^2} g = 0 \tag{7.78}$$

On the other hand, we can transform the hypergeometric equation in standard form in normal form

$$\frac{d^2}{dz^2} g + \frac{2c[z(a+b-1)+1] + z\left[-z(a-b)^2 - 4ab + z\right] - c^2}{4(z-1)^2 z^2} g = 0 \tag{7.79}$$

Sending also $z \to z+1$ we get the values of hypergeometric parameters in table 7.1. We notice that we never get poles skipping points, as right zeros $b = -n_R^z$ and left poles $a - c + 1 = -n_L^p$ or left zeros $a = -n_L^z$ and right poles $b - c + 1 = -n_R^p$ never simultaneously appear.[17]

7.7. $SU(2)$ $N_f = 4$ gauge theory and its gravity counterpart

The quantum SW curve for $SU(2)$ $N_f = 4$ is

$$-\hbar^2 \frac{d^2}{dy^2}\psi + \Bigg\{ -\exp(2y)\left(q\left(q\left(m_1^2 + m_2^2 + m_3^3 + m_4^2\right) - 24(m_1 m_2 + m_3 m_4)\right) + 16(q+4)u\right) \tag{7.80}$$

$$+ 4\sqrt{q}\exp(3y)\left(m_1^2 q - m_1 m_2 (q+8) + m_2^2 q - m_3 m_4 q + 8u\right) \tag{7.81}$$

$$+ 4\sqrt{q}\exp(y)\left(-m_1 m_2 q + m_3^2 q - m_3 m_4 (q+8) + m_4^2 q + 8u\right) - \tag{7.82}$$

$$- 4q\exp(4y)(m_1 - m_2)^2 - 4q(m_3 - m_4)^2 \Bigg\} \frac{\exp(-2y)}{4\left(-4\sqrt{q}\cosh(y) + q + 4\right)^2}\psi + \tag{7.83}$$

$$+ \frac{\hbar^2\left(\sqrt{q}\exp(-y)\left(q\exp(2y) - 8\sqrt{q}\exp(y) + 4\exp(2y) + q + 4\right)\right)}{2\left(-4\sqrt{q}\cosh(y) + q + 4\right)^2}\psi = 0 \tag{7.84}$$

Viceversa, by letting $q \to 4q$ and $y \to \ln\left(\frac{z}{\sqrt{q}}\right)$ equation (7.80) becomes a Heun equation in canonical form

$$-\hbar^2 \frac{d^2}{dz^2}\phi + \frac{1}{4z^2(z-1)^2(z-q)^2}\Big[+z^4\left(m_1^2 - 2m_1 m_2 + m_2^2 - \hbar^2\right) + m_3^2 q^2 - 2m_3 m_4 q^2 + m_4^2 q^2 - q^2\hbar^2 \tag{7.85}$$

$$+ z^2\left(m_1^2 q^2 - 6m_1 m_2 q + m_2^2 q^2 + m_3^3 q^2 - 6m_3 m_4 q + m_4^2 q^2 + \left(-q^2 - 1\right)\hbar^2 + 4qu + 4u\right) \tag{7.86}$$

$$+ z^3\left(-2m_1^2 q + 2m_1 m_2 q + 4m_1 m_2 - 2m_2^2 q + 2m_3 m_4 q + (q+1)\hbar^2 - 4u\right) \tag{7.87}$$

$$+ z\left(2m_1 m_2 q^2 - 2m_3^2 q^2 + 2m_3 m_4 q^2 + 4m_3 m_4 q - 2m_4^2 q^2 - 4qu + (q+1)q\hbar^2\right)\Big]\phi = 0 \tag{7.88}$$

[17] This at least holds for $k = 1, 4$, which is Stroganov's case, but if k could be $k = 9$ there would be some poles skipping for $n = 0$: $a \pm n$, $b - c + 1 = \mp n$ or $b \pm n$, $a - c + 1 = \mp n$.

7.7.1. Naive ODE/IM

We have the symmetries

$$\begin{array}{llllllll}\Omega_- : & \tau=\tau+1 & y\to y-i\pi & m_1\to m_1 & m_2\to m_2 & m_3\to -m_3 & m_4\to -m_4 \\ \Omega_+ : & \tau=\tau+1 & y\to y+i\pi & m_1\to -m_1 & m_2\to -m_2 & m_3\to m_3 & m_4\to m_4\end{array} \tag{7.89}$$

The asymptotic solutions at $y \to \mp\infty$ are

$$\psi_{-,0} \sim \frac{\hbar\theta(m_3-m_4)}{\sqrt{m_3-m_4}}\exp\left\{\frac{m_3-m_4}{2\hbar}(y-i\pi\tau)\right\}\left\{1+e^y\frac{(8m_3^2+8m_4^2-2m_3m_4q+2m_1m_2q-16u-(4+q)\hbar^2)}{(8\sqrt{q}\hbar(m_3-m_4+\hbar))}\right\} \tag{7.90}$$

$$+\frac{\hbar\theta(m_4-m_3)}{\sqrt{m_4-m_3}}\exp\left\{\frac{m_4-m_3}{2\hbar}(y-i\pi\tau)\right\}\left\{1+e^y\frac{(8m_3^2+8m_4^2-2m_3m_4q+2m_1m_2q-16u-(4+q)\hbar^2)}{(8\sqrt{q}\hbar(m_4-m_3+\hbar))}\right\} \tag{7.91}$$

$$\Re y \to -\infty \tag{7.92}$$

$$\psi_{+,0} \sim \frac{\hbar\theta(m_1-m_2)}{\sqrt{m_1-m_2}}\exp\left\{-\frac{m_1-m_2}{2\hbar}(y+i\pi\tau)\right\}\left\{1+e^{-y}\frac{(8m_1^2+8m_2^2-2m_1m_2q+2m_3m_4q-16u-(4+q)\hbar^2)}{(8\sqrt{q}\hbar(m_1-m_2+\hbar))}\right\} \tag{7.93}$$

$$+\frac{\hbar\theta(m_2-m_1)}{\sqrt{m_2-m_1}}\exp\left\{-\frac{m_2-m_1}{2\hbar}(y+i\pi\tau)\right\}\left\{1+e^{-y}\frac{(8m_1^2+8m_2^2-2m_1m_2q+2m_3m_4q-16u-(4+q)\hbar^2)}{(8\sqrt{q}\hbar(m_2-m_1+\hbar))}\right\} \tag{7.94}$$

$$\Re y \to +\infty \tag{7.95}$$

The symmetries act on them as

$$\begin{aligned}\psi_{-,1} &= \Omega_-\psi_{-,0}\,, \qquad \Omega_+\psi_{-,0}=\psi_{-,0}\\ \psi_{+,1} &= \Omega_+\psi_{+,0}\,, \qquad \Omega_-\psi_{+,0}=\psi_{+,0}\end{aligned} \tag{7.96}$$

The solution are normalized so that

$$W[\psi_{-,1},\psi_{-,0}] = -i \tag{7.97}$$

$$W[\psi_{+,1},\psi_{+,0}] = i \tag{7.98}$$

We can try to define a kind of Q function as usual as

$$Q(\tau) = W[\psi_{+,0},\psi_{-,0}] \tag{7.99}$$

We notice there is no Stokes behaviour with an equation with only regular singularities. That implies that the kind of Q function obtained through this "naive" ODE/IM has τ in the parameter range $\Re(\tau'-\tau) \leq 2$. Besides we cannot define a T function. We have the connection relations

$$\begin{aligned}\psi_{-,0} &= -iQ(\tau+1,-m_1,-m_2,m_3,m_4)\psi_{+,0}+iQ_4(\tau,m_1,m_2,m_3,m_4)\psi_{+,1}\\ \psi_{-,1} &= -iQ(\tau+2,-m_1,-m_2,-m_3,-m_4)\psi_{+,0}+iQ(\tau+1,m_1,m_2,-m_3,-m_4)\psi_{+,1}\end{aligned} \tag{7.100}$$

taking their wronskian we get s kind of QQ system

$$\begin{aligned} &Q(\tau, m_1, m_2, m_3, m_4)Q(\tau+2, -m_1, -m_2, -m_3, -m_4) \\ &- Q(\tau+1, -m_1, -m_2, m_3, m_4)Q(\tau+1, m_1, m_2, -m_3, -m_4) = +1 \end{aligned} \tag{7.101}$$

One could try to define a Y function as

$$Y(\tau, m_1, m_2, m_3, m_4) = Q(\tau, -m_1, -m_2, -m_3, -m_4)Q(\tau, m_1, m_2, m_3, m_4) \tag{7.102}$$

so the Y system would be

$$\begin{aligned} &[1 + Y(\tau+1, -m_1, -m_2, m_3, m_4)][1 + Y(\tau+1, m_1, m_2, -m_3, -m_4)] \\ &= Y(\tau, m_1, m_2, m_3, m_4)Y(\tau+2, -m_1, -m_2, -m_3, -m_4) \end{aligned} \tag{7.103}$$

The Q function can be concretely computed as the limit

$$Q(\tau, m_1, m_2, m_3, m_4) = -i \lim_{y\to+\infty} \frac{\psi_{-,0}}{\psi_{+,1}} \tag{7.104}$$

7.7.2. Gravity realization

The Regge-Wheeler equation for the gravitational perturbation (with spin s) of the four dimensional asymptotically AdS_4 (with cosmological constant $\Lambda < 0$) Schwarschild black holes is

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr} + \omega^2 - V(r)\right]\phi(r) = 0 \tag{7.105}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 \tag{7.106}$$

and

$$V(r) = f(r)\left[\frac{l(l+1)}{r^2} + (1-s^2)\left(\frac{2M}{r^3} - \frac{4-s^2}{6}\Lambda\right)\right]. \tag{7.107}$$

After the redefinition $\phi(r) = \Phi(r)/\sqrt{f(r)}$ the ODE becomes in normal form as

$$\Phi''(r) + U(r)\Phi(r) = 0\,, \qquad U(r) = \frac{\omega^2 - V(r)}{f(r)^2} + \frac{f'(r)^2}{4f(r)^2} - \frac{f''(r)}{2f(r)} \tag{7.108}$$

which the potential which explicitly reads

$$U(r) = \frac{1}{2r^2\left(6M + \Lambda r^3 - 3r\right)^2}\Big[r^6\left(\Lambda^2\left(-s^2\right)\left(s^2-5\right) - 4\Lambda^2\right) + r^4\left(18\Lambda + 6\Lambda l^2 + 6\Lambda l + 3\Lambda s^2\left(s^2-5\right) + 18\omega^2\right) \tag{7.109}$$

$$- 6\Lambda M r^3\left(s^4 - 3s^2 + 8\right) + \left(-18l^2 - 18l\right)r^2 + 36Mr\left(l^2 + l + s^2\right) + M^2\left(18 - 72s^2\right)\Big] \tag{7.110}$$

We want to map this equation into a Heun equation, which we can relate to the gauge theory one (7.85). We start by changing variable as $v = 1/r$ and we get

$$\frac{d^2}{dv^2}\hat{\Phi} + \hat{Q}(v)\hat{\Phi} = 0 \tag{7.111}$$

$$\hat{Q}(v) = \frac{1}{2v^2\left(\Lambda + 6Mv^3 - 3v^2\right)^2}\Big[18M^2\left(1-4s^2\right)v^6 + v^5\left(36l^2M + 36lM + 36Ms^2\right) + \left(-18l^2 - 18l\right)v^4 \tag{7.112}$$

$$-6\Lambda M\left(s^4 - 3s^2 + 8\right)v^3 + v^2\left(6\Lambda l^2 + 6\Lambda l + 3\Lambda\left(s^4 - 5s^2 + 6\right) + 18\omega^2\right) + \Lambda^2\left(-\left(s^4 - 5s^2 + 4\right)\right)\Big] \tag{7.113}$$

We notice that for $s = 1, 2$ the constant term in the denominator is zero and the v^2 in the denominator simplifies, so for $s = 1, 2$ $v = 0$, that is $r = \infty$ is not a singular point and the equation has only 4 regular singularities, so it is a indeed a Heun equation like (7.85). For $s = 0$ instead it has 5 regular singularities.

The 4 regular singularities for $s = 1, 2$ are

$$\begin{aligned}
v_\infty &= \infty \\
v_0 &= \frac{\sqrt[3]{-972\Lambda M^2 + \sqrt{\left(54 - 972\Lambda M^2\right)^2 - 2916} + 54}}{18\sqrt[3]{2}M} + \frac{1}{2^{2/3}M\sqrt[3]{-972\Lambda M^2 + \sqrt{\left(54 - 972\Lambda M^2\right)^2 - 2916} + 54}} + \frac{1}{6M} \\
v_1 &= -\frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{-972\Lambda M^2 + \sqrt{\left(54 - 972\Lambda M^2\right)^2 - 2916} + 54}}{36\sqrt[3]{2}M} - \frac{1 + i\sqrt{3}}{2\,2^{2/3}M\sqrt[3]{-972\Lambda M^2 + \sqrt{\left(54 - 972\Lambda M^2\right)^2 - 2916} + 54}} + \frac{1}{6M} \\
v_2 &= -\frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{-972\Lambda M^2 + \sqrt{\left(54 - 972\Lambda M^2\right)^2 - 2916} + 54}}{36\sqrt[3]{2}M} - \frac{1 - i\sqrt{3}}{2\,2^{2/3}M\sqrt[3]{-972\Lambda M^2 + \sqrt{\left(54 - 972\Lambda M^2\right)^2 - 2916} + 54}} + \frac{1}{6M}
\end{aligned} \tag{7.114}$$

So that the denominator of $\hat{Q}(v)$ is proportional to

$$\left(v^3 - \frac{1}{2M}v^2 + \frac{\Lambda}{6M}\right)^2 = [(v - v_0)(v - v_1)(v - v_2)]^2 \tag{7.115}$$

and we can write it as

$$\hat{Q}(v) = \frac{1}{[(v - v_0)(v - v_1)(v - v_2)]^2}\sum_{n=0}^{4}\hat{c}_n v^n \tag{7.116}$$

We can now change variable as $v' = v - v_0$, so that the potential becomes

$$\hat{Q}_1(v') = \frac{1}{[v'(v' + v_0 - v_1)(v' + v_0 - v_2)]^2}\sum_{n=0}^{4}\hat{c}'_n {v'}^n \tag{7.117}$$

with coefficients

$$
\begin{aligned}
\hat{c}'_0 &= \sum_{k=0}^{4} \hat{c}_k v_0^k \\
\hat{c}'_1 &= \sum_{k=1}^{4} k\hat{c}_k v_0^{k-1} \\
\hat{c}'_2 &= \sum_{k=2}^{4} \frac{k(k-1)}{2}\hat{c}_k v_0^{k-2} \\
\hat{c}'_3 &= \sum_{k=3}^{4} \frac{1}{6}k(k-1)(k-2)\hat{c}_k v_0^{k-3} \\
\hat{c}'_4 &= \hat{c}_4
\end{aligned} \tag{7.118}
$$

Now we can change again variable as $v'' = \frac{v'}{v_1 - v_0}$ so that the potential becomes

$$
\hat{Q}_2(v'') = \frac{1}{[v''(v''-1)(v''-a)]^2}\frac{1}{(v_1-v_0)^4}\sum_{n=0}^{4}\hat{c}''_n v''^n \tag{7.119}
$$

with

$$
\hat{c}''_n = \hat{c}'_n (v_1 - v_0)^n \tag{7.120}
$$

and

$$
a = \frac{v_2 - v_0}{v_1 - v_0} \tag{7.121}
$$

Comparing with (7.85) we have $v'' = z$ and $a = q$. We give the dictionary implicitly, without making explicit the gravity parameters, since they would be otherwise very cumbersome expressions.

$$
\hat{c''}_0 = -q^2(v_0 - v_1)^4\left(m_3^2 - 2m_3m_4 + m_4^2 - \hbar^2\right), \tag{7.122}
$$

$$
\hat{c''}_1 = q(v_0 - v_1)^4\left(-2m_1m_2q + 2m_3^2q - 2m_3m_4q - 4m_3m_4 + 2m_4^2q - q\hbar^2 + 4u - \hbar^2\right), \tag{7.123}
$$

$$
\hat{c''}_2 = -(v_0 - v_1)^4\left(m_1^2q^2 - 6m_1m_2q + m_2^2q^2 + m_3^3q^2 - 6m_3m_4q + m_4^2q^2 - q^2\hbar^2 + 4qu + 4u - \hbar^2\right), \tag{7.124}
$$

$$
\hat{c''}_3 = (v_0 - v_1)^4\left(2m_1^2q - 2m_1m_2q - 4m_1m_2 + 2m_2^2q - 2m_3m_4q - q\hbar^2 + 4u - \hbar^2\right), \tag{7.125}
$$

$$
\hat{c''}_4 = -(v_0 - v_1)^4\left(m_1^2 - 2m_1m_2 + m_2^2 - \hbar^2\right) \tag{7.126}
$$

7.7.3. Poles skipping

The Green function is contructed from the $r \to \infty$ asymptotic, for which we can the exact expression

$$
-i\frac{\psi_{-,0}}{\psi_{+,1}} = Q(\tau, m_1, m_2, m_3, m_4) - Q(\tau+1, -m_1, -m_2, m_3, m_4)\frac{\psi_{+,0}}{\psi_{+,1}} \tag{7.127}
$$

with $\psi_{+,0} \to 0$ as $r \to \infty$. So the Green function is proportional to

$$G_R \propto \frac{Q(\tau + 1, -m_1, -m_2, m_3, m_4)}{Q(\tau, m_1, m_2, m_3, m_4)} \,. \tag{7.128}$$

We recover thus the interpretation of poles skipping as simultaneous zeros of Q functions. However, now we have Q functions with different parameters, as for the Heun equation connection coefficients are not known explicitly as for the hypergeometric equation. The consistency of this characterization with the other is explained in section 7.4.

A. Fibre bundles and connections in gauge theory

In this appendix we show how some differential geometry and differential topology notions are applied to gauge theories in physics. We follow mainly [105], [106].

Usually, one thinks at a field as map $\phi : \mathcal{M} \to V$ from space-time to a vector space V; or, as in the non-linear σ-model, to a quotient group: $\phi : \mathcal{M} \to G/H$. However, a more general kind of situation is when the space in which the field takes its values varies form point to point of spacetime: then we have a family (bundle) of target spaces $\mathcal{N}_x$ and $\phi(x) \in \mathcal{N}_x$. In Yang-Mills theory we have a bundle of copies of the internal symmetry group, one for each point in spacetime.

A.1. Fibre bundles in general

Definition 1 *A* bundle *is a triple* $(E, \pi, \mathcal{M})$ *where* E *and* $\mathcal{M}$ *are topological spaces and* $\pi : E \to \mathcal{M}$ *is a continuous map.*

The space E is called *total space*, *bundle space* or, loosely speaking, bundle; $\mathcal{M}$ is the *base space*; π is the *projection* and we can assume it to be surjective. We often denote the triple with a greek letter ξ or η. The inverse image $\pi^{-1}(x)$ is the *fibre* over $x \in \mathcal{M}$.

In all existing applications in physics the bundles that arise have the property that all the fibres $\pi^{-1}(x)$ are homeomorphic to a common space F, called fibre of the bundle. This kind of bundles are called *fibre bundles*.

A bundle $\xi = (E, \pi, \mathcal{M})$ with fibre F it is often written as $F \to E \xrightarrow{\pi} \mathcal{M}$ or

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & \mathcal{M} \end{array}$$

Definition 2 *A* cross section *of a bundle* $(E, \pi; \mathcal{M})$ *is a map* $s : \mathcal{M} \to E$ *such that the image of each point* $x \in \mathcal{M}$ *lies in the fibre* $\pi^{-1}(x)$ *over* x*:*

$$\pi \circ s = id_{\mathcal{M}} \tag{A.1}$$

Definition 3 *A* bundle map *between a pair of bundles* $(E, \pi, \mathcal{M})$ *and* $(E', \pi', \mathcal{M}')$ *is a pair of maps* (u, f)*, with* $u : E \to E'$*,* $f : \mathcal{M} \to \mathcal{M}'$ *such that following diagram is commutative:*

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow \pi & & \downarrow \pi' \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M}' \end{array}$$

The pair of maps (u, f) maps fibres into fibres:

$$\forall x \in \mathcal{M} \quad u(\pi^{-1}(x)) \subset \pi'^{-1}(f(x)) \tag{A.2}$$

Since π is surjective, the map f is completely determined by the map u.

An *isomorphism* between a pair of bundles $(E, \pi, \mathcal{M})$ and $(E', \pi', \mathcal{M}')$ is bundle map (u, f) from $(E, \pi, \mathcal{M})$ to $(E', \pi', \mathcal{M}')$ for which there is another bundle map (u', f') from $(E', \pi', \mathcal{M}')$ to $(E, \pi, \mathcal{M})$ such that:

$$u' \circ u = \mathrm{id}_E \quad u' \circ u = \mathrm{id}_{E'}$$
$$f' \circ f = \mathrm{id}_{\mathcal{M}} \quad f \circ f = \mathrm{id}_{\mathcal{M}'}$$

Two bundles ξ and η with the same base space $\mathcal{M}$ are said to be *locally isomorphic* if, for each $x \in \mathcal{M}$, there exists an open neighborhood U of x such that $\xi|_U$ and $\eta|_U$ are U-isomorphic. The relation of being locally isomorphic is an equivalence relation on the set of all bundles over the topological space $\mathcal{M}$.

Definition 4 *A fibre bundle $(E, \pi, \mathcal{M})$ is* trivial *if it is $\mathcal{M}$-isomorphic to the product bundle $(\mathcal{M} \times F, pr_1, \mathcal{M})$. It is* locally trivial *if it is locally isomorphic to the product bundle.*

We can now give an alternative definition of fibre bundle. A triple $(E, \pi, \mathcal{M})$ is a fibre bundle with fibre F if and only if for each $x \in \mathcal{M}$, there exists an open neighborhood $U \subset \mathcal{M}$ of x and a homeomorphism $h : U \times F \to \pi^{-1}(U)$, called *trivialization*, such that

$$\pi(h(x, y)) = x \quad x \in U, y \in F \tag{A.3}$$

The idea in fibre bundle theory is to study spaces which are not globally products but only locally products.

An example of bundle i the *tangent bundle*. The base space $\mathcal{M}$ is a generic differentiable manifold, the total space is given by the union of all tangent spaces $T_p(\mathcal{M})$ to all points p of $\mathcal{M}$:

$$T(\mathcal{M}) = \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M}) \tag{A.4}$$

the fibre at any point $p \in \mathcal{M}$ is the tangent space $T_p(\mathcal{M})$, the projection $\pi : T(\mathcal{M}) \to \mathcal{M}$ associates to each tangent space $T_p(\mathcal{M})$ the point p to which it is attached.

A.2. Principal bundles and fundamental gauge bundles

Definition 5 *A bundle $(E, \pi, \mathcal{M})$ is a G-bundle if E is a right G-space and if $(E, \pi, \mathcal{M})$ is isomorphic to the bundle $(E, \rho, E/G)$:*

$$\begin{array}{ccc} E & \xrightarrow{\;u\;} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ \mathcal{M} & \xrightarrow{\;f\;} & E/G \end{array}$$

If the action of G is free, the G-bundle is said to be a principal G-bundle*. G is called* structure group *of the bundle.*

The fibres of the G-bundle (the orbits of the G-action) in general are not homeomorphic to each other, but when the action is free they are: a principal G-bundle is a fibre bundle with fibre G.

For example, let H be a closed Lie subgroup of a Lie group G. H acts on the right on G with a free action; the orbit space is the space G/H of cosets. Thus we get a principal H-bundle $(G, \pi, G/H)$ whose fibre is H.

In Yang-Mills theory with internal symmetry group $SU(2)$ the bundle (at least at instanton number one) is a principal $SU(2)$ bundle whose base space is S^4, the one-point compactification of Euclidean spacetime: $S^3 \to S^7 \to S^4$. It is not a product bundle since S^7 is not isomorphic to $S^3 \times S^4$.

Consider an m-dimensional differentiable manifold $\mathcal{M}$. A *linear frame* at a point $x \in \mathcal{M}$ is an ordered set $(b_1, b_2, ..., b_m)$ of basis vectors for the tangent space $T_x\mathcal{M}$. The *bundle of frames* $B(\mathcal{M})$ of $\mathcal{M}$ is the principal bundle with $\mathcal{M}$ as base space, the set of all frames at all points of $\mathcal{M}$ as total space, the function $\pi : B(\mathcal{M}) \to \mathcal{M}$ that takes a frame into the point in $\mathcal{M}$ to which it is attached as projection map. The free right action of $GL(m, \mathbb{R})$ on $B(\mathcal{M})$ is defined by:

$$(b_1, b_2, ..., b_m)g = \big(\sum_{j_1=1}^{m} b_{j_1} g_{j_1 1}, \sum_{j_2=1}^{m} b_{j_2} g_{j_2 2}, ... \sum_{j_m=1}^{m} b_{j_m} g_{j_m m}\big) \tag{A.5}$$

for all $g \in GL(m, \mathbb{R})$.

Definition 6 *A bundle map (u, f) between a pair of principal G-bundles $(P, \pi, \mathcal{M})$ and $(P', \pi', \mathcal{M}')$ is a* principal bundle map *if u is G-equivariant in the sense that:*

$$u(pg) = u(p)g \quad p \in P, g \in G \tag{A.6}$$

The set of all principal bundle maps from a principal G-bundle to itself form a group. It is called the automorphism group *of the bundle.*

If $\mathcal{A}$ is the set of all Yang-Mills potentials of the theory, the gauge group $\mathcal{G}$ is the group of automorphisms of the bundle. The physical configurations of the theory are identified with the orbits of the action of the gauge group, that is, they are elements of the quotient space $\mathcal{A}/\mathcal{M}$. It can be shown that this action is free and therefore $\mathcal{A}$ is a principal $\mathcal{G}$-bundle with base space $\mathcal{A}/\mathcal{M}$.

Theorem 1 *If ξ is the product bundle $(\mathcal{M} \times G, pr_1, \mathcal{M})$, then the automorphisms $u : \mathcal{M} \times G \to \mathcal{M} \times G$ are in one-to-one correspondence with the maps $\chi : \mathcal{M} \to G$ such that $u(x, g) = (x, \chi(x)g)$.*

In other words, if the bundle is trivial Aut(ξ) is isomorphic to the group $C^\infty(\mathcal{M}, G)$ of the usual smooth gauge transformations.

Theorem 2 *A principal G-bundle $(P, \pi, \mathcal{M})$ is trivial if and only if it possesses a continuous cross-section*

In Yang-Mills theory, choosing a cross-section of the bundle $\mathcal{G} \to \mathcal{A} \to \mathcal{A}/\mathcal{G}$ corresponds physically to *choosing a gauge*. This principal bundle is not trivial and hence no smooth cross-sections exist. This is responsible for the *Gribov effect*: there is an intrinsic obstruction to choosing a gauge that works for all physical configurations.

A.3. Connections and Yang Mills field

Let G be a Lie group that has a right action $g \to \delta_g$ on a differentiable manifold $\mathcal{M}$. Then the vector field X^A on $\mathcal{M}$ induced by the action of the one-parameter subgroup $t \to \exp tA$, $A \in T_eG$ is defined as:

$$X_p^A(f) = \frac{d}{dt} f(p \exp tA)\Big|_{t=0} \tag{A.7}$$

where $f \in \mathcal{C}^\infty(\mathcal{M})$, and $\delta_g(p)$ has been abbreviated to pg.

The map $i : A \to X^A$, which associated to each $A \in T_eG$ the induced vector field X^A on $\mathcal{M}$, is a homomorphism of group of left invariant vector fields $L(G) \simeq T_eG$ into the infinite-dimensional Lie algebra of all vector fields on $\mathcal{M}$ VFlds $\mathcal{M}$:

$$[X^A, X^B] = X^{[A,B]} \tag{A.8}$$

for all $A, B \in T_eG \simeq L(G)$.

Consider a principal bundle P with fibre G. Both the Lie group G and the base space $\mathcal{M}$ are differentiable manifolds, so P also is differentiable manifold. Therefore we can consider the tangent and cotangent bundles TP, T^*P. We decompose each tangent space T_pP (a point of TP) into *vertical* and *horizontal subspaces*:

$$T_pP \simeq V_pP \oplus H_pP \quad \forall p \in P \tag{A.9}$$

$$V_pP = \{\tau \in T_pP | \pi_* \tau = 0\} \tag{A.10}$$

Thus any $\tau \in T_pP$ can be decomposed uniquely into a sum of *vertical* and *horizontal* components ver(τ) and hor(τ).

Definition 7 *A* connection *in a principal bundle $G \to P \to \mathcal{M}$ is a smooth assignment to each point $p \in P$ of a subspace H_pP of T_pP such that*

$$T_pP \simeq V_pP \oplus H_pP \quad \forall p \in P \tag{A.11}$$

$$\delta_{g^*}(H_pP) = H_{pg}P \quad \forall g \in G, p \in P \tag{A.12}$$

A connection can be equivalently characterized as a T_eG-valued one-form ω on P defined as

$$\omega_p(\tau) = i^{-1}(ver(\tau)). \tag{A.13}$$

and with the following properties

$$\omega_p(\tau)(X^A) = A \quad \forall p \in P, A \in T_eG \tag{A.14}$$

$$(\delta_{g^*}\omega)_p(\tau) = \boldsymbol{Ad}_{g^{-1}}(\omega_p(\tau))\,, \forall \tau \in T_pP \tag{A.15}$$

$$\tau \in H_pP \iff \omega_p(\tau) = 0 \tag{A.16}$$

Let $\sigma : U \subset \mathcal{M} \to P$ be a local section of a principal bundle $G \to P \to \mathcal{M}$ which is equipped with a connection one-form ω. The *local σ-representative* of ω is the T_eG valued one form on the open set $U \subset \mathcal{M}$ given by $\sigma^*\omega$. $\sigma^*\omega$ corresponds to the *Yang Mills field*:

$$(\sigma^*\omega)_x = A(x) = \sum_{\mu=1}^{m} \sum_{a=1}^{\dim G} A_\mu^a(x) E_a (dx^\mu)_x \tag{A.17}$$

where $\{E_1, E_2, ..., E_{\dim G}\}$ is a basis set for T_eG. In particular:

$$A_\mu(x) = (\sigma^*\omega)_x(\partial_\mu) \tag{A.18}$$

Theorem 3 *Let $h : U \times G \to \pi^{-1}(U) \subset P$ be the local trivialisation of P induced by σ: $h(x, g) = \sigma(x)g$. Let $(\alpha, \beta) \in T_{(x,g)}(U \times G) \simeq T_xU \oplus T_gG$, the local representative $h^*\omega$ of ω on $U \times G$ can be written in terms of the local Yang Mills field $\sigma^*\omega$ as:*

$$(h^*\omega)_{(x,g)}(\alpha, \beta) = g^{-1}(\sigma^*\omega_x)(\alpha)\, g + \Xi_g(\beta) \tag{A.19}$$

where Ξ is the Cartan-Maurer T_eG-valued one form on G.

Thus the connection one-form ω can be decomposed locally as the sum of a Yang-Mills field on $\mathcal{M}$ plus a *fixed* T_eG-valued one-form on G. Hence, at least locally, specifying a connection is equivalent to giving a Yang-Mills field.

There are two ways of interpreting a gauge transformation: the active view and the passive view. According to the active view, a *gauge transformation* in the principal bundle $G \to P \to \mathcal{M}$ is any principal automorphism of the bundle. In fact, let $\phi : P \to P\,, \phi \in$ Aut (P) and let ω be a connection on P, then $\phi^*(\omega)$ is a connection. $\phi^*(\omega)$ is the *gauge transform* of ω under the gauge transformation ϕ. On the other hand, in the passive view, a gauge transformation is simply a change of coordinates of the fibre, as the following theorem describes.

Theorem 4 *Let $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$ be two local trivialisations with $U_1 \cap U_2 \neq \emptyset$, thus $A_\mu^{(1)} = \sigma_1^*\omega$ and $A_\mu^{(2)} = \sigma_2^*\omega$. If $\Omega : U_1 \cap U_2 \to G$ is the unique local gauge function defined by:*

$$\sigma_2(x) = \sigma_1(x)\Omega(x) \tag{A.20}$$

the local representatives are related on $U_1 \cap U_2$ by:

$$A_\mu^{(2)}(x) = \Omega(x)^{-1}A_\mu^{(1)}(x)\,\Omega(x) + (\Omega^*\Xi)_\mu(x) \tag{A.21}$$

If G is a group of matrices, $\Omega^*\Xi = \Omega^{-1}d\Omega$, therefore:

$$A_\mu^{(2)}(x) = \Omega(x)^{-1}A_\mu^{(1)}(x)\,\Omega(x) + \Omega(x)^{-1}\partial_\mu\Omega(x) \tag{A.22}$$

The previous equation relates *two* Yang-Mills fields, whose regions of definitions may be different. One can recover the usual equation of the gauge transform by considering an active gauge transformation $\phi : P \to P$, which induces a transformation $A \to \sigma^*(\phi^*\omega) = (\phi \circ \sigma)^*\omega$. There exists some $\Omega : U \to G$ such that, for all $x \in U$, $\sigma(x) = \phi \circ \sigma(x)\Omega(x)$, then it follows that:

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega^{-1}(x) + \Omega(x)\partial_\mu\Omega(x)^{-1} \tag{A.23}$$

Only if the bundle is trivial (A.23) refers to a globally defined T_eG valued one form on $\mathcal{M}$. If the bundle is not trivial, one must cover $\mathcal{M}$ with local trivializing charts, then local Yang-Mills fields associated with any pair of overlapping charts U_i, U_j will be related on $U_i \cap U_j$ by (A.22), with corresponding gauge function $\omega_{ij}(x)$ such that $\sigma_i(x) = \sigma_j(x)\Omega_{ij}(x)$.

A.4. Associated bundles and matter fields

Let X and Y be a pair of right G-spaces (they can also be both left G-spaces or one right and one left). The G-product of X and Y is the space of orbits of the G-action on $X \times Y$. Thus we define an equivalence relation on $X \times Y$: $(x, y) \sim (x', y')$ if there exists $g \in G$ such that $x' = xg$ and $y' = yg$. The G-product is denoted $X \times_G Y$ and the equivalence class of $(x, y) \in X \times Y$ $[x.y]$.

Definition 8 *Let $\xi = (P, \pi, \mathcal{M})$ be a principal G-bundle and let F be a left G-space. Define $P_F = P \times_G F$ where $(p, v)g = (pg, g^{-1}v)$ and define a map $\pi_F : P_F \to \mathcal{M}$ by $\pi_F([p, v]) = \pi(p)$. Then $\xi[F] = (P_F, \pi_F, \mathcal{M})$ is a fibre bundle over $\mathcal{M}$ with fibre F that is said to be* associated *with the principal bundle ξ via the action of the group G on F.*

In fact, it can be proven that for each $x \in \mathcal{M}$, the space $\pi_F^{-1}(\{x\})$is isomorphic to F.

Let $V = \mathbb{R}^m \otimes \mathbb{R}^m \otimes \cdots \otimes \mathbb{R}^m \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \otimes \cdots \otimes (\mathbb{R}^m)^*$ where the first tensor product is taken k times and the second l times, let $a \in GL(m, \mathbb{R})$ act on $v \in V$ by a representation $\rho : GL(m, \mathbb{R}) \to AutV$ as follows:

$$(\rho(a)v)^{i_1 \dots i_l}_{j_1 \dots j_k} = (\det a)^\omega \sum_{k_1 \dots k_l = 1}^{m} \sum_{h_1 \dots h_k = 1}^{m} a^{i_1}{}_{k_1} \dots a^{i_l}{}_{k_l} a_{j_1}{}^{h_1} \dots a_{j_k}{}^{h_k} v^{k_1 \dots k_l}_{h_1 \dots h_k} \tag{A.24}$$

The associated bundle to the bundle of frames $B[V] = B(\mathcal{M}) \times_{GL(m,\mathbb{R})} V$ is the *bundle of tensors densities of weight ω, contravariant rank k and covariant rank l*. A particular case is tangent bundle (A.4) illustrated above.

A *vector bundle* is an associated bundle in which the fibre is a vector space. All tensor bundles are vector bundles. The space $\Gamma(E)$ of all cross-sections of a vector bundle $(E, \pi, \mathcal{M})$ is equipped with a natural module structure over the ring $\mathcal{C}(\mathcal{M})$ of continuous, real-valued functions on $\mathcal{M}$, that is:

$$(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x) \quad \forall x \in \mathcal{M}, \psi_1, \psi_2 \in \Gamma(E) \tag{A.25}$$

$$(f\psi)(x) = f(x)\psi(x) \quad \forall x \in \mathcal{M}, f \in \mathcal{C}(\mathcal{M}), \psi \in \Gamma(E) \tag{A.26}$$

In Yang-Mills theory, *matter fields* (that is, all fields with the exception of the Yang-Mills field) have the property that they belong to some vector space V which is acted on by the group G via a representation ρ. Matter fields are identified with cross sections of various vector (tensor) bundles associated with Yang-Mills principal fibre bundle.

Let (u, f) be a principal bundle map between a pair of principal G-bundles $\xi = (P, \pi, \mathcal{M})$ and $\xi' = (P', \pi', \mathcal{M}')$. An *associated bundle map* between the associated bundles $P \times_G F$ and $P' \times_G F$ can be defined by $u_F([p, v]) = [u(p), v]$. A *vector bundle map* is a bundle map (u, f) in which the restriction of $u : E \to E'$ to each fibre is a linear map. An *automorphism* of an associated bundle $\xi[F]$ is a bundle map u_F defined by $u_F([p, v]) = [u(p), v]$ where $u \in \text{Aut}\,\xi$ is an automorphism of the principal bundle.

Theorem 5 *If $(P_F, \pi_F, \mathcal{M})$ is an associated fibre bundle then its cross sections are in bijective correspondence with maps $\phi : P(\xi) \to F$ that satisfy $\phi(pg) = g^{-1}\phi(p) \quad \forall p \in P(\xi), g \in G$. Let*

$i_p : F \to \pi_F^{-1}(\{x\})$ *be defined by* $i_p(v) = [p, v]$*, then cross sections* ψ *and maps* ϕ *are related by:*

$$\psi_\phi(x) = [p, \phi(p)] \tag{A.27}$$
$$\phi_\psi(p) = i_p^{-1}(\psi(x)) \tag{A.28}$$

with $p \in \pi^{-1}(\{x\})$.

An associated fibre bundle $\xi[F]$ is trivial if its underlying principal bundle ξ is trivial.
If $\sigma : U \subset \mathcal{M} \to P(\xi)$ is a local trivializing cross section of the principal bundle ξ, the *local representative* $\psi_U : U \to P$ of a section ψ of P_F is defined by:

$$\psi_U(x) = \phi_\psi(\sigma(x)) \tag{A.29}$$

With these definitions, we can understand how gauge transformations act on the matter fields of the system. If $\sigma_1 : U_1 \to P$ and $\sigma_2 \to P$, are two local sections of P with $U_1 \cap U_2 \neq \emptyset$, then there exists some local *gauge function* $\Omega : U_1 \cap U_2 \to G$ such that $\sigma_2(x) = \sigma_1(x)\Omega(x)$ for all $x \in U_1 \cap U_2$. Then

$$\psi_{U_2}(x) = \phi_\psi(\sigma_2(x)) = \phi_\psi(\sigma_1(x)\Omega(x)) = \Omega^{-1}(x)\phi_\psi(\sigma_1(x)) \tag{A.30}$$

Then the local representatives are related by the *gauge transformation*:

$$\psi_{U_1}(x) = \Omega(x)\psi_{U_2}(x) \tag{A.31}$$

A.5. Parallel transport and covariant differentiation

Let us consider again a principal bundle P. $\pi_* : H_pP \to T_{\pi(p)}\mathcal{M}$ is an isomorphism, therefore:

Definition 9 *To each vector vector field* X *on* $\mathcal{M}$ *there exists a unique vector field* $X^\uparrow$ *on* P *, the* horizontal lift *of* X *such that, for all* $p \in P$*,*

$$\pi_*(X_p^\uparrow) = X_{\pi(p)} \tag{A.32}$$
$$ver(X_p^\uparrow) = 0 \tag{A.33}$$

Horizontal lifting is G-equivariant:

$$\delta_{g*}(X_p^\uparrow) = X_{pg}^\uparrow \tag{A.34}$$

Let α be a smooth curve that maps a closed interval $[a, b] \subset \mathbb{R}$ into $\mathcal{M}$. A *horizontal lift* of α is a curve $\alpha^\uparrow : [a, b] \to P$ which is horizontal (ver$[\alpha^\uparrow] = 0$) and such that $\pi(\alpha^\uparrow(t)) = \alpha(t)$ for all $t \in [a, b]$. For each point $p \in \pi^{-1}\{\alpha(a)\}$ there exists a unique horizontal lift of α such that $\alpha^\uparrow(a) = p$.

Definition 10 *Let* $\alpha : [a, b] \to \mathcal{M}$ *be a curve. The* parallel translation *along* α *is the map* $\tau : \pi^{-1}(\{\alpha(a)\}) \to \pi^{-1}(\{\alpha(b)\})$ *obtained by associating with each point* $p \in \pi^{-1}(\{\alpha(a)\})$ *the point* $\alpha^\uparrow(b) \in \pi^{-1}(\{\alpha(b)\})$ *where* $\alpha^\uparrow(a) = p$.

Given the associated bundle $\xi[F] = (P_F, \pi_F, \mathcal{M})$, the *vertical subspace* of the tangent space $T_y(P_F)$, is defined as:

$$V_y(P_F) = \{\tau \in T_y(P_F) \,|\, \pi_{F*}\tau = 0\} \tag{A.35}$$

Let $k_v : P \to P_F\,, \quad v \in F$ be defined by $k_v(p) = [p, v]$. Then the *horizontal subspace* of $T_{[p,v]}$ is defined as:

$$H_{[p,v]}(P_F) = k_{v*}(H_p P) \tag{A.36}$$

Let $\alpha : [a, b] \to \mathcal{M}$, $[p, v] \in \pi_F^{-1}(\{\alpha(a)\})$, $\alpha^{\uparrow}$ such that $\alpha^{\uparrow}(a) = p$, then the curve

$$\alpha_F^{\uparrow}(t) = k_v(\alpha^{\uparrow}(t)) = [\alpha^{\uparrow}(t), v] \tag{A.37}$$

is the *horizontal lift* of α to P_F that passes through $[p, v]$ at $t = a$. The *parallel translation* in the associated bundle is the map $\tau_F : \pi_F^{-1}(\{\alpha(a)\}) \to \pi_F^{-1}(\{\alpha(b)\})$.

Definition 11 *Let P_V be a vector bundle, $\psi : \mathcal{M} \to P_V$ be a cross section, $\alpha : [0, \epsilon] \to \mathcal{M}$ be a curve in $\mathcal{M}$ such that $\alpha(0) = x_0$. The* covariant derivative *of ψ in the direction α at x_0 is:*

$$\nabla_\alpha \psi = \lim_{t\to 0}\left(\frac{\tau_V^t \psi(\alpha(t)) - \psi(x_0)}{t}\right) \in \pi_V^{-1}(\{x_0\}) \tag{A.38}$$

In a local bundle chart, it can be shown that:

$$(\nabla_\alpha \psi)_U = \frac{d}{dt}(g(t)^{-1}\psi_U(\alpha(t)))\bigg|_{t=0} = \sum_{\mu=1}^{m}(\partial_\mu \psi_U(x_0) + A_\mu(x_0))\frac{dx^\mu(\alpha(t))}{dt}\bigg|_{t=0} \tag{A.39}$$

If $v \in T_x\mathcal{M}$ the covariant derivative of the section along v is defined by $\nabla_v \psi = \nabla_\alpha \psi$, $\forall \alpha \in [v]$. Analogously, if X is a vector field on $\mathcal{M}$ the covariant derivative along X is the linear operator $\nabla_X : \Gamma(P_V) \to \Gamma(P_V)$ defined by:

$$(\nabla_X \psi)(x) = \nabla_{X_x}\psi \tag{A.40}$$

A particular case is when $\nabla_\mu = \nabla_{\partial_\mu}$:

$$(\nabla_\mu \psi)(x) = \partial_\mu \psi(x) + A_\mu(x)\psi(x) \tag{A.41}$$

The linear operator $\nabla_X : \Gamma(P_V) \to \Gamma(P_V)$ possesses also the following properties:

$$\nabla_X(f\psi) = f\nabla_X(\psi) + X(f)\psi \tag{A.42}$$

$$\nabla_{X+Y}\psi = \nabla_X(\psi) + \nabla_Y(\psi) \tag{A.43}$$

$$\nabla_{fX}(\psi) = f\nabla_X\psi \tag{A.44}$$

A.6. The curvature two-form or gauge field

Definition 12 *If ω is a k-form on a principal bundle space $P(\xi)$, the* exterior covariant derivative *of ω is the horizontal $(k+1)$-form $D\omega$ defined by:*

$$D\omega = d\omega \circ hor \tag{A.45}$$

If ω is a connection one-form on $P(\xi)$, the curvature two-form *of ω is defined as $G = D\omega$.*

Theorem 6 *If X, Y are arbitrary vector fields on $P(\xi)$ we have, for all $p \in P(\xi)$ the* Cartan structural equation*:*

$$G_p(X,Y) = d\omega_p(X,Y) + [\omega_p(X), \omega_p(Y)] \tag{A.46}$$

If $\{E_1, ..., E_{dimG}\}$ is a basis for the Lie algebra T_eG, $\omega = \omega^a E_a$ then the equation becomes:

$$G^a = d\omega^a + \frac{1}{2}\sum_{b,c=1}^{dimG} C^a_{bc}\omega^b \wedge \omega^c \tag{A.47}$$

If $\sigma : U \to P$ is a local section of the principal bundle, the local representative $A = \sigma^*\omega$ is supplemented with the local representative $F = \sigma^* G$. Then $F^a = dA^a + \frac{1}{2}\sum_{b,c=1}^{\dim G} C^a_{bc}A^b_\mu A^c_\nu$ or

$$F^a_{\mu\nu} = \frac{1}{2}(A^a_{\mu,\nu} - A^a_{\nu,\mu} + \sum_{b,c=1}^{\dim G} C^a_{bc}A^b_\mu A^c_\nu) \tag{A.48}$$

The *Bianchi identity* holds

$$DF = 0 \tag{A.49}$$

If $\sigma_1 : U_1 \to P$ and $\sigma_2 : U_2 \to P$ are a pair of local sections with $U_1 \cap U_2 \neq \emptyset$, there exists some local gauge function $\Omega : U_1 \cap U_2 \to G$ such that $\sigma_2(x) = \sigma_1(x)\Omega(x)$. If $F^{(1)} = \sigma_1^* G$ and $F^{(2)} = \sigma_2^* G$ it can be shown that:

$$F^{(2)}_\mu(x) = \Omega(x)^{-1}F^{(1)}_{\mu\nu}(x)\Omega(x) \tag{A.50}$$

Introducing the *dual gauge field*

$${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{A.51}$$

we can express the pure Yang-Mills action as

$$S = \int_{\mathcal{M}} \frac{-1}{2} tr(F_{\mu\nu}F^{\mu\nu}) = -\int_{\mathcal{M}} tr(F \wedge^* F) \tag{A.52}$$

form which the Euler Lagrange equations follow:

$$D^*F = 0 \tag{A.53}$$

Thanks to the Bianchi identities If one can find a connection A such that F is proportional to $*F$:

$$F = \lambda^* F \tag{A.54}$$

then the Euler Lagrange equations are automatically satisfied. *Instantons* are such solutions with $\lambda = \pm 1$ in Euclidean space, $\lambda = \pm i$ in Minkowski space. The so called *istanton number* is the integer k which labels the forth cohomology group

$$k \in H^4(\mathcal{M}, \pi_3(S^3)) \simeq \mathbb{Z} \tag{A.55}$$

where $\pi_3(S^3) = \mathbb{Z}$ is the third homotopy group of the three-sphere.

B. One-step large energy/WKB recursion

This Section contains a general two-fold result concerning the (modified) Schrödinger equation: an efficient technique of one-step-recursion for computing the asymptotic expansion of the wave function/periods, for high energy and small Planck constant. In the Section 4.3, we will apply this result for efficiently compute the NS-deformed periods modes (4.107), while in section 4.2 we will apply to the computation of the local integrals of motion.

B.1. Large energy expansion

Let us first start by the large energy expansion of the wave function which we will apply for computing the local integrals of motion for Liouville theory in section 4.2. Consider a general *modified Schrödinger equation*, with energy $e^{2\theta}$ which multiplies the modification $\phi(x)$ and potential $v(x)$

$$\left\{-\frac{d^2}{dx^2}+v(x)-e^{2\theta}\phi(x)\right\}\psi(x)=0 \tag{B.1}$$

By the transformation $dw=\sqrt{\phi}dx$, $\chi=\sqrt[4]{\phi}\psi$, the modified Schrödinger equation can be transformed into an ordinary Schrödinger equation

$$\left\{-\frac{d^2}{dw^2}+U(w)-e^{2\theta}\right\}\chi(w)=0, \qquad U=\frac{v}{\phi}+\frac{1}{4}\frac{\phi''}{\phi^2}-\frac{5}{16}\frac{\phi'^2}{\phi^3} \ . \tag{B.2}$$

As usual, we define $\Pi(w)=-i\frac{d}{dw}\ln\chi(w)\doteq\sum_{n=-1}^{\infty}e^{-n\theta}\Pi_n(x)$ (the last equality is *a*symptotical for large energy, $\Re\theta\to+\infty$) satisfying the usual Riccati equation

$$\Pi^2(w)-i\frac{d\Pi(w)}{dw}=e^{2\theta}-U(w) \tag{B.3}$$

which is solved by $\Pi_{-1}=1$, $\Pi_0=0$, $\Pi_1=-\frac{1}{2}U$ and the recursion relation for the high energy modes[18]

$$\Pi_{n+1}=+\frac{1}{2}\left\{i\frac{1}{\sqrt{\phi(x)}}\frac{d\Pi_n}{dx}-\sum_{m=1}^{n-1}\Pi_m\Pi_{n-m}\right\} \qquad n=1,2,\dots \ . \tag{B.4}$$

Eventually the wave function $\psi=(\phi)^{-1/4}\chi$ can be written and then expanded at large energy $\Re\theta\to+\infty$

$$\psi(x;\theta)=\frac{1}{\sqrt[4]{\phi(x)}}\exp\left\{i\int^x\sqrt{\phi(x')}\,\Pi(x')dx'\right\}\doteq\frac{1}{\sqrt[4]{\phi(x)}}\exp\left\{i\sum_{n=-1}^{\infty}e^{-n\theta}\int^x\sqrt{\phi(x')}\,\Pi_n(x')dx'\right\}, \tag{B.5}$$

[18]Equation (B.3) is solved also by the other solution generated by $\Pi_{-1}=-1$, $\Pi_0=0$, $\Pi_1=\frac{1}{2}U$, with a recursion given by (B.4) with a relative minus sign.

from which we can read off the quantum momentum of the (modified) Schrödinger equation (B.1):

$$\mathcal{P}(x) = -i\frac{d}{dx}\ln\psi(x) = \frac{i}{4}\frac{\phi'}{\phi} + \sqrt{\phi(x)}\,\Pi(x) \doteq \sum_{n=-1}^{\infty} e^{-n\theta}\mathcal{P}_n(x)\ . \tag{B.6}$$

Then, we split $\Pi(x) = \Pi_{odd}(x) + \Pi_{even}(x)$ into odd and even n powers. In terms of even and odd part, the Riccati equation becomes splitted in two equations

$$\Pi^2_{\text{even}}(w) + \Pi^2_{\text{odd}}(w) - i\frac{d\Pi_{\text{even}}}{dw} = e^{2\theta} - U(w) \tag{B.7}$$

$$2\Pi_{\text{even}}(x)\Pi_{\text{odd}}(w) - i\frac{d\Pi_{\text{odd}}}{dw} = 0 \tag{B.8}$$

From the latter equation emerges that Π_{even} is a total derivative

$$\Pi_{\text{even}} = \frac{i}{2}\frac{d}{dw}\ln\Pi_{\text{odd}}\ , \tag{B.9}$$

which becomes irrelevant when integrating under special circumstances, for example on the real axis with suitable asymptotic conditions or on a period. Forgetting about the even modes Π_{2n} (total derivatives), an important fact happens for the large energy expansion of (B.2), *i.*e. the arising of the Gelfand-Dikii (differential) polynomials, $R_n[U]$ [107].

B.1.1. Gelfand-Dikii polynomials

To see how this happens, we substitute Π_{even} in equation (B.7), obtaining (we use the prime $'$ to indicate the w derivative)

$$2\Pi''_{\text{odd}}\Pi_{\text{odd}} - 3\Pi'^2_{\text{odd}} + 4\Pi^4_{\text{odd}} = 4\left(e^{2\theta} - U\right)\Pi^2_{\text{odd}} \tag{B.10}$$

We define the function R as the algebraic inverse of Π_{odd}

$$R(x) = \frac{1}{\Pi_{\text{odd}}(x)} \tag{B.11}$$

such function $R(x)$ expands asymptotically for $\hbar \to 0$ in terms with some modes R_n defined by

$$R(x) = \sum_{n=0}^{\infty} R_n(x)e^{-\theta(2n+1)} \tag{B.12}$$

with $R_0 = 1$. Continuing the previous calculations, by (B.10) we obtain the function $R(x)$ satisfies the equivalent equation

$$-2R''R + R'^2 = 4\left(e^{2\theta} - U\right)R^2 - 4 \tag{B.13}$$

We apply the w-derivative and find

$$R''' = -4\left(e^{2\theta} - U\right)R' + 2U'R \tag{B.14}$$

In terms of the modes R_n this equation means a *one-step* recursion relation (we restore $' = \frac{d}{dw}$)

$$\frac{dR_{n+1}}{dw} = -\frac{1}{4}\frac{d^3}{dw^3}R_n + U\frac{dR_n}{dw} + \frac{1}{2}\frac{dU}{dw}R_n \tag{B.15}$$

or, in terms of the x variable (now $' = \frac{d}{dx}$)

$$\begin{aligned}\frac{dR_{n+1}}{dx} &= -\frac{1}{4}\frac{1}{\phi}\frac{d^3}{dx^3}R_n + \frac{3}{8}\frac{\phi'}{\phi^2}\frac{d^2}{dx^2}R_n + \left[\frac{v}{\phi} + \frac{3}{8}\frac{\phi''}{\phi^2} - \frac{9}{16}\frac{\phi'^2}{\phi^3}\right]\frac{d}{dx}R_n \\ &+ \left[\frac{1}{2}\frac{v'}{\phi} - \frac{1}{2}\frac{v\phi'}{\phi^2} + \frac{1}{8}\frac{\phi'''}{\phi^2} - \frac{9}{16}\frac{\phi''\phi'}{\phi^3} + \frac{15}{32}\frac{\phi'^3}{\phi^4}\right]R_n\end{aligned} \tag{B.16}$$

with initial condition $R_0 = 1$.

The first Gelfand-Dikii polynomial are [107]

$$R_0[U] = 1 \tag{B.17}$$

$$R_1[U] = \frac{1}{2}U \tag{B.18}$$

$$R_2[U] = \frac{3}{8}U^2 - \frac{1}{8}U'' \tag{B.19}$$

$$= \frac{3}{8}U^2 - \frac{1}{8}\frac{1}{\phi}\frac{d^2}{dx^2}U + \frac{1}{16}\frac{1}{\phi^2}\frac{d\phi}{dx}\frac{d}{dx}U \tag{B.20}$$

$$R_3[U] = \frac{5}{16}U^3 - \frac{5}{32}U'^2 - \frac{5}{16}U''U + \frac{1}{32}U^{iv} \tag{B.21}$$

$$= \frac{5}{16}U^3 - \frac{5}{32}\frac{1}{\phi}\left(\frac{dU}{dx}\right)^2 - \frac{5}{16}\frac{1}{\phi}U\frac{d^2U}{dx^2} + \frac{5}{32}\frac{\frac{d\phi}{dx}}{\phi^2}U\frac{dU}{dx} \tag{B.22}$$

$$+ \frac{1}{32}\frac{1}{\phi^2}\frac{d^4U}{dx^4} - \frac{3}{32}\frac{\frac{d\phi}{dx}}{\phi^3}\frac{d^3U}{dx^3} + \left[-\frac{1}{16}\frac{\frac{d^2\phi}{dx^2}}{\phi^3} + \frac{19}{128}\frac{(\frac{d\phi}{dx})^2}{\phi^4}\right]\frac{d^2}{dx^2}U \tag{B.23}$$

$$+ \left[-\frac{1}{64}\frac{\frac{d^3\phi}{dx^3}}{\phi^3} + \frac{13}{128}\frac{\frac{d\phi}{dx}\frac{d^2\phi}{dx^2}}{\phi^4} - \frac{7}{64}\frac{(\frac{d\phi}{dx})^3}{\phi^5}\right]\frac{d}{dx}U \tag{B.24}$$

$$\begin{aligned}R_4[U] &= \frac{35}{128}U^4 - \frac{35}{64}UU'^2 - \frac{35}{64}U^2U'' + \frac{21}{128}U''^2 + \frac{14}{64}U'U''' + \frac{7}{64}UU^{(4)} \\ &- \frac{1}{128}U^{(6)}\end{aligned} \tag{B.25}$$

It can be proven (see for instance [108]) that the Geldand-Dikii polynomials R_n are proportional to the modes $\sqrt{\phi(x)}\Pi_{2n-1}$ up to a x-total derivative:

$$\Pi_{2n-1}(x) = \frac{-1}{2n-1}R_n(x) + \frac{d}{dw}(\text{local fields}). \tag{B.26}$$

The advantage of using the equivalent R_n integrands (which are *equivalent* under integration, if one can neglect total derivatives) is that their recursion (B.16) is far simpler than that (B.4) for the original integrands $\mathcal{P}_{2n-1}$. In fact, in (B.16), to compute the $n+1$-th term, it is sufficient to know *only the first preceding* n-th term, *not all* the preceding, as in (B.4).

B.1.2. Equivalence proof for the energy-WKB integrands

In this paragraph we report the proof of (B.26) which is in [108]. Define $k = e^{\theta}$ for simplicity of notation. By definition Π_{odd} is expanded as

$$\Pi_{\text{odd}}(w) \doteq k + \sum_{n=1}^{\infty} \frac{\Pi_{2n-1}(w)}{k^{2n-1}}\,, \tag{B.27}$$

while $R = \frac{1}{\Pi_{\text{odd}}}$ is expanded as

$$R(w) = \sum_{n=0}^{\infty} \frac{R_n(w)}{k^{2n+1}}\,. \tag{B.28}$$

Now derive Π_{odd} with respect to k

$$\frac{\partial \Pi_{\text{odd}}(w)}{\partial k} = 1 - \sum_{n=1}^{\infty} (2n-1) \frac{\Pi_{2n-1}(w)}{k^{2n}} \tag{B.29}$$

and define a new quantity σ as

$$\sigma(w) = \frac{1}{k} \frac{\partial \Pi_{\text{odd}}(w)}{\partial k} = \frac{1}{k} - \sum_{n=1}^{\infty} (2n-1) \frac{\Pi_{2n-1}(w)}{k^{2n+1}} \tag{B.30}$$

With the aid of heuristic examples, we can conjecture

$$\sigma(w) = R(w) + \text{t.d.} \tag{B.31}$$

where t.d. is a total w derivative. Now we prove (B.31). We rewrite in the new notation the Riccati equation (B.10) for the odd part of Π

$$+2\Pi_{\text{odd}}\Pi''_{\text{odd}} - 3\Pi'^{2}_{\text{odd}} + 4\Pi^4_{\text{odd}} - 4(k^2 - U)\Pi^2_{\text{odd}} = 0 \tag{B.32}$$

We differentiate (B.32) with respect to k

$$16\Pi^3_{\text{odd}} \frac{\partial \Pi_{\text{odd}}}{\partial k} - 6\Pi'_{\text{odd}} \frac{\partial \Pi'_{\text{odd}}}{\partial k} + 2\frac{\partial \Pi_{\text{odd}}}{\partial k}\Pi''_{\text{odd}} + 2\Pi_{\text{odd}}\left(\frac{\partial \Pi_{\text{odd}}}{\partial k}\right)'' - 8k\Pi^2_{\text{odd}} - 8(k^2-U)\Pi_{\text{odd}}\frac{\partial \Pi_{\text{odd}}}{\partial k} = 0 \tag{B.33}$$

then divide by $2k\Pi_{\text{odd}}$

$$8\Pi^2_{\text{odd}}\sigma - 3\frac{\Pi'_{\text{odd}}}{\Pi_{\text{odd}}}\sigma' + \sigma\frac{\Pi''_{\text{odd}}}{\Pi_{\text{odd}}} + \sigma'' - 4\Pi_{\text{odd}} - 4(k^2-U)\sigma = 0 \tag{B.34}$$

Now, since

$$\frac{\Pi'_{\text{odd}}}{\Pi_{\text{odd}}} = -\frac{R'}{R} \qquad \frac{\Pi''_{\text{odd}}}{\Pi_{\text{odd}}} = 2\left(\frac{R'}{R}\right)^2 - \frac{R''}{R} \tag{B.35}$$

we can replace Π_{odd} by R to obtain

$$8\frac{1}{R^2}\sigma + 3\frac{R'}{R}\sigma' + 2\sigma\left(\frac{R'}{R}\right)^2 - \frac{R''}{R}\sigma + \sigma'' - 4\frac{1}{R} - 4(k^2 - U)\sigma = 0 \tag{B.36}$$

$$8\sigma + 3R'R\sigma' + 2R'^2\sigma - RR''\sigma + R^2\sigma'' - 4R - 4(k^2 - U)R^2\sigma = 0 \tag{B.37}$$

$$4(k^2 - U)R^2\sigma + 4R = 8\sigma + 3RR'\sigma' + 2R'^2\sigma - RR''\sigma + R^2\sigma'' \tag{B.38}$$

Recalling also equation (B.13) for R

$$-2R''R + R'^2 + 4 = 4(k^2 - U)R^2 \tag{B.39}$$

we can write

$$\begin{aligned} 4R &= 4\sigma + 3RR'\sigma' + R'^2\sigma + RR''\sigma + R^2\sigma'' \\ 4\sigma - 4R &= -3RR'\sigma' - R'^2\sigma - RR''\sigma - R'^2\sigma'' \end{aligned} \tag{B.40}$$

We note that the r.h.s. is a total derivative

$$3RR'\sigma' + R'^2\sigma + RR''\sigma + R^2\sigma'' = (\sigma R^2)'' - (\sigma RR')' \tag{B.41}$$

We have proven the conjecture (B.31), which, in terms of the modes is

$$\Pi_{2n-1}(w) = -\frac{1}{2n-1}R_n(w) + \text{t.d.} \tag{B.42}$$

We have thus proved that, under integration over a period or over the entire dominion, integrating the standard mode Π_{2n-1} is equivalent to integrating the Gelfand-Dikii polynomial R_n, up to a simple numerical n dependent factor.

B.2. Small $\hbar$ recursion

We show now that these results can be adapted for the usual small $\hbar$ WKB asymptotic expansion of a generic Schrödinger equation

$$+\frac{d^2}{dx^2}\psi + \frac{\phi(x)}{\hbar^2}\psi = 0 \qquad \text{with} \quad \phi(x) = 2m(E - V(x))\,. \tag{B.43}$$

In fact, the usual WKB analysis envisages the exact quantum momentum $\mathcal{P}(x) = -i\frac{d}{dx}\ln\psi(x) \doteq \sum_{n=-1}^{\infty}\hbar^n\mathcal{P}_n(x)$ verifying the Riccati equation and modes recursion relation, respectively

$$\mathcal{P}^2(x) - i\frac{d\mathcal{P}(x)}{dx} = \frac{\phi(x)}{\hbar^2}\ , \quad \mathcal{P}_{n+1} = \frac{1}{2\sqrt{\phi}}\left(i\frac{d}{dx}\mathcal{P}_n - \sum_{m=0}^{n}\mathcal{P}_m\mathcal{P}_{n-m}\right)\ , \tag{B.44}$$

with initial condition the classical momentum $\mathcal{P}_{-1} = \sqrt{\phi}$ [19]. As above, we split $\mathcal{P}(x) = \mathcal{P}_{odd}(x) + \mathcal{P}_{even}(x)$: then $\mathcal{P}_{even} = -\frac{1}{2}(\ln\mathcal{P}_{odd})'$ and $\mathcal{P}_{2n}$ are total derivatives, which, under

[19] As above there is also the solution with $\mathcal{P}_{-1} = -\sqrt{\phi}$.

specific circumstances, can be forgetten about. Now we wish to think of (B.43) as the particular case, $v = 0$, of the previous modified Schrödinger equation (B.1) with *e*nergy $= 1/\hbar^2$. Thus, we obtain the usual Schrödinger equation (B.2) with potential $U = \frac{1}{4}\frac{\phi''}{\phi^2} - \frac{5}{16}\frac{\phi'^2}{\phi^3}$: in this manner small $\hbar$ is interpreted as large energy. And we can make us of the Gelfand-Dikii polynomials [107], with recursion relation (B.16) with $v = 0$

$$R'_{n+1} = -\frac{1}{4\phi}R'''_n + \frac{3}{8}\frac{\phi'}{\phi^2}R''_n + \left(\frac{3}{8}\frac{\phi''}{\phi^2} - \frac{9}{16}\frac{\phi'^2}{\phi^3}\right)R'_n + \left(\frac{1}{8}\frac{\phi'''}{\phi^2} - \frac{9}{16}\frac{\phi''\phi'}{\phi^3} + \frac{15}{32}\frac{\phi'^3}{\phi^4}\right)R_n\,, \tag{B.45}$$

and initial condition $R_0 = 1$. In fact, we have seen above $\mathcal{P}_{2n-1} = \sqrt{\phi(x)}\Pi_{2n-1}$ which, in its turn, is expressible by $\sqrt{\phi(x)}R_n$ up to a x-total derivative:

$$\mathcal{P}_{2n-1}(x) = \frac{-1}{2n-1}\sqrt{\phi(x)}R_n(x) + \frac{d}{dx}(\text{local fields})\ . \tag{B.46}$$

The advantage of using the R_n integrands (which are *e*quivalent as they give the same integral, under suitable boundary conditions) is that their recursion (B.45) is far simpler than that (B.44) for the original integrands $\mathcal{P}_{2n-1}$. In fact, using recusion (B.45) to compute the $n+1$-th term, requires to know *only the first preceding* n-th term, *not all* the preceding, as in recursion (B.44).

B.3. homogeneous operators

We give the first homogeneous operators in table B.3.

$n \downarrow k \rightarrow$	0	1	2	3	4	5	6	7
1	$\frac{1}{48}$	$\frac{1}{24}$						
2	$\frac{5}{1536}$	$\frac{1}{192}$	$\frac{7}{5760}$					
3	$\frac{41}{57344}$	$\frac{153}{143360}$	$\frac{79}{215040}$	$\frac{31}{967680}$				
4	$\frac{15229}{70778880}$	$\frac{9539}{30965760}$	$\frac{517}{4128768}$	$\frac{13}{716800}$	$\frac{127}{154828800}$			
5	$\frac{484249}{5813305344}$	$\frac{5049503}{43599790080}$	$\frac{8430053}{163499212800}$	$\frac{780341}{81749606400}$	$\frac{61729}{81749606400}$	$\frac{73}{3503554560}$		
6	$\frac{520147}{13153337344}$	$\frac{1970237719}{36623823667200}$	$\frac{24135991171}{952219415347200}$	$\frac{97132361}{17854114037760}$	$\frac{764801}{1335885824000}$	$\frac{4232933}{148784283648000}$	$\frac{1414477}{2678117105664000}$	
7	$\frac{44689671097}{2009078326886400}$	$\frac{2720394516799}{91413063873331200}$	$\frac{222004327651}{15235510645555200}$	$\frac{1184129396719}{342798989524992000}$	$\frac{819210613}{1883510931456000}$	$\frac{841389097}{28566582460416000}$	$\frac{1592519}{1587032358912000}$	$\frac{8191}{612141052723200}$

Table B.1: The homogeneous coefficients $h_{n,k}$ until $n = 7$

For $n = 8$:

$$\begin{aligned} h_{8,0} &= \frac{167130500455}{11556416963739648} & h_{8,1} &= \frac{1269525458269}{66482228271513600} \\ h_{8,2} &= \frac{1195089081731507}{124321766867730432000} & h_{8,3} &= \frac{19019473001273}{7770110429233152000} \\ h_{8,4} &= \frac{65112995081339}{186482650301595648000} & h_{8,5} &= \frac{30582560027}{105956051307724800 0} \\ h_{8,6} &= \frac{31743234689}{23310331287699456000} & h_{8,7} &= \frac{17569063}{5203198948147200 00} \\ h_{8,8} &= \frac{16931177}{49950709902213120000}\,. \end{aligned} \tag{B.47}$$

For $n = 9$:

$$\begin{aligned} h_{9,0} &= \frac{433703134460825}{40652173045017870336} & h_{9,1} &= \frac{4214294136530702387}{302350537022320410624000} \\ h_{9,2} &= \frac{270779932823697631}{37793817127790051328000} & h_{9,3} &= \frac{3118931647444703}{1619735019762430771200} \\ h_{9,4} &= \frac{7032053466443077}{23343239990693855232000} & h_{9,5} &= \frac{5692637782272613}{198417539920897769472000} \\ h_{9,6} &= \frac{95887701568193}{57235828823335895040000} & h_{9,7} &= \frac{2405464260517}{41336987483520368640000} \\ h_{9,8} &= \frac{272028221291}{248021924901122211840000} & h_{9,9} &= \frac{5749691557}{669659197233029971968000}\,. \end{aligned} \tag{B.48}$$

C. $N_f = 1, 2$ Seiberg-Witten periods

In this appendix we define and give some relations for the Seiberg-Witten periods for the $SU(2)$ $N_f = 1, 2$ theories, that is, the leading $\hbar \to 0$ of the quantum (or deformed) exact periods which we prove are connected to integrability exact Y and T functions.

C.1. Massless $N_f = 1$ SW periods

The massless $N_f = 2$ gauge periods are just the $N_f = 0$ gauge periods already dealt with in [1]. Hence we treat here the (much more complex) $N_f = 1$ massless $m = 0$ case, following and extending [83]. In that case the low energy effective action has three finite $\mathbb{Z}_3$ symmetric singularities, corresponding to dyon BPS particles becoming massless. If we set $\Lambda_1 = \Lambda_1^*$ with

$$\Lambda_1^* = \sqrt[6]{\frac{256}{27}}\,, \tag{C.1}$$

those singularities are situated at

$$u_0 = -1 \qquad u_1 = -e^{2\pi i/3} \qquad u_2 = -e^{-2\pi i/3}\,. \tag{C.2}$$

The massless $m = 0$ $N_f = 1$ SW curve is

$$y_{SW}^2(u, \Lambda_1) = x^3 - ux^2 - \frac{\Lambda_1^6}{64}\,, \tag{C.3}$$

and it gives the SW periods through the integrals

$$\begin{pmatrix} a^{(0)}(u, \Lambda_1) \\ a_D^{(0)}(u, \Lambda_1) \end{pmatrix} = \frac{\sqrt{2}}{8\pi} \oint_{A,B} dx\, \frac{2u - 3x}{\sqrt{x^3 - ux^2 - \frac{\Lambda_1^6}{64}}}\,. \tag{C.4}$$

It can be shown then that $\Pi^{(0)} = a^{(0)}, a_D^{(0)}$ satisfy the SW Picard-Fuchs equation

$$\left(\frac{27\Lambda_1^6}{256} + u^3\right) \frac{\partial^2 \Pi^{(0)}(u)}{\partial u^2} + \frac{u}{4} \Pi^{(0)}(u) = 0\,, \tag{C.5}$$

with boundary condition as $u \to \infty$ as

$$\begin{aligned} a^{(0)}(u, \Lambda_1) &\simeq \sqrt{\frac{u}{2}} \qquad u \to \infty \\ a_D^{(0)}(u, \Lambda_1) &\simeq -i \left[\frac{1}{2\pi} a^{(0)}(u, 0, \Lambda_1) \left(-i\pi - 3\ln\frac{16u}{\Lambda_1^2}\right) + \frac{3}{\pi}\sqrt{\frac{u}{2}}\right] \qquad u \to \infty\,. \end{aligned} \tag{C.6}$$

The massless SW Picard-Fuchs equation can be mapped into an hypergeometric equation and then explicit formulas for $a^{(0)}, a_D^{(0)}$ follow:

$$
\begin{aligned}
a^{(0)}(u,\Lambda_1) &= \sqrt{\frac{u}{2}}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6};1;-\frac{27\Lambda_1^6}{256u^3}\right)\\
a_D^{(0)}(u,\Lambda_1) &= \begin{cases} -a^{(0)}(u,\Lambda_1)+e^{-i\pi/3}f_D(u,\Lambda_1) & 0<\arg(u)\leq\frac{2\pi}{3}\\ f_D(u,\Lambda_1)-2a^{(0)}(u,\Lambda_1) & \frac{2\pi}{3}<\arg(u)\leq\pi\\ a^{(0)}(u,\Lambda_1)-f_D(u,\Lambda_1) & -\pi<\arg(u)<-\frac{2\pi}{3}\\ \exp\left(-\frac{2\pi i}{3}\right)f_D(u,\Lambda_1) & -\frac{2\pi}{3}\leq\arg(u)\leq 0\end{cases}
\end{aligned}
\tag{C.7}
$$

(sectors given assuming $\Lambda_1 > 0$) where

$$
f_D(u,\Lambda_1) = \frac{\Lambda_1\left(\frac{256u^3}{27\Lambda_1^6}+1\right)\,{}_2F_1\left(\frac{5}{6},\frac{5}{6};2;\frac{256u^3}{27\Lambda_1^6}+1\right)}{4\sqrt[3]{2}\sqrt{2}\sqrt{3}}
\tag{C.8}
$$

So defined, $a^{(0)}$ has a branch cut for $u<0$ (due to the square root and three other cuts from the origin $u=0$ to u_0, u_1 and u_2 (due to the hypergeometric function). Instead, $a_D^{(0)}$ so defined has a branch cut for $u<0$ and from $u=0$ to u_2.

C.1.1. $\mathbb{Z}_3$ R-symmetry

We find the following $\mathbb{Z}_3$ R-symmetry relations

$$
\begin{aligned}
a^{(0)}(e^{2\pi i/3}u) &= -e^{-2\pi i/3}a^{(0)}(u) && -\pi<\arg u\leq\pi/3\\
a^{(0)}(e^{2\pi i/3}u) &= e^{-2\pi i/3}a^{(0)}(u) && \pi/3<\arg u\leq\pi\\
a^{(0)}(e^{-2\pi i/3}u) &= -e^{2\pi i/3}a^{(0)}(u) && -\pi/3<\arg u\leq\pi\\
a^{(0)}(e^{-2\pi i/3}u) &= e^{2\pi i/3}a^{(0)}(u) && -\pi<\arg u\leq-\pi/3\\
a_D^{(0)}(e^{2\pi i/3}u) &= -e^{-2\pi i/3}\left[a_D^{(0)}(u)-a^{(0)}(u)\right] && -\pi<\arg u\leq\pi/3\\
a_D^{(0)}(e^{-2\pi i/3}u) &= -e^{2\pi i/3}\left[a_D^{(0)}(u)+a^{(0)}(u)\right] && -\pi/3<\arg u\leq\pi
\end{aligned}
\tag{C.9}
$$

C.2. Massive $N_f = 1, 2$ SW periods

The massive $N_f = 1$ SW curve is [84]

$$
y_{SW}^2 = x^3 - ux^2 + \frac{\Lambda_1^3}{4}m_1 x - \frac{\Lambda_1^6}{64}
\tag{C.10}
$$

The SW differential is

$$
\lambda = \frac{\sqrt{2}}{4\pi}\left[-\left(3x-2u+\frac{\Lambda_1^3}{4}\frac{m}{x}\right)\frac{dx}{2y_{SW}}\right]
\tag{C.11}
$$

The SW periods $a_1^{(0)}$, $a_2^{(0)}$ are given by the integrals

$$\int_{\gamma_i} \lambda = \frac{\sqrt{2}}{4\pi}\left[uI_1^{(i)} - 3I_2^{(i)} - \frac{\Lambda_1^3}{4} m I_3^{(i)}\left(-\frac{u}{3}\right)\right] \tag{C.12}$$

Define e_k as the roots of the Seiberg-Witten curve in canonical form

$$\begin{aligned} y_{SW}^2(x = \xi + \frac{u}{3}) &= (\xi - e_1)(\xi - e_2)(\xi - e_3) \\ &= -\frac{\Lambda_1^6}{64} + \xi\left(\frac{\Lambda_1^3 m}{4} - \frac{u^2}{3}\right) + \frac{1}{12}\Lambda_1^3 m u + \xi^3 - \frac{2u^3}{27}\,, \end{aligned} \tag{C.13}$$

Basic integrals over the cycle γ_1

$$\begin{aligned} I_1^{(1)} &= 2\int_{e_3}^{e_2} \frac{d\xi}{\eta} = \frac{2}{(e_1 - e_3)^{1/2}} K(k) \\ I_2^{(1)} &= 2\int_{e_3}^{e_2} \frac{\xi d\xi}{\eta} = \frac{2}{(e_1 - e_3)^{1/2}}[e_1 K(k) + (e_3 - e_1)E(k)] \\ I_3^{(1)} &= 2\int_{e_3}^{e_2} \frac{d\xi}{\eta(\xi - c)} = \frac{2}{(e_1 - e_3)^{3/2}}\left[\frac{1}{1 - \tilde{c} + k'} K(k) + \frac{4k'}{1 + k'}\frac{1}{(1 - \tilde{c})^2 k'^2}\Pi_1\left(\nu(c), \frac{1 - k'}{1 + k'}\right)\right]. \end{aligned} \tag{C.14}$$

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} \quad k'^2 = 1 - k^2$$
$$\tilde{c} = \frac{c - e_3}{e_1 - e_3} \quad \nu(c) = -\left(\frac{1 - \tilde{c} + k'}{1 - \tilde{c} - k'}\right)^2 \left(\frac{1 - k'}{1 + k'}\right)^2 \tag{C.15}$$

Elliptic integrals of the first, second and third kind:

$$\begin{aligned} K(k) &= \int_0^1 \frac{dx}{[(1 - x^2)(1 - k^2x^2)]^{1/2}} \\ E(k) &= \int_0^1 dx \left(\frac{1 - k^2x^2}{1 - x^2}\right)^{1/2} \\ \Pi_1(\nu, k) &= \int_0^1 \frac{dx}{[(1 - x^2)(1 - k^2x^2)]^{1/2}(1 + \nu x^2)} \end{aligned} \tag{C.16}$$

The corresponding integrals $I_i^{(2)}$ over the cycle γ_2 are obtained by exchaning in $I_i^{(1)}$ e_1 and e_3.

The massive $N_f = 1$ SW periods also satisfy the Picard-Fuchs equation [109]

$$\begin{aligned} &\frac{\partial^3 \Pi^{(0)}(u,m)}{\partial u^3} + \frac{81\Lambda_1^6 - 2048m^4u - 384\Lambda_1^3m^3 + 3840m^2u^2 - 1536u^3}{(4m^2 - 3u)\left(27\Lambda_1^6 + 256u^2\left(u - m^2\right) + 32\Lambda_1^3 m\left(8m^2 - 9u\right)\right)}\frac{\partial^2 \Pi^{(0)}(u,m)}{\partial u^2} \\ &- \frac{8\left(32m^4 - 72m^2u + 9\Lambda_1^3 m + 24u^2\right)}{(4m^2 - 3u)\left(27\Lambda_1^6 + 256u^2\left(u - m^2\right) + 32\Lambda_1^3 m\left(8m^2 - 9u\right)\right)}\frac{\partial \Pi^{(0)}(u,m)}{\partial u} = 0 \end{aligned} \tag{C.17}$$

with boundary conditions

$$a^{(0)}(u,m,\Lambda_1) \simeq \left[\sqrt{\frac{u}{2}} - \frac{m}{2^4\sqrt{2}u^{3/2}}\Lambda_1^3\right] \qquad u\to\infty$$

$$a_D^{(0)}(u,m,\Lambda_1) \simeq -i\frac{1}{2\pi\sqrt{2}}\left[\sqrt{2}a^{(0)}(u,m,\Lambda_1)\left(-i\pi - 3\ln\frac{16u}{\Lambda_1^2}\right) + 6\sqrt{u} + \frac{m^2}{\sqrt{u}} + \frac{\frac{m^4}{6} - \frac{m}{4}\Lambda_1^3}{u^{3/2}}\right] \qquad u\to\infty \tag{C.18}$$

Notice however that the periods $a^{(0)}$ and $a_D^{(0)}$ so defined are in principle different from the periods $a_1^{(0)}$ and $a_2^{(0)}$ defined as integrals. They are in fact linear combinations of each other, which also possible separate mass term contribution.

For $N_f = 2$ we have similarly (in the cubic SW curve conventions [84])

$$y_{SW}^2 = x^3 - ux^2 - \frac{\Lambda_2^4}{64}(x-u) + \frac{\Lambda_2^2}{4}m_1m_2x - \frac{\Lambda_2^4}{64}(m_1^2+m_2^2)\,. \tag{C.19}$$

$$\begin{aligned}\lambda &= -\frac{\sqrt{2}}{4\pi}\frac{dx}{y_{SW}}\left[x - u - \frac{\Lambda_2^2}{16}\frac{(m_1-m_2)^2}{x-\frac{\Lambda_2^2}{8}} + \frac{\Lambda_2^2}{16}\frac{(m_1+m_2)^2}{x+\frac{\Lambda_2^2}{8}}\right] \\ &= -\frac{\sqrt{2}}{4\pi}\frac{y_{SW}\,dx}{x^2-\frac{\Lambda_2^4}{64}}\end{aligned} \tag{C.20}$$

$$\int\lambda = \frac{\sqrt{2}}{4\pi}\left[\frac{4}{3}uI_1 - 2I_2 + \frac{\Lambda_2^2}{8}(m_1-m_2)^2I_3\left(\frac{\Lambda_2^2}{8}-\frac{u}{3}\right) - \frac{\Lambda_2^2}{8}(m_1+m_2)^2I_3\left(-\frac{\Lambda_2^2}{8}-\frac{u}{3}\right)\right]\,. \tag{C.21}$$

C.3. Relations between alternatively defined periods

We show now the relation between $a^{(0)}, a_D^{(0)}$ and $a_1^{(0)}, a_2^{(0)}$ in the massless case. Assuming $u > 0$ and with small $|u|$ we have

$$\begin{aligned}a^{(0)}(u) &= a_1^{(0)}(u) \qquad \Re a^{(0)}(u) > 0 \\ a_D^{(0)}(u) &= -a_2^{(0)}(u)) \qquad \Re a_D^{(0)}(u) < 0 \\ a^{(0)}(e^{2\pi i/3}u) &= a_1^{(0)}(e^{2\pi i/3}u) - a_2^{(0)}(e^{2\pi i/3}u) \\ a_D^{(0)}(e^{2\pi i/3}u) &= -a_1^{(0)}(e^{2\pi i/3}u) + 2a_2^{(0)}(e^{2\pi i/3}u) \\ a^{(0)}(e^{-2\pi i/3}u) &= a_1^{(0)}(e^{-2\pi i/3}u) - a_2^{(0)}(e^{-2\pi i/3}u) \\ a_D^{(0)}(e^{-2\pi i/3}u) &= -a_2^{(0)}(e^{-2\pi i/3}u)\end{aligned} \tag{C.22}$$

with their inverses

$$\begin{aligned}a_1^{(0)}(u) &= a^{(0)}(u) & \Re a_1^{(0)}(u) > 0 \\ a_2^{(0)}(u) &= -a_D^{(0)}(u) & \Re a_2^{(0)}(u) > 0 \\ a_1^{(0)}(e^{2\pi i/3}u) &= a_D^{(0)}(e^{2\pi i/3}u) + 2a^{(0)}(e^{2\pi i/3}u) & \Re e^{2\pi i/3}a_1^{(0)}(e^{2\pi i/3}u) < 0 \\ a_2^{(0)}(e^{2\pi i/3}u) &= a_D^{(0)}(e^{2\pi i/3}u) - a^{(0)}(e^{2\pi i/3}u) & \Re e^{2\pi i/3}a_2^{(0)}(e^{2\pi i/3}u) > 0 \\ a_1^{(0)}(e^{-2\pi i/3}u) &= a^{(0)}(e^{-2\pi i/3}u) - a_D^{(0)}(e^{-2\pi i/3}u) & \Re e^{-2\pi i/3}a_1^{(0)}(e^{-2\pi i/3}u) < 0 \\ a_2^{(0)}(e^{-2\pi i/3}u) &= -a_D^{(0)}(e^{-2\pi i/3}u) & \Re e^{-2\pi i/3}a_2^{(0)}(e^{-2\pi i/3}u) > 0\end{aligned} \tag{C.23}$$

Also

$$\begin{aligned} a^{(0)}(-u) &= -a_1^{(0)}(-u) + a_2^{(0)}(-u) \\ a_D^{(0)}(-u) &= 3a_1^{(0)}(-u) - 2a_2^{(0)}(-u) \\ a^{(0)}(-e^{2\pi i/3}u) &= a_2^{(0)}(-e^{2\pi i/3}u) \\ a_D^{(0)}(-e^{2\pi i/3}u) &= -a_1^{(0)}(-e^{2\pi i/3}u) + a_2^{(0)}(-e^{2\pi i/3}u) \\ a^{(0)}(-e^{-2\pi i/3}u) &= a_2^{(0)}(-e^{-2\pi i/3}u) \\ a_D^{(0)}(-e^{-2\pi i/3}u) &= a_1^{(0)}(-e^{-2\pi i/3}u) - 2a_2^{(0)}(-e^{-2\pi i/3}u) \end{aligned} \tag{C.24}$$

In the massive case, similar relations can be found by looking at the large u asymptotics (C.18) and, if the small u region is of interest, also to the continuous behaviour of the functions involved.

D. Connection to Heun equations

D.1. Doubly confluent Heun equation

Let us now show that the equations for $N_f = 0, 1, 2$ are just particular cases of the *doubly confluent Heun* equation[20]:

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z^2} + \frac{\delta}{z} + \epsilon\right)\frac{dw}{dz} + \frac{\alpha z - q}{z^2}w = 0 \tag{D.1}$$

It's general solution is given by Mathematica as

$$w = c_1 \text{HeunD}[q, \alpha, \gamma, \delta, \epsilon, z] + c_2 z^{2-\delta} e^{\frac{\gamma}{z} - z\epsilon} \text{HeunD}[\delta + q - 2, \alpha - 2\epsilon, -\gamma, 4 - \delta, -\epsilon, z] \tag{D.2}$$

It is enough to just change variable as $z = e^y$

$$\frac{d^2w}{dy^2} + (\delta + \gamma e^{-y} + e^y\epsilon - 1)\frac{dw}{dy} + (\alpha e^y - q)w = 0 \tag{D.3}$$

and transforming the solution as

$$\psi(y) = \exp\left\{\frac{1}{2}\left(\gamma e^{-y} + (1-\delta)y - \epsilon e^y\right)\right\} w(y) \tag{D.4}$$

to get

$$\frac{d^2\psi}{dy^2} - \frac{1}{4}\left[\gamma^2 e^{-2y} + 2\gamma(\delta - 2)e^{-y} + \left(2\gamma\epsilon + (\delta-1)^2 + 4q\right) + e^y(2\delta\epsilon - 4\alpha) + \epsilon^2 e^{2y}\right]\psi(y) = 0 \tag{D.5}$$

By comparing with the quantum SW curve for $N_f = 2$

$$-\hbar^2 \frac{d^2}{dy_2^2}\psi + \left(\frac{\Lambda_2^2}{16}e^{2y_2} + \frac{\Lambda_2 m_1}{2}e^{y_2} + \frac{\Lambda_2 m_2}{2}e^{-y_2} + \frac{\Lambda_2^2}{16}e^{-2y_2} + u\right)\psi = 0 \tag{D.6}$$

[20] in the Mathematica's notation, let $\delta \leftrightarrow \gamma$ and set $\epsilon = 1$

we get the parameter dictionary

$$\begin{aligned}
&\gamma = \pm\frac{\Lambda_2}{2\hbar} \qquad \epsilon = \frac{\Lambda_2}{2\hbar} \\
&\delta = \frac{2(1 \pm m_2)}{\hbar} \\
&\alpha = \frac{1}{2\hbar^2}(\Lambda_2\hbar - m_1\Lambda_2 \pm m_2\Lambda_2) \\
&q = \frac{1}{8\hbar^2}[-2\hbar^2 + 8u - 8m_2^2 \mp 8m_2\hbar \mp \Lambda_2^2]
\end{aligned} \tag{D.7}$$

or

$$\begin{aligned}
&\gamma = \pm\frac{\Lambda_2}{2\hbar} \qquad \epsilon = -\frac{\Lambda_2}{2\hbar} \\
&\delta = \frac{2(1 \pm m_2)}{\hbar} \\
&\alpha = \frac{1}{2\hbar^2}(-\Lambda_2\hbar - m_1\Lambda_2 \mp m_2\Lambda_2) \\
&q = \frac{1}{8\hbar^2}[-2\hbar^2 + 8u - 8m_2^2 \mp 8m_2\hbar \pm \Lambda_2^2]
\end{aligned} \tag{D.8}$$

By comparing with the quantum SW curve for $N_f = 1$ with $y \to -y_1$

$$-\hbar^2\frac{d^2}{dy_1^2}\psi + \left(\frac{\Lambda_1^2}{4}e^{-y_1} + \Lambda_1 m_1 e^{y_1} + \frac{\Lambda_1^2}{4}e^{2y_1} + u\right)\psi = 0 \tag{D.9}$$

we get the parameter dictionary

$$\begin{aligned}
&\gamma = \pm\frac{\Lambda_1}{\hbar} \\
&\epsilon = 0 \\
&\delta = \frac{2(\hbar \pm m_1)}{\hbar} \\
&\alpha = -\frac{\Lambda_1^2}{4} \\
&q = \frac{1}{4\hbar^2}[-\hbar^2 + 4u - 4m_1^2 \mp 4m_1\hbar]
\end{aligned} \tag{D.10}$$

By comparing with the quantum SW curve for $N_f = 0$, after also change of variable $y \to y_0/2$[21]

$$-\hbar^2\frac{d^2}{dy_0^2}\psi + \left(\frac{\Lambda_0^2}{2}e^{y_0} + \frac{\Lambda_0^2}{2}e^{-y_0} + u\right)\psi = 0 \tag{D.11}$$

$$\begin{aligned}
&\gamma = \pm\frac{2\sqrt{2}\Lambda_0}{\hbar} \qquad \epsilon = \frac{2\sqrt{2}\Lambda_0}{\hbar} \qquad \alpha = \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
&q = \frac{1}{4\hbar^2}[-\hbar^2 \mp 16\Lambda_0^2 + 16u] \qquad \delta = 2
\end{aligned} \tag{D.12}$$

[21] Notice though that as for the $N_f = 1, 2$ theories in this paper, with respect to $N_f = 0$ in [1] we use make the rescaling $\hbar \to \sqrt{2}\hbar$.

or

$$\begin{aligned}
&\gamma = \pm\frac{2\sqrt{2}\Lambda_0}{\hbar} \qquad \epsilon = -\frac{2\sqrt{2}\Lambda_0}{\hbar} \qquad \alpha = -\frac{2\sqrt{2}\Lambda_0}{\hbar} \\
&q = \frac{1}{4\hbar^2}[-\hbar^2 \pm 16\Lambda_0^2 + 16u] \qquad \delta = 2
\end{aligned} \tag{D.13}$$

D.1.1. Alternative form

In the book on Heun equations [77] it is given another form for the doubly confluent Heun equation, namely

$$z\frac{d}{dz}z\frac{d}{dz}w + \alpha\left(z+\frac{1}{z}\right)z\frac{d}{dz}w + \left[(\beta_1+\frac{1}{2})\alpha z + \left(\frac{\alpha^2}{2}-\gamma\right) + (\beta_{-1}-\frac{1}{2})\frac{\alpha}{z}\right]w = 0 \tag{D.14}$$

Transforming in normal form, then changing variable as $z = e^y$ and transforming again into normal form we get

$$-\frac{d^2}{dy^2}\psi + \left(\gamma + \frac{1}{4}\alpha^2 e^{-2y} + \frac{1}{4}\alpha^2 e^{2y} - \alpha\beta_{-1}e^{-y} - \alpha\beta_1 e^y\right)\psi = 0 \tag{D.15}$$

We have

$$w(z) = e^{-\frac{\alpha}{2}\left(z-\frac{1}{z}\right)}\psi(y) \tag{D.16}$$

We get the parameters map for $N_f = 2$

$$\alpha = \pm\frac{\Lambda_2}{2\hbar} = \pm 2e^{\theta} \qquad \beta_1 = \mp\frac{m_1}{\hbar} = \mp M_1 \qquad \beta_{-1} = \mp\frac{m_2}{\hbar} = \mp M_1 \qquad \gamma = \frac{u}{\hbar^2} = P^2 \tag{D.17}$$

The authors [77] in particolar have solutinos corresponding to the lower sign convention

$$w_{\infty,1}(y) \simeq (-2e^{\theta+y})^{-(\frac{1}{2}+M_1)} \simeq e^{e^{\theta+y}}e^{-i\pi(\frac{1}{2}+M_1)}\psi_{+,0}(y) \qquad y\to+\infty \tag{D.18}$$

$$w_{\infty,2}(y) \simeq e^{2e^{\theta+y}}(-2e^{\theta+y})^{M_1-\frac{1}{2}} \simeq e^{e^{\theta+y}}\psi_{+,1} \qquad y\to+\infty \tag{D.19}$$

with

$$W[w_{\infty,2}, w_{\infty,1}] = 1 \tag{D.20}$$

Define

$$\lambda = \gamma - \alpha^2/2 \tag{D.21}$$

The DCHE has a countable number of eigenvalues, denoted $\lambda_\mu(\alpha,\beta)$ with

$$\mu \in \nu + \mathbb{Z} \tag{D.22}$$

where ν is the Floquet characteristic exponent. The eigenvalues have expansion

$$\lambda_\mu(\alpha,\beta) = \mu^2 + \sum_{m=1}^{\infty}\lambda_{\mu,m}(\beta)\alpha^{2m}\,. \tag{D.23}$$

The first coefficient is [77]

$$\lambda_{\mu,1}(\beta) = -\frac{1}{2} + \frac{2\beta_{-1}\beta_1}{4\mu^2-1}\,. \tag{D.24}$$

D.2. Confluent Heun equation

We also connect the equation for $N_f = 3$ to *confluent Heun* equation and $N_f = 4$ ($s = 1, 2$) to *Heun* equation...

Let us now show that the equations for $N_f = 0, 1, 2$ are just particular cases of the *doubly confluent Heun* equation[22]:

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)}w = 0 \tag{D.25}$$

It's general solution is given by Mathematica as

$$w = c_1 \text{HeunC}[q, \alpha, \gamma, \delta, \epsilon, z] + c_2 z^{1-\gamma}\text{HeunC}[(1-\gamma)(\epsilon-\delta) + q, \alpha + (1-\gamma)\epsilon, 2-\gamma, \delta, \epsilon, z] \tag{D.26}$$

It is enough to just change variable as $z = e^y$

$$w''(y) + \frac{(-\gamma + e^y(\gamma + \delta + (e^y - 1)\epsilon - 1) + 1)}{e^y - 1}w'(y) + \frac{e^y(\alpha e^y - q)}{e^y - 1}w(y) = 0 \tag{D.27}$$

and transform the solution as

$$\psi(y) = (1 - e^y)^{-\frac{\delta}{2}} \exp\left\{\frac{1}{2}[(1-\gamma)y - \epsilon e^y]\right\} w(y) \tag{D.28}$$

The equation becomes

$$\begin{aligned}&\frac{d^2}{dy^2}\psi(y) - \frac{(\gamma-1)^2 + e^{2y}(4\alpha - 2\epsilon(2\gamma+\delta) + (\gamma+\delta-1)^2 + 4q + \epsilon^2) - 2e^y(\gamma(\gamma+\delta-\epsilon-2) + 2q + 1)}{4(e^y-1)^2}\psi(y)\\ &- \frac{-2e^{3y}(2\alpha - \epsilon(\gamma+\delta) + \epsilon^2) + e^{4y}\epsilon^2}{4(e^y-1)^2}\psi(y) = 0\end{aligned} \tag{D.29}$$

Send $y \to -y$ and then $y \to y - \ln 2 + \frac{1}{2}\ln\Lambda_3$ then we obtain the quantum SW curve for $N_f = 3$ with

$$\begin{aligned}\gamma &= 1 + m1 - m2\\ \delta &= \sqrt{m_1^2 + 2m_1m_2 + m_2^2 - \hbar^2 + 1} + 1\\ \epsilon &= \frac{\Lambda_3}{4}\\ \alpha &= \frac{1}{8}\Lambda_3(\delta + m_1 - m_2 - 2m_3 + 1)\\ q &= \frac{1}{8}\left(-4\delta + \Lambda_3 - 4m_1^2 - 4\delta m_1 + \Lambda_3 m_1 - 4m_2^2 + 4\delta m_2 - \Lambda_3 m_2 - \Lambda_3 m_3 + 8u + 2\hbar^2\right)\end{aligned} \tag{D.30}$$

[22] in the Mathematica's notation, let $\delta \leftrightarrow \gamma$ and set $\epsilon = 1$

E. Numerical wave functions

$$-\frac{d^2}{dy_0^2}\psi + \left[e^{2\theta}(e^y + e^{-y}) + P^2\right]\psi = 0 \tag{E.1}$$

its general solution is given in terms of even C and odd S Mathieu functions as

$$\psi_0 = c_1 C\left[4P^2, -4e^{2\theta}, \frac{iy}{2}\right] - c_2 S\left[4P^2, -4e^{2\theta}, \frac{iy}{2}\right] \tag{E.2}$$

The solutions $\psi_{0,\pm,0}$ are defined as

$$\begin{aligned} &\lim_{y\to-\infty}\psi_{0,-,0}(y) = 0\,, \quad \lim_{y\to-\infty}\frac{d\psi_{0,-,0}(y)}{dy} = 0 \\ &\lim_{y\to+\infty}\psi_{0,+,0}(y) = 0\,, \quad \lim_{y\to+\infty}\frac{d\psi_{0,+,0}(y)}{dy} = 0 \end{aligned} \tag{E.3}$$

So they are given by

$$\begin{aligned} \psi_{0,+,0} &= c_1(\theta, P)\left[iC\left(4P^2, -4e^{2\theta}, \frac{iy}{2}\right) + S\left(4P^2, -4e^{2\theta}, \frac{iy}{2}\right)\right] \\ \psi_{0,-,0} &= c_1(\theta, P)\left[iC\left(4P^2, -4e^{2\theta}, \frac{iy}{2}\right) - S\left(4P^2, -4e^{2\theta}, \frac{iy}{2}\right)\right] \end{aligned} \tag{E.4}$$

We see the behaviour (E.3) confirmed numerically. In particular we find that $\psi_{0,\pm,0}$ is many orders of magnitude smaller at $y_0 \to \pm\infty$ than at $y_0 \to \mp\infty$. However, the values we numerically get are always large because the Mathieu functions C and S are defined to be divergent at infinity and so subtracting them even when they asymptotic to each other gives a large difference because of the finite number of digits used.

We notice however that the normalization is not fixed. Even computing the wronskian gives just a functional relation for the normalization

$$\begin{aligned} W[\psi_{0,-,1}, \psi_{0,-,0}] = -i = -ic_0(\theta, P)c_0(\theta + i\pi/2, P)\Big\{ &W[C\left(4P^2, -4e^{2\theta}, \frac{iy}{2}\right), S\left(4P^2, 4e^{2\theta}, \frac{i(y - i\pi)}{2}\right)] \\ &- W[C\left(4P^2, 4e^{2\theta}, \frac{i(y - i\pi)}{2}\right), S\left(4P^2, -4e^{2\theta}, \frac{iy}{2}\right)]\Big\} \end{aligned} \tag{E.5}$$

This normalization problem is avoided by taking the logarithmic derivative of ψ and studying the solution of the Riccati equation $\mathcal{P}$ as we do in the main text.

For instance we get the Floquet exponent as

$$2\pi i\nu = \ln\frac{\psi_{0,+,0}(y + 2\pi i)}{\psi_{0,+,0}(y)} = -\ln\frac{\psi_{0,-,0}(y + 2\pi i)}{\psi_{0,-,0}(y)} \tag{E.6}$$

so we can identify the solution $\psi_{0,\pm,0}$ with the positive (negative) Floquet solution

$$\psi_{0,\pm,0}(y + 2\pi i) = e^{\pm 2\pi i\nu}\psi_{0,\pm,0}(y) \tag{E.7}$$

If we used instead the doubly confluent Heun function given by Mathematica we would not see this $y \to y + 2\pi i$ Floquet monodromy since those functions are defined for $z = e^y \in \mathbb{C}$ rather than on the Riemann surface.

F. Floquet exponent through Hill determinant

Consider the more general $N_f = 2$ equation and change variable as $z = iy$. We get

$$\frac{d^2}{dz^2}\psi + [\theta_0 + \theta_2 e^{2iz} + \theta_{-2}e^{-2iz} + \theta_1 e^{iz} + \theta_{-1}e^{-iz}]\psi = 0 \tag{F.1}$$

with

$$\theta_0 = P^2 \qquad \theta_{\pm 2} = e^{2\theta} \qquad \theta_{\pm 1} = 2e^{\theta} q_{1,2} \tag{F.2}$$

We search for Floquet solutions, such that

$$\psi_+(z + 2\pi) = e^{2\pi\nu}\psi_+(z) \qquad \psi_-(z + 2\pi) = e^{-2\pi\nu}\psi_-(z) \tag{F.3}$$

that implies they can be expanded in Fourier series as

$$\psi(z) = e^{\nu z}\sum_{n=-\infty}^{\infty} b_n e^{niz} \tag{F.4}$$

From the equation we get the recursion

$$(\nu + in)^2 b_n + \sum_{m=-2}^{2}\theta_m b_{n-m} = 0 \tag{F.5}$$

Dividing by $\theta_0 - n^2$ we get the matrix with convergent determinant

$$\begin{pmatrix} \vdots & & & & & & \vdots \\ \cdots & \xi_{n,n-1} & 1 & \xi_{n,n+1} & \xi_{n,n+2} & 0 & \cdots \\ \cdots & \xi_{n+1,n-1} & \xi_{n+1,n} & 1 & \xi_{n+1,n+2} & \xi_{n+1,n+3} & \cdots \\ \cdots & 0 & \xi_{n+2,n} & \xi_{n+2,n+1} & 1 & \xi_{n+2,n+3} & \cdots \\ \cdots & 0 & 0 & \xi_{n+3,n+1} & \xi_{n+3,n+2} & 1 & \cdots \\ \vdots & & & & & & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ b_{n-1} \\ b_n \\ b_{n+1} \\ b_{n+2} \\ \vdots \end{pmatrix} = 0. \tag{F.6}$$

with

$$\xi_{mn} = \frac{-\theta_{m-n}}{(m - i\nu)^2 - \theta_0} \qquad \xi_{m,m} = 1 \tag{F.7}$$

Defining $\mathcal{A}_n$ as the finite $2n + 1 \times 2n + 1$ submatrix We also introduce a $(2n + 1) \times (2n + 1)$ matrix

$$\mathcal{A}_n = \begin{pmatrix} 1 & \xi_{-n,-n+1} & \xi_{-n,-n+2} & & & & & & & & & & \\ \xi_{-n+1,-n} & 1 & \xi_{-n+1,-n+2} & & & & & & & & & & \\ \xi_{-n+2,-n} & \xi_{-n+2,-n+1} & 1 & & & & & & & & & & \\ & & & \vdots & \cdots & & & & & & & & \\ & & & & \chi_{-1} & 1 & \xi_{-1} & 0 & 0 & & & & \\ & & & & \xi_{0,-2} & \xi_{0,-1} & 1 & \xi_{0,1} & \xi_{0,2} & & & & \\ & & & & 0 & \xi_{1,-1} & \xi_{1,0} & 1 & \xi_{1,2} & \xi_{1,3} & & & \\ & & & & & & & & & \cdots & \vdots & & \\ & & & & & & & & & & 1 & \xi_{n-2,n-1} & \xi_{n-2,n} \\ & & & & & & & & & & \xi_{n-1,n-2} & 1 & \xi_{n-1,n} \\ & & & & & & & & & & \xi_{n,n-2} & \xi_{n,n-1} & 1 \end{pmatrix} \tag{F.8}$$

and

$$\Delta(i\nu) = \lim_{n\to\infty} \det \mathcal{A}_n \tag{F.9}$$

by ordinary methods [92] we arrive a this relation

$$\Delta(i\nu) = \Delta(0) - \frac{\sin^2(\pi i\nu)}{\sin^2 \pi\sqrt{\theta_0}} \tag{F.10}$$

The Floquet exponent is then given by the roots of the equation

$$\sin^2(\pi i\nu) = \Delta(0)\sin^2 \pi\sqrt{\theta_0} \tag{F.11}$$

or

$$\cosh(2\pi\nu) = 1 - 2\Delta(0)\sin^2 \pi P \tag{F.12}$$

In particular for $N_f = 2$ $\xi_{m,n}$ are given by

$$\xi^{(2)}_{m,m\mp 2} = -\frac{e^{2\theta}}{(m-i\nu)^2 - P^2} \qquad \xi^{(2)}_{m,m\mp 1} = -\frac{2e^{\theta}q_{1,2}}{(m-i\nu)^2 - P^2} \tag{F.13}$$

while for $N_f = 1$

$$\xi^{(1)}_{m,m-2} = -\frac{e^{2\theta}}{(m-i\nu)^2 - P^2} \qquad \xi^{(1)}_{m,m+1} = -\frac{e^{2\theta}}{(m-i\nu)^2 - P^2} \qquad \xi^{(1)}_{m,m-1} = -\frac{2e^{\theta}q_1}{(m-i\nu)^2 - P^2} \tag{F.14}$$

G. Renormalization flow from higher to lower N_f

From the quantum SW curve for $N_f = 4$

$$- \hbar^2 \frac{d^2}{dy_4^2}\psi + \Big\{ - \exp(2y_4)\left(q\left(q\left(m_1^2+m_2^2+m_3^3+m_4^2\right)-24(m_1m_2+m_3m_4)\right)+16(q+4)u\right) \tag{G.1}$$

$$+ 4\sqrt{q}\exp(3y_4)\left(m_1^2q - m_1m_2(q+8)+m_2^2q - m_3m_4q+8u\right) \tag{G.2}$$

$$+ 4\sqrt{q}\exp(y_4)\left(-m_1m_2q+m_3^2q-m_3m_4(q+8)+m_4^2q+8u\right) - \tag{G.3}$$

$$- 4q\exp(4y_4)(m_1-m_2)^2 - 4q(m_3-m_4)^2 \Big\} \frac{\exp(-2y_4)}{4\left(-4\sqrt{q}\cosh(y_4)+q+4\right)^2}\psi + \tag{G.4}$$

$$+ \frac{\hbar^2\left(\sqrt{q}\exp(-y_4)\left(q\exp(2y_4)-8\sqrt{q}\exp(y_4)+4\exp(2y_4)+q+4\right)\right)}{2\left(-4\sqrt{q}\cosh(y_4)+q+4\right)^2}\psi = 0 \tag{G.5}$$

Since we have

$$qm_4 = \Lambda_3 \qquad m_4 \to \infty \qquad q \to 0 \tag{G.6}$$

we can set

$$y_4 = y_3 + \frac{1}{2}\ln\Lambda_3 - \frac{1}{2}\ln q = y_3 + \frac{1}{2}\ln m_4 \to +\infty \tag{G.7}$$

and exchanging the masses $m_3 \leftrightarrow m_2$ we arrive to

$$- \hbar^2 \frac{d^2}{dy_3^2}\psi + \frac{4e^{2y_3}\Lambda_3(m_1-m_3)^2 + 4e^{y_3}\sqrt{\Lambda_3}\left(-2\hbar^2+8m_1m_3+\Lambda_3m_2-8u\right)}{16\left(\sqrt{\Lambda_3}e^{y_3}-2\right)^2}\psi \tag{G.8}$$

$$+ \frac{\Lambda_3^2+64u-24\Lambda_3m_2+4e^{-y_3}\sqrt{\Lambda_3}(8m_2-\Lambda_3)+4\Lambda_3e^{-2y_3}}{16\left(\sqrt{\Lambda_3}e^{y_3}-2\right)^2}\psi = 0 \tag{G.9}$$

which is the qSW curve for $N_f = 3$.

Since we have

$$\Lambda_3 m_3 = \Lambda_2^2 \qquad m_3 \to \infty \qquad \Lambda_3 \to 0 \tag{G.10}$$

we can set

$$y_3 = y_2 - \frac{1}{2}\ln m_3 \to -\infty \tag{G.11}$$

we get

$$- \hbar^2 \frac{d^2}{dy_2^2}\psi + \frac{e^{2y_2}\left(4\Lambda_3\frac{m_1^2}{m_3} - 8\Lambda_3m_1 + 4\Lambda_3m_3\right) + e^{y_2}\left(-8\frac{\sqrt{\Lambda_3}}{\sqrt{m_3}}\hbar^2 + 32\sqrt{\Lambda_3}m_1\sqrt{m_3} + 4\frac{\Lambda_3^{3/2}m_2}{\sqrt{m_3}} - 32\frac{\sqrt{\Lambda_3}u}{\sqrt{m_3}}\right)}{16\left(\frac{\sqrt{\Lambda_3}}{\sqrt{m_3}}e^{y_2}-2\right)^2}\psi \tag{G.12}$$

$$+ \frac{\Lambda_3^2+64u-24\Lambda_3m_2+e^{-y_2}\left(32\sqrt{\Lambda_3}\sqrt{m_3}m_2 - 4\Lambda_3^{3/2}\sqrt{m_3}\right)+4\Lambda_3m_3e^{-2y_2}}{16\left(\frac{\sqrt{\Lambda_3}}{\sqrt{m_3}}e^{y_2}-2\right)^2}\psi = 0 \tag{G.13}$$

which in the limit precisely reduce to

$$-\hbar^2 \frac{d^2}{dy_2^2}\psi + \left[\frac{1}{16}\Lambda_2^2(e^{2y_2}+e^{-2y_2}) + \frac{1}{2}\Lambda_2 m_1 e^{y_2} + \frac{1}{2}\Lambda_2 m_2 e^{-y_2} + u\right]\psi = 0 \tag{G.14}$$

which is the quantum SW curve for $SU(2)$ $N_f = 2 = (1,1)$.

Alternatively, exchange

$$m_3 \leftrightarrow m_2 \tag{G.15}$$

and let

$$y_3 = y_2' - \frac{1}{2}\ln\Lambda_3 + \ln\Lambda_2 = y_2' + \frac{1}{2}\ln m_3 \to +\infty \tag{G.16}$$

$$-\hbar^2 \frac{d^2}{dy_2'^2}\psi + \frac{e^{2y_2'}\Lambda_2^2\left(m_1-m_2\right)^2 + e^{y_2'}\Lambda_2\left(-2\hbar^2+8m_1m_2+\Lambda_2^2-8u\right)+16u-6\Lambda_2^2+8\Lambda_2 e^{-y_2'}}{4\left(\Lambda_2 e^{y_2'}-2\right)^2}\psi = 0 \tag{G.17}$$

which is the quantum SW curve for $SU(2)$ $N_f = 2 = (0,2)$.

Since we have

$$\Lambda_2^2 m_2 = \Lambda_1^3 \qquad m_2 \to \infty \qquad \Lambda_2 \to 0 \tag{G.18}$$

we can set

$$y_2 = y_1 + \frac{1}{2}\ln m_2 \to +\infty \tag{G.19}$$

then the equation becomes

$$-\hbar^2 \frac{d^2}{dy_1^2}\psi + \left[\frac{1}{16}\Lambda_2^2\left(m_2 e^{2y_1} + \frac{1}{m_2}e^{-2y_1}\right) + \frac{1}{2}\Lambda_2\sqrt{m_2}m_1 e^{y_2} + \frac{1}{2}\Lambda_2\sqrt{m_2}e^{-y_2} + u\right]\psi = 0 \tag{G.20}$$

which in the limit reduces to the $N_f = 1$ equation:

$$-\hbar^2 \frac{d^2}{dy_1^2}\psi + \left[\frac{1}{16}\Lambda_1^3 e^{2y_1} + \frac{1}{2}\Lambda_1^{3/2}e^{-y_1} + \frac{1}{2}\Lambda_1^{3/2}m_1 e^{y_1} + u\right]\psi = 0\,. \tag{G.21}$$

Since we have

$$\Lambda_1^3 m_1 = \Lambda_0^4 \qquad m_1 \to \infty \qquad \Lambda_1 \to 0 \tag{G.22}$$

Let

$$y_1 = y_0 - \frac{1}{2}\ln m_1 \to -\infty \tag{G.23}$$

then we get

$$-\hbar^2 \frac{d^2}{dy_0^2}\psi + \left[\frac{1}{16}\frac{\Lambda_1^3}{m_1}e^{2y_0} + \frac{1}{2}\Lambda_1^{3/2}m_1^{1/2}e^{-y_0} + \frac{1}{2}\Lambda_1^{3/2}m_1^{1/2}e^{y_0} + u\right]\psi = 0 \tag{G.24}$$

that precisely reduce to the $N_f = 0$ equation:

$$-\hbar^2 \frac{d^2}{dy_0^2}\psi + (\Lambda_0^2\cosh y_0 + u)\psi = 0\,. \tag{G.25}$$

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