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Low codimensional intrinsic regular submanifolds in the Heisenberg group $\mathbb{H}^n$

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Abstract

The thesis mainly concerns the study of intrinsically regular submanifolds of low codimension in the Heisenberg group $\mathbb{H}^n$, called $\mathbb{H}$-regular surfaces of low codimension, from the point of view of geometric measure theory. We consider an $\mathbb{H}$-regular surface of $\mathbb{H}^n$ of codimension k, with k between 1 and n, parametrized by a uniformly intrinsically differentiable map acting between two homogeneous complementary subgroups of $\mathbb{H}^n$, with target subgroup horizontal of dimension k. In particular the considered submanifold is the intrinsic graph of the parametrization. We extend various results of Ambrosio, Serra Cassano and Vittone, available for the case when $k = 1$. We prove that the uniform intrinsic differentiability of the parametrizing map is equivalent to the existence and continuity of its intrinsic differential, to the local existence of a suitable approximating family of Euclidean regular maps, and, when the domain and the codomain of the map are orthogonal, to the existence and continuity of suitably defined intrinsic partial derivatives of the function. Successively, we present a series of area formulas, proved in collaboration with V. Magnani. They allow to compute the $(2n + 2 - k)$-dimensional spherical Hausdorff measure and the $(2n+2-k)$-dimensional centered Hausdorff measure of the parametrized $\mathbb{H}$-regular surface, with respect to any homogeneous distance fixed on $\mathbb{H}^n$.

Furthermore, we focus on $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$, where $\mathbb{G}$ and $\mathbb{M}$ are two arbitrary Carnot groups. Suitable implicit function theorems ensure the local existence of an intrinsic parametrization of such a set, at any of its points. We prove that it is uniformly intrinsically differentiable.

Finally, we prove a coarea-type inequality for a continuously Pansu differentiable function acting between two Carnot groups endowed with homogeneous distances. We assume that the level sets of the function are uniformly lower Ahlfors regular and that the Pansu differential is everywhere surjective.

Sintesi

La tesi riguarda principalmente lo studio delle sottovarietà intrinsecamente regolari nel gruppo di Heisenberg $\mathbb{H}^n$, chiamate superfici $\mathbb{H}$-regolari di codimensione bassa, dal punto di vista della teoria geometrica della misura. Consideriamo una superfice $\mathbb{H}$-regolare di codimensione k, con k compreso tra 1 ed n, parametrizzata da una funzione uniformemente intrinsecamente differenziabile che agisce tra due sottogruppi omogenei complementari di $\mathbb{H}^n$, con sottogruppo di arrivo orizzontale di dimensione k. In particolare, la superficie considerata è il grafico intrinseco della parametrizzazione. Generalizziamo vari risultati di Ambrosio, Serra Cassano e Vittone, validi per il caso $k = 1$. Dimostriamo che la differenziabilità intrinseca uniforme della parametrizzazione è equivalente all'esistenza e continuità del suo differenziale intrinseco, all'esistenza locale di una successione di mappe regolari dal punto di vista Euclideo che approssimano opportunamente la funzione, e, quando dominio e codominio della mappa sono ortogonali, all'esistenza e continuità di derivate parziali intrinseche, opportunamente definite, della funzione. Successivamente, presentiamo una serie di formule dell'area, ottenute in collaborazione con V. Magnani. Esse permettono di calcolare la misura di Hausdorff sferica $(2n + 2 - k)$-dimensionale e la misura di Hausdorff centrata $(2n + 2 - k)$-dimensionale della superficie $\mathbb{H}$-regolare parametrizzata, rispetto ad una qualsiasi distanza omogenea fissata su $\mathbb{H}^n$.

In seguito, ci concentriamo sugli insiemi $(\mathbb{G}, \mathbb{M})$-regolari di $\mathbb{G}$, dove $\mathbb{G}$ e $\mathbb{M}$ sono due gruppi di Carnot arbitrari. Adeguati teoremi della funzione implicita, assicurano l'esistenza locale di una parametrizzazione intrinseca per un tale insieme, ad ognuno dei suoi punti. Nella tesi dimostriamo che tale mappa è uniformemente intrinsecamente differenziabile.

Infine, dimostriamo una disuguaglianza di tipo coarea per una mappa continuamente Pansu differenziabile che agisce tra due gruppi di Carnot dotati di distanze omogenee. Assumiamo che gli insiemi di livello della funzione siano uniformemente Ahlfors regolari dal basso e che il differenziale di Pansu sia ovunque suriettivo.

Introduction

A Carnot group $\mathbb{G}$ is a finite dimensional connected, simply connected, nilpotent Lie group such that the Lie algebra of the left invariant vector fields on $\mathbb{G}$, $\mathrm{Lie}(\mathbb{G})$, is stratified, namely it can be written as a finite direct sum of linear subspaces $V_1, \dots, V_\kappa$ such that

$$\mathrm{Lie}(\mathbb{G}) = V_1 \oplus \cdots \oplus V_\kappa$$

and

$$[V_1, V_i] = V_{i+1} \text{ for } 1 \leq i \leq \kappa - 1, \qquad V_\kappa \neq \{0\}, \qquad [V_1, V_\kappa] = \{0\}.$$

In the literature, Carnot groups are also referred to as stratified groups. A Carnot group $\mathbb{G}$ is naturally endowed with a family of left translations $\{l_x\}_{x\in\mathbb{G}}$, where $l_x(y) = x \cdot y$ for every $y \in \mathbb{G}$. We equip $\mathbb{G}$ with a graded left invariant Riemannian metric g, namely a Riemannian metric with respect to which the subspaces $V_1, \dots, V_\kappa$ are orthogonal and all left translations are isometries. The stratification of the Lie algebra allows to introduce a family of anisotropic dilations on $\mathbb{G}$, $\{\delta_t\}_{t>0}$, that makes $\mathbb{G}$ a homogeneous group in the sense of [FS82]. The Lie algebra of $\mathbb{G}$ can be identified with the tangent space at the identity element of the group $T_e\mathbb{G}$, so that the first layer V_1 of $\mathrm{Lie}(\mathbb{G})$ individuates by left translation a subbundle $H\mathbb{G}$ of the tangent bundle called the horizontal bundle of $\mathbb{G}$. More precisely, if we fix a basis $(X_1, \dots, X_{m_1})$ of V_1, the fiber of $H\mathbb{G}$ at x is defined as

$$H_x\mathbb{G} = \mathrm{span}(X_1(x), \dots, X_{m_1}(x)) \subset T_x\mathbb{G};$$

in other words $H\mathbb{G}$ is a smooth distribution of subspaces. By the definition of Carnot group, the subspace V_1 generates by Lie bracket the whole $\mathrm{Lie}(\mathbb{G})$. This Lie-generating condition is related to various mathematical research fields, like subelliptic PDEs, non-holonomic mechanics and optimal control theory, and it has been referred to in the literature as Chow's condition, Hörmander's condition, total non-holonomicity, or, simply, bracket generating condition. A horizontal curve on $\mathbb{G}$ is an absolutely continuous curve $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma'(t) \in H_{\gamma(t)}\mathbb{G}$ for almost every $t \in [0,1]$. We consider the horizontal curves as the unique admissible ones along which it is possible to move on $\mathbb{G}$. More in general, given a smooth manifold M and a prescribed smooth distribution of subspaces $\Delta \subset TM$, an admissible curve is an absolutely continuous curve tangent at almost every point to Δ. The relation between the possibility of connecting two points of M through an admissible curve and the validity of the Chow's condition for the set of smooth vector fields whose value lies at every point in Δ, has been investigated for a long time. During the same period, Rashevsky and Chow prove indipendently in [Ras38] and [Cho39] that if Δ satisfies the Chow's condition, then any two points of M can be joined by an admissible curve. Therefore, since in our context V_1 Lie-generates $\mathrm{Lie}(\mathbb{G})$, for any couple of points $x, y \in \mathbb{G}$, there exists a horizontal curve that connects them. This connectivity property allows to equip $\mathbb{G}$ with a so-called Carnot-Carathéodory distance, denoted by d_c, that is defined for every couple of points $x, y \in \mathbb{G}$ as the infimum of the lengths of the horizontal curves joining x and y. Thanks to the results of Chow and Rashevsky, this set of curves is non empty and d_c is finite. The metric space $(\mathbb{G}, d_c)$ is non-Riemannian at any scale: $\mathbb{G}$ is a sub-Riemannian manifold. The topological dimension q of $(\mathbb{G}, d_c)$ is always smaller than or equal to its Hausdorff dimension, that is $Q = \sum_{i=1}^{\kappa} i\dim(V_i)$: this is a peculiarity of sub-Riemannian geometry, in fact the two dimensions coincide if and and only if $\kappa = 1$, hence if $\mathbb{G}$ is commutative.

The origin of the terminology "Carnot group" goes back to a paper of Carathéodory of 1909, [Car09], where the author models a thermodynamic process as a curve on $\mathbb{R}^n$ connecting two states and then he considers the adiabatic curves as the admissible ones. The heath exchanged during a process is represented by the integral of a one-form θ along the curve representing the involved process, therefore an admissible curve is a curve along which θ vanishes at every point. This model represents a Carnot-Carathéodory space, where Δ can be defined as the distribution of the subspaces of $\mathbb{R}^n$ on which θ vanishes. In this context Carnot raised the first accessibility problem, individuating two states that cannot be connected by an adiabatic process; this observation related his name to Carnot groups.

We need to stress that Carnot groups are not just examples of sub-Riemannian manifolds, they play a privileged role in this context. They can be seen as infinitesimal models for sub-Riemannian manifolds, as Euclidean spaces are local models for Riemannian manifolds. In fact, a suitable blow-up limit of a sub-Riemannian manifold at a regular point is a Carnot group, therefore Carnot groups are the natural tangent spaces to sub-Riemannian manifolds [Mit85, Bel96, MM00]. For valuable introductions to sub-Riemannian geometry and Carnot groups the reader can refer to [Str86, Gro96, Mon02, BLU07, CDPT07, LD15, LD17, ABB20] and to the references therein.

The research in analysis on sub-Riemannian manifolds, and, in particular, on Carnot groups, has been developed along various directions. We recall below some of the main investigated themes, providing for any topic some related references. Surely the following lists are incomplete, our aim is to give a flavour of the wide available related literature, without any ambition of completeness. Among the various research lines we highlight the study of solutions of elliptic PDEs [H$\ddot{6}$7, RS76, BLU07] and degenerate elliptic PDEs [Bon69, Fol73], optimal control theory [AR04, Rig05, FR10, Jea14, Rif14], the problem about the existence and regularity of geodesics [Mon02, LS95, LM08, Mon14b, LD15], the theory of Sobolev spaces and its connection with Poincarè-type inequalities [Jer86, FGW94, FSSC97, HK00, CMPSC16], the theory of singular integrals [CM14, Fä19, CFO19a], contact and CR geometry [DT06], fractal geometry [BRSC03, BTW09, MV19], the development of mathematical models for portions of the visual cortex [CS06, SCP08, CS10] and the application of Carnot groups to computer science [NY17, NY18, NY20]. Indeed, many researches have focused on the extension of the concepts of geometric measure theory on Carnot groups and, in general, on sub-Riemannian manifolds. In particular we recall the study about the Besicovitch covering property [Rig04, LDR17, GR19, LDR19], the theory of BV functions and Caccioppoli sets [CDG94, FGW94, GN96, Amb01, FSSC01, Amb02, FSSC03a, AKLD09, AGM15] and the subsequent investigations about isoperimetric inequalities and isoperimetric sets, [Pan82b, GN96, LR03, LM05, RR06, DGN08b, RR08, MR09, Mon14a], the theory of minimal surfaces [Pau04a, BASCV07, CHY07, DGN08a, MSCV08, DGNP09, CCM09, Rit09, DGNP10, HRR10, SCV14, Mon14a], the various approaches to the study of the regularity of submanifolds [FSSC03b, ASCV06, CM06, FSSC07, Vit08, BV10, BSC10a, LM11, Mag13, FMS14, Koz15, MTV15, BCSC15, FS16, SC16, DD17, ADDDLD20, Vit20] and to the study of rectifiable sets [AK00, MSC01, FSSC01, FSSC02, Mag06b, Ser08, AKLD09, MSSC10, CT15, Rig19, CFO19b, Fä19, DLDMV19, FOR20, CMT20], the theory of currents and differential forms [Rum94, FT15, BFP16, BFP20], the development of suitable area and coarea formulas in Carnot groups [Pan82a, MSC01, Mag01, Mag02b, FSSC07, MV08, LDM10, Mag11b, KV13, FSSC15, Mag15, MV15, Mag19, JNGV20]. The contributions of this thesis fall into the two categories related to the study of the regularity of submanifolds and to the development of correlated area and coarea formulas. The main characters of our study are intrinsically regular submanifolds in Carnot groups, that are usually referred to as *intrinsic regular submanifolds*, with particular attention to the ones of the Heisenberg group $\mathbb{H}^n$, $n \geq 1$, studied from the point of view of geometric measure theory. Therefore the material, both the known one and the new one, is organized in order to emphasize the role of the several presented concepts in relation with this main topic. We present the original results provided by the thesis explaining at the same time its structure. The original contributions are proved in Chapters 4, 5, 6 and 7.

In Chapter 1 we collect some preliminary classical notions before introducing the specific setting of Carnot groups. In particular, we present the definition and a brief introduction to the space of vector fields on a smooth manifold, the notion of abstract Lie algebra, the notion of Lie group and the connection between the two, through the concept of left invariant vector field, the definition and some fundamental properties of the exponential map, the notions of homogeneous Lie algebra and homogeneous Lie group and, finally, we provide a synthetic exposition about sub-Riemannian manifolds.

In Chapter 2 we introduce Carnot groups and, in particular, the Heisenberg group $\mathbb{H}^n$, that is the simplest example of a non commutative Carnot group. For more detailed introductions to this theme the reader can refer to [FS82, CDPT07, BLU07, SC16] and to the references therein. Through the identification of $\mathbb{G}$ with $\mathrm{Lie}(\mathbb{G})$, based on the properties of the exponential map and on the Baker-Campbell-Hausdorff formula, we consider a Carnot group $\mathbb{G} = (\mathbb{G}, [\cdot,\cdot], \cdot)$ as a finite dimensional, nilpotent, stratified Lie algebra $(\mathbb{G}, [\cdot,\cdot])$, $\mathbb{G} = V_1 \oplus \cdots \oplus V_\kappa$, endowed with the group product $\cdot$ defined on $\mathbb{G}$ by the Baker-Campbell-Hausdorff operation associated with $(\mathbb{G}, [\cdot,\cdot])$. In particular the group $\mathbb{G}$ is considered as a vector space. When necessary, it is possible to fix suitable coordinates of $\mathbb{G}$ by choosing a basis adapted to the stratification, so that $\mathbb{G}$ can be identified with the space $\mathbb{R}^q$, endowed with a polynomial group product and a Lie algebra structure. In Section 2.3 we describe the peculiar structure of the Heisenberg group $\mathbb{H}^n$ and we introduce suitable coordinates on it, then in Section 2.4 we present the definition and the main properties of homogeneous distances on Carnot groups, namely distances such that $d(x,y) = d(z \cdot x, z \cdot y)$ for every $x, y, z \in \mathbb{G}$ and $d(\delta_t(x), \delta_t(y)) = td(x,y)$ for every $x, y \in \mathbb{G}$ and $t > 0$. Given a homogeneous distance d, we define the norm $\|x\| := d(x,0)$ for $x \in \mathbb{G}$. Notice in particular that the Carnot-Carathéodory distance d_c is homogeneous and then it is equivalent to any other homogeneous distance defined on $\mathbb{G}$. In Section 2.5, we consider $\mathbb{G}$ as a measure space, hence we present some measures that can naturally be taken in consideration on a Carnot group. We present the notions of α-Hausdorff measure $\mathcal{H}^\alpha$ and α-spherical Hausdorff measure $\mathcal{S}^\alpha$, built through Carathéodory's construction with respect to a fixed homogeneous distance, and the one of α-centered Hausdorff measure $\mathcal{C}^\alpha$, due to [RT88]. In addition, we introduce the α-Euclidean Hausdorff measure $\mathcal{H}^\alpha_E$, again built through Carathéodory's construction but now with respect to the Riemannian norm associated with the fixed Riemannian metric g on $\mathbb{G}$. We conclude the chapter presenting two abstract differentiation theorems, proved in [FSSC15] and [Mag15]. They will play a fundamental role in Chapter 6.

Chapter 3 is devoted to a detailed description of the structure of a Carnot group $\mathbb{G}$, both *from outside*, in relation with other Carnot groups through appropriate morphisms, and *from within*, in regard with homogeneous subgroups, which are natural structures arising within $\mathbb{G}$ according to its group structure. In particular, we illustrate some fundamental tools of the differential calculus, on and within a Carnot group, developed in the last thirty years. When it is possible, we organize these tools in a scheme that retraces the plan of the corresponding concepts in the Euclidean setting. For the material about h-homomorphisms and Pansu differentiability we mainly refer to [FS82, Pan89, Mag02a, Mag13], while for the references concerning homogeneous subgroups, intrinsic graphs, intrinsic Lipschitz continuity and intrinsic differentiability we follow [FSSC03b, FSSC05, FSSC06, FSSC07, AS09, FSSC11, FMS14, FS16, SC16]. We start by presenting, in Section 3.1, the definition of homogeneous subgroup: a (Lie) subgroup $\mathbb{W} \subset \mathbb{G}$ is homogeneous if $\delta_t(w) \in \mathbb{W}$ for every $w \in \mathbb{W}$ and $t > 0$. Two complementary subgroups of $\mathbb{G}$ are two homogeneous subgroups $\mathbb{W}, \mathbb{V} \subset \mathbb{G}$ such that $\mathbb{W} \cap \mathbb{V} = \{0\}$ and $\mathbb{W} \cdot \mathbb{V} = \mathbb{G}$. If, in addition, $\mathbb{W}$ is normal, we write $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$. If $\mathbb{G}$ is the product of two complementary subgroups $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$, any element $x \in \mathbb{G}$ can be written as the (ordered) product of a unique element $\pi_{\mathbb{W}}(x) \in \mathbb{W}$, and a unique element $\pi_{\mathbb{V}}(x) \in \mathbb{V}$. As a consequence, the two group projections $\pi_{\mathbb{W}} : \mathbb{G} \to \mathbb{W}$ and $\pi_{\mathbb{V}} : \mathbb{G} \to \mathbb{V}$ related to the splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ are well defined. In Section 3.1.1 we focus on the properties of some suitable Hausdorff measures concentrated on a homogeneous subgroup. Contextually, we present some results related to the splitting of $\mathbb{G} = V \cdot \mathbb{W}$ as a product of a homogeneous linear subspace V, that is not necessarily a subgroup, and a homogeneous normal subgroup $\mathbb{W}$. These results, which retrace analogous results proved in [Mag20], will be useful in Chapter 7. In Section 3.2 we consider two Carnot groups, $\mathbb{G}$ and $\mathbb{M}$, equipped with homogeneous distances, and we introduce the family of h-homomorphisms, i.e. homogeneous group homomorphisms, between $\mathbb{G}$ and $\mathbb{M}$. We state and discuss the fundamental notion of Pansu differentiability, due to Pansu [Pan89]: given an open set $\Omega \subset \mathbb{G}$ and a point $x \in \Omega$, a map $f : \Omega \to \mathbb{M}$ is Pansu differentiable at x if it can be approximated, close to x, with respect to the homogeneous distances, by a h-homomorphism $Df(x) : \mathbb{G} \to \mathbb{M}$, which is called the Pansu differential of f at x. If, in particular, f is Pansu differentiable at every point of Ω and the Pansu differential is continuous, we say that $f \in C^1_h(\Omega, \mathbb{M})$. Section 3.3 contains a brief introduction to the theory of functions of bounded variation and locally finite perimeter sets in Carnot groups. The contributions of the thesis do not rely in particular on these concepts, but they are preliminary to understand the state of the art about intrinsic regular hypersurfaces in Carnot groups. Sections 3.4 and 3.5 are devoted to the notion and the properties of intrinsic graphs, due to Franchi, Serapioni and Serra Cassano [FSSC03b, FSSC07]. Actually, a theory of regularity has been developed for maps acting between two complementary subgroups of a Carnot group. We sketch here just the concepts underlying the main definitions. We consider two complementary subgroups of a Carnot group $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ and a map $\phi : U \to \mathbb{V}$, with $U \subset \mathbb{W}$. The intrinsic graph of ϕ is the set

$$\mathrm{graph}(\phi) = \{w \cdot \phi(w) : w \in U\}.$$

We call graph map of ϕ the map $\Phi : U \to \mathbb{G}$, $\Phi(w) := w \cdot \phi(w)$. The term "intrinsic" has a precise meaning in this context. It is used to emphasize concepts that are invariant with respect to left translations and anisotropic dilations, therefore, in this sense, an intrinsic object fits the Carnot group structure. For instance, if we fix a point $x \in \mathbb{G}$ there exists a well defined map $\phi_x : U_x \to \mathbb{V}$, where $U_x \subset \mathbb{W}$ is a suitable set depending on U and x, such that $l_x(\mathrm{graph}(\phi)) = \mathrm{graph}(\phi_x)$. Assume now that U is (relatively) open. A map ϕ is said to be intrinsic Lipschitz if there exists an appropriately defined homogeneous cone, whose (left) cosets by points of $\mathrm{graph}(\phi)$ do not intersect the intrinsic graph of ϕ, except for the vertex. In other words, we say that ϕ is intrinsic C-Lipschitz, for some positive number C, if for every $w, w' \in U$, $\|\pi_{\mathbb{V}}(\Phi(w')^{-1} \cdot \Phi(w))\| \leq C\|\pi_{\mathbb{W}}(\Phi(w')^{-1} \cdot \Phi(w))\|$. A map $L : \mathbb{W} \to \mathbb{V}$ is intrinsic linear if graph(L) is a homogeneous subgroup. If we fix a point $\bar{w} \in U$, we say that the function ϕ is intrinsically differentiable at $\bar{w}$ if there exists (a left coset of) a homogeneous subgroup that approximates graph(ϕ) at $\Phi(\bar{w})$ with respect to the homogeneous distance fixed on $\mathbb{G}$, namely if there exists an intrinsic linear map $d\phi_{\bar{w}} : \mathbb{W} \to \mathbb{V}$, called the intrinsic differential of ϕ at $\bar{w}$, such that

$$\|d\phi_{\bar{w}}(w)^{-1} \cdot \phi_{\Phi(\bar{w})^{-1}}(w)\| = o(\|w\|) \tag{1}$$

as $\|w\| \to 0$, for $w \in U_{\Phi(\bar{w})^{-1}}$. Moreover, if the approximation (1) holds uniformly close to $\bar{w}$, we say that ϕ is uniformly intrinsically differentiable at $\bar{w}$ (Definition 3.5.29). We end Chapter 3 with Section 3.6, which is dedicated to some recent remarkable progresses presented in [FMS14, FS16, NY18, Vit20], arised from the study of intrinsic Lipschitz maps ϕ acting between two complementary subgroups, with one dimensional target space.

In Chapter 4 we present the main characters of our investigation: intrinsic regular submanifolds within a Carnot group. This research topic originated by the necessity of stating "good" notions of submanifold and rectifiable set in a Carnot group, that is to say searching for definitions that suit the algebraic homogeneous structure of a Carnot group. In fact, the classical notions of regular surface and rectifiable set in a general metric space, due to [Fed69], are not appropriate to achieve this aim (for instance one can refer to [AK00]). This consideration opened the path towards the introduction of several innovative notions. We start by presenting $\mathbb{H}$-regular surfaces in the Heisenberg group $\mathbb{H}^n$, following [FSSC07, SC16] and the references therein. We distinguish low dimensional $\mathbb{H}$-regular surfaces and low codimensional ones. Since the thesis will mainly focus on the latter ones, here we sketch only their definition.

Definition 1. A set $\Sigma \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular surface of codimension k, with $1 \leq k \leq n$, if for every $\bar{x} \in \Sigma$ there exist an open set Ω such that $\bar{x} \in \Omega \subset \mathbb{H}^n$, and a continuously Pansu differentiable map $f \in C^1_h(\Omega, \mathbb{R}^k)$ such that $\Sigma \cap \Omega = f^{-1}(0)$ and such that the Pansu differential $Df(x)$ is a surjective map for every $x \in \Omega$.

A suitable implicit function theorem by Franchi, Serapioni, and Serra Cassano permits to deduce that for any $\mathbb{H}$-regular surface $\Sigma \subset \mathbb{H}^n$ there exists locally a homogeneous horizontal subgroup $\mathbb{V}$ of $\mathbb{H}^n$ such that, for every homogeneous subgroup $\mathbb{W} \subset \mathbb{H}^n$ complementary to $\mathbb{V}$, Σ is locally the intrinsic graph of a unique continuous function acting between $\mathbb{W}$ and $\mathbb{V}$. The notion of $\mathbb{H}$-regular surface has then been extended in [Mag13] from the Heisenberg group to a generic Carnot group, through the notions of $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ and of $\mathbb{M}$, presented in Section 4.2, where $\mathbb{G}$ and $\mathbb{M}$ are arbitrary Carnot groups. We mainly focus on $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$.

Definition 2. If $\mathbb{G}$ and $\mathbb{M}$ are Carnot groups, a set $\Sigma \subset \mathbb{G}$ is a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ if for every $\bar{x} \in \Sigma$ there exist an open set Ω, such that $\bar{x} \in \Omega \subset \mathbb{G}$, and a function $f \in C^1_h(\Omega, \mathbb{M})$ such that $\Sigma \cap \Omega = f^{-1}(0)$ and, for every $x \in \Omega$, the Pansu differential $Df(x)$ is surjective and there exists a homogeneous subgroup $\mathbb{V}$ complementary to $\ker Df(x)$.

Thanks to an implicit function theorem proved by Magnani, according to the notation of Definition 2, Σ is locally the intrinsic graph of an intrinsic Lipschitz map $\phi : U \to \mathbb{V}$, where U is a relatively open subset of $\ker(Df(\bar{x}))$. In Section 4.3, still according to the notation of Definition 2, we assume that $\mathbb{V}$ is a homogeneous subgroup complementary to $\ker(Df(\bar{x}))$ for every $\bar{x} \in \Sigma$ and that Σ is parametrized as the intrinsic graph of a function $\psi : U \to \mathbb{V}$, where $U \subset \mathbb{W}$ is a relatively open set and $\mathbb{W}$ is an homogeneous subgroup complementary to $\mathbb{V}$. Notice that the existence of such a parametrizing map ψ

is ensured by a recent result in [JNGV20]. Then, we prove that ψ is uniformly intrinsically differentiable on U, namely that it is uniformly intrinsically differentiable at every point $w \in U$ (Theorem 4.3.7). The proof relies on the explicit verification of the definition, and on a geometric characterization, of uniform intrinsic differentiability in Carnot groups. This is the first original contribution of this thesis. We dedicate Section 4.4 to the family of regular submanifolds investigated by Kozhevnikov in [Koz15], we call these sets $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds. They can be seen as a generalization of $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$. The research about these submanifolds is still at an early stage and, for this reason, it is a rich source of open questions and interesting phenomena. In Section 4.5 we see how the definitions of $\mathbb{H}$-regular surface and intrinsic Lipschitz graph have been applied to state suitable definitions of intrinsically rectifiable set, often referred to in the literature as *intrinsic rectifiable set*, in the Heisenberg group. In particular, we present $(k, \mathbb{H})$-rectifiable sets of $\mathbb{H}^n$ and $(k, \mathbb{H}_L)$-rectifiable ones. We sketch how analogous definitions can be introduced in a generic Carnot group. We end the chapter with Section 4.5.1 providing a presentation of the very recent available results about a theory of uniform, or quantitative, rectifiability in the Heisenberg group. In particular, we expose some details about the results presented in [NY18], that can be considered the starting point of the development of this theory. For more details please refer to [Fä19] and to the references therein.

In Chapter 5 we present the results of [Cor19]. Arena and Serapioni in [AS09] characterized locally any k-codimensional $\mathbb{H}$-regular surface, with $1 \leq k \leq n$, as the intrinsic graph of a uniformly intrinsically differentiable map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ acting between two complementary subgroups, $\mathbb{W}$ and $\mathbb{V}$, of $\mathbb{H}^n$, with $\mathbb{W}$ normal of dimension $2n+1-k$, $\mathbb{V}$ horizontal of dimension k and $U \subset \mathbb{W}$ open set. Comparing this result with the Euclidean implicit function theorem, it is quite natural to consider the uniform intrinsic differentiability of the parametrizing map ϕ as the analogue of the Euclidean C^1-regularity of a map acting between two linear subspaces whose direct sum is the whole Euclidean space $\mathbb{R}^n$. Then, it is quite natural to conjecture the existence of a characterization of the uniform intrinsic differentiability of ϕ in terms of the existence and continuity of the intrinsic differential of ϕ or of suitably defined intrinsic partial derivatives of ϕ. This conjecture has been positively aswered for $k = 1$ in [ASCV06, BSC10a, BSC10b, SC16, DD20a], the results of which are presented in Section 5.2. The analogous problem has been addressed to in [CMPSC14, BCSC15, ABC16a, ABC16b, DD20b] for the case when ϕ is an intrinsic Lipschitz continuous map. Exploiting techniques similar to the ones used in [ASCV06] and in [BSC10b], we generalize various results of Ambrosio, Serra Cassano and Vittone, that were proved for a uniformly intrinsically differentiable function $\phi : U \to \mathbb{V}$, where $U \subset \mathbb{W}$ is a relatively open set and $\mathbb{V}$ is a horizontal homogeneous subgroup of dimension 1, orthogonal and complementary to $\mathbb{W}$. In particular, we extend various outcomes of [ASCV06] and [SC16] to the case when the target homogeneous subgroup $\mathbb{V}$ of ϕ is still horizontal and orthogonal to $\mathbb{W}$, but of dimension k with $1 \leq k \leq n$. Notice that any horizontal homogeneous subgroup of $\mathbb{H}^n$ is necessarily of dimension between 1 and n. We start by setting suitable coordinates, in Section 5.1: we fix an orthonormal basis $\mathcal{B}$ of $\mathbb{H}^n$ and we identify $\mathbb{W}$ with $\mathbb{R}^{2n+1-k}$ and $\mathbb{V}$ with $\mathbb{R}^k$ through the two bases $\mathcal{B} \cap \mathbb{W}$ and $\mathcal{B} \cap \mathbb{V}$. The map ϕ is then suitably identified in coordinates with a function, here again denoted by ϕ, that acts from an open subset of $\mathbb{R}^{2n+1-k}$ to $\mathbb{R}^k$. Any intrinsic linear map $L : \mathbb{W} \to \mathbb{V}$ can be identified with a $k \times (2n-k)$ real matrix $M_L \in M_{k,2n-k}(\mathbb{R})$. Therefore, if a map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable at a point $w \in U$ we can consider the matrix associated in this way with the intrinsic differential $d\phi_w$ of ϕ at w, that is $D^\phi \phi(w) := M_{d\phi_w}$, and we call it the intrinsic Jacobian matrix of ϕ at w. If a map ϕ is C^1 (in the Euclidean sense), then it is uniformly intrinsically differentiable on U and for every $w \in U$,

$$D^{\phi}\phi(w) = \begin{bmatrix} W_1^{\phi}\phi_1(w) & \dots & W_{2n-k}^{\phi}\phi_1(w) \\ \dots & \dots & \dots \\ W_1^{\phi}\phi_k(w) & \dots & W_{2n-k}^{\phi}\phi_k(w) \end{bmatrix},$$

where $\{W_j^{\phi}\}_{j=1,\dots,2n-k}$ is a family of $2n-k$ vector fields on $\mathbb{R}^{2n+1-k}$ (Definition 5.1.14). The first and the last $n-k$ vector fields of this family are smooth and they are simply the corresponding smooth vector fields of the basis of Lie($\mathbb{W}$) suitably read in coordinates. The k central ones, instead, are nonlinear vector fields, whose coefficients depend on ϕ, and whose behaviour reminds to the Burger's operator. The main results of Chapter 5 are presented in Section 5.3 (precisely in Proposition 5.3.21 and in Theorem 5.3.24) and they can be summarized in the following characterization. Let $\mathbb{W}$, $\mathbb{V}$ be two orthogonal complementary homogeneous subgroups of $\mathbb{H}^n$, with $\mathbb{V}$ horizontal subgroup of dimension k, with $1 \le k \le n$. Let $U \subset \mathbb{W}$ be a relatively open set, let $\phi : U \to \mathbb{V}$ be a function and set $\Sigma = \text{graph}(\phi)$, then the following conditions are equivalent.

(i) ϕ is uniformly intrinsically differentiable on U.

(ii) There exist an open set Ω of $\mathbb{H}^n$ and a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ such that $\Sigma = \{x \in \Omega : f(x) = 0\}$ and $Df(x)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k$ is a homogeneous group isomorphism for every $x \in \Sigma$.

(iii) $\phi \in C^0(U)$ and, for every $a \in U$ and $j \in \{1, \dots, 2n-k\}$, there exist $\partial^{\phi_j}\phi(a) \in \mathbb{R}^k$, i.e. a k-dimensional vector of real numbers $\begin{pmatrix}\alpha_{1,j} & \dots & \alpha_{k,j}\end{pmatrix} \in \mathbb{R}^k$ such that for every integral curve $\gamma^j : (-\delta, \delta) \to \Omega$ of W_j^{ϕ} such that $\gamma^j(0) = a$, the limit $\lim_{t\to 0} \frac{\phi(\gamma^j(t)) - \phi(a)}{t}$ exists and it is equal to $\begin{pmatrix}\alpha_{1,j} & \dots & \alpha_{k,j}\end{pmatrix}^T$ and for every $j = 1, \dots 2n-k$ the map

$$\partial^{\phi_j}\phi : U \to \mathbb{R}^k,$$

is continuous. For every $a \in U$, we call the elements $\partial^{\phi_j}\phi_i(a)$, for $i = 1, \dots, k$, of the vectors $\partial^{\phi_j}\phi(a)$, for $j = 1, \dots, 2n-k$, the intrinsic partial derivatives of ϕ at a.

(iv) ϕ is intrinsically differentiable on U and the function $D^{\phi}\phi : U \to M_{k,2n-k}(\mathbb{R})$ is continuous.

(v) For every point $a \in U$, there exist a positive number $\delta > 0$, a family of maps $\{\phi_\epsilon\}_{\epsilon>0} \subset C^1(I_\delta(a), \mathbb{R}^k)$, where $I_\delta(a) = \{x \in \mathbb{R}^{2n+1-k} : |a_i - x_i| < \delta \text{ for } i = 1, \dots, 2n+1-k\}$, and a continuous matrix-valued function $M : I_\delta(a) \to M_{k,2n-k}(\mathbb{R})$ such that $\phi_\epsilon \to \phi$ and $D^{\phi_\epsilon}\phi_\epsilon \to M$ uniformly on $I_\delta(a)$ as $\varepsilon \to 0$.

In this direction of research some studies have been carried out after [Cor19]. We dedicate Section 5.4 to a brief summary of the recent results available in the literature, presented in [ADDDLD20, ADDD20]. The last section of Chapter 5, Section 5.5, is devoted to the proof of the first area formula presented in this thesis. We consider on $\mathbb{H}^n$ the $(2n+2-k)$-centered Hausdorff measure $\mathcal{C}_\infty^{2n+2-k}$, built with respect to the homogeneous distance d_∞, introduced in [FSSC07]. We take in consideration, as before, a uniformly intrinsically differentiable map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ with $\mathbb{W}$ and $\mathbb{V}$ orthogonal complementary subgroups of $\mathbb{H}^n$, $\mathbb{V}$ horizontal subgroup of dimension k, with $1 \le k \le n$, and $U \subset \mathbb{W}$ relatively open set, and we set $\Sigma = \text{graph}(\phi)$. Combining the characterization above with results in [FSSC07] and [FSSC15], we obtain an area formula for the measure $\mathcal{C}_\infty^{2n+2-k} \llcorner \Sigma$ in terms of the intrinsic derivatives of the components of ϕ along the integral curves of the vector fields W^{ϕ_j}, namely in terms of the intrinsic derivatives of ϕ on U (Theorem 5.5.5). For every Borel set $B \subset \Sigma$

$$\mathcal{C}_{\infty}^{2n+2-k}(\Sigma \cap B) = \int_{\Phi^{-1}(B)} J^{\phi}\phi(w)\ d\mathcal{H}_E^{2n+1-k}(w) \tag{2}$$

where, for every $w \in U$, $J^{\phi}\phi(w)$ is the intrinsic Jacobian of ϕ (Definition 5.5.3) at w defined as

$$J^{\phi}\phi(w) = \sqrt{1 + \sum_{\ell=1}^{k} \sum_{I \in \mathcal{I}_\ell} (M_I^{\phi}(w))^2},$$

where for every $\ell \in \{1, \dots, k\}$, $\mathcal{I}_\ell$ is the set of multi-indexes

$$\{(i_1, \dots, i_\ell, j_1, \dots, j_\ell) \in \mathbb{N}^{2\ell} : 1 \le i_1 < i_2 < \cdots < i_\ell \le 2n-k,\ 1 \le j_1 < j_2 \cdots < j_\ell \le k\}$$

and for $I = (i_1, \dots, i_\ell, j_1, \dots, j_\ell) \in \mathcal{I}_\ell$

$$M_I^{\phi}(w) = \det \begin{pmatrix} \partial^{\phi_{i_1}}\phi_{j_1}(w) & \dots & \partial^{\phi_{i_\ell}}\phi_{j_1}(w) \\ \dots & \dots & \dots \\ \partial^{\phi_{i_1}}\phi_{j_\ell}(w) & \dots & \partial^{\phi_{i_\ell}}\phi_{j_\ell}(w) \end{pmatrix}.$$

In Chapter 6 we present the results of [CM20], obtained in collaboration with Prof. V. Magnani of the University of Pisa. The main aim of the chapter is to present some area formulas for the spherical Hausdorff measure of a regularly parametrized $\mathbb{H}$-regular surface of low codimension, namely of a uniformly intrinsically differentiable graph of low codimension in $\mathbb{H}^n$. In Sections 6.1 and 6.2 we introduce some preliminary notions, among which the concept of extrinsic differentiability for a map $f : \mathbb{W} \to \mathbb{R}^k$, where $\mathbb{W}$ is a normal homogeneous subgroup of $\mathbb{H}^n$. From a formal point of view, this notion has been obtained through a slight modification of the notion of intrinsic differentiability, nevertheless it allows to prove a suitable chain rule (Theorem 6.2.2), that makes extrinsic differentiability a bridge between the two notions of Pansu and intrinsic differentiability. In Section 6.3 we prove the core of the area formulas (Theorem 6.3.4). We consider a uniformly intrinsically differentiable map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ acting between two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$ of $\mathbb{H}^n$, with $\mathbb{V}$ horizontal subgroup of dimension k, $1 \le k \le n$, and $U \subset \mathbb{W}$ relatively open. The intrinsic graph of ϕ, $\Sigma = \text{graph}(\phi)$, can always be written as a level set of a map $f \in C_h^1(\Omega, \mathbb{R}^k)$, with $\Omega \subset \mathbb{H}^n$ open set, such that for every $y \in \Sigma$, $Df(y)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k$ is a homogeneous group isomorphism, and consequently $J_{\mathbb{V}} f(y) := J(Df|_{\mathbb{V}}(y)) > 0$. Let us fix an orthonormal basis $(v_1, \dots v_k)$ of $\mathbb{V}$ and an orthonormal basis $(w_{k+1}, \dots, w_{2n}, e_{2n+1})$ of $\mathbb{W}$, and let us set $V = v_1 \wedge \cdots \wedge v_k$ and $W = w_{k+1} \wedge \cdots \wedge w_{2n} \wedge e_{2n+1}$. We introduce a measure μ associated to Σ on $\mathbb{H}^n$: for every Borel set $B \subset \mathbb{H}^n$

$$\mu(B) := \|V \wedge W\|_g \int_{\Phi^{-1}(B)} \frac{J_H f(\Phi(w))}{J_{\mathbb{V}} f(\Phi(w))}\ d\mathcal{H}_E^{2n+1-k}(w),$$

where $\|\cdot\|_g$ is the Riemannian norm associated with the metric g on the multivectors of $\mathbb{H}^n$ and for every $x \in \Omega$

$$J_H f(x) := J(Df(x)) = \|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g,$$

where $\nabla_H f_i(x)$ denotes the horizontal gradient of the i-th component of f at x. We compute the $(2n+2-k)$-Federer density of the measure μ at any fixed point of the surface $x \in \Sigma$, $\theta^{2n+2-k}(\mu, x)$, with respect to an arbitrary homogeneous distance d on $\mathbb{H}^n$. We prove that it is equal to the spherical factor $\beta_d(\mathrm{Tan}(\Sigma, x))$ of the intrinsic subgroup tangent to the surface at the point x, that is

$$\theta^{2n+2-k}(\mu, x) = \beta_d(\mathrm{Tan}(\Sigma, x)) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(z, 1)),$$

where $\mathbb{B}(z, r)$ denotes the metric closed ball centered at z of radius r with respect to the distance d. The proof of this "upper blow-up" theorem is preceded by a delicate lemma about the Jacobian of a restricted group projection related to two semidirect splittings of the Heisenberg group $\mathbb{H}^n$, $\mathbb{W} \rtimes \mathbb{V}$ and $\mathbb{M} \rtimes \mathbb{V}$ sharing a common horizontal homogeneous subgroup $\mathbb{V}$. Slightly simplifying the computation of the Federer density of μ on Σ, we compute the value of the $(2n+2-k)$-centered density of the measure μ at any fixed point $x \in \Sigma$, $\theta_c^{2n+2-k}(\mu, x)$, proving that

$$\theta_c^{2n+2-k}(\mu, x) = \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(0, 1)).$$

Finally, Section 6.4 contains a series of area formulas for Σ. In fact, combining the value of the Federer density of μ on Σ with a result of [Mag15], we deduce that (Theorem 6.4.1), for every homogeneous distance d on $\mathbb{H}^n$, for every measurable set $B \subset \Sigma$

$$\mu(B) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x)) \, d\mathcal{S}^{2k+2-k}(x). \tag{3}$$

If, in addition, the distance d preserves some symmetries, namely d is $(2n+1-k)$-vertically symmetric (Definition 6.1.2) or multiradial (Definition 6.1.5), the formula simplifies. In this case, by suitable results of [Mag18, Mag20], the spherical factor $\beta_d(\mathrm{Tan}(\mu, x))$ is a geometric constant $\omega_d(2n+1-k)$ that depends only on the distance d and on the topological dimension of $\mathrm{Tan}(\Sigma, x)$, that in our case equals $2n+1-k$ for every $x \in \Sigma$. Normalizing the spherical Hausdorff measure as $\mathcal{S}_d^{2n+2-k} := \omega_d(2n+1-k)\mathcal{S}^{2n+2-k}$, formula (3) simplifies to

$$\mathcal{S}_d^{2n+2-k} \llcorner \Sigma = \mu \llcorner \Sigma.$$

Moreover, if we assume $\mathbb{W}$ and $\mathbb{V}$ to be orthogonal, it is possible, adopting suitable coordinates, to rewrite the measure μ uniquely in terms of the intrinsic derivatives of the parametrization ϕ (Theorem 6.4.4), as we did in the proof of (2). Actually, in this case, if the distance d is $(2n+1-k)$-vertically symmetric or multiradial, the spherical Hausdorff measure of Σ can be computed, for any Borel set $B \subset \Sigma$, as

$$\mathcal{S}_d^{2n+2-k}(B) = \int_{\Phi^{-1}(B)} J^\phi \phi(w) \, d\mathcal{H}_E^{2n+1-k}(w).$$

Exploiting an analogous path, combining the value of the centered density of the measure μ on Σ with a result of [FSSC15], we obtain an area formula that joins μ and the $(2n+2-k)$-centered Hausdorff measure of Σ, with respect to any homogeneous distance d (Theorem 6.4.5): for every Borel set $B \subset \Sigma$

$$\mu(B) = \int_B \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(0, 1)) \, d\mathcal{C}^{2k+2-k}(x).$$

In addition, we observe that for every homogeneous distance whose unit metric ball $\mathbb{B}(0, 1)$ is convex

$$\mathcal{S}^{2n+2-k} \llcorner \Sigma = \mathcal{C}^{2n+2-k} \llcorner \Sigma.$$

The last chapter of the thesis, Chapter 7, is devoted to the proof of a coarea-type inequality for a class of continuously Pansu differentiable mappings with everywhere surjective differential acting between two Carnot groups, presented in [Cor20]. We started to deepen this theme during a visiting period at Université Paris-Sud, under the supervision of Prof. P. Pansu. The validity of the coarea formula for Lipschitz maps between two arbitrary Carnot groups, endowed with homogeneous distances, is an open problem in the context of geometric measure theory in Carnot groups. In this setting, the more general available result, besides a general coarea estimate for Lipschitz maps between metric spaces, due to [Fed69], is a coarea-type inequality, proved in [Mag02b]. In particular, if $\mathbb{G}$ and $\mathbb{M}$ are two Carnot groups, of Hausdorff dimension Q and P and topological dimension q and p, respectively, endowed with homogeneous distances, if $A \subset \mathbb{G}$ is a measurable set and $f : A \to \mathbb{M}$ is a Lipschitz map, by [Mag02b],

$$\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap A) d\mathcal{S}^P(m) \leq \int_A C_P(Df(x)) d\mathcal{S}^Q(x), \tag{4}$$

where $C_P(Df(x))$ is the coarea factor of $Df(x)$ (Definition 7.2.1) that plays the role of the Jacobian of the Pansu differential and incorporates the contribute of the homogeneous distances fixed on $\mathbb{G}$ and $\mathbb{M}$. In this context, the main challenge consists of replacing inequality (4) by an equality. After collecting, in Sections 7.1 and 7.2, some preliminary notions about packing measures, coarea factor and local Ahlfors regularity, in Section 7.3 we prove the following coarea-type inequality (Theorem 7.3.4). Let $f \in C_h^1(\mathbb{G}, \mathbb{M})$ be a function and assume that $Df(x)$ is surjective at every $x \in \mathbb{G}$. Assume that there exist two constants $\tilde{r}, C > 0$ such that for $\mathcal{S}^P$-a.e. $m \in \mathbb{M}$ the level set $f^{-1}(m)$ is $\tilde{r}$-locally C-lower Ahlfors $(Q-P)$-regular with respect to $\mathcal{S}^{Q-P}$, i.e. that for every $0 < r < \tilde{r}$ and $x \in f^{-1}(m)$, $\mathcal{S}^{Q-P}(f^{-1}(m) \cap \mathbb{B}(x, r)) \geq Cr^{Q-P}$. Then there exists a constant $L = L(C, \mathbb{G}, p)$ such that for every measurable set $A \subset \mathbb{G}$,

$$\int_A C_P(Df(x)) d\mathcal{S}^Q(x) \leq L \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap A) d\mathcal{S}^P(m). \tag{5}$$

The proof of this inequality has been inspired by an abstract procedure presented in [Pan20], where it is used to prove a coarea inequality for maps acting from a metric space to a measure space, for packing-type measures. In Section 7.4, as an application of inequality (5), we propose some new results about the slicing of a subset of $\mathbb{G}$ by the level sets of a map f for which (5) holds. In addition, we compare our result with some available related ones presented in [Koz15] and [JNGV20], stressing that the assumption about the uniform local lower Ahlfors regularity of the level sets of the map f is not pointless and that it can be meant as a substitute of the existence of a suitable splitting of $\mathbb{G}$.

Contents

Chapter 1

Preliminary notions

The aim of this chapter is to introduce, in a synthetic way, some useful definitions and results needed to understand the setting in which the contributions of the thesis take place. In particular we collect some information about vector fields, Lie algebras, Lie groups and sub-Riemannian manifolds. One can refer to any book of differential geometry. Good introductions, aimed at subsequent topics, are also provided by the first chapters of [Ric03] and [BLU07].

1.1 Vector fields

In this section we recall some classical notions of differential geometry.

Definition 1.1.1. A topological space M is *locally Euclidean of dimension* n if for every point $p \in M$ there is a neighbourhood U of p such that there exists a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$, $\phi : U \to \mathbb{R}^n$. We call the pair (U, ϕ) a *chart* or a *coordinate chart* (at p).

Definition 1.1.2. A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Definition 1.1.3. Two charts (U, ϕ), (V, ψ) of a topological manifold M are C^∞*-compatible* if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \quad \text{and} \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are smooth.

Definition 1.1.4. Given a locally Euclidean space M, an *atlas* of M is a collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e. such that $M = \bigcup_\alpha U_\alpha$.

In particular, an atlas is said to be *maximal* if it is not contained in a larger atlas, in other words, if $\mathcal{A}'$ is any other atlas of M containing $\mathcal{A}$, then $\mathcal{A}' = \mathcal{A}$.

Definition 1.1.5. A *smooth manifold* is a topological manifold M endowed with a maximal atlas.

Definition 1.1.6. If M and N are smooth manifolds, of dimension m and n, respectively, a map $f : M \to N$ is smooth if, for every chart (U, ϕ) of M and (V, ψ) of N, the map

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \mathbb{R}^n$$

is smooth.

The map f is a *diffeomorphism* if it is a bijective smooth map whose inverse f^{-1} is smooth.

Let us fix a smooth manifold M of dimension n. We recall the notion of smooth vector bundle over M. Roughly speaking it is a smoothly varying family of vector spaces, parametrized by M.

Definition 1.1.7. Given any map $\pi : E \to M$, for every point $p \in M$ the *fiber* of E at p is the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$. The fiber of E at p is often denoted by E_p. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M a map $\phi : E \to E'$ is said to be *fiber-preserving* if $\phi(E_p) = E'_p$ for all $p \in M$.

Definition 1.1.8. A surjective smooth map $\phi : E \to M$ between two manifolds is said to be *locally trivial of rank* r if

(i) each fiber $\pi^{-1}(p)$ has the structure of a vector space of dimension r;

(ii) for each $p \in M$, there are an open neighbourhood $U \subset M$ of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \times \mathbb{R}^k$ such that for every $q \in U$ the restriction $\phi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$ is a vector space isomorphism.

Definition 1.1.9. A *smooth vector bundle* of rank r is a triple (E, M, π) consisting of two manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. By a slight abuse of language, we say that E is a vector bundle over M.

Definition 1.1.10. Let E be a vector bundle over M. A *(smooth) subbundle* of E is a collection $\{\Delta_p\}_{p\in M}$ such that for every $p \in M$, Δ_p is a linear subspace of the vector fiber E_p of E such that $\sqcup_{p\in M}\Delta_p$ is a smooth vector bundle over M, and we denote it by Δ.

Fix now a point $p \in M$ and consider the set of all open neighbourhoods of p in M

$$\mathfrak{U}_p =: \{U \subset M : U \text{ open set},\ p \in U\}.$$

Definition 1.1.11. Let p be a point of M and consider two open sets $U,\ V \in \mathfrak{U}_p$. If we consider two functions $f \in C^\infty(U),\ g \in C^\infty(V)$, we say that $f \sim_p g$ if and only if there exists $W \in \mathfrak{U}_p$ such that $f|_W = g|_W$.

Notice that $\sim_p$ is an equivalence relation on the union of the families of functions belonging to $C^\infty(U)$, for U ranging over the elements of $\mathfrak{U}_p$. We denote the quotient of this union with respect to the relation $\sim_p$ by $C^\infty_p(M)$. This object has a natural structure of vector space. The equivalence class of a function f with respect to $\sim_p$, $[f]_{\sim_p} \in C^\infty_p(M)$, will be denoted by f. Essentially then, by f we denote the class of those functions that coincide close to p.

Definition 1.1.12. Let $p \in M$, we say that a function $D : C^\infty_p(M) \to \mathbb{R}$ is a *derivation* at p if it is $\mathbb{R}$-linear and satisfies the Leibniz rule, i.e. for all $f, g \in C^\infty_p(M)$,

$$D(fg) = D(f)g(p) + f(p)D(g).$$

Definition 1.1.13. The set of derivations at p is called the *tangent space* of M at p

$$T_pM := \{D : C^\infty_p(M) \to \mathbb{R} : D \text{ is a derivation at } p\}. \tag{1.1}$$

For each $p \in M$, T_pM is a vector space over $\mathbb{R}$. Let us briefly recall how the choice of a chart (U, ϕ) at a point p of the manifold gives in a standard way a local system of coordinates on M close to p. Let us denote by r_i the usual i-th coordinate map on

$\mathbb{R}^n$, $r_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, \ldots, n$, then we denote the maps $x_i := r_i \circ \phi : U \to \mathbb{R}$, for $i = 1, \ldots, n$, and we introduce for every $f \in C_p^\infty(M)$

$$\frac{\partial}{\partial x_i}\Big|_p f := \frac{\partial}{\partial r_i}\Big|_{\phi(p)} (f \circ \phi^{-1}).$$

For every $i = 1, \ldots, n$, $\frac{\partial}{\partial x_i}\big|_p$ is a derivation at p. In addition, $\left(\frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_n}\big|_p\right)$ is a basis of T_pM.

Definition 1.1.14. A *curve* on M is a continuous function $c : (a, b) \to M$, with $(a, b) \subset \mathbb{R}$. One usually requires that $0 \in (a, b)$ and in this case the point $p = c(0)$ is called the *initial point*, or *starting point*, of the curve c.

Let (U, ϕ) be a coordinate chart at some point $p \in M$ and let $a < 0 < b$. Let $c : (a, b) \to U$ be a smooth curve and fix $t \in (a, b)$. The *velocity vector* of c at t, denoted by $c'(t)$, is a derivation at $c(t)$ defined as follows

$$c'(t) := \sum_{i=1}^{n} (x_i \circ c)'(t) \frac{\partial}{\partial x_i}\Big|_{c(t)} = \sum_{i=1}^{n} y_i'(t) \frac{\partial}{\partial x_i}\Big|_{c(t)},$$

where $y_i := x_i \circ c = r_i \circ \phi \circ c : (a, b) \to \mathbb{R}$ for $i = 1, \ldots, n$.
There is a one-to-one correspondence between the smooth curves on M of initial point p and the derivations at p, namely the tangent vectors of M at p. The correspondence is realized associating with any derivation $D \in T_pM$, a smooth curve $c : (-\varepsilon, \varepsilon) \to \mathbb{R}$, for some $\varepsilon > 0$, of initial point p such that $c'(0) = D$. It is enough to choose a curve c of initial point p such that $Df = \frac{d}{dt}\big|_{t=0} f(c(t))$ for $f \in C_p^\infty(M)$. Notice that a similar c is not unique.

Definition 1.1.15. The disjoint union of the tangent spaces of M is a bundle called the *tangent bundle* of M and it is denoted by TM

$$TM := \coprod_{p \in M} T_pM.$$

We denote by $\pi_{TM} : TM \to M$ the canonical projection from the tangent bundle to the manifold

$$\pi_{TM} : TM \to M, \; \pi_{TM}(v) = p \quad \text{if } v \in T_pM.$$

Remark 1.1.16. We defined TM as a set. Actually, one endows TM with a structure of smooth manifold that makes it a smooth vector bundle over M. Since we will not need more details, we do not report here the whole explicit construction, the reader can refer for example to [Tu11, Sections 12.1, 12.2].

Definition 1.1.17. A subbundle Δ of the tangent bundle TM is also called a *distribution of subspaces* on M.

Definition 1.1.18. Let M and N be two smooth manifolds. Consider a smooth map $F : M \to N$ and a point $p \in M$. The *differential* of F at p is the linear map

$$d_E F(p) : T_pM \to T_{F(p)}N$$

such that for every $D \in T_pM$ and $f \in C_{F(p)}^\infty(N)$

$$d_E F(p)(D)(f) = D(f \circ F).$$

Remark 1.1.19. A small remark about our choice of notation: $d_E F(p)$ stands for *Euclidean* differential of F at p. Someone might argue that this writing is unusual or superabundant, but in the next chapters many different notions of differentiability will be introduced, so we reserve simpler notation for the more recent concepts that will be some of the main characters of the thesis.

Definition 1.1.20. A *vector field* on M is a map $X : M \to TM$ such that $\pi_{TM} \circ X = Id_M$, i.e. such that

$$X(p) \in T_pM \quad \text{for every } p \in M.$$

We denote the set of vector fields on M by $\mathrm{Vect}(M)$.

Remark 1.1.21. In the literature the name "vector field" is sometimes reserved for the objects that here will be called smooth vector fields. We do not make any preliminary assumption on the regularity of vector fields. The choice of this line of presentation is motivated by the fact that, in the next chapters, a family of vector fields whose coefficients are only continuous will play a crucial role to state our results.

If we fix a chart (U, ϕ) of M at a point p_0, then a vector field X can be locally written in coordinates, i.e. we can consider for every $p \in U$

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}\Big|_p,$$

where $a_i(p)$, $i = 1, \dots, n$, are suitable real numbers. The maps $a_i : U \to \mathbb{R}$ for $i = 1, \dots, n$ are called the *coefficients* of X. Then, if we denote by $\tilde{X}$ the vector field X considered with respect to a variable varying on the space of parameters $\phi(U) \subset \mathbb{R}^n$, i.e. $\tilde{X} = X \circ \phi^{-1} : \phi(U) \to TM$, we obtain for every $p \in U$

$$\begin{aligned}
\tilde{X}(\phi(p)) = (X \circ \phi^{-1})(\phi(p)) = X(p) &= \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}\Big|_p \\
= \sum_{i=1}^{n} (a_i \circ \phi^{-1})(\phi(p)) \frac{\partial}{\partial x_i}\Big|_p &= \sum_{i=1}^{n} b_i(\phi(p)) \frac{\partial}{\partial x_i}\Big|_p,
\end{aligned}$$

where we have introduced the maps $b_i := a_i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ for $i = 1, \dots, n$. In particular, the maps b_i are the coefficients of X read in the system of coordinates given by the fixed coordinate chart.

Definition 1.1.22. The regularity of a vector field $X \in \mathrm{Vect}(M)$ is the regularity of its coefficients a_i for $i = 1, \dots, n$, i.e. it is the Euclidean regularity of the maps b_i for $i = 1, \dots, n$.

More explicitly, if the coefficients a_i are smooth, we call X a *smooth vector field*, and then X is what in the literature is called a smooth section of the tangent bundle. If the coefficients a_i are continuous we call X a *continuous vector field*, and so on. We denote by $\mathrm{Vect}^\infty(M)$ the set of smooth vector fields on M.

Remark 1.1.23. There exists an alternative interpretation of the space $\mathrm{Vect}^\infty(M)$. If $X \in \mathrm{Vect}^\infty(M)$, then X can be seen as an $\mathbb{R}$-linear function

$$X : C^\infty(M) \to C^\infty(M)$$

that satisfies the Liebniz rule, i.e. for every $f, g \in C^\infty(M)$

$$X(fg) = X(f)g + fX(g).$$

More precisely, if we fix a coordinate chart (U, ϕ), a vector field $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \in \mathrm{Vect}^\infty(M)$ and a map $f \in C^\infty(M)$, Xf is the smooth map on M

$$Xf(q) = \sum_{i=1}^q a_i(q) \frac{\partial f}{\partial x_i}\Big|_q \quad \text{for every } q \in U.$$

Definition 1.1.24. Let $X \in \mathrm{Vect}(M)$ and let $p_0 \in M$, an *integral curve of X starting at p_0* is an everywhere differentiable curve $c : (a, b) \to M$, with $a < 0 < b$, that is a solution of the following Cauchy problem

$$\begin{cases} c'(t) = X(c(t)) & t \in (a, b) \\ c(0) = p_0 \end{cases} \tag{1.2}$$

If c is a solution of (1.2), we denote c by $\gamma_{p_0}^X$. An integral curve $c : (a, b) \to M$ is said *maximal* if it does not exists any real open interval I such that $(a, b) \subset I$ and such that there exists an integral curve of X starting at p_0 defined on I.

Remark 1.1.25. Notice that $\gamma_{p_0}^X$ can be non-unique.

Consider a vector field $X \in \mathrm{Vect}(M)$ and let us fix a coordinate chart (U, ϕ) at a point $p_0 \in M$ such that $\phi(p_0) = 0$ (clearly, this assumption is not restrictive). We read X in local coordinates as above $\tilde{X} : \phi(U) \to TM$, $\tilde{X}(\phi(p)) = \sum_{i=1}^n b_i(\phi(p)) \frac{\partial}{\partial x_i}\big|_p$. If we consider a curve $c : (a, b) \to M$, we can read also c in coordinates as follows

$$\phi \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, \ \phi \circ c(t) = (y_1(t), \ldots, y_n(t)) := y(t).$$

Then we can restate the condition given by the system (1.2) in coordinates: we can write for every $t \in (a, b)$

$$c'(t) = \sum_{i=1}^n y_i'(t) \frac{\partial}{\partial x_i}\Big|_{c(t)}$$

$$X(c(t)) = \tilde{X}(\phi \circ c(t)) = \sum_{i=1}^n b_i(\phi \circ c(t)) \frac{\partial}{\partial x_i}\Big|_{c(t)} = \sum_{i=1}^n b_i(y(t)) \frac{\partial}{\partial x_i}\Big|_{c(t)}.$$

Therefore, since $\{\frac{\partial}{\partial x_i}\big|_p\}_{i=1,\ldots,n}$ is a basis of T_pM for every $p \in M$, we can determine the solutions of the Cauchy problem (1.2) by solving the following system of ODEs

$$\begin{cases} y_i'(t) = b_i(y(t)) & t \in (a, b), \quad \text{for } i = 1, \ldots, n \\ y_i(0) = 0 & \text{for } i = 1, \ldots, n. \end{cases} \tag{1.3}$$

Thanks to the formulation in (1.3), the classical theory of ODEs provides some information about the existence and uniqueness of the integral curves of the vector field X. We highlight some of this information that will be needed later on. First of all, by the Cauchy-Lipschitz Theorem about the existence and uniqueness of the solutions of ODEs, it is possible to state some facts. If we assume $X \in \mathrm{Vect}(M)$ to be smooth (or even just locally Lipschitz), once fixed a starting point $p_0 \in M$, there exists a unique maximal integral curve of X starting at p_0, $\gamma_{p_0}^X$, that is a solution of (1.2) and is defined on a maximal interval I_{p_0} containing 0. Moreover, for every compact subset $K \subset U$, there exists a positive $\varepsilon_K > 0$ such that $\gamma_{p_0}^X$ is defined for every $|t| < \varepsilon_K$ for every $p_0 \in K$ (i.e $(-\varepsilon_K, \varepsilon_K) \subset I_{p_0}$ for every $p_0 \in K$). Moreover we know that $\gamma_{p_0}^X$ is continuous with respect both to $p_0 \in U$ and to $X \in \mathrm{Vect}^\infty(M)$. On the other side, by the Peano's existence Theorem about the solutions of ODEs, we can deduce some information about the existence of integral curves also for the case when the vector field X is less regular than locally Lipschitz. In fact, if we assume the vector field X to be continuous, then, once fixed an initial point p_0, the existence of an integral curve $\gamma_{p_0}^X$ solution of (1.2) is still ensured, nevertheless its uniqueness is not guaranteed any more. We refer the reader to [Mus05, Theorem 1], where some useful precise information about the length of the intervals on which the integral curves are defined is also collected.

Definition 1.1.26. Let us consider two vector fields $X, Y \in \mathrm{Vect}^\infty(M)$. The *commutator* between X and Y is the smooth vector field $[X, Y]$ obtained as follows: for every $f \in C^\infty(M)$

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Hence, for every chart (U, ϕ) of M, if $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}$, with coefficients $a_i, b_i : U \to \mathbb{R}$, $i = 1, \dots, n$.

$$[X, Y] := XY - YX = \sum_{j=1}^n \left(\sum_{k=1}^n \left(a_k \frac{\partial}{\partial x_k} b_j - b_k \frac{\partial}{\partial x_k} a_j \right) \right) \frac{\partial}{\partial x_j}.$$

Definition 1.1.27. Let M and N be smooth manifolds. Let $F : M \to N$ be a smooth map and let X, Y be smooth vector fields respectively on M and N. The vector fields X and Y are said *F-related* if, for every $p \in M$,

$$d_E F(p)(X(p)) = Y(F(p)).$$

Lemma 1.1.28. *[BLU07, Lemma 2.1.37] Let M, N be smooth manifolds and let $F : M \to N$ be a smooth map. Let X_1, Y_1 be two smooth F-related vector fields and let X_2, Y_2 be two smooth F-related vector fields, then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related.*

1.2 Lie algebras and Lie groups

Definition 1.2.1. A *Lie algebra* $\mathfrak{a}$ (over $\mathbb{R}$) is a (real) vector space endowed with a bilinear map, called *commutator* or *Lie bracket*

$$[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$$

that satisfies the following properties

(i) antisymmetry: $[a, b] = -[b, a]$, for every $a, b \in \mathfrak{a}$;

(ii) Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$, for every $a, b, c \in \mathfrak{a}$.

Definition 1.2.2. A *subalgebra* of $\mathfrak{a}$ is a linear subspace $\mathfrak{s} \subset \mathfrak{a}$ closed with respect to the Lie bracket, i.e. such that for all $a, b \in \mathfrak{s}$, $[a, b] \in \mathfrak{s}$.

Given a set of elements $a_1, \dots, a_k \in \mathfrak{a}$, the smallest subalgebra of $\mathfrak{a}$ containing $a_1, \dots, a_k$ is called the *Lie subalgebra generated by* $a_1, \dots, a_k$. We denote it by $\mathfrak{s}_{\mathfrak{a}}(a_1, \dots, a_k)$.

We collect below some classical definitions about Lie algebras. Let $\mathfrak{a}$ be a Lie algebra. Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathfrak{a}$ we introduce $[\mathcal{A}, \mathcal{B}] := \{[a, b] : a \in \mathcal{A},\ b \in \mathcal{B}\}$.

(i) $\mathfrak{a}$ is *abelian* if $[a, b] = 0$ for every $a, b \in \mathfrak{a}$.

(ii) $\mathfrak{a}$ is *nilpotent* if there is a natural number κ such that $[\dots[[a_1, a_2], a_3], \dots], a_\kappa] = 0$ for every $a_1, \dots, a_\kappa \in \mathfrak{a}$. The minimum number κ for which this property is satisfied is called *the step of* $\mathfrak{a}$.

(iii) The *center* of $\mathfrak{a}$ is the set of elements $a \in \mathfrak{a}$ such that $[a, b] = 0$ for every $b \in \mathfrak{a}$.

(iv) $\mathfrak{a}$ is *graded* if it is the direct sum of a finite number of linear subspaces, $\mathfrak{a} = V_1 \oplus \dots \oplus V_\kappa$ such that $[V_i, V_j] \subset V_{i+j}$ if $i + j \leq \kappa$ and $[V_i, V_j] = 0$, otherwise.

(v) $\mathfrak{a}$ is *stratified* if it is graded, thus $\mathfrak{a} = V_1 \oplus \dots \oplus V_\kappa$, and the first layer V_1 generates the other layers by Lie bracket, i.e. if

$$V_{j+1} = [V_1, V_j]$$

for every integer $j \geq 1$, and $[V_1, V_\kappa] = 0$.

Definition 1.2.3. A *Lie group* $(G, \cdot)$ is a smooth manifold endowed with a group operation $\cdot$ such that the product map $\cdot : G \times G \to G,\ (x, y) \to x \cdot y$ and the inverse map with respect to the product $i : G \to G,\ i(x) = x^{-1}$ are smooth.

We denote by e the identity element of G with respect to the group product.

Definition 1.2.4. A *(Lie) subgroup* $H \subset G$ is a subgroup such that, if we consider the inclusion $J : H \hookrightarrow G$, the differential of J at every $x \in H$, $d_E J(x) : T_x H \to T_x G$, is an injective mapping, and such that the product map and the inverse map of G restricted to M are smooth.

Definition 1.2.5. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras. A *homomorphism of Lie algebras* is a linear application $L : \mathfrak{a} \to \mathfrak{b}$ such that

$$[L(a), L(b)]_{\mathfrak{b}} = L([a, b]_{\mathfrak{a}}) \quad \forall a, b \in \mathfrak{a},$$

where by $[\cdot, \cdot]_{\mathfrak{a}}$ and $[\cdot, \cdot]_{\mathfrak{b}}$ we denote the Lie brackets of $\mathfrak{a}$ and $\mathfrak{b}$, respectively. If L is bijective, it is an *isomorphism of Lie algebras.*

Given two Lie groups with their group operations $(G_1, \star_1)$ and $(G_2, \star_2)$ a *homomorphism of Lie groups* is a smooth map $L : G_1 \to G_2$ such that

$$L(g_1 \star_1 g_2) = L(g_1) \star_2 L(g_2) \qquad \forall g_1, g_2 \in G_1.$$

Moreover L is an *isomorphism* if it is a diffeomorphism and a group isomorphism.

Definition 1.2.6. Given a Lie group G, we naturally associate with any point $x \in G$ the *left translation* denoted by l_x

$$l_x : G \to G,\ l_x(y) = x \cdot y.$$

Notice that l_x is a diffeomorphism for every $x \in G$.

Definition 1.2.7. Let G be a Lie group and let X be a smooth vector field on G, X is said *left invariant* if it is l_x-related with itself for every $x \in G$. Equivalently, if for every $x \in G$,

$$d_E l_x(X) = X,$$

namely if for every $x, y \in G$

$$d_E l_x(y)(X(y)) = X(x \cdot y) = X(l_x(y)).$$

By Lemma 1.1.28, the linear subspace of $\mathrm{Vect}^\infty(G)$ of the left invariant vector fields on a Lie group G is closed with respect to the Lie bracket, hence it is a Lie (sub)algebra. It is called the *Lie algebra of* G and we denote it by $\mathrm{Lie}(G)$. Depending on the situation, we will indicate by $\mathrm{Lie}(G)$ also the vector space underlying the Lie algebra.

Let us recall a standard fact.

Proposition 1.2.8. *[Ric03, Theorem 4.3] When two Lie groups G_1, G_2 are connected and simply connected, they are isomorphic if and only if their Lie algebras are isomorphic.*

We call a simply connected Lie group *nilpotent* if its Lie algebra is nilpotent, *abelian* if its Lie algebra is abelian and, in analogous way, all definitions about Lie algebras are transferred on the corresponding Lie groups.

Definition 1.2.9. Let G be a Lie group. A *left invariant measure* on G is a measure μ such that for every measurable set $B \subset G$ and every $x \in G$

$$\mu(l_x(B)) = \mu(B).$$

Proposition 1.2.10. *For any locally compact topological group G there exists, up to positive scalar multiples, a unique non-zero Radon left invariant measure μ on G. The measure μ is called the Haar measure of G.*

Example 1.2.11. Since any Lie group G is locally compact, if one individuates a non-zero left invariant Radon measure on G, necessarily it is the Haar measure of G.

1.3 Exponential map and homogeneous Lie groups

In this section we consider a Lie group G and we focus on the relation between G and its Lie algebra $\mathrm{Lie}(G)$. The family of left translations $\{l_x\}_{x\in G}$ allows to individuate some canonical one-to-one correspondences between the tangent space T_eG at the unit element e of the group G, the tangent space T_xG at any point $x \in G$ and the Lie algebra $\mathrm{Lie}(G)$ of the left invariant vector fields on G. Let us describe more explicitly these correspondences.

(i) For every $x \in G$, the differential of the left translation l_x at the origin

$$d_E l_x(e) : T_eG \to T_xG$$

is an isomorphism that moves canonically the vectors of T_eG onto the vectors of T_xG.

(ii) By the left invariance of the elements of $\mathrm{Lie}(G)$, the following map is a canonical isomorphism

$$\Psi : T_eG \to \mathrm{Lie}(G),\ \ \Psi(v) = X,$$

where X is the unique smooth left invariant vector field such that $X(e) = v$.
Notice that, by definition, a left invariant vector field is completely determined by the value that it takes at the unit element, hence Ψ is well defined.

The correspondences above allow, when needed, to identify in a natural way T_eG with T_xG or with $\mathrm{Lie}(G)$. Moreover, they allow to equip in a natural way any tangent space

T_xG with a structure of Lie algebra. In particular, the commutator $[v,w]_e$ between two vectors $v,w \in T_eG$ is defined considering the commutator between the corresponding left invariant vector fields in the Lie algebra as

$$[v,w]_e := [\Psi(v), \Psi(w)].$$

Thus, it is immediate to guess the definition of the commutator $[\cdot,\cdot]_x$ on T_xG, that can be naturally introduced through the differential of left translations: for every $v,w \in T_xG$

$$[v,w]_x := [d_E l_{x^{-1}}(x)(v), d_E l_{x^{-1}}(x)(w)]_e.$$

Let us go back for a while to consider some definitions on a generic smooth manifold M of dimension n.

Definition 1.3.1. A vector field $X \in \mathrm{Vect}(M)$ is said to be *complete* if for every $p \in M$ there exists an integral curve of X starting at p defined on the whole $\mathbb{R}$.

Definition 1.3.2. Let X be a smooth complete vector field on M. The *flow* of X is a family of functions

$$\{\varphi_t : M \to M \mid t \in \mathbb{R}\}$$

such that

(i) $\varphi_0(p) = p$ for every $p \in M$;

(ii) $\frac{d}{dt}\varphi_t(p)\big|_{t=t_0} = X(\varphi_{t_0}(p))$ for every $p \in M$ and $t_0 \in \mathbb{R}$.

Roughly speaking, $\varphi_t(p)$ denotes the point of M that we reach starting from a point $p \in M$ and moving for a time t on the integral curve of X that passes through p, that is unique, since we have assumed X to be smooth. By [BLU07, Lemma 1.2.23] one can deduce the following Lemma.

Lemma 1.3.3. *Any left invariant vector field on G is complete.*

Definition 1.3.4. A *one-parameter group* of G is a smooth Lie group homomorphism $\gamma : \mathbb{R} \to G$, i.e. it is a smooth map γ such that $\gamma(s+t) = \gamma(s) \cdot \gamma(t)$ for every $s,t \in \mathbb{R}$.

By [Ric03, Theorem 3.4] (refer also to [BLU07, Corollary 1.2.24]) the following result holds.

Proposition 1.3.5. *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the flow generated on G by a left invariant vector field X. Then φ_t is defined at every point of G for every $t \in \mathbb{R}$. Moreover $\gamma_e^X(t) = \varphi_t(e)$ is a one-parameter group and*

$$\varphi_t(x) = x \cdot \varphi_t(e) = x \cdot \gamma_e^X(t) \quad \forall x \in G,\ \forall t \in \mathbb{R}.$$

On the other side, given any one-parameter group $\gamma(t)$ of G, there is a left invariant vector field X whose flow is given by $\varphi_t(x) = x \cdot \gamma(t)$ for every $t \in \mathbb{R}$ and $x \in G$.

Proposition 1.3.5 states a remarkable one-to-one correspondence between the Lie algebra of G, $\mathrm{Lie}(G)$, and the space of the smooth Lie group homomorphisms from $\mathbb{R}$ to G. In particular, we can associate any vector field $X \in \mathrm{Lie}(G)$ with the (unique) integral curve of X starting at the unit element e, that we have denoted by $\gamma_e^X(t) := \varphi_t(e)$ for $t \in \mathbb{R}$.

Notice that γ_e^X is a homomorphism from $\mathbb{R}$ to G. According to these considerations, we have substantially described the following function, that is called the *exponential map*

$$\begin{aligned}\exp : \mathrm{Lie}(G) &\to G\\ X &\mapsto \gamma_e^X(1).\end{aligned}$$

The exponential map exp is a local diffeomorphism close to the origin, since its differential at the unit element is $d_E\exp(e) = \Psi^{-1}$ and then clearly it coincides with $Id_{\mathrm{Lie}(G)}$, once that T_eG is identified with $\mathrm{Lie}(G)$ through Ψ, as described above.

Remark 1.3.6. For any left invariant vector field X, for all $t, s \in \mathbb{R}$, it holds

$$\gamma_e^{tX}(s) = \gamma_e^X(ts). \tag{1.4}$$

Therefore, from (1.4) we have for all $t \in \mathbb{R}$

$$\exp(tX) = \gamma_e^{tX}(1) = \gamma_e^X(t).$$

Let us introduce the notions of *homogeneous Lie algebra* and *homogeneous Lie group.*

Definition 1.3.7. Let V be a real vector space. A family of endomorphisms of V, $\{\delta_t\}_{t>0}$, is called a *set of dilatations* on V if there are real numbers $\alpha_j > 0$ and linear subspaces W_{α_j} of V such that V is the direct sum of the W_{α_j}, and

$$\delta_t|_{W_{\alpha_j}} = t^{\alpha_j} Id_{W_{\alpha_j}}$$

for every j.

Definition 1.3.8. Let $\mathfrak{a}$ be a Lie algebra and let $\{\delta_t\}_{t>0}$ be a set of dilatations on the vector space underlying $\mathfrak{a}$, that we still denote by $\mathfrak{a}$. If, for every $t > 0$, δ_t is an automorphism of $\mathfrak{a}$, the pair $(\mathfrak{a}, \{\delta_t\}_{t>0})$ is called a *homogeneous Lie algebra.* A *homogeneous Lie group* is a connected Lie group G endowed with a family of automorphisms $\{D_t\}_{t>0}$ such that $\mathrm{Lie}(G)$ is homogeneous with respect to the set of dilations $\{\delta_t\}_{t>0}$, where for $t > 0$, $\delta_t = d_E(D_t)(e)$.

Remark 1.3.9. If $\mathfrak{a} = V_1 \oplus \cdots \oplus V_\kappa$ is a graded Lie algebra, the linear dilations

$$\delta_t(v) = t^j v \quad \text{for } v \in V_j,$$

for $j = 1, \ldots, \kappa$, for $t > 0$, are automorphisms, hence any graded Lie algebra canonically inherits a homogeneous structure.

1.4 Sub-Riemannian manifolds

Let us consider a smooth manifold M and consider a smooth subbundle Δ of the tangent bundle TM. In other words, Δ is a distribution of subspaces, namely, we recall, any point $p \in M$ is associated with a subspace of the tangent space $\Delta_p \subset T_pM$ that varies continuously as p varies on M.

Definition 1.4.1. Let M be a smooth manifold and consider a smooth subbundle Δ of the tangent bundle TM. If $I \subset \mathbb{R}$ is a real interval, a curve $\gamma : I \subset \mathbb{R} \to M$ is called Δ-*admissible* if it is absolutely continuous and

$$\dot\gamma(t) \in \Delta_{\gamma(t)}$$

for almost every $t \in I$.

Definition 1.4.2. Let M be a smooth manifold and consider a smooth subbundle Δ of the tangent bundle TM. If for every couple of points $p, q \in M$ there exists a Δ-admissible curve that connects p and q, we call M a Δ-*connected* manifold. In this case M is called a *Carnot-Carathéodory space* or a *CC-space*.

Definition 1.4.3. Let M be a smooth manifold and consider a smooth subbundle Δ of the tangent bundle TM. A quadratic form g on TM,

$$g : TM \to [0, \infty), (v, w) \to g(p)(v, w) \quad \text{for } v, w \in T_pM,\ p \in M$$

such that the restriction $g|_\Delta$ is Lipschitz regular on Δ is a *sub-Riemannian metric* on M. The triple (M, Δ, g) is called a *sub-Riemannian manifold.*

One can think of a sub-Riemannian manifold as a couple (M, Δ) composed of a smooth manifold M and a distribution of subspaces Δ, endowed with a metric g on M that is regular enough to state in a reasonable way a notion of length of a Δ-admissible curve. This notion of length, in turn, allows to define a sub-Riemannian distance on M.

Definition 1.4.4. Let (M, Δ, g) be a sub-Riemannian manifold and let $\gamma : I \subset \mathbb{R} \to M$ be a Δ-admissible curve. We define the *length of* γ as

$$\text{length}(\gamma) := \int_I \sqrt{g(\gamma(t))(\dot\gamma(t), \dot\gamma(t))}dt.$$

Remark 1.4.5. Notice that Definition 1.4.4 is just one of the possible definitions that one can give of the length of a Δ-admissible curve. Alternative natural possibilities have been explored in [Mon01], where, at the same time, it is proved that most of them are equivalent.

Definition 1.4.6. Let M be a smooth manifold and let Δ be a smooth subbundle of the tangent bundle TM. We denote by $\Gamma(\Delta)$ the set

$$\Gamma(\Delta) = \{X \in \text{Vect}^\infty(M) : X(p) \in \Delta_p \text{ for every } p \in M\}.$$

Let us consider the Lie algebra generated by $\Gamma(\Delta)$ with respect to the commutator of smooth vector fields, namely $\mathfrak{s}_{\text{Vect}^\infty(M)}(\Gamma(\Delta))$. We say that Δ satisfies the *Chow-Hörmander's condition* if at any point $p \in M$,

$$\{X(p) : X \in \mathfrak{s}_{\text{Vect}^\infty(M)}(\Gamma(\Delta))\} = T_pM.$$

In a natural way, we can translate this condition on a set of vector fields as follows.

Definition 1.4.7. Given $X_1, X_2, \dots, X_\ell$ smooth vector fields on a smooth manifold M of dimension n. We denote by $\Delta(X_1, \dots, X_\ell)$ the distribution of subspaces such that for every $p \in M$

$$(\Delta(X_1, \dots X_\ell))_p = \text{span}(X_1(p), \dots, X_\ell(p)).$$

We say that the set of vector fields $(X_1, X_2, \dots, X_\ell)$ satisfies the Chow-Hörmander's condition if $\Delta(X_1, \dots, X_\ell)$ satisfies the Chow-Hörmander's condition.

The following accessibility theorem has been independently proved by Chow and Rashevsky in [Cho39] and [Ras38], respectively.

Theorem 1.4.8 (Chow-Rashevsky's theorem)**.** *Let M be a smooth manifold and consider a smooth subbundle Δ of the tangent bundle TM that satisfies the Chow-Hörmander's condition. Then M is Δ-connected, hence it is a Carnot-Carathéodory space.*

Those Carnot-Carathéodory spaces which are also sub-Riemannian manifolds are interesting objects in the study of metric spaces, since one can equip them with a distance in a natural way, following a path analogous to the one that permits to define the classical Riemannian distance. In fact, if we consider a CC-space that is also a sub-Riemannian manifold (M, Δ, g) and if we consider two points $p, q \in M$, by Theorem 1.4.8, the set of Δ-admissible curves connecting p and q is not empty and we have introduced a suitable definition that allows to evaluate the length of each of these curves. Hence it is quite natural to introduce the following definition.

Definition 1.4.9. Let (M, Δ, g) be a sub-Riemannian manifold. Let us consider two points $p, q \in M$; we denote by $\Gamma_{p,q}$ the set of Δ-admissible curves $\gamma : [0, T] \to M$ for some $T > 0$ such that $\gamma(0) = p$, $\gamma(T) = q$. We call *Carnot-Carathéodory distance* or CC-distance between p and q the number

$$d_c(p, q) = \inf\{\text{length}(\gamma) : \gamma \in \Gamma_{p,q}\}.$$

The study of sub-Riemannian manifolds is very intriguing. In this thesis, we focus on some particular examples of CC-manifolds, non-Riemannian at any scale from the metric point of view, but at the same time endowed with a rich algebraic structure of translations and dilations. In particular, we focus on *Carnot groups* endowed with *homogeneous distances*. The leading motivation to develop geometric measure theory on these metric spaces is that they are the natural starting point towards the comprehension of more general settings: homogeneous Carnot groups are infinitesimal models for sub-Riemannian manifolds. In fact, the blow-up at a regular point of a sub-Riemannian manifold is a Carnot group [Mit85, Bel96].

Chapter 2

Carnot groups

In this chapter we present the definition of Carnot group, following [Pan89]. We introduce the notion of homogeneous distance on a Carnot group, and, successively, mainly following [SC16], we collect the definitions and some properties of various interesting measures that can be naturally considered on these metric spaces. About our strategy of presentation, when we introduce a definition, we generally specialize it below in the setting of the Heisenberg group, $\mathbb{H}^n$, that is the simplest example of a non-commutative Carnot group. This stylistic choice is supported by two main motivations. First, most of the original results proved in the next chapters will be valid limited to the setting of the Heisenberg group, so we think that it is worthy to make the reader familiarize with the special form of the concepts in this setting, that allows to work in more explicit terms. The second point is that we think that explaining at once what happens in a particular simple model can be an useful tool in order to better understand how things work in more general situations.

2.1 Carnot groups

Definition 2.1.1. A *graded group* $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group whose Lie algebra $\mathrm{Lie}(\mathbb{G})$ is graded, i.e. there exist linear subspaces $V_1,\ V_2,\ \dots, V_\kappa$ such that $\mathrm{Lie}(\mathbb{G}) = V_1 \oplus \cdots \oplus V_\kappa$ and $[V_i, V_j] \subset V_{i+j}$ for every i, j positive integers, $V_\kappa \neq \{0\}$ and $V_j = \{0\}$ for $j > \kappa$.

Definition 2.1.2. A *Carnot group* $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group whose Lie algebra $\mathrm{Lie}(\mathbb{G})$ is stratified, i.e. there exist linear subspaces $V_1,\ V_2,\ \dots, V_\kappa$ such that

$$\mathrm{Lie}(\mathbb{G}) = V_1 \oplus \cdots \oplus V_\kappa$$

and

$$[V_1, V_i] = V_{i+1} \quad \text{for } 1 = 1, \dots, \kappa - 1, \qquad V_\kappa \neq \{0\}, \qquad [V_1, V_\kappa] = \{0\}.$$

We introduce some notions related to a generic Carnot group $\mathbb{G}$, assuming the notation of Definition 2.1.2 and the following one, to be valid throughout the whole thesis, when nothing different is specified.

The number κ is called the *step* of the group. We set $m_0 = 0$ and, for $j = 1, \dots, \kappa$, we set

$$m_j := \dim(V_j).$$

For $j = 0, \dots, \kappa$, we set

$$h_j := \sum_{i=0}^{j} \mathrm{m}_i.$$

The *topological dimension* of $\mathbb{G}$ is denoted by $q := h_\kappa = \sum_{i=1}^{\kappa} m_j$.

Definition 2.1.3. The natural number $Q := \sum_{i=1}^{\kappa} im_i = \sum_{i=1}^{\kappa} i\dim(V_i)$ is called the *homogeneous dimension* of $\mathbb{G}$.

Theorem 2.1.4. *[CG90, Theorem 1.2.1] If $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group, the exponential map*

$$\exp : \mathrm{Lie}(\mathbb{G}) \to \mathbb{G}$$

is an analytic global diffeomorphism.

Let us introduce an important operation on a nilpotent Lie algebra.

Definition 2.1.5. [BLU07, Definition 2.2.11] Given a nilpotent Lie algebra $(\mathfrak{a}, [\cdot,\cdot])$, we define the *Baker-Campbell-Hausdorff operation* associated with $(\mathfrak{a}, [\cdot,\cdot])$ as

$$H : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a},$$

$$H(X,Y) = X + Y + \frac{1}{2}[X,Y] + \sum_{j\geq 3} c_j(X,Y) \tag{2.1}$$

where for every $j \in \mathbb{N}$, $c_j(X,Y)$, up to constants, is a finite sum of commutators of X and Y of length j.

We define also $c_2(X,Y) = \frac{1}{2}[X,Y]$, so that, if necessary, $H(X,Y)$ can be written in a more compact form. Notice that, since we have assumed $\mathfrak{a}$ to be nilpotent, the sum $H(X,Y)$ is finite for every $X, Y \in \mathfrak{a}$. For the precise form of the functions c_j one can refer to [BLU07, Definition 2.2.11] or to [Var84, Lemma 2.15.3]. For the sake of simplicity, we do not report it here, since we will not need more explicit details. We just recall that if $\mathfrak{a} = V_1 \oplus \cdots \oplus V_\kappa$ is a graded Lie algebra, then the maps $c_j : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ are homogeneous, i.e. for every $j = 2, \dots, \kappa$ and for every $X, Y \in \mathfrak{a}$ and $t > 0$,

$$c_j(tX, tY) = t^j c_j(X,Y).$$

The following proposition illustrates the fundamental role of the Baker-Campbell-Hausdorff operation in the study of Carnot groups. It will be, together with Theorem 2.1.4, a key result in order to represent $\mathbb{G}$ in a convenient way.

Proposition 2.1.6. *[CG90, Theorem 1.2.1] Let $\mathbb{G}$ be a Carnot group, then the Baker-Campbell-Hausdorff formula holds, i.e. for every $X, Y \in$* Lie($\mathbb{G}$)

$$\exp(X) \cdot \exp(Y) = \exp(H(X,Y))$$

where H is the Baker-Campbell-Hausdorff operation (2.1) *associated with* Lie($\mathbb{G}$).

The stratification of Lie($\mathbb{G}$) naturally induces a one-parameter family of anisotropic dilations that makes Lie($\mathbb{G}$) a homogeneous Lie algebra.

Definition 2.1.7. If $\mathbb{G}$ is a Carnot groups, for every $t > 0$ the *dilation* δ_t associated with t is the linear automorphism of Lie($\mathbb{G}$) such that $\delta_t(v) = t^i v$ if $v \in V_i$ for $i = 1, \dots, \kappa$. Through the exponential map, one can transfer the notion of dilation on the group $\mathbb{G}$: we denote again by δ_t the map $\exp \circ \delta_t \circ \exp^{-1} : \mathbb{G} \to \mathbb{G}$.

Remark 2.1.8. For every $v, w \in \mathbb{G}$ and for every $s, t > 0$ the following properties hold

(i) $\delta_t \circ \delta_s = \delta_{ts}$;

(ii) $\delta_t(vw) = \delta_t(v) \cdot \delta_t(w)$.

2.2 Identifications of Carnot groups

The main goal of this section is to describe how we think of a Carnot group $\mathbb{G}$.

Definition 2.2.1. A basis $(X_1, \dots, X_q)$ of Lie($\mathbb{G}$) is called *adapted* (to the stratification) if $(X_{h_{j-1}+1}, \dots, X_{h_j})$ is a basis of V_j for every $j = 1, \dots, \kappa$.

If we fix a Carnot group $\mathbb{G}$ and an adapted basis $(X_1, \dots, X_q)$ of Lie($\mathbb{G}$), the group $\mathbb{G}$ can be identified with $\mathbb{R}^q$ through the following map

$$\varphi : \mathbb{G} \to \mathbb{R}^q, \ \varphi \left(\exp \left(\sum_{i=1}^{q} x_i X_i \right) \right) = (x_1, \dots, x_q). \tag{2.2}$$

More explicitly, one can identify $\mathbb{G}$ with $\mathbb{R}^q$ identifying any point $x \in \mathbb{G}$ with $\varphi(x) = (x_1, \dots, x_q)$, so that one denotes by $x = (x_1, \dots, x_q)$ the generic element of $\mathbb{G}$. These coordinates are called *exponential coordinates*.

The described identification between $\mathbb{G}$ and $\mathbb{R}^q$ is the most traditional one used in the literature. Nevertheless we prefer to adopt a slightly different compatible approach. Although perhaps it was already implied in many related papers, like [Mag01], the explicit employement of this point of view in our line of research is quite recent and we think it is worthwile, since an initial effort of abstraction is rewarded by the possibility of working on a Carnot group without setting a specific coordinate system, when it is not needed. Moreover, it allows to dispose directly on the group of some tools that are generally proper of its Lie algebra, like, for example, the concept of orthogonality. This point of view is shared for example by the papers [Mag11b, LM11, MTV15, Mag19, JNGV20, LDMR20, Mag20]. We describe below in detail the identifications that will be used in this thesis.

2.2.1 Carnot groups as vector spaces

Let us consider a Carnot group $\mathbb{G} = (\mathbb{G}, \cdot)$ and let us denote the Lie algebra of $\mathbb{G}$ by Lie($\mathbb{G}$) $= (\mathfrak{g}, [\cdot, \cdot])$. Notice that in this and in the following subsection, we need to distinguish between Lie algebras and their own underlying vector spaces. Let us introduce the following binary operation $*$ on $\mathfrak{g}$: for every $X, Y \in \mathfrak{g}$,

$$X * Y := H(X, Y), \tag{2.3}$$

where $H : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Baker-Campbell-Hausdorff operation in (2.1) associated with Lie($\mathbb{G}$). The following two results justify our successive point of view.

(i) By [BLU07, Theorem 2.2.13] (see also [Ric03, Theorem 4.2]) the space

$$\mathbb{M} := (\mathfrak{g}, *)$$

is a Lie group and it is isomorphic to the group $\mathbb{G}$. Moreover, the exponential map $\exp : \mathbb{M} \to \mathbb{G}$ is a Lie group isomorphism since it is a diffeomorphism and, by

Theorem 2.1.6, for every $X, Y \in \mathbb{M}$,

$$\exp(X * Y) = \exp(X) \cdot \exp(Y).$$

The identity element of $\mathbb{M}$ with respect to $*$ is the null vector, and we denote it by 0.

(ii) By [BLU07, Corollary 2.2.15] (see also [Ric03, Theorem 4.2]), the Lie algebra of $\mathbb{M}$, $\mathrm{Lie}(\mathbb{M}) = (\mathfrak{m}, [\cdot,\cdot])$ is isomorphic to $(\mathfrak{g}, [\cdot,\cdot])$ through the Lie algebras isomorphism

$$\Psi : \mathfrak{g} \to \mathfrak{m}, \ \ \Psi(v) = X, \tag{2.4}$$

where $X \in \mathfrak{m}$ is the unique $*$-left invariant vector field such that $X(0) = v$ (we have identified $\mathfrak{g}$ with $T_0\mathbb{G}$ as usual).

Remark 2.2.2. Let us consider the exponential map associated with $\mathbb{M}$

$$\exp_{\mathbb{M}} : \mathrm{Lie}(\mathbb{M}) \to \mathbb{M}.$$

For every $v \in \mathbb{M}$, denoting by $V := \Psi(v)$, the $*$-left invariant vector field such that $V(0) = v$, we get that $\exp_{\mathbb{M}}(tV) = tv$ for every $t \in \mathbb{R}$. In fact if we consider the curve $\gamma(t) := tv$, it is immediate to observe that $\gamma(0) = 0$ and that for every t,

$$\begin{aligned} &\dot\gamma(t) = v \\ V(\gamma(t)) &= V(tv) = d_E l_{tv}(0)(V(0)) = d_E l_{tv}(0)(v) = \frac{d}{ds}(l_{tv}(sv))\Big|_{s=0} \\ &= \frac{d}{ds}(tv * sv)\Big|_{s=0} = \frac{d}{ds}H(tv, sv)\Big|_{s=0} = \frac{d}{ds}((t+s)v)\Big|_{s=0} = v. \end{aligned}$$

Hence $\gamma(t)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dot\gamma(t) = V(\gamma(t)) & t \in \mathbb{R} \\ \gamma(0) = v \end{cases}$$

hence, by the definition of exponential map, $\exp_{\mathbb{M}}(V) = \gamma(1) = 1v = v$. For more information about the map $\exp_{\mathbb{M}}$, we refer the reader to [BLU07, Theorem 2.2.24].

Remark 2.2.3. As a combination of Theorems 2.1.4 and 2.1.6 applied to $\mathbb{M}$, with Remark 2.2.2 and with the definition and the properties of the map Ψ in (2.4), one can deduce that $\exp_{\mathbb{M}}$, as a map between the spaces $\mathfrak{m}$ and $\mathfrak{g}$, coincide with the inverse of Ψ. In addition, it is both a Lie algebra and a Lie group isomorphism, once $\mathrm{Lie}(\mathbb{M})$ is endowed with the Baker-Campbell-Hausdorff operation associated to the standard commutator of $\mathrm{Vect}^\infty(\mathbb{M})$.

As a consequence of all these observations, we are allowed to identify the two groups $\mathbb{M}$ and $\mathbb{G}$, so that the exponential map exp reduces to the identity map and the elements of the group can be thought as vectors of a vector space. To summarize, the Carnot group $\mathbb{G}$ can be thought as the object $(\mathfrak{g}, *, [\cdot,\cdot])$, where, if needed, we can identify $(\mathfrak{g}, [\cdot,\cdot])$ with the Lie algebra of the Lie group $(\mathfrak{g}, *)$. We resume our considerations as follows.

We think any Carnot group $\mathbb{G}$ as a finite dimensional vector space $(\mathbb{G}, \cdot, [\cdot,\cdot])$ equipped with both a Lie algebra structure and a Lie group structure. For every $x, y \in \mathbb{G}$, the product $x \cdot y$ equals $H(x, y)$, where H is the Baker-Campbell-Hausdorff operation in (2.1) associated with $(\mathbb{G}, [\cdot,\cdot])$. As it is usual in vector

spaces, $\mathbb{G}$ is naturally identified with the tangent space $T_0\mathbb{G}$. If necessary we can identify $(\mathbb{G}, [\cdot,\cdot])$ with the Lie algebra of $(\mathbb{G}, \cdot)$ through the map Ψ in (2.4).

2.2.2 Carnot groups in coordinates

Let us see how one can fix suitable coordinates on a Carnot group $\mathbb{G}$.

Fix an adapted basis $\mathcal{B} = (v_1, \dots, v_q)$ of $\mathbb{G}$ (i.e. a basis such that $(v_{h_{j-1}+1}, \dots, v_{h_j})$ is a basis of V_j for every $j = 1, \dots, \kappa$) and consider the isomorphism of vector spaces

$$\pi_{\mathcal{B}} : \mathbb{G} \to \mathbb{R}^q, \ \pi_{\mathcal{B}}\left(\sum_{i=1}^{q} x_i v_i\right) = (x_1, \dots, x_q). \tag{2.5}$$

We define through $\pi_{\mathcal{B}}$ a binary operation $\Diamond$ on $\mathbb{R}^q$: for $x, y \in \mathbb{R}^q$

$$x \Diamond y := \pi_{\mathcal{B}}(\pi_{\mathcal{B}}^{-1}(x) \cdot \pi_{\mathcal{B}}^{-1}(y)).$$

We denote again by δ_t the automorphism $\pi_{\mathcal{B}} \circ \delta_t \circ \pi_{\mathcal{B}}^{-1}$ of $\mathbb{R}^q$.

The product $\Diamond$ on $\mathbb{R}^q$ has the polynomial form presented in the next proposition, deduced by the form of the Baker-Campbell-Hausdorff formula (Definition 2.1.5). We refer the reader to [BLU07, Proposition 2.2.22] and [SC16, Proposition 2.3] for more details.

Proposition 2.2.4. *For all $x, y \in \mathbb{R}^q$*

$$x \Diamond y = x + y + Q(x, y) \tag{2.6}$$

where

$$Q = (Q_1, \ Q_2, \dots, \ Q_q) : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q$$

with components $Q_j : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ for every $j \in \{1, \dots, q\}$ that are homogeneous polynomials with respect to the intrinsic dilations such that for every $x, y \in \mathbb{R}^q$ and $t > 0$

$$Q_j(\delta_t(x), \delta_t(y)) = t^i Q_j(x, y) \quad \text{if j} \in \{\mathrm{h_{i-1}} + 1, \dots, \mathrm{h_i}\}.$$

Moreover, for every $x, y \in \mathbb{R}^q$, the mappings Q_j satisfy the following properties

(i) $Q_j(x, y) = 0$ for $j = 1, \dots, h_1$ and $Q_j(x, 0) = Q_j(0, y) = Q_j(x, x) = Q_j(x, -x) = 0$ for $j = h_1 + 1, \dots, q$.

(ii) Q is antisymmetric, i.e. $Q(x, y) = -Q(-y, -x)$.

(iii) $Q_j(x, y) = Q_j(x_1, \dots, x_{h_{i-1}}, y_1, \dots, y_{h_{i-1}})$ if $\mathrm{j} \in \{\mathrm{h_{i-1}} + 1, \dots, \mathrm{h_i}\}$.

As a consequence, the inverse of an element $x = (x_1, \dots, x_q) \in \mathbb{R}^q$ with respect to the product $\Diamond$ is $x^{-1} = (-x_1, \dots, -x_q)$, while the identity element is the null vector, that we denote again by 0.

Again by [BLU07, Theorem 2.2.22], one can deduce the following facts, that justify our successive identifications

(i) $\mathbb{L} := (\mathbb{R}^q, \Diamond)$ is a Lie group.

(ii) $(\mathbb{G}, \cdot)$ is isomorphic, as a Lie group, to $\mathbb{L}$ through the map $\pi_{\mathcal{B}}$.

(iii) The Lie algebra Lie($\mathbb{G}$) is isomorphic to the Lie algebra of $\mathbb{L}$, Lie($\mathbb{L}$) $= (\mathfrak{l}, [\cdot,\cdot])$ through $d_E\pi_{\mathcal{B}}$. Moreover, if, for every $i = 1, \dots, q$, we denote by $P_i \in \mathfrak{l}$ the $\Diamond$-left invariant vector field such that $P_i(0) = \mathbf{e}_i$, where $(\mathbf{e}_1, \dots, \mathbf{e}_q)$ is the canonical basis of $\mathbb{R}^q$, then

$$d_E\pi_{\mathcal{B}}(\mathcal{V}_i) = P_i.$$

where $\mathcal{V}_i$ is the $\cdot$-left invariant vector field of $\mathrm{Lie}(\mathbb{G})$ such that $\mathcal{V}_i(0) = v_i$.

(iv) The exponential map $\exp_{\mathbb{L}} : \mathfrak{l} \to \mathbb{L}$ is a linear isomorphism (of vector spaces) such that

$$\exp_{\mathbb{L}}\left(\sum_{i=1}^{q} p_i P_i\right) = (p_1, \dots, p_q) = \sum_{i=1}^{q} p_i \mathbf{e}_i.$$

We equip $\mathbb{R}^q$ with the following structure of Lie bracket $[\cdot,\cdot]_{\mathbb{L}}$, so that $(\mathfrak{l}, [\cdot,\cdot])$ and $(\mathbb{R}^q, [\cdot,\cdot]_{\mathbb{L}})$ are isomorphic Lie algebras through $\exp_{\mathbb{L}}$: for $(x_1, \dots, x_q), (y_1, \dots, y_q) \in \mathbb{R}^q$,

$$[(x_1, \dots, x_q), (y_1, \dots, y_q)]_{\mathbb{L}} = [\sum_{i=1}^{q} x_i \mathbf{e}_i, \sum_{i=1}^{q} y_i \mathbf{e}_i]_{\mathbb{L}} := \exp_{\mathbb{L}}([\sum_{i=1}^{q} x_i P_i, \sum_{i=1}^{q} y_i P_i]). \tag{2.7}$$

Remark 2.2.5. If we identify through $\exp_{\mathbb{L}}$ the two vector spaces $\mathfrak{l} \simeq \mathbb{L}$, and $\mathrm{Lie}(\mathbb{G})$ with $(\mathbb{G}, [\cdot,\cdot])$ through Ψ, we can read $d_E\pi_{\mathcal{B}}(0)$ as equal to $\pi_{\mathcal{B}} : \mathbb{G} \to \mathbb{R}^q$. Observe then that the commutator (2.7) equals

$$[(x_1, \dots, x_q), (y_1, \dots, y_q)]_{\mathbb{L}} = \pi_{\mathcal{B}}([\pi_{\mathcal{B}}^{-1}(x_1, \dots, x_q), \pi_{\mathcal{B}}^{-1}(y_1, \dots, y_q)]),$$

so that $\Diamond$ coincides with the binary operation that can be defined on $\mathbb{R}^q$ by the Baker-Campbell-Hausdorff operation associated with the Lie bracket $[\cdot,\cdot]_{\mathbb{L}}$, i.e. for any $x, y \in \mathbb{R}^q$, $x \Diamond y = H(x, y)$, where H here denotes the map in (2.1) associated with $(\mathbb{R}^q, [\cdot,\cdot]_{\mathbb{L}})$.

To summarize, we can identify both the two isomorphic Lie groups $(\mathbb{R}^q, \Diamond)$ and $(\mathbb{G}, \cdot)$ and the two Lie algebras $(\mathbb{R}^q, [\cdot,\cdot]_{\mathbb{L}})$ and $(\mathbb{G}, [\cdot,\cdot])$ through the map $\pi_{\mathcal{B}}$. We resume our considerations as follows.

When we fix an adapted basis $\mathcal{B}$ of a Carnot group $\mathbb{G}$, by $(\mathbb{G}, \cdot, [\cdot,\cdot])$ we mean $(\mathbb{R}^q, \Diamond, [\cdot,\cdot]_{\mathbb{L}}))$ and we call it $\mathbb{G}$ *in adapted coordinates with respect to $\mathcal{B}$*. More explicitly, we consider that we have identified $\mathbb{G}$ with $\mathbb{R}^q$ through $\pi_{\mathcal{B}}$, where $\mathbb{R}^q$ is endowed, at the same time, with the Lie group structure given by the polynomial product in (2.6) and the Lie algebra structure given by the commutator (2.7).

Remark 2.2.6. Recall that, analogously to what we said in the previous section, if necessary we identify the Lie algebra of $(\mathbb{R}^q, \Diamond)$ with $(\mathbb{R}^q, [\cdot,\cdot]_{\mathbb{L}})$ through the map Ψ relative to $(\mathbb{R}^q, \Diamond)$, that associates any vector $v \in \mathbb{R}^q$ with the $\Diamond$-left invariant vector field X such that $X(0) = v$.
Moreover we can observe that, given a Carnot group $\mathbb{G}$ in adapted coordinates with respect to a fixed adapted basis $\mathcal{B} = (v_1, \dots, v_q)$, one can always find, again through Ψ, an adapted basis of the Lie algebra $\mathrm{Lie}(\mathbb{G})$ composed of vector fields with polynomial coefficients. In fact, by the left invariance of the vector fields and by the polynomial form of the group product given by Proposition 2.2.4 it is enough to consider, for $i = 1, \dots, q$, the left invariant vector field $X_i \in \mathrm{Lie}(\mathbb{G})$ such that $X_i(0) = v_i$ (and we recall that v_i is identified with the i-th vector $\mathbf{e}_i$ of the canonical basis of $\mathbb{R}^q$).

If we consider $\mathbb{G}$ in adapted coordinates with respect to an adapted basis, for any $t > 0$

the dilation δ_t is the map $\delta_t : \mathbb{G} \to \mathbb{G}$,

$$\delta_t(x_1, \ldots, x_q) = (tx_1, \ldots, tx_{h_1}, t^2 x_{h_1+1}, \ldots, t^2 x_{h_2}, \ldots, t^\kappa x_{h_{\kappa-1}+1}, \ldots, t^\kappa x_q).$$

From now on, in order to simplify the notation, given a Carnot group $\mathbb{G}$, we denote the product of two points $x, y \in \mathbb{G}$ as

$$xy := x \cdot y.$$

2.3 The Heisenberg group

In this section we introduce the Heisenberg group $\mathbb{H}^n$, along with special suitable adapted coordinates through which we identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. Often, people in the literature refer to Heisenberg groups using the plural form, stressing the fact that every natural number $n \in \mathbb{N}$ is associated with a different group $\mathbb{H}^n$. We prefer to use the singular form, implicitly assuming that we have fixed a priori an arbitrary value of n. This choice is not restrictive for our purposes, since from our point of view, neither the algebraic structure nor the group structure substantially changes when n changes. When we need to refer to $\mathbb{H}^1$, we call it the first Heisenberg group. The Heisenberg group is the simplest example of a non-commutative Carnot group. Our presentation follows for instance the approach of [Mag11b].

The Heisenberg group can be represented as a direct sum of two linear subspaces

$$\mathbb{H}^n = H_1 \oplus H_2$$

with $\dim(H_1)= 2n$ and $\dim(H_2)= 1$, endowed with a symplectic form ω on H_1 and a fixed nonvanishing element

$$e_{2n+1} \in H_2. \tag{2.8}$$

We denote by π_{H_1} and π_{H_2} the canonical projections on H_1 and H_2, respectively, associated with the direct sum. We can give to $\mathbb{H}^n$ a structure of Lie algebra by setting

$$[x, y] = \omega(\pi_{H_1}(x), \pi_{H_1}(y))\ e_{2n+1}. \tag{2.9}$$

Then the Baker-Campbell-Hausdorff formula ensures that

$$xy = x + y + \frac{[x, y]}{2} \tag{2.10}$$

defines a Lie group operation on $\mathbb{H}^n$. We fix a symplectic basis $(e_1, \ldots, e_{2n})$ of (H_1, ω), namely

$$\omega(e_i, e_{n+j}) = \delta_{ij}, \qquad \omega(e_i, e_j) = \omega(e_{n+i}, e_{n+j}) = 0$$

for every $i, j = 1, \ldots, n$, where δ_{ij} is the Kronecker delta. Thus, considering the vector e_{2n+1} given in (2.8), we obtain a basis

$$\mathcal{B} = (e_1, \ldots, e_{2n+1}). \tag{2.11}$$

We call any basis of the form (2.11) a *Heisenberg basis.* We will use $\mathcal{B}$ to consider $\mathbb{H}^n$ in adapted coordinates, through the linear isomorphism associated with $\mathcal{B}$

$$\pi_{\mathcal{B}} : \mathbb{H}^n \to \mathbb{R}^{2n+1}, \quad \pi_{\mathcal{B}}(x) = (x_1, \ldots, x_{2n+1}) \tag{2.12}$$

for $x = \sum_{j=1}^{2n+1} x_j e_j$. We can read the given Lie product on $\mathbb{H}^n$ in adapted coordinates with respect to $\mathcal{B}$ (i.e. on $\mathbb{R}^{2n+1}$) as follows

$$
\begin{aligned}
[(x_1,\dots,x_{2n+1}),(y_1,\dots,y_{2n+1})] &= \pi_{\mathcal{B}}\left([\sum_{i=1}^{2n+1} x_i e_i, \sum_{i=1}^{2n+1} y_i e_i]\right)\\
&= \left(0,\dots,0,\sum_{i=1}^{n}(x_i y_{i+n} - x_{i+n} y_i)\right)
\end{aligned}
$$

then the group product takes in coordinates the following form

$$
(x_1,\dots,x_{2n+1})(y_1,\dots,y_{2n+1}) = \left(x_1+y_1,\dots,x_{2n+1}+y_{2n+1}+\sum_{i=1}^{n}\frac{x_i y_{i+n} - x_{i+n} y_i}{2}\right). \quad (2.13)
$$

Moreover, taking in consideration the product (2.13), in our coordinates we obtain the following basis of left invariant vector fields: for $x = (x_1,\dots,x_{2n+1})$

$$
\begin{aligned}
X_j(x) &= \partial_{x_j} - \frac{1}{2}x_{j+n}\partial_{x_{2n+1}} \qquad j=1,\dots,n\\
Y_j(x) &= \partial_{x_{n+j}} + \frac{1}{2}x_j\partial_{x_{2n+1}} \qquad j=1,\dots,n \qquad (2.14)\\
T(x) &= \partial_{x_{2n+1}}.
\end{aligned}
$$

They clearly constitute a basis $(X_1,\dots,X_{2n+1})$ of Lie($\mathbb{H}^n$) such that $X_j(0)=e_j$ for every $j=1,\dots,2n+1$. Any linear combination of $X_1,\dots,X_{2n}$ is a *left invariant horizontal vector field of* $\mathbb{H}^n$. For $t>0$, the dilation δ_t is the linear mapping $\delta_t:\mathbb{H}^n\to\mathbb{H}^n$ such that

$$
\delta_t(x_1,\dots,x_{2n+1}) = (tx_1,\dots,tx_{2n},t^2x_{2n+1}).
$$

2.4 Metrics on Carnot groups

We endow Carnot groups with suitable distances, so that we consider them as metric spaces.

Definition 2.4.1. Let $\mathbb{G}$ be a Carnot group. A *left invariant metric* on $\mathbb{G}$ is a Riemannian metric such that all left translations of the group are isometries.

If g is a left invariant metric on $\mathbb{G}$, we denote the Riemannian left invariant norm associated with g by $\|\cdot\|_g$. Since $\mathbb{G}$ is identified with $T_0\mathbb{G}$, a left invariant metric can be naturally defined fixing a scalar product on $\mathbb{G}$. In particular, we fix a *graded* scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{G}$, i.e. a scalar product that makes the subspaces $V_1,\dots,V_\kappa$ orthogonal. By left translation, we extend in a left invariant way the fixed scalar product to a graded left invariant metric that we denote by g on $\mathbb{G}$: for every $x\in\mathbb{G}$ and $v,w\in T_x\mathbb{G}$ we set

$$
g(x)(v,w) := \langle d_E(l_{x^{-1}})(x)(v), d_E(l_{x^{-1}})(x)(w)\rangle.
$$

If we fix an orthonormal adapted basis $(e_1,\dots,e_q)$ of $\mathbb{G}$ and we consider $\mathbb{G}$ in adapted coordinates with respect to the fixed basis, then, the Riemannian norm $\|\cdot\|_g$ coincides on the vectors of $\mathbb{G}$ with the Euclidean norm, that we denote by $|\cdot|$. We recall that $|\cdot|$ is well defined since $\mathbb{G}$ is automatically meant as $\mathbb{R}^q$ in adapted coordinates with respect to the fixed adapted basis.

Now we are ready to follow the very natural path towards the introduction of a metric that makes $\mathbb{G}$ a non-Riemannian metric space at any scale. As usual, we adopt the notation of Definition 2.1.2. As we observed in Section 1.3, it is natural to identify the Lie algebra Lie($\mathbb{G}$) with the tangent space $T_0\mathbb{G}$. As a consequence, the first layer, V_1, that is a linear subspace of Lie($\mathbb{G}$), can be identified with a linear subspace of $T_0\mathbb{G}$. Since all left translations are diffeomorphisms, we can consider for every $x \in \mathbb{G}$ the following subspace of $T_x\mathbb{G}$

$$H_x\mathbb{G} := d_E l_x(0)(V_1).$$

The disjoint union of these subspaces, for $x \in \mathbb{G}$, $H\mathbb{G} = \coprod_{x\in\mathbb{G}} d_E l_x(0)(V_1)$ is a subbundle of the tangent bundle $T\mathbb{G}$ and we call it the horizontal bundle. We introduce below a more precise definition.

Definition 2.4.2. Let $\mathbb{G}$ be a Carnot group. Let $(X_1, X_2, \dots, X_q)$ be an adapted basis of Lie($\mathbb{G}$). We call *horizontal bundle* of $\mathbb{G}$ the bundle

$$H\mathbb{G} := \operatorname{span}(X_1,\ X_2,\ \dots, X_{m_1}),$$

hence for every $x \in \mathbb{G}$ the *fiber of* $H\mathbb{G}$ *at* x is the subspace of $T_x\mathbb{G}$

$$H_x\mathbb{G} = \operatorname{span}(X_1(x),\ X_2(x), \dots,\ X_{m_1}(x)).$$

Remark 2.4.3. Notice that by previous observations $H\mathbb{G}$ is an object independent of the choice of the adapted basis fixed in Definition 2.4.2.

Definition 2.4.4. A vector field $X \in$ Lie($\mathbb{G}$) is *horizontal* if $X(x) \in H_x\mathbb{G}$ for every $x \in \mathbb{G}$.

Definition 2.4.5. Let $\mathbb{G}$ be a Carnot group and let $H\mathbb{G}$ be its horizontal bundle. An absolutely continuous curve on $\mathbb{G}$ is called *horizontal* if it is $H\mathbb{G}$-admissible.

In the notation of Definition 2.1.2, for any adapted basis $(X_1, \dots, X_q)$ of Lie($\mathbb{G}$), the set of the first m_1 vector fields $(X_1,\ X_2,\ \dots,\ X_{m_1})$ is a basis of V_1 that generates the whole Lie algebra of $\mathbb{G}$ by Lie bracket, i.e.

$$\mathfrak{s}_{\mathrm{Lie}(\mathbb{G})}(X_1, X_2, \dots, X_{m_1}) = \mathrm{Lie}(\mathbb{G}),$$

or, equivalently, for every $x \in \mathbb{G}$, $\mathfrak{s}_{T_x\mathbb{G}}(X_1(x),\ X_2(x),\ \dots,\ X_{m_1}(x)) = T_x\mathbb{G}$. This means that the set of vector fields $(X_1, \dots, X_{m_1})$ satisfies the Chow-Hörmander's condition, then, by Theorem 1.4.8, $\mathbb{G}$ is $H\mathbb{G}$-connected, thus $(\mathbb{G}, H\mathbb{G})$ is a Carnot-Carathéodory space. The left invariant metric g satisfies all the properties to say that $(\mathbb{G}, H\mathbb{G}, g)$ is a sub-Riemannian manifold. Hence a Carnot-Carathéodory distance d_c associated with $H\mathbb{G}$ is naturally well defined on $\mathbb{G}$, according to Definition 1.4.9. The metric space $(\mathbb{G}, d_c)$ is conceptually easy to interpret. One can imagine that it is possible to move on $\mathbb{G}$ only along a family of selected admissible directions, i.e. horizontal vector fields, which correspond to admissible paths, i.e. horizontal curves. Then the distance d_c between two points of $\mathbb{G}$ can be regarded as the minimum possible time one needs to move from one of the two points to the second one, walking exclusively on admissible paths. Nevertheless, for practical computations, the distance d_c is truly not very convenient. Luckily, for many of our purposes, d_c can be regarded just as the first remarkable example of a homogeneous distance on $\mathbb{G}$.

Definition 2.4.6. A distance $d : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ is called a *homogeneous distance* if it is left invariant, i.e. $d(zx, zy) = d(x, y)$ for every $x, y, z \in \mathbb{G}$, and homogeneous of degree one with respect to anisotropic dilations, i.e. $d(\delta_t(x), \delta_t(y)) = td(x, y)$ for every $x, y \in \mathbb{G}$ and $t > 0$.

Given a homogeneous distance d on $\mathbb{G}$, we introduce the following norm

$$\|\cdot\| : \mathbb{G} \to \mathbb{R}, \ \|x\| := d(x, 0).$$

By the left invariance of d, $\|x\| = \|x^{-1}\|$ for every $x \in \mathbb{G}$ and by the homogeneity of d, $\|\delta_t(x)\| = t\|x\|$ for every $x \in \mathbb{G}$ and $t > 0$. We call a norm that satisfies these two properties a *homogeneous norm.*

All homogeneous distances are equivalent.

Proposition 2.4.7. *[Mag02a, Proposition 2.3.37] Let d_1, d_2 be two homogeneous distances on $\mathbb{G}$, then they are equivalent, i.e. there exist two positive constants $K_1, K_2 > 0$ such that for every $x, y \in \mathbb{G}$*

$$K_1 d_1(x, y) \le d_2(x, y) \le K_2 d_1(x, y).$$

Proposition 2.4.8. *[Mag02a, Proposition 2.3.39] The Carnot-Carathéodory distance is a homogeneous distance.*

One of the mostly used homogeneous distances in the literature is the metric d_∞ defined for all $x, y \in \mathbb{G}$ as

$$d_\infty(x, y) = \|y^{-1}x\|_\infty,$$

where the homogeneous norm $\|\cdot\|_\infty$ is defined, for $x = \sum_{i=1}^{\kappa} x^j \in \mathbb{G}$, with $x^j \in V_j$, as

$$\|x\|_\infty := \max\{\varepsilon_j |x^j|^{1/j}, \ j = 1, \dots, \kappa\}$$

where $\varepsilon_1 = 1$, $\varepsilon_j \in (0, 1]$ are positive constants depending on the group structure (see [FSSC03a, Theorem 5.1]). This homogeneous distance is very convenient to deal with explicit calculations.

Any homogeneous distance induces on $\mathbb{G}$ the Euclidean topology, nevertheless the metric space $(\mathbb{G}, d_c)$ or, equivalently, $\mathbb{G}$ endowed with any fixed homogeneous distance, is non-Riemannian at any scale. Later on, we will discuss more carefully this observation, but a first hint of this non-equivalence can be already seen in the following proposition. We recall that we endowed $\mathbb{G}$ with the left invariant Riemannian graded metric g, hence, in order to interpret the following proposition, we fix an adapted orthonormal basis of $\mathbb{G}$, $(e_1, \dots, e_q)$ so that we can think $\mathbb{G}$ in adapted coordinates. Then, as we said, we denote by $|\cdot|$ the Euclidean distance on $\mathbb{G}$ (equivalently it is the left invariant norm $\|\cdot\|_g$ on $T_0\mathbb{G}$, once $\mathbb{G}$ is identified with $T_0\mathbb{G}$).

Proposition 2.4.9. *[SC16, Proposition 2.15] Let $\mathbb{G}$ be a Carnot group of step κ endowed with a homogeneous distance d. Then*

(i) $A \subset (\mathbb{G}, d)$ is bounded if and only if $A \subset (\mathbb{G}, |\cdot|)$ is bounded.

(ii) For each compact set $F \subset \mathbb{G}$ there exists a positive constant C_F such that

$$C_F^{-1}|x| \le d(x, 0) \le C_F |x|^{\frac{1}{\kappa}} \qquad \forall x \in F.$$

(iii) The identity map $id : (\mathbb{G}, d) \to (\mathbb{G}, |\cdot|)$ is a homeomorphism.

More precisely, the following estimate holds.

Proposition 2.4.10. *[BLU07, Proposition 5.15.1] Let $\mathbb{G}$ be a Carnot group of step κ and let d be a homogeneous distance. Then, for every compact set $F \subset \mathbb{G}$, there is a constant $c_F > 0$ such that for every $x, y \in F$*

$$\frac{1}{c_F}|x - y| \le d(x, y) \le c_F |x - y|^{\frac{1}{\kappa}}.$$

2.4.1 Metrics on the Heisenberg group

We fix on $\mathbb{H}^n$ a scalar product $\langle\cdot,\cdot\rangle$ that makes the Heisenberg basis $\mathcal{B} = (e_1, \ldots, e_{2n+1})$ fixed in (2.11) orthonormal. In the sequel, any Heisenberg basis will be understood to be orthonormal. We denote by $|\cdot|$ the norm induced by $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^n$; notice that it coincides with the Euclidean norm on $\mathbb{H}^n$ considered in adapted coordinates, i.e. on $\mathbb{R}^{2n+1}$. The symmetries of the Heisenberg group are detected through the isometry

$$J : H_1 \to H_1,$$

that is defined on the Heisenberg basis

$$J(e_i) = e_{n+i} \quad \text{and} \quad J(e_{n+i}) = -e_i$$

for all $i = 1, \ldots, n$. It is then easy to check that

$$\langle x, y\rangle = \omega(x, Jy) \quad \text{and} \quad J^2 = -I$$

for all $x, y \in H_1$.

As we described above for a general setting, by identifying $T_0\mathbb{H}^n$ with $\mathbb{H}^n$ and by left translating the fixed scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^n$ we obtain a graded left invariant Riemannian metric g on $\mathbb{H}^n$ and as usual the associated Riemannian norm is denoted by $\|\cdot\|_g$.
The norm $\|\cdot\|_\infty$ on $\mathbb{H}^n$ is defined for every $x \in \mathbb{H}^n$ as

$$\|(x_1, \ldots, x_{2n+1})\|_\infty := \max\{|(x_1, \ldots, x_{2n})|, |x_{2n+1}|^{\frac{1}{2}}\}.$$

Remark 2.4.11. A homogeneous norm frequently used in the literature in the Heisenberg group is the Cygan-Korányi norm: for every $(x_1, \ldots, x_{2n+1}) \in \mathbb{H}^n$,

$$\|(x_1, \ldots, x_{2n+1})\|_K := \sqrt{|(x_1, \ldots, x_{2n})|^4 + |x_{2n+1}|^2}. \tag{2.15}$$

Remark 2.4.12. Notice that the identification of $\mathbb{H}^n$ in adapted coordinates with $\mathbb{R}^{2n+1}$ is somehow independent of the choice of the Heisenberg basis. More precisely, if we consider two different Heisenberg bases, the group $\mathbb{H}^n$ in adapted coordinates with respect to the two bases gives the same $\mathbb{R}^{2n+1}$, in the sense that both the identifications give the product (2.13) on $\mathbb{R}^{2n+1}$. In other words, $\mathbb{R}^{2n+1}$ endowed with the product (2.13) and with the Euclidean norm, represents $\mathbb{H}^n$ in adapted coordinates with respect to any arbitrary Heisenberg basis.

2.5 Measures on Carnot groups

Consider a Carnot group $\mathbb{G}$ equipped with a homogeneous distance d. We denote by $B_d(x, r)$ and $\mathbb{B}_d(x, r)$ the metric open ball and the metric closed ball, respectively, centered at $x \in \mathbb{G}$ of radius $r > 0$

$$B_d(x, r) := \{y \in \mathbb{G} : d(x, y) < r\}, \qquad \mathbb{B}_d(x, r) := \{y \in \mathbb{G} : d(x, y) \le r\}.$$

When it is clear from the context, we drop the index d.
We denote the Euclidean open and closed balls on $\mathbb{G}$ by $B_E(x, r)$ and $\mathbb{B}_E(x, r)$, respectively.

Given a set $A \subset \mathbb{G}$, we define its diameter as

$$\operatorname{diam}_d(A) = \sup\{d(x, y) : x, y \in A\}.$$

Also in this case, when it is possible the index d will be omitted.

We consider now $\mathbb{G}$ in adapted coordinates with respect to a fixed adapted basis $(e_1, \dots, e_q)$. From the point of view of measures, first of all we can consider the Lebesgue measure $\mathcal{L}^q$ on $\mathbb{G}$.

Remark 2.5.1. We will refer to the measure $\mathcal{L}^q$ also for a Carnot group $\mathbb{G}$ not explicitly in adapted coordinates. In this case, when we refer to the measure $\mathcal{L}^q$ on $\mathbb{G}$ we consider the measure $(\pi_{\mathcal{B}}^{-1})_\sharp \mathcal{L}^q$, implicitly assuming that we have fixed an orthonormal adapted basis $\mathcal{B}$ of $\mathbb{G}$, so that the map $\pi_{\mathcal{B}}$ in (2.5) is well defined.

The Lebesgue measure $\mathcal{L}^q$ is left invariant and homogeneous.

Proposition 2.5.2. *[SC16, Proposition 2.19] If $\mathbb{G}$ is a Carnot group of topological dimension q and homogeneous dimension Q, the Lebesgue measure $\mathcal{L}^q$ is the Haar measure of the group, then for every measurable set $E \subset \mathbb{G}$ and for every $x \in \mathbb{G}$*

$$\mathcal{L}^q(l_x(E)) = \mathcal{L}^q(E).$$

Moreover, for every $t > 0$,

$$\mathcal{L}^q(\delta_t(E)) = t^Q \mathcal{L}^q(E).$$

By Proposition 2.5.2, it follows that for every $x \in \mathbb{G}$ and $r > 0$

$$\mathcal{L}^q(\mathbb{B}_d(x, r)) = \mathcal{L}^q(l_x(\delta_r(\mathbb{B}_d(0, 1))) = r^Q \mathcal{L}^q(\mathbb{B}_d(0, 1)).$$

Moreover, since $\mathcal{L}^q$ is a Radon measure homogeneous with respect to the family of anisotropic dilations, the Lebesgue measure of the boundary of balls is null, i.e. $\mathcal{L}^q(\partial\mathbb{B}_d(x, r)) = 0$ for every $x \in \mathbb{G}$ and $r > 0$ (see [SC16, Proposition 2.8]).

Remark 2.5.3. Notice that for every homogeneous distance d, $\operatorname{diam}_d(\mathbb{B}_d(x, r)) = 2r$ for all $x \in \mathbb{G}$ and $r > 0$ (see again [SC16, Proposition 2.8]).

Remark 2.5.4. If g is the left invariant graded Riemannian metric that we have fixed on $\mathbb{G}$, the Riemannian volume v_g on $\mathbb{G}$ coincides with $\mathcal{L}^q$ (for more details refer to [Mag02a, Proposition 2.3.47]).

Let us introduce various Hausdorff-type measures on a Carnot group. These measures arise as results of different applications of the process that we are going to describe below, called *Carathéodory's construction.* For more details please refer to [Fed69, Section 2.10].

Definition 2.5.5 (Carathéodory's construction)**.** Let $\mathcal{F} \subset \mathcal{P}(\mathbb{G})$ be a non-empty family of closed subsets of a Carnot group $\mathbb{G}$, equipped with a homogeneous distance d. Let $\zeta : \mathcal{F} \to \mathbb{R}^+$ be a function such that $0 \leq \zeta(S) < \infty$ for any $S \in \mathcal{F}$. If $\delta > 0$ and $A \subset \mathbb{G}$, we define

$$\phi_{\delta,\zeta}(A) = \inf\left\{\sum_{j=0}^{\infty} \zeta(B_j) : A \subset \bigcup_{j=0}^{\infty} B_j,\ \frac{\operatorname{diam}(B_j)}{2} \leq \delta,\ B_j \in \mathcal{F}\right\}, \tag{2.16}$$

the measure of A resulting by Carathéodory's construction is the limit

$$\lim_{\delta \to 0} \phi_{\delta,\zeta}(A) = \sup_{\delta > 0} \phi_{\delta,\zeta}(A).$$

If $\mathcal{F}$ coincides with the family of all closed subsets of $\mathbb{G}$ and $\zeta(B) = \left(\frac{\mathrm{diam}_d(B)}{2}\right)^\alpha$ we call

$$\mathcal{H}^\alpha(A) := \sup_{\delta>0} \phi_{\delta,\zeta}(A)$$

the α-*Hausdorff measure* of A.

If $\mathcal{F}$ coincides with the family of closed balls with respect to the distance d, that we denote by $\mathcal{F}_b$, and $\zeta(\mathbb{B}_d(x,r)) = r^\alpha$ we call

$$\mathcal{S}^\alpha(A) := \sup_{\delta>0} \phi_{\delta,\zeta}(A)$$

the α-*spherical Hausdorff measure* of A.

If $\mathcal{F}$ coincides with the family of all closed sets, and for $\alpha \in \{1,\dots,q\}$ we set

$$c_\alpha = \mathcal{L}^\alpha(\{x \in \mathbb{R}^\alpha : |x| \le 1\}) = \mathcal{L}^\alpha(\mathbb{B}_E(0,1)) \quad \text{and} \quad \zeta(B) = c_\alpha \left(\frac{\mathrm{diam}_E(B)}{2}\right)^\alpha,$$

we call

$$\mathcal{H}^\alpha_E(A) := \sup_{\delta>0} \phi_{\delta,\zeta}(A)$$

the α-*Euclidean Hausdorff measure.*

Remark 2.5.6. Notice that generally, in the literature, in the definition of Carathéodory's construction, (2.16) is substituted by

$$\psi_{\delta,\zeta}(A) = \inf \left\{ \sum_{j=0}^{\infty} \zeta(B_j) : A \subset \bigcup_{j=0}^{\infty} B_j,\ \mathrm{diam}(B_j) \le \delta,\ B_j \in \mathcal{F} \right\}.$$

Clearly, for every ζ and $\delta > 0$, $\phi_{\delta,\zeta}(A) = \psi_{2\delta,\zeta}(A)$, hence the limit measures built through Carathéodory's construction with respect to $\phi_{\delta,\zeta}$ or $\psi_{\delta,\zeta}$ are equal. Nevertheless, we prefer to state the formulation of Definition 2.5.5 through condition (2.16) since later on, for instance in the last chapter of the thesis, we will need to compare various different measures introduced through Carathéodory's construction, with $\mathcal{F} = \mathcal{F}_b$, with some *packing-type* measures. These comparisons will be easier if we put the constraint on the length of the radius of covering balls instead of on their diameter.

All measures obtained through Carathéodory's construction are Borel regular. Since d is a homogeneous distance, for all $m > 0$, for all measurable set $A \subset \mathbb{G}$ and for all $x \in \mathbb{G}$

$$\mathcal{H}^m(A) = \mathcal{H}^m(l_x(A)),$$
$$\mathcal{S}^m(A) = \mathcal{S}^m(l_x(A)).$$

Moreover, for every $t > 0$ we have

$$\mathcal{H}^m(\delta_t(A)) = t^m \mathcal{H}^m(A),$$
$$\mathcal{S}^m(\delta_t(A)) = t^m \mathcal{S}^m(A),$$

refer to [SC16, (2.40)] for more details.
Let us now recall the definition of Hausdorff (or metric) dimension of a a set.

Definition 2.5.7. Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d. Let

A be a subset of $\mathbb{G}$. We call *Hausdorff dimension*, or *metric dimension*, of A the number

$$\dim_H(A) = \inf\{t \in [0,\infty) : \mathcal{S}^t(A) = 0\}.$$

The Hausdorff dimension of $\mathbb{G}$ can be individuated through the Ahlfors regularity of $\mathbb{G}$. It coincides with the homogeneous dimension of $\mathbb{G}$.

Definition 2.5.8. Let (X,d) be a metric space, let μ be a Radon measure on X and fix $N \in \mathbb{N}$. The measure space (X,d,μ) is said *Ahlfors regular of dimension N* if there exist two real numbers $c_1, c_2 > 0$ such that for every $x \in X$ and $r > 0$

$$c_1 r^N \leq \mu(\mathbb{B}_d(x,r)) \leq c_2 r^N.$$

Proposition 2.5.9. *Let $\mathbb{G}$ be a Carnot group of homogeneous dimension Q equipped with a homogeneous distance d, then $(\mathbb{G}, d, S^Q)$ is Ahlfors regular of dimension Q.*

Directly, it follows that $\dim_H(\mathbb{G}) = Q$.

Remark 2.5.10. The measures $\mathcal{H}^Q$ and $\mathcal{S}^Q$ are left invariant on $\mathbb{G}$. Moreover, by Proposition 2.5.9, they are Radon and non-zero and then they are Haar measures on $\mathbb{G}$. We observed above that the Haar measure of $\mathbb{G}$ is unique, up to a constant. As a consequence, $\mathcal{H}^Q$ and $\mathcal{S}^Q$ are equal, up to positive constants, to $\mathcal{L}^q$.

For our purposes, it is useful to recall also a less known Hausdorff-type measure, first introduced in [RT88]. Given $\alpha \in [0,\infty)$ we define the *α-centered Hausdorff measure $\mathcal{C}^\alpha$* of a set $A \subset \mathbb{G}$ as

$$\mathcal{C}^\alpha(A) = \sup_{E \subset A} \mathcal{D}^\alpha(E)$$

where $\mathcal{D}^\alpha(E) = \lim_{\delta \to 0+} \mathcal{D}^\alpha_\delta(E)$, and, in turn, for every $\delta \in (0,\infty)$, $\mathcal{D}^\alpha_\delta(E) = 0$ if $E = \emptyset$ and if $E \neq \emptyset$

$$\mathcal{D}^\alpha_\delta(E) = \inf \left\{ \sum_i r_i^\alpha : E \subset \cup_i \mathbb{B}_d(x_i, r_i),\ x_i \in E,\ \operatorname{diam}_d(\mathbb{B}_d(x_i,r_i)) \leq \delta \right\}.$$

For every $\alpha > 0$, also the centered Hausdorff measure $\mathcal{C}^\alpha$ is Borel regular ([FSSC15, Lemma 2.2]), left invariant and homogeneous of degree α with respect to intrinsic dilations δ_t. In addition, all the homogeneous measures that we have introduced are equivalent. More precisely, for every $\alpha > 0$

$$\mathcal{H}^\alpha \leq \mathcal{S}^\alpha \leq \mathcal{C}^\alpha \leq 2^\alpha \mathcal{H}^\alpha \tag{2.17}$$

(see [FSSC15, Remark 2.3] and references therein). Let us recall some important results, that will be fundamental to make comparisons between different measures obtained through Carathéodory's construction.

Proposition 2.5.11. *[Fed69, Theorem 2.10.17(i)] Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d. Assume that $\mathcal{F}$ is the family of the closed subsets of $\mathbb{G}$ or the family of the closed metric balls of $\mathbb{G}$ with respect to d. Let us consider ζ and $\phi_{\delta,\zeta}$ as in Definition 2.5.5. Consider a subset $A \subset \mathbb{G}$. If there exist $t > 0$ and $\delta > 0$ such that*

$$\mu(A \cap S) \leq t\zeta(S)$$

whenever $S \in \mathcal{F}$ and $\operatorname{diam}(S) \leq \delta$, *then*

$$\mu(A) \leq t \sup_{\delta > 0} \phi_{\delta,\zeta}(A).$$

Now we introduce two notions of density of a Borel regular measure in a Carnot group and we report below two corresponding abstract differentiation theorems.

Definition 2.5.12. Let us consider a Carnot group $\mathbb{G}$ endowed with a homogeneous distance d. Let $\alpha > 0$, $x \in \mathbb{G}$ and let μ be a Borel regular measure on $\mathbb{G}$. We define the *(upper) α-centered density of μ at x*

$$\theta_c^\alpha(\mu, x) := \limsup_{r \to 0} \frac{\mu(\mathbb{B}_d(x, r))}{r^\alpha}.$$

Theorem 2.5.13. *[FSSC15, Theorem 3.1] Let $\alpha > 0$ and let μ be a Borel regular measure on $\mathbb{G}$ such that there exists a countable open covering of $\mathbb{G}$, whose elements have μ-finite measure. Let $A \subset \mathbb{G}$ be a Borel set. If $\mathcal{C}^\alpha(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}^\alpha \llcorner A$, then $\theta_c^\alpha(\mu, \cdot) : \mathbb{G} \to [0, \infty]$ is a Borel measurable function on A and for every Borel set $B \subset A$*

$$\mu(B) = \int_B \theta_c^\alpha(\mu, x) \, d\mathcal{C}^\alpha(x).$$

Let us introduce the notion of Federer density. As the centered density, it is a suitable limit superior of the ratio between the measure we wish to differentiate and the gauge of the spherical measure with respect to the class of balls, see [Mag15].

Definition 2.5.14. Let $\mathcal{F}_b$ be the family of closed metric balls with positive radius in $\mathbb{G}$ endowed with a homogeneous distance d. Let $\alpha > 0$, $x \in \mathbb{G}$ and let μ be a Borel regular measure on $\mathbb{G}$. We call *(spherical) α-Federer density of μ at x* the real number

$$\theta^\alpha(\mu, x) = \inf_{\varepsilon > 0} \sup \left\{ \frac{\mu(\mathbb{B})}{r(\mathbb{B})^\alpha} : x \in \mathbb{B} \in \mathcal{F}_b, \ \operatorname{diam}(\mathbb{B}) < \varepsilon \right\}.$$

This density naturally appears in representing Borel regular measures absolutely continuous with respect to the α-spherical Hausdorff measure. Actually the following theorem is obtained by applying [Mag15, Theorem 11] to the metric space $(\mathbb{G}, d)$, as in [Mag19, Theorem 7.2].

Theorem 2.5.15 ([Mag15, Theorem 11])**.** *Let $\alpha > 0$ and let μ be a Borel regular measure on $\mathbb{G}$ such that there exists a countable open covering of $\mathbb{G}$ whose elements have μ finite measure. If $A \subset \mathbb{G}$ is a Borel set, then $\theta^\alpha(\mu, \cdot)$ is a Borel function on A. In addition, if $\mathcal{S}^\alpha(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner A$, then for every Borel set $B \subset A$ we have*

$$\mu(B) = \int_B \theta^\alpha(\mu, x) \, d\mathcal{S}^\alpha(x).$$

Remark 2.5.16. Centered density and Federer density do not always coincide. In [Mag15] Magnani provides an example of this phenomenon: in the first Heisenberg group $\mathbb{H}^1$, equipped with its Carnot-Carathéodory metric d_c, there are a Radon measure μ, a set $A \subset \mathbb{H}^1$ and two constants $0 < k_1 < k_2$ such that $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^2 \llcorner A$ and for all $x \in A$

$$\theta_c^2(\mu, x) = k_1 < k_2 = \theta^2(\mu, x)$$

and for all $t \in (k_1, k_2)$

$$\mu(A) > t\mathcal{S}^2(A).$$

Chapter 3

Differential Calculus on and within Carnot groups

The goal of this chapter is to present many deep notions and results that nowadays are considered well-established tools to do research about geometric analysis in Carnot groups. Our presentation is organized in order to retrace the path of the classical calculus in Euclidean spaces, replacing the Euclidean notions with corresponding generalized definitions in Carnot groups, when they are available. We start from the concept of homogeneous subgroup, that can be thought as the analogue of the concept of linear subspace in Euclidean spaces. Then we introduce the notion of splitting of a Carnot group as the product of two complementary subgroups. According to our comparison, this notion is the analogue of writing an Euclidean space as a direct sum of two linear subspaces. We introduce then the family of the h-homomorphisms, whose name stands for homogeneous homomorphisms, between two Carnot groups. They generalize the concept of linear map between two Euclidean spaces. We individuate two particular families of injective and surjective h-homomorphisms, called h-monomorphisms and h-epimorphisms, respectively. Successively, we focus our attention on the notion of Pansu differentiability, introduced in [Pan89], that is a relevant generalization of the Euclidean differentiability to maps acting between two Carnot groups. Afterwords, we collect the definition and some fundamental properties of continuously Pansu differentiable maps. Then we provide a very short introduction to the theory of functions of bounded variation and of Caccioppoli sets in Carnot groups. The aim of this section is to provide, without any ambition of completeness, the basic ideas of this deep theory in order to prepare the reader to understand the state of the art about the study of regular hypersurfaces in Carnot groups, that will be, for instance, the starting point of the last section of this chapter. We move then to introduce intrinsic graphs, i.e. group-theoretical graphs of mappings acting between two complementary subgroups. We report and discuss various notions of regularity for intrinsic graphs. They have been introduced by Franchi, Serapioni and Serra Cassano and deeply investigated in the last twenty years in a long series of papers among which [FSSC06, ASCV06, FSSC07, AS09, FSSC11, FMS14, FS16, SC16]. This part of the presentation will be quite detailed, since intrinsic graphs will be the main characters of our research in the next chapters. We reserve the final section to describe some recent available results about intrinsic graphs of intrinsically regular functions, keeping the main focus on the family of maps whose target space is a one dimensional homogeneous subgroup. We devote special attention to this setting since, how we will try to convey to the reader, it can often be considered a promising starting point for future researches towards more general settings.

When nothing different is specified, $\mathbb{G}$ denotes a Carnot group endowed with a homogeneous distance d. By $\|\cdot\|$ we denote the homogeneous norm associated with d.

3.1 Homogeneous subgroups and complementary subgroups

Homogeneous subgroups are Lie subgroups closed with respect to anisotropic dilations.

Definition 3.1.1 (Homogeneous subgroup)**.** Let $\mathbb{G}$ be a Carnot group. A *homogeneous subgroup* $\mathbb{W} \subset \mathbb{G}$ is a Lie subgroup such that

$$\delta_t(w) \in \mathbb{W}$$

for every $w \in \mathbb{W}$ and $t > 0$.

More in general, an *homogeneous set* is a set $E \subset \mathbb{G}$ such that $\delta_t(E) \subset E$ for every $t > 0$.

Remark 3.1.2. By the classical theory of Lie groups, since $\mathbb{G}$ is simply connected and nilpotent, the exponential map gives a standard one-to-one correspondence between Lie subgroups of a Carnot group and Lie subalgebras of its Lie algebra. Hence, according to the identifications that we introduced in the previous chapter, the set of the homogeneous subgroups of a Carnot group $\mathbb{G}$ coincides with the set of the homogeneous subalgebras of $\mathbb{G}$.

Remark 3.1.3. Homogeneous subgroups are in particular homogeneous linear subspaces of $\mathbb{G}$. In addition, every homogeneous subgroup $\mathbb{W}$ of a Carnot group $\mathbb{G} = V_1 \oplus \cdots \oplus V_\kappa$ can be written as a direct sum of linear subspaces

$$\mathbb{W} = N_1 \oplus \cdots \oplus N_\kappa,$$

where N_i is a subspace of V_i for $i = 1, \dots, \kappa$ (see for instance [Mag13, Theorem 7.2]). The homogeneous dimension of $\mathbb{W}$ is the number $\sum_{i=1}^{\kappa} i(\dim(N_i))$.

The topological dimension of a subgroup is its dimension as vector space and it is always smaller or equal to its homogeneous dimension, that, in turn, coincides with its Hausdorff dimension with respect to any homogeneous distance fixed on $\mathbb{G}$.

Definition 3.1.4. A homogeneous subgroup $\mathbb{W}$ is called *horizontal* if $\mathbb{W} \subset V_1$.

Remark 3.1.5. Any horizontal subgroup $\mathbb{V} \subset \mathbb{G}$ is commutative. More precisely, a horizontal subgroup $\mathbb{V}$ is a subalgebra contained in the first layer V_1, hence, necessarily it is abelian, then, by the form of the product (2.3), $\mathbb{V}$ is commutative as a subgroup. In particular, the topological dimension of a horizontal subgroup coincides with the metric one. Hence, if we denote by k this value, $\mathbb{V}$ is isomorphic and isometric to $\mathbb{R}^k$.

Definition 3.1.6. A homogeneous subgroup $\mathbb{W}$ is called *vertical* if for some $1 \le \ell \le \kappa$,

$$\mathbb{W} = N_\ell \oplus V_{\ell+1} \oplus \cdots \oplus V_\kappa,$$

where N_ℓ is a linear subspace of V_ℓ.

Remark 3.1.7. Every vertical subgroup is normal, namely $xwx^{-1} \in \mathbb{W}$ for every $x \in \mathbb{G}$ and $w \in \mathbb{W}$. An homogeneous subgroup $\mathbb{W}$ is normal if and only if it is an ideal with respect to the Lie algebra structure, i.e. $[x, w] \in \mathbb{W}$ for every $x \in \mathbb{G}$, $w \in \mathbb{W}$ (refer for instance to [Var84, Theorem 2.13.5]).

Remark 3.1.8. Throughout the whole thesis, if $\mathbb{W}$ is a homogeneous subgroup of a Carnot group $\mathbb{G}$, when we consider an open (resp. closed) set $U \subset \mathbb{W}$ we mean a relatively open (resp. closed) set in $\mathbb{W}$.

Definition 3.1.9 (Complementary subgroups)**.** Given two subgroups $\mathbb{W}$, $\mathbb{V}$ of a $\mathbb{G}$. We call them *complementary subgroups* if they are two homogeneous subgroups such that $\mathbb{W} \cap \mathbb{V} = \{0\}$ and $\mathbb{G} = \mathbb{W}\mathbb{V}$.

If $\mathbb{W}$ is a normal subgroup, the product $\mathbb{W}\mathbb{V}$ is semidirect. In this case we write $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$. If $\mathbb{W}$ and $\mathbb{V}$ are both normal, $\mathbb{W}\mathbb{V}$ is a direct product and we write $\mathbb{G} = \mathbb{W} \times \mathbb{V}$.

Remark 3.1.10. According to our identifications, for instance by Proposition [Mag13, Proposition 7.6], two subsets $\mathbb{W}, \mathbb{V} \subset \mathbb{G}$ are complementary subgroups such that $\mathbb{G} = \mathbb{W}\mathbb{V}$ if and only if they are two homogeneous subalgebras such that $\mathbb{G} = \mathbb{W} \oplus \mathbb{V}$.

Remark 3.1.11. By Remark 3.1.10, if a Carnot group $\mathbb{G}$ is the product of two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$ and $\mathbb{V}$ is one-dimensional, then $\mathbb{V}$ is necessarily horizontal and the product is semidirect.

Remark 3.1.12. If a Carnot group $\mathbb{G} = \mathbb{W}\mathbb{V}$ is the product of two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$ such that $\mathbb{W}$ is normal, then $\mathbb{V}$ is a Carnot group. See [Mag13, Proposition 8.2] or [AM20a, Remark 2.1] for more details.

Definition 3.1.13. If a Carnot group $\mathbb{G} = \mathbb{W}\mathbb{V}$ is the product of two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$, then for every $x \in \mathbb{G}$ there exist unique two elements $x_{\mathbb{W}} \in \mathbb{W}$ and $x_{\mathbb{V}} \in \mathbb{V}$ such that $x = x_{\mathbb{W}} x_{\mathbb{V}}$. We call *group projections* on $\mathbb{W}$ and $\mathbb{V}$, respectively, the two mappings

$$\pi_{\mathbb{W}} : \mathbb{G} \to \mathbb{W},\ \pi_{\mathbb{W}}(x) = x_{\mathbb{W}}, \quad \pi_{\mathbb{V}} : \mathbb{G} \to \mathbb{V},\ \pi_{\mathbb{V}}(x) = x_{\mathbb{V}}.$$

Remark 3.1.14. It is a standard fact that if $\mathbb{G} = \mathbb{W}\mathbb{V}$, then also $\mathbb{G} = \mathbb{V}\mathbb{W}$, hence the notion of being complementary subgroups does not depend on the order under which the two homogeneous subgroups are taken. Nevertheless, the definition of the projection mappings $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}$ relative to a splitting $\mathbb{G} = \mathbb{W}\mathbb{V}$ depends on that order. Later on it will be necessary to introduce some more precise notation to better specify the splitting with respect to which the group projections are meant.

Remark 3.1.15. Given a generic splitting of $\mathbb{G}$ as the product of two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$, the group projections $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}$ are not necessarily Lipschitz maps, with respect to the homogeneous distance of $\mathbb{G}$ restricted to subgroups. For instance, by [FS16, Proposition 2.19], the Lipschitz continuity of the projection $\pi_{\mathbb{V}}$ is guaranteed only when $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$ is a semidirect product and, vice versa, $\pi_{\mathbb{W}}$ is Lipschitz continuous when $\mathbb{V}$ is normal. Nevertheless, notice that by [FS16, Proposition 2.17] the group projections $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}$, read in coordinates, are always polynomial maps, hence they are in particular C^∞ maps.

Remark 3.1.16 (Subgroups in the Heisenberg group)**.** It is easy to realize that any homogeneous subgroup $\mathbb{W}$ of the Heisenberg group $\mathbb{H}^n$ is either horizontal, i.e. $\mathbb{W} \subset H_1$, or vertical, i.e. $H_2 \subset \mathbb{W}$. If $\mathbb{W}$ is horizontal and if we denote its dimension by k, surely $1 \leq k \leq n$. On the other side, $\mathbb{W}$ is normal if and only if it is vertical.

Assume now that $\mathbb{H}^n$ is the product of two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$, $\mathbb{H}^n = \mathbb{W}\mathbb{V}$. The structure of $\mathbb{H}^n$ permits to deduce some features of $\mathbb{V}$ and $\mathbb{W}$. Necessarily, one of the two subgroups, for instance $\mathbb{W}$, contains the second layer H_2, then it is vertical, and consequently normal. As a consequence, the second one, $\mathbb{V}$, is surely a horizontal subgroup, and then it is commutative. Then the topological dimension of $\mathbb{V}$, that we denote by k, equals the homogeneous one, and k has to be strictly smaller than $n+1$. As a consequence, the topological dimension of $\mathbb{W}$ equals $2n+1-k$, while the homogeneous one is equal to $2n+2-k$.

When we have a splitting, the following useful propositions hold.

Proposition 3.1.17. *[FS16, Proposition 2.12] If $\mathbb{G} = \mathbb{W}\mathbb{V}$ is the product of two complementary subgroups, there exists a constant $c_0 = c_0(\mathbb{W}, \mathbb{V}) > 0$ such that*

$$c_0(\|w\| + \|v\|) \leq \|wv\| \leq \|w\| + \|v\| \tag{3.1}$$

for all $w \in \mathbb{W}$, $v \in \mathbb{V}$.

Proposition 3.1.18. *[FS16, Corollary 2.15] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, then*

$$c_0\|\pi_{\mathbb{V}}(x)\| \leq \operatorname{dist}(x, \mathbb{W}) \leq \|\pi_{\mathbb{V}}(x)\|$$

for all $x \in \mathbb{G}$, where $c_0 = c_0(\mathbb{W}, \mathbb{V})$ is the constant given by Proposition 3.1.17 and $\operatorname{dist}(x, \mathbb{W}) = \inf\{d(x, w) : w \in \mathbb{W}\}$.

3.1.1 Measures on homogeneous subgroups

We collect in this section some considerations about suitable measures concentrated on homogeneous subgroups. Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d and let $\mathbb{W} \subset \mathbb{G}$ be a homogeneous subgroup of topological dimension n and Hausdorff dimension N. Since $\mathbb{W}$ is a linear subspace of $\mathbb{G}$, the Lebesgue measure $\mathcal{L}^n$ coincides with the Hausdorff measure $\mathcal{H}^n_E$ on the measurable subsets of $\mathbb{W}$.

Remark 3.1.19. We slightly abused of the notation in the previous sentence, so let us explain it more carefully. In particular, we can fix an orthonormal basis $(b_1, \dots, b_n)$ of $\mathbb{W}$ and then, considering the isometry

$$i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^n, \ i_{\mathbb{W}}\left(\sum_{i=1}^n x_i b_i\right) = (x_1, \dots, x_n), \tag{3.2}$$

we obtain the relation

$$(i_{\mathbb{W}})_\sharp(\mathcal{H}^n_E \llcorner \mathbb{W}) = \mathcal{L}^n. \tag{3.3}$$

Clearly this process can be exploited for any homogeneous subgroup. Hence, we say that the measures $\mathcal{L}^n$ and $\mathcal{H}^n_E$ coincide in the sense expressed by (3.3). Actually, when we refer to the measure $\mathcal{L}^n$ on the subsets of $\mathbb{W}$, we implicitly refer to the measure $\mathcal{H}^n_E \llcorner \mathbb{W}$.

By taking in consideration the explicit polynomial form of the product (given in adapted coordinates by Proposition 2.2.4), it is clear that the Jacobian of any left translation of the group $l_x : \mathbb{G} \to \mathbb{G}$ is unitary at any point, for every $x \in \mathbb{G}$. Clearly, this keeps on being true for the restricted family of translations $l_w|_{\mathbb{W}} : \mathbb{W} \to \mathbb{W}$, with $w \in \mathbb{W}$, and this implies that $\mathcal{H}^n_E$ is a left invariant measure on $\mathbb{W}$, equipped with the group product of $\mathbb{G}$ restricted to $\mathbb{W}$, i.e.

$$\mathcal{H}^n_E(B) = \mathcal{H}^n_E(l_w(B))$$

for every measurable set $B \subset \mathbb{W}$ and every $w \in \mathbb{W}$. Again working by coordinates, in [FS16, Lemma 2.20] (see also [Ser08, Lemma 4.7]) it is proved by explicit computation that the Jacobian of the restricted dilation $\delta_t|_{\mathbb{W}}$ is t^N, at any point of $\mathbb{W}$, for every $t > 0$. Thus, the measure $\mathcal{H}^n_E \llcorner \mathbb{W}$ is N-homogeneous with respect to the family of dilations $\delta_t|_{\mathbb{W}}$, for $t > 0$, hence for every $x \in \mathbb{W}$ we have

$$\mathcal{H}^n_E(\mathbb{B}(x,t) \cap \mathbb{W}) = t^N \mathcal{H}^n_E(\mathbb{B}(x,1) \cap \mathbb{W}).$$

Since d is a left invariant distance, also the Hausdorff measures $\mathcal{S}^N \llcorner \mathbb{W}$ and $\mathcal{H}^N \llcorner \mathbb{W}$ are left invariant on $\mathbb{W}$. Moreover they are Radon measures. In fact, since the measure $\mathcal{H}^n_E$ is N-homogeneous, it is Ahlfors N-regular on $\mathbb{W}$, therefore there are two constants $0 < c < C$ such that

$$c\mathcal{H}^n_E(B) \leq \mathcal{H}^N(B) \leq C\mathcal{H}^n_E(B)$$

for every Borel set $B \subset \mathbb{W}$ (see for instance [Hei01, Exercise 8.11]). Taking into account the basic comparisons between the Hausdorff measure and the spherical Hausdorff measure (refer to (2.17)), we infer that both $\mathcal{S}^N \llcorner \mathbb{W}$ and $\mathcal{H}^N \llcorner \mathbb{W}$ are non-zero and locally finite. For a reference about the previous considerations, one can refer also to [JNGV20, Lemma 3.1]. Then, by the uniqueness of the Haar measure of $\mathbb{W}$, there exist two constants C_1 and C_2 such that for every measurable subset $B \subset \mathbb{W}$,

$$\mathcal{H}^n_E(B) = C_1 \mathcal{S}^N(B) \quad \mathcal{H}^n_E(B) = C_2 \mathcal{H}^N(B). \tag{3.4}$$

By left invariance and homogeneity, the two constants in (3.4) can be computed as

$$C_1 = \frac{\mathcal{H}^n_E(\mathbb{B}(0,1) \cap \mathbb{W})}{\mathcal{S}^N(\mathbb{B}(0,1) \cap \mathbb{W})} \quad \text{and} \quad C_2 = \frac{\mathcal{H}^n_E(\mathbb{B}(0,1) \cap \mathbb{W})}{\mathcal{H}^N(\mathbb{B}(0,1) \cap \mathbb{W})}.$$

Again by [FS16, Lemma 2.20], the Jacobian of the map

$$\sigma_x : \mathbb{W} \to \mathbb{W}, \ \sigma_x(w) := \pi_{\mathbb{W}}(xw)$$

is unitary at any point for every $x \in \mathbb{G}$. This consideration and the homogeneity of $\mathcal{H}^n_E$ imply that, if $\mathbb{W}$ is complemented, namely if there exists a homogeneous subgroup $\mathbb{V} \subset \mathbb{G}$ complementary to $\mathbb{W}$, setting $c := \mathcal{L}^n(\pi_{\mathbb{W}}(\mathbb{B}(0,1)))$ we have

$$\mathcal{L}^n(\pi_{\mathbb{W}}(\mathbb{B}(x,r)) = cr^N \tag{3.5}$$

for all balls $\mathbb{B}(x,r) \subset \mathbb{G}$. The property (3.5) gives a control on the measure of projected sets through the group projection $\pi_{\mathbb{W}}$ relative to the splitting $\mathbb{G} = \mathbb{W}\mathbb{V}$ and this is quite surprising, since, if $\mathbb{V}$ is not a normal subgroup, the group projection $\pi_{\mathbb{W}}$ can be a non-Lipschitz map between $\mathbb{G}$ and $\mathbb{W}$, endowed with the restriction of the homogeneous distance d (see Remark 3.1.15).

Let us now collect three lemmas about a splitting of a Carnot group as the product of a normal homogeneous subgroup $\mathbb{W}$ and a homogeneous linear subspace V, that is not necessarily required to be a subgroup. These results have been proved in [Mag18, Lemmas 9.2, 9.3, 9,4] (see also [Mag20, Lemmas 3.3, 3.4, 3.5]), where nevertheless $\mathbb{W}$ was assumed to be a vertical subgroup. Since we slightly weaken the hypotheses of the original three lemmas, we report their proofs.

Lemma 3.1.20. *Let $\mathbb{G}$ be a Carnot group, let $V \subset \mathbb{G}$ be a homogeneous linear subspace and let $\mathbb{W}$ be a normal homogeneous subgroup such that $\mathbb{G} = V \oplus \mathbb{W}$, then the map*

$$F : V \times \mathbb{W} \to \mathbb{G}, \ F(v,w) = vw$$

is a diffeomorphism. Moreover, its inverse map $T : \mathbb{G} \to V \times \mathbb{W}$ *is defined by*

$$T(x) = (P_V(x), \Pi_{\mathbb{W}}(x)),$$

where $P_V : \mathbb{G} \to V$ *is the linear projection related to the direct sum* $V \oplus \mathbb{W}$ *and* $\Pi_{\mathbb{W}}(x) := P_V(x)^{-1}v$.

Proof. Since the differential of F at $(0,0)$ is a linear isomorphism, F is a diffeomorphism from a neighbourhood of $(0,0)$ to a neighbourhood of $0 \in \mathbb{G}$. Since $F(\delta_t v, \delta_t w) = \delta_t(F(v,w))$ for every $t > 0$, $v \in V$ and $w \in \mathbb{W}$, F is a surjective diffeomorphism onto $\mathbb{G}$.

Now we consider

$$vw = v'w',$$

where $v, v' \in V$ and $w, w' \in \mathbb{W}$. By the Baker-Campbell-Hausdorff formula, we have

$$v + w + \sum_{j=2}^{\kappa} c_j(v,w) = v' + w' + \sum_{j=2}^{\kappa} c_j(v',w'). \tag{3.6}$$

Since $\mathbb{W}$ is a normal subgroup, it is an ideal of $\mathbb{G}$ and then

$$w + \sum_{j=2}^{\kappa} c_j(v,w),\ w' + \sum_{j=2}^{\kappa} c_j(v',w') \in \mathbb{W}.$$

Now we apply P_V to equation (3.6) and we obtain that

$$v = P_V(vw) = P_V(v'w') = v'.$$

Then one can write any element $x \in \mathbb{G}$ as $x = P_V(x)\Pi_{\mathbb{W}}(x)$. □

Lemma 3.1.21. *Let* $\mathbb{G}$ *be a Carnot group, let* $V \subset \mathbb{G}$ *be a homogeneous linear subspace and let* $\mathbb{W}$ *be a normal homogeneous subgroup such that* $\mathbb{G} = V \oplus \mathbb{W}$*, then for every* $v \in V$

$$v + \mathbb{W} = v\mathbb{W}.$$

Proof. By the Baker-Campbell-Hausdorff formula and the fact that $\mathbb{W}$ is an ideal with respect to the Lie algebra structure of $\mathbb{G}$ for every $w \in \mathbb{W}$ it holds that $c_j(v,w) \in \mathbb{W}$ for every $j = 2, \dots, \kappa$ and then

$$vw = v + w + \sum_{j=2}^{\kappa} c_j(v,n) \in v + \mathbb{W}.$$

Hence $v\mathbb{W} \subset v + \mathbb{W}$. On the other side we can apply Lemma 3.1.20 to $v + w$ obtaining that $v + w = P_V(v+w)\Pi_{\mathbb{W}}(v+w) = v\Pi_{\mathbb{W}}(v+w) \in V\mathbb{W}$, then $v + \mathbb{W} \subset v\mathbb{W}$. □

Proposition 3.1.22. *If* $\mathbb{G}$ *is a Carnot group and* $\mathbb{W}$ *is a normal homogeneous subgroup of topological dimension* n*, then for every* $x \in \mathbb{G}$*, we have*

$$\mathcal{H}^n_E(B) = \mathcal{H}^n_E(l_x(B))$$

for every measurable set $B \subset \mathbb{W}$.

Proof. We consider $x \in \mathbb{G}$ and the map $l_x|_{\mathbb{W}} : \mathbb{W} \to \mathbb{G}$. First of all, by Lemma 3.1.21 we know that $l_x(\mathbb{W}) = x\mathbb{W} = x + \mathbb{W}$. In fact, by the Baker-Campbell-Hausdorff formula, for

$w \in \mathbb{W}$ we get

$$l_x(w) = l_x|_{\mathbb{W}}(w) = x + w + \sum_{j=2}^{\kappa} c_j(x,w) \in x + \mathbb{W}.$$

Since $\mathbb{W}$ is normal, it is an ideal with respect to the Lie algebra structure of $\mathbb{G}$ and then

$$c_j(x,w) \in \mathbb{W} \text{ for every } j = 2, \dots, \kappa. \tag{3.7}$$

The subgroup $\mathbb{W}$ can be written as

$$\mathbb{W} = W_1 \oplus \cdots \oplus W_\kappa$$

with W_i linear subspace of V_i for every $i = 1, \dots, \kappa$. Clearly $\sum_{j=1}^{\kappa} \dim(W_j) = \dim(\mathbb{W})$. Let us now consider the homogeneous subspace $L := \mathbb{W}^\perp$ and notice that

$$L = L_1 \oplus \cdots \oplus L_k$$

where L_i, for every $i = 1, \dots, \kappa$ is a linear subspace of V_i such that $V_i = L_i \oplus W_i$.
Let us now fix an adapted orthonormal basis $(e_1 \dots, e_q)$ of $\mathbb{G}$ such that for every $\ell = 0, \dots, \kappa - 1$

$$(e_{h_\ell+1}, \dots, e_{h_\ell+\dim(L_{\ell+1})}) \text{ is a basis of } L_\ell$$

and

$$(e_{h_\ell+\dim(L_{\ell+1})+1}, \dots, e_{h_{\ell+1}}) \text{ is a basis of } W_\ell.$$

We denote by I_L and I_W the sets of indexes

$$I_L := \{j \in \{1, \dots, q\} : e_j \in L\} \qquad I_W := \{1, \dots, q\} \setminus I_L.$$

From now one we consider $\mathbb{G}$ in adapted coordinates with respect to the fixed basis $(e_1, \dots, e_q)$. Let us compute explicitly the left translation restricted to $\mathbb{W}$, $l_x|_{\mathbb{W}}$, where $x = \sum_{j=1}^{q} x_j e_j \in \mathbb{G}$. We consider $w \in \mathbb{W}$

$$w = \sum_{j \in I_W} w_j e_j = \sum_{\ell=0}^{\kappa-1} \sum_{j=h_\ell+\dim(L_{\ell+1})+1}^{h_{\ell+1}} w_j e_j \in \mathbb{W}.$$

By reading the Baker-Campbell-Hausdorff formula in adapted coordinates, i.e. by applying Proposition 2.2.4, the translation $l_x(w)$ has the following form

$$l_x(w) = x + w + Q(x,w) = \sum_{j \in I_L} (x_j + Q_j(x,w)) e_j + \sum_{j \in I_W} (x_j + w_j + Q_j(x,w)) e_j,$$

where the map Q is

$$Q = (Q_1, Q_2, \dots, Q_q) : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$$

where $Q_j : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ for $j \in \{1, \dots, q\}$ are the components of Q.
By the form of the product in coordinates, for every $j \in \{1, \dots, q\}$, if $j \in \{h_\ell+1, \dots, h_{\ell+1}\}$ for some $\ell \in \{0, \dots, \kappa - 1\}$, then

$$Q_j(x,w) \text{ depends only on } \{x_s\}_{s \le h_\ell}, \{w_i\}_{i \in I_W, i \le h_\ell}. \tag{3.8}$$

Moreover, by (3.7) we can deduce that

$$Q_j(x,w)=0 \text{ for every } j\in I_L.$$

Hence we can explicitly rewrite l_x as $l_x(w)=((xw)_1,\dots,(xw)_q)$, with

$$\begin{aligned}(xw)_j&=x_j \quad \text{for } j\in I_L\\ (xw)_j&=x_j+w_j+Q_j(x,w) \quad \text{for } j\in I_W.\end{aligned} \tag{3.9}$$

Now we consider the two isometries

$$T:\mathbb{W}\to\mathbb{R}^n$$

$$T(w)=(w_{\dim(L_1)+1},\dots,w_{h_1},w_{h_1+\dim(L_2)+1},\dots,w_{h_2},\dots,\dots,w_{h_{\kappa-1}+\dim(L_\kappa)+1},\dots,w_q),$$

for $w=\sum_{j\in I_W} w_je_j$ and

$$R:x+\mathbb{W}\to\mathbb{R}^n,$$

$$R(x+w)=(w_{\dim(L_1)+1},\dots,w_{h_1},w_{h_1+\dim(L_2)+1},\dots,w_{h_2},\dots,\dots,w_{h_{\kappa-1}+\dim(L_\kappa)+1},\dots,w_q),$$

if $w=\sum_{j\in I_W} w_je_j$ and we take in consideration the map

$$F:=R\circ l_x\circ T^{-1}:\mathbb{R}^n\to\mathbb{R}^n.$$

In particular, setting $\eta=(\eta_1,\dots,\eta_n)\in\mathbb{R}^n$, we explicitly have

$$\begin{aligned}F(\eta_1,\dots,\eta_n)=&(\eta_1,\dots,\eta_{\dim(W_1)},\\ &\eta_{\dim(W_1)+1}+Q_{h_1+\dim(L_2)+1}(x,T^{-1}(\eta)),\dots,\eta_{\dim(W_1)+\dim(W_2)}+Q_{h_2}(x,T^{-1}(\eta)),\\ &\dots,\\ &\eta_{\sum_{i=1}^{\kappa-1}\dim(W_i)+1}+Q_{h_{\kappa-1}+\dim(L_\kappa)+1}(x,T^{-1}(\eta)),\dots,\eta_n+Q_q(x,T^{-1}(\eta))).\end{aligned}$$

Hence, for $i=1,\dots,n$, if $j\in\{1,\dots,\dim(W_1)\}$

$$\frac{\partial F_j}{\partial\eta_i}(\cdot)=\delta_{i,j}.$$

On the other hand, if $j\geq\dim(W_1)+1$, there is an integer $\ell=\ell(j)\in\{1,\dots,\kappa-1\}$ such that $\sum_{p=1}^{\ell}\dim(W_p)+1\leq j\leq\sum_{p=1}^{\ell+1}\dim(W_p)$ and

$$\frac{\partial F_j}{\partial\eta_i}(\cdot)=\delta_{i,j}+\frac{\partial Q_{h_\ell+\dim(L_\ell)+m}(x,T^{-1}(\cdot))}{\partial\eta_i},$$

where m is the integer $m=j-\sum_{p=1}^{\ell}\dim(W_p)$ (surely $1\leq m\leq\dim(W_{\ell+1})$) and $\delta_{i,j}$ is the Kronecker delta. The crucial observation now is that, by the definition of the map T and by (3.8), if $i>\sum_{p=1}^{\ell}\dim(W_p)$,

$$\frac{\partial Q_{h_\ell+\dim(L_\ell)+m}(x,T^{-1}(\cdot))}{\partial\eta_i}=0,$$

then, if $i\geq j$, since $j>\sum_{p=1}^{\ell}\dim(W_p)$, surely

$$\frac{\partial F_j}{\partial\eta_i}(\cdot)=\delta_{i,j}.$$

Then we can conclude that at any point of $\mathbb{R}^n$ the Jacobian of F is $JF = 1$.
Since T and R are isometries we have

$$R_\sharp \mathcal{H}^n_E = \mathcal{L}^n \qquad (T^{-1})_\sharp \mathcal{L}^n = \mathcal{H}^n_E.$$

Then F preserves the Lebesgue measure $\mathcal{L}^n$ and we can conclude the proof as follows

$$\begin{aligned}\mathcal{H}^n_E(l_x(B)) &= R_\sharp \mathcal{H}^n_E(F \circ T(B)) = \mathcal{L}^n(F(T(B))) = \mathcal{L}^n(T(B)) \\ &= (T^{-1})_\sharp \mathcal{L}^n(B) = \mathcal{H}^n_E(B).\end{aligned} \tag{3.10}$$

□

3.2 Pansu differentiability

The notions and the results about Pansu differentiability in this section originate mainly by the work of Pansu [Pan89] and by the ideas presented therein. In this section we consider two Carnot groups endowed with homogeneous distances $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$, of topological dimension p and q and Hausdorff dimension Q and P, respectively. The groups $\mathbb{G}$ and $\mathbb{M}$ are direct sums of linear subspaces as follows

$$\mathbb{G} = V_1 \oplus \cdots \oplus V_\kappa \qquad \mathbb{M} = W_1 \oplus \cdots \oplus W_\vartheta.$$

We denote by δ^1_t and δ^2_t the intrinsic anisotropic dilations of $\mathbb{G}$ and $\mathbb{M}$, respectively, associated with the scale $t > 0$. As usual we consider both $\mathbb{G}$ and $\mathbb{M}$ equipped, respectively, with a graded left invariant Riemannian metric.

3.2.1 Homogeneous homomorphisms

Definition 3.2.1. A map $L : \mathbb{G} \to \mathbb{M}$ is a *h-homomorphism*, that stands for *homogeneous homomorphism*, if it is a group homomorphism such that $L(\delta^1_t(x)) = \delta^2_t(L(x))$ for any $x \in \mathbb{G}$ and $t > 0$.

Remark 3.2.2. According to our usual identifications, $L : \mathbb{G} \to \mathbb{M}$ is a group homomorphism if and only if it is a Lie algebra homomorphism. Hence, any group homomorphism is, in particular, a linear map.

We denote by $\mathcal{L}_h(\mathbb{G}, \mathbb{M})$ the family of h-homomorphisms from $\mathbb{G}$ to $\mathbb{M}$.

Remark 3.2.3. For every $t > 0$, $\delta^1_t \in \mathcal{L}_h(\mathbb{G}, \mathbb{G})$.

Remark 3.2.4. If $L : \mathbb{G} \to \mathbb{M}$ is a h-homomorphism, it is immediate to verify that $\ker(L)$ is a normal homogeneous subgroup.

Given two h-homomorphisms $L, T \in \mathcal{L}_h(\mathbb{G}, \mathbb{M})$, we define the distance

$$d_{\mathcal{L}_h(\mathbb{G},\mathbb{M})}(L, T) := \sup_{x \in \mathbb{B}(0,1)} d_2(L(x), T(x))$$

and we denote by $\|L\|_{\mathcal{L}_h(\mathbb{G},\mathbb{M})} := d_{\mathcal{L}_h(\mathbb{G},\mathbb{M})}(L, I)$, where $I : \mathbb{G} \to \mathbb{M}$ is the map that associates with any point of $\mathbb{G}$ the unit element of $\mathbb{M}$, $I(x) \equiv 0$ for every $x \in \mathbb{G}$.

Notice that, if we fix two adapted orthonormal bases $(v_1, \dots, v_q) \subset \mathbb{G}$ and $(w_1, \dots, w_p) \subset \mathbb{M}$, we can consider $\mathbb{G}$ and $\mathbb{M}$ in adapted coordinates with respect to the two fixed bases, as we said above. Any h-homomorphism L is in particular a linear map from $\mathbb{G}$ to $\mathbb{M}$. When $q \geq p$, we call the *Jacobian of* L the number

$$JL := \sqrt{\det(LL^*)},$$

where L^* denotes the adjoint map $L^* : \mathbb{M} \to \mathbb{G}$ of L. The Jacobian JL is the Euclidean algebraic Jacobian of L from $\mathbb{R}^q$ to $\mathbb{R}^p$, or, equivalently, it is the Jacobian of L between the two Lie algebras $\mathbb{G}$ and $\mathbb{M}$, with respect to the graded scalar products fixed on $\mathbb{G}$ and $\mathbb{M}$.

Proposition 3.2.5. *[Mag02a, Theorem 3.1.12] Any h-homomorphism* $L : \mathbb{G} \to \mathbb{M}$ *is Lipschitz and satisfies the following contact property*

$$L(V_j) \subset W_j \quad \text{for } j = 1, \dots, \kappa. \tag{3.11}$$

Remark 3.2.6. In particular, if $\mathbb{M}$ is the commutative group $\mathbb{R}^p$, hence if $\mathbb{M} = W_1 = \mathbb{R}^p$, then

$$L(V_j) = \{0\} \quad \text{for } j = 2, \dots, \kappa.$$

In this case, $L : \mathbb{G} \to \mathbb{R}^p$ can be identified in coordinates with a $p \times m_1$ matrix M_L with real coefficients such that for every $x \in \mathbb{G}$,

$$L(x) = M_L \cdot (\pi(x))^T, \tag{3.12}$$

where $\cdot$ denotes the matrix product and π denotes the orthogonal projection on the first layer V_1 of $\mathbb{G}$: $\pi : \mathbb{G} \to V_1,\ \pi(x_1, \dots, x_q) := (x_1, \dots, x_{m_1})$.

Remark 3.2.7. Any Lie algebra homomorphism $L : \mathbb{G} \to \mathbb{M}$ that satisfies the contact property (3.11) is a h-homomorphism.

Definition 3.2.8 (h-isomorphism)**.** An invertible h-homomorphism $L : \mathbb{G} \to \mathbb{M}$ is called a *h-isomorphism.*

For more details about the properties of h-homomorphisms, we refer the reader to [Mag02a, Section 3.1] or to [Mag01, Section 3.1]. Now we introduce two particular families of injective and surjective h-homomorphisms presented in [Mag13, Definition 2.5].

Definition 3.2.9 (h-monomorphism)**.** Let $L : \mathbb{G} \to \mathbb{M}$ be a h-homomorphism. L is a *h-monomorphism* if it has a left inverse that is also a h-homomorphism.

These objects have been characterized by their property of factorizing codomain. Actually, requiring the existence of a left inverse is a stronger condition than requiring injectivity.

Proposition 3.2.10. *[Mag13, Lemma 7.11] Let* $L : \mathbb{G} \to \mathbb{M}$ *be a h-homomorphism and let* $\mathbb{H}$ *be its image, then* L *is a h-monomorphism if and only if* L *is injective and there exists a normal subgroup* $\mathbb{K} \subset \mathbb{G}$ *complementary to* $\mathbb{H}$*.*

Definition 3.2.11 (h-epimorphism)**.** Let $L : \mathbb{G} \to \mathbb{M}$ be a h-homomorphism. L is a *h-epimorphism* if it has a right inverse that is also a h-homomorphism.

These objects have been characterized by their property of factorizing domain. Actually, requiring the existence of a left inverse is a stronger condition than requiring surjectivity.

Proposition 3.2.12. *[Mag13, Lemma 7.10] Let $L : \mathbb{G} \to \mathbb{M}$ be a h-homomorphism and let $\mathbb{K}$ be its kernel, then L is an h-epimorphism if and only if L is surjective and there exists a homogeneous subgroup $\mathbb{H} \subset \mathbb{G}$ complementary to $\mathbb{K}$.*

If $L : \mathbb{G} \to \mathbb{M}$ is a h-epimorphism, the restriction $L|_{\mathbb{H}}$ is a h-isomorphism.

Remark 3.2.13. If $L : \mathbb{G} \to \mathbb{M}$ is a h-homomorphism and there exists a homogeneous subgroup $\mathbb{V} \subset \mathbb{G}$ such that $T := L|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is a h-isomorphism, then L is a h-epimorphism since T^{-1} is a right inverse of L. In particular $\mathbb{V}$ and $\ker(L)$ are complementary subgroups of $\mathbb{G}$ and any point $x \in \mathbb{G}$ can be written as $x(T^{-1}(L(x)))^{-1}T^{-1}(L(x))$, with $x(T^{-1}(L(x)))^{-1} \in \ker(L)$ and $T^{-1}(L(x)) \in \mathbb{V}$.

For more details about h-monomorphisms and h-epimorphisms, please refer to [Mag13].

3.2.2 Pansu differentiable functions

We set $\|x\|_1 := d_1(x, 0)$, for every $x \in \mathbb{G}$, and $\|x\|_2 := d_2(x, 0)$, for every $x \in \mathbb{M}$.

Definition 3.2.14. Let Ω be an open set in $\mathbb{G}$, let $f : \Omega \to \mathbb{M}$ be a function and let $\bar{x} \in \Omega$. We say that f is *Pansu differentiable* at $\bar{x}$ if there exists a h-homomorphism $L : \mathbb{G} \to \mathbb{M}$ that satisfies

$$\|L(\bar{x}^{-1}y)^{-1}f(\bar{x})^{-1}f(y)\|_2 = o(\|\bar{x}^{-1}y\|_1) \quad \text{as} \quad \|\bar{x}^{-1}y\|_1 \to 0. \tag{3.13}$$

If condition (3.13) is verified, we call L the *Pansu differential* of f at $\bar{x}$ and we denote it by $Df(\bar{x})$.

Proposition 3.2.15. *[Mag02a, Proposition 3.2.2] If f is Pansu differentiable at a point $\bar{x}$, the Pansu differential $Df(\bar{x})$ is unique, and consequently it is well defined.*

Pansu differentiability is a natural generalization of the Euclidean differentiability of maps acting between two Euclidean spaces to functions acting between two Carnot groups. A convincing evidence in support of this fact is a Rademacher-type theorem available in this context, the following *Pansu-Rademacher Theorem.*

Theorem 3.2.16. *[Pan89, Theorem 2] Let $\Omega \subset \mathbb{G}$ be an open set and let $f : \Omega \to \mathbb{M}$ be a function. If f is Lipschitz, then it is Pansu differentiable at $\mathcal{L}^q$-almost every point $x \in \Omega$.*

Remark 3.2.17. Theorem 3.2.16 has been generalized in [Mag02a] to the case when Ω is any measurable subset of $\mathbb{G}$.

An immediate consequence of Theorem 3.2.16 is the following Corollary.

Corollary 3.2.18. *[Sem96, Theorem 7.1] A non-commutative Carnot group $\mathbb{G}$ of topological dimension q, endowed with a homogeneous distance d, is not bi-Lipschitz equivalent to the Euclidean space $\mathbb{R}^q$.*

We sketch the proof of this result, that is an immediate consequence of Theorem 3.2.16. If we assume, by contradiction, the existence of a bi-Lipschitz map $f : \mathbb{G} \to \mathbb{R}^q$, by Theorem 3.2.16 f is Pansu differentiable almost everywhere on $\mathbb{G}$ and $Df(x)$ is a bi-Lipschitz h-homomorphism such that $Df(x)(V_j) = \{0\}$ for every $2 \le j \le \kappa$, for almost every $x \in \mathbb{G}$. Then, for almost every $x \in \mathbb{G}$, $\dim(\ker(Df(x))) = \dim(V_2 \oplus \cdots \oplus V_\kappa) > 0$ and we get a contradiction with the fact that $Df(x)$ is bi-Lipschitz. Hence, we can conclude that it cannot exist a bi-Lipschitz map between $\mathbb{G}$ and $\mathbb{R}^q$.

Definition 3.2.19. Let us consider an open set $\Omega \subset \mathbb{G}$. We say that $f : \Omega \to \mathbb{M}$ is *continuously Pansu differentiable* on Ω if f is Pansu differentiable at every point of Ω and the function $Df : \Omega \to \mathcal{L}_h(\mathbb{G}, \mathbb{M})$ is continuous. The family of all continuously Pansu differentiable mappings is denoted by $C^1_h(\Omega, \mathbb{M})$.

Continuously Pansu differentiable functions are a generalization of C^1-regular Euclidean mappings. In respect with continuously Pansu differentiable maps, the following holds.

Proposition 3.2.20. *Let $\Omega \subset \mathbb{G}$ be an open set and let $f \in C^1_h(\Omega, \mathbb{M})$, then the function $\|Df\|_{\mathcal{L}_h(\mathbb{G},\mathbb{M})} : \Omega \to \mathbb{R},\ x \to \|Df(x)\|_{\mathcal{L}_h(\mathbb{G},\mathbb{M})}$ is continuous on Ω.*

Now, we collect some results about the case when $\mathbb{M} = \mathbb{R}^p$.

Definition 3.2.21. Let $\Omega \subset \mathbb{G}$ be an open set and let $f : \Omega \to \mathbb{R}$ be a continuous function. Fix a point $x \in \Omega$ and a vector $v \in V_1$. Let $X_v \in \mathrm{Lie}(\mathbb{G})$ be the unique left invariant vector field on $\mathbb{G}$ such that $X_v(0) = v$. If there exists the limit

$$\lim_{t\to 0} \frac{f(x\exp(tX_v)) - f(x)}{t} = \lim_{t\to 0} \frac{f(x(tv)) - f(x)}{t} \in \mathbb{R} \tag{3.14}$$

we call it the *horizontal partial derivative at x along X_v* and we denote the value of the limit (3.14) by $X_v f(x)$.

The continuity of the Pansu differential has been characterized in terms of the existence and continuity of horizontal partial derivatives.

Proposition 3.2.22. *If $\Omega \subset \mathbb{G}$ is an open set and $f : \Omega \to \mathbb{R}^p$ is a map, f is continuously Pansu differentiable on Ω if and only if it is continuous and, for every $x \in \Omega$ and for every $j = 1, \dots, p$, for every horizontal vector field $X_v \in \mathrm{Lie}(\mathbb{G})$ (or equivalently for every $v \in V_1$), the horizontal derivative $X_v f_j(x)$ exists and the mapping $X_v f_j : \Omega \to \mathbb{R}$ is continuous on Ω.*

A slightly different analogous characterization has also been proved. The map $f : \Omega \to \mathbb{R}^p$ is continuously Pansu differentiability if and only, for any horizontal vector field X_v, the horizontal derivatives $X_v f_j$, for $j = 1, \dots, p$, are continuous in the sense of distributions. The proofs of the two characterizations are analogous and can be obtained through adaptations of the proof of [FSSC01, Proposition 5.8] (see also [Mag13, Theorem 1.1]).

Definition 3.2.23. Let $\Omega \subset \mathbb{G}$ be an open set and let $f \in C^1_h(\Omega, \mathbb{R})$. We call *horizontal gradient* of f at $x \in \Omega$ the unique horizontal vector $\nabla_H f(x) \in V_1$ such that

$$Df(x)(z) = \langle \nabla_H f(x), z \rangle$$

for every $z \in V_1$.

Remark 3.2.24. Assume $(v_1, \dots, v_q)$ to be any adapted orthonormal basis of $\mathbb{G}$ and let us call $X_j \in \mathrm{Lie}(\mathbb{G})$ the left invariant vector field such that $X_j(0) = v_j$, for every $j = 1, \dots, q$, then

$$\nabla_H f(x) = \sum_{i=1}^{m_1} X_i f(x) v_i.$$

Definition 3.2.25 (Horizontal Jacobian matrix)**.** Let us consider an open set $\Omega \subset \mathbb{G}$, a map $f : \Omega \to \mathbb{R}^p$ and a point $x \in \Omega$. Assume $(v_1, \dots, v_q)$ to be any adapted orthonormal basis of $\mathbb{G}$ and let us denote by $X_j \in \mathrm{Lie}(\mathbb{G})$ the left invariant vector field such that $X_j(0) = v_j$ for $j = 1, \dots, q$. Then we call *horizontal Jacobian matrix* of f at x, and we denote it by $J_{\mathbb{H}} f(x)$, the matrix $M_{Df(x)} \in M_{p,m_1}(\mathbb{R})$ individuated by equation (3.12) in Remark 3.2.6 applied to $L = Df(x)$, i.e.

$$J_{\mathbb{H}} f(x) = \begin{pmatrix} X_1 f_1(x) & \dots & X_{m_1} f_1(x) \\ \dots & \dots & \dots \\ X_1 f_p(x) & \dots & X_{m_1} f_p(x) \end{pmatrix}.$$

Remark 3.2.26. Consider an orthonormal basis $(v_1, \dots, v_q)$ of $\mathbb{G}$. We can identify, as usual, $\mathbb{G}$ with $\mathrm{Lie}(\mathbb{G})$ through the map that associates any vector $v \in \mathbb{G}$ with the left invariant vector field X_v such that $X_v(0) = v$. Then, if we have a function $f \in C^1_h(\Omega, \mathbb{R})$, the horizontal gradient $\nabla_H f \in C^0(\Omega, V_1)$, $\nabla_H f(x) = \sum_{i=1}^{m_1} X_{v_i} f(x) v_i$ is automatically identified with the continuous horizontal left invariant vector field

$$\nabla_H f \in C^0(\Omega, H\mathbb{G}),\ \nabla_H f(x) = \sum_{i=1}^{m_1} X_{v_i} f(x) X_{v_i}(x) \in H_x\mathbb{G}.$$

Let us set some preparatory definitions to state Theorem 3.2.30, that will be used multiple times throughout the thesis. Roughly speaking, it states that continuous Pansu differentiability implies in some sense the *uniform* convergence of limit (3.13) on *small enough* sets (the meaning of the terms "uniform" and "small enough" can be better understood by comparing (3.13) with (3.16)).

Definition 3.2.27. [Mag13, Definition 4.11] Let $f : K \to Y$ be a continuous function from a compact metric space (K, d_1) to a metric space (Y, d_2). We define the *modulus of continuity* of f on K as

$$\omega_{K,f}(t) = \max_{\substack{x,y \in K \\ d_1(x,y) \le t}} d_2(f(x), f(y)).$$

Remark 3.2.28. In the statement of Theorem 3.2.30, and also later on, we will refer to two geometrical constants $c = c(\mathbb{G}, d)$ and $H = H(\mathbb{G}, d)$, that can be associated with any Carnot group $\mathbb{G}$ endowed with a homogeneous distance d. In order to achieve our results, the exact value of these constants will not be relevant. Hence, since their definition is very technical, for the sake of accuracy, we refer to [Mag13, Lemma 4.9, Definition 4.10] for a precise evaluation; here we limit ourselves to explain heuristically their role. Roughly speaking, they are needed to state condition (3.15) in Theorem 3.2.30 that can be translated, in the notation of the theorem, as follows: given any two points $x, y \in \Omega_1$, one wants to be sure that any horizontal curve connecting x and y is completely contained in Ω_2. The two constants are then defined in order to be the right ones to ensure that (3.15) implies this desired geometric property. Here we just highlight that c and H depend only on the group $\mathbb{G}$ and on the distance d.

Remark 3.2.29. Consider an open set $\Omega \subset \mathbb{G}$ and a mapping $f : \Omega \subset \mathbb{G} \to \mathbb{M}$; for $j = 1, \dots, \vartheta$, we call $F_j := \pi_{W_j} \circ f$, where $\pi_{W_j} : \mathbb{M} \to W_j$ is the orthogonal projection onto the j-th layer of $\mathbb{M}$. If f is Pansu differentiable at $x \in \Omega$, by [Mag13, Theorem 4.12], the maps F_j are Pansu differentiable at x and, in particular, $DF_1(x) = \pi_{W_1} \circ Df(x)$ (where $DF_1 : \Omega \to W_1$).

Theorem 3.2.30. *[Mag13, Theorem 1.2] Let $(\mathbb{G}, d_1)$ and $(\mathbb{M}, d_2)$ be two Carnot groups endowed with homogeneous distances. Let κ be the step of $\mathbb{G}$, let $\Omega \subset \mathbb{G}$ be an open set and consider a map $f \in C^1_h(\Omega, \mathbb{M})$. Let $\Omega_1, \Omega_2 \subset \mathbb{G}$ be two open subsets of $\mathbb{G}$ such that Ω_2 is compactly contained in Ω and*

$$\{x \in \mathbb{G} : \operatorname{dist}(x, \Omega_1) \leq cH \operatorname{diam}(\Omega_1)\} \subset \Omega_2, \tag{3.15}$$

where $c = c(\mathbb{G}, d_1)$ and $H = H(\mathbb{G}, d_1)$ are the geometric constants discussed in Remark 3.2.28. Then there exists a constant C, only depending on $\mathbb{G}$, $\max_{x \in \overline{\Omega_2}} \|DF_1(x)\|_{\mathcal{L}_h(\mathbb{G}, W_1)}$ and on $\omega_{\overline{\Omega_2}, DF_1}(\operatorname{diam}(\Omega_2))$ such that

$$\frac{\|Df(x)(x^{-1}y)^{-1}f(x)^{-1}f(y)\|_2}{\|x^{-1}y\|_1} \leq C[\omega_{\overline{\Omega_2}, DF_1}(cHd_1(x, y))]^{1/\kappa^2} \tag{3.16}$$

for every $x, y \in \overline{\Omega_1}$ with $x \neq y$.

Remark 3.2.31. By [Mag13, Theorem 1.1], if f is continuously Pansu differentiable, then $x \to DF_1(x)$ is a continuous map from Ω to $\mathcal{L}_h(\mathbb{G}, W_1)$, and so by Proposition 3.2.20, the modulus of continuity $\omega_{\overline{\Omega_2}, DF_1}(s)$ goes to zero as s goes to zero.

One can use Theorem 3.2.30 to prove the following.

Proposition 3.2.32. *If $\Omega \subset \mathbb{G}$ is an open set, any map $f \in C^1_h(\Omega, \mathbb{M})$ is locally Lipschitz.*

Proof. Let us fix $\bar{x} \in \Omega$. By Theorem 3.2.30 we have

$$\lim_{\varepsilon \to 0} \sup_{x, y \in B(\bar{x}, \varepsilon)} \frac{\|Df(x)(x^{-1}y)^{-1}f(x)^{-1}f(y)\|_2}{\|x^{-1}y\|_1} = 0 \tag{3.17}$$

By the continuity of the Pansu differential it holds that, when ε goes to zero, every $x \in B(\bar{x}, \varepsilon)$ goes to $\bar{x}$ and then for every $x, y \in B(\bar{x}, \varepsilon)$

$$\frac{\|Df(\bar{x})(x^{-1}y)^{-1}Df(x)(x^{-1}y)\|_2}{\|x^{-1}y\|_1} \leq d_{\mathcal{L}_h(\mathbb{G}, \mathbb{M})}(Df(\bar{x}), Df(x)) \to 0 \tag{3.18}$$

as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon \to 0} \sup_{x, y \in B(\bar{x}, \varepsilon)} \frac{\|Df(\bar{x})(x^{-1}y)^{-1}f(x)^{-1}f(y)\|_2}{\|x^{-1}y\|_1} = 0. \tag{3.19}$$

Combining (3.17) and (3.18), we can choose $\varepsilon = \varepsilon(\bar{x}) > 0$ such that for every $x, y \in B(\bar{x}, \varepsilon)$

$$\frac{\|Df(\bar{x})(x^{-1}y)^{-1}f(x)^{-1}f(y)\|_2}{\|x^{-1}y\|_1} \leq 1,$$

we can deduce that f is Lipschitz on $B(\bar{x}, \varepsilon)$, since for every $x, y \in B(\bar{x}, \varepsilon)$

$$d_2(f(x), f(y)) \leq (1 + \|Df(\bar{x})\|_{\mathcal{L}_h(\mathbb{G}, \mathbb{M})})d_1(x, y).$$

□

Remark 3.2.33. Property (3.19) has been referred to in [JNGV20, Proposition 2.4] as *strictly Pansu differentiability* of f at $\bar{x}$. In particular, Julia, Nicolussi Golo and Vittone proved that $f \in C^1_h(\Omega, \mathbb{M})$ if and only if it is strictly Pansu differentiable on Ω, i.e. if

(3.19) holds for every $\bar{x} \in \Omega$. We refer also the reader to the interesting discussion in the introduction of [ADDDLD20] about the analogous property of Euclidean C^1-regular maps in the Euclidean setting.

We complete this subsection by reporting an extension theorem for continuously Pansu differentiable maps. We will see that the following Whitney-type theorem is an useful tool in order to study intrinsic submanifolds in Carnot groups. We report below the more general available version of the result, that works for functions acting from a closed subset of a generic Carnot group $F \subset \mathbb{G}$ to any Euclidean space $\mathbb{R}^p$. The first version of the theorem was presented for the case when $\mathbb{G} = \mathbb{H}^n$ and $p = 1$ in [FSSC01, Theorem 6.8], and successively it was extended in [FSSC03a, Theorem 2.14] to the case when $\mathbb{G}$ is a generic Carnot group and $p = 1$. A version of the latter result involving a control on the modulus of continuity of horizontal derivatives is proved in [VP06]. For a proof for the case of maps between two Euclidean spaces refer for instance to [EG92, Theorem 6.5].

Theorem 3.2.34. *[DD17, Theorem 2.3.8] Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d. Let $F \subset \mathbb{G}$ be a closed set. Let $f : F \to \mathbb{R}^p$ be a continuous function and let $g : F \to M_{p,m_1}(\mathbb{R})$ be a continuous matrix-valued map. We define for every $x, y \in \mathbb{G}$*

$$R(x,y) := \frac{f(x) - f(y) - g(y) \cdot (\pi(y^{-1}x))^T}{\|y^{-1}x\|},$$

where $\cdot$ denotes the usual matrix product. If $K \subset F$ is a compact set and $\delta > 0$, we define

$$\rho_K(\delta) := \sup\{|R(x,y)| : x, y \in K,\ 0 < \|x^{-1}y\| < \delta\}.$$

Assume that

$$\rho_K(\delta) \to 0 \text{ as } \delta \to 0 \quad \text{for every compact set } K \subset F,$$

then there exists $\tilde{f} \in C^1_h(\mathbb{G}, \mathbb{R}^p)$ such that,

$$\tilde{f}|_F = f, \quad \nabla_H \tilde{f}|_F = g.$$

3.3 BV functions and finite perimeter sets on Carnot group

In this section we provide a coincise introduction to the theory of sets of locally finite perimeter in Carnot groups, usually called locally finite H-perimeter sets. To be precise, the new results we are going to present in the following chapters do not rely on the theory of locally finite H-perimeter sets. Nevertheless, we think that the concepts of this section may help the reader to understand the state of the art in the context of regular submanifolds in Carnot groups. In fact, powerful tools of this deep theory have often been used in the literature to study the features of intrinsic regular hypersurfaces, i.e. one-codimensional intrinsic regular submanifolds. Nowadays the same tools can be seen as one of the frontiers of research on regular submanifolds in Carnot groups. For example, many projects about extending the validity of a result proved for regular hypersurfaces to higher codimensional regular submanifolds, can be translated in the research of alternative tools that can substitute results coming from the theory of locally finite H-perimeter sets. More material about these sets can be found in [DG54] and [DG55] for subsets of Euclidean spaces and in [CDG94, FGW94, GN96, FSSC96, FSSC01, FSSC02, FSSC03a] for finite H-perimeter subsets of Carnot groups.

Let $\Omega \subset \mathbb{G}$ be an open subset of a Carnot group $\mathbb{G}$. Let $(v_1, \ldots, v_q)$ be an adapted orthonormal basis of $\mathbb{G}$ and denote as above by $X_j \in \mathrm{Lie}(\mathbb{G})$ the left invariant vector field such that $X_j(0) = v_j$ for every $j = 1, \dots, q$. We denote the set of smooth horizontal compactly supported sections on Ω by

$$C_0^\infty(\Omega, H\mathbb{G}) := \left\{ \phi = \sum_{i=1}^{m_1} \phi_i X_i : \phi_i \in C_h^\infty(\Omega, \mathbb{R}),\ \mathrm{spt}(\phi) \text{ is compact } \right\}.$$

Analogously, we can denote by $C_0^k(\Omega, H\mathbb{G})$, for $k \in \mathbb{N}$, the set of the compactly supported sections with components in $C_h^k(\Omega, \mathbb{R})$.

Definition 3.3.1. Let ϕ be a section in $C_0^1(\Omega, H\mathbb{G})$, where Ω is an open set of $\mathbb{G}$. If $X_j\phi \in L^1_{loc}(\mathbb{G})$ for every $j = 1, \dots, m_1$, we define the *horizontal divergence* of ϕ as

$$\mathrm{div}_H \phi := \sum_{i=1}^{m_1} X_j \phi_j.$$

Definition 3.3.2. A function $f : \Omega \to \mathbb{R}$ is said a *function of bounded variation in* Ω if $f \in L^1(\Omega, H\mathbb{G})$ and

$$\|\nabla_H f\|(\Omega) := \sup \left\{ \int_\Omega f \mathrm{div}_H \phi \, d\mathcal{L}^q : \phi \in C_0^1(\Omega, H\mathbb{G}), |\phi(x)| \le 1 \text{ for all } x \in \Omega \right\} < \infty.$$

The normed space of bounded variation functions in Ω is denoted by $BV_H(\Omega)$. The space $BV_{H,\mathrm{loc}}(\Omega)$ is the set of functions in $BV_H(U)$ for each $U \Subset \Omega$.

Riesz's representation theorem provides the proof of the following structure Theorem for $BV_{H,\mathrm{loc}}$ functions, namely that if $f \in BV_{H,\mathrm{loc}}$, the total variation $\|\nabla_H f\|$ is a Radon measure ([CDG94, FSSC96]).

Theorem 3.3.3. *If $f \in BV_{H,\mathrm{loc}}(\Omega)$, then $\|\nabla_H f\|$ induces a Radon measure on Ω, still denoted by $\|\nabla_H f\|$. Moreover, if $f \in BV_{H,\mathrm{loc}}(\Omega)$ there exists a $\|\nabla_H f\|$-measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f(y)|_y = 1$ for $\|\nabla_H f\|$-a.e. $y \in \Omega$, and*

$$\int_\Omega f \mathrm{div}_H \phi \, d\mathcal{L}^n = \int_\Omega \langle \phi, \sigma_f \rangle \, d\|\nabla_H f\|$$

for every $\phi \in C_0^1(\Omega, H\mathbb{G})$.

Remark 3.3.4. As a consequence of Theorem 3.3.3, the notion of horizontal gradient $\nabla_H f$ can be extended from regular functions to functions $f \in BV_H(\Omega)$ defining $\nabla_H f$ as the vector valued measure

$$\nabla_H f := \sigma_f \llcorner \|\nabla_H f\| = (-(\sigma_f)_1 \llcorner \|\nabla_H f\|, \dots, -(\sigma_f)_{m_1} \llcorner \|\nabla_H f\|)$$

where $(\sigma_f)_j$ are the components of σ_f with respect to the moving basis X_{v_j}.

One of the key advantages provided by the use of bounded variation functions is that the space $BV_{H,\mathrm{loc}}$ maintains the properties of compactness [GN96] and lower semicontinuity of the total variation with respect to L^1 convergence [FSSC96] that hold for the corresponding maps in Euclidean metric spaces.

Theorem 3.3.5. *If $\mathbb{G}$ is a Carnot group, $BV_{H,\mathrm{loc}}(\mathbb{G})$ is compactly embedded in $L^m_{loc}(\mathbb{G})$ for $1 \le m < \frac{Q}{Q-1}$ where Q is the homogeneous dimension of $\mathbb{G}$.*

Theorem 3.3.6. *Let $\mathbb{G}$ be a Carnot group and $\Omega \subset \mathbb{G}$ be an open set. Let $f, f_k \in L^1(\Omega)$, $k \in \mathbb{N}$, be such that $f_k \to f$ in $L^1(\Omega)$, then*

$$\liminf_{k\to\infty} \|\nabla_H f_k\|(\Omega) \geq \|\nabla_H f\|(\Omega).$$

Definition 3.3.7. Let $\mathbb{G}$ be a Carnot group and let $\Omega \subset \mathbb{G}$ be an open set. A measurable set $E \subset \mathbb{G}$ is said of *locally finite H-perimeter* in Ω or is called a *H-Caccioppoli set* if its characteristic function $\mathbf{1}_E$ belongs to $BV_{H,\mathrm{loc}}(\Omega)$. In this case we call *$H$-perimeter* of E the measure

$$|\partial E|_H := \|\nabla_H \mathbf{1}_E\|,$$

and we call *generalized horizontal outward H-normal* to ∂E in Ω the horizontal vector field

$$\nu_E := \sigma_{\mathbf{1}_E}. \tag{3.20}$$

An important tool about Caccioppoli sets in geometric measure theory is the isoperimetric inequality, that is valid also in this context. It has been proved in [GN96].

Theorem 3.3.8. *Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance and denote by q the topological dimension of $\mathbb{G}$ and by Q the metric one. Then, there exists a positive constant C such that for any H-Caccioppoli set $E \subset \mathbb{G}$, for any $x \in \mathbb{G}$ and $r > 0$*

$$\min\{\mathcal{L}^q(E \setminus B(x,r)), \mathcal{L}^q(B(x,r) \setminus E)\}^{\frac{Q-1}{Q}} \leq C|\partial E|_H(B(x,r)).$$

A global version of Theorem 3.3.8 is also available (see [SC16, Theorem 3.41] for more details). We recall also the notion of reduced boundary in this sub-Riemannian context.

Definition 3.3.9. Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d and let $E \subset \mathbb{G}$ be a set of locally finite H-perimeter. Let ν_E be the generalized inward H-normal. A point $x \in \mathbb{G}$ belongs to the *H-reduced boundary* of E, denoted by $\partial^*_H E$, if the following conditions hold.

(i) For any $r > 0$, $|\partial E|_H(B(x,r)) > 0$.

(ii) There exists the limit

$$\lim_{r\to 0} \frac{1}{|\partial E|_H(B(x,r))} \int_{B(x,r)} \nu_E \, d|\partial E|_H.$$

(iii) The value of the limit in (ii) is

$$\lim_{r\to 0} \frac{1}{|\partial E|_H(B(x,r))} \left| \int_{B(x,r)} \nu_E \, d|\partial E|_H \right| = 1.$$

The H-reduced boundary of E is invariant under left translation, in the sense that for $x_0, x \in \mathbb{G}$

$$x_0 \in \partial^*_H E \Leftrightarrow l_x(x_0) \in \partial^*_H(l_x E)$$

and also

$$\nu_E(x_0) = \nu_{l_x E}(l_x(x_0)).$$

In this sense the notion of H-reduced boundary is intrinsic.

3.4 Intrinsic graphs

In this section we recall the definitions and results about intrinsic graphs of functions acting between two homogeneous complementary subgroups. We present them as preparatory tools towards the definition of regular submanifold in Carnot groups. From this perspective, intrinsic graphs will be presented endowed with well-established properties and suitable notions of regularity. Nonetheless we stress that, from an historical point of view, they originated from [FSSC07, Theorem 1], and, more precisely, for the case of one-codimensional graphs in the Heisenberg group, from [FSSC01, Theorem 6.5] and in Carnot groups from [FSSC03b, Theorem 2.1]. Then, their introduction is actually successive to the definitions of $\mathbb{G}$-regular hypersurface (Definition 4.2.1) and $\mathbb{H}$-regular surface (Definitions 4.1.1 and 4.1.8). We decided to postpone the presentation of regular submanifolds to the next chapter, collocating it after the one regarding intrinsic graphs, since we prefer to follow, as much as possible, an analogy with the Euclidean path. For more information about the theory of intrinsic graphs one can refer for instance to [SC16], to [FS16] and to the references therein. These two references provide a rich introduction to the theory.

By $\mathbb{G}$ we denote, as usual, a generic Carnot group endowed with a homogeneous distance d.

Definition 3.4.1. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, let $U \subset \mathbb{W}$ be a set and let us consider a function $\phi : U \to \mathbb{V}$. The *intrinsic graph of* ϕ is the set

$$\text{graph}(\phi) := \{w\phi(w) : w \in U\}.$$

The map $\Phi : U \to \mathbb{G}$, $\Phi(w) := w\phi(w)$ is called the *graph map* of ϕ.

Remark 3.4.2. Actually, Definition 3.4.1 is not the most general definition of intrinsic graph available in the literature. In particular, it requires the existence of a splitting of $\mathbb{G}$ as the product of two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$. The original definition of intrinsic graph is satisfied by a larger family of objects, we report it here for the sake of completeness. Let $\mathbb{V}$ be a homogeneous subgroup of a Carnot group $\mathbb{G}$. A subset $\mathcal{S} \subset \mathbb{G}$ is called a $\mathbb{V}$*-graph* if $\mathcal{S}$ intersects any (left) coset of $\mathbb{V}$ at at most at one point, i.e. if for every $\xi \in \mathbb{G}$ there exists at most one point $x = x(\xi) \in \mathbb{G}$ such that

$$\mathcal{S} \cap \xi\mathbb{V} = \{x\},$$

where $\xi\mathbb{V} = \{\xi v : v \in \mathbb{V}\}$. If $\mathbb{V}$ admits a complementary homogeneous subgroup $\mathbb{W}$, this definition coincides with the given one (Definition 3.4.1). Nevertheless, not every $\mathbb{V}$-graph is the intrinsic graph of some function acting between two complementary subgroups. For example we can consider the family of $\mathbb{T}$-graphs in the Heisenberg group $\mathbb{H}^n$, where $\mathbb{T} = H_2$ is the center of $\mathbb{H}^n$. A $\mathbb{T}$-graph has the following form in coordinates

$$\mathcal{S} = \{(x_1, x_2, \ldots, x_{2n}, \psi(x_1, x_2, \ldots, x_{2n}))\} \subset \mathbb{H}^n,$$

with $\psi : H_1 \to \mathbb{T}$. The left cosets of $\mathbb{T}$ are parametrized on H_1, nevertheless $\mathbb{T}$ does not have any complementary subgroup since it is a vertical homogeneous subgroup of dimension one, then any homogeneous subgroup candidate to be complementary to $\mathbb{T}$ should be horizontal of topological dimension $2n$, but this is not possible (see Remark 3.1.16). This means that it does not exist any function ϕ acting from a complementary subgroup of $\mathbb{T}$ to $\mathbb{T}$ such that $\mathcal{S} = \text{graph}(\phi)$. Nevertheless, $\mathbb{T}$-graphs remain interesting objects of research, see for instance [SCV14, NGSC19].

The notion of graph is *intrinsic*, in the sense that both left translations and dilations send an intrinsic graph on a new intrinsic graph, i.e. on the intrinsic graph of a well defined function, different from the one defining the original graph, but acting between the same complementary subgroups.

Proposition 3.4.3. *[FS16, Proposition 2.21] Let us consider $\mathbb{G} = \mathbb{W}\mathbb{V}$ as the product of two complementary subgroups. Let $U \subset \mathbb{W}$ be a set and let us consider a map $\phi : U \to \mathbb{V}$. Then the following hold*

(i) for all $t > 0$, $\delta_t(\mathrm{graph}(\phi)) = \mathrm{graph}(\phi_t)$, with

$$\phi_t : \delta_t(U) \to \mathbb{V},\ \ \phi_t(\eta) = \delta_t(\phi(\delta_{1/t}(\eta)));$$

(ii) for all $x \in \mathbb{G}$, $l_x(\mathrm{graph}(\phi)) = \mathrm{graph}(\phi_x)$, with

$$\phi_x : U_x \to \mathbb{V},\ \ \phi_x(\eta) = \pi_{\mathbb{V}}(x^{-1}\eta)^{-1}\phi(\pi_{\mathbb{W}}(x^{-1}\eta))$$

where $U_x = \{\pi_{\mathbb{W}}(l_x(w)) \in U : w \in U\} \subset \mathbb{W}$. We call the map ϕ_x the translation of ϕ at x.

If $\mathbb{W}$ is a normal subgroup, the formula of the translation ϕ_x can be simplified (refer for example to [FS16, Remark 2.23]).

Definition 3.4.4. Let $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$ be the semidirect product of two complementary subgroups. Let us consider a point $x \in \mathbb{G}$ and define the map $\sigma_x : \mathbb{W} \to \mathbb{W}$ as follows

$$\sigma_x(\eta) := \pi_{\mathbb{W}}(l_x(w)) = xw(\pi_{\mathbb{V}}(x))^{-1}.$$

Given a set $U \subset \mathbb{W}$ and a function $\phi : U \to \mathbb{V}$, the translation of ϕ at x, ϕ_x, can be simplified as

$$\phi_x : \sigma_x(U) \to \mathbb{V},\ \ \phi_x(w) = \pi_{\mathbb{V}}(x)\phi(x^{-1}\eta\pi_{\mathbb{V}}(x)) = \pi_{\mathbb{V}}(x)\phi(\sigma_{x^{-1}}(\eta)). \tag{3.21}$$

Remark 3.4.5. Notice that the map σ_x is invertible on $\mathbb{W}$ and we have

$$\sigma_{x^{-1}}(\eta) = x^{-1}\eta\pi_{\mathbb{V}}(x^{-1})^{-1} = x^{-1}\eta\pi_{\mathbb{V}}(x) = \sigma_x^{-1}(\eta),$$

then we may also write for $\eta \in \sigma_x(U)$

$$\phi_x(\eta) = \pi_{\mathbb{V}}(x)\phi(\sigma_x^{-1}(\eta)) = \pi_{\mathbb{V}}(x)\phi(x^{-1}\eta\pi_{\mathbb{V}}(x)). \tag{3.22}$$

3.5 Regularity of intrinsic graphs

Various notions of regularity have been introduced by Franchi, Serapioni and Serra Cassano for functions acting between two complementary subgroups and, consequently, for their respective intrinsic graphs (see [FSSC03b, FSSC05, ASCV06, FSSC06, FSSC07, AS09, FSSC11, FMS14, FS16]). The leading common idea is again to introduce *intrinsic* notions, hence notions that respect the group's structure, i.e. notions invariant under left translations and dilations. The three authors developed the notions of *intrinsic Lipschitz continuity* and *intrinsic differentiability*. Roughly speaking, a function acting between two complementary subgroups is intrinsic Lipschitz if its intrinsic graph does not intersect the left cosets, by points of the graph, of an appropriately defined homogeneous cone, except that at the vertex. A function is intrinsically differentiable when its intrinsic graph admits an appropriately defined tangent homogeneous subgroup at a considered point.

3.5.1 Cones and intrinsic Lipschitz functions

Definition 3.5.1. Let $\mathbb{V}$ be a homogeneous subgroup of a Carnot group $\mathbb{G}$. Let x be a point of $\mathbb{G}$ and $\alpha \in [0,1]$. We call *cone* with *axis* $\mathbb{V}$, *vertex* x and *opening* α the set

$$X(x, \mathbb{V}, \alpha) := xX(0, \mathbb{V}, \alpha),$$

where

$$X(0, \mathbb{V}, \alpha) := \{y \in \mathbb{G} : \operatorname{dist}(y, \mathbb{V}) \leq \alpha \|y\|\},$$

where $\operatorname{dist}(y, \mathbb{V}) = \inf\{\|v^{-1}y\| : v \in \mathbb{V}\}$.

Following the path established by the related literature, we state another definition of intrinsic (closed) cone, that is, in a precise way, equivalent to the previous one. The first assumption in this case is that $\mathbb{G}$ can be splitted as a product of two complementary subgroups.

Definition 3.5.2. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let x be a point of $\mathbb{G}$ and let $\beta \geq 0$. We call $(\mathbb{W}, \mathbb{V})$-*cone* with *basis* $\mathbb{W}$, *axis* $\mathbb{V}$, *opening* β and *vertex* x the set

$$C_{\mathbb{W},\mathbb{V}}(x, \beta) := xC_{\mathbb{W},\mathbb{V}}(0, \beta),$$

where

$$C_{\mathbb{W},\mathbb{V}}(0, \beta) := \{y \in \mathbb{G} : \|\pi_{\mathbb{W}}(y)\| \leq \beta \|\pi_{\mathbb{V}}(y)\|\}.$$

Remark 3.5.3. We can observe that

$$\begin{aligned} \mathbb{V} &= X(0, \mathbb{V}, 0) = C_{\mathbb{W},\mathbb{V}}(0, 0), \\ \mathbb{G} &= X(0, \mathbb{V}, 1) = \overline{\cup_{\beta > 0} C_{\mathbb{W},\mathbb{V}}(0, \beta)}. \end{aligned}$$

The $(\mathbb{W}, \mathbb{V})$-cones are homogeneous sets, i.e. they are invariant under dilations, since for every $\alpha > 0$ and $t > 0$,

$$\delta_t(C_{\mathbb{W},\mathbb{V}}(0, \alpha)) = C_{\mathbb{W},\mathbb{V}}(0, \alpha). \tag{3.23}$$

The two families of cones that we have introduced are equivalent in the following sense.

Proposition 3.5.4. *[FS16, Proposition 3.1] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Then for every $\alpha \in (0,1)$, there exist $\beta \geq 1$, depending on α, $\mathbb{W}$ and $\mathbb{V}$, such that*

$$C_{\mathbb{W},\mathbb{V}}\left(x, \frac{1}{\beta}\right) \subset X(x, \mathbb{V}, \alpha) \subset C_{\mathbb{W},\mathbb{V}}(x, \beta).$$

for every $x \in \mathbb{G}$.

Remark 3.5.5. In particular, we want to highlight a property proved as a step of the proof of [FS16, Proposition 3.1]. The authors show that for every $\beta > 0$ it is possible to choose $\alpha \in (0,1)$ such that for every $x \in \mathbb{G}$

$$X(x, \mathbb{V}, \alpha) \subset C_{\mathbb{W},\mathbb{V}}(x, \beta).$$

An intrinsic Lipschitz map is a function whose intrinsic graph does not intersect the cosets of an intrinsic cone of positive opening with vertex on the graph, except for the vertex.

Definition 3.5.6. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups and L be a positive constant. Let $U \subset \mathbb{W}$ be an open set, a function $\phi : U \to \mathbb{V}$ is called *intrinsic L-Lipschitz* if for every $\tilde{L} > L$,

$$\operatorname{graph}(\phi) \cap C_{\mathbb{W},\mathbb{V}}\left(x, \frac{1}{\tilde{L}}\right) = \{x\} \quad \forall x \in \operatorname{graph}(\phi).$$

We denote by $\operatorname{Lip}(\phi)$ the infimum of the positive constants L such that ϕ is L-Lipschitz.

We say that $\operatorname{graph}(\phi)$ is intrinsic L-Lipschitz if ϕ is intrinsic L-Lipschitz. Moreover, we simply say that a function $\phi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz if there exists some constant $L > 0$ such that ϕ is intrinsic L-Lipschitz.

Remark 3.5.7. By Proposition 3.5.4 (refer also to Remark 3.5.5) the fact that ϕ is intrinsic Lipschitz is equivalent to say that there exists $\alpha \in (0, 1)$ such that

$$\operatorname{graph}(\phi) \cap X(x, \mathbb{V}, \alpha) = \{x\} \quad \forall x \in \operatorname{graph}(\phi).$$

According to [FS16, Theorem 3.2], left translations and intrinsic dilations of intrinsic Lipschitz graphs are intrinsic Lipschitz graphs, hence the use of the word "intrinsic" in this context is appropriate.

Remark 3.5.8. By taking in consideration the definition of $(\mathbb{W}, \mathbb{V})$-cone, Definition 3.5.2, we can rewrite Definition 3.5.6 as follows. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups and let L be a positive constant. Let $U \subset \mathbb{W}$ be an open set, a function $\phi : U \to \mathbb{V}$ is intrinsic L-Lipschitz if and only if

$$\|\pi_{\mathbb{V}}(\Phi(w')^{-1}\Phi(w))\| \leq L\|\pi_{\mathbb{W}}(\Phi(w')^{-1}\Phi(w))\| \tag{3.24}$$

for every $w, w' \in U$. Moreover, if $\mathbb{W}$ is a normal subgroup, (3.24) can be rephrased as

$$\|\phi(w')^{-1}\phi(w)\| \leq L\|\phi(w')^{-1}w'^{-1}w\phi(w')\| \tag{3.25}$$

for every $w, w' \in U$.

The object $\|\phi(w')^{-1}w'^{-1}w\phi(w')\|$ will play a role in our results, this justify the following definition. This object is usually called the graph distance between w and w', for $w, w' \in U$. Nevertheless, notice that it is not a distance (it is not even symmetric).

Definition 3.5.9. Let $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$ be the semidirect product of two complementary subgroups. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a continuous function. We call *graph distance* the following map

$$d_\phi : U \times U \to \mathbb{R}^+,\ d_\phi(w, w') = \|\pi_{\mathbb{W}}(\Phi(w')^{-1}\Phi(w))\| = \|\phi(w')^{-1}w'^{-1}w\phi(w')\| \tag{3.26}$$

Remark 3.5.10. In the literature it is often used the following symmetrized variant of the graph distance, in the notation of Definition 3.5.9

$$D_\phi : U \times U \to \mathbb{R}^+$$

$$D_\phi(w, w') = \frac{1}{2}\left(\|\pi_{\mathbb{W}}(\Phi(w)^{-1}\Phi(w'))\| + \|\pi_{\mathbb{W}}(\Phi(w')^{-1}\Phi(w))\|\right) = \frac{1}{2}(d_\phi(w', w) + d_\phi(w, w')).$$

For our purposes, working with the non symmetric version d_ϕ is equivalent to work with D_ϕ. In fact, by [SC16, Proposition 4.60], the notion of intrinsic Lipschitz continuity can be equivalently stated in terms of D_ϕ or in terms of d_ϕ since the following conditions are equivalent

(i) ϕ is intrinsic L-Lipschitz for some positive constant L.

(ii) There is a positive constant L' such that for every $w, w' \in U$

$$\|\phi(w')^{-1}\phi(w)\| \leq L' D_\phi(w, w').$$

By [FS16, Remark 3.6], if ϕ is an intrinsic Lipschitz function, D_ϕ is a quasi-distance (i.e. a distance satisfying a weaker triangular inequality) on any compact subset of U.

Intrinsic Lipschitz continuity has been characterized in various ways.

Proposition 3.5.11. *[FS16, Proposition 3.3] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups and let L be a positive constant. Let $U \subset \mathbb{W}$ be an open set, a function $\phi : U \to \mathbb{V}$ is intrinsic L-Lipschitz if and only if*

$$\|\phi_{x^{-1}}(w)\| \leq L\|w\|$$

for every $x \in \mathrm{graph}(\phi)$ and $w \in U_{x^{-1}}$.

Intrinsic Lipschitz maps are not metric Lipschitz, but they are $\frac{1}{\kappa}$-Hölder continuous.

Proposition 3.5.12. *[FS16, Proposition 3.8] Let $\mathbb{W}$, $\mathbb{V}$ be complementary subgroups of a Carnot group $\mathbb{G}$ of step κ. If $\phi : U \subset \mathbb{W} \to \mathbb{V}$ is an intrinsic L-Lipschitz function, for some $L > 0$, then*

(i) for all $R > 0$, there is a constant $C_1 = C_1(\mathbb{W}, \mathbb{V}, R, \phi) > 0$ such that

$$\phi(\mathbb{B}(0, R) \cap U) \subset \mathbb{B}(0, C_1)$$

(ii) ϕ is $\frac{1}{\kappa}$-Hölder continuous on the bounded subsets of U, i.e. for every $R > 0$, there exist a constant $C_2 = C_2(\mathbb{G}, \mathbb{W}, \mathbb{V}, R, L, C_1) > 0$ such that

$$\|\phi(\bar{w})^{-1}\phi(w)\| \leq C_2\|\bar{w}^{-1}w\|^{\frac{1}{\kappa}} \qquad \forall w, \bar{w} \in U \cap \mathbb{B}(0, R).$$

Let us recall a crucial property of intrinsic Lipschitz graphs: they are Ahlfors regular.

Proposition 3.5.13. *[FS16, Theorem 3.9] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let N be the Hausdorff dimension of $\mathbb{W}$ and let $L > 0$. If $\phi : \mathbb{W} \to \mathbb{V}$ is an intrinsic L-Lipschitz function, there is a constant $c = c(\mathbb{W}, \mathbb{V}) > 0$ such that*

$$\left(\frac{c_0}{1+L}\right)^N r^N \leq \mathcal{S}^N(\mathrm{graph}(\phi) \cap \mathbb{B}(x, r)) \leq c(1+L)^N r^N$$

for all $x \in \mathrm{graph}(\phi)$ and $r > 0$, where the constant $c_0 = c_0(\mathbb{W}, \mathbb{V})$ is the constant given by Proposition 3.1.17.

Remark 3.5.14. The proof of Proposition 3.5.13 in [FS16] is based on local arguments. Let us now assume that ϕ is defined from an open set $U \subset \mathbb{W}$ to $\mathbb{V}$ and set $w \in U$. Assume that, for a positive constant $\tilde{r} > 0$, $\pi_{\mathbb{W}}(B(\Phi(w), \tilde{r})) \subset U$, then for any $0 < r < \tilde{r}$ we get that $\mathcal{S}^N(\text{graph}(\phi) \cap \mathbb{B}(\Phi(w), r)) \geq \left(\frac{c_0(\mathbb{W},\mathbb{V})}{1+L}\right)^N r^N$. For instance one can refer to [Ser08, Lemma 4.8].

As a consequence of Proposition 3.5.13, if $U \subset \mathbb{W}$ is an open set, N is the Hausdorff dimension of $\mathbb{W}$ and $\phi : U \to \mathbb{V}$ is an intrinsic Lipschitz map, the Hausdorff dimension of graph(ϕ) equals N, i.e. it coincides with the Hausdorff dimension of the homogeneous subgroup that contains the domain of ϕ. This consideration is not true anymore if U is not open, thus if it has Hausdorff dimension lower that N.

Remark 3.5.15. Intrinsic Lipschitz continuity is a property somehow independent of the choice of the homogeneous subgroup on which a fixed graph is parametrized. Assume that, as in Definition 3.5.6, $\mathbb{G} = \mathbb{W}\mathbb{V}$ is a product of two complementary subgroups and $\phi : U \subset \mathbb{W} \to \mathbb{V}$ is an intrinsic L-Lipschitz map for some $L > 0$. Assume now that $\mathbb{L}$ is an other homogeneous subgroup of $\mathbb{G}$ complementary to $\mathbb{V}$ such that $\mathbb{G} = \mathbb{L}\mathbb{V}$, and assume that there exist an open set $U' \subset \mathbb{L}$ and a function $\psi : U' \to \mathbb{V}$ such that

$$\text{graph}(\phi) = \text{graph}(\psi).$$

Then, by Remark 3.5.7 ψ is intrinsic $\tilde{L}$-Lipschitz for some positive constant $\tilde{L}$ (that can depend on $\mathbb{W}$, $\mathbb{V}$, $\mathbb{L}$ and L).

Remark 3.5.16. Since all homogeneous distances are bi-Lipschitz equivalent on a Carnot group, replacing a homogeneous norm with another one in our definitions of cones gives rise to the same class of intrinsic Lipschitz functions. A different definition of intrinsic Lipschitz continuity, limited to intrinsic graphs of codimension 1 in the Heisenberg group, i.e. to maps acting between two complementary subgroups with one dimensional target space, has been proposed by Naor and Young in [NY18, Section 2.3]. We will give some more details about their notion later on, in Section 3.6.

3.5.2 Intrinsic differentiability and uniform intrinsic differentiability

In this section, we focus on the notion of intrinsic differentiability for maps acting between two complementary subgroups. As we said above, a map is intrinsically differentiable if its intrinsic graph can be approximated by (a left coset of) a tangent homogeneous subgroup. This requirement is quite natural if the reader accepts that the right analogue of linear subspaces in Euclidean spaces are homogeneous subgroups in Carnot groups.

Definition 3.5.17. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. A function $L : \mathbb{W} \to \mathbb{V}$ is said *intrinsic linear* if L is defined on the whole $\mathbb{W}$ and graph(L) is a homogeneous subgroup of $\mathbb{G}$.

Intrinsic linear functions have been algebraically characterized.

Proposition 3.5.18. *[FMS14, Proposition 3.1.3] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. A function $L : \mathbb{W} \to \mathbb{V}$ is intrinsic linear if and only if*

(i) $L(\delta_t(w)) = \delta_t(L(w)) \quad \forall w \in \mathbb{W},\ \forall t > 0$*;*

(ii) $L(w_1 w_2) = (\pi_{\mathbb{V}}(L(w_1)^{-1} w_2))^{-1} L(\pi_{\mathbb{W}}(L(w_1)^{-1} w_2))$ *for every* $w_1, w_2 \in \mathbb{W}$.

Proposition 3.5.19. *[AS09, Proposition 3.23] If $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ is the semidirect product of two complementary subgroups, a map $L : \mathbb{W} \to \mathbb{V}$ is intrinsic linear if and only if it is a h-homomorphism, i.e. if* *(i)* $L(\delta_t(w)) = \delta_t(L(w))$ *for every* $w \in \mathbb{W}$ *and* $t > 0$.

(ii) $L(w_1 w_2) = L(w_1)L(w_2)$ *for all* $w_1, w_2 \in \mathbb{W}$.

We recall some properties of intrinsic linear functions that will be useful later on.

Proposition 3.5.20. *[FMS14, Proposition 3.1.5] Let* $\mathbb{G} = \mathbb{W}\mathbb{V}$ *be the product of two complementary subgroups.*

(i) If $L : \mathbb{W} \to \mathbb{V}$ *is intrinsic linear, then graph*(L) *is a homogeneous subgroup complementary to* $\mathbb{V}$*, hence* $\mathbb{G} = \mathrm{graph}(L)\mathbb{V}$.

(ii) If $\mathbb{L}$ *is a homogeneous subgroup of* $\mathbb{G}$ *such that* $\mathbb{L}$ *and* $\mathbb{V}$ *are complementary subgroups of* $\mathbb{G}$*, then there is a unique intrinsic linear function* $L : \mathbb{W} \to \mathbb{V}$ *such that* $\mathbb{L} = \mathrm{graph}(L)$.

Proposition 3.5.21. *[FMS14, Proposition 3.1.6] Let* $\mathbb{G} = \mathbb{W}\mathbb{V}$ *be the product of two complementary subgroups and* $L : \mathbb{W} \to \mathbb{V}$ *be an intrinsic linear map. Then,* L *is intrinsic Lipschitz continuous and* $\mathrm{Lip}(L) = sup_{\|w\|=1}\|L(w)\|$.

Definition 3.5.22. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. Let $\bar{w} \in U$ and define $x = \bar{w}\phi(\bar{w})$. The function ϕ is *intrinsically differentiable* at $\bar{w}$ if there exists an intrinsic linear map $L : \mathbb{W} \to \mathbb{V}$ such that

$$\|L(w)^{-1}\phi_{x^{-1}}(w)\| = o(\|w\|) \tag{3.27}$$

as $\|w\| \to 0$, $w \in U_{x^{-1}}$. If the function L exists, it is unique. We call it the *intrinsic differential* of ϕ at $\bar{w}$ and we denote it by $d\phi_{\bar{w}}$.

Remark 3.5.23. Let us focus on the case when $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$ is a semidirect product of complementary subgroups. In this case through a quite standard change of variables, performed for instance in [AS09, Proposition 3.25(ii)], condition (3.27) is equivalent to ask that

$$\|d\phi_{\bar{w}}(\phi(\bar{w})^{-1}\bar{w}^{-1}w\phi(\bar{w}))^{-1}\phi(\bar{w})^{-1}\phi(w)\| = o(d_\phi(w, \bar{w}))$$

as $d_\phi(w, \bar{w}) \to 0$. For explicit computations one can refer also to [SC16, Proposition 4.76].

Intrinsic differentiability has been geometrically characterized by the existence of a tangent homogeneous (affine) subgroup to the intrinsic graph of a function.

Definition 3.5.24. [FMS14, Definition 3.2.6] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let U be an open subset of $\mathbb{W}$ and let $\phi : U \to \mathbb{V}$ be a function. Let us fix $\bar{w} \in U$ and consider the point $\bar{x} = \bar{w}\phi(\bar{w}) \in \mathrm{graph}(\phi)$. Let $\mathbb{T}$ be a homogeneous subgroup of $\mathbb{G}$. The coset $\bar{x}\mathbb{T}$ is a *tangent (affine) subgroup* or *tangent coset* to $\mathrm{graph}(\phi)$ at $\bar{x}$ if for every $\varepsilon > 0$ there is $\lambda = \lambda(\varepsilon) > 0$ such that

$$\mathrm{graph}(\phi) \cap \{y \in \mathbb{G} : \|\pi_{\mathbb{W}}(\bar{x}^{-1}y)\| < \lambda(\varepsilon)\} \subset X(\bar{x}, \mathbb{T}, \varepsilon).$$

Remark 3.5.25. The notion of tangent coset to $\mathrm{graph}(\phi)$ is invariant under left translations. This means that $\bar{x}\mathbb{T}$ is the tangent coset to $\mathrm{graph}(\phi)$ at $\bar{x}$ if and only if $\mathbb{T}$ is the tangent coset to $\mathrm{graph}(\phi_{\bar{x}^{-1}})$ at 0. Therefore, the notion of tangent coset (and then, in light of the following proposition, also the notion of intrinsic differentiability) is intrinsic.

Theorem 3.5.26. *[FMS14, Theorem 3.2.8] Let* $\mathbb{G} = \mathbb{W}\mathbb{V}$ *be the product of two complementary subgroups. Let* U *be open subset of* $\mathbb{W}$ *and let* $\phi : U \to \mathbb{V}$ *be a map. Then the following hold.*

(i) *If* ϕ *is intrinsically differentiable at* $\bar{w} \in U$*, we set* $\mathbb{T} := \mathrm{graph}(d\phi_{\bar{w}})$*, then*

- *(i_1)* $\mathbb{T}$ *is a homogeneous subgroup;*
- *(i_2)* $\mathbb{T}$ *and* $\mathbb{V}$ *are complementary subgroups of* $\mathbb{G}$*;*
- *(i_3)* $\bar{x}\mathbb{T}$ *is the tangent coset to graph(ϕ) at* $\bar{x} = \bar{w}\phi(\bar{w})$*.*

(ii) *Conversely, if* $\bar{x} = \bar{w}\phi(\bar{w}) \in$ *graph(ϕ) and if there is a set* $\mathbb{T}$ *such that (i_1), (i_2), (i_3) hold, then* ϕ *is intrinsically differentiable at* $\bar{w}$ *and* $d\phi_{\bar{w}}$ *is the unique intrinsic linear function such that* $\mathrm{graph}(d\phi_{\bar{w}}) = \mathbb{T}$*.*

We report also the following geometrical characterization of intrinsic differentiability in the Heisenberg group in terms of the existence and uniqueness of the blow-up limit of the intrinsic graph of the considered function. It is related to Theorem 3.5.26 and it has been proved in [FSSC11, Theorem 4.15], We think that it illustrates a fruitful point of view to visualize the notion of intrinsic differentiability.

Theorem 3.5.27. *Let* $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ *be the product of two complementary subgroups. Let* U *be open subset of* $\mathbb{W}$ *and let* $\phi : U \to \mathbb{V}$ *be a map. Let* $\bar{w} \in \mathrm{graph}(\phi)$ *and set* $\bar{x} = \bar{w}\phi(\bar{w})$*. Then* ϕ *is intrinsically differentiable at* $\bar{w}$ *if and only if there is a vertical subgroup* $\mathbb{T}$*, complementary to* $\mathbb{V}$*, such that*

$$\lim_{t\to 0+} \delta_{\frac{1}{t}}(l_{\bar{x}^{-1}}(\mathrm{graph}(\phi)) = \mathbb{T},$$

in the sense of the Hausdorff convergence on compact subsets on $\mathbb{H}^n$*.*

Remark 3.5.28. The proof of Theorem 3.5.27 substantially relies on Theorem 3.5.26, therefore it could be extended verbatim to a generic Carnot group.

Now we introduce a strengthened notion of intrinsic differentiability, the *uniform intrinsic differentiability*, introduced for the first time in [AS09, Definition 3.16]. For the form that we adopt of this concept in the following definition, please refer to [DD20a, Definition 3.3].

Definition 3.5.29. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let $U \subset \mathbb{W}$ be an open set, let $\bar{w} \in U$ and let $\phi : U \to \mathbb{V}$ be a function. The map ϕ is *uniformly intrinsically differentiable at* $\bar{w}$ if there exists an intrinsic linear map $L : \mathbb{W} \to \mathbb{V}$ such that

$$\lim_{\delta\to 0} \sup_{\|\bar{w}^{-1}w'\|<\delta} \sup_{0<\|w\|<\delta} \frac{\|L(w)^{-1}\phi_{\Phi(w')^{-1}}(w)\|}{\|w\|} = 0 \tag{3.28}$$

with $w' \in U, w \in U_{\Phi(w')^{-1}}$, where Φ is the graph map of ϕ. We say that the map ϕ is *uniformly intrinsically differentiable on* U if it is uniformly intrinsically differentiable at any point of U.

Remark 3.5.30. Clearly, in the notation of Definition 3.5.29, if ϕ is uniformly intrinsically differentiable at $\bar{w}$, then ϕ is intrinsically differentiable at $\bar{w}$ and $L = d\phi_{\bar{w}}$.

Remark 3.5.31. When we refer to Definition 3.5.29, we will not always specify that $w' \in U$ and $w \in U_{\Phi(w')^{-1}}$. In fact, since we focus on an infinitesimal limit, we can implicitly assume that δ is small enough so that $B(\bar{w}, \delta) \cap \mathbb{W} \subset U$ and $B(0, \delta) \cap \mathbb{W} \subset U_{\Phi(w')^{-1}}$.

Remark 3.5.32. If $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$ is the semidirect product of two complementary subgroups, by a quite standard change of variables, to which we referred to also in Remark 3.5.23, one can deduce that a map $\phi : U \to \mathbb{V}$ is uniformly intrinsically differentiable at a point $\bar{w} \in U$ if and only if there exists an intrinsic linear map $L : \mathbb{W} \to \mathbb{V}$ such that for $w, w' \in U$

$$\lim_{r\to 0} \sup_{w'\in B(\bar{w},r)} \sup_{\{w:\ 0<d_\phi(w,w')<r\}} \frac{\|L(\phi(w')^{-1}w'^{-1}w\phi(w'))^{-1}\phi(w')^{-1}\phi(w)\|}{d_\phi(w,w')} = 0. \tag{3.29}$$

This is equivalent to ask that ϕ is intrinsically differentiable at $\bar{w}$ and for $w, w' \in U$

$$\lim_{r\to 0} \sup_{w'\in B(\bar{w},r)} \sup_{\{w:\ 0<d_\phi(w,w')<r\}} \frac{\|d\phi_{\bar{w}}(\phi(w')^{-1}w'^{-1}w\phi(w'))^{-1}\phi(w')^{-1}\phi(w)\|}{d_\phi(w,w')} = 0. \tag{3.30}$$

Remark 3.5.33. Observe that by the estimates of [FS16, Lemma 2.13], for every compact subset $F \subset U$ there exist two constants $C_1, C_2 > 0$, such that for every $w, w' \in F$

$$C_1\|w'^{-1}w\|^\kappa \le d_\phi(w,w') \le C_2\|w'^{-1}w\|^{\frac{1}{\kappa}}.$$

Hence, condition (3.29) turns out to be equivalent to the following

$$\lim_{r\to 0} \sup_{\substack{w,w'\in B(\bar{w},r)\cap U\\ w\neq w'}} \frac{\|L(\phi(w')^{-1}w'^{-1}w\phi(w'))^{-1}\phi(w')^{-1}\phi(w)\|}{d_\phi(w,w')} = 0. \tag{3.31}$$

Uniform intrinsic differentiability implies local intrinsic Lipschitz continuity.

Proposition 3.5.34. *[AS09, Proposition 3.17] Let $\mathbb{W}$, $\mathbb{V}$ be complementary subgroups of a Carnot group $\mathbb{G}$. Let $U \subset \mathbb{W}$ be an open set. Consider a function $\phi : U \to \mathbb{V}$ uniformly intrinsically differentiable on U. Then for every $\bar{w} \in U$, there is $r > 0$ such that ϕ is intrinsic Lipschitz on $\mathbb{B}(\bar{w}, r) \cap U$.*

3.5.3 Intrinsic difference quotients

In this section we report a characterization of intrinsic Lipschitz continuity presented in [Ser17]. In the Euclidean setting the notion of Lipschitz function can be characterized through the boundedness of the difference quotients of the function itself. Serapioni has investigated an analogous point of view for intrinsic Lipschitz maps developing the notion of intrinsic difference quotients. For more details about the characterization of intrinsic Lipschitz continuity presented in this section please refer, beyond [Ser17], to [FSSC06, Section 3.2], [AS09, Section 5] or [FS16, Section 3.2].

Definition 3.5.35. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a map. Let $W \in \mathrm{Lie}(\mathbb{W})$ and set $\bar{w} = W(0)$. Let us assume that $0 \in U$ and $\phi(0) = 0$. The *intrinsic difference quotient* $\Delta_W\phi(0, t)$ *of* ϕ *at* $0 \in U$ *along* W is defined as

$$\Delta_W\phi(0; t) := \delta_{\frac{1}{t}}\phi(\delta_t(\exp(W))) = \delta_{\frac{1}{t}}\phi(\delta_t(\bar{w}))$$

for all $t > 0$ such that $\delta_t(\bar{w}) \in U$. This definition is then extended to any point $w \in U$. Fix $w \in U$ and consider the corresponding point on the graph $y := w\phi(w) \in \mathrm{graph}(\phi)$, then $0 \in U_{y^{-1}}$ and $\phi_{y^{-1}}(0) = 0$, and we define the *intrinsic difference quotient of* ϕ *at* w

along W as

$$\begin{aligned}\Delta_W\phi(w;t) :&= \Delta_W\phi_{y^{-1}}(0;t)\\ &= \delta_{\frac{1}{t}}(\phi_{y^{-1}}(\delta_t(\exp(W)))\\ &= \delta_{\frac{1}{t}}(\phi_{y^{-1}}(\delta_t(\bar{w})))\end{aligned}$$

for all $t>0$ such that $\delta_t(\bar{w})\in U_{y^{-1}}$.

Remark 3.5.36. We can write this definition more explicitly through the formula that relates $\phi_{y^{-1}}$ and ϕ. In the notation of Definition 3.5.35, the intrinsic difference quotient of ϕ at w along W is

$$\Delta_W\phi(w;t) = \delta_{\frac{1}{t}}\left((\pi_{\mathbb{V}}(\phi(w)\delta_t(\bar{w})))^{-1}\phi(w\pi_{\mathbb{W}}(\phi(w)\delta_t(\bar{w})))\right)$$

for all $t>0$ such that $w\pi_{\mathbb{W}}(\phi(w)\delta_t(\bar{w}))\in U$. For explicit computations refer to [Ser17, Definition 3.7].

Proposition [Ser17, Proposition 3.11], reported below, gives us a characterization of intrinsic Lipschitz maps in terms of the boundedness of their intrinsic difference quotients.

Proposition 3.5.37. *Let $\mathbb{G}=\mathbb{W}\mathbb{V}$ be the product of two complementary subgroup. Let $U\subset\mathbb{W}$ be an open set and $\phi: U\to\mathbb{V}$ be a function. The following statements are equivalent.*

(i) ϕ is intrinsic L-Lipschitz on U.

(ii) There is a constant $L>0$ such that, for all $W\in\mathrm{Lie}(\mathbb{W})$ and for all $w\in U$, setting $\bar{w}=W(0)$,

$$\|\Delta_W\phi(w,t)\|\le L\|\bar{w}\|,$$

for all $t>0$ such that $w\pi_{\mathbb{W}}(\phi(w)\delta_t(\bar{w}))\in U$.

Basically then, intrinsic Lipschitz functions correspond to functions with equibounded intrinsic difference quotients, as metric Lipschitz functions correspond to functions with equibounded incremental ratios.

Remark 3.5.38. For some Carnot groups, in order to prove the intrinsic Lipschitz continuity of a function, it is enough to prove that the intrinsic difference quotients along the vector fields corresponding to horizontal vectors in $\mathbb{W}\cap V_1$ are bounded. The Heisenberg group, for instance, is a group in which this phenomenon can be observed [Ser17, Theorem 3.21]. This is not obvious in general, since if $\mathbb{W}$ is a homogeneous subgroup, it is surely graded but it is not always a Carnot group, i.e. $\mathbb{W}\cap V_1$ does not always Lie bracket-generate the whole $\mathbb{W}$. For more details about this idea please refer to [Ser17].

Moving a step forward according to the leading idea of Serapioni's approach, one can observe that the natural definition of continuously (Euclidean) differentiable function can be stated through the existence and continuity of partial derivatives. Exploiting an analogous point of view in Carnot groups for uniform intrinsic differentiability is a very complicated problem and it has been deeply studied in a long series of papers. One step was taken in [Ser17, Section 3.3] (see Remark 5.1.16). A contribute to this line of research is provided by Chapter 5 of this thesis.

3.6 The one-codimensional case and beyond

In this section our initial focus are intrinsic graphs of maps $\phi : U \to \mathbb{V}$ acting between two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$ of a generic Carnot group $\mathbb{G}$, *with* $\mathbb{V}$ *of dimension one*, and $U \subset \mathbb{W}$ open set. We reserve a proper section for this class of graphs since they have been the main objects of research of a huge amount of studies. One of the reasons that fuels this interest arises from a simple key observation. When we assume that a homogeneous subgroup $\mathbb{V}$ is one-dimensional, it can be naturally identified with $(\mathbb{R}, +)$, so that it is natural to endow the subgroup $\mathbb{V}$ with an order relation. This observation allows us to use many suitable tools to work in this situation and makes this setting a forerunner for the study of geometric measure theory on generic intrinsic graphs. Consider two complementary subgroups $\mathbb{W}$, $\mathbb{V}$ of a generic Carnot group $\mathbb{G} = \mathbb{W}\mathbb{V}$ with $\mathbb{V}$ one dimensional. Consider an open subset $U \subset \mathbb{W}$, a function $\phi : U \to \mathbb{V}$ and a vector $v \in V_1$ such that $\mathbb{V} = \mathrm{span}(v)$. The map ϕ can be identified with the unique real-valued mapping $\varphi : U \to \mathbb{R}$ that satisfies

$$\phi(w) = \varphi(w)v$$

for every $w \in U$. We can define the *supergraph* E_ϕ^+ and the *subgraph* E_ϕ^- of ϕ as follows

$$E_\phi^+ = \{w(tv) : w \in U \ , \ t > \varphi(w)\},$$

$$E_\phi^- = \{w(tv) : w \in U \ , \ t < \varphi(w)\}.$$

Now we are ready to take a brief round-up of the main interesting recent results that have been proved in this setting. A *Mc Shane-type extension theorem* for intrinsic Lipschitz graphs is available.

Theorem 3.6.1. *[FS16, Theorem 4.1] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, with $\mathbb{V}$ of dimension one. Let $B \subset \mathbb{W}$ be a Borel subset of $\mathbb{W}$ and let $\phi : B \to \mathbb{V}$ be an intrinsic L-Lipschitz function. Then, there exist a map $\tilde{\phi} : \mathbb{W} \to \mathbb{V}$ and a constant $\tilde{L} = \tilde{L}(L, \mathbb{G}, \mathbb{W}, \mathbb{V}) \geq L$ such that $\tilde{\phi}$ is intrinsic $\tilde{L}$-Lipschitz on $\mathbb{W}$ and $\tilde{\phi}(w) = \phi(w)$ for all $w \in B$.*

An analogous extension theorem has been proved in the Heisenberg group by Naor and Young [NY18, Theorem 27], according to a slightly modified notion of intrinsic Lipschitz continuity, introduced through a small change in the definition of intrinsic cone. For many homogeneous distances the family of intrinsic Lipschitz maps considered by Naor and Young coincides with the one that we have considered in this thesis. Naor and Young's extension Theorem exactly retraces Theorem 3.6.1. The improvement consists of the fact that the intrinsic Lipschitz constant of the extended function $\tilde{\phi}$ can be chosen equal to the intrinsic Lipschitz constant of the starting map ϕ, so this constant does not increase (see also Remark 4.5.11).

An other generalized Mc-Shane type extension theorem has recently been proved by Vittone [Vit20, Theorem 1.5]. It is a generalization of Theorem 3.6.1 to the case of *co-horizontal* intrinsic Lipschitz graphs in a generic Carnot group $\mathbb{G}$, i.e. intrinsic graphs of a map $\phi : U \subset \mathbb{W} \to \mathbb{V}$, where $\mathbb{G}$ is the product of two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$ and $\mathbb{V}$ is horizontal of dimension k. The proof of [Vit20, Theorem 1.5] is obtained through a new innovative characterization of intrinsic Lipschitz continuity for maps whose target space is a horizontal homogeneous subgroup $\mathbb{V}$, proved in [Vit20, Theorem 1.4]. More explicitly, the author proved that any co-horizontal graph, like $\mathrm{graph}(\phi)$, coincides with the level set of an Euclidean Lipschitz map $f : \mathbb{G} \to \mathbb{R}^k$ satisfying a condition of uniform coercivity, i.e. such that there exists some $\delta > 0$ such that

$$\langle f(xv) - f(x), v\rangle \geq \delta |v|^2 \quad \text{for every } v \in \mathbb{V} \text{ and } x \in \mathbb{G},$$

(for the case $k = 1$ previous results had been proved in this direction, see [Vit12, Definition 1.1, Theorem 3.2]). This approach opens the possibility of dealing with intrinsic Lipschitz graphs *from outside*. Indeed, this is an interesting starting point to extend arguments and techniques typically used for low codimensional intrinsic regular submanifolds, like $\mathbb{H}$-regular surfaces of low codimension (see Chapter 4), to lower regular submanifolds, i.e. to low codimensional intrinsic Lipschitz graphs.
For the sake of completeness, we mention also a Mc-Shane extension theorem very recently proved by Di Donato and Fässler [DDF20, Theorem 1.2]. The two authors proved that, if $\mathbb{W} \subset \mathbb{H}^n$ is a horizontal subgroup and $\mathbb{V}$ is a homogeneous subgroup complementary to $\mathbb{W}$, for every $L \geq 0$ there exists a constant L', depending on n, on $\dim(\mathbb{V})$ and on L such that every intrinsic Lipschitz function $\phi : U \subset \mathbb{W} \to \mathbb{V}$, $U \subset \mathbb{W}$, can be extended to an intrinsic L'-Lipschitz map $\tilde{\phi} : \mathbb{W} \to \mathbb{V}$, such that $\phi = \tilde{\phi}$ on U.

The subgraph of an intrinsic Lipschitz map with one dimensional target space is a set of locally finite H-perimeter.

Theorem 3.6.2. *[FMS14, Theorem 4.2.9] Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, with $\mathbb{V}$ of dimension one. If $\phi : \mathbb{W} \to \mathbb{V}$ is an intrinsic Lipschitz map, its subgraph E^-_ϕ is a set of locally finite H-perimeter.*

Theorem 3.6.2, along with Theorem 3.6.1, was the key starting point to prove a series of interesting results, the most relevant of which is surely the following Rademacher-type Theorem for intrinsic Lipschitz maps. It has been proved in [FSSC11, Theorem 4.29] in the setting of the Heisenberg group and then it has been generalized to groups of type $\star$ in [FMS14].

Definition 3.6.3. A Carnot group $\mathbb{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$ is of type $\star$ if there exists a basis $(v_1, \dots, v_{m_1})$ of V_1 such that

$$[v_j, [v_j, v_i]] = 0 \quad \text{for every } i, j = 1, \dots, m_1.$$

All step-2 Carnot groups are of type $\star$. One can prove that there exist Carnot groups of type $\star$ of any step.

Theorem 3.6.4. *[FMS14, Theorem 4.3.5] Let $\mathbb{G}$ be a Carnot group of type $\star$ of topological dimension q and let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, with $\mathbb{V}$ of dimension one. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be an intrinsic Lipschitz function, then ϕ is intrinsically differentiable $(\mathcal{L}^{q-1} \llcorner \mathbb{W})$-almost everywhere on U.*

One of the main tools used to prove Theorem 3.6.4 is a blow-up result valid at almost every point of the intrinsic graph of the considered intrinsic Lipschitz map ϕ. In particular, the authors prove that the blow-up of the subgraph of ϕ, E^-_ϕ, at each point of its H-reduced boundary exists, is unique and is a vertical half-space (i.e. an half-space whose boundary is a vertical subgroup). This permits to deduce the intrinsic differentiability of ϕ at all the points of the form $\pi_{\mathbb{W}}(x) \in \mathbb{W}$ for some point $x \in \partial^\star_H E^-_\phi$. Finally, the proof is substantially completed by observing that almost every point of the graph of ϕ belongs to the H-reduced boundary of the subgraph and observing that the group projection $\pi_{\mathbb{W}}$ on $\mathbb{W}$ preserves full-measure sets.
Theorem 3.6.4 cannot be easily extended to a generic Carnot group. For instance, a well-known counterexample [FSSC03a, Example 3.2] describes a locally finite H-perimeter set

E in the Engel group (that is the simplest step-3 Carnot group) such that $0 \in \partial_H^\star E$ but the blow-up at 0 fails, in the sense that the blow-up of E at 0 is not a vertical half-space. By the scheme of its proof it is also clear that Theorem 3.6.4 relies on deep results coming from the theory of locally finite H-perimeter sets. As a consequence, its proof cannot be easily extended to intrinsic Lipschitz graphs of codimension higher than one, even in the Heisenberg group. One more problem is that, unlike what happens in the Euclidean setting, working by coordinates is not fruitful, since the coordinates of an intrinsic Lipschitz function are not necessarily intrinsic Lipschitz.
Nevertheless, recently Vittone proved a generalization in the Heisenberg group $\mathbb{H}^n$ of Theorem 3.6.4 to intrinsic Lipschitz maps with target space a horizontal subgroup $\mathbb{V}$ with $1 \leq \dim(\mathbb{V}) \leq n$; see [Vit20, Theorem 1.1]. His innovative and very deep proof substantially relies on three main ingredients (on top of the extension theorem for intrinsic Lipschitz maps that we already discussed):

(i) an approximation result for intrinsic Lipschitz graphs by (Euclidean) smooth uniformly intrinsic Lipschitz maps whose domain is an entire homogeneous subgroup [Vit20, Theorem 1.6];

(ii) the first original result available for one-codimensional intrinsic Lipschitz graphs, [FSSC11, Theorem 4.29];

(iii) some delicate tools coming from the theory of currents in the Heisenberg group, that are objects defined through the complex of differential forms introduced by Rumin in [Rum94].

The involvement of the language of Heisenberg currents, that are very difficult to handle, makes the proof quite technical. For good introductions to this theme one can also refer to [FSSC07, Section 5] and [FT15].
For the sake of completeness, we recall that a Rademacher-type theorem has recently been proved by Antonelli and Merlo in [AM20a]. It holds for intrinsic Lipschitz functions $\phi : U \subset \mathbb{W} \to \mathbb{V}$, where $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups of a Carnot group $\mathbb{G}$, such that $\mathbb{V}$ is a normal homogeneous subgroup. Its proof substantially relies on the Pansu-Rademacher theorem, that in this context is applied to the graph map $\Phi : U \to \mathbb{G}$. For example, [AM20a, Theorem 1.1] permits to deduce that if $\mathbb{H}^n = \mathbb{V} \rtimes \mathbb{W}$ is the semidirect product of two complementary subgroups, and then $\mathbb{W}$ is horizontal of dimension $1 \leq \dim(\mathbb{W}) \leq n$, any intrinsic Lipschitz map $\phi : U \to \mathbb{V}$, where $U \subset \mathbb{W}$ is an open set, is $\mathcal{L}^{\dim(\mathbb{W})}$-a.e. intrinsically differentiable.
In view of the described context, one could naturally wish to extend the proof of [Vit20, Theorem 1.1] to every intrinsic Lipschitz graph in an arbitrary Carnot group, so that to prove the general validity of the Rademacher's theorem for intrinsic Lipschitz maps. In this respect, a first observation could be that, in order to deal with Carnot groups not of type $\star$, one should first provide an alternative proof of the original Theorem 3.6.4. In fact, a proof that can be extended to more general settings would be needed since, how we already stressed, relying on a blow-up theorem at the points of the H-reduced boundary of the subgraph of an intrinsic Lipschitz map of codimension 1 is not possible even in the Engel group, that is the simplest step-3 Carnot group. One could then reconsider the project, and decide to focus first on those settings in which Theorem 3.6.4 is verified to hold, hence, for instance, Carnot groups of step 2. Nevertheless even a similar project is destined to fail: very recently Julia, Nicolussi Golo and Vittone in [JNGV21] showed explicitly that is possible to build nowhere intrinsically differentiable intrinsic Lipschitz maps in suitable Carnot groups, among which suitable Carnot groups of step 2. For instance, it is possible

to build a 2-codimensional intrinsic Lipschitz graph in $\mathbb{H}^1 \times \mathbb{R}$ such that, at any of its points, there exist infinite blow-ups, and none of them is a homogeneous subgroup. Hence, not only Vittone's proof cannot be extended to the very general setting of Carnot groups but, actually, the conjecture about the general validity of the Rademacher's theorem for all intrinsic Lipschitz functions is false, even in Carnot groups of step 2.

Further available results about intrinsic regular one-codimensional graphs will be presented later, precisely in Sections 4.5.1 and 5.2. We decided not to gather all these results together for two different motivations. First, Section 4.5.1 focuses on results connected with uniform intrinsic rectifiability in the Heisenberg group, hence we prefer to collocate all the related discussions close to the definition of intrinsic rectifiable set. On the other side, we postpone the presentation of the results of Section 5.2 since Chapter 5 will be entirely devoted to the generalization of those results. We will adopt a quite different point of view from the one that characterized the current section. The underlying spirit will be to prove various characterizations of the intrinsic regularity of a map ϕ in terms of the regularity of a vector-valued function (or, depending on the involved dimensions, of a matrix-valued function), called the *intrinsic gradient* (or, respectively, the *intrinsic Jacobian matrix*), denoted by $\nabla^{\phi}\phi$ (or, respectively, by $D^{\phi}\phi$), that represents the intrinsic differential of ϕ.

Chapter 4

Regular submanifolds in Carnot groups

A typical problem of geometric measure theory in metric spaces is to individuate an appropriate definition of regular submanifold, or regular surface. The classical notion goes back to Federer, that defined a "good" surface in a metric space as the image of an open subset of an Euclidean space through a (metric) Lipschitz map.
A strong motivation supporting the need of understanding how regular submanifolds in Carnot groups can be considered, is the necessity of stating a suitable definition of rectifiability in these spaces. Rectifiable sets were introduced in the 1920's by Besicovitch and in 1947 in general dimension by Federer [Fed69, 3.2.14]. Successively this theme has been deepened in general metric spaces by Ambrosio and Kircheim in [AK00]. According to Federer's original definition, a k-dimensional rectifiable set is a set that can be covered, up to a negligible set, by a countable union of Lipschitz images of subsets of $\mathbb{R}^k$ (i.e. k-dimensional Federer's "good" submanifolds). Unfortunately, these classical notions of regular surface and rectifiable set do not suit Carnot group geometry because, roughly speaking, open subsets of $\mathbb{R}^k$ are not always appropriate to be used as parameter spaces for k-dimensional submanifolds within a Carnot group $\mathbb{G}$. For example, Ambrosio and Kirchheim in [AK00, Theorem 7.2] showed that in $\mathbb{H}^1$, for $k = 2, 3, 4$, for any Lipschitz map $f : A \subset \mathbb{R}^k \to \mathbb{H}^1$, it holds that $\mathcal{H}^k(f(A)) = 0$, namely $\mathbb{H}^1$ is k-purely unrectifiable (see also [Mag04b]).

In the Euclidean space $\mathbb{R}^n$, as a consequence of the Rademacher's theorem, an equivalent way to define a k-dimensional rectifiable set is to require that it can almost all be covered by a countable union of k-dimensional C^1-regular submanifolds, i.e. images through C^1-regular maps of open subsets of $\mathbb{R}^k$. In analogous way, we would like to state a suitable notion of rectifiability in Carnot groups following (at least separately) both the path involving Lipschitz continuity and the one involving C^1-regularity. Therefore we need first to individuate suitable intrinsic notions of C^1 and Lipschitz regular submanifolds.
One of the first ideas that could come to mind is to define regular submanifolds as Euclidean C^1-regular submanifolds embedded in a Carnot group. For our purposes, this is not a good point of view, since these objects are not always regular from the point of view of the group structure. If we consider, for instance, a C^1 (Euclidean) regular hypersurface Σ embedded in a Carnot group $\mathbb{G}$, there could exist on Σ the so-called *characteristic points*, i.e. points $x \in \Sigma$ where the horizontal fiber at x is contained in the Euclidean tangent space, i.e. $H_x\mathbb{G} \subset T_x\Sigma$, hence a sort of transversality condition fails. Characteristic points are singular with respect to the metric structure of $\mathbb{G}$, in particular, the blow-up of Σ at a characteristic point is not a subgroup, and this is an unpleasant occurrence according

to Carnot group natural structure. We search for a definition of regular submanifold that respects more carefully the structure of the group. In the Euclidean space $\mathbb{R}^n$, a regular submanifold of arbitrary dimension k can be equivalently defined as a subset of $\mathbb{R}^n$ that is, locally, either the Euclidean graph of a C^1-regular function $\phi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ or a level set of a C^1-regular function $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ with continuous surjective differential. In Carnot group both these two strategies can be mimicked, so that regular submanifolds can be defined as level sets (or images, depending on the dimension) of continuously non-degenerate Pansu differentiable functions f acting between two Carnot groups or as intrinsic regular graphs of functions acting between two complementary subgroups of a Carnot group. At first sight, it could seem that these two approaches, even if developed through appropriate notions of regularity, do not give equivalent results. Nevertheless, in many cases, this first impression is disproved by suitable implicit function theorems.

In this chapter we provide a summary of the available definitions of intrinsic regular submanifolds in Carnot groups. They have been introduced mimicking the Euclidean notions through the concept of Pansu differentiability. We enrich the picture proving some new related results. The material is organized in order to introduce first the available definition in the Heisenberg group, and successively the notions in a generic Carnot group. In this perspective, $\mathbb{H}$-*regular surfaces* will be our starting point. We focus on the notions both of low dimensional and low codimensional $\mathbb{H}$-regular surface (Definitions 4.1.1 and 4.1.8). Introduced and deeply investigated by Franchi, Serapioni and Serra Cassano in a long series of papers (the first of which is [FSSC07]), they have been the first objects to be candidate for the role of regular submanifolds in the Heisenberg group. The leading idea of this definition is to individuate subsets of the Heisenberg group equipped at any point with a tangent plane and a transversal plane, and to require that both of them are left cosets of homogeneous subgroups that vary in a continuous way as the considered point varies on the surface.
Following the same philosophy, we move then to focus on more general settings and we present the notions of $(\mathbb{G}, \mathbb{M})$-*regular sets* of $\mathbb{G}$ and of $\mathbb{M}$, where $\mathbb{G}$ and $\mathbb{M}$ are two generic Carnot groups (Definitions 4.2.8 and 4.2.6). These objects have been described in an organic way by Magnani in [Mag13], where their properties have been deeply studied. They can be considered as the natural generalization of the concept of $\mathbb{H}$-regular submanifold from the Heisenberg group to any generic Carnot group. We dedicate a preliminary section to present $\mathbb{G}$-regular hypersurfaces of a generic Carnot group, introduced in [FSSC03b]. In this thesis we consider them as $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$, for $\mathbb{M} = \mathbb{R}$, but, for the sake of accuracy, we stress that theirs is the first notion of intrinsic regular submanifold in a Carnot group presented in the literature.
We prove then Theorem 4.3.7 that is an original contribution of this thesis. Starting from the implicit function Theorem 4.2.13, we prove that, given two Carnot groups $\mathbb{G}$ and $\mathbb{M}$, any $(\mathbb{G}, \mathbb{M})$-regular set Σ of $\mathbb{G}$ is locally the intrinsic graph of a uniformly intrinsically differentiable function. In particular, we prove that the intrinsic parametrization of Σ provided by the implicit function Theorem 4.2.15 is uniformly intrinsically differentiable.
Successively, we briefly present the wider and wilder family of regular submanifolds introduced by Kozhevnikov in [Koz15], we will call them $(\mathbb{G}, \mathbb{M})_K$-*regular submanifolds* (Definition 4.4.1). In [Koz15] the author approaches the formulation of a definition of regular submanifold within a Carnot group in a way slightly different from the one described above. In broad terms, he retraces the previous definitions giving up the requirement that the transversal space to the submanifold has to be a homogeneous subgroup complementary to the tangent subgroup. When compared to the available investigations about $(\mathbb{G}, \mathbb{M})$-regular sets, the research about $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds is at a very initial and early stage. In particular, up to now, it has been deepened only for a subclass of

$(\mathbb{G}, \mathbb{M})_K$-regular submanifolds called *co-Abelian surfaces*, introduced in [BK14].
We conclude the chapter by presenting some notions of intrinsically rectifiable set, usually referred to as *intrinsic rectifiable sets*, in the Heisenberg group (Definitions 4.5.2 and 4.5.4). They are a natural application of the definitions of $\mathbb{H}$-regular surface and intrinsic Lipschitz graph, respectively. We follow the approach proposed by Franchi, Serapioni, Serra Cassano, that, up to now, is the most developed one in this line of research. In particular for the definition of intrinsic rectifiable set in the Heisenberg group, and, more in general, in Carnot groups, one can refer to [FSSC01] for sets of codimension 1 and to [FSSC07, FSSC11, MSSC10] for sets of generic dimension. We give also a glance of the available notions of intrinsic rectifiability in a generic Carnot group. For the sake of completeness, we stress that a different formulation of rectifiability in Carnot groups has been proposed by Pauls, paired with a different concept of regular submanifold (see [Pau04b, Definition 4.1]). In particular, Pauls considers as regular submanifolds the images in $\mathbb{G}$ of Lipschitz maps defined, not on some Euclidean space $\mathbb{R}^k$, as was required in Federer's definition, but on a subgroup of $\mathbb{G}$. We will not deepen this theme in the thesis but we want to highlight that the relation between the two proposed definitions of intrinsic rectifiable set is still far from being clear and it is an open problem on which many researches are nowadays actively concentrated. In particular, Antonelli and Le Donne have recently shown, in [ALD20], that in a generic Carnot group Pauls' rectifiability is not equivalent to Franchi, Serapioni, Serra Cassano's one. On the other side, it has been proved that on some families of regular sets the two notions coincide (refer to [CP06, BV10, DDFO19]).
In continuity with the topic of intrinsic rectifiability, the last section is dedicated to some very recent remarkable results regarding the starting development of a theory of *uniform* or *quantitative* intrinsic rectifiability in the Heisenberg group.

4.1 $\mathbb{H}$-regular surfaces

An initial detailed study of $\mathbb{H}$-regular surfaces has been carried out in [FSSC07], where an implicit function theorem and an area formula for the centered Hausdorff measure of these objects, with respect to the distance d_∞, have been proved. In the same work, following the Rumin complex [Rum94], the authors exploit their results to introduce the first notions of the theory of Heisenberg currents. It is necessary to distinguish $\mathbb{H}$-regular surfaces, according to their dimensions, in the two families of low dimensional and low codimensional ones. We will discuss why the two definitions cannot be extended one to cover the other one. Nevertheless, at the end of this section it will be clear to the reader that it is possible to see all the objects satisfying one of the two definitions, under a common light. In fact, all these objects have been characterized as sets that are locally the intrinsic graph of a uniformly intrinsically differentiable map. This allows to unify the two definitions of low dimensional and low codimensional $\mathbb{H}$-regular surfaces under a common theoretical approach. Actually, the concept of intrinsic graph was individuated by Franchi, Serapioni and Serra Cassano proving an implicit function theorem for $\mathbb{H}$-regular hypersurfaces, i.e. $\mathbb{H}$-regular surfaces of codimension one ([FSSC01, Theorem 6.5]). The authors realized that the natural parametrization of an $\mathbb{H}$-regular hypersurface can be exactly realized by the notion of intrinsic graph, and therefore they understood that this notion is the right one to be stated in homogeneous group contexts to generalize the definition of Euclidean graph. A small remark about terminology: a careful reader may already noticed that we use the term “surface” as a synonym of the term “submanifold”, this will be done in the whole thesis. This choice is quite common also in the related literature. To indicate one-codimensional submanifolds people usually use the term *hypersurface*.

4.1.1 $\mathbb{H}$-regular surfaces of low dimension

Low dimensional $\mathbb{H}$-regular surfaces retrace Federer's original idea of surface. In fact, the intrinsic regular surfaces of $\mathbb{H}^n$ of dimension k, with $1 \le k \le n$, are images of continuously Pansu differentiable functions, with injective Pansu differential, defined from an open subset of the Euclidean commutative group $\mathbb{R}^k$ to $\mathbb{H}^n$. Since continuously Pansu differentiable maps are locally Lipschitz, these objects really satisfy Federer's idea of submanifold.

Definition 4.1.1. Let $1 \le k \le n$, a set $\Sigma \subset \mathbb{H}^n$ is a $\mathbb{H}$-*regular surface of dimension* k if for every $\bar{x} \in \Sigma$ there are two open sets $V \subset \mathbb{R}^k$, $\Omega \subset \mathbb{H}^n$ with $\bar{x} \in \Omega$, and a function $f : V \to \Omega$ such that f is injective, $f \in C^1_h(V, \Omega)$ and the Pansu differential $Df(x) : \mathbb{R}^k \to \mathbb{H}^n$ is injective for each $x \in V$, such that

$$\Sigma \cap \Omega = f(V).$$

Remark 4.1.2. Definition 4.1.1 cannot be extended to the case when $k \ge n+1$. If fact, if we assume by contradiction that $k \ge n+1$, the Pansu differential at any point $x \in V$, $Df(x) : \mathbb{R}^k \to \mathbb{H}^n$ is an injective h-homomorphism and $Df(x)(\mathbb{R}^k) \subset H_1$ is a homogeneous commutative subalgebra, then $k \le n$ so that we reach a contradiction.

Let us introduce the definition of homogeneous tangent cone, that can be thought as the set of all those vectors in some sense tangent to a set A at a point x.

Definition 4.1.3. Let $A \subset \mathbb{H}^n$ and choose a point $x \in A$. We call *homogeneous tangent cone* of A at x the set

$$\begin{aligned}\mathrm{Tan}(A, x) = \Big\{ \nu \in \mathbb{H}^n : \nu = \lim_{h\to\infty} \delta_{r_h}(x^{-1}x_h), \text{ for some sequences } (r_h)_{h\in\mathbb{N}} > 0, \\ (x_h)_{h\in\mathbb{N}} \subset A, \ \lim_{h\to\infty} x_h = x \Big\}.\end{aligned} \tag{4.1}$$

The homogeneous tangent cone of an $\mathbb{H}$-regular surface of low dimension, at any of its points, is individuated by the following proposition.

Proposition 4.1.4. *[FSSC07, Theorem 3.5] Let $1 \le k \le n$ and let $\Sigma \subset \mathbb{H}^n$ be a k-dimensional $\mathbb{H}$-regular surface. Moreover, let $x \in \Sigma \cap \Omega$ and f be as in Definition 4.1.1. If $f(\bar{w}) = x$, then*

$$\mathrm{Tan}(\Sigma, x) = \{Df(\bar{w})(w) : w \in \mathbb{R}^k\} = Df(\bar{w})(\mathbb{R}^k).$$

Moreover, $\mathbb{H}$-regular surfaces of low dimension are Euclidean regular surfaces and the homogeneous tangent cone of a low dimensional $\mathbb{H}$-regular surface, at any of its point, coincides with the Euclidean tangent space. Infact, again by [FSSC07, Theorem 3.5], the following holds.

Proposition 4.1.5. *Let $\Sigma \subset \mathbb{H}^n$ be a k-dimensional $\mathbb{H}$-regular surface, with $1 \le k \le n$, then Σ is an Euclidean k-dimensional submanifold of $\mathbb{H}^n$ of class C^1. Moreover, for every $x \in \Sigma$, the Euclidean tangent space of Σ at x coincides with the homogeneous tangent cone of Σ at x, i.e. it holds that for every $x \in \Sigma$*

$$T_x\Sigma = \mathrm{Tan}(\Sigma, x).$$

If Σ is an $\mathbb{H}$-regular surface of low dimension, its spherical Hausdorff measure is comparable to its Euclidean Hausdorff measure.

Proposition 4.1.6. *[FSSC07, Theorem 3.5] Let $\Sigma \subset \mathbb{H}^n$ be a k-dimensional $\mathbb{H}$-regular surface, with $1 \leq k \leq n$, then $\mathcal{S}^k \llcorner \Sigma$ is comparable to $\mathcal{H}^k_E \llcorner \Sigma$.*

Remark 4.1.7. Notice that the proof of Proposition 4.1.6 is based on a general area formula for Lipschitz maps acting between two Carnot groups, proved in [Mag02a, Theorem 4.3.4] (see also [Pau04b, Theorem 3.3]). In fact, if f is a defining map for Σ at a point $\bar{x} \in \Sigma$, as in Definition 4.1.1, f is a (injective) locally Lipschitz map between $\mathbb{R}^k$ and $\mathbb{H}^n$.

4.1.2 $\mathbb{H}$-regular surfaces of low codimension

Unlike low dimensional $\mathbb{H}$-regular surfaces, low codimensional $\mathbb{H}$-regular surfaces, i.e. regular submanifolds of low codimension in the Heisenberg group, are not Euclidean regular surfaces in $\mathbb{H}^n$. A set $\Sigma \subset \mathbb{H}^n$ is a regular surface of codimension k, with $1 \leq k \leq n$, if it is locally a level set of a Pansu differentiable function f from $\mathbb{H}^n$ to $\mathbb{R}^k$, with Pansu differential both continuous and surjective.

Definition 4.1.8. Let $\Sigma \subset \mathbb{H}^n$ be a set and let $1 \leq k \leq n$. We say that Σ is a $\mathbb{H}$-*regular surface of codimension* k if for every $\bar{x} \in \Sigma$ there exist an open set $\Omega \subset \mathbb{H}^n$ such that $\bar{x} \in \Omega$ and a function $f = (f_1, \dots, f_k) \in C^1_h(\Omega, \mathbb{R}^k)$ such that

(i) $\Sigma \cap \Omega = \{x \in \Omega : f(x) = 0\}$;

(ii) $\nabla_H f_1(x) \wedge \dots \wedge \nabla_H f_k(x) \neq 0$ for all $x \in \Omega$.

Remark 4.1.9. If $\Sigma \subset \mathbb{H}^n$ is a k-codimensional $\mathbb{H}$-regular surface, with $1 \leq k \leq n$, we will also call it a $(2n+1-k)$-dimensional $\mathbb{H}$-regular surface.

Remark 4.1.10. Definition 4.1.8 could theoretically be extended to the case when $k \geq n+1$. Nevertheless, the constraint usually required to k, i.e. that $1 \leq k \leq n$, originates more from the ideas underlying the definition than from the definition itself. Let us explain more carefully this concept. As we explained in the introduction of this chapter, we want to individuate regular surfaces as sets equipped at any point with a tangent plane and a transversal plane such that both of them are left cosets of homogeneous subgroups. Requiring (ii) of Definition 4.1.8, we require that at any point $x \in \Omega$ the Pansu differential $Df(x) : \mathbb{H}^n \to \mathbb{R}^k$ is surjective and then that $\ker(Df(x))$, that plays the role of the tangent space (see Proposition 4.1.13), is a homogeneous vertical subgroup of $\mathbb{H}^n$ of dimension $2n+1-k$. In addition, when $k \leq n$, by Proposition 4.1.21, requiring (ii) is equivalent to ask that $\ker(Df(x))$ admits a complementary homogeneous subgroup, which plays the role of the transversal homogeneous subspace. By the form of the couples of complementary subgroups of $\mathbb{H}^n$, any subgroup complementary to $\ker(Df(x))$ has to be horizontal, and then commutative, and if we denote its dimension by k, necessarily k has to be smaller or equal than n.

Remark 4.1.11. As we said, if $1 \leq k \leq n$, a k-dimensional $\mathbb{H}$-regular surface Σ is an Euclidean submanifold of $\mathbb{R}^{2n+1}$ of class C^1. Instead, a k-codimensional $\mathbb{H}$-regular surface, can be very irregular from the Euclidean point of view. For instance, in [KSC04], it is shown the construction of an $\mathbb{H}$-regular hypersurface that is an Euclidean fractal. In particular, the authors build an $\mathbb{H}$-regular surface with Euclidean Hausdorff dimension equal to 2.5 in $(\mathbb{H}^1, |\cdot|)$.

Remark 4.1.12. On the other side, there exist embedded Euclidean C^1-regular surfaces Σ in $\mathbb{H}^n$ that are not $\mathbb{H}$-regular. For example, if Σ is an Euclidean regular embedded hypersurface, on Σ could exist the so-called *characteristic points*, that are points $x \in \Sigma$

for which $H_x\mathbb{H}^n \subset T_x\Sigma$. In fact, if Σ is the zero level set of some function $f \in C^1(\Omega,\mathbb{R}) \subset C^1_h(\Omega,\mathbb{R})$ and $x \in \Omega$ with $f(x) = 0$, it is not guaranteed that $\nabla_H f(x) \neq 0$, even if $\nabla f(x) \neq 0$. We will return on characteristic points in Remark 4.2.20.

The homogeneous tangent cone of an $\mathbb{H}$-regular surface of codimension k, with $1 \leq k \leq n$, has been characterized as follows.

Proposition 4.1.13. *[FSSC07, Proposition 3.29] If Σ is an $\mathbb{H}$-regular surface of codimension k, with $1 \leq k \leq n$, and $f \in C^1_h(\Omega,\mathbb{R}^k)$ is a defining function of Σ as in Definition 4.1.8, then for all $x \in \Sigma \cap \Omega$, we have*

$$\mathrm{Tan}(\Sigma, x) = \ker Df(x).$$

We refer to Chapter 5, Section 5.5, and to Chapter 6 for results about the Hausdorff measure of low codimensional $\mathbb{H}$-regular surfaces in $\mathbb{H}^n$. Here we limit ourself to say that a suitable area formula to compute the $(2n+2-k)$-centered Hausdorff measure an $\mathbb{H}$-regular surface of codimension k, $1 \leq k \leq n$, with respect to the distance d_∞ is proved in [FSSC07, Theorem 4.1] (see also [SC16, Theorem 4.50]).

4.1.3 From $\mathbb{H}$-regular surfaces to intrinsic graphs

Low-dimensional $\mathbb{H}$-regular surfaces are locally intrinsic regular graphs.

Theorem 4.1.14. *[AS09, Theorem 4.2] Let $1 \leq k \leq n$. The following conditions are equivalent:*

(i) $\Sigma \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular surface of dimension k.

(ii) For every $x \in \Sigma$ there exists an open set Ω such that $x \in \Omega$ and such that $\Sigma \cap \Omega$ is the intrinsic graph of a uniformly intrinsically differentiable function ϕ acting from a subset of a k-dimensional horizontal subgroup $\mathbb{V}$ to a homogeneous subgroup complementary to $\mathbb{V}$ in $\mathbb{H}^n$.

We will focus in a more detailed way on the result analogous to Theorem 4.1.14 available for low codimensional $\mathbb{H}$-regular surfaces, since they will be the main characters of the original contributions presented in the next chapters. Let us introduce some notions, preliminary to state an implicit function theorem for low-codimensional regular surfaces in the Heisenberg group.

Definition 4.1.15. Given an open subset $\Omega \subset \mathbb{H}^n$ and a function $f : \Omega \to \mathbb{R}^k$ Pansu differentiable at a point $x \in \Omega$, we define the *horizontal Jacobian* of f at x as

$$J_H f(x) := \|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g,$$

where $\|\cdot\|_g$ denotes the Riemannian norm associated with our fixed left invariant metric g on the multivectors of $\mathbb{H}^n$.

Definition 4.1.16. Given an open subset $\Omega \subset \mathbb{H}^n$ and a function $f : \Omega \to \mathbb{R}$ Pansu differentiable at a point $x \in \Omega$, if $\mathbb{V}$ is a homogeneous subgroup, we define $\nabla_{\mathbb{V}} f(x) \in \mathbb{V}$ the unique vector such that

$$Df(x)(z) = \langle \nabla_{\mathbb{V}} f(x), z\rangle \quad \text{for every } z \in \mathbb{V}.$$

Definition 4.1.17. Given an open subset $\Omega \subset \mathbb{H}^n$ and a function $f : \Omega \to \mathbb{R}^k$ Pansu differentiable at a point $x \in \Omega$, if $\mathbb{V} \subset \mathbb{G}$ is a homogeneous k-dimensional subgroup, we call the *horizontal Jacobian of f with respect to* $\mathbb{V}$ at x the number

$$J_{\mathbb{V}} f(x) := \|\nabla_{\mathbb{V}} f_1(x) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(x)\|_g.$$

Remark 4.1.18. Assume that $\mathbb{V} \subset H_1$ is a horizontal k-dimensional homogeneous subgroup and consider an orthonormal basis $(v_1, \ldots, v_{2n}, e_{2n+1})$ of $\mathbb{H}^n$ such that $(v_1, \ldots, v_k)$ is a basis of $\mathbb{V}$. Set $X_j \in \mathrm{Lie}(\mathbb{G})$ the left invariant vector field such that $X_j(0) = v_j$, for $j = 1, \ldots, 2n$. Let $f : \Omega \subset \mathbb{H}^n \to \mathbb{R}$ be a Pansu differentiable map at $x \in \Omega$, then

$$\nabla_{\mathbb{V}} f(x) = \sum_{i=1}^{k} X_i f(x) v_i. \tag{4.2}$$

Moreover, it is immediate to verify that $J_H f(x) = J(Df(x))$ and, taking (4.2) in consideration, that

$$J_{\mathbb{V}} f(x) = \left| \det \begin{pmatrix} X_1 f_1(x) & \ldots & X_k f_1(x) \\ \ldots & \ldots & \ldots \\ X_k f_1(x) & \ldots & X_k f_k(x) \end{pmatrix} \right| = |\det([X_i f_j(x)]_{i,j=1,\ldots,k})| = J(Df(x)|_{\mathbb{V}}).$$

The following implicit function theorem is proved in [FSSC07, Theorem 3.27]. Some previous versions are available: the theorem has first been presented in [FSSC03b, Theorem 2.1] for one-codimensional intrinsic regular hypersurfaces in Carnot groups (see also [FSSC01, Theorem 6.5] where it is limited to the setting of the Heisenberg group). A more general proof is presented in [CM06, Theorems 1.1, 1.2]. A remarkable result of this type in general Carnot groups is [Mag13, Theorem 1.4]. We will report and use it later on. The most general analogous result in the framework of Carnot groups is the recent [JNGV20, Lemma 2.10]. It will be useful later on, in the last chapter of this thesis.

Theorem 4.1.19 (Implicit function theorem)**.** *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $f \in C_h^1(\Omega, \mathbb{R}^k)$ be a function and consider a point $x_0 \in \Omega$ such that $J_H f(x_0) > 0$. Then there exists a horizontal k-dimensional subgroup $\mathbb{V}$ such that $J_{\mathbb{V}} f(x_0) > 0$. We set $\Sigma = \{x \in \Omega : f(x) = f(x_0)\}$ and we fix a homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$. Setting $\pi_{\mathbb{W}}(x_0) = w_0$ and $\pi_{\mathbb{V}}(x_0) = v_0$, there exist an open set $\Omega' \subset \Omega \subset \mathbb{H}^n$, with $x_0 \in \Omega'$, an open set $U \subset \mathbb{W}$ with $w_0 \in U$ and a unique continuous function $\phi : U \to \mathbb{V}$ such that $\phi(w_0) = v_0$ and*

$$\Sigma \cap \Omega' = \{w\phi(w) : w \in U\}.$$

We stress once more that the introduction of the concept of intrinsic graph originated by the proof of this implicit function theorem. In particular, the proof of Theorem 4.1.19 is the result of the combination of [FSSC07, Proposition 3.13] and [FSSC07, Proposition 3.25], that we report below under the names of Proposition 4.1.20 and Proposition 4.1.21, respectively.

Proposition 4.1.20. *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $f \in C_h^1(\Omega, \mathbb{R}^k)$ be a function and consider a point $x_0 \in \Omega$. Assume there exists a horizontal k-dimensional subgroup $\mathbb{V}$ such that $J_{\mathbb{V}} f(x_0) > 0$. We set $\Sigma = \{x \in \Omega : f(x) = f(x_0)\}$ and we fix a homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$. Setting $\pi_{\mathbb{W}}(x_0) = w_0$ and $\pi_{\mathbb{V}}(x_0) = v_0$, there exist an open set $\Omega' \subset \Omega \subset \mathbb{H}^n$, with $x_0 \in \Omega'$, an open set $U \subset \mathbb{W}$ with $w_0 \in U$ and a unique continuous function $\phi : U \to \mathbb{V}$ such that $\phi(w_0) = v_0$ and*

$$\Sigma \cap \Omega' = \{w\phi(w) : w \in U\}.$$

Proposition 4.1.21. *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $f \in C^1_h(\Omega, \mathbb{R}^k)$ be a function and consider a point $x_0 \in \Omega$ such that $J_H f(x_0) > 0$. Then there exist a horizontal k-dimensional subgroup $\mathbb{V}$ and an open neighbourhood of x_0, $\Omega' \subset \mathbb{H}^n$ such that $J_{\mathbb{V}} f(x) > 0$ for every $x \in \Omega'$.*

Let us focus on the regularity of the parametrization ϕ individuated by the Theorem 4.1.19. In particular, being an $\mathbb{H}$-regular surface of codimension k, with $1 \leq k \leq n$ is truly equivalent to be locally the intrinsic graph of a uniformly intrinsically differentiable map defined on a normal homogeneous subgroup of $\mathbb{H}^n$ of topological dimension $2n+1-k$.

Theorem 4.1.22. *[AS09, Theorem 4.2] Let $1 \leq k \leq n$, the following conditions are equivalent:*

(i) *$\Sigma \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular surface of codimension k.*

(ii) *For all $x \in \Sigma$ there exists an open set Ω such that $x \in \Omega$ and $\Sigma \cap \Omega$ is the intrinsic graph of a uniformly intrinsically differentiable function ϕ acting from a subset of a normal subgroup $\mathbb{W}$ of $\mathbb{H}^n$ of topological dimension $2n+1-k$ to a homogeneous subgroup complementary to $\mathbb{W}$ in $\mathbb{H}^n$.*

Then in particular, by Theorem 4.1.22 one can deduce the following.

Theorem 4.1.23. *In the hypotheses of Theorem 4.1.19, ϕ is uniformly intrinsically differentiable on U.*

By Theorems 4.1.14 and 4.1.22, the objects satisfying either Definition 4.1.1 or Definition 4.1.8 can be locally seen as intrinsic graphs of uniformly intrinsically differentiable maps. In this sense the point of view of intrinsic regular graphs can be used to introduce a unitary definition of regular submanifold in the Heisenberg group. This point of view has been explicitly practised by Serapioni in [Ser08, Section 5], where a suitable notion of intrinsic regular submanifold has been proposed, already set in a generic Carnot group, [Ser08, Definition 5.1]. In particular, in a generic Carnot group the form of homogeneous subgroups and of the possible couples of complementary subgroups can be various. Hence, a unique number is not sufficient to carry all the information about the dimension of a regular submanifold (or, equivalently, of a rectifiable set), as it is not enough to carry all the information about the dimension of a homogeneous subgroup. This is the reason for which, in the following definition, Serapioni identifies the dimension of a regular submanifold through a pair of natural numbers. This stresses the discrepancy between the topological and the Hausdorff dimension of a homogeneous subgroup.

Definition 4.1.24. Let $\mathbb{G}$ be a Carnot group. Let n and N be natural numbers such that $n \leq N$.

(i) A set $\Sigma \subset \mathbb{G}$ is an *intrinsic Lipschitz (n, N)-submanifold* if for every $x \in \Sigma$, there are a positive $r > 0$, two complementary subgroups $\mathbb{W}, \mathbb{V}$ of $\mathbb{G}$, where $\mathbb{W}$ is a homogeneous subgroup of topological dimension n and metric dimension N, and an intrinsic Lipschitz function $\phi : U \to \mathbb{V}$, with $U \subset \mathbb{W}$, such that

$$\Sigma \cap \mathbb{B}(x, r) = \mathrm{graph}(\phi).$$

(ii) A set $\Sigma \subset \mathbb{G}$ is an *intrinsic (n, N)-submanifold* if for every $x \in \Sigma$, there are a positive $r > 0$, two complementary subgroups $\mathbb{W}, \mathbb{V}$ of $\mathbb{G}$, where $\mathbb{W}$ is a homogeneous subgroup of topological dimension n and metric dimension N, and a uniformly intrinsically differentiable function $\phi : U \to \mathbb{V}$, with $U \subset \mathbb{W}$, such that

$$\Sigma \cap \mathbb{B}(x, r) = \mathrm{graph}(\phi).$$

By Proposition 3.5.34, any intrinsic (n, N)-submanifold of $\mathbb{G}$ is a intrinsic Lipschitz (n, N)-submanifold.

Remark 4.1.25. According to Definition 4.1.24 and to Theorems 4.1.14 and 4.1.22, if $1 \le k \le n$, an $\mathbb{H}$-regular surface of dimension k is an intrinsic (k, k)-submanifold. An $\mathbb{H}$-regular surface of codimension k, instead, is an intrinsic $(2n + 1 - k, 2n + 2 - k)$-submanifold. In addition, taking into account the form of the homogeneous subgroups in $\mathbb{H}^n$ (Remark 3.1.16), these two families of intrinsic submanifolds exhaust all the possible intrinsic submanifolds of the Heisenberg group $\mathbb{H}^n$ satisfying Definition 4.1.24, i.e. the family of the subsets $\Sigma \subset \mathbb{H}^n$ for which there exist a couple of natural numbers n and N such that Σ is a (n, N)-intrinsic submanifold of $\mathbb{H}^n$. Since, again by Theorems 4.1.14 and 4.1.22, any intrinsic submanifold of $\mathbb{H}^n$ is an $\mathbb{H}$-regular surface, the family of $\mathbb{H}$-regular surfaces coincide with the one of intrinsic submanifolds in $\mathbb{H}^n$, according to Definition 4.1.24.

We conclude this section with the definition of parametrized $\mathbb{H}$-regular surface ([CM20, Definition 2.12]).

Definition 4.1.26 (Parametrized $\mathbb{H}$-regular surface)**.** Let $\Sigma \subset \Omega$ be an $\mathbb{H}$-regular surface and assume that there exist a factorization of $\mathbb{H}^n$ as the product of two complementary subgroups $\mathbb{H}^n = \mathbb{W}\mathbb{V}$, an open set $U \subset \mathbb{W}$ and a continuous mapping $\phi : U \to \mathbb{V}$ such that $\Sigma = \{w\phi(w) : w \in U\}$. We say that Σ is a *parametrized* $\mathbb{H}$*-regular surface with respect to* $(\mathbb{W}, \mathbb{V})$. We call ϕ a *parametrization of* Σ.

Moreover, we can highlight some sufficient conditions that one can require on a defining map f of a low codimensional $\mathbb{H}$-regular surface Σ in order to be sure that Σ is parametrized (refer to [CM20, Proposition 2.5]).

Proposition 4.1.27. *Let $\Omega \subset \mathbb{H}^n$ be an open set. Let $f \in C^1_h(\Omega, \mathbb{R}^k)$, with $1 \le k \le n$, let $x_0 \in \Omega$, set $\Sigma = f^{-1}(f(x_0))$ and suppose that for some k-dimensional horizontal subgroup, $\mathbb{V} \subset \mathbb{H}^n$, $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$. If $\mathbb{W} \subset \mathbb{H}^n$ is any normal subgroup such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, then Σ is a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$. In addition, if $\phi : U \to \mathbb{V}$ is a parametrization of Σ, with $U \subset \mathbb{W}$ open set, ϕ is unique and uniformly intrinsically differentiable.*

Proof. The proof follows by exploiting the local parametrization given by the implicit function Theorem 4.1.19 (more precisely by Proposition 4.1.20). The uniqueness of the parametrization follows again from Theorem 4.1.19. The uniform intrinsic differentiability of the parametrization follows from Theorem 4.1.22 (more precisely, see Theorem 4.1.23). □

4.2 $(\mathbb{G}, \mathbb{M})$-regular sets

Before introducing $(\mathbb{G}, \mathbb{M})$-regular sets, we briefly recall the very first example of a definition of an intrinsic regular surface of a generic Carnot group presented in the literature.

4.2.1 $\mathbb{G}$-regular hypersurfaces

From the historical point of view, $\mathbb{H}$-regular surfaces have been preceded by the notion of $\mathbb{G}$-regular hypersurface, introduced by Franchi, Serapioni and Serra Cassano in [FSSC03b, 1.6] in a generic Carnot group $\mathbb{G}$ (and before, limited to the setting of the Heisenberg group, in [FSSC01, Definition 6.1]). Basically they are $(q-1)$-dimensional intrinsic regular surfaces of $\mathbb{G}$.

Definition 4.2.1. A set $\Sigma \subset \mathbb{G}$ is a $\mathbb{G}$-regular hypersurface if for every $\bar{x} \in \Sigma$, there exists an open set $\Omega \subset \mathbb{G}$ such that $\bar{x} \in \Omega$, and a function $f \in C^1_h(\Omega, \mathbb{R})$ such that

(i) $\Sigma \cap \Omega = f^{-1}(0)$;

(ii) $\nabla_H f(x) \neq 0$ for every $x \in \Omega$.

Any $\mathbb{G}$-regular hypersurface is the boundary of a H-Caccioppoli set ([FSSC03b, Theorem 2.1]). In fact, this definition of hypersurface is the starting point of a series of papers [FSSC01, FSSC02, FSSC03b, FSSC03a] whose main aim is the proof of a fundamental De Giorgi's structure theorem for locally finite H-perimeter sets in a Carnot group of step 2 [FSSC03a, Theorem 3.9]. In particular the three authors show that if $\mathbb{G}$ is a Carnot group of step 2 and $E \subset \mathbb{G}$ is a H-Caccioppoli set, then $\partial^\star_H E$ is $(Q-1)$-dimensional $\mathbb{G}$-rectifiable, according to [FSSC03a, Definition 2.33], namely $\partial^\star_H E$ can be covered, up to a $\mathcal{H}^{Q-1}$-negligible set, by a contable union of compacts subsets of $\mathbb{G}$-regular hypersurfaces. $\mathbb{G}$-regular hypersurfaces share many of the properties of $\mathbb{H}$-regular surfaces of low codimension in $\mathbb{H}^n$, since, in particular, $\mathbb{H}^n$-regular hypersurfaces coincide with $\mathbb{H}$-regular surfaces of codimension 1. An implicit function theorem for $\mathbb{G}$-regular hypersurfaces is available [FSSC03b, Theorem 2.1], the homogeneous tangent cone to a $\mathbb{G}$-regular hypersurface has been characterized [FSSC03b], suitable area formulas for the perimeter measure, the $(Q-1)$-centered Hausdorff measure and $(Q-1)$-spherical Hausdorff measure, with respect to a homogeneous distance, of these sets have been discussed [FSSC01, FSSC03a, FSSC03b, Mag05, FSSC15, Mag17], and a study of the relation between C^1 Euclidean hypersurfaces embedded in $\mathbb{G}$ and $\mathbb{G}$-regular hypersurfaces has been carried out [Bal03, FSSC03b, Mag06b] (see also Remark 4.2.20). Nevertheless, we do not want to discuss these objects on their behalf, but we want to include them in a larger drawing about regular submanifolds in Carnot groups. Precisely, we consider them as a particular case of $(\mathbb{G}, \mathbb{M})$-regular sets, defined in the next section (refer to Remark 4.2.10). The following two remarks aim at this direction.

Remark 4.2.2. Let us consider a $\mathbb{G}$-regular hypersurface $\Sigma \subset \mathbb{G}$. Consider a point $\bar{x} \in \Sigma$ and a defining function f of Σ at $\bar{x}$, as in Definition 4.2.1. By (ii), for every $x \in \Omega$, $\nabla_H f(x) \neq 0$, hence $\operatorname{span}(\nabla_H f(x)) \subset V_1$ is a homogeneous subgroup complementary to $\ker Df(x)$. Therefore, surely $Df(x)$ is a h-epimorphism.

Remark 4.2.3. Notice that the family of $\mathbb{G}$-regular hypersurfaces is non-empty for every Carnot group $\mathbb{G}$. This is due to the fact that there exists always a one-dimensional subalgebra, namely a one-dimensional subgroup, contained in the first layer of $\mathbb{G}$, V_1. It is enough to consider the span of a non-null vector $v \in V_1$.

4.2.2 $(\mathbb{G}, \mathbb{M})$-regular sets

We present the generalization of $\mathbb{H}$-regular surfaces to regular submanifolds of Carnot groups proposed in [Mag13]. We restrict definitions and results of [Mag13] to the case when the group that we will denote by $\mathbb{M}$ is stratified, i.e. is a Carnot group. They were originally stated for the case when $\mathbb{M}$ is just graded (see Remark 4.2.11).

Definition 4.2.4 (h-quotient and h-embeddings)**.** Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups. We say that $\mathbb{M}$ is an *h-quotient* of $\mathbb{G}$ if there exists a normal homogeneous subgroup $\mathbb{W} \subset \mathbb{G}$ such that $\mathbb{G}/\mathbb{W}$ is h-isomorphic to $\mathbb{M}$. We say that $\mathbb{G}$ *h-embeds* into $\mathbb{M}$ if there exists a homogeneous subgroup $\mathbb{V}$ of $\mathbb{M}$ which is h-isomorphic to $\mathbb{G}$.

Remark 4.2.5. By [Mag13, Proposition 11.1], $\mathbb{M}$ is a h-quotient of $\mathbb{G}$ if and only if there exists a surjective h-homomorphism $L : \mathbb{G} \to \mathbb{M}$. It is also easy to check that $\mathbb{G}$ h-embeds into $\mathbb{M}$ if and only if there exists an injective h-homomorphism $T : \mathbb{G} \to \mathbb{M}$.

Definition 4.2.6 ($(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{M}$)**.** Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups such that $\mathbb{G}$ h-embeds into $\mathbb{M}$. We say that a subset $\Sigma \subset \mathbb{M}$ is $(\mathbb{G}, \mathbb{M})$*-regular*, or that Σ is a $(\mathbb{G}, \mathbb{M})$*-regular set of* $\mathbb{M}$ if for every point $\bar{x} \in \Sigma$, there exist two open neighbourhoods $\Omega \subset \mathbb{M}$ of $\bar{x}$ and $V \subset \mathbb{G}$ of $0 \in \mathbb{G}$ and a continuously Pansu differentiable topological embedding $f : V \to \mathbb{M}$, such that $\Sigma \cap \Omega = f(V)$ and $Df(x) : \mathbb{G} \to \mathbb{M}$ is an h-monomorphism for every $x \in V$.

Remark 4.2.7. $\mathbb{H}$-regular surfaces of dimension k, with $1 \leq k \leq n$, coincide with $(\mathbb{R}^k, \mathbb{H}^n)$-regular sets of $\mathbb{H}^n$.

Definition 4.2.8 ($(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$)**.** Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups such that $\mathbb{M}$ is a h-quotient of $\mathbb{G}$. We say that a subset $\Sigma \subset \mathbb{G}$ is $(\mathbb{G}, \mathbb{M})$*-regular*, or that Σ is a $(\mathbb{G}, \mathbb{M})$*-regular set of* $\mathbb{G}$ if for every point $\bar{x} \in \Sigma$, there exist an open neighbourhood $\Omega \subset \mathbb{G}$ of $\bar{x}$ and a continuously Pansu differentiable map $f : \Omega \to \mathbb{M}$ such that $\Sigma \cap \Omega = f^{-1}(0)$ and $Df(x) : \mathbb{G} \to \mathbb{M}$ is a h-epimorphism for every $x \in \Omega$.

Remark 4.2.9. $\mathbb{H}$-regular surfaces of codimension k, with $1 \leq k \leq n$, coincide with $(\mathbb{H}^n, \mathbb{R}^k)$-regular sets of $\mathbb{H}^n$. In fact, the definition of $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ is a generalization of Definition 4.1.8. In the Heisenberg group, if we assume that $\Omega \subset \mathbb{H}^n$ is an open set and we consider a map $f \in C^1_h(\Omega, \mathbb{R}^k)$, with $1 \leq k \leq n$, by Proposition 4.1.21 assuming that $Df(x)$ is surjective at some $x \in \Omega$ is equivalent to assume the existence of k horizontal vectors $v_1, \dots, v_k \in H_1$ such that $[v_i, v_j] = 0$ for all $i, j \in \{1, \dots, k\}$ and such that $\mathbb{V} = \mathrm{span}(v_1, \dots, v_k)$ is a horizontal subgroup complementary to $\ker(Df(x))$. This ensures that for every $x \in \Omega' \cap f^{-1}(0)$, for some open set $\Omega' \subset \Omega$, if we assume $Df(x)$ to be surjective, it is automatically verified that $Df(x)$ is a h-epimorphism; this is not guaranteed in a generic setting, hence for maps acting between two Carnot groups (see Section 4.4).

Remark 4.2.10. By Remark 4.2.3, $\mathbb{R}$ is a h-quotient of any Carnot group $\mathbb{G}$. Moreover, by Remark 4.2.2, $\mathbb{G}$-regular hypersurfaces coincide with $(\mathbb{G}, \mathbb{R})$-regular sets of $\mathbb{G}$.

Remark 4.2.11. As we hinted above, all the definitions of this section were originally stated, in [Mag13], for the case when $\mathbb{G}$ is a Carnot group and $\mathbb{M}$ is just a graded group. In particular, the definition of Pansu differentiability (Definition 3.2.14) can be easily extended to maps acting between two graded groups. Also Theorems 3.2.30 and 4.2.13 have been proved in [Mag13] for continuously Pansu differentiable maps from a Carnot group to a graded group. Nevertheless, in the sequel we will mainly focus on $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ and, by [Mag13, Proposition 8.2], for these sets it is not restrictive to assume that $\mathbb{M}$ is stratified, i.e. that $\mathbb{M}$ is a Carnot group. We stress that for $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{M}$ this is not true.

If a set Σ is a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ or of $\mathbb{M}$, we simply call it a $(\mathbb{G}, \mathbb{M})$*-regular set*.

The definitions and the characterizations about homogeneous tangent cones have been generalized to $(\mathbb{G}, \mathbb{M})$-regular sets.

Definition 4.2.12. Let $A \subset \mathbb{G}$ and consider a point $x \in A$, we call *homogeneous tangent cone* of A at x the set

$$\begin{aligned}\operatorname{Tan}(A,x) = \Big\{\nu \in \mathbb{G} :\ \nu = \lim_{h\to\infty} \delta_{r_h}(x^{-1}x_h), \text{ for some sequences } (r_h)_{h\in\mathbb{N}} > 0,\\ (x_h)_{h\in\mathbb{N}} \subset A,\ \lim_{h\to\infty} x_h = x\Big\}.\end{aligned} \tag{4.3}$$

By [Mag13, Theorem 1.7], if $\Sigma \subset \mathbb{M}$ is a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{M}$, $\bar{x} \in \Sigma$, $f \in C^1_h(V,\mathbb{M})$ is a defining function for Σ at $\bar{x}$ as in Definition 4.2.6 and $f(\bar{w}) = \bar{x}$, then

$$\operatorname{Tan}(\bar{x},\Sigma) = Df(\bar{w})(\mathbb{G}).$$

If $\Sigma \subset \mathbb{G}$ is a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$, $\bar{x} \in \Sigma$ and $f \in C^1_h(\Omega,\mathbb{M})$ is a defining function for Σ at $\bar{x}$ as in Definition 4.2.8, then

$$\operatorname{Tan}(\bar{x},\Sigma) = \ker(Df(\bar{x})).$$

Every $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ is locally an intrinsic Lipschitz graph. This is ensured by the following implicit function theorem.

Theorem 4.2.13. *[Mag13, Theorem 1.4] Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups endowed with homogeneous distances. Let $\Omega \subset \mathbb{G}$ be an open set and let $f \in C^1_h(\Omega,\mathbb{M})$. Consider $x_0 \in \Sigma := \{x \in \Omega : f(x) = 0\}$ and assume that $Df(x_0)$ is a h-epimorphism. Let us set $\mathbb{W} := \ker Df(x_0)$ and let $\mathbb{V}$ be a homogeneous subgroup complementary to $\mathbb{W}$. Then there exist two open sets $\Omega' \subset \mathbb{G}$, $U \subset \mathbb{W}$ and a map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ such that*

$$\operatorname{graph}(\phi) = \Sigma \cap \Omega'.$$

In addition, there exists a constant $k > 0$ such that for every $w, w' \in U$

$$\|\phi(w')^{-1}\phi(w)\| \le k\|\phi(w')^{-1}w'^{-1}w\phi(w')\|, \tag{4.4}$$

where $\|\cdot\|$ denotes the homogeneous norm associated to the homogeneous distance fixed on $\mathbb{G}$.

Remark 4.2.14. Since $\mathbb{W}$ is a normal homogeneous subgroup, by Remark 3.5.8 condition (4.4) can be rephrased saying that ϕ is intrinsic k-Lipschitz.

Theorem 4.2.13 has been recently improved by the following theorem, whose content follows from [JNGV20, Lemma 2.10] (read in light of Proposition 3.5.20).

Theorem 4.2.15. *Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups and let $\Omega \subset \mathbb{G}$ be open. Let $f \in C^1_h(\Omega,\mathbb{M})$ be a function and fix $x_0 \in \Omega$. Assume that $Df(x_0)$ is a h-epimorphism and consider a subgroup $\mathbb{V}$ complementary to $\ker(Df(x_0))$. Fix a homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$. Write $x_0 = w_0v_0$ with respect to the splitting $\mathbb{W}\mathbb{V}$. Then there exist an open set $U \subset \mathbb{W}$, with $w_0 \in U$, and a continuous map $\phi : U \to \mathbb{V}$ such that $f(w\phi(w)) = f(x_0)$ for every $w \in U$.*

Remark 4.2.16. The main improvement brought from our point of view by Theorem 4.2.15 to Theorem 4.2.13, in the notation of Theorem 4.2.15, is the possibility of choosing $\mathbb{W}$ as any arbitrary homogeneous subgroup complementary to $\mathbb{V}$: $\mathbb{W}$ does not need to coincide with $\ker(Df(x_0))$.

Remark 4.2.17. For the sake of completeness, we remark that Theorem 4.2.13 has been extended by Kozhevnikov in [Koz10, Theorem 2] to prove the local existence of a suitable intrinsic graph parametrization of the level sets of a suitably regular mapping acting between two Carnot manifolds endowed with their Carnot-Carathéodory distances. Carnot manifolds are connected smooth Riemannian manifolds that extend the concept of Carnot group. Many notions regarding Carnot groups have been extended to these structures. In fact, from a local viewpoint, the geometry of a Carnot manifold, in the first order approximation with respect to the Carnot-Carathéodory metric, is modelled as the geometry of a nilpotent graded Lie group. Moreover, the tangent space at each point of a Carnot manifolds has the structure of Carnot group.

Remark 4.2.18. Actually, Theorems 4.2.13 and 4.2.15 imply that, according to Definition 4.1.24, any $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ is an intrinsic Lipschitz $(q-p, Q-P)$-submanifold of $\mathbb{G}$, where q and p are the topological dimensions and Q and P the metric dimensions of $\mathbb{G}$ and $\mathbb{M}$, respectively. We will improve this remark in the next section, Section 4.3, showing that any $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ is an intrinsic $(q-p, Q-P)$-submanifold.

For the sake of completeness, we recall that an implicit function theorem analogous to 4.2.13, is available also for $(\mathbb{G},\mathbb{M})$-regular sets of $\mathbb{M}$ but we do not report it here since it will not be needed in the sequel. For more details please refer to [Mag13, Theorem 1.5].

Remark 4.2.19. If $\mathbb{G}$ is a Carnot group and $1 \leq k \leq n$, where n is the maximum of the dimensions of the commutative subalgebras of $\mathbb{G}$, then any $(\mathbb{R}^k,\mathbb{G})$-regular set of $\mathbb{G}$ is a k-dimensional Euclidean C^1-submanifold ([Mag13, Theorem 12.1]).

Remark 4.2.20. Let $\Sigma \subset \mathbb{G}$ be a C^1-regular (Euclidean) hypersurface. As we said above, a point $x \in \Sigma$ is called a characteristic point if $H_x\mathbb{G} \subset T_x\Sigma$. The presence of characteristic points is the reason why regular (Euclidean) hypersurfaces are not always $(\mathbb{G},\mathbb{R})$-regular sets (i.e. $\mathbb{G}$-regular hypersurfaces). Many researches about the negligibility of the characteristic set, i.e. the set of characteristic points, in various contexts have been carried on.
Since the horizontal vector fields on $\mathbb{G}$, generate a non-integrable distribution, by the Frobenius theorem the set of characteristic points is small, namely has empty interior. Many results about its smallness have been obtained assuming different regularity hypotheses on the surfaces and considering different measures to estimate the size of the characteristic set. For instance one can refer to [Der71, Der72, Bal03, FSSC03a, DGN03, Mag06b]. In particular, in [Mag06b], it has been proved that for all C^1-regular embedded hypersurfaces, the set of the characteristic points is negligible with respect to the $(Q-1)$-dimensional (homogeneous) Hausdorff measure.
Successively, developing a theory of *degree* of vectors and multivectors in Carnot groups, the notion of characteristic point has been extended, first, through the notion of *horizontal point*, [Mag06b] to $(\mathbb{G},\mathbb{R}^k)$-regular sets (necessarily for small vales of k), and then, finally, through the notion of *point of not-maximum pointwise degree*, to C^1-regular submanifolds of any dimension embedded in a Carnot group (for more details refer to [MV08, Section 2], [Mag19, Section 2] and to the references therein). This latter definition allowed to prove various area formulas for the spherical Hausdorff measures of those C^1-regular Euclidean embedded submanifolds of a Carnot group that belong to some classes which are individuated by suitable algebraic conditions that ensure the negligibility of the set of points of not-maximum degree with respect to the homogeneous Hausdorff measure. This is exactly the spirit of the approaches adopted for example in [MV08, LDM10, Mag19] (the first two papers concern $C^{1,1}$-regular submanifolds). In fact, all the proofs of the area formulas presented in the cited papers rely on two main steps: a blow-up result for the considered embedded regular submanifold at the points of maximum degree and a negligibility condition about the set of points of not-maximum degree.

Remark 4.2.21. Given two Carnot groups $\mathbb{G}$ and $\mathbb{M}$, if $\Sigma \subset \mathbb{M}$ is a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{M}$, by the definition of $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{M}$, the Hausdorff measure of Σ can be computed exploiting the available area formula for Lipschitz maps acting between two Carnot groups [Mag02a, Theorem 4.3.4] (whose proof relies on the Pansu-Rademacher theorem, Theorem 3.2.16).
If $\Sigma \subset \mathbb{G}$ is a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$, various area formulas for Σ are available for the case when $\mathbb{M} = \mathbb{R}$, i.e. for $(\mathbb{G}, \mathbb{R})$-regular sets [FSSC15, Mag17], where $\mathbb{G}$ is a generic Carnot group (see also [FSSC03b, ASCV06, DD20a, ADDDLD20]). Notice that these specific regular hypersurfaces are boundary of finite H-perimeter sets [FSSC03b], in fact many of the cited papers develop area formulas for the H-perimeter of Σ. In Section 5.5 of Chapter 5 (relying on results of [FSSC07]) and in Chapter 6 (developing new tecniques) we prove various area formulas for regularly parametrized $(\mathbb{H}^n, \mathbb{R}^k)$-regular sets of $\mathbb{H}^n$, for $1 \leq k \leq n$. In respect to the latter setting, the unique previous available result is an area formula for the centered Hausdorff measure of Σ [FSSC07]. Recently, a very general area formula for parametrized $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ has been proved in [JNGV20] (for more information we refer the reader to Section 6.5).

4.3 Uniform intrinsic differentiability of parametrizations

By Remark 4.1.25, the family of the $\mathbb{H}$-regular surfaces of $\mathbb{H}^n$ coincides with the one of the intrinsic submanifolds of $\mathbb{H}^n$ satisfying Definition 4.1.24, i.e. the family of those subsets $\Sigma \subset \mathbb{H}^n$ for which there exists a couple of natural numbers n and N such that Σ is a (n, N)-intrinsic submanifold of $\mathbb{H}^n$. In this section we focus on the analogous relationship between $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ and intrinsic $(q-p, Q-P)$-submanifolds of $\mathbb{G}$, considering two Carnot group $\mathbb{G}$ and $\mathbb{M}$ of topological dimensions p and q and homogeneous dimensions Q and P, respectively.

Let us synthesize the situation emerged by the previous sections. Both $\mathbb{H}$-regular surfaces of low dimension (Definition 4.1.1) and low-codimension (Definition 4.1.8) can be locally seen as intrinsic graphs of uniformly intrinsically differentiable maps and both $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ and $\mathbb{M}$ (Definitions 4.2.6 and 4.2.8) can be locally seen as intrinsic graphs. Nevertheless, we do not know much about the regularity of the intrinsic graph-parametrization of $(\mathbb{G}, \mathbb{M})$-regular sets, and, in particular, of $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$. The strongest information we have is condition (4.4) of Theorem 4.2.13, which ensures the intrinsic Lipschitz continuity of the parametrizing map individuated by the theorem. By this information we know that any $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ is an intrinsic Lipschitz $(q-p, q-P)$-submanifold of $\mathbb{G}$.

From this picture a natural question arises: **is it true that, analogously to what happens in the Heisenberg group, being a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ is *equivalent* to be locally the intrinsic graph of a uniformly intrinsically differentiable map?** We answer to one half of this question proving that any $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ is locally the intrinsic graph of a uniformly intrinsically differentiable map (and consequently it is an intrinsic $(q-p, Q-P)$-submanifold of $\mathbb{G}$). Nevertheless, we do not manage to provide a complete answer to the question, proving the conjectured equivalence. In other words we leave to be understood if any uniformly intrinsically differentiable graph of $\mathbb{G}$ is a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ for some suitable Carnot group $\mathbb{M}$.

Let us start by recalling [DD20a, Theorem 4.1] which is the most general complete available answer to our question in the literature. A previous answer limited to the setting of the Heisenberg group was provided by [AS09, Theorem 4.2] (namely Theorems 4.1.14 and 4.1.22). In particular, the following theorem gives positive answer to the question for $(\mathbb{G}, \mathbb{R}^k)$-regular sets of $\mathbb{G}$, for k not larger than the maximum possible dimension of a horizontal homogeneous subgroup of $\mathbb{G}$.

Theorem 4.3.1. *Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, with $\mathbb{V}$ horizontal of dimension k. Let $v_1, \dots, v_k$ be commuting vectors in $\mathbb{G}$ such that $\mathbb{V} = \mathrm{span}(v_1, \dots, v_k)$ and denote by $X_i \in \mathrm{Lie}(\mathbb{G})$ the left invariant vector field such that $X_i(0) = v_i$, for $i = 1, \dots, k$. Let $U \subset \mathbb{W}$ be an open set, let $\phi : U \to \mathbb{V}$ be a function and consider $\Sigma = \mathrm{graph}(\phi)$. Then, the following conditions are equivalent.*

(i) There are $\Omega \subset \mathbb{G}$ open set, and $f = (f_1, ..., f_k) \in C^1_h(\Omega, \mathbb{R}^k)$ such that $\Sigma = \{x \in \Omega : f(x) = 0\}$ and $\det([X_i f_j(y)]_{i,j=1,\dots,k}) \neq 0$, for all $y \in \Sigma$.

(ii) ϕ is uniformly intrinsically differentiable on U.

Remark 4.3.2. The proof of $(ii) \Rightarrow (i)$ of Theorem 4.3.1 relies on the Whitney-type extension Theorem 3.2.34. Actually, the main relevant obstacle towards a complete answer to our question in a general setting is the lack of a more general Whitney-type extension theorem, i.e. a theorem like Theorem 3.2.34 valid for functions acting from a closed subset of a Carnot group $\mathbb{G}$ to a second Carnot group $\mathbb{M}$. In fact, if we consider the intrinsic graph of a uniformly intrinsically differentiable map ϕ acting between two generic complementary subgroups $\mathbb{W}$, $\mathbb{V}$ of $\mathbb{G}$, $\phi : U \subset \mathbb{W} \to \mathbb{V}$, in order to retrace the argument used to prove Theorem 4.3.1, we would need a suitable Whitney-type theorem to prove that the intrinsic graph of ϕ is contained in the zero level set of a map $f \in C^1_h(\Omega, \mathbb{M})$, with $\Omega \subset \mathbb{G}$ an open set and $\mathbb{M}$ a suitable stratified group, such that $Df(y)$ is a h-epimorphism for every $y \in \Omega \cap \mathrm{graph}(\phi)$.

Let us sketch our contribution, whose proof is the goal of the current section. We consider Σ a $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$. By Theorem 4.2.13, according to the notation therein, Σ is locally parametrized, in a neighbourhood of any point $x_0 \in \Sigma$, by an intrinsic Lipschitz map $\phi : U \to \mathbb{V}$ with $\mathbb{V} \subset \mathbb{G}$ homogeneous subgroup complementary to $\mathbb{W} := \ker(Df(x_0))$ ($\mathbb{V}$ not necessarily commutative) and $U \subset \mathbb{W}$ open set. In Theorem 4.3.7, we prove that ϕ is uniformly intrinsically differentiable on U. Moreover, we prove that this is true also if $\mathbb{W}$ is any homogeneous subgroup of $\mathbb{G}$ complementary to $\mathbb{V}$. The proof consists of a direct verification of the definition of uniform intrinsic differentiability (Theorem 4.3.3) and on a geometrical characterization of uniform intrinsic differentiability (Theorem 4.3.6).

From now on, we denote by $B_{\mathbb{W}}(x, r)$ the open ball relative to $\mathbb{W}$ centered at $x \in \mathbb{W}$ of radius $r > 0$, $B_{\mathbb{W}}(x, r) = B(x, r) \cap \mathbb{W} = \{w \in \mathbb{W} : d(w, x) < r\}$ and we set also $B^*_{\mathbb{W}}(x, r) = B_{\mathbb{W}}(x, r) \setminus \{x\}$. Moreover, when nothing different is specified, by $\|\cdot\|$ we denote the homogeneous norm associated with the homogeneous distance fixed on $\mathbb{G}$.

Theorem 4.3.3. *In the hypotheses of Theorem 4.2.13, ϕ is uniformly intrinsically differentiable at $w_0 = \pi_{\mathbb{W}}(x_0)$ and the intrinsic differential of ϕ at w_0, $d\phi_{w_0} : \mathbb{W} \to \mathbb{V}$, is $d\phi_{w_0} \equiv 0$.*

Proof. We want to prove that there is an intrinsic linear function $L : \mathbb{W} \to \mathbb{V}$ such that

$$\sup_{w' \in B_{\mathbb{W}}(w_0, r)} \sup_{w \in B^*_{\mathbb{W}}(0, r)} \frac{\|L(w)^{-1} \phi_{\Phi(w')^{-1}}(w)\|}{\|w\|} \to 0$$

as $r \to 0$. By a change of variables, taking into account that $\mathbb{W} = \ker(Df(x_0))$ is a normal subgroup and recalling that we have set $d_\phi(w, w') = \|\phi(w')^{-1} w'^{-1} w \phi(w')\|$, we can reformulate our goal as proving that

$$\sup_{w'\in B_{\mathbb{W}}(w_0,r)} \sup_{\{w:0<d_\phi(w,w')<r\}} \frac{\|L(\phi(w')^{-1}w'^{-1}w\phi(w'))^{-1}\phi(w')^{-1}\phi(w)\|}{d_\phi(w,w')} \to 0$$

as $r \to 0$. Observe that this is equivalent to prove

$$\sup_{\substack{w,w'\in B_{\mathbb{W}}(w_0,r)\\ w\neq w'}} \frac{\|L(\phi(w')^{-1}w'^{-1}w\phi(w'))^{-1}\phi(w')^{-1}\phi(w)\|}{d_\phi(w,w')} \to 0 \tag{4.5}$$

as $r \to 0$. Let us consider two points $w, w' \in B_{\mathbb{W}}(w_0, r)$, $w \neq w'$ with r small enough. Surely

$$f(\Phi(w'))^{-1}f(\Phi(w)) = 0$$

and then

$$Df(x_0)(\Phi(w')^{-1}\Phi(w))^{-1}f(\Phi(w'))^{-1}f(\Phi(w)) = Df(x_0)(\Phi(w')^{-1}\Phi(w))^{-1} \tag{4.6}$$

hence

$$\|Df(x_0)(\Phi(w')^{-1}\Phi(w))^{-1}f(\Phi(w'))^{-1}f(\Phi(w))\| = \|Df(x_0)(\Phi(w')^{-1}\Phi(w))\|. \tag{4.7}$$

Now consider that

$$\|Df(x_0)(\Phi(w')^{-1}\Phi(w))\| = \|Df(x_0)(\phi(w')^{-1}\phi(w))\| \geq \eta\|\phi(w')^{-1}\phi(w)\|, \tag{4.8}$$

since $\|Df(x_0)(v)\| \geq \eta\|v\|$ for some $\eta > 0$, for some $\delta > 0$ for any $v \in \mathbb{V} \cap \mathbb{B}(0, \delta)$. In fact if we assume by contradiction for every $n \in \mathbb{N}$ there is $v_n \in \mathbb{V} \cap \mathbb{B}(0, \frac{1}{n})$ such that

$$\frac{1}{\|v_n\|}\|Df(x_0)(v_n)\| \leq \frac{1}{n} \tag{4.9}$$

then

$$\|Df(x_0)(\delta_{1/\|v_n\|}(v_n))\| \leq \frac{1}{n} \tag{4.10}$$

hence by compactness one would find an element $\bar{v} \in \mathbb{V}$ such that $\|\bar{v}\| = 1$ and $\|Df(\bar{v})\| = 0$ so that $\bar{v} \in \ker(Df(x_0))$ and this is not possible since $\mathbb{V}$ and $\mathbb{W}$ are complementary.

On the other side

$$\begin{aligned}
&\sup_{\substack{w,w'\in B_{\mathbb{W}}(w_0,r)\\ w\neq w'}} \frac{\|Df(x_0)(\Phi(w')^{-1}\Phi(w))^{-1}f(\Phi(w'))^{-1}f(\Phi(w))\|}{\|\Phi(w')^{-1}\Phi(w)\|} \\
&\leq \sup_{\substack{w,w'\in B_{\mathbb{W}}(w_0,r)\\ w\neq w'}} \Bigg(\frac{\|Df(x_0)(\Phi(w')^{-1}\Phi(w))^{-1}Df(\Phi(w'))(\Phi(w')^{-1}\Phi(w))\|}{\|\Phi(w')^{-1}\Phi(w)\|} \\
&\qquad + \frac{\|Df(\Phi(w'))(\Phi(w')^{-1}\Phi(w))^{-1}f(\Phi(w'))^{-1}f(\Phi(w))\|}{\|\Phi(w')^{-1}\Phi(w)\|} \Bigg) \to 0
\end{aligned} \tag{4.11}$$

as $r \to 0$ by the continuity of Df and by Theorem 3.2.30 applied with $\Omega_1 = B(x_0, Dr)$ for some constant $D > 0$ and r small enough. By the intrinsic Lipschitz continuity of ϕ, stated in (4.4) of Theorem 4.2.13, for any $w, w' \in U$, if we denote by $c_0 = c_0(\mathbb{W}, \mathbb{V})$ the constant given by Proposition 3.1.17, we have

$$\begin{aligned} d_\phi(w,w') &\leq \frac{1}{c_0} d(\Phi(w),\Phi(w')) \\ &\leq \frac{1}{c_0}(\|\phi(w')^{-1}\phi(w)\| + d_\phi(w,w')) \leq \frac{k+1}{c_0} d_\phi(w,w') \end{aligned} \tag{4.12}$$

then it holds that

$$\begin{aligned} \sup_{\substack{w,w'\in B_{\mathbb{W}}(w_0,r)\\ w\neq w'}} & \frac{\|Df(x_0)(\Phi(w')^{-1}\Phi(w))^{-1}Df(\Phi(w'))(\Phi(w')^{-1}\Phi(w))\|}{d_\phi(w,w')} \\ &+\frac{\|Df(\Phi(w'))(\Phi(w')^{-1}\Phi(w))^{-1}f(\Phi(w'))^{-1}f(\Phi(w))\|}{d_\phi(w,w')} \to 0 \end{aligned} \tag{4.13}$$

as $r \to 0$.

Hence, combining (4.7), (4.8) and (4.13), we have

$$\sup_{\substack{w,w'\in B_{\mathbb{W}}(w_0,r)\\ w\neq w'}} \frac{\|\phi(w')^{-1}\phi(w)\|}{d_\phi(w,w')} \to 0 \tag{4.14}$$

as r goes to zero. Hence, by the comparison between (4.5) and (4.14), ϕ is uniformly intrinsically differentiable at w_0 and $d\phi_{w_0}$ is the constant function $d\phi_{w_0} \equiv 0$. □

Our next step is to prove Theorem 4.3.7. We want to extend the uniform intrinsic differentiability of ϕ at w_0, ensured by Theorem 4.3.3, and to prove the uniform intrinsic differentiability of ϕ on its whole domain. Moreover, we want to prove the uniform intrinsic differentiability of *any* parametrization ϕ of Σ, not necessarily defined on $\ker(Df(x_0))$ but on any possible homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$. In order to do this we prove a geometrical characterization of uniform intrinsic differentiability analogous to the characterization of intrinsic differentiability given by Theorem 3.5.26. Let us start by introducing a suitable definition of uniform tangent coset.

Definition 4.3.4. Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. Let us fix $w_0 \in U$ and consider a point $x_0 = w_0\phi(w_0) \in \operatorname{graph}(\phi)$. Let $\mathbb{T}$ be a homogeneous subgroup of $\mathbb{G}$; the coset $x_0\mathbb{T}$ is the *uniform tangent (affine) subgroup* or *uniform tangent coset* to $\operatorname{graph}(\phi)$ at x_0 if for all $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every $y \in \Phi(B_{\mathbb{W}}(w_0,\delta))$

$$\operatorname{graph}(\phi_{y^{-1}}) \cap \{x \in \mathbb{G} : \|\pi_{\mathbb{W}}(x)\| < \delta\} \subset X(0,\mathbb{T},\varepsilon). \tag{4.15}$$

Remark 4.3.5. If we assume ϕ to be intrinsic Lipschitz, condition (4.15) can be rephrased as follows

$$\operatorname{graph}(\phi_{y^{-1}}) \cap B(0,\delta) \subset X(0,\mathbb{T},\varepsilon). \tag{4.16}$$

since by Proposition 3.5.11 we have

$$c_0\|w\| \leq \|w\phi_{y^{-1}}(w)\| \leq (1+Lip(\phi))\|w\| \tag{4.17}$$

for every $w \in \sigma_{y^{-1}}(U)$ and $y \in \operatorname{graph}(\phi)$.

Theorem 4.3.6. *Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups. Let $U \subset \mathbb{W}$ be an open set, let $\phi : U \to \mathbb{V}$ be a function, let us consider a point $w_0 \in U$ and* *set $x_0 = \Phi(w_0) \in graph(\phi)$. The following conditions are equivalent.*

(i) ϕ is uniformly intrinsically differentiable at $w_0 \in U$.

(ii) There exists a set $\mathbb{T}$ such that

(ii_1) $\mathbb{T}$ is a homogeneous subgroup;

(ii_2) $\mathbb{T}$ and $\mathbb{V}$ are complementary subgroups of $\mathbb{G}$;

(ii_3) $x_0\mathbb{T}$ is the uniform tangent coset to graph(ϕ) at x_0.

In particular if (i) (and then (ii)) holds, the intrinsic differential $d\phi_{w_0} : \mathbb{W} \to \mathbb{V}$ is the unique intrinsic linear function such that graph($d\phi_{w_0}$) $= \mathbb{T}$.

Proof. $(i) \Rightarrow (ii)$. Conditions (ii_1) and (ii_2) follow from Proposition 5.3.21 (i). Then we are left to prove (ii_3). We know that

$$\begin{aligned} &\sup_{y\in\Phi(B_{\mathbb{W}}(w_0,r))}\ \sup_{w\in B^*_{\mathbb{W}}(0,r)} \frac{\|d\phi_{w_0}(w)^{-1}\phi_{y^{-1}}(w)\|}{\|w\|} = \\ &\sup_{w'\in B_{\mathbb{W}}(w_0,r)}\ \sup_{w\in B^*_{\mathbb{W}}(0,r)} \frac{\|d\phi_{w_0}(w)^{-1}\phi_{\Phi(w')^{-1}}(w)\|}{\|w\|} \to 0 \end{aligned} \tag{4.18}$$

as r goes to zero. Hence for every $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that for every $y \in \Phi(B_{\mathbb{W}}(w_0, r))$ and $w \in B^*_{\mathbb{W}}(0, r)$,

$$\frac{\|d\phi_{w_0}(w)^{-1}\phi_{y^{-1}}(w)\|}{\|w\|} < \varepsilon, \tag{4.19}$$

thus, for every $y \in \Phi(B_{\mathbb{W}}(w_0, r))$ and $w \in B^*_{\mathbb{W}}(0, r)$,

$$\begin{aligned} \operatorname{dist}(w\phi_{y^{-1}}(w), \mathbb{T}) &\le \|d\phi_{w_0}(w)^{-1}w^{-1}w\phi_{y^{-1}}(w)\| \\ &= \|d\phi_{w_0}(w)^{-1}\phi_{y^{-1}}(w)\| \\ &\le \varepsilon\|w\| \le \frac{\varepsilon}{c_0}\|w\phi_{y^{-1}}(w)\|, \end{aligned} \tag{4.20}$$

where $c_0 = c_0(\mathbb{W}, \mathbb{V})$ is the constant given by Proposition 3.1.17. Now we chose $\delta = \delta(\varepsilon) = r(\varepsilon c_0)$ and we get that for every $y \in \Phi(B_{\mathbb{W}}(w_0, \delta))$ and w such that $\|w\| < \delta$

$$\operatorname{dist}(w\phi_{y^{-1}}(w), \mathbb{T}) \le \varepsilon\|w\phi_{y^{-1}}(w)\|. \tag{4.21}$$

If fact, when $\|w\| = 0$, $w = 0$ then (4.21) is automatically verified for every $y \in \Phi(B_{\mathbb{W}}(w_0, r)$. Hence for every $y \in \Phi(B_{\mathbb{W}}(w_0, \delta))$

$$\operatorname{graph}(\phi_{y^{-1}}) \cap \{x \in \mathbb{G} : \|\pi_{\mathbb{W}}(x)\| < \delta\} \subset X(0, \mathbb{T}, \varepsilon). \tag{4.22}$$

$(ii) \Rightarrow (i)$. By Proposition 5.3.21 (ii), it is well defined a unique intrinsic linear function $L : \mathbb{W} \to \mathbb{V}$ such that graph$(L) = \mathbb{T}$. We want to prove that

$$\begin{aligned} &\sup_{w'\in B_{\mathbb{W}}(w_0,r)}\ \sup_{w\in B^*_{\mathbb{W}}(0,r)} \frac{\|L(w)^{-1}\phi_{\Phi(w')^{-1}}(w)\|}{\|w\|} \\ &= \sup_{y\in\Phi(B_{\mathbb{W}}(w_0,r))}\ \sup_{w\in B^*_{\mathbb{W}}(0,r)} \frac{\|L(w)^{-1}\phi_{y^{-1}}(w)\|}{\|w\|} \to 0 \end{aligned} \tag{4.23}$$

as r goes to zero. By our hypothesis, we know that for every $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that for every $y \in \Phi(B_{\mathbb{W}}(w_0, r))$ and for every w such that $\|w\| < r$,

$$\operatorname{dist}(w\phi_{y^{-1}}(w), \mathbb{T}) \leq \varepsilon \|w\phi_{y^{-1}}(w)\|. \tag{4.24}$$

Now we observe that $w\phi_{y^{-1}}(w) = wL(w)L(w)^{-1}\phi_{y^{-1}}(w)$ hence $wL(w)$ and $L(w)^{-1}\phi_{y^{-1}}(w)$ are, respectively, the components along $\mathbb{T}$ and $\mathbb{V}$ of $w\phi_{y^{-1}}(w)$, in the decomposition $\mathbb{G} = \mathbb{T}\mathbb{V}$, and so by setting $\tilde{c_0} = c_0(\mathbb{T}, \mathbb{V})$ the constant given by Proposition 3.1.17 and by taking into account Remark 3.1.18,

$$\tilde{c_0}\|L(w)^{-1}\phi_{y^{-1}}(w)\| \leq \operatorname{dist}(w\phi_{y^{-1}}(w), \mathbb{T}) \leq \varepsilon \|w\phi_{y^{-1}}(w)\|. \tag{4.25}$$

Now notice that for every $\varepsilon \leq \frac{1}{2}$, for every $y \in \Phi(B(w_0, r))$ and for every w such that $\|w\| < r$

$$\|\phi_{y^{-1}}(w)\| \leq c\|w\|$$

for some positive constant c. In fact, let us observe that, by Proposition 3.5.21,

$$\begin{aligned} \|\phi_{y^{-1}}(w)\| &= \|L(w)L(w)^{-1}\phi_{y^{-1}}(w)\| \leq \|L(w)\| + \|L(w)^{-1}\phi_{y^{-1}}(w)\| \\ &\leq \operatorname{Lip}(L)\|w\| + \varepsilon\|w\phi_{y^{-1}}(w)\| \leq \|w\|(\operatorname{Lip}(L) + \varepsilon) + \varepsilon\|\phi_{y^{-1}}(w)\|, \end{aligned} \tag{4.26}$$

where $\operatorname{Lip}(L)$ is the intrinsic Lipschitz constant of L. Hence our claim is achieved by posing $c := 2(2\operatorname{Lip}(L) + 1) \geq \frac{\operatorname{Lip}(L)+\varepsilon}{1-\varepsilon}$.

In conclusion, for any $\varepsilon \leq \frac{1}{2}$, for every $y \in \Phi(B(w_0, r))$, for every w such that $\|w\| < r$,

$$\|L(w)^{-1}\phi_{y^{-1}}(w)\| \leq \varepsilon\|w\phi_{y^{-1}}(w)\| \leq \frac{\varepsilon c}{\tilde{c_0}}\|w\|. \tag{4.27}$$

Hence, by posing $\delta = r(\frac{\varepsilon\tilde{c_0}}{c})$, (i) is proved. □

We are now ready to prove the uniform intrinsic differentiability of the parametrization ϕ. In particular, now we assume that ϕ is defined on $\mathbb{W}$, that is required to be any homogeneous subgroup complementary to $\mathbb{V}$, which in turn is assumed to be a homogeneous subgroup complementary to $\ker(Df(x))$, for every $x \in \Sigma$. The result substantially follows from the combination of Theorem 4.3.3 and Theorem 4.3.6.

Theorem 4.3.7. *Let $\mathbb{G}$, $\mathbb{M}$ be two Carnot groups endowed with homogeneous distances and let $\Omega \subset \mathbb{G}$ be an open set. Consider a function $f \in C^1_h(\Omega, \mathbb{M})$ and set $\Sigma = \{x : f(x) = 0\}$. Let us consider a homogeneous subgroup $\mathbb{V} \subset \mathbb{G}$ such that $Df(x)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is an h-isomorphism for every $x \in \Sigma$ and consider a homogeneous subgroup $\mathbb{W} \subset \mathbb{G}$ complementary to $\mathbb{V}$. Assume that $U \subset \mathbb{W}$ is an open set and let $\phi : U \to \mathbb{V}$ be a function such that for some open set $\Omega' \subset \Omega \subset \mathbb{G}$, $\Sigma \cap \Omega' = \Phi(U)$. Then ϕ is uniformly intrinsically differentiable at any point of U.*

Proof. Let us consider a point $y \in U$. Surely $Df(\Phi(y))$ is surjective by our hypothesis and it is a h-epimorphism, since it is surjective and $\mathbb{L} := \ker(Df(\Phi(y)))$ and $\mathbb{V}$ are complementary subgroups. By Theorem 4.2.13, there exist an open set $U'' \subset \mathbb{L}$, a function $\psi : U'' \to \mathbb{V}$ and an open set $\Omega'' \subset \mathbb{G}$ with $\Phi(y) \in \Omega''$ such that $\operatorname{graph}(\psi) = \Omega'' \cap \Sigma$. Clearly $\Omega' \cap \Omega'' \cap \operatorname{graph}(\psi) = \Omega' \cap \Omega'' \cap \operatorname{graph}(\phi)$. We denote by Ψ the graph map of ψ. By Theorem 4.3.3, ψ is uniformly intrinsically differentiable at $w_y := \pi_{\mathbb{L}}(\Phi(y))$ and $d\psi_{w_y} \equiv 0$.

Notice that $\operatorname{graph}(d\psi_{w_y}) = \mathbb{L}$, then by Theorem 4.3.6, $\Psi(w_y)\mathbb{L}$ is the uniform tangent coset of $\operatorname{graph}(\phi)$ at $\Psi(w_y) = \Phi(y)$. By (4.4) of Theorem 4.2.13, we know that ψ is intrinsic Lipschitz and Remark 3.5.15 ensures then that also ϕ is intrinsic Lipschitz (even if the

Lipschitz constant could change). Therefore, combining Remark 4.3.5 with Definition 4.3.4 (or better with Definition 4.3.4 read in the equivalent form given by Remark 4.3.5) applied to ϕ we obtain that $\Psi(w_y)\mathbb{L} = \Phi(y)\mathbb{L}$ is the uniform tangent coset of $\mathrm{graph}(\phi)$ at $\Phi(y)$ and so by applying once more Theorem 4.3.6, ϕ is uniformly intrinsically differentiable at y and $d\phi_y$ is the unique intrinsic linear function from $\mathbb{W}$ to $\mathbb{V}$ whose intrinsic graph coincides with $\mathbb{L} = \ker(Df(\Phi(y))$. Since this argument is independent of the choice of the point $y \in U$, ϕ is uniformly intrinsically differentiable on U. □

Corollary 4.3.8. *Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups and let $\Omega \subset \mathbb{G}$ be an open set. Let $f \in C^1_h(\Omega, \mathbb{M})$ be a function and set $\Sigma = f^{-1}(0)$. Assume that there exists a homogeneous subgroup $\mathbb{V}$ of $\mathbb{G}$ such that $Df(x)|_{\mathbb{V}}$ is a h-isomorphism for every $x \in \Sigma$. Then, for any homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$, Σ is parametrized by a uniformly intrinsically differentiable map $\phi : U \to \mathbb{V}$ with $U \subset \mathbb{W}$ open set, i.e. $\Sigma = \mathrm{graph}(\phi)$.*

Proof. The existence of the map ϕ follows by applying, multiple time if necessary, Theorem 4.2.15. Its uniform intrinsic differentiability follows from Theorem 4.3.7. □

4.4 $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds

In [Koz15] (and partially before, in [BK14]) a more general definition of regular submanifold of a Carnot group has been proposed. We report and briefly discuss it here, since this point of view could be a valuable starting point for future investigations.

Definition 4.4.1. Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups. A set $\Sigma \subset \mathbb{G}$ is a $(\mathbb{G}, \mathbb{M})_K$*-regular submanifold* if for every point $\bar{x} \in \Sigma$ there exist an open neighbourhood of $\bar{x}$, $\Omega \subset \mathbb{G}$, and a function $f \in C^1_h(\Omega, \mathbb{M})$ such that $\Sigma \cap \Omega \subset f^{-1}(0)$ and $Df(x)$ is surjective for every $x \in \Omega$.

Remark 4.4.2. Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups. By a direct comparison between the relative definitions, any $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ is a $(\mathbb{G}, \mathbb{M})_K$-regular submanifold. The opposite is not true. As an example, we can consider a map

$$f : \mathbb{H}^1 \to \mathbb{R}^2, \; f(x_1, x_2, x_3) = (ax_1 + bx_1, cx_1 + dx_2)$$

with $a, b, c, d \in \mathbb{R}$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. Let us consider $\Sigma = f^{-1}(0) = \mathrm{span}(e_3)$. For every $x \in \Sigma$, $J_H f(x) = |\det J_{\mathbb{H}} f(x)| = \left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \neq 0$, hence $Df(x)$ is surjective. Therefore, Σ is an $(\mathbb{H}^1, \mathbb{R}^2)_K$-regular submanifold. At the same time, for every $x \in \Sigma$, $\ker(Df(x)) = \mathrm{span}(e_3)$ and we have already discussed (see Remark 3.4.2) that this subgroup, that is the center of the Heisenberg group, cannot be complemented by any homogeneous subgroup. Thus, Σ is not a $(\mathbb{H}^1, \mathbb{R}^2)$-regular set.

By [Koz15, Theorem 3.1.1], $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds are ε*-Reifenberg flat* with respect to the kernel of the Pansu differential of the defining map f, i.e. in the notation of Definition 4.4.1 there is an increasing function $\varepsilon : (0, \infty) \to (0, \infty)$, $\varepsilon(t) \to 0$ for $t \to 0^+$ such that for every $x \in \Omega \cap \Sigma$ and $r > 0$,

$$\frac{\mathrm{dist}_{d_1}(B(x, r) \cap \Sigma, B(x, r) \cap x \ker(Df(x)))}{r} \leq \varepsilon(r), \tag{4.28}$$

where for every subsets $E_1, E_2 \subset \mathbb{G}$, the Hausdorff distance between E_1 and E_2 is defined as

$$\operatorname{dist}_{d_1}(E_1, E_2) := \max \left\{ \sup_{a \in E_2} d_1(a, E_1), \sup_{a \in E_1} d_1(a, E_2) \right\},$$

where $d_1(a, E_i) = \inf\{d(a, b) : b \in E_i)$.

Remark 4.4.3. The ε-Reifenberg flatness (4.28) allows to prove that the Hausdorff dimension of the level set Σ in Definition 4.4.1 is $Q - P$, where Q and P are the Hausdorff dimensions of $\mathbb{G}$ and $\mathbb{M}$, respectively ([Koz15, Theorem 3.5.1]). Nevertheless, it is not a sufficient condition to deduce any regularity of the measures $\mathcal{H}^{Q-P}$ on the level set Σ. The author, in fact, presents in [Koz15, Section 6.2] some irregular examples, showing that it is possible to build a level set of a continuously Pansu differentiable non-degenerate (i.e. with surjective Pansu differential) function with zero or infinite Hausdorff measure. If we consider a $(\mathbb{G}, \mathbb{M})_K$-regular submanifold Σ, it is still an open conjecture to understand if Σ is locally homeomorphic to $\ker(Df(x))$ (see [Koz15, Conjecture 3.4.1]). In [Koz15, Corollary 6.2.1] the author gives positive answer for $\mathbb{G} = \mathbb{H}^n$, $\mathbb{M} = \mathbb{R}^k$ for $1 \leq k \leq 2n$.

Now we restrict our attention on a particular class of $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds by assuming that $\mathbb{M}$ coincides with the commutative Euclidean group $\mathbb{R}^N$, for some natural number N. The author calls the family of the $(\mathbb{G}, \mathbb{R}^N)_K$-regular submanifolds, for any $N \in \mathbb{N}$, *co-Abelian submanifolds* (or, more precisely, co-Abelian intrinsic submanifolds). They have been investigated by Bigolin and Kozhevnikov in [BK14], and successively by the latter author in [Koz15]. We first report below Theorem 3.1.12 in [Koz15], which clarifies why, at this state of the art, it is interesting to focus on the particular case when $\mathbb{M}$ is commutative, i.e. on co-Abelian submanifolds.

Theorem 4.4.4. *Let $\Sigma \subset \mathbb{G}$ be a connected locally closed set. Assume that to each point $y \in \Sigma$ corresponds a closed homogeneous set $\mathbb{W}_y$, and assume that $\mathbb{W}_x$ is a vertical subgroup of codimension N for some $x \in \Sigma$. Assume that for every relatively compact subset $\Sigma' \subset \Sigma$ there is an increasing function $\varepsilon(t) \to 0$ for $t \to 0^+$ such that for every $y \in \Sigma'$ and $r > 0$,*

$$\frac{\operatorname{dist}_{d_1}(B(y, r) \cap \Sigma, B(y, r) \cap y\mathbb{W}_y)}{r} \leq \varepsilon(r), \tag{4.29}$$

then there exist an open neighbourhood Ω of Σ and a map $f \in C^1_h(\Omega, \mathbb{R}^N)$, such that

$$\Sigma = f^{-1}(0) \text{ and } \ker(Df(y)) = \mathbb{W}_y \text{ for every } y \in \Sigma.$$

By combining (4.28) and Theorem 4.4.4, it is immediate to observe that co-Abelian submanifolds are completely characterized in terms of flatness. On the other side, when the target space $\mathbb{M}$ is a generic Carnot group not necessarily commutative, a suitable Whitney-type extension theorem is not available. This lack is the main obstacle towards the proof of a generalization of Theorem 4.4.4 to the case when $\mathbb{M}$ is not commutative. A compact set $\Sigma \subset \mathbb{G}$ satisfying (4.29) may be not a level set of a function $f \in C^1_h(\Omega, \mathbb{M})$ with surjective differential on Σ, i.e. may be not a $(\mathbb{G}, \mathbb{M})_K$-regular submanifold.

Many other properties of $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds, and, in particular, of co-Abelian submanifolds, have been investigated in [BK14] and [Koz15, Section 3]. For instance an intensive study about alternative notions of homogeneous tangent cones (Definition 4.2.12) has been carried out, namely *positive* and *negative tangent* and *paratangent* cones have been considered. Substantial differences among these various notions regard the possibility of exploring the use of the quantifiers relative to the sequences involved in Definition 4.2.12.

Definition 4.4.5. [BK14, Definition 2.4] Let $\Sigma \subset \mathbb{G}$ and $x \in \Sigma$.

- $v \in \mathbb{G}$ belongs to $\mathrm{Tan}^+_{\mathbb{G}}(\Sigma, x)$ (v is an *upper tangent vector* to Σ in x) if and only if there exist two sequences $(r_m)_m \subset \mathbb{R}^+$, with $\lim_{m\to\infty} r_m = 0$, and $(x_m)_m \subset \Sigma$, with $\lim_{m\to\infty} x_m = x$, such that
$$\lim_{m\to\infty} \delta_{1/r_m}(x^{-1}x_m) = v.$$
- $v \in \mathbb{G}$ belongs to $\mathrm{Tan}^-_{\mathbb{G}}(\Sigma, x)$ (v is a *lower tangent vector* to Σ in x) if and only if for every sequence $(r_m)_m \subset \mathbb{R}^+$ with $\lim_{m\to\infty} r_m = 0$ there exists a sequence $(x_m)_m \subset \Sigma$ with $\lim_{m\to\infty} x_m = x$ such that
$$\lim_{m\to\infty} \delta_{1/r_m}(x^{-1}x_m) = v.$$
- $v \in \mathbb{G}$ belongs to $\mathrm{pTan}^+_{\mathbb{G}}(\Sigma, x)$ (an *upper paratangent vector* to Σ in x) if and only if there exist three sequences $(r_m)_m \subset \mathbb{R}^+$, with $\lim_{m\to\infty} r_m = 0$, and $(x_m)_m$, $(y_m)_m \subset \Sigma$, with $\lim_{m\to\infty} x_m = x$, such that
$$\lim_{m\to\infty} \delta_{1/r_m}(y_m^{-1}x_m) = v.$$
- $v \in \mathbb{G}$ belongs to $\mathrm{pTan}^-_{\mathbb{G}}(\Sigma, x)$ (a *lower paratangent vector* to Σ in x) if and only if for every sequence $(r_m)_m \subset \mathbb{R}^+$, with $\lim_{m\to\infty} r_m = 0$ and every sequence $(x_m)_m \subset \Sigma$, with $\lim_{m\to\infty} x_m = x$, there exists a sequence $(y_m)_m \subset \Sigma$ such that
$$\lim_{m\to\infty} \delta_{1/r_m}(y_m^{-1}x_m) = v.$$

Remark 4.4.6. Comparing Definitions 4.2.12 and 4.4.5, it is immediate to observe that $\mathrm{Tan}^+_{\mathbb{G}}(\Sigma, x) = \mathrm{Tan}(\Sigma, x)$.

This multivariate definitions of tangent cones are aimed to distinguish the cases when at a point of a $(\mathbb{G}, \mathbb{M})_K$-regular surface the blow-up limit of the surface exists or does not exist, in a uniform or non-uniform way, and, at the same time, if it is or it is not unique.

Remark 4.4.7. The following inclusions follow from definitions
$$\mathrm{pTan}^-_{\mathbb{G}}(\Sigma, x) \subset \mathrm{Tan}^-_{\mathbb{G}}(\Sigma, x) \subset \mathrm{Tan}^+_{\mathbb{G}}(\Sigma, x) \subset \mathrm{pTan}^+_{\mathbb{G}}(\Sigma, x).$$

These four definitions of cones are then used to provide various interesting geometric characterizations of $(\mathbb{G}, \mathbb{R}^N)_K$-regular submanifolds, $N \in \mathbb{N}$, the most relevant of which is [BK14, Theorem 1.2], also stated as [Koz15, Theorem 3.3.5]. It is referred to as *four cones theorem*. Roughly speaking, this result characterizes co-Abelian submanifolds in terms of the existence of a unique and “uniform” blow-up at any point of a co-Abelian surface; moreover, at least at one point of the submanifold the blow-up limit is a vertical homogeneous subgroup.

Theorem 4.4.8. *[BK14, Theorem 1.2] Let $\Sigma \subset \mathbb{G}$ be a closed connected set. The following conditions are equivalent:*

(i) Σ is a co-Abelian submanifold of codimension N, namely is a $(\mathbb{G}, \mathbb{R}^N)_K$-submanifold;

(ii) Tangent cones coincide at every point $x \in \Sigma$:
$$pTan^+_{\mathbb{G}}(\Sigma, x) = pTan^-_{\mathbb{G}}(\Sigma, x). \tag{4.30}$$

and there is some point x_0 such that $pTan^+_{\mathbb{G}}(\Sigma, x_0)$ is a vertical subgroup homogeneous of codimension N.

Remark 4.4.9. A partial analogous result to Theorem 4.4.8 is also available for $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds [Koz15, Theorem 3.3.3]. In particular, if Σ is a $(\mathbb{G}, \mathbb{M})_K$-regular submanifold, where $\mathbb{M}$ is a generic Carnot group, and $x \in \Sigma$ condition (4.30) is valid. Nevertheless, in perfect analogy with the previously discussed results concerning ε-Reifenberg flatness, a complete characterization of geneneral $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds in terms of the coincidence of the four cones at every point of the surface is not available yet.

It is interesting to highlight that the techniques and the tools adopted to work on $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds are different from the ones "classically" used. This is mainly due to the lack of an implicit function theorem for $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds. In fact, the more general available implicit function theorem providing the existence of an intrinsic graph parametrization for a submanifold is [JNGV20, Lemma 2.10], which, in the notation of Definition 4.4.1, still requires that $\ker(Df(x))$ can be complemented with a homogeneous subgroup (or, in equivalent words, that $Df(x)$ is a h-epimorphism). If we consider a continuously Pansu differentiable function $f \in C^1_h(\mathbb{G}, \mathbb{M})$ with everywhere surjective differential, we are sure that, for every $x \in \mathbb{G}$, $\ker(Df(x))$ is a normal homogeneous subgroup of $\mathbb{G}$, but the existence of a complementary subgroup is not guaranteed. One could try to bypass this lack considering a homogeneous subspace complementary to $\ker(Df(x))$ (see for instance Lemma 3.1.20), as Kozhevnikov did in order to prove ε-Reifenberg flatness (4.28), but the existence of an intrinsic graph parametrization is not ensured for such a splitting.

4.5 Intrinsic rectifiability in the Heisenberg group

The classical notion of rectifiable set in a metric space goes back to Federer.

Definition 4.5.1. [Fed69, 3.2.14] If (X, d) is a metric space, a set $E \subset (X, d)$ is said *(countably) $\mathcal{H}^k_d$-rectifiable* or, simply, *k-rectifiable* if there is a sequence of Lipschitz functions $(f_i)_{i\in\mathbb{N}}$, with $f_i : A_i \subset \mathbb{R}^k \to (X, d)$ such that

$$\mathcal{H}^k_d \left(E \setminus \bigcup_{i\in\mathbb{N}} f_i(A_i) \right) = 0,$$

where $\mathcal{H}^k_d$ is the Hausdorff measure on (X, d) with respect to the distance d.

The notion of $\mathbb{H}$-regular surface allows to introduce, mimicking Definition 4.5.1, an intrinsic notion of rectifiable set in the Heisenberg group. The first definition of intrinsic rectifiable set has been proposed in the literature for one-codimensional sets, in relation with the notion of $\mathbb{G}$-regular hypersurface, in connection with the study of the H-reduced boundary of H-Caccioppoli sets in the Heisenberg group [FSSC01, Definition 6.4] and successively in a generic Carnot group [FSSC03a, Definition 2.33]. After the development of the definitions of $\mathbb{H}$-regular surface of any dimension, it has been natural to extend the first definition to a notion of rectifiable set of arbitrary dimension.

Definition 4.5.2. [MSSC10, Definition 3.11] We say that a set $E \subset \mathbb{H}^n$ is *k-dimensional $\mathbb{H}$-rectifiable*, or *$(k, \mathbb{H})$-rectifiable*, if there exists a sequence of k-dimensional $\mathbb{H}$-regular surfaces $(\Sigma_i)_{i\in\mathbb{N}}$ such that

$$\mathcal{S}^{k_m} \left(E \setminus \bigcup_{i\in\mathbb{N}} \Sigma_i \right) = 0,$$

where $k_m = k$ if $1 \le k \le n$ and $k_m = k+1$ if $n+1 \le k \le 2n$.

Remark 4.5.3. [MSSC10, Remark 3.12] From the relations between $\mathbb{H}$-regular submanifolds and Euclidean regular submanifolds , described in the previous sections, it is quite natural to observe that, if $1 \le k \le n$, $(k, \mathbb{H})$-rectifiable sets are Euclidean k-rectifiable and, if $n+1 \le k \le 2n$, Euclidean k-rectifiable sets are $(k, \mathbb{H})$-rectifiable. The converse of both these sentences is false.

Through the notions of $(\mathbb{G}, \mathbb{M})$-regular set, mimicking Definitions 4.5.2, one can analogously introduce notions of intrinsic rectifiable set in a general Carnot group. In particular, one can introduce $(\mathbb{G}, \mathbb{M})$-*rectifiable sets* of $\mathbb{G}$ as those sets that, up to a negligible set, can be covered by a countable union of $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$. In this case the negligibility condition would be considered with respect to the measure $\mathcal{S}^{Q-P}$ (or equivalently $\mathcal{H}^{Q-P}$), where $Q = \dim_H(\mathbb{G})$ and $P = \dim_H(\mathbb{M})$. An analogous definition can be stated for $(\mathbb{G}, \mathbb{M})$-rectifiable sets of $\mathbb{M}$. For precise references, in the literature one can refer to the definition of $(\mathbb{G}, \mathbb{R}^k)$-rectifiable set (of $\mathbb{G}$) [Mag06b, Definition 3.2], to other references in [Mag06b] and to the definition of countably $(\mathbb{G}, \mathbb{M})$-rectifiable set (of $\mathbb{G}$) in [JNGV20, Definitions 2.18]. Observe that, by the same arguments valid in the Heisenberg group, discussed in Remark 4.5.3, for small values of k (in particular when k is smaller that the maximum of the dimensions of the commutative subalgebras contained in the first layer of $\mathbb{G}$), $(\mathbb{R}^k, \mathbb{G})$-rectifiable sets of $\mathbb{G}$ are Euclidean k-rectifiable [Mag13, Theorem 12.1] and Euclidean k-rectifiable subsets of $\mathbb{G}$ are $(\mathbb{G}, \mathbb{R}^k)$-rectifiable sets [Mag06b, Theorem 3.8]. Moreover, a detailed study of k-rectifiable sets of $\mathbb{G}$ according to Federer's definition, for small k (and $(\mathbb{R}^k, \mathbb{G})$-rectifiable sets of $\mathbb{G}$ are in this category), has been recently carried out in [IMM20] (see also Remark 4.5.6).

By Theorems 4.1.14 and 4.1.22, it is immediate to observe that, in the notation of Definition 4.5.2, the notion of $(k, \mathbb{H})$-rectifiable set can be restated requiring that a set that can be covered, up to a set of $\mathcal{S}^{k_n}$- measure zero, by the union of a countable family of k-dimensional intrinsic graphs of uniformly intrinsically differentiable maps. On the other hand, as we said above, Franchi, Serapioni and Serra Cassano developed also the notion of intrinsic Lipschitz graph (Definition 3.5.6), which permits to mimic once more Definition 4.5.1.

Definition 4.5.4. [SC16, Definition 4.106] Consider a subset $E \subset \mathbb{H}^n$. We say that E is k-dimensional $\mathbb{H}_L$-*rectifiable*, or $(k, \mathbb{H}_L)$-*rectifiable*

- for $1 \le k \le n$, if there exists a sequence of Lipschitz maps $(f_i)_{i \in \mathbb{N}}$, $f_i : \Omega_i \to \mathbb{H}^n$, with $\Omega_i \subset \mathbb{R}^k$ open set, such that
$$\mathcal{S}^k \left(E \setminus \bigcup_{i \in \mathbb{N}} f_i(\Omega_i) \right) = 0.$$
- for $n+1 \le k \le 2n$, if there exists a sequence of k-dimensional intrinsic Lipschitz graphs $(\Sigma_i)_{i \in \mathbb{N}}$, $\Sigma_i = \mathrm{graph}(\phi_i)$ with $\phi_i : U_i \to \mathbb{V}_i$, with $U_i \subset \mathbb{W}_i$ such that
$$\mathcal{S}^{k+1} \left(E \setminus \bigcup_{i \in \mathbb{N}} \Sigma_i \right) = 0,$$
where, for every $i \in \mathbb{N}$, $\mathbb{W}_i$ and $\mathbb{V}_i$ are complementary subgroups of $\mathbb{H}^n$ of dimension $2n+1-k$ and k, respectively.

Remark 4.5.5. For $1 \leq k \leq n$, $(k, \mathbb{H})_L$-rectifiable sets of $\mathbb{H}^n$ are k-rectifiable in the sense of Federer.

Remark 4.5.6. Intrinsic rectifiable sets have been characterized in several geometrical ways, according to infinitesimal local arguments. In [MSSC10], when $1 \leq k \leq n$, $(k, \mathbb{H}_L)$-rectifiable sets in $\mathbb{H}^n$ have been characterized by Mattila, Serapioni and Serra Cassano, by the almost everywhere existence of approximate tangent homogeneous subgroups, or by the almost everywhere existence of suitable tangent measures of the spherical measure concentrated on the considered set; when $n+1 \leq k \leq 2n$ an analogous characterization is proved for $(k, \mathbb{H})$-rectifiable sets assuming on the set the additional hypothesis of the positive lower density of the spherical Hausdorff measure concentrated on the set. The first of these two characterizations has been extended by Idu, Magnani and Maiale in [IMM20] to the family of k-rectifiable sets of $\mathbb{G}$ according to Federer's definition, i.e. sets that can be covered, up to a $\mathcal{H}^k$-negligible sets, by Lipschitz images of $\mathbb{R}^k$ on $\mathbb{G}$, where $\mathbb{G}$ is a generic Carnot group (with k smaller or equal than the maximum possible dimension of a horizontal homogeneous subgroup in $\mathbb{G}$). Namely, the authors prove that these sets can be characterized by the fact that, at almost every of their points, the tangent measures, of the Hausdorff measure concentrated on the set, are multiples of the Haar measure supported on horizontal subgroups or, equivalently, by the fact that there exists an approximate tangent horizontal subgroup almost everywhere.
In [Mer20], Merlo showed that, in a generic Carnot group $\mathbb{G}$, if μ is a Radon measure on $\mathbb{G}$ absolutely continuous with respect to $\mathcal{H}^{Q-1}$, the fact that $\mathbb{G}$ can be μ-almost all covered by countably many $(\mathbb{G}, \mathbb{R})$-regular sets can be characterized in terms of suitable controls almost everywhere on the lower and upper densities of μ and of the almost everywhere flatness of its tangent measures, namely in terms of the *rectifiability* of the measure μ, according to the author's definition, which extends the classical Euclidean definition of rectifiable Radon measure. As an application, the author reaches the first extension of Preiss' rectifiability theorem in the Heisenberg group $\mathbb{H}^n$. In the same line of research, one can refer also to the recent paper by Antonelli and Merlo [AM20b], where the authors deeply investigated the measures satisfying this new weaker general notion of rectifiability. Let us state it more explicitely: for a positive integer h, a Radon measure on $\mathbb{G}$ is said to be h-rectifiable, according to Merlo's definition, if almost everywhere it has positive lower density and finite upper density and its tangent measures are multiples of the Haar measure of a h-dimensional homogeneous subgroup (eventually, when necessary, one can also require that this subgroup belongs to a suitable selected family of homogeneous subgroups, so that to individuate a precise sub-class of h-rectifiable measures). The authors show that this notion of rectifiability is strictly weaker than the other ones discussed in this chapter. As an application of their results, the authors reach the one-dimensional analogue of the Preiss' theorem in the first Heisenberg group $\mathbb{H}^1$.
These and other characterizations are among the motivations for which Franchi, Serapioni, Serra Cassano's notion of intrinsic rectifiability seems to fit better than the other proposed notions the structure of Carnot groups (refer to the introduction of this chapter for a discussion about alternative notions of intrinsic rectifiability).

The two Definitions 4.5.2 and 4.5.4 have been extended also to the context of a generic Carnot group $\mathbb{G}$ in [Ser08, Definition 5.2], according to the notions of intrinsic submanifold stated in Definition 4.1.24.

Definition 4.5.7. Let $\mathbb{G}$ be a Carnot group and let n and N be natural numbers, let $n \leq N$.

(i) A set $E \subset \mathbb{G}$ is *intrinsic* $(n,N)_L$*-rectifiable* if there is a sequence of intrinsic Lipschitz (n,N)-submanifolds $(\Sigma_i)_{i\in\mathbb{N}}$ such that

$$\mathcal{S}^N\left(E \setminus \bigcup_{i\in\mathbb{N}} \Sigma_i\right) = 0.$$

(ii) A set $E \subset \mathbb{G}$ is *intrinsic* (n,N)*-rectifiable* if there is a sequence of intrinsic (n,N)-submanifolds $(\Sigma_i)_{i\in\mathbb{N}}$ such that

$$\mathcal{S}^N\left(E \setminus \bigcup_{i\in\mathbb{N}} \Sigma_i\right) = 0.$$

By Proposition 3.5.34, surely intrinsic (n,N)-rectifiable sets are intrinsic $(n,N)_L$-rectifiable. In other words in particular, if $E \subset \mathbb{H}^n$ is a $(k,\mathbb{H})$-rectifiable set, it is clearly $(k,\mathbb{H}_L)$-rectifiable. We would like to know when the opposite is true, and then when the two intrinsic definitions of rectifiability are equivalent. Even limited to the Heisenberg group, this is a difficult question. Let us limit ourselves to the Heisenberg group, to discuss the case when $n+1 \le k \le 2n$. A first partial answer to the question in this setting was furnished by the Rademacher-type Theorem 3.6.4. As we said above, it ensures that any one-codimensional intrinsic Lipschitz graph is almost everywhere intrinsically differentiable. This property, combined with Lusin's and Egoroff's Theorems, allows to prove that $(2n,\mathbb{H}_L)$-rectifiable sets are always $(2n,\mathbb{H})$-rectifiable (and this holds also for Carnot groups of type $\star$ [FMS14, Proposition 4.4.4]). Hence for $k = 2n$ the two definitions coincide. Recently Vittone extended the proof of the Rademacher's theorem to all the intrinsic Lipschitz graphs of low codimension, as we discussed above. Exploiting this result, he managed to extend the equivalence of the two definitions of intrinsic $(k,\mathbb{H})$ and $(k,\mathbb{H}_L)$-rectifiable sets to the case when $n+1 \le k \le 2n-1$, see [Vit20, Corollary 7.4], finally providing a complete answer to the problem in the Heisenberg group for $n+1 \le k \le 2n$. In a generic Carnot group the problem is basically open, except for the few examples of Carnot groups for which a Rademacher's theorem is available, as Carnot groups of type $\star$, that we discussed in Section 3.6 (see [FSSC11] and [FMS14]).

Remark 4.5.8. In [DLDMV19], a recent new line of research is presented. Roughly speaking, it consists of defining *cones* in Carnot groups $\mathbb{G}$ in a very general way as nonempty sets $E \subset \mathbb{G}$ satisfying $\delta_t(E) \subset E$ for every $t > 0$. According to this new more general definition of cones, mimicking the definition of intrinsic Lipschitz graph, the authors define in this setting the family of *sets that satisfy an outer cone property*, i.e. sets $\Gamma \subset \mathbb{G}$ for which there exists an open cone E whose translations by elements in Γ do not intersect Γ. The authors prove that the H-reduced boundary of every locally finite H-perimeter set can be covered by countably many sets that satisfy the outer cone property. These new definitions open a path leading to an innovative definition of rectifiable set in Carnot groups, possibly weaker than the "classical" one presented by Franchi, Serapioni and Serra Cassano.

4.5.1 Towards uniform intrinsic rectifiability

According to the line of research that gives origin to Definition 4.5.4, one-codimensional intrinsic Lipschitz graphs have been recently used as cornerstones of the starting development of a theory of *uniform* or *quantitative rectifiability* in Carnot groups, up to now limited to the setting of the Heisenberg group. The mathematical motivations leading this research arises mainly from harmonic analysis. For example, in Euclidean spaces, quantitative rectifiability is strongly related to the problem of characterizing those sets that are removable for Lipschitz harmonic functions or those sets $E \subset \mathbb{R}^n$ for which the Dirichlet problem is solvable with boundary value belonging to a proper space $L^p(\mathcal{H}^{n-1}_E \llcorner \partial E)$ space, for some $p < \infty$. An accessible introduction to this theme is provided by [Fä19], where the reader can find also a rich collection of references about the motivations and the main outcomes of the theory. We limit ourselves to give a flavour of the first innovative techniques used in this setting. One of the first steps in this direction has been moved in [NY18]. In order to determine the approximation ratio of the Goemans-Linial algorithm for the Sparsest Cut Problem, which was a long-standing open problem about numerical optimization, Naor and Young developed powerful tools about geometric measure theory in the Heisenberg group. Regarding [NY18] from this perspective, the more significant result of the paper is that, considering a subset $E \subset \mathbb{H}^n$, *if* $(E, E^c, \partial E)$ *is locally a* $(2n+1)$*-Ahlfors regular triple,* ∂E *admits an intrinsic corona decomposition.* Roughly speaking, this means that one can associate with any small scale, up to infinitesimal scales, a suitable partition of the boundary ∂E, in such a way that each element of any partition (called *cube*) is quantitatively close to an intrinsic Lipschitz graph, so that, overall, ∂E is locally close, in a scale-invariant quantitative manner, to an intrinsic Lipschitz graph. In addition, the intrinsic Lipschitz constants of the approximating intrinsic Lipschitz graphs can be uniformly controlled, introducing suitable quantitative control parameters. In the Euclidean settings, the existence of a corona decomposition is one of the possible definitions of uniformly rectifiable set, hence one can think of uniform rectifiability as of a quantitative scale-invariant notion of rectifiability. We dedicate below some space to explain the innovative techniques used in [NY18] more in detail. We try to maintain as much as possible the notation of the original paper, even if it does not always coincide with the one mostly used in this thesis.

By $\mathbb{H}^n$ we will denote as usual the Heisenberg group and we consider it endowed with a homogeneous distance d. As usual $\|\cdot\|$ denotes the homogeneous norm associated with d. A *horizontal line* is a coset of the form $x\text{span}(v)$, with $x \in \mathbb{H}^n$ and $v \in H_1$. Analogously, an *horizontal plane* is a coset hH_1 with $h \in \mathbb{H}^n$. Planes that are not horizontal are called *vertical.*

Remark 4.5.9. Notice that horizontal lines are cosets of one dimensional homogeneous subgroups and vertical planes are cosets of $2n$-dimensional vertical subgroups (while horizontal planes are never subgroups or cosets of homogeneous subgroups).

Definition 4.5.10. Let $V \subset \mathbb{H}^n$ be a vertical plane. Let $W = V^\perp = \text{span}(\nu)$, $\nu \in H_1$ be the horizontal line orthogonal to V passing through the origin. For every $\lambda \in (0,1)$, we define the cone

$$C_\lambda(V) = \{y \in \mathbb{H}^n : \|p_W(y)\| > \lambda \|y\|\} \tag{4.31}$$

where p_W is the orthogonal projection on the line W.
Let $U \subset V$ be a open set. Given a map $\phi : U \to W$, we set $\Gamma := \text{graph}(\phi) = \{w\phi(w) : w \in U\}$. The set $\Gamma \subset \mathbb{H}^n$ is an *intrinsic NY-Lipschitz graph* (on V) if there exists a constant $\lambda \in (0,1)$ such that for every $x \in \Gamma$,

$$(xC_\lambda(V)) \cap \Gamma = \emptyset.$$

We call the map ϕ intrinsic NY-Lipschitz.

If we consider a map $\phi : U \subset V \to W$, it can be identified with the real-valued map $\psi : U \to \mathbb{R}$ such that

$$\psi(w)\nu = \phi(w)$$

for every $w \in U$. Then the intrinsic graph of ϕ

$$\Gamma = \{w\phi(w) : w \in U\} = \{w(\psi(w)\nu) : w \in U\}$$

bounds two half-space of $\mathbb{H}^n$, one of which is the *positive* half-space

$$\Gamma^+ = \{w(t\nu) : t \geq \psi(w)\}.$$

If ϕ is intrinsic NY-Lipschitz, we say that Γ^+ is an *intrinsic NY-Lipschitz half-space*.

Remark 4.5.11. In [FOR20, Remark 2.5], the authors observe that Definition 4.5.10 is not equivalent to Definition 3.5.6 for any homogeneous distance on $\mathbb{H}^n$. For instance, if we compare the two definitions for a map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ with $\mathbb{W}$ and $\mathbb{V}$ complementary orthogonal subgroups of $\mathbb{H}^n$, with $\mathbb{V}$ one dimensional, the two definitions of intrinsic Lipschitz continuity coincide only if $\mathbb{H}^n$ is equipped with a homogeneous distance d that satisfies a sort of convexity property, i.e. if there does not exists any point $x \in \mathbb{H}^n$ such that $d(x, 0) = \|\pi_{\mathbb{V}}(x)\|$ and $\pi_{\mathbb{W}}(x) \neq 0$. For example, the distance d_∞ does not satisfy this property, while the distance d_c does.

Remark 4.5.12. As we hinted in the previous section, the notion of intrinsic NY-Lipschitz function allows to prove, through suitable properties of semigroup of the new cones defined in (4.31), an extension theorem for one-codimensional intrinsic Lipschitz graphs, [NY18, Theorem 27]. As we said, this theorem can be considered as a strengthening of the extension theorem presented by Franchi, Serapioni and Serra Cassano in [FSSC11], that is Theorem 3.6.1.

Limited to this section, we call intrinsic Lipschitz graphs the intrinsic NY-Lipschitz graphs.

From now until the end of the thesis, when we write $a \lesssim b$, we mean that there exists some positive constant C such that $a \leq Cb$. If C depends on some parameter p, it will be specified with a subscript. For instance, by $a \lesssim_p b$ we mean that there exists a constant C depending on p such that $a \leq Cb$. Analogous notations are assumed for $\gtrsim$.

Definition 4.5.13 (Regular triple)**.** Fix $C, r, s \in (0, \infty)$. A Borel subset $A \subset \mathbb{H}^n$ is r-locally C-Ahlfors s-regular if for every $x \in A$ and $\rho \in (0, r]$ we have

$$\frac{\rho^s}{C} \leq \mathcal{H}^s(B(x, \rho) \cap A) \lesssim C\rho^s.$$

In particular we call $(E, E^c, \partial E)$ a (C, r)-*regular triple* if E, E^c are r-locally C-Ahlfors $(2n + 2)$-regular and ∂E is r-locally C-Ahlfors $(2n + 1)$-regular.

Now, we introduce the precise definition of *intrinsic corona decomposition* step by step through a series of nested definitions.

Definition 4.5.14. Fix $K, s, r \in (0, \infty)$. Let $m \in \mathbb{Z}$ be such that $2^m \leq r < 2^{m+1}$. Consider a Borel set $A \subset \mathbb{H}^n$. A *s-dimensional* (K, r)*-cubical patchwork* for A is a sequence of Borel partitions $(\Delta_i)_{i=-\infty}^m$ such that

(i) for every integer $i \leq m$ and every $Q \in \Delta_i$, we have

$$\frac{2^i}{K} < \operatorname{diam}(Q) < K2^i \qquad \frac{2^{is}}{K} < \mathcal{H}^s(Q) < K2^{is};$$

(ii) for every two integers $i, j \leq m$ with $i \leq j$ the partition Δ_i is a refinement of the partition Δ_j, i.e. for every $Q \in \Delta_i$ and $Q' \in \Delta_j$, either $Q \cap Q' = \emptyset$ or $Q \subset Q'$;

(iii) for every integer $i \leq m$, $Q \in \Delta_i$ and $t > 0$ we have

$$\mathcal{H}^s(\partial_{\leq t2^i} Q) \leq K t^{\frac{1}{K}} 2^{is},$$

where for every $\rho > 0$,

$$\partial_{\leq \rho} Q := \{x \in Q : d(x, A \setminus Q) \leq \rho\} \cup \{x \in A \setminus Q : d(x, Q) \leq \rho\}.$$

We denote by $\Delta = \bigsqcup_{i=-\infty}^{m} \Delta_i$. We call any element of Δ a *cube*. For every integer $i \leq m$, if $Q \in \Delta_i$, we set $\sigma(Q) = 2^i$. It can be thought as the side-length of the cube Q. We will consider $A = \partial E$ for some set $E \subset \mathbb{H}^n$. Hence $s = 2n + 1$.

Definition 4.5.15. Let $\mathcal{D} \subset \Delta$ be a collection of cubes; $\mathcal{D}$ *is N-Carleson* if for every cube $Q \in \Delta$, we have

$$\sum_{\substack{R \in \mathcal{D} \\ R \subset Q}} \sigma(R)^{2n+1} \leq N\ \sigma(Q)^{2n+1}.$$

Roughly speaking, this means that if Δ is a cubical patchwork for A, $\mathcal{H}^{2n+1}$-almost every point $x \in A$ is contained in finitely many elements of $\mathcal{D}$.

Definition 4.5.16. A collection of cubes $\mathcal{S} \subset \Delta$ is said to be *coherent* if the following conditions hold

(i) $\mathcal{S}$ has a maximal element with respect to the inclusion, i.e. there exists a unique cube $\mathbf{Q}(\mathcal{S}) \in \mathcal{S}$ such that $Q \subset \mathbf{Q}(\mathcal{S})$ for every $Q \in \mathcal{S}$;

(ii) if $Q \in \mathcal{S}$ and $Q' \in \Delta$ satisfies $Q \subset Q' \subset \mathbf{Q}(\mathcal{S})$, then $Q' \in \mathcal{S}$;

(iii) if $Q \in \mathcal{S}$ then either all the children of Q in $\{\Delta_i\}_{i=-\infty}^{m}$ belong to $\mathcal{S}$ or none of them does, i.e. if $\sigma(Q) = 2^i$ then either $\{Q' \in \Delta_{i+1} : Q' \subset Q\} \subset \mathcal{S}$ or $\{Q' \in \Delta_{i+1} : Q' \subset Q\} \cap \mathcal{S} = \emptyset$.

Definition 4.5.17. Fix $K, N, r \in (0, \infty)$ and $E \subset \mathbb{H}^n$. A (K, N, r)-*coronization* of ∂E is a triple $(\mathcal{B}, \mathcal{G}, \mathcal{F})$ with the following properties. There exists a (K, r)-cubical patchwork $\{\Delta_i\}_{i=-\infty}^{m}$ for ∂E such that $\mathcal{B} \subset \Delta$ (*bad cubes*) and $\mathcal{G} \subset \Delta$ (*good cubes*) partition Δ into two disjoint sets, i.e. $\mathcal{B} \cup \mathcal{G} = \Delta$ and $\mathcal{B} \cap \mathcal{G} = \emptyset$, and $\mathcal{F} \subset 2^{\mathcal{G}}$ is a collection of sub-collections of $\mathcal{G}$, called *stopping time regions.* These sets are required to have the following properties.

(i) $\mathcal{B}$ is N-Carleson;

(ii) The elements of $\mathcal{F}$ are pairwise disjoint and their union is $\mathcal{G}$;

(iii) each $\mathcal{S} \in \mathcal{F}$ is coherent;

(iv) the set of maximal cubes $\{\mathbf{Q}(\mathcal{S}) : \mathcal{S} \in \mathcal{F}\}$ is N-Carleson.

Definition 4.5.18. If $U, V, W \subset \mathbb{H}^n$, we define the U-local distance between V and W by

$$d_U(V, W) = \inf\{r > 0 : (V \triangle W) \cap U \subset \mathrm{nbhd}_r(\partial V) \cap \mathrm{nbhd}_r(\partial W)\},$$

where $\mathrm{nbhd}_r(A) = \{h \in \mathbb{H}^n : d(h, A) < r\}$ for every $A \subset \mathbb{H}^n$.

If $Q \in \Delta$ and $\rho > 0$, we define

$$N_\rho(Q) = nbhd_{\rho\sigma(Q)}(Q).$$

Definition 4.5.19 (Intrinsic corona decomposition)**.** Fix $K, N, r, \lambda, \theta > 0$. Given $E \subset \mathbb{H}^n$, the pair $(E, \partial E)$ admits a $(K, N, r, \lambda, \theta)$-*intrinsic corona decomposition* if there is a (K, N, r)-coronization $(\mathcal{B}, \mathcal{G}, \mathcal{F})$ of ∂E such that for each $\mathcal{S} \in \mathcal{F}$ there is an intrinsic λ-Lipschitz graph $\Gamma(\mathcal{S})$ that bounds a Lipschitz half-space $\Gamma^+(\mathcal{S})$, such that for all $Q \in \mathcal{S}$ we have

$$d_{N_4(Q)}(\Gamma^+(\mathcal{S}), E) \leq \theta\sigma(Q).$$

We say that $(E, \partial E)$ admits a (K, r)-intrinsic corona decomposition if for every $\lambda, \theta > 0$ there exists $N = N(\lambda, \theta)$ such that the pair $(E, \partial E)$ admits a $(K, N, r, \lambda, \theta)$-intrinsic corona decomposition. The main result in [NY18], from the point of view of geometric measure theory is the following theorem.

Theorem 4.5.20. *[NY18, Theorem 57] Fix $C, r \in (0, \infty)$. Let $(E, E^c, \partial E)$ be a (C, r)-Ahlfors regular triple in $\mathbb{H}^n$. Then there exists $K = K(C, n) \in (0, \infty)$ such that the pair $(E, \partial E)$ admits a (K, r)-intrinsic corona decomposition.*

Roughly speaking, to give a flavour of the proof, given a set $E \subset \mathbb{H}^n$ such that $(E, E^c, \partial E)$ is an Ahlfors regular triple, Naor and Young, adapting ideas by Cheeger and Kleiner (see for instance [CKN11]), have shown how to build an intrinsic corona decomposition of ∂E governed by the so-called *non-monotonicity* of E, i.e. an instrument that measures how much E is quantitatively locally close to be a half-space and, in particular, to be a vertical half-space. The concept of non-monotonicity is defined by taking in consideration, in a quantitative way, some measure-theoretic features of the set of the intersections of all the horizontal lines of the group with the set E. The strength of Theorem 4.5.20, in our opinion, relies in the fact that it can be an useful tool to generalize properties of intrinsic Lipschitz graphs to Ahlfors regular triples, or, even more, to larger families of sets. For instance, Naor and Young followed this approach in order to prove an isoperimetric-type inequality that involves the vertical perimeter and the measure $\mathcal{H}^{2n+1} \llcorner \partial E$ of any measurable sets. To better explain our viewpoint, let us concisely explain what did they prove and let us sketch how the proof of their result is carried out. For $E \subset \mathbb{H}^n$ and $s \in \mathbb{R}$, let us define

$$D_s E := E \triangle E(2^{2s} e_{2n+1}) \subset \mathbb{H}^n.$$

If $E,\ U \subset \mathbb{H}^n$ are measurable sets, we introduce the function

$$\bar{v}_U(E) : \mathbb{R} \to \mathbb{R},\ \bar{v}_U(E)(s) := \frac{\mathcal{H}^{2n+2}(D_s E \cap U)}{2^s}.$$

The local *vertical perimeter of E in U* is the quantity

$$\|\bar{v}_U(E)\|_{L^2(\mathbb{R})} = \sqrt{\int_{-\infty}^{+\infty} (\bar{v}(E)(s))^2 ds}.$$

If $U = \mathbb{H}^n$, we denote the global vertical perimeter by dropping the sub-index.
The main general result of [NY18] is the following theorem.

Theorem 4.5.21. *[NY18, Theorem 36] If $n \geq 2$ and $E \subset \mathbb{H}^n$ is a measurable set, then*

$$\|\bar{v}(E)\|_{L_2(\mathbb{R})} \lesssim_n \mathcal{H}^{2n+1}(\partial E). \tag{4.32}$$

Heuristically speaking, Theorem 4.5.21 says that if it is difficult to leave the set E in the horizontal directions, then it is difficult to leave E in the vertical direction. This is substantially due to the fact that the vertical direction e_{2n+1} is the commutator of couples of horizontal directions, that can be chosen in n "independent" ways.
The logical scheme of the proof of Theorem 4.5.21 is the following

(i) approximate, in a measure-theoretical way, any finite H-perimeter set with *cellular sets* ([NY18, Lemma 21]). A set is said cellular if it is the union of cosets of the fundamental closed cell $F := \left[-\frac{1}{2}, \frac{1}{2}\right]^{2n+1}$ by elements of the discrete Heisenberg group $\mathbb{H}^n_{\mathbb{Z}}$, zF, with $z \in \mathbb{H}^n_{\mathbb{Z}}$.

(ii) "decompose" cellular sets into Ahlfors regular triples ([NY18, Lemma 58]).

(iii) show that any Ahlfors regular triple admits an intrinsic corona decomposition ([NY18, Theorem 57]).

(iv) prove the inequality (4.32) for intrinsic Lipschitz subgraphs ([NY18, Proposition 41]).

(v) extend the inequality (4.32) to sets locally close to intrinsic Lipschitz subgraphs, i.e. to sets whose boundary admits an intrinsic corona decomposition ([NY18, Proposition 55]).

We reported this scheme in order to express that it is a faithful hope that a similar path could be exploited in order to extend different properties in analogous way.

Remark 4.5.22. For the interested reader, an isoperimetric-type inequality analogous to the one of Theorem 4.5.21 has been recently proved in [NY20] for the case when $E \subset \mathbb{H}^1$. In this case, the norm $\|\cdot\|_{L^2(\mathbb{R})}$ of the vertical perimeter has to be replaced by the norm $\|\cdot\|_{L^4(\mathbb{R})}$.

In the Euclidean setting, concepts analogous to the ones collected in this section had been introduced and deeply investigated by David and Semmes ([DS91, DS93a, DS93b]). In particular, in $\mathbb{R}^n$ it is completely clear how for an Ahlfors k-regular closed set E, for any $1 \leq k \leq n$, the existence of a corona decomposition can be compared with other quantitative geometrical concepts such as the *weak geometric lemma*, the properties of having *big projections* or of having *big pieces of Lipschitz graphs*. For precise definitions, please refer to [DS91]. In the context of the Heisenberg group the algebraic structure of the group makes things more delicate. In particular there is a strong difference between vertical and horizontal objects (planes, lines). Strongly inspired by Naor and Young's techniques, Chousionis, Fässler, Orponen, Rigot (see for instance [FOR20, CFO19b, Rig19]) moved on to develop quantitative concepts analogous to the Euclidean ones, on top of the one of intrinsic corona decomposition, related to quantitative scale-invariant rectifiability in $\mathbb{H}^n$. For instance we refer to the *weak geometric lemma*, to the properties of having *big vertical projections* or *big pieces of intrinsic Lipschitz graphs*. The complete knowledge of the relations among these notions is still far from being clear, even for one-codimensional sets. The line of research devoted to get a complete development of a theory of uniform rectifiability in Carnot groups is now very active.

Chapter 5

$\mathbb{H}$-regular surfaces of low codimension

By the comparison between the Euclidean implicit function theorem and Theorem 4.1.19, the uniform intrinsic differentiability of maps acting between two complementary subgroups of the Heisenberg group, seems to be the analogue of the Euclidean C^1-regularity of maps acting between two linear subspaces whose direct sum is the Euclidean space $\mathbb{R}^n$. Then, it is natural to conjecture that, analogously to what happens for Euclidean C^1-regular functions, uniform intrinsic differentiability can be characterized in terms of the existence and continuity of appropriately defined intrinsic partial derivatives. In this chapter, we give a positive answer to this conjecture in the Heisenberg group, presenting the results of [Cor19]. After fixing in Section 5.1 a suitable choice of coordinates, we reserve Section 5.2 to describe the state of the art in this direction of research for the case when $k = 1$. For this situation in fact many results are available in the literature. In Section 5.3 we generalize some of the results presented in Section 5.2. In particular, we provide various characterizations of uniformly intrinsically differentiable maps $\phi : U \subset \mathbb{W} \to \mathbb{V}$ acting between two complementary subgroups such that $\mathbb{H}^n = \mathbb{W}\mathbb{V}$, with U open set and $\mathbb{V}$ horizontal subgroup: if we denote by k the dimension of $\mathbb{V}$, our new results regard the case when $1 < k \le n$. Roughly speaking, considering such a map ϕ, in Proposition 5.3.21, we prove the equivalence between the uniform intrinsic differentiability of ϕ and the existence, at any point $a \in U$, of a family of C^1-regular maps that approximate uniformly the map ϕ, and whose intrinsic differentials approximate uniformly the intrinsic differential of ϕ, on a proper neighbourhood of a. The second main result is Theorem 5.3.24 that states that the uniform intrinsic differentiability of ϕ is equivalent to the existence and continuity of the *intrinsic partial derivatives* of ϕ (Definition 5.3.8), that are suitably defined derivatives of the components of ϕ along a family of nonlinear vector fields (Definition 5.1.14) whose coefficients depend on the map ϕ. Finally, still in Theorem 5.3.24, we prove that the uniform intrinsic differentiability of the map ϕ is equivalent to the existence and continuity of the intrinsic differential of ϕ, $d\phi_w : \mathbb{W} \to \mathbb{V}$, with respect to $w \in U$. The proofs of the main results are quite nested and articulated, then we have organized them in multiple subsections, the titles of which are meant to lead the reader towards the main final goals. A precise generalization to arbitrary Carnot groups of the results in [Cor19] has been recently proposed in [ADDDLD20]. In this paper, the authors proved characterizations, analogous to the ones presented in this chapter, of uniformly intrinsic differentiability for maps acting from a normal homogeneous subgroup $\mathbb{W}$ to a horizontal subgroup $\mathbb{V}$ complementary to $\mathbb{W}$ in an arbitrary Carnot group $\mathbb{G}$. We collect some more details about [ADDDLD20] in a proper section, Section 5.4, where we report also a brief summary of the results of [ADDD20]. We conclude the chapter presenting in Section 5.5 an area formula for the centered Hausdorff measure of the intrinsic graph of a uniformly intrinsically differentiable map, with respect to the distance d_∞. It is the first area formula, for intrinsic regular graphs of codimension higher than 1, involving uniquely the intrinsic partial derivatives of the map that parametrizes the considered graph.

We highlight that the results established before [Cor19] for intrinsic regular maps $\phi : U \subset \mathbb{W} \to \mathbb{V}$ acting between two complementary subgroups of $\mathbb{H}^n$, $\mathbb{W}$ and $\mathbb{V}$, with $\mathbb{V}$ horizontal such that $1 < \dim(\mathbb{V}) \leq n$, are a few. They have been proved by Kozhevnikov in [Koz15, Chapter 4] and by Di Donato in [DD20a, Section 4]. As we said in Section 3.6, recently Vittone presented in [Vit20] an extension theorem, an approximation theorem and a Rademacher-type theorem valid in this situation.

In this chapter we assume that $\mathbb{H}^n$ is endowed with the homogeneous distance $d = d_\infty$, then $\| \cdot \|$ stands for $\| \cdot \|_\infty$. This choice is related to the necessity of setting a homogeneous distance that permits to deal with explicit computations in coordinates. Since all homogeneous distances are equivalent, for our purposes it will not be restrictive to have fixed a particular one.

5.1 Setting and notation

We start by restating the notion of uniform intrinsic differentiability and the related concepts in the Heisenberg group in terms of appropriately fixed coordinates. Throughout the whole chapter, we assume that $\mathbb{H}^n$ is the semidirect product of two complementary subgroups $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ *with* $\mathbb{W}$ *orthogonal to* $\mathbb{V}$. Since $\mathbb{W}$ is normal, by Remark 3.1.16, $\mathbb{V}$ has to be horizontal, thus dimension $1 \leq \dim(\mathbb{V}) \leq n$. The following proposition, that is [CM20, Proposition 2.8], ensures that under these assumptions we can always find a Heisenberg basis adapted, in some sense, to the factorization.

Proposition 5.1.1. *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product. Assume that the horizontal subgroup $\mathbb{V}$ is spanned by an orthonormal basis of horizontal vectors $v_1, \dots, v_k \in H_1$ and assume that $\mathbb{W}$ is orthogonal to $\mathbb{V}$. Then there exist $2n - k$ horizontal vectors $v_{k+1}, \dots, v_n, w_1, \dots, w_n \in H_1$ such that $(v_{k+1}, \dots, v_n, w_1, \dots, w_n, e_{2n+1})$ is an orthonormal basis of $\mathbb{W}$ and $(v_1, \dots, v_n, w_1, \dots, w_n, e_{2n+1})$ is a Heisenberg basis of $\mathbb{H}^n$.*

Proof. Since $\mathbb{V}$ is commutative, an element $v = J(w)$, with $v, w \in \mathbb{V}$, satisfies

$$|v|^2 = \langle v, J(w) \rangle = -\omega(v, w) = 0,$$

therefore $\mathbb{V} \cap J(\mathbb{V}) = \{0\}$. We set $w_i = J(v_i) \in \mathbb{W}$ for $i = 1, \dots, k$ and define the $2k$-dimensional subspace

$$\mathbb{S}_1 = \mathbb{V} \oplus J(\mathbb{V}) \subset H_1.$$

We notice that $\dim(\mathbb{S}_1^\perp \cap H_1) = 2(n - k)$. If $k < n$, we pick a vector $v_{k+1} \in \mathbb{S}_1^\perp \cap H_1$ of unit norm and define $w_{k+1} = J(v_{k+1})$. It is easily observed that both w_{k+1} and v_{k+1} are orthogonal to $\mathbb{S}_1$, so that $(v_1, \dots, v_{k+1}, w_1, \dots, w_{k+1}, e_{2n+1})$ is a Heisenberg basis of

$$\mathbb{S}_2 \oplus \operatorname{span} \{e_{2n+1}\},$$

where we have defined $\mathbb{S}_2 = \mathbb{V} \oplus \operatorname{span} \{v_{k+1}\} \oplus J(\mathbb{V} \oplus \operatorname{span} \{v_{k+1}\})$. Indeed, the previous subspace has the structure of a $(2k + 3)$-dimensional Heisenberg group. One can iterate this process until a Heisenberg basis of $\mathbb{H}^n$ is found. $\square$

From now on, we fix the Heisenberg basis $(v_1, \dots, v_n, w_1, \dots, w_n, e_{2n+1})$ provided by Proposition 5.1.1 and we consider $\mathbb{H}^n$ in adapted coordinates with respect to this basis. Since we will work by coordinates, without loss of generality, according to Remark 2.4.12 we can assume that $v_i = e_i$ and $w_i = e_{n+i}$ for $i = 1, \dots, n$, where $\mathcal{B} = (e_1, \dots, e_{2+1})$ is the Heisenberg basis we had fixed at the beginning (in (2.11)).
We set, as in (2.14)

$$\begin{aligned} X_i &= X_{e_i} && \text{for } i = 1, \dots, n \\ Y_i &= X_{e_{i+n}} && \text{for } i = 1, \dots, n \\ T &= X_{e_{2n+1}} \end{aligned} \tag{5.1}$$

where $X_{e_i} \in \mathrm{Lie}(\mathbb{H}^n)$ denotes the unique left invariant vector field such that $X_{e_i}(0) = e_i$. We can identify $\mathbb{V}$ with $\mathbb{R}^k$ and $\mathbb{W}$ with $\mathbb{R}^{2n+1-k}$ through the following diffeomorphisms

$$i_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k,\ i_{\mathbb{V}}\left(\sum_{i=1}^k x_i e_i\right) = (x_1, \dots, x_k),$$

$$i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^{2n+1-k},$$

$$i_{\mathbb{W}}\left(\tau e_{2n+1} + \sum_{i=k+1}^{n} (x_i e_i + y_i e_{i+n}) + \sum_{i=1}^{k} \eta_i e_{i+n}\right) = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau).$$

Furthermore, we move on $\mathbb{R}^{2n+1-k}$ and $\mathbb{R}^k$ the structures of homogeneous group of $\mathbb{W}$ and $\mathbb{V}$, respectively, in such a way that $i_{\mathbb{W}}$ and $i_{\mathbb{V}}$ are group isomorphisms. Therefore, we may identify the subgroup $(\mathbb{W}, \cdot, \delta_t|_{\mathbb{W}})$ with the homogeneous group $(\mathbb{R}^{2n+1-k}, \star, \delta_t^\star)$, where for every $a, b \in \mathbb{R}^{2n+1-k}$ and $t > 0$

$$a \star b := i_{\mathbb{W}}(i_{\mathbb{W}}^{-1}(a) i_{\mathbb{W}}^{-1}(b)) \qquad \delta_t^\star(a) := i_{\mathbb{W}}(\delta_t(i_{\mathbb{W}}^{-1}(a))).$$

Analogously, we may identify $(\mathbb{V}, \cdot, \delta_t|_{\mathbb{V}})$ with the commutative group $(\mathbb{R}^k, +, d_t)$, where by d_t we denote the Euclidean isotropic dilation associated with the parameter $t > 0$.

Besides transferring the structure of homogeneous group from $\mathbb{W}$ to $\mathbb{R}^{2n+1-k}$, we can also push forward on $\mathbb{R}^{2n+1-k}$, through $i_{\mathbb{W}}$, the linear vector fields that generate $\mathrm{Lie}(\mathbb{W})$. In fact, we set

$$\begin{aligned} \tilde{X}_j &= (i_{\mathbb{W}})_*(X_j) = \partial_{x_j} - \frac{1}{2} y_j \partial_\tau && \text{for } j = k+1, \dots, n \\ \tilde{Y}_j &= (i_{\mathbb{W}})_*(Y_j) = \partial_{\eta_j} && \text{for } j = 1, \dots, k \\ \tilde{Y}_j &= (i_{\mathbb{W}})_*(Y_j) = \partial_{y_j} + \frac{1}{2} x_j \partial_\tau && \text{for } j = k+1, \dots, n \\ \tilde{T} &= (i_{\mathbb{W}})_*(T) = \partial_\tau. \end{aligned}$$

Let us consider now an open set $\tilde{U} \subset \mathbb{W}$ and a map $\tilde{\phi} : \tilde{U} \to \mathbb{V}$. Let U denote the open subset of $\mathbb{R}^{2n+1-k}$ corresponding to $\tilde{U}$, that is

$$U := i_{\mathbb{W}}(\tilde{U}) \subset \mathbb{R}^{2n+1-k}$$

and let ϕ denote the map corresponding to $\tilde{\phi}$, that is

$$\phi := i_{\mathbb{V}} \circ \tilde{\phi} \circ i_{\mathbb{W}}^{-1} : U \to \mathbb{R}^k.$$

Vice versa, if we start by considering an open subset $U \subset \mathbb{R}^{2n+1-k}$ and a map $\phi : U \to \mathbb{R}^k$, we denote by $\tilde{U}$ the open set of $\mathbb{W}$ corresponding to U, that is $\tilde{U} := i_{\mathbb{W}}^{-1}(U) \subset \mathbb{W}$, and we set $\tilde{\phi} : \tilde{U} \to \mathbb{V}$, $\tilde{\phi} := i_{\mathbb{V}}^{-1} \circ \phi \circ i_{\mathbb{W}}$ to denote the map corresponding to ϕ.

Remark 5.1.2. The map ϕ is basically $\tilde{\phi}$ read in some specific coordinates, associated to the fixed basis, through the maps $i_{\mathbb{V}}$ and $i_{\mathbb{W}}$ (compare for example these maps with the map considered in (3.2)). In the next chapter we will not need this distinction any more, hence we will basically identify ϕ with $\tilde{\phi}$, directly denoting by ϕ the map acting between two homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$. Actually in fact, we will mainly work without fixing coordinates.

We want to interpret, in the coordinates that we have fixed, the notions we set in the previous chapters for maps $\tilde{\phi}$ acting between $\mathbb{W}$ and $\mathbb{V}$. More precisely, we consider a mapping ϕ, that acts from an open set $U \subset \mathbb{R}^{2n+1-k}$ to $\mathbb{R}^k$. It corresponds as described above to the map $\tilde{\phi}$ from $\tilde{U} \subset \mathbb{W}$ to $\mathbb{V}$. We want to focus on the definitions of intrinsic differentiability, and on related notions, and to individuate for each notion the right condition one has to ask to the map ϕ in order to be sure that $\tilde{\phi}$ satisfies the considered definition.

Consider an open set $U \subset \mathbb{R}^{2n+1-k}$ and a function ϕ

$$\phi : U \to \mathbb{R}^k,\ (x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau) \longmapsto (\phi_1, , \dots, \phi_k),$$

where we have set $\phi_j = \phi_j(x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau)$ for every $j \in \{1, \dots, k\}$. We restate the notion of graph map for ϕ as

$$\Phi := \tilde{\Phi} \circ i_{\mathbb{W}}^{-1},$$

where $\tilde{\Phi}$ denotes the usual graph map of $\tilde{\phi}$, $\tilde{\Phi}(w) = w\tilde{\phi}(w)$, for $w \in \tilde{U}$. More explicitly

$$\Phi : U \to \mathbb{H}^n,\ \Phi(a) = \tilde{\Phi}(i_{\mathbb{W}}^{-1}(a)) = i_{\mathbb{W}}^{-1}(a)(i_{\mathbb{V}}^{-1}(\phi(a)) \tag{5.2}$$

then, if $a = (x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau)$,

$$\begin{aligned}
&\Phi(x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau)\\
&= i_{\mathbb{W}}^{-1}(x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau)\tilde{\phi}(i_{\mathbb{W}}^{-1}(x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau))\\
&= i_{\mathbb{W}}^{-1}(x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau)\\
&\quad (i_{\mathbb{V}}^{-1} \circ \phi \circ i_{\mathbb{W}})(i_{\mathbb{W}}^{-1}(x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau))\\
&= (0, \dots, 0, x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau)(\phi_1, \dots, \phi_k, 0, \dots, 0)\\
&= \left(\phi_1, \dots, \phi_k, x_{k+1} \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau - \frac{1}{2}\sum_{i=1}^{k} \eta_i \phi_i\right).
\end{aligned}$$

Notice that $\Phi(U) = \tilde{\Phi} \circ i_{\mathbb{W}}^{-1}(i_{\mathbb{W}}(\tilde{U})) = \tilde{\Phi}(\tilde{U}) = \mathrm{graph}(\tilde{\phi})$. We call this object the *intrinsic graph of* ϕ and we denote it by $\mathrm{graph}(\phi) := \Phi(U)$, so that $\mathrm{graph}(\phi) = \mathrm{graph}(\tilde{\phi})$.

Now, we translate the notion of graph distance $d_{\tilde{\phi}} : \tilde{U} \times \tilde{U} \to \mathbb{R}$ in the corresponding definition of graph distance on $U \times U$, $d_\phi : U \times U \to \mathbb{R}$: for every $a, b \in U$

$$d_\phi(a, b) := d_{\tilde{\phi}}(i_{\mathbb{W}}^{-1}(a), i_{\mathbb{W}}^{-1}(b)) = \|\pi_{\mathbb{W}}(\tilde{\Phi}(i_{\mathbb{W}}^{-1}(b))^{-1}\tilde{\Phi}(i_{\mathbb{W}}^{-1}(a))\| = \|\pi_{\mathbb{W}}(\Phi(b)^{-1}\Phi(a))\|. \tag{5.3}$$

Thus, in particular, if we have

$$\begin{aligned} a &= (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1} \dots, y_n, \tau), \\ b &= (x'_{k+1}, \dots, x'_n, \eta'_1, \dots, \eta'_k, y'_{k+1}, \dots, y'_n, \tau'), \end{aligned} \tag{5.4}$$

and, for $j \in \{1, \dots, k\}$, we set $\phi_j = \phi_j(a)$, $\phi'_j = \phi_j(b)$ and

$$\xi := (x_{k+1} - x'_{k+1}, \dots, x_n - x'_n, \eta_1 - \eta'_1, \dots, \eta_k - \eta'_k, y_{k+1} - y'_{k+1}, \dots, y_n - y'_n) \in \mathbb{R}^{2n-k},$$

then, by direct computations we get

$$d_\phi(a, b) = \max\{\ |\xi|, |\tau - \tau' + \sum_{j=1}^{k} \phi'_j(\eta'_j - \eta_j) + \sigma(x, y, x', y')|^{\frac{1}{2}}\}, \tag{5.5}$$

where $x, y, x', y' \in \mathbb{R}^{n-k}$ are the vectors $x = (x_{k+1}, \dots, x_n)$, $y = (y_{k+1}, \dots, y_n)$, $x' = (x'_{k+1}, \dots, x'_n)$, $y' = (y'_{k+1}, \dots, y'_n) \in \mathbb{R}^{n-k}$ and $\sigma(x, y, x', y') := \frac{1}{2} \sum_{j=k+1}^{n} (x_j y'_j - x'_j y_j)$.

Remark 5.1.3. By the equality (5.5), one can deduce that for any compact subset $F \subset U$, there is a positive constant c, depending on F, such that for every $a, b \in F$ with $|a-b| < 1$

$$d_\phi(a, b) \le c|b - a|^{\frac{1}{2}}.$$

In fact, if we define $\Delta := \max_{p \in F} |p|$ and for every $j = 1, \dots, k$, $M_j := \max_{p \in F} |\phi_j(p)|$, by direct computations

$$\begin{aligned} d_\phi(a, b) &\le \sqrt{|\xi|} + \sqrt{|\tau - \tau'|} + \sum_{j=1}^{k} \sqrt{|\phi'_j(\eta'_j - \eta_j)|} + \sqrt{\frac{1}{2} \sum_{j=k+1}^{n} |x_j y'_j - x'_j y_j|} \\ &\le \left(2 + \sum_{j=1}^{k} \sqrt{M_j} + \sqrt{(n-k)\Delta}\right) \sqrt{|b - a|}. \end{aligned}$$

Definition 5.1.4. We call a function $L : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$ *$\star$-linear* if it is a homogeneous homomorphism between $(\mathbb{R}^{2n+1-k}, \star, \delta^\star_t)$ and $(\mathbb{R}^k, +, d_t)$, i.e. if for every $a, b \in \mathbb{R}^{2n+1-k}$ and $t > 0$,

$$L(a \star b) = L(a) + L(b) \quad \text{and} \quad L(\delta^\star_t(a)) = tL(a).$$

Remark 5.1.5. By taking in consideration Remark 3.5.19, it is immediate to verify that a map $L : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$ is $\star$-linear if and only if the corresponding map $\tilde{L} = i_{\mathbb{V}}^{-1} \circ L \circ i_{\mathbb{W}} : \mathbb{W} \to \mathbb{V}$ is intrinsic linear.

We are finally ready to restate in coordinates the notions of intrinsic differentiability and uniform intrinsic differentiability. By $|\cdot|$ we denote the Euclidean norm on $\mathbb{R}^k$.

Definition 5.1.6. Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and $\phi : U \to \mathbb{R}^k$ be a map. Consider a point $a_0 \in U$, we say that ϕ *is intrinsically differentiable at* a_0 if there exists a $\star$-linear function $L : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$ such that

$$\lim_{a \to a_0} \frac{|\phi(a) - \phi(a_0) - L(a_0^{-1} \star a)|}{d_\phi(a, a_0)} = 0. \tag{5.6}$$

We call the function L the intrinsic differential of ϕ at a_0 and we denote it by $d\phi_{a_0}$. We say that ϕ is intrinsically differentiable on U if it is intrinsically differentiable at any point $a \in U$.

Remark 5.1.7. Observe that, for $k = 1$, this definition coincides with the definition of W^ϕ-differentiability stated in [ASCV06].

Let $a = (a_1, \dots, a_{2n+1-k}) \in \mathbb{R}^{2n+1-k}$ be a point and let us pick a positive constant $\delta > 0$. We define the square open δ-neighbourhood centered at a in $\mathbb{R}^{2n+1-k}$ as

$$I_\delta(a) := \{p = (p_1, \dots, p_{2n+1-k}) \in \mathbb{R}^{2n+1-k} : |p_i - a_i| < \delta \text{ for } i = 1, \dots, 2n+1-k\}.$$

We use this notation to re-state in coordinates the stronger notion of uniform intrinsic differentiability as well.

Definition 5.1.8. Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a function. Consider a point $a_0 \in U$, we say that ϕ is *uniformly intrinsically differentiable* at a_0 if there exists a $\star$-linear function $L : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$ such that

$$\lim_{r \to 0} \sup_{\substack{a,b \in I_r(a_0),\\ a \neq b}} \left\{ \frac{|\phi(b) - \phi(a) - L(a^{-1} \star b)|}{d_\phi(b, a)} \right\} = 0. \tag{5.7}$$

We say that the map ϕ is uniformly intrinsically differentiable on U if it is uniformly intrinsically differentiable at every point $a \in U$.

Remark 5.1.9. The role of the squares $I_r(a)$ could be equivalently played by the Euclidean balls of $\mathbb{R}^{2n+1-k}$. Nevetheless, we maintain the use of square neighbourhoods in continuity with [ASCV06].

Remark 5.1.10. It is immediate to verify that Definitions 5.1.6 and 5.1.8 satisfy the desired correspondences: if $\tilde{U} \subset \mathbb{W}$ is an open set, $\tilde{\phi} : \tilde{U} \to \mathbb{V}$ is a map and we fix a point $w_0 \in \tilde{U}$, then $\tilde{\phi}$ is (uniformly) intrinsically differentiable at w_0 if and only if ϕ is (uniformly) intrinsically differentiable at $a_0 = i_{\mathbb{W}}(w_0) \in U$. Moreover, in this case, $d\phi_{a_0} = i_{\mathbb{V}} \circ d\tilde{\phi}_{w_0} \circ i_{\mathbb{W}}^{-1}$. To better compare Definitions 3.5.29 and 5.1.8, we refer the reader to Remark 3.5.33 and to the fact that for every $0 < r < 1$, and for every $a_0 \in U$, by direct computations, the following inclusions holds

$$I_r(a_0) \subset i_{\mathbb{W}}(B(a_0, (2n-k)\sqrt{r}) \cap \mathbb{W}) \qquad i_{\mathbb{W}}(B(a_0, r) \cap \mathbb{W}) \subset I_r(a_0).$$

Moreover, observe that, since $\mathbb{V}$ is horizontal, and then commutative, it is isometric to $\mathbb{R}^k$ and then the homogeneous norm $\|\cdot\|$ restricted to $\mathbb{V}$ coincides with the Euclidean norm on $\mathbb{R}^k$, which we have denoted by $|\cdot|$ in (5.6) and (5.7).

By [DD20a, Proposition 3.4], any $\star$-linear function $L : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$ is identified in a natural way with a well defined $k \times (2n-k)$ matrix, which we denote by M_L, such that for every $a \in \mathbb{R}^{2n+1-k}$

$$L(a) = M_L \cdot (\pi(a))^T,$$

where $\cdot$ denotes the matrix product and π denotes the orthogonal projection from $\mathbb{R}^{2n+1-k}$ to $\mathbb{R}^{2n-k}$ on the first $2n-k$ components.

Then, if we consider an open set $U \subset \mathbb{R}^{2n+1-k}$ and a function $\phi : U \to \mathbb{R}^k$ intrinsically differentiable at a point $a \in U$, we define the *intrinsic Jacobian matrix* of ϕ at a as the matrix associated with the intrinsic differential of ϕ at a

$$D^\phi \phi(a) := M_{d\phi_a}.$$

If ϕ is intrinsically differentiable on U, we can take in consideration the function that associates every point of $a \in U$ with the intrinsic Jacobian matrix of ϕ at a

$$D^{\phi}\phi : U \to M_{k,2n-k}(\mathbb{R}).$$

Uniform intrinsic differentiability implies the continuity of the intrinsic Jacobian matrix.

Proposition 5.1.11. *[DD20a, Proposition 3.7] Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a function uniformly intrinsically differentiable on U, then the function*

$$D^{\phi}\phi : U \to M_{k,2n-k}(\mathbb{R})$$

is continuous.

Now we specialize Theorem 4.3.1 to the case of the Heisenberg group, to which we are interested to in this chapter. The content of the following result (limited to the Heisenberg group) was substantially contained already in [AS09, Theorem 4.2] (see Theorem 4.1.22 and Proposition 4.1.27), where, nevertheless, it was not explicitly stated condition (5.8), that will be useful later on for our purposes.

Theorem 5.1.12. *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be the product of the two complementary subgroups*

$$\mathbb{W} = \mathrm{span}(e_{k+1}, \dots, e_{2n+1}) \qquad \mathbb{V} = \mathrm{span}(e_1, \dots, e_k).$$

Let $\tilde{U} \subset \mathbb{W}$ be an open set, $\tilde{\phi} : \tilde{U} \to \mathbb{V}$ be a continuous function and $\Sigma = \mathrm{graph}(\tilde{\phi})$. Then the following conditions are equivalent

(i) there are $\Omega \subset \mathbb{H}^n$ open set and $f = (f_1, ..., f_k) \in C_h^1(\Omega, \mathbb{R}^k)$ such that $\Sigma = \{x \in \Omega : f(x) = 0\}$ and $J_{\mathbb{V}} f(x) = |\det([X_i f_j(x)]_{i,j=1,\dots,k})| > 0$ for all $x \in \Sigma$.

(ii) $\tilde{\phi}$ is uniformly intrinsically differentiable on $\tilde{U}$.

We report a brief sketch of the scheme of the proof of Theorem 5.1.12, following the proofs in [DD17, Theorem 3.1.1] (one can refer also to [DD20a, Theorem 4.1]. Our aim is to highlight some details that will be useful later on.

Let us start by $(i) \Rightarrow (ii)$. Consider an open set Ω in $\mathbb{H}^n$ and a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ as in (i). By Theorem 4.1.19, there exist an open set $\tilde{U} \subset \mathbb{W}$ and a unique intrinsic continuous parametrization $\tilde{\phi} : \tilde{U} \to \mathbb{V}$, that corresponds as above to a map $\phi : U \to \mathbb{R}^k$, with $U \subset \mathbb{R}^{2n+1-k}$, such that $\Sigma = \mathrm{graph}(\phi)$. By (i), at any point $a \in U$ the horizontal Jacobian matrix $J_{\mathbb{H}} f(\Phi(a))$ of f at $\Phi(a)$ is of maximum rank k and, in particular, the following $k \times k$ matrix (that represents $Df(\Phi(a))|_{\mathbb{V}}$) is invertible

$$\mathbf{X} f(\Phi(a)) := \begin{pmatrix} X_1 f_1(\Phi(a)) & \dots & X_k f_1(\Phi(a)) \\ \dots & \dots & \dots \\ X_1 f_k(\Phi(a)) & \dots & X_k f_k(\Phi(a)) \end{pmatrix}.$$

We introduce also the $k \times (2n-k)$ matrix (that represents $Df(\Phi(a))|_{\mathbb{W}}$)

$$\mathbf{Y} f(\Phi(a)) := \begin{pmatrix} X_{k+1} f_1(\Phi(a)) & \dots & X_n f_1(\Phi(a)) & Y_1 f_1(\Phi(a)) & \dots & Y_n f_1(\Phi(a)) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{k+1} f_k(\Phi(a)) & \dots & X_n f_k(\Phi(a)) & Y_1 f_1(\Phi(a)) & \dots & Y_n f_k(\Phi(a)) \end{pmatrix}.$$

By the combination of some direct computations with a Morrey-type inequality, it is proved that the parametrization ϕ is uniformly intrinsically differentiable at every point $a \in U$ and that for every $a \in U$

$$D^{\phi}\phi(a) = -(\mathbf{X} f(\Phi(a)))^{-1} \mathbf{Y} f(\Phi(a)). \tag{5.8}$$

On the other side, to prove that $(ii) \Rightarrow (i)$ we start by considering a function $\tilde{\phi} : \tilde{U} \subset \mathbb{W} \to \mathbb{V}$ that corresponds as above to a map $\phi : U \to \mathbb{R}^k$, with $U \subset \mathbb{R}^{2n+1-k}$, uniformly intrinsically differentiable on U. Thus, by applying the Whitney-type Theorem 3.2.34, it is possible to individuate a function $f \in C^1_h(\Omega, \mathbb{R}^k)$, where $\Omega \subset \mathbb{H}^n$ is an open set containing $\Phi(U)$, such that for every $a \in U$

$$f \circ \Phi(a) = 0$$

and such that the horizontal Jacobian matrix of f at every point of the intrinsic graph of ϕ, $\Phi(a)$ with $a \in U$, has the following form

$$J_{\mathbb{H}} f(\Phi(a)) = (\ \mathbb{I}_k \mid -D^{\phi}\phi(a)\ ), \tag{5.9}$$

where by $\mathbb{I}_k$ we indicate the identity matrix of dimension k. Equivalently $\mathbf{X} f(\Phi(a)) = \mathbb{I}_k$ (hence $J_{\mathbb{V}} f(\Phi(a)) = 1$) and $\mathbf{Y} f(\Phi(a)) = -D^{\phi}\phi(a)$.

Remark 5.1.13. If we consider an open set $U \subset \mathbb{R}^{2n+1-k}$ and a continuously (Euclidean) differentiable function $\phi : U \to \mathbb{R}^k$, then ϕ is uniformly intrinsically differentiable on U. It is not difficult to convince ourselves of this claim. In fact, let us consider the map

$$f : \mathbb{H}^n \to \mathbb{R}^k,\ f_i(x) := x_i - \phi_i(i_{\mathbb{W}}(\pi_{\mathbb{W}}(x))) \quad \text{for } i = 1, \dots, k.$$

Observe that $f(\text{graph}(\phi)) = 0$ and that $f \in C^1(\mathbb{H}^n, \mathbb{R}^k)$. By the continuity of its partial derivatives, and consequently of its horizontal derivatives, clearly $f \in C^1_h(\mathbb{H}^n, \mathbb{R}^k)$. Then, combining the uniqueness of the intrinsic parametrization, ensured by Theorem 4.1.19, with Theorem 5.1.12 it is immediate to conclude that ϕ is uniformly intrinsically differentiable on U.

Definition 5.1.14. Given an open set $U \subset \mathbb{R}^{2n+1-k}$ and a continuous function $\phi : U \to \mathbb{R}^k$, let us define the following family of $2n - k$ first order operators

$$W_j^{\phi} := \begin{cases} \tilde{X}_{j+k} & j = 1, \dots, n-k \\ \nabla^{\phi_i} := \partial_{\eta_i} + \phi_i \partial_{\tau} & j = n-k+1, \dots, n \quad i = j - (n-k) \\ \tilde{Y}_{j-(n-k)} & j = n+1, \dots, 2n-k. \end{cases} \tag{5.10}$$

We can identify them with vector fields of $\mathbb{R}^{2n+1-k}$ in the usual way. Notice that the first and the last $n-k$ vector fields have smooth coefficients, while the k central ones have just continuous coefficients and their form reminds to the Burger's operator.

Remark 5.1.15. The vector fields $\{W_j^{\phi}\}_{j=1,\dots,2n-k}$ defined in (5.10) are *projected vector fields* in the sense that they correspond to the projections of the vector fields of the canonical basis of the Lie algebra of $\mathbb{W}$, projected through the (Euclidean) differential of the group projection $\pi_{\mathbb{W}}$ at the points of the intrinsic graph of ϕ: for every $a \in U$

$$W_j^{\phi}(a) = \begin{cases} d_E\pi_{\mathbb{W}}(\Phi(a))(X_{j+k}(\Phi(a)) & \text{for } j = 1, \dots, n-k \\ d_E\pi_{\mathbb{W}}(\Phi(a))(Y_{j-k}(\Phi(a)) & \text{for } j = n-k+1, \dots, 2n-k \end{cases}$$

This observation was presented in [Koz15, Definition 4.2.12] (in a coordinate free manner), where a family of continuous projected vector fields analogous to (5.10) had been defined for any intrinsic Lipschitz map from $\mathbb{W}$ to $\mathbb{V}$, where $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups of a generic Carnot group $\mathbb{G}$ such that $\mathbb{G} = \mathbb{W} \rtimes \mathbb{V}$. This is a crucial point in order to extend the arguments adopted in the Heisenberg group to the settings of generic Carnot groups (see for instance [ADDDLD20]). Please refer to [Koz15, Section 4.6] or [ADDDLD20, Example 3.4] for explicit computations.

Remark 5.1.16. The family of the vector fields (5.10) was individuated also by Serapioni in [Ser17, Section 3.3], computing, in the notation of Definition 5.1.14, the limit as t goes to zero of the intrinsic difference quotients $\Delta_W\phi(w;t)$ at the points $w \in U$ of the map ϕ along a vector field W that varies in the set $\{X_{k+1},\dots,X_n,Y_1,\dots,Y_n\}$. In particular, Serapioni calls for $w \in U$ and $W \in \mathrm{Lie}(\mathbb{W})$

$$D_W\phi(w) = \lim_{t\to 0^+} \Delta_W\phi(w;t) = \lim_{t\to 0^+} \Delta_{-W}\phi(w;t) \tag{5.11}$$

the *intrinsic directional derivative of ϕ at w along W*, if the two limits in (5.11) exist and are equal. The author proved, through explicit computations, that, if the intrinsic directional derivatives $D_W\phi(w)$ exists along every $W \in \{X_{k+1},\dots,X_n,Y_1,\dots,Y_n\}$ at a point $w \in U$, then for every $j = 1,\dots,k$

$$\begin{aligned}
\left[D_{X_i}\phi(w)\right]_j &= D_{X_i}\phi_j(w) = W^{\phi}_{i-n}\phi_j(w) \quad \text{for } i = k+1,\dots,n\\
\left[D_{Y_i}\phi(w)\right]_j &= D_{Y_i}\phi_j(w) = W^{\phi}_{i+(n-k)}\phi_j(w) \quad \text{for } i = 1,\dots,k\\
\left[D_{Y_i}\phi(w)\right]_j &= D_{Y_i}\phi_j(w) = W^{\phi}_{n+i}\phi_j(w) \quad \text{for } i = k+1,\dots,n
\end{aligned}$$

The vector fields of the family in (5.10) can be individuated also through the following proposition, that can be meant as a confirm of the fact that they individuate the right candidate directions on which one could define suitable intrinsic partial derivatives of ϕ.

Proposition 5.1.17. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set. If $\phi : U \to \mathbb{R}^k$ is a continuously (Euclidean) differentiable function on U and $a \in U$, then*

$$D^{\phi}\phi(a) = \begin{pmatrix} W^{\phi}_1\phi_1(a) & \dots & W^{\phi}_{2n-k}\phi_1(a)\\ \dots & \dots & \dots\\ W^{\phi}_1\phi_k(a) & \dots & W^{\phi}_{2n-k}\phi_k(a)\end{pmatrix}. \tag{5.12}$$

Proof. Since ϕ is continuously differentiable in the Euclidean sense, by the smoothness of the group product, the graph map Φ has the same regularity. Moreover, by Remark 5.3.5, ϕ is uniformly intrinsically differentiable on U. Then in particular we can choose a function $f \in C^1(\Omega,\mathbb{R}^k)$, and then $f \in C^1_h(\Omega,\mathbb{R}^k)$, defined on an open neighbourhood Ω containing $\Phi(U)$, such that $f(\Phi(a)) = 0$ for every $a \in U$ and such that $\mathbf{X}f(\Phi(a))$ is invertible at every point $a \in U$ (for instance f can be chosen as in Remark 5.1.13). More explicitly we get that for every point $a = (x_{k+1},\dots x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau) \in U$

$$f\left(\phi_1,\dots,\phi_k,\dots,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau - \frac{1}{2}\sum_{j=1}^{k}\phi_j\eta_j\right) = 0, \tag{5.13}$$

where for $j \in \{1,\dots,k\}$ we have set $\phi_j = \phi_j(a)$. Let us first briefly expose our strategy. We fix some point $a \in U$ and we differentiate the equation (5.13) at a with respect to the variables x_{k+1}, $\dots$, x_n, η_1, $\dots$, η_k, y_{k+1}, $\dots$, y_n, τ. Then we re-order the equations that we obtain, in order to recover the derivatives of ϕ along the vector fields W^{ϕ}_j (notice that they exist in the classical sense since ϕ is continuously Euclidean differentiable). We complete the proof observing that the left hand side of (5.12) is equal to the left hand side of (5.8), and then it coincides with the intrinsic Jacobian matrix of ϕ at a.

Let us move to explicit computations. In order to simplify the notation, we will not specify that the derivatives of the components of f, the f_j, are computed at $\Phi(a) \in \Omega$ while the derivatives of the components of ϕ, the ϕ_j, are computed at $a \in U$, this clearly follows from the (Euclidean) chain rule. Moreover, we think that it is clear to which coordinates the symbol ∂_{x_i} refers, depending if it is applied to a coordinate of the map $f(x_1, \dots, x_{2n+1})$ in $\mathbb{H}^n$ or to a coordinate of the mapping $\phi(x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots y_n, \tau)$ in $\mathbb{R}^{2n+1-k}$. Let us differentiate (5.13), for $\ell = k+1, \dots, n$ and $j = 1, \dots, k$ we get

$$\partial_{x_\ell}(f_j \circ \Phi)(a) = \sum_{i=1}^{k} (\partial_{x_i} f_j)(\partial_{x_\ell}\phi_i) + \partial_{x_\ell} f_j + \partial_{x_{2n+1}} f_j \left(-\frac{1}{2} \sum_{i=1}^{k} \eta_i \partial_{x_\ell} \phi_i \right) = 0 \tag{5.14}$$

$$\partial_{y_\ell}(f_j \circ \Phi)(a) = \sum_{i=1}^{k} (\partial_{x_i} f_j)(\partial_{y_\ell}\phi_i) + \partial_{x_{\ell+n}} f_j + \partial_{x_{2n+1}} f_j \left(-\frac{1}{2} \sum_{i=1}^{k} \eta_i \partial_{y_\ell} \phi_i \right) = 0 \tag{5.15}$$

for $j, \ell = 1, \dots, k$

$$\partial_{\eta_\ell}(f_j \circ \Phi)(a) = \sum_{i=1}^{k} (\partial_{x_i} f_j)(\partial_{\eta_\ell}\phi_i) + \partial_{x_{\ell+(n-k)}} f_j + \partial_{x_{2n+1}} f_j \left(-\frac{1}{2} \sum_{i=1}^{k} (\eta_i \partial_{\eta_\ell} \phi_i + \phi_\ell) \right) = 0 \tag{5.16}$$

$$\partial_{\tau}(f_j \circ \Phi)(a) = \sum_{i=1}^{k} (\partial_{x_i} f_j)(\partial_{\tau}\phi_i) + \partial_{x_{2n+1}} f_j + \partial_{x_{2n+1}} f_j \left(-\frac{1}{2} \sum_{i=1}^{k} \eta_i \partial_{\tau} \phi_i \right) = 0. \tag{5.17}$$

now we consider for $\ell = 1, \dots, k$, the equation obtained computing (5.16)+ϕ_ℓ(5.17)

$$\begin{aligned}
&\sum_{i=1}^{k} (\partial_{x_i} f_j)(\phi_\ell \partial_\tau \phi_i + \partial_{\eta_\ell}\phi_i) + \partial_{x_{\ell+(n-k)}} f_j + \phi_\ell \partial_{x_{2n+1}} f_j \\
&- \frac{1}{2} \partial_{x_{2n+1}} f_j \left(\sum_{i=1}^{k} \left(\eta_i \partial_{\eta_\ell} \phi_i + \phi_\ell + \phi_\ell \sum_{i=1}^{k} \eta_i \partial_\tau \phi_i \right) \right) \\
= &\sum_{i=1}^{k} (\partial_{x_i} f_j)(\nabla^{\phi_\ell} \phi_i) + \partial_{x_{\ell+(n-k)}} f_j + \phi_\ell \partial_{x_{2n+1}} f_j - \frac{1}{2} \partial_{x_{2n+1}} f_j \left(\sum_{i=1}^{k} \eta_i \nabla^{\phi_\ell} \phi_i + \phi_\ell \right) = 0
\end{aligned}$$

that coincides with

$$\sum_{j=1}^{k} (X_i f_j(\Phi(a)) \nabla^{\phi_\ell} \phi_i(a) + \partial_{x_{\ell+(n-k)}} f_j + \frac{1}{2} \phi_\ell (\partial_{x_{2n+1}} f_j(\Phi(a)))$$

$$= \sum_{j=1}^{k} (X_i f_j(\Phi(a)) \nabla^{\phi_\ell} \phi_i(a) + Y_{\ell+n} f_j(\Phi(a)) = 0,$$

We have assumed that that $\det([X_i f_j(\Phi(a))]_{i,j=1,\dots,k}) \neq 0$ and then that this system can be solved, thus, for every $\ell = 1, \dots, k$, one can explicitly compute the column

$$\nabla^{\phi_j} \phi(a) = \begin{pmatrix} \nabla^{\phi_\ell} \phi_1(a) \\ \dots \\ \nabla^{\phi_\ell} \phi_k(a) \end{pmatrix} = -(\mathbf{X} f(\Phi(a)))^{-1} \begin{pmatrix} Y_{\ell+n} f_1(\Phi(a)) \\ \dots \\ Y_{\ell+n} f_k(\Phi(a)) \end{pmatrix}.$$

Now, proceeding in an analogous way computing for $\ell = k+1, \dots, n$, the equations (5.14)$-\frac{1}{2}\eta_{\ell-k}$(5.17) and (5.15)$+\frac{1}{2}\phi_{\ell-k}$(5.17), we obtain respectively

$$\sum_{i=1}^{k} X_i f_j(\Phi(a)) \tilde{X}_\ell \phi_i(a) + X_\ell f_j(\Phi(a)) = 0$$

and

$$\sum_{i=1}^{k} X_i f_j(\Phi(a))\tilde{Y}_\ell \phi_i(a) + Y_\ell f_j(\Phi(a)) = 0.$$

Thus, for $\ell = k+1, \dots, n$, we get

$$\tilde{X}_\ell \phi = \begin{pmatrix} \tilde{X}_\ell \phi_1(a) \\ \dots \\ \tilde{X}_\ell \phi_k(a) \end{pmatrix} = -(\mathbf{X}f(\Phi(a)))^{-1} \begin{pmatrix} X_\ell f_1(\Phi(a)) \\ \dots \\ X_\ell f_k(\Phi(a)) \end{pmatrix}$$

and

$$\tilde{Y}_\ell \phi = \begin{pmatrix} \tilde{Y}_\ell \phi_1(a) \\ \dots \\ \tilde{Y}_\ell \phi_k(a) \end{pmatrix} = -(\mathbf{X}f(\Phi(a)))^{-1} \begin{pmatrix} Y_\ell f_1(\Phi(a)) \\ \dots \\ Y_\ell f_k(\Phi(a)) \end{pmatrix}.$$

□

5.2 One-codimensional intrinsic regular graphs

Now we resume the state of the art about the case when $\mathbb{V}$ is of dimension one. Then, according to the previous stated correspondences, we consider an open set $U \subset \mathbb{R}^{2n}$ and a map $\phi : U \to \mathbb{R}$. In this case the definition of intrinsic differentiability for ϕ equals the definition of W^ϕ-differentiability stated in [ASCV06]. The intrinsic Jacobian matrix at a point $a \in U$, $D^\phi\phi(a)$, is a $(2n-1)$-vector that we call the *intrinsic gradient* of ϕ at a, and we denote it by $\nabla^\phi\phi(a)$. Relevant results have been presented by Ambrosio, Bigolin, Serra Cassano and Vittone [ASCV06, BSC10a, BSC10b]; we resume them in the following theorem.

Theorem 5.2.1. *Let $U \subset \mathbb{R}^{2n}$ be an open set and let $\phi : U \to \mathbb{R}$ be a continuous function. Then, the following conditions are equivalent.*

(i) ϕ is uniformly intrinsically differentiable on U.

(ii) There exists $\mathfrak{w} \in C^0(U, \mathbb{R}^{2n-1})$ such that

$$(W_1^\phi \phi, \dots, W_{2n-1}^\phi \phi) = \mathfrak{w}$$

in distributional sense on U.

(iii) There exists $\mathfrak{w} \in C^0(U, \mathbb{R}^{2n-1})$ and a family of functions $\{\phi_\varepsilon\}_{\varepsilon>0} \subset C^1(U)$ such that

$$\phi_\varepsilon \to \phi \quad \text{and} \quad \nabla^{\phi_\varepsilon}\phi_\varepsilon \to \mathfrak{w}$$

uniformly on every open set $U' \Subset U$ as ε goes to zero.

Remark 5.2.2. Notice in particular that in point (*ii*) of Theorem 5.2.1, the intrinsic gradient of ϕ is considered in distributional sense, i.e. the derivatives $W_j^\phi \phi$ are interpreted in distributional sense for $j = 1, \dots, 2n-1$. This is immediate for $j = 1, \dots, n-1$ and $j = n+1, \dots, 2n-1$. In order to see $W_n^\phi \phi$ as a distribution, it is enough to exploit the Leibniz rule, and to rewrite

$$W_n^\phi \phi = \partial_\eta \phi + \phi \partial_\tau \phi = \partial_\eta \phi + \partial_\tau \left(\frac{\phi^2}{2} \right).$$

Remark 5.2.3. Please refer to [ADDDLD20, Remark 4.14] for a careful discussion about point (*iii*) of Theorem 5.2.1. In fact, the authors noticed an imprecision in the proof of (*i*) $\Rightarrow$ (*ii*) of [ASCV06, Theorem 5.1]. Nevertheless the proof can be fixed adapting the approximation argument explained in the proof of [MV12, Theorem 1.2].

In [SC16] (relying on results of [ASCV06, BSC10a, BCSC15]), the author prove two further relevant characterizations, that we resume in the following theorem.

Theorem 5.2.4. *[SC16, Theorem 4.95] Let $U \subset \mathbb{R}^{2n}$ be an open set and let $\phi : U \to \mathbb{R}$ be a function. The following conditions are equivalent:*

(i) ϕ is uniformly intrinsically differentiable on U.

(ii) $\phi \in C^0(U)$ and for every $a \in U$, for every $j \in \{1, \dots, 2n-1\}$, there exists $\partial^{\phi_j}\phi(a)$, i.e. a real number $\partial^{\phi_j}\phi(a)$ such that for every $\gamma^j : (-\delta, \delta) \to \Omega$ integral curve of W_j^ϕ with $\gamma^j(0) = a$, the limit $\lim_{t\to 0} \frac{\phi(\gamma^j(t))-\phi(\gamma^j(0))}{t}$ exists, it is equal to $\partial^{\phi_j}\phi(a)$ and the map $\partial^{\phi_j}\phi : U \to \mathbb{R}$ is continuous.

(iii) ϕ is intrinsically differentiable on U and the map $\nabla^\phi\phi : U \to \mathbb{R}^{2n-1}$ is continuous.

In particular, notice that Theorems 5.2.1 and 5.2.4 provide characterizations of the uniform intrinsic differentiability of a map ϕ in terms of different suitable notions of continuous weak solution $\phi : U \to \mathbb{R}$ of the first order non-linear system

$$(W_1^\phi\phi, \dots, W_{2n-1}^\phi\phi) = \mathfrak{w} \quad \text{on } U, \tag{5.18}$$

for a prescribed continuous map $\mathfrak{w} \in C^0(U, \mathbb{R}^{2n-1})$. In fact, condition (*ii*) of Theorem 5.2.4, can be rephrased, according to the literature, saying that ϕ is a *broad solution* of the system (5.18).
A generalization of these results to one-codimensional uniformly intrinsically differentiable graphs in Carnot groups of step 2 has been proved by Di Donato [DD20a]. The goal of the current chapter is to generalize Theorems 5.2.1 and 5.2.4 in the Heisenberg group to the case when $1 \leq \dim(\mathbb{V}) \leq n$. In particular our main results will be presented in Proposition 5.3.21 and Theorem 5.3.24.
Characterizations analogous to the ones in Theorem 5.2.4 have been proved for intrinsic Lipschitz functions. A result in the Heisenberg group about the existence of a smooth approximation for intrinsic Lipschitz maps with target homogeneous subgroup of dimension 1, along with their intrinsic gradient, (analogous to (*iii*) of Theorem 5.2.1) has been proved by Citti, Manfredini, Pinamonti and Serra Cassano [CMPSC14, Theorem 1.7]. On the other side, in [BCSC15] Bigolin, Caravenna and Serra Cassano have considered suitably defined weak continuous solutions $\phi : U \to \mathbb{R}$ of the system (5.18), assuming now that $\mathfrak{w}$ is not a continuous function any more, but just a bounded map $\mathfrak{w} \in L^\infty(U, \mathbb{R}^{2n-1})$. We summarize in the following theorem the results of [BCSC15]. In order to do this, we need to precise the distinction between the space $L^\infty(U, \mathbb{R}^{2n-1})$ and the space of all the possible representatives $\mathfrak{L}^\infty(U, \mathbb{R}^{2n-1})$, i.e. the space of measurable bounded functions from U to $\mathbb{R}^{2n-1}$.

Theorem 5.2.5. *[BCSC15, Theorem 1.2] Let $U \subset \mathbb{R}^{2n}$ be an open set and let $\phi : U \to \mathbb{R}$ be a continuous function. The following conditions are equivalent*

(i) ϕ is intrinsic locally Lipschitz.

(ii) *There exists* $\mathfrak{w} \in L^\infty(U, \mathbb{R}^{2n-1})$ *such that*

$$(W_1^\phi \phi, \dots, W_{2n-1}^\phi \phi) = \mathfrak{w} \tag{5.19}$$

in distributional sense on U.

(iii) ϕ *is a broad solution of system*

$$(W_1^\phi \phi, \dots, W_{2n-1}^\phi \phi) = \mathfrak{w}$$

on U*, i.e. there exists a Borel function* $\hat{\mathfrak{w}} \in \mathfrak{L}^\infty(U, \mathbb{R}^{2n-1})$ *such that:*

(iii$_1$*)* $\mathfrak{w}(a) = \hat{\mathfrak{w}}(a)$ *for* $\mathcal{L}^{2n}$*-a.e.* $a \in U$*,*

(iii$_2$*)* *for every continuous vector field* W_j^ϕ *(for* $j = 1, \dots, 2n-1$*) and for every integral curve* $\gamma^j : (-\delta, \delta) \to U$ *of* W_j^ϕ*,* $\phi \circ \gamma^j$ *is absolutely continuous and satisfies for a.e.* $s \in (-\delta, \delta)$

$$\frac{d}{ds}\phi(\gamma^j(s)) = \hat{\mathfrak{w}}_i(\gamma^j(s)).$$

A very deep correlated investigation about the weak continuous solutions of the system (5.19), from many different points of view, has been carried on also by Alberti, Bianchini and Caravenna in [ABC16a] and [ABC16b]. They approach system (5.19) as a scalar balance law with quadratic flux and bounded source term. In fact, the authors reduce the study of the intrinsic gradient, and consequently of the system (5.19), to the study of the central equation involving the Burger's type first order operator where the non-linearity concentrates, that is

$$W_n^\phi \phi = \partial_\eta \phi + \phi \partial_\tau \phi = \mathfrak{w}_n \in L^\infty(U, \mathbb{R}) \tag{5.20}$$

which in distributional terms reads as

$$\partial_\eta \phi + \partial_\tau \left(\frac{\phi^2}{2}\right) = \omega,$$

where ω is a prescribed map in $L^\infty(U, \mathbb{R})$. In particular, the main results of [BCSC15, ABC16a, ABC16b] follow from the comparison, in respect with (5.20), between the Eulerian viewpoint, obtained interpreting the equation in distributional sense, the broad viewpoint (as in *(iii)* of Theorem 5.2.5), obtained reducing the equation to an infinite dimensional system of ODEs along characteristics, and the Lagrangian viewpoint, obtained reducing the equation to an infinite dimensional system of ODEs along a chosen family of characteristics, which is called a Lagrangian parametrization.
A partial generalization of Theorem 5.2.5 to one-codimensional intrinsic Lipschitz graphs in Carnot groups of step 2 has been provided by Di Donato in [DD20b]. How one can imagine, one of the main tools that permits to generalize results concerning intrinsic differentiability to results about intrinsic Lipschitz continuity is Theorem 3.6.4. This is why we can unbalance ourselves to foresee that the Rademacher-type result recently presented by Vittone in [Vit20] will soon allow to extend the results of the next section to the case of intrinsic Lipschitz functions (or, equivalently, to extend Theorem 5.2.5, at least partially, to the case when $1 < \dim(\mathbb{V}) \leq n$).

5.3 Characterizations of uniform intrinsic differentiability

This section is devoted to the extension of the characterizations presented in Theorems 5.2.1 and 5.2.4 to the case when $1 < \dim(\mathbb{V}) \leq n$. In particular, in this section we present the results of [Cor19].

5.3.1 Uniform intrinsic differentiability implies the existence of a local approximation

We start by proving that the uniform intrinsic differentiability of a map ϕ implies the existence of a suitable approximating family of Euclidean C^1-regular functions at any point of the domain.

Proposition 5.3.1. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a function uniformly intrinsically differentiable on U. Then, for every $a \in U$ there are a number $\delta = \delta(a) > 0$ such that $I_\delta(a) \Subset U$ and a family of functions $\{\phi_\varepsilon\}_{\varepsilon>0} \in C^1(\overline{I_\delta(a)}, \mathbb{R}^k)$ such that*

$$\phi_\varepsilon \to \phi,$$
$$D^{\phi_\varepsilon}\phi_\varepsilon \to D^\phi \phi$$

uniformly on $\overline{I_\delta(a)}$ as $\varepsilon \to 0$.

Proof. The proof mirrors the one of [FSSC03b, Theorem 2.1]. Without any loss of generality, one can assume that $a = 0$ and $\Phi(a) = 0$. By Theorem 5.1.12, any uniformly intrinsically differentiable function ϕ locally parametrizes a ℍ-regular submanifold, hence there exist $r > 0$ and a function $f \in C^1_h(B(0,r), \mathbb{R}^k)$ such that

$$f \circ \Phi(w) = 0 \text{ for every } w \in \overline{I_{\bar\delta}(0)},$$

where $\bar\delta > 0$ is small enough to have the inclusion $\Phi(\overline{I_{\bar\delta}(0)}) \subset B(0,r)$. Moreover, again by Theorem 5.1.12, the horizontal Jacobian matrix of f has rank k and, in particular, we can assume that $\det((\mathbf{X}f)(y)) > 0$, for every $y \in B(0,r')$, for some $r' \leq r$. Then we consider the map $f : B(0,r') \to \mathbb{R}^k,\ x \longmapsto (f_1(x), \ldots, f_k(x))$ and an Euclidean Friedrichs' mollifier ρ_ε. For every $0 < \varepsilon < \operatorname{dist}(B(0,r'), \mathbb{H}^n \setminus B(0,r))$, we convolve the components of the function f with ρ_ε and we set

$$f_\varepsilon : B(0,r') \to \mathbb{R}^k,\ x \longmapsto (f_{\varepsilon,1}(x), \ldots, f_{\varepsilon,k}(x)),$$

where $f_{\varepsilon,i}(x) := f_i * \rho_\varepsilon$ for $i = 1, \ldots, k$. Since the f_i are continuous mappings we know that the maps $f_{\varepsilon,i}$ converge uniformly on $B(0,r')$ to f_i, for any $i = 1, \ldots, k$, as ε goes to zero, hence the f_ε converge uniformly to f. In this proof, in order to have a simpler notation, we set $X_j = Y_{j-(n-k)}$ for $j = n-k+1 \ldots, 2n-k$. We want to prove that for $i = 1, \ldots, k$ and $j = 1, \ldots, 2n-k$ the derivatives $X_j f_{\varepsilon,i}$ uniformly converge to $X_j f_i$ on $B(0,r')$ as ε goes to zero. Surely $(X_j f_i) * \rho_\varepsilon$ converge to $X_j f_i$ uniformly on $B(0,r')$, as ε goes to zero. Then we can write

$$X_j f_{\varepsilon,i} = (X_j f_i) * \rho_\varepsilon - ((X_j f_i) * \rho_\varepsilon - X_j f_{\varepsilon,i}),$$

and in order to conclude our claim it is enough to prove that $(X_j f_i) * \rho_\varepsilon - X_j f_{\varepsilon,i}$ go uniformly to zero as ε goes to zero. We report the prove for $i = 1,\ j = 1$, for other choices of i and j the argument works in analogous way. Consider a point $x \in B(0,r')$, then

$$(X_1 f_1) * \rho_\varepsilon(x) - X_1 f_{\varepsilon,1}(x) =$$

$$
\begin{aligned}
&= \int_{|x-y|<\varepsilon} \left(\partial_{y_1} - \frac{1}{2} y_{n+1} \partial_{y_{2n+1}}\right) f_1(y) \rho_\varepsilon(x-y) dy \\
&\quad - \left(\partial_{x_1} - \frac{1}{2} x_{n+1} \partial_{x_{2n+1}}\right) \int_{|x-y|<\varepsilon} f_1(y) \rho_\varepsilon(x-y) dy
\end{aligned}
$$

integrating by parts,

$$
\begin{aligned}
&= \int_{|x-y|<\varepsilon} f_1(y) \left(\partial_{x_1} - \frac{1}{2} y_{n+1} \partial_{x_{2n+1}}\right) \rho_\varepsilon(x-y) dy \\
&\quad - \int_{|x-y|<\varepsilon} f_1(y) \left(\partial_{x_1} - \frac{1}{2} x_{n+1} \partial_{x_{2n+1}}\right) \rho_\varepsilon(x-y) dy \\
&= \int_{|x-y|<\varepsilon} f_1(y) \left(-\frac{1}{2} y_{n+1} \partial_{x_{2n+1}} + \frac{1}{2} x_{n+1} \partial_{x_{2n+1}}\right) \rho_\varepsilon(x-y) dy \\
&= \int_{|x-y|<\varepsilon} (f_1(y) - f_1(x)) \left(-\frac{1}{2} y_{n+1} \partial_{x_{2n+1}} + \frac{1}{2} x_{n+1} \partial_{x_{2n+1}}\right) \rho_\varepsilon(x-y) dy + \\
&\quad + \ f_1(x) \int_{|x-y|<\varepsilon} \left(-\frac{1}{2} y_{n+1} \partial_{x_{2n+1}} + \frac{1}{2} x_{n+1} \partial_{x_{2n+1}}\right) \rho_\varepsilon(x-y) dy \\
&= I_1(x) + f_1(x) I_2(x).
\end{aligned}
$$

First, we consider $I_2(x)$. We change the sign and we put the derivative on the variable y,

$$
\begin{aligned}
I_2(x) &= \int_{|x-y|<\varepsilon} \frac{1}{2}(-y_{n+1} + x_{n+1}) \partial_{x_{2n+1}} \rho_\varepsilon(x-y) dy \\
&= -\int_{|x-y|<\varepsilon} \frac{\partial}{\partial y_{2n+1}} \left(\frac{1}{2}(-y_{n+1} + x_{n+1}) \rho_\varepsilon(x-y)\right) dy = 0
\end{aligned}
$$

since the support of $y \longmapsto \frac{1}{2}(-y_{n+1} + x_{n+1}) \rho_\varepsilon(x-y)$ is contained in $B(x, \varepsilon)$. If we call w_1 the modulus of continuity of f_1, we get

$$
\begin{aligned}
|I_1(x)| &= \left| \int_{|x-y|<\varepsilon} (f_1(y) - f_1(x)) \left(\frac{1}{2}(-y_{n+1} + x_{n+1})\right) \frac{\partial}{\partial y_{2n+1}} \rho_\varepsilon(x-y) dy \right| \\
&\leq w_1(\varepsilon) \frac{1}{2} \varepsilon \max_{|x-y|\leq\varepsilon} \left| \frac{\partial}{\partial y_{2n+1}} \rho_\varepsilon(x-y) \right| C \varepsilon^{2n+1} \to 0
\end{aligned}
$$

when $\varepsilon \to 0$. This concludes the proof, then $X_j f_{\varepsilon,i} \to X_j f_i$ uniformly on $B(0, r')$ as $\varepsilon \to 0$.

To summarize, up to now, we have built a family of functions $\{f_\varepsilon\}_{\varepsilon>0} \in C^1(B(0, r'), \mathbb{R}^k)$ such that

$$
\begin{gathered}
f_\varepsilon \to f \\
J_{\mathbb{H}} f_\varepsilon \to J_{\mathbb{H}} f,
\end{gathered}
$$

uniformly on $B(0, r')$ as ε goes to zero. Now we want to use the (Euclidean) implicit function theorem on the mappings f_ε in order to obtain a family of functions $\phi_\varepsilon \in C^1(\overline{I_\delta(0)})$ such that $\phi_\varepsilon \to \phi$ uniformly on $\overline{I_\delta(0)}$, for some δ suitably smaller that $\bar{\delta}$. For appropriately small $h_1, \ldots, h_k$, δ, the following map γ

$$
\begin{aligned}
&\gamma(s_1, \ldots, s_k, x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau) := \\
&= (0, \ldots, 0, x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau)(s_1 e_1) \ldots (s_k e_k) \\
&= (s_1, \ldots, s_k, x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau - \frac{1}{2} \sum_{j=1}^{k} s_j \eta_j)
\end{aligned}
$$

is a diffeomorphism from a neighbourhood $[-h_1,h_1]\times,\dots,\times[-h_k,h_k]\times\overline{I_\delta(0)}$ onto a neighbourhood of $0\in\mathbb{R}^{2n+1}$. We set the two compositions on $[-h_1,h_1]\times,\dots,\times[-h_k,h_k]\times\overline{I_\delta(0)}$

$$\begin{aligned}&g_\varepsilon(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau):=\\&f_\varepsilon(\gamma(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))\end{aligned}$$

and

$$\begin{aligned}&g(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau):=\\&f(\gamma(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)).\end{aligned}$$

Let us compute, for every $i\in\{1,\dots,k\}$,

$$\begin{aligned}&\frac{\partial g_\varepsilon}{\partial s_i}(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\\
=&\frac{\partial f_\varepsilon}{\partial s_i}(\gamma(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))=\\
=&\frac{\partial f_\varepsilon}{\partial s_i}\left(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau-\frac{1}{2}\sum_{j=1}^k s_j\eta_j\right)\\
=&\begin{pmatrix}\frac{\partial f_{\varepsilon,1}}{\partial s_i}(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau-\frac{1}{2}\sum_{j=1}^k s_j\eta_j)\\ \dots\\ \frac{\partial f_{\varepsilon,k}}{\partial s_i}(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau-\frac{1}{2}\sum_{j=1}^k s_j\eta_j)\end{pmatrix}\\
=&\begin{pmatrix}\frac{\partial f_{\varepsilon,1}}{\partial x_i}-\frac{1}{2}\eta_i\frac{\partial f_{\varepsilon,1}}{\partial x_{2n+1}}\\ \dots\\ \frac{\partial f_{\varepsilon,k}}{\partial x_i}-\frac{1}{2}\eta_i\frac{\partial f_{\varepsilon,k}}{\partial x_{2n+1}}\end{pmatrix}\\
=&\begin{pmatrix}X_i f_{\varepsilon,1}(\gamma(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))\\ \dots\\ X_i f_{\varepsilon,k}(\gamma(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))\end{pmatrix}.\end{aligned}$$

Now, if we set

$$w=(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\in[-h_1,h_1]\times,\dots,\times[-h_k,h_k]\times\overline{I_\delta(0)},$$

we can summarize our computation as

$$\begin{aligned}\begin{pmatrix}\frac{\partial g_{\varepsilon,1}}{\partial s_1}(w)&\dots&\frac{\partial g_{\varepsilon,1}}{\partial s_k}(w)\\ \dots&\dots&\dots\\ \frac{\partial g_{\varepsilon,k}}{\partial s_1}(w)&\dots&\frac{\partial g_{\varepsilon,k}}{\partial s_k}(w)\end{pmatrix}&=\begin{pmatrix}\frac{\partial f_{\varepsilon,1}}{\partial s_1}(\gamma(w))&\dots&\frac{\partial f_{\varepsilon,1}}{\partial s_k}(\gamma(w))\\ \dots&\dots&\dots\\ \frac{\partial f_{\varepsilon,k}}{\partial s_1}(\gamma(w))&\dots&\frac{\partial f_{\varepsilon,k}}{\partial s_k}(\gamma(w))\end{pmatrix}\\
&=\begin{pmatrix}X_1 f_{\varepsilon,1}(\gamma(w))&\dots&X_k f_{\varepsilon,1}(\gamma(w))\\ \dots&\dots&\dots\\ X_1 f_{\varepsilon,k}(\gamma(w))&\dots&X_k f_{\varepsilon,k}(\gamma(w))\end{pmatrix}\end{aligned}$$

and, according to the previously introduced notation, we have denoted this matrix by $\mathbf{X}f_\varepsilon(\gamma(w))$. By the previously proved uniform convergence of the horizontal derivatives, we know that $\mathbf{X}f_\varepsilon\to\mathbf{X}f$ on $B(0,r')$ as $\varepsilon\to0$. Hence if the parameters $h_1,\dots,\ h_k,\ \delta$ are small enough to guarantee that $\gamma([-h_1,h_1]\times\dots\times[-h_k,h_k]\times\overline{I_\delta(0)})\subset B(0,r')$, for every point

$$(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\in\overline{I_\delta(0)},$$

the function

$$(s_1,\dots,s_k)\in[-h_1,h_1]\times\dots\times[-h_k,h_k]\to g(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)$$

is a diffeomorphism, since

$$\det(\mathbf{X}f(\gamma(w))>0$$

for $w=(s_1,\dots,s_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\in[-h_1,h_1]\times\dots\times[-h_k,h_k]\times\overline{I_\delta(0)}$. By the uniform convergence of the horizontal derivatives, at least for ε small enough, we can suppose that

$$\det(\mathbf{X}f_\varepsilon(\gamma(w))\neq 0$$

for every $w\in[-h_1,h_1]\times\dots\times[-h_k,h_k]\times\overline{I_\delta(0)}$.
Now, we can apply to the maps $g_\varepsilon=f_\varepsilon\circ\gamma:[-h_1,h_1]\times\dots\times[-h_k,h_k]\times\overline{I_\delta(0)}\to\mathbb{R}^k$ the Euclidean implicit function theorem and we get that for every $\varepsilon>0$ there exists a function

$$\phi_\varepsilon:\overline{I_\delta(0)}\to[-h_1,h_1]\times\dots\times[-h_k,h_k]\subset\mathbb{R}^k,$$

$$(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\to(\phi_{\varepsilon,1},\dots,\phi_{\varepsilon,k}).$$

with $\phi_\varepsilon\in C^1(\overline{I_\delta(0)},\mathbb{R}^k)$, where we have denoted for $i=1,\dots,k$,

$$\phi_{\varepsilon,i}=\phi_{\varepsilon,i}(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau),$$

such that for every $(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\in\overline{I_\delta(0)}$,

$$\begin{aligned}
&g_\varepsilon(\phi_{\varepsilon,1},\dots,\phi_{\varepsilon,k},x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\\
=&f_\varepsilon(\gamma(\phi_{\varepsilon,1},\dots,\phi_{\varepsilon,k},x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))\\
=&f_\varepsilon(\phi_{\varepsilon,1},\dots,\phi_{\varepsilon,k},x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau-\frac{1}{2}\sum_{j=1}^k\eta_j\phi_{\varepsilon,j}))\\
=&f_\varepsilon(\Phi_\varepsilon(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))
\end{aligned}$$

where Φ_ε is the graph map of ϕ_ε, hence for every ε,

$$f_\varepsilon(\mathrm{graph}(\phi_\varepsilon))=0.$$

Now we want to prove that the family of maps $\phi_\varepsilon\in C^1(\overline{I_\delta(0)},\mathbb{R}^k)$ converges uniformly to ϕ on $\overline{I_\delta(0)}$ as $\varepsilon\to 0$. We proceed assuming by contradiction that there exist $\sigma>0$, a sequence

$$((x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h))_{h\in\mathbb{N}}\subset\overline{I_\delta(0)}$$

and a sequence $(\varepsilon_h)_{h\in\mathbb{N}}$, converging to zero as h going to infinity, such that

$$|\phi_{\varepsilon_h}(x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h)-\phi(x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h)|\geq\sigma.$$

By compactness, we suppose that, up to a subsequence, we have the convergences

$$(x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h)\to(\bar x_{k+1},\dots,\bar x_n,\bar\eta_1,\dots,\bar\eta_k,\bar y_{k+1},\dots,\bar y_n,\bar\tau)\in\overline{I_\delta(0)}$$

and that

$$\phi_{\varepsilon_k}(x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h)\to(\phi_{0,1},\dots,\phi_{0,k})\in[-h_1,h_1]\times\dots\times[-h_k,h_k]$$

as h goes to infinity. Hence

$$\begin{aligned}0 &= g_{\varepsilon_h}(\phi_{\varepsilon_h,1},\dots,\phi_{\varepsilon_h,k},x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h)\\ &= f_{\varepsilon_h}(\gamma(\phi_{\varepsilon_h,1},\dots,\phi_{\varepsilon_h,k},x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h))\\ &\to f(\gamma(\phi_{0,1},\dots,\ \phi_{0,k},\bar{x}_{k+1},\dots,\bar{x}_n,\bar{\eta}_1,\dots,\bar{\eta}_k,\bar{y}_{k+1},\dots,\bar{y}_n,\bar{\tau}))\end{aligned}$$

as h goes to ∞.
But at the beginning we had chosen f such that for every point $(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau)\in\overline{I_\delta(0)}$,

$$\begin{aligned}&f(\Phi(x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))\\ =&f(\gamma(\phi_1,\dots,\phi_k,x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau))=0.\end{aligned}$$

Since, once fixed the map f, the intrinsic parametrization ϕ is unique, then necessarily

$$\phi_{0,j}=\phi_j(\bar{x}_{k+1},\dots,\bar{x}_n,\bar{\eta}_1,\dots,\bar{\eta}_k,\bar{y}_{k+1},\dots,\bar{y}_n,\bar{\tau})$$

for any $j=1,\dots,k$. At the same time this is not possible, since we have assumed that ϕ_{ε_k} and ϕ keep a distance of at least σ on the sequence $(x^h_{k+1},\dots,x^h_n,\eta^h_1,\dots,\eta^h_k,y^h_{k+1},\dots,y^h_n,\tau^h)$, $h\in\mathbb{N}$, and then this has to be valid also at the limit. Then we have reached a contradiction and we can conclude that ϕ_ε converges to ϕ uniformly on $\overline{I_\delta(0)}$ as $\varepsilon\to 0$.

Let us now prove the uniform convergence of the intrinsic Jacobian matrices of the maps ϕ_ε. Observe that for every $\varepsilon>0$, ϕ_ε is the intrinsic parametrization of f_ε, hence we know that for every $b\in\overline{I_\delta(0)}$,

$$D^{\phi_\varepsilon}\phi_\varepsilon(b)=-(\mathbf{X}f_\varepsilon(\Phi_\varepsilon(b)))^{-1}(\mathbf{Y}f_\varepsilon(\Phi_\varepsilon(b))).$$

Since we have proved that for every $i=1,\dots,k$ and $j=1,\dots,2n-k$ the derivatives $X_jf_{\varepsilon,i}$ converge uniformly to X_jf_i on $B(0,r')$ as $\varepsilon\to 0$, and since we are sure that $\det((\mathbf{X}f_\varepsilon(y))$ is far from being zero for every $y\in\gamma([-h_1,h_1]\times\dots\times[-h_k,h_k]\times\overline{I_\delta(0)})\subset B(0,r')$, we get that $D^{\phi_\varepsilon}\phi_\varepsilon(b)=-(\mathbf{X}f_\varepsilon(\Phi_\varepsilon(b)))^{-1}(\mathbf{Y}f_\varepsilon)(\Phi_\varepsilon(b))$ converge to $-(\mathbf{X}f(\Phi(b)))^{-1}(\mathbf{Y}f(\Phi(b)))$ uniformly on $\overline{I_\delta(0)}$. Now, to conclude the proof, it is enough to observe that, according to (5.8) of the proof of Theorem 5.1.12,

$$-(\mathbf{X}f(\Phi(b)))^{-1}(\mathbf{Y}f(\Phi(b)))=D^\phi\phi(b).$$

□

5.3.2 The existence of an approximation implies the existence of a family of exponential maps

We need to give meaning to the action of the vector fields W^ϕ_j on the components of ϕ. In order to do this, we consider the behaviour of ϕ along the integral curves of the vector fields W^ϕ_j. When we assume that the map ϕ is only continuous, the integral curves of the vector fields W^ϕ_j for $j=n-k+1,\dots,n$, are not unique in general. Nevertheless, once we fix an initial point, the existence of at least one of these integral curves is ensured by the Peano's Theorem. The reader can refer for instance to [Mus05, Theorem 1]. For this reason, the authors in [ASCV06] introduced the notion of a family of exponential maps. We generalize the definition to our setting.

Definition 5.3.2 (Family of exponential maps)**.** Let $U\subset\mathbb{R}^{2n+1-k}$ be an open set and let $\phi:U\to\mathbb{R}^k$ be a continuous function. We assume that for any $a\in U$ there exist

$0 < \delta_2 < \delta_1$ such that for each $j = 1, \dots, 2n-k$ there exists a map

$$\gamma^j : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)} \to \overline{I_{\delta_1}(a)}$$
$$(s, b) \longmapsto \gamma_b^j(s)$$

such that:

(i) $\gamma_b^j := \gamma^j(\cdot, b) \in C^1([-\delta_2, \delta_2], \overline{I_{\delta_1}(a)})$ for any $b \in \overline{I_{\delta_2}(a)}$;

(ii) $\dot\gamma_b^j(s) = W_j^\phi(\gamma_b^j(s))\ \forall s \in [-\delta_2, \delta_2]$, $\gamma_b^j(0) = b$.

(iii) There exist $k \times (2n-k)$ continuous functions $\omega_{i,j} : U \to \mathbb{R}$, with $i = 1, \dots, k$, $j = 1, \dots, 2n-k$, such that for each $s \in [-\delta_2, \delta_2]$,

$$\phi_i(\gamma_b^j(s)) - \phi_i(\gamma_b^j(0)) = \int_0^s \omega_{i,j}(\gamma_b^j(r))dr. \tag{5.21}$$

From now on, $\gamma_b^j(s)$ will also be denoted by $\exp_a(sW_j^\phi)(b)$. The family of mappings $\{\gamma^j\}_{j=1,\dots,2n-k}$ is called a *family of exponential maps at a*.

Remark 5.3.3. Because of the left invariance of the vector fields W_i^ϕ for $j = 1, \dots, n-k, n+1, \dots, 2n-k$, one must have that for $j \in \{1, \dots, n-k\}$

$$\exp_a(sW_j^\phi)(b) = b \star i_{\mathbb{W}}(\exp(sX_{j+k})) = b \star s\mathbf{e}_j$$

and for $j \in \{n+1, \dots, 2n-k\}$

$$\exp_a(sW_j^\phi)(b) = b \star i_{\mathbb{W}}(\exp(sY_{j-(n-k)})) = b \star s\mathbf{e}_j,$$

where, by $(\mathbf{e}_1, \dots, \mathbf{e}_{2n-k}, \mathbf{e}_{2n-k+1})$ we denote the canonical basis of $\mathbb{R}^{2n-k+1}$.

Remark 5.3.4. If we have a family of exponential maps at a, for any $c \in \mathbb{R}^{2n-2k}$ such that $c = (c_1, \dots, c_{n-k}, c_{n+1}, \dots, c_{2n-k})$, there is an exponential map at a also for the vector field

$$V_c = \sum_{j=k+1}^{n} c_{j-k}\tilde X_j + c_{j+n-k}\tilde Y_j = \sum_{j=k+1}^{n} c_{j-k}W_{j-k}^\phi + c_{j+n-k}W_{j-k+n}^\phi$$

in the sense that there is a continuous map $\gamma_c : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)} \to \overline{I_{\delta_1}(a)}$ and there are k continuous maps $\omega_{c,i} : U \to \mathbb{R}$, for any $i \in \{1, \dots, k\}$, such that for every $s \in [-\delta_2, \delta_2]$ and $b \in \overline{I_{\delta_2}(a)}$,

$$\begin{aligned}
\dot\gamma_c(s, b) = V_c(\gamma_c(s, b)) &= \sum_{j=k+1}^{n} c_{j-k}\tilde X_j(\gamma_c(s, b)) + c_{j+n-k}\tilde Y_j(\gamma_c(s, b))\\
\gamma_c(0, b) &= b\\
\phi_i(\gamma_c(s, B)) - \phi_i(\gamma_c(0, b)) &= \int_0^s \omega_{c,i}(\gamma_c(r, b))dr.
\end{aligned}$$

For instance, one can take $\exp_a(sV_c)(b) = \gamma_c(s, b) := b\star\left(\sum_{j=1}^{n-k} s(c_j\mathbf{e}_j) + \sum_{j=n+1}^{2n-k} s(c_j\mathbf{e}_j)\right)$ and $\omega_{c,i} = \sum_{j=1}^{n-k} c_j\omega_{i,j} + \sum_{j=n+1}^{2n-k} c_j\omega_{i,j}$.

Remark 5.3.5. If the function ϕ is continuously differentiable (in the Euclidean sense) on U, there exists a family of exponential maps at every point $a \in U$. In fact, once we fix an initial point $b \in U$, by the Cauchy-Lipschitz theorem, for any $j = 1, \dots, 2n-k$ there exists a unique maximal integral curve $\gamma_b^j : (-\delta, \delta) \to U$ of W_j^ϕ starting at b (for some $\delta > 0$). Then, to verify (i) and (ii) of Definition 5.3.2, it is enough to define $\exp_a(tW_j^\phi)(b) = \gamma_b^j(t)$ and to observe that for every j the map $\exp_a(tW_j^\phi)(b)$ is defined on $[-\delta_{2,j}, \delta_{2,j}] \times \overline{I_{\delta_{2,j}}(a)}$, for a sufficiently small $\delta_{2,j}$, with values in $\overline{I_{\delta_1}(a)} \Subset U$ (for a fixed $\delta_1 > 0$). Then one can chose δ_2 as the minimum of the $\delta_{2,j}$. In this case the role of the function $\omega_{i,j}$ in (iii) of Definition 5.3.2 is played by the derivative of the i-th component of ϕ, ϕ_i, read on the unique integral curve γ_b^j: the maps $\omega_{i,j}$ can be defined for $i = 1, \dots, k$, $j = 1, \dots, 2n-k$ for every $(\bar{s}, b) \in [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)}$ as

$$\omega_{i,j}(\gamma_b^j(\bar{s})) = \frac{d}{ds}\phi_i(\gamma_b^j(s))\Big|_{s=\bar{s}}$$

so that they verify (iii) of Definition 5.3.2.

If a continuous function $\phi : U \to \mathbb{R}^k$, along with its intrinsic Jacobian matrix, can be uniformly approximated by a family of continuously Euclidean differentiable functions, along with their intrinsic Jacobian matrices, respectively, then for every point a in U there exists a family of exponential maps at a.

Proposition 5.3.6. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function. Let us assume that there exist a family of functions $\{\phi_\varepsilon\}_{\varepsilon>0} \subset C^1(U)$ and a continuous matrix-valued function $M \in C^0(U, M_{k,2n-k}(\mathbb{R}))$*

$$M : U \to M_{k,2n-k}(\mathbb{R}),$$
$$a \longmapsto M(a) = \begin{pmatrix} m_{1,1}(a) & \dots & m_{1,2n-k}(a) \\ \dots & \dots & \dots \\ m_{k,1}(a) & \dots & m_{k,2n-k}(a) \end{pmatrix}$$

such that

$$\begin{aligned} \phi_\varepsilon &\to \phi \text{ uniformly } \overline{U'} \\ D^{\phi_\varepsilon}\phi_\varepsilon &\to M \text{ uniformly on } \overline{U'} \end{aligned} \tag{5.22}$$

for every $U' \Subset U$ as $\varepsilon \to 0$. Then for every $a \in U$ there exists a family of exponential maps at a, hence there exist $0 < \delta_2 < \delta_1$ such that for each $\ell = 1, \dots, 2n-k$ and for all $(s, b) \in [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)}$, there exists $\exp_a(sW_\ell^\phi)(b) \in \overline{I_{\delta_1}(a)} \subset U$. Moreover the continuous functions of (iii) in Definition 5.3.2 coincide with the corresponding elements of the matrix M, i.e.

$$\omega_{i,\ell}(b) := m_{i,\ell}(b) = \frac{d}{ds}\phi_i(\exp_a(sW_\ell^\phi)(b))\big|_{s=0}$$

for $i = 1, \dots, k$ and $\ell = 1, \dots, 2n-k$.

Proof. The proof mirrors the one of of [ASCV06, Lemma 5.6]. We work by taking in consideration separately the coordinates of ϕ, so that we have the following convergence

$$\phi_{\varepsilon,i} \to \phi_i,$$

uniformly on $\overline{U'}$ for every open subset $U' \Subset U$ as ε goes to zero, for every $i = 1, \dots, k$.

Fix a point $a \in U$. For any of the first and the last $(n-k)$ vector fields W_j^ϕ, i.e. for W_j^ϕ, for $j = 1, \ldots, n-k, n+1, \ldots, 2n-k$, the exponential map at a, $\exp_a(sW_j^\phi(b))$ can be defined as the unique integral curve starting at b of the vector field $W_j^{\phi_\varepsilon} = W_j^\phi$ for every ε since the coefficients of these vector fields do not depend on the maps ϕ_ε, since they coincide, for every ε, with the vector fields $\tilde{X}_j$ and $\tilde{Y}_j$ for $j = k+1, \ldots, n$. Of course, their coefficients are smooth and hence locally Lipschitz so, once fixed an initial point $b = (x_{k+1,b}, \ldots, x_{n,b}, \eta_{1,b}, \ldots, \eta_{k,b}, y_{k+1,b}, \ldots, y_{n,b}, \tau_b)$, there exists one unique maximal well defined integral curve of W_j^ϕ starting from b (refer to Remark 5.3.3). Let us work explicitly, as an example, on the field $\tilde{X}_j$ for a fixed $j \in \{n+1, \ldots, 2n-k\}$. We can define

$$\begin{aligned}\exp_a(tW_{j-n}^\phi)(b) &= \exp_a(t\tilde{X}_j)(b) = \gamma_b^j(t) := b \star t\mathbf{e}_{j-k}\\ &= (x_{k+1,b}, \ldots, x_{j,b}+t, \ldots, x_{n,b}, \eta_{1,b}, \ldots, \eta_{k,b}, y_{k+1,b}, \ldots, y_{n,b}, \tau_b - \frac{1}{2}y_{j,b}t).\end{aligned}$$

Moreover, $\exp_a(tW_{j-n}^\phi)(b)$ is defined on $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)}$ for a sufficiently small δ_2 with values in $\overline{I_{\delta_1}(a)} \subset U$ (with $\delta_1 > 0$ fixed), then (i) and (ii) of Definition 5.3.2 are satisfied by construction. By the fundamental theorem of calculus, for every $\varepsilon > 0$ and $(t,b) \in [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)}$ we have

$$\begin{aligned}\phi_{\varepsilon,i}(\gamma_b^j(t)) - \phi_{\varepsilon,i}(\gamma_b^j(0)) &= \int_0^t \tilde{X}_j \phi_{\varepsilon,i}(\gamma_b^j(r))dr\\ &= \int_0^t W_j^{\phi_\varepsilon}\phi_{\varepsilon,i}(\gamma_b^j(r))dr = \int_0^t [D^{\phi_\varepsilon}\phi_\varepsilon(\gamma_b^j(r))]_{i,j}dr.\end{aligned} \tag{5.23}$$

Letting ε going to zero, thanks to the hypothesis of uniform convergence on $\overline{I_{\delta_1}(a)}$, we get

$$\phi_i(\gamma_b^j(s)) - \phi_i(\gamma_b^j(0)) = \int_0^s m_{i,j}(\gamma_b^j(r))dr,$$

hence, thanks to the fundamental theorem of calculus we can set for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k, n+1, \ldots, n+k$,

$$\omega_{i,j}(b) := \frac{d}{dt}\phi_i(\exp_a(tW_j^\phi)(b))\big|_{t=0} = m_{i,j}(b)$$

and they satisfy (iii) of Definition 5.3.2.

We consider now the k "central" vector fields $W_\ell^\phi = \nabla^{\phi_j} = \partial_{\eta_j} + \phi_j\partial_\tau$ for $\ell = n-k+1, \ldots, n$ (we have set $j = \ell - (n-k)$). Fix $j \in \{1, \ldots, k\}$. Let us consider for every $\varepsilon > 0$ the solution $(\gamma_\varepsilon^j)_b$ of the Cauchy problem:

$$\begin{cases}(\dot{\gamma}_\varepsilon^j)_b(s) = \partial_{\eta_j} + \phi_{\varepsilon,j}((\gamma_\varepsilon^j)_b(s))\partial_\tau = \nabla^{\phi_{\varepsilon,j}}((\gamma_\varepsilon^j)_b(s))\\ (\gamma_\varepsilon^j)_b(0) = b\end{cases} \tag{5.24}$$

We are dealing for every ε with vector fields with C^1 coefficients, hence we can uniquely solve the Cauchy problem (5.24), getting for each fixed ε and b a unique maximal solution. Then there is a solution map

$$\gamma_\varepsilon^j : [-\delta_{2,j}(\varepsilon), \delta_{2,j}(\varepsilon)] \times \overline{I_{\delta_{2,j}(\varepsilon)}(a)} \to \overline{I_{\delta_1}(a)},$$

with $\delta_1 > 0$ and $\delta_{2,j}(\varepsilon) \geq \max\left\{\frac{\delta_1}{2}, \frac{\delta_1}{2\|\phi_{\varepsilon,j}\|_{L^\infty(\overline{I_{\delta_1}(a)})}}\right\}$. This estimate on the parameter $\delta_{2,j}(\varepsilon)$ follows from the Peano's estimate on the existence interval of the solution to (5.24).

Remark 5.3.7. Let us give some more details about the way in which one can fix the parameters δ_1 and $\delta_{2,j}(\varepsilon)$, referring to the classical Peano's theorem (refer for instance to [Mus05, Theorem 1]). One can fix $0 < \delta_1 < 1$ such that $I_{\delta_1}(a) \Subset U$. Fix then $0 < \frac{\delta_1}{2} < \delta_1$. Then for every $b \in I_{\frac{\delta_1}{2}}(a)$, $B_E(b, \frac{\delta_1}{2}) \subset I_{\frac{\delta_1}{2}}(b) \Subset I_{\delta_1}(a) \Subset U$. We set $\|\phi_{\varepsilon,j}\| = \|\phi_{\varepsilon,j}\|_{L^\infty(\overline{I_{\delta_1}(a)})} = \max_{\overline{I_{\delta_1}(a)}} |\phi_{\varepsilon,j}(y)|$. Then for every $\varepsilon > 0$ the time of existence of the maximal solution is at least $\frac{\delta_1}{2\max\{1,\|\phi_{\varepsilon,j}\|\}}$, so that we can fix $\delta_{2,j}(\varepsilon) = \min\{\frac{\delta_1}{2}, \frac{\delta_1}{2\|\phi_{\varepsilon,j}\|}\}$.

Now, if we fix a positive number $M > 0$, there exists some $\varepsilon_0 = \varepsilon_0(j) > 0$ such that for every $\varepsilon \leq \varepsilon_0$

$$\|\phi_{\varepsilon,j} - \phi_j\| \leq M,$$

where $\|\cdot\|$ denotes the norm in $L^\infty(\overline{I_{\delta_1(a)}})$. Hence for every $\varepsilon \leq \varepsilon_0$ we have

$$\|\phi_{\varepsilon,j}\| \leq \|\phi_{\varepsilon,j} - \phi_j\| + \|\phi_j - \phi_{\varepsilon_0}\| + \|\phi_{\varepsilon_0}\| < M + M + \max_{\overline{U}} |\phi_{\varepsilon_0}| := N_j. \tag{5.25}$$

Hence the maps $\{\phi_{\varepsilon,j}\}_{\varepsilon\leq\varepsilon_0}$ are equibounded. We get that, for every $0 < \varepsilon \leq \varepsilon_0$, $\delta_{2,j}(\varepsilon) \geq \frac{\delta_1}{2\max\{1,\|\phi_{\varepsilon,j}\|\}} \geq \frac{\delta_1}{2\max\{1,N_j\}}$, so we can chose a common value of existence time $0 < \delta_{2,j} < \delta_{2,j}(\varepsilon)$ for every $\varepsilon \leq \varepsilon_0$. Hence we can consider the family of maps $\{\gamma_\varepsilon^j\}_{\varepsilon\leq\varepsilon_0}$ defined on a common non-degenerating compact set: $[-\delta_{2,j}, \delta_{2,j}] \times \overline{I_{\delta_{2,j}}(a)}$.

More explicitly, if $b = (x_{k+1,b}, \dots, x_{n,b}, \eta_{1,b}, \dots, \eta_{k,b}, y_{k+1,b}, \dots, y_{n,b}, \tau_b) \in \overline{I_{\delta_{2,j}}(a)}$, and $t \in [-\delta_{2,j}, \delta_{2,j}]$ we have

$$(\gamma_\varepsilon^j)_b(t) = \left(x_{k+1,b}, \dots, x_{n,b}, \eta_{1,b}, \dots, \eta_{j,b} + t, \dots, \eta_{k,b}, y_{k+1,b}, \dots, y_{n,b}, \tau_b + \int_0^t \phi_{\varepsilon,j}((\gamma_\varepsilon^j)_b(r))dr\right),$$

then it is immediate to observe that each component of $(\gamma_\varepsilon^j)_b(t)$ can be bounded independently of ε, b and t on $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)}$. Hence the $\{\gamma_\varepsilon^j\}_{\varepsilon\leq\varepsilon_0}$ are equibounded. Since, by the uniform convergence of their derivatives on $\overline{I_{\delta_1}(a)}$, they are also uniformly continuous on $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)}$, we can then apply the Ascoli-Arzelà Theorem and we can extract a subsequence of $\{\gamma_\varepsilon^j\}_{\varepsilon\leq\varepsilon_0}$, $\{\gamma_{\varepsilon_h}^j\}_{h\in\mathbb{N}}$ that converges uniformly on $[-\delta_{2,j}, \delta_{2,j}] \times \overline{I_{\delta_{2,j}}(a)}$ to a function γ^j, when h goes to infinity. For every h we have

$$\begin{aligned}(\gamma_{\varepsilon_h}^j)_b(s) &= (\gamma_{\varepsilon_h}^j)_b(0) + \int_0^s (\dot\gamma_{\varepsilon_h}^j)_b(r)dr \\ &= b + \int_0^s \nabla^{\phi_{\varepsilon_h},j}((\gamma_{\varepsilon_h}^j)_b(r))dr\end{aligned}$$

and

$$\phi_{\varepsilon_h,i}((\gamma_{\varepsilon_h}^j)_b(s)) - \phi_{\varepsilon_h,i}((\gamma_{\varepsilon_h}^j)_b(0) = \int_0^s \nabla^{\phi_{\varepsilon_h},j}\phi_{\varepsilon_h,i}((\gamma_{\varepsilon_h}^j)_b(r))dr$$

for $i = 1, \dots, k$. Now, letting h go to infinity, since all the convergences are uniform on $[-\delta_{2,j}, \delta_{2,j}] \times \overline{I_{\delta_{2,j}}(a)}$ as $h \to \infty$ we can exchange limits and integrals and we get that

$$\gamma_b^j(s) = b + \int_0^s \nabla^{\phi_j}(\gamma_b^j(r))dr$$

and that
$$\phi_i(\gamma_b^j(s)) - \phi_i(\gamma_b^j(0)) = \int_0^s m_{i,j}(\gamma_b^j(r))dr,$$
where $m_{i,j}$ is the (i,j)-th element of the continuous matrix-valued function to which by hypothesis the sequence $D^{\phi_\varepsilon}\phi_\varepsilon$ converges uniformly on $\overline{I_{\delta_1}(a)}$ as ε goes to zero.

The whole path can be carried out for every $j = 1, \dots, k$ in order to individuate a map γ^j associated with ∇^{ϕ_j} defined from $[-\delta_{2,j}, \delta_{2,j}] \times \overline{I_{\delta_{2,j}}(a)}$ to $\overline{I_{\delta_1}(a)}$ for suitable $\delta_{2,j} > 0$. Since we have k possible values of j, to satisfy Definition 5.3.2 it is enough to choose δ_2 as the minimum of the $\delta_{2,j}$. By the fundamental theorem of calculus we can finally define for $i = 1, \dots, k$, $j = n-k+1, \dots, n$ and $t \in [\delta_2, \delta_2]$, $b \in \overline{I_{\delta_2}(a)}$
$$\exp_a(tW_j^\phi)(b) := \gamma_b^{j-(n-k)}(t)$$
$$\omega_{i,j}(b) := \frac{d}{dt}\phi_i(\exp_a(tW_j^\phi)(b))\big|_{t=0} = m_{i,j}(\exp_a(0W_j^\phi)(b)) = m_{i,j}(b).$$
These maps satisfy condition (iii) of Definition 5.3.2.

Finally, it is enough to observe that choosing δ_2 as the minimum between the one individuated in the first part of the proof and the one individuated in the second part of the proof, we have built a family of exponential maps at a. $\square$

5.3.3 Intrinsic derivatives

Let us stress again that for $j = 1, \dots, n-k, n+1, \dots, 2n-k$, once we fix an initial point $a \in U$, the integral curve of W_j^ϕ starting at a is unique by the Cauchy-Lipschitz theorem. For $j = n-k+1, \dots, n$, instead, we lose the uniqueness; the existence is ensured by the Peano's Theorem, since the coefficients of W_j^ϕ are continuous. Hence, if we only assume that ϕ is continuous, *the value of the limit (5.26) depends a priori on the choice of the integral curve.* Then, it makes sense to introduce the following definition.

Definition 5.3.8. Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function. Let a be a point in U. Given $j \in \{1, \dots, 2n-k\}$, we say that ϕ *has ∂^{ϕ_j}-derivative at a* if and only if there exists a vector $\begin{pmatrix}\alpha_{1,j} & \dots & \alpha_{k,j}\end{pmatrix} \in \mathbb{R}^k$ such that for any $\gamma^j : (-\delta, \delta) \to U$ integral curve of W_j^ϕ such that $\gamma^j(0) = a$, the limit $\lim_{s\to 0} \frac{\phi(\gamma^j(s)) - \phi(a)}{s}$ exists and is equal to $\begin{pmatrix}\alpha_{1,j} & \dots & \alpha_{k,j}\end{pmatrix}^T$. We denote it by
$$\partial^{\phi_j}\phi(a) = \begin{pmatrix}\partial^{\phi_j}\phi_1(a) \\ \dots \\ \partial^{\phi_j}\phi_k(a)\end{pmatrix} := \begin{pmatrix}\alpha_{1,j} \\ \dots \\ \alpha_{k,j}.\end{pmatrix}.$$
for $j = 1, \dots, 2n-k$.

Nevertheless, if the function ϕ is intrinsically differentiable at a point $a \in U$, the limit $\lim_{s\to 0} \frac{\phi_i(\gamma^j(s)) - \phi_i(\gamma^j(0))}{s}$ does not depend on the choice of the integral curve of W_j^ϕ, γ^j, starting at $a = \gamma^j(0)$.

Proposition 5.3.9. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function. Let $a \in U$, assume that ϕ is intrinsically differentiable at a and let $D^\phi\phi(a)$ be the intrinsic Jacobian matrix of ϕ at a. Let $j \in \{1, \dots, 2n-k\}$ and let*
$$\gamma^j : [-\delta, \delta] \to U$$

be an arbitrary integral curve of the vector field W_j^ϕ, with $\gamma^j(0) = a$. Then, for any $i \in \{1, \dots, k\}$ we have

$$\lim_{s \to 0} \frac{\phi_i(\gamma^j(s)) - \phi_i(\gamma^j(0))}{s} = [D^\phi \phi(a)]_{i,j}. \tag{5.26}$$

Proof. Let us consider the point

$$a = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau).$$

If $j = 1, \dots, n-k$, W_j^ϕ is $\tilde{X}_{j+k}$, if $j = n+1, \dots, 2n-k$, W_j^ϕ is $\tilde{Y}_{j-(n-k)}$. In both cases the integral curve γ^j of W_j^ϕ with $\gamma^j(0) = a$ is unique; it is then immediate to verify that for $s \in [-\delta, \delta]$

$$d_\phi(\gamma^j(s), a) = d_\phi(\gamma^j(s), \gamma^j(0)) = |s|.$$

In fact, let us consider for instance $j \in \{1, \dots, n-k\}$, then we have

$$\gamma^j(s) = \left(x_{k+1}, \dots, x_j + s, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau - \frac{1}{2} y_j s \right)$$

and

$$\begin{aligned} d_\phi(\gamma^j(s), a) &= d_\phi(\gamma^j(s), \gamma^j(0)) \\ &= \max\{|s|, |\tau - \frac{1}{2} y_j s - \tau \\ &\quad + \sigma((x_{k+1}, \dots, x_j + s, \dots, x_n), (y_{k+1} \dots, y_n), (x_{k+1}, \dots, x_n), (y_{k+1} \dots, y_n))|^{\frac{1}{2}}\} \\ &= \max\{|s|, |-\frac{1}{2} y_j s + \frac{1}{2}(x_j + s) y_j - \frac{1}{2} x_j y_j|^{\frac{1}{2}}\} \qquad (5.27) \\ &= |s|. \end{aligned}$$

Let us now assume that $j \in \{n-k+1, \dots, n\}$ and then that γ^j is an integral curve of the vector field $\nabla^{\phi_\ell} = \partial_{\eta_\ell} + \phi_\ell \partial_\tau$ for $\ell = j - (n-k)$. As already pointed out, the integral curve γ^j can fail to be unique, nevertheless, it has the following integral form for $s \in [-\delta, \delta]$

$$\gamma^j(s) = \left(x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_\ell + s, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau + \int_0^s \phi_\ell(\gamma^j(r)) dr \right).$$

On the other hand, ϕ is intrinsically differentiable at a, then

$$\lim_{w \to a} \frac{|\phi(w) - \phi(a) - D^\phi \phi(a) \cdot (\pi(a^{-1} w))^T|}{d_\phi(w, a)} = 0.$$

Hence, there exist two positive constants C, r such that

$$|\phi(w) - \phi(a)| \le C d_\phi(w, a) \qquad \forall w \in B(a, r) \cap \mathbb{W} \tag{5.28}$$

(for more details about (5.28) please refer to [SC16, Remark 4.75, Proposition 4.76]). Then, unless we restrict the domain of the curve γ^j, we can assume that γ^j is defined on an interval $[-\delta_j, \delta_j]$ such that the previous inequality holds for $w = \gamma^j(s)$ for any $s \in [-\delta_j, \delta_j]$:

$$|\phi(\gamma^j(s)) - \phi(\gamma^j(0))| \le C d_\phi(\gamma^j(s), a) \qquad \forall s \in [-\delta_j, \delta_j].$$

Hence, for every $i = 1, \dots, k$

$$|\phi_i(\gamma^j(s)) - \phi_i(\gamma^j(0))| \leq |\phi(\gamma^j(s)) - \phi(\gamma^j(0))| \ \leq \ C d_\phi(\gamma^j(s), a),$$

$\forall s \in [-\delta_j, \delta_j]$. Then we study $d_\phi(\gamma^j(s), a)$ for $s \in [-\delta_j, \delta_j]$

$$\begin{aligned} d_\phi(\gamma^j(s), a) = & \ \max\{|s|, |\int_0^s \phi_\ell(\gamma^j(r))dr + \phi_\ell(a)(-s)|^{\frac{1}{2}}\} \\ = & \ \max\{|s|, |\int_0^s \phi_\ell(\gamma^j(r)) - \phi_\ell(a)dr|^{\frac{1}{2}}\} \\ \leq & \ \max\{|s|, C^{\frac{1}{2}}|s|^{\frac{1}{2}}(\sup_{s\in[-\delta_j,\delta_j]} d_\phi(\gamma^j(s), a))^{\frac{1}{2}}\} \\ \leq & \ \max\{|s|, \frac{C}{2}|s| + \frac{1}{2}\sup_{s\in[-\delta_j,\delta_j]} d_\phi(\gamma^j(s), a)\}. \end{aligned} \tag{5.29}$$

Therefore

$$d_\phi(\gamma^j(s), a) \ \leq \ C_2|s|, \tag{5.30}$$

where $C_2 = \max\{1, C\}$. Hence

$$\begin{aligned} & \frac{\left| \begin{pmatrix} \phi_1(\gamma^j(s)) - \phi_1(a) - [D^\phi\phi(a)]_{1,j}s \\ \dots \\ \phi_k(\gamma^j(s)) - \phi_k(a) - [D^\phi\phi(a)]_{k,j}s \end{pmatrix} \right|}{|s|} \\ = & \frac{\left|\phi(\gamma^j(s)) - \phi(\gamma^j(0)) - D^\phi\phi(a)\cdot(s\mathbf{f}_{j-n+k})\right|}{|s|} \\ = & \frac{|\phi(\gamma^j(s)) - \phi(\gamma^j(0)) - D^\phi\phi(a)\cdot\pi(a^{-1}\gamma^j(s))^T|}{|s|} \\ \leq & \ C_2\frac{|\phi(\gamma^j(s)) - \phi(\gamma^j(0)) - D^\phi\phi(a)\cdot\pi(a^{-1}\gamma^j(s))^T|}{d_\phi(\gamma^j(s), a)}. \end{aligned} \tag{5.31}$$

where $\mathbf{f}_{j-n+k} \in M_{2n-k,1}(\mathbb{R})$ denotes the $(j-n+k)$-th element of the canonical basis of $\mathbb{R}^{2n-k}$. Now, thanks to the intrinsic differentiability of ϕ at a, (5.31) goes to zero as s tends to zero and we get the thesis. □

By Proposition 5.3.9, it is not difficult to deduce the following conditions.

Corollary 5.3.10. *Given an open set $U \subset \mathbb{R}^{2n+1-k}$ and a C^1-regular (Euclidean) function $\phi : U \to \mathbb{R}^k$, then for every $a \in U$ we have*

$$D^\phi\phi(a) = \begin{pmatrix} \omega_{1,1}(a) & \dots & \omega_{1,2n-k}(a) \\ \dots & \dots & \dots \\ \omega_{k,1}(a) & \dots & \omega_{k,2n-k}(a) \end{pmatrix}, \tag{5.32}$$

where $\omega_{i,j} : U \to \mathbb{R}$, for $i = 1, \dots, k$, is the map

$$\begin{aligned} \omega_{i,j}(a) = & \begin{cases} \tilde{X}_{j+k}\phi_i(a) & \text{for } j = 1, \dots, n-k \\ \nabla^{\phi_\ell}\phi_i(a) = \partial_{\eta_\ell}\phi_i(a) + \phi_\ell(a)\partial_\tau\phi_i(a) & \text{for } j = n-k+1, \dots, n, \ \ell = j-(n-k) \\ \tilde{Y}_{j-(n-k)}\phi_i(a) & \text{for } j = n+1, \dots, 2n-k. \end{cases} \\ = & \frac{d}{ds}\phi_i(\gamma^j(s))\Big|_{s=0} \end{aligned}$$

where $\gamma^j : [-\delta, \delta] \to U$ is an integral curve of W_j^ϕ such that $\gamma^j(0) = a$.

Corollary 5.3.11. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function. Let $a \in U$ and assume that ϕ is intrinsically differentiable at a. Then, for every $j = 1, \dots, 2n-k$, there exists the intrinsic partial derivative $\partial^{\phi_j}\phi(a)$ and it equals the j-th column of the intrinsic Jacobian matrix $D^\phi\phi(a)$*

$$\partial^{\phi_j}\phi(a) = \begin{pmatrix} \partial^{\phi_j}\phi_1(a) \\ \dots \\ \partial^{\phi_j}\phi_k(a) \end{pmatrix} = \begin{pmatrix} [D^\phi\phi(a)]_{1,j} \\ \dots \\ [D^\phi\phi(a)]_{k,j}. \end{pmatrix}$$

5.3.4 The existence of an approximation implies a Hölder-type regularity

A uniformly intrinsically differentiable map satisfies locally a Hölder-type regularity of order $\frac{1}{2}$.

Proposition 5.3.12. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set, let $\phi : U \to \mathbb{R}^k$ be a function and assume that ϕ is uniformly intrinsically differentiable on U. Then for every open set $U' \Subset U$*

$$\lim_{r\to 0+} \sup\left\{ \frac{|\phi(b') - \phi(b)|}{|b'-b|^{\frac{1}{2}}} : b, b' \in U',\ 0 < |b'-b| \le r \right\} = 0. \tag{5.33}$$

Proof. Let us consider an arbitrary point $a \in U$. By the uniform intrinsic differentiability of ϕ and by Proposition 3.5.34, there is a positive $R = R(a) > 0$ such that $I_R(a) \Subset U$ and ϕ is intrinsic Lipschitz on $I_R(a)$. Then, by Remark 5.1.3, for every $b, b' \in I_R(a)$, there is a positive constant $c = c(a)$ such that, if $|b'-b| < 1$, $d_\phi(b, b') \le c|b'-b|^{\frac{1}{2}}$. Now, let us observe that

$$\lim_{r\to 0+} \sup\left\{ \frac{|\phi(b') - \phi(b)|}{|b'-b|^{\frac{1}{2}}} : b, b' \in I_R(a),\ 0 < |b'-b| \le r \right\} = 0. \tag{5.34}$$

In fact, assume by contradiction that (5.34) is not true. Then there exist some positive constant $\overline{\varepsilon} > 0$ such that for every $n \in \mathbb{N}$ there are $b_n, b_n' \in I_R(a)$ such that $0 < |b_n' - b_n| \le \frac{1}{n}$ and

$$\frac{|\phi(b_n') - \phi(b_n)|}{|b_n' - b_n|^{\frac{1}{2}}} \ge \overline{\varepsilon} > 0.$$

By compactness, up to a subsequence, we can assume that b_n and b_n' converge to some point $b^* \in \overline{I_R(a)} \subset U$ as n goes to ∞. Let us now consider for every $n \in \mathbb{N}$

$$\frac{|\phi(b_n') - \phi(b_n)|}{|b_n' - b_n|^{\frac{1}{2}}} \le \frac{|\phi(b_n') - \phi(b_n) - d\phi_a(b_n^{-1} \star b_n')|}{d_\phi(b_n', b_n)} \frac{d_\phi(b_n', b_n)}{|b_n' - b_n|^{\frac{1}{2}}} + \frac{|d\phi_a(b_n^{-1} \star b_n')|}{|b_n' - b_n|^{\frac{1}{2}}}.$$

It converges to zero as n goes to ∞ by the intrinsic differentiability of ϕ at a and by the following estimate

$$\begin{aligned}\frac{|d\phi_{b^*}(b_n^{-1} \star b_n')|}{|b'-b|^{\frac{1}{2}}} &= \frac{|D^\phi\phi(b^*) \cdot (\pi(b_n^{-1} \star b_n'))^T|}{|b'-b|^{\frac{1}{2}}} \\ &\le \frac{\|D^\phi\phi(b^*)\| |b_n' - b_n|}{\sqrt{|b_n' - b_n|}} \le \|D^\phi\phi(b^*)\| \sqrt{|b_n' - b_n|} \to 0\end{aligned}$$

as n goes to ∞; by $\|D^\phi\phi(b^*)\|$ we have denoted the norm of the matrix $D^\phi\phi(b^*)$. The thesis (5.33) follows from a standard compactness argument. □

Remark 5.3.13. In the literature condition (5.33) of the previous proposition has been also referred to as *little Hölder continuity of order* $\frac{1}{2}$ on U' or $\frac{1}{2}$-little Hölder continuity of the map ϕ on U', for instance one can refer to [Lun95, Section 0.2]. The space of functions defined on an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^k$ for some $n, k \in \mathbb{N}$, which satisfy condition (5.67) on every open set $U' \Subset U$, is usually denoted by $h^{\frac{1}{2}}_{\mathrm{loc}}(U, \mathbb{R}^k)$.

The following theorem is the key tool to prove that the existence of a local uniform approximation of a function ϕ through a family of C^1-regular functions as the one in Proposition 5.3.1, implies a further regularity in every direction and, in particular, gives a Hölder-type control of order $\frac{1}{2}$ along the vertical direction.

Proposition 5.3.14. *Let $I \subset \mathbb{R}^{2n+1-k}$ be a rectangle, let $\phi \in C^1(I, \mathbb{R}^k)$ and consider the map intrinsic Jacobian matrix $D^\phi\phi \in C^0(I, M_{k,2n-k}(\mathbb{R}))$. For $a \in I$*

$$D^\phi\phi(a) = \begin{pmatrix} \omega_{1,1}(a) & \dots & \omega_{1,2n-k}(a) \\ \dots & \dots & \dots \\ \omega_{k,1}(a) & \dots & \omega_{k,2n-k}(a) \end{pmatrix},$$

where, for any $i = 1, \dots k$,

$$\omega_{i,\ell}(a) = \begin{cases} \tilde{X}_{\ell+k}\phi_i(a) & \text{for } \ell = 1, \dots, n-k \\ \nabla^{\phi_j}\phi_i(a) = \partial_{\eta_j}\phi_i(a) + \phi_j(a)\partial_\tau\phi_i(a) & \text{for } \ell = n-k+1, \dots, n, \quad j = \ell - (n-k) \\ \tilde{Y}_{\ell-(n-k)}\phi_i(a) & \text{for } \ell = n+1, \dots, 2n-k. \end{cases}$$

Given a fixed rectangle $I' \Subset I$, for any other rectangle I'' such that $I' \Subset I'' \Subset I$, there exists a function

$$\alpha : (0, \infty) \to [0, \infty)$$

depending on k, on I'', on $\{\|\phi_j\|_{L^\infty(I'')}\}_{j=1,\dots,k}$, on $\|D^\phi\phi\|_{L^\infty(I'')}$ and on the modulus of continuity of the maps $\{\omega_{j,j+(n-k)}\}_{j=1,\dots,k}$ on I'', such that the limit $\lim_{r\to 0}\alpha(r) = 0$ and, for r sufficiently small,

$$\sup\left\{ \frac{|\phi(a) - \phi(b)|}{|a-b|^{1/2}} : a, b \in I',\ 0 < |a-b| \le r \right\} \le \alpha(r).$$

Proof. The proof is inspired to the proof of [ASCV06, Proposition 5.8].

For $\ell = n-k+1, \dots, n$ we set $j = \ell - (n-k)$, so that $j \in \{1, \dots, k\}$. We set $K := \sup_{a \in I''}|a|$, $M_j := \|\phi_j\|_{L^\infty(I'')}$ and $N := \|D^\phi\phi\|_{L^\infty(I'')}$. We call β_j the modulus of continuity of $\omega_{j,j+(n-k)}$, on I'', that is a continuous increasing function $\beta_j : (0,\infty) \to [0, \infty)$ such that $|\omega_{j,j+(n-k)}(a) - \omega_{j,j+(n-k)}(b)| \le \beta_j(|a-b|)$ for all $a, b \in I''$, with $\lim_{r\to 0}\beta_j(r) = 0$. We introduce $k+1$ rectangles such that $I' \Subset J_1 \Subset J_2 \cdots \Subset J_{k+1} \Subset I''$. We set $I' = J_0$, and, for any $a = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau) \in J_i$ (for $i = 0, \dots, k-1$), for $j \in \{1, \dots, k\}$, we consider the integral curves:

$$\begin{cases} \dot{\gamma}^j_a(t) = (\frac{\partial}{\partial\eta_j} + \phi_j(\gamma^j_a(t))\frac{\partial}{\partial\tau})(\gamma^j_a(t)) = \nabla^{\phi_j}(\gamma^j_a(t)) \\ \gamma^j_a(\eta_j) = a. \end{cases} \tag{5.35}$$

Thanks to the Cauchy-Lipschitz theorem, these are well defined and $\gamma^j_a \in C^1([\eta_j - \varepsilon_{i+1,j}, \eta_j + \varepsilon_{i+1,j}])$ for a certain constant $\varepsilon_{i+1,j}$ that depends on J_i and J_{i+1} (in particular

$\varepsilon_{i+1,j}$ depends on the distance between the two boundaries ∂J_i and ∂J_{i+1}). We can choose, for all $j = 1, \ldots, k$, $\varepsilon_{i+1,j} > 0$ such that $\gamma_a^j(t)([\eta_j - \varepsilon_{i+1,j}, \eta_j + \varepsilon_{i+1,j}]) \subset J_{i+1}$ for every $a \in J_i$ (the choice is uniform in a). If $a = (x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau) \in J_i$ we get, for $t \in [\eta_j - \varepsilon_{i+1,j}, \eta_j + \varepsilon_{i+1,j}]$,

$$\gamma_a^j(t) = \left(x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_j + (t - \eta_j), \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau + \int_{\eta_j}^t \phi_j(\gamma_a^j(s))ds \right). \tag{5.36}$$

Denoting $\tau_a^j(t) = \tau + \int_{\eta_j}^t \phi_j(\gamma_a^j(s))ds$ we also have

$$\dot{\tau}_a^j(t) = \phi_j(\gamma_a^j(t)), \qquad \ddot{\tau}_a^j(t) = \frac{d^2}{d^2t}\tau_a^j(t) = \frac{d}{dt}\phi_j(\gamma_a^j(t)) = \omega_{j,j+(n-k)}(\gamma_a^j(t)).$$

Let us now set

$$\delta_j(r) := \max\{r^{1/4}, 2\sqrt{2k\beta_j(r + 4kM_j r^{1/4})}\}.$$

We will prove that

$$\begin{aligned} \theta(r) :&= \sup\left\{ \frac{|\phi(a) - \phi(b)|}{|a-b|^{1/2}} : a, b \in I',\ 0 < |a-b| \le r \right\} \\ &\le \left(\sum_{j=1}^k \delta_j(r) \right) + \sqrt{\sum_{j=1}^k M_j} \left(\sum_{j=1}^k \delta_j(r) \right) + kNr^{1/2} \end{aligned} \tag{5.37}$$

for r sufficiently small. The thesis will follow directly from this inequality.
In order to prove (5.37), we proceed by contradiction.
Let us first consider a and b as below. Later on, the result will be extended to a and b in I' of generic coordinates. Set

$$\begin{aligned} a &= (x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau) \in I' \\ b &= (x_{k+1}, \ldots, x_n, \eta_1', \ldots, \eta_k', y_{k+1}, \ldots, y_n, \tau') \in I' \end{aligned} \tag{5.38}$$

such that $|a - b|$ is sufficiently small and assume that

$$\frac{|\phi(a) - \phi(b)|}{|a-b|^{1/2}} > \sum_{j=1}^k \delta_j + \sqrt{\sum_{j=1}^k M_j} \left(\sum_{j=1}^k \delta_j \right) + k^2 N r^{1/2} \tag{5.39}$$

where $\delta_j = \delta_j(|a - b|)$. Notice that the functions δ_j are monotonically increasing.
For $j = 1, \ldots, k$, we call $\delta_j' = \delta_j(|\tau' - \tau|) \le \delta_j$. We obtain that, thanks to the definition of δ_j,

$$\begin{aligned} \frac{\beta_j(|\tau' - \tau| + 4kM_j|\tau' - \tau|^{1/2}/\delta_j)}{\delta_j^2} &\le \frac{\beta_j(|\tau' - \tau| + 4kM_j|\tau' - \tau|^{1/2}/\delta_j')}{\delta_j'^2} \\ &\le \frac{\beta_j(|\tau' - \tau| + 4kM_j|\tau' - \tau|^{1/4})}{\delta_j'^2} \\ &\le \frac{\delta_j'^2}{8k}\frac{1}{\delta_j'^2} = \frac{1}{8k}. \end{aligned} \tag{5.40}$$

We now consider

$$c = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau') \in I'.$$

Notice that a and c differ only for the vertical coordinate and c and b for the horizontal ones. In particular

$$\begin{aligned} |a - c|^{1/2} &= |\tau - \tau'|^{1/2} \\ |c - b|^{1/2} &= |(\eta_1 - \eta_1', \dots, \eta_k - \eta_k')|^{1/2}. \end{aligned} \tag{5.41}$$

Since we are proceeding by contradiction let us continue from (5.39)

$$\begin{aligned} &\sum_{j=1}^k \delta_j + \sqrt{\sum_{j=1}^k M_j} \left(\sum_{j=1}^k \delta_j \right) + kNr^{1/2} \\ &< \frac{|\phi(a) - \phi(b)|}{|a - b|^{1/2}} \\ &\le \frac{|\phi(a) - \phi(c)|}{|a - b|^{1/2}} + \frac{|\phi(c) - \phi(b)|}{|a - b|^{1/2}} \\ &\le \frac{|\phi(a) - \phi(c)|}{|\tau - \tau'|^{1/2}} + \frac{|\phi(c) - \phi(b)|}{|(\eta_1 - \eta_1', \dots, \eta_k - \eta_k')|^{1/2}} \\ &\le \frac{\sum_{j=1}^k |\phi_j(a) - \phi_j(c)|}{|\tau - \tau'|^{1/2}} + \frac{\sum_{j=1}^k |\phi_j(c) - \phi_j(b)|}{|(\eta_1 - \eta_1', \dots, \eta_k - \eta_k')|^{1/2}} \\ &:= R_1 + R_2. \end{aligned} \tag{5.42}$$

We reach a contradiction by showing that

(i) $R_1 \le \sum_{j=1}^k \delta_j$;

(ii) $R_2 \le \sqrt{\sum_{j=1}^k M_j} (\sum_{j=1}^k \delta_j) + kNr^{1/2}$.

Let us prove (i). We show (i) showing that for any $a, c \in J_k$ (hence in particular, for $a, c \in I'$), when a and c differ only for the vertical coordinate, $R_1 \le \sum_{j=1}^k \delta_j$. In particular, we prove that for any j the following holds,

$$\frac{|\phi_j(a) - \phi_j(c)|}{|\tau - \tau'|^{1/2}} \le \delta_j.$$

Then let us fix $j \in \{1, \dots, k\}$ and consider a and $c \in J_k$ as before and let us assume that $\tau > \tau'$ and let us assume by contradiction that

$$\frac{|\phi_j(a) - \phi_j(c)|}{|\tau - \tau'|^{\frac{1}{2}}} > \delta_j. \tag{5.43}$$

Consider the maps γ_a^j and γ_c^j. For any $t \in [\eta_j - \varepsilon_{k+1,j}, \eta_j + \varepsilon_{k+1,j}]$ we have

$$\begin{aligned} &\tau_a^j(t) - \tau_c^j(t) \\ &= \tau - \tau' + \int_{\eta_j}^t [\dot{\tau}_a^j(\eta_j) - \dot{\tau}_c^j(\eta_j) + \int_{\eta_j}^s [\ddot{\tau}_a^j(r) - \ddot{\tau}_c^j(r)] dr] ds \\ &= \tau - \tau' + (t - \eta_j)(\phi_j(a) - \phi_j(c)) + \int_{\eta_j}^t \int_{\eta_j}^s \omega_{j,j+(n-k)}(\gamma_a^j(r)) - \omega_{j,j+(n-k)}(\gamma_c^j(r)) dr ds \end{aligned}$$

$$\begin{aligned}
&\le \tau - \tau' + (t-\eta_j)(\phi_j(a) - \phi_j(c)) + (t-\eta_j)^2 \sup_{r\in[\eta_j,t]} \beta_j(|\gamma_a^j(r) - \gamma_c^j(r)|) \\
&\le \tau - \tau' + (t-\eta_j)(\phi_j(a) - \phi_j(c)) + (t-\eta_j)^2 \beta_j(|\tau-\tau'| + 2M_j|t-\eta_j|)
\end{aligned} \tag{5.44}$$

since by the fundamental theorem of calculus and by the triangle inequality the following holds

$$\begin{aligned}
|\gamma_a^j(r) - \gamma_c^j(r)| &\le |\gamma_a^j(\eta_j) - \gamma_c^j(\eta_j)| + |r-\eta_j|(\|\dot\tau_a^j\| + \|\dot\tau_c^j\|) \\
&\le |\tau-\tau'| + 2M_j|t-\eta_j|,
\end{aligned} \tag{5.45}$$

where by $\|\cdot\|$ here we have denoted the norm in $\|\cdot\|_{L^\infty(I'')}$.
Now, if $(\phi_j(a) - \phi_j(c)) > 0$, we set

$$t := \eta_j - 2k\frac{(\tau-\tau')^{1/2}}{\delta_j} \tag{5.46}$$

or otherwise

$$t := \eta_j + 2k\frac{(\tau-\tau')^{1/2}}{\delta_j}. \tag{5.47}$$

We can take a and b close enough so that $2k(\tau-\tau')^{1/4} \le \varepsilon_{k+1,j}$, since, according to the definition of δ_j, we have

$$\begin{aligned}
2k(\tau-\tau')^{1/4} &\ge 2k\frac{(\tau-\tau')^{1/2}}{\delta_j} \\
&= |t-\eta_j|
\end{aligned} \tag{5.48}$$

Hence, for r small enough, let us start assuming that $(\phi(a) - \phi(c)) > 0$ and then, for t given by (5.46), the last terms in (5.44) equals

$$\begin{aligned}
&\tau - \tau' + \left(-2k\frac{(\tau-\tau')^{1/2}}{\delta_j}\right)(\phi_j(a) - \phi_j(c)) \\
&\quad + \left(-2k\frac{(\tau-\tau')^{1/2}}{\delta_j}\right)^2 \beta_j\left(|\tau-\tau'| + 2M_j\left|-2k\frac{(\tau-\tau')^{1/2}}{\delta_j}\right|\right).
\end{aligned} \tag{5.49}$$

By contradiction we had assumed (5.43) to be true, i.e. that $\frac{|\phi_j(a)-\phi_j(c)|}{|\tau-\tau'|^{1/2}} > \delta_j$. Then (5.49) can be estimated from above by

$$\begin{aligned}
&\tau - \tau' + (-2k((\tau-\tau')^{1/2})\frac{|\tau-\tau'|^{1/2}}{(\phi_j(a) - \phi_j(c))}(\phi_j(a) - \phi_j(c)) \\
&+ 4k^2\frac{(\tau-\tau')}{\delta_j^2}\beta_j\left(|\tau-\tau'| + 2M_j\left|-2k\frac{(\tau-\tau')^{1/2}}{\delta_j}\right|\right) \\
&= \tau - \tau' + (-2k((\tau-\tau')^{1/2})(\tau-\tau')^{1/2} + 4k^2(\tau-\tau')\frac{\beta_j\left(|\tau-\tau'| + 4kM_j\frac{(\tau-\tau')^{1/2}}{\delta_j}\right)}{\delta_j^2} \\
&\le \ \tau - \tau' - 2k(\tau-\tau') + \left(4k^2\frac{(\tau-\tau')}{8k}\right) \\
&= \ \tau - \tau' - 2k(\tau-\tau') + \frac{1}{2}k(\tau-\tau') \\
&= \frac{2-3k}{2}(\tau-\tau') < 0
\end{aligned} \tag{5.50}$$

since $k \ge 1$. This is not possible since it would imply that the two integral curves of ∇^{ϕ_j}

starting at a and c meet each other at some point on the plane (η_j, τ).
The study of the case (5.47) for $(\phi_j(a) - \phi_j(c)) < 0$ gives an identical result.
Hence, for our choice of a and c, $R_1 \leq \sum_{j=1}^k \delta_j$.
Let us now prove (ii). By contradiction we assume that

$$R_2 > \sqrt{\sum_{j=1}^k M_j \left(\sum_{j=1}^k \delta_j \right)} + kNr^{1/2}. \tag{5.51}$$

First of all we define for $j = 2, \ldots, k$

$$\begin{aligned} d_1 &:= \gamma_b^1(\eta_1) \\ d_j &:= \gamma_{d_{j-1}}^j(\eta_j). \end{aligned} \tag{5.52}$$

(remember that $\gamma_b^j(\eta_j') = b$).
The points $b, d_1, \ldots, d_k$ are vertices of a piecewise regular "polygonal" curve connecting b and d_k. The segments of this curve are built following the integral curves of the vector fields ∇^{ϕ_j} for time $\eta_j' - \eta_j$, for $j = 1, \ldots, k$. It turns out that

$$d_k = (x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, \tau'')$$

for a certain well defined τ''.
If, for every $j \in \{1, \ldots, k\}$, $|\eta_j' - \eta_j|$ is sufficiently small, we get that d_k is well defined and belongs to J_k. We can compute for every $i = 1, \ldots, k$,

$$\begin{aligned} |\phi_i(b) - \phi_i(d_k)| &\leq |\phi_i(b) - \phi_i(d_1)| + |\phi_i(d_1) - \phi_i(d_2)| + \cdots + |\phi_i(d_{k-1}) - \phi_i(d_k)| \\ &= \left| \int_{\eta_1'}^{\eta_1} \omega_{i,1}(\gamma_b^1(t)) dt \right| + \cdots + \left| \int_{\eta_k'}^{\eta_k} \omega_{i,k}(\gamma_{d_{k-1}}^k(t)) dt \right| \\ &\leq N|\eta_1 - \eta_1'| + \cdots + N|\eta_k - \eta_k'| \\ &\leq kN|(\eta_1 - \eta_1', \ldots, \eta_k - \eta_k')|. \end{aligned} \tag{5.53}$$

Let us set now $b = d_0$ and compute

$$\begin{aligned} |\tau' - \tau''| &= \left| \tau' - \tau - \sum_{j=1}^k \int_{\eta_j'}^{\eta_j} \phi_j(\gamma_{d_{j-1}}^j(t)) dt \right| \\ &= \left| \sum_{j=1}^k \int_{\eta_j'}^{\eta_j} \phi_j(\gamma_{d_{j-1}}^j(t)) \right| \\ &\leq \sum_{j=1}^k M_j |\eta_j - \eta_j'| \\ &\leq \left(\sum_{j=1}^k M_j \right) |(\eta_1 - \eta_1', \ldots, \eta_k - \eta_k')|. \end{aligned} \tag{5.54}$$

Now, by (5.51) and (5.53), we get

$$
\begin{aligned}
\sum_{i=1}^{k} |\phi_i(c)-\phi_i(d_k)| \geq\ & \sum_{i=1}^{k}(|\phi_i(c)-\phi_i(b)| - |\phi_i(b)-\phi_i(d_k)|) \\
> & \left(\sqrt{\sum_{j=1}^{k} M_j}\left(\sum_{j=1}^{k}\delta_j\right) + kNr^{1/2}\right)|(\eta_1'-\eta_1,\dots,\eta_k'-\eta_k)|^{1/2} \\
& - kN|(\eta_1'-\eta_1,\dots,\eta_k'-\eta_k)| \\
\geq & \left(\sqrt{\sum_{j=1}^{k} M_j}\left(\sum_{j=1}^{k}\delta_j\right) + kNr^{1/2} - kN|(\eta_1'-\eta_1,\dots,\eta_k'-\eta_k)|^{1/2}\right) \\
& |(\eta_1'-\eta_1,\dots,\eta_k'-\eta_k)|^{1/2}.
\end{aligned}
\tag{5.55}
$$

If $|(\eta_1'-\eta_1,\dots,\eta_k'-\eta_k)|^{1/2} \leq |a-b|^{\frac{1}{2}} \leq r^{1/2}$, we get that the last term in (5.55) can be estimated from below by

$$
\sqrt{\sum_{j=1}^{k} M_j}\left(\sum_{j=1}^{k}\delta_j\right)|(\eta_1'-\eta_1,\dots,\eta_k'-\eta_k)|^{1/2} \geq\ |\tau'-\tau''|^{1/2}\left(\sum_{j=1}^{k}\delta_j\right). \tag{5.56}
$$

Therefore, we have proved that for $c, d_k \in J_k$,

$$
\frac{\sum_{i=1}^{k}|\phi_i(c)-\phi_i(d_k)|}{|\tau'-\tau''|^{1/2}} > \sum_{j=1}^{k}\delta_j,
$$

which is not possible (for what we proved before) for any $a, c \in J_k$.
Let us now consider the more general case when when $a, b \in I'$

$$
\begin{aligned}
a &= (x_{k+1},\dots,x_n,\eta_1,\dots,\eta_k,y_{k+1},\dots,y_n,\tau), \\
b &= (x_{k+1}',\dots,x_n',\eta_1',\dots,\eta_k',y_{k+1}',\dots,y_n',\tau').
\end{aligned}
$$

We want to exploit what we have proved before. In order to do this we move along the integral curves of the vector fields $\tilde{X}_j,\ \tilde{Y}_j$ for $j = k+1,\dots,n$ in order to make coincide the variables x_j and y_j. We then define

$$
\begin{aligned}
a^* :&= \exp_a\left(\sum_{j=k+1}^{n}((x_j'-x_j)W_{j-k}^{\phi} + (y_j'-y_j)W_{j+(n-k)}^{\phi})\right)(a) \\
&= a \star \left(\sum_{j=k+1}^{n}((x_j'-x_j)\mathbf{e}_j + (y_j'-y_j)\mathbf{e}_{j+n})\right),
\end{aligned}
$$

where $x = (x_{k+1},\dots,x_n),\ y = (y_{k+1},\dots,y_n), x'-x = (x_{k+1}'-x_{k+1},\dots,x_n'-x_n),\ y'-y = (y_{k+1}'-y_{k+1},\dots,y_n'-y_n) \in \mathbb{R}^{n-k}$. Hence

$$
a^* = (x_{k+1}',\dots,x_n',\eta_1,\dots,\eta_k,y_{k+1}',\dots,y_n',\tau+\sigma(x,y,x'-x,y'-y))
$$

and

$$
\begin{aligned}
|\phi_i(a)-\phi_i(a^*)| = |\int_0^1 \sum_{j=k+1}^{n} &((x'_j - x_j)\omega_{i,j-k}(\exp_a(tW^{\phi}_{j-k})(a)) \\
&+ (y'_j - y_j)\omega_{i,j+(n-k)}(\exp_a(tW^{\phi}_{j+(n-k)})(a))dt| \\
&\le N(n-k)(|x'-x|+|y'-y|) \\
&\le 2N(n-k)|a-b|.
\end{aligned}
\tag{5.57}
$$

Hence

$$
\begin{aligned}
|\phi(a)-\phi(a^*)| &\le \sum_{i=1}^{k} |\phi_i(a)-\phi_i(a^*)| \\
&\le 2k(n-k)N|a-b|.
\end{aligned}
\tag{5.58}
$$

If we consider

$$
|\sigma(x,y,x'-x,y'-y)| = \left| \frac{1}{2}\sum_{j=k+1}^{n}((x'_j-x_j)y_j - x_j(y'_j-y_j)) \right| \le K(n-k)|a-b|,
$$

since it is controlled by the norm $|a-b|$, we can assume r sufficiently small, and hence a, b sufficiently close, such that $a^* \in I'$. Then we get

$$
\begin{aligned}
|a^*-b| &\le |(\eta'_1-\eta_1,\dots,\eta'_k-\eta_k)| + |\tau'-\tau| + |\sigma(x,y,x'-x,y'-y)| \\
&\le 2|a-b| + K(n-k)|a-b| \\
&= (2+K(n-k))|a-b|.
\end{aligned}
\tag{5.59}
$$

Now

$$
\begin{aligned}
\frac{|\phi(a)-\phi(b)|}{|a-b|^{1/2}} &\le \frac{|\phi(a)-\phi(a^*)|}{|a-b|^{1/2}} + \frac{|\phi(a^*)-\phi(b)|}{|a-b|^{1/2}} \\
&\le \frac{2(n-k)kN|a-b|}{|a-b|^{1/2}} + \left(\frac{1}{2+K(n-k)}\right)\frac{|\phi(a^*)-\phi(b)|}{|a^*-b|^{1/2}},
\end{aligned}
\tag{5.60}
$$

so we are in the particular case we had at the beginning. The last term of (5.60) can then be estimated from above by

$$
\begin{aligned}
&2(n-k)kN|a-b|^{1/2} + \left(\frac{1}{2+K(n-k)}\right)\alpha'(|a^*-b|^{1/2}) \\
\le\ &2(n-k)kN|a-b|^{1/2} + \left(\frac{1}{2+K(n-k)}\right)\alpha'((2+K(n-k))|a-b|^{1/2}),
\end{aligned}
\tag{5.61}
$$

which goes to zero when b goes to a. This concludes our proof. □

5.3.5 The continuity of intrinsic derivatives implies a Hölder-type regularity

Let us modify the hypotheses of Proposition 5.3.14. Notice that the existence and continuity of the intrinsic derivatives of ϕ, $\partial^{\phi_j}\phi : I \to \mathbb{R}^k$ for $j = 1,\dots,2n-k$ would be enough to satisfy the hypotheses of the next proposition.

Proposition 5.3.15. *Let $I \subset \mathbb{R}^{2n+1-k}$ be a rectangle. Let $\phi : I \to \mathbb{R}^k$ be a continuous function and assume that there exist $k \times (2n-k)$ continuous functions $w_{i,\ell} : I \to \mathbb{R}$ such* *that for every $i, \dots, k$ and $\ell = 1, \dots, 2n-k$, for every $\gamma^\ell : [-\delta, \delta] \to I$ integral curve of the vector field W_ℓ^ϕ the following holds*

$$\frac{d}{dt}\phi_i(\gamma^\ell(t)) = w_{i,\ell}(\gamma^\ell(t)) \tag{5.62}$$

for any $t \in [-\delta, \delta]$. Given a fixed rectangle $I' \Subset I$, for any other rectangle I'' such that $I' \Subset I'' \Subset I$, there exists a function

$$\alpha : (0, \infty) \to [0, \infty)$$

which depends on k, on I'', on $\{\|\phi_j\|_{L^\infty(I'')}\}_{j=1,\dots,k}$, on $\|[\omega_{i,j}]_{i,j}\|_{L^\infty(I'')}$ and on the modulus of continuity of the maps $\{\omega_{j,j+(n-k)}\}_{j=1,\dots,k}$, on I'', such that $\lim_{r\to 0}\alpha(r) = 0$ and, for r sufficiently small,

$$\sup\left\{\frac{|\phi(a)-\phi(b)|}{|a-b|^{1/2}} : a, b \in I',\ 0 < |a-b| \le r\right\} \le \alpha(r).$$

Proof. The proof is analogous to the one of Proposition 5.3.14, therefore we keep the same notation. Unique change is that now we reset the variable N to denote the norm in $L^\infty(I'')$ of the matrix containing as (i,j)-th element the value of the maps $\omega_{i,j}$ at the considered point, $N := \|[\omega_{i,j}]_{i,j}\|_{L^\infty(I'')}$. In this setting we lose the uniqueness of the integral curves of ∇^{ϕ_j} for $j = 1, \dots, k$. This lack of uniqueness is replaced by the condition (5.62) on the integral curves of the vector fields ∇^{ϕ_j}. We still denote by γ_a^j an arbitrarily chosen integral curve of ∇^{ϕ_j} ($j = \ell-(n-k)$) of initial point $a = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau) \in J_i$ such that $\gamma_a^j(\eta_j) = a$. We assume as before that γ_a^j it is defined on $[\eta_j - \varepsilon_{i+1,j}, \eta_j + \varepsilon_{i+1,j}]$ such that $\gamma^j([\eta_j - \varepsilon_{i+1,j}, \eta_j + \varepsilon_{i+1,j}]) \subset J_{i+1}$. The loss of uniqueness implies that two such integral curves could indeed meet each other, so the previous contradiction obtained in (5.50) would no longer hold in this case. Therefore, we have to replace it with a different contradiction. We present a new suitable contradiction argument inspired by an argument used in [BSC10b]. Suppose, for the sake of simplicity, that $j = 1$.

As we did in the proof of Proposition 5.3.14, in order to obtain (5.50), we fix as above $a, c \in J_k$ and we assume that they only differ for their vertical coordinate (we assume again $\tau_a = \tau > \tau' = \tau_c$). Then in the proof of Proposition 5.3.14, we proved that if $\phi_1(a) - \phi_1(c) < 0$, there exists $\bar t \in [\eta_1, \eta_1 + \varepsilon_{k+1,j}]$ (or $\bar t \in [\eta_1 - \varepsilon_{k+1,j}, \eta_1]$ if $\phi_1(a) - \phi_1(c) > 0$) such that

$$\tau_a^1(\bar t) - \tau_c^1(\bar t) < 0,$$

while $\tau_a^1(\eta_1) = \tau > \tau' = \tau_c^1(\eta_1)$. We can then define

$$t^* := \sup\{t \in [\eta_1, \eta_1 + \varepsilon_{k+1,1}] : t \le \bar t,\ \tau_a^1(t) > \tau_c^1(t)\}.$$

We have $0 < t^* < \bar t \le \eta_1 + \varepsilon_{k+1,1}$ and, by continuity, $\tau_a^1(t^*) = \tau_c^1(t^*)$, hence

$$\gamma_a^1(t^*) = \gamma_c^1(t^*).$$

Let us prove that $\phi_1(\gamma_a^1(t^*)) \neq \phi_1(\gamma_c^1(t^*))$, so that we will obtain a new contradiction and the thesis can be obtained mirroring verbatim the remaining part of the proof of Proposition 5.3.14. Clearly, the second order derivatives of τ_a^j and τ_c^j are replaced by the maps $\omega_{j,j+(n-k)}$. Remember that if $\phi_1(a) - \phi_1(c) < 0$, we assumed that (5.43), i.e. that

$\phi_1(a) - \phi_1(c) < -\delta_1\sqrt{\tau - \tau'}$, hence

$$\begin{aligned}
\phi_1(\gamma_a^1(t^*)) - \phi_1(\gamma_c^1(t^*)) &= \phi_1(a) - \phi_1(c) + \int_{\eta_1}^{t^*} \omega_{1,n-k+1}(\gamma_a^1(s)) - \omega_{1,n-k+1}(\gamma_c^1(s))ds \\
&\le \phi_1(a) - \phi_1(c) + (t^* - \eta_1)\beta_1(|\tau - \tau'| + 2M_1|t^* - \eta_1|) \\
&\le \phi_1(a) - \phi_1(c) + (\bar{t} - \eta_1)\beta_1(|\tau - \tau'| + 2M_1|\bar{t} - \eta_1|) \\
&\quad \text{keeping in mind (5.43) and (5.47),} \\
&< -\delta_1\sqrt{\tau - \tau'} + 2k\frac{\beta_1(|\tau - \tau'| + 2M_1|\bar{t} - \eta_1|)}{\delta_1}\sqrt{\tau - \tau'} \\
&\le -\delta_1\sqrt{\tau - \tau'} + 2k\frac{\beta_1(|\tau - \tau'| + 4kM_1\sqrt{|\tau - \tau'|/\delta_1})}{\delta_1}\sqrt{\tau - \tau'} =
\end{aligned}$$

if $\delta_1 < 1$ (and we can choose r small enough such that $\delta_j < 1$ for any $j = 1, \dots, k$)

$$\begin{aligned}
&= 2\delta_1\sqrt{\tau - \tau'}\left(-\frac{1}{2} + k\frac{\beta_1(|\tau - \tau'| + 4kM_1\sqrt{|\tau - \tau'|/\delta_1(r)})}{\delta_1^2}\right) \\
&< 2\delta_1\sqrt{\tau - \tau'}\left(-\frac{1}{2} + \frac{k}{8k}\right) < 0.
\end{aligned} \tag{5.63}$$

This proof, after small modification, works also for the case when $\phi_1(a) - \phi_1(c) > 0$, starting from $\phi_1(\gamma_c^1(t^*)) - \phi_1(\gamma_a^1(t^*))$ and using hypotheses (5.43), and (5.46); of course it works also for the curves γ_a^j and γ_c^j, for $j = 2, \dots, k$, so that

$$\frac{\sum_{j=1}^k |\phi_j(a) - \phi_j(c)|}{|\tau - \tau'|^{\frac{1}{2}}} \le \sum_{j=1}^k \delta_j.$$

Hence (ii) has to be valid, and we can resume verbatim the proof of Proposition 5.3.14 from (5.51). □

By a standard compactness argument we get the following.

Proposition 5.3.16. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function such that there are $k \times (2n-k)$ continuous functions $\omega_{i,j} : U \to \mathbb{R}$ such that for every $i = 1, \dots, k$ and $j = 1, \dots, 2n-k$, for every integral curve $\gamma^j : [-\delta, \delta] \to U$ integral curve of the vector field W_j^ϕ the following holds*

$$\frac{d}{dt}\phi_i(\gamma^j(t)) = w_{i,j}(\gamma^j(t)),$$

for any $t \in [-\delta, \delta]$. Then, if we fix an open set $U' \Subset U$, we know that for any open U'' such that $U' \Subset U'' \Subset U$ there exists a function

$$\alpha : (0, \infty) \to [0, \infty)$$

which depends on U'', on k, on $\{\|\phi_j\|_{L^\infty(U'')}\}_{j=1,\dots,k}$, on $\|[\omega_{i,j}]_{i,j}\|_{L^\infty(U'')}$ and on the modulus of continuity of the maps $\{\omega_{j,j+(n-k)}\}_{j=1,\dots,k}$ on U'', such that $\lim_{r\to 0}\alpha(r) = 0$ and, for r sufficiently small,

$$\sup\left\{\frac{|\phi(a) - \phi(b)|}{|a-b|^{1/2}} : a, b \in U',\ 0 < |a-b| \le r\right\} \le \alpha(r). \tag{5.64}$$

5.3.6 Characterizations of uniform intrinsic differentiability

Proposition 5.3.17. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set, let $\phi : U \to \mathbb{R}^k$ be a continuous function and let $a, b \in U$ be two generic points*

$$a = (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau),\ b = (x'_{k+1}, \dots, x'_n, \eta'_1, \dots, \eta'_k, y'_{k+1}, \dots, y'_n, \tau').$$

Set

$$\xi := (x_{k+1} - x'_{k+1}, \dots, x_n - x'_n, \eta_1 - \eta'_1, \dots, \eta_k - \eta'_k, y_{k+1} - y'_{k+1}, \dots, y_n - y'_n),$$

and consider the following function, ρ_ϕ, that is analogous of the one considered in [ASCV06]

$$\rho_\phi(a, b) := \max\{|\xi|, |\tau - \tau' + \frac{1}{2}\sum_{j=1}^{k}(\phi'_j + \phi_j)(\eta'_j - \eta_j) + \sigma(x, y, x', y')|^{\frac{1}{2}}\}, \tag{5.65}$$

where $\sigma(x, y, x', y') := \frac{1}{2}\sum_{j=k+1}^{n}(x_j y'_j - x'_j y_j)$, $\phi_j := \phi_j(a)$ and $\phi'_j := \phi_j(b)$ for $j = 1, \dots, k$. If there exists a constant $c > 0$ such that

$$|\phi(a) - \phi(b)| \le c\ \rho_\phi(a, b)$$

for every $a, b \in U$, then ϕ is intrinsic Lipschitz.

Proof. If

$$|\phi(a) - \phi(b)| \le c|(x_{k+1} - x'_{k+1}, \dots, x_n - x'_n, \eta_1 - \eta'_1, \dots, \eta_k - \eta'_k, y_{k+1} - y'_{k+1}, \dots, y_n - y'_n)|$$

the thesis is valid. Let us then consider the case when

$$\begin{aligned}
|\phi(a) - \phi(b)| &\le c\ |\tau - \tau' + \frac{1}{2}\sum_{j=1}^{k}(\phi'_j + \phi_j)(\eta'_j - \eta_j) + \sigma(x, y, x', y')|^{\frac{1}{2}} \\
&= c\ |\tau - \tau' + \sum_{j=1}^{k}\phi'_j(\eta'_j - \eta_j) + \frac{1}{2}\sum_{j=1}^{k}(\phi_j - \phi'_j)(\eta'_j - \eta_j) + \sigma(x, y, x', y')|^{\frac{1}{2}} \\
&\quad \text{for any } \varepsilon > 0 \\
&\le c\left(d_\phi(a, b) + \frac{1}{2}\sum_{j=1}^{k}\left|(\phi_j - \phi'_j)\ \varepsilon\left(\frac{\eta'_j - \eta_j}{\varepsilon}\right)\right|^{\frac{1}{2}}\right) \\
&\le c\left(d_\phi(a, b) + \sum_{j=1}^{k}\left(\frac{1}{4}\varepsilon|\phi_j - \phi'_j| + \frac{1}{4}\frac{|\eta'_j - \eta_j|}{\varepsilon}\right)\right) \\
&\le c\left(d_\phi(a, b) + k\frac{1}{4}\varepsilon|\phi(a) - \phi(b)| + k\frac{1}{4}\frac{d_\phi(a, b)}{\varepsilon}\right).
\end{aligned}$$

We denoted by $|\cdot|$ both the Euclidean norm on $\mathbb{R}$ and on $\mathbb{R}^k$. If we now fix $\varepsilon = \frac{2}{ck}$, we finally get

$$|\phi(a) - \phi(b)| \le 2\left(c + \frac{k^2c^2}{8}\right) d_\phi(a, b).$$

□

Proposition 5.3.18. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be an intrinsic Lipschitz function. Then, there exists a constant $c > 0$ such that*

$$\rho_\phi(a,b) \leq c\ d_\phi(a,b)$$

for every $a, b \in U$

Proof. By direct computations , we have

$$\begin{aligned}\rho_\phi(a,b) \leq&\ d(\Phi(a),\Phi(b)) \\ &= \|\Phi(b)^{-1}\Phi(a)\| \\ &\leq d_\phi(a,b) + |\phi(a)-\phi(b)| \\ &\leq (1+\mathrm{Lip}(\phi))\ d_\phi(a,b).\end{aligned} \tag{5.66}$$

□

The existence of a family of exponential maps and a $\frac{1}{2}$-Hölder-type regularity of the map are sufficient to ensure uniform intrinsic differentiability.

Theorem 5.3.19. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function. If for a certain $a \in U$ there exist $0 < \delta_2 < \delta_1$ and a family of exponential maps at a*

$$\exp_a(sW_\ell^\phi)(b) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(a)} \to \overline{I_{\delta_1}(a)}$$

for $\ell = 1, \dots, 2n-k$, and if

$$\lim_{r\to 0+} \sup\left\{\frac{|\phi(b')-\phi(b)|}{|b'-b|^{1/2}} : b, b' \in U',\ 0 < |b'-b| \leq r\right\} = 0 \tag{5.67}$$

for every open set $U' \Subset U$, then ϕ is uniformly intrinsically differentiable at a. Moreover, in this case, the (i,ℓ)-th component $[D^\phi\phi(a)]_{i,\ell}$ of the intrinsic Jacobian matrix of ϕ at a equals

$$[D^\phi\phi(a)]_{i,\ell} = \frac{d}{ds}\phi_i(\exp_a(sW_\ell^\phi)(a))\big|_{s=0}$$

for $i = 1, \dots, k$ and $\ell = 1, \dots, 2n-k$.

Remark 5.3.20. Notice that, according to the proof below, hypothesis (5.67) can be localized in the sense that one can substitute (5.67) with the following condition

$$\lim_{r\to 0+} \sup\left\{\frac{|\phi(b')-\phi(b)|}{|b'-b|^{1/2}} : b, b' \in I_r(a),\ b \neq b'\right\} = 0. \tag{5.68}$$

Condition (5.68) better highlights the fact that the uniform intrinsic differentiability of ϕ at a is a local property.

Proof. We set

$$\begin{aligned} a &= (\bar{x}_{k+1}, \dots, \bar{x}_n, \bar{\eta}_1, \dots, \bar{\eta}_k, \bar{y}_{k+1}, \dots, \bar{y}_n, \bar{\tau}) \in U, \\ b &= (x_{k+1}, \dots, x_n, \eta_1, \dots, \eta_k, y_{k+1}, \dots, y_n, \tau) \in I_{\delta_0}(a), \\ b' &= (x'_{k+1}, \dots, x'_n, \eta'_1, \dots, \eta'_k, y'_{k+1}, \dots, y'_n, \tau') \in I_{\delta_0}(a), \end{aligned} \tag{5.69}$$

for δ_0 small; surely

$$|(x'_{k+1}-x_{k+1}, \dots, x'_n - x_n, \eta'_1 - \eta_1, \dots, \eta'_k - \eta_k, y'_{k+1} - y_{k+1}, \dots, y'_n - y_n)| \leq 2(2n-k)\delta_0. \tag{5.70}$$

Just to simplify the computation we assume that $\eta_i' \geq \eta_i$, for $i = 1, \ldots, k$.
Let us define the smooth vector field

$$\bar{X} := \sum_{j=k+1}^{n} (x_j' - x_j) W_{j-k}^{\phi} + (y_j' - y_j) W_{j+(n-k)}^{\phi} = \sum_{j=k+1}^{n} (x_j' - x_j)\tilde{X}_j + (y_j' - y_j)\tilde{Y}_j.$$

We start moving from b to the point

$$b_0^* := \exp_a(\bar{X})(b) = b \star \left(\sum_{j=k+1}^{n} (x_j' - x_j)\mathbf{e}_{j-k} + (y_j' - y_j)\mathbf{e}_{j+(n-k)} \right)$$

(recall that $\mathbf{e}_j$ denotes the j-th vector of the canonical basis of $\mathbb{R}^{2n+1-k}$, see Remark 5.3.4). Then we move for a time $\eta_1' - \eta_1$ along the exponential map at a of $W_{n-k+1}^{\phi} = \nabla^{\phi_1}$ with initial point b_0^*. We reach a point b_1^* and then we move for a time $\eta_2' - \eta_2$ along the exponential map at a of $W_{n-k+2}^{\phi} = \nabla^{\phi_2}$ with initial point b_1^*. We denote by b_2^* the endpoint of this two-piecewise integral curve and we iterate this process setting for $j = 0, \ldots, k-1$

$$\begin{aligned} b_{j+1}^* : &= \exp_a((\eta_{j+1}' - \eta_{j+1}) W_{(n-k)+j+1}^{\phi})(b_j^*) \\ &= \exp_a((\eta_{j+1}' - \eta_{j+1}) \nabla^{\phi_{j+1}})(b_j^*). \end{aligned}$$

The coordinates of b_k^* equal the ones of b', except for the vertical one denoted by τ_k^*:

$$\tau_k^* = \tau + \sum_{j=1}^{k} \int_0^{\eta_j' - \eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr + \sigma(x, y, x', y'), \tag{5.71}$$

where $\sigma(x, y, x', y')$ is the same defined in (5.5). The point b_k^* belongs to a square $I_{C\delta_0 + D\delta_0^2}(a)$ for some positive constant C and D. In fact

$$\begin{aligned} &|\tau_k^* - \bar{\tau}| = |\tau + \sum_{j=1}^{k} \int_0^{\eta_j' - \eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr + \sigma(x, y, x', y') - \bar{\tau}| \\ \leq &|\tau - \bar{\tau}| + \sum_{i=1}^{k} (\eta_i' - \eta_i) \max_{I_{\delta_1(a)}} |\phi_i| + \frac{1}{2} | \sum_{i=k+1}^{n} (x_i(y_i' - y_i) - y_i(x_i' - x_i))| \\ \leq &|\tau - \bar{\tau}| + \sum_{i=1}^{k} (\eta_i' - \eta_i) \max_{I_{\delta_1(a)}} |\phi_i| + \frac{1}{2} | \sum_{i=k+1}^{n} ((x_i - \bar{x}_i + \bar{x}_i)(y_i' - y_i) - (y_i - \bar{y}_i + \bar{y}_i)(x_i' - x_i))| \\ \leq &|\tau - \bar{\tau}| + \sum_{i=1}^{k} |\eta_i' - \eta_i| \max_{I_{\delta_1(a)}} |\phi_i| + \frac{1}{2} \sum_{i=k+1}^{n} (|x_i - \bar{x}_i| + |\bar{x}_i|)|y_i' - y_i| + (|y_i - \bar{y}_i| + |\bar{y}_i|)|x_i' - x_i| \\ \leq &|\tau - \bar{\tau}| + \sum_{i=1}^{k} (\eta_i' - \eta_i) \max_{I_{\delta_1(a)}} |\phi_i| \\ &+ \frac{1}{2} \sum_{i=k+1}^{n} ((|\bar{x}_i| + \delta_0)(|y_i' - \bar{y}_i| + |y_i - \bar{y}_i|) + (|\bar{y}_i| + \delta_0)(|x_i' - \bar{x}_i| + |x_i - \bar{x}_i|)) \\ \leq &\ \delta_0 + \sum_{i=1}^{k} \delta_0 \max_{I_{\delta_1(a)}} |\phi_i| \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=k+1}^{n}((|\bar{x}_i|+\delta_0)(2\delta_0)+(|\bar{y}_i|+\delta_0)(2\delta_0))\\
&< C\delta_0+D\delta_0^2.
\end{aligned}
$$

Now, we can consider that

$$
\begin{aligned}
\phi(b')-\phi(b) &= \phi(b')-\phi(b_k^*)+\sum_{i=1}^{k}(\phi(b_i^*)-\phi(b_{i-1}^*))+\phi(b_0^*)-\phi(b)\\
&= \phi(b')-\phi(b_k^*)+\sum_{i=1}^{k}(\phi(\exp_a((\eta_i'-\eta_i)\nabla^{\phi_i})(b_{i-1}^*))-\phi(b_{i-1}^*))+\phi(b_0^*)-\phi(b)\\
&=\phi(b')-\phi(b_k^*)+\sum_{j=1}^{k}\begin{pmatrix}\int_0^{\eta_j'-\eta_j}\omega_{1,j+(n-k)}(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr\\ \cdots\\ \int_0^{\eta_j'-\eta_j}\omega_{k,j+(n-k)}(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr\end{pmatrix} \qquad (5.72)\\
&\quad+\sum_{j=k+1}^{n}\begin{pmatrix}\int_0^1(x_j'-x_j)\omega_{1,j-k}(\exp_a(r\bar{X})(b))dr\\ \cdots\\ \int_0^1(x_j'-x_j)\omega_{k,j-k}(\exp_a(r\bar{X})(b))dr\end{pmatrix}\\
&\quad+\sum_{j=k+1}^{n}\begin{pmatrix}\int_0^1(y_j'-y_j)\omega_{1,j+(n-k)}(\exp_a(r\bar{X})(b))dr\\ \cdots\\ \int_0^1(y_j'-y_j)\omega_{k,j+(n-k)}(\exp_a(r\bar{X})(b))dr\end{pmatrix}.
\end{aligned}
$$

For $i=1,\dots,k$ and $\ell=1,\dots,2n-k$, we denote as usual by $\omega_{i,\ell}$ the maps in (iii) of Definition 5.3.2.

Claim 1. For any $i=1,\dots,k$, for $\ell=n-k+1,\dots,n$, $j=\ell-(n-k)$ (and then for $j=1,\dots,k$)

$$
\int_0^{\eta_j'-\eta_j}\omega_{i,\ell}(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr=\omega_{i,\ell}(a)(\eta_j'-\eta_j)+o(|\eta_j'-\eta_j|) \quad \text{as } \delta_0\to 0.
$$

Let us prove Claim 1. Fix $i\in\{1,\dots,k\}$ and consider for every j

$$
\int_0^{\eta_j'-\eta_j}\omega_{i,\ell}(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))-\omega_{i,\ell}(a)dr+\omega_{i,\ell}(a)(\eta_j'-\eta_j).
$$

We want to prove that

$$
\lim_{\delta_0\to 0}\frac{1}{\eta_j'-\eta_j}\int_0^{\eta_j'-\eta_j}\omega_{i,\ell}(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))-\omega_{i,\ell}(a)dr=0.
$$

Let us first show that

$$
|\omega_{i,\ell}(b_0^*)-\omega_{i,\ell}(a)|=o(1) \qquad \text{as } \delta_0\to 0.
$$

In fact, we have

$$
\begin{aligned}
|\omega_{i,\ell}(b_0^*)-\omega_{i,\ell}(a)| &\le |\omega_{i,\ell}(b_0^*)-\omega_{i,\ell}(b)|+|\omega_{i,\ell}(b)-\omega_{i,\ell}(a)|\\
&\le \beta_{i,\ell}(|b_0^*-b|)+\beta_{i,\ell}(|b-a|),
\end{aligned}
$$

where $\beta_{i,\ell}$ is the modulus of continuity of $\omega_{i,\ell}$. Let us now observe that the two terms go to zero. Indeed, $\omega_{i,\ell}$ is continuous by hypothesis. Since (5.70) holds, we know that $|(x',y')-(x,y)| \to 0$ as $\delta_0 \to 0$, and we can then find a real number $\bar{\delta} > 0$ such that $|(x',y')-(x,y)| \le c\delta_0 < \delta$ for $\delta_0 < \bar{\delta}$. Hence

$$\lim_{\delta_0 \to 0} |\omega_{i,\ell}(b_0^*) - \omega_{i,\ell}(b)| = 0.$$

Moreover, when δ_0 goes to zero, b and b' get closer and closer to a, so when δ_0 goes to zero, $|b-a|$ goes to zero too.
Once we fix $p \in \{1,\dots,k\}$, we get

$$\begin{aligned}
&\frac{1}{\eta_p'-\eta_p}\int_0^{\eta_p'-\eta_p} \omega_{i,n-k+p}(\exp_a(r\nabla^{\phi_p})(b_{p-1}^*)) - \omega_{i,n-k+p}(a)dr \\
&= \frac{1}{\eta_p'-\eta_p}\int_0^{\eta_p'-\eta_p} \omega_{i,n-k+p}(\exp_a(r\nabla^{\phi_p})(b_{p-1}^*)) - \omega_{i,n-k+p}(b_{p-1}^*)dr \\
&\quad + \sum_{i=2}^{p}(\omega_{i,n-k+p}(b_{k-1}^*) - \omega_{i,n-k+p}(b_{k-2}^*))) + \omega_{i,n-k+p}(b_0^*) - \omega_{i,n-k+p}(a) \\
&\le \sup_{r\in[0,\eta_p'-\eta_p]} |\omega_{i,n-k+p}(\exp_a(r\nabla^{\phi_p})(b_{p-1}^*)) - \omega_{i,p}(b_{p-1}^*)| \\
&\quad + \sum_{i=2}^{p} |(\omega_{i,n-k+p}(b_{k-1}^*) - \omega_{i,n-k+p}(b_{k-2}^*)))| + |\omega_{i,n-k+p}(b_0^*) - \omega_{i,n-k+p}(a)|,
\end{aligned}$$

which goes to zero as δ_0 tends to zero, by what we have already proved and by the fact that if δ_0 goes to zero, then $|\eta_j'-\eta_j|$ goes to zero for $j=1,\dots,p$. Hence $|\omega_{i,n-k+p}(b_{k-1}^*) - \omega_{i,n-k+p}(b_{k-2}^*)| \le \beta_{i,n-k+p}(|b_{k-1}^* - b_{k-2}^*|)$ goes to zero. We finally reach the conclusion from the absolute continuity of $\omega_{i,n-k+p}(\exp_a(r\nabla^{\phi_1})(b_{p-1}^*))$ on $[0,\eta_p'-\eta_p]$ and then Claim 1 is proved.
Since Claim 1 holds, we can rewrite (5.72) as

$$\begin{aligned}
&\phi(b') - \phi(b_k^*) + \sum_{j=1}^{k}\begin{pmatrix}\omega_{1,j+(n-k)}(a)(\eta_j'-\eta_j) + o(|\eta_j'-\eta_j|)\\ \dots \\ \omega_{k,j+(n-k)}(a)(\eta_j'-\eta_j) + o(|\eta_j'-\eta_j|)\end{pmatrix} \\
&\qquad + \sum_{j=k+1}^{n}\begin{pmatrix}\omega_{1,j-k}(a)(x_j'-x_j) + o(|x_j'-x_j|)\\ \dots \\ \omega_{k,j-k}(a)(x_j'-x_j) + o(|x_j'-x_j|)\end{pmatrix} \\
&\qquad + \sum_{j=k+1}^{n}\begin{pmatrix}\omega_{1,j+(n-k)}(a)(y_j'-y_j) + o(|y_j'-y_j|)\\ \dots \\ \omega_{k,j+(n-k)}(a)(y_j'-y_j) + o(|y_j'-y_j|)\end{pmatrix} \\
&= \phi(b') - \phi(b_k^*) + \begin{pmatrix}\omega_{1,1}(a) & \omega_{1,2}(a) & \dots & \omega_{1,2n-k}(a)\\ \omega_{2,1}(a) & \omega_{2,2}(a) & \dots & \omega_{2,2n-k}(a)\\ \dots & & & \\ \dots & & & \\ \dots & & & \\ \omega_{k,1}(a) & \omega_{k,2}(a) & \dots & \omega_{k,2n-k}(a)\end{pmatrix}\begin{pmatrix}x_{k+1}'-x_{k+1}\\ \dots \\ x_n'-x_n\\ \eta_1'-\eta_1\\ \dots \\ \eta_k'-\eta_k\\ y_{k+1}'-y_{k+1}\\ \dots \\ y_n'-y_n\end{pmatrix}
\end{aligned}$$

$$
+\begin{pmatrix} \sum_{j=k+1}^{n} o(|x'_j - x_j|) + o(|y'_j - y_j|) + \sum_{j=1}^{k} o(|\eta'_j - \eta_j|) \\ \cdots \\ \sum_{j=k+1}^{n} o(|x'_j - x_j|) + o(|y'_j - y_j|) + \sum_{j=1}^{k} o(|\eta'_j - \eta_j|) \end{pmatrix}
$$

$$
= \phi(b') - \phi(b_k^*) + \begin{pmatrix} \omega_{1,1}(a) & \omega_{1,2}(a) & \dots & \omega_{1,2n-k}(a) \\ \omega_{2,1}(a) & \omega_{2,2}(a) & \dots & \omega_{2,2n-k}(a) \\ \dots & & & \\ \dots & & & \\ \dots & & & \\ \omega_{k,1}(a) & \omega_{k,2}(a) & \dots & \omega_{k,2n-k}(a) \end{pmatrix} \begin{pmatrix} x'_{k+1} - x_{k+1} \\ \dots \\ x'_n - x_n \\ \eta'_1 - \eta_1 \\ \dots \\ \eta'_k - \eta_k \\ y'_{k+1} - y_{k+1} \\ \dots \\ y'_n - y_n \end{pmatrix}
$$

$$
+\begin{pmatrix} o(d_\phi(b,b')) \\ \dots \\ o(d_\phi(b,b')) \end{pmatrix}
$$

as δ_0 goes to zero, since $|x'_j - x_j| \le d_\phi(b,b')$, $|y'_j - y_j| \le d_\phi(b,b')$, $|\eta'_j - \eta_j| \le d_\phi(b,b')$. The same argument yields that

$$
\phi(b') - \phi(b) \le \phi(b') - \phi(b_k^*)
$$

$$
+\begin{pmatrix} \omega_{1,1}(a) & \omega_{1,2}(a) & \dots & \omega_{1,2n-k}(a) \\ \omega_{2,1}(a) & \omega_{2,2}(a) & \dots & \omega_{2,2n-k}(a) \\ \dots & & & \\ \dots & & & \\ \dots & & & \\ \omega_{k,1}(a) & \omega_{k,2}(a) & \dots & \omega_{k,2n-k}(a) \end{pmatrix} \begin{pmatrix} x'_{k+1} - x_{k+1} \\ \dots \\ x'_n - x_n \\ \eta'_1 - \eta_1 \\ \dots \\ \eta'_k - \eta_k \\ y'_{k+1} - y_{k+1} \\ \dots \\ y'_n - y_n \end{pmatrix} + \begin{pmatrix} o(\rho_\phi(b,b')) \\ \dots \\ o(\rho_\phi(b,b')) \end{pmatrix}. \tag{5.73}
$$

In order to get the thesis, we are left to prove that

$$
|\phi(b') - \phi(b_k^*)| = o(d_\phi(b,b')) \text{ as } \delta_0 \to 0. \tag{5.74}
$$

To prove this, it is enough to show that

$$
|\phi(b') - \phi(b_k^*)| = o(\rho_\phi(b,b')) \text{ as } \delta_0 \to 0. \tag{5.75}
$$

In fact, if (5.75) holds, we can apply (5.73) to get that

$$
\lim_{\delta_0 \to 0} \sup_{\substack{b,b' \in I_{\delta_0}(a) \\ b \neq b'}} \left\{ \frac{|\phi(b') - \phi(b) - M(a) \cdot (\pi(b^{-1} \cdot b'))^T|}{\rho_\phi(b',b)} \right\} = 0, \tag{5.76}
$$

where $M(a)$ is the $k \times (2n-k)$ matrix $[M(a)]_{i,j} = \omega_{i,j}(a)$ for $i = 1, \dots, k$, $j = 1, \dots, 2n-k$. Now, this implies (for instance refer to [AS09, Proposition 3.17]) that for every $b, b' \in I_{\delta_0}(a)$, there exists a constant $c > 0$ such that

$$
|\phi(b) - \phi(b')| \le c\, \rho_\phi(b,b'). \tag{5.77}
$$

By Propositions 5.3.17 and 5.3.18, inequality (5.77) implies that there exists a constant

$c_2 > 0$ such that for every $b, b' \in I_{\delta_0}(a)$, $\rho_\phi(b,b') \le c_2 d_\phi(b,b')$, and therefore for every $b, b' \in I_{\delta_0}(a)$,

$$0 \le \frac{1}{c_2}\frac{|\phi(b)-\phi(b_k^*)|}{d_\phi(b,b')} \le \frac{|\phi(b)-\phi(b_k^*)|}{\rho_\phi(b,b')}.$$

This means that if we prove (5.75), then (5.74) will follow.
Let us start by adapting an argument by [ASCV06, Theorem 5.7]:

$$\begin{aligned}
\frac{|\phi(b')-\phi(b_k^*)|}{\rho_\phi(b,b')} &= \frac{|\phi(b')-\phi(b_k^*)|}{|\tau'-\tau_k^*|^{1/2}}\frac{|\tau'-\tau_k^*|^{1/2}}{\rho_\phi(b,b')} \\
&= \frac{|\phi(b')-\phi(b_k^*)|}{|b'-b_k^*|^{1/2}}\frac{|\tau'-\tau_k^*|^{1/2}}{\rho_\phi(b,b')} \\
&\le \upsilon_\phi(C\delta_0+D\delta_0^2)\frac{|\tau'-\tau_k^*|^{1/2}}{\rho_\phi(b,b')},
\end{aligned} \tag{5.78}$$

where the function

$$\upsilon_\phi(\delta) := \sup\left\{\frac{|\phi(a')-\phi(a'')|}{|a'-a''|^{1/2}} : a' \neq a'', a', a'' \in I_\delta(a)\right\} \tag{5.79}$$

goes to zero if $\delta \to 0$ by the second hypothesis, (5.67).
In order to achieve the proof of (5.75), we need to show that $\frac{|\tau'-\tau_k^*|^{1/2}}{\rho_\phi(b,b')}$ is bounded close to a. By (5.65) and (5.71),

$$\begin{aligned}
|\tau-\tau_k^*| &= |\tau'-\tau-\sigma(x,y,x',y') - \sum_{j=1}^k \int_0^{\eta_j'-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr| \\
&= |\tau'-\tau-\sigma(x,y,x',y') - \sum_{j=1}^k \int_0^{\eta_j'-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr \\
&\quad + \frac{1}{2}\sum_{j=1}^k(\phi_j(b')+\phi_j(b))(\eta_j'-\eta_j) - \frac{1}{2}\sum_{j=1}^k(\phi_j(b')+\phi_j(b))(\eta_j'-\eta_j)| \\
&\le \rho_\phi(b,b')^2 + |-\sum_{j=1}^k \int_0^{\eta_j'-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr \\
&\quad + \frac{1}{2}\sum_{j=1}^k(\phi_j(b')+\phi_j(b))(\eta_j'-\eta_j)| \\
&= \rho_\phi(b,b')^2 + |-\sum_{j=1}^k \int_0^{\eta_j'-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr \\
&\quad + \frac{1}{2}(\sum_{j=1}^k(\phi_j(b')-\phi_j(b_j^*)+\phi_j(b_j^*)+\phi_j(b_{j-1}^*)-\phi_j(b_{j-1}^*)+\phi_j(b))(\eta_j'-\eta_j)| \\
&\le \rho_\phi(b,b')^2 \\
&\quad + |-\sum_{j=1}^k(\int_0^{\eta_j'-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b_{j-1}^*))dr + \frac{1}{2}(\phi_j(b_j^*)+\phi_j(b_{j-1}^*))(\eta_j'-\eta_j))| \\
&\quad + |\frac{1}{2}(\sum_{j=1}^k(\phi_j(b')-\phi_j(b_j^*)+\phi_j(b)-\phi_j(b_{j-1}^*))(\eta_j'-\eta_j)|.
\end{aligned} \tag{5.80}$$

For any j, by Claim 1, at least for δ_0 small enough, we have

$$
\begin{aligned}
&|-\int_0^{\eta'_j-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b^*_{j-1}))dr + \frac{1}{2}(\phi_j(b^*_j)+\phi_j(b^*_{j-1}))(\eta'_j-\eta_j))| \\
&= |-\int_0^{\eta'_j-\eta_j} \phi_j(\exp_a(r\nabla^{\phi_j})(b^*_{j-1})) - \phi_j(b^*_{j-1})dr + \frac{1}{2}(\phi_j(b^*_j)-\phi_j(b^*_{j-1}))(\eta'_j-\eta_j))| \\
&= |-\int_0^{\eta'_j-\eta_j} \int_0^r \omega_{j,j+(n-k)}(\exp_a(s\nabla^{\phi_j})(b^*_{j-1}))ds\ dr \\
&\quad + \frac{1}{2}(\eta'_j-\eta_j)\int_0^{\eta'_j-\eta_j} \omega_{j,j+(n-k)}(\exp_a(r\nabla^{\phi_j})(b^*_{j-1}))dr| \\
&= O(|\eta'_j-\eta_j|^2) \\
&= O(\rho_\phi(b,b'))^2.
\end{aligned}
$$

Hence we can estimate the last line of (5.80) from above by

$$
\rho_\phi(b,b')^2 + C\rho_\phi(b,b')^2 + |\frac{1}{2}(\sum_{j=1}^k(\phi_j(b')-\phi_j(b^*_j)+\phi_j(b)-\phi_j(b^*_{j-1}))(\eta'_j-\eta_j)|.
$$

We are left to estimate

$$
\begin{aligned}
&|\frac{1}{2}(\sum_{j=1}^k(\phi_j(b')-\phi_j(b^*_j)+\phi_j(b)-\phi_j(b^*_{j-1}))(\eta'_j-\eta_j)| \\
&= |\frac{1}{2}\sum_{j=1}^k\{(\phi_j(b')-\phi_j(b^*_k)) + \sum_{i=j}^{k-1}(\phi_j(b^*_{i+1})-\phi_j(b^*_i)) \\
&\quad + (\phi_j(b)-\phi_j(b^*_0)) + \sum_{i=0}^{j-2}(\phi_j(b^*_i)-\phi_j(b^*_{i+1}))\}(\eta'_j-\eta_j)| \qquad (5.81)\\
&\le \frac{1}{2}\sum_{j=1}^k\{|\phi_j(b')-\phi_j(b^*_k)| + \sum_{i=j}^{k-1}|\phi_j(b^*_{i+1})-\phi_j(b^*_i)| \\
&\quad + |\phi_j(b)-\phi_j(b^*_0)| + \sum_{i=0}^{j-2}|\phi_j(b^*_i)-\phi_j(b^*_{i+1})|\}(\eta'_j-\eta_j).
\end{aligned}
$$

Let us then estimate the different components of (5.81).

- First of all, for a fixed $j=1,\dots,k$

$$
\begin{aligned}
&|\frac{1}{2}(\phi_j(b')-\phi_j(b^*_k))(\eta'_j-\eta_j)| \\
&= \frac{1}{2}\frac{|\phi_j(b')-\phi_j(b^*_k)|}{|\tau'-\tau^*_k|^{1/2}}|\tau'-\tau^*_k|^{1/2}|\eta'_j-\eta_j| \qquad (5.82)\\
&\le \frac{1}{2}\upsilon_{\phi_j}(C\delta_0+D\delta_0^2)|\tau'-\tau^*_k|^{1/2}|\eta'_j-\eta_j|.
\end{aligned}
$$

  Of course the function $\upsilon_{\phi_j}(\delta)$ goes to 0 when δ goes to 0 again by the second hypothesis.

We can estimate the last line of (5.82) from above by

$$\frac{1}{4}(\upsilon_{\phi_j}(C\delta_0+D\delta_0^2)^2|\tau'-\tau_k^*|+|\eta_j'-\eta_j|^2). \tag{5.83}$$

If b and b' become sufficiently close, then also b, b', a become sufficiently close, as well as b, to b_k^*. In other words, for every $\varepsilon>0$, there exists $\delta_{\varepsilon,j}>0$ such that if $\delta\in(0,\delta_{\varepsilon,j}]$, $\upsilon_{\phi_j}(\delta)^2\le\varepsilon$, then, when $\delta_0<\delta_{\varepsilon,j}$ is small enough, we can estimate (5.83) from above by

$$\frac{1}{4}(\varepsilon|\tau'-\tau_k^*|+|\eta_j'-\eta_j|^2)\le\ \frac{1}{4}\varepsilon|\tau'-\tau_k^*|+\text{const }(\rho_\phi(b,b'))^2.$$

For instance, we can fix $\varepsilon=2$, and if we take δ small enough, we can carry this contribute to the left hand side of (5.81).

- We can now consider for any fixed $j=1,\dots,k$

$$\begin{aligned}
&\frac{1}{2}|(\phi_j(b)-\phi_j(b_0^*))(\eta_j'-\eta_j)|\\
=&\frac{1}{2}|\eta_j'-\eta_j||\phi_j(b)-\phi_j(b_0^*)|\\
=&\frac{1}{2}|\eta_j'-\eta_j|\sum_{j=k+1}^{n}(|x_j'-x_j|(\omega_{i,j}(a)+o(1))+|y_j'-y_j|(\omega_{i,n+j}(a)+o(1)))\\
\le&\frac{1}{2}c_2|\eta_j'-\eta_j||(x'-x,y'-y)|\\
\le&\frac{1}{2}c_2|\eta'-\eta||(x'-x,y'-y)|\\
\le&\frac{1}{4}c_2|\eta'-\eta|^2+\frac{1}{4}|(x'-x,,y'-y)|^2\\
\le&C_2(\rho_\phi(b,b'))^2.
\end{aligned}$$

- Let us now fix $j\in\{1,\dots,k\}$ and $i\in\{0,\dots,j-2,j,\dots,k-1\}$ and we want to estimate

$$\begin{aligned}
&|\frac{1}{2}(\phi_j(b_{i+1}^*)-\phi_j(b_i^*))(\eta_j'-\eta_j)|=\ \frac{1}{2}|\phi_j(\exp_a((\eta_{i+1}'-\eta_{i+1})\nabla^{\phi_{i+1}})(b_i^*))-\phi_j(b_i^*)||\eta_j'-\eta_j|\\
=&\frac{1}{2}|\int_0^{\eta_{i+1}'-\eta_{i+1}}\omega_{j,i+1+(n-k)}(\exp_a(r\nabla^{\phi_{i+1}})(b_i^*))dr||\eta_j'-\eta_j|\\
\le&\frac{1}{2}|\int_0^{\eta_{i+1}'-\eta_{n+1}}\omega_{j,i+1+(n-k)}(\exp_a(r\nabla^{\phi_{i+1}})(b_i^*))-\omega_{j,i+1+(n-k)}(b_i^*)dr||\eta_j'-\eta_j|\\
&+|\omega_{j,i+1+(n-k)}(b_i^*)||\eta_{i+1}'-\eta_{i+1}||\eta_j'-\eta_j|\\
\le&\frac{1}{2}\Big(\sup_{r\in[0,\eta_{i+1}'-\eta_{i+1}]}|\omega_{j,i+1+(n-k)}(\exp_a(r\nabla^{\phi_{i+1}})(b_i^*))-\omega_{j,i+1+(n-k)}(b_i^*)|\\
&+|\omega_{j,i+1+(n-k)}(b_i^*)|\Big)|\eta_{i+1}'-\eta_{i+1}||\eta_j'-\eta_j|\\
\le&\frac{1}{4}(|\eta_{i+1}'-\eta_{i+1}|^2+|\eta_j'-\eta_j|^2)\cdot\\
&\cdot\Big(\sup_{r\in[0,\eta_{i+1}'-\eta_{i+1}]}|\omega_{j,i+1+(n-k)}(\exp_a(r\nabla^{\phi_{i+1}})(b_i^*))-\omega_{j,i+1+(n-k)}(b_i^*)|+|\omega_{j,i+1+(n-k)}(b_i^*)|\Big)\\
\le&C_3(\rho_\phi(b,b'))^2(o(1)+\omega_{j,i+1+(n-k)}(b_i^*))\qquad\text{as }\delta_0\to0.
\end{aligned}$$

Combining the three estimates that we obtained with equation (5.81), we finally get (5.75), from which the thesis follows. □

We are now ready to prove the first characterization.

Proposition 5.3.21. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set and let $\phi : U \to \mathbb{R}^k$ be a continuous function. Then the following statements are equivalent.*

(i) ϕ is uniformly intrinsically differentiable on U.

(ii) For every $a \in U$, there exist $\delta > 0$ such that $I_\delta(a) \Subset U$, a family of functions $\{\phi_\varepsilon\}_{\varepsilon>0} \subset C^1(I_\delta(a), \mathbb{R}^k)$ and a matrix-valued function $M \in C^0(I_\delta(a), M_{k,2n-k}(\mathbb{R}))$ such that

$$\phi_\varepsilon \to \phi$$

$$D^{\phi_\varepsilon}\phi_\varepsilon \to M$$

uniformly on $I_\delta(a)$ as ε goes to zero.

Proof. $(i) \Rightarrow (ii)$. This follows from Proposition 5.3.1.

$(ii) \Rightarrow (i)$. Since (ii) implies that the hypotheses of Theorem 5.3.19 are satisfied at any point $a \in U$, Theorem 5.3.19 will conclude the proof. In fact, the existence of a family of exponential maps at any point $a \in U$ is a direct consequence of Proposition 5.3.6 applied to $U = I_\delta(a)$. Condition (5.67) instead follows substantially from Proposition 5.3.14. The crucial observation is that, by (ii), we can estimate uniformly with respect to ε both $\|\phi_{\varepsilon,i}\|_{L^\infty(I'')}$, for $i = 1, \ldots, k$, and $\|D^{\phi_\varepsilon}\phi_\varepsilon\|_{L^\infty(I'')}$ for any rectangle $I'' \Subset I_\delta(a)$. Moreover for every point a and any rectangle $I'' \Subset I_\delta(a)$ we can choose a modulus of continuity for $D^{\phi_\varepsilon}\phi_\varepsilon$ on I'' independent of ε by the uniform convergence of the intrinsic Jacobian matrices $D^{\phi_\varepsilon}\phi_\varepsilon$. Then for every two rectangles $I' \Subset I'' \Subset I_\delta(a)$ there exists a function not depending on ε

$$\alpha : (0, \infty) \to [0, \infty)$$

such that $\lim_{r\to 0} \alpha(r) = 0$ and for every ε

$$\sup\left\{\frac{|\phi_\varepsilon(b') - \phi_\varepsilon(b)|}{|b' - b|^{1/2}} : b, b' \in I',\ 0 < |b' - b| \le r\right\} \le \alpha(r). \tag{5.84}$$

Hence

$$\sup\left\{\frac{|\phi(b') - \phi(b)|}{|b' - b|^{1/2}} : b, b' \in I',\ 0 < |b' - b| \le r\right\} \le \alpha(r), \tag{5.85}$$

Then, in order to prove that (5.67) is satisfied one can conclude applying a standard compactness argument (anyway notice that, by Remark 5.3.20, condition (5.85) would be sufficient to apply Theorem 5.3.19). □

Theorem 5.3.22. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set, let $\phi : U \to \mathbb{R}^k$ be a function and set $\Sigma = \mathrm{graph}(\phi)$. Then the following are equivalent.*

(i) ϕ is uniformly intrinsically differentiable on U.

(ii) $\phi \in C^0(U)$ and for every $a \in U$ there exist the intrinsic derivative $\partial^{\phi_j}\phi(a)$ for every $j = 1, \ldots 2n - k$, the functions

$$\partial^{\phi_j}\phi : U \to \mathbb{R}^k,$$

are continuous on U and for every open set $U' \Subset U$ it holds that

$$\lim_{r\to 0+} \sup\left\{ \frac{|\phi(b')-\phi(b)|}{|b'-b|^{1/2}} : b, b' \in U',\ 0<|b'-b|\le r\right\} = 0. \tag{5.86}$$

(iii) ϕ is intrinsically differentiable on U, the map $D^{\phi}\phi : U \to M_{k,2n-k}(\mathbb{R})$ is continuous on U and for every open set $U' \Subset U$ it holds that

$$\lim_{r\to 0+} \sup\left\{ \frac{|\phi(b')-\phi(b)|}{|b'-b|^{1/2}} : b, b' \in U',\ 0<|b'-b|\le r\right\} = 0. \tag{5.87}$$

(iv) There are an open set $\Omega \subset \mathbb{H}^n$ and a map $f \in C^1_h(\Omega, R^k)$ such that $\Sigma = \{x \in \Omega : f(x) = 0\}$, $J_{\mathbb{V}} f(x) = |\det([X_i f_j(x)]_{i,j=1,\dots,k})| > 0$, for all $x \in \Sigma$.

Proof. $(i) \Leftrightarrow (iv)$. This equivalence is exactly the content of Theorem 5.1.12.

$(i) \Rightarrow (iii)$. By Proposition 5.1.11 the map ϕ is intrinsically differentiable at every point of U and $D^{\phi}\phi : U \to M_{k,2n-k}(\mathbb{R})$ is a continuous map. The Hölder-type condition (5.87) follows from Proposition 5.3.12.

$(iii) \Rightarrow (ii)$. The map ϕ is continuous since it is intrinsically differentiable (see for instance [FMS14, Proposition 3.2.3]). The existence of the intrinsic partial derivative $\partial^{\phi_j}\phi(a) \in \mathbb{R}^k$ for every $j = 1, \dots, 2n-k$, at every point $a \in U$, follows from the intrinsic differentiability of ϕ at every point of U and, in particular, by Proposition 5.3.9. The continuity of the intrinsic partial derivatives $\partial^{\phi_j}\phi : U \to \mathbb{R}^k$ follows again from Proposition 5.3.9 combined with the fact that we have assumed $D^{\phi}\phi(a)$ to be a continuous map with respect to $a \in U$.

$(ii) \Rightarrow (i)$. It follows from Theorem 5.3.19, since (ii) implies that the hypotheses of Theorem 5.3.19 are satisfied. In particular, we have to convince ourselves that the existence and continuity of the intrinsic derivatives $\partial^{\phi_j}\phi : U \to \mathbb{R}^k$ implies the existence of a family of exponential maps at every $a \in U$. Fix some $a \in U$. For any point b belonging to any neighbourhood $I_\delta(a) \Subset U$ of $a \in U$, there exists at least one integral curve of the vector field W^{ϕ}_j, $j \in \{1, \dots, 2n-k\}$ starting at b on which we can use the chain rule (5.21) setting the continuous maps needed to satisfy (iii) of Definition 5.3.2 as $\omega_{i,j} = \partial^{\phi_j}\phi_i : U \to \mathbb{R}$. In fact, since the chain rule holds for all the integral curves by hypothesis, we can choose, for every starting point b, an arbitrary curve that will play the role of the exponential map at a $\exp_a(\,\cdot W^{\phi}_j)(\mathrm{b})$ (this can be done for every $j = 1, \dots, 2n-k$) and we can use this family of arbitrarily chosen integral curves as a family of exponential map at a in order to apply Theorem 5.3.19. In particular, since U is open, once we fix $a \in U$, we fix an arbitrary $\delta_1 > 0$ such that $I_{\delta_1}(a) \Subset U$ and one can choose $0 < \delta_3 < \frac{1}{2}\delta_1$ such that for every point $b \in I_{\delta_3}(a)$, the integral curves starting at b exist on a common interval of time $[-\delta_2, \delta_2]$ for $\delta_2 > 0$ appropriately small (how much small will depends on δ_3, surely $\delta_2 \le \delta_3$). Hence, any integral curve starting at $b \in I_{\delta_2}(a)$ exists at least for an interval of time $[-\delta_2, \delta_2]$. □

Remark 5.3.23. The equivalence $(iv) \Leftrightarrow ((ii) + (iii))$ had already been proved by Kozhevnikov in his Phd thesis (see [Koz15, Theorem 4.3.1]), in the more general context of $(\mathbb{G}, \mathbb{R}^k)$-regular sets of $\mathbb{G}$, with $\mathbb{G}$ generic Carnot group and k sufficiently small. We have reported here our proof, which is more direct. Moreover, here we proved also that (ii) and (iii) are independentely equivalent for low codimensional $\mathbb{H}$-surfaces in $\mathbb{H}^n$. Moreover, taking into account Di Donato's results, we manage to relate explicitly the results in [Koz15] to the notions of intrinsic differentiability and uniform intrinsic differentiability.

Moreover, by taking in consideration Proposition 5.3.16, we obtain a stronger result.

Theorem 5.3.24. *Let $U \subset \mathbb{R}^{2n+1-k}$ be an open set, let $\phi : U \to \mathbb{R}^k$ be a function and set $\Sigma = \mathrm{graph}(\phi)$. Then the following conditions are equivalent*

(i) *ϕ is uniformly intrinsically differentiable on U.*

(ii) *$\phi \in C^0(U)$, for every $a \in U$ there exist the intrinsic partial derivative $\partial^{\phi_j}\phi(a)$, for $j = 1, \dots 2n-k$, and the functions*

$$\partial^{\phi_j}\phi : U \to \mathbb{R}^k,$$

are continuous on U.

(iii) *ϕ is intrinsically differentiable on U and the map $D^\phi\phi : U \to M_{k,2n-k}(\mathbb{R})$ is continuous on U.*

(iv) *There are an open set $\Omega \subset \mathbb{H}^n$ and a map $f = (f_1, ..., f_k) \in C^1_h(\Omega, \mathbb{R}^k)$ such that $\Sigma = \{x \in U : f(x) = 0\}$ and $J_{\mathbb{V}} f(x) = |\det([X_i f_j(x)]_{i,j=1,\dots,k})| > 0$ for all $x \in \Sigma$.*

Proof. The proof is analogous to the one of Theorem 5.3.22: by taking into account Proposition 5.3.16 we can simplify conditions (*ii*) and (*iii*). In particular, if ϕ is uniformly intrinsically differentiable, by Proposition 5.1.11 ϕ is intrinsically differentiable on U and its intrinsic Jacobian matrix $D^\phi\phi : U \to M_{k,2n-k}(\mathbb{R})$ is a continuous function. Hence, by Proposition 5.3.9, ϕ is differentiable on every integral curve γ^j of the vector field W^ϕ_j, for every $j = 1, \dots, 2n-k$, and, for every $i = 1, \dots, k$ and $j = 1, \dots, 2n-k$, $\frac{d}{dt}\phi_i(\gamma^j(t)) = [D^\phi\phi(\gamma^j(t))]_{i,j}$, then the intrinsic partial derivative $\partial^{\phi_j}\phi_i : U \to \mathbb{R}$ exists and $\partial^{\phi_j}\phi_i(a) = [D^\phi\phi(a)]_{i,j}$ for every $a \in U$ and $[D^\phi\phi(a)]_{i,j} : U \to \mathbb{R}$ is a continuous map. The continuity of the intrinsic partial derivatives guarantees that the hypotheses of Proposition 5.3.16 are satisfied and then in particular permits to deduce that ϕ satisfies the condition of $\frac{1}{2}$-little Hölder continuity in (5.86) so that one can finally conclude by applying Theorem 5.3.19 (in particular, the existence of a family of exponential maps at every point $a \in U$ can be deduced by the existence and continuity of the intrinsic partial derivatives of ϕ, repeating verbatim the argument used to prove (*ii*) $\Rightarrow$ (*i*) in Theorem 5.3.22). $\square$

Remark 5.3.25. Let us notice that Theorem 5.2.1 (*i*),(*iii*) can be seen as a corollary of Theorem 5.3.21. Analogously Theorem 5.2.4 can be seen as a corollary of Theorem 5.3.24.

Remark 5.3.26. We did not work from a distributional point of view. In particular, we did not obtain any characterization analogous to (*ii*) of Theorem 5.2.1. This is due to the fact that we did not manage to perform considerations analogous to the ones in Remark 5.2.2. In particular, we fail to give distributional meaning to all the writings of the form $W^\phi_j\phi_i$, or to suitable linear combinations of them with a view to reducing a system of the form $W^\phi_j\phi_i = \omega_{i,j}, j = 1, \dots, 2n-k,\ i = 1, \dots, k$, for prescribed continuous maps $\omega_{i,j} \in C^0(U, \mathbb{R})$, to an equivalent ones that could be read in distributional sense. One can refer to [BSC10a, BCSC15, ABC16a, ABC16b] where this point of view in codimension one has been fully explored, as we discussed in Section 5.2.

5.4 Further advances in the literature

Further results in this direction of research have been presented after [Cor19]. We provide a summary of the more recent available ones, proved by Antonelli, Di Donato, Don and Le Donne in [ADDDLD20] and by the first three authors in [ADDD20]. We expose their results exploiting the concepts introduced in this chapter. Our aim is to give to the reader a flavour of the main outcomes of the two papers without any ambition of completeness or strictness. We take now advantage of the opportunity to express our gratitude to the four authors for having pointed out to us an imprecision in the first version of [Cor19]. The presentation has been explicitly fixed in the previous chapter: please refer to [ADDDLD20, Remark 4.14] for more details. We explain the results of [ADDDLD20] in coordinates, while in the original papers many definitions and the main results are stated in a free-coordinates fashion.

The main generalization of our results is [ADDDLD20, Theorem 1.6] which can be seen as an extension of the content of Proposition 5.3.21 and Theorem 5.3.22 to the setting of a generic Carnot group $\mathbb{G}$ splitted as the product of two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$, with $\mathbb{V}$ horizontal. First of all, the authors introduce, in [ADDDLD20, Definition 1.3], a family of projected vector fields associated with a continuous map defined on an open set $U \subset \mathbb{W}$, $\phi : U \to \mathbb{V}$ acting between two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$ of a generic Carnot group $\mathbb{G}$, with $\mathbb{V}$ horizontal (the same vector fields had been individuated in [Koz15, Definition 4.2.12]). For every $W \in \mathrm{Lie}(\mathbb{W})$ the *projected vector field* on U corresponding to W is defined as

$$D_W^\phi(w) := d_E(\pi_{\mathbb{W}})(\Phi(w))(W(\Phi(w)),$$

for every $w \in U$.
Then, we consider an adapted basis $(b_1, \dots, b_q)$ of $\mathbb{G}$ such that $(b_1, \dots, b_k)$ is a basis of $\mathbb{V}$ and $(b_k, \dots, b_q)$ is a basis of $\mathbb{W}$, where by q we denote as usual the topological dimension on $\mathbb{G}$ and by k the topological dimension of $\mathbb{V}$. We identify, as we did in our context in Section 5.1, $\mathbb{V}$ with $\mathbb{R}^k$ and $\mathbb{W}$ with $\mathbb{R}^{q-k}$ through the fixed bases. For the sake of simplicity, here we do not distinguish between ϕ and $\tilde{\phi}$, $\mathbb{V}$ and $\mathbb{R}^k$ and $\mathbb{W}$ and $\mathbb{R}^{q-k}$, we identify them and we denote them respectively by ϕ, $\mathbb{W}$ and $\mathbb{V}$. Consider the associated basis $\{W_j\}_{j=1,\dots,q-k}$ of $\mathrm{Lie}(\mathbb{W})$, such that, for $j = 1, \dots, q-k$, W_j is the left invariant vector field such that $W_j(0) = b_{j+k}$ so that a precise ordered family of continuous projected vector fields $\{D^\phi_{W_j}\}_{j=1,\dots,q-k}$ is individuated, and $W_j \in \mathrm{Lie}(\mathbb{W}) \cap V_1$ if and only if $j = 1, \dots, m_1 - k$.

Then, mimicking for instance [BSC10b, Definition 1.1], the authors substantially say that a continuous map $\phi : U \to \mathbb{V}$ is a *broad⋆ solution* of the system

$$D^\phi_{W_j}\phi_i = \omega_{i,j} \text{ for } i = 1, \dots, k,\ j = 1, \dots m_1 - k \quad \text{on } U \tag{5.88}$$

for prescribed maps $\omega_{i,j} \in C^0(U, \mathbb{R})$, if there exists a family of exponential maps relative to the family of projected vector fields $\{D^\phi_{W_j}\}_{W_j \in \mathrm{Lie}(\mathbb{W})\cap V_1}$ at any point of U. Here, for the sake of simplicity, if there exist some maps $\omega_{i,j} \in C^0(U, \mathbb{R})$ such that ϕ is a broad⋆ solution of the system (5.88), we say that ϕ is *broad⋆ regular* on U.
The authors introduce, in [ADDDLD20, Definition 1.5] the notion of *vertically broad⋆ hölder regularity* for the map ϕ (refer also to [Koz15, Theorem 4.3.1]). A function ϕ is vertically broad⋆ hölder if for every point a_0 in the domain U of ϕ there exist a neighbourhood U_{a_0} of a_0 and a positive $\delta > 0$ such that for every $a \in U_{a_0}$ and for every projected vector fields $D^\phi_{W_j}$ with $W_j \in \mathrm{Lie}(\mathbb{W}) \cap V_d$ for every $d > 1$, there exists an integral curve $\gamma : [-\delta, \delta] \to U$ of $D^\phi_{W_j}$ starting at a, such that

$$\lim_{r\to 0} \left\{ \frac{|\phi(\gamma(t)) - \phi(\gamma(s))|}{|t-s|^{\frac{1}{d}}} : t, s \in [-\delta, \delta],\ 0 < |t-s| \le r \right\} = 0, \tag{5.89}$$

where by $\|\cdot\|$ we denote a generic homogeneous norm on $\mathbb{G}$.
We are now ready to state the characterization. By [DD20a, Theorem 4.1] it was already known that being a $(\mathbb{G}, \mathbb{R}^k)$-regular set of $\mathbb{G}$ is equivalent to be locally the intrinsic graph of a uniformly intrinsically differentiable map $\phi : U \subset \mathbb{W} \to \mathbb{V}$, with $\mathbb{G} = \mathbb{W}\mathbb{V}$, $\mathbb{V}$ horizontal and k-dimensional. In [ADDDLD20, Theorem 1.6], the authors prove that a map $\phi : U \to \mathbb{V}$ is uniformly intrinsically differentiable if and only if ϕ is vertically broad* hölder and, either it is broad* regular on U, or there are continuous maps $\omega_{i,j} \in C^0(U, \mathbb{R})$, for every $j = 1, \dots, m_1 - k$ and $i = 1, \dots, k$, such that at any point $a \in U$ there exist a positive $\delta > 0$ and a family of smooth functions $\phi_\varepsilon \subset C^\infty(B(a, \delta), \mathbb{V})$ such that ϕ_ε converges uniformly over $B(a, \delta)$ to ϕ and, $D^{\phi_\varepsilon}_{W_j}(\phi_\varepsilon)_i$ converges uniformly to $\omega_{i,j}$ as ε goes to zero. This result are a generalization of Propositions 5.3.21 and 5.3.22.

Successively, in the same paper, the authors present [ADDDLD20, Theorem 1.7], that is a generalization of Theorem 5.3.24 to the setting of Carnot groups of step 2, where such a group $\mathbb{G}$ is seen as the product of two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$, with $\mathbb{V}$ horizontal of dimension one. In this precise case, they prove that, as we proved Theorem 5.3.24 through Proposition 5.3.15, it is possible to repeat all the characterizations of Theorem [ADDDLD20, Theorem 1.6] for a uniformly intrinsically differentiable map $\phi : U \subset \mathbb{W} \to \mathbb{V}$, removing, from all the items where it appears, the hypothesis of vertically broad* hölder regularity. As a consequence, in Carnot groups of step 2, as happens in the Heisenberg group, the uniform intrinsic differentiability of hypersurfaces is uniquely determined by the behaviour of their parametrizing map ϕ along the horizontal directions (more precisely, along the directions individuated by the vector fields projected by $\mathrm{Lie}(\mathbb{W}) \cap V_1$). The key point of the proof is that the broad* regularity along the projected horizontal directions (that, in turn, is guaranteed by the existence of a family of smooth locally approximating maps) automatically allows to prove the needed broad* Hölder-type regularity of ϕ along the vertical directions. Please refer to [ADDDLD20, Section 5] for a careful comparison between [ADDDLD20, Proposition 5.2], [BSC10b, Theorem 3.2] and Proposition 5.3.15, that is [Cor19, Proposition 4.9]. A counterexample in [Koz15, Example 4.5.1] set in the Engel group, that is the simplest example of a Carnot group of step 3, shows that it is not possible to extend this method in order to remove the vertically broad* hölder regularity hypothesis in a generic Carnot group.

Moving a step further, in [ADDD20] the authors consider $\mathbb{G}$ a step-2 Carnot group splitted as the product of two complementary subgroups $\mathbb{W}$ and $\mathbb{V}$, where $\mathbb{V}$ is a one-dimensional subgroup. They consider maps $\phi : U \subset \mathbb{W} \to \mathbb{V}$, with $U \subset \mathbb{W}$ open set, and they give a distributional meaning to the system (5.88) for prescribed maps $\omega_{i,j} \in C^0(U, \mathbb{R})$. The authors prove that ϕ is a distributional solution of (5.88) if and only if it is a broad* solution of (5.88) if and only if it is a broad solution of (5.88), i.e. roughly speaking, if the existence and continuity of the *intrinsic derivatives of* ϕ along the vector fields $D^{\phi}_{W_j}$, for $W_j \subset \mathrm{Lie}(\mathbb{W}) \cap V_1$, are guaranteed. By intrinsic derivatives here we mean derivatives opportunely defined in this context analogously to how we defined the intrinsic derivatives in our setting, that is the Heisenberg group.

5.5 Centered Hausdorff measure of low codimensional $\mathbb{H}$-regular surfaces

We prove in this subsection an area formula for parametrized $\mathbb{H}$-regular surfaces of low codimension. In the next chapter, by the introduction of new arguments, we will provide a detailed generalization of this formula and really more. We will prove area formulas for both the centered and, especially, for the spherical Hausdorff measures of low codimensional parametrized $\mathbb{H}$-regular surfaces, with respect to any homogeneous distance. We decided anyway to report briefly this first partial result, whose proof easily follows reinterpreting [SC16, Theorem 4.50](see also [FSSC07, Theorem 4.1] and [FSSC15]) in the light of the results proved in the current chapter. This choice is due to the fact that the one we are going to present is the very first area formula completely stated in terms of the intrinsic derivatives of a parametrization ϕ with target space horizontal of dimension higher than one. It permits to compute the centered Hausdorff measure of the intrinsic graph of a uniformly intrinsically differentiable map with respect to the homogeneous distance d_∞. It can be considered as a generalization of [ASCV06, Theorem 1.2], where it is presented a formula for the centered Haudorff measure of one-codimensional $\mathbb{H}$-regular surfaces (based on a previous formula for the H-perimeter presented in [FSSC01, Theorem 6.5]). Other extensions of [ASCV06, Theorem 1.2] in the literature are the formulas for the spherical Hausdorff measure of $\mathbb{H}$-regular hypersurfaces in the Heisenberg group in [FSSC15, Mag17], of one-codimensional intrinsic Lipschitz graphs in the Heisenberg group [CMPSC14, Theorem 1.6] and of one-codimensional uniformly intrinsically differentiable graphs in Carnot groups of step 2 [DD20a, Theorem 5.4].

Let us introduce for any $m \in \{1, \dots, n, n+2, \dots, 2n+2\}$ the normalized measure $\mathcal{C}^m_\infty = \zeta_\infty(m)\mathcal{C}^m$, where

$$\zeta_\infty(m) := \begin{cases} \omega_m & \text{if } 1 \le m \le n \\ 2\omega_{m-2} & \text{if } n+2 \le m \le 2n+2, \end{cases}$$

where ω_m denotes the Lebesgue measure of the unit ball in $\mathbb{R}^m$.

Remark 5.5.1. We do not need to define the normalized measure for $m = n+1$, since $\mathbb{H}$-regular surfaces of Hausdorff dimension $n+1$ cannot exist. In fact, $\mathbb{H}$-regular surfaces of dimension k, have Hausdorff dimension k when $1 \le k \le n$, and $k+1$ when $n+1 \le k \le 2n+1$.

Remark 5.5.2. Notice that for any normal homogeneous subgroup $\mathbb{W} \subset \mathbb{H}^n$ of Hausdorff dimension $P \neq n+1$ and topological dimension p it holds

$$\zeta_\infty(P) = \mathcal{H}^p_E(\mathbb{B}_\infty(0,1) \cap \mathbb{W}),$$

where $\mathbb{B}_\infty(0,1)$ is the metric closed ball centered at 0 of radius 1 with respect to d_∞.

Now, keeping in mind that, given a map $\tilde{\phi} : \tilde{U} \subset \mathbb{W} \to \mathbb{V}$, with $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ product of complementary subgroups, by ϕ we denote the corresponding map $\phi : U \to \mathbb{R}^k$, with $U \subset \mathbb{R}^{2n+1-k}$, in coordinates as we explained above in the chapter, in Section 5.1, we can state the following definition.

Definition 5.5.3. Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect orthogonal product of complementary subgroups, with $\mathbb{W}$ of dimension $2n+1-k$. Let $\tilde{U} \subset \mathbb{W}$ be an open set and let $\tilde{\phi} : \tilde{U} \to \mathbb{V}$ be an intrinsically differentiable function at a point $\bar{w} \in \tilde{U}$. We define the *intrinsic Jacobian of $\tilde{\phi}$ at $\bar{w}$* as

$$J^{\tilde{\phi}}\tilde{\phi}(\bar{w}) = \sqrt{1 + \sum_{\ell=1}^{k} \sum_{I \in \mathcal{I}_\ell} (M^\phi_I(\bar{w}))^2},$$

where for every $\ell \in \{1, \dots, k\}$, $\mathcal{I}_\ell$ is the set of multi-indexes

$$\{(i_1, \dots, i_\ell, j_1, \dots, j_\ell) \in \mathbb{N}^{2\ell} : 1 \le i_1 < i_2 < \dots < i_\ell \le 2n-k,\ 1 \le j_1 < j_2 \dots < j_\ell \le k\}$$

and for $I = (i_1, \dots, i_\ell, j_1, \dots, j_\ell) \in \mathcal{I}_\ell$ the minor $M_I^\phi(\bar{w})$ is defined as

$$M_I^\phi(\bar{w}) = \det \begin{pmatrix} [D^\phi\phi(\bar{a})]_{j_1,i_1} & \dots & [D^\phi\phi(\bar{a})]_{j_1,i_\ell} \\ \dots & \dots & \dots \\ [D^\phi\phi(\bar{a})]_{j_\ell,i_1} & \dots & [D^\phi\phi(\bar{a})]_{j_\ell,i_\ell} \end{pmatrix} = \det \begin{pmatrix} \partial^{\phi_{i_1}}\phi_{j_1}(\bar{a}) & \dots & \partial^{\phi_{i_\ell}}\phi_{j_1}(\bar{a}) \\ \dots & \dots & \dots \\ \partial^{\phi_{i_1}}\phi_{j_\ell}(\bar{a}) & \dots & \partial^{\phi_{i_\ell}}\phi_{j_\ell}(\bar{a}) \end{pmatrix},$$

where $\bar{a} = i_{\mathbb{W}}(\bar{w})$ and $i_{\mathbb{W}}$ is the map defined in Section 5.1.

By Theorems 4.1.19 and 4.1.23, any $\mathbb{H}$-regular surface of low codimension can be locally seen as the intrinsic orthogonal graph of a uniformly intrinsically differentiable map $\tilde{\phi} : \tilde{U} \subset \mathbb{W} \to \mathbb{V}$ between two homogeneous subgroups such that $\mathbb{W} \rtimes \mathbb{V}$. The map $\tilde{\phi}$ corresponds as described above to a map $\phi : U \to \mathbb{R}^k$, with $U \subset \mathbb{R}^{2n+1-k}$. We can then focus on computing the area of a regularly parametrized surface, that is the intrinsic graph of $\tilde{\phi}$, $\Sigma := \mathrm{graph}(\tilde{\phi}) = \mathrm{graph}(\phi)$. If we are able to do this, by a standard covering argument is it possible to compute the area of any low codimensional $\mathbb{H}$-regular surface.

According to Theorem 4.1.22, we know that there exist an open set Ω of $\mathbb{H}^n$, with $\Phi(U) \subset \Omega$, and a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ such that $\Sigma = \{x \in U : f(x) = 0\}$ and such that $J_{\mathbb{V}} f(x) = |\det(([X_i f_j(x)]_{i,j=1,\dots,k})| > 0$ for all $x \in \Sigma$. Moreover, by the proof of Theorem 5.1.12 one can choose f such that

$$\begin{aligned} f \circ \Phi = 0 \quad &\text{on } U, \\ J_{\mathbb{H}} f(\Phi(a)) = \begin{pmatrix} \mathbb{I}_k & | & -D^\phi\phi(a) \end{pmatrix} &\in M_{k,2n-k}(\mathbb{R}) \quad \forall a \in U. \end{aligned} \tag{5.90}$$

By the choice of f in (5.90), and by the results of Theorem 5.3.24, it turns out that the horizontal Jacobian matrix of f at every $\Phi(a) \in \Phi(U)$ is given by

$$J_{\mathbb{H}} f(\Phi(a)) = \begin{pmatrix} 1 & \dots & 0 & -\partial^{\phi_1}\phi_1(a) & \dots & -\partial^{\phi_{2n-k}}\phi_1(a) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -\partial^{\phi_1}\phi_k(a) & \dots & -\partial^{\phi_{2n-k}}\phi_k(a) \end{pmatrix}. \tag{5.91}$$

From the form of this matrix it is clear that $J_{\mathbb{V}} f(\Phi(a)) = 1$ for every $a \in U$.

Let us now resume the substantial content of [FSSC07, Theorem 4.1] (in respect with the results of [AS09]).

Theorem 5.5.4. *Let $\mathbb{H}^n$ be the Heisenberg group equipped with the distance d_∞. Let $\mathbb{W}$ and $\mathbb{V}$ be orthogonal complementary subgroups such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, and let k be the dimension of $\mathbb{V}$. Let $\tilde{U} \subset \mathbb{W}$ be an open set. Let $\tilde{\phi} : \tilde{U} \to \mathbb{V}$ be a uniformly intrinsically differentiable map and set $\Sigma := \mathrm{graph}(\tilde{\phi})$. The map $\tilde{\phi}$ corresponds as described above to a map $\phi : U \subset \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$. Consider a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ such that $\Sigma = f^{-1}(0)$ and such that $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$. Then, the $(2n+2-k)$-centered Hausdorff measure of $\Sigma = \mathrm{graph}(\phi) = \mathrm{graph}(\tilde{\phi})$ can be computed as*

$$C_\infty^{2n+2-k} \llcorner \Sigma = \tilde{\Phi}_\sharp \left(\left(\frac{J_H f}{J_{\mathbb{V}} f} \circ \tilde{\Phi} \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{W} \right) = \Phi_\sharp \left(\left(\frac{J_H f}{J_{\mathbb{V}} f} \circ \Phi \right) \mathcal{L}^{2n+1-k} \llcorner \mathbb{R}^{2n+1-k} \right). \tag{5.92}$$

Hence, combining (5.90) with (5.92), it is not difficult to convince ourself of the validity of the following result.

Theorem 5.5.5. *Let $\mathbb{H}^n$ be the Heisenberg group equipped with the distance d_∞. Let $\mathbb{W}$ and $\mathbb{V}$ be orthogonal complementary subgroups such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, and let k be the dimension of $\mathbb{V}$. Let $\tilde{U} \subset \mathbb{W}$ be an open set. Let $\tilde{\phi} : \tilde{U} \to \mathbb{V}$ be a uniformly intrinsically* *differentiable map. If we set* $\Sigma := \mathrm{graph}(\tilde{\phi})$*, then for every Borel set* $B \subset \Sigma$*,*

$$C_\infty^{2n+2-k}(B) = \int_{\tilde{\Phi}^{-1}(B)} J^{\tilde{\phi}}\tilde{\phi}(w)\ d\mathcal{H}_E^{2n+1-k}(w). \tag{5.93}$$

Proof. We call as usual $\phi : U \to \mathbb{R}^k$ the map that corresponds to $\tilde{\phi}$. We denote for any generic point $w \in \mathbb{W}$, the corresponding point in coordinates $a = i_{\mathbb{W}}(w) \in U$, and vice versa. Notice that, by definition, for every $a \in U$, $\Phi(a) = \tilde{\Phi}(w)$. We know by Proposition 5.1.11 that, since ϕ is a uniformly intrinsically differentiable function, $D^\phi\phi$ is a continuous matrix-valued function on U, hence it makes sense to integrate its components which, by Theorem 5.3.24, coincide with the elements $[D^\phi\phi(a)]_{i,j} = \partial^{\phi_j}\phi_i(a)$ for every $i = 1, \dots, k$, $j = 1, \dots, 2n-k$, $a \in U$. By Theorem 5.1.12, we know that, given the uniformly intrinsically differentiable function ϕ, its intrinsic graph Σ is the zero-level set of a function $f \in C_h^1(\Omega, \mathbb{R}^k)$, for some open set Ω with $\Phi(U) \subset \Omega$, such that $J_{\mathbb{V}}f(x) = |\det([X_i f_j(x)]_{i,j=1,\dots,k})| > 0$ for $x \in \Sigma$. In particular, we choose a map f that satisfies the relations of condition (5.90). This choice is allowed by the proof of Theorem 5.1.12. Hence, for every $a \in U$, the horizontal Jacobian matrix of f is

$$J_{\mathbb{H}}f(\Phi(a)) = \begin{pmatrix} 1 & \dots & 0 & -\partial^{\phi_1}\phi_1(a) & \dots & -\partial^{\phi_{2n-k}}\phi_1(a) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -\partial^{\phi_1}\phi_k(a) & \dots & -\partial^{\phi_{2n-k}}\phi_k(a) \end{pmatrix} \tag{5.94}$$

and clearly $J_{\mathbb{V}}f(\Phi(a)) = 1$. Then, for every $a \in U$, we compute directly

$$J_H f(\Phi(a)) = \|(\nabla_H f_1 \wedge \dots \wedge \nabla_H f_k)(\Phi(a))\|_g = J(Df(\Phi(a)),$$

observing that, for every $i = 1, \dots, k$, we have

$$\begin{aligned} \nabla_H f_i(\Phi(a)) &= \sum_{j=1}^{n}(X_j f_i(\Phi(a)))e_j + \sum_{j=1}^{n}(Y_j f_i(\Phi(a)))e_j \\ &= \sum_{j=1}^{k}\delta_{i,j}e_j - \sum_{j=1}^{2n-k}\partial^{\phi_j}\phi_i(a)e_{j+k} \in H_1, \end{aligned}$$

where $\delta_{i,j}$ is the Knonecker's delta. According to the notation in Definition 5.5.3, for every $a \in U$, we get

$$J_H f(\Phi(a)) = \sqrt{1 + \sum_{\ell=1}^{k}\sum_{I \in \mathcal{I}_\ell}(M_I^\phi(w))^2}. \tag{5.95}$$

Let us give some more details about the computation leading to (5.95). We need to compute the Jacobian of $J_{\mathbb{H}}f(\Phi(a))$. The constant 1 in equation (5.95) stands for the determinant of the identity matrix $\mathbb{I}_k$ (it coincides with the coefficient of the k-vector $e_1 \wedge \dots \wedge e_k$). Let us now focus on the second addend in the square root of (5.95). The index $\ell \in \{1, \dots, k\}$ in equation (5.95) highlights the fact that, for every $I \in \mathcal{I}_\ell$, we are computing the minor $M_I^\phi(w)$ of a $k \times k$ sub-matrix of $J_{\mathbb{H}}f(\Phi(a))$ composed by choosing

(i) the first $k - \ell$ columns among the first k columns of $J_{\mathbb{H}}f(\Phi(a))$ (in particular, for a given multi-index $I = (i_1, \dots i_\ell, j_1, \dots, j_\ell)$, these are the columns whose indexes belong to the set $\{1, \dots, k\} \setminus \{j_1, \dots, j_\ell\}$)

(ii) the last ℓ columns among the $2n - k$ last columns of $J_{\mathbb{H}}f(\Phi(a))$ (in particular, for a given multi-index $I = (i_1, \dots i_\ell, j_1, \dots, j_\ell)$, these are the columns whose indexes belong to the set $\{k + i_1, \dots, k + i_\ell\}$).

Hence, we can finally rewrite the area formula (5.92) in the light of our computations, so that we essentially obtain the following formula

$$C_\infty^{2n+2-k}(B) = \int_{\tilde{\Phi}^{-1}(B)} \sqrt{1 + \sum_{\ell=1}^{k} \sum_{I \in \mathcal{I}_\ell} (M_I^\phi(w))^2} \, d\mathcal{H}_E^{2n+1-k}(w), \tag{5.96}$$

which, according to the definition of intrinsic Jacobian (Definition 5.5.3, is an exact rephrasing of formula (5.93).

□

Chapter 6

Area formulas for $\mathbb{H}$-regular surfaces of low codimension

In this chapter we present the results of [CM20], obtained in collaboration with Prof. V. Magnani, of the University of Pisa. We will not need anymore to distinguish between the map $\tilde{\phi} : \mathbb{W} \to \mathbb{V}$, where $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ is product of complementary subgroups, and the corresponding function $\phi : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$, that was introduced in the previous chapter to work in suitable coordinates. Hence, from now on, to simplify the notation we will reserve the notation ϕ for a generic map acting between two complementary homogeneous subgroups $\mathbb{W}$ to $\mathbb{V}$.

Finding explicit area formulas to compute the Hausdorff measure of regular surfaces in Carnot groups represents an intriguing problem. Due to the delicate algebraic structure of groups, the classical methods in general cannot be blindly applied and this is the main reason why many researches have been recently developed in this direction. We cite below some contributions organized according to the different approaches adopted by the authors. The following lists may result incomplete, but the cited results are just meant as some examples of the wide available related literature. A significative line of research established general abstract paths, or schemes, that can be followed in order to derive area formulas for the Hausdorff measure of regular submanifolds in various precise cases [Mag01, Mag15, FSSC15, LM20]. Some authors proved area formulas to compute the H-perimeter measure of locally finite H-perimeter sets and compared the H-perimeter measure of these regular sets with their Hausdorff-type measures [FSSC01, FSSC02, FSSC03a, FSSC03b, ASCV06, Mag06b, CMPSC14, Mag17, DD20a, DD20b, ADDDLD20]. Some others authors have focused on submanifolds satisfying intrinsic-type regularity, of low dimension [Mag01, Mag02a, Pau04b, FSSC07, AM20a] and of low codimension [FSSC07, Cor19, CM20, JNGV20, Vit20] while other ones focused on computing the area of Euclidean C^1-regular submanifolds, embedded in Carnot groups endowed with homogeneous distances, satisfying a negligibility condition on the generalized set of characteristic points (that, roughly speaking, are the points at which the blow-up of the surfaces does not behave well, in the sense that it is not necessarily a subgroup, see [Mag19, (1.8)] and Remark 4.2.20) [MV08, LDM10, Mag11a, Mag19].

The main result of this chapter is a series of area formulas that permit to compute the spherical Hausdorff measure, with respect to any fixed homogeneous distance, of $\mathbb{H}$-regular surfaces of low codimension in the Heisenberg group. Precisely, we choose $1 \leq k \leq n$ and we fix a regularly parametrized $\mathbb{H}$-regular surface Σ of codimension k (Definition 4.1.26), we associate with Σ a "parametrized measure" μ, using a defining continuously Pansu differentiable mapping f and an intrinsic regular parametrizing map ϕ, according to (6.10).

As we said in the previous chapter, the measure μ already appeared in [FSSC07], where the authors introduced it to prove an area formula for the centered Hausdorff measure of Σ, see [SC16, Theorem 4.5]. The choice of the measure μ is furthermore justified by Theorem 5.5.5 ([Cor19, Theorem 6.1]), where it is shown how μ can be rewritten uniquely in terms of the intrinsic derivatives of the parametrizing mapping ϕ. The main step towards the proof of our area formula is an upper blow-up theorem (Theorem 6.3.4) to compute the Federer density of the measure μ at any point of Σ. The terminology "upper blow-up" goes back to [Mag17], where a Federer density was first computed with applications to sets of finite H-perimeter in Carnot groups. In our higher codimensional framework, the proof of the upper blow-up involves some tools: it relies on three key aspects. First, the intrinsic differentiability of the parametrizing map ϕ (Theorem 4.1.22) is crucial in establishing the limit of the set (6.13) in the proof of the upper blow-up. Second, we prove an "intrinsic chain rule" (Theorem 6.2.2) that permits us to connect the kernel of Df with the intrinsic differential of ϕ, according to (6.15). However, to make our chain rule work we have slightly modified the well known notion of intrinsic differentiability associated with a factorization, introducing the notion of extrinsic differentiability (Definition 6.2.1). Third, we establish a delicate algebraic lemma for computing the Jacobian of projections between vertical subgroups, that are associated with two semidirect factorizations with the same horizontal complementary subgroup (Lemma 6.3.3).

By combining Theorem 6.3.4 with the abstract measure-theoretic area formula given by Theorem 2.5.15 ([Mag15, Theorem 11]), we obtain an area formula for Σ (Theorem 6.4.1), involving μ and the spherical measure $\mathcal{S}^{2n+2-k}$ with respect to any homogeneous distance d. In the assumptions of Theorem 6.3.4, for any Borel set $B \subset \Sigma$ we have

$$\mu(B) = \int_B \beta_d(\operatorname{Tan}(\Sigma, x)) \, d\mathcal{S}^{2k+2-k}(x).$$

If the factors of the semidirect product are orthogonal the measure μ can be written in terms of intrinsic Jacobian, and then in terms of intrinsic partial derivatives of the parametrization ϕ of Σ (Theorem 6.4.2). If the distance d is invariant under some classes of symmetries (Definition 6.1.2) or it is multiradial (Definition 6.1.5), then the area formula simplifies (Theorem 6.4.4). Precisely, in these cases the spherical factor only depends on the distance, on the dimension of the surface and on the fixed scalar product on $\mathbb{H}^n$, becoming a geometric constant. Some additional applications follow from our results. By a slight modification of the proof of Theorem 6.3.4, we obtain a standard blow-up theorem computing the centered density of μ at any point of Σ (Theorem 6.3.8). By combining it with the measure-theoretic area formula for the centered density Theorem 2.5.13 ([FSSC15, Theorem 3.1]), we obtain an area formula for the centered Hausdorff measure of Σ, that extends the one of [FSSC07] to any homogeneous distance (Theorem 6.4.5). We finally provide the cases when the spherical measure and the centered Hausdorff measure do coincide (Corollary 6.4.7).

6.1 Some preliminary notions

We introduce some definitions and known results that will be specifically useful in this chapter.

Definition 6.1.1 (Spherical factor)**.** Let d be a homogeneous distance on $\mathbb{H}^n$. If $\mathbb{W}$ is a linear subspace of topological dimension p of $\mathbb{H}^n$, then we define the *spherical factor* of $\mathbb{W}$

with respect to d as

$$\beta_d(\mathbb{W}) = \max_{z\in\mathbb{B}(0,1)} \mathcal{H}^p_E(\mathbb{W}\cap\mathbb{B}(z,1)).$$

When we deal with a homogeneous distance d that preserves suitable symmetries the spherical factor can become a geometric constant. The following definition detects those homogeneous distances giving a constant spherical factor. It extends [Mag17, Definition 6.1] to higher codimension.

Definition 6.1.2. Let d be a homogeneous distance on $\mathbb{H}^n$ and let $p = 1,\dots,2n+1$. If $p = 1$ or $p = 2n+1$, then we automatically say that d is *p-vertically symmetric*. If $2 \le p \le 2n$, we say that d is *p-vertically symmetric* if the following conditions hold. We refer to the fixed graded scalar product $\langle\cdot,\cdot\rangle$ and we assume that there exists a family $\mathcal{E} \subset O(H_1)$ of isometries such that for any couple of p-dimensional subspaces $S_1, S_2 \subset H_1$, there exists $L \in \mathcal{E}$ that satisfies the condition $L(S_1) = S_2$. Taking into account that H_1 and H_2 are orthogonal, we introduce the class of isometries

$$\mathcal{O} = \{T \in O(\mathbb{H}^n) : T|_{H_2} = \mathrm{Id}_{H_2},\ T|_{H_1} \in \mathcal{E}\}.$$

Then we say that d is p-vertically symmetric if the following holds:

- $\pi_{H_1}(\mathbb{B}(0,1)) = \mathbb{B}(0,1)\cap H_1 = \{h \in H_1 : \theta(|\pi_{H_1}(h)|) \le r_0\}$ for some monotone non-decreasing function $\theta : [0,+\infty) \to [0,+\infty)$ and $r_0 > 0$,
- $T(\mathbb{B}(0,1)) = \mathbb{B}(0,1)$ for all $T \in \mathcal{O}$.

More information on p-vertically symmetric distances in general stratified homogeneous groups can be found in [Mag18], or in the recent derived paper [Mag20]. For instance, the sub-Riemannian distance in the Heisenberg group $\mathbb{H}^1$ is 2-vertically symmetric.

The next theorem specializes to the Heisenberg group [Mag20, Theorem 1.1]. In fact, according to the terminology of [Mag20, Definition 1.2], Theorem 6.1.3 states that any homogeneous p-vertically symmetric distance is rotationally symmetric with respect to the family of p-dimensional vertical homogeneous subgroups.

Theorem 6.1.3. *If $p = 1,\dots,2n+1$ and d is a homogeneous p-vertically symmetric distance on $\mathbb{H}^n$, then the spherical factor $\beta_d(\mathbb{W})$ is constant on every p-dimensional vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$.*

The previous theorem motivates the following definition.

Definition 6.1.4. If we have the class of the p-dimensional homogeneous subgroups $\mathcal{D}_p$ and $\beta_d(S)$ remains constant as $S \in \mathcal{D}_p$, then we denote the spherical factor by $\omega_d(p)$, without indicating the special class of subgroups.

Definition 6.1.5 ([Mag20, Definition 5.1])**.** Let d be a homogeneous distance on $\mathbb{H}^n$. We say that d is *multiradial* if there exists a function $\theta : [0,+\infty)^2 \to [0,+\infty)$, which is continuous and monotone non-decreasing on each single variable, with

$$d(x,0) = \theta(|\pi_{H_1}(x)|, |\pi_{H_2}(x)|).$$

The function θ is also assumed to be coercive in the sense that $\theta(x) \to +\infty$ as $|x| \to +\infty$.

Proposition 6.1.6. *If $d : \mathbb{H}^n \times \mathbb{H}^n \to [0,\infty)$ is multiradial, then it is also p-vertically symmetric for every $p = 1,\dots,2n+1$.*

A more general statement can be found in [Mag20, Proposition 5.1]. One may also check that both d_∞ and the Cygan-Korányi distance (2.15) are multiradial.

It is also possible to find conditions under which the spherical factor has a simpler representation. The next theorem is established in [Mag18] (or refer to [Mag20, Theorem 1.4]).

Theorem 6.1.7. *If $p = 1, \ldots, 2n+1$ and d is a homogeneous distance on $\mathbb{H}^n$ whose unit ball $\mathbb{B}(0,1)$ is convex, then for every p-dimensional vertical subgroup $\mathbb{W}$*

$$\beta_d(\mathbb{W}) = \mathcal{H}^p_E(\mathbb{W} \cap \mathbb{B}(0,1)).$$

6.2 Extrinsic differentiability in Heisenberg groups

Now we introduce the notion of *extrinsic differentiability* for a map acting from a normal homogeneous subgroup $\mathbb{W} \subset \mathbb{H}^n$ to $\mathbb{R}^k$. It can be considered as a bridge between intrinsic differentiability and Pansu differentiability. In particular, this notion permits to prove a useful chain rule (Theorem 6.2.2). By a direct comparison, one can observe that extrinsic differentiability is a slight modification of the notion of intrinsic differentiability.

Definition 6.2.1. Let $\mathbb{W}$ be a vertical subgroup of $\mathbb{H}^n$, let $U \subset \mathbb{W}$ be an open set and let $F : U \to \mathbb{R}^k$ with $\bar{w} \in U$. We fix any horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$ such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ and we choose $v \in \mathbb{V}$. We define $x = \bar{w}v$ in $\mathbb{H}^n$ and the corresponding translated function

$$F_{x^{-1}}(w) = F(\sigma_x(w)) - F(\bar{w})$$

for $w \in \sigma_{x^{-1}}(U)$, where for any $y \in \mathbb{H}^n$, σ_y is the map introduced in Definition 3.4.4, that is $\sigma_y(m) = \pi_{\mathbb{W}}(ym) = yw(\pi_{\mathbb{V}}(y))^{-1}$, for every $m \in \mathbb{W}$. We say that F is *extrinsically differentiable at $\bar{w}$ with respect to* $(\mathbb{W}, \mathbb{V}, x)$ if there exists an h-homomorphism $L : \mathbb{W} \to \mathbb{R}^k$ such that

$$\frac{|F_{x^{-1}}(w) - L(w)|}{\|w\|} \to 0 \quad \text{as } w \to 0. \tag{6.1}$$

The uniqueness of L allows us to denote it by $d_x^{\mathbb{W},\mathbb{V}} F$.

The terminology *extrinsic differentiabilty* arises from the fact that, in the notation of Definition 6.2.1, the subgroup $\mathbb{V}$ and the point x cannot be detected by the information we have on F. They are actually artifically added from outside.

One could refer to Definition 6.2.1 and think that the factor $\mathbb{V}$ as metric space replaces $\mathbb{R}^k$. Then in this case essentially one considers a map F as $F : U \subset \mathbb{W} \to \mathbb{V}$, $\bar{w} \in U$ and one can choose the point v as $v = F(\bar{w}) \in \mathbb{V}$, so that $x = \bar{w}F(\bar{w}) \in \mathbb{H}^n$. In this case the numerator of (6.1) could be rewritten as $d(F_{x^{-1}}(w), L(w))$, then condition 6.1 would coincide precisely with the condition of intrinsic differentiability of the map F at the point $\bar{w}$. Then, in this situation, the "extrinsic differentiability" of F at $\bar{w}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$, would express the intrinsic differentiability of F at $\bar{w}$.

We have introduced this notion in order to make sense of the following chain rule involving intrinsic differentiability. Somehow extrinsic and intrinsic differentiability compensate each other in the following theorem.

Theorem 6.2.2 (Chain rule). *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product. Let us consider two open sets $U \subset \mathbb{W}$, $\Omega \subset \mathbb{H}^n$ and two functions $f : \Omega \to \mathbb{R}^k$, $\phi : U \to \mathbb{V}$. Assume that $\Phi(U) \subset \Omega$, where Φ, as usual, denotes the graph function of ϕ. Let us consider $x_{\mathbb{W}} \in U$*

and set $x = \Phi(x_{\mathbb{W}})$. If f and ϕ are Pansu differentiable at x and intrinsically differentiable at $x_{\mathbb{W}}$, respectively, then the composition $F = f \circ \Phi : U \to \mathbb{R}^k$, given by

$$F(u) = f(u\phi(u)) \quad \textit{for all } u \in U,$$

is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$. For every $w \in \mathbb{W}$ the formula

$$d_x^{\mathbb{W},\mathbb{V}} F(w) = Df(x)(wd\phi_{x_{\mathbb{W}}}(w)) \tag{6.2}$$

holds. If in addition $f(w\phi(w)) = c$ for every $w \in U$ and some $c \in \mathbb{R}$, then we obtain

$$\ker(Df(x)) = \mathrm{graph}(d\phi_{x_{\mathbb{W}}}). \tag{6.3}$$

Proof. Let us first show that F is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$. We define

$$L(w) = Df(x)(wd\phi_{x_{\mathbb{W}}}(w)) = Df(x)(w) + Df(x)(d\phi_{x_{\mathbb{W}}}(w))$$

for $w \in \mathbb{W}$, that is an h-homomorphism. For w small enough, we have

$$\begin{aligned}
\frac{|F_{x^{-1}}(w) - L(w)|}{\|w\|} &= \frac{|f(xwx_{\mathbb{V}}^{-1}\phi(xwx_{\mathbb{V}}^{-1})) - f(x) - L(w)|}{\|w\|} \\
&= \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(wd\phi_{x_{\mathbb{W}}}(w))|}{\|w\|} \\
&\leq \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\|} \\
&+ \frac{|Df(x)(w\phi_{x^{-1}}(w)) - Df(x)(wd\phi_{x_{\mathbb{W}}}(w))|}{\|w\|}.
\end{aligned}$$

Let us consider the last two addends separately:

$$\begin{aligned}
&\frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\|} \\
&= \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\phi_{x^{-1}}(w)\|} \frac{\|w\phi_{x^{-1}}(w)\|}{\|w\|} \to 0
\end{aligned}$$

as $\|w\| \to 0$, by the Pansu differentiability of f at x and by the validity of

$$\frac{\|w\phi_{x^{-1}}(w)\|}{\|w\|} \leq 1 + \frac{\|\phi_{x^{-1}}(w)\|}{\|w\|} = 1 + \left\| d\phi_{x_{\mathbb{W}}}\left(\frac{w}{\|w\|}\right)\right\| + \frac{\|d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w)\|}{\|w\|} \leq C_x$$

for all $w \neq 0$ and sufficiently small. It is indeed a consequence of the intrinsic differentiability of ϕ at $x_{\mathbb{W}}$. For the second addend, the previous intrinsic differentiability yields

$$\frac{|Df(x)(d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w))|}{\|w\|} = \left| Df(x)\left(\frac{d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w))}{\|w\|}\right)\right| \to 0$$

as $w \to 0$. This complete the proof of the first claim and also establishes formula (6.2).

Let us now assume the constancy of $w \to f(w\phi(w))$ on U. Since we have proved that F is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$, being in this case $F_{x^{-1}}$ identically vanishing, we obtain

$$d_x^{\mathbb{W},\mathbb{V}} F(w) = o(\|w\|)$$

as $w \to 0$. Therefore, for any $u \in \mathbb{W}$, we have

$$\|Df(x)(\delta_t u d\phi_{x_{\mathbb{W}}}(\delta_t u))\| = o(t)$$

as $t \to 0$. Due to the h-linearity, it follows that

$$Df(x)(ud\phi_{x_{\mathbb{W}}}(u)) = 0.$$

We have proved the inclusion $\operatorname{graph}(d\phi_{x_{\mathbb{W}}}) \subset \ker(Df(x))$ of homogeneous subgroups with the same dimension, hence formula (6.3) is established. □

The proof of the following proposition is a simple application of Theorem 6.2.2.

Proposition 6.2.3. *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ with $\mathbb{V}$ horizontal k-dimensional homogeneous subgroup. Let $\phi : U \to \mathbb{V}$, where $U \subset \mathbb{W}$ be open, and assume that ϕ is everywhere intrinsically differentiable. Let $\Sigma = \{w\phi(w) : w \in U\}$ and let $f : \Omega \to \mathbb{R}^k$ be everywhere Pansu differentiable with*

$$\Sigma = f^{-1}(f(x_0)) \cap (U\mathbb{V})$$

for some $x_0 \in \Omega$. If $J_H f(x) > 0$ for all $x \in \Sigma$, then $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$.

Proof. We consider $x = w\phi(w)$, so by Theorem 6.2.2 the function $F = f \circ \Phi$ is extrinsically differentiable at w with respect to $(\mathbb{W}, \mathbb{V}, x)$ and

$$0 = d_x^{\mathbb{W},\mathbb{V}} F(v) = Df(x)(vd\phi_w(v)) = Df|_{\mathbb{W}}(x)(v) + Df|_{\mathbb{V}}(x)(d\phi_w(v))$$

where $v \in \mathbb{W}$. If by contradiction $Df|_{\mathbb{V}}(x) : \mathbb{V} \to \mathbb{R}^k$ would not be a isomorphism, then its image T would have linear dimension less than k. Then the previous equalities would imply that the image of $Df|_{\mathbb{W}}(x)$ would be contained in T, hence the same would hold for the image of $Df(x)$. This conflicts with the fact that $Df(x)$ is surjective. □

Proposition 6.2.4. *Let $\phi : U \to \mathbb{V}$, where $U \subset \mathbb{W}$ is open, and assume that ϕ is everywhere intrinsically differentiable. Let $\Sigma = \{w\phi(w) : w \in U\}$ and assume that there is $f \in C_h^1(\Omega, \mathbb{R}^k)$ such that $\Sigma = f^{-1}(f(x_0)) \cap (U\mathbb{V})$ for some $x_0 \in \Omega$ and $J_H f(x) > 0$ for all $x \in \Sigma$, then ϕ in uniformly intrinsically differentiable.*

Proof. By Proposition 6.2.3 $J_{\mathbb{V}} f(x) > 0$ for every $x \in \Sigma$. Then, by Proposition 4.1.27, ϕ is the unique parametrization of Σ with respect to $(\mathbb{W}, \mathbb{V})$ and it is uniformly intrinsically differentiable. □

6.3 Low codimensional blow-up in the Heisenberg group

Definition 6.3.1. Let $\mathbb{M}$, $\mathbb{W}$ and $\mathbb{V}$ be homogeneous subgroups of $\mathbb{H}^n$ such that

$$\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V} = \mathbb{W} \rtimes \mathbb{V}. \tag{6.4}$$

The semidirect product $\mathbb{W} \rtimes \mathbb{V}$ automatically yields the classical group-projections

$$\pi_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W} \quad \text{and} \quad \pi_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V}$$

such that $x = \pi_{\mathbb{W}}(x)\pi_{\mathbb{V}}(x)$ for every $x \in \mathbb{H}^n$. To emphasize the dependence on the semidirect factorization now we introduce the notation $\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{V}}$. The same holds for $\mathbb{M} \rtimes \mathbb{V}$. We define also the following restrictions

$$\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}|_{\mathbb{M}} : \mathbb{M} \to \mathbb{W} \quad \text{and} \quad \pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}} = \pi_{\mathbb{M}}^{\mathbb{M},\mathbb{V}}|_{\mathbb{W}} : \mathbb{W} \to \mathbb{M}$$

Remark 6.3.2. The uniqueness of the factorizations (6.4) implies that both the restrictions $\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}$ and $\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}}$ are invertible and

$$\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}} = (\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}})^{-1}. \tag{6.5}$$

In fact, if $m \in \mathbb{M}$ we can write

$$m = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}(m)\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{V},\mathbb{M}}(m)$$

so that

$$m(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{V},\mathbb{M}}(m))^{-1} = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}(m).$$

Then, by the uniqueness of projections

$$m = \pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}}(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}(m)),$$

and the thesis is proved.

The main result of this section needs the following algebraic lemma.

Lemma 6.3.3. *We consider two vertical subgroups $\mathbb{M}$, $\mathbb{W}$ of $\mathbb{H}^n$ and a k-dimensional horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$ such that*

$$\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V} = \mathbb{W} \rtimes \mathbb{V}.$$

We introduce the multivectors

$$V = v_1 \wedge \cdots \wedge v_k, \quad W = w_1 \wedge \cdots \wedge w_{2n-k} \wedge e_{2n+1}, \quad M = m_1 \wedge \cdots \wedge m_{2n-k} \wedge e_{2n+1},$$

where $(v_1, \dots, v_k)$, $(w_1, \dots w_{2n-k}, e_{2n+1})$ and $(m_1, \dots, m_{2n-k}, e_{2n+1})$ are orthonormal bases of $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{M}$, respectively. Then for every Borel set $B \subset \mathbb{M}$, we have

$$(\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}})_\sharp \mathcal{H}^{2n+1-k}_E(B) = \mathcal{H}^{2n+1-k}_E(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}(B)) = \frac{\|V \wedge M\|_g}{\|V \wedge W\|_g}\mathcal{H}^{2n+1-k}_E(B),$$

where the projections $\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}}$ and $\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}$ have been introduced in Definition 6.3.1. The norms of $V \wedge M$ and $V \wedge W$ are taken with respect to the Hilbert structure of $\Lambda_{2n+1}(\mathbb{H}^n)$ induced by our scalar product on $\mathbb{H}^n$.

Proof. It is clearly not restrictive to relabel the bases of $\mathbb{M}$ and $\mathbb{W}$ as $w_{k+1}, \dots, w_{2n}, e_{2n+1}$ and $m_{k+1}, \dots, m_{2n}, e_{2n+1}$. We define the isomorphisms $i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^{2n+1-k}$,

$$i_{\mathbb{W}}\left(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i w_i\right) = (x_{k+1}, \dots, x_{2n+1}),$$

$i_{\mathbb{M}} : \mathbb{M} \to \mathbb{R}^{2n+1-k}$,

$$i_{\mathbb{M}}\left(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i m_i\right) = (x_{k+1}, \dots, x_{2n+1})$$

and $i_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k$

$$i_{\mathbb{V}}\left(\sum_{i=i}^{k} x_i v_i\right) = (x_1, \dots, x_k).$$

We introduce the map $\Psi_1 : \mathbb{R}^{2n+1} \to \mathbb{H}^n$,

$$\Psi_1(x_1,\dots,x_{2n+1}) = \left(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i w_i\right)\left(\sum_{i=1}^{k} x_i v_i\right), \tag{6.6}$$

and we now verify that $J\Psi_1(x) = \|V \wedge W\|_g$ for every $x = (x_1,\dots,x_{2n+1}) \in \mathbb{R}^{2n+1}$. In fact,

$$\begin{aligned}\Psi_1(x) &= \left(x_{2n+1}e_{2n+1} + \sum_{j=k+1}^{2n} x_j w_j\right)\left(\sum_{j=1}^{k} x_j v_j\right)\\ &= x_{2n+1}e_{2n+1} + \sum_{j=k+1}^{2n} x_j w_j + \sum_{j=1}^{k} x_j v_j + \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x_j w_j, \sum_{j=1}^{k} x_j v_j\right) e_{2n+1},\end{aligned}$$

hence

$$\partial_{x_i}\Psi_1(x) = v_i + \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x_j w_j, v_i\right) e_{2n+1} = v_i + c_i e_{2n+1}$$

for suitable constants $c_i \in \mathbb{R}$, $i = 1,\dots,k$ and

$$\partial_{x_j}\Psi_1(x) = w_j + \frac{1}{2}\omega\left(w_j, \sum_{\ell=1}^{k} x_\ell v_\ell\right) e_{2n+1} = w_j + d_j e_{2n+1}$$

for suitable constants $d_j \in \mathbb{R}$, $j = k+1,\dots,2n$. We may write

$$\begin{aligned}&\partial_{x_1}\Psi_1(x) \wedge \partial_{x_2}\Psi_1 \wedge \cdots \wedge \partial_{x_{2n+1}}\Psi_1(x)\\ =&(v_1 + c_1e_{2n+1}) \wedge \cdots \wedge (v_k + c_k e_{2n+1}) \wedge (w_{k+1} + d_{k+1}e_{2n+1}) \wedge \cdots \wedge (w_{2n} + d_{2n}e_{2n+1}) \wedge e_{2n+1}\\ =&v_1 \wedge \cdots \wedge v_k \wedge w_{k+1} \wedge \cdots \wedge w_{2n} \wedge e_{2n+1}.\end{aligned}$$

We have proved that

$$\partial_{x_1}\Psi_1(x) \wedge \partial_{x_2}\Psi_1(x) \wedge \cdots \wedge \partial_{x_{2n+1}}\Psi_1(x) = v_1 \wedge \cdots \wedge v_k \wedge w_{k+1} \wedge \cdots \wedge w_{2n} \wedge e_{2n+1}.$$

Taking into account that $V = v_1 \wedge \cdots \wedge v_k$ and $W = w_{k+1} \wedge \cdots \wedge w_{2n} \wedge e_{2n+1}$, and that

$$d_E\Psi_1(x) \wedge d_E\Psi_1(x) \wedge \cdots \wedge d_E\Psi_1(x)(e_1 \wedge \cdots \wedge e_{2n+1}) = \partial_{x_1}\Psi_1(x) \wedge \partial_{x_2}\Psi_1(x) \wedge \cdots \wedge \partial_{x_{2n+1}}\Psi_1(x),$$

we have proved that

$$J\Psi_1(x) = \|V \wedge W\|_g.$$

We define another map $\Psi_2 : \mathbb{R}^{2n+1} \to \mathbb{H}^n$,

$$\Psi_2(x_1,\dots,x_{2n+1}) = \left(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i m_i\right)\left(\sum_{j=1}^{k} x_i v_i\right),$$

and we observe in the same way that for every $x = (x_1,\dots,x_{2n+1}) \in \mathbb{R}^{2n+1}$,

$$J\Psi_2(x) = \|V \wedge M\|_g.$$

We introduce the embedding $q : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n+1}$,

$$q(x_1, \dots, x_{2n+1-k}) = (0, \dots, 0, x_1, \dots, x_{2n+1-k})$$

and the projection $p : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1-k}$,

$$p(x_1, \dots, x_{2n+1}) = (x_{k+1}, \dots, x_{2n+1}).$$

For every $z \in \mathbb{H}^n$, we observe that

$$\Psi_1^{-1}(z) = \left(i_{\mathbb{V}} \circ \pi_{\mathbb{V}}(z), i_{\mathbb{W}} \circ \pi_{\mathbb{W}}(z)\right).$$

It follows that

$$i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} = \pi_{\mathbb{W}}.$$

If we take any $m \in \mathbb{M}$, then

$$\begin{aligned} \pi_{\mathbb{W}}(m) &= i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ \Psi_2^{-1}(m) \\ &= i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ q \circ i_{\mathbb{M}}(m) \\ &= \pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}(m). \end{aligned} \tag{6.7}$$

Let us start by considering the map

$$\Psi_1^{-1} \circ \Psi_2,$$

and let us represent Ψ_1 as follows

$$\begin{aligned} \Psi_1(x) &= \sum_{j=k+1}^{2n} x_j w_j + \sum_{j=1}^{k} x_j v_j + \left(x_{2n+1} + \frac{1}{2}\omega \left(\sum_{j=k+1}^{2n} x_j w_j, \sum_{j=1}^{k} x_j v_j \right) \right) e_{2n+1} \\ &= \sum_{j=1}^{k} \tilde{x}_j v_j + \sum_{\ell=k+1}^{2n} \tilde{x}_\ell w_\ell + \tilde{x}_{2n+1} e_{2n+1} \end{aligned}$$

for suitable constants $\tilde{x}_j \in \mathbb{R}$, $j = 1, \dots, 2n+1$. Then clearly we have the following explicit form

$$\begin{aligned} &\Psi_1^{-1} \left(\sum_{j=1}^{k} \tilde{x}_j v_j + \sum_{\ell=k+1}^{2n} \tilde{x}_\ell w_\ell + \tilde{x}_{2n+1} e_{2n+1} \right) \\ &= \left(\tilde{x}_1, \dots, \tilde{x}_{2n}, \tilde{x}_{2n+1} - \frac{1}{2}\omega \left(\sum_{j=k+1}^{2n} \tilde{x}_j w_j, \sum_{j=1}^{k} \tilde{x}_j v_j \right) \right). \end{aligned} \tag{6.8}$$

Now we can consider the change of basis for $j = k+1, \dots, 2n$

$$m_j = \sum_{s=1}^{k} b_j^s v_s + \sum_{\ell=k+1}^{2n} b_j^\ell w_\ell + a_j e_{2n+1},$$

where b_j^s, b_j^ℓ, $a_j \in \mathbb{R}$ are suitable real numbers for $s = 1, \dots, k$, $\ell = k+1, \dots, 2n$.

Hence

$$\begin{aligned}\Psi_2(x) =& \sum_{j=1}^{k} x_j v_j + \sum_{j=k+1}^{2n} x_j m_j + \left(x_{2n+1} + \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x_j m_j, \sum_{j=1}^{k} x_j v_j\right)\right) e_{2n+1} \\ =& \sum_{s=1}^{k}\left(x_s + \sum_{j=k+1}^{2n} x_j b_j^s\right) v_s + \sum_{\ell=k+1}^{2n}\left(\sum_{j=k+1}^{2n} x_j b_j^\ell\right) w_\ell \\ &+ \left(x_{2n+1} + \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x_j m_j, \sum_{j=1}^{k} x_j v_j\right) + \sum_{j=k+1}^{2n} x_j a_j\right) e_{2n+1}.\end{aligned}$$

Finally we set

$$\begin{aligned} x'_s &= x'_s(x_1,\dots,x_{2n}) = x_s + \sum_{j=k+1}^{2n} x_j b_j^s \quad \text{for } s = 1,\dots,k, \\ x'_\ell &= x'_\ell(x_1,\dots,x_{2n}) = \sum_{j=k+1}^{2n} x_j b_j^\ell \qquad \text{for } \ell = k+1,\dots,2n, \\ x'_{2n+1} &= x_{2n+1} + \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x_j m_j, \sum_{j=1}^{k} x_j v_j\right) + \sum_{j=k+1}^{2n} x_j a_j \end{aligned}$$

and considering (6.8) we may write

$$\begin{aligned}\Psi_1^{-1}\circ\Psi_2(x) &= \left(x_1 + \sum_{j=k+1}^{2n} x_j b_j^1, \dots, x_k + \sum_{j=k+1}^{2n} x_j b_j^k, \sum_{j=k+1}^{2n} x_j b_j^{k+1}, \dots, \sum_{j=k+1}^{2n} x_j b_j^{2n}, \gamma(x)\right) \\ &= (x'_1,\dots,x'_{2n},\gamma(x)),\end{aligned}$$

where

$$\begin{aligned}\gamma(x) &= x'_{2n+1} - \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x'_j w_j, \sum_{j=1}^{k} x'_j v_j\right) \\ &= x_{2n+1} + \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x_j m_j, \sum_{j=1}^{k} x_j v_j\right) + \sum_{j=k+1}^{2n} x_j a_j - \frac{1}{2}\omega\left(\sum_{j=k+1}^{2n} x'_j w_j, \sum_{j=1}^{k} x'_j v_j\right) \\ &= x_{2n+1} + S(x),\end{aligned}$$

with $S(x) = S(x_1,\dots,x_{2n})$. The Jacobian matrix of $\Psi_2^{-1}\circ\Psi_1$ at any point x has the following form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b^1_{k+1} & b^1_{k+2} & \cdots & b^1_{2n} & 0 \\ 0 & 1 & \cdots & \vdots & b^2_{k+1} & b^2_{k+2} & \cdots & b^2_{2n} & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b^k_{k+1} & b^k_{k+2} & \cdots & b^k_{2n} & 0 \\ 0 & 0 & \cdots & 0 & b^{k+1}_{k+1} & b^{k+1}_{k+2} & \cdots & b^{k+1}_{2n} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b^{2n}_{k+1} & b^{2n}_{k+2} & \cdots & b^{2n}_{2n} & 0 \\ \partial_{x_1}S(x) & \partial_{x_2}S(x) & \cdots & \partial_{x_k}S(x) & \partial_{x_{k+1}}S(x) & \partial_{x_{k+2}}S(x) & \cdots & \partial_{x_{2n}}S(x) & 1 \end{pmatrix}.$$

We may write this matrix emphasizing its blocks hence the map $\Psi_1^{-1} \circ \Psi_2$ is a polynomial diffeomorphism, whose Jacobian matrix at x has the following form

$$\mathcal{J}(\Psi_1^{-1} \circ \Psi_2)(x) = \begin{pmatrix} I & R_1 & 0 \\ 0 & R_2 & 0 \\ L_1(x) & L_2(x) & 1 \end{pmatrix} \in M_{2n+1,2n+1}(\mathbb{R}),$$

where $I \in M_{k,k}(\mathbb{R})$ is the identity matrix $\mathbb{I}_k$, $R_1 \in M_{k,2n-k}(\mathbb{R})$, $R_2 \in M_{2n+1-k,2n+1-k}(\mathbb{R})$ and $L_1 : \mathbb{R}^{2n+1} \to \mathbb{R}^k$, $L_2 : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n-k}$ are affine functions.
From the definition of $q : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n+1}$ and $p : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1-k}$, if we consider the Jacobian matrix corresponding to the composition $\Psi_1^{-1} \circ \Psi_2 \circ q$ at any point $y \in \mathbb{R}^{2n+1-k}$, $\mathcal{J}(\Psi_1^{-1} \circ \Psi_2 \circ q)(y)$, it is the following matrix

$$\begin{pmatrix} R_1 & 0 \\ R_2 & 0 \\ L_2(q(y)) & 1 \end{pmatrix}.$$

Therefore we have

$$\mathcal{J}(p \circ \Psi_1^{-1} \circ \Psi_2 \circ q)(y) = \begin{pmatrix} R_2 & 0 \\ L_2(q(y)) & 1 \end{pmatrix}$$

and then it follows that

$$J(p \circ \Psi_1^{-1} \circ \Psi_2 \circ q)(y) = J(\Psi_1^{-1} \circ \Psi_2)(q(y)) = |\det R_2|. \tag{6.9}$$

As a consequence, taking into account that

$$\frac{\|V \wedge M\|_g}{\|V \wedge W\|_g} = J(\Psi_1^{-1} \circ \Psi_2),$$

the following equalities hold

$$\begin{aligned} \mathcal{H}_E^{2n+1-k}(B) &= \mathcal{L}^{2n+1-k}(i_{\mathbb{M}}(B)) \\ &= \frac{\|V \wedge W\|_g}{\|V \wedge M\|_g} \mathcal{L}^{2n+1-k}(p(\Psi_1(\Psi_2^{-1}(q(i_{\mathbb{M}}(B)))))) \\ &= \frac{\|V \wedge W\|_g}{\|V \wedge M\|_g} \mathcal{H}_E^{2n+1-k}(i_{\mathbb{W}}^{-1}(p(\Psi_1(\Psi_2^{-1}(q(i_{\mathbb{M}}(B)))))) \\ &= \frac{\|V \wedge W\|_g}{\|V \wedge M\|_g} \mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}(B)). \end{aligned}$$

□

We are now ready to compute the upper blow-up theorem.

Theorem 6.3.4 (Upper blow-up). *Let $\mathbb{H}^n$ be equipped with a homogeneous distance d. Consider an $\mathbb{H}$-regular surface Σ, consider an open set $\Omega \subset \mathbb{H}^n$ and a function $f \in C^1_h(\Omega, \mathbb{R}^k)$, with $1 \le k \le n$ and assume that $\Sigma \subset f^{-1}(0)$ and that for a k-dimensional horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$, $J_{\mathbb{V}} f(x) > 0$ for every $x \in \Sigma$. Then if we fix an homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$ such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, by Proposition 4.1.27 Σ is a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$. We call $\phi : U \to \mathbb{V}$ its parametrization, where $U \subset \mathbb{W}$ is an open set and $\Sigma = \Phi(U) = \mathrm{graph}(\phi)$, where $\Phi : U \to \mathbb{H}^n$ is the graph mapping of ϕ. Let $(v_1, \dots v_k) \subset H_1$ be an orthonormal basis of $\mathbb{V}$ and set $V = v_1 \wedge \dots \wedge v_k$. Consider an orthonormal basis $(w_{k+1}, \dots, w_{2n}, e_{2n+1})$ of $\mathbb{W}$ and define $W = w_{k+1} \wedge \dots \wedge w_{2n} \wedge e_{2n+1}$. Let us introduce the following measure*

$$\mu(B) = \|V \wedge W\|_g \int_{\Phi^{-1}(B)} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))} \, d\mathcal{H}^{2n+1-k}_E(n) \tag{6.10}$$

for every Borel set $B \subset \mathbb{H}^n$. Then for every $x \in \Sigma$ we have

$$\theta^{2n+2-k}(\mu, x) = \beta_d(\mathrm{Tan}(\Sigma, x)).$$

Remark 6.3.5. The factor $\|V \wedge W\|_g$ can be thought as the contribute given by the angle between the two subgroups $\mathbb{V}$ and $\mathbb{W}$. In fact, when $\mathbb{V}$ and $\mathbb{W}$ are orthogonal, $\|V \wedge W\|_g = 1$.

Remark 6.3.6. The theorem can be equivalently formulated assuming initially that Σ is a regularly parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$, with $\mathbb{V}$ of dimension k and $1 \le k \le n$ and calling $\phi : U \to \mathbb{V}$ its parametrization, where $U \subset \mathbb{W}$ is open set and $\Sigma = \Phi(U) = \mathrm{graph}(\phi)$. If we assume that ϕ is everywhere intrinsically differentiable and we take a map $f \in C^1_h(\Omega, \mathbb{R}^k)$ such that $\Sigma \subset f^{-1}(0)$ and $J_H f(x) > 0$ for all $x \in \Sigma$, then, by Proposition 6.2.3, $J_{\mathbb{V}} f(x) > 0$ for any $y \in \Sigma$. Hence we are in the same hypotheses of Theorem 6.3.4.

Remark 6.3.7. Theorem 6.3.4 holds also if we consider a splitting $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ with $\mathbb{V}$ horizontal subgroup of dimension k and we consider a uniformly intrinsically differentiable map $\phi : U \to \mathbb{V}$, with $U \subset \mathbb{W}$ open set. In fact, under these hypotheses Theorem 5.1.12 guarantees the existence of a defining function $f \in C^1_h(\Omega, \mathbb{R}^k)$, with $\Omega \subset \mathbb{H}^n$ open set such that $f(\mathrm{graph}(\phi)) = 0$ and $J_{\mathbb{V}} f(x) > 0$ for every $x \in \mathrm{graph}(\phi)$.

Proof of Theorem 6.3.4. Let us consider $x \in \Sigma$. By formula (6.10), for any $y \in \Omega$, taking $t > 0$ sufficiently small, we can write

$$\mu(\mathbb{B}(y,t)) = \|V \wedge W\|_g \int_{\Phi^{-1}(\mathbb{B}(y,t))} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))} \, d\mathcal{H}^{2n+1-k}_E(n). \tag{6.11}$$

We denote by $\zeta \in U$ the element such that

$$x = \Phi(\zeta) = \zeta\phi(\zeta).$$

We now perform the change of variables

$$n = \sigma_x(\Lambda_t(\eta)) = x(\Lambda_t\eta)(\pi_{\mathbb{V}}(x))^{-1} = x(\Lambda_t\eta)(\phi(\zeta))^{-1},$$

where $\Lambda_t = \delta_t|_{\mathbb{W}}$. The Jacobian of Λ_t is t^{2n+2-k}. It is well known that σ_x has unitary Jacobian (see for instance [FS16, Lemma 2.20]). Setting $\alpha(x) = J_H f(x)/J_{\mathbb{V}} f(x)$, we obtain that

$$\frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}} = \|V \wedge W\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))} (\alpha \circ \Phi)(\sigma_x(\Lambda_t(\eta)))) \, d\mathcal{H}_E^{2n+1-k}(\eta).$$

By the general definition of Federer density we obtain that

$$\begin{aligned}\theta^{2n+2-k}(\mu, x) &= \inf_{r>0} \sup_{\substack{y\in\mathbb{B}(x,t)\\ 0<t<r}} \frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}} \\ &= \inf_{r>0} \sup_{\substack{y\in\mathbb{B}(x,t)\\ 0<t<r}} \|V \wedge W\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))} (\alpha \circ \Phi)(\sigma_x(\Lambda_t(\eta))) \, d\mathcal{H}_E^{2n+1-k}(\eta).\end{aligned}$$

There exists $R_0 > 0$ such that for $t > 0$ and $y \in \mathbb{B}(x,t)$ we have the following inclusion

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))) \subset \mathbb{B}_{\mathbb{W}}(0, R_0), \tag{6.12}$$

where we have set

$$\mathbb{B}_{\mathbb{W}}(0, R_0) = \mathbb{B}(0, R_0) \cap \mathbb{W}.$$

To see (6.12), we write more explicitly the set $\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))$, that is

$$\left\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \left\|y^{-1}x(\Lambda_t\eta)\phi(\zeta)^{-1}\phi(x(\Lambda_t\eta)\phi(\zeta)^{-1})\right\| \le t\right\}.$$

It can be written as follows

$$\left\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \left\|(\delta_{1/t}(y^{-1}x))\eta\left(\frac{\phi(\zeta)^{-1}\phi(x(\Lambda_t\eta)\phi(\zeta)^{-1})}{t}\right)\right\| \le 1\right\}.$$

According to (3.21), the translated function of ϕ at x^{-1} is

$$\phi_{x^{-1}}(\eta) = \pi_{\mathbb{V}}(x^{-1})\phi(x\eta\pi_{\mathbb{V}}(x^{-1})) = \phi(\zeta)^{-1}\phi(x\eta\phi(\zeta)^{-1}).$$

We finally get

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))) = \left\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \left\|(\delta_{1/t}(y^{-1}x))\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right)\right\| \le 1\right\}, \tag{6.13}$$

hence for $\eta \in \Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))$, taking into account the previous equality, we get

$$\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right) \in \mathbb{B}(0,2).$$

By the estimate (3.1), we know that

$$c_0\left(\|\eta\| + \left\|\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right\|\right) \le \left\|\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right)\right\| \le 2,$$

hence the inclusion (6.12) holds with $R_0 = 2/c_0$. As a consequence, we have

$$\theta^{2n+2-k}(\mu, x) < \infty.$$

There exist a positive sequence $(t_p)_p$ converging to zero and $y_p \in \mathbb{B}(x, t_p)$, for every $p \in \mathbb{N}$ such that

$$\|V \wedge W\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \frac{J_H f(\Phi(\sigma_x(\Lambda_{t_p}(\eta))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_{t_p}(\eta)))} \, d\mathcal{H}_E^{2n+1-k_E}(\eta) \to \theta^{2n+2-k}(\mu, x)$$

as $p \to \infty$. Up to extracting a subsequence, since $y_p \in \mathbb{B}(x, t_p)$ for every $p \in \mathbb{N}$, there exists $z \in \mathbb{B}(0,1)$ such that

$$\lim_{p\to\infty} \delta_{1/t_p}(x^{-1}y_p) = z.$$

For the sake of simplicity, we use the notation

$$\mathbb{M}_x = \ker Df(x).$$

Using the projection introduced in Definition 6.3.1, we set

$$S_z = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z,1)) \subset \mathbb{W}.$$

Claim 1. For each $\omega \in \mathbb{W} \setminus S_z$, there exists

$$\lim_{p\to\infty} \mathbf{1}_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))}(\omega) = 0.$$

By contradiction, if we had a subsequence of the integers p such that

$$(\delta_{1/t_p}(y_p^{-1}x))\omega\left(\frac{\phi_{x^{-1}}(\Lambda_{t_p}\omega)}{t}\right) \in \mathbb{B}(0,1),$$

then by a slight abuse of notation, we could still call $(t_p)_p$ the sequence such that

$$(\delta_{1/t_p}(y_p^{-1}x))\omega d\phi_\zeta(\omega)\left(\frac{(d\phi_\zeta(\Lambda_{t_p}\omega))^{-1}\phi_{x^{-1}}(\Lambda_{t_p}\omega)}{t_p}\right) \in \mathbb{B}(0,1) \tag{6.14}$$

for all p, where we have used the homogeneity of the intrinsic differential $d\phi_\zeta$ of ϕ. Indeed, by Theorem 4.1.27, the function ϕ is in particular intrinsically differentiable at ζ (see also Theorems 4.1.22 and 4.1.23). Due to the intrinsic differentiability, taking into account (6.14) as $p \to \infty$, it follows that

$$\omega d\phi_\zeta(\omega) \in \mathbb{B}(z,1).$$

It is now interesting to observe that the chain rule of Theorem 6.2.2 yields

$$\operatorname{graph}(d\phi_\zeta) = \ker(Df(x)) = \mathbb{M}_x. \tag{6.15}$$

As a consequence, $\omega d\phi_\zeta(\omega) \in \mathbb{B}(z,1) \cap \mathbb{M}_x$ and then

$$\omega = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\omega d\phi_\zeta(\omega)) \in \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z,1)) = S_z, \tag{6.16}$$

that is not possible by our assumption. This concludes the proof of Claim 1.

Now we introduce the density function

$$\alpha(t,\eta) = \frac{J_H f(\Phi(\sigma_x(\Lambda_t(\eta))))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_t(\eta)))}$$

to write

$$\|V \wedge W\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) = I_p + J_p.$$

The sequence I_p, defined in the following equality, satisfies the estimate

$$\begin{aligned} I_p &= \|V \wedge W\|_g \int_{S_z \cap \Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\leq \|V \wedge W\|_g \int_{S_z} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned}$$

Analogously for J_p, we find

$$\begin{aligned} J_p &= \|V \wedge W\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))\setminus S_z} \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\leq \|V \wedge W\|_g \int_{\mathbb{B}_{\mathbb{W}}(0,R_0)\setminus S_z} \mathbf{1}_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))}(\eta)\, \alpha(t_p,\eta)\, d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned}$$

Claim 1 joined with the dominated convergence theorem prove that $J_p \to 0$ as $p \to \infty$, hence $I_p \to \theta^{2n+2-k}(\mu, x)$. To study the asymptotic behavior of I_p, we first observe that

$$\alpha(t_p,\eta) \to \frac{J_H f(x)}{J_{\mathbb{V}} f(x)} = c(x)$$

as $p \to \infty$. It follows that

$$\theta^{2n+2-k}(\mu,x) = \lim_{p\to\infty} I_p \leq \|V \wedge W\|_g\; c(x)\; \mathcal{H}_E^{2n+1-k}(S_z). \tag{6.17}$$

Claim 2. We set $\mathbb{M}_x = \ker(Df(x))$ and we consider $N_x = m_{k+1} \wedge \cdots \wedge m_{2n} \wedge e_{2n+1}$, where $(m_{k+1}, \ldots, m_{2n}, e_{2n+1})$ is an orthonormal basis of $\mathbb{M}_x$. We have

$$c(x) = \frac{J_H f(x)}{J_{\mathbb{V}} f(x)} = \frac{1}{\|V \wedge N_x\|_g}. \tag{6.18}$$

Since $\mathrm{span}\{\nabla_H f_1(x), \ldots, \nabla_H f_k(x)\}$ is orthogonal to $\mathbb{M}_x$, it is a standard fact that

$$m_{k+1} \wedge \cdots \wedge m_{2n} \wedge e_{2n+1} = *(\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x))\lambda \tag{6.19}$$

for some $\lambda \in \mathbb{R}$, see for instance [Mag08, Lemma 5.1]. Here we have defined the Hodge operator $*$ in $\mathbb{H}^n$ with respect to the fixed orientation

$$\mathbf{e} = e_1 \wedge \ldots e_{2n} \wedge e_{2n+1}$$

and the fixed scalar product $\langle \cdot, \cdot \rangle$. Precisely, we are referring to the Heisenberg basis $(e_1, \ldots, e_{2n}, e_{2n+1})$, according to Sections 2.3 and 2.4.1. Therefore $*\eta$ is the unique $(2n+1-k)$-vector such that

$$\xi \wedge *\eta = \langle \xi, \eta \rangle\ \mathbf{e} \tag{6.20}$$

for all k-vectors ξ. Since the Hodge operator is an isometry, we get

$$|\lambda| = \frac{1}{\|\nabla_H f_1(x) \wedge \ldots \nabla_H f_k(x)\|_g}. \tag{6.21}$$

Due to (6.20) and (6.21), we have

$$\begin{aligned}
\|V \wedge N_x\|_g &= |\lambda| \| \|v_1 \wedge \cdots \wedge v_k \wedge (*(\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)))\|_g \\
&= \frac{\|\langle v_1 \wedge \cdots \wedge v_k, \nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\rangle \mathbf{e}\|_g}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{|\langle v_1 \wedge \cdots \wedge v_k, \nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\rangle|}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{\|\nabla_{\mathbb{V}} f_1(x) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(x)\|_g}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{J_{\mathbb{V}} f(x)}{J_H f(x)},
\end{aligned}$$

hence establishing Claim 2.

As a result, taking into account (6.17), we have proved that

$$\theta^{2n+2-k}(\mu, x) \leq \frac{\|V \wedge W\|_g}{\|V \wedge N_x\|_g}\ \mathcal{H}_E^{2n+1-k}(S_z). \tag{6.22}$$

By Lemma 6.3.3 applied to $B = \mathbb{M}_x \cap \mathbb{B}(z,1)$, the following formula holds

$$\mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}_x}^{\mathbb{W},\mathbb{V}}(\mathbb{M}_x \cap \mathbb{B}(z,1))) = \frac{\|V \wedge N_x\|_g}{\|V \wedge W\|_g} \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z,1)). \tag{6.23}$$

It follows that

$$\theta^{2n+2-k}(\mu, x) \leq \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z,1)) \leq\ \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z_0,1)), \tag{6.24}$$

where $z_0 \in \mathbb{B}(0,1)$ is chosen such that $\beta_d(\mathbb{M}_x) = \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z_0,1))$.

For the opposite inequality, we follow the scheme in the proof of [Mag17, Theorem 3.1]. We consider a specific family of points $y_t^0 = x\delta_t z_0 \in \mathbb{B}(x,t)$ and fix $\lambda > 1$. We have

$$\sup_{0<t<r} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} \leq \sup_{\substack{y \in \mathbb{B}(x,t), \\ 0<t<\lambda r}} \frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}}$$

for every $r > 0$, therefore

$$\limsup_{t \to 0^+} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} \leq \theta^{2n+2-k}(\mu, x). \tag{6.25}$$

We introduce the set

$$\begin{aligned}
A_t^0 &= \Lambda_{1/\lambda t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_t^0, \lambda t))) \\
&= \left\{ \eta \in \Lambda_{1/\lambda t}(\sigma_x^{-1}(U)) : \eta \left(\frac{\phi_{x^{-1}}(\Lambda_{\lambda t} \eta)}{\lambda t} \right) \in \mathbb{B}(\delta_{1/\lambda} z_0, 1) \right\}.
\end{aligned}$$

The second equality can be deduced by (6.13). Then we can rewrite

$$\begin{aligned}
\frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} &= \|V \wedge W\|_g \int_{A_t^0} \alpha(\lambda t, \eta) d\mathcal{H}_E^{2n+1-k}(\eta) \\
&= \frac{\|V \wedge W\|_g}{\lambda^{2n+2-k}} \int_{\delta_\lambda A_t^0} \alpha(\lambda t, \delta_{1/\lambda} \eta) d\mathcal{H}_E^{2n+1-k}(\eta)
\end{aligned} \tag{6.26}$$

The domain of integration satisfies

$$\delta_\lambda A_t^0 = \left\{ \eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \eta\left(\frac{\phi_{x^{-1}}(\Lambda_t \eta)}{t}\right) \in \mathbb{B}(z_0,\lambda) \right\}.$$

Due to (6.12) and the definition of A_t^0, it holds

$$\delta_\lambda A_t^0 \subset \mathbb{B}_{\mathbb{W}}(0,\lambda R_0).$$

Claim 3. For every $\eta \in \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap B(z_0,\lambda))$, we have

$$\lim_{t\to 0^+} \mathbf{1}_{\delta_\lambda A_t^0}(\eta) = 1. \tag{6.27}$$

The intrinsic differentiability of ϕ at ζ shows that

$$\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right) \to \eta d\phi_\zeta(\eta) \quad \text{as} \quad t\to 0.$$

Taking into account (6.5) and (6.16), we get

$$\pi^{\mathbb{M}_x,\mathbb{V}}_{\mathbb{M}_x,\mathbb{W}}(\eta) = \eta d\phi_\zeta(\eta),$$

hence our assumption on η can be written as follows

$$d\left(\eta d\phi_\zeta(\eta), z_0\right) < \lambda.$$

We conclude that $\eta \in \delta_\lambda A_t^0$ for any $t>0$ sufficiently small, therefore the limit (6.27) holds and the proof of Claim 3 is complete.

By Fatou's lemma, taking into account (6.25) and (6.26) we get

$$\frac{\|V\wedge W\|_g}{\lambda^{2n+2-k}} \int_{\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x\cap B(z_0,\lambda))} \liminf_{t\to 0}\left(\mathbf{1}_{\delta_\lambda A_t^0}(\eta)\alpha(\lambda t,\delta_{1/\lambda}\eta)\right) d\mathcal{H}_E^{2n+1-k}(\eta) \le \theta^{2n+2-k}(\mu,x).$$

Claim 3 joined with (6.18) yields

$$\frac{1}{\lambda^{2n+2-k}} \frac{\|V\wedge W\|_g}{\|V\wedge N_x\|_g} \mathcal{H}_E^{2n+1-k}\left(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x\cap B(z_0,\lambda))\right) \le \theta^{2n+2-k}(\mu,x).$$

Applying again (6.23), we obtain

$$\frac{1}{\lambda^{2n+2-k}} \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x\cap B(z_0,\lambda)) \le \theta^{2n+2-k}(\mu,x).$$

Taking the limit as $\lambda\to 1^+$ and considering the opposite inequality (6.24), the proof is complete. In fact, it is enough to notice that by Proposition 4.1.13 $\mathbb{M}_x = \mathrm{Tan}(\Sigma,x)$. □

Slightly modifying the proof of the previous theorem, we obtain the following "centered blow-up" Theorem. We present the proof without repeating verbatim those parts of the proof that coincide with the ones of the proof of Theorem 6.3.4, to which we refer for explicit computations. For this reason, we have decided to name the claims of the following proof " Claim 1_c" and "Claim 3_c", since they mirror Claim 1 and Claim 3 of the proof of Theorem 6.3.4, respectively; the letter "c" stands for *centered.* Claim 2 will still be needed but its formulation, its proof and its role do not change at all in the proof of Theorem 6.3.8, then we will not rename it, considering it verbatim valid and directly referring the reader to the computations of Claim 2 of the proof of Theorem 6.3.4.

Theorem 6.3.8. *In the assumptions of Theorem 6.3.4, for every $x \in \Sigma$, we have*

$$\theta_c^{2n+2-k}(\mu, x) = \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(0,1)).$$

Proof. By the definition of θ_c^{2n+2-k}, performing a change of variable analogous to the one performed in the proof of the Theorem 6.3.4, we obtain

$$\begin{aligned}\theta_c^{2n+2-k}(\mu, x) &= \limsup_{t\to 0} \frac{\mu(\mathbb{B}(x,t))}{t^{2n+2-k}} \\ &= \limsup_{t\to 0} \|V \wedge W\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t))))} (\alpha \circ \Phi)(\sigma_x(\Lambda_t(\eta)))\, d\mathcal{H}_E^{2n+1-k}(\eta).\end{aligned}$$

There exists $R_0 > 0$ such that for $t > 0$ we have the following inclusion

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t)))) \subset \mathbb{B}_{\mathbb{W}}(0, R_0), \tag{6.28}$$

where we have set

$$\mathbb{B}_{\mathbb{W}}(0, R_0) := \mathbb{B}(0, R_0) \cap \mathbb{W}.$$

To see this we write, as in the previous proof, more explicitly the set $\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))$ as

$$\left\{\eta \in \Lambda_{1/t}\sigma_x^{-1}(U) : \left\|\eta \frac{\phi(\zeta)^{-1}\phi(\Phi(\zeta)\Lambda_t\eta\phi(\zeta)^{-1})}{t}\right\| \le 1\right\}.$$

Again by the definition of the translated map we obtain that

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t)))) = \left\{\eta \in \Lambda_{1/t}\sigma_x^{-1}(U) : \left\|\eta \frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right\| \le 1\right\}, \tag{6.29}$$

hence for $\eta \in \Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t)))$, taking into account (6.29) we have established that

$$\eta \frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t} \in \mathbb{B}(0,1).$$

By the estimate (3.1.17), we have

$$c_0\left(\|\eta\| + \left\|\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right\|\right) \le \left\|\eta\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right\| \le 1,$$

hence the inclusion (6.28) holds with $R_0 = 1/c_0$. As a consequence, we have

$$\theta_c^{2n+2-k}(\mu, x) < \infty.$$

Then there exists a positive sequence $(t_p)_p$ converging to zero such that

$$\|V \wedge W\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t_p))))} \frac{J_H f(\Phi(\sigma_x(\Lambda_{t_p}(\eta))}{J_V f(\Phi(\sigma_x(\Lambda_{t_p}(\eta)))}\, d\mathcal{H}_E^{2n+1-k}(\eta) \to \theta_c^{2n+2-k}(\mu, x)$$

as $p \to \infty$. Using the special projection of Definition 6.3.1, we set $\mathbb{M}_x := \ker Df(x)$ and we introduce the set

$$S = \pi_{\mathbb{W},\mathbb{M}_x}^{\mathbb{W},\mathbb{V}}(\mathbb{M}_x \cap \mathbb{B}(0,1)).$$

Following verbatim the proof of Theorem 6.3.4 one can easily verify the following claim.

Claim 1_c. For each $\omega \in \mathbb{W} \setminus S$, we have

$$\lim_{p\to\infty} 1_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t_p)))}(\omega) = 0.$$

Then, we can introduce as before the density function $\alpha(t,\eta) = \frac{J_H f(\Phi(\sigma_x(\Lambda_t(\eta)))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_t(\eta)))}$ and we can write for every $p \in \mathbb{N}$

$$\|V \wedge W\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t_p))))} \alpha(t_p,\eta)\; d\mathcal{H}_E^{2n+1-k}(\eta) = I_p + J_p,$$

where

$$\begin{aligned} I_p &= \|V \wedge W\|_g \int_{S\cap\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t_p))))} \alpha(t_p,\eta)\; d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\le \|V \wedge W\|_g \int_S \alpha(t_p,\eta)\; d\mathcal{H}_E^{2n+1-k}(\eta) \end{aligned}$$

and

$$\begin{aligned} J_p &= \|V \wedge W\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t_p))))\setminus S} \alpha(t_p,\eta)\; d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\le \|V \wedge W\|_g \int_{\mathbb{B}_{\mathbb{W}}(0,R_0)\setminus S} 1_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,t_p))))}(\eta)\; \alpha(t_p,\eta)\; d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned}$$

Again following verbatim the previous proof one can directly verify that $J_p \to 0$ as $p \to \infty$ and that

$$\lim_{p\to\infty} I_p \le \mathcal{H}_E^{2n+1-k}(S).$$

Hence, exploiting the same arguments adopted in the proof of the previous theorem, we get

$$\theta_c^{2n+1-k}(\mu,x) \le \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(0,1)). \tag{6.30}$$

In order to prove the opposite inequality, again we adapt the proof of the previous theorem. We start by fixing $\lambda > 1$. We have

$$\limsup_{t\to 0^+} \frac{\mu(\mathbb{B}(x,\lambda t))}{(\lambda t)^{2n+2-k}} \le \theta_c^{2n+2-k}(\mu,x). \tag{6.31}$$

Then, in analogy with the previous proof, we introduce the set

$$\begin{aligned} A_t &= \Lambda_{1/\lambda t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(x,\lambda t))) \\ &= \left\{\eta \in \Lambda_{1/\lambda t}(\sigma_x^{-1}(U)) : \eta\left(\frac{\phi_{x^{-1}}(\Lambda_{\lambda t}\eta)}{\lambda t}\right) \in \mathbb{B}(0,1)\right\}, \end{aligned}$$

where the second equality can be deduced by (6.29). Then we can rewrite

$$\begin{aligned} \frac{\mu(\mathbb{B}(x,\lambda t))}{(\lambda t)^{2n+2-k}} &= \|V \wedge W\|_g \int_{A_t} \alpha(\lambda t,\eta) d\mathcal{H}_E^{2n+1-k}(\eta) \\ &= \frac{\|V \wedge W\|_g}{\lambda^{2n+2-k}} \int_{\delta_\lambda A_t} \alpha(\lambda t, \delta_{1/\lambda}\eta) d\mathcal{H}_E^{2n+1-k}(\eta) \end{aligned} \tag{6.32}$$

The domain of integration satisfies

$$\delta_\lambda A_t = \left\{ \eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \eta \left(\frac{\phi_{x^{-1}}(\Lambda_t \eta)}{t} \right) \in \mathbb{B}(0,\lambda) \right\}.$$

Due to (6.28) and the definition of A_t, it holds that

$$\delta_\lambda A_t \subset \mathbb{B}_{\mathbb{W}}(0, \lambda R_0).$$

Clearly, the following Claim 3_c, which mirrors Claim 3 of the proof of Theorem 6.3.4, holds.

Claim 3_c. For every $\eta \in \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap B(0,\lambda))$, we have

$$\lim_{t \to 0^+} \mathbf{1}_{\delta_\lambda A_t}(\eta) = 1. \tag{6.33}$$

Then, by Fatou's lemma, taking into account (6.31) and (6.32) we get

$$\frac{\|V \wedge W\|_g}{\lambda^{2n+2-k}} \int_{\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap B(0,\lambda))} \liminf_{t \to 0} \left(\mathbf{1}_{\delta_\lambda A_t^0}(\eta) \alpha(\lambda t, \delta_{1/\lambda}\eta) \right) d\mathcal{H}_E^{2n+1-k}(\eta) \le \theta_c^{2n+2-k}(\mu, x).$$

Claim 3_c joined with (6.18) (considering that Claim 2 of the previous proof remains verbatim valid) yield

$$\frac{1}{\lambda^{2n+2-k}} \frac{\|V \wedge W\|_g}{\|V \wedge N_x\|_g} \mathcal{H}_E^{2n+1-k} \left(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap B(0,\lambda)) \right) \le \theta_c^{2n+2-k}(\mu, x).$$

Applying again Lemma 6.3.3 as in (6.23), we obtain

$$\frac{1}{\lambda^{2n+2-k}} \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap B(0,\lambda)) \le \theta_c^{2n+2-k}(\mu, x).$$

Taking the limit as $\lambda \to 1^+$ and considering the opposite inequality (6.30), the proof is complete. □

6.4 Area formulas

Combining Theorems 2.5.15 and Theorem 6.3.4 we immediately get the following area formula.

Theorem 6.4.1 (Area formula). *In the assumptions of Theorem 6.3.4, for any Borel set $B \subset \Sigma$ we have*

$$\mu(B) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x))\, d\mathcal{S}^{2k+2-k}(x). \tag{6.34}$$

If the factors of the semidirect product are orthogonal, the measure μ can be written in terms of the intrinsic partial derivatives of the parametrization ϕ of Σ. In the proof of the following theorem the map ϕ is though, as in Section 5.1, in coordinates with respect to a fixed Heisenberg basis. We assume $\mathbb{W}$ and $\mathbb{V}$ to be orthogonal subgroups, so that the basis can be chosen as in Proposition 5.1.1, to be sure that the product of $\mathbb{H}^n$ read in coordinates maintains the form (2.13). Thus, we can use all the results of Chapter 5.

Theorem 6.4.2. *In the assumptions of Theorem 6.3.4, if in addition $\mathbb{W}$ is orthogonal to*

$\mathbb{V}$, *then for every Borel set* $B \subset \Sigma$ *we have*

$$\int_B \beta_d(\mathrm{Tan}(\Sigma, x))\; d\mathcal{S}^{2k+2-k}(x) = \int_{\Phi^{-1}(B)} J^\phi \phi(w)\; d\mathcal{H}_E^{2n+1-k}(w), \tag{6.35}$$

where $J^\phi \phi$ *is the intrinsic Jacobian of* ϕ*, introduced in Definition 5.5.3.*

Proof. Since $\mathbb{W}$ and $\mathbb{V}$ are orthogonal, by Proposition 5.1.1 we can fix a Heisenberg basis $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n, w_1, \ldots w_{2n}, e_{2n+1})$ such that $\mathbb{V} = \mathrm{span}(v_1, \ldots, v_k)$ and $\mathbb{W} = \mathrm{span}(v_{k+1}, \ldots, v_n, w_i, \ldots, w_n, e_{2n+1})$. Our claim then follows by representing the measure μ in terms of the intrinsic partial derivatives of the parametrization ϕ of Σ, arguing as in the proof of Theorem 5.5.5. □

Remark 6.4.3. Taking into account Remark 6.3.7, formula (6.35) holds for any uniformly intrinsically differentiable function $\phi : U \subset \mathbb{W} \to \mathbb{V}$, where $\mathbb{W}$ and $\mathbb{V}$ are two complementary orthogonal subgroups such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$.

We now restrict our attention to homogeneous distances satisfying particular symmetries. Exploiting both Theorem 6.1.3 and Proposition 6.1.6 we obtain simpler versions of the area formula.

Theorem 6.4.4. *Let* d *be either a* $(2n+1-k)$*-vertically symmetric distance or a multiradial distance of* $\mathbb{H}^n$*. Then in the assumptions of Theorem 6.3.4, we have*

$$\mu = \omega_d(2n+1-k)\mathcal{S}^{2k+2-k} \llcorner \Sigma. \tag{6.36}$$

Therefore, by defining $\mathcal{S}_d^{2n+2-k} := \omega_d(2n+1-k)\mathcal{S}^{2n+1-k}$*, we have*

$$\mathcal{S}_d^{2n+2-k} \llcorner \Sigma = \|V \wedge W\|_g\; \Phi_\sharp \left(\frac{J_H f}{J_{\mathbb{V}} f} \circ \Phi \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{W}. \tag{6.37}$$

In the assumptions of the previous theorem, assuming in addition that $\mathbb{W}$ and $\mathbb{V}$ are orthogonal, equation (6.37) can be rewritten for any Borel set $B \subset \Sigma$ as

$$\mathcal{S}_d^{2n+2-k}(B) = \int_{\Phi^{-1}(B)} J^\phi \phi(w)\; d\mathcal{H}_E^{2n+1-k}(w), \tag{6.38}$$

where $J^\phi \phi$ is the intrinsic Jacobian of ϕ defined in 5.5.3.

By Theorem 2.5.13 and Theorem 6.3.8 we obtain an area formula for the centered Hausdorff measure of Σ. It is the analogue of Theorem 6.4.1.

Theorem 6.4.5. *In the assumptions of Theorem 6.3.4, for any Borel set* $B \subset \Sigma$ *we have*

$$\mu(B) = \int_B \mathcal{H}_E^{2n+1-k}(\mathrm{Tan}(\Sigma, x) \cap \mathbb{B}(0,1))\; d\mathcal{C}^{2k+2-k}(x). \tag{6.39}$$

Remark 6.4.6. When $d = d_\infty$, Theorem 6.4.5 recovers [FSSC07, Theorem 4.1].

Corollary 6.4.7. *Let* d *be a homogeneous distance on* $\mathbb{H}^n$ *such that* $\mathbb{B}(0,1)$ *is convex. In the assumptions of Theorem 6.3.4, for every* $x \in \Sigma$ *we obtain*

$$\theta_c^{2n+2-k}(\mu, x) = \theta^{2n+2-k}(\mu, x) \quad \text{and} \quad \mathcal{C}^{2n+2-k} \llcorner \Sigma = \mathcal{S}^{2n+2-k} \llcorner \Sigma. \tag{6.40}$$

Proof. By Theorem 6.1.7 and Theorem 6.3.8, for every $x \in \Sigma$ we have

$$\beta_d(\ker(Df(x)) = \mathcal{H}_E^{2n+1-k}(\ker(Df(x)) \cap \mathbb{B}(0,1)) = \theta_c^{2n+2-k}(\mu, x).$$

Finally, the area formulas (6.34) and (6.39) conclude the proof. □

Remark 6.4.8. If we consider the distance d_∞ on the Heisenberg group $\mathbb{H}^n$, its metric unit ball $\mathbb{B}(0,1) = \mathbb{B}_\infty(0,1)$ is convex. Hence, taking in consideration Remark 5.5.2 and Theorem 6.1.7, the area formula in Theorem 5.5.5 is consistent with the results of this section.

Remark 6.4.9. Notice that, combining Theorem 6.4.4 with [Mag11b, Theorem 1.1], we obtain the following useful coarea formula.
Fix a natural number $1 \leq k \leq n$. Let d be a $(2n+1-k)$-vertically symmetric distance on $\mathbb{H}^n$ and $f : \mathbb{H}^n \to \mathbb{R}^k$ be a Lipschitz mapping. Then we have

$$\int_{\mathbb{H}^n} u(x) J_H f(x)\, dx = \int_{\mathbb{R}^k} \left(\int_{f^{-1}(y)} u(x)\, d\mathcal{S}_d^{2n+2-k}(x) \right) dy, \tag{6.41}$$

where $u : \mathbb{H}^n \to [0,\infty]$ is a non-negative measurable function.

6.5 Recent results in the literature

Some progresses in developing formulas to compute the Hausdorff measure of intrinsic regular graphs in Carnot groups have been done after [CM20]. We collect below, in particular, some results that we consider the more connected with our direction of research. A very general area formula for suitably parametrized $(\mathbb{G},\mathbb{M})$-regular sets of a Carnot group $\mathbb{G}$, for a suitable Carnot group $\mathbb{M}$, has been proved by Julia, Nicolussi Golo and Vittone in [JNGV20, Theorem 1.1]. The authors consider a splitting of a Carnot group $\mathbb{G}$ as the product of two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$, with $\mathbb{W}$ normal, and a function $\phi : U \to \mathbb{V}$, with $U \subset \mathbb{W}$ open set such that $\Sigma = \mathrm{graph}(\phi)$ is a suitable $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ for some Carnot group $\mathbb{M}$. In particular, they assume that for every point $\bar{x} \in \Sigma$, there is a function $f \in C_h^1(\Omega,\mathbb{M})$, with $\Omega \subset \mathbb{G}$ open neighbourhood of $\bar{x}$, such that $\Sigma \cap \Omega = f^{-1}(0)$ and such that, for every $x \in \Omega$, $Df(x)$ is a h-epimorphism and $\ker(Df(x))$ is complementary to $\mathbb{V}$ (i.e. $Df(x)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is a h-isomorphism). They define for $x \in \Sigma \cap \Omega$, $T_x^H\Sigma := \ker(Df(x))$ and they notice that it does not depend on the choice of f. Then, if N is the homogeneous dimension of $\mathbb{W}$ and $h : \Sigma \to [0,\infty)$ is a Borel function, it holds that

$$\int_\Sigma h(y) d\psi^N(y) = \int_U h(\Phi(w)) \mathcal{A}(T^H_{\Phi(w)}\Sigma) d\psi^N(w), \tag{6.42}$$

where ψ^N denotes either the Hausdorff measure $\mathcal{H}^N$, or the spherical Hausdorff measure $\mathcal{S}^N$ with respect to the homogeneous distance of $\mathbb{G}$. The function $\mathcal{A}(\cdot)$ is a continuous map defined on Σ to $\mathbb{R}$ and depends only on the homogeneous distance fixed on $\mathbb{G}$, on $\mathbb{W}$, $\mathbb{V}$ and on the homogeneous tangent space $T^H_{\Phi(w)}\Sigma$ at the point $\Phi(w)$ of the graph. The map $\mathcal{A}$ is called *area factor*, see [JNGV20, Definition 3.2]. The theorem in [JNGV20] is valid for a wide generality of settings but we remark that in formula (6.42) the area factor remains only implicitly defined. The natural hope is to be able to compute the area factor in terms of suitably defined intrinsic derivatives of the parametrizing map ϕ, as it has been done for intrinsic regular graphs in the Heisenberg group. Notice, in fact, that, by Theorem 4.3.7, in the hypotheses of [JNGV20, Theorem 1.1], the map ϕ is uniformly intrinsically differentiable on U.

In [ADDDLD20, Proposition 1.8] the authors extend the area formula presented in [DD20a, Proposition 5.4] to Carnot groups of step 2 (see also the preceding formulas in

[ASCV06, Proposition 2.22] and [FSSC01, Theorem 6.5]). In particular they consider a map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ uniformly intrinsically differentiable, with $\mathbb{G} = \mathbb{W}\mathbb{V}$ a Carnot group of step 2 and $\mathbb{V}$ one-dimensional. They prove an area formula to compute the H-perimeter measure of graph(ϕ) in terms of the intrinsic derivatives of ϕ.

Finally, as a consequence of his Rademacher's theorem that we already discussed, Vittone derived a Lusin-type Theorem [Vit20, Theorem 1.2], that permits to the author to generalize to low codimensional intrinsic Lipschitz graphs formula (6.35), which by Remark 6.4.3 holds for all low codimensional uniformly intrinsically differentiable graphs in the Heisenberg group. The formula is directly obtained by combining the Lusin-type result [Vit20, Theorem 1.2] with [CM20, Theorem 1.2]. For a precise statement we refer the reader to [Vit20, Theorem 1.3].

Chapter 7

A Coarea-type inequality between Carnot groups

The purpose of this chapter is to present the results of [Cor20], whose development has been started during a visiting period at Université Paris-Sud, under the supervision of Prof. P. Pansu. A long-standing open problem about geometric measure theory in Carnot groups is the validity of the coarea formula for (metric) Lipschitz maps acting between two Carnot groups, endowed with homogeneous distances. A very general coarea-type inequality for Lipschitz maps between two metric spaces is due to Federer [Fed69, 2.10.25]. Restricting the focus on Lipschitz maps between two Carnot groups, Magnani proved a coarea-type inequality in [Mag02b]. Some stronger results, i.e. some coarea formulas have been proved for specific situations. For instance, one can refer to [FSSC96, MSC01] for bounded variation maps acting on an arbitrary Carnot group, to [Pan82a, Pan82b, Mag04a, Mag06a, Mag08, KV13] for Euclidean (or Riemannian) Lipschitz maps from a Carnot group to some $\mathbb{R}^k$, to [Koz15, MST18] for $C_h^{1,\alpha}$-regular mappings from the Heisenberg group $\mathbb{H}^n$ to some $\mathbb{R}^k$, with $\alpha > 0$ (refer to Remark 7.4.4 for a precise definition), and to [Mag05, Mag11b] for Lipschitz maps from $\mathbb{H}^n$ to $\mathbb{R}^k$, for $1 \leq k \leq n$. As usual, these references are only intended to give a flavour of the available results in the literature. Moreover, a general result has recently been proved in [JNGV20]. The authors consider two Carnot groups $\mathbb{G}$ and $\mathbb{M}$, endowed with homogeneous distances, an open set $\Omega \subset \mathbb{G}$ and a map $f : \Omega \to \mathbb{M}$, with Pansu differential $Df(x)$ continuous on Ω. Then, the coarea formula holds for f if, at every point $x \in \Omega$, either $Df(x)$ is a h-epimorphism or $Df(x)$ is not surjective. A key step in the proof of the coarea formula [JNGV20, Theorem 1.3] is the possibility of exploiting Theorem 4.2.15. In fact, if we fix a value $m \in \mathbb{M}$ and we consider a point $x \in f^{-1}(m)$ such that $Df(x)$ is surjective, if we assume that there exists a homogeneous subgroup $\mathbb{V}$ complementary to $\ker(Df(x))$ and we choose any homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$, then there exist an open neighbourhood $\Omega \subset \mathbb{G}$ of x, an open set $U \subset \mathbb{W}$ and a map $\phi : U \subset \mathbb{W} \to \mathbb{V}$ such that $f^{-1}(m) \cap \Omega$ is the intrinsic graph of ϕ. How we discussed in Section 4.4, it is still not clear how to prove the existence of an analogous parametrization if we assume the Pansu differential $Df(x)$ only to be surjective. Nevertheless in this chapter we bypass this lack and we prove a weaker coarea-type inequality, which permits, under a further regularity condition, to deal with more general situations. More precisely we prove the following theorem.

Theorem 7.0.1. *Let $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$ be two Carnot groups endowed with homogeneous distances, of Hausdorff dimension Q and P and topological dimension q and p, respectively. Let $f \in C_h^1(\mathbb{G}, \mathbb{M})$ be a function and assume that $Df(x)$ is surjective at every point $x \in \mathbb{G}$. Assume that there exist two constants $\tilde{r}, C > 0$ such that for $\mathcal{S}^P$-a.e $m \in \mathbb{M}$, the level set*

$f^{-1}(m)$ is $\tilde{r}$-locally C-lower Ahlfors $(Q-P)$-regular with respect to the measure $\mathcal{S}^{Q-P}$. Then there exists a constant $L = L(C, \mathbb{G}, p)$ such that, if Ω is a closed bounded subset of $\mathbb{G}$,

$$\int_\Omega C_P(Df(x))d\mathcal{S}^Q(x) \le L \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega)d\mathcal{S}^P(m).$$

The factor $C_P(Df(x))$ is the coarea factor of the Pansu differential $Df(x)$ (see Definition 7.2.1) and it plays the role of the Jacobian of the Pansu differential at x. Clearly $\mathcal{S}^P$ denotes the spherical Hausdorff measure built on $\mathbb{M}$ with respect to d_2, while by $\mathcal{S}^Q$ and $\mathcal{S}^{Q-P}$ we denote the spherical Hausdorff measures on $\mathbb{G}$ with respect to d_1. Refer to Definition 7.2.4 for the notion of locally lower Ahlfors regular set.

The proof of Theorem 7.0.1 is inspired to an abstract procedure presented in [Pan20], where it is used to prove a coarea-type inequality for functions acting from a metric space to a measure space, for packing-type measures. An analogous argument involving suitable packing measures is adapted here to prove Claim 1 of Theorem 7.3.3. It is immediate to extend Theorem 7.0.1, to the case when Ω is a measurable subset of $\mathbb{G}$ (Theorem 7.3.4). As an example of its generality, notice that Theorem 7.0.1 can be applied to any continuously Pansu differentiable functions $f : \mathbb{H}^1 \to \mathbb{R}^2$ satisfying the requirements. As a corollary of Theorem 7.0.1, we deduce new results about the slicing of measurable functions by the level sets of f (Corollaries 7.4.2 and 7.4.3).

We can compare [JNGV20, Theorem 3.1] with Theorem 7.0.1 in terms of regular submanifolds, exploiting the various notions of regularity introduced in Chapter 4. Both the results involve two Carnot groups $\mathbb{G}$ and $\mathbb{M}$, an open set $\Omega \subset \mathbb{G}$ and a function $f \in C^1_h(\Omega, \mathbb{M})$. In Theorem 7.0.1 we assume that the Pansu differential $Df(x)$ is surjective at every $x \in \mathbb{G}$. This implies that, for every $m \in \mathbb{M}$, the level set $f^{-1}(m)$ is a $(\mathbb{G}, \mathbb{M})_K$-regular submanifold of $\mathbb{G}$. Then we restrict ourselves to the family of functions for which $\mathcal{S}^P$-almost all level sets $f^{-1}(m)$ satisfy our hypothesis of uniform local Ahlfors regularity. In fact, as we already stressed in Remark 4.4.3, $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds do not necessarily satisfy this measure-theoretic property. The result in [JNGV20, Theorem 3.1], instead, can be applied, in particular, assuming that the level sets $f^{-1}(m)$ are $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$, and therefore that they are, at least locally, regular intrinsic graphs. Hence, in Theorem 7.0.1 the assumption about the uniform local lower Ahlfors regularity of the level sets of the map f can then be read as a substitute of the existence of a suitable splitting of $\mathbb{G}$. In fact, this condition is automatically verified if one assumes the existence of a homogeneous subgroup $\mathbb{V} \subset \mathbb{G}$ complementary to $\ker(Df(x))$ for every point $x \in \mathbb{G}$ (Corollary 7.4.9), since, we remark, in this case every level set is locally an intrinsic Lipschitz graph and, by the results in [FS16], intrinsic Lipschitz graphs are Ahlfors regular, and then in particular lower Ahlfors regular. We stress that Corollary 7.4.9, by our point of view, is just an example of an application of Theorem 7.0.1. In fact, as we discussed above, it can be immediately derived also by the coarea formula in [JNGV20, Theorem 1.3].

7.1 Setting and packings

When nothing different is specified, in this chapter we consider two Carnot groups endowed with homogeneous distances $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$, of topological dimension q and p, and Hausdorff dimension Q and P, respectively. The groups $\mathbb{G}$ and $\mathbb{M}$ are direct sums of linear subspaces as follows

$$\mathbb{G} = V_1 \oplus \cdots \oplus V_\kappa \qquad \mathbb{M} = W_1 \oplus \cdots \oplus W_\vartheta.$$

By $\mathbb{B}(x,r)$ and $\mathbb{B}_{\mathbb{M}}(x,r)$ we denote the closed metric balls (of center x and radius r) in $\mathbb{G}$ and $\mathbb{M}$, respectively. Analogously, we denote by $B(x,r)$ and $B_{\mathbb{M}}(x,r)$ the corresponding open balls. By δ_t^1 and δ_t^2 we denote the intrinsic anisotropic dilations of $\mathbb{G}$ and $\mathbb{M}$, respectively. If we have a generic metric ball $\mathbb{B}$ of $\mathbb{G}$, hence $\mathbb{B} = \mathbb{B}(x,r)$ for some $x \in \mathbb{G}$ and $r > 0$, for any number $\ell > 0$, we call the concentric ball $\ell\mathbb{B} := \mathbb{B}(x, \ell r)$.

Definition 7.1.1 (Packing)**.** Let N and ℓ be two natural numbers, with $\ell \geq 1$. Let X be a metric space. An *ℓ-packing* is a countable collection of closed balls $\{\mathbb{B}_i\}$ such that the concentric balls $\ell\mathbb{B}_i$ are pairwise disjoint. An (N,ℓ)-packing is a collection of balls $\{\mathbb{B}_i\}$ which is the union of at most N ℓ-packings

In the previous definition, and from now on, by $\{\mathbb{B}_i\}$ we mean $\{\mathbb{B}_i\}_{i\in\mathbb{N}}$.

Remark 7.1.2. In a doubling metric space it is not restrictive to assume that once fixed a number $\ell \geq 1$, there exists a natural number N, only depending on ℓ, such that, for every δ small enough, there exist (N,ℓ)-packings made of balls of radius smaller that δ that cover the whole space. For instance, in [Pan20, Remark 3.2] it is proved that if a metric space X is doubling at small scales, fine coverings that are (N,ℓ)-packings exist, with N depending only on ℓ. We report here the argument: we fix $\ell \geq 1$, for any $\delta > 0$ we can fix some $0 < \varepsilon < \delta$ and consider a maximal packing of disjoint balls of radius $\frac{\varepsilon}{2}$, $\left\{\mathbb{B}(x, \frac{\varepsilon}{2})\right\}_{x\in I}$ (I is just a suitable countable set of points of the space). Observe that the corresponding balls of double radius $\{\mathbb{B}(x,\varepsilon)\}_{x\in I}$ cover the space. Now we fix one of these balls $\mathbb{B}(x,\varepsilon)$, hence a center $x \in I$, and we assume that for some $x' \in I$, $\mathbb{B}(x,\ell\varepsilon) \cap \mathbb{B}(x',\ell\varepsilon) \neq \emptyset$. Clearly it must be true that $\mathbb{B}(x', \frac{\varepsilon}{2}) \subset \mathbb{B}(x,(2\ell+1)\varepsilon)$ and this implies that the number of the balls $\mathbb{B}(x',\ell\varepsilon)$, with $x' \in I$, overlapping $\mathbb{B}(x,\ell\varepsilon)$ is necessarily bounded by the number of disjoint balls of radius $\frac{\varepsilon}{2}$ contained in $\mathbb{B}(x,(2\ell+1)\varepsilon)$ that in a doubling metric space is bounded above depending only on ℓ. Then, for some N depending on ℓ, $\{\mathbb{B}(x,\varepsilon)\}_{x\in I}$ is a (N,ℓ)-packing of balls of radius smaller than δ. Of course, for our purposes, it is enough that the doubling property is satisfied for small values of ε. The argument extends to packings of balls centered on a subset Ω of the space. In particular, the number N, that we individuate for packings of the whole space, still works for packings of balls centered on Ω.

Definition 7.1.3 (Packing pre-measure)**.** Let $\ell \geq 1$ and N be natural numbers. Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d and let $\delta > 0$ and $\alpha > 0$. Let $E \subset \mathbb{G}$, we introduce

$$\mathcal{P}^{\alpha}_{N,\ell,\delta}(E) = \sup\Big\{ \sum_{i=1}^{\infty} r(B_i)^{\alpha} : \ \{B_i\} \ (N,\ell)\text{-packing of } E, \ E \subset \bigcup_{i=1}^{\infty} B_i, \\ B_i \text{ centered on } E, \ r(B_i) \leq \delta \Big\}$$

and we define

$$\mathcal{P}^{\alpha}_{N,\ell}(E) := \inf_{\delta>0} \mathcal{P}^{\alpha}_{N,\ell,\delta}(E).$$

We define also the following packing-type pre-measure

$$\tilde{\mathcal{P}}^{\alpha}_{N,\ell,\delta}(E) = \sup\left\{ \sum_{i=1}^{\infty} r(B_i)^{\alpha} : \{B_i\} \ (N,\ell)\text{-packing of } E, B_i \text{ centered on } E, r(B_i) \leq \delta \right\},$$

and

$$\tilde{\mathcal{P}}^{\alpha}_{N,\ell}(E) := \inf_{\delta>0} \tilde{\mathcal{P}}^{\alpha}_{N,\ell,\delta}(E).$$

In particular, notice that in this case we have not required the packings of E to cover E.

Remark 7.1.4. Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d and let $E \subset \mathbb{G}$ and $\alpha > 0$. Let $\ell \geq 1$ and let N be a natural number such that there exist fine (N, ℓ)-packings that cover $\mathbb{G}$ (see Remark 7.1.2). Then

$$\mathcal{S}^\alpha(E) \leq \mathcal{P}^\alpha_{N,\ell}(E). \tag{7.1}$$

In fact, for every $\delta > 0$, any (N, ℓ)-packing of E that covers E made of balls centered on E of radius smaller that δ is a covering of E made of balls of radius smaller than δ, hence for any $\delta > 0$ we have

$$\phi_{\delta,\zeta}(E) \leq \mathcal{P}^\alpha_{N,\ell,\delta}(E),$$

where $\phi_{\delta,\zeta}$ is built as in (2.16) setting $\mathcal{F}$ as the family of closed balls $\mathcal{F}_b$ and $\zeta(\mathbb{B}_d(x, r)) = r^\alpha$. Letting δ go to zero, we get (7.1).

7.2 Coarea factor and some known results

Definition 7.2.1 (Coarea factor)**.** Let $L : \mathbb{G} \to \mathbb{M}$ be a h-homomorphism and let $Q \geq P$. We call *coarea factor* of L, $C_P(L)$, the unique positive constant such that

$$\mathcal{S}^Q(\mathbb{B}(0, 1))C_P(L) = \int_{\mathbb{M}} \mathcal{S}^{Q-P}(L^{-1}(m) \cap \mathbb{B}(0, 1))d\mathcal{S}^P(m). \tag{7.2}$$

Proposition 7.2.2. *[Mag02b, Proposition 1.12] In the notation of Definition 7.2.1, the coarea factor $C_P(L)$ is well defined. Moreover, $C_P(L)$ is not equal to zero if and only if L is surjective and in this case it can be computed as follows*

$$\begin{aligned} C_P(L) &= \frac{\mathcal{S}^{Q-P}(\ker(L) \cap \mathbb{B}(0, 1))}{\mathcal{H}^{q-p}_E(\ker(L) \cap \mathbb{B}(0, 1))} \frac{\mathcal{S}^P(\mathbb{B}_{\mathbb{M}}(0, 1))}{\mathcal{L}^p(\mathbb{B}_{\mathbb{M}}(0, 1))} \frac{\mathcal{L}^q(\mathbb{B}(0, 1))}{\mathcal{S}^Q(\mathbb{B}(0, 1))} JL \\ &= Z \frac{\mathcal{S}^{Q-P}(\ker(L) \cap \mathbb{B}(0, 1))}{\mathcal{H}^{q-p}_E(\ker(L) \cap \mathbb{B}(0, 1))} JL, \end{aligned} \tag{7.3}$$

where we set $Z := \frac{\mathcal{S}^P(\mathbb{B}_{\mathbb{M}}(0,1))}{\mathcal{L}^p(\mathbb{B}_{\mathbb{M}}(0,1))} \frac{\mathcal{L}^q(\mathbb{B}(0,1))}{\mathcal{S}^Q(\mathbb{B}(0,1))}$.

Remark 7.2.3. Observe that Z is a geometrical constant not depending on L.

Proof. Surely, for any $r > 0$, the restricted dilation $\delta^2_r|_{L(\mathbb{G})}$ has Jacobian $r^{P'}$, where $P' = \sum_{i=1}^k i\dim(L(V_i))$. Then for every $r > 0$

$$\mathcal{H}^{p'}_E(\mathbb{B}_{\mathbb{M}}(0, r) \cap L(\mathbb{G})) = r^{P'}\mathcal{H}^{p'}_E(\mathbb{B}_{\mathbb{M}}(0, 1) \cap L(\mathbb{G})),$$

where p' is the topological dimension of $L(\mathbb{G})$.
If L is not surjective, $P' < \sum_{i=1}^{\vartheta} i\dim(W_i) = P$, then the Hausdorff dimension of $L(\mathbb{G})$ is less than P and then by (2.17) and (7.2), $C_P(L) = 0$.
Assume now that L is surjective, and set $\mathbb{K} := L^{-1}(0) = \ker L$. Surely $\mathbb{K}$ is a homogeneous normal subgroups of topological dimension $q - p$ and $\mathbb{K}$ is graded $\mathbb{K} = \mathbb{K}_1 \oplus \cdots \oplus \mathbb{K}_\kappa$, with $\mathbb{K}_i$ linear subspace of V_i for $i = 1, \ldots, \kappa$. Reasoning as above for $\delta^1_r|_{\mathbb{K}}$, we have

$$\mathcal{H}^{q-p}_E(\mathbb{B}(0, r) \cap \mathbb{K}) = r^{Q'}\mathcal{H}^{q-p}_E(\mathbb{B}(0, 1) \cap \mathbb{K}),$$

where $Q' = \sum_{i=1}^\kappa i\dim(\mathbb{K}_i)$. L is a h-homomorphism, and then $L(V_i) \subset W_i$ for every $i = 1, \ldots, \kappa$. Hence, since we have assumed that L is surjective we know that $\kappa \geq \vartheta$,

$\dim(V_i) \geq \dim(W_i)$ and $\dim(\mathbb{K}_i) = \dim(V_i) - \dim(W_i)$ for every $i = 1, \dots, \kappa$. Then

$$Q' = \sum_{i=1}^{\kappa} i\dim(\mathbb{K}_i) = \sum_{i=1}^{\kappa} i(\dim(V_i) - \dim(W_i)) = \sum_{i=1}^{\kappa} i\dim(V_i) - \sum_{i=1}^{\kappa} i\dim(W_i) = Q - P.$$

Since $\mathcal{S}^{Q-P} \llcorner \mathbb{K}$ and $\mathcal{H}_E^{q-p} \llcorner \mathbb{K}$ are both Haar measures on $\mathbb{K}$ (endowed with the product of $\mathbb{G}$ restricted to $\mathbb{K}$) they must coincide up to a constant $\alpha_{Q,P}$, that is

$$\mathcal{S}^{Q-P} \llcorner \mathbb{K} = \alpha_{P,Q} \mathcal{H}_E^{q-p} \llcorner \mathbb{K}, \tag{7.4}$$

where $\alpha_{P,Q} = \frac{\mathcal{S}^{Q-P}(\mathbb{B}(0,1)\cap\mathbb{K})}{\mathcal{H}_E^{q-p}(\mathbb{B}(0,1)\cap\mathbb{K})}$. Now, we fix an arbitrary $m \in \mathbb{M}$. We can observe that $L^{-1}(m) = x\mathbb{K} = x + \mathbb{K}$ for any fixed $x \in \mathbb{G}$ such that $L(x) = m$. By the left invariance of the homogeneous distance on $\mathbb{G}$, the measure $\mathcal{S}^{Q-P}$ satisfies the following left invariance property

$$(l_x)_\sharp \mathcal{S}^{Q-P} \llcorner \mathbb{K} = \mathcal{S}^{Q-P} \llcorner x\mathbb{K}. \tag{7.5}$$

At the same time, by Proposition 3.1.22, since $\mathbb{K}$ is a normal homogeneous subgroup it holds that for any $x \in L^{-1}(m)$,

$$(l_x)_\sharp \mathcal{H}_E^{q-p} \llcorner \mathbb{K} = \mathcal{H}_E^{q-p} \llcorner x\mathbb{K}. \tag{7.6}$$

Hence, by (7.5) and (7.6), we can substitute $\mathbb{K}$ with $L^{-1}(m)$ in (7.4) and, without modifying $\alpha_{Q,P}$ we obtain that for every $m \in \mathbb{M}$, and for every $x \in \mathbb{G}$ such that $L(x) = m$, it holds that

$$\mathcal{S}^{Q-P} \llcorner L^{-1}(m) = \mathcal{S}^{Q-P} \llcorner x\mathbb{K} = \alpha_{P,Q} \mathcal{H}_E^{q-p} \llcorner x\mathbb{K} = \alpha_{P,Q} \mathcal{H}_E^{q-p} \llcorner L^{-1}(m). \tag{7.7}$$

Let us now consider the *coarea measure* ν_L associated with L on $\mathbb{G}$, defined for every Borel set $B \subset \mathbb{G}$ as

$$\nu_L(B) = \int_{\mathbb{M}} \mathcal{S}^{Q-P}(B \cap L^{-1}(m)) d\mathcal{S}^P(m). \tag{7.8}$$

In particular, by our previous considerations, ν is positive on open bounded sets. Moreover, by [Fed69, 2.10.25] (see, for instance, [Mag02b, Theorem 1.4]), ν_L is finite on the sets with $\mathcal{S}^Q$-finite measure. Moreover, by a change of variables involving left translations, ν_L is left invariant on $\mathbb{G}$. Then ν_L is a Haar measure on $\mathbb{G}$ and then it coincides with $\mathcal{S}^Q$ on $\mathbb{G}$ up to a positive constant $C_P(L)$, that is

$$\nu_L = C_P(L)\mathcal{S}^Q.$$

Let us compute $C_P(L)$ explicitly. By the previous observations and by the fact that $\mathcal{S}^P$ is proportional to $\mathcal{L}^p$ on $\mathbb{M}$ we can write

$$\int_{\mathbb{M}} \mathcal{S}^{Q-P}(\mathbb{B}(0,1) \cap L^{-1}(m)) d\mathcal{S}^P(m) = \alpha_{Q,P}\beta_P \int_{\mathbb{M}} \mathcal{H}_E^{q-p}(\mathbb{B}(0,1) \cap L^{-1}(m)) d\mathcal{L}^p(m),$$

where $\beta_P := \frac{\mathcal{S}^P(\mathbb{B}_{\mathbb{M}}(0,1))}{\mathcal{L}^p(\mathbb{B}_{\mathbb{M}}(0,1))}$ and then, by the Euclidean coarea formula, we get

$$\int_{\mathbb{M}} \mathcal{S}^{Q-P}(\mathbb{B}(0,1) \cap L^{-1}(m)) d\mathcal{S}^P(m) = \alpha_{Q,P}\beta_P\gamma_Q \; JL \; \mathcal{S}^Q(\mathbb{B}(0,1)), \tag{7.9}$$

where $\gamma_Q := \frac{\mathcal{L}^q(\mathbb{B}(0,1))}{\mathcal{S}^Q(\mathbb{B}(0,1))}$. By the comparison between (7.9) and the definition of the coarea factor, we get that $C_P(L) = \beta_P \gamma_Q \alpha_{Q,P} JL = Z\alpha_{Q,P} JL$ and then the claimed formula (7.2) is proved. □

Definition 7.2.4. Let (X, d, μ) be a metric measure space, consider a subset $E \subset X$ and two positive numbers $\alpha, C > 0$. We say that E is *locally C-lower Ahlfors α-regular with respect to* μ if there is $\tilde{r} > 0$ such that for every $x \in E$ and $0 < r < \tilde{r}$,

$$\mu(\mathbb{B}(x, r) \cap E) \geq Cr^\alpha.$$

If we need to stress the value of $\tilde{r}$, we say that E is *$\tilde{r}$-locally C-lower Ahlfors α-regular with respect to μ*. If $\tilde{r} = \infty$, we say that E is *C-lower Ahlfors α-regular with respect to μ*.

In [Mag02b], also relying on a coarea estimate for Lipschitz maps in arbitrary metric spaces due to Federer [Fed69, 2.10.25], the author proved a coarea-type inequality. We recall it here, adapting it to our context.

Theorem 7.2.5. *[Mag02b, Theorem 2.6] Let $A \subset \mathbb{G}$ be a measurable set and $f : A \to \mathbb{M}$ be a Lipschitz map, then*

$$\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap A) d\mathcal{S}^P(m) \leq \int_A C_P(Df(x)) d\mathcal{S}^Q(x). \tag{7.10}$$

7.3 Coarea-type Inequality

We will need the following simple proposition in the proof of our main theorem.

Proposition 7.3.1. *Let $\mathbb{G}$ be a Carnot group endowed with a homogeneous distance d. Let $\mathbb{W} \subset \mathbb{G}$ be a homogeneous subgroup of topological dimension n and Hausdorff dimension N. Then for every Borel set $B \subset \mathbb{W}$ we have*

$$\mathcal{H}^n_E(B) = \sup_{w \in \mathbb{B}(0,1)} \mathcal{H}^n_E(\mathbb{B}(w, 1) \cap \mathbb{W})\ \mathcal{S}^N(B)$$

Proof. Let us consider the measure $\mu_{\mathbb{W}} := \mathcal{H}^n_E \llcorner \mathbb{W}$. Since $\mu_{\mathbb{W}}$ and $\mathcal{S}^N \llcorner \mathbb{W}$ are both Haar measures on $\mathbb{W}$, they coincide up to a positive constant and therefore we can apply Theorem 2.5.15. Hence we know that $\mu_{\mathbb{W}}(A) = \int_A \theta^N(\mu_{\mathbb{W}}, x) \mathcal{S}^N(x)$. Let us then compute for any $x \in \mathbb{W}$

$$\theta^N(\mu_{\mathbb{W}}, x) = \inf_{r>0} \sup_{\substack{\{z: x \in \mathbb{B}(z,t)\} \\ 0<t<r}} \frac{\mu_{\mathbb{W}}(\mathbb{B}(z, t))}{t^N}$$

notice that for every $t > 0$ and $z \in B(x, t)$,

$$\begin{aligned} \frac{\mu_{\mathbb{W}}(\mathbb{B}(z, t))}{t^N} &= \frac{\mathcal{H}^n_E(z\delta_t(\mathbb{B}(0, 1)) \cap \mathbb{W})}{t^N} \\ &= \frac{\mathcal{H}^n_E(x^{-1}z\delta_t(\mathbb{B}(0, 1)) \cap x^{-1}\mathbb{W})}{t^N} \\ &= \mathcal{H}^n_E(\delta_{1/t}(x^{-1}z)\mathbb{B}(0, 1) \cap \mathbb{W}). \end{aligned}$$

We have used the left invariance of $\mathcal{H}^n_E \llcorner \mathbb{W}$ with respect to the product restricted to $\mathbb{W}$, and the fact that the Jacobian of the dilation δ_t restricted to $\mathbb{W}$ is t^N for every $t > 0$. Hence

$$\theta^N(\mu_{\mathbb{W}}, x) = \inf_{r>0} \sup_{\{w: d(w,0) \leq 1\}} \mathcal{H}^n_E(\mathbb{B}(w, 1) \cap \mathbb{W}) = \sup_{w \in \mathbb{B}(0,1)} \mathcal{H}^n_E(\mathbb{B}(w, 1) \cap \mathbb{W}).$$

□

Before proving our main result, that is Theorem 7.0.1, we make some preliminary observations referring to the statement of the theorem.

Remark 7.3.2. It is immediate to observe that any continuously Pansu differentiable function is locally metric Lipschitz (by Proposition 3.2.32), hence by [Mag11b, Theorem 2.1] we know that for every measurable set $A \subset \mathbb{G}$ the function $m \to \mathcal{S}^{Q-P}(A \cap f^{-1}(m))$ is $\mathcal{S}^P$-measurable. Notice that the measurability of the map $x \to C_P(Df(x))$ follows from [Mag02b, Theorem 2.6].

Theorem 7.0.1 is a direct consequence of the following result. In fact, it is analogous to Theorem 7.3.3, where the hypotheses have been slightly weakened even if they are definitely more technical.

Theorem 7.3.3. *Let $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$ be two Carnot groups, endowed with homogeneous distances, of Hausdorff dimension Q, P and topological dimension q, p, respectively. Let Ω' be an open subset of $\mathbb{G}$. Let $f \in C_h^1(\Omega', \mathbb{M})$ be a function and assume that $Df(x)$ is surjective at every $x \in \Omega'$. Let $\Omega \Subset \Omega'$ be a closed bounded set such that there exists an open set Ω'' and a positive number $s > 0$ such that Ω'' is compactly contained in Ω and, setting $\Omega_s := \{x \in \mathbb{G} : \operatorname{dist}(x, \Omega) < s\}$ and $R := cH\operatorname{diam}(\Omega_s)$, we have*

$$\Omega_R^s := \{x \in \mathbb{G} : \operatorname{dist}(x, \Omega_s) \le R\} \subset \Omega'', \tag{7.11}$$

where $H = H(\mathbb{G}, d_1)$ and $c = c(\mathbb{G}, d_1)$ are the geometric constants that depend on $\mathbb{G}$ and d_1 as in Theorem 3.2.30. Assume that there exist two constants $\tilde{r}, C > 0$ such that for $\mathcal{S}^P$-a.e. $m \in \mathbb{M}$, the level set $f^{-1}(m)$ is $\tilde{r}$-locally C-lower Ahlfors $(Q-P)$-regular with respect to the measure $\mathcal{S}^{Q-P}$. Then there exists a constant $L = L(C, \mathbb{G}, p)$ such that

$$\int_\Omega C_P(Df(x)) d\mathcal{S}^Q(x) \le L \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega) d\mathcal{S}^P(m).$$

Proof. For any $\delta > 0$ and $E \subset \mathbb{G}$, we set

$$\mathcal{T}_\delta(E) = \inf \left\{ \sum_{i=1}^\infty \zeta_T(\mathbb{B}_i) : \mathbb{B}_i \in \mathcal{F}_b,\ E \subset \bigcup_{i=1}^\infty \mathbb{B}_i,\ r(\mathbb{B}_i) \le \delta \right\}$$

with $\zeta_T(\mathbb{B}(x,r)) = r^{Q-P} \mathcal{S}^P(f(\mathbb{B}(x,r))$. We define the measure resulting by Carathéodory's construction as

$$\mathcal{T}(E) := \sup_{\delta > 0} \mathcal{T}_\delta(E).$$

Moreover, we define for any $\delta > 0$ and $E \subset \mathbb{G}$

$$\mathcal{K}_{N,\ell,\delta}(E) = \sup \Big\{ \sum_{i=1}^\infty \zeta_T(\mathbb{B}_i) : \{\mathbb{B}_i\}\ (N,\ell)\text{-packing of } E,\ E \subset \bigcup_{i=1}^\infty \mathbb{B}_i,$$
$$\mathbb{B}_i \text{ centered on } E,\ r(\mathbb{B}_i) \le \delta \Big\}$$

and the resulting measure as

$$\mathcal{K}_{N,\ell}(E) := \inf_{\delta > 0} \mathcal{K}_{N,\ell,\delta}(E)$$

Fix $N = N(3, \mathbb{G})$ the minimum natural number such that there exists fine $(N, 3)$-packings of $\mathbb{G}$ that cover $\mathbb{G}$ itself (see Remark 7.1.2).
Claim 1. There exists a constant $T = T(C, \mathbb{G})$ such that

$$\mathcal{T}(\Omega) \leq \mathcal{K}_{N,3}(\Omega) \leq T \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega) d\mathcal{S}^P(m).$$

The first inequality of the claim follows analogously to the usual comparison between the packing measure and the spherical measure (see Remark 7.1.4).

Let us focus on the second inequality of the claim; we follow the scheme of [Pan20, Proposition 4.2]. Let $\bar{\delta} > 0$ and set $\Omega_{\bar{\delta}} := \{x \in \mathbb{G} : \operatorname{dist}(x, \Omega) < \bar{\delta}\}$. Let $\delta > 0$ be such that $0 < \delta < \min\left\{\frac{\bar{\delta}}{3}, \tilde{r}\right\}$. Let $\{\mathbb{B}_i\}$ be a $(N, 3)$-packing of Ω such that $r(\mathbb{B}_i) \leq \delta$ and $\mathbb{B}_i$ are centered on Ω. We fix $m \in \mathbb{M}$ and we consider every index i such that $m \in f(\mathbb{B}_i)$. We distinguish the two following cases:

- if $m \in f(\mathbb{B}_i \cap \Omega)$, we pick $x_i \in \mathbb{B}_i \cap f^{-1}(m) \cap \Omega$ and we call $\mathbb{B}_{i,m}$ the smallest ball centered at x_i that contains $\mathbb{B}_i$. Observe that $\mathbb{B}_{i,m} \subset 3\mathbb{B}_i$, so that, for each $m \in \mathbb{M}$, the collection $\{\mathbb{B}_{i,m}\}_{i \in \{i : m \in f(\mathbb{B}_i \cap \Omega)\}}$ is a $(N, 1)$-packing of $f^{-1}(m) \cap \Omega$ consisting of balls with radius less or equal than 3δ centered on $f^{-1}(m) \cap \Omega$, that covers $f^{-1}(m) \cap \Omega$.

- if $m \notin f(\mathbb{B}_i \cap \Omega)$, we pick $x_i \in \mathbb{B}_i \cap f^{-1}(m)$ and we call $\mathbb{B}'_{i,m}$ the smallest ball centered at x_i that contains $\mathbb{B}_i$. Observe that $x_i \in f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)$ and that $\mathbb{B}'_{i,m} \subset 3\mathbb{B}_i$, so that, for each $m \in \mathbb{M}$, the collection $\{\mathbb{B}'_{i,m}\}_{i \in \{i : m \in f(\mathbb{B}_i), m \notin f(\mathbb{B}_i \cap \Omega)\}}$ is a $(N, 1)$-packing of $f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)$ consisting of balls with radius less or equal than 3δ centered on $f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)$. Notice that it does not necessarily cover $f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)$.

Then

$$\begin{aligned} \sum_i r(\mathbb{B}_i)^{Q-P} \mathcal{S}^P(f(\mathbb{B}_i)) &= \sum_i \left(\int_{\mathbb{M}} \mathbf{1}_{f(\mathbb{B}_i)}(m) d\mathcal{S}^P(m) \right) (r(\mathbb{B}_i))^{Q-P} \\ &= \int_{\mathbb{M}} \sum_i \mathbf{1}_{f(\mathbb{B}_i)}(m) (r(\mathbb{B}_i))^{Q-P} d\mathcal{S}^P(m). \end{aligned} \tag{7.12}$$

Now we can observe that

$$\begin{aligned} \sum_i \mathbf{1}_{f(\mathbb{B}_i)}(m)(r(\mathbb{B}_i))^{Q-P} &= \sum_{i \in \{i : m \in f(\mathbb{B}_i \cap \Omega)\}} (r(\mathbb{B}_i))^{Q-P} + \sum_{i \in \{i : m \in f(\mathbb{B}_i),\ m \notin f(\mathbb{B}_i \cap \Omega)\}} (r(\mathbb{B}_i))^{Q-P} \\ &\leq \sum_{i \in \{i : m \in f(\mathbb{B}_i \cap \Omega)\}} (r(\mathbb{B}_{i,m}))^{Q-P} + \sum_{i \in \{i : m \in f(\mathbb{B}_i),\ m \notin f(\mathbb{B}_i \cap \Omega)\}} (r(\mathbb{B}'_{i,m}))^{Q-P} \\ &\leq \mathcal{P}^{Q-P}_{N,1,3\delta}(f^{-1}(m) \cap \Omega) + \tilde{\mathcal{P}}^{Q-P}_{N,1,3\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)). \end{aligned} \tag{7.13}$$

Let us now prove that for $\mathcal{S}^P$-a.e. $m \in \mathbb{M}$ the two following conditions hold

$$\mathcal{P}^{Q-P}_{N,1,3\delta}(f^{-1}(m) \cap \Omega) \leq 2\frac{N}{C} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{\Omega_\delta}), \tag{7.14}$$

$$\tilde{\mathcal{P}}^{Q-P}_{N,1,3\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)) \leq 2\frac{N}{C} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{(\Omega_{\bar{\delta}} \setminus \Omega)_\delta}). \tag{7.15}$$

The proofs of (7.14) and (7.15) rely on the hypothesis of Ahlfors regularity of the level sets $f^{-1}(m)$ for $\mathcal{S}^P$-a.e. $m \in \mathbb{M}$. Let us prove (7.14). Surely we have

$$\mathcal{P}^{Q-P}_{N,1,\delta}(f^{-1}(m) \cap \Omega) \leq N \tilde{\mathcal{P}}^{Q-P}_{1,1,\delta}(f^{-1}(m) \cap \Omega).$$

Now, as we said, we want to exploit our hypothesis about the Ahlfors-regularity of the level sets $f^{-1}(m)$ for $\mathcal{S}^P$-a.e $m \in \mathbb{M}$. Remember that we have assumed that $0 < \delta < \tilde{r}$. For any fixed $m \in \mathbb{M}$ for which the hypothesis of lower Ahlfors-regularity holds, we can consider a $(1,1)$-packing $\{\mathbb{P}_i\}$ of balls of radius smaller than δ, centered on $f^{-1}(m) \cap \Omega$ such that

$$\sum_i r(\mathbb{P}_i)^{Q-P} \geq \frac{1}{2} \tilde{\mathcal{P}}^{Q-P}_{1,1,\delta}(f^{-1}(m) \cap \Omega).$$

Then by the Ahlfors-regularity of $f^{-1}(m)$ we have

$$\begin{aligned}
\mathcal{P}^{Q-P}_{N,1,\delta}(f^{-1}(m) \cap \Omega) &\leq N \tilde{\mathcal{P}}^{Q-P}_{1,1,\delta}(f^{-1}(m) \cap \Omega) \\
&\leq 2N \sum_i r(\mathbb{P}_i)^{Q-P} \\
&\leq 2\ \frac{N}{C}\ \mathcal{S}^{Q-P}(f^{-1}(m) \cap \mathbb{P}_i) \\
&= 2\ \frac{N}{C}\ \mathcal{S}^{Q-P}(f^{-1}(m) \cap \mathbb{P}_i \cap \overline{\Omega_\delta}) \\
&\leq 2\ \frac{N}{C}\ \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{\Omega_\delta}),
\end{aligned}$$

where $\Omega_\delta := \{x \in \mathbb{G} : \operatorname{dist}(x, \Omega) < \delta\}$. Let us now focus on the proof of (7.15), that is quite similar. Analogously to what we did before, surely we have

$$\tilde{\mathcal{P}}^{Q-P}_{N,1,\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)) \leq N \tilde{\mathcal{P}}^{Q-P}_{1,1,\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)).$$

We want to exploit again our hypothesis about the Ahlfors regularity of the level sets $f^{-1}(m)$ for $\mathcal{S}^P$-a.e $m \in \mathbb{M}$. For any fixed $m \in \mathbb{M}$ for which the hypothesis of Ahlfors regularity holds we can consider a $(1,1)$-packing $\{\mathbb{D}_i\}$ of balls of radius $r(\mathbb{D}_i) \leq \delta$, centered on $f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)$ such that

$$\sum_i r(\mathbb{D}_i)^{Q-P} \geq \frac{1}{2} \tilde{\mathcal{P}}^{Q-P}_{1,1,\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)).$$

Then by the Ahlfors-regularity of $f^{-1}(m)$ we have

$$\begin{aligned}
\tilde{\mathcal{P}}^{Q-P}_{N,1,\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)) &\leq N \tilde{\mathcal{P}}^{Q-P}_{1,1,\delta}(f^{-1}(m) \cap (\Omega_{\bar{\delta}} \setminus \Omega)) \\
&\leq 2N \sum_i r(\mathbb{D}_i)^{Q-P} \\
&\leq 2\ \frac{N}{C}\ \mathcal{S}^{Q-P}(f^{-1}(m) \cap \mathbb{D}_i) \\
&= 2\ \frac{N}{C}\ \mathcal{S}^{Q-P}(f^{-1}(m) \cap \mathbb{D}_i \cap \overline{(\Omega_{\bar{\delta}} \setminus \Omega)_\delta}) \\
&\leq 2\ \frac{N}{C}\ \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{(\Omega_{\bar{\delta}} \setminus \Omega)_\delta}),
\end{aligned}$$

where $(\Omega_{\bar{\delta}} \setminus \Omega)_\delta = \{x \in \mathbb{G} : \operatorname{dist}(x, \Omega_{\bar{\delta}} \setminus \Omega) < \delta\}$.
Now then we can continue by (7.12), and combining it with (7.14) and (7.15) we obtain

$$\sum_i r(\mathbb{B}_i)^{Q-P} \mathcal{S}^P(f(\mathbb{B}_i)) \leq 2\ \frac{N}{C} \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{(\Omega_{\bar{\delta}} \setminus \Omega)_\delta}) + \mathcal{S}^{Q-P}(f^{-1}(m) \cap \overline{\Omega_\delta}) d\mathcal{S}^P(m).$$

Since this holds for every fixed $(N,3)$-packing $\{\mathbb{B}_i\}$ of Ω centered on Ω that covers Ω, we obtain

$$\mathcal{K}_{N,3,\delta}(\Omega) \leq 2\,\frac{N}{C}\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap \overline{(\Omega_{\bar\delta}\setminus\Omega)_\delta}) + \mathcal{S}^{Q-P}(f^{-1}(m)\cap\overline{\Omega_\delta})d\mathcal{S}^P(m).$$

Now we let δ go to zero. By monotone convergence theorem, since Ω is closed, for every $m\in\mathbb{M}$, $f^{-1}(m)\cap\overline{\Omega_\delta}\searrow f^{-1}(m)\cap\Omega$ and $f^{-1}(m)\cap\overline{(\Omega_{\bar\delta}\setminus\Omega)_\delta}\searrow f^{-1}(m)\cap\overline{(\Omega_{\bar\delta}\setminus\Omega)}$ as $\delta\to 0$. Thus, we get

$$\mathcal{K}_{N,3}(\Omega) \leq 2\,\frac{N}{C}\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap \overline{(\Omega_{\bar\delta}\setminus\Omega)}) + \mathcal{S}^{Q-P}(f^{-1}(m)\cap\Omega)d\mathcal{S}^P(m). \tag{7.16}$$

Finally, we let $\bar\delta$ go to zero and, again since Ω is closed, by monotone convergence theorem we get

$$\begin{aligned}\mathcal{K}_{N,3}(\Omega) &\leq 2\,\frac{N}{C}\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap\partial\Omega) + \mathcal{S}^{Q-P}(f^{-1}(m)\cap\Omega)d\mathcal{S}^P(m)\\ &\leq 2\,\frac{N}{C}\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap\Omega) + \mathcal{S}^{Q-P}(f^{-1}(m)\cap\Omega)d\mathcal{S}^P(m)\\ &\leq 4\,\frac{N}{C}\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap\Omega)d\mathcal{S}^P(m).\end{aligned} \tag{7.17}$$

Then the second inequality of Claim 1 holds for $T = T(C,N) = T(C,\mathbb{G}) = 4\frac{N}{C}$. Hence the proof of Claim 1 is concluded.
From now on, we denote by δ_t^i the intrinsic dilations by $t>0$, on $\mathbb{G}$ for $i=1$ and on $\mathbb{M}$ for $i=2$, respectively.
Claim 2. If κ is the step of $\mathbb{G}$,

$$\mathcal{T}(\Omega) \gtrsim_{\kappa,p} \int_\Omega J(Df(x))d\mathcal{S}^Q(x).$$

The proof of Claim 2 is composed of two main steps.
First, we can observe that $Df(x)$ is a continuous function on Ω', so we can consider the following measure on Ω': for any $A\subset\Omega'$, $\mu(A) := \int_A J(Df(x))d\mathcal{S}^Q(x)$. We want to compare μ with the Carathéodory's measure built through coverings of arbitrary closed balls weighted by the function

$$\zeta_R(\mathbb{B}(x,r)) := J(Df(x))r^Q.$$

We denote this measure by $\mathcal{R} = \sup_{\delta>0}\phi_{\delta,\zeta_R}$, where, as we said, in the definition of ϕ_{δ,ζ_R} we set $\mathcal{F}=\mathcal{F}_b$.
We want to prove that there exists $\bar r>0$ such that for every $0<r\leq\bar r$ and for every $x\in\Omega$, $\mu(\mathbb{B}(x,r))\lesssim\zeta_R(\mathbb{B}(x,r))$. In fact, by [Fed69, 2.10.17 (1)], this implies that $\mu(A)\lesssim\mathcal{R}(A)$ for any $A\subset\Omega$ (and then also for $A=\Omega$).
Since Ω is closed and bounded, it is compact. The function $J(Df(\cdot)) : \overline{\Omega_s}\to\mathbb{R}$, $x\to J(Df(x))$ is a continuous function on a compact set. Let us fix $\varepsilon = \min_{x\in\overline{\Omega_s}} J(Df(x))>0$; it is positive since $Df(x)$ is everywhere surjective by hypothesis. Moreover, the map $J(Df(\cdot))$ is uniformly continuous, then there exists $r'>0$ such that $|J(Df(x))-J(Df(y))|\leq\varepsilon$ for every $|x-y|\leq r'$, $x,y\in\overline{\Omega_s}$.

Let us assume that $\bar{r} < \min\{r', s\}$, $x \in \Omega$ and $0 < r \le \bar{r}$, and compute

$$\begin{aligned}
\frac{\mu(\mathbb{B}(x,r))}{J(Df(x))r^Q} &= \frac{1}{r^Q}\int_{\mathbb{B}(x,r)} \frac{J(Df(y))}{J(Df(x))} d\mathcal{S}^Q(y) \\
&\le \frac{1}{r^Q}\int_{\mathbb{B}(x,r)} \frac{|J(Df(y)) - J(Df(x))|}{J(Df(x))} d\mathcal{S}^Q(y) + \frac{1}{r^Q}\int_{\mathbb{B}(x,r)} \frac{J(Df(x))}{J(Df(x))} d\mathcal{S}^Q(y) \\
&\le \frac{1}{r^Q}\int_{\mathbb{B}(x,r)} \frac{\varepsilon}{\min_{x\in\overline{\Omega_s}} J(Df(x))} \mathcal{S}^Q(y) + \mathcal{S}^Q(\mathbb{B}(0,1)) = 2\mathcal{S}^Q(\mathbb{B}(0,1)) = 2b,
\end{aligned}$$

where $0 < b = \mathcal{S}^Q(\mathbb{B}(0,1)) < \infty$. Hence for any $x \in \Omega$ and $0 < r \le \bar{r}$, we have

$$\frac{\mu(\mathbb{B}(x,r))}{J(Df(x))r^Q} \le 2b$$

and as above $\mu(\Omega) \le 2b\mathcal{R}(\Omega)$, so that $\mu(\Omega) \lesssim \mathcal{R}(\Omega)$.

In the second part of the proof, we want to compare $\mathcal{T}(\Omega)$ with $\mathcal{R}(\Omega)$, and, in particular, we want to prove that

$$\mathcal{R}(\Omega) \lesssim_{\kappa,p} \mathcal{T}(\Omega). \tag{7.18}$$

As we observed above, the measure $\mathcal{T}$ is defined as the resulting measure of Carathéodory's construction defined with respect to coverings of closed balls weighted by the function

$$\zeta_T(B(x,r)) = r^{Q-P}\mathcal{S}^P(f(B(x,r)).$$

Our strategy relies on the comparison between ζ_T and ζ_R. In particular we fix $h > 0$ such that $h < s$ and we want to prove that there exists $\bar{r} > 0$ such that for every $0 < r \le \bar{r}$ and every $x \in \Omega_h := \{y \in \mathbb{G} : \mathrm{dist}(y, \Omega) < h\}$,

$$\zeta_T(B(x,r)) \gtrsim_{k,p} \zeta_R(B(x,r)). \tag{7.19}$$

This would give the desired thesis (7.18).

Let us first define for any $x \in \Omega_h$ and $r > 0$, the two sets

$$A_{x,r} := \delta^2_{\frac{1}{r}}(f(x)^{-1}f(\mathbb{B}(x,r)) \quad \text{and} \quad A_x := Df(x)(\mathbb{B}(0,1)).$$

The proof will be composed of various steps, and it will be useful to individuate the two following conditions:

$$\lim_{r\to 0}\sup_{x\in\Omega_h} \left|\mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m)\right| = 0 \quad \text{for any } m \in \mathbb{M}; \tag{7.20}$$

$$\lim_{r\to 0}\sup_{x\in\Omega_h} \left|\mathcal{S}^P(A_{x,r}) - \mathcal{S}^P(A_x)\right|. \tag{7.21}$$

Our strategy consists of proving that (7.20) $\Rightarrow$ (7.21) $\Rightarrow$(7.19), and then we will conclude the proof by proving (7.20). Let us start by proving that (7.21) $\Rightarrow$ (7.19).

Let $x \in \Omega_h$ and consider the homogeneous subspace $V(x) := (\ker(Df(x))^{\perp}$. For every r small enough

$$\begin{aligned}
\zeta_T(B(x,r)) = r^{Q-P}\mathcal{S}^P(\mathbb{B}(x,r)) = r^Q\frac{\mathcal{S}^P(f(\mathbb{B}(x,r))}{r^P} &= r^Q\frac{\mathcal{S}^P(f(x)^{-1}f(\mathbb{B}(x,r))}{r^P} \\
&= r^Q\mathcal{S}^P(\delta^2_{1/r}(f(x)^{-1}f(\mathbb{B}(x,r))),
\end{aligned}$$

hence

$$\frac{\zeta_T(\mathbb{B}(x,r))}{\zeta_R(\mathbb{B}(x,r))} = \frac{\mathcal{S}^P(\delta^2_{1/r}(f(x)^{-1}f(\mathbb{B}(x,r))}{J(Df(x))}.$$

Observe that the map $Df(x)|_{V(x)} : V(x) \to \mathbb{M}$ is injective and surjective and that $J(Df(x)) = J(Df(x)|_{V(x)})$, thanks to our choice of $V(x)$. If we denote by $\pi_{V(x)}$ the orthogonal projection on $V(x)$, for some geometric positive constant β_P (the same used in the proof of Proposition 7.2.2) we have

$$\begin{aligned}\mathcal{S}^P(Df(x)(\mathbb{B}(0,1))) &= \beta_P\ \mathcal{L}^p(Df(x)(\mathbb{B}(0,1)) \\ &= \beta_p\ \mathcal{L}^p(Df(x)(\pi_{V(x)}(\mathbb{B}(0,1))) \\ &= \beta_P\ J(Df(x))\ \mathcal{L}^p(\pi_{V(x)}(\mathbb{B}(0,1))\end{aligned} \tag{7.22}$$

by the classical Euclidean area formula. In fact, since $\ker(Df(x))$ is a homogeneous normal subgroup, Lemma 3.1.20 ensures that one can see any element $y \in \mathbb{G}$ as $y = (\pi_{V(x)}(y))m_y$, for some $m_y \in \ker(Df(x))$. Observe that for every $x \in \Omega_h$, $V(x) = (\ker(Df(x))^\perp$ is a linear (homogeneous) subspace of constant topological dimension $p > 0$, since $Df(x)$ is surjective at any point $x \in \Omega_h$, therefore the factor $\mathcal{L}^p(V(x) \cap \mathbb{B}_E(0,1))$ does not depend on x. Remember now that by Proposition 2.4.9 applied to $F = \mathbb{B}(0,1)$ there exists a constant $C_{\mathbb{B}(0,1)}$ such that

$$\frac{1}{C_{\mathbb{B}(0,1)}}|x| \le \|x\|_1 \le C_{\mathbb{B}(0,1)}|x|^{\frac{1}{\kappa}},$$

where κ is the step of $\mathbb{G}$. Hence

$$\begin{aligned}\mathcal{L}^p(\pi_{V(x)}(\mathbb{B}(0,1))) &\ge \mathcal{L}^p(V(x) \cap \mathbb{B}(0,1)) \\ &\ge \mathcal{L}^p\left(V(x) \cap B_E\left(0, \frac{1}{(C_{\mathbb{B}(0,1)})^\kappa}\right)\right) \\ &= \frac{1}{(C_{\mathbb{B}(0,1)})^{\kappa p}}\mathcal{L}^p(V(x) \cap \mathbb{B}_E(0,1)) := D(\kappa,p) > 0.\end{aligned} \tag{7.23}$$

Hence for every $x \in \Omega_h$, we have

$$\frac{\mathcal{S}^P(A_x)}{J(Df(x))} \ge GD(\kappa,p) := D'(\kappa,p) = D' > 0.$$

If we now assume (7.21) to be true, and we fix $\varepsilon = \frac{D' \min_{x \in \overline{\Omega_h}} J(Df(x))}{2} > 0$, there exists $0 < \bar{r} \le s - h$ such that for every $0 < r \le \bar{r}$ and every $x \in \Omega_h$,

$$\left|\mathcal{S}^P(A_{x,r}) - \mathcal{S}^P(A_x)\right| \le \sup_{x \in \Omega_h} \left|\mathcal{S}^P(A_{x,r}) - \mathcal{S}^P(A_x)\right| \le \varepsilon.$$

Thus, for every $0 < r \le \bar{r}$ and every $x \in \Omega_h$,

$$\mathcal{S}^P(A_{x,r}) \ge \mathcal{S}^P(A_x) - \varepsilon,$$

so that for every $x \in \Omega_h$ and $0 < r \leq \bar{r}$

$$\begin{aligned}\frac{\zeta_T(\mathbb{B}(x,r))}{\zeta_R(\mathbb{B}(x,r))} &= \frac{\mathcal{S}^P(A_{x,r})}{J(Df(x))} \geq \frac{\mathcal{S}^P(A_x)}{J(Df(x))} - \frac{\varepsilon}{J(Df(x))} \\ &\geq D' - \frac{\varepsilon}{J(Df(x))} \geq D' - \frac{\varepsilon}{\min_{x\in\overline{\Omega_h}} J(Df(x))} = \frac{D'}{2} > 0\end{aligned} \tag{7.24}$$

by the choice of ε. This concludes the proof of the fact that (7.21) $\Rightarrow$(7.19).

Second, we prove that (7.20) $\Rightarrow$ (7.21). Surely, the following holds

$$\begin{aligned}&\lim_{r\to 0} \sup_{x\in\Omega_h} \left|\mathcal{S}^P(A_{x,r}) - \mathcal{S}^P(A_x)\right| \\ &\leq \lim_{r\to 0} \sup_{x\in\Omega_h} \left|\int_{\mathbb{M}} \mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m) d\mathcal{S}^P(m)\right| \\ &\leq \lim_{r\to 0} \int_{\mathbb{M}} \sup_{x\in\Omega_h} \left|\mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m)\right| d\mathcal{S}^P(m).\end{aligned} \tag{7.25}$$

We want now to apply the Lebesgue dominated convergence theorem exploiting (7.20). In order to do this, we prove that for $r \leq s-h$, for any $m \in \mathbb{M}$

$$\sup_{x\in\Omega_h} \left|\mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m)\right| \leq 2\mathbf{1}_{\mathbb{B}(0,W)}(m) \tag{7.26}$$

for some constant $W > 0$. Notice that $2\mathbf{1}_{\mathbb{B}(0,W)} \in L^1_{\mathcal{S}^P}(\mathbb{M})$.
First, consider that $\sup_{x\in\Omega_h} \left|\mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m)\right| \leq \sup_{x\in\Omega_h} \left|\mathbf{1}_{A_{x,r}}(m)\right| + \sup_{x\in\Omega_h} |\mathbf{1}_{A_x}(m)|$.
For any $x \in \Omega_h$ and $m \in \mathbb{M}$, if we assume $\mathbf{1}_{A_x}(m) = 1$, it implies that $m = Df(x)(\eta)$ for some $\eta \in B(0,1)$, then

$$\|m\|_2 = \|Df(x)(\eta)\|_2 \leq \|Df(x)\|_{\mathcal{L}_h(\mathbb{G},\mathbb{M})} \leq \max_{x\in\overline{\Omega_h}} \|Df(x)\|_{\mathcal{L}_h(\mathbb{G},\mathbb{M})} =: \|Df\|_{\overline{\Omega_h}}.$$

For any $x \in \Omega_h$, $r \leq s-h$ and $m \in \mathbb{M}$, if $\mathbf{1}_{A_{x,r}}(m) = 1$, then $m = \delta^2_{1/r}(f(x)^{-1}f(q_r))$ for some $q_r \in B(x,r) \subset \Omega_s$. Hence

$$\begin{aligned}\|m\|_2 &= \|\delta^2_{1/r}(f(x)^{-1}f(q_r))\|_2 = \|Df(x)(\delta^1_{1/r}(x^{-1}q_r))\delta^2_{1/r}(Df(x)(x^{-1}q_r))^{-1}f(x)^{-1}f(q_r))\|_2 \\ &\leq \|Df(x)(\delta^1_{1/r}(x^{-1}q_r))\|_2 + \|\delta^2_{1/r}(Df(x)(x^{-1}q_r))^{-1}f(x)^{-1}f(q_r)\|_2 \\ &\leq \|Df(x)\|_{\mathcal{L}_h(\mathbb{G},\mathbb{M})} + K(\omega_{\overline{\Omega''},DF_1}(Hc(s-h)))^{\frac{1}{\kappa^2}} \\ &\leq \|Df\|_{\overline{\Omega_h}} + K(\omega_{\overline{\Omega''},DF_1}(Hc(s-h)))^{\frac{1}{\kappa^2}},\end{aligned}$$

where $\omega_{\overline{\Omega''},DF_1}$ is the modulus of continuity of $x \to DF_1(x)$ defined in Definition 3.2.27 and K is a constant that plays the role of C of Theorem 3.2.30.
Hence

$$\begin{aligned}\sup_{x\in\Omega_h} \left|\mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m)\right| &\leq \sup_{x\in\Omega_h} \mathbf{1}_{A_{x,r}}(m) + \sup_{x\in\Omega_h} \mathbf{1}_{A_x}(m) \\ &\leq \mathbf{1}_{\mathbb{B}(0,\|Df\|_{\overline{\Omega_h}} + K(\omega_{\overline{\Omega''},DF_1}(Hc(s-h)))^{\frac{1}{\kappa^2}})}(m) + \mathbf{1}_{B(0,\|Df\|_{\overline{\Omega_h}})}(m) \\ &\leq 2\mathbf{1}_{\mathbb{B}(0,\|Df\|_{\overline{\Omega_h}} + K(\omega_{\overline{\Omega''},DF_1}(Hc(s-h)))^{\frac{1}{\kappa^2}})}(m)\end{aligned}$$

and this implies that (7.26) is true, with $W = \|Df\|_{\overline{\Omega_h}} + (\omega_{\overline{\Omega''},DF_1}(Hc(s-h)))^{\frac{1}{\kappa^2}})$.

We can then apply the Lebesgue dominated convergence Theorem to (7.25), and since we have assumed (7.20) to be true, we obtain (7.21).

We are left to prove (7.20).
By contradiction, we assume (7.20) to be false. Then, there exists at least one element $m \in \mathbb{M}$ such that the limit

$$\lim_{r \to 0} \sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m) \right|$$

does not exist or

$$\lim_{r \to 0} \sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{x,r}}(m) - \mathbf{1}_{A_x}(m) \right| > 0.$$

In both cases, since all the considered elements are positive, there exists at least a positive infinitesimal sequence $(r_n)_n$ such that

$$\lim_{n \to \infty} \sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{x,r_n}}(m) - \mathbf{1}_{A_x}(m) \right| > 0.$$

This implies that there exists $\tilde{n} > 0$ such that for every $n \geq \tilde{n}$,

$$\sup_{x \in \Omega_h} \left| \mathbf{1}_{A_{x,r_n}}(m) - \mathbf{1}_{A_x}(m) \right| > 0 \quad \text{and} \quad r_n \leq s - h.$$

Hence for every $n \geq \tilde{n}$ there exists at least an element $x_n \in \Omega_h \subset \overline{\Omega_h}$ such that

$$\left| \mathbf{1}_{A_{x_n,r_n}}(m) - \mathbf{1}_{A_{x_n}}(m) \right| > 0$$

and then

$$\left| \mathbf{1}_{A_{x_n,r_n}}(m) - \mathbf{1}_{A_{x_n}}(m) \right| = 1. \tag{7.27}$$

Since $\overline{\Omega_h}$ is a compact set, the sequence $(x_n)_n$ converges up to a subsequence to some $\bar{x} \in \overline{\Omega_h}$. Let us first prove that there exists some $\bar{n}$ such that for every $n \geq \bar{n}$

$$\left| \mathbf{1}_{A_{x_n,r_n}}(m) - \mathbf{1}_{A_{\bar{x}}}(m) \right| = 1. \tag{7.28}$$

Let us then assume by contradiction that there exists a subsequence $(x_{n_k})_k$ such that

$$\lim_{k \to \infty} \left| \mathbf{1}_{A_{x_{n_k},r_{n_k}}}(m) - \mathbf{1}_{A_{\bar{x}}}(m) \right| = 0 \tag{7.29}$$

then, on this subsequence, we have

$$\left| \mathbf{1}_{A_{x_{n_k},r_{n_k}}}(m) - \mathbf{1}_{A_{x_{n_k}}}(m) \right| \leq \left| \mathbf{1}_{A_{x_{n_k},r_{n_k}}}(m) - \mathbf{1}_{A_{\bar{x}}}(m) \right| + \left| \mathbf{1}_{A_{x_{n_k}}}(m) - \mathbf{1}_{A_{\bar{x}}}(m) \right|. \tag{7.30}$$

Let us prove that (7.30) goes to zero as $k \to \infty$ and this would give a contradiction with (7.27). The fact that (7.30) goes to zero follows from the assumption (7.29) and by the fact that

$$\left| \mathbf{1}_{A_{x_{n_k}}}(m) - \mathbf{1}_{A_{\bar{x}}}(m) \right| \to 0 \text{ as } k \to \infty. \tag{7.31}$$

Let us prove (7.31). Assume by contradiction that on a subsequence of $(x_{n_k})_k$, $(x_{n_{k_\ell}})_\ell$, for ℓ sufficiently large,

$$\left| \mathbf{1}_{A_{x_{n_{k_\ell}}}}(m) - \mathbf{1}_{A_{\bar{x}}}(m) \right| = 1. \tag{7.32}$$

Let us assume $m \notin A_{\bar{x}}$, then $m \in A_{x_{n_{k_\ell}}}$ so that $m = Df(x_{n_{k_\ell}})(\eta_{n_{k_\ell}})$ for $\eta_{n_{k_\ell}} \in \mathbb{B}(0,1)$,

hence we can extract a converging subsequence such that $(\eta_{n_{k_{\ell_t}}})_t \to \bar{\eta} \in \mathbb{B}(0,1)$ and letting t go to infinity, by the continuity of the differential, we know that $m = Df(\bar{x})(\bar{\eta}) \in A_{\bar{x}}$, but this is not possible.
So it must be true that $m \in A_{\bar{x}}$. Hence, by the definition of $A_{\bar{x}}$ and the continuity of the Pansu differential, it is true that for some $\eta \in \mathbb{B}(0,1)$, $m = Df(\bar{x})(\eta) = \lim_{\ell\to\infty} Df(x_{n_{k_\ell}})(\eta) = \lim_{\ell\to\infty} X_\ell$ where for every ℓ, $X_\ell := Df(x_{n_{k_\ell}})(\eta) \in A_{x_{n_{k_\ell}}}$. Hence $m \in \limsup_{\ell\to\infty} A_{x_{n_{k_\ell}}}$, and then $\mathbf{1}_{\limsup_{\ell\to\infty} A_{x_{n_{k_\ell}}}}(m) = \limsup_{\ell\to\infty} \mathbf{1}_{A_{x_{n_{k_\ell}}}}(m) = 1$. At the same time, by (7.32), $m \notin A_{x_{n_{k_\ell}}}$ for ℓ sufficiently large and this implies that $\lim_{\ell\to\infty} \mathbf{1}_{A_{x_{n_{k_\ell}}}}(m) = 0 = \limsup_{\ell\to\infty} \mathbf{1}_{A_{x_{n_{k_\ell}}}}(m)$, which gives a contradiction.
Let us then continue from (7.28). We need to prove that (7.27) is not possible. Since m is fixed, there are only two possibilities:

$$m \in A_{\bar{x}}; \tag{7.33}$$
$$m \notin A_{\bar{x}}. \tag{7.34}$$

We show that neither (7.33) nor (7.34) can be true. Assume that (7.33) is true, then $m = Df(\bar{x})(\eta)$ for some $\eta \in B(0,1)$. Then by (7.28) $m \notin A_{x_n,r_n}$ for $n \geq \bar{n}$ and so clearly there exists the following limit

$$\lim_{n\to\infty} \mathbf{1}_{A_{x_n,r_n}}(m) = 0. \tag{7.35}$$

Let us define for any n, $q_n := x_n\delta^1_{r_n}(\eta) \in B(x_n,r_n) \subset \Omega_s$. Consider for any $n \geq \bar{n}$

$$\delta^2_{1/r_n}(f(x_n)^{-1}f(q_n))$$
$$= Df(\bar{x})(\eta)\delta^2_{1/r_n}(Df(\bar{x})(x_n^{-1}q_n)^{-1}Df(x_n)(x_n^{-1}q_n))\delta^2_{1/r_n}(Df(x_n)(x_n^{-1}q_n)^{-1}f(x_n)^{-1}f(q_n))$$

and observe that by Theorem 3.2.30, since $x_n, q_n \in \Omega_s$ and (7.11) holds, then

$$\|\delta^2_{1/r_n}(Df(x_n)(x_n^{-1}q_n)^{-1}f(x_n)^{-1}f(q_n))\|_2 \leq K(\omega_{\overline{\Omega''},DF_1}(cHr_n))^{\frac{1}{\kappa^2}} \to 0$$

as $n \to \infty$ and

$$\|(Df(\bar{x})(\eta))^{-1}Df(x_n)(\eta)\|_2 \leq d_{\mathcal{L}_h(\mathbb{G},\mathbb{M})}(Df(x_n),Df(\bar{x})) \to 0$$

as $n \to \infty$ by the continuity of $Df(x)$. Hence $\lim_{n\to\infty} \delta^2_{1/r_n}(f(x_n)^{-1}f(q_n)) = m$. This permits to conclude that $m \in \limsup_{n\to\infty} A_{x_n,r_n}$ and so that

$$\limsup_{n\to\infty} \mathbf{1}_{A_{x_n,r_n}}(m) = \mathbf{1}_{\limsup_{n\to\infty} A_{x_n,r_n}}(m) = 1.$$

At the same time, (7.35) implies that there exists the limit

$$\limsup_{n\to\infty} \mathbf{1}_{A_{x_n,r_n}}(m) = \lim_{n\to\infty} \mathbf{1}_{A_{x_n,r_n}}(m) = 0,$$

so we reach a contradiction.
Assume now (7.34), then $m \in A_{x_n,r_n}$ for every $n \geq \bar{n}$ and then by (7.28), $m \notin A_{\bar{x}}$. For every $n \geq \bar{n}$ there exists $q_n \in \mathbb{B}(x_n,r_n) \subset \Omega_s$ such that

$$m = \delta^2_{1/r_n}(f(x_n)^{-1}f(q_n)) = Df(\bar{x})(\delta^1_{1/r_n}(x_n^{-1}q_n))Df(\bar{x})(\delta^1_{1/r_n}(x_n^{-1}q_n))^{-1}$$

$$Df(x_n)(\delta^1_{1/r_n}(x_n^{-1}q_n))\delta^2_{1/r_n}(Df(x_n)(x_n^{-1}q_n)^{-1}f(x_n)^{-1}f(q_n))$$

and again by Theorem 3.2.30 and by continuity of Df we obtain that

$$m = \lim_{n\to\infty} \delta^2_{1/r_n}(f(x_n)^{-1}f(q_n)) = \lim_{n\to\infty} Df(\bar{x})(\delta^1_{1/r_n}(x_n^{-1}q_n))$$

that up to a subsequence is equal to $Df(\bar{x})(\eta)$ for some $\eta \in B(0,1)$. Thus $m \in A_{\bar{x}}$, which contradicts (7.27). Hence, finally (7.20) is proved and this concludes the proof of Claim 2.
Claim 3. For every $x \in \Omega'$,

$$C_P(Df(x)) \lesssim_{q,p,\kappa} J(Df(x)).$$

Since we have assumed that $Df(x)$ is surjective at every x, by (7.3), we have

$$C_P(Df(x)) = Z\frac{\mathcal{S}^{Q-P}(\ker(Df(x)) \cap \mathbb{B}(0,1))}{\mathcal{H}_E^{q-p}(\ker(Df(x)) \cap \mathbb{B}(0,1))}J(Df(x)).$$

By Proposition 7.3.1, for any $x \in \Omega'$ and for any Borel set $B \subset \ker(Df(x))$

$$\mathcal{S}^{Q-P}(B) = \frac{1}{\sup_{w\in\mathbb{B}(0,1)} \mathcal{H}_E^{q-p}(\ker(Df(x)) \cap \mathbb{B}(w,1))}\mathcal{H}_E^{q-p}(B).$$

Hence, by taking into account Proposition 2.4.9, we have

$$\begin{aligned}
&\frac{\mathcal{S}^{Q-P}(\ker(Df(x)) \cap \mathbb{B}(0,1))}{\mathcal{H}_E^{q-p}(\ker(Df(x)) \cap \mathbb{B}(0,1))}\\
&= \frac{1}{\sup_{w\in\mathbb{B}(0,1)} \mathcal{H}_E^{q-p}(\mathbb{B}(w,1) \cap \ker(Df(x))}\\
&\le \frac{1}{\mathcal{H}_E^{q-p}(\mathbb{B}(0,1) \cap \ker(Df(x))}\\
&\le \frac{1}{\mathcal{H}_E^{q-p}(\mathbb{B}_E(0, \frac{1}{(C_{\mathbb{B}(0,1)})^\kappa})) \cap \ker(Df(x))}\\
&= \frac{1}{\mathcal{L}^{q-p}(\mathbb{B}_E(0, \frac{1}{(C_{\mathbb{B}(0,1)})^\kappa}) \cap \ker(Df(x))} =: D''(q,p,\kappa) > 0.
\end{aligned}$$

In the last passage we considered that $\ker(Df(x))$ is a linear subspace of constant topological dimension $q-p$.

By combining all the claims the proof is achieved. □

It is easy to extend Theorem 7.0.1 to the case in which Ω is a measurable set but not necessarily compact.

Theorem 7.3.4. *Let $(\mathbb{G}, d_1)$, $(\mathbb{M}, d_2)$ be two Carnot groups, endowed with homogeneous distances, of Hausdorff dimension Q, P and topological dimension q, p, respectively. Let $f \in C_h^1(\mathbb{G}, \mathbb{M})$ be a function and assume $Df(x)$ to be surjective at any point $x \in \mathbb{G}$. Assume that there exist two constants $\tilde{r}, C > 0$ such that for $\mathcal{S}^P$-a.e. $m \in \mathbb{M}$ the level set $f^{-1}(m)$ is $\tilde{r}$-locally C-lower Ahlfors $(Q-P)$-regular with respect to the measure $\mathcal{S}^{Q-P}$. Then there exists a constant $L = L(C, \mathbb{G}, p)$ such that, if $A \subset \mathbb{G}$ is a measurable set,*

$$\int_A C_P(Df(x))d\mathcal{S}^Q(x) \le L\int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap A)\, d\mathcal{S}^P(m).$$

Proof. Let us consider an increasing sequence of compact sets in $(\Omega_n)_{n\in\mathbb{N}} \subset A$ such that $\Omega_n \nearrow A$. Hence by Theorem 7.0.1, there exists $L = L(C, \mathbb{G}, p)$ such that for every $n \in \mathbb{N}$

$$\begin{aligned}\int_{\Omega_n} C_P(Df(x))d\mathcal{S}^Q(x) &\le L \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap\Omega_n)\ d\mathcal{S}^P(m)\\ &\le L \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m)\cap A)\ d\mathcal{S}^P(m),\end{aligned}$$

so if we let n go to ∞, by monotone convergence theorem we get the thesis. □

7.4 Applications

Our main theorem allows to prove the following corollaries. We focus on a measurable set $A \subset \mathbb{G}$. Essentially, we slice A by the level sets of a map f satisfying the hypotheses of Theorem 7.3.4, so that we find out some properties related to A. In particular, we remark that, doing so, we slice A by $(\mathbb{G},\mathbb{M})_K$-regular submanifolds of $\mathbb{G}$.

Corollary 7.4.1. *In the hypotheses of Theorem 7.3.4, let $u : A \to \mathbb{R}$ be a non-negative measurable function, then there exists a constant $L = L(C, \mathbb{G}, p)$ such that*

$$\int_A u(x)C_P(Df(x))d\mathcal{S}^Q(x) \le L\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A} u(x)d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m).$$

Proof. By [EG92, Theorem 7], can write $u = \sum_{k=1}^{\infty}\frac{1}{k}\mathbf{1}_{A_k}$ with A_k measurable sets. By monotone convergence theorem we have

$$\begin{aligned}\int_A u(x)C_P(Df(x))d\mathcal{S}^Q(x) &= \sum_{k=1}^{\infty}\frac{1}{k}\int_{A\cap A_k} C_P(Df(x))d\mathcal{S}^Q(x)\\ &\le \sum_{k=1}^{\infty}\frac{1}{k}L\int_{\mathbb{M}}\mathcal{S}^{Q-P}(f^{-1}(m)\cap A\cap A_k)d\mathcal{S}^P(m)\\ &\le \sum_{k=1}^{\infty}\frac{1}{k}L\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A}\mathbf{1}_{A_k}(x)d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m) \qquad (7.36)\\ &= L\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A}\sum_{k=1}^{\infty}\frac{1}{k}\mathbf{1}_{A_k}(x)d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m)\\ &= L\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A}u(x)d\mathcal{S}^{Q-P}(x)\ d\mathcal{S}^P(m).\end{aligned}$$

□

Corollary 7.4.2. *In the hypotheses of Theorem 7.3.4, let $u : A \to \mathbb{R}$ be a measurable function. If we assume that*

(i) u is $\mathcal{S}^{Q-P}$-summable on $f^{-1}(m)\cap A$ for $\mathcal{S}^P$-a.e. $m \in \mathbb{M}$,

(ii) $\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A}|u(x)|d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m) < \infty$,

then u is summable on A.

Proof. We can write $u = u^+ - u^-$. By proceeding analogously to (7.36), considering Theorem 7.2.5 instead of Theorem 7.0.1, we then obtain that

$$-\int_A u^-(x)C_P(Df(x))d\mathcal{S}^Q(x) \le -\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A} u^-(x)d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m). \tag{7.37}$$

Hence by (7.36) applied to u^+, (7.37), and by our hypotheses, we get the following

$$\begin{aligned}
&\int_A u(x)C_P(Df(x))d\mathcal{S}^Q(x) = \int_A u^+(x)C_P(Df(x))d\mathcal{S}^Q(x) - \int_A u^-(x)C_P(Df(x))d\mathcal{S}^Q(x)\\
&\le L\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A} u^+(x)d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m) - \int_{\mathbb{M}}\int_{f^{-1}(m)\cap A} u^-(x)d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m)\\
&\le L\int_{\mathbb{M}}\int_{f^{-1}(m)\cap A} |u(x)|d\mathcal{S}^{Q-P}(x)d\mathcal{S}^P(m) < \infty.
\end{aligned}$$

Now, it is enough to prove that $C_P(Df(x)) > 0$, for every $x \in A$. This follows from the facts that $J(Df(x)) > 0$ for every $x \in A$ and that, taking into consideration Proposition 2.4.9, for every x we have the following

$$\begin{aligned}
&\frac{\mathcal{S}^{Q-P}(\ker(Df(x)) \cap \mathbb{B}(0,1))}{\mathcal{H}_E^{q-p}(\ker(Df(x)) \cap \mathbb{B}(0,1))}\\
&= \frac{1}{\sup_{w\in\mathbb{B}(0,1)} \mathcal{H}_E^{q-p}(\mathbb{B}(w,1) \cap \ker(Df(x)))}\\
&\ge \frac{1}{\mathcal{H}_E^{q-p}(\mathbb{B}(0,2) \cap \ker(Df(x)))}\\
&= \frac{1}{\mathcal{L}^{q-p}(\mathbb{B}(0,2) \cap \ker(Df(x)))}\\
&\ge \frac{1}{\mathcal{L}^{q-p}(\mathbb{B}_E(0,2C_{\mathbb{B}(0,2)}) \cap \ker(Df(x)))}\\
&= \frac{1}{(2C_{\mathbb{B}(0,2)})^{q-p}} > 0.
\end{aligned} \tag{7.38}$$

□

Corollary 7.4.3. *In the hypotheses of Theorem 7.3.4, if $\mathbf{1}_A(x) = 0$ for $\mathcal{S}^{Q-P}$-a.e. $x \in f^{-1}(m)$, for $\mathcal{S}^P$-a.e $m \in \mathbb{M}$, then $\mathbf{1}_A(x) = 0$ for $\mathcal{S}^Q$-a.e $x \in \mathbb{G}$.*

Proof. It follows from Theorem 7.3.4 and (7.38). □

Remark 7.4.4. The hypothesis of Theorem 7.0.1 about the uniform Ahlfors regularity of the level sets of the map f is not pointless: we stress again that, if we consider f continuously Pansu differentiable with Pansu differential everywhere surjective on $\mathbb{G}$, the lower Ahlfors regularity of the level sets is not always guaranteed, even locally. One can refer to [Koz15, Corollary 6.2.4], where explicit examples of this phenomenon are presented. Nevertheless, it is still not known if pathological level sets are exceptional. Hopefully those types of irregularities do not reflect the generic behaviour of level sets. In addition, in [Koz15], a class of mappings of higher regularity from the Heisenberg group $\mathbb{H}^n$ to the Euclidean space $\mathbb{R}^{2n}$ is studied. More precisely, the author considers functions $f \in C_h^{1,\alpha}(\mathbb{H}^n, \mathbb{R}^{2n})$ with $\alpha > 0$, i.e. given a homogeneous distance d on $\mathbb{H}^n$, continuously

Pansu differentiable maps such that, for every $a, b \in \mathbb{H}^n$

$$d_{\mathcal{L}_h(\mathbb{H}^n,\mathbb{R}^{2n})}(Df(a), Df(b)) \lesssim d(a,b)^\alpha.$$

By [Koz15, Corollary 5.5.6], if we assume that the Pansu differential of f is everywhere surjective, the level sets of f are uniformly locally Ahlfors 2-regular with respect to $\mathcal{S}^2$. Therefore, the validity of the inequality given by Theorem 7.0.1 for this class of regular functions is ensured by our result. This somehow confirms the coarea-type equality proved in [Koz15, Theorem 6.2.5]. To summarize, in this setting, we have weakened the hypotheses adopted in [Koz15, Theorem 6.2.5] about the required regularity of the considered map, passing from $C_h^{1,\alpha}$-regular maps, with $\alpha > 0$, to continuously Pansu differentiable functions. We compensate the lower regularity of the map with the more geometrical hypothesis about the Ahlfors regularity of its level sets (to be precise Kozhevnikov substantially could ignore the characteristic sets of the points where the Pansu differential is not surjective thanks to the results of [Mag02b]). We need to remark that these considerations are limited, up to now, to maps from the Heisenberg group $\mathbb{H}^n$ to $\mathbb{R}^{2n}$, about which more results are available. We recall that for maps $f \in C_h^{1,\alpha}(\Omega, \mathbb{R}^{2n})$, for some open set $\Omega \subset \mathbb{H}^n$, for any $\alpha > 0$, a coarea formula is proved also in [MST18, Theorem 8.2].

Now we fix again two Carnot groups $(\mathbb{G}, d_1)$ and $(\mathbb{M}, d_2)$, endowed with two homogeneous distances, of Hausdorff dimension Q, P and topological dimension q, p, respectively. From now on, for any set $\Omega \subset \mathbb{G}$, and any real number $D > 0$ we set

$$\Omega_D := \{y \in \mathbb{G} : \mathrm{dist}(y, \Omega) < D\}.$$

We will consider a map $f \in C_h^1(\mathbb{G}, \mathbb{M})$ with Pansu differential everywhere surjective and a compact set $\Omega \subset \mathbb{G}$. We want to see how to apply Theorem 7.0.1 to the particular geometrical case in which there exists a p-dimensional homogeneous subgroup $\mathbb{V}$ complementary to $\ker(Df(x))$ for any point x of a neighbourhood of Ω, i.e. to the case when the level sets $f^{-1}(m)$ are not only $(\mathbb{G}, \mathbb{M})_K$-regular submanifolds but $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$. The key point is the possibility to apply in this situation the implicit function theorem stated in Theorem 4.2.15. Therefore any level set $f^{-1}(m)$, for $m \in \mathbb{M}$, can be seen as an intrinsic Lipschitz graph, and this is a crucial observation, since, by Proposition 3.5.13, intrinsic Lipschitz graphs are lower Ahlfors regular. In particular, then, since we want to exploit Theorem 7.3.3, we need to focus on the intrinsic Lipschitz constants of the intrinsic parametrizing maps of the level sets of f. Our strategy will be to individuate two universal positive constants R and L, independent of the choice of $m \in \mathbb{M}$ and of $x \in f^{-1}(m)$, such that for every $m \in \mathbb{M}$ and $x \in f^{-1}(m)$, $f^{-1}(m) \cap \mathbb{B}(x, R)$ is an intrinsic L-Lipschitz graph. Successively, we will deduce through Proposition 3.5.13 the existence of a positive constant C such that for every $m \in \mathbb{M}$, $f^{-1}(m)$ is R-locally C-lower Ahlfors $(Q-P)$-regular with respect to $\mathcal{S}^{Q-P}$.

First of all, by modifying the proof of [JNGV20, Lemma 2.9], combining it with an easy compactness argument and with Theorem 3.2.30, the following proposition immediately follows.

Proposition 7.4.5. *Let us consider a map $f \in C_h^1(\mathbb{G}, \mathbb{M})$ and a compact set $\Omega \subset \mathbb{G}$. Assume that there exists a p-dimensional homogeneous subgroup $\mathbb{V}$ such that $Df(x)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is a h-isomorphism for every $x \in \Omega$. Then there exists a constant $R > 0$ such that for every $x \in \Omega$, for every $y \in \mathbb{B}(x, R)$, and for every $v \in \mathbb{V}$ such that $yv \in \mathbb{B}(x, R)$,*

$$d_2(f(y), f(yv)) \geq R\|v\|_1.$$

Proof. Notice that our hypotheses imply that $\mathbb{V}$ is complementary to $\ker(Df(x))$ for every $x \in \Omega$. Let us proceed by contradiction. Assume that for every $n \in \mathbb{N}$ there exist $x_n \in \Omega$, $y_n \in \mathbb{B}(x_n, \frac{1}{n})$, $v_n \in \mathbb{V} \setminus \{0\}$ such that $y_n v_n \in \mathbb{B}(x_n, \frac{1}{n})$ and

$$\|f(y_n)^{-1} f(y_n v_n)\|_2 < \frac{1}{n}\|v_n\|_1.$$

Since Ω is compact, up to a subsequence, there exists some $\bar{x} \in \Omega$ such that $x_n \to \bar{x}$. Again up to a subsequence, we can assume that there exists $\bar{v} \in \mathbb{V}$, with $\|\bar{v}\|_1 = 1$ such that $\frac{v_n}{\|v_n\|_1} \to \bar{v}$ as $n \to \infty$. Now we can consider for every n

$$\frac{\|Df(\bar{x})(v_n)Df(\bar{x})(v_n)^{-1}Df(y_n)(v_n)Df(y_n)(v_n)^{-1}f(y_n)^{-1}f(y_n v_n)\|_2}{\|v_n\|_1} < \frac{1}{n}.$$

Consider that by Pansu differentiability,

$$\frac{Df(y_n)(v_n)^{-1}f(y_n)^{-1}f(y_n v_n)}{\|v_n\|_1} \to 0 \in \mathbb{M},$$

as n goes to ∞ and by the continuity of the Pansu differential on Ω

$$\begin{aligned}\frac{\|Df(\bar{x})(v_n)^{-1}Df(y_n)(v_n)\|_2}{\|v_n\|_1} &= Df(\bar{x})\left(\frac{v_n}{\|v_n\|_1}\right)^{-1} Df(y_n)\left(\frac{v_n}{\|v_n\|_1}\right) \\ &\leq d_{\mathcal{L}_h(\mathbb{G},\mathbb{M})}(Df(\bar{x}), Df(y_n)) \to 0\end{aligned} \tag{7.39}$$

as n goes to ∞, then we can deduce that, letting n go to ∞,

$$\|Df(\bar{x})\left(\frac{v_n}{\|v_n\|_1}\right)\|_2 \to 0$$

but at the same time it converges also to $\|Df(\bar{x})(\bar{v})\|_2$, hence $Df(\bar{x})(\bar{v}) = 0$, and then $\bar{v} \in \ker(Df(\bar{x}))$. Therefore $\bar{v} \in \ker(Df(\bar{x})) \cap \mathbb{V}$ and $\|\bar{v}\|_1 = 1$ and this is not possible since $\mathbb{V}$ and $\ker(Df(\bar{x}))$ are complementary subgroups. □

The following proposition is a reformulation of [JNGV20, Corollary 2.16] according to our notation. Proposition 7.4.6 concerns the intrinsic regularity of the level sets of the considered map f: it allows to deduce the intrinsic Lipschitz continuity of the local parametrizing maps of all level sets of f. In particular, it provides quantitative information about the intrinsic Lipschitz constant of the intrinsic parametrizations. From a qualitative point of view, we could deduce the intrinsic Lipschitz continuity of the parametrization also by Corollary 4.3.8.

Proposition 7.4.6. *Let us consider a map $f \in C^1_h(\mathbb{G}, \mathbb{M})$, a compact set $\Omega \subset \mathbb{G}$, and assume that there exists a p-dimensional homogeneous subgroup $\mathbb{V}$ such that $Df(x)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is a h-isomorphism for every $x \in \Omega$. Let R be the constant associated with Ω provided by Proposition 7.4.5. Fix $\bar{x} \in \Omega$ and consider $\Sigma := \{y : f(y) = f(\bar{x})\} \cap \mathbb{B}(\bar{x}, R) = f^{-1}(f(\bar{x})) \cap \mathbb{B}(\bar{x}, R)$. Set $L = \mathrm{Lip}(f|_{\mathbb{B}(\bar{x},R)})$. If we define*

$$\mathcal{C} := \{0\} \cup \bigcup_{v \in \mathbb{V}} B\left(v, \frac{R}{L}\|v\|_1\right),$$

we have

$$\Sigma \cap x\mathcal{C} = \{x\} \quad \text{for every } x \in \Sigma. \tag{7.40}$$

Proof. We know that $d_2(f(y), f(yv)) \leq R\|v\|_1$ for every $y \in \mathbb{B}(\bar{x}, R)$ and for every $v \in \mathbb{V}$ such that $yv \in \mathbb{B}(\bar{x}, R)$. Let $y \in \Sigma$ and $z \in y\mathcal{C}$, $z \neq y$, then there exists some $v \in \mathbb{V}$ such that $z \in yB\left(v, \frac{R}{L}\|v\|_1\right) = B\left(yv, \frac{R}{L}\|v\|_1\right)$, and then $d_1(z, yv) < \frac{R}{L}\|v\|_1$. Then

$$d_2(f(y), f(z)) \geq d_2(f(y), f(yv)) - d_2(f(yv), f(z)) \geq R\|v\|_1 - Ld_2(z, yv) > 0$$

hence $f(z) \neq f(y) = f(\bar{x})$, then $z \notin \Sigma$, and then for every $y \in \Sigma$, $\Sigma \cap y\mathcal{C} = \{y\}$. $\square$

Remark 7.4.7. In the notation of Corollary 7.4.6, the previous result implies that Σ is an intrinsic Lipschitz graph, whose intrinsic Lipschitz constant depends on $\mathbb{W}$, $\mathbb{V}$, R and L. In fact, let us set $\alpha := \frac{R}{R+L}$ so that $\frac{\alpha}{1-\alpha} = \frac{R}{L}$. Then, for every $\gamma < \alpha$,

$$X\left(0, \mathbb{V}, \gamma\right) \subset \mathcal{C}. \tag{7.41}$$

In fact, if $y \in X(0, \mathbb{V}, \gamma)$, then $\operatorname{dist}(y, \mathbb{V}) \leq \gamma\|y\|_1 < \alpha\|y\|_1$. Hence there exists some $\bar{v} \in \mathbb{V}$ such that $d_1(y, \bar{v}) < \alpha\|y\|_1$, but by triangle inequality, $\|y\|_1 \leq d(y, \bar{v}) + \|\bar{v}\|_1 < \alpha\|y\|_1 + \|\bar{v}\|_1$. Hence $\|y\|_1(1-\alpha) < \|\bar{v}\|_1$ and so $\|y\|_1 < \frac{\|\bar{v}\|_1}{1-\alpha}$. Therefore $d(y, \bar{v}) < \frac{\alpha}{1-\alpha}\|\bar{v}\|_1 = \frac{R}{L}\|\bar{v}\|_1$ and hence $y \in B(\bar{v}, \frac{R}{L}\|\bar{v}\|_1) \subset \mathcal{C}$. Hence, condition (7.40) implies that for every $\gamma < \frac{R}{R+L}$,

$$\Sigma \cap X\left(x, \mathbb{V}, \gamma\right) = \{x\} \quad \text{for every } x \in \Sigma. \tag{7.42}$$

Then, in addition, if there exist two complementary subgroups $\mathbb{G} = \mathbb{W}\mathbb{V}$, an open set $U \subset \mathbb{W}$ and a parametrizing function $\phi : U \subset \mathbb{W} \to \mathbb{V}$ such that $\operatorname{graph}(\phi) \subset \Sigma$, combining Proposition 3.5.4, or better Remark 3.5.5, with (7.42), ϕ is intrinsic Lipschitz and its intrinsic Lipschitz constant depends on $\mathbb{W}$, $\mathbb{V}$ and α, then on $\mathbb{W}$, $\mathbb{V}$, R and L.

Combining Proposition 7.4.5 with Proposition 7.4.6, we get the following.

Proposition 7.4.8. *Let us consider a map $f \in C^1_h(\mathbb{G}, \mathbb{M})$ and a compact set $\Omega \subset \mathbb{G}$. Let us assume that there exists a p-dimensional homogeneous subgroup $\mathbb{V}$ such that $Df(x)|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is a h-isomorphism for every $x \in \overline{\Omega_D}$ for some $D > 0$. Then there exists a constant L, such that for every $m \in \mathbb{M}$, $x \in f^{-1}(m) \cap \Omega$, the set $f^{-1}(m) \cap \mathbb{B}(x, R)$ is an intrinsic L-Lipschitz graph, where R is the constant of Proposition 7.4.5 applied to Ω.*

Proof. By hypothesis, at any point $x \in \overline{\Omega_D}$, $\ker Df(x)$ is a normal homogeneous subgroup complementary to $\mathbb{V}$, hence $Df(x)$ is a h-epimorphism. Assume that R is smaller than D. Let us fix a homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$. For every $m \in \mathbb{M}$ and $x \in f^{-1}(m) \cap \Omega$, the set $f^{-1}(m) \cap \mathbb{B}(x, R)$ is contained in the intrinsic graph of a function $\phi_{m,x} : U_{m,x} \subset \mathbb{W} \to \mathbb{V}$, for some open set $U_{m,x} \subset \mathbb{W}$. The map $\phi_{m,x}$ is given by Theorem 4.1.19, repeatedly applied to different points of $f^{-1}(m) \cap \mathbb{B}(x, R)$, if necessary.

By Remark 7.4.7, $f^{-1}(m) \cap \mathbb{B}(x, R)$ is the intrinsic Lipschitz graph of an intrinsic L-Lipschitz function $\phi_{m,x}$ for some constant L depending on R, and on the Lipschitz constant of $f|_{\mathbb{B}(x,R)}$, that can be uniformly controlled by the $\sup_{x\in\Omega} \operatorname{Lip}(f|_{\mathbb{B}(x,R)}) \leq \operatorname{Lip}(f|_{\overline{\Omega_D}}) < \infty$. As a consequence, the sets $f^{-1}(m) \cap \mathbb{B}(x, R)$, for every $m \in \mathbb{M}$ and $x \in f^{-1}(m) \cap \Omega$, are intrinsic L-Lipschitz for some positive L independent of x and m. $\square$

Corollary 7.4.9. *Let $f \in C^1_h(\mathbb{G}, \mathbb{M})$ be a function, assume that $Df(x)$ is surjective for every $x \in \mathbb{G}$ and let $\Omega \subset \mathbb{G}$ be a compact set. Assume that there exists a p-dimensional subgroup $\mathbb{V}$ of $\mathbb{G}$ such that $Df(x)|_{\mathbb{V}}$ is an h-isomorphism for every $x \in \overline{\Omega_D}$ for some $D > 0$. Set $\lambda = \sup_{x\in\Omega} \operatorname{Lip}(f|_{\mathbb{B}(x,R)})$, where R is the constant given by Proposition 7.4.5 applied to Ω. Then there exists a constant $1 \leq T(\mathbb{G}, \lambda, R, p) < \infty$, such that*

$$\int_\Omega C_P(Df(x))d\mathcal{S}^Q(x) \leq T \int_{\mathbb{M}} \mathcal{S}^{Q-P}(f^{-1}(m) \cap \Omega)d\mathcal{S}^P(m).$$

Proof. We can assume that $R < D$. Set $\mathbb{W}$ any homogeneous subgroup complementary to $\mathbb{V}$. By Propositions 7.4.8 and 3.5.13, there exists a constant $K > 0$ such that for every $m \in \mathbb{M}$ and $x \in f^{-1}(m) \cap \Omega$, for every $0 < r < R$, $\mathcal{S}^{Q-P}(f^{-1}(m) \cap \mathbb{B}(x,r)) \geq K r^{Q-P}$, where K is a constant depending on $c_0(\mathbb{W},\mathbb{V}) > 0$ and on the intrinsic Lipschitz constants of the parametrizing maps $\phi_{m,x} : U_{m,x} \subset \mathbb{W} \to \mathbb{V}$ of $\{f^{-1}(m) \cap \mathbb{B}(x,r)\}_{m \in \{m \in \mathbb{M}, x \in f^{-1}(m) \cap \Omega\}}$. Moreover observe that by Proposition 7.4.8, the maps $\phi_{m,x}$ are intrinsic L-Lipschitz, for some constant L independent of m and x. Now notice that our hypotheses imply that the hypothesis that the level sets $f^{-1}(m)$ are uniformly locally lower Ahlfors $(Q-P)$-regular with respect to $\mathcal{S}^{Q-P}$ in Theorem 7.0.1 is satisfied (more precisely in Claim 1 on Theorem 7.3.3), hence we can apply our result to this situation, and we directly get the thesis. □

Remark 7.4.10. We have seen, in the proof of Corollary 7.4.9, that the existence of a p-dimensional homogeneous subgroup $\mathbb{V}$ complementary to $\ker(Df(x))$ for every point $x \in \mathbb{G}$, implies that the level sets of f are R-locally C-lower Ahlfors $(Q-P)$-regular with respect to $\mathcal{S}^{Q-P}$, for some positive constants C and R, locally independent of the choice of the level set. We want to highlight that the opposite may be false. In fact, there exist continuously Pansu differentiable maps between two Carnot groups, with everywhere surjective differential, such that their level sets are lower Ahlfors regular, but at the same time $\ker(Df(x))$ does not admit any complementary subgroup.

We present a very simple example of this fact in the first Heisenberg group $\mathbb{H}^1$, that is the simplest non-commutative Carnot group. As usual we consider $\mathbb{H}^1$ in adapted coordinates with respect to a orthonormal graded basis (e_1, e_2, e_3) such that $[e_1, e_2] = e_3$. Let us consider the map

$$f : \mathbb{H}^1 \to \mathbb{R}^2,\ f(x,y,z) = (ax + by, cx + dy), \text{ with } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Observe that $f \in \mathcal{L}_h(\mathbb{H}^1, \mathbb{R}^2)$ and that the Pansu differential of f is constant on $\mathbb{H}^1$: for every $\bar{x} \in \mathbb{H}^1$,

$$Df(\bar{x}) = f,$$

hence, $\ker(Df(\bar{x})) = \mathrm{span}(e_3)$ for every $\bar{x} \in \mathbb{H}^1$. Notice that $\mathrm{span}(e_3)$ is a normal homogeneous subgroup of Hausdorff dimension 2 that does not admit any complementary subgroup. Let us now focus on the level sets of f. If we fix $v \in \mathbb{R}^2$ we know that $f^{-1}(v) = w\mathrm{span}(e_3)$ for some fixed $w \in H_1$ such that $f(w) = v$, hence any level set is a coset of $\mathrm{span}(e_3)$. Then, by the left invariance and the homogeneity of the homogeneous distance, the level sets $f^{-1}(v)$ are C-lower Ahlfors 2-regular with respect to $\mathcal{S}^2$, for some positive constant C, independent of the choice of v.

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