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AUTOMORPHISMS ON ALGEBRAIC VARIETIES: K3 SURFACES, HYPERKÄHLER MANIFOLDS, AND APPLICATIONS ON ULRICH BUNDLES

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Abstract

One of the main tools to study the geometry of complex algebraic varieties is the group of automorphisms. The first part of this thesis concerns the study of symplectic automorphisms of finite order on K3 surfaces, and birational symplectic maps of finite order on projective hyperkähler manifolds which are deformation equivalent to the Hilbert scheme of K3 surfaces. In the second part of this thesis, the automorphism groups of rational homogeneous spaces are used to study Ulrich bundles in smooth projective varieties.

Dedicata alla memoria dei miei genitori

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Introduction

"Seguire la gioia ed evitare la tristezza"
G.W. Leibnitz

In this thesis, we study the geometry of some algebraic varieties admitting a non-trivial automorphism group. For simplicity, we assume that *algebraic varieties* are irreducible algebraic sets in affine or projective space over the complex numbers, and an *automorphism* is a bijective holomorphic map where the inverse is also holomorphic. To go further, we can study the action of (birational maps) automorphisms in cohomology, *or* we can find obstructions on the main object of study that can be reflected in the description of the group of automorphisms. In some cases, the natural action in the singular cohomology of some finite groups of (birational maps) automorphisms determines the geometry of the variety being studied. While in other cases, the presence of a large group of automorphisms characterizes varieties admitting objects (e.g., vector bundles) with specific conditions. It is not our purpose to study the group of automorphisms of some algebraic varieties, instead, we use the action of automorphisms in different contexts, to restrict the geometry of the variety or in some cases the existence of certain objects. In other words, we use the group of automorphisms as one of the main ingredients to solve the question to be addressed.

One of the algebraic varieties that we study in depth are the hyperkähler manifolds. A *hyperkähler manifold* is a smooth simply connected compact Kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2–form ω_X. Two–dimensional examples of hyperkähler manifolds are called K3 surfaces. In fact, a *K3 surface* is defined as a compact complex surface with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$. The unicity of the holomorphic 2–form ω_X (up to scalar multiplication) determines the complex structure of X and its period. The *period* of a hyperkähler manifold is given by the second cohomology group endowed with its Hodge structure and the Beauville–Bogomolov quadratic form. To distinguish two hyperkähler varieties, it is enough to distinguish their complex structures or what is equivalent to distinguish their periods. The group of automorphisms of a hyperkähler manifold plays an important role in the study of some of their geometrical properties. For example, a particular type of automorphisms called *symplectic* are biholomorphic maps that preserve the period of the hyperkähler manifold. This means that the action of symplectic automorphisms on $H^{2,0}(X) = H^0(X, \Omega_X^2)$ is trivial. These particular automorphisms on K3 surfaces have been fully classified, starting from the foundational work of Nikulin [Nik79], Mukai [Muk88] and later by Hashimoto [Has12].

Higher dimensional hyperkähler manifolds share a lot of features with K3 surfaces and in fact the latter were often used as main motivation. As we saw previously, the period of a hyperkähler manifold determines its Hodge structure, which is essentially what Torelli's theorem claims for K3 surfaces and later proved in higher dimensional hyperkähler manifolds but in a *weaker form* (i.e., in terms of monodromy operators). Then, the geometry of a hyperkähler manifold X is encoded in the cohomology group $H^2(X, \mathbb{Z})$. We emphasize that Torelli's theorem on K3 surfaces is much stronger than Torelli's theorem for higher dimensional hyperkähler manifolds in the following way: if the Hodge structures are isometric, then the hyperkähler manifolds are birational, however, in the case of K3 surfaces this implies that they are isomorphic (see Theorem 1.4.9 for a precise statement). The cohomology group $H^2(X, \mathbb{Z})$ of K3 surfaces is isometric to an even unimodular lattice of signature $(3, 19)$. For higher dimensional hyperkähler manifolds, the cohomology group $H^2(X, \mathbb{Z})$ is isometric to a lattice of signature $(3, b_2(X) - 3)$ and not unimodular in general. This lattice theoretical approach is a powerful technique that allows studying many geometrical properties of hyperkähler manifolds in family.

Examples in higher dimensions of hyperkähler manifolds are extremely hard to construct. All the known hyperkähler manifolds are deformation of:

- a Hilbert scheme of a projective K3 surface;
- the generalized Kummer associated to an Abelian surface;
- the O'Grady's 10–dimensional example;
- the O'Grady's 6–dimensional example.

Some hyperkähler manifolds of the first type of deformation come as examples of moduli spaces of (twisted) stable sheaves on projective K3 surfaces. The numerical conditions on the sheaves, encoded in the first and the second Chern class, define good moduli spaces even with models in some projective space. Such moduli spaces reflect the geometry of the underlying K3 surface: they can be used to reveal properties of the K3 surface, or they may be studied as interesting spaces in their own right.

In the first part of this thesis, we adopt the lattice approach introduced by Nikulin in [Nik76] to the study of symplectic automorphisms of finite order on K3 surfaces, and to the study of symplectic birational maps of finite order on hyperkähler manifolds of $K3^{[n]}$–type. The technique that we address for both cases is based on the main facts of the lattice theory. However, the questions we develop converge to two different kind of problems.

K3 surfaces: In [Nik76], Nikulin introduced the finite order symplectic automorphisms of K3 surfaces and studied their properties considering both their action on the surface (in particular determining their fixed locus) and the action that they induce on the second cohomology group of the K3 surfaces. It was proved that both the topology of the fixed locus and the action induced in cohomology are *unique* and depend only on the order of the automorphism, see [Nik76, Theorem 4.7]. One of the main reasons of interest in these particular automorphisms is that their existence induces a natural relation between two different (families of) K3 surfaces: the family of K3 surfaces admitting a symplectic automorphism of order n and the one of the desingularization of the quotient of the K3 surfaces by a symplectic automorphism of order n. The latter surfaces are still K3 surfaces, but in general not isomorphic to the original ones. From [Nik76], [GS07], and [GS08] it is possible to describe the family $\mathcal{S}$ (resp. $\mathcal{T}$) of the projective K3 surfaces X admitting a symplectic automorphism σ_n of order n for prime numbers n (resp. of the projective K3 surfaces Y obtained as desingularization of the quotient of a K3 surface by a symplectic automorphism of order n), in terms of families of the lattice polarized K3 surfaces. Both these families have countable many irreducible components. It remains an open problem to determine the relationship between the components of these two families. More explicitly, if one considers a K3 surface X admitting a symplectic automorphism σ_n of order n, then X is contained in a specific component of $\mathcal{S}$. This information determines the component of $\mathcal{T}$ which contains Y (resolution of $X/\langle\sigma_n\rangle$). By the theory of the moduli space of lattice polarized K3 surfaces in [Dol96], this problem is equivalent to determine the relation between the Néron–Severi group of X and the one of Y.

For symplectic involutions, this problem was resolved thanks to the results contained in [Mor84], [vGS07] and [GS08]. In [Mor84], Morrison studied the isometry induced by a symplectic involution on the lattice $\Lambda_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}$.

The action in the cohomology group of any K3 surface can be described as an action on Λ_{K3}. In fact, Morrison proved that symplectic involutions act as the identity on the direct sum $U^{\oplus 3}$ and switch the two copies of E_8 in the summand $E_8^{\oplus 2}$. Thanks to this result, Morrison identified a nice and strong relation, called Shioda–Inose structure, between certain K3 surfaces and an associated Abelian surface. By using Morrison's result, van Geemen and Sarti described the family $\mathcal{S}$ and the maps π^* and π_*, where π is the rational map $\pi : X \dashrightarrow Y$ induced by the quotient map $X \to X/\langle\sigma_2\rangle$. As a consequence, in [GS07] the description of the family $\mathcal{T}$ and the explicit relation between components of $\mathcal{S}$ and of $\mathcal{T}$ are obtained. This result is applied in [CG20] to construct infinite towers of isogenous K3 surfaces.

The main results in this thesis concerning symplectic automorphisms on K3 surfaces, are generalization of these known results on symplectic automorphisms of order 2 to the order 3 case, see [GPM21]. As mentioned above, the results of the order 2 case mainly depend on the description of the isometry induced by a symplectic involution on the second cohomology group. Hence our first goal is to establish a similar result for the order 3 case. It turns out that the action induced by a symplectic automorphism of order 3 on Λ_{K3} is not compatible with the direct summands $U^{\oplus 3} \oplus E_8^{\oplus 2}$. So we give a different description of the lattice Λ_{K3} and we provide the action on such a lattice, see Theorem 2.2.5:

Main Theorem 1 ([GPM21])**.** *The lattice Λ_{K3} is an overlattice of index 3^2 of the lattice $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$. The action induced by an order 3 symplectic automorphism on $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ is the identity on the summands $A_2(-1) \oplus U$ and a cyclic permutation of the three summands in $E_6^{\oplus 3}$. The action on the overlattice Λ_{K3} is induced by the one on $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ by $\mathbb{Q}$-linear extension.*

This result is achieved in a geometric way: thanks to [Nik76, Theorem 4.7] the isometry induced on Λ_{K3} by a specific symplectic automorphism σ of order 3 on a specific K3 surface X is essentially unique (i.e. it does not depend on X and σ). This allows us to describe the isometry induced by any automorphism of order 3 on the lattice Λ_{K3}.

In Chapter 2, we describe a particular K3 surface S with a specific automorphism σ of order 3: the surface S is a classic K3 surface of Picard number 20 studied in [SI77] and [Vin83], and σ is a symplectic automorphism of order 3 that acts on the generators classes of the Néron–Severi group of S by permuting three copies of E_6. In Chapter 3, we describe the maps π_* and π^* induced by the quotient map, while in Chapter 4, we describe the families $\mathcal{S}$ and $\mathcal{T}$, listing all their components, see Theorems 4.1.4 and 4.2.4 respectively. In Chapter 5, merging the results on the families $\mathcal{S}$ and $\mathcal{T}$ with the ones about the maps π^* and π_*, we deduce the relations between the components of $\mathcal{S}$ and the ones of $\mathcal{T}$ and in particular we relate the Néron–Severi group of a projective K3 surface X admitting a symplectic automorphism σ of order 3 with the one of the desingularization Y of $X/\langle\sigma\rangle$. This result is proved under the assumptions of generality for X and Y, it is stated in Theorem 5.1.2 and it can be summarized in the following theorem:

Main Theorem 2 ([GPM21])**.** *Let X be a projective K3 surface admitting an order 3 symplectic automorphism σ, then $\rho(X) \geq 13$ and if $\rho(X) = 13$ the Néron–Severi group of X determines uniquely the one of the minimal resolution of $X/\langle\sigma\rangle$ and viceversa.*

The application of these results to specific families of K3 surfaces allows us to exhibit equations of families of projective K3 surfaces admitting symplectic automorphisms of order 3 and of their quotients in Section 5.3. For example we show that there is exactly one component of $\mathcal{S}$ which corresponds to a family of quartic hypersurfaces in $\mathbb{P}^3$ admitting a symplectic automorphism of order 3 (and in this case the quotient is a singular surface in $\mathbb{P}^7$). But there are two different components of $\mathcal{S}$ which correspond to two families of complete intersections of type $(2,3)$ in $\mathbb{P}^4$ admitting a symplectic automorphism of order 3. The quotient of the surfaces contained in one of these components is a singular double cover of $\mathbb{P}^2$ and it corresponds to a component in $\mathcal{T}$. The quotient of the surfaces in the other case has a singular model in $\mathbb{P}^{10}$ and it corresponds to another component in $\mathcal{T}$.

In Chapter 6, we briefly present two applications of the Main Theorems 1 and 2. By Main Theorem 2, we are able to construct infinite towers of isogenous K3 surfaces by considering iterated quotients by symplectic automorphisms of order 3. The analogous construction which uses symplectic involutions was presented in [CG20]. By Main Theorem 1, the definition and previous results on Classical Shioda–Inose structures in [Mor84] can be generalized to symplectic automorphisms of order 3 of K3 and Abelian surfaces.

Hyperkähler manifolds of K3$^{[n]}$–type: In higher dimensional hyperkähler manifolds, the maps preserving the period can be also obtained by birational maps. The induced maps in cohomology of symplectic birational maps have a similar behavior of induced maps in cohomology of symplectic automorphisms. When a hyperkähler manifold is deformation equivalent to a Hilbert scheme of n

points on a K3 surface, symplectic automorphisms are actually characterized as subgroups of Conway's group, see [Mon16]. Moreover, the action of these groups in cohomology can be explicitly described by studying these actions on the Leech lattice. In [HM16], Höhn and Mason computed the orbit of fixed–point sublattices of the Leech lattice with respect to the action of the Conway group. This allows to classify all finite automorphism groups of hyperkähler manifolds preserving the period. Let X be a projective hyperkähler manifold of K3$^{[n]}$–type. Markman in [Mar10] proved that the Monodromy group of X is the subgroup of isometries of $H^2(X,\mathbb{Z})$ preserving the orientation and acting via multiplication by Id or $-\operatorname{Id}$ on the discriminant group $A_X = H^2(X,\mathbb{Z})^\vee/H^2(X,\mathbb{Z})$.

In this thesis, we study symplectic birational maps of finite order admitting a non-trivial action on the discriminant group A_X. Some examples of hyperkähler manifolds with this kind of maps were provided by Markman in [Mar13] as moduli spaces of sheaves of general K3 surfaces (i.e., a K3 surface with Picard number one) with a particular Mukai vector. Moreover, these examples show evidence that any symplectic birational map with non-trivial action on A_X can be obtained as the composition of an involution with non-trivial action on A_X and a map with a trivial action on A_X. The involutions in these particular examples are described by reflection maps of classes in cohomology associated to divisors which are not prime exceptional. Our main result in this thesis on the study of these maps can be summarized as follows:

Main Theorem 3. *Let X be a projective hyperkähler manifold of K3$^{[n]}$–type admitting a symplectic birational map of finite order with a non-trivial action on A_X. Then, X is birational to a moduli space of (twisted) sheaves on a K3 surface.*

The result imposes strict conditions in the moduli space of hyperkähler admitting symplectic groups (i.e., groups of isometries preserving the period). In chapter 7, we provide several properties of this kind of birational maps. Most of these results were obtained from a theoretical lattice viewpoint. Unfortunately, it does not give a big contribution to the problem of factorization of symplectic birational maps presented above. We expect that at least from a lattice theoretical viewpoint, the existence of the involution is strictly related with the existence of hyperbolic 2-elementary lattices (see Proposition 7.1.4).

Another type of varieties that we study are smooth projective varieties admitting non-trivial vector bundles with a large number of vanishing cohomology groups. One approach to the study of these vector bundles comes with a flavor in commutative algebra. In fact, the vector bundles that we are interested (i.e., Ulrich bundles) were introduced in an algebraic setting, see [Ulr84]. Let $X \subset \mathbb{P}^N$ be a smooth projective variety. We say that a vector bundle E of rank r is an *Ulrich bundle* on X if E admits a linear resolution of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-N+n)^{\oplus a_{N-n}} \to \cdots \to \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus a_1} \to \mathcal{O}_{\mathbb{P}^N}^{\oplus a_0} \to E \to 0.$$

In particular, Ulrich bundles on X are arithmetically Cohen-Macaulay (aCM for short), i.e., $H^i(X, E(jH)) = 0$ for all $j \in \mathbb{Z}$ and $1 \leq i \leq n$ which are objects intensely studied in algebraic geometry. Hence, Ulrich bundles can be roughly defined as aCM vector bundles on a projective variety that have the largest permitted number of global sections. The first geometric manifestations of Ulrich bundles go back to seminal works such as [Bea00, ES03], where the authors relate in a very precise way the existence of such bundles on X to some measurement of the complexity of the underlying polarized variety $(X, \mathcal{O}_X(1))$. In [Bea00], Beauville made a systematically study of the relation between Ulrich bundles and determinantal representation of hypersurfaces (i.e., write a homogeneous polynomial f as the determinant of a matrix with linear entries) which was a problem first stated by Hesse in [Hes55].

The existence of an Ulrich bundle on a projective variety has strong consequences for the underlying variety. For instance, we have the following (non-exhaustive) list of smooth projective complex varieties that are known to support Ulrich bundles:

- Algebraic curves. See [ES03, Section 4].
- Complete intersections in the projective space. See [BHU87, HUB91].
- Veronese varieties. See [ES03, Section 5].
- Grassmannians and, more generally, many rational homogeneous spaces of Picard number one carry an equivariant Ulrich bundle. See [CMR15, Fon16, LP21].
- Many geometrically ruled surfaces, and del Pezzo surfaces. See [ES03, Cas17, ACMR18, Cas19, ACC+20, Section 6] .
- Abelian surfaces, bi-elliptic surfaces, K3 surfaces and Enriques surfaces. See [Bea16, AFO17, Cas17, Bea18, BN18, Fae19].
- Elliptic and some quasi-elliptic surfaces. See [MRPL19, Lop21].

- Some surfaces of general type. See [Cas17, Bea18, Cas18, Cas19, Lop19, Lop21].

- Some Fano threefolds of Picard number one. See [Bea18].

There are several techniques that have been developed in order to produce Ulrich bundles. Among them, we can mention the use of tools from commutative algebra (cf. [HUB91]), representation theory and Borel–Bott–Weil theorem (cf. [ES03]), the study of some Noether–Lefschetz loci (cf. [AFO17]), the Hartshorne–Serre construction (cf. [Bea18, Section 6]), and the use of the Cayley–Bacharach property for suitable zero-dimensional subschemes on surfaces (cf. [Bea18, Section 5]). Additionally, these techniques have been modified in order to produce certain torsion-free sheaves which, in some cases, can be deformed inside their corresponding moduli space of semistable sheaves to obtain the desired Ulrich bundles (cf. [Fae19]).

In practice, many of these constructions are not explicit, and they depend on some choices (e.g., suitable codimension two subschemes to be used in the Cayley–Bacharach or Hartshorne–Serre construction). Because of this, even for varieties where the existence of Ulrich bundles is known to be true, it is a natural and challenging problem to classify Ulrich bundles with fixed numerical invariants and to determine the *Ulrich complexity* of a given smooth projective variety X (i.e., the minimum integer $r \in \mathbb{N}_{\geq 1}$ such that there exists a rank r Ulrich bundle on X). See [BES17, FK20] for some results towards the Ulrich complexity of smooth hypersurfaces.

In this thesis, we adopt the new approach to the study of Ulrich bundles that was recently initiated by Lopez and his collaborators in [Lop20, LMn21, LS21]. More precisely, they study the positivity properties of Ulrich bundles and give classification results for projective varieties carrying Ulrich bundles for which these positivity conditions fail. Along the same lines, it is natural to try to understand (by means of positivity techniques) smooth projective manifolds that enjoy the property of having Ulrich bundles that are canonically attached to them.

The main results in this thesis on Ulrich bundles state that if $X \subseteq \mathbb{P}^N$ is a smooth projective variety of dimension $n \geq 1$, then the cotangent bundle Ω^1_X is never an Ulrich bundle and for the tangent bundle we have the following:

Main Theorem 4 ([MPTB21]). *Let X be a smooth projective variety of dimension $n \geq 1$. If the tangent bundle T_X is an Ulrich bundle, then X is isomorphic to the twisted cubic in $\mathbb{P}^3$ or to the Veronese surface in $\mathbb{P}^5$.*

Our main inspiration comes from the systematic study of the positivity of the tangent bundle, initiated by the solutions of Mori [Mor79] and Siu–Yau [SY80] to the Hartshorne and Frankel conjectures and pushed further by many authors in order to give structure results for manifolds whose tangent bundle satisfies weaker positivity assumptions. It turns out that smooth projective varieties with Ulrich tangent bundle fit very well into this picture, and we show that they are rational homogeneous spaces with rather a large automorphism group.

A first ingredient for our analysis, that we believe is interesting in its own right, is that there are many numerical restrictions on the Chern classes of Ulrich bundles. This has already been observed in the case of surfaces (cf. [ES03, Section 6]), and used notably by Casnati (cf. [Cas17, Cas19]) in order to give a numerical characterization of Ulrich bundles on surfaces. We extend this characterization to the case of threefolds in Proposition 8.3.1, and we observe in Lemma 8.2.3 a useful restriction concerning the first Chern class of Ulrich bundles in any dimension.

In Chapter 8 we revisit some known results concerning the Chern classes of Ulrich bundles, and we prove Proposition 8.3.1 and Lemma 8.2.3 reported above. In Chapter 9, we prove Main Theorem 4 and Theorem 9.3.2. To do so, we carry out an analysis depending on the dimension in Section 9.1, Section 9.2 and Section 9.3 to reduce the problem to analyze higher dimensional varieties of Picard rank at least two. This last case can be settled by means of Lie algebra computations.

Contents

1 | Basic Notions

1.1 Lattice Theory

Definitions and examples

A free $\mathbb{Z}$-module L of finite rank is a **lattice** if it is endowed with a non degenerate symmetric $\mathbb{Z}$-bilinear form

$$(\ \cdot\ )_L : L \times L \longrightarrow \mathbb{Z}.$$

Two lattices L and K are **isometric** if there is an isomorphism φ of L onto K which preserves the bilinear forms, i.e., $(x \cdot y)_L = (\varphi(x) \cdot \varphi(y))_K$. The map φ is called an **isometry** and the **group of isometries** of L is denoted by $\mathrm{O}(L)$.

A lattice L is **even** if $(x \cdot x)_L \in 2\mathbb{Z}$ for all $x \in L$. The **signature** of L is a pair $(l_{(+)}, l_{(-)})$ where $l_{(\pm)}$ denotes the multiplicity of the eigenvalue ± 1 for the $\mathbb{R}$-bilinear form of $L \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $L^\vee$ the dual module $\mathrm{Hom}(L, \mathbb{Z})$ of L, and by A_L the **discriminant group** $L^\vee / L$ of L. The discriminant group is a finite Abelian group that comes with a symmetric $\mathbb{Q}$-bilinear form $(\ \cdot\ )_{A_L} : A_L \times A_L \longrightarrow \mathbb{Q}/\mathbb{Z}$ induced by $(\ \cdot\ )_L$:

$$(x + L \cdot y + L)_{A_L} = (x \cdot y)_L \mod \mathbb{Z} \text{ for all } x,\ y \in L.$$

The **quadratic form of** L is a map $q_L : A_L \longrightarrow \mathbb{Q}/2\mathbb{Z}$ defined as

$$q_L(x + L) = (x \cdot x)_L \mod 2\mathbb{Z} \text{ for all } x \in L.$$

We remark that $q_L(x + y) - q_L(x) - q_L(y) \equiv 2(x \cdot y)_L \mod 2\mathbb{Z}$. A subgroup $H \subset A_L$ such that $q_{L_{|H}} = 0$ is called q_L-**isotropic**.

The **group of isometries** of A_L is denoted by $\mathrm{O}(A_L)$. A lattice L is **unimodular** if A_L is trivial. An injective map $L \hookrightarrow M$ is a **primitive embedding** if M/L is a torsion free $\mathbb{Z}$-module. An even lattice M is an **overlattice** of L if there is an embedding $L \hookrightarrow M$ such that M/L is a finite Abelian group.

Example 1.1.1 (Hyperbolic plane)**.** *We denote by $U \cong \mathbb{Z} \cdot e \oplus \mathbb{Z} \cdot f$ the lattice of* *rank* 2 *with the Gram matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is an even unimodular lattice of signature $(1,1)$*.*

Example 1.1.2 (ADE Lattices)**.** *We denoted by* A_n*,* D_n*,* E_6*,* E_7*, and* E_8 *the lattices associated to the Dynkin diagrams where each vertex corresponds to a generator* v *with* $(v \cdot v) = -2$*, and* $(v \cdot w) = 1$ *if* v *and* w *are connected by an edge, otherwise* $(v \cdot w) = 0$*. We use the convention that ADE lattices are negative definite and the root ordering follows Bourbaki's notation. For example, consider the lattice* A_5 *generated by* $a_1, ..., a_5$*, where* $(a_i \cdot a_{i+1}) = 1$*,* $i = 1, ..., 4$ *and* $(a_i \cdot a_j) = 0$*,* $j \neq i+1$*. Then, a basis for the dual module* $A_5^\vee \subset A_5 \otimes \mathbb{Q}$ *is given by* $\{\frac{a_1+2a_2+3a_3+4a_4+5a_5}{6}, a_2, a_3, a_4, a_5\}$*, and so* $A_5^\vee/A_5 = \langle g \rangle \cong \mathbb{Z}/6\mathbb{Z}$ *where* $g = \frac{a_1+2a_2+3a_3+4a_4+5a_5}{6}$*. Note that* $q_{A_5}(g) = \frac{1}{36}(a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 \cdot a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5) = -5/6$*, and so the discriminant form* $A_5 \cong \mathbb{Z}_6(\frac{1}{6})$*.*

We record the discriminant form for all lattices of ADE type in Table 9.2.

Example 1.1.3 (The lattice K_{12})**.** *We denote by* K_{12} *the* $\mathbb{Z}$*-module with the following bilinear Gram Matrix form:*

$$\begin{pmatrix}
-8 & -3 & 3 & 0 & -3 & 0 & 0 & -3 & 3 & 0 & -3 & 0 \\
-3 & -4 & 2 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
3 & 2 & -4 & 2 & 0 & 2 & 0 & 1 & -2 & 1 & 0 & 1 \\
0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
-3 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 2 & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 \\
-3 & -2 & 1 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 \\
3 & 1 & -2 & 1 & 0 & 1 & 0 & 2 & -4 & 2 & 0 & 2 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 2 & -4 & 2 & 0 \\
-3 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 2 & -4 & 0 \\
0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & -4
\end{pmatrix}$$

The lattice K_{12} *is an even negative definite lattice of rank* 12 *and discriminant group isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^6$*. Set* k_i*,* $i = 1, ..., 12$*, be a basis of* K_{12}*. A basis of the discriminant group* $A_{K_{12}}$ *can be written as*

$$g_1 = \frac{k_7 + 2k_8 + k_{10} + 2k_{11}}{3}, \; g_2 = \frac{k_6 + k_{12}}{3}, \; g_3 = \frac{k_5 + k_{11}}{3}$$

$$g_4 = \frac{k_4 + k_{10}}{3}, \ g_5 = \frac{k_3 + k_9}{3}, \ g_6 = \frac{k_2 + k_8}{3}.$$

In this basis, the quadratic form of $A_{K_{12}}$ can be written as

$$\begin{array}{rcl} q_{K_{12}} : A_{K_{12}} & \longrightarrow & \mathbb{Q}/2\mathbb{Z} \\ x_1 g_1 + \ldots + x_6 g_6 & \mapsto & \frac{4}{3}x_1^2 + \frac{2}{3}(x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) \\ & & +\frac{4}{3}(x_2x_5 + x_3x_4 + x_4x_5 + x_5x_6). \end{array} \tag{1.1}$$

Remark 1.1.4. *The lattice K_{12} is also denoted the Coxeter–Todd lattice. Our notation is slightly different from that of [[CS93],Ch.4, Section 9], cf. [NS14], since they define K_{12} as a positive definite lattice. In this case, we say that K_{12} is the opposite of the Coxeter–Todd lattice.*

Example 1.1.5 (The lattice $M_{\mathbb{Z}/3\mathbb{Z}}$)**.** *We denote by $M_{\mathbb{Z}/3\mathbb{Z}}$ the $\mathbb{Z}$-module of rank* 12 *with the following Gram Matrix form*

$$\begin{pmatrix} -4 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

The quadratic form of $A_{M_{\mathbb{Z}/3\mathbb{Z}}} \cong (\mathbb{Z}/3\mathbb{Z})^4$ can be written as

$$q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}(v) = -\frac{2}{3}(x_3^2 + x_4^2) - \frac{1}{3}(2x_1x_2 + 2x_2x_3 - 2x_2x_4 - x_3x_4).$$

Some properties are given without proofs, see [[Nik79]; [BHPVdV04], I.2] for further details. Suppose that $L \hookrightarrow M$ is an embedding:

- If $\operatorname{rank} L = \operatorname{rank} M$, then $(M : L)^2 = |A_L| \cdot |A_M|^{-1}$.
- $\operatorname{rank} L + \operatorname{rank} L^{\perp} = \operatorname{rank} M$.

- If $L \subset M$ is primitive, every basis of L can be complemented to a basis of M.
- If M is unimodular and $L \subset M$ is primitive, then $|A_L| = |A_{L^\perp}|$

Let L be a lattice and $G \subset \mathrm{O}(L)$. Denote by L^G the invariant sublattice of L and by $S_G(L) = (L^G)^\perp \subset L$ the co-invariant sublattice of L.

Lemma 1.1.6. *Let L be a lattice and $G \subset \mathrm{O}(L)$. Then the following hold*

1. *L^G contains $\sum_{g \in G} gv$ for all $v \in L$.*
2. *$S_G(L)$ contains $v - gv$ for all $v \in L$ and all $g \in G$.*
3. *$L/(L^G \oplus S_G(L))$ is of $|G|$–torsion.*

Proof. The proof of (1) is straightforward. To deduce (2), consider $w \in L^G$ and $g \in G$, then $(w \cdot v) = (gw \cdot gv) = (w \cdot gv)$ for all $v \in L$. This gives $(w \cdot v - gv) = 0$ for all $v \in L$ and all $g \in G$.

Let $l \in L$. The result follows from $|G| \cdot l = \sum_{g \in G} g(l) + \sum_{g \in G} (l - g(l))$ where the first term lies in L^G and the second in $S_G(L)$ by (1) and (2). ∎

The following lemma is useful to understand the isometries of indefinite lattices.

Lemma 1.1.7. *[Nik79, Proposition 1.14.2], cf. [Dol83, Proposition 1.4.7] Let L be an indefinite lattice of $\operatorname{rank} L \geq l(A_L) + 2$. Then the canonical morphism $\mathrm{O}(L) \longrightarrow \mathrm{O}(A_L)$ defined as $\varphi \mapsto \bar{\varphi}$ where*

$$\bar{\varphi}(x + L) = (\varphi^{-1})^\vee(x) + L$$

is surjective.

In the next definition, we follow the notation of [Bri83]. Let L be an even lattice of signature $(l_{(+)}, l_{(-)})$ and A_L its discriminant group with quadratic form $q_L : A_L \longrightarrow \mathbb{Q}/2\mathbb{Z}$.

Definition 1.1.8. *The signature modulo 8 of q_L is*

$$sign(q_L) := l_{(+)} - l(-) \mod 8.$$

Set p be a prime.

Suppose that p is odd. We denote by $w^\epsilon_{p,\alpha}$, $\epsilon = -1, +1$, two finite quadratic forms on $\mathbb{Z}/p^\alpha\mathbb{Z}$: the quadratic form $w^{+1}_{p,\alpha}$ has generator value $q_L(1) = a/p^\alpha \mod 2\mathbb{Z}$

where a is the smallest positive even number which is a quadratic residue modulo p; and for $w_{p,\alpha}^{-1}$ we have $q_L(1) = a/p^\alpha \mod 2\mathbb{Z}$ with a the smallest positive even number such that is not a quadratic residue modulo p.

If $p = 2$, we also define finite quadratic forms $w_{2,\alpha}^{\epsilon}$ on $\mathbb{Z}/2^\alpha\mathbb{Z}$, $\alpha \geq 1$ but $\epsilon = \pm 1, \pm 5$. The quadratic form $w_{2,\alpha}^{\epsilon}$ has generator value $q_L(1) = \epsilon/2^\alpha$. We denote by u_α, v_α two additional finite quadratic forms on $\mathbb{Z}/2^\alpha\mathbb{Z} \oplus \mathbb{Z}/2^\alpha\mathbb{Z}$:

$$u_\alpha = \begin{pmatrix} 0 & 1/2^\alpha \\ 1/2^\alpha & 0 \end{pmatrix}, \quad v_\alpha = \begin{pmatrix} 1/2^{\alpha-1} & 1/2^\alpha \\ 1/2^\alpha & 1/2^{\alpha-1} \end{pmatrix}$$

Example 1.1.9. *The signatures modulo* 8 *of the four quadratic forms* $w_{2,1}^{1}$, $w_{2,1}^{-1}$, u_1 *and* v_1 *are* 1, −1, 0, *and* 4 *respectively. These follow by taking* $L = (2)$, (-2), $U(2)$, *and* D_4 *respectively.*

Theorem 1.1.10. *[Nik79, Theorem 1.8.1] The semigroup of non-degenerate p–adic finite quadratic forms is generated by*

- $w_{p,\alpha}^{\pm 1}$ $(\alpha \geq 1)$ *if* p *is odd;*
- $w_{2,\alpha}^{\pm 1}$, $w_{2,\alpha}^{\pm 5}$, u_α, *and* v_α $(\alpha \geq 1)$ *if* $p = 2$.

In particular, q_L *is isomorphic to an orthogonal direct sum of the forms* $w_{p,\alpha}^{\epsilon}$, u_α, v_α.

The following Theorem introduces the notation of p–adic lattices which will be very useful in the characterization of the existence of a lattice.

Theorem 1.1.11. *Let* p *be a prime and* q_p *be a quadratic form on a finite Abelian* p*–group* A_p*. Then there exists a* p*–adic lattice* $K(q_p)$ *of rank* $l(A_p)$ *and discriminant form* $q_{K(q_p)} \simeq q_p$*. The* p*–adic lattice* $K(q_p)$ *is unique (up to isometries) unless* $p = 2$ *and there exists a finite quadratic form* q_2' *such that* $q_2 \simeq w_{2,1}^{\pm 1} \oplus q_2'$.

Extending lattices

We will be interested in even lattices that can be obtained as overlattices. This technique involves the computation of discriminant forms and the vanishing of subgroups of the discriminant group associated to a fixed lattice.

Suppose that L is an even lattice and let M be an even overlattice of L. There exists a natural chain of embeddings

$$L \hookrightarrow M \hookrightarrow M^\vee \hookrightarrow L^\vee,$$

such that $H := M/L \subset M^\vee/L \subset L^\vee/L = A_L$, and $(M^\vee/L)/H = A_M$.

Lemma 1.1.12. *Let q_L be the quadratic form of L. The set of even overlattices of L is in $1-1$ correspondence with the set of subgroups of A_L where q_L vanishes.*

Proof. Let H be an isotropic subgroup of A_L.

Set

$$L_H := \{x \in L^\vee \mid x + L \in H\}.$$

Then $(L_H, (\ \cdot\ )_L \otimes \mathbb{Q})$ is an even lattice since $q_L(x+L) \in 2\mathbb{Z}$. It follows from the definition that

$$L \subset L_H \subset L_H^\vee \subset L^\vee,$$

and so $|A_L| = |A_{L_H}|[L_H : L]^2$, $A_{L_H} = H^\perp/H$ and $q_{L_H} = q_L|_{H^\perp}/H$.

Conversely, if M is an overlattice of L, then $H := M/L$ is a subgroup of A_L. Moreover, $q_L(m+L) = q(m \cdot m)_L \in 2\mathbb{Z}$ since M is an even overlattice of L. This implies $q_L(m+L) = 0$ for all $m \in M$. ■

Note that the signature and the rank of the overlattices are preserved but the order of the new discriminant group decreases if the isotropic subgroup is not trivial.

Lemma 1.1.13. *Let L be an unimodular lattice and $T \subset L$ be a primitive sublattice. Then, as groups*

$$A_T \cong A_{T^\perp} \cong \frac{L}{T \oplus T^\perp}.$$

Proof. Since $T \subset L$ is a primitive embedding, then L is an overlattice of $T \oplus T^\perp$ with same rank, and in particular $|A_L|[L : T \oplus T^\perp]^2 = |A_T||A_{T^\perp}|$. In fact, the quotient $L/(T \oplus T^\perp)$ is isomorphic to a subgroup $M \subset A_T \oplus A_{T^\perp}$ which is isotropic (with respect to $q_{A_{T \oplus T^\perp}}$). Now, assuming that L is unimodular, we get $|M| = |A_T| = |A_{T^\perp}|$. Let us consider $p_T : A_T \oplus A_{T^\perp} \longrightarrow A_T$ and $p_{T^\perp} : A_T \oplus A_{T^\perp} \longrightarrow A_{T^\perp}$ be the projections maps, then $p_T(M) \subset A_T$ has the same order of M which is the order of A_T and so $p_T : M \xrightarrow{\sim} A_T$ is an isomorphism. Analogously, $p_{T^\perp} : M \xrightarrow{\sim} A_{T^\perp}$ is an isomorphism. ■

Example 1.1.14. *Let e, f be a basis of $L = U(2)$ such that $(e \cdot e) = (f \cdot f) = 0$ and $(e \cdot f) = 2$. Then, $e/2$ and $f/2$ are a basis of $A_{U(2)}$ with the $\mathbb{Q}$-bilinear form:*

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

Set $H_1 = \langle e/2\rangle$ and $H_2 = \langle f/2\rangle$. Then, H_1 and H_2 are isotropic groups. The lattice obtained from $U(2)$ by adding the vector $e/2$ (resp. $f/2$) is isomorphic to U:

$$\begin{aligned}\pi^{-1}(H_1) &= \{\frac{e}{2}+bf \mid bf \in U(2)\}\\ &= \{\frac{e}{2}+bf \mid b\in\mathbb{Z}\}\\ &= \langle\frac{e}{2}, f\rangle\end{aligned}$$

where $\pi : U(2)^\vee \longrightarrow U(2)^\vee/U(2)$ is the quotient map, and $(e/2\cdot f) = 1$ and $(e/2\cdot e/2) = (f\cdot f) = 0$ (resp. $(f/2\cdot e) = 1$ and $(f/2\cdot f/2) = (e\cdot e) = 0$).

Existence, uniqueness, and embeddings of lattices

We recall some standard facts on the existence and uniqueness of even lattices. We also restrict our attention in primitive embeddings of lattices. Roughly speaking, the signature and the parity of a lattice can determine the lattice up to isometries. At least in the unimodular case, the lattice can be written in terms of E_8 and U. In the case of non-unimodular lattices, the genus associated to a discriminant quadratic form plays a fundamental role in the classification of these lattices.

Theorem 1.1.15 ([Mil58]). *Let L be an even unimodular lattice of signature $(l_{(+)}, l_{(-)})$.*

- *L with these invariants exists if and only $l_{(+)} - l_{(-)} \equiv 0 \mod 8$.*
- *If $l_{(+)},\ l_{(-)} > 0$, then L is the unique lattice with these invariants (up to isometries).*

Corollary 1.1.16 ([Mil58]). *If L is an even unimodular lattice of signature $(l_{(+)}, l_{(-)})$, then*

$$L \cong U^{\oplus m} \oplus E_8(\pm 1)^{\oplus n} \text{ where } m = l_{\pm},\ n = \mp\frac{l_{(+)} - l_{(-)}}{8}.$$

Lemma 1.1.17. *[Nik79, Theorem 1.10.1] An even lattice L of signature $(l_{(+)}, l_{(-)})$ exists if and only if the following conditions hold:*

1. *$l_{(+)} - l_{(-)} \equiv sign\ q \mod 8$;*
2. *$l_{(+)},\ l_{(-)} \geq 0$, $l(A_L) \leq \operatorname{rank} L$;*

3. *for all odd prime p such that* $\operatorname{rank} L = l(A_p)$,

$$(-1)^{l_{(-)}}|A_L| \equiv discr\ K(q_p) \mod (\mathbb{Z}_p^*)^2;$$

4. *if* $\operatorname{rank}(L) = l(A_2)$ *and* $q_2 \neq w_{2,1}^{\pm 1} \oplus q_2'$ *for any finite quadratic form* q_2',

$$|A_L| \equiv \pm discr\ K(q_2) \mod (\mathbb{Z}_2^*)^2.$$

The following lemmas guarantees the uniqueness of a lattice with some fix invariants.

Lemma 1.1.18 (Kneser [Kne56], Nikulin [Nik79], Corollary 1.13.3)**.** *Let L be an even lattice. If $0 < l_{(+)}, l_{(-)}$ and $l(A_L) + 2 \leq \operatorname{rank}(L)$, then L is the unique lattice (up to isometries) with signature $(l_{(+)}, l_{(-)})$ and discriminant quadratic form q_L.*

Lemma 1.1.19. *[CS93, Chapter 15, Theorem 21] If the genus of an indefinite lattice L with $\operatorname{rank}(L) = n$ and discr L, contains more than one isometry class, then $4^{\frac{n}{2}}$ discr L is divisible by $k^{\binom{n}{2}}$ for a non-square natural number $k \equiv 0, 1 \mod 4$.*

The following lemmas are some results on the existence and uniqueness of primitive embeddings.

Lemma 1.1.20 ([Nik79], Theorem 1.14.4)**.** *Let M be an even lattice of signature $(t_{(+)}, t_{(-)})$ and L be an even unimodular lattice of signature $(s_{(+)}, s_{(-)})$. There exists a unique primitive embedding of M into L if the following conditions hold*

- $t_{(+)} < s_{(+)}$, $t_{(-)} < s_{(-)}$;
- $l(A_M) \leq \operatorname{rank} L - \operatorname{rank} M - 2$.

Lemma 1.1.21 ([Nik79], Theorem 1.12.4)**.** *Let $(t_{(+)}, t_{(-)})$ and $(l_{(+)}, l_{(-)})$ be two pairs of non-negative integers. For all even lattices S of signature $(t_{(+)}, t_{(-)})$, there exists an even unimodular lattice L of signature $(l_{(+)}, l_{(-)})$ and $S \hookrightarrow L$ a primitive embedding, if and only if*

$$l_{(+)} - l_{(-)} \equiv 0 \mod 8,\ t_{(+)} \leq l_{(+)},\ t_{(-)} \leq l_{(-)} \text{ and } t_{(+)} + t_{(-)} \leq \frac{1}{2}(l_{(+)} + l_{(-)}).$$

2-elementary lattices

We start by recalling a few basic facts of 2-elementary and hyperbolic lattices.

Definition 1.1.22. *A lattice L is called* ***hyperbolic*** *if the signature of L is $(1,*)$. A lattice L is called $2-$**elementary** if $A_L \cong (\mathbb{Z}/2)^l$ for some $l \geq 0$. We define an* ***invariant*** *δ of a 2-elementary lattice L: if the image of the discriminant quadratic form q_L is contained in $\mathbb{Z}/2$, then $\delta(L) = 0$, otherwise $\delta(L) = 1$.*

Note that if L is a 2–elementary, by Theorem 1.1.10 the quadratic form q_L can be represented as a direct sum of quadratic forms $w^{\pm 1}_{2,\alpha}$, $w^{\pm 5}_{2,\alpha}$, u_α, and v_α $(\alpha \geq 1)$.

All hyperbolic even 2-elementary lattices are determined by the invariants rank, length of the discriminant group and the delta invariant:

Lemma 1.1.23. *[Nik81, Theorem 4.3.2] There exists a hyperbolic, even, 2-elementary lattice L with invariants $r = \operatorname{rank} L$, $a = l(A_L)$ and $\delta = \delta(L)$ if and only if the following conditions are satisfied*

- *$a \leq r$;*
- *$a + r \equiv 0 \mod 2$;*
- *if $\delta = 0$, then $r \equiv 2 \mod 4$;*
- *if $a \leq 1$, then $r \equiv 2 \pm a \mod 8$;*
- *if $a = 0$, then $\delta = 0$;*
- *if $a = 2$, $r \equiv 6 \mod 8$, then $\delta = 0$;*
- *if $a = r$ and $\delta = 0$, then $r \equiv 2 \mod 8$.*

Example 1.1.24 (2-elementary hyperbolic lattices of rank 2)**.** *Let L be a 2-elementary lattice of signature $(1,1)$. The existence of L implies that $|(A_L)| \leq 2$. In particular, if $|(A_L)| = 0$ then L is unimodular and so $L \cong U$. When $|(A_L)| = 2$, then L is isometric to $U(2)$ or $\langle 2\rangle \oplus \langle -2\rangle$.*

Proposition 1.1.25. *[Dol83, Proposition 1.5.1] Let L be an even unimodular lattice, $i : L \longrightarrow L$ be an isometry of order 2 in $\mathrm{O}(L)$. The lattices $L^i = \{x \in L \mid i(x) = x\}$ and $S_i(L) = \{x \in L \mid i(x) = -x\}$ are 2-elementary lattices.*

Proof. It follows of Lemma 1.1.13 and Lemma 1.1.6 since $|G| = \langle \iota \rangle$ has order two. ∎

1.2 K3 surfaces

In this section, we collect some well known facts of K3 surfaces and automorphisms of K3 surfaces, particularly the symplectic case.

A **K3 surface** is a simply connected compact complex manifold of dimension two such that it admits a non-degenerate holomorphic 2-form. Among classical examples of K3 surfaces are the Kummer surfaces, double coverings of $\mathbb{P}^2$ branched along a smooth curve of $\mathbb{P}^2$ of degree six, and smooth complete intersection of type: 4 in $\mathbb{P}^3$, or $(2,3)$ in $\mathbb{P}^4$, or $(2,2,2)$ in $\mathbb{P}^5$ (this yields examples of K3 surfaces of degree four, six, and eight respectively).

The fact that X is a compact surface implies by Poincare's duality that $H^2(X,\mathbb{Z})$ is unimodular. A computation involving the Wu formula shows that $H^2(X,\mathbb{Z})$ is an even lattice, see [Wu50]. By Hirzebruch's index theorem, we obtain that the signature of $H^2(X,\mathbb{Z})$ is $(3,19)$ since K3 surfaces are Kähler manifolds, see [Siu83].

Theorem 1.2.1 ([Mil58])**.** *If X is a K3 surface, then the $\mathbb{Z}$–module $H^2(X,\mathbb{Z})$ with the cup product is an even unimodular lattice of signature $(3,19)$ isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$.*

Set Λ_{K3} be the abstract lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$.

Theorem 1.2.2 (Torelli type theorem for K3 surfaces)**.** *Let X, Y be K3 surfaces and let ω_X, ω_Y be non-degenerate holomorphic 2-forms on X and Y respectively. Suppose that there exists an isometry of lattices*

$$\varphi : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$$

satisfying

- *the extension $\varphi_{\mathbb{C}} = \varphi \otimes_{\mathbb{Z}} \mathbb{C}$ acts on $H^2(X,\mathbb{C})$ such that $\varphi(\omega_X) = c \cdot \omega_Y$ for some $c \in \mathbb{C}^*$;*
- *φ sends a Kähler class of X to a Kähler class of Y.*

Then there exists a unique isomorphism f such that $f^* = \varphi$.

Roughly speaking, the Torelli-type theorem implies that the isomorphism class of a K3 surface is determined by its period, see Theorem 1.2.4. A strong consequence of this theorem is the approach to study the geometry of K3 surfaces from a lattice theoretical viewpoint. In order to formalize the connection between these two approaches we introduce a fundamental ingredient: the period map for K3 surfaces.

Let $\Delta(X)$ be the set

$$\Delta(X) = \{\delta \in H^{1,1}(X) \cap H^2(X,\mathbb{Z}) \mid \delta^2 = -2\}.$$

In particular, non-singular rational curves belong to $\Delta(X)$. For each $\delta \in \Delta(X)$, we define the reflection map s_δ on $H^2(X,\mathbb{Z})$ given by $x \mapsto x + (x \cdot \delta)$ which is an isometry of $H^2(X,\mathbb{Z})$. In fact, it is a reflection with respect to the hyperplane $\delta^\perp \subset H^2(X,\mathbb{R})$. Denote by $W(X)$ the subgroup of $\mathrm{O}(H^2(X,\mathbb{Z}))$ generated by all reflections s_δ with $\delta \in \Delta$. Let $P(X)$ be the connected component of the set

$$\{x \in H^{1,1}(X) \mid x^2 > 0\}$$

that contains a Kähler class. Then $W(X)$ acts on $P(X)$ and each connected component of $P(X) \setminus \bigcup_{\delta\in\Delta} \delta^\perp$ is a fundamental domain of $W(X)$. The connected component containing a Kähler class is called the *Kähler cone* of X.

Remark 1.2.3. *The second condition of Theorem 1.2.2 is equivalent to the following*

- $\varphi(\Delta(X)) = \Delta(Y)$*;*
- φ *preserves the Kähler cones;*
- φ *preserves the effective divisors.*

Let $\mathcal{D}$ be the space

$$\mathcal{D} = \{\omega \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \mid (\omega \cdot \omega) = 0 \text{ and } (\omega \cdot \bar{\omega}) > 0\}$$

called the period domain.

A *marked K3 surface* is a pair (X,ϕ) where X is a K3 surface and ϕ is an isometry $\phi : H^2(X,\mathbb{Z}) \longrightarrow \Lambda_{\mathrm{K3}}$. By theorem 1.2.2, for each marked K3 surface (X,ϕ) we have that $\phi \otimes \mathbb{C}(\omega_X)$ is in $\mathcal{D}$. The point $(\phi \otimes \mathbb{C})(\omega_X) \in \mathcal{D}$ is called the *period* of a marked K3 surface (X,ϕ). Denote by $\mathcal{M}$ the set of marked K3 surfaces.

Theorem 1.2.4 (Surjectivity of period map, [Tod80])**.** *The map*

$$\begin{aligned} p : \mathcal{M} &\longrightarrow \mathcal{D} \\ (X,\phi) &\mapsto (\phi \otimes \mathbb{C})(\omega_X) \end{aligned}$$

is surjective.

The map p is called the *period map* for K3 surfaces.

Symplectic automorphisms

Let X be a K3 surface. We denote by $\mathrm{Aut}(X)$ the group of all biholomorphic maps. Essentially, there are two kinds of automorphisms: symplectic and non-symplectic automorphisms. Let f be an automorphism of finite order. Since $Aut(X)$ has the structure of a complex Lie group, we say that f is **symplectic** if the induced action on $H^{2,0}(X) \cong \mathbb{C} \cdot \omega_X$ for some non-degenerate 2-holomorphic form ω is the identity, i.e., $f^*\omega_X = \omega_X$. We say that f is **non-symplectic** if $f^*\omega_X = \zeta_n \omega_X$ where $1 \neq \zeta_n$ is a nth root of unity in $\mathbb{C}$. We say that $G \subset Aut(X)$ is symplectic if all elements of G are symplectic automorphisms of finite order. It is possible to get finite groups of symplectic automorphisms of K3 surfaces to embed in the Mathieu group M_{23} by using the classification of Niemeier lattices but we will not develop this point here, see [Muk88], [Nie73], cf. [Kon98].

One of the main reasons of interest in these particular automorphisms is that their presence relates two different (families of) K3 surfaces: (the family of) K3 surfaces admitting a symplectic automorphism of order n and the one (family) of the desingularization of the quotient of a K3 surface by a symplectic automorphism of order n, which is still a K3 surface, but in general not isomorphic to the original one. We will prove this fact in Proposition 1.2.9.

Let $f : X \longrightarrow X$ be a symplectic automorphism. Denote by T_X the minimal sub-Hodge structure of $H^2(X, \mathbb{Z})$ containing $H^{2,0}(X)$. Since $H^0(X, \Omega^2_X) \simeq H^{2,0}(X) \subset T_X \otimes \mathbb{C}$, we get

Proposition 1.2.5. *[Nik76, Theorem 3.1] The automorphism f is symplectic if and only if $f^* = \mathrm{Id}$ on T_X.*

Proof. Suppose that f is symplectic, then $\ker((f^* - \mathrm{Id})|_{T_X}) \subset T_X$ is a sub-Hodge structure containing $H^{2,0}(X)$. Since $T_X / \ker((f^* - \mathrm{Id})|_{T_X})$ is torsion free, then $\ker((f^* - \mathrm{Id})|_{T_X}) = T_X$ and so $f^* = \mathrm{Id}^*$ on T_X. ∎

Lemma 1.2.6 ([Muk88], Proposition 1.5)**.** *If $G \subset \mathrm{Aut}(X)$ be a symplectic group with a fixed point, then G is isomorphic to a subgroup of $SL(2, \mathbb{C})$*

In particular, if G is a cyclic group generated by a symplectic automorphism of order n, there exists a local holomorphic coordinate system z_1, z_2 around x such that $f(z_1, z_2) = (\zeta_x z_1, \zeta_x^{-1} z_2)$ where ζ_x is a primitive nth root of unity, cf. [Huy16, Lemma 1.4].

Here are some elementary facts about symplectic automorphisms of finite order.

Proposition 1.2.7. *[[Muk88], Proposition 1.2] Let f be a symplectic automorphism of finite order $n \neq 1$. Then,*

- $1 \leq |\operatorname{Fix}(f)| \leq 8$;
- $n \leq 8$;
- $\rho(X) = \operatorname{rank} \operatorname{NS}(X) \geq 8$.

In particular, the number of fixed points of f depends only on the order n as in Table 1.1 , see [Nik76, Section 5] for more details.

n	2	3	4	5	6	7	8
$\|\operatorname{Fix}(f)\|$	8	6	4	4	2	3	2
$\rho(X) \geq$	8	12	14	16	16	18	18

Table 1.1

The following results are elementary observations about symplectic automorphisms in the seminal paper of Nikulin, [Nik76]. He studied the problem of existence and uniqueness of the action of a symplectic automorphism in the second cohomology group from a lattice theoretical viewpoint.

Let $G \subset \mathrm{O}(H^2(X, \mathbb{Z}))$. Denote by

$$H^2(X, \mathbb{Z})^G \text{ and } S_G(X) := (H^2(X, \mathbb{Z})^G)^{\perp} \subset H^2(X, \mathbb{Z}),$$

the invariant and the co-invariant lattices of $H^2(X, \mathbb{Z})$ respectively.

Theorem 1.2.8 ([Nik76], Lemma 4.2)**.** *If $G \subset Aut(X)$ is a symplectic group, then the following assertions are true:*

- *$S_G(X)$ is non-degenerate and negative definite;*
- *$S_G(X)$ does not contain elements with square (-2);*
- *$T_X \subset H^2(X, \mathbb{Z})^G$ and $S_G(X) \subset \operatorname{NS}(X)$;*
- *the group G acts trivially on the discriminant group $A_{S_G(X)}$.*

Nikulin's approach to the classification of groups of symplectic automorphisms comes from the following observation: essentially, the set of finite groups acting faithfully and symplectically on K3 surfaces is closed under quotients.

Quotients of K3 surfaces by symplectic groups

By Proposition 1.2.7, the number of fixed points of a symplectic automorphism is finite and it depends of the order of the automorphism. Since $G \subset SL(2,\mathbb{C})$, then the quotient X/G is a normal surface with isolated singularities of ADE-type, see [BHPVdV04, Chapter III.7]. Let Y be the minimal desingularization of X/G which is obtained by introducing an exceptional ADE set of curves for the corresponding type of the singularity, and π be the induced rational map $\pi = \beta^{-1} \circ q$:

$$\begin{tikzcd} & X \arrow[loop right, "G"] \arrow[dl, dashed, "\pi"'] \arrow[d, "q"] \\ Y \arrow[r, "\beta"] & X/G \end{tikzcd} \tag{1.2}$$

where q is the quotient map, β is the birational map in the minimal desingularization.

Proposition 1.2.9. *Under the above assumptions,*

- *Y is a K3 surface;*
- *π is a generically finite map of degree $|G|$.*

Proof. Let ω_X be a non-degenerate two holomorphic form on X. Since $\sigma^*\omega_X = \omega_X$, it defines a non-degenerate two holomorphic form $\tilde{\omega}$ on the smooth part of X/G. The surface $Y \setminus \cup_i\{E_i\}$, where E_i's are the exceptional curves of the blow ups, is isomorphic to $X/G \setminus \mathrm{Sing}(X/G)$. Then $\tilde{\omega}|_{Y\setminus\cup_i\{E_i\}}$ is a nowhere vanishing holomorphic 2-form on $Y \setminus \cup_i\{E_i\}$. This one can be extended on all Y, and so Y admits a non-degenerate holomorphic 2-form, see [[BHPVdV04] Proposition 3.5]. Since Y has exceptional (-2)-curves arising from the resolution of X/G, then Y cannot be a torus, and so Y is a K3 surface.

The fact that π is generically finite is because q is a covering map $|G| : 1$ and β is the composition of birational maps given in the resolution of $\mathrm{Sing}(X/G)$. ∎

Example 1.2.10 (Cyclic groups)**.** *Let σ be a symplectic automorphisms of prime order n and let p be a fixed point of σ. By Lemma 1.2.6, there exists local coordinates (z_1, z_2) around p such that $(z_1,z_2) \overset{\sigma}{\mapsto} (\zeta_n z_1, \zeta_n^{-1} z_2)$. Let $p(z_1,z_2) \in \mathbb{C}[z_1,z_2]$ be a polynomial of degree d such that $p \circ \sigma = p$. Without losing generality, suppose that $p(z_1,z_2) = z_1^i z_2^{d-i}$. Since p is σ-invariant, then $i = \frac{d+nk}{2}$, and so $p(z_1,z_2) = z_1^{\frac{d+nk}{2}} z_2^{\frac{d-nk}{2}}$. It is straightforward that $p(z_1,z_2) = z_1z_2$, or $p(z_1,z_2) = z_1^n$,* *or* $p(z_1, z_2) = z_2^n$ *are* σ*-invariant and generate* $\mathbb{C}[z_1, z_2]^\sigma$. *Set* $w_1 = z_1^n$, $w_2 = z_2^n$, *and* $w_3 = z_1 z_2$. *Then,* $w_3^n = w_1 w_2$. *Using the change of variables* $w_3 \mapsto z, w_2 \mapsto -x + iy, w_1 \mapsto x + iy$, *we get*

$$\mathbb{C}^2/\sigma = V(x^2 + y^2 + z^{(n-1)+1})$$

which is the local expression of a singularity of A_{n-1}*-type.*

In [Nik79], Nikulin introduced the symplectic automorphisms of K3 surfaces studying the action on the surface and the action on the second cohomology group of a K3 surface. Roughly speaking, he proved that the topology of the fixed locus and the action induced in cohomology are unique and depend only on the order of the automorphism.

We say that a group G has a unique action on the 2-dimensional integral cohomology of K3 surfaces if given any two embeddings $i : G \hookrightarrow \mathrm{Aut}(X)$ and $i' : G \hookrightarrow \mathrm{Aut}(X')$ under which G is an algebraic automorphism group for the K3 surfaces X and X', there exists an isomorphism $\varphi : H^2(X, \mathbb{Z}) \longrightarrow H^2(X', \mathbb{Z})$ (preserving intersection index) such that $\iota'(g^*) = \varphi \circ \iota(g)^* \circ \varphi^{-1}$ for any $g \in G$.

Theorem 1.2.11 ([Nik79], Theorem 4.7). *Any Abelian group* G *with a symplectic action has a unique action (up to isometries) on the* 2*-dimensional integral cohomology of K3 surfaces.*

Theorem 1.2.12 ([Nik79], Theorem 4.15). *Let* X *be a K3 surface and* G *be a subgroup of* $\mathrm{Aut}(X)$. *Then,* G *is a symplectic group if and only if there exists a primitive embedding*

$$S_G(H^2(X, \mathbb{Z})) \subset \mathrm{NS}(X).$$

We recall an useful formula which relates the bilinear form of two lattices for the case of finite maps.

Lemma 1.2.13 ([EH16],Theorem 1.23; cf. [Ful98], Example 8.1.7). *Let* $\pi : X \longrightarrow Y$ *be a map of smooth quasi-projective varieties.*

- *There is a unique map of groups* $\pi^* : H^c(Y, \mathbb{Z}) \longrightarrow H^c(X, \mathbb{Z})$ *such that whenever* $A \subset Y$ *is a subvariety with* $\pi^{-1}(A)$ *generically reduced and* $\mathrm{cod}_X(\pi^{-1}(A)) = \mathrm{cod}_Y(A) = c$, *we have*

$$\pi^*([A]) = [\pi^{-1}(A)].$$

- ***(Push-pull formula or Projection formula)*** *The map* $\pi_* : H_*(X,\mathbb{Z}) \longrightarrow H_*(Y,\mathbb{Z})$ *is a map of graded modules over the graded ring* $H_*(Y,\mathbb{Z})$. *More explicitly, if* $\alpha \in H^k(Y,\mathbb{Z})$ *and* $\beta \in H_l(X,\mathbb{Z})$, *then*

$$\pi_*(\pi^*\alpha \cdot \beta) = \alpha \cdot \pi_*\beta \in H_{l-k}(Y).$$

Remark 1.2.14. *The last statement of this lemma is the result of applying appropriate multiplicities to the set-theoretic equality* $\pi(\pi^{-1}(A) \cap B) = A \cap \pi(B)$. *In particular, if* $\pi = \phi^{-1} \circ q$ *as in 1.2, by Poincare's duality we obtain,*

$$\pi_*(\pi^*\alpha \cdot \beta) = |G|(\alpha \cdot \pi_*\beta) \in \mathbb{Z}. \tag{1.3}$$

for all $\alpha \in H^2(Y,\mathbb{Z})$ *and* $\beta \in H^2(X,\mathbb{Z})$.

1.3 Elliptic surfaces

Let X be a surface and $\pi : X \longrightarrow C$ be a holomorphic map to a smooth curve C. We say that X is an **elliptic surface** if any fiber, except over finitely many points of C, is an elliptic curve. The map π is called an **elliptic fibration** of X. An elliptic surface is called **relatively minimal** if no fiber contains an exceptional curve. We are interested in the particular case when X is a K3 surface (or a complex torus). Elliptic K3 surfaces come with a rich geometrical data which is reflected in their Nerón-Severi group. Popular sources as for elliptic surfaces include [Mir89], and [SS10] for a modern view point.

As in elliptic curves, an useful tool in the study of elliptic surfaces is the Weierstrass form. One can show that there exists a 1-1 correspondence between Weierstrass forms over C and smooth minimal elliptic surfaces over C with a section s_0, see [[Mir89], Corollary II.1.3, and II.5.5]. Then, elliptic surfaces are elliptic curves over the function field $k(C)$ of C. Its Weierstrass form is usually given by an equation

$$y^2 = x^3 + A(t)x + B(t),$$

where $A(t),\ B(t) \in k(C)$.

A triple $(\mathcal{L}, A, B)$ will be called **Weierstrass data** over C, if $\mathcal{L}$ is a line bundle on C and A, B are global sections of $\mathcal{L}^4$ and $\mathcal{L}^6$ respectively such that the section $\Delta = 4A^3 + 27B^2$ of $\mathcal{L}^{12}$(called the discriminant of the data) is not identically 0. A fiber of π over a point $c \in C$ is singular if and only if the discriminant section Δ is zero at c.

There are at most finitely many singular fibers for relatively minimal elliptic surfaces. This classification is due by Kodaira in [Kod63] using functional and topological invariants. Since we assume the existence of a section, the case of multiple fibers are excluded.

Let a, b, δ be the order of vanishing of A, B and Δ respectively at a point $c \in C$. One can show that if a, b, and δ are as in Table 9.3, then the Kodaira type of the fiber $\pi^{-1}(c)$ is as indicated, see [[Mir89], Proposition IV.3.1]. For each fiber, we also recall the multiplicities of each component of the fiber, which corresponds in the Table 9.3 to the number on each vertex, see [cf. [Mir89], Proposition I.4.2], and its Euler characteristic.

Let X be an elliptic surface. Let $g := g(C)$ be the genus of C and $q := h^{1,0}(X)$ be the irregularity of X.

Lemma 1.3.1 ([Mir89], III.4.1).

$$q = \begin{cases} g, & \textit{if } X \textit{ is not a product of curves} \\ g+1, & \textit{if } X \textit{ is a product of curves} \end{cases}$$

By the adjunction formula, we can deduce

$$\omega_X \cong \pi^*(\omega_C \otimes \mathcal{L}). \tag{1.4}$$

In particular, $K_X^2 = 0$, and using Noether's formula we can deduce

$$e(X) = 12 \deg(\mathcal{L}). \tag{1.5}$$

Proposition 1.3.2. *Let X be an elliptic K3 surface. Then,*

- *$C \cong \mathbb{P}^1$;*
- *$A(t)$ and $B(t)$ are polynomials of degree 8 and 12 respectively.*

Proof. It follows from the lemma 1.3.1 that $g = q = 0$, then $C \cong \mathbb{P}^1$ since C is a smooth curve. Since $e(X) = 24$ for X a K3 surface, by the Equation 1.5 we get that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2)$. Hence, if $(\mathcal{L}, A, B)$ is a Weierstrass data over $\mathbb{P}^1$, then $\deg(A) = 8$ and $\deg(B) = 12$. ∎

The Mordell Weil group and the Shioda–Tate formula

The **Mordell Weil group** is the group of the sections of π. It is denoted by $\mathrm{MW}(X)$ and the fixed section s_0 corresponds to the neutral element of $\mathrm{MW}(X)$. In particular, the Mordell Weil group of an elliptic surface is a finitely generated Abelian group, see [Mir89, Corollary VII.2.2].

Let X_c be the fiber at $c \in C$. Define by $X_c^{\#}$ the subset of all components of X_c obtained by deleting the singular points of the fiber, and $X_{c,0}^{\#} \subset X_c^{\#}$ obtained by deleting also the component meeting the zero section. The operation on $\mathrm{MW}(X)$ induces an operation on $X_c^{\#}$ such that $X_c^{\#}$ is an Abelian group. For each type of fiber, we describe the group $X_c^{\#}/X_{c,0}^{\#}$, see Table 9.3. Since an automorphism of a reducible fiber X_c has to be compatible with the group structure of the fiber, $X_c^{\#}/X_{c,0}^{\#}$ enforces strict conditions on the automorphism group of an elliptic surface that contains X_c.

Let $\rho(X)$ be the Picard number of X. A practical formula which relates the Picard number of X and the rank $\mathrm{MW}(X)$ is the following which is known as the *Shioda-Tate's formula.*

Lemma 1.3.3. *[Mir89, Corollary VII.2.4] The Picard number ρ of an elliptic surface X is*

$$\rho = 2 + r + \sum_{c \in \Delta} r_c,$$

where $r_c = \#$ of components of $X_c - 1$ and r is the rank of $\mathrm{MW}(X)$.

Note that each section $s \in \mathrm{MW}(X)$ intersects one component of each singular fiber in a point of multiplicity one, i.e., $(s \cdot F) = 1$. Moreover, if s, and s' are two different sections in $\mathrm{MW}(X)$, then the components which each section intersect the singular fiber are different.

Another useful formula is given in the following lemma. Set $r_c^{(m)} := \#$ of components of X_c with multiplicity m. In particular, if $m = 1$, then $r_c^{(1)}$ counts the simple components on X_c.

Lemma 1.3.4 ([Shi72], Lemma 1.3). *If $\mathrm{MW}(X)$ has rank 0 and order n, then*

$$|A_{T_X}| = \frac{1}{n^2} \prod_{c \in \Delta} r_c^{(1)}.$$

Constructing a basis of the Nerón–Severi lattice

Let F be the class of a singular fiber. Since the fibers are all algebraically equivalent, we get that $(F \cdot F) = 0$. The zero section s_0 intersects any fiber in one point, then $(F \cdot s_0) = 1$.

Proposition 1.3.5 ([Shi90], Section 7). *Let X be an elliptic surface and a its arithmetic genus. The following hold*

- $(s_0 \cdot s_0) = -a$
- *There exists an embedding of*

$$\bar{U} := (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}) \subset \mathrm{NS}(X).$$

Corollary 1.3.6. *Let X be a K3 surface. If X admits an elliptic fibration, then $U \subset \mathrm{NS}(X)$.*

Let Tr_X be the lattice generated by F, the class of the zero section s_0, and the classes of irreducible components of reducible fibers which do not intersect s_0. This lattice is called the **trivial lattice** of X.

One can show that the Nerón Severi lattice of X over $\mathbb{Q}$ is generated by Tr_X and the sections of $\mathrm{MW}(X)$.

Theorem 1.3.7 ([Shi90], Theorem 1.3)**.** *Let X be an elliptic surface. The Mordell Weil group is isomorphic to* $\mathrm{NS}(X)/\mathrm{Tr}_X$.

If X admits a section t of n-torsion, then t is linearly equivalent to a combination of the classes generating Tr_X with coefficients in $\frac{1}{n}\mathbb{Z}$. This implies that $\mathrm{NS}(X)$ is an overlattice of index at least n of the lattice Tr_X.

Proposition 1.3.8. *Let X be an elliptic K3 surface, and t be a torsion section of order n. The translation by the section t is a symplectic automorphism of X of order n.*

Proof. By Formula 1.4 we know that the nowhere vanishing holomorphic two form ω of an elliptic K3 surface is locally written as $\omega = \pi(t)(dx/y) \wedge dt$ where $\pi(t)$ is a nowhere vanishing holomorphic function. Let $dz = dx/y$. Then dz is a holomorphic form on each fiber E_c. Since the base of the elliptic fibration is fixed by the automorphism of translation, then it acts as the identity on t, and fixes dz. Then, ω is fixed by the automorphism. ∎

1.4 Hyperkähler manifolds

Definition 1.4.1. *A* ***hyperkähler manifold*** *is a simply connected compact Kähler manifold X that admits a unique (up to scalars) non-degenerate holomorphic 2-form $\omega_X \in H^{2,0}(X)$.*

Hyperkähler manifolds are also known as irreducible holomorphic symplectic manifolds (IHS). Such a form ω_X is called symplectic. Note that ω_X defines a skew-symmetric isomorphism on the holomorphic tangent bundle T_X and the holomorphic cotangent bundle ω_X. Hence the complex dimension of X is always even. A 2-dimensional hyperkähler manifold is nothing else but a K3 surface. Examples in higher dimension are hard to construct. Fujiki and Beauville were the first to provide examples of Hyperkähler manifolds in dimension greater than 2: they show that the Hilbert scheme $S^{[n]}$ parametrizing subschemes of length n on a projective K3 surface S is a Hyperkähler manifold of dimension $2n$, see [Bea83]. For instance, we have the following list of known examples, where manifolds of the same deformation type are not distinguished:

- The Hilbert scheme $S^{[n]}$ of a K3 surface S. Its dimension is $2n$ and for $n > 1$ its second Betti number is $b_2(S^{[n]}) = 23$.
- The generalized Kummer variety $K_n(A)$ of a 2–complex torus A. Its dimension is $2n$ and for $n > 2$ its second Betti number is $b_2(K_n(A)) = 7$, see [Bea83].
- The O'Grady's 10–dimensional example OG'10. Its second Betti number is 24, see [O'G99].
- The O'Grady's 6–dimensional example OG'6. Its second Betti number is 8, see [O'G03].

There are other examples of hyperkähler in the literature but they have all turned out to be deformation equivalent to one of the above list. As we will see later, moduli spaces of stable sheaves (and also twisted sheaves) on K3 surfaces can be example of hyperkähler manifolds. We will restrict our attention to some of these examples which have the same deformation type of Hilbert scheme of K3 surfaces.

The main motivation from the definition of Hyperkähler manifolds comes of the well known decomposition of compact Kähler manifold with $c_1(X) = 0$ in terms of compact complex torus, Calabi-Yau varieties and Hyperkähler manifolds. This fact is known as Beauville-Bogomolov decomposition:

Theorem 1.4.2. *[Bea83], [Bog74] Any compact Kähler simply connected manifolds with trivial canonical bundle can be written as a product of irreducible holomorphic symplectic manifolds and Calabi–Yau manifolds.*

Example 1.4.3 (K3$^{[2]}$ manifolds)**.** *Let S be a projective K3 surface. The* ***Hilbert scheme*** $S^{[2]}$ *of* 0*-dimensional subschemes of length* 2 *on* S *is a smooth projective irreducible manifold of dimension* 4*. One can show that the symplectic form of* S *lifts to a symplectic form on* $S^{[2]}$*, see [Bea83, Proposition 5]. Moreover, the fact that* S *is simply connected implies that* $S^{[2]}$ *is also simply connected, see [Bea83, Lemma 1].*

The variety $S^{[2]}$ *admits a morphism*

$$\epsilon : S^{[2]} \longrightarrow S^{(2)},$$

where $S^{(2)}$ *denotes the symmetric product of* S*. This morphism associates to a point in* $S^{[2]}$ *(i.e., a closed subscheme of dimension* 0 *and length* 2 *on* S*) in the pair of points of the support of* x*. Hence, the map* ϵ *corresponds to the blow–up of* $S^{(2)}$ *along the diagonal* $\Delta = \{(x,x) \mid x \in S\}$*. The pullback* ϵ^* *in* $H^2(X,\mathbb{Z})$ *is an injective morphism of Hodge structures of weight two, see [Bea83, Lemma 2]. The group* $H^2(S^{(2)},\mathbb{C})$ *is Hodge isometric to the invariant part (under the action of the involution) of* $H^2(S^2,\mathbb{C})$*. This shows*

$$H^2(S^{[2]},\mathbb{C}) = H^2(S,\mathbb{C}) \oplus \mathbb{C} \cdot E,$$

where E *denotes the exceptional divisor of* ϵ*. At the level of lattices, there exists an integral class* δ *such that* $2\delta = E$*, and so*

$$H^2(S^{[2]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z} \cdot \delta.$$

In particular, $H^2(S^{[2]},\mathbb{Z})$ *is torsion free and* $H^2(S,\mathbb{Z})$ *is orthogonal to* δ *under the symmetric bilinear pairing of* $H^2(S^{[2]},\mathbb{Z})$*. The restriction of this bilinear pairing to* $H^2(S,\mathbb{Z})$ *coincides with the intersection product on* S*, and* $\delta^2 = -2$*, then by Theorem 1.2.1 we obtain* $H^2(S^{[2]},\mathbb{Z})$ *is isometric to* $U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$*.*

Theorem 1.4.4. *[Bea83], [Fuj88] Let* X *be a hyperkähler manifold of dimension* $2n$*. Then there exists an integral non-degenerate quadratic form* q_X *on* $H^2(X,\mathbb{Z})$ *of signature* $(3, b_2(X) - 3)$ *and a positive constant* c_X *such that*

$$\int_X \alpha^n = c_X q_X(\alpha)^{2n}, \tag{1.6}$$

for every $\alpha \in H^2(X,\mathbb{Z})$*.*

The quadratic form q_X is called the *Beauville–Bogomolov–Fujiki* form and the constant c_X is called *Fujiki's constant.* The lattice $(H^2(X,\mathbb{Z}), q_X)$ is known as the *Beauville–Bogomolov–Fujiki* lattice.

We recall in Table 1.2 the Fujiki constant and Beauville–Bogomolov–Fujiki form of the known hyperkähler manifolds.

X	c_X	$H^2(X,\mathbb{Z})$
$S^{[n]}$	1	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle$
$K_n(A)$	n+1	$U^{\oplus 3} \oplus \langle -2(n+1) \rangle$
OG'10	1	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$
OG'6	4	$U^{\oplus 3} \oplus (-2)^{\oplus 2}$

Table 1.2

Definition 1.4.5. *Let* $b_2 \geq 3$ *be an integer, and* Λ *be an even lattice of signature* $(3, b_2-3)$*. A* ***marked pair*** (X,η) *consists of an hyperkähler manifold* X *and an isometry* $\eta : H^2(X,\mathbb{Z}) \longrightarrow \Lambda$*. Two marked pairs* (X,η_1)*,* (Y,η_2) *are* ***isomorphic*** *if there exists an isomorphism* $f : X \longrightarrow Y$ *such that* $\eta_1 \circ f^* = \eta_2$*.*

The *period* of the marked pair (X,η) is a point p in $\mathbb{P}(\Lambda \otimes \mathbb{C})$ that satisfies $(p \cdot p) = 0$ and $(p \cdot \bar{p}) > 0$ (i.e., the line $\eta(H^{2,0}(X))$).

One can show that there exists a coarse moduli space $\mathcal{M}_\Lambda$ parametrizing isomorphism classes of marked pairs, see [Huy99]. In fact, $\mathcal{M}_\Lambda$ is a smooth complex manifold of dimension $b_2 - 2$ but it is not a Hausdorff space.

Let $\mathcal{M}_\Lambda^0$ be a connected component of $\mathcal{M}_\Lambda$ and

$$\Omega_\Lambda = \{p \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (p \cdot p) = 0 \text{ and } (p \cdot \bar{p}) > 0\}$$

be the period domain. Note that Ω_Λ is an open subset of the quadric in $\mathbb{P}(\Lambda \otimes \mathbb{C})$ of isotropic lines with the classical topology, see [Bea83].

The *period map* is given by

$$\begin{array}{rcl} P_0 : \mathcal{M}_\Lambda^0 & \longrightarrow & \Omega_\Lambda \\ (X,\eta) & \mapsto & \eta(H^{2,0}(X)) \end{array}$$

Theorem 1.4.6. *[Bea83], cf. [Huy99, Theorem 8.1] The period map* P_0 *is a local isomorphism.*

This means that each point $p \in \Omega_\Lambda$ determines a weight 2 Hodge structure on $\Lambda \otimes \mathbb{C}$ such that the marking map η is an isomorphism of Hodge structure. A strongest version of Theorem 1.4.6 is the following

Theorem 1.4.7. *[Ver13, Theorem 2.2], [Huy12]If $P_0(X, \eta_1) = P_0(Y, \eta_2)$, then X and Y are birational.*

The Theorem 1.4.6 is known as the Local Torelli Theorem of hyperkähler manifolds while the Theorem 1.4.7 is known as its global version.

Monodromy group

Let X_1 and X_2 be hyperkähler manifolds which are deformation equivalent. An isomorphism $f : H^*(X_1, \mathbb{Z}) \longrightarrow H^*(X_2, \mathbb{Z})$ is a **parallel transport operator** if there exist a smooth and proper family $\pi : \chi \longrightarrow T$ of hyperkähler manifolds, over an analytic base T, points $t_i \in T$, isomorphisms $\psi_i : X_i \longrightarrow \chi_{t_i} = \pi^{-1}(t_i)$, i=1,2, and a continuous path $\gamma : [0, 1] \longrightarrow T$ such that $\gamma(0) = t_1$, $\gamma(1) = t_2$, such that the parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ induces the homomorphism $\psi_{2_*} \circ f \circ \psi_{1_*} : H^*(\chi_{t_1}, \mathbb{Z}) \longrightarrow H^*(\chi_{t_2}, \mathbb{Z})$.

Definition 1.4.8. *Let X be a hyperkähler manifold. An isometry g on $H^2(X, \mathbb{Z})$ is called a **monodromy operator** if there exists a family $\mathcal{X} \longrightarrow T$ of hyperkähler manifolds having X as a fiber over a point $t_0 \in T$, and such that g belongs to the image of $\pi_1(T, t_0)$ under the monodromy representation. The **monodromy group** $Mon^2(X)$ of X is the subgroup of $\mathrm{O}(H^2(X, \mathbb{Z}))$ generated by all monodromy operators.*

Note that the operation in $\mathrm{Mon}^2(X)$ is given by composition of two operators g_1 and g_2: if $p_i : \mathcal{X} \longrightarrow T_i$ $i = 1, 2$ is a family of hyperkähler manifolds with $[\gamma_i] \in \pi_1(T_i)$, then we can form a family $p : \mathcal{X} \longrightarrow T$ and a loop $[\gamma] \in \pi_1(T)$ by gluing T_1 and T_2 along the point t_i corresponding to X, $\mathcal{X}_1$ and $\mathcal{X}_2$ along $\mathcal{X}_{t_i}$, and concatenating the loops γ_1 and γ_2.

Let $\mathcal{K}_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ be the Kähler cone of X (i.e., the set of all Kähler classes associated to any Kähler structure on X) and let C_X be the positive cone of X (i.e., the connected component of the cone $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid (\alpha \cdot \alpha) > 0\}$ containing the Kähler cone $\mathcal{K}_X$).

The following theorem combines the Global Torelli Theorem with results on the Kähler cone of hyperkähler manifolds, see [Huy03] and [Bou01] for a proof.

Theorem 1.4.9 (A Hodge Theoretic Torelli Theorem)**.** *Let X and Y be hyperkähler manifolds which are deformation equivalents.*

1. *X and Y are birational, if and only if there exists a parallel transport operator $g : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$ which is a Hodge isometry.*

2. *Let $g : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$ be a parallel transport operator which is a Hodge isometry. There exists an isomorphism $\widetilde{g} : X \longrightarrow Y$ such that $g = \widetilde{g}_*$, if and only if g maps some Kähler class on X to a Kähler class on Y.*

Given a birational map $f : X \longrightarrow Y$ of hyperkähler manifolds, the map

$$f_* : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$$

is a homomorphism induced by the closure $X \times Y$ of the graph of f. Moreover, it is an isometry by [O'G97, Proposition 1.6.2], cf. [GHJ03, Proposition 21.6].

Hyperkähler manifolds of $K3^{[n]}$–type

The Example 1.4.3 can be generalized for $n > 2$.

Let ϵ be the Hilbert–Chow map,

$$\begin{aligned} \epsilon : S^{[n]} &\longrightarrow S^{(n)} \\ Z &\mapsto \sum_{p \in S} l(\mathcal{O}_{Z,p})p, \end{aligned} \tag{1.7}$$

where the sum is a formal sum and $S^{(n)}$ denotes the symmetric n-th product of S (i.e., the quotient of S^n by the natural action of the symmetric group of n elements σ_n). One can show that the Hilbert-Chow map is a the blow-up of $S^{(n)}$ along the large diagonal $D_n = \bigcup_{i \neq j} \{(x_1, ..., x_n) \in S^n \mid x_i = x_j\}$ and it is a resolution of singularities, see [Hai01, Proposition 3.8.4]. We denote by Δ_n is the exceptional set of ϵ. In [Bea83] also proves that $S^{[n]}$ is a Hyperkähler manifold of dimension $2n$, and the second integral cohomology of X is a lattice with respect to the Beauville-Bogomolov pairing isometric to

$$H^2(X,\mathbb{Z}) \cong H^2(S,\mathbb{Z}) \oplus \langle -2(n-1) \rangle,$$

where δ is a class such that $\Delta_n = 2\delta$ and $(\delta \cdot \delta) = -2(n-1)$. Moreover, $H^2(X,\mathbb{Z})$ is abstractly isometric to

$$\Lambda_{\mathrm{K3}^{[n]}} := U \oplus U \oplus U \oplus E_8 \oplus E_8 \oplus \langle -2(n-1) \rangle.$$

Since U and E_8 are unimodular lattices, the discriminant group $A_X := H^2(X,\mathbb{Z})^\vee / H^2(X,\mathbb{Z})$ is a cyclic group of order $2(n-1)$.

Definition 1.4.10. *A hyperkähler manifold X is called* ***hyperkähler manifold of K3$^{[n]}$–type*** *if X is deformation equivalent to a Hilbert scheme of n points $S^{[n]}$ of some projective K3 surface S.*

Remark 1.4.11. *Note that if X is a hyperkähler manifold of K3$^{[n]}$–type, then $H^2(X,\mathbb{Z}) \cong \Lambda_{\mathrm{K3}^{[n]}}$. In general, the property to be deformation equivalent to some Hilbert scheme of n points on a K3 surface does not imply an isomorphism. We will see some examples in the next section.*

Theorem 1.4.12. *[Mar10, Theorem 1.2, Lemma 4.2]Let X be a hyperkähler manifold of K3$^{[n]}$–type. The monodromy group $Mon^2(X)$ is equal to the subgroup of $\mathrm{O}^+(H^2(X,\mathbb{Z}))$ which acts via multiplication by 1 or -1 on the discriminant group A_X.*

Let $e \in H^2(X,\mathbb{Z})$ be a primitive class of negative degree.

The reflection R_e associated to e is:

$$\begin{aligned} R_e : H^2(X,\mathbb{Q}) &\longrightarrow H^2(X,\mathbb{Q}) \\ x &\mapsto x - 2\frac{(x\cdot e)}{(e\cdot e)}e \end{aligned} \tag{1.8}$$

Lemma 1.4.13. *[GHS07, Corollary 3.4] Let X be a hyperkähler manifold of K3$^{[n]}$–type. The reflection R_e in (1.8) belongs to $Mon^2(X)$ if and only if e has one of the following properties:*

1. *$(e\cdot e) = -2$, or*
2. *$(e\cdot e) = -2(n-1)$, and $n-1$ divides the class $(e,-) \in H^2(X,\mathbb{Z})^\vee$.*

Denote by $\widetilde{\Lambda}$ the abstract lattice $U^{\oplus 4} \oplus E_8^{\oplus 2}$.

Let $\mathrm{O}(\Lambda_{\mathrm{K3}^{[n]}}, \widetilde{\Lambda})$ be the set of primitive isometric embeddings. The group of isometries $\mathrm{O}(\widetilde{\Lambda})$ acts on the group $\mathrm{O}(\Lambda_{\mathrm{K3}^{[n]}}, \widetilde{\Lambda})$ by compositions. Note that, if $n-1$ is a prime power, then $\mathrm{O}(\Lambda_{\mathrm{K3}^{[n]}}, \widetilde{\Lambda})$ consists of a single $\mathrm{O}(\widetilde{\Lambda})$–orbit, otherwise

Lemma 1.4.14. *[Mar10, Lemma 4.3]Let $n > 2$. There are $2^{\eta-1}$ distinct $\mathrm{O}(\widetilde{\Lambda})$–orbits in $\mathrm{O}(\Lambda_{\mathrm{K3}^{[n]}}, \widetilde{\Lambda})$ where η corresponds to the number of distinct primes in the factorization of $n-1$.*

Example 1.4.15. *Suppose that X is a hyperkähler of K3$^{[7]}$–type. By Lemma 1.4.14, there exist two different $\mathrm{O}(\widetilde{\Lambda})$–orbits $i, j : \Lambda_{\mathrm{K3}^{[n]}} = \Lambda \oplus \langle -2(n-1)\rangle \hookrightarrow \widetilde{\Lambda} \cong \Lambda \oplus U$, where Λ is isomorphic to $U^{\oplus 3} \oplus E_8^{\oplus 2}$. In particular, choosing $H^2(X,\mathbb{Z}) \hookrightarrow \Lambda$*

for some isometry, it suffices to find all primitive embeddings $\mathbb{Z}\delta \hookrightarrow U$ *where* $\delta^2 = -12$*: set* $\begin{pmatrix} e \\ f \end{pmatrix} \in U$ *such that* $f \cdot e = -6$ *and* $\gcd(e, f) = 1$.

Hence

$$\begin{pmatrix} e \\ f \end{pmatrix} \in \{u_1 = \begin{pmatrix} 1 \\ -6 \end{pmatrix},\ u_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix},\ u_3 = \begin{pmatrix} 2 \\ -3 \end{pmatrix},\ u_4 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}\}$$

are all possible primitive embeddings. It follows from the action of $\mathrm{O}(\widetilde{\Lambda})$ *that* $u_1 = r^*(u_4)$ *and* $u_2 = r^*(u_3)$ *where* r^* *is the reflection isometry (up to a sign).*

Theorem 1.4.16. *[Mar10, Theorem 1.10] Let* X *be a hyperkähler manifold of* $K3^{[n]}$*-type,* $n \geq 2$*. Then* X *comes with a natural choice of an* $\mathrm{O}(\widetilde{\Lambda})$*-orbit of primitive isometric embeddings of* $H^2(X, \mathbb{Z})$ *in* $\widetilde{\Lambda}$*. This orbit is monodromy invariant which means* $\iota : H^2(X, \mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$ *belongs to this orbit if and only if* $\iota \circ g$ *does, for all* $g \in Mon^2(X)$*.*

Set $\iota_X : H^2(X, \mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$ the primitive isometric embedding provided in Theorem 1.4.16. Since the orthogonal complement of $\iota_X(H^2(X, \mathbb{Z}))$ in $\widetilde{\Lambda}$ is a lattice of rank 1, we can choose a generator v of $\iota_X(H^2(X, \mathbb{Z}))^{\perp}$. It follows from the unimodularity of $\widetilde{\Lambda}$ and $\delta^2 = -2(n-1)$ that $(v \cdot v) = 2(n-1)$.

Definition 1.4.17. *Let* $e \in H^2(X, \mathbb{Z})$ *with* $(e \cdot e) = -2(n-1)$*. The* ***divisibility*** *of* e *is an integer number* $div(e, -) \in \{n-1, 2(n-1)\}$ *such that* $(e, -)/div(e, -)$ *is an integral primitive class in* $H^2(X, \mathbb{Z})^{\vee}$*. The invariants* $\boldsymbol{\rho}$ *and* $\boldsymbol{\sigma}$ *associated to* e *are the largest positive integers such that* $(e + v)/\rho$ *and* $(e - v)/\sigma$ *are integral primitive classes in* $\widetilde{\Lambda}$*, where* v *is a generator of* $\iota_X(H^2(X, \mathbb{Z}))^{\perp}$*.*

Let $\widetilde{L}$ be the smallest primitive sublattice of $\widetilde{\Lambda}$ that contains $\langle v \rangle \oplus \langle e \rangle$. This lattice is known as the saturation of $\langle v \rangle \oplus \langle e \rangle$ in $\widetilde{\Lambda}$.

In the following lemma, Markman characterizes the pairs $(\widetilde{L}, e)$ in terms of the invariants $\mathrm{div}(e, -)$, ρ and σ.

Proposition 1.4.18. *[Mar13, Proposition 6.3] The isometry class of the lattice* $\widetilde{L}$ *is determined as follows:*

$$\widetilde{L} = \begin{cases} U & \text{if } div(e, \cdot) = 2n - 2, \\ (2) \oplus (-2) & \text{if } div(e, \cdot) = n - 1 \text{ and } n \text{ is even}, \\ & \text{if } div(e, \cdot) = n - 1,\ n \equiv 1 \mod 8 \text{ and } \rho\sigma = n - 1, \\ U(2) & \text{if } div(e, \cdot) = n - 1,\ n \text{ is odd},\ n \not\equiv 1 \mod 8, \\ & \text{if } div(e, \cdot) = n - 1,\ n \equiv 1 \mod 8 \text{ and } \rho\sigma = 2(n - 1). \end{cases}$$

Example 1.4.19 (Continuation of Example 1.4.15)**.** *Let e be a primitive class of $H^2(X,\mathbb{Z})$ with $(e\cdot e) = -2(3\times 2)$. Since n is odd, $\{\rho,\sigma\}$ is one of the pair $\{6,4\},\{2,12\}$ or $\{2,6\}$. Let see each possible case:*

- *Suppose that $\rho = 6$ and $\sigma = 4$, then $\tilde{L}$ is the lattice that contains $(-2)\oplus(2)$ and the classes $\frac{e+v}{6}$ and $\frac{e+v}{4}$. Since the discriminant of $\tilde{L}$ is one and its signature is $(1,1)$, then $\tilde{L}\cong U$.*

- *Suppose that $\rho = 2$ and $\sigma = 12$, then $\tilde{L}$ is the lattice that contains $(-2)\oplus(2)$ and the classes $\frac{e+v}{2}$ and $\frac{e+v}{12}$. Since the discriminant of $\tilde{L}$ is one and its signature is $(1,1)$, then $\tilde{L}\cong U$.*

- *Suppose that $\rho = 2$ and $\sigma = 6$, then $\tilde{L}$ is the lattice that contains $(-2)\oplus(2)$ and the classes $\frac{e+v}{2}$ and $\frac{e+v}{6}$. Since the discriminant of $\tilde{L}$ is equal to the discriminant form of $U(2)$, then $\tilde{L}\cong U(2)$.*

Definition 1.4.20. *Let D be a divisor of X and T_D the primitive rank 2 lattice containing v and D in $\widetilde{\Lambda}$. The divisor D is a wall divisor such that there exists $r_D\in T_D$ if*

$$r_D^2 = -2 \text{ and } 0\le (v\cdot r_D)\le v^2/2.$$

Note that the condition to be a wall divisor is equivalent to

$$0\le r_D^2 v^2 \le (v\cdot r_D)^2 < (v^2/2)^2.$$

The set of wall divisors restricts the geometry of X. In particular,

Theorem 1.4.21. *[Mon15, Theorem 1.3 and Proposition 1.5] Let $\mathcal{W}$ be the set of wall divisors on a hyperkähler X of $K3^{[n]}$–type. Then the Kähler cone of X is one of the connected components of the following set*

$$\{x\in H^2(X,\mathbb{R}) \mid (x\cdot x)>0 \text{ and } (x\cdot w)\neq 0\ \forall w\in\mathcal{W}\}.$$

Moduli space of (twisted) sheaves on K3 surfaces

Firstly, we recall some properties of the sheaves in consideration. Secondly, we consider moduli spaces of sheaves on K3 surfaces. We will recall that moduli spaces of (semi) stable sheaves on projective K3 surfaces provide examples of projective hyperkähler manifolds. Then, we consider the case of moduli space of twisted sheaves on K3 surfaces which also provides examples (in some cases) of hyperkähler manifolds. Both type of moduli spaces are of K3$^{[n]}$–type deformation.

Definitions on coherent sheaves

Let X be a compact complex variety of dimension n and $\mathcal{F}$ be a coherent sheaf of X of dimension d. The **torsion filtration** of $\mathcal{F}$ is the unique filtration

$$T(\mathcal{F}):\ 0 \subset T_0(\mathcal{F}) \subset \ldots \subset T_d(\mathcal{F}) = \mathcal{F}$$

where $T_i(\mathcal{F})$ is the maximal subsheaf of dimension $\leq i$.

We say that $\mathcal{F}$ is **torsion free** if the dimension of $\mathcal{F}$ coincides with the dimension of X and $T_{n-1}(\mathcal{F}) = 0$.

The category of coherent sheaves of X can be seen as a generalization of vector bundles on X. In fact, one can show that the category of **locally free** coherent sheaves of finite rank is in correspondence with the category of vector bundles on X. When X is a curve, a vector bundle on X has two numerical invariants: the rank and the degree. It follows from the torsion filtration that the property of being torsion free is equivalent to being locally free. However, in higher dimensions this equivalence is not true, and in general torsion free coherent sheaves are not automatically locally free.

Here and subsequently, (S, H) denotes a surface with an ample divisor H on S.

Definition 1.4.22. *Let $\mathcal{F}$ be a torsion free coherent sheaf on S. The **slope** of $\mathcal{F}$ is*

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot H}{\operatorname{rank} \mathcal{F}},$$

where c_1 is the first Chern class of $\mathcal{F}$.

*A torsion free coherent sheaf $\mathcal{F}$ is **μ-stable** (resp. **μ-semistable**) if $\mu(\mathcal{E}) < \mu(\mathcal{F})$ (resp. $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$) for all coherent subsheaves $\mathcal{E} \subset \mathcal{F}$ with $0 < \operatorname{rank} \mathcal{E} < \operatorname{rank} \mathcal{F}$.*

Recall that the Euler characteristic of a torsion free coherent sheaf $\mathcal{F}$ is

$$\chi(F) = \sum (-1)^i \dim H^i(X, \mathcal{F}).$$

Definition 1.4.23. *The **normalized Hilbert polynomial** of a torsion free coherent sheaf $\mathcal{F}$ is*

$$p_{H,\mathcal{F}}(n) = \frac{\chi(\mathcal{F} \otimes H^n)}{\operatorname{rank} \mathcal{F}}.$$

*A torsion free coherent sheaf $\mathcal{F}$ is **stable** (resp. **semistable**) if $p_{H,\mathcal{E}}(n) < p_{H,\mathcal{F}}(n)$ (resp. $p_{H,\mathcal{E}}(n) \leq p_{H,\mathcal{F}}(n)$) for all proper subsheaves $\mathcal{E} \subset \mathcal{F}$ and $n \gg 0$.*

Lemma 1.4.24. *[Saw16, Lemma 2] Let $\mathcal{F}$ be a torsion free coherent sheaf on S.*

1. *If $\mathcal{F}$ is stable, then $\mathcal{F}$ is simple (i.e., $Hom(\mathcal{F},\mathcal{F}) \cong \mathbb{C}$).*
2. *We have the following implications:*

$$\mathcal{F} \text{ is } \mu\text{-stable} \implies \mathcal{F} \text{ is stable} \implies \mathcal{F} \text{ is semistable} \implies \mathcal{F} \text{ is } \mu\text{-semistable.}$$

For our purpose, we define the Brauer group $\mathrm{Br}(X)$ of a K3 surface X as the torsion part of $H^2(S,\mathcal{O}_X^*)$.

Definition 1.4.25. *A twisted K3 surface (X,α) consists of a K3 surface X with a class $\alpha \in H^2(X,\mathcal{O}_X^*)$.*

Let $\alpha \in H^2(X,\mathcal{O}_X^*)$. We represent to α by a Čech 2–cocycle

$$\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

with respect to an open analytic covering $X = \bigcup U_i$. An $\{\alpha_{ijk}\}$–twisted sheaf E consists of pairs $(\{E_i\},\{\varphi_{ij}\})$ such that E_i is a coherent sheaf on U_i and φ_{ij} : $E_j|_{U_i\cap U_j} \longrightarrow E_i|_{U_i \cap U_j}$ are isomorphisms satisfying the following

$$\varphi_{ii} = \mathrm{Id}, \;\; \varphi_{ji} = \varphi_{ij}^{-1}, \;\; \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk}\,\mathrm{Id}\,.$$

The non-twisted case

The **Mukai lattice** of a K3 surface S is defined as

$$\widetilde{H}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}), \tag{1.9}$$

with the pairing

$$((r,l,s)\cdot(r',l',s')) = -rs' + (l\cdot l') - sr'.$$

Note that the Mukai lattice of S is abstractly isometric to $\widetilde{\Lambda}$. In particular, it is an unimodular lattice and its signature is $(4,20)$.

One can give it a Hodge structure of weight 2 induced by the Hodge structure of $H^2(S,\mathbb{Z})$:

$$\widetilde{H}(S)^{0,2} := H^{0,2}(S) \cong \overline{H^{2,0}(S)} =: \overline{\widetilde{H}(S)^{2,0}}$$
$$\widetilde{H}(S)^{1,1} := H^0(S) \oplus H^{1,1}(S) \oplus H^4(S).$$

Definition 1.4.26. *A vector $\nu = (r,l,s) \in \widetilde{H}(S)$ is called a Mukai vector (not necessarily primitive) if:*

- $r > 0$ *or*
- $r = 0$, $0 \neq l \in \mathrm{NS}(S)$ *is effective and* $s \neq 0$, *or*
- $r = 0 = l$ *and* $s < 0$.

Let $\mathcal{F}$ be a stable sheaf on (S, H). The Mukai vector associated to $\mathcal{F}$ is given by

$$(\operatorname{rank}\mathcal{F}, c_1(\mathcal{F}), \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) + \operatorname{rank}\mathcal{F}),$$

where c_1 and c_2 are the first and second Chern class respectively.

Set $v \in \widetilde{H}(S, \mathbb{Z})$ be a Mukai vector. Gieseker in [Gie77] constructs the moduli space $M_H(v)$ of H–stable sheaves on S with Mukai vector equals to v, as a quasi-projective scheme. This moduli space parameterizes H-stable sheaves $\mathcal{F}$ on S with Mukai vector $v(\mathcal{F}) = v$. If $M_H(v)$ is not already compact, then one can compactify by adding H–semistable sheaves and obtain a new moduli space $\bar{M}_H(v)$ such that it is projective and $M_H(v) \subset \bar{M}_H(v)$ is open, see [HL10, Chapter 4]. We emphasize that the ample divisor H of S does not lie in a wall of the ample cone of S (i.e., H is v–general) and the compactification by adding H–semistable sheaves is actually obtained under S–equivalence classes (i.e., two semistable sheaves are said to be S–equivalent if they have the same graded factors in their Jordan–Hölder filtration with the same normalized Hilbert polynomials).

Example 1.4.27. *Set* $v = (1, 0, 1 - n)$ *be a Mukai vector. Let* $\mathcal{F}$ *be a torsion free sheaf with* $v(\mathcal{F}) = v$. *Hence* $\mathcal{F}$ *has rank one,* $c_1(\mathcal{F}) = 0$ *and* $c_2(\mathcal{F}) = n$. *Since there are no coherent subsheaves* $\mathcal{E} \subset \mathcal{F}$ *such that* $0 < \operatorname{rank}\mathcal{E} < \operatorname{rank}\mathcal{F} = 1$, *then* $\mathcal{F}$ *is automatically* μ*-stable.*

Suppose that H *is a fixed polarization of* S. *If* $\mathcal{E} \subset \mathcal{F}$ *is a proper subsheaf, then* $\operatorname{rank} E = \operatorname{rank}\mathcal{F} = 1$, *and* $\mathcal{F}/\mathcal{E}$ *is free torsion. This implies that* $\mathcal{F}$ *is also stable, and thus*

$$p_{H,\mathcal{F}}(n) - p_{H,\mathcal{E}}(n) = \chi(\mathcal{F} \otimes H^n) - \chi(\mathcal{E} \otimes H^n) = \chi(\mathcal{F}/\mathcal{E} \otimes H^n) > 0$$

for $n \gg 0$. *By [Har88, Corollary 1.4],* $\mathcal{F}^{\vee\vee}$ *is a reflexive sheaf on* S *and therefore locally free; thus* $\mathcal{F}^{\vee\vee} \cong \mathcal{O}_S$. *Since the cokernel of the inclusion* $\mathcal{F} \hookrightarrow F^{\vee\vee}$ *is the structure sheaf* $\mathcal{O}_Z$ *of a* 0*-dimensional subscheme* $Z \subset X$ *of length* n, *we can identify* $\mathcal{F} \cong l_Z$ *the ideal sheaf of* Z. *Then,*

$$0 \longrightarrow \mathcal{F} \cong l_Z \longrightarrow \mathcal{F}^{\vee\vee} \cong \mathcal{O}_S \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and the moduli space $M_H(1,0,1-n)$ is identified with the Hilbert scheme $S^{[n]}$. In particular, for $n=2$ we get the Example 1.4.3.

The following result is due to several people: Mukai [Muk87], Göttsche–Huybrechts [GH96], O'Grady [O'G97], Yoshioka [Yos01]. It can be found in its final form in [Yos01, Proposition 5.1, Theorem 8.1]. It gives some restrictions on the choice of the Mukai vector in order to get a Moduli space of sheaves on a K3 surface that is a hyperkähler manifold of K3$^{[n]}$-type.

Theorem 1.4.28. *Let S be a projective K3 surface, v be a primitive Mukai vector with $(v\cdot v)^2 \geq -2$, and H be a v–general ample class. The moduli space $M_H(v)$ of H-stable sheaves on S with class v is a smooth projective hyperkähler manifold of K3$^{[n]}$–type with $2n = (v\cdot v)+2$.*

One can show that the Hodge structure of $H^2(M_H(v),\mathbb{Z})$ with the Beauville–Bogomolov pairing is determined by the Hodge structure of $v^\perp \subset \widetilde{H}(S,\mathbb{Z})$ since there exist a Hodge isometry $\theta : v^\perp \longrightarrow H^2(M_H(v),\mathbb{Z})$ given by

$$x \overset{\theta}{\mapsto} c_1(\pi_{2_!}(\pi_1^!(x^\vee)\otimes[\mathcal{F}])), \tag{1.10}$$

where $\mathcal{F}$ is a universal sheaf over $S\times M_H(v)$, $[\mathcal{F}]$ denotes the class of $\mathcal{F}$ in $K(S\times M_H(v))$ and π_i is the projection from $S\times M_H(v)$ onto the ith factor. The isometry θ is known as *Mukai's Hodge isometry*, see [Yos01].

The inverse θ^{-1} induces a primitive isometric embedding ι of $H^2(M_H(v),\mathbb{Z})$ in $\widetilde{\Lambda}$ by

$$\iota : H^2(X,\mathbb{Z}) \xrightarrow{\theta^{-1}} v^\perp \subset \widetilde{H}(S,\mathbb{Z}) \cong \widetilde{\Lambda}. \tag{1.11}$$

which is monodromy invariant. In fact, ι is the one described in Theorem 1.4.16 when $M_H(v)$ is a hyperkähler manifold of K3$^{[n]}$–type.

The twisted case

Let α in $H^2(S,\mathrm{O}_S^*)_{\mathrm{tors}}$ be a *Brauer class*. The lattice $\widetilde{H}(S,\alpha,\mathbb{Z})$ is a $\mathbb{Z}$-module isomorphic to the Mukai lattice $\widetilde{H}(S,\mathbb{Z})$ defined in (1.9) but with the following Hodge structure:

$$\widetilde{H}^{2,0}(S,\alpha) = \mathbb{C}\cdot(\sigma + B\wedge\sigma) \text{ and } \widetilde{H}^{1,1}(S,\alpha) = \exp(B)\cdot\widetilde{H}^{1,1}(S),$$

where $0\neq\sigma\in H^{2,0}(S)$ and $B\in H^2(S,\mathbb{Q})$ maps to α under the exponential map

$$H^2(S,\mathbb{Q}) \longrightarrow H^2(S,\mathcal{O}_S) \xrightarrow{\exp} H^2(S,\mathcal{O}_S^*). \tag{1.12}$$

Note that if $\alpha = 1$ the neutral element, we obtain $\widetilde{H}(S,1,\mathbb{Z}) = \widetilde{H}(S,\mathbb{Z})$.

It is possible to construct moduli spaces of twisted sheaves of K3 surfaces as in Theorem 1.4.28. To do this, one can define a twisted Mukai vector v_B associated to an ordinary Mukai vector $v = (r, H, s) \in \widetilde{H}(S,\mathbb{Z})$ by

$$v_B = (r, H + rB, s + B \wedge H + rB^2/2),$$

which is an element of $\widetilde{H}(S,\alpha,\mathbb{Z})$, if and only if n divides r. Then, a moduli space of twisted sheaves $M_H(v_B)$ with Mukai vector v_B exists and has dimension $v_B^2 + 2$, see [Yos06] for more details.

Moreover, for $v_B^2 \geq 2$ there is a Hodge isometry

$$H^2(M_H(v_B),\mathbb{Z}) \simeq v_B^{\perp} \subset \widetilde{\Lambda}.$$

One can show that different choices of a B–lift determine equivalent twisted Mukai vectors v_B in the category of α-twisted coherent sheaves.

Set

$$Q = \{x \in \mathbb{P}(\widetilde{\Lambda} \otimes \mathbb{C}) \mid (x \cdot x) = 0 \text{ and } (x \cdot \bar{x}) > 0\}$$

be the period domain of generalized K3 surfaces.

Definition 1.4.29. *A period $x \in Q$ is of twisted K3 type if there exists a twisted K3 surface (S,α) such that the Hodge structure on $\widetilde{\Lambda}$ defined by x is Hodge isometric to $\widetilde{H}(S,\alpha,\mathbb{Z})$.*

The set of periods of twisted K3 type is denoted by $Q_{K3'}$ while the set of periods of K3 type (where the Brauer class is the class of the identity) is denoted by Q_{K3}.

Note that $Q_{\mathrm{K3}} \subset Q_{\mathrm{K3}'} \subset Q$.

The proof of the following lemma consists in the construction of Moduli space of twisted sheaves on a K3 surface from a lattice theoretical viewpoint, see [Huy17, Lemma 2.6].

Lemma 1.4.30. *Let $x \in Q$. Then, x is a period of twisted K3 type if and only if there exists a (not necessarily primitive) embedding $U(n) \hookrightarrow \widetilde{\Lambda}$ for some $n \neq 0$ into the $(1,1)$ part of the Hodge structure defined by x.*

In particular, when $n = 1$ the period x is of K3 type.

Proof. Let (S,α) be a twisted K3 surface and pick a lift $B \in H^2(S,\mathbb{Q})$ of α. Since $\widetilde{H}^{1,1}(S,\alpha,\mathbb{Z}) = \exp(B) \cdot \widetilde{H}^{1,1}(S,\mathbb{Q}) \cap \widetilde{H}(S,\mathbb{Z})$, the lattice generated by $(0,0,1)$ and $(n, nB, nB^2/2)$, where n is the minimal integer such that in $(n, nB, nB^2/2) \in \widetilde{H}(S,\mathbb{Z})$, is a sublattice of $\widetilde{H}(S,\mathbb{Z})$ isomorphic to $U(n)$.

Conversely, assume $U(n) \subset \widetilde{\Lambda}$ is in $\widetilde{\Lambda}_x^{1,1}$ the algebraic part with respect to x. Let n be the minimal integer where $e_n = e$ is primitive in $\widetilde{\Lambda}$. Then, $e \in U(n)$ can be completed to a sublattice of $\widetilde{\Lambda}$ which is isomorphic to U generated by e and a class f. This induces an orthogonal decomposition of $\widetilde{\Lambda} \simeq \Lambda \oplus U$. The second basis vector f_n of $U(n)$ can be written as $f_n = \gamma + nf + ke$ where $\gamma \in \Lambda$. The generator of the $(2,0)$–part of the Hodge structure determined by x is orthogonal to e and so it can be written as $\sigma + \lambda e$ for some $\sigma \in \Lambda \otimes \mathbb{C}$ and $\lambda \in \mathbb{C}$. Since the generator $(2,0)$–part is orthogonal to f_n, we obtain $(\gamma \cdot \sigma) = -n\lambda$. Set $B = -(1/n)\gamma$, then $\sigma + \lambda e = \sigma + B \wedge \sigma$ where $B \wedge \sigma$ stands for $(B \cdot \sigma)e$.

The surjectivity of the period map implies that $\sigma \in \Lambda \otimes \mathbb{C}$ can be obtained as the period of some K3 surface S, and so there exists a Hodge isometry $H^2(S, \mathbb{Z}) \cong \Lambda$ identifying $H^{2,0}(S)$ with $\mathbb{C} \cdot \sigma \subset \Lambda \otimes \mathbb{C}$. Set α be the Brauer class induced by $B = -(1/n)\gamma$ under the exponential map in (1.12). It induces a Hodge isometry between $\widetilde{H}(S, \alpha, \mathbb{Z})$ and the Hodge structure $\widetilde{\Lambda}_x$. ∎

1.5 Ulrich bundles

Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety over $\mathbb{C}$, and let H be a very ample divisor on X such that $\mathcal{O}_X(H) \cong \mathcal{O}_{\mathbb{P}^N}(1)|_X =: \mathcal{O}_X(1)$. We denote by $d := \deg(X) = H^n \geq 1$ the degree of X, where $n = \dim(X)$.

Given $m \in \mathbb{Z}$ and a coherent sheaf E on X, we write $E(mH) := E \otimes \mathcal{O}_X(mH)$. We say that E is **initialized** if $\mathrm{H}^0(X, E) \neq 0$ and $\mathrm{H}^0(X, E(-H)) = 0$.

Let us recall the following result from [ES03, Section 2] (see also [Bea18, Theorem 2.3]).

Theorem 1.5.1 (Eisenbud–Schreyer–Weyman)**.** *Let E be a rank $r \geq 1$ vector bundle on X. The following are equivalent conditions:*

1. *E admits a linear resolution of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-N+n)^{\oplus a_{N-n}} \to \cdots \to \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus a_1} \to \mathcal{O}_{\mathbb{P}^N}^{\oplus a_0} \to E \to 0.$$

   In particular, $a_0 = r\deg(X)$ and $a_i = \binom{N-n}{i} a_0$ for all i.

2. $\mathrm{H}^i(X, E(-jH)) = 0$ *for all $i \geq 0$ and all $j \in \{1, \dots, n\}$.*

3. $\mathrm{H}^i(X, E(-iH)) = \mathrm{H}^j(X, E(-(j+1)H)) = 0$ *for every $i > 0$ and $j < n$.*

4. *For all finite linear projections $\pi : X \to \mathbb{P}^n$, the sheaf $\pi_* E$ is the trivial sheaf $\mathcal{O}_{\mathbb{P}^n}^{\oplus t}$ for some t.*

Definition 1.5.2. *The vector bundle E is called an* **Ulrich bundle** *if it satisfies any of the equivalent conditions in Theorem 1.5.1.*

As a consequence of the previous result, it can be show that Ulrich bundles enjoy several good properties (see e.g. [Bea18, Section 3]). Let us recall some of them.

- If E is a rank r Ulrich bundle on X, then E is **aCM**, i.e., $\mathrm{H}^i(X, E(jH)) = 0$ for all $j \in \mathbb{Z}$ and $1 \leq i \leq n-1$. Moreover, $h^0(X, E) = r\deg(X)$.

- An Ulrich bundle is 0–regular in the sense of Castelnuovo–Mumford, and hence it is globally generated (see e.g. [Laz04, Section 1.8.A]). In particular, an Ulrich bundle is nef.

- If E is an Ulrich bundle on X with respect to $\mathcal{O}_X(H)$, and $Y \in |H|$ is a smooth hyperplane section, then $E|_Y$ is an Ulrich bundle on Y with respect to $\mathcal{O}_X(H)|_Y$.

- If E is a rank r Ulrich bundle on X, then E is **semistable** with respect to H, i.e., for every non-zero subsheaf $\mathcal{F} \subseteq E$ we have that $\mu_H(\mathcal{F}) \le \mu_H(E)$, where
$$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{F})} \in \mathbb{Q}.$$
This follows from Theorem 1.5.1(4) (see also [CHGS12, Theorem 2.9]).

Definition 1.5.3. *A rank 2 vector bundle E on X is called a* **special Ulrich bundle** *(or Ulrich special) if it is an Ulrich bundle with respect to H and we have that*
$$\det(E) = \omega_X \otimes \mathcal{O}_X((n+1)H).$$

Definition 1.5.4. *Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(H)$ a very ample line bundle. For a vector bundle E on X, we define its* **Ulrich dual** *with respect to H by*
$$E^{\mathrm{ul}} := E^\vee \otimes \mathcal{O}_X(K_X + (n+1)H).$$
In particular, it follows from Serre duality and Theorem 1.5.1(3), that E is an Ulrich bundle with respect to H if and only if E^{ul} is an Ulrich bundle with respect to H.

Remark 1.5.5. *By Serre duality, the fact that $\mathrm{H}^0(X, E^{\mathrm{ul}}(-H)) = 0$ (cf. the* initialized *condition) is equivalent to $\mathrm{H}^n(X, E(-nH)) = 0$. Moreover, since for every rank 2 vector bundle E on a smooth projective variety X we have that $E \cong E^\vee \otimes \det(E)$, it follows that a rank 2 Ulrich bundle E is special (see Definition 1.5.3) if and only if $E \cong E^{\mathrm{ul}}$.*

We recall classical tools in order to construct special Ulrich bundles.

One of this tools is by applying the known property as the Cayley–Bacharach property. Roughly speaking, a set of points A in a n–dimensional affine or projective space satisfies the *Cayley–Bacharach property* of degree d if any hypersurface of degree d containing all points of A but one automatically contains the last point.

Definition 1.5.6. *Let $Z \subset X$ be a local complete intersection of codimension two, and L and M be line bundles on X. We say that Z satisfies the Cayley–Bacharach property if for all $Z' \subset Z$ a subscheme with $l(Z') = l(Z) - 1$ and $s \in H^0(X, L^\vee \otimes M \otimes K_X)$ with $s|_{Z'} = 0$, then $s|_Z = 0$.*

An application of this Property can be found in [HL10, Theorem 5.1.1].

The second one tools is known as the *Hartshorne–Serre construction* that can be stated as follows

Theorem 1.5.7. *[Arr07, Theorem 1.1] Let X be a smooth algebraic variety and let Y be a local complete intersection subscheme of codimension two in X. Let $N_{Y/X}$ be the normal bundle of Y in X and let L be a line bundle on X such that $H^2(X, L^\vee) = 0$. Assume that $\bigwedge^2 N_{Y/X} \otimes L^\vee_{|Y}$ admits $r-1$ generating global sections $s_1, ..., s_{r-1}$. Then there exists a rank r vector bundle E over X such that*

- *$\bigwedge^r E = L$:*
- *E has $r-1$ global sections $\alpha_1, ..., \alpha_{r-1}$ whose dependency locus is Y and such that $s_1\alpha_{1|Y} + ... + s_{r-1}\alpha_{r-1|Y} = 0$.*

Moreover, if $H^1(X, L^\vee) = 0$, E is unique (up to isomorphisms).

This technique allows to construct vector bundles from local complete intersection subschemes of codimension two. This will be done, as in the correspondence of hypersurfaces and line bundles, by patching together local determinantal equations in order to produce sections of a vector bundle. For a deeper discussion of this technique we refer to [Arr07].

1.6 Uniruled and rationally connected varieties

We refer the reader to [Deb01, Ch. 4] for an introduction to uniruled and rationally connected varieties, as well as their main properties.

Definition 1.6.1. *A variety X is* ***uniruled*** *if there exists on X a rational curve whose deformations cover a dense open subset of X.*

A non-trivial fact is that for Fano varieties there always exists a rational curve through every point. In particular, Fano varieties are uniruled.

Definition 1.6.2. *A variety X of dimension n is* ***rationally connected*** *if it is proper and if there exists a variety M of dimension $n-1$ and a rational map $e : \mathbb{P}^1 \times M \dashrightarrow X$ such that the rational map*

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 \times M & \dashrightarrow & X \times X \\ (t, t', z) & \mapsto & (e(t,z), e(t',z)) \end{array}$$

is dominant.

When the variety is proper, rational connectedness is a birational property. This means that if X is rationally connected variety and $X \dashrightarrow Y$ is a dominant rational map (i.e., the image is dense), then Y is rationally connected. In particular, a proper unirational variety is rationally connected.

Numerical characterization

Let us first recall the main results concerning the semi-stability of sheaves with respect to a movable curve class, a notion introduced in [CP11] and further developed in [GKP16].

Set $\mathrm{N}_1(X)_{\mathbb{R}}$ be the real vector space of numerical curve classes on X.

Definition 1.6.3. *A curve class $\alpha \in \mathrm{N}_1(X)_{\mathbb{R}}$ is called* **movable** *if $D \cdot \alpha \geq 0$ for every effective Cartier divisor D on X. The set of movable classes form a closed convex cone $\mathrm{Mov}_1(X) \subseteq \mathrm{N}_1(X)_{\mathbb{R}}$, called the* **movable cone** *of X.*

Remark 1.6.4. *Since X is smooth and projective, it follows from [BDPP13] that $\mathrm{Mov}_1(X)$ is the closure of the convex cone in $\mathrm{N}_1(X)_{\mathbb{R}}$ generated by classes of curves whose deformations cover a dense subset of X. Moreover, a numerical divisor class $[D] \in \mathrm{N}^1(X)_{\mathbb{R}}$ is pseudo-effective if and only if $D \cdot \alpha \geq 0$ for all $\alpha \in \mathrm{Mov}_1(X)$.*

Let $\mathcal{F}$ be a non-zero torsion free coherent sheaf on X. Recall that

$$c_1(\mathcal{F}) = (\bigwedge^r \mathcal{F})^{\vee\vee},$$

where $r = \mathrm{rk}(\mathcal{F}) \geq 1$ is the (generic) rank of $\mathcal{F}$. The **slope** of $\mathcal{F}$ with respect to a movable curve class $\alpha \in \mathrm{Mov}_1(X)$ is defined by

$$\mu_\alpha(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot \alpha}{\mathrm{rk}(\mathcal{F})} \in \mathbb{R}.$$

As before, we say that $\mathcal{F}$ is **semistable** with respect to α if

$$\mu_\alpha(\mathcal{G}) \leq \mu_\alpha(\mathcal{F})$$

for every non-zero subsheaf $\mathcal{G} \subseteq \mathcal{F}$.

As it was already observed in [CP11] (cf. [GKP16]), many of the properties of classical slope semi-stability (with respect to an ample divisor) extend to this setting. For instance, the following quantities

$$\mu_\alpha^{\max}(\mathcal{F}) := \sup \{\mu_\alpha(\mathcal{G}),\ \mathcal{G} \subseteq \mathcal{F} \text{ non-zero coherent subsheaf}\},$$
$$\mu_\alpha^{\min}(\mathcal{F}) := \inf \{\mu_\alpha(\mathcal{Q}),\ \mathcal{F} \twoheadrightarrow \mathcal{Q} \text{ non-zero torsion-free quotient}\},$$

are finite, they satisfy $\mu_\alpha^{\max}(\mathcal{F}) = -\mu_\alpha^{\min}(\mathcal{F}^\vee)$, and they can be computed by the **Harder–Narasimhan filtration** of $\mathcal{F}$ with respect to α. Namely, there exists a unique filtration

$$\mathrm{HN}_\bullet^\alpha(\mathcal{F}): \quad 0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_\ell = \mathcal{F},$$

where each quotient $\mathcal{Q}_i := \mathcal{F}_i / \mathcal{F}_{i-1}$ is torsion-free, semistable with respect to α, and where we have $\mu_\alpha^{\max}(\mathcal{F}) = \mu_\alpha(\mathcal{Q}_1) > \mu_\alpha(\mathcal{Q}_2) > \cdots > \mu_\alpha(\mathcal{Q}_\ell) = \mu_\alpha^{\min}(\mathcal{F})$. In particular, $\mathcal{F}$ is semistable with respect to α if and only if $\mu_\alpha^{\max}(\mathcal{F}) = \mu_\alpha^{\min}(\mathcal{F})$.

Using the above notation, we can state the following remarkable results by Boucksom, Demailly, Păun and Peternell in [BDPP13, Theorem 2.6] and by Campana and Păun in [CP19, Theorem 4.7] (see also [Cla17, Section 1.5]).

Theorem 1.6.5. *Let X be a smooth projective manifold and T_X be the tangent bundle of X. Then*

1. *X is uniruled if and only if there exists $\alpha \in \mathrm{Mov}_1(X)$ with $\mu_\alpha^{\max}(T_X) > 0$;*

2. *X is rationally connected if and only if there exists $\alpha \in \mathrm{Mov}_1(X)$ with $\mu_\alpha^{\min}(T_X) > 0$.*

1.7 Rational homogeneous spaces

A **rational homogeneous space** is a projective manifold X given by the quotient G/P of a semi-simple complex Lie group G by a parabolic subgroup $P \subseteq G$. In particular, if G/P has Picard number one, then it follows that G is a *simple* complex Lie group and P is a *maximal* parabolic subgroup of G (see e.g. [Tev05, Section 7.4.1]).

These manifolds can be classified in terms of the associated simple complex Lie algebra $\mathfrak{g}$ together with the marking of a single node in the corresponding Dynkin diagram. The dimension of the manifold G/P_r, where P_r denotes parabolic subgroup associated to the r-th node of the Dynkin diagram of $\mathfrak{g}$, can be found in [Sno89, Section 9.3].

We summarize the relevant information for us in Table 9.1 and we refer the reader to [MnOSC+15, Table 2] for the geometric description of each manifold G/P_r.

Recall that if X is a smooth projective variety, then $\mathrm{Lie}(\mathrm{Aut}^{\circ}(X)) \cong \mathrm{H}^0(X, T_X)$, where $\mathrm{Aut}^{\circ}(X)$ is the connected component of the identity in $\mathrm{Aut}(X)$.

The automorphism group of rational homogeneous manifolds G/P which are quotient of simple complex Lie groups G have been extensively studied. More precisely, following Demazure [Dem77], we say that the pair (G, P) is **non-exceptional** if $\mathrm{Aut}^{\circ}(G/P) \cong G$. The exceptional cases (i.e., for which there is a different pair (G', P') such that $G'/P' \cong G/P$) are well-known (see e.g. [Tit63, Footnote 6] and [Dem77, Section 2]): they correspond geometrically to the odd-dimensional projective space $\mathbb{P}^{2\ell-1}$, the Spinor variety $\mathbb{S}_{\ell}$, and the smooth quadric hypersurface $\mathbb{Q}^5 \subseteq \mathbb{P}^6$.

Lemma 1.7.1. *Let $X \cong G/P$ be a rational homogeneous space of Picard number one and dimension n. Then, X is isomorphic to $\mathbb{P}^n$, $\mathbb{Q}^n$ or $\mathrm{Gr}(2,5)$ if and only if*

$$\dim \mathrm{H}^0(X, T_X) \geq \frac{n(n+2)}{2}.$$

Proof. Following the notation of Table 9.1, a straightforward case-by-case analysis shows that if G is a classical Lie group of type

- A_{ℓ}, then the parabolic subgroup P_r is associated to the node $r = 1$ or $r = \ell$ (i.e., $X \cong \mathbb{P}^{\ell}$), unless $(r, \ell) \in \{(2,3), (2,4), (3,4)\}$. The latter cases correspond to $\mathrm{Gr}(2,4) \cong \mathbb{Q}^4 \subseteq \mathbb{P}^5$ and $\mathrm{Gr}(2,5) \cong \mathrm{Gr}(3,5)$.

- B_{ℓ}, then the parabolic subgroup P_r is associated to the node $r = 1$ (i.e.,

$X \cong \mathbb{Q}^{2\ell}$), unless $(r,\ell) = (2,2)$. The latter case corresponds to $\mathbb{S}_2 \cong \mathbb{P}^3$ (cf. [IP99, Examples 2.1.9 (a)]).

- C_ℓ, then the parabolic subgroup P_r is associated to the node $r = 1$ (i.e., $X \cong \mathbb{P}^{2\ell-1}$).
- D_ℓ, then the parabolic subgroup P_r is associated to the node $r = 1$ (i.e., $X \cong \mathbb{Q}^{2\ell-1}$), unless $(r,\ell) = (4,4)$. The latter case corresponds to $\mathbb{S}_3 \cong \mathbb{Q}^6 \subseteq \mathbb{P}^7$ (cf. [IP99, Examples 2.1.9 (b)]).

Similarly, we note that G cannot be of type E_ℓ, F_4 or G_2. ■

Definition 1.7.2. *Let V be a minimal dominating family of rational curves on X, and $x \in X$ be a general point. The rational* ***tangent map*** *is*

$$\begin{aligned} \tau_x : V_x &\dashrightarrow \mathbb{P}(T_x) && (1.13)\\ l &\mapsto \mathbb{P}(T_l|_x^\vee) && (1.14) \end{aligned}$$

associating to a curve through x its tangent direction at x.

The ***Variety of Minimal Rational Tangents*** *at a general point x is $\overline{\tau_x(V_x)}$.*

Remark 1.7.3. *If X is an n-dimensional Fano manifold of Picard number one (not necessarily rational homogeneous), then it is expected that* $\dim \mathrm{H}^0(X,T_X) \le n^2+2n$ *with equality if and only if $X \cong \mathbb{P}^n$ (see [HM05, Conjecture 2]). A positive result in this direction was recently obtained in [FOX18, Theorem 1.2] (cf. [HM05, Theorem 1.3.2]), where the authors prove that if the* Variety of Minimal Rational Tangents *(VMRT) at a general point of X is smooth irreducible and linearly non-degenerate, then* $\dim \mathrm{H}^0(X,T_X) > n(n+1)/2$ *if and only if X is isomorphic to $\mathbb{P}^n$, $\mathbb{Q}^n$ or* $\mathrm{Gr}(2,5)$.

This condition on the VMRT holds true for the remarkable class of rational homogeneous spaces which are **irreducible Hermitian symmetric spaces** *of compact type (see e.g. [FH12, Main Theorem]). These manifolds were classified by Cartan and they correspond to Grassmannians* $\mathrm{Gr}(r,n)$, *smooth quadric hypersurfaces $\mathbb{Q}^n \subseteq \mathbb{P}^n$, Lagrangian Grassmannians* $\mathrm{Lag}(2n)$, *Spinor varieties $\mathbb{S}_n$, the Cayley plane $\mathbb{OP}^2$, and the rational homogeneous space E_7/P_1 of dimension* 27. *However, there are rational homogeneous spaces of Picard number one whose VMRT is linearly degenerate (see e.g. [Rus12, Table 2] and [Hwa01, Section 1.4.6]). We refer to [KSC06] and [Hwa01] for comprehensive surveys of the theory of Varieties of Minimal Rational Tangents, developed by Hwang, Mok and Kebekus in [HM99, Keb02, HM04].*

Let us recall a few basic facts about such quotients G/P:

Set by $\mathfrak{g}$ the Lie algebra of G and by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra and $\Phi \subset \mathfrak{h}^\vee \cong \mathfrak{h}$ is the set of roots of $\mathfrak{g}$. Moreover $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ will be a basis of simple roots of Φ with Bourbaki's notation, and we will denote by $\omega_1, \ldots, \omega_n \in \mathfrak{h}$ the corresponding set of fundamental weights.

Any parabolic subgroup $P = P(\Sigma)$ is uniquely defined by a subset $\Sigma \subset \Delta \cong \{1, \cdots, n\}$ of the set of vertices of the Dynkin diagram associated to G. The Lie algebra $\mathfrak{p}$ of (a conjugate of) $P(\Sigma)$ is given by

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+(\Sigma)} \mathfrak{g}_\alpha,$$

where $\Phi^+(\Sigma) := \{\alpha \in \Phi^+ \mid \alpha = \sum_{\alpha_i \notin \Sigma} c_i \alpha_i\}$.

Homogeneous vector bundles on G/P are in one to one correspondence with representations of P. For any simple root α_i in Σ, there exists a homogeneous line bundle L_i which corresponds to the one dimensional representation of P whose highest weight with respect to $\mathfrak{h}$ is ω_i. With the choices we have made, L_i is a positive line bundle. The Picard group of G/P is equal to

$$\mathrm{Pic}(G/P) = \bigoplus_{i \in \Sigma} \mathbb{Z} L_i.$$

A line bundle over G/P is thus a linear combination of the bundles L_i. Let $L = \sum_{i \in \Sigma} a_i L_i$ be such a line bundle. It is positive (i.e. globally generated) if and only if $a_i \geq 0$ for all $i \in \Sigma$; it is ample if and only if it is very ample if and only if $a_i > 0$ for all $i \in \Sigma$. Since $-K_{G/P}$ is ample, we know that

$$\det(T_{G/P}) = -K_{G/P} = \sum_{i \in \Sigma} j_i L_i$$

for some integers $j_i > 0$ for all $i \in \Sigma$.

Lemma 1.7.4. *Let $X = G/P(\Sigma)$ be a homogeneous rational projective variety and anti-canonical bundle $-K_X = \sum_{i \in \Sigma} j_i L_i$. Then $j_i < \dim(X)$ for all $i \in \Sigma$, except when $X = \mathbb{P}^n$, $X = \mathbb{Q}^n$ or $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$.*

The proof we will give of this lemma will be essentially combinatorial, but before let us make a few remarks.

Remark 1.7.5. *When the Picard number of X is equal to one the result is clear, since the only Fano varieties whose index i_X is greater or equal than* $\dim(X)$ *are projective spaces or quadrics by [KO73]. So we are reduced to the case when the Picard number is greater than one.*

Remark 1.7.6. *When G is classical, the lemma can be derived from the explicit description of $X = G/P$ as a flag manifold. Let us suppose that G is of type A_{n-1} and the Picard number of $X = G/P$ is greater than one. Then X is a flag manifold $X = \mathrm{Fl}(i_1, \dots, i_k, n)$, where $\Sigma = \{i_1, \dots, i_k\}$. The line bundle L_h is the determinant $L_h = \det(\mathcal{U}_h^\vee)$ of the dual of the tautological bundle of rank i_h. It is easy to deduce the explicit formula for the canonical bundle:*

$$-K_X = L_1^{i_2} \otimes L_2^{i_3 - i_1} \otimes \cdots \otimes L_k^{n - i_{k-1}}.$$

Since $\dim(X) \geq n$, *it is straightforward to check that $j_h = i_{k+1} - i_{k-1}$ is strictly smaller than* $\dim(X)$ *for $h \in \{1, \dots, k\}$.*

For the other classical groups, one could proceed similarly using the fact that the corresponding homogeneous varieties $X = G/P$ are zero loci of a general section of $\wedge^2 \mathcal{U}_k^\vee$ (type C_m) or $S^2 \mathcal{U}_k^\vee$ (type B_m and D_m) inside $\mathrm{Fl}(i_1, \dots, i_k, n)$, *which allows to use adjunction in order to understand $-K_X$. However, this strategy does not generalize straightforwardly to the exceptional groups.*

Proof. Let us assume that $X = G/P(\Sigma)$ is a homogeneous rational projective variety with Picard number greater than one (see Remark 1.7.5). The tangent bundle of X is homogeneous, and it corresponds to a P-representation $\mathfrak{T}$. Since the action of G on X is homogeneous, we get that

$$\mathfrak{T} \cong \mathfrak{p}^\perp \cong \bigoplus_{\alpha \in \Phi^+, \alpha \notin \Phi^+(\Sigma)} \mathfrak{g}_\alpha,$$

where the last expression is the decomposition of $\mathfrak{T}$ in irreducible $\mathfrak{h}$-modules. As a result the $\mathfrak{h}$-weight of $\det(\mathfrak{T})$ is equal to $c_\Sigma := \sum_{\alpha \in \Phi^+, \alpha \notin \Phi^+(\Sigma)} \alpha$. Since this should be a weight of a one dimensional representation of P (corresponding to the line bundle $\det(T_X) = -K_X$), one can write it as

$$c_\Sigma = \sum_{i \in \Sigma} j_i \omega_i,$$

where the j_i's are the same as those appearing in the expression of $-K_X$. Notice that $j_i = (c_\Sigma, H_{\alpha_i})$, where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{h}$ and $H_{\alpha_i} = 2\frac{\alpha_i}{(\alpha_i, \alpha_i)}$ is the co-root of α_i. Recall finally that $(\alpha_i, H_{\alpha_j}) = 2$ if $i = j$, while it is negative if $i \neq j$

(and strictly negative if α_i, α_j are not orthogonal).

Let us focus our attention on one of the exponents j_i for $i \in \Sigma$. Since the Picard number of X is greater than one, $\dim(X) = \dim(G/P(\Sigma)) > \dim(G/P(\{i\}))$. For any positive root $\alpha \in \Phi^+$, if $\alpha \notin \Phi^+(\{i\})$ then $\alpha \notin \Phi^+(\Sigma)$. Moreover if $\alpha \in \Phi^+(\{i\})$ then $(\alpha, H_{\alpha_i}) \leq 0$. Now we will distinguish two cases.

The first case is when there exists $h \in \Sigma$, $h \neq i$ which is contained in the same connected component of the Dynkin diagram of G that contains the node i. Then one can check easily that there exists $\alpha \in \Phi^+(\{i\})$ but $\alpha \notin \Phi^+(\Sigma)$ such that $(\alpha, H_{\alpha_i}) < 0$. Putting everything together with the fact that $(c_{\{i\}}, H_{\alpha_i}) = i_{G/P(\{i\})} \leq \dim(G/P(\{i\})) + 1$, we obtain that

$$j_i = (c_\Sigma, H_{\alpha_i}) < (c_{\{i\}}, H_{\alpha_i}) \leq \dim(G/P(\{i\})) + 1 \leq \dim(G/P(\Sigma)),$$

thus proving the inequality we wanted.

The second case is when i is the only element in Σ which is contained in its own connected component inside the Dynkin diagram of G. In such a situation and contrary to what happened in the first case, we deduce that $(c_\Sigma, H_{\alpha_i}) = (c_{\{i\}}, H_{\alpha_i})$, which in general gives $j_i \leq \dim(G/P(\Sigma))$. Moreover we deduce that $G/P(\Sigma) = G/P(\{i\}) \times G/P(\Sigma \setminus \{i\})$. Therefore, if $\dim(G/P(\Sigma \setminus \{i\})) > 1$ or if $i_{G/P(\{i\})} < \dim(G/P(\{i\})) + 1$ we obtain $j_i < \dim(G/P(\Sigma))$; the two conditions are not satisfied only when $G/P(\Sigma \setminus \{i\}) = \mathbb{P}^1$ and $G/P(\{i\}) = \mathbb{P}^l$, i.e., when $X = \mathbb{P}^1 \times \mathbb{P}^l$. ■

2 | An elliptic K3 surface with a symplectic automorphism of order three

The aim of this chapter is to describe a specific elliptic K3 surface S with a symplectic automorphism σ of order 3. We will describe the geometry of S, and we determine the action of $\sigma^* : \mathrm{NS}(S) \longrightarrow \mathrm{NS}(S)$. From this description, we extend the action of σ^* on $H^2(S,\mathbb{Z})$ passing to Λ_{K3}. This K3 surface is well known and it is one of the so called two most algebraic K3 surface, see [SI77] and [Vin83].

2.1 The elliptic K3 surface S

Let ζ_3 be a third root of unity and $E_{\zeta_3} = \mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$ be the elliptic curve with complex multiplication of order 3. Its Weierstrass equation is given by $E_{\zeta_3} : y^2 = x^3 - 1$, see [[Har77], Example 4.20.2].

Proposition 2.1.1. *Let $\alpha : E_{\zeta_3} \longrightarrow E_{\zeta_3}$ be the map $z \mapsto \zeta_3 z$. The map α has 3 fixed points.*

Proof. The map α can be seen as follows $z = e^{\phi i} \mapsto \alpha(z) = e^{(\phi + \frac{2\pi}{3})i}$, where clearly has 3 fixed points v_1, v_2, v_3, see Fig.2.1.

■

Proposition 2.1.2. *Let S be the surface obtained as the minimal resolution of the quotient $(E_{\zeta_3} \times E_{\zeta_3})/\langle \alpha \times \alpha^2 \rangle$. Then, S is an elliptic K3 surface that comes with a natural elliptic fibration induced by E_{ζ_3}.*

Proof. For simplicity, we denote $E_{\zeta_3} \times E_{\zeta_3}$ by A. Since the map α has 3 fixed points v_1, v_2, v_3, then (v_i, v_j) $(i,j = 1,2,3)$ are the fixed points of $\alpha \times \alpha^2$. This implies that the quotient $A/\langle \alpha \times \alpha^2 \rangle$ has 9 singular points at $p_{ij} = [(v_i, v_j)]$. Each singularity p_{ij} is a rational double point which locally corresponds to a singularity

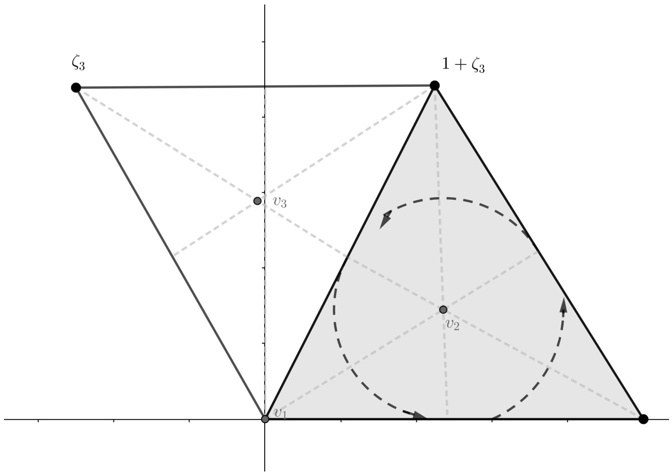


Figure 2.1

of A_2-type. The minimal resolution S of $A/\langle\alpha \times \alpha^2\rangle$ is given by introducing two non-singular rational curves C_{ij}, C'_{ij} for each p_{ij} such that $(C_{ij})^2 = (C'_{ij})^2 = -2$ and $(C_{ij} \cdot C'_{ij}) = 1$, see [[BHPVdV04], Theorem 7.1].

First, we show that there exists a nowhere-vanishing holomorphic 2-form on S. Since A is a 2-dimensional complex torus, it has a nowhere-vanishing holomorphic 2-form invariant under the action of $\langle\alpha \times \alpha^2\rangle$. Let us denote it by ω. Then, ω induces a holomorphic 2-form on the open set deleting the singular point that can be extended to a holomorphic 2-form on the minimal resolution since the singularities are double points, see [[BHPVdV04], Proposition 3.5, Theorem 7.2]. The fact that S is a K3 surface and not a complex torus follows immediately from the non-empty collection of rational curves in S.

The surface S is naturally endowed with an elliptic fibration induced by one of the projection (which is trivially an elliptic fibration) on $E_{\zeta_3} \times E_{\zeta_3}$, see Figure 2.2.

$$\begin{tikzcd} & A \arrow[r, "p"] \arrow[d, "q_A"] \arrow[dl, dashed, "3:1"'] & E_{\zeta_3} \arrow[d, "q_{E_{\zeta_3}}"] \\ S \arrow[r, "\epsilon"] & A/\langle\alpha \times \alpha^2\rangle \arrow[r, "\bar{p}"'] & \mathbb{P}^1 \end{tikzcd}$$

Figure 2.2

where p and $\bar{p}$ are one of the projection maps in A and $A/\langle\alpha \times \alpha^2\rangle$ respectively, $q_{E_{\zeta_3}}$ and q_A are the quotient maps in E_{ζ_3} and A respectively. ■

The Weierstrass equation of S

Let us find the Weierstrass equation of S with $\pi := \bar{p} \circ \epsilon$: let $v_i^2 = u_i^3 - 1$ be the equation of the i-th copy of E_{ζ_3} in $E_{\zeta_3} \times E_{\zeta_3}$ and we assume the automorphism α to be $(v_i, u_i) \mapsto (v_i, \zeta_3 u_i)$. Hence, the functions $y = v_1 u_2^6$, $x = u_1 u_2^4$ are invariant for the action of $\langle \alpha \times \alpha^2 \rangle$ and satisfies the equation

$$y^2 = x^3 - (v_2^2 + 1)^4, \tag{2.1}$$

which is the Weierstrass equation of $(E_{\zeta_3} \times E_{\zeta_3})/\langle \alpha \times \alpha^2 \rangle$.

Its homogeneous form is

$$y^2 = x^3 - (v_2^2 + w_2^2)^4 w_2^4. \tag{2.2}$$

By applying the change of coordinates $t := v_2 + \sqrt{3} w_2$ and $s := v_2 - \sqrt{3} w_2$ (and multiplying the last term by $6\sqrt{3}$) one obtains the Weierstrass equation of S

$$S : y^2 = x^3 - (t^3 - s^3)^4. \tag{2.3}$$

Taking $s = 1$, we obtain

$$A(t) = 0 \in H^0(\mathbb{P}^1, \mathcal{O}(2)^{\otimes 4}), \ B(t) = (t^3 - 1)^4 \in H^0(\mathbb{P}^1, \mathcal{O}(2)^{\otimes 6}),$$

$$\Delta(t) = 27(t^3 - 1)^8 \in H^0(\mathbb{P}^1, \mathcal{O}(2)^{\otimes 12}).$$

The discriminant $\Delta(t) = 0$ if and only if $t = 1, \zeta_3, -\zeta_3$. This implies, that there are three singular fibers over each value of t. Let $a(t)$, $b(t)$, $\delta(t)$ be the order vanishing of A, B and Δ at t respectively. Then, $a(t) = \infty$, $b(t) = 4$, and $\delta(t) = 8$ for all $t \in \{1, \zeta_3, -\zeta_3\}$. By Table 9.3, we can conclude that π has three reducible fibers of type IV^* ($C^{(j)}$, $j = 1, 2, 3$), where each reducible fiber with components $C_i^{(j)}$, $i = 0, ..., 6$, intersect as in (2.4).

$$\begin{tikzcd}[arrows=dash] C_0^{(j)} \arrow[r] & C_1^{(j)} \arrow[r] & C_2^{(j)} \arrow[r] \arrow[d] & C_5^{(j)} \arrow[r] & C_6^{(j)} \\ & & C_3^{(j)} \arrow[d] & & \\ & & C_4^{(j)} & & \end{tikzcd} \tag{2.4}$$

We also obtain that the Equation 2.3 admits the sections

$$s_\pm : t \mapsto (x(t), y(t)) = (0, \pm(t^3 - 1)^2).$$

In particular, $T_1 := Z(s_+)$ and $T_2 := Z(s_-)$ are 3-torsion sections.

Set $\mathcal{O} = Z(s_0)$ the zero section of the fibration π. Hence $\{\mathcal{O}, T_1, T_2\}$ with the group structure of $\mathrm{MW}(S)$ is a copy of $\mathbb{Z}/3\mathbb{Z}$.

The three fibers satisfy the following intersections:

$$(C_0^{(j)} \cdot C_1^{(j)}) = (C_1^{(j)} \cdot C_2^{(j)}) = (C_2^{(j)} \cdot C_3^{(j)}) = (C_3^{(j)} \cdot C_4^{(j)}) = (C_2^{(j)} \cdot C_5^{(j)}) = (C_5^{(j)} \cdot C_6^{(j)}) = 1,$$

$$(C_0^{(j)} \cdot \mathcal{O}) = (C_4^{(j)} \cdot T_1) = (C_6^{(j)} \cdot T_2) = 1,$$
$$(C_i^{(j)})^2 = \mathcal{O}^2 = T_1^2 = T_2^2 = -2,$$

and all the other intersections are 0, see Figure 2.3.

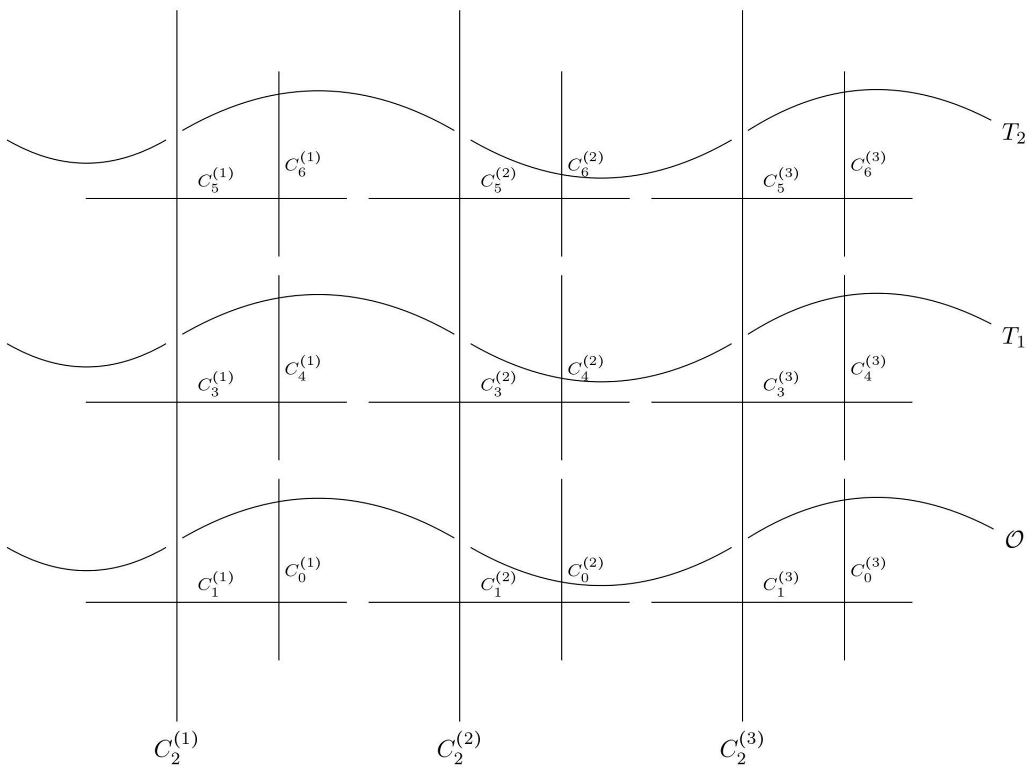


Figure 2.3

Let F be the class of the fiber. By Table 9.3, we can write F as

$$F = C_0^{(j)} + 2C_1^{(j)} + 3C_2^{(j)} + 2C_3^{(j)} + C_4^{(j)} + 2C_5^{(j)} + C_6^{(j)}, \ j = 1, 2, 3.$$

Let us write T_1 and T_2 in terms of the classes F, $\mathcal{O}$, and $C_i^{(j)}$ for $i = 1, ..., 6$, $j = 1, 2, 3$.

Proposition 2.1.3.

$$T_1 = 2F + \mathcal{O} - \frac{1}{3}\sum_{j=1}^{3}(3C_1^{(j)} + 6C_2^{(j)} + 5C_3^{(j)} + 4C_4^{(j)} + 4C_5^{(j)} + 2C_6^{(j)}),$$

$$T_2 = 2F + \mathcal{O} - \frac{1}{3}\sum_{j=1}^{3}(3C_1^{(j)} + 6C_2^{(j)} + 4C_3^{(j)} + 2C_4^{(j)} + 5C_5^{(j)} + 4C_6^{(j)}).$$

Proof. Let $T_1 = \alpha F + \beta\mathcal{O} + \sum_{i=1}^{6}\sum_{j=1}^{3}\gamma_{ij}C_i^{(j)}$.

From $(T_1 \cdot F) = 1$, we get $\beta = 1$. Since $(T_1 \cdot \mathcal{O}) = 0$, then $\alpha - 2\beta = 0$ and so $\alpha = 2$. Now, for all $j = 1, 2, 3$ we have

$$(T_1 \cdot C_k^{(j)}) = \begin{cases} 1, & \text{if } k = 4 \\ 0, & \text{if } k \neq 4. \end{cases}$$

Then, the following linear system

$$\begin{aligned} -2\gamma_{1j} + \gamma_{2j} &= 0 \ (k = 1) \\ \gamma_{1j} - 2\gamma_{2j} + \gamma_{3j} + \gamma_{5j} &= 0 \ (k = 2) \\ \gamma_{2j} - 2\gamma_{3j} + \gamma_{4j} &= 0 \ (k = 3) \\ \gamma_{3j} - 2\gamma_{4j} &= 1 \ (k = 4) \\ \gamma_{2j} - 2\gamma_{5j} + \gamma_{6j} &= 0 \ (k = 5) \\ \gamma_{5j} - 2\gamma_{6j} &= 0 \ (k = 6) \end{aligned}$$

has unique solution at $\gamma_{1j} = -1, \gamma_{2j} = -2, \gamma_{3j} = -\frac{5}{3}, \gamma_{4j} = \gamma_{5j} = -\frac{4}{3}, \gamma_{6j} = -\frac{2}{3}$ for all $j = 1, 2, 3$.

In a similar way, we obtain

$$T_2 = 2F + \mathcal{O} - \frac{1}{3}\sum_{j=1}^{3}(3C_1^{(j)} + 6C_2^{(j)} + 4C_3^{(j)} + 2C_4^{(j)} + 5C_5^{(j)} + 4C_6^{(j)}).$$

∎

2.2 The symplectic automorphism σ of order 3

Let us now define the symplectic automorphism of order three induced by a non-zero section of $\mathrm{MW}(X)$. By Proposition 1.3.8, the translation by T_1 is a well defined automorphism of S of order 3 and it is symplectic.

We know that σ preserves each fiber and it acts on each fiber as a translation by T_1. Hence, its action σ^* on the classes $\mathcal{O}$, T_1, T_2, $C_i^{(j)}$ for $i = 1, ..., 6$, $j = 1, 2, 3$ is uniquely determined by the intersection form in 2.3. In other words, since σ^* is an isometry

$$(D_1 \cdot D_2) = (\sigma^*(D_1) \cdot \sigma^*(D_2)),$$

we have that the action σ^* is as in (2.5).

$$\begin{tikzcd}
\mathcal{O} \arrow[r, dash] \arrow[dddrrr, mapsto, bend right] & C_0^{(j)} \arrow[r, dash] \arrow[ddrr, mapsto, bend right] & C_1^{(j)} \arrow[r, dash] \arrow[dr, mapsto, bend right] & C_2^{(j)} \arrow[r, dash] \arrow[d, dash] & C_5^{(j)} \arrow[r, dash] \arrow[ll, mapsto, bend right] & C_6^{(j)} \arrow[r, dash] \arrow[llll, mapsto, bend right] & T_2 \arrow[llllll, mapsto, bend right] \\
 & & & C_3^{(j)} \arrow[ur, mapsto, bend right] \arrow[d, dash] & & & \\
 & & & C_4^{(j)} \arrow[uurr, mapsto, bend right] \arrow[d, dash] & & & \\
 & & & T_1 \arrow[uuurrr, mapsto, bend right] & & &
\end{tikzcd} \tag{2.5}$$

The following three orthogonal copies of E_6 are permuted by σ^*:

$$
\begin{tikzcd}
C_1^{(1)} \arrow[r, dash] & C_0^{(1)} \arrow[r, dash] & \mathcal{O} \arrow[r, dash] \arrow[d, dash] & C_0^{(2)} \arrow[r, dash] & C_1^{(2)} \\
 & & C_0^{(3)} & & \\
C_3^{(1)} \arrow[r, dash] & C_4^{(1)} \arrow[r, dash] & T_1 \arrow[r, dash] \arrow[d, dash] & C_4^{(2)} \arrow[r, dash] & C_3^{(2)} \\
 & & C_4^{(3)} & & \\
C_5^{(1)} \arrow[r, dash] & C_6^{(1)} \arrow[r, dash] & T_2 \arrow[r, dash] \arrow[d, dash] & C_6^{(2)} \arrow[r, dash] & C_5^{(2)} \\
 & & C_6^{(3)} & &
\end{tikzcd}
\tag{2.6}
$$

The Nerón–Severi group of S and the isometry σ^* on $\mathrm{NS}(S)$

The components of the three copies of E_6 in 2.6 are 18 classes in $\mathrm{NS}(S)$. Since the discriminant of $E_6 \oplus E_6 \oplus E_6$ is not zero, then these classes are independent (cf. Example 1.1.2).

By Lemma 1.3.3, we have $\rho(S) = 20$ since $r_c = 6$ for singular fiber of type IV^* (See Table 9.3) and $\operatorname{rank} \mathrm{MW}(S) = 0$. In order to get a basis of $\mathrm{NS}(S)$, we add two independent classes.

Set $D := 3\mathcal{O} + C_1^{(1)} + C_1^{(2)} + C_1^{(3)} + 2(C_0^{(1)} + C_0^{(2)} + C_0^{(3)})$.

Note that:

$$
(D \cdot C) = \begin{cases}
3(3\mathcal{O}^2 + 2(C_0^{(1)} \cdot \mathcal{O}) + 2(C_0^{(2)} \cdot \mathcal{O}) + 2(C_0^{(3)} \cdot \mathcal{O})) = 0 & \text{if } C = 3\mathcal{O}, \\
(C_1^{(1)})^2 + 2(C_0^{(1)} \cdot C_1^{(1)}) = 0 & \text{if } C = C_1^{(1)}, \\
(C_1^{(2)})^2 + 2(C_0^{(2)} \cdot C_1^{(2)}) = 0 & \text{if } C = C_1^{(2)}, \\
(C_1^{(3)})^2 + 2(C_0^{(3)} \cdot C_1^{(3)}) = 0 & \text{if } C = C_1^{(3)}, \\
2(3(\mathcal{O} \cdot C_0^{(1)}) + (C_1^{(1)} \cdot C_0^{(1)}) + 2(C_0^{(1)})^2) = 0 & \text{if } C = 2C_0^{(1)}, \\
2(3(\mathcal{O} \cdot C_0^{(2)}) + (C_1^{(2)} \cdot C_0^{(2)}) + 2(C_0^{(2)})^2) = 0 & \text{if } C = 2C_0^{(2)}, \\
2(3(\mathcal{O} \cdot C_0^{(3)}) + (C_1^{(3)} \cdot C_0^{(3)}) + 2(C_0^{(3)})^2) = 0 & \text{if } C = 2C_0^{(3)},
\end{cases}
$$

and

$$(D \cdot C_2^{(3)}) = (C_1^{(3)} \cdot C_2^{(3)}) = 1.$$

The lattice generated by the classes $C_2^{(3)}$ and D has intersection form $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ which is isometric to U.

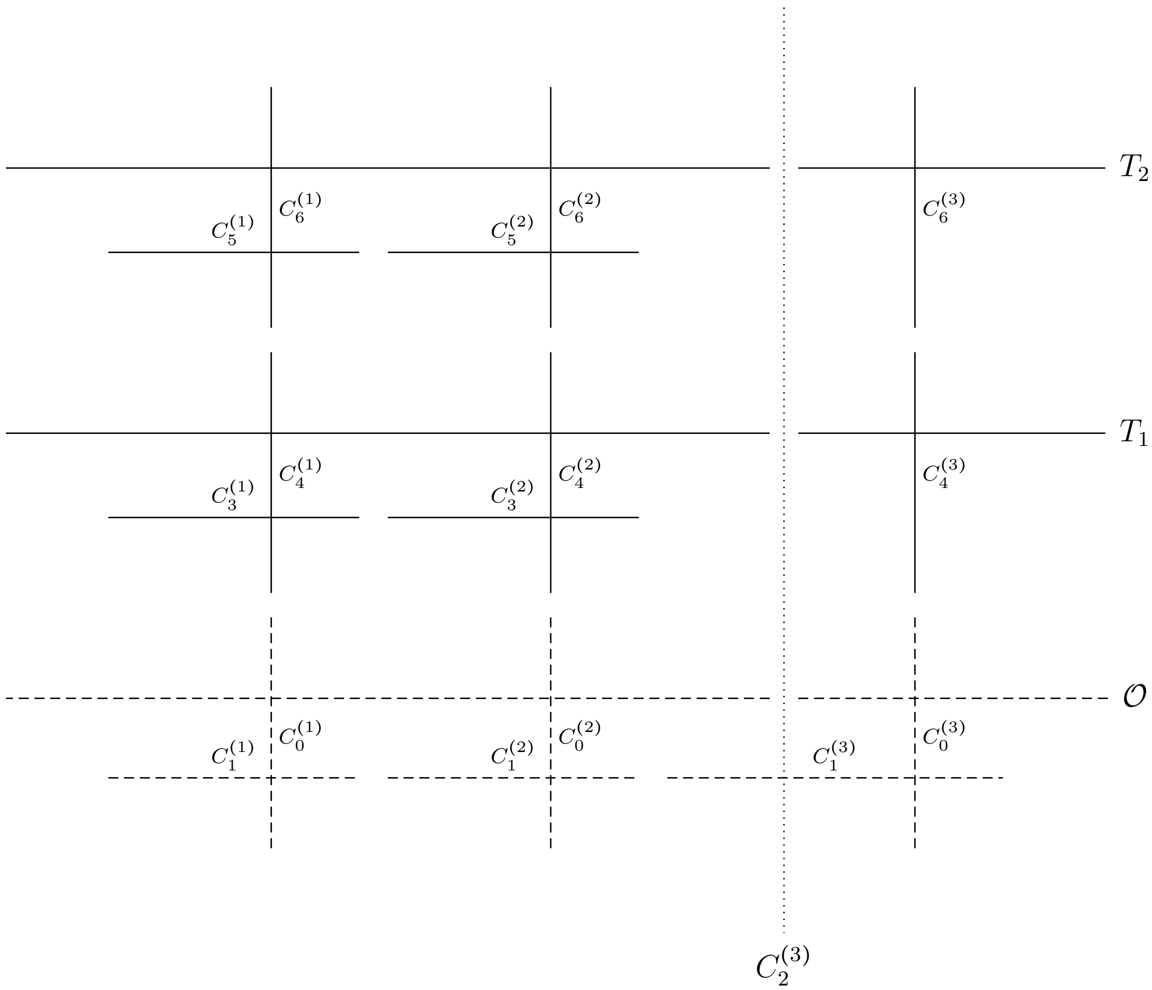
Proposition 2.2.1. *The lattice generated by $C_2^{(3)}$ and D is orthogonal to $E_6 \bigoplus E_6 \bigoplus E_6$.*

Proof. It is straightforward that the lattice generated by D and $C_2^{(3)}$ is orthogonal to the second and third copy of E_6, see (2.7). One can directly check that it is also orthogonal to the first copy of E_6.

Let C be a component of the first copy of E_6. By (2.7),

$$(D \cdot C) = \begin{cases} (C_1^{(1)})^2 + 2(C_0^{(1)} \cdot C_1^{(1)}) = 0 & \text{if } C = C_1^{(1)}, \\ 3(\mathcal{O} \cdot C_0^{(1)}) + (C_1^{(1)} \cdot C_0^{(1)}) + 2(C_0^{(1)})^2 = 0 & \text{if } C = C_0^{(1)}, \\ 3\mathcal{O}^2 + 2(C_0^{(1)} \cdot \mathcal{O}) + 2(C_0^{(2)} \cdot \mathcal{O}) + 2(C_0^{(3)} \cdot \mathcal{O}) = 0 & \text{if } C = \mathcal{O}, \\ 3(\mathcal{O} \cdot C_0^{(3)}) + (C_1^{(3)} \cdot C_0^{(3)}) + 2(C_0^{(3)})^2 = 0 & \text{if } C = C_0^{(3)}, \\ 3(\mathcal{O} \cdot C_0^{(2)}) + (C_1^{(2)} \cdot C_0^{(2)}) + 2(C_0^{(2)})^2 = 0 & \text{if } C = C_0^{(2)}, \\ (C_1^{(2)})^2 + 2(C_0^{(2)} \cdot C_1^{(2)}) = 0 & \text{if } C = C_1^{(2)}. \end{cases}$$

■



(2.7)

Proposition 2.2.2. *The classes appearing in (2.6), $C_2^{(3)}$ and D with the bilinear form in 2.7 form a $\mathbb{Q}$-basis of* $\mathrm{NS}(S)$.

Proof. Let $N := E_6 \oplus E_6 \oplus E_6 \oplus U$, then N has discriminant $3^3 \neq 0$. As $\mathbb{Z}$-modules $N \subset \mathrm{NS}(S)$ and $\operatorname{rank} N = 20$, then $\mathrm{NS}(S) \otimes \mathbb{Q} = N \otimes \mathbb{Q}$. ∎

Since $\{\mathcal{O}, T_1, T_2\}$ with the group structure of $\mathrm{MW}(S)$ is a copy of $\mathbb{Z}/3\mathbb{Z}$, applying Lemma 1.3.4, we get $|A_{T_S}| = 3$. Since $H^2(S,\mathbb{Z})$ is unimodular, we get $|A_{\mathrm{NS}(S)}| = |A_{T_S}| = 3$. Note that as lattices, N and $\mathrm{NS}(S)$ are not isometric because their discriminant groups are not isomorphic. However, $\mathrm{NS}(S)$ is an even overlattice of N of finite index.

Proposition 2.2.3. *The transcendental lattice of S is isomorphic to $A_2(-1)$, i.e.,*

$$T_S = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Proof. Since σ is symplectic, ω and $\bar{\omega}$ are in $T_S \otimes \mathbb{R}$. Moreover, $\mathrm{sign}(T_S) = (2,0)$ because the $\mathrm{sign}(\mathrm{NS}(S)) = (1,19)$. We saw that $|A_{T_S}| = 3$, then T_S is a positive definite lattice of rank two and discriminant group isomorphic to $\mathbb{Z}/3\mathbb{Z}$, i.e., $T_S \cong A_2(-1)$. ∎

In order to get a $\mathbb{Z}$-basis of $\mathrm{NS}(S)$ from the classes in Proposition 2.2.2, we consider an extra class in $\mathrm{NS}(S)$.

Set

$$G := \frac{C_1^{(1)} + 2C_0^{(1)} + C_0^{(2)} + 2C_1^{(2)} + C_3^{(1)} + 2C_4^{(1)} + C_4^{(2)} + 2C_3^{(2)} + C_5^{(1)} + 2C_6^{(1)} + C_6^{(2)} + 2C_5^{(2)}}{3}. \tag{2.8}$$

Note that $G \in A_{E_6} \oplus A_{E_6} \oplus A_{E_6}$ is an isotropic class.

Recalling that $C_0^{(j)} = F - 2C_1^{(j)} - 3C_2^{(j)} - 2C_3^{(j)} - C_4^{(j)} - 2C_5^{(j)} - C_6^{(j)}$, one obtains

$$G = F - C_1^{(1)} - 2C_2^{(1)} - C_3^{(1)} - C_5^{(1)} - C_2^{(2)}.$$

Thus G is a linear combination with integer coefficients of the classes contained in $\mathrm{NS}(S)$.

Set $(E_6^{\oplus 3})' := E_6 \oplus E_6 \oplus E_6 + \langle G \rangle$. This is an overlattice of index 3 of $E_6^{\oplus 3}$ contained in $\mathrm{NS}(S)$.

Proposition 2.2.4. *The Nerón Severi lattice* $\mathrm{NS}(S)$ *is isometric to the lattice* $U \oplus (E_6^{\oplus 3})'$.

Proof. We have proved that $U \oplus (E_6^{\oplus 3})' \subset \mathrm{NS}(S)$ since we exhibit the classes of $\mathrm{NS}(S)$ which generate $U \oplus (E_6^{\oplus 3})'$. The inclusion $U \oplus (E_6^{\oplus 3})' \subset \mathrm{NS}(S)$ has finite index, the lattices have the same ranks, the same signatures and the same discriminants, so

$$\mathrm{NS}(S) \simeq U \oplus (E_6^{\oplus 3})'.$$

∎

The isometry σ^* on $H^2(S, \mathbb{Z})$

In the previous section, we saw that the action of σ^* on $U \oplus (E_6)^{\oplus 3}$ is easily described: it acts as the identity on U and it permutes the three copies of E_6.

Since

$$\begin{aligned} \sigma^*(G) &= \sigma(F - C_1^{(1)} - 2C_2^{(1)} - C_3^{(1)} - C_5^{(1)} - C_2^{(2)}) \\ &= F - C_3^{(1)} - 2C_2^{(1)} - C_5^{(1)} - C_1^{(1)} - C_2^{(2)} = G, \end{aligned}$$

it means that G is preserved by σ^*.

The fact that σ is symplectic implies that it acts as the identity on the generator of $H^{2,0}(S)$, and so it acts as the identity on $T_S \simeq A_2(-1)$.

Note that $H^2(S,\mathbb{Z})$ is an overlattice of $T_S \oplus \mathrm{NS}(S)$ of index 3 since $H^2(S,\mathbb{Z})$ is unimodular and $A_{T_S} \cong A_{\mathrm{NS}(S)} \cong \mathbb{Z}/3\mathbb{Z}$. Hence, we obtain that $H^2(S,\mathbb{Z})$ is an overlattice of index 3^2 of $A_2(-1) \oplus U \oplus E_6 \oplus E_6 \oplus E_6$ because $\mathrm{NS}(S)$ is an overlattice of index 3 of $U \oplus (E_6)^{\oplus 3}$.

The action of σ^* on $A_2(-1) \oplus U \oplus E_6 \oplus E_6 \oplus E_6$ can be written in an explicit way:

$$\begin{array}{ccccccccccccc} \sigma^*: & A_2(-1)\oplus & U\oplus & E_6\oplus & E_6\oplus & E_6 & \longrightarrow & A_2(-1)\oplus & U\oplus & E_6\oplus & E_6\oplus & E_6 \\ & (a, & u, & e, & f, & g) & \mapsto & (a, & u, & g, & e, & f) \end{array} \tag{2.9}$$

Since $H^2(S,\mathbb{Z})$ is an overlattice of finite index of $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$, the action of σ^* on $H^2(S,\mathbb{Z})$ is induced by the $\mathbb{Q}$-linear extension of (2.9). In order to obtain $H^2(S,\mathbb{Z})$ from $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ one has to add two classes, which are contained in the discriminant group of $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$: let us fix the following generators for the discriminant group of E_6 and $A_2(-1)$ respectively:

$$v^{(i)} = \frac{e_1^{(i)} + 2e_2^{(i)} + e_4^{(i)} + 2e_5^{(i)}}{3}, \text{ and } w = \frac{a_1 + 2a_2}{3} \tag{2.10}$$

where a_i are the generators of $A_2(-1)$ and $e_j^{(i)}$, $j = 1, \ldots, 6$, $i = 1, 2, 3$ are the generators of E_6 whose intersections are as in the diagram:

$$\begin{tikzcd} e_1^{(i)} \arrow[r, dash] & e_2^{(i)} \arrow[r, dash] & e_3^{(i)} \arrow[r, dash] \arrow[d, dash] & e_4^{(i)} \arrow[r, dash] & e_5^{(i)} \\ & & e_6^{(i)} & & \end{tikzcd} \tag{2.11}$$

We consider the overlattice $E_6^{\oplus 3}$ obtained adding to the abstract lattice $E_6^{\oplus 3}$ the class

$$x := v^{(1)} + v^{(2)} + v^{(3)}.$$

It coincides with the lattice $(E_6^{\oplus 3})'$ constructed before (an explicit isometry is obtained mapping the generators of $E_6^{\oplus 3}$ to the curves which appear in (2.6) and the vector x to the class G defined in (2.8)).

Hence we constructed the lattice $A_2(-1) \oplus U \oplus (E_6^{\oplus 3})'$ as an overlattice of index 3 of $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ by adding the class x.

Since $\left(v^{(i)}\right)^2 = -4/3$ and $w^2 = 2/3$ (cf. Table 9.2), the class $y = w + v^{(1)} - v^{(2)}$ is such that $y^2 = -2$ (which is of course equivalent to 0 mod $2\mathbb{Z}$). In particular $y \in A_{A_2(-1)\oplus U\oplus (E_6^{\oplus 3})'}$ is isotropic. Hence, by adding y to $A_2(-1) \oplus U \oplus (E_6^{\oplus 3})'$ one obtains an even overlattice of $A_2(-1) \oplus U \oplus (E_6^{\oplus 3})'$ which is even, unimodular and whose signature is $(3, 19)$. This implies then this lattice is Λ_{K3}. Since $A_2(-1) \simeq T_S$ and $U \oplus (E_6^{\oplus 3})' \simeq \mathrm{NS}(S)$, y is the gluing vector, which enlarges $T_S \oplus \mathrm{NS}(S)$ to $H^2(S,\mathbb{Z})$.

To resume the results of this section we proved the following (with the previous notations):

Theorem 2.2.5. *There is an even overlattice of index 3 of $E_6^{\oplus 3}$ which is obtained by adding to $E_6^{\oplus 3}$ the class $x = v^{(1)} + v^{(2)} + v^{(3)}$ where $v^{(i)}$ are as in* (2.10)*.*

The Néron–Severi group of the K3 surface S is isometric to $U \oplus \left(E_6^{\oplus 3}\right)'$ and the action of σ^* on $U \oplus E_6^{\oplus 3}$ is a cyclic permutation of order 3 on $E_6^{\oplus 3}$ and is the identity on U.

The lattice $H^2(S,\mathbb{Z})$ is an overlattice of index 3^2 of $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ obtained by adding x and $y := w + v^{(1)} - v^{(2)}$ where w and $v^{(i)}$ are as in (2.10)*.*

The action of σ^* on $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ is the identity on $A_2(-1) \oplus U$ and a cyclic permutation of order 3 on $E_6^{\oplus 3}$; the one on $H^2(S,\mathbb{Z})$ is obtained extending this one to x and y.

3 | The cohomological action of symplectic automorphisms of order 3

In the previous Section we described the action of a particular symplectic automorphism of order 3 on a particular K3 surface and the isometry it induces. Applying Theorem 1.2.11, the action of a finite order symplectic automorphism on the second cohomology group of a K3 surface is unique, one obtains general results by the ones described above in a specific example. The aim of this section is to state and prove these more general results: first, in Section 3.1 we describe the isometry induced by a symplectic automorphism of order 3 on the lattice Λ_{K3}; then, in Sections 3.2 and 3.3 we describe the maps π_* and π^*, where π is induced by the quotient map. The description of these maps is the main technical result of our thesis and it is the analogous of the results proved in [vGS07] for the order 2 case. Some properties of the action of σ^* on Λ_{K3} were already known. In particular the lattice $\left(H^2(S,\mathbb{Z})^{\sigma^*}\right)^{\perp}$ is isometric to the lattice K_{12}, by [GS07, Theorem 4.1].

3.1 The action of a symplectic automorphism of order 3 on Λ_{K3}

We denote by Λ_{K3} the unique even unimodular lattice of signature $(3,19)$. It is known to be isometric to $U^3 \oplus E_8^2$. Since for each K3 surface X, $H^2(X,\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$, we have that $H^2(X,\mathbb{Z})$ is isometric to Λ_{K3}. By Theorem 2.2.5, we have an alternative description of Λ_{K3}: it is the overlattice of index 3^2 of $A_2(-1) \oplus U \oplus E_6^{\oplus 3}$ obtained by adding the classes x and y.

Let σ be a symplectic automorphism of order 3 on a K3 surface X, then σ^* acts on $H^2(X,\mathbb{Z})$ and this action is unique, by 1.2.11. So σ^* is an order 3 isometry of Λ_{K3}. In Theorem 2.2.5, we described this isometry by considering a very special

K3 surface and by the uniqueness of this action we conclude that

$$\begin{array}{cccccccccccccc} \sigma^*: & A_2(-1)\oplus & U\oplus & E_6\oplus & E_6\oplus & E_6 & \to & A_2(-1)\oplus & U\oplus & E_6\oplus & E_6\oplus & E_6 \\ & (a, & u, & e, & f, & g) & \mapsto & (a, & u, & g, & e, & f). \end{array} \tag{3.1}$$

The isometry σ^* can be extended to an action of $(A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6)\otimes\mathbb{Q}$ and so to an action of the overlattice of index 3^2 of $A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6$ isometric to Λ_{K3}. In particular $\sigma^*(x)=x$ and $\sigma^*(y)=w+v^{(2)}-v^{(3)}$, with the notation of Theorem 2.2.5.

The invariant lattice $\Lambda_{K3}^{\sigma^*}$ and its orthogonal complement

The action of σ^* splits Λ_{K3} in two sublattices: the invariant sublattice $\Lambda_{K3}^{\sigma^*}$ and its orthogonal complement (denoted by $\Omega_{\mathbb{Z}/3\mathbb{Z}}$ in [GS07]). By the description of σ^*, one obtains that the σ-invariant sublattice of $A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6$ is spanned by the classes $(a_h,0,0,0,0)$, $(0,u_k,0,0,0)$, $(0,0,e_i,e_i,e_i)$ where $h,k=1,2$, $i=1,\ldots,6$ and a_h, u_k and e_i are generators of the lattices $A_2(-1)$, U and E_6 respectively. So

$$(A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6)^{\sigma^*}\simeq A_2(-1)\oplus U\oplus E_6(3),$$

where the generators of $E_6(3)$ are $e_i^{(1)}+e_i^{(2)}+e_i^{(3)}$, $i=1,\ldots,6$ (the sum of the generators of the three copies of E_6). Moreover also the class $x=(\sum_{j=1}^3(e_1^{(j)}+2e_2^{(j)}+e_4^{(j)}+2e_5^{(j)}))/3$ is contained in $(\Lambda_{K3})^{\sigma^*}$. Hence this lattice is an overlattice of index 3 of $A_2(-1)\oplus U\oplus E_6(3)$, often denoted by $A_2(-1)\oplus U\oplus E_6^*(3)$ where E_6^* is the dual $\mathbb{Q}$-lattice of E_6.

Let us now consider the orthogonal complement of $(\Lambda_{K3})^{\sigma^*}$.

Proposition 3.1.1. *The lattice $\left(\Lambda_{K3}^{\sigma^*}\right)^{\perp}$ is isometric to the lattice K_{12} and it is spanned by $k_i:=e_i^{(1)}-e_i^{(2)}$, $k_{i+6}:=e_i^{(1)}-e_i^{(3)}$, $i=1,\ldots,6$ and by the class*

$$z:=(k_1+k_4+k_7+k_{10}+2(k_2+k_5+k_8+k_{11}))/3. \tag{3.2}$$

Proof. We already described Λ_{K3} as an overlattice of $A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6$ and we described the sublattice $(\Lambda_{K3})^{\sigma^*}$ as the lattice generated by $(a_i,0,0,0,0)$, $(0,u_i,0,0,0)$, $(0,0,e_j,e_j,e_j)$, $i=1,2$, $j=1,...,6$, and x. So, $\left(\Lambda_{K3}^{\sigma^*}\right)^{\perp}$ is spanned, at least over $\mathbb{Q}$, by the classes k_i, $i=1,\ldots,12$. The intersection matrix of these classes is the block matrix

$$\left[\begin{array}{c|c} E_6(2) & E_6 \\ \hline E_6 & E_6(2) \end{array}\right] \tag{3.3}$$

We denote by $\widetilde{K_{12}}$ the lattice whose bilinear form is given by the previous matrix. Since the determinant of this matrix is 3^8 and the discriminant of K_{12} is 3^6, we deduce that K_{12} is an overlattice of index 3 of the lattice $\widetilde{K_{12}}$. This overlattice is obtained by adding to $\{k_i\}_{i=1,\ldots,12}$ the class z.

Note that the class z in terms of generators of $\mathrm{NS}(S)$ (where S is the specific K3 surface considered in Section 2.1)corresponds to the class

$$2F-2C_1^{(1)}-4C_2^{(1)}-3C_3^{(1)}-2C_4^{(1)}-3C_5^{(1)}-2C_6^{(1)}-2C_2^{(2)}-2C_3^{(2)}-C_4^{(2)}-2C_5^{(2)}-C_6^{(2)},$$

which is contained in $\left(\mathrm{NS}(S)^{\sigma^*}\right)^{\perp}$ (since it is orthogonal to the generators of $\mathrm{NS}(S)^{\sigma^*}$). As a consequence, the lattice $K_{12} \simeq \left(H^2(S,\mathbb{Z})^{\sigma^*}\right)^{\perp}$ is the overlattice of index 3 of $\widetilde{K_{12}}$ obtained by adding z to $\widetilde{K_{12}}$, cf. Example 1.1.3. ■

3.2 The map π_*

Given a K3 surface X with a symplectic automorphism σ, the quotient $X/\langle\sigma\rangle$ is a singular surface, whose desingularization is a K3 surface Y, see Section 1.2.9. Hence there is a generically 3:1 rational map $\pi : X \dashrightarrow Y$, induced by the quotient map $q : X \longrightarrow X/\langle\sigma\rangle$.

Hence π_* maps $H^2(X,\mathbb{Z})$ to a sublattice of $H^2(Y,\mathbb{Z})$. Since both these surfaces are K3 surfaces, π_* maps $\Lambda_{K3} \simeq H^2(X,\mathbb{Z})$ to a sublattice of $\Lambda_{K3} \simeq H^2(Y,\mathbb{Z})$.

Proposition 3.2.1. *The map π_* acts on $A_2(-1) \oplus U \oplus E_6 \oplus E_6 \oplus E_6$ as follows:*

$$\begin{array}{cccccccccc} \pi_*: & A_2(-1)\oplus & U\oplus & E_6\oplus & E_6\oplus & E_6 & \to & A_2(-3)\oplus & U(3)\oplus & E_6 \\ & (a, & u, & e, & f, & g) & \mapsto & (a, & u, & e+f+g) \end{array}.$$

Its extension to Λ_{K3} is such that

$$\pi_*(H^2(X,\mathbb{Z})) \simeq A_2^*(-3) \oplus U(3) \oplus E_6 \simeq A_2(-1) \oplus U(3) \oplus E_6,$$

which is an overlattice of index 3 of $A_2(-3) \oplus U(3) \oplus E_6$. It is a lattice of rank 10*, signature* $(3,7)$ *and discriminant group* $(\mathbb{Z}/3\mathbb{Z})^4$*.*

Proof. If $\alpha \in H^2(X,\mathbb{Z})^{\sigma^*}$ is a σ^*-invariant class, applying the push-pull formula in

Lemma 1.2.13, one obtains

$$(\pi_*(\alpha)\cdot\pi_*(\alpha)) = 3\alpha^2$$

indeed $\pi^*\pi_*(\alpha) = \alpha + \sigma^*(\alpha) + (\sigma^*)^2(\alpha) = 3\alpha$.

Given $\alpha_1, \alpha_2 \in H^2(X,\mathbb{Z})^{\sigma^*}$, one obtains

$$(\pi_*\alpha_1\cdot\pi_*\alpha_2) = \frac{1}{3}(\pi^*\pi_*\alpha_1\cdot\pi^*\pi_*\alpha_2) = \frac{1}{3}(3\alpha_1\cdot 3\alpha_2) = 3(\alpha_1\cdot\alpha_2).$$

Hence if $\alpha \in A_2(-1)\oplus U \subset H^2(X,\mathbb{Z})^{\sigma^*}$, then π^* multiplies the form by 3, so $\pi_*(A_2(-1)\oplus U) = A_2(-3)\oplus U(3)$.

Consider the image of the classes of the form $(0,0,e,0,0) \in A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6$: let us denote, as before, by $e_i^{(j)}$, $i=1,\dots,6$ the basis of the j-th copy of E_6. Then

$$(\pi_*e_i^{(1)}\cdot\pi_*e_j^{(1)}) = \frac{1}{3}(\pi^*\pi_*e_i^{(1)}\cdot\pi^*\pi_*e_j^{(1)}) = \frac{1}{3}((e_i^{(1)}+e_i^{(2)}+e_i^{(3)})\cdot(e_j^{(1)}+e_j^{(2)}+e_j^{(3)})) = (e_i^{(1)}\cdot e_j^{(1)}),$$

where we used:

$$\pi^*\pi_*(e_i^{(1)}) = e_i^{(1)} + \sigma^*(e_i^{(1)}) + (\sigma^*)^2(e_i^{(1)}) = e_i^{(1)} + e_i^{(2)} + e_i^{(3)},$$
$$(e_i^{(h)}\cdot e_j^{(h)}) = (e_i^{(1)}\cdot e_j^{(1)}),\ \text{and}\ (e_i^{(h)}\cdot e_j^{(k)}) = 0,\ \text{if}\ h\neq k.$$

Hence $\pi_*(E_6\oplus E_6\oplus E_6) \simeq E_6$.

So we obtain $\pi_*(A_2(-1)\oplus U\oplus E_6\oplus E_6\oplus E_6) \simeq A_2(-3)\oplus U(3)\oplus E_6$. In order to find $\pi_*(H^2(X,\mathbb{Z}))$, it remains to understand the images of the classes x and y.

The class $x = \left(\sum_j(e_1^{(j)} + 2e_2^{(j)} + e_4^{(j)} + 2e_5^{(j)})\right)/3$ is mapped to

$$\pi_*(x) = \pi_*(e_1^{(1)} + 2e_2^{(1)} + e_4^{(1)} + 2e_5^{(1)})$$

since $\pi_*(e_i^{(1)} + e_i^{(2)} + e_i^{(3)}) = 3\pi_*(e_i^{(1)})$.

The class $y = \left(a_1 + 2a_2 + e_1^{(1)} + 2e_2^{(1)} + e_4^{(1)} + 2e_5^{(1)} - e_1^{(2)} - 2e_2^{(2)} - e_4^{(2)} - 2e_5^{(2)}\right)/3$ is mapped to

$$\pi_*(y) = \pi_*((a_1+2a_2)/3) = (\pi_*a_1 + 2\pi_*a_2)/3$$

since $\pi_*e_i^{(1)} - \pi_*e_i^{(2)} = \pi_*e_i^{(1)} - \pi_*e_i^{(1)} = 0$. The vectors π_*a_i are the generators of $A_2(-3)$ and hence we are constructing an overlattice of index 3 of $A_2(-3)$, often denoted by $A_2^*(-3)$. This lattice is isometric to $A_2(-1)$, with basis

$$a_1' := (\pi_*a_1 + 2\pi_*a_2)/3,\quad a_2' := \pi_*(a_2) - ((\pi_*a_1 + 2\pi_*a_2)/3) = (-\pi_*a_1 + \pi_*a_2)/3.$$

We observe that $a'_1 = \pi_*(y)$ and $a'_2 = \pi_*(a_2) - \pi_*(y)$. ∎

The cohomology of the quotient K3 surface Y

Let X be a K3 surface admitting a symplectic automorphism σ of order 3, then the surface $X/\langle\sigma\rangle$ has 6 singularities of type A_2 (see Example 1.2.10). We denote by Y the desingularization of $X/\langle\sigma\rangle$, which introduces 12 irreducible curves, 6 disjoint pairs of rational curves meeting in a point. We call the resolution map $\beta : Y \longrightarrow X/\langle\sigma\rangle$ and we denote by $M_i^{(j)}$, $i = 1, 2$, $j = 1, \dots, 6$ the curves introduced by β, with the following conventions:

$$M_1^{(j)} M_2^{(j)} = 1, \quad \left(M_i^{(j)}\right)^2 = -2 \text{ and } M_i^{(j)} M_h^{(k)} = 0 \text{ if } j \neq k.$$

By Theorem 1.2.12, the minimal primitive sublattice of $H^2(Y,\mathbb{Z})$ which contains the curves $M_i^{(j)}$ contains also the class $\hat{M} := \sum_{j=1}^{6}(M_1^{(j)} + 2M_2^{(j)})/3$. Thus it is an overlattice of index 3 of $A_2^{\oplus 6}$ and so it is a negative definite lattice of rank 12 and discriminant group $(\mathbb{Z}/3\mathbb{Z})^4$. This lattice is denoted by $M_{\mathbb{Z}/3\mathbb{Z}}$.

The lattice $\pi_*(H^2(X,\mathbb{Z}))$ is naturally embedded in $H^2(Y,\mathbb{Z})$. The curves introduced by β are orthogonal to $\beta^*(D)$ for each divisor $D \in \pi_*(\mathrm{NS}(X))$ and of course they are also orthogonal to each class in $\pi_*(T_X)$. So we have the following orthogonal decomposition (which holds over the rational field, but not over $\mathbb{Z}$):

$$H^2(Y,\mathbb{Q}) = \left(M_{\mathbb{Z}/3\mathbb{Z}} \oplus \pi_*(H^2(X,\mathbb{Z}))\right) \otimes \mathbb{Q}. \tag{3.4}$$

So $M_{\mathbb{Z}/3\mathbb{Z}}$ and $\pi_*(H^2(X,\mathbb{Z}))$ are two sublattices embedded in Λ_{K3}.

Gluing $\pi_*(H^2(X,\mathbb{Z}))$ and $M_{\mathbb{Z}/3\mathbb{Z}}$ to obtain $H^2(Y,\mathbb{Z})$

By (3.4), the orthogonal complement of $M_{\mathbb{Z}/3\mathbb{Z}}$ in $H^2(Y,\mathbb{Z})$ is either $\pi_*(H^2(X,\mathbb{Z}))$ or an overlattice of finite index of $\pi_*(H^2(X,\mathbb{Z}))$.

By definition $M_{\mathbb{Z}/3\mathbb{Z}}$, is primitively embedded in $H^2(Y,\mathbb{Z})$. Since $H^2(Y,\mathbb{Z})$ is unimodular, the discriminant of the orthogonal complement of $M_{\mathbb{Z}/3\mathbb{Z}}$ in $H^2(Y,\mathbb{Z})$ is $-d(M_{\mathbb{Z}/3\mathbb{Z}})$. So, if $d(\pi_*((H^2(X,\mathbb{Z})))) = -d(M_{\mathbb{Z}/3\mathbb{Z}})$, we conclude that $\pi_*((H^2(X,\mathbb{Z})))$ is the orthogonal complement of $M_{\mathbb{Z}/3\mathbb{Z}}$. One immediately checks that

$$|d(M_{\mathbb{Z}/3\mathbb{Z}})| = 3^4 = |d(\pi_*(H^2(X,\mathbb{Z})))|,$$

so $\pi_*((H^2(X,\mathbb{Z})))$ is the orthogonal complement of $M_{\mathbb{Z}/3\mathbb{Z}}$.

We recall that the orthogonal complement of a given sublattice inside a bigger lattice Λ, is necessarily primitively embedded in Λ. In particular, $\pi_*(H^2(Y,\mathbb{Z})) \simeq M_{\mathbb{Z}/3\mathbb{Z}}^{\perp}$ is primitively embedded in $H^2(Y,\mathbb{Z})$. Therefore, to construct the unimodular lattice $H^2(Y,\mathbb{Z})$ one has to glue these two primitive sublattices $\pi_*(H^2(Y,\mathbb{Z}))$ and $M_{\mathbb{Z}/3\mathbb{Z}}$.

To this purpose, we need the following description of the discriminant group of $M_{\mathbb{Z}/3\mathbb{Z}}$.

Lemma 3.2.2. *With the notation of Section 3.2, the discriminant group of $M_{\mathbb{Z}/3\mathbb{Z}}$ is generated by*

$$\begin{aligned} b_1 &:= z_1+z_2+z_3, & b_2 &:= z_1+z_2+z_4,\\ b_3 &:= z_2-z_3+z_4-z_5, & b_4 &:= -z_1+z_3-z_4+z_5, \end{aligned}$$

where

$$z_j := \left(M_1^{(j)}+2M_2^{(j)}\right)/3.$$

Proof. It suffices to check that $b_kM_i^{(j)} \in \mathbb{Z}$ and that b_i are independent in $(\mathbb{Z}/3\mathbb{Z})^4 \simeq A_{M_{\mathbb{Z}/3\mathbb{Z}}}$. The latter statement can be checked by observing that the discriminant form computed on $\{b_1,b_2,b_3,b_4\}$ is non degenerate, and indeed it is the opposite of the discriminant form of $U(3)\oplus A_2(-1)\oplus E_6$. ■

Proposition 3.2.3. *Denoted by a'_i, u'_i, e'_j, $i=1,2$, $j=1,\dots,6$ the standard generators of $A_2(-1)$, $U(3)$ and E_6 respectively.*

The overlattice $H^2(Y,\mathbb{Z})$ of $\pi_*(H^2(X,\mathbb{Z}))\oplus M_{Z/3\mathbb{Z}} \simeq A_2(-1)\oplus U(3)\oplus E_6\oplus M_{\mathbb{Z}/3\mathbb{Z}}$ is obtained by adding the classes

$$\begin{aligned} n_1 &:= \tfrac{a'_1+2a'_2}{3}+b_3, & n_2 &:= \tfrac{e'_1+2e'_2+e'_4+2e'_5}{3}+b_4,\\ n_3 &:= \tfrac{u'_1}{3}+b_1, & n_4 &:= \tfrac{u'_2}{3}+b_2 \end{aligned}$$

to $A_2(-1)\oplus U(3)\oplus E_6\oplus M_{\mathbb{Z}/3\mathbb{Z}}$.

Proof. It suffices to check that the lattice obtained by adding to the generators of $A_2(-1)\oplus U(3)\oplus E_6\oplus M_{\mathbb{Z}/3\mathbb{Z}}$ the classes n_1, n_2, n_3 and n_4 is even and unimodular. It follows that it is an even unimodular lattice of signature $(3,19)$ and then it is isometric to the lattice $\Lambda_{K3} \simeq H^2(Y,\mathbb{Z})$. ■

Here we are interested in describing the classes n_i in terms of the curves contained $M_{\mathbb{Z}/3\mathbb{Z}}$. In order to construct the overlattice of $A_2(-1)\oplus U(3)\oplus E_6\oplus M_{\mathbb{Z}/3\mathbb{Z}}$, it is equivalent if one adds to it the class n_i or the class $n_i+\sum_i \alpha_{i,j}M_i^{(j)}$, $\alpha_{i,j}\in\mathbb{Z}$, indeed these two classes are equivalent in the discriminant group.

We can rewrite the classes n_i in the following form

$$n_1 \sim \frac{2\pi_*(a_2) - \pi_*(y) + M_1^{(2)} + 2M_2^{(2)} + 2M_1^{(3)} + M_2^{(3)} + M_1^{(4)} + 2M_2^{(4)} + 2M_1^{(5)} + M_2^{(5)}}{3},$$

$$n_2 \sim \frac{\pi_*(e_1 + 2e_2 + e_4 + 2e_5) + 2M_1^{(1)} + M_2^{(1)} + M_1^{(3)} + 2M_2^{(3)} + 2M_1^{(4)} + M_2^{(4)} + M_1^{(5)} + 2M_2^{(5)}}{3},$$

$$n_3 \sim \frac{\pi_*(u_1) + M_1^{(1)} + 2M_2^{(1)} + M_1^{(2)} + 2M_2^{(2)} + M_1^{(3)} + 2M_2^{(3)}}{3},$$

$$n_4 \sim \frac{\pi_*(u_2) + M_1^{(1)} + 2M_2^{(1)} + M_1^{(2)} + 2M_2^{(2)} + M_1^{(4)} + 2M_2^{(4)}}{3},$$

where we used that $a_1' = \pi_*(y)$ and $a_2' = \pi_*(a_2) - \pi_*(y)$, by the proof of Proposition 3.3.1.

3.3 The map π^*

We now consider the map π^*, dual of the map π_* considered above.

Proposition 3.3.1. *The map π^* acts as follows*

$$\begin{array}{cccccccccc} \pi^*: & A_2(-3)\oplus & U(3)\oplus & E_6 & \to & A_2(-1)\oplus & U\oplus & E_6\oplus & E_6\oplus & E_6 \\ & (\alpha, & \mu, & e) & \mapsto & (3\alpha, & 3\mu, & e, & e, & e). \end{array}$$

With the notation of Proposition 3.2.3, $H^2(Y,\mathbb{Z}) \simeq \Lambda_{K3}$ is obtained by adding to $A_2(-3) \oplus U(3) \oplus E_6 \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ the classes a_1' and n_h, $h = 1,2,3,4$, so the extension of π^* to $H^2(Y,\mathbb{Z})$ is given by

$$\pi^*(a_1') = a_1 + 2a_2, \quad \pi^*(n_1) = 2a_2 - y, \quad \pi^*(n_2) = x, \quad \pi^*(n_3) = u_1, \quad \pi^*(n_4) = u_2$$

where a_i and u_i are generators of $A_2(-1)$ and U respectively.

Proof. By Proposition 3.2.1, $\pi_*(\gamma) = \gamma$ if $\gamma \in A_2(-1) \oplus U \subset H^2(X,\mathbb{Z})^{\sigma^*}$ and the map $\pi_{*|H^2(X,\mathbb{Z})^{\sigma^*}}$ multiplies the form by 3. For every $\gamma \in A_2(-1) \oplus U \subset H^2(X,\mathbb{Z})^{\sigma^*}$ and $\alpha \in \pi_*(H^2(X,\mathbb{Z})^{\sigma^*}) \simeq A_2(-3) \oplus U(3) \oplus E_6$, it holds $(\pi^*\alpha \cdot \gamma) = (\alpha \cdot \pi_*\gamma)$. So we obtain

$$(\pi^*\alpha \cdot \gamma) = (\alpha \cdot \pi_*\gamma) = 3(\alpha \cdot \gamma)$$

and hence $\pi^*(\alpha) = 3\alpha$ for every $\alpha \in A_2(-3) \oplus U(3)$.

Let $e \in E_6 \subset H^2(Y,\mathbb{Z})$ and $(f,0,0) \in E_6^{\oplus 3} \subset H^2(X,\mathbb{Z})$. By push-pull formula in

Lemma 1.2.13 and Proposition 3.2.1,

$$(\pi^* e \cdot (f,0,0)) = (e \cdot \pi_*(f,0,0)) = (e \cdot f).$$

Similarly $(\pi^* e \cdot (0,f,0)) = (e \cdot f)$ and $(\pi^* e \cdot (0,0,f)) = (e \cdot f)$. Hence

$$\pi^*(e) = (e,e,e) \in E_6 \oplus E_6 \oplus E_6 \subset H^2(X,\mathbb{Z}).$$

Now we want to extend this map to the lattice $A_2(-1) \oplus U(3) \oplus E_6$, obtained by adding to $A_2(-3) \oplus U(3) \oplus E_6$ the class $a_1' = \pi_*((a_1 + 2a_2)/3) = \pi_*(y)$. Since $\pi^*(\alpha) = 3\alpha$ for every $\alpha \in A_2(-3)$, we obtain $\pi^*(a_1') = \pi^*(\pi_*(a_1 + 2a_2)/3) = a_1 + 2a_2$.

In order to extend this map to $H^2(Y,\mathbb{Z}) \simeq \Lambda_{K3}$, it remains to compute $\pi^*(n_h)$ for $h = 1,2,3,4$. We observe that $\pi^*(M_i^{(j)}) = 0$ indeed if one reconstruct X from Y, ones consider the (minimal resolution of the) triple cover Y branched on the curves $M_i^{(j)}$. This surface is not minimal and then one contracts some of the (-1)–curves. Among these contracted curves, one finds the curves which are covers of the curves $M_i^{(j)}$. In particular, there are no curves on X which are the inverse image of curves $M_i^{(j)} \subset Y$.

So,

$$\begin{aligned}
\pi^*(n_1) &= \pi^*(2\pi_*(a_2) - \pi_*(y))/3 = (2\pi^*\pi_*(a_2) - \pi^*\pi_*(y))/3 = 2a_2 - y \\
\pi^*(n_2) &= \sum_{j=1}^{3} \left(e_1^{(j)} + 2e_2^{(j)} + e_4^{(j)} + 2e_5^{(j)}\right)/3 = x, \\
\pi^*(n_3) &= u_1, \\
\pi^*(n_4) &= u_2.
\end{aligned} \tag{3.5}$$

■

4 | Families of projective K3 surfaces admitting a symplectic automorphism of order 3 and of their quotients

We describe the families of the projective K3 surfaces admitting a symplectic automorphism of order 3 and the families of their quotients, in terms of families of lattices polarized K3 surfaces. So, we want to reduce the geometric property "to admit a symplectic automorphism of order 3" (resp. "to be a quotient of a K3 surface by a symplectic automorphism of order 3") into a lattice theoretic property. It is already known that this is possible by considering overlattices of the lattices K_{12} and $M_{\mathbb{Z}/3\mathbb{Z}}$ with certain properties (see [Nik76], cf. [GS07], [Gar17]). Here we classify these overlattices providing a complete description of the two families considered, see Theorems 4.1.4 and 4.2.4 and Corollaries 4.1.6 and 4.2.5.

4.1 Projective K3 surfaces with a symplectic automorphism of order 3

Lemma 4.1.1. *Let $q_{A_{K_{12}}}$ be the quadratic form on the discriminant group $A_{K_{12}} \simeq (\mathbb{Z}/3\mathbb{Z})^6$, then $q_{A_{K_{12}}}(v)$ for $v \in A_{K_{12}}$ is one of the following three values $0, \frac{2}{3}, \frac{4}{3}$.*

The actions of $O(A_{K_{12}})$ has four orbits: the orbit which contains 0, and the orbits:

- $o_0 = \{v \in A_{K_{12}} \mid q_{A_{K_{12}}}(v) = 0$ *and* $v \neq 0\}$*;*
- $o_1 = \{v \in A_{K_{12}} \mid q_{A_{K_{12}}}(v) = \frac{2}{3}\}$*;*
- $o_2 = \{v \in A_{K_{12}} \mid q_{A_{K_{12}}}(v) = \frac{4}{3}\}$*.*

Proof. The result follows by [CS83, Section 3]. Nevertheless, we give an idea of the proof. The isometries of K_{12} induce isometries of its discriminant group. If two vectors $a \neq 0$ and $b \neq 0$ in $A_{K_{12}}$ are in the same orbit for the action of

$\mathrm{O}(A_{K_{12}})$, then $q_{A_{K_{12}}}(a) = q_{A_{K_{12}}}(b)$. In order to reverse the implication, i.e., to prove that the orbit of a non-zero vector is completely determined by the value of the discriminant form on it, it suffices to find an isometry in $O(A_{K_{12}})$ which maps a in b if $q_{A_{K_{12}}}(a) = q_{A_{K_{12}}}(b)$. By Example 1.1.3, the vectors

$$g_1 = \frac{k_7 + 2k_8 + k_{10} + 2k_{11}}{3},\ g_2 = \frac{k_6 + k_{12}}{3},$$

$$g_3 = \frac{k_5 + k_{11}}{3},\ g_4 = \frac{k_4 + k_{10}}{3},\ g_5 = \frac{k_3 + k_9}{3},\ g_6 = \frac{k_2 + k_8}{3}$$

form a basis of $A_{K_{12}}$ such that the quadratic form $q_{A_{K_{12}}}$ is

$$q_{A_{K_{12}}}(v) = \frac{4}{3}x_1^2 + \frac{2}{3}(x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) + \frac{4}{3}(x_2x_5 + x_3x_4 + x_4x_5 + x_5x_6),$$

where $v = x_1g_1 + ... + x_6g_6$ is an arbitrary element of $A_{K_{12}}$.

Let φ be the order 2 isometry of K_{12} defined as $k_1 \leftrightarrow k_5$, $k_2 \leftrightarrow k_4$, $k_7 \leftrightarrow k_{11}$, $k_8 \leftrightarrow k_{10}$, $k_i \leftrightarrow k_i$ for all $i = 3, 6, 9, 12$. Then $\varphi \in \mathrm{O}(K_{12})$ induces $\varphi_A \in \mathrm{O}(A_{K_{12}})$ as follows

$$\varphi_A(v) = -x_1g_1 + x_2g_2 + x_3g_3 + (-x_3 + x_6)g_4 + x_5g_5 + (x_3 + x_4)g_6.$$

By considering $\langle\varphi_A, -\mathrm{Id}\rangle \in \mathrm{O}(A_{K_{12}})$, one shows that the vectors $g_3, -g_3, g_3 - g_4 + g_6, -g_3 + g_4 - g_6$ are in the same orbit for $\mathrm{O}(A_{K_{12}})$. A complete table of all 729 isometries, needed to show the result, can be computed as in Appendix 9.3. ■

Proposition 4.1.2. *Let d be a positive integer.*

If $d \not\equiv 0 \mod 3$, then there exist no even overlattices of finite index of $\langle 2d\rangle \oplus K_{12}$ such that $\langle 2d\rangle$ and K_{12} are primitively embedded in it.

If $d \equiv 0 \mod 3$, then there exists a unique overlattice of finite index of $\langle 2d\rangle \oplus K_{12}$ such that $\langle 2d\rangle$ and K_{12} are primitively embedded in it. The index is 3 and the generators of the overlattices are the ones of $\langle 2d\rangle \oplus K_{12}$ and $\frac{L+g}{3}$ where L is a generator of $\langle 2d\rangle$ and $g \in K_{12}$ can be chosen as follows:

if $d \equiv 0 \mod 9$, then $g = k_1 + k_3 + k_5 + k_7 + k_9 + k_{11}$;

if $d \equiv 3 \mod 9$, then $g = k_1 + k_3 + k_7 + k_9$;

if $d \equiv 6 \mod 9$, then $g = k_1 + k_7$.

Proof. Let Λ be a proper overlattice of finite index of $\langle 2d\rangle \oplus K_{12}$ in which K_{12} and $\langle 2d\rangle$ are primitively embedded. Let f be one generator of $\Lambda/(\langle 2d\rangle \oplus K_{12})$, then

$f \in A_{(\langle 2d\rangle \oplus K_{12})}$ and hence

$$f = (\alpha L + \beta g)/t, \text{ with } \alpha L/t \in A_{\langle 2d\rangle}, \ \beta g/t \in A_{K_{12}}.$$

Since $A_{K_{12}} \simeq (\mathbb{Z}/3\mathbb{Z})^6$, t has to be equal to 3 and hence $\alpha, \beta \in \{0,1,2\}$.

Since $\langle 2d\rangle = \mathbb{Z}L$ (resp. K_{12}) is primitively embedded in Λ, $\beta \neq 0$ (resp. $\alpha \neq 0$). Moreover, since $\Lambda/(\langle 2d\rangle \oplus K_{12}) \subset (\mathbb{Z}/3\mathbb{Z})^c$, one can freely substitute g (resp. f) by $2g$ (resp. $2f$) because 2 is invertible in $(\mathbb{Z}/3\mathbb{Z})$, so one can assume that $\beta = 1$ (resp. $\alpha = 1$) and $g/3 \in A_{K_{12}}$.

So $f = (L+g)/3$. The bilinear form on Λ takes integer values and the lattice is even, which implies

$$fL = 2d/3 \in \mathbb{Z} \Leftrightarrow d \equiv 0 \mod 3, \quad f^2 = (2d+g^2)/9 \in 2\mathbb{Z} \Leftrightarrow 2d \equiv -g^2 \mod 18.$$

In particular the existence of f implies that $d \equiv 0 \mod 3$. Moreover
if $d \equiv 3 \mod 9$, then $g/3 \in A_{K_{12}}$ satisfies $q_{A_{K_{12}}}(\frac{g}{3}) = -\frac{2}{3}$,
if $d \equiv 6 \mod 9$, then $g/3 \in A_{K_{12}}$ satisfies $q_{A_{K_{12}}}(\frac{g}{3}) = -\frac{4}{3}$,
if $d \equiv 0 \mod 9$, then $g/3 \in A_{K_{12}}$ satisfies $q_{A_{K_{12}}}(\frac{g}{3}) = 0$.

By Lemma 4.1.1 one can choose arbitrarily an element $g \in K_{12}$ such that $g/3 \in K_{12}^\vee/K_{12}$ is non trivial and $q_{A_{12}}(g)$ is the required one, since the choice of the value of $q(g)$ determines a unique orbit. In the statement, we made a specific choice for g. Any other is equivalent up to isometries of K_{12}.

It remains to prove that $\Lambda/(\langle 2d\rangle \oplus K_{12})$ is a cyclic group and thus the index of $\langle 2d\rangle \oplus K_{12} \hookrightarrow \Lambda$ is 3. Suppose there exist two independent generators f_1 and f_2 of $\Lambda/(\langle 2d\rangle \oplus K_{12})$. Both of them can be chosen to be of the form $f_i = (L+g_i)/3$. So $f_1 - f_2 = (g_1 - g_2)/3 \in \Lambda$ and $(g_1-g_2)/3 \in A_{K_{12}}$ are non trivial. Then K_{12} is not primitively embedded in Λ, which is absurd. ■

Definition 4.1.3. *If $d \equiv 0 \mod 3$ we denote by $(\langle 2d\rangle \oplus K_{12})'$ the overlattice constructed in the previous proposition.*

Theorem 4.1.4. ([GS07, Proposition 5.1]) *Let X be a K3 surface. Then X admits a symplectic automorphism of order 3 if and only if K_{12} is primitively embedded in $\mathrm{NS}(X)$. If X is a projective K3 surface which admits a symplectic automorphism of order 3, then $\rho(X) \geq 13$. Assume $\rho(X) = 13$ and let L be a generator of $K_{12}^{\perp_{NS(X)}}$, so $L^2 = 2d$, $d \in \mathbb{N}_{>0}$. Then*

- *if $d \not\equiv 0 \mod 3$, then $\mathrm{NS}(X) = \langle 2d\rangle \oplus K_{12}$;*
- *if $d \equiv 0 \mod 3$, then $\mathrm{NS}(X)$ is either $\langle 2d\rangle \oplus K_{12}$ or $(\langle 2d\rangle \oplus K_{12})'$.*

Proof. The statement was already proved in [GS07, Proposition 5.1], the unique difference is that in [GS07, Proposition 5.1], one proved that if $d \equiv 0 \mod 3$ the Nerón–Severi group of X could be an overlattice of index 3 of $\langle 2d \rangle \oplus K_{12}$. By Proposition 4.1.2, we know that this overlattice exists, it is unique up to isometries and it coincides with the lattice of the Definition 4.1.3.

Let ζ be a primitive third root of the unity and σ be a symplectic automorphism of order 3. The vector space $H^2(X, \mathbb{C})$ can be decomposed in eigenspaces of the eigenvalues 1, ζ and ζ^2:

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{C})^{\sigma^*} \oplus H^2(X, \mathbb{C})_\zeta \oplus H^2(X, \mathbb{C})_{\zeta^2} \tag{4.1}$$

where the non-rational eigenvalues ζ and ζ^2 have the same multiplicity.

Set $a :=$ multiplicity of the eigenvalue 1, and $b :=$ multiplicity of the eigenvalue ζ. In the following we find a and b.

Proposition 4.1.5. *Let X be a K3 surface an σ a symplectic automorphism of order three. Then, $a = 10$, $b = 6$, and the dimension of the moduli space of algebraic K3 surfaces admitting a symplectic automorphism of order 3 is at most 7.*

Proof. By Proposition 1.2.7 (cf. Table 1.1) , σ has exactly 6 fixed points. Using the Lefschetz fixed point formula applied in K3 surfaces:

$$6 = 1 + 0 + \mathrm{trace}(\sigma^*|_{H^2(X,\mathbb{C})}) + 0 + 1,$$

and so

$$a + b(\zeta + \zeta^2) = 4.$$

Since $\zeta^2 = -(1 + \zeta)$ and $\dim H^2(X, \mathbb{C}) = 22$, we obtain a, b have to satisfy:

$$\begin{cases} a - b = 4 \\ a + 2b = 22. \end{cases}$$

We have $\dim H^2(X, \mathbb{C}) = 10 = a$ and $\dim H^2(X, \mathbb{C})_{\zeta^2} = \dim H^2(X, \mathbb{C})_\zeta = 6 = b$.

Since $T_X \otimes \mathbb{C} \subset H^2(X, \mathbb{C})^{\sigma^*}$,

$$((H^2(X, \mathbb{C}))^{\sigma^*})^\perp = H^2(X, \mathbb{C})_\zeta \oplus H^2(X, \mathbb{C})_{\zeta^2} \subset \mathrm{NS}(X) \otimes \mathbb{C}.$$

We consider only algebraic K3 surfaces and so we have an ample class h on X, so $h + \sigma^* h + (\sigma^*)^2 h$ is a σ-invariant class, hence in $H^2(X, \mathbb{C})^{\sigma^*}$.

By Theorem 4.1.4, $\rho(X) \geq 13$ and so $\mathrm{rank}\, T_X \leq 9$. The dimension of the moduli space is at most $h^{1,1}(X) - 13 = 7$. ∎

Corollary 4.1.6. (See [GS07, Proposition 5.2]) *Let X be a $(\langle 2d\rangle \oplus K_{12})$-polarized K3 surface or $(\langle 2d\rangle \oplus K_{12})'$-polarized K3 surface). Then X is a projective K3 surface and admits a symplectic automorphism of order 3.*

Let

$$\mathcal{S} = \{\textit{projective K3 surfaces admitting an order 3 symplectic automorphism}\}/\simeq$$

where $\simeq$ is the isomorphisms of polarized K3 surfaces. Then $\mathcal{S}$ is

$$\bigcup_{d\in\mathbb{N}_{>0}} \left(\{(\langle 2d\rangle \oplus K_{12})\text{-polarized K3s }\}_{/\simeq}\right) \bigcup_{\substack{d\in\mathbb{N}_{>0},\\ d\equiv 0 \mod 3}} \left(\{(\langle 2d\rangle \oplus K_{12})'\text{-polarized K3s}\}_{/\simeq}\right).$$

In particular $\mathcal{S}$ is the union of countably many components all of dimension 7.

4.2 Quotients of projective K3 surfaces by a symplectic automorphism of order 3

Lemma 4.2.1. *Let $q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}$ be the quadratic form of the discriminant group $A_{M_{\mathbb{Z}/3\mathbb{Z}}} \simeq (\mathbb{Z}/3\mathbb{Z})^4$, then for $v \in A_{M_{\mathbb{Z}/3\mathbb{Z}}}$, $q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}(v)$ is one of the following three values $0, \frac{2}{3}, \frac{4}{3}$.*

The actions induced by $O(A_{M_{\mathbb{Z}/3\mathbb{Z}}})$ on $A_{M_{\mathbb{Z}/3\mathbb{Z}}}$ has four orbits: the orbit which contains 0, and the orbits:

- $p_0 = \{v \in A_{M_{\mathbb{Z}/3\mathbb{Z}}} \mid q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}(v) = 0 \text{ and } v \neq 0\}$;
- $p_1 = \{v \in A_{M_{\mathbb{Z}/3\mathbb{Z}}} \mid q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}(v) = \frac{2}{3}\}$;
- $p_2 = \{v \in A_{M_{\mathbb{Z}/3\mathbb{Z}}} \mid q_{M_{\mathbb{Z}/3\mathbb{Z}}}(v) = \frac{4}{3}\}$.

Proof. We sketch the proof which is analogous to the one of Lemma 4.1.1. By Example 1.1.5, the quadratic form $q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}$ can be written as

$$q_{A_{M_{\mathbb{Z}/3\mathbb{Z}}}}(v) = -\frac{2}{3}(x_3^2 + x_4^2) - \frac{1}{3}(2x_1x_2 + 2x_2x_3 - 2x_2x_4 - x_3x_4)$$

where $v = x_1b_1 + x_2b_2 + x_3b_3 + x_4b_4$ and b_i, $i = 1, \dots, 4$ are a basis of $A_{M_{\mathbb{Z}/3\mathbb{Z}}}$. As example, we observe that the isometry of $M_{\mathbb{Z}/3\mathbb{Z}}$ which permutes $M_i^{(3)}$ with $M_i^{(4)}$, $i = 1, 2$, and acts as the identity on $M_i^{(j)}$, for $i = 1, 2$, $j = 1, 2, 5, 6$, induces on $A_{M_{\mathbb{Z}/3\mathbb{Z}}}$ an isometry which maps b_1 to b_2. A complete table of all 81 isometries needed to show the result can be computed as in Appendix 9.3. ∎

Proposition 4.2.2. *[Gar17, Proposition 5.5] Let e be a positive integer.*

If $e \not\equiv 0 \mod 3$, then there exists no even overlattices of finite index of $\langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ such that $\langle 2e \rangle$ and $M_{\mathbb{Z}/3\mathbb{Z}}$ are primitively embedded in it.

If $e \equiv 0 \mod 3$, then there exists a unique overlattice of finite index of $\langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ such that $\langle 2e \rangle$ and $M_{\mathbb{Z}/3\mathbb{Z}}$ are primitively embedded in it. The index is 3 and the generators of the overlattices are the ones of $\langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ and $\frac{H+g}{3}$ where H is a generator of $\langle 2e \rangle$ and $g \in M_{\mathbb{Z}/3\mathbb{Z}}$ can be chosen as follows:

if $e \equiv 0 \mod 9$, then $g = \sum_{i=1}^{3} \left(M_1^{(i)} + 2M_2^{(i)} \right)$;

if $e \equiv 3 \mod 9$, then $g = \sum_{i=1}^{2} \left(2M_1^{(i)} + M_2^{(i)} \right) + \sum_{j=3}^{4} \left(M_1^{(j)} + 2M_2^{(j)} \right)$;

if $e \equiv 6 \mod 9$, then $g = M_1^{(1)} + 2M_2^{(1)} + 2M_1^{(2)} + M_2^{(2)}$.

Proof. The proof is analogous to the one of Proposition 4.2.2 and it is based on Lemma 4.2.1. ∎

Definition 4.2.3. *If $e \equiv 0 \mod 3$ we denote by $\left(\langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'$ the overlattice constructed in the previous proposition.*

Theorem 4.2.4. *[Gar17, Proposition 5.5] Let Y be a K3 surface. It is the minimal resolution of the quotient of a K3 surface by a symplectic automorphism of order 3 if and only if $M_{\mathbb{Z}/3\mathbb{Z}}$ is primitively embedded in $\mathrm{NS}(Y)$. If Y is projective, then $\rho(Y) \geq 13$. Assume $\rho(Y) = 13$ and let H be a generator of $M_{\mathbb{Z}/3\mathbb{Z}}^{\perp_{NS(Y)}}$, so $H^2 = 2e$, $e \in \mathbb{N}_{>0}$. Then*

- *if $e \not\equiv 0 \mod 3$, then $\mathrm{NS}(Y) = \langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$;*
- *if $e \equiv 0 \mod 3$, then $\mathrm{NS}(Y)$ is either $\langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ or $(\langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}})'$.*

Proof. The proof is analogous to the one of Theorem 4.1.4 and it is based on Proposition 4.2.2. ∎

Corollary 4.2.5. *Let Y be a $\left(\langle 2d \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)$-polarized K3 surface (or a $\left(\langle 2d \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'$-polarized K3 surface). Then Y is a projective K3 surface and it is the desingularization of the quotient of a K3 surface by a symplectic automorphism of order 3.*

Let

$$\mathcal{T} = \{\textit{quotient of K3s by an order 3 symplectic automorphism}\}/\simeq$$

where $\simeq$ is the isomorphisms of K3 surfaces. Then $\mathcal{T}$ is

$$\bigcup_{e\in\mathbb{N}_{>0}} \{\left(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)\text{-polarized K3s }\}_{/\simeq} \quad \bigcup_{\substack{e\in\mathbb{N}_{>0},\\ e\equiv 0 \mod 3}} \{\left(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'\text{-polarized K3s}\}_{/\simeq}.$$

In particular $\mathcal{T}$ is the union of countably many components all of dimension 7.

We now introduce some divisors which will be interesting in the following.

Remark 4.2.6. Let $e \equiv 0 \mod 9$ and Y be a K3 surface such that $\mathrm{NS}(Y) \simeq \left(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'$. Then the following divisors are contained in $\mathrm{NS}(Y)$:

$$\begin{aligned} D_1 &:= \frac{H-(M_1^{(1)}+2M_2^{(1)}+M_1^{(2)}+2M_2^{(2)}+M_1^{(3)}+2M_2^{(3)})}{3},\\ D_2 &:= \frac{H-(2M_1^{(4)}+M_2^{(4)}+2M_1^{(5)}+M_2^{(5)}+2M_1^{(6)}+M_2^{(6)})}{3},\\ D_3 &:= \frac{H-\left(\sum_{i=1}^{3}(2M_1^{(i)}+M_2^{(i)})+\sum_{j=4}^{6}(M_1^{(j)}+2M_2^{(j)})\right)}{3}. \end{aligned} \tag{4.2}$$

Indeed the divisor D_1 is contained in $\mathrm{NS}(Y)$ by Proposition 4.2.2, the divisor D_2 (resp. D_3) is obtained from D_1 by adding first the class $2\hat{M}$ (resp. $\hat{M}$) and then a linear combination with integer coefficients of the curves $M_i^{(j)}$, i.e. by adding to D_1 classes in $M_{\mathbb{Z}/3\mathbb{Z}} \subset \mathrm{NS}(Y)$.

Similarly, if $e \equiv 3 \mod 9$ and Y is a K3 surface such that $\mathrm{NS}(Y) \simeq \left(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'$, the following divisors are contained in $\mathrm{NS}(Y)$:

$$\begin{aligned} D_1 &:= \left(H-\left(2M_1^{(1)}+M_2^{(1)}+2M_1^{(2)}+M_2^{(2)}+M_1^{(3)}+2M_2^{(3)}+M_1^{(4)}+2M_2^{(4)}\right)\right)/3,\\ D_2 &:= \left(H-\left(M_1^{(1)}+2M_2^{(1)}+M_1^{(2)}+2M_2^{(2)}+M_1^{(5)}+2M_2^{(5)}+M_1^{(6)}+2M_2^{(6)}\right)\right)/3,\\ D_3 &:= \left(H-\left(2M_1^{(3)}+M_2^{(3)}+2M_1^{(4)}+M_2^{(4)}+M_1^{(5)}+2M_2^{(5)}+M_1^{(6)}+2M_2^{(6)}\right)\right)/3. \end{aligned} \tag{4.3}$$

If $e \equiv 6 \mod 9$ and Y is a K3 surface such that $\mathrm{NS}(Y) \simeq \left(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'$, the following divisors are contained in $\mathrm{NS}(Y)$:

$$\begin{aligned} D_1 &:= \left(H-\left(M_1^{(1)}+2M_2^{(1)}+2M_1^{(2)}+M_2^{(2)}\right)\right)/3,\\ D_2 &:= \left(H-\left(M_1^{(2)}+2M_2^{(2)}+\sum_{i=3}^{6}(2M_1^{(i)}+M_2^{(i)})\right)\right)/3,\\ D_3 &:= \left(H-\left(2M_1^{(1)}+M_2^{(1)}+\sum_{i=3}^{6}(M_1^{(i)}+2M_2^{(i)})\right)\right)/3. \end{aligned} \tag{4.4}$$

If $e \not\equiv 0$ and Y is a K3 surface such that $\mathrm{NS}(Y) \simeq \langle 2e \rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$, the following divisors are contained in $\mathrm{NS}(Y)$:

$$D_1 := H, \quad D_2 := H - \left(\sum_{i=i}^{6} (2M_1^{(i)} + M_2^{(i)}) \right) /3, \quad D_3 := H - \left(\sum_{i=i}^{6} (M_1^{(i)} + 2M_2^{(i)}) \right) /3. \tag{4.5}$$

5 | Relation between the families of projective K3 surfaces admitting a symplectic automorphism and their quotients

The main results of this section are Theorem 5.1.2, where we determine the relation between $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$ and Theorem 5.2.1. In the latter we determine the big and nef divisors on Y which are associated by the map π^* to specific ample divisors on X. This gives the relation between the dimension of the projective space in which we are embedding X and the ones were $X/\langle\sigma\rangle$ has natural models.

5.1 Relation between $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$

We first fix an embedding of $\mathrm{NS}(X)$ in $H^2(X,\mathbb{Z}) \simeq \Lambda_{K3}$ and then we apply the map π_* described in Proposition 3.2.1 in order to find $\mathrm{NS}(Y)$. To embed $\mathrm{NS}(X)$ in Λ_{K3}, we consider a specific embedding of K_{12} in Λ_{K3} constructed by embedding $\widetilde{K_{12}}$ in $U \oplus A_2(-1) \oplus E_6 \oplus E_6 \oplus E_6$ and then extending this embedding to the overlattices K_{12} and Λ_{K3}.

We fix a basis $\{k_i\}_{i=1,\dots,12}$ of the lattice $\widetilde{K_{12}}$ on which the bilinear form is the one given by the matrix (3.3). Then

$$\begin{array}{ccccccccc}
\widetilde{K_{12}} & \overset{\lambda}{\to} & U\oplus & A_2(-1)\oplus & E_6\oplus & E_6\oplus & E_6 & \\
k_i & \mapsto & (\underline{0}, & \underline{0}, & \underline{e_i^{(1)}}, & -\underline{e_i^{(2)}}, & \underline{0}), & \text{if } i=1,\dots,6 \\
k_i & \mapsto & (\underline{0}, & \underline{0}, & \underline{e_i^{(1)}}, & \underline{0}, & -\underline{e_i^{(3)}}), & \text{if } i=7,\dots,12
\end{array}$$

is an embedding of lattices. It extends to an embedding of K_{12} into Λ_{K3}, indeed the overlattice K_{12} is obtained by adding to $\lambda(\widetilde{K_{12}})$ the class $\lambda(z)$, written in (3.2). This is a class contained also in the overlattice Λ_{K3} of $U \oplus A_2(-1) \oplus E_6^{\oplus 3}$, so the enlargements of lattices are compatible.

This gives an embedding, still denoted by λ, of K_{12} in Λ_{K3}, which is the $\mathbb{Q}$-linear extension of the embedding λ defined above. By Lemma 1.1.20, the embedding of K_{12} into Λ_{K3} is unique up to isometries.

Both the lattices $\langle 2d\rangle \oplus K_{12}$ and $(\langle 2d\rangle \oplus K_{12})'$ admit a unique primitive embedding in Λ_{K3} up to isometries (by Lemma 1.1.20) and in the following proposition we exhibit one possible choice.

Proposition 5.1.1. *The embedding*

$$\begin{array}{ccccccc} \langle 2d\rangle & \stackrel{j}{\to} & U\oplus & A_2(-1)\oplus & E_6\oplus & E_6\oplus & E_6 \\ L & \mapsto & (\begin{pmatrix} 1 \\ d \end{pmatrix}, & \underline{0}, & \underline{0}, & \underline{0}, & \underline{0}) \end{array}$$

is such that $(j,\lambda) : \langle 2d\rangle \oplus K_{12} \to \Lambda_{K3}$ *is a primitive embedding of* $\langle 2d\rangle \oplus K_{12}$ *in* Λ_{K3}.

If $d \equiv 0 \mod 3$*, we consider the embedding*

$$\begin{array}{ccccccc} \langle 2d\rangle & \stackrel{\widetilde{j}}{\to} & U\oplus & A_2(-1)\oplus & E_6\oplus & E_6\oplus & E_6 \\ L & \mapsto & (\begin{pmatrix} 3 \\ 3k \end{pmatrix}, & \underline{0}, & \underline{g}^{(1)}, & \underline{g}^{(2)}, & \underline{g}^{(3)}), \end{array}$$

where

$$\underline{g}^{(i)} = \begin{cases} e_1^{(i)} + e_3^{(i)} + e_5^{(i)} & \text{if } d \equiv 0 \mod 9 \\ e_1^{(i)} + e_3^{(i)} & \text{if } d \equiv 3 \mod 9 \\ e_1^{(i)} & \text{if } d \equiv 6 \mod 9 \end{cases} \quad \text{and } k \text{ is s. t. } d = \begin{cases} 9k-9 & \text{if } d \equiv 0 \mod 9, \\ 9k-3 & \text{if } d \equiv 3 \mod 9, \\ 9k-3 & \text{if } d \equiv 6 \mod 9. \end{cases}$$

Then $(\widetilde{j},\lambda) : \langle 2d\rangle \oplus K_{12} \to \Lambda_{K3}$ *is an embedding whose primitive closure is* $(\langle 2d\rangle \oplus K_{12})'$.

Proof. It is straightforward that (j,λ) and $(\widetilde{j},\lambda)$ are embeddings of $\langle 2d\rangle \oplus K_{12}$ in Λ_{K3}. One can directly check that the first one is primitive, for example by observing that it coincides with the orthogonal complement of its orthogonal complement. The embeddings $\widetilde{j} : \langle 2d\rangle \to \Lambda_{K3}$ and $\lambda : K_{12} \to \Lambda_{K3}$ are primitive. Nevertheless, $\widetilde{j}(L)+(g^{(1)}-g^{(2)})+(g^{(1)}-g^{(3)}) \in (\widetilde{j},\lambda)(\langle 2d\rangle \oplus K_{12})$ and it is divisible by 3 in Λ_{K3}. So the primitive closure of $(\widetilde{j},\lambda)(\langle 2d\rangle \oplus K_{12})$ is an overlattice of index 3 of $(\widetilde{j},\lambda)(\langle 2d\rangle \oplus K_{12})$ which contains primitively K_{12} and $\langle 2d\rangle$, and thus it is $(\langle 2d\rangle \oplus K_{12})'$. ∎

Theorem 5.1.2. *Let* X *be a projective K3 surface admitting a symplectic automorphism* σ *of order* 3 *and let* Y *be the resolution of* $X/\langle\sigma\rangle$*. Let* $\rho(X) = 13$*. Then:*

$\mathrm{NS}(X) \simeq \langle 2d\rangle \oplus K_{12}$ *if and only if* $\mathrm{NS}(Y) \simeq \left(\langle 6d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'$;
$\mathrm{NS}(X) \simeq (\langle 6e\rangle \oplus K_{12})'$ *if and only if* $\mathrm{NS}(Y) \simeq \langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$.

Proof. We apply π_* to $\mathrm{NS}(X) = (j,\lambda)(\langle 2d\rangle, K_{12}) \subset \Lambda_{K3}$ (resp. $(\widetilde{j},\lambda)(\langle 2d\rangle, K_{12})$). By Proposition 3.2.1,

$$\pi_*(\lambda(K_{12})) = \underline{0} \in H^2(Y,\mathbb{Z}) \simeq \Lambda_{K3}.$$

It remains to consider the image of L under the embeddings we are considering. In particular we have:

$$\pi_*(j(L)) = \begin{pmatrix} 1 \\ d \end{pmatrix} \subset U(3) \subset H^2(Y,\mathbb{Z}).$$

Denote by $H = \pi_*(j(L))$, we have $H^2 = 6d$. When we extend $U(3) \oplus A_2(-3) \oplus E_6$ to $H^2(Y,\mathbb{Z})$, the class $u_1' + du_2'$ glues with elements in $M_{\mathbb{Z}/3\mathbb{Z}}$ and gives the element $n_3 + dn_4 \in H^2(Y,\mathbb{Z})$, see Proposition 3.2.3. Moreover $3(n_3 + dn_4) \in \mathrm{NS}(Y)$ because it is a linear combination of classes in $\mathrm{NS}(Y)$, that is of $H = \pi_*(j(L))$ and of classes in $M_{\mathbb{Z}/3\mathbb{Z}}$. Since $n_3 + dn_4 \in H^2(Y,\mathbb{Z})$, $3\,(n_3 + dn_4) \in \mathrm{NS}(Y)$ and $\mathrm{NS}(Y)$ is primitively embedded in $H^2(Y,\mathbb{Z})$, it follows that $n_3 + dn_4 \in \mathrm{NS}(Y)$. Hence $\mathrm{NS}(Y)$ is spanned, over $\mathbb{Z}$, by H, the generators of $M_{\mathbb{Z}/3\mathbb{Z}}$ and an extra class, i.e., $n_3 + dn_4$. So $\mathrm{NS}(Y)$ is an overlattice of index 3 of $\langle 6d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ and in particular it is necessarily the unique overlattice of index 3 described in Proposition 4.2.2.

We consider now the embedding $\widetilde{j}$:

$$\pi_*(\widetilde{j}(L)) = \left(\begin{pmatrix} 3 \\ 3k \end{pmatrix}, \underline{0}, 3\underline{g}\right) \subset U(3) \oplus A_2(-1) \oplus E_6 \subset H^2(Y,\mathbb{Z}),$$

where $\underline{g}$ is the vector $\underline{g}^{(i)}$ of Proposition 5.1.1. It is clear that $\pi_*(\widetilde{j}(L))$ is not primitive, since it is 3 divisible. So

$$\pi_*(\widetilde{j}(L))/3 = \left(\begin{pmatrix} 1 \\ k \end{pmatrix}, \underline{0}, \underline{g}\right) \in \mathrm{NS}(Y) \subset H^2(Y,\mathbb{Z}).$$

and we define H to be $\pi_*(L)/3$. So $H^2 = 6k - (\underline{g})^2$. Even enlarging $U(3) \oplus A_2(-3) \oplus E_6$ to $H^2(Y)$ by gluing the classes of $M_{\mathbb{Z}/3\mathbb{Z}}$ we do not find new classes in $\mathrm{NS}(Y)$, so $\mathrm{NS}(Y)$ is generated, over $\mathbb{Z}$, by H and by the classes generating $M_{\mathbb{Z}/3\mathbb{Z}}$. The intersection form is $\langle 2d/3\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$, which concludes the proof. ∎

Corollary 5.1.3. *The K3 surface X is a generic member of the family of the*

$(\langle 2d\rangle \oplus K_{12})$-polarized K3 surfaces if and only if Y is a generic member of the family of the $\big(\langle 2d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\big)'$-polarized K3 surfaces.

The K3 surface X is a generic member of the family of the $(\langle 6e\rangle \oplus K_{12})'$-polarized K3 surfaces if and only if Y is a generic member of the family of the $\big(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\big)$-polarized K3 surfaces.

To apply the map π^* to NS(Y), we first fix an embedding of $M_{\mathbb{Z}/3\mathbb{Z}}$ in Λ_{K3}: we constructed $\Lambda_{K3} \simeq H^2(Y,\mathbb{Z})$ as overlattice of index 3^4 of $A_2(-1)\oplus U(3)\oplus E_6\oplus M_{\mathbb{Z}/3\mathbb{Z}}$ in Proposition 3.2.3. The natural embedding of $M_{\mathbb{Z}/3\mathbb{Z}}$ in $A_2(-1)\oplus U\oplus E_6\oplus M_{\mathbb{Z}/3\mathbb{Z}}$,

$$\begin{array}{cccccccc} \mu: & M_{\mathbb{Z}/3\mathbb{Z}} & \to & A_2(-1)\oplus & U(3)\oplus & E_6\oplus & M_{\mathbb{Z}/3\mathbb{Z}} & \hookrightarrow \Lambda_{K3} \\ & m_i & \mapsto & (\underline{0}, & \underline{0}, & \underline{0}, & m_i) & \end{array}$$

extends to a primitive embedding $\mu : M_{\mathbb{Z}/3\mathbb{Z}} \to \Lambda_{K3}$.

Both the lattices $\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ and $(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}})'$ admit a unique primitive embedding in Λ_{K3} up to isometries and in the following proposition we exhibit one possible choice.

Proposition 5.1.4. *The embedding*

$$\begin{array}{cccccc} \langle 2e\rangle & \xrightarrow{h} & U(3)\oplus & A_2(-1)\oplus & E_6\oplus & M_{\mathbb{Z}/3\mathbb{Z}} \\ H & \mapsto & (\begin{pmatrix} 1 \\ k \end{pmatrix}, & \underline{0}, & \underline{f}, & \underline{0}) \end{array}$$

where

$$\underline{f} = \begin{cases} e_1 + e_3 + e_5 & \text{if} \quad 2e = 6k-6, \\ e_1 + e_3 & \text{if} \quad 2e = 6k-4, \\ e_1 & \text{if} \quad 2e = 6k-2 \end{cases}$$

is such that $(h,\mu) : \langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}} \to \Lambda_{K3}$ is a primitive embedding of $\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ in Λ_{K3}.

If $e \equiv 0 \mod 3$, then there exists k such that $e = 3k$ and the embedding

$$\begin{array}{cccccc} \langle 2e\rangle & \xrightarrow{\widetilde{h}} & U(3)\oplus & A_2(-1)\oplus & E_6\oplus & M_{\mathbb{Z}/3\mathbb{Z}} \\ H & \mapsto & (\begin{pmatrix} 1 \\ k \end{pmatrix}, & \underline{0}, & \underline{0}, & \underline{0}), \end{array}$$

is such that $(\widetilde{h},\lambda) : \langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}} \to \Lambda_{K3}$ is an embedding whose primitive closure is $\big(\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\big)'$.

Proof. The proof is analogous to the one of Proposition 5.1.1. ∎

Corollary 5.1.5. *Let Y be a K3 surface such that $\mathrm{NS}(Y) \simeq \langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ and the embedding of $\mathrm{NS}(Y)$ in Λ_{K3} is (h,μ). Then*

$$\pi^*((h,\mu)(D_i)) = \widetilde{j}(L)$$

where D_i, $i = 1,2,3$ are the divisors defined in (4.5).

Let Y be a K3 surface such that $\mathrm{NS}(Y) \simeq (\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}})'$ and the embedding of $\langle 2e\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ in Λ_{K3} is $(\widetilde{h},\mu)$. Then

$$\pi^*((\widetilde{h},\mu)(D_i)) = j(L)$$

where D_i, $i = 1,2,3$ are the divisors defined in (4.2), (4.3), (4.4).

Proof. We first consider π^* of $h(H)$ and of $\widetilde{h}(H)$:

$$\pi^*(h(H)) = \pi^*\left(\left(\begin{array}{c}1\\k\end{array}\right),\underline{0},\underline{f},\underline{0}\right) = \left(\left(\begin{array}{c}3\\3k\end{array}\right),\underline{0},\underline{g}^{(1)},\underline{g}^{(2)},\underline{g}^{(3)}\right) \in U\oplus A_2(-1)\oplus E_6\oplus E_6\oplus E_6$$

and

$$\pi^*(\widetilde{h}(H)) = \pi^*\left(\left(\begin{array}{c}1\\k\end{array}\right),\underline{0},\underline{0},\underline{0}\right) = \left(\left(\begin{array}{c}3\\3k\end{array}\right),\underline{0},\underline{0},\underline{0},\underline{0},\right) \in U\oplus A_2(-1)\oplus E_6\oplus E_6\oplus E_6.$$

In particular we observe that

$$\pi^*(h(H)) = \widetilde{j}(L) \text{ and } \pi^*(\widetilde{h}(H)) = 3j(L).$$

The divisors D_i defined in Remark 4.2.6 are embedded in $\mathrm{NS}(Y) \subset \Lambda_{K3}$ by the embedding (h,μ) or $(\widetilde{h},\mu)$ (according to the properties of $\mathrm{NS}(Y)$). We observe that $\pi^*(M_i) = \underline{0} \in H^2(X,\mathbb{Z})$, which allows to conclude. As example we consider the divisor D_1 described in Remark 4.2.6, (4.2):

$$\pi^*((\widetilde{h},\mu)(D_1)) = \pi^*\left(\frac{\widetilde{h}(H) - \mu(M_1^{(1)} + 2M_2^{(1)} + M_1^{(2)} + 2M_2^{(2)} + M_1^{(3)} + 2M_2^{(3)})}{3}\right) = j(L).$$

■

5.2 Relation of projective models for both families

We give the relation between the dimension of the projective space in which we are embedding X and the ones were $X/\langle\sigma\rangle$ has natural models. Examples of the geometric application of this result are provided in the next section.

Theorem 5.2.1. *Let X be a projective K3 surface admitting a symplectic automorphism of order 3 such that $\rho(X) = 13$, let L be the ample generator of $K_{12}^{\perp_{NS(X)}}$, let Y be the minimal resolution of $X/\langle\sigma\rangle$, and D_i, $i = 1, 2, 3$, the divisors defined in Remark (4.2.6).*

Then,

$$H^0(X, L) = \pi^* H^0(Y, D_1) \oplus \pi^* H^0(Y, D_2) \oplus \pi^* H^0(Y, D_3)$$

and the previous decomposition corresponds to the decomposition of $H^0(X, L)$ in eigenspaces with respect to the action of σ^* on $H^0(X, L)$.

Proof. We recall that the action of σ^* on the divisor L is the identity (since $K_{12} \simeq (\mathrm{NS}(X)^{\sigma^*})^{\perp}$). Hence σ^* acts on the vector space $V := H^0(X, L)$ and its action splits V in the direct sum of three eigenspaces, i.e., $V := V_{+1} \oplus V_{\zeta_3} \oplus V_{\zeta_3^2}$.

By Corollary 5.1.5, the pullbacks of the sections in $H^0(Y, D_i)$ are sections in $H^0(X, L)$. Moreover, D_i are divisors on Y, so their sections are well defined on the quotient surface $X/\langle\sigma\rangle$. Hence, given a basis $\{s_1, \dots s_r\}$ of $H^0(Y, D_i)$, $\pi^*(s_1), \dots, \pi^*(s_r)$ lie in the same eigenspace for the action of σ^* on $H^0(X, L)$, otherwise they would not be contained in the same $H^0(Y, B)$ for a divisor $B \in \mathrm{NS}(Y)$. The images of the exceptional curves $M_i^{(k)}$ under the maps $\varphi_{|D_i|} : Y \to \mathbb{P}(H^0(Y, D_i)^{\vee})$ change if one considers the divisors D_i or D_j with $i \neq j$ (since the intersection number of the divisors $M_h^{(k)}$ with D_i and D_j are not the same). So the sections of D_i and D_j lie on different eigenspaces $V_\epsilon \subset V$ where $\epsilon = +1, \zeta_3, \zeta_3^2$. It remains to prove that $\pi^* H^0(Y, D_1)$ coincides with one eigenspace V_ϵ and it is not only contained in it. To this purpose, it suffices to prove that

$$\dim(H^0(X, L)) = \dim(H^0(Y, D_1)) + \dim(H^0(Y, D_2)) + \dim(H^0(Y, D_3)).$$

We already proved that $\pi^* H^0(Y, D_1)$ is contained in an eigenspace, so we have

$$\dim(H^0(X, L)) \geq \dim(H^0(Y, D_1)) + \dim(H^0(Y, D_2)) + \dim(H^0(Y, D_3))$$

and it remains to prove that

$$\dim(H^0(X, L)) \leq \dim(H^0(Y, D_1)) + \dim(H^0(Y, D_2)) + \dim(H^0(Y, D_3)).$$

If $\mathrm{NS}(X) \simeq \langle 2d \rangle \oplus K_{12}$ and L is the ample generator of $\langle 2d \rangle$, then

$$\dim(H^0(X,L)) = d+2 \text{ and } \mathrm{NS}(Y) \simeq \left(\langle 6d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}\right)'.$$

By Riemann-Roch theorem, we obtain:

- if $3d \equiv 0 \mod 9$ then $\chi(D_1) = \frac{d}{3}+1$, $\chi(D_2) = \frac{d}{3}+1$, $\chi(D_3) = \frac{d}{3}$;
- if $3d \equiv 3 \mod 9$ then $\chi(D_1) = \frac{d}{3}+\frac{2}{3}$, $\chi(D_2) = \frac{d}{3}+\frac{2}{3}$, $\chi(D_3) = \frac{d}{3}+\frac{2}{3}$;
- if $3d \equiv 6 \mod 9$ then $\chi(D_1) = \frac{d}{3}+\frac{4}{3}$, $\chi(D_2) = \frac{d}{3}+\frac{1}{3}$, $\chi(D_3) = \frac{d}{3}+\frac{1}{3}$.

In all the listed cases $\chi(D_1)+\chi(D_2)+\chi(D_3) = d+2 = \dim(H^0(X,L))$.

The divisors D_i have positive intersection with the pseudoample divisor H and their self intersection is bigger or equal than -2. Then $h^0(D_i) > 0$ and $h^2(D_i) = h^0(-D_i) = 0$. Hence $h^0(D_i) \geq \chi(D_i)$. So

$$h^0(Y,D_1)+h^0(Y,D_2)+h^0(Y,D_3) \geq \chi(D_1)+\chi(D_2)+\chi(D_3) = h^0(X,L) = d+2,$$

which concludes the proof if $\mathrm{NS}(Y) = \langle 2d\rangle \oplus K_{12}$.

If $\mathrm{NS}(X) \simeq (\langle 2d\rangle \oplus K_{12})'$ and L is the ample generator of $\langle 2d\rangle$, then we argue as above, observing that $\dim(H^0(X,L)) = d+2$, $\mathrm{NS}(Y) \simeq \langle \frac{2d}{3}\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$ and $\chi(D_1) = \frac{d}{3}+2$, $\chi(D_2) = \frac{d}{3}$, $\chi(D_3) = \frac{d}{3}$. ■

5.3 Examples

We saw in Theorem 4.1.4 that a K3 surface X such that $\mathrm{NS}(X)$ is either $\langle 2d\rangle \oplus K_{12}$ or $(\langle 2d\rangle \oplus K_{12})'$ is projective and admits an order 3 symplectic automorphism σ. In Theorem 5.2.1, we observed that σ^* acts on the space $H^0(X,L)^\vee$, where L is the ample generator of $\langle 2d\rangle$ in the Néron–Severi group. Hence, X admits a projective model in $\mathbb{P}(H^0(X,L)^\vee)$ such that the automorphism σ is the restriction to $\varphi_{|L|}(X)$ of a projective transformation of $\mathbb{P}(H^0(X,L)^\vee)$. As an application of Theorem 5.2.1, we can determine this projective transformation on $\mathbb{P}(H^0(X,L)^\vee)$, in particular the dimension of its eigenspaces.

The surface Y, minimal resolution of $X/\langle\sigma\rangle$, has a polarization H induced by L and it is described in Theorem 5.1.2. It is orthogonal to all the classes $M_i^{(j)}$, so $\varphi_{|H|}(Y)$ is the singular model of Y where all the curves $M_i^{(j)}$ are contracted, thus it is the projective model of $X/\langle\sigma\rangle$.

As an application of Theorem 5.1.2, we have a relation between the dimension of the projective space $\mathbb{P}(H^0(X,L)^\vee)$ where the surface X is embedded, and the dimension of the projective space $\mathbb{P}(H^0(Y,H)^\vee)$ where the quotient surface $X/\langle\sigma\rangle$

is embedded. The aim of this section is to apply the previous results to obtain explicit projective equations of X and $X/\langle\sigma\rangle$ for certain values of d.

Moreover, by Theorem 5.2.1 we find other projective models of the surface Y, related with the existence of the divisors D_i.

The case $d = 1$

In this case there is a unique possibility for $\mathrm{NS}(X)$, which is $\mathrm{NS}(X) = \langle 2\rangle \oplus K_{12}$; $\varphi_{|L|} : X \to \mathbb{P}^2 = \mathbb{P}(H^0(X, L)^\vee)$ and σ is induced by an automorphism of $\mathbb{P}^2$. By Theorem 5.1.2, $\mathrm{NS}(Y) \simeq (\langle 6\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}})'$ and the definition of the divisors D_i on Y is the one given in Remark 4.2.6, (4.3). So $\dim(H^0(Y, D_i)) = 1$, for $i = 1, 2, 3$, and by Theorem 5.2.1, the eigenspaces V_ϵ have dimension 1.

Therefore there exists a choice of coordinates of $\mathbb{P}^2 = \mathbb{P}(H^0(X, L)^\vee)$ such that the action of σ on $\mathbb{P}^2$ is

$$(x_0 : x_1 : x_2) \overset{\sigma}{\mapsto} (x_0 : \zeta_3 x_1 : \zeta_3^2 x_2).$$

Let us consider the equation f_6 of the plane sextic curves invariant for σ:

$$a_1x_0^6+a_2x_0^4x_1x_2+a_3x_0^3x_1^3+a_4x_0^3x_2^3+a_5x_0^2x_1^2x_2^2+a_6x_0x_1^4x_2+a_7x_0x_1x_2^4+a_8x_1^6+a_9x_1^3x_2^3+a_{10}x_2^6.$$

The double cover of $\mathbb{P}^2$ branched on these sextic curves has equation

$$w^2 = f_6(x_0 : x_1 : x_2) \subset \mathbb{P}(3, 1, 1, 1).$$

This gives a family of K3 surfaces X admitting a symplectic automorphism of order 3, which is a lift of σ. The family depends on 10 parameters, but it is defined up to projective transformations of $\mathbb{P}^2$ which commute with σ. These are the diagonal transformations, so the dimension of the family, up to projective transformations is $(10-1)-(3-1) = 7$, which is the expected dimension of a family of projective K3 surface with automorphisms. The points of $\mathbb{P}^2$ fixed by σ are $(1:0:0)$, $(0:1:0)$, $(0:0:1)$. None of them lies on the branch sextic curve, hence they correspond to 6 points on the K3 surfaces and there exists a lift of σ which fixes all these points. So the automorphism σ acts on the K3 surface fixing 6 isolated points, and hence it is symplectic, therefore we found an explicit equation of the family of K3 surfaces whose Néron–Severi group is $\langle 2\rangle \oplus K_{12}$.

By Theorem 5.1.2, the divisor H on Y has self intersection 6 and then $\varphi_{|H|}(Y) \subset$

$\mathbb{P}^4$. The following functions are invariant for σ

$$z_0 := w, z_1 := x_0^3, z_2 := x_1^3, z_3 := x_2^3, z_4 := x_0x_1x_2$$

and satisfy the equations

$$\begin{cases} z_1z_2z_3 = z_4^3 \\ z_0^2 = a_1z_1^2 + a_2z_1z_4 + a_3z_1z_2 + a_4z_1z_3 + a_5z_4^2 + a_6z_2z_4 + a_7z_3z_4 + a_8z_2^2 + a_9z_2z_3 + a_{10}z_3^2. \end{cases} \tag{5.1}$$

Hence (5.1) are the equations of $X/\langle\sigma\rangle$ in $\mathbb{P}^4$, i.e., the equations of $\varphi_{|H|}(Y)$.

The case $d = 2$

In this case there is a unique possibility for $NS(X)$, which is $NS(X) = \langle 4\rangle \oplus K_{12}$, $\varphi_{|L|} : X \to \mathbb{P}^3 = \mathbb{P}(H^0(X, L)^\vee)$ and σ is induced by an automorphism of $\mathbb{P}^3$. By Theorem 5.1.2, $NS(Y) \simeq (\langle 12\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}})'$ and the definition of the divisors D_i on Y is the one given in Remark 4.2.6, (4.4). So $\dim(H^0(Y, D_1)) = 2$, $\dim(H^0(Y, D_i)) = 1$ for $i = 2, 3$ (cf. proof of Theorem 5.2.1) hence one eigenspace has dimension 2 and the other two have dimension 1.

So there exists a choice of coordinates of $\mathbb{P}^3 = \mathbb{P}(H^0(X, L)^\vee)$ such that the action of σ on $\mathbb{P}^3$ is

$$(x_0 : x_1 : x_2 : x_3) \overset{\sigma}{\mapsto} (x_0 : x_1 : \zeta_3x_2 : \zeta_3^2x_3).$$

The quartic equations invariant for σ are

$$f_4(x_0 : x_1) + f_2(x_0 : x_1)x_2x_3 + f_1(x_0 : x_1)x_2^3 + g_1(x_0 : x_1)x_3^3 + \alpha x_2^2x_3^2 = 0 \tag{5.2}$$

where f_i and g_i are homogeneous polynomials of degree i.

This defines a family of K3 surfaces admitting an automorphism induced by σ. The family depends on 13 parameters but it is defined up to the actions of the projective transformation of $\mathbb{P}^3$ which commutes with σ. So the family depends on $(13-1)-(6-1) = 7$ parameters. There are 4 fixed points in the eigenspace V_{+1}, which are defined by $f_4(x_0 : x_1) = 0$, $x_2 = x_3 = 0$; 1 fixed point in V_{ζ_3}, i.e., $(0 : 0 : 1 : 0)$, and 1 in $V_{\zeta_3^2}$, i.e., $(0 : 0 : 0 : 1)$. In particular, the automorphism σ fixes 6 isolated points on the K3 surfaces in the family and hence it is a symplectic automorphism of each member of this family.

By Theorem 5.1.2, the divisor H has self intersection 12 and then $\varphi_{|H|}(Y) \subset \mathbb{P}^7$. We now show that the ideal defining $\varphi_{|H|}(Y) \subset \mathbb{P}^7$ is generated by 10 quadrics and we determine a set of generators.

The Néron–Severi group $\mathrm{NS}(Y)$ is isometric to $(\langle 12\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}})'$ and hence each class in $\mathrm{NS}(Y)$ can be written as $\alpha H + \beta m$ where $m \in M_{\mathbb{Z}/3\mathbb{Z}}$ and $\alpha,\ \beta \in \frac{1}{3}\mathbb{Z}$. Let C be an irreducible curve on Y, hence $H \cdot C \geq 0$ and so the class of $C \in \mathrm{NS}(Y)$ is $\alpha H + \beta m$ with $\alpha \geq 0$. The intersection $H \cdot C$ is either 0, if $\alpha = 0$ or $12\alpha \geq 4$. In particular there are no curves C such that $H \cdot C = 2$. By [SD74, Theorem 5.2], the linear system H is not hyperelliptic and by [SD74, Theorem 7.2], the ideal of $\varphi_{|H|}(Y) \subset \mathbb{P}^7$ is generated by quadrics. Since $h^0(Y,H) = 10$, $\dim S^2(H^0(Y,H)) = \binom{9}{2} = 36$ and, by $(2H)^2 = 28$, it follows $h^0(Y,2H) = 26$. Hence the ideal of $\varphi_{|H|}(Y) \subset \mathbb{P}^7$ is generated by 10 quadrics and we are going to determine them.

We preliminary observe that the number n_h of monomials of degree h in the variables x_2 and x_3, which are invariant for the action of σ, are as in the following table

h	0	1	2	3	4	5	6
n_h	1	0	1	2	1	2	3

This allows to compute the number of number m_k of monomials of degree k in the variables x_i, $i = 0,1,2,3,4$, which are invariant for the action of σ are as in the following table

k	0	1	2	3	4	5	6
m_k	1	2	4	8	13	22	30

.

There are 8 invariant monomials of degree 3 in the variables x_i:

$$a_i = x_i^3, i = 0,1,2,3, \quad a_4 = x_0^2 x_1,\ a_5 = x_0 x_1^2,\ a_6 = x_0 x_2 x_3,\ a_7 = x_1 x_2 x_4.$$

We choose them as coordinates of the projective space $\mathbb{P}^7_{a_i}$ such that $\varphi_{|H|}(Y) \subset \mathbb{P}^7_{a_i}$.

The following quadric equations are satisfied by the variables a_i:

$$a_4^2 = a_0 a_5,\ a_4 a_5 = a_0 a_1,\ a_4 a_6 = a_0 a_7,\ a_5^2 = a_1 a_4,\ a_5 a_6 = a_4 a_7,\ a_5 a_7 = a_1 a_6 \qquad (5.3)$$

so they are contained in the ideal of quadrics defining $\varphi_{|H|}(Y) \subset \mathbb{P}^7_{a_i}$.

We now determine the other 4 quadrics generating the ideals of $\varphi_{|H|}(Y)$.

The 4 monomials in x_i which are of degree 2 and are invariant for σ, are x_0^2, x_1^2, $x_0 x_1$, $x_2 x_3$. Since the quartic (5.2) is invariant for the action of σ, multiplying it for each of the invariant monomials of degree 2 written above one finds an invariant sextic in the variables x_i, i.e.,

$$\begin{aligned} &x_i^2\left(f_4(x_0:x_1)+f_2(x_0:x_1)x_2x_3+f_1(x_0:x_1)x_2^3+g_1(x_0:x_1)x_3^3+\alpha x_2^2x_3^2\right)\\ &x_hx_k\left(f_4(x_0:x_1)+f_2(x_0:x_1)x_2x_3+f_1(x_0:x_1)x_2^3+g_1(x_0:x_1)x_3^3+\alpha x_2^2x_3^2\right)\end{aligned} \tag{5.4}$$

with $i=1,2$ and $(h,k)=(0,1),(2,3)$. The monomials appearing in these sextic are invariant monomials of degree 6. Each of them can be expressed as a monomial of degree 2 in the a_i's, since the space of the quadric in the variables a_i has dimension 30 (there 36 quadrics and 6 relations, as seen above) which is the same dimension of the space of the invariant polynomials of degree 6 in the x_i's.

So each of the four sextics in (5.4) can be expressed as a quadric equation in the variables a_i and this provides other four quadrics in $\mathbb{P}^7$ which vanish on the surface $\varphi_{|H|}(Y)$.

These four quadrics together with the ones listed in (5.3) are 10 generators of the ideal of $\varphi_{|H|}(Y)$.

The map induced by D_1 is an elliptic fibration on the quotient surface, i.e.,

$$\varphi_{|D_1|}:Y\to\mathbb{P}^1$$

is an elliptic fibration given by the projection of the quartic surface from the space $x_0=x_1=0$ to the linear subspace $\mathbb{P}^1_{(x_0:x_1)}\subset\mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$.

The case $d=3$

In this case there are two possibilities for $\mathrm{NS}(X)$, which are $\mathrm{NS}(X)=\langle 6\rangle\oplus K_{12}$ and $\mathrm{NS}(X)=(\langle 6\rangle\oplus K_{12})'$, in both cases $\varphi_{|L|}:X\to\mathbb{P}^4=\mathbb{P}(H^0(X,L)^\vee)$ and σ is induced by an automorphism of $\mathbb{P}^4$.

The case $\mathrm{NS}(X)=\langle 6\rangle\oplus K_{12}$

By Theorem 5.1.2, $\mathrm{NS}(Y)\simeq(\langle 18\rangle\oplus M_{\mathbb{Z}/3\mathbb{Z}})'$ and the definition of the divisors D_i on Y is the one given in Remark 4.2.6, (4.2). So $\dim(H^0(Y,D_i))=2$, $i=1,2$, $\dim(H^0(Y,D_3))=1$ (cf. proof of Theorem 5.2.1) hence two eigenspaces have dimension 2 and the other one has dimension 1.

So there exists a choice of coordinates of $\mathbb{P}^4=\mathbb{P}(H^0(X,L)^\vee)$ such that the action of σ on $\mathbb{P}^4$ is

$$(x_0:x_1:x_2:x_3:x_4)\overset{\sigma}{\mapsto}(x_0:x_1:\zeta_3x_2:\zeta_3x_3:\zeta_3^2x_4).$$

The K3 surfaces embedded in $\mathbb{P}^4$ are complete intersections of a quadric and a cubic, hence one has to find equations for cubic and quadric hypersurfaces such that both the hypersurfaces are invariant for the action of σ and the restriction of σ to their intersection fixes exactly 6 points. A choice which satisfies all the required conditions is:

$$\begin{cases} a_1(x_0:x_1)b_1(x_2:x_3)+\alpha x_4^2=0, \\ c_3(x_0:x_1)+d_1(x_0:x_1)e_1(x_2:x_3)x_4+f_3(x_2:x_3)+\beta x_4^3=0, \end{cases}$$

where all the polynomials are homogeneous and their degrees are their subscripts. There are 3 fixed points in $x_0=x_1=x_4=0$ (the ones which satisfy $f_3(x_2:x_3)=0$) and other 3 fixed points in $x_2=x_3=x_4=0$ (the ones which satisfy $c_3(x_0:x_1)=0$).

By Theorem 5.1.2, the divisor H has self intersection 18, and then $\varphi_{|H|}(Y)\subset\mathbb{P}^{10}$. The maps $\varphi_{|D_1|}:Y\rightarrow\mathbb{P}^1$ and $\varphi_{|D_2|}:Y\rightarrow\mathbb{P}^1$ define two elliptic fibrations, each of them has 3 independent sections corresponding to certain curves $M_i^{(j)}$ and contracts all the other curves $M_i^{(j)}$, which are necessarily irreducible components of reducible fibers. The elliptic fibrations correspond to the projections of the surfaces to the subspaces $\mathbb{P}^1_{(x_0:x_1)}$ and $\mathbb{P}^1_{(x_3:x_4)}$ respectively.

The case $\mathrm{NS}(X)=(\langle 6\rangle\oplus K_{12})'$

By Theorem 5.1.2, $\mathrm{NS}(Y)\simeq\langle 2\rangle\oplus M_{\mathbb{Z}/3\mathbb{Z}}$ and the definition of the divisors D_i on Y is the one given in Remark 4.2.6, equation (4.5). So $\dim(H^0(Y,D_1))=3$, $\dim(H^0(Y,D_i))=1$, $i=2,3$ (cf. proof of Theorem 5.2.1) hence one eigenspace has dimension 3 and the others have dimension 1.

So there exists a choice of coordinates of $\mathbb{P}^4=\mathbb{P}(H^0(X,L)^\vee)$ such that the action of σ is

$$(x_0:x_1:x_2:x_3:x_4)\overset{\sigma}{\mapsto}(x_0:x_1:x_2:\zeta_3x_3:\zeta_3^2x_4).$$

The K3 surface X is the complete intersection of a quadric and a cubic whose equations are invariant for σ:

$$\begin{cases} q_2(x_0:x_1:x_2)+\alpha x_3x_4=0, \\ c_3(x_0:x_1:x_2)+l_1(x_0:x_1:x_2)x_3x_4+\beta x_3^3+\gamma x_4^3=0, \end{cases}$$

where all the polynomials are homogeneous and their degrees are the subscript numbers. The 6 fixed points are all contained in the eigenspace V_{+1}, which are the intersections of $c_3=0$ and $q_2=0$ in the plane $x_3=x_4=0$.

By Theorem 5.1.2, the divisor H has self intersection 2 and then $\varphi_{|H|}: Y \to \mathbb{P}^2$ is a double cover. This map is induced by the projection of X on the invariant plane $\mathbb{P}^2_{(x_0:x_1:x_2)}$. The projection to $\mathbb{P}^2_{(x_0:x_1:x_2)}$ is the projection from the line $x_0 = x_1 = x_2 = 0 \subset \mathbb{P}^4$. Let us denote by $p: \mathbb{P}^4 \to \mathbb{P}^2$ this projection and by p_X its restriction to X. Given a point $(\overline{x_0} : \overline{x_1} : \overline{x_2}) \in \mathbb{P}^2$, its inverse image $p_X^{-1}((\overline{x_0} : \overline{x_1} : \overline{x_2}))$ consists of six points. Indeed in the affine chart $x_4 = 1$, the inverse image of the point for the map p is the plane whose parametric equations are

$$\begin{cases} x_0 = s\overline{x_0}, \\ x_1 = s\overline{x_1}, \\ x_2 = s\overline{x_2}, \\ x_3 = t. \end{cases}$$

The intersection of this plane with X is

$$\begin{cases} t = -s^2 q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})/\alpha, \\ s^6 \beta \left(q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})/\alpha\right)^3 + s^3 \left(c_3(\overline{x_0} : \overline{x_1} : \overline{x_2}) - l_1(\overline{x_0} : \overline{x_1} : \overline{x_2}) q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})/\alpha\right) + \gamma = 0. \end{cases}$$

If

$$\left(c_3(\overline{x_0} : \overline{x_1} : \overline{x_2}) - \frac{l_1(\overline{x_0} : \overline{x_1} : \overline{x_2}) q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})}{\alpha}\right)^2 + 4\gamma\beta \frac{q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})^3}{\alpha^3} \neq 0,$$

then one obtains two 2 solutions for s^3 in the last equation. So one obtains 6 solutions for s and each choice for s determines a choice for t and thus a point in X. Hence $p_x^{-1}(\overline{x_0} : \overline{x_1} : \overline{x_2})$ consists of 6 points. These six points form two orbits for the action of σ (indeed σ acts by multiplying s by a root of unity, so it identifies values of s which have the same third power). Hence these six points on X correspond to 2 points on the quotient surface $X/\langle\sigma\rangle$.

On the other hand if

$$\left(c_3(\overline{x_0} : \overline{x_1} : \overline{x_2}) - \frac{l_1(\overline{x_0} : \overline{x_1} : \overline{x_2}) q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})}{\alpha}\right)^2 + 4\beta \frac{q_2(\overline{x_0} : \overline{x_1} : \overline{x_2})^3}{\alpha^3} = 0,$$

then one obtains one solution (with multiplicity 2) for s^3. Hence in this case $p_X^{-1}(\overline{x_0} : \overline{x_1} : \overline{x_2})$ consists of 3 points which are in the same orbit for σ and thus to a point in $X/\langle\sigma\rangle$.

So this projection describes $X/\langle\sigma\rangle$ as a double cover of $\mathbb{P}^2_{(x_0:x_1:x_2)}$ branched on the sextic

$$\alpha^3 c_3^2 + \alpha l_1^2 q_2^2 - 2\alpha^2 c_3 l_1 q_2 + 4\beta q_2^3 = 0$$

where we wrote c_3 (resp. l_1 and q_2) instead of $c_3(x_0 : x_1 : x_2)$ (resp. $l_1(x_0 : x_1 : x_2)$ and $q_2(x_0 : x_1 : x_2)$). We observe that the branch locus is a sextic with 6 singularities of type A_2, that are the points where $c_3(x_0 : x_1 : x_2) = q_2(x_0 : x_1 : x_2) = 0$. These are the images of the 6 points in X fixed by σ.

6 | Applications

In Section 3, we described the action induced in cohomology by a symplectic automorphism of order 3, while in Section 5, we described the relation between the Néron–Severi group of projective K3 surfaces admitting an order 3 symplectic automorphism and the Néron–Severi group of its quotient. Hence, we are able to generalize some of the results obtained for the involutions to the order 3 automorphisms. In particular, we consider two different aspects: the construction of isogenies of K3 surfaces which are not quotient maps and the Shioda–Inose structures, which are relations between the K3 surfaces that are quotients of Abelian surfaces and other K3 surfaces.

6.1 Isogenies of K3 surfaces

In [CG20, Theorem 3.9], it is proved the following result

Proposition 6.1.1. *There exists the following lattice isometry*

$$(\langle 6d\rangle \oplus K_{12})' \xrightarrow{\simeq} \langle 6d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}.$$

By the previous proposition if X_d is a K3 surface such that $\mathrm{NS}(X_d) \simeq (\langle 6d\rangle \oplus K_{12})' \simeq \langle 6d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$, then:

- X_d admits a symplectic automorphism σ_d of order 3 (by Theorem 4.1.4),
- there exists a K3 surface S which admits a symplectic automorphism σ_S of order 3 such that X_d is the minimal resolution of S/σ_S (by Theorem 4.2.4).

Corollary 6.1.2. *There exists an infinite tower of isogenies*

$$X_d \overset{\alpha_1}{\dashleftarrow} X_{3d} \overset{\alpha_2}{\dashleftarrow} X_{3^2d} \overset{\alpha_3}{\dashleftarrow} X_{3^3d} \ldots X_{3^{h-1}d} \overset{\alpha_h}{\dashleftarrow} X_{3^hd} \ldots .$$

Each map α_i is the composition of a quotient map of order 3 automorphism and a birational map which resolves the singularities of the quotient.

The compositions

$$\alpha_i \circ \alpha_{i-1} \circ \ldots \circ \alpha_{i-k}$$

are not induced by quotients.

Proof. To construct the tower, we start with a K3 surface X_d such that $\mathrm{NS}(X_d) \simeq \langle 6d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$. Since $M_{\mathbb{Z}/3\mathbb{Z}}$ is primitively embedded in $\mathrm{NS}(X_d)$, there exists a K3 surface S such that S admits a symplectic automorphism σ and the minimal resolution of S/σ is X_d. By Theorem 5.1.2, the Néron–Severi group of S is $\mathrm{NS}(S) \simeq (\langle 18d\rangle \oplus K_{12})' \simeq \langle 18d\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$, hence S is the surface that we denote by X_{3d}. Since the lattice $M_{\mathbb{Z}/3\mathbb{Z}}$ is primitively embedded in $\mathrm{NS}(X_{3d})$, there exists a K3 surface which admits a symplectic automorphism of order 3 such that the minimal resolution of the quotient of the surface by the automorphism is X_{3d}. This surface is X_{3^2d}, since its Néron–Severi group is isometric to $(\langle 54d\rangle \oplus K_{12})' \simeq \langle 54\rangle \oplus M_{\mathbb{Z}/3\mathbb{Z}}$. This process can be iterated infinitely many times. As explained in [CG20, Proposition 3.11], it is not possible that the maps $X_h \dashrightarrow X_k$ are induced by quotient maps if $k/h > 3$. ■

6.2 Generalized Shioda-Inose structures

We recall that given an Abelian surface A, there exists an involution ι_A (which maps each point to its opposite w.r.t the group law of A) such that the minimal resolution of $A/\langle\iota_A\rangle$ is a K3 surface, called Kummer surface of A and denoted by $Km(A)$.

Definition 6.2.1. *A* ***Shioda–Inose structure*** *is the triple* (A, X, ι) *where:* A *is an Abelian surface,* X *is a K3 surface,* ι_X *a symplectic involution such that the minimal resolution of* $X/\langle\iota_X\rangle$ *is the Kummer surface* $Km(A)$ *and the transcendental lattice of* X *and* A *are isometric.*

One can show that a Shioda–Inose structure is associated to the following diagram

$$\begin{tikzcd} A \arrow[dr, dashed] & & X \arrow[dl, dashed] \\ & Km(A) & \end{tikzcd}$$

where the arrows are rational generically $2:1$ maps and there are the following relations between the transcendental lattices of the surfaces:

$$T_A \simeq T_X, \quad T_{Km(A)} \simeq T_A(2) \simeq T_X(2).$$

These structures are defined and studied in [Mor84].

We first generalize the definition of Shioda-Inose structure. The present definition is an extension of the previous definition given by [OS99] and [CO00]

Definition 6.2.2. *A generalized Shioda–Inose structure of order* 3 *is given by* $(A, \sigma_A, X, \sigma_X)$ *where:*

- A *is a 2-dimensional torus admitting a symplectic automorphism* σ_A *of order* 3 *such that the minimal resolution of* $A/\langle\sigma_A\rangle$ *is a K3 surface* $Km_3(A)$*;*
- X *is a K3 surface admitting a symplectic automorphism* σ_X *of order* 3 *such that the minimal resolution of* $X/\langle\sigma_X\rangle$ *is the K3 surface* $Km_3(A)$*;*
- $T_A \simeq T_X$.

In particular a generalized Shioda–Inose structure of order 3 is associated to the following diagram

$$\begin{tikzcd} A \arrow[dr, dashed] & & X \arrow[dl, dashed] \\ & Km_3(A) & \end{tikzcd}$$

where the dash-arrows correspond to $3:1$ rational maps, and $T_A \simeq T_X$.

Remark 6.2.3. *Note that all the Abelian surfaces admit an involution* ι_A *such that* $A/\langle\iota_A\rangle$ *is birational to a K3 surface, but not all the Abelian surfaces admit an automorphism* σ_A *of order* 3 *such that* $A/\langle\sigma_A\rangle$ *is birational to a K3 surface. The classification of the Abelian surfaces admitting such an automorphism of order* 3 *is due to Fujiki, [Fuj88] and their lattice theoretic characterization is recalled in Proposition 6.2.5.*

The geometric and the lattice theoretic characterizations of the K3 surfaces which are obtained as minimal resolution of the quotient of an Abelian surface by an automorphism of order 3 can be found in [Bar98] and [Ber88] respectively. In particular there is a lattice, denoted by K_3, which characterizes the K3 surfaces which are birational to the quotient of an Abelian surface by a symplectic automorphism of order 3 and which is the order 3 analogous of the Kummer lattice.

Let A is a 2-dimensional torus with a symplectic automorphism σ_A of order 3 such that the minimal resolution of $A/\langle\sigma_A\rangle$ is a K3 surface, then $A/\langle\sigma_A\rangle$ has 9 singularities of type A_2.

Definition 6.2.4. *The lattice K_3 is the minimal primitive sublattice of* $\mathrm{NS}(Km_3(A))$ *containing the curves arising from the resolution of the singularities of* $A/\langle\sigma_A\rangle$.

In the following Proposition, we summarize some results on tori and Abelian surfaces A which admit a symplectic automorphism σ_A of order 3.

Set $Km_3(A)$ by the minimal resolution of $A/\langle\sigma_A\rangle$ and by π_{A*} and π_A^* the map induced in cohomology by the rational $3:1$ map $\pi_A : A \dashrightarrow Km_3(A)$.

We recall that for every torus A, $H^2(A,\mathbb{Z}) \simeq U \oplus U \oplus U$, see Corollary 1.1.16.

Proposition 6.2.5. *a) A 2-dimensional torus A admits a symplectic automorphism σ_A of order 3 if and only if its transcendental lattice T_A is primitively embedded in $U \oplus A_2(-1)$.*

b) A K3 surface Y is birational to the quotient of a torus by a symplectic automorphism of order 3 if and only if K_3 is primitively embedded in $\mathrm{NS}(Y)$.

c) The lattice K_3 is a negative definite lattice of rank 18 with the discriminant form which is the opposite of the one of $U(3) \oplus A_2(-1)$.

d) If A is a 2-dimensional torus endowed with a symplectic automorphism σ_A as above,

$$\pi_{A*}(H^2(A,\mathbb{Z})) = \pi_{A*}(U \oplus U \oplus U) = U(3) \oplus A_2(-1).$$

e) Let A, σ_A and $Km_3(A)$ be as above, then $T_{Km_3(A)} = \pi_{A*}(T_A)$.

Proof. a) is proved in [Fuj88, Theorem 6.4]; b) is proved in [Ber88]. To prove c), we recall that in [Ber88] it is also proved that K_3 is negative definite of rank 18 and that $A_{K_3} \simeq (\mathbb{Z}/3\mathbb{Z})^3$. So K_3 is a 3-elementary lattice, i.e., its discriminant group is generated by a finite number of copies of $\mathbb{Z}/3\mathbb{Z}$. By b) we have that K_3 is primitively embedded in Λ_{K3} and this implies that the orthogonal of K_3 in Λ_{K3} is an indefinite 3-elementary lattice of rank 4. Since these lattices are completely determined by the rank, the signature and the length, we deduce that the orthogonal to K_3 in Λ_{K3} is $U(3) \oplus A_2(-1)$ which implies that the discriminant form of K_3 and $U(3) \oplus A_2(-1)$ are opposite each other.

We now prove d). By a) and [Fuj88] one deduces the action of σ_A^* on the second cohomology group of A. Since σ_A is symplectic σ_A^* acts as the identity on T_A. In particular if A is a generic torus admitting a symplectic automorphism of order 3, $(H^2(A,\mathbb{Z}))^{\sigma_A^*} \simeq T_A \simeq U \oplus A_2(-1)$. So $((H^2(A,\mathbb{Z}))^{\sigma_A^*})^\perp \simeq \mathrm{NS}(A) \simeq A_2$. Therefore one can describe $H^2(A,\mathbb{Z}) \simeq U \oplus U \oplus U$ as overlattice of index 3 of $U \oplus A_2(-1) \oplus A_2$

obtained by adding the class $y_A = (\alpha_1+2\alpha_2+\beta_1+2\beta_2)/3$ where α_i are the generators of $A_2(-1)$ and β_i the ones of A_2. The isometry σ_A^* of $U\oplus A_2(-1)\oplus A_2$ is the identity on $U\oplus A_2(-1)$ and acts as follows on A_2: $\sigma_A^*(\beta_1)=\beta_2$, $\sigma_A^*(\beta_2)=-\beta_1-\beta_2$. The action of σ_A^* on $U^{\oplus 3}$ is obtained by $\mathbb{Q}$-linear extension. As in Proposition 3.2.1, this implies that π_* acts on $U\oplus A_2(-1)\oplus A_2$ as follows

$$\begin{array}{cccccccc} \pi_{A*}: & U\oplus & A_2(-1)\oplus & A_2 & \to & U(3)\oplus & A_2(-3) \\ & (\underline{u}, & \underline{\alpha}, & \underline{\beta}) & \to & (\underline{u} & \underline{\alpha}). \end{array}$$

Its $\mathbb{Q}$-linear extension to $U^{\oplus 3}$ is computed by considering the image of y_A. Since $\pi_*(y_A)=\pi_*(\alpha_1+2\alpha_2)/3$, one obtains, as in Proposition 3.2.1, that

$$\pi_{A*}(U\oplus A_2(-1)) = U(3)\oplus A_2(-1)\subset H^2(Km_3(A),\mathbb{Z})\simeq \Lambda_{K3}$$

where the embedding $\pi_{A*}(U\oplus A_2(-1))\hookrightarrow H^2(Km_3(A),\mathbb{Z})$ is primitive.

To prove $e)$ we observe that by construction $\pi_{A*}(U^{\oplus 3})$ and K_3 are orthogonal in Λ_{K3}, K_3 is primitive, they have the opposite discriminant form. Hence both of them are primitive and if $T_A\simeq U\oplus A_2(-1)$, then $T_{Km_3(A)}=U(3)\oplus A_2(-1)=\pi_{A*}(T_A)$. If T_A is primitively embedded in $U\oplus A_2(-1)$ we obtain that $\pi_{A*}(T_A)$ is primitively embedded in Λ_{K3} and hence $T_{Km_3(A)}=\pi_{A*}(T_A)$. ■

We now consider a special case of symplectic automorphism of order 3 on a K3 surface. We know that the action induced in cohomology by a symplectic automorphism of order 3 permutes three copies of E_6, but for a general K3 surface, these copies of E_6 are not necessarily contained in the Néron–Severi group. We are now interested in K3 surfaces whose Néron–Severi groups contain three copies of E_6 and hence admit a symplectic automorphism of order 3 which permutes these copies inside the Néron–Severi group. Of course the surface S considered in Section 2.1 is an example of this kind of surfaces. To state our result, we need to define a certain overlattice of $M_{\mathbb{Z}/3\mathbb{Z}}\oplus E_6$:

Definition 6.2.6. *The lattice $(M_{\mathbb{Z}/3\mathbb{Z}}\oplus E_6)'$ is the overlattice of index 3 of $M_{\mathbb{Z}/3\mathbb{Z}}\oplus E_6$ obtained by adding the class n_2 (defined in Proposition 3.2.3) as*

$$n_2=\frac{e_1+2e_2+e_4+2e_5}{3}-\frac{M_1^{(1)}+2M_2^{(1)}-M_1^{(3)}-2M_2^{(3)}+M_1^{(4)}+2M_2^{(4)}-M_1^{(5)}-2M_2^{(5)}}{3},$$

where e_i are the standard generators of E_6, and $M_1^{(j)}$ and $M_2^{(j)}$ are the generators of the j–copy of A_2.

Proposition 6.2.7. *a) The lattice $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ is a negative definite lattice of rank 18 whose discriminant form is the opposite to the one of $U(3) \oplus A_2(-1)$.*

b) Let X be a K3 surface with a symplectic automorphism σ of order 3 and Y be the minimal resolution of $X/\langle\sigma\rangle$. The automorphism σ permutes three copies of E_6 contained in $\mathrm{NS}(X)$ if and only if $\mathrm{NS}(Y)$ contains primitively $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$.

c) If X and Y are as in b), then T_X is primitively embedded in $U \oplus A_2(-1)$ and $T_Y = \pi_*(T_X) \subset U(3) \oplus A_2(-1)$.

Proof. a) follows by a direct computation, since one has an explicit basis of $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$. Alternatively, one can observe that the orthogonal to $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ in Λ_{K3} is $U(3) \oplus A_2(-1)$.

To prove b), we first assume that there is an embedding of $E_6^{\oplus 3}$ in $\mathrm{NS}(X)$ and that σ_3 is a symplectic automorphism which permutes cyclically these three copies of E_6. By Theorem 2.2.5, the lattice $(E_6 \oplus E_6 \oplus E_6)'$ is primitively embedded in Λ_{K3}, and since $\mathrm{NS}(X)$ is primitive in $H^2(X,\mathbb{Z}) \simeq \Lambda_{K3}$, there is a primitive embedding of $(E_6 \oplus E_6 \oplus E_6)'$ in $\mathrm{NS}(X)$. The action of σ^* on $\mathrm{NS}(X)$ is the restriction of the action of σ^* on $H^2(X,\mathbb{Z})$ described in Equation (3.1) and the action of π_* is the one described in Proposition 3.3.1. Since $\pi_*(\mathrm{NS}(X)) \subset \mathrm{NS}(Y)$, $\pi_*((E_6 \oplus E_6 \oplus E_6)') = E_6 \subset \mathrm{NS}(Y)$, where the embedding is primitive. Moreover $M_{\mathbb{Z}/3\mathbb{Z}} \subset \mathrm{NS}(Y)$, because Y is, by construction, the desingularization of $X/\langle\sigma\rangle$. By Proposition 3.2.3, the class n_2 is contained in $H^2(Y,\mathbb{Z})$ and since $\mathrm{NS}(Y)$ is primitive in $H^2(Y,\mathbb{Z})$, it has to be contained also in $\mathrm{NS}(Y)$. It follows that the lattice $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ is primitively embedded in $\mathrm{NS}(Y)$. Vice versa, if $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ is primitively embedded in $\mathrm{NS}(Y)$, we construct the triple cover of Y branched on the curves contained in $M_{\mathbb{Z}/3\mathbb{Z}}$ and then we contract some (-1)-curves to obtain X. This induces the map $\pi^* : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, described in Proposition 3.2.1 which restricts to a map $\pi^* : \mathrm{NS}(Y) \to \mathrm{NS}(X)$. Since $E_6 \subset \mathrm{NS}(Y)$, we obtain $\pi^*(E_6) = E_6 \oplus E_6 \oplus E_6 \subset \mathrm{NS}(X)$. Moreover, the description of the maps π_* and π^* are obtained by the assumption that σ^* permutes the three copies of E_6 in $E_6 \oplus E_6 \oplus E_6 \subset \Lambda_{K3}$. It follows that the order 3 symplectic automorphism acting on X in such a way that Y is birational to X/σ, permutes the three copies of E_6 in $\pi^*(E_6) \subset \mathrm{NS}(X)$.

To prove c) we observe that the orthogonal to $(E_6 \oplus E_6 \oplus E_6)'$ in Λ_{K3} is $U \oplus A_2(-1)$ (either by Theorem 2.2.5, or by the unicity of the indefinite 3-elementary lattices with a given signature and length). Hence $(E_6 \oplus E_6 \oplus E_6)'$ is primitively embedded in $\mathrm{NS}(X)$ if and only if T_X is primitively embedded in $U \oplus A_2(-1)$. The application of the map π_* to $T_X \subset U \oplus A_2(-1)$, implies the relation between T_X and T_Y. ■

Now we can generalize the results by Morrison in [Mor84, Theorem 6.3] to the generalized Shioda–Inose structure of order 3.

Theorem 6.2.8. *Let X be a projective K3 surface. The following conditions are equivalent:*

a) T_X is primitively embedded in $U \oplus A_2(-1)$;

b) X admits an order 3 symplectic automorphism permuting three copies of E_6 contained in $\mathrm{NS}(X)$;

c) there exists an Abelian surface A such that $T_A \simeq T_X$ and A admits a symplectic automorphism of order 3, denoted by σ_A, such that $A/\langle\sigma_A\rangle$ is birational to a K3 surface;

d) there exist an Abelian surface A, an order 3 symplectic automorphism σ_A on A and an order 3 symplectic automorphism σ_X on X such that $(A, \sigma_A, X, \sigma_X)$ is a generalized Shioda–Inose structure of order 3.

Proof. We first prove that *a*) implies *b*). The lattice T_X admits a unique embedding in Λ_{K3}, up to isometries. We constructed Λ_{K3} as an index 3 overlattice of $U \oplus A_2(-1) \oplus (E_6 \oplus E_6 \oplus E_6)'$ where $U \oplus A_2(-1)$ is primitively embedded. So we fixed a primitive embedding of $U \oplus A_2(-1)$ in Λ_{K3} whose orthogonal is $(E_6 \oplus E_6 \oplus E_6)'$. Hence

$$T_X \hookrightarrow U \oplus A_2(-1) \Leftrightarrow \mathrm{NS}(X) = (T_X)^\perp \hookleftarrow (U \oplus A_2(-1))^\perp = (E_6 \oplus E_6 \oplus E_6)'.$$

If $\mathrm{NS}(X)$ contains primitively $(E_6 \oplus E_6 \oplus E_6)'$ it contains K_{12}, which is primitively contained in $(E_6 \oplus E_6 \oplus E_6)'$, and hence admits an order 3 symplectic automorphism σ. The action of σ^* such that $K_{12} \simeq (\mathrm{NS}(X)^{\sigma^*})^\perp$ is primitively embedded in $(E_6 \oplus E_6 \oplus E_6)'$ is the one described in 3.1 and hence σ^* permutes three copies of E_6 contained in $\mathrm{NS}(X)$.

By Proposition 6.2.7, *b*) implies *a*).

The equivalence between *a*) and *c*) follows by Proposition 6.2.5, point *a*). Indeed, since $U \oplus A_2(-1)$ is primitively embedded in $U \oplus U \oplus U$, if T_X is primitively embedded in $U \oplus A_2(-1)$, then it is primitively embedded in $U \oplus U \oplus U$. This implies that there exists an Abelian surface A such that $T_A \simeq T_X$. This Abelian surface admits a symplectic automorphism σ_A by Proposition 6.2.5. Viceversa, if A is an

Abelian surface admitting an automorphism σ_A as in c), then T_A is primitively embedded $U \oplus A_2(-1)$, by Proposition 6.2.5. Then, if T_X is isometric to T_A, T_X is primitively embedded in $U \oplus A_2(-1)$. This shows the equivalence between a) and c).

If b) holds, X is a K3 surface endowed with a symplectic automorphism σ_X acting as described. Denote by Y the minimal resolution of $X/\langle\sigma_X\rangle$, $\mathrm{NS}(Y)$ contains primitively $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ by Proposition 6.2.7. Since the lattices $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ and K_3 have the same rank, signature, discriminant group and form, $(M_{\mathbb{Z}/3\mathbb{Z}} \oplus E_6)'$ is primitively embedded in $\mathrm{NS}(Y)$ if and only if K_3 is primitively embedded in $\mathrm{NS}(Y)$, by [Mor84, Lemma 2.3]. So there exists an Abelian surface A and a symplectic automorphism of order 3 on it, σ_A, such that $Y = Km_3(A)$. In order to conclude that $(A, \sigma_A, X, \sigma_X)$ is an order 3 generalized Shioda–Inose structure, one has to show that $T_A \simeq T_X$.

We observe that there are two maps acting on $U \oplus A_2(-1)$: the map $(\pi_{A*})_{|U\oplus A_2(-1)}$, where π_{A*} is the map described in Proposition 6.2.5, and the map $(\pi_*)_{|U\oplus A_2(-1)}$, where $\pi_* : A_2(-1) \oplus U \oplus E_6^{\oplus 3} \longrightarrow A_2(-3) \oplus U(3) \oplus E_6$ is the map described in Proposition 3.2.1. The action of these two maps coincides on $U \oplus A_2(-1)$:

$$(\pi_{A*})_{|U\oplus A_2(-1)} = (\pi_*)_{|U\oplus A_2(-1)} \text{ and}$$

$$\pi_{A*}(U \oplus A_2(-1)) = U(3) \oplus A_2(-1) = \pi_*(U \oplus A_2(-1)).$$

The lattice T_X is primitively embedded in $U \oplus A_2(-1)$ and the transcendental lattice of Y is $T_Y = \pi_*(T_X)$. On the other hand, since Y is $Km_3(A)$ for a certain A, $T_Y = \pi_{A*}(T_A)$. Hence

$$\pi_*(T_X) = \pi_{A*}(T_A).$$

Since π_* and π_{A*} coincide on $U \oplus A_2(-1)$ it follows that $T_X = T_A$. So we proved that b) implies d).

Assuming d), $T_A \simeq T_X$ and A admits an order 3 symplectic automorphism σ_A such that $A/\langle\sigma_A\rangle$ is birational to a K3 surface, by definition of generalized Shioda–Inose structure. So T_A is primitively embedded in $U \oplus A_2(-1)$ by Proposition 6.2.5 and hence T_X is primitively embedded in $U \oplus A_2(-1)$ which shows that d) implies a). This concludes the proof. ∎

Remark 6.2.9. *If $(A, \sigma_A, X, \sigma_X)$ is a generalized Shioda–Inose structure of order 3 and the surface A is projective, then there exists an involution ι_X on X such that (A, X, ι_X) is a "classical" Shioda–Inose structure (of order 2). Indeed if $(A, \sigma_A, X, \sigma_X)$ is a generalized Shioda–Inose structure of order 3, then $T_X \simeq T_A$ and*

X and A are projective. Hence by [Mor84, Theorem 6.3] X admits the required involution.

7 | Symplectic birational maps on hyperkähler manifolds of K3$^{[n]}$–type

Let X be a projective hyperkähler manifold of K3$^{[n]}$–type. Denote by $\mathrm{Bir}(X)$ the group of birational maps of X. There are natural maps

$$\mathrm{Bir}(X) \longrightarrow \mathrm{O}(H^2(X,\mathbb{Z})) \text{ and } \mathrm{O}(H^2(X,\mathbb{Z})) \longrightarrow \mathrm{O}(A_X)$$

where the first map sends a birational map g of X to its action g^* on cohomology, and the second one sends an isometry φ of $H^2(X,\mathbb{Z})$ to an isometry $\bar{\varphi}$ of A_X. Note that the second map is surjective by Lemma 1.1.7.

The following lemma characterizes when an isometry of $H^2(X,\mathbb{Z})$ comes from a symplectic automorphism group.

Theorem 7.0.1. *[Mon16, Theorem 26] Let G be a finite subgroup of* $\mathrm{O}(H^2(X,\mathbb{Z}))$*. Then G is induced by a symplectic subgroup of* $\mathrm{Aut}(X)$ *if and only if the following hold:*

- *$S_G(X)$ is non degenerate and negative definite;*
- *$S_G(X)$ contains no numerical wall divisors;*
- *G acts trivially on A_X.*

7.1 Cohomological action of symplectic birational maps

We say that G is a *symplectic group* if it is a finite subgroup of $\mathrm{Bir}(X)$ such that for all $g \in G$, $g^*_{|_{H^{2,0}(X)}} = \mathrm{Id}^*$. Define the map α as the composition of the inclusion $G \subset \mathrm{Bir}(X)$ with the natural maps $\mathrm{Bir}(X) \longrightarrow \mathrm{O}(H^2(X,\mathbb{Z}))$ and $\mathrm{O}(H^2(X,\mathbb{Z})) \longrightarrow \mathrm{O}(A_X)$. As we saw in Theorem 1.4.12, the action of $\mathrm{Mon}^2(X)$ on A_X can be either Id or $-\mathrm{Id}$.

We will restrict our attention to symplectic groups G where there exist at least one $g \in G$ such that g^* is a monodromy operator and g^* acts on A_X as $-\mathrm{Id}$. Since $\mathrm{O}(A_X) \simeq (\mathbb{Z}/2)^r$ for some r, we get $\alpha(G) = \mathbb{Z}/2\mathbb{Z}$. Hence, there exists a short exact sequence

$$1 \longrightarrow H \hookrightarrow G \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

where $H \cong \ker(\alpha)$ acts on A_X by Id^*. So in particular, $\mathrm{ord}(g)$ is even. Let us say that $\mathrm{ord}(g) = 2^k m$ for some $k > 0$ and $2 \nmid m$. It follows easily that g^m has order 2^k and so the existence of g implies that always existence a map of order 2^k with non–trivial action on A_X since m is odd.

Proposition 7.1.1. *If $g^*_{|A_X} = -\mathrm{Id}$, then $g^*\delta = -\delta + 2(n-1)w$ for some $w \in H^2(X,\mathbb{Z})$.*

Proof. let $g^* : H^2(X,\mathbb{Z})^\vee \longrightarrow H^2(X,\mathbb{Z})^\vee$ be the $\mathbb{Q}$-linear extension of g^*, and q be the quotient map. By the following commutative diagram,

$$\begin{array}{ccc} H^2(X,\mathbb{Z})^\vee & \xrightarrow{q} & A_X \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle -Id} \\ H^2(X,\mathbb{Z})^\vee & \xrightarrow[q]{} & A_X \end{array}$$

the class $\delta/2(n-1)$ is mapped to $-\delta/2(n-1) + H^2(X,\mathbb{Z})$ and also to $g^*\delta/2(n-1) + H^2(X,\mathbb{Z})$. This implies $\frac{1}{2(n-1)}(g^*(\delta)+\delta) = w$ for some $w \in H^2(X,\mathbb{Z})$, and so $g^*\delta = -\delta + 2(n-1)w$. ∎

Let ι be the primitive isometric embedding in (1.11). We want to extend g^* to the Mukai lattice $\widetilde{\Lambda}$:

Proposition 7.1.2. *There exists a unique extension of g^* on $\widetilde{\Lambda}$ such that $g^*_{|A_X} = -\mathrm{Id}$.*

Proof. Note that a natural extension of g^* in $\widetilde{\Lambda}$ such that $g^*|_{A_X} = -\mathrm{Id}$ implies that $g^*v = -v$. In fact, if $(v\cdot v) = (g^*v \cdot g^*v) = (\lambda v\cdot \lambda v) = \lambda^2(v\cdot v)$ for some $\lambda \in \mathbb{Z}$. Then, $(\lambda^2-1)(v\cdot v) = 0$ implies that $\lambda = \pm 1$. Suppose that $\lambda = 1$, then $g^*|_{A_{\langle v\rangle}} = \mathrm{Id}$ and so $g^*_{A_X} = \mathrm{Id}$ which is a contradiction with the choice of g. ∎

Denote by $H^2(X,\mathbb{Z})^{g^*}$ and $\widetilde{\Lambda}^{g^*}$ the invariant sublattices of $H^2(X,\mathbb{Z})$ and $\widetilde{\Lambda}$ respectively, and $S_{g^*}(X) = (H^2(X,\mathbb{Z})^{g^*})^\perp \subset H^2(X,\mathbb{Z})$ and $S_{g^*}(\widetilde{\Lambda}) = (\widetilde{\Lambda}^{g^*})^\perp \subset \widetilde{\Lambda}$ the co-invariant sublattices of $H^2(X,\mathbb{Z})$ and $\widetilde{\Lambda}$ respectively. By Proposition 7.1.2, we get $H^2(X,\mathbb{Z})^{g^*} = \widetilde{\Lambda}^{g*}$. It is an elementary but useful observation that g^2 is a birational map on X of order $n/2$ and acting on A_X as Id. It follows from Proposition 1.1.25 that if $\operatorname{ord} g = 2$, then $S_{g^*}(\widetilde{\Lambda})$ is automatically a 2–elementary lattice.

Proposition 7.1.3. *Suppose that* $\operatorname{ord}(g) = 2$. *Then,* $S_{g^*}(\widetilde{\Lambda}) = U \oplus K$ *if and only if the following hold*

- $l(A_{S_{g^*}(\widetilde{\Lambda})}) \leq \operatorname{rank} S_{g^*}(\widetilde{\Lambda}) - 2$*;*
- *if* $\delta(S_{g^*}(\widetilde{\Lambda})) = 0$ *and* $l(A_{S_{g^*}(\widetilde{\Lambda})}) = \operatorname{rank} S_{g^*}(\widetilde{\Lambda}) - 2$*, then* $\operatorname{rank} S_{g^*}(\widetilde{\Lambda}) \equiv 2 \mod 8$*,*

where K *is a negative definite, even, 2-elementary lattice of* $l(K) = l(S_{g^*}(\widetilde{\Lambda}))$ *and* $\delta(K) = \delta(S_{g^*}(\widetilde{\Lambda}))$.

Proof. Applying Lemma 1.1.23, we obtain the desired splitting since both lattices are 2–elementary. ■

For an arbitrary order of g, it is not immediate a splitting of $S_{g^*}(\widetilde{\Lambda})$ as $U \oplus K$ for some K. However, it can be possible to obtain a similar decomposition in terms of $U(2)$ or $(2) \oplus (-2)$ instead of U. Let us note some properties of $S_{g^*}(X)$ and $S_{g^*}(\widetilde{\Lambda})$ under the above conditions of g.

Proposition 7.1.4. *The lattice* $S_{g^*}(X)$ *is negative definite and the lattice* $S_{g^*}(\widetilde{\Lambda})$ *has signature* $(1, r)$ *where* $r = \operatorname{rank} S_{g^*}(X)$.

Proof. Let ω be a non degenerate holomorphic 2-form of X. Since g is symplectic, $g^*(\omega) = \omega$ and $g^*(\bar{\omega}) = \bar{\omega}$. Let $\alpha \in H^{1,1}(X)$ be a Kähler class of X. The class $\alpha + g^*(\alpha) + (g^2)^*(\alpha) + \ldots + (g^{\operatorname{ord}(g)-1})^*(\alpha)$ is in $H^2(X,\mathbb{Z})^{g^*}$ and it is different from ω and $\bar{\omega}$. Hence, $\operatorname{sign}(H^2(X,\mathbb{Z})^{g^*}) = (3, \operatorname{rank} H^2(X,\mathbb{Z})^{g^*} - 3)$, and so

$$\operatorname{sign}(S_{g^*}(X)) = (0, 23 - \operatorname{rank} H^2(X,\mathbb{Z})^{g^*}).$$

Since $H^2(X,\mathbb{Z}) \subset \widetilde{\Lambda}$ primitively, and $v \in S_{g^*}(\widetilde{\Lambda})$ then $S_{g^*}(X) \oplus \langle v \rangle \subset S_{g^*}(\widetilde{\Lambda})$. The proof is completed by showing $\operatorname{rank} S_{g^*}(\widetilde{\Lambda}) = 24 - \operatorname{rank} H^2(X,\mathbb{Z})^{g^*} = 1 + r$. ■

Remark 7.1.5. *We saw that* $\operatorname{ord}(g) = 2^k m$ *where* m *is odd. Suppose that* $m = 1$. *By Lemma 1.1.13,* $A_{S_{g^*}(\widetilde{\Lambda})}$ *is isomorphic to* $\widetilde{\Lambda}/(\widetilde{\Lambda}^{g^*} \oplus S_{g^*}(\widetilde{\Lambda}))$ *and by Lemma 1.1.6,*

$$A_{S_{g^*}(\widetilde{\Lambda})} \cong (\mathbb{Z}/2\mathbb{Z})^{\alpha_1} \oplus (\mathbb{Z}/2^2\mathbb{Z})^{\alpha_2} \oplus \ldots \oplus (\mathbb{Z}/2^k\mathbb{Z})^{\alpha_k}, \tag{7.1}$$

where $l := l(A_{S_{g^*}(\widetilde{\Lambda})}) = \alpha_1 + \ldots + \alpha_k \leq 1 + r$ *and at least* $\alpha_k \neq 0$.

Note that the existence of $S_{g^*}(\widetilde{\Lambda})$ implies that $l \leq 1+r$ and if $l+2 \leq r+1$, then automatically $S_{g^*}(\widetilde{\Lambda})$ is a unique lattice with these invariants and it decomposes in a copy of U and its complement Q in $S_{g^*}(\widetilde{\Lambda})$. By Proposition 7.1.4, the lattice $S_{g^*}(\widetilde{\Lambda})$ is hyperbolic. However, it is not always 2-elementary. Nevertheless, in our particular case we can find a copy of a 2-elementary lattice of signature $(1,1)$ in $S_{g^*}(\widetilde{\Lambda})$.

Proposition 7.1.6. *Let g be a symplectic birational map of finite order. Then, the lattice $S_{g^*}(\widetilde{\Lambda})$ is unique in its genus.*

Proof. We only need to show that $S_{g^*}(\widetilde{\Lambda})$ is uniqueness for any symplectic birational map g of order 2^k. Since $x^{\frac{r(1+r)}{2}} \nmid 4^{\frac{1+r}{2}}|A_{S_{g^*}(\widetilde{\Lambda})}| = 2^{1+r+\alpha_1+\ldots+k\alpha_k}$ for all non–square $x \equiv 0,1 \mod 4$, then the assertion follows of Lemma 1.1.19. ∎

Proposition 7.1.7. *The lattice $S_{g^*}(\widetilde{\Lambda})$splits as $\widetilde{L} \oplus Q$ where $\widetilde{L}$ is a 2-elementary lattice of signature $(1,1)$ and $Q \cong \widetilde{L}^{\perp} \subset \widetilde{\Lambda}$ is a lattice of signature $(0,r-1)$.*

Proof. The existence of this splitting depends of the invariants of $S_{g^*}(\widetilde{\Lambda})$. The lattice Q exists if the conditions of Lemma 1.1.17 are satisfied. Condition (1) follows by $\operatorname{sign} q_Q = 1-r \equiv \operatorname{sign} q_{S_{g^*}(\widetilde{\Lambda})} \mod 8$. By Remark 7.1.5, conditions (3) and (4) follows from $\operatorname{ord}(g) = 2^k$. Suppose that $l \leq r-1$, then $l_Q := l(A_Q) \leq l \leq \operatorname{rank} Q$. This implies that $\tilde{L} = U$, and so $S_{g^*}(\widetilde{\Lambda}) = U \oplus Q$. Now, for $l \leq r+1$, we get $l-2 \leq r-1 = \operatorname{rank} Q$, and so $S_{g^*}(\widetilde{\Lambda}) = U(2) \oplus Q$ or $S_{g^*}(\widetilde{\Lambda}) = (2) \oplus (-2) \oplus Q$.

The unicity of the splitting of $S_{g^*}(\widetilde{\Lambda}) = \widetilde{L} \oplus Q$ follows by Proposition 7.1.6.

The proof is completed by showing that $S_{(g^m)^*}(\widetilde{\Lambda}) \subset S_{g^*}(\widetilde{\Lambda})$ where m is odd and $\operatorname{ord}(g) = 2^k m$. ∎

Note that if $\widetilde{L} = (2) \oplus (-2)$, $\widetilde{L}$ is an overlattice of $U(4)$ by adding the class $(e_4 - f_4)/2$ where e_4 and f_4 are the generators of $U(4)$.

Since $\widetilde{L}$ can be obtained as an overlattice of $U(n)$ for some $n \neq 0$, and $S_{g^*}(\widetilde{\Lambda})$ is in the algebraic part of $\widetilde{\Lambda}$, we obtain that $U(n) \hookrightarrow \widetilde{\Lambda}$ into the $(1,1)$ part of the Hodge structure. By Lemma 1.4.30, if x corresponds to the period of a hyperkähler manifold X of K3$^{[n]}$–type admitting a symplectic birational map with non-trivial action on A_X, then

1. $\widetilde{L} = U \hookrightarrow \widetilde{\Lambda}$ iff there exists a K3 surface S such that the Hodge structure on $\widetilde{\Lambda}_x \simeq \widetilde{H}(S,\mathbb{Z})$;

2. $\widetilde{L} = U(2) \hookrightarrow \widetilde{\Lambda}$ iff there exists a K3 surface S and a Brauer class α induced by $B = -(1/2)\gamma$ such that the Hodge structure on $\widetilde{\Lambda}_x \simeq \widetilde{H}(S,\alpha,\mathbb{Z})$;

3. $U(4) \subset \widetilde{L} = (2) \oplus (-2) \hookrightarrow \widetilde{\Lambda}$ iff there exists a K3 surface S and a Brauer class α induced by $B = -(1/4)\gamma$ such that the Hodge structure on $\widetilde{\Lambda}_x \simeq \widetilde{H}(S, \alpha, \mathbb{Z})$.

The class γ in (2) and (3), is explicitly described in Lemma 1.4.30 and its depend of n that can be 2 or 4.

Since the period of X corresponds to the period of a moduli space of coherent sheaves on a (twisted) K3 surface, by the Hodge Theoretic Torelli theorem in 1.4.9 we obtain

Theorem 7.1.8. *Let X be a projective hyperkähler manifold of $K3^{[n]}$–type admitting a symplectic birational map g with $g^*_{|A_X} = -\mathrm{Id}$. Then X is birational to the Moduli space of (twisted) sheaves on a K3 surface.*

We want to conclude this section to mention that a hyperkähler manifold admitting a symplectic birational map with the conditions of the previous theorem is an evidence (at least from a lattice theoretical viewpoint) for the following:

Conjecture 7.1.9. *Let X be a projective hyperkähler manifold of $K3^{[n]}$–type admitting a symplectic birational map g with $g^*_{|A_X} = -\mathrm{Id}$. There exists a reflection map R_e on X and h a map with $h^*|_{A_X} = \mathrm{Id}$ such that $g = R_e \circ h$ and $e \in H^2(X, \mathbb{Z})$ is the class of an irreducible divisor.*

Moreover, by Theorem 7.0.1, we know that the Divisor E associated to the class e can not be a prime exceptional divisor (i.e., A PED is a reduced, irreducible divisor of negative Beauville–Bogomolov degree). One of the next examples is a geometrical evidence of this fact.

The example $M_H(r, 0, -s)$

Let S be a projective K3 surface of Picard number one. Let H be an ample line bundle and r, s two integers satisfying $s \geq r \geq 1$ and $\gcd(r, s) = 1$. Set by $M_H(r, 0, -s)$ the Moduli space of sheaves of S with Mukai vector $(r, 0, -s)$. By Theorem 1.4.28, X is a hyperkähler manifold of K3$^{[n]}$–type with $n = 1 + rs$. Let $e = \theta(r, 0, s)$ where θ is defined in (1.10). The weight two integral Hodge structure $H^2(M_H(r, 0, -s), \mathbb{Z})$ is Hodge isometric to $H^2(S, \mathbb{Z}) \oplus \mathbb{Z} \cdot e$ where e is a monodromy reflective class of Hodge–type $(1, 1)$, degree $-2(n-1)$ and $\mathrm{div}(e, -) = 2(n-1)$. In some cases e corresponds to a primitive class coming from an exceptional prime divisor, and in other cases the divisor is not a prime exceptional divisor:

- $\mathbf{r = 1}$: In this case, $X = M_H(1, 0, -s) = S^{[1+s]}$ and $e = [E]/2$ where $E \subset S^{[1+s]}$ is the diagonal, so E corresponds to the prime divisor which is the exceptional locus of the Hilbert–Chow morphism $\epsilon : S^{[1+s]} \longrightarrow S^{(1+s)}$ defined in (1.7).

- **r = 2**: In this case, $e = [E]$ where $E \subset M_H(2,0,-s)$ is the locus of H–stable sheaves which are not locally free. By [Li93] (cf. [Mar13, Lemma 10.16]), E is a prime divisor which is the exceptional locus of Jun Li's morphism from $M_H(2,0,-s)$ onto the Uhlenbeck–Yau compactification of the moduli space of H–slope stable vector bundles.

- **r ≥ 3**: In this case, $e = [E]$ where $E \subset M_H(r,0,-s)$ is the locus of H–stable sheaves which are not locally free or not H–slope stable. Denote by $U = X \setminus E$ the locally free H–slope stable sheaves and $\iota : U \longrightarrow U$ the map that sends $\mathcal{F}$ in its dual sheaf $\mathcal{F}^\vee$. By [Mar13, Lemma 9.5], E is a closed subset of codimension ≥ 2 in $M_H(r,0,-s)$ and so $\iota : M_H(r,0,-s) \longrightarrow M_H(r,0,-s)$ is a birational involution. By [Mar13, Proposition 11.1], the induced map ι^* in cohomology corresponds to the reflection map R_e. Note that e is not $\mathbb{Q}$–effective, and so E is not a prime exceptional divisor (cf. [Mar13, Observation 1.12]).

The last case shows geometrical evidence of the existence of R_e as in Conjecture 7.1.9: the map R_e is given by $(x,y,z) \mapsto (-x,y,-z)$ and its action on $A_{M_H(r,0,-s)} = \langle g = e/2(n-1) \rangle$ is $-\mathrm{Id}$. Any symplectic birational map g on $M_H(r,0,-s)$ admitting a non-trivial action on the discriminant group $A_{M_H(r,0,-s)}$ is the composition of R_e and a map h acting trivially on $A_{M_H(r,0,-s)}$.

8 | Chern classes of Ulrich bundles

We can restate the following characterization of Ulrich bundles on surfaces obtained by Casnati in [Cas17, Proposition 2.1, Corollary 2.2].

8.1 Ulrich bundles on surfaces

Proposition 8.1.1 (Casnati)**.** *Let S be a smooth projective surface and $\mathcal{O}_S(H)$ be a very ample line bundle. For any vector bundle E of rank r on S, the following assertions are equivalent:*

(a) *E is an Ulrich bundle.*

(b) *E is an aCM bundle and*

$$c_1(E)\cdot H = \frac{r}{2}(K_S+3H)\cdot H \quad \text{and} \quad c_2(E) = \frac{1}{2}(c_1^2(E)-c_1(E)\cdot K_S)-r(H^2-\chi(S,\mathcal{O}_S)). \tag{8.1}$$

(c) $h^0(S,E(-H)) = h^0(S,E^{\mathrm{ul}}(-H)) = 0$ *and the identities* (8.1) *hold.*

In particular, a rank two vector bundle E on S is a special Ulrich bundle if and only if E is initialized and the identities

$$c_1(E) = K_S + 3H \quad \text{and} \quad c_2(E) = \frac{1}{2}(5H^2 + 3H\cdot K_S) + 2\chi(S,\mathcal{O}_S) \tag{8.2}$$

hold.

Along the same lines, we observe that we can follow verbatim the proof of Casnati's formulas in order to obtain the following vanishing of certain twisted Euler characteristic of Ulrich bundles. Together with the Hirzebruch–Riemann–Roch Theorem, they give many restrictions on the Chern classes of Ulrich bundles.

8.2 Ulrich bundles on higher dimensional varieties

Let us begin with the following observation concerning certain aCM bundles, which in the case of Ulrich bundles is a direct consequence of Theorem 1.5.1 (2).

Lemma 8.2.1. *Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(H)$ be a very ample line bundle. Let E be an aCM bundle on X with respect to H such that $h^0(X, E(-H)) = h^n(X, E(-nH)) = 0$, then*

$$\chi(X, E(-H)) = \chi(X, E(-2H)) = \cdots = \chi(X, E(-nH)) = 0.$$

Proof. Since E is an aCM vector bundle, we have that

$$h^1(X, E(-jH)) = \cdots = h^{n-1}(X, E(-jH)) = 0 \text{ for } j = 1, \dots, n.$$

On the other hand, it follows from the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

that $h^0(X, E(-nH)) \leq h^0(X, E(-(n-1)H)) \leq \cdots \leq h^0(X, E(-H)) = 0$. Moreover, the vanishing $h^n(X, E(-nH)) = 0$ implies that $\chi(X, E(-nH)) = 0$.

Similarly, Serre duality and the above short exact sequence give us that

$$h^n(X, E(-jH)) = h^0(X, E^\vee(K_X+jH)) \leq h^0(X, E^\vee(K_X+(j+1)H)) = h^n(X, E(-(j+1)H))$$

for every $j \in \mathbb{Z}$, and hence $h^n(X, E(-H)) \leq h^n(X, E(-2H)) \leq \cdots \leq h^n(X, E(-nH)) = 0$. We conclude that $\chi(X, E(-H)) = \chi(X, E(-2H)) = \cdots = \chi(X, E(-(n-1)H)) = 0$ as well. ∎

It is worth noting that aCM bundles satisfy

Lemma 8.2.2. *Let X be a smooth projective variety of* even *(resp.* odd*) dimension n and $\mathcal{O}_X(H)$ a very ample line bundle, E be an aCM bundle (resp.* initialized *aCM bundle) on X with respect to H. If $\chi(X, E(-H)) = \chi(X, E(-nH)) = 0$, then E is an Ulrich bundle with respect to H.*

Proof. By definition of aCM bundle, for every $i = 1, \dots, n-1$, we have that

$$\mathrm{H}^i(X, E(jH)) = 0$$

for all $j \in \mathbb{Z}$. By Theorem 1.5.1, we are left to prove that

$$\mathrm{H}^0(X, E(-H)) = \mathrm{H}^n(X, E(-nH)) = 0.$$

If both quantities

$$\chi(X, E(-H)) = h^0(X, E(-H)) + (-1)^n h^n(X, E(-H))$$

$$\chi(X, E(-nH)) = h^0(X, E(-nH)) + (-1)^n h^n(X, E(-nH))$$

are zero, then the above vanishing conditions follow immediately when n is even. On the other hand, when n is odd, we obtain instead that

$$h^0(X, E(-H)) = h^n(X, E(-H)) \text{ and } h^0(X, E(-nH)) = h^n(X, E(-nH)).$$

As we already discussed during the proof of Lemma 8.2.1, we have the inequalities

$$h^0(X, E(-nH)) \leq h^0(X, E(-H)) \text{ and } h^n(X, E(-H)) \leq h^n(X, E(-nH)).$$

In particular, if E is initialized then $h^0(X, E(-H)) = 0$, and then $h^n(X, E(-nH)) = 0$ as well. ■

In order to extend Casnati's characterization of Ulrich bundles on surfaces to threefolds, let us recall that if E is a rank r vector bundle on a smooth projective threefold X, then the Hirzebruch–Riemann–Roch theorem takes the following form

$$\begin{aligned}\chi(X, E) &= \int_X \mathrm{ch}(E)\,\mathrm{td}(X)\\ &= r\chi(X, \mathcal{O}_X) + \frac{1}{12} c_1(E) \cdot (K_X^2 + c_2(X)) + \frac{1}{4}(2c_2(E) - c_1^2(E)) \cdot K_X\\ &\quad + \frac{1}{6}(c_1^3(E) - 3c_1c_2(E) + 3c_3(E)),\end{aligned}$$

where $\chi(X, \mathcal{O}_X) = -\frac{1}{24} K_X \cdot c_2(X)$. Additionally, if $L \cong \mathcal{O}_X(D)$ is a line bundle on X, then

$$c_i(E \otimes \mathcal{O}_X(D)) = \sum_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(E) D^j \quad \text{in } \mathrm{H}^*(X, \mathbb{R})$$

for every $i \geq 0$ (see e.g. [Ful98, Example 3.2.2]). In particular, for $j \in \mathbb{Z}$ and

$D = -jH$, we get the following relations

$$c_1(E(-jH)) = c_1(E) - jrH,\ c_2(E(-jH)) = c_2(E) - j(r-1)c_1(E)H + j^2\frac{r(r-1)}{2}H^2,$$
$$c_3(E(-jH)) = c_3(E) - j(r-2)c_2(E)H + j^2\frac{(r-1)(r-2)}{2}c_1(E)H^2 - j^3\frac{r(r-1)(r-2)}{6}H^3.$$

The first identity in (8.1) and (8.2) (i.e., the one related with the *first* Chern class of an Ulrich bundle) can be made more precise in some higher dimensional cases. More precisely, Lopez showed in [Lop20, Lemma 3.2] that if X is a smooth projective variety of dimension $n \geq 2$ such that $\mathrm{Pic}(X) \cong \mathbb{Z}$, then every rank r Ulrich bundle on X (with respect to a very ample divisor H) satisfies

$$c_1(E) = \frac{r}{2}(K_X + (n+1)H).$$

We remark that we can drop the assumption on the Picard rank if we restrict ourselves to compute $c_1(E)\cdot H^{n-1}$, instead of $c_1(E)$.

Lemma 8.2.3. *Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(H)$ a very ample line bundle. Let E be an Ulrich bundle on X with respect to H, then*

$$c_1(E)\cdot H^{n-1} = \frac{r}{2}(K_X + (n+1)H)\cdot H^{n-1}.$$

Proof. Let $H_1, \dots, H_{n-2}$ be general members in the linear system $|H|$ and let $Y_{n-j} := H_1 \cap H_2 \cap \dots \cap H_j$ for $j = 1, \dots, n-2$. By Bertini theorem, each Y_j is smooth irreducible of dimension j. First, we recall that (topological) Chern classes commute with arbitrary pullback and hence $c_1(E|_{Y_j}) = c_1(E)|_{Y_j}$ in $\mathrm{H}^2(Y_j, \mathbb{R})$. In particular, applying [Deb01, Proposition 1.8(b)] inductively, we have that

$$c_1(E|_S)\cdot H|_S = c_1(E)\cdot H^{n-1},$$

where $S := Y_2$. On the other hand, we know that each $E|_{Y_j}$ is an Ulrich bundle with respect to $H|_{Y_j}$ (see Section1.5) and hence Casnati's formulas (8.1) in Proposition 8.1.1 give

$$c_1(E|_S)\cdot H|_S = \frac{r}{2}(K_S + 3H|_S)\cdot H|_S.$$

Finally, since $\mathcal{N}_{S/X} \cong \mathcal{O}_X(H)|_S^{\oplus(n-2)}$, we deduce from the adjunction formula, and [Deb01, Proposition 1.8(b)] that

$$(K_S + 3H|_S)\cdot H|_S = (K_X + (n+1)H)\cdot H^{n-1}.$$

■

8.3 Application on threefolds

From the previous formulas, we can deduce the following characterization of Ulrich bundles on smooth projective threefolds.

Proposition 8.3.1. *Let X be a smooth projective threefold and $\mathcal{O}_X(H)$ a very ample line bundle. For any rank r vector bundle E on X, the following are equivalent:*

(a) E is an Ulrich bundle.

(b) E is an initialized aCM bundle and the identities

$$\begin{aligned} c_1(E)\cdot H^2 &= \frac{r}{2}H^2(K_X+4H), \\ c_2(E)\cdot H &= \frac{r}{12}(K_X^2+c_2(X)-22H^2)\cdot H+\frac{1}{2}(c_1(E)-K_X)\cdot c_1(E)H, \\ &\quad -\frac{1}{6}c_1(E)(K_X^2+c_2(X))+2r(H^3-\chi(X,\mathcal{O}_X)), \end{aligned} \tag{8.3}$$

hold.

(c) $h^0(X,E(-H)) = h^1(X,E(-H)) = h^1(X,E^{\mathrm{ul}}(-H)) = h^1(X,E(-2H)) = 0$ and the identities (8.3) hold.

In particular, a rank two vector bundle E on X is a special Ulrich bundle if and only if $h^0(X,E(-H)) = h^1(X,E(-H)) = h^1(X,E(-2H)) = 0$ and the identities

$$\det(E) = \mathcal{O}_X(K_X+4H) \quad \text{and} \quad c_2(E)\cdot H = \frac{1}{6}(K_X^2+c_2(X))\cdot H+2H^2\cdot K_X+\frac{13}{3}H^3$$

hold.

Proof. If we assume that E is Ulrich, then it follows from Theorem 1.5.1 and Lemma 8.2.1 that E is an aCM bundle and that

$$\chi(X,E(-H)) = \chi(X,E(-2H)) = \chi(X,E(-3H)) = 0.$$

On the other hand, it follows from the Hirzebruch–Riemann–Roch theorem and the discussion above that $\chi(X, E(-jH)) = \chi(X, E) + \Delta_j$, where

$$\begin{aligned}\Delta_j := \frac{1}{12} jH \cdot (12c_2(E) + 6c_1(E)K_X - 6c_1^2(E) - rK_X^2 - rc_2(X)) \\ + \frac{1}{4} j^2 H^2 \cdot (2c_1(E) - rK_X) - \frac{1}{6} j^3 H^3 r\end{aligned}$$

for every $j \in \mathbb{Z}$. In particular, the identities (8.3) are obtained simply by solving the linear system which is determined by the relations $\Delta_2 - \Delta_1 = \Delta_3 - \Delta_2 = \chi(X, E) + \Delta_1 = 0$. We conclude in this way that (a) implies (b), and that (a) implies (c).

Suppose now that E is an initialized aCM bundle and that the identities (8.3) hold. In particular, the identities (8.3) imply that $\chi(X, E(-jH)) = 0$ for $j = 1, 2, 3$. The fact that E is Ulrich follows from Lemma 8.2.2. In other words, (b) implies (a).

It is clear that (b) implies (c), since $h^1(X, E^{\mathrm{ul}}(-H)) = h^2(X, E(-3H))$ by Serre duality, and hence the vanishing of higher cohomology follows directly from the aCM assumption.

Finally, let us check that (c) implies (a). To do so, by Theorem 1.5.1 and our assumptions, we are left to check the vanishing conditions

$$\mathrm{H}^2(X, E(-2H)) = \mathrm{H}^3(X, E(-3H)) = 0.$$

Moreover, as in the proof of Lemma 8.2.1, we have that $h^0(X, E(-H)) = 0$ implies $h^0(X, E(-jH)) = 0$ for all $j \in \mathbb{N}_{\geq 1}$. As before, the identities (8.3) give us $\chi(X, E(-jH)) = 0$ for $j = 1, 2, 3$, and in particular we have

$$0 = \chi(X, E(-3H)) = -h^1(X, E(-3H)) - h^3(X, E(-3H)).$$

It follows that $h^3(X, E(-3H)) = 0$ and $h^3(X, E(-2H)) = 0$ as in the proof of Lemma 8.2.1. The identity $\chi(X, E(-2H)) = 0$ implies thus $h^2(X, E(-2H)) = h^1(X, E(-2H)) = 0$, where the last vanishing holds by assumption. ∎

Fano threefolds

Let us illustrate an application of Proposition 8.3.1 by considering the special Ulrich bundles on Fano threefolds of even index, constructed by Beauville in [Bea18, Section 6].

Let X be a smooth Fano threefold such that $-K_X = 2H$ with H a very ample divisor, and let us consider the associated embedding $X \subseteq \mathbb{P}^{d+1}$, where $d = \deg(X) = H^3$. If we assume that the Fano index of X is exactly 2, then it follows that $d = 3, 4, 5, 6$, or 7, and all possible threefolds are classified (see [IP99, Theorem 3.3.1]).

If one tries to construct a special Ulrich bundle E by means of the Cayley-Bacharach property (see Definition 1.5.6) or the Hartshorne-Serre construction (see Theorem 1.5.7) , then we need to find a local complete intersection curve $\Gamma \subseteq X$ whose ideal sheaf $\mathcal{I}_\Gamma$ fits in an exact sequence of sheaves

$$0 \longrightarrow \mathcal{L} \longrightarrow E \longrightarrow \mathcal{M} \otimes \mathcal{I}_\Gamma \longrightarrow 0,$$

where $\mathcal{L}$, $\mathcal{M} \in \mathrm{Pic}(X)$ are line bundles. If we consider Γ to be the zero locus of a general section $s \in \mathrm{H}^0(X, E)$, then we would have

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot s} E \longrightarrow \mathcal{O}_X(2H) \otimes \mathcal{I}_\Gamma \longrightarrow 0,$$

since $\det(E) = \mathcal{O}_X(K_X + 4H) = \mathcal{O}_X(2H)$. The Ulrich condition implies that Γ is an elliptic curve (see [Bea18, Remark 6.3.(3)]), and the identities (8.3) give us in this case

$$\deg(\Gamma) = c_2(E)\cdot H = \frac{1}{6}(K_X^2 + c_2(X))\cdot H + 2H^2\cdot K_X + \frac{13}{3}H^3 = \frac{1}{6}(4d+12) - 4d + \frac{13}{3}d = d+2.$$

The existence of such elliptic curve of degree $d+2$ is proved in [Bea18, Lemma 6.2] using deformation theory, while the vanishing conditions $h^0(X, E(-H)) = h^1(X, E(-H)) = h^1(X, E(-2H)) = 0$ are verified in [Bea18, Proposition 6.1].

Corollary 8.3.2. *Let X be a smooth projective threefold with $c_1(X) = 0$ (i.e., K_X is numerically trivial). Assume that E is a special Ulrich bundle with respect to a very ample line bundle $\mathcal{O}_X(H)$, then*

$$12c_2(E) \cdot H - 13c_1(E) \cdot H^2 = 2c_2(X) \cdot H.$$

In particular, if X is an abelian threefold then $12c_2(E) \cdot H = 13c_1(E) \cdot H^2$ and the general section $s \in \mathrm{H}^0(X, E)$ defines a smooth connected curve $\Gamma = V(s) \subseteq X$ of genus $g(\Gamma) = 2\deg(\Gamma) + 1$.

Proof. The identity $12c_2(E) \cdot H - 13c_1(E) \cdot H^2 = 2c_2(X) \cdot H$ follows directly from Proposition 8.3.1, since $K_X \cdot H^2 = 0$ and $c_1(E) = 4H$ in this case. If we assume moreover that X is an Abelian threefold, then we have that $c_2(X) = 0$, and hence $12c_2(E) \cdot H = 13c_1(E) \cdot H^2$ in that case. Finally, if we consider $s \in \mathrm{H}^0(X, E)$ a general section and we define $\Gamma := V(s)$, then it follows from Bertini theorem that Γ is a smooth projective curve. We observe that in this case, we have a short exact sequences of sheaves

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot s} E \longrightarrow \mathcal{O}_X(4H) \otimes \mathcal{I}_\Gamma \longrightarrow 0,$$

as $\det(E) = \mathcal{O}_X(4H)$ in $\mathrm{Pic}(X)$. From the long exact sequence in cohomology associated to the twisted short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-4H) \longrightarrow E(-4H) \longrightarrow \mathcal{I}_\Gamma \longrightarrow 0,$$

we deduce that $h^1(X, \mathcal{I}_\Gamma) = 0$ and $h^2(X, \mathcal{I}_\Gamma) = \frac{26d}{3} + h^3(X, \mathcal{I}_\Gamma)$. Indeed, the Ulrich condition implies that $h^i(X, E(-4H)) = 0$ for $i \leq 2$, Kodaira vanishing implies that $h^i(X, \mathcal{O}_X(-4H)) = 0$ for $i \leq 2$, and by Serre duality and Hirzebruch–Riemann–Roch (resp. since $E \cong E^{\mathrm{ul}}$ is Ulrich special), we compute

$$h^3(X, \mathcal{O}_X(-4H)) = \frac{32d}{3} \text{ (resp. } h^3(X, E(-4H)) = 2d).$$

Similarly, with the short exact sequence

$$0 \longrightarrow \mathcal{I}_\Gamma \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_\Gamma \longrightarrow 0$$

associated to the closed embedding $\iota : \Gamma \hookrightarrow X$, we get $h^0(\Gamma, \mathcal{O}_\Gamma) = h^3(X, \mathcal{I}_\Gamma) = 1$ and $h^1(\Gamma, \mathcal{O}_\Gamma) = h^2(X, \mathcal{I}_\Gamma)$. From this, we deduce that Γ is connected of genus

$$g(\Gamma) = \frac{26d}{3} + 1 = 2c_2(E) \cdot H + 1 = 2\deg(\Gamma) + 1.$$

■

9 | Projective manifolds whose tangent bundle is Ulrich

In this section, we address the main problem of [MPTB21]. Namely, we would like to classify all the pairs $(X, \mathcal{O}_X(1))$ such that the tangent bundle T_X (resp. the cotangent bundle Ω^1_X) is an Ulrich bundle with respect to H, where $X \subseteq \mathbb{P}^N$ is a smooth projective variety of dimension n and where H is a very ample divisor on X such that $\mathcal{O}_X(H) \cong \mathcal{O}_X(1)$. As usual, we will denote by $d := \deg(X) = H^n \geq 1$ the degree of X, and we carry out an analysis depending on the dimension n.

9.1 Curves

Let us consider a smooth projective curve $C \subseteq \mathbb{P}^N$ of degree $d = \deg(H)$, and let E be a vector bundle on C. It follows from Theorem 1.5.1 that E is an Ulrich bundle if and only if $h^0(C, E(-H)) = h^1(C, E(-H)) = 0$. Since $T_C \cong \omega_C^\vee$ and $\Omega^1_C \cong \omega_C$ in this case, where ω_C is the canonical bundle of C, we can easily deduce the following.

Proposition 9.1.1. *Let $(C, \mathcal{O}_C(1))$ as above. Then Ω^1_C is never an Ulrich bundle, and T_C is an Ulrich bundle if and only if C is the* twisted cubic *in $\mathbb{P}^3$,*

i.e.,

$$(C, \mathcal{O}_C(1)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)).$$

Proof. In the case of $\Omega^1_C \cong \omega_C$, the Ulrich condition reduces to check the two vanishing conditions

$$h^0(C, \omega_C(-H)) = h^1(C, \omega_C(-H)) = 0.$$

They are equivalent, by Serre duality, to $h^1(C, \mathcal{O}_C(H)) = h^0(C, \mathcal{O}_C(H)) = 0$. The latter vanishing is impossible since $\mathcal{O}_C(H) \cong \mathcal{O}_C(1)$ is very ample, and hence Ω^1_C cannot be Ulrich with respect to any H.

In the case of $T_C \cong \omega_C^\vee$, we are left to check the two vanishing conditions

$$h^0(C, \omega_C^\vee(-H)) = h^1(C, \omega_C^\vee(-H)) = 0.$$

Let $g := g(C)$ be the genus of C. If $C \cong \mathbb{P}^1$, then C has $g = 0$ and by Serre duality the vanishing conditions reduce to

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-d)) = 0 \text{ and } h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-d)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-4)) = 0.$$

The first vanishing implies that $d \geq 3$, while the second one implies that $d \leq 3$. We obtain therefore that in this case $(C, \mathcal{O}_C(1)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. On the other hand, if $g \geq 1$ then the first vanishing $h^0(C, \omega_C^\vee(-H)) = 0$ follows directly, since $\deg(\omega_C^\vee(-H)) = 2 - 2g - d < 0$. However, the second vanishing is equivalent to $h^0(C, \omega_C^{\otimes 2}(H)) = 0$ by Serre duality. Since $h^1(C, \omega_C^{\otimes 2}(H)) = h^0(C, \omega_C^\vee(-H)) = 0$, the Riemann-Roch theorem yields

$$h^0(C, \omega_C^{\otimes 2}(H)) = 4g - 4 + d - g + 1 = 3g - 3 + d \geq 1$$

and hence T_C is not an Ulrich bundle. ∎

9.2 Surfaces

Let us consider a smooth projective surface $S \subseteq \mathbb{P}^N$ of degree $d = H^2 \geq 1$, and let E be a vector bundle on S. Again, by Theorem 1.5.1, we have that E is an Ulrich bundle if and only if

$$h^1(S, E(-H)) = h^2(S, E(-2H)) = h^0(S, E(-H)) = h^1(S, E(-2H)) = 0.$$

Before treating the general case, let us take a look at the following motivating example.

Example 9.2.1. *Let us recall that a vector bundle E on S is* **Ulrich special** *if it is an Ulrich bundle,* $\mathrm{rk}(E) = 2$ *and* $c_1(E) = K_S + 3H$*. Then, T_S is an Ulrich special bundle if and only if S is the* Veronese surface *in $\mathbb{P}^5$, i.e., $(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*

Indeed, if we assume that T_S is an Ulrich special bundle then the condition $-K_S = c_1(T_S) = 3H + K_S$ in $\mathrm{NS}(S)$ implies that $-2K_S = 3H$ is very ample. In particular, S is a del Pezzo surface and thus $\mathrm{NS}(S) \cong \mathbb{Z}^{10-m}$ is torsion-free, where $m = K_S^2 \in \{1, \ldots, 9\}$ is the anti-canonical degree of S. Hence, the equality $-2K_S =$

$3H$ implies that $H = 2A$ for some ample divisor A, and in particular $-K_S = 3A$ for some ample divisor A. In other words, the Fano index $i_S = 3$ of S is maximal, i.e., $S \cong \mathbb{P}^2$ by the Kobayashi–Ochiai theorem [KO73]. Since $\omega_{\mathbb{P}^2}^\vee \cong \mathcal{O}_{\mathbb{P}^2}(3)$ we have that $\mathcal{O}_S(H) \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and hence $(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is the Veronese surface.

We are left to check that $T_{\mathbb{P}^2}$ is Ulrich with respect to $\mathcal{O}_{\mathbb{P}^2}(H) \cong \mathcal{O}_{\mathbb{P}^2}(2)$. This is already stated in [ES03, Proposition 5.9] (see also [CG17, Theorem 5.2] and [AHMPL19, Example 3.1], where the authors show moreover that $T_{\mathbb{P}^2}$ is actually the unique *Ulrich bundle on the Veronese surface in $\mathbb{P}^5$), but we include a short proof here for the sake of completeness.*

First of all, for any smooth projective surface S we have that $T_S \cong \Omega_S^1 \otimes \omega_S^\vee$ and hence $T_{\mathbb{P}^2} \cong \Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(3)$. If follows therefore that $h^1(\mathbb{P}^2, T_{\mathbb{P}^2}(-H)) = h^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = 0$ by Bott vanishing. Secondly, we note that

$$h^2(\mathbb{P}^2, T_{\mathbb{P}^2}(-2H)) = h^2(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-4)) = h^2(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = h^0(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)),$$

by the same reason as before and by Serre duality. The long exact sequence in cohomology induced by the twisted Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \to 0$$

gives us the vanishing $h^0(\mathbb{P}^2, T_{\mathbb{P}^2}(-H)) = h^0(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) = 0$. Finally, the last condition

$$h^1(\mathbb{P}^2, T_{\mathbb{P}^2}(-2H)) = h^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-4)) = h^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = 0$$

follows from Serre duality and Bott vanishing.

In order to treat the general case, let us recall the following result by Reider in [Rei88, Theorem 1, Remark 1.2].

Theorem 9.2.2 (Reider)**.** *Let $\mathcal{L} \cong \mathcal{O}_S(D)$ be a nef line bundle on a smooth projective surface S. If $D^2 \geq 9$, then the adjoint line bundle $\omega_S \otimes \mathcal{L}$ is* very ample *unless there exists a non-zero effective divisor E verifying one of the following conditions:*

(a) $D \cdot E = 0$ and $E^2 \in \{-1, -2\}$.

(b) $D \cdot E = 1$ and $E^2 \in \{0, -1\}$.

(c) $D \cdot E = 2$ and $E^2 = 0$.

(d) $D \cdot E = 3$, $D \equiv 3E$ in $\mathrm{NS}(S)$, and $E^2 = 1$.

We will also need the following observation.

Lemma 9.2.3. *Let* $(S, \mathcal{O}_S(1))$ *as above. If* T_S *is an Ulrich bundle with respect to* H*, then* $\kappa(S) = -\infty$*,* $K_S \cdot H = -6$ *and* $\deg(S) = 4$*.*

Proof. The identities (8.1) in Proposition 8.1.1 imply that $c_1(T_S) \cdot H = -K_S \cdot H = 3H^2 + H \cdot K_S$ and hence $2K_S \cdot H = -3H^2 < 0$. In particular, since H is very ample it follows that K_S is not pseudo-effective and therefore $\kappa(S) = -\infty$ (see e.g. [Bad01, Lemma 14.6]). Finally, if follows from Bertini theorem that a general curve $C \in |H|$ is smooth irreducible and hence

$$g(C) = 1 + \frac{1}{2}(H^2 + K_S \cdot H) = 1 - \frac{1}{4}H^2 \in \mathbb{Z}_{\geq 0},$$

from which we deduce that $\deg(S) = H^2 = 4$, and thus $K_S \cdot H = -6$. ∎

We are now ready to state the main result of this section.

Theorem 9.2.4. *Let* $(S, \mathcal{O}_S(1))$ *as above. Then* Ω^1_S *is never an Ulrich bundle, and* T_S *is an Ulrich bundle if and only if* S *is the* Veronese surface *in* $\mathbb{P}^5$*,*
i.e.,

$$(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)).$$

Proof. First, if we assume that the cotangent bundle Ω^1_S of S is an Ulrich bundle, then the identities (8.1) in Proposition 8.1.1 imply that $c_1(\Omega^1_S) \cdot H = K_S \cdot H = 3H^2 + K_S \cdot H$ and hence $H^2 = 0$, which is impossible since H is very ample. Therefore, the cotangent bundle Ω^1_S is never an Ulrich bundle.

Second, if we assume that T_S is an Ulrich bundle on S, the identities (8.1) in Proposition 8.1.1 together with Lemma 9.2.3 imply that

$$\chi_{\text{top}}(S) = K_S^2 - 8 + 2\chi(S, \mathcal{O}_S).$$

Combining this with Noether's formula

$$12\chi(S, \mathcal{O}_S) = K_S^2 + \chi_{\text{top}}(S),$$

we get that $\chi(S, \mathcal{O}_S) = \frac{1}{5}(K_S^2 - 4)$.

Let us notice that the divisor $K_S + 3H$ is very ample. Indeed, the divisor $D := 3H$ is very ample with $D^2 = 36$ and hence the fact that $K_S + D$ is very ample as well follows from Theorem 9.2.2: since $D \cdot E \geq 3$ for every non-zero effective divisor, we only need to consider the case (d) in Reider's theorem, but in that case we would have $D^2 = 9$.

Since $K_S + 3H$ is very ample, Bertini theorem implies that a general curve $C \in |K_S + 3H|$ is smooth irreducible and hence

$$g(C) = 1 + \frac{1}{2}(K_S^2 + 6K_S \cdot H + 9H^2 + K_S^2 + 3K_S \cdot H) = K_S^2 - 8,$$

since $H^2 = 4$ and $K_S \cdot H = -6$, by Lemma 9.2.3. It follows that $K_S^2 \geq 8$.

Since $\kappa(S) = -\infty$ by Lemma 9.2.3, we know that S is a ruled surface and hence birationally isomorphic to $\Gamma \times \mathbb{P}^1$, for some smooth projective curve Γ. In particular, $p_g(S) = 0$ and $q(S) = g(\Gamma)$, from which we deduce that

$$\chi(S, \mathcal{O}_S) = 1 - g(\Gamma) = \frac{1}{5}(K_S^2 - 4) \Leftrightarrow g(\Gamma) = \frac{1}{5}(9 - K_S^2) \geq 0.$$

We conclude therefore that $K_S^2 \leq 9$. By divisibility reasons, we have that $K_S^2 = 9$ and in particular S is rational by Castelnuovo's criterion, since $q(S) = g(\Gamma) = 0$.

Finally, it follows from the classification of minimal rational surfaces that the unique rational surface S with $K_S^2 = 9$ is $S \cong \mathbb{P}^2$. The fact that $\mathcal{O}_S(H) \cong \mathcal{O}_{\mathbb{P}^2}(2)$ follows from $\deg(S) = H^2 = 4$. ∎

Remark 9.2.5. *It is worth mentioning that one could use a similar method as in the higher dimensional cases to prove directly that S is rational, and in particular $q(S) = g(\Gamma) = 0$ (cf. Theorem 9.3.2). However, we preferred to give an alternative proof only based on classical results for algebraic surfaces.*

9.3 Threefolds and higher dimension varieties

Let us first remark that we cannot expect a similar answer as in the lower dimensional cases. More precisely, we have the following observation (cf. [ES03, Section 5]).

Lemma 9.3.1. *Let $n \geq 3$ be a positive integer. Then the tangent bundle of $\mathbb{P}^n$ is never Ulrich.*

Proof. It follows directly from the Euler exact sequence for $T_{\mathbb{P}^n}$ that

$$h^0(\mathbb{P}^n, T_{\mathbb{P}^n}) = n(n+2).$$

On the other hand, if $T_{\mathbb{P}^n}$ is Ulrich with respect to the very ample line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ then we would have $h^0(\mathbb{P}^n, T_{\mathbb{P}^n}) = n \deg(\mathbb{P}^n) = nd^n$ and hence $d^n = n+2$. In particular, $d \geq 2$ in that case. This is impossible, since $d^n \geq 2^n > n+2$ for $n \geq 3$. ∎

The numerical characterization of rationally connected varieties discussed in Section 1.6 together with the restrictions on the first Chern class of Ulrich bundles treated in Section 8 give us the following result in higher dimensions.

Theorem 9.3.2. *Let X be a smooth projective variety of dimension $n \geq 3$ and $\mathcal{O}_X(H)$ a very ample line bundle. Then,*

(a) The cotangent bundle Ω^1_X is never an Ulrich bundle with respect to H.

(b) Assume that the tangent bundle T_X is an Ulrich bundle with respect to H, then $X \cong G/P$ is a rational *homogeneous space, where G is a semi-simple complex Lie group and $P \subseteq G$ is a parabolic subgroup. In this case,* $\deg(X)$ *is a positive multiple of* $(n+2)/\gcd(n^2+n, n+2)$.

Proof. We know by Lemma 8.2.3 that if E is a rank r Ulrich bundle with respect to H then

$$c_1(E) \cdot H^{n-1} = \frac{r}{2}(K_X + (n+1)H) \cdot H^{n-1}.$$

In particular, if we assume in (a) that Ω^1_X is an Ulrich bundle, then we get that

$$c_1(\Omega^1_X) \cdot H^{n-1} = \frac{n}{2}(K_X + (n+1)H) \cdot H^{n-1} \Leftrightarrow \frac{n-2}{n(n+1)} K_X \cdot H^{n-1} = -H^n.$$

Since $n \geq 3$, we deduce that $K_X \cdot H^{n-1} < 0$. On the other hand, Ω^1_X is semistable with respect to the movable class $\alpha := H^{n-1}$ (see Section 1.6) and thus

$$\mu_\alpha^{\max}(\Omega^1_X) = \mu_\alpha(\Omega^1_X) = \frac{K_X \cdot H^{n-1}}{n} < 0,$$

or equivalently $\mu_\alpha^{\min}(T_X) > 0$, and therefore X must be rationally connected by Theorem 1.6.5. This is impossible, since in that case we would have that $\mathrm{H}^0(X, \Omega^1_X) = 0$ (see e.g. [Deb01, Corollary 4.18]), which contradicts the fact that Ω^1_X is an Ulrich bundle.

Assume now that T_X is an Ulrich bundle. First of all, the fact that X is a homogeneous manifold (i.e., that admits a transitive action of an algebraic group) follows from the fact that T_X is globally generated (see Section 1.5) and [MnOSC+15, Proposition 2.1]. Moreover, we know by a classical result of Borel and Remmert (see [BR62]) that in this case $X \cong A \times G/P$, where A is an abelian variety, G is a semi-simple complex Lie group and $P \subseteq G$ is a parabolic subgroup.

In order to rule out the factor A, we proceed as in (a). More precisely, Lemma

8.2.3 implies in this case that

$$\frac{n+2}{n(n+1)}(-K_X \cdot H^{n-1}) = H^n,$$

and hence $\mu_\alpha^{\min}(T_X) = \mu_\alpha(T_X) > 0$. It follows from Theorem 1.6.5, that X is rationally connected and thus $X \cong G/P$. The previous computation shows that $\deg(X) = H^n$ is a positive multiple of $(n+2)/\gcd(n^2+n, n+2)$ (by divisibility reasons). This shows (b). ■

Remark 9.3.3. *The case of threefolds with Ulrich tangent bundle can be treated using classification results. Indeed, if* $\dim(X) = 3$ *and* T_X *is an Ulrich bundle with respect to* H*, then we would have by Theorem 9.3.2 (b) that* $\deg(X) \geq 5$*. In particular, we would have that*

$$\dim \mathrm{Aut}^\circ(X) = h^0(X, T_X) = \dim(X)\deg(X) \geq 15,$$

where $\mathrm{Aut}^\circ(X)$ *denotes the connected component of the identity in the automorphism group of* X*. On the other hand, it is known that rational homogeneous varieties are* Fano*, i.e., the anti-canonical bundle* $-K_X$ *is ample (see e.g. [MnOSC+15, Proposition 2.3]). Using the classification of smooth Fano threefolds by Iskovskikh [Isk77, Isk78, Isk79] and Mori–Mukai [MM83, MM03], together with the recent results on infinite automorphism groups on Fano threefolds, we can perform a case-by-case analysis that give us the desired result. More precisely, it follows from [KPS18, Theorem 1.1.2] and [PCS19, Theorem 1.2] (see also [BFT21, Appendix A]) that* $\dim \mathrm{Aut}^\circ(X) \leq 15$*, with equality if and only if* $X \cong \mathbb{P}^3$ *or* $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$*. The first case is ruled out by Lemma 9.3.1, while the second variety is not a rational homogeneous threefold by [CP91, Theorem 6.1].*

We observe that the fact that the tangent bundle is Ulrich imposes that the manifold has a rather big automorphism group. More precisely, we have the following consequence of the above result and the discussion in Section 1.7.

Corollary 9.3.4. *Let* X *be a smooth projective variety of dimension* $n \geq 4$ *and Picard number one. Then,* T_X *is never an Ulrich bundle.*

Proof. Let $\mathcal{O}_X(H)$ be a very ample line bundle on X, and assume by contradiction that T_X is an Ulrich bundle with respect to H. It follows from Theorem 9.3.2 (b) that $\deg(X)$ is a positive multiple of ℓ, where

$$\ell := \begin{cases} n+2 & \text{if } n \text{ is odd,} \\ \dfrac{n+2}{2} & \text{if } n \text{ is even.} \end{cases}$$

In particular, we have that $h^0(X, T_X) = n\deg(X) \geq n\ell \geq \frac{n(n+2)}{2}$.

On the other hand, it follows from Lemma 1.7.1 that X is isomorphic to $\mathbb{P}^n$, the smooth projective quadric hypersurface $\mathbb{Q}^n \subseteq \mathbb{P}^{n+1}$ or the Grassmannian $\mathrm{Gr}(2,5)$. Moreover, if n is odd then $h^0(X, T_X) \geq n^2 + 2n$ and hence $X \cong \mathbb{P}^n$ in that case.

Since we know that the tangent bundle of $\mathbb{P}^n$ is not Ulrich by Lemma 9.3.1, we will henceforth assume that $X \cong \mathbb{Q}^{2m} \subseteq \mathbb{P}^{2m+1}$ is an even dimensional smooth quadric hypersurface or that $X \cong \mathrm{Gr}(2,5)$.

If $X \cong \mathbb{Q}^{2m}$ we have that $h^0(X, T_X) = \dim \mathfrak{so}_{2m+2}(\mathbb{C}) = (2m+1)(m+1)$.

On the other hand, the Ulrich condition and the previous discussion impose that

$$h^0(X, T_X) = \dim(X)\deg(X) = 2mk\ell = 2m(m+1)k$$

for some $k \in \mathbb{N}^{\geq 1}$, which is impossible by parity reasons.

Similarly, if $X \cong \mathrm{Gr}(2,5)$ we have, on one hand, that $h^0(X, T_X) = \dim \mathfrak{sl}_5(\mathbb{C}) = 24$. On the other hand, we would have that $h^0(X, T_X) = \dim(X)\deg(X) = 6\deg(X)$ and hence $\deg(X) = 4$. Since $\mathrm{Pic}(\mathrm{Gr}(2,5)) \cong \mathbb{Z}$ is generated by the class of $\mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}^9}(1)$, where $\varphi : \mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9$ is the Plücker embedding and where $\deg(\mathcal{L}) = 5$, we have that $\deg(X)$ has to be a multiple of 5, which leads to a contradiction. ∎

The following question naturally arises.

Question 9.3.5. *Is there a rational homogeneous space $X \cong G/P$ of dimension $n \geq 4$ and Picard number at least two such that T_X is an Ulrich bundle?*

Answering Question 9.3.5 above, we first prove the following lemmas.

Lemma 9.3.6. *The tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^l$, $l \geq 1$, is never Ulrich.*

Proof. Let us suppose that $T_{\mathbb{P}^1\times\mathbb{P}^l}$ is Ulrich with respect to $H := \mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^l}(b)$, with $a, b \geq 1$; notice that $-K_{\mathbb{P}^1\times\mathbb{P}^l} = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^l}(l+1)$. By applying Lemma 8.2.3 to $T_{\mathbb{P}^1\times\mathbb{P}^l}$, we obtain

$$0 = \frac{(l+1)(l+2)}{l+3}H^{l+1} + K_X \cdot H^l =$$
$$= ab^l\Big(\frac{(l+1)^2(l+2)}{l+3} - \frac{a(l+1)+2b}{ab}\Big) \geq \frac{(l+1)^2(l+2)-(l+3)^2}{l+3},$$

where the last inequality is a consequence of the fact that $a, b \geq 1$. From Theorem 9.2.4 we can assume that $l \geq 2$; then it is easy to check that $(l+1)^2(l+2) - (l+3)^2 > 0$, which gives a contradiction with the equation in Lemma 8.2.3, thus showing that the tangent bundle is not Ulrich. ∎

Lemma 9.3.7. *There exists no polarized variety (X, H) with Picard number greater than one whose tangent bundle is Ulrich.*

Proof. By Proposition 9.1.1, Theorem 9.2.4, Theorem 9.3.2 and Corollary 9.3.4, we can suppose that $X = G/P$ is a rational homogeneous projective variety with Picard number equal to $k > 1$. From Lemma 1.7.4 and Lemma 9.3.6 we know that

$$\det(T_X) = -K_X = \sum_i j_i L_i,$$

with $j_i < n := \dim(X)$ for $i = 1, \ldots, k$. Let us suppose that $H = \sum_i a_i L_i$ with $a_i > 0$ for $i = 1, \ldots, k$. By applying Lemma 8.2.3 to the tangent bundle, we obtain

$$\frac{n(n+1)}{n+2} H^n + K_X \cdot H^{n-1} = 0.$$

For any k-partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of n, the coefficient of $L_1^{\lambda_1} \cdots L_k^{\lambda_k}$ is equal to

$$\binom{n}{\lambda} a_1^{\lambda_1} \cdots a_k^{\lambda_k} \left(\frac{n(n+1)}{n+2} - \sum_i \frac{\lambda_i j_i}{n a_i} \right) \geq \frac{n(n+1)}{n+2} - \sum_i \frac{\lambda_i j_i}{n} \geq$$

$$\frac{n(n+1)}{n+2} - \sum_i \frac{\lambda_i (n-1)}{n} = \frac{n(n+1)}{n+2} - (n-1) = \frac{2}{n+2} > 0.$$

Since for any λ, $L_1^{\lambda_1} \cdots L_k^{\lambda_k} > 0$, we deduce that $\frac{n(n+1)}{n+2} H^n + K_X \cdot H^{n-1} > 0$, thus the equation in Lemma 8.2.3 is never satisfied; therefore there exists no rational homogeneous projective variety G/P with Picard number greater than one such that its tangent bundle is Ulrich. ∎

By Proposition 9.1.1, Theorem 9.2.4, Corollary 9.3.4, and Lemma 9.3.7, we can prove the main result:

Theorem 9.3.8. *Let X be a smooth projective variety of dimension $n \geq 1$. If T_X is an Ulrich bundle, then X is isomorphic to the twisted cubic in $\mathbb{P}^3$ or to the Veronese surface in $\mathbb{P}^5$.*

Appendices

Dynkin diagrams

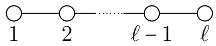
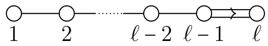
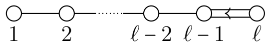
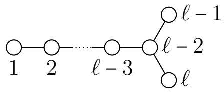
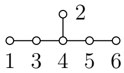
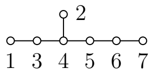
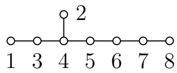
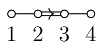
Lie algebra $\mathfrak{g}$	Dynkin diagram	$\dim_{\mathbb{C}} \mathfrak{g}$	$n = \dim_{\mathbb{C}}(G/P_r)$								
A_ℓ $(\ell \geq 1)$		$\ell^2 + 2\ell$	$r(\ell+1-r)$								
B_ℓ $(\ell \geq 2)$		$2\ell^2 + \ell$	$\frac{r}{2}(4\ell+1-3r)$								
C_ℓ $(\ell \geq 3)$		$2\ell^2 + \ell$	$\frac{r}{2}(4\ell+1-3r)$								
D_ℓ $(\ell \geq 4)$		$2\ell^2 - \ell$	$\frac{r}{2}(4\ell-1-3r)$								
E_6		78	r	1	2	3	4	5	6		
			n	16	21	25	29	25	16		
E_7		133	r	1	2	3	4	5	6	7	
			n	33	42	47	53	50	42	27	
E_8		248	r	1	2	3	4	5	6	7	8
			n	78	92	98	106	104	97	83	57
F_4		52	r	1	2	3	4				
			n	15	20	20	15				
G_2		14	r	1	2						
			n	5	5						

Table 9.1: Rational homogeneous spaces of Picard number one.

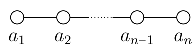
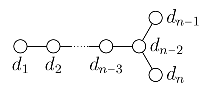
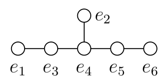
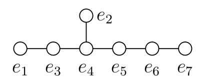
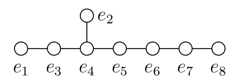
L	A_n	D_n	E_6	E_7	E_8
Generators A_L	$\frac{a_1+2a_2+\ldots+na_n}{n+1}$	$g_1 = \frac{d_{n-1}+d_n}{2}$ $g_2 = \begin{cases} \frac{d_n + \sum_{i=0}^{\frac{n}{2}-2} d_{2i+1}}{2}, \text{ if } n \text{ is even} \\ \frac{3d_n+d_{n-1}}{4} + \frac{\sum_{i=0}^{\lceil \frac{n}{2} \rceil -2} d_{2i+1}}{2}, \text{ if } n \text{ is odd} \end{cases}$	$\frac{e_1+2e_3+e_5+2e_6}{3}$	$\frac{e_2+e_5+e_7}{2}$	1
Discriminant form A_L	$\mathbb{Z}_{n+1}(\frac{1}{n+1})$	$(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix})$, if $n \equiv 0 \mod 4$ $(\mathbb{Z}_4, (\frac{3}{4}))$, if $n \equiv 1 \mod 4$ $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix})$, if $n \equiv 2 \mod 4$ $(\mathbb{Z}_4, (\frac{1}{4}))$, if $n \equiv 3 \mod 4$	$\mathbb{Z}_3(\frac{2}{3})$	$\mathbb{Z}_2(\frac{1}{2})$	$\{0\}$
Dynkin diagram					

Table 9.2: A-D-E lattices

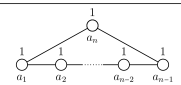
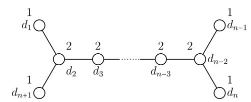
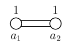
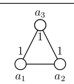
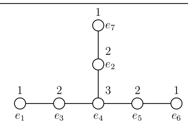
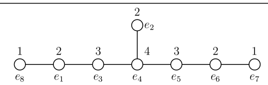
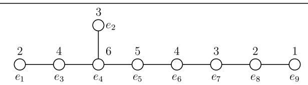
Name fiber X_c	Description	Extended Dynkin diagram	r_c	a	b	δ	Euler characteristic	$X_c^{\#}/X_{c,0}^{\#}$
I_0	a smooth elliptic curve		0	0 $a \geq 1$ 0	0 0 $b \geq 1$	0 0 0	0	0
I_1	a nodal rational curve		0	0	0	1	1	0
$I_{n\geq 2}$	n smooth rational curves meeting with dual graph $\tilde{A}_{n-1}$		$n-1$	0	0	n	n	$\mathbb{Z}/n\mathbb{Z}$
$I^*_{n\geq 0}$	$n+5$ smooth rational curves meeting with dual graph $\tilde{D}_{n+4}$		$n+4$	$n=0$: 2 $a \geq 3$ 2 $n \geq 0$: 2	3 3 $b \geq 4$ 3	n+6	n+6	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if n is even $\mathbb{Z}/4\mathbb{Z}$ if n is odd
II	a cuspidal rational curve		0	$a \geq 1$	1	2	2	0
III	two smooth rational curves meeting at one point to order 2		1	1	$b \geq 2$	3	3	$\mathbb{Z}/2\mathbb{Z}$
IV	three smooth rational curves all meeting at one point		2	$a \geq 2$	2	4	4	$\mathbb{Z}/3\mathbb{Z}$
IV^*	seven smooth rational curves meeting with dual graph $\tilde{E}_6$		6	$a \geq 3$	4	8	8	$\mathbb{Z}/3\mathbb{Z}$
III^*	eight smooth rational curves meeting with dual graph $\tilde{E}_7$		7	3	$b \geq 5$	9	9	$\mathbb{Z}/2\mathbb{Z}$
II^*	nine smooth rational curves meeting with dual graph $\tilde{E}_8$		8	$a \geq 4$	5	10	10	0

Table 9.3: Classification of singular fibers of an elliptic surface

Isometries of K_{12}

The automorphism group of K_{12} was computed by Mitchell in 1914 in [Mit14] while the lattice itself was first explicitly described by Coxeter and Todd in 1954 [CT53].

Having this in mind, and the description of K_{12} in Proposition 3.1.1, we compute the isometries of K_{12} preserving the orbits in Lemma 4.1.1: fix $a \neq 0$ be an element of $o_i \subset A_{K_{12}}$, $i = 0, 1, 2$ and $b \in o_i$, we looking for an isometry $\varphi \in \mathrm{O}(K_{12})$ such that $\bar{\varphi} \in \mathrm{O}(A_{K_{12}})$ (the map induced in the quotient $A_{K_{12}}$) satisfies $\bar{\varphi}(b) = a$. The search can be successful (or not) and it depends if the natural map $\mathrm{O}(K_{12}) \longrightarrow \mathrm{O}(A_{K_{12}})$ is surjective (or is not). A useful criterion that determines when this map is surjective is given by Nikulin in [Nik76, Theorem 1.14.2] but it applies just in the case of indefinite lattices (which is not our case). Nevertheless, the automorphism group of K_{12} has order $2^{10} \cdot 3^7 \cdot 5 \cdot 7 = 78382080$ and finding an explicit isometry was possible by using the integral lattices package of Sage [Sag].

Here we report the code to use in order to compute the 729 isometries.

Step 1: Define the lattice K_{12}:

```
basis_K=[[-4 , 2 , 0 , 0 , 0 , 0 , -2 , 1 , 0 , 0 , 0 , 0 ],
[2 , -4 , 2 , 0 , 0 , 0 , 1 , -2 , 1 , 0 , 0 , 0 ],
[0 , 2 , -4 , 2 , 0 , 2 , 0 , 1 , -2 , 1 , 0 , 1 ],
[0 , 0 , 2 , -4 , 2 , 0 , 0 , 0 , 1 , -2 , 1 , 0 ],
[0 , 0 , 0 , 2 , -4 , 0 , 0 , 0 , 0 , 1 , -2 , 0 ],
[0 , 0 , 2 , 0 , 0 , -4 , 0 , 0 , 1 , 0 , 0 , -2 ],
[-2 , 1 , 0 , 0 , 0 , 0 , -4 , 2 , 0 , 0 , 0 , 0 ],
[1 , -2 , 1 , 0 , 0 , 0 , 2 , -4 , 2 , 0 , 0 , 0 ],
[0 , 1 , -2 , 1 , 0 , 1 , 0 , 2 , -4 , 2 , 0 , 2 ],
[0 , 0 , 1 , -2 , 1 , 0 , 0 , 0 , 2 , -4 , 2 , 0 ],
[0 , 0 , 0 , 1 , -2 , 0 , 0 , 0 , 0 , 2 , -4 , 0 ],
[0 , 0 , 1 , 0 , 0 , -2 , 0 , 0 , 2 , 0 , 0 , -4]]

K12_Tilde=IntegralLattice(Matrix(ZZ,basis_K))
k1,k2,k3,k4,k5,k6,k7,k8,k9,k10,k11,k12=K12_Tilde.gens()
N=(1/3)*(k1+k7) + (2/3)*(k2+k8)+(1/3)*(k4 + k10) +(2/3)*(k5 + k11)

K12=IntegralLattice(Matrix(ZZ,(K12_Tilde.overlattice([N])).gram_matrix()))
```

Step 2: Compute the discriminant form of $A_{K_{12}}$:

```
Disc_K12=K12.discriminant_group()
a1,a2,a3,a4,a5,a6=Disc_K12.gens()
```

Step 3: Compute the automorphism group of K_{12}:

```
Aut_K12=K12.automorphisms()
```

Step 4: List the elements of an orbit v_i of value $v \in \{0, \frac{2}{3}, \frac{4}{3}\}$:

```
v=4/3
Y=[]
for x in Disc_K12:
  if x.quadratic_product()==v:
    Y=Y+[x]
```

Step 5: Check the existence of an isometry for two elements in v_i and show the isometry $\varphi = A$. We distinguish two cases.

If $v \neq 0$:

```
while len(Y)>0:
  x0=Y[0];YY=[]
  for y in Y:
    if y != x0:
      cont=0
      for A in Aut_K12:
        if y*A==x0:
          A
          break
        else:
          if cont==Aut_K12.cardinality():
            YY=YY+[y]
          else:
            cont=cont+1
  Y=YY
```

If $v = 0$:

```
while len(Y)>0:
  x0=Y[1];YY=[]
  for y in Y:
```

```
    if y != x0 and y != Y[0]:
      cont=0
      for A in Aut_K12:
        if y*A==x0:
          A
          break
        else :
          if cont==Aut_K12.cardinality():
            YY=YY+[y]
          else :
            cont=cont+1
  Y=YY
```

Isometries of $M_{\mathbb{Z}/3\mathbb{Z}}$

```
#Construction of MZ3:
basis_M3Z=[[-4, -1, 0, -1, 0, -1, 0, -1, 0, -1, 0, -1],
[-1, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
[ 0, 0, -2, 1, 0, 0, 0, 0, 0, 0, 0, 0],
[-1, 0, 1, -2, 0, 0, 0, 0, 0, 0, 0, 0],
[ 0, 0, 0, 0, -2, 1, 0, 0, 0, 0, 0, 0],
[-1, 0, 0, 0, 1, -2, 0, 0, 0, 0, 0, 0],
[ 0, 0, 0, 0, 0, 0, -2, 1, 0, 0, 0, 0],
[-1, 0, 0, 0, 0, 0, 1, -2, 0, 0, 0, 0],
[ 0, 0, 0, 0, 0, 0, 0, 0, -2, 1, 0, 0],
[-1, 0, 0, 0, 0, 0, 0, 0, 1, -2, 0, 0],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, 1],
[-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -2]]

M3Z=IntegralLattice(Matrix(ZZ,basis_M3Z))
#Group of isometries of M3Z:
Aut_M3Z=M3Z.automorphisms()
#Discriminant group of M3Z:
Disc_M3Z=M3Z.discriminant_group()
v=4/3
Y=[]
```

```
for x in Disc_M3Z:
  if x.quadratic_product()==v:
    Y=Y+[x]

#If v is different to 0:

while len(Y)>0:
  x0=Y[0];YY=[]
  x0
  for y in Y:
    if y != x0:
      cont=0
      for A in Aut_M3Z:
        if y*A==x0:
          A
          break
        else:
          if cont==Aut_M3Z.cardinality():
            YY=YY+[y]
          else:
            cont=cont+1
  Y=YY

#If v is zero:
while len(Y)>0:
  x0=Y[1];YY=[]
  for y in Y:
    if y != x0 and y != Y[0]:
      cont=0
      for A in Aut_M3Z:
        if y*A==x0:
          A
          break
        else:
          if cont==Aut_M3Z.cardinality():
            YY=YY+[y]
          else:
```

```
            cont=cont+1
Y=YY
```

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