Alma Mater Studiorum - Universitá di Bologna

DOTTORATO DI RICERCA IN MATEMATICA

Ciclo XXXIII

Settore Concorsuale: 01/A3
Settore Scientifico Disciplinare: Analisi Matematica MAT/05

INTERPOLATION PROBLEMS IN DIRICHLET TYPE SPACES

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Esame finale anno 2020

Contents

Preface

This thesis is the fruit of my 3 years stay at the University of Bologna as a PhD student[1] under the supervision of professor Nicola Arcozzi. Essentially it comprises three preprints "Onto Interpolation for the Dirichlet Space and for $H_2(\mathbb{D})$", "Random Interpolating Sequences for Dirichlet Spaces" and "Totally Null Sets and Capacity in Dirichlet Type Spaces" [33, 34, 35]. Although they are all independent works they revolve around the same problems and ideas and I think that they fit organically in this thesis.

For his help, encouragement and for his enthusiasm about mathematics in our endless hours of discussions I would like to thank my supervisor Nicola Arcozzi. This PhD thesis wouldn't have been possible without my collaborators, Andreas Hartmann, Karim Kellay, Brett Wick and Michael Hartz. I am also indebted to Pavel Mozolyako, Dimitris Betsakos, Aristomenis Siskakis, Matteo Levi and Filippo Sarti for insightful comments and interesting discussions in various points in the past three years. I would also like to thank the referees for their invaluable comments on the manuscript.

[1] My PhD was funded by the project INdAM-DP-COFUND-2015, Grant number 713485 of the "Istituto Nazionale di Alta Matematica" cofunded by Marie Skłodowska-Curie Actions.

Nomenclature

$Int(\mathcal{Z})$	p. 48	$S(I)$	p. 17
$K_s(z,w)$	p. 37	$S(z)$	p. 46
$k_s(z,w)$	p. 39	$S^\eta(z)$	p. 47
$P(\alpha)$	p. 33	$S^\gamma_\circ(z)$	p. 72
$p(\alpha)$	p. 33	$StrongSep(\mathcal{Z})$	p. 48
$Q_\delta(\zeta)$	p. 38	$T_\mathcal{H}$	p. 27
$S(\alpha)$	p. 33	V_f	p. 107
$s(\alpha)$	p. 33	z^*	p. 47

Notation

For two quantities $M, N \geq 0$ which depend on some parameters we write $M \lesssim N$, if there exists some constant $C > 0$, not depending on the parameters, such that $M \leq C \cdot N$. We will also write $M \approx N$ if $M \lesssim N$ and $N \lesssim M$. In statements of lemmas, propositions or theorems the dependence of constants on the parameters is denoted by subscripts. We usually write c_0 for an absolute constant. When we write C we mean a general positive constant which might change from appearance to appearance, but it always depends on the same parameters.

Chapter 1

Introduction

Interpolation problems by analytic functions is a subject of more than a century old that is rich of interesting and deep results but also continues to stimulate new research. Interpolation has come a long way since the fundamental papers of Pick [70] and Nevanlinna [64, 65], but the spirit of the problems is invariant. One is given a subset of analytic functions in the unit disc $\varepsilon \subset \mathcal{O}(\mathbb{D})$, a sequence (finite of infinite) $\{z_i\} \subset \mathbb{D}$ of *interpolating nodes* and a *target space* $\mathcal{X}$, i.e. a set of sequences to be interpolated. The problem is then to determine whether or not for any data $\{w_i\} \in \mathcal{X}$ there exists a function $f \in \varepsilon$ such that

$$f(z_i) = w_i, \ \ \forall i.$$

The interpolation problem considered by Pick [70] is the following. Suppose that we have a finite number of nodes $z_1, \dots, z_n \in \mathbb{D}$ and some data $w_1, \dots, w_n \in \mathbb{D}$. Is there a way to determine if there exists a holomorphic function $f : \mathbb{D} \mapsto \mathbb{D}$ such that $f(z_i) = w_i, i = 1, \dots n$? Pick gave a necessary and sufficient condition in terms of the positivity of a matrix that now is referred to as the *Pick matrix.* He showed that the interpolation problem has a solution if and only if the matrix

$$\left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{i,j=1}^{n}$$

is positive semi definite. Nevanlinna also considered the same problem [64] and probably unaware of Pick's work gave a parametric representation of the solutions of the interpolation problem in the case that the matrix is strictly positive definite [65].

The work of Pick and Nevanlinna remained relatively unknown until Carleson [27] considered a related interpolation problem. Carleson gave a characterization of all sequences $\mathcal{Z} = \{z_i\} \subset \mathbb{D}$ such that for any bounded data $\{w_i\} \in \ell^\infty$ there exists a bounded analytic function $f \in H^\infty(\mathbb{D})$ such that $f(z_i) = w_i, i = 1, 2, \dots$

Carleson's result had a profound impact in the function theory of the Banach algebra $H^\infty(\mathbb{D})$ and in particular it lies in the heart of the proof of the Corona theorem [25] for $H^\infty(\mathbb{D})$.

Despite their apparent similarity though, the connection between the two interpolation problems remained a mystery until later works of Shapiro and Shields [79] and Sarason [76]. More details on the modern approach to the two problems can be found in Section 2.4. But it worth saying that the connection has everything to do with the Hardy space $H^2(\mathbb{D})$ and the fact that $H^\infty(\mathbb{D})$ is the space of multipliers of $H^2(\mathbb{D})$ (for the relevant definitions see Sections 2.3 and 2.2). In fact the interpolating sequences for $H^\infty(\mathbb{D})$ as defined by Carleson coincide with the interpolating sequences for $H^2(\mathbb{D})$ for a weighted interpolation problem. A sequence $\mathcal{Z} = \{z_i\}$ is called interpolating for $H^2(\mathbb{D})$ if for any $\alpha \in \ell^2$ there exists $f \in H^2(\mathbb{D})$ such that

$$f(z_i) = \alpha_i(1 - |z_i|^2)^{-1/2}, i = 1, 2, \dots$$

Shapiro and Shields [79] showed that the two notions coincide; a sequence is interpolating for $H^2(\mathbb{D})$ if and only if it is interpolating for $H^\infty(\mathbb{D})$.

With the development of function theory in spaces other than the Hardy space the results about interpolating sequences were generalized in various directions. In two yet unpublished preprints Marshall and Sundberg [60] and Bishop [21] developed a theory of interpolating sequences for the Dirichlet space $\mathcal{D}$, the Hilbert space of analytic functions with finite Dirichlet integral;

$$\int_{\mathbb{D}} |f'|^2 dA < +\infty.$$

In particular they gave a characterization of interpolating sequences for the multiplier space of $\mathcal{D}$ analogous to that of Carleson for $H^\infty(\mathbb{D})$. While the characterization of interpolating sequences for $\mathcal{D}$, i.e. sequences $\mathcal{Z} = \{z_i\}$ such that for any $\alpha \in \ell^2$ there exists $f \in \mathcal{D}$ such that

$$f(z_i) = \alpha_i\Big(\log\frac{e}{1 - |z_i|^2}\Big)^{-1/2}$$

still remains an open problem. In Theorem 3.1.1 we give a characterization of such sequences under the additional hypothesis that the so-called Shapiro-Shields condition is satisfied, i.e.

$$\sum_{i=1}^{\infty}\Big(\log\frac{e}{1 - |z_i|^2}\Big)^{-1} < +\infty.$$

It should be mentioned that by no means the Dirichlet space is the only setting in which such questions have been investigated. In the Bergman space interpolating sequences have a very satisfactory characterization due to Seip [77] in terms of Beurling densities. For Paley-Wiener type spaces interpolating sequences is an old topic related to the celebrated Whittaker-Kotelnikov-Shannon sampling theorem [68, 56, 67].

A different approach, which aims to study interpolation problems collectively for large family of spaces that have some common properties is that suggested by Shapiro and Shields [79]. They suggest that a general interpolation problem can be phrased for all *reproducing kernel Hilbert spaces*, a notion introduced by Aronszajn [16]. This approach has been proven very fruitful and is the basis for the modern theory of interpolating sequences.

Another concept of interpolation that we explore in Chapter 4 is that of a random interpolating sequence. There are many ways that one can rigorously interpret the notion of a random sequence in the unit disc. One of them is the so called Steinhaus sequence of random variables. For a prescribed sequence of radii r_n one considers a sequence

$$r_1 e^{i\Theta_1}, r_2 e^{i\Theta_2} \dots$$

of random variables such that Θ_i are independent and uniformly distributed in $[0, 2\pi]$.

In such a situation one would be interested in the probability that a sequence is interpolating for a given space. For the Hardy space these question have been answered by Cochran [39], Bogdan [23] and Rudowicz [75]. In Chapter 4 we shall prove a Kolmogorov 0-1 law for interpolating sequences for the Dirichlet space and for certain weighted versions of the Dirichlet space which also include the classical Hardy space.

Finally in Chapter 5 we are interested in a problem which superficially looks rather different but it turns out that is also related to a kind of interpolation. The central concept there is that of an exceptional set.

Looking again at the Hardy space a fundamental theorem of Fatou says that a function $f \in H^2(\mathbb{D})$ has radial limits at almost every boundary point, that is for almost every $\zeta \in \mathbb{T} := \partial\mathbb{D}$ the limit

$$\lim_{r \to 1^-} f(\zeta r)$$

exists and is finite. Hence subsets of $\mathbb{T}$ of measure zero can be considered negligible as far as the function theory of the Hardy space is concerned. This intuition has been verified in many different situations [45, 50, 73].

Changing the underling space though usually calls for a different notion of negligibility. For the Dirichlet space this notion comes from potential theory and is that of the logarithmic capacity (see Section 2.8). Logarithmic capacity has been an invaluable tool of function theory for the Dirichlet space. For example Beurling [18] proved that the radial limits of a function in the Dirichlet space exist everywhere except a set of a logarithmic capacity zero.

In Chapter 5 we compare the notion of logarithmic capacity to a notion of negligibility which has been introduced in connection with the study of

$A(\mathbb{B}_d)$ the algebra of analytic functions in the unit ball of $\mathbb{C}^d$ which extend continuously to the boundary, as described by Rudin [74]. The notion is that of a *totally null set* (see Section 5.4.2 for the rigorous definition). In fact in Theorem 5.2.1 we show that for compact subsets of $\mathbb{T}$ the notion of logarithmic capacity zero and totally null set coincide. Our results apply not only to the Dirichlet space but also to a large family of spaces called Hardy-Sobolev spaces, when "logarithmic capacity zero" is replaced by an appropriate notion of capacity. From this identification we draw a number of consequences that include some results on boundary interpolation which improve upon previous work of Cohn and Verbitsky [41].

Chapter 2

Preliminaries

2.1 Reproducing kernel Hilbert spaces

Let X be a set and $\mathcal{H}$ a (complex) Hilbert space of complex valued functions defined on X, with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ such that the linear functionals $\ell_x : f \to f(x)$ are bounded. Such spaces are usually called *reproducing kernel Hilbert spaces* (or RKHS for short). The reason for this terminology is that by Riesz's representation Theorem there exists $k_x \in \mathcal{H}$ which represents ℓ_x, i.e.

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}}, \ \ \forall f \in \mathcal{H}.$$

We can store all the information that k_x contains in a single function of two variables which we shall call *reproducing kernel* defined as follows

$$k(x, y) := k_y(x), \ \ x, y \in X.$$

We shall further assume that the Hilbert function spaces that we work with have the property that the kernel does not vanish on the diagonal, for that would imply that all functions in the space vanish at that point.

Lemma 2.1.1. *Suppose that k is a reproducing kernel. The following hold.*

(i) $k(x, y) = \overline{k(y, x)}, \ \ x, y \in X.$

(ii) k is positive semidefnite.

These properties of a reproducing kernel characterize reproducing kernels completely. Any function of two variables $k : X \times X \mapsto \mathbb{C}$ satisfying properties (i) and (ii) is the reproducing kernel of a RKHS of functions defined on X [2, Theorem 2.23].

Definition 2.1.2. A RKHS with kernel k is called *irreducible if*

(i) For every $x, y \in X$ the vectors k_x, k_y are linearly independent.

(ii) For every $x, y \in X, k(x, y) \neq 0$.

2.2 Multipliers space

For a given RKHS one can define the corresponding multiplier algebra, usually denoted by $\mathcal{M}(\mathcal{H})$ as the set

$$\{\varphi : X \to \mathbb{C}, \varphi \cdot f \in \mathcal{H}, \ \ \forall f \in \mathcal{H}\}.$$

If $\varphi \in \mathcal{M}(\mathcal{H})$ an application of the closed graph theorem gives that the operator

$$M_\varphi f := \varphi f,$$

is a bounded linear operator on $\mathcal{H}$. Pulling back the operator norm to the space $\mathcal{M}(\mathcal{H})$ we can define a norm on the multiplier space. If $\varphi \in \mathcal{M}(\mathcal{H})$

$$\|\varphi\|_{\mathcal{M}(\mathcal{H})} := \|M_\varphi\|.$$

As a result there is a fundamental characterization of the multipliers space in purely operator theoretic terms.

Theorem 2.2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ (a bounded linear operator on $\mathcal{H}$). Then T is a multiplication operator if and only if every kernel vector k_x is an eigenvector of T^*.*

Proof. Suppose that $T = M_\varphi$ for some $\varphi \in \mathcal{M}(\mathcal{H})$. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}\langle M_\varphi^* k_x, f\rangle_{\mathcal{H}} &= \langle k_x, M_\varphi f\rangle_{\mathcal{H}}\\ &= \langle k_x, \varphi f\rangle_{\mathcal{H}}\\ &= \overline{\varphi(x) f(x)}\\ &= \overline{\varphi(x)}\langle k_x, f\rangle_{\mathcal{H}}\\ &= \langle \overline{\varphi(x)} k_x, f\rangle_{\mathcal{H}}.\end{aligned}$$

Hence,

$$M_\varphi^* k_x = \overline{\varphi(x)} k_x. \tag{2.2.1}$$

The converse statement follows by the same calculation and the fact that $\vee\{k_x : x \in X\} = \mathcal{H}$, where $\vee$ denotes the closed linear span of the vectors. □

As the modulus of an eigenvalue is always bounded by the corresponding operator norm we immediately get the following corrolary.

Corollary 2.2.2. *For any $\varphi \in \mathcal{M}(\mathcal{H})$,*

$$\sup_{x\in X} |\varphi(x)| \le \|\varphi\|_{\mathcal{M}(\mathcal{H})}.$$

Next we shall introduce tensor multipliers. This is done not only for the shake of generalizing the notion of the multiplier space, but it turns out to be the right way to formulate a fundamental property of many RKHS.

Let μ be an at most countable cardinal, i.e. $\mu = 1, 2, \ldots, \aleph_0$. we denote by ℓ^2_μ either the Hilbert space $\mathbb{C}^\mu$ with the standard Hermitian inner product when μ is finite, or $\ell^2(\mathbb{N})$ when $\mu = \aleph_0$. Then the tensor product $\mathcal{H} \otimes \ell^2_\mu$ can be thought of as the space of column vectors

$$F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix}, \quad f_i \in \mathcal{H}, \ \ 0 \leq i < \mu \tag{2.2.2}$$

Pure tensors are vectors of the form

$$f \otimes v = \begin{bmatrix} v_0 f \\ v_1 f \\ \vdots \end{bmatrix}.$$

The norm is given by

$$\|F\|^2_{\mathcal{H}\otimes\ell^2_\mu} := \sum_{i=0}^{\mu-1} \|f_i\|^2_{\mathcal{H}}.$$

We define now tensor multipliers as follows, for any cardinals μ, ν as before the space of multipliers $\mathcal{M}(\mathcal{H} \otimes \ell^2_\mu, \mathcal{H} \otimes \ell^2_\nu)$ is defined as the space of functions $\Phi : X \to \mathcal{B}(\ell^2_\mu, \ell^2_\nu)$ such that

$$X \ni x \mapsto \Phi(x)F(x)$$

is in $\mathcal{H} \otimes \ell^2_\nu$ for all $F \in \mathcal{H} \otimes \ell^2_\mu$. In an analogous fashion as for scalar multipliers on can define a norm by pulling back the norm of the operator that the multiplier defines. A similar statement as equation (2.2.1) holds for tensor multipliers.

Proposition 2.2.3.

$$M^*_\Phi(k_x \otimes v) = k_x \otimes \Phi^*(x)v, \ \ \forall x \in X, v \in \ell^2_\mu. \tag{2.2.3}$$

Proof. Let $f \otimes v \in \ell^2_\nu$ a pure tensor and $\Phi = (\varphi_{ij})_{i,j=0}^{\mu-1,\nu-1}$. We have

$$\begin{aligned} \langle M^*_\Phi(k_x \otimes u), f \otimes v\rangle &= \sum_{j=0}^{\nu-1} \langle k_x v_j, \sum_{i=0}^{\mu-1} f u_i \varphi_{ij}\rangle_{\mathcal{H}} \\ &= \sum_{i=0}^{\mu-1} \langle k_x \big(\sum_{j=0}^{\nu-1} \overline{\varphi_{ij}(x)} v_j \big), u_i\rangle_{\mathcal{H}} \\ &= \langle k_x \otimes \Phi^*(x)v, f \otimes u\rangle. \end{aligned}$$

Since pure tensors span a dense subset of $\mathcal{H} \otimes \ell^2_\nu$ we have proved the desired identity. $\square$

As before the identity has the following consequence. If $x \in X$ and $\Phi \in \mathcal{M}(\mathcal{H} \otimes \ell^2_\mu, \mathcal{H} \otimes \ell^2_\nu)$

$$\|M_\Phi(k_x \otimes u)\| = \|k_x \otimes \Phi^*(x)u\| = \|k_x\|\|\Phi^*(x)u\|.$$

Hence,

$$\frac{\|\Phi^*(x)u\|}{\|u\|} \leq \|M_\Phi\|.$$

Or equivalently,

$$\sup_{x \in X} \|\Phi(x)\|_{\mathcal{B}(\ell^2_\mu, \ell^2_\nu)} \leq \|M_\Phi\| = \|\Phi\|.$$

2.3 Hardy, Dirichlet and beyond

So far we have we have seen many interesting facts about reproducing kernel Hilbert spaces but we are lacking concrete examples on which to test this abstract theory. Since the cornerstone of every interesting theory are the examples we introduce in this section a list of concrete RKHS. All of them are going to be spaces consisting of analytic functions in the unit disc $\mathbb{D}$ of the complex plane or the unit ball of $\mathbb{C}^d$ which we denote by $\mathbb{B}_d$.

2.3.1 The Hardy space

The Hardy space H^2 consists of functions $f \in \mathcal{O}(\mathbb{D})$ (holomorphic in the unit disc) such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} < +\infty. \tag{2.3.1}$$

The theory of the Hardy space is rich and beautiful but a full exposition of it is beyond the scope of this thesis. The classic monograph [45] contains more information on the Hardy space. A classic theorem of Fatou [45, Theorem 1.2] says that a function in the Hardy space has non tangential limits almost everywhere. More precisely let $0 < \sigma < 1$, and let $\Gamma_\sigma(\zeta)$ be the interior of the convex hull of the point $\zeta \in \mathbb{T}$ and the disc with center 0 and radius σ. Then the limit

$$\lim_{\Gamma_\sigma(\zeta) \ni z \to \zeta} f(z) := f^*(\zeta)$$

exists (and is finite) for almost every $\zeta \in \mathbb{T}$ Furthermore the boundary function f^* defined in this way belongs to $L^2(\mathbb{T})$ and all its negative Fourier coefficients vanish. In this way one has a canonical unitary operator from the Hardy space into the space $\mathcal{H}^2(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0, \forall n < 0\}$. It can be proven that this operator is also surjective, so this is an isometric identification of the spaces $H^2(\mathbb{D})$ and $\mathcal{H}^2(\mathbb{T})$.

Seen as a linear manifold the Hardy space is a Hilbert space (with the norm defined by (2.3.1)) and it has a reproducing kernel. This follows by the growth estimate for H^2 functions [45, p. 36]

$$|f(z)| \leq \|f\|_{H^2}(1-|z|^2)^{\frac{1}{2}},$$

which implies that point evaluation functionals are continuous. After a computation with the Cauchy formula one can derive the following expression for the reproducing kernel of the Hardy space

$$S(z,w) := \frac{1}{1-z\overline{w}}, \forall z, w \in \mathbb{D}.$$

The letter "S" stands for Szëgo, which is the name by which this kernel usually goes. The computation of column and row multipliers of the Hardy space is a feasible task.

Proposition 2.3.1. *Let μ an at most countable cardinal and $\varphi_i \in \mathcal{O}(\mathbb{D})$. Then*

$$\left\| \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \end{bmatrix} \right\|^2_{\mathcal{M}(H^2, H^2\otimes \ell^2_\mu)} = \|[\varphi_1 \ \varphi_2 \ \ldots]\|^2_{\mathcal{M}(H^2\otimes\ell^2_\mu, H^2)} = \sup_{z\in\mathbb{D}} \sum_{i=1}^{\mu} |\varphi_i(z)|^2.$$

This is a special case of [2, Theorem 4.14]. In particular the scalar multipliers of the Hardy space is just the algebra of bounded analytic functions. Therefore the Hardy space represents an extreme case in the sense that its multipliers space is the largest possible. This property has a profound impact on the function theory of the Hardy space. We will see that many results that apply to the Hardy space fail for other spaces due to this peculiar property.

Another tool that is of great importance in the study of the Hardy space (and other RKHS as well) is that of Carleson measures. A positive Borel measure μ on the unit disc is called Carleson if

$$H^2 \subset L^2(\mathbb{D}, \mu).$$

Or equivalently, if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) \leq C\|f\|^2_{H^2}, \forall f \in H^2.$$

Such measures have a neat characterization due to Carleson. The characterization involves the notion of a Carleson box. Let $I \subset \mathbb{T}$ an arc we denote by $S(I)$ the closed hyperbolic half-plane which has the same endpoints with I and contains I in its boundary (see Figure 2.3.1). Although there are more "square" versions of the Carleson box we prefer this definition which is easier expressed in terms of hyperbolic geometry.

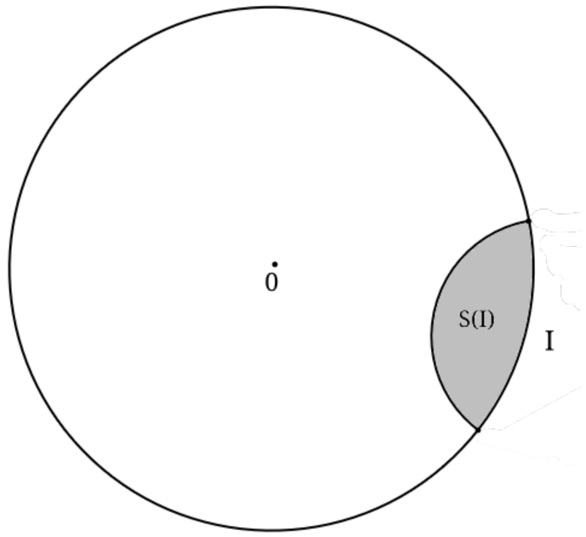

Figure 2.1: A Carleson Box

Theorem 2.3.2 (Carleson). *A positive Borel measure μ is Carleson for the Hardy space if and only if there exists $C > 0$ such that for all arcs $I \subset \mathbb{T}$*

$$\mu(S(I)) \leq C|I|.$$

2.3.2 The Dirichlet space

The Dirichlet space consists of functions $f \in \mathcal{O}(\mathbb{D})$ with finite Dirichlet integral, i.e.

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

Where dA is the normalized area measure such that $dA(\mathbb{D}) = 1$. The above quantity defines only a semi-norm. To render it a proper norm one usually ads to the above quantity the Hardy norm of f or just $|f(0)|^2$. So we end up with two equivalent norms

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

and

$$\|f\|_{\mathcal{D},1}^2 := \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

The corresponding reproducing kernels are given by the formulas

$$k(z, w) = \log \frac{e}{1 - z\overline{w}}.$$

and

$$k_1(z,w) = \frac{1}{z\overline{w}} \log \frac{1}{1 - z\overline{w}}$$

It is interesting to notice that

$$\|k_z\|_{\mathcal{D}}^2 = 1 + \log \frac{1}{1-|z|^2} \approx 1 + d_h(0,z)$$

Where by d_h we denote the hyperbolic distance in the unit disc, Möbius invariant distance such that

$$d_h(r,0) = \log_2 \frac{1+r}{1-r}, \;\; 0 < r < 1.$$

We now turn to the multipliers space $\mathcal{M}(\mathcal{D})$. This turns out to be a proper subset of H^∞. In this section we shall reduce the problem of characterizing $\mathcal{M}(\mathcal{D})$ to the problem of characterizing Carleson measures for the Dirichlet space and later we will also give the characterization of Carleson measures.

Let

$$\mathcal{X} := \{f \in \mathcal{O}(\mathcal{D}) : |f'(z)|^2 dA(z) \text{ is } \mathcal{D} - Carleson\}$$

By $\mathcal{D}$ - Carleson measure we a positive Borel measure μ such that $\mathcal{D} \subset L^2(d\mu, \mathbb{D})$.

Theorem 2.3.3. *[61, Theorem 5.1.7]*

$$\mathcal{M}(\mathcal{D}) = H^\infty \cap \mathcal{X}.$$

The function theory in the Dirichlet space is in many respects less well developed than the corresponding one for the Hardy space. We shall illustrate this with an example.

The following definition applies to all RKHS

Definition 2.3.4. A sequence $\{x_i\} \subset X$ is called *zero sequence* for a RKHS $\mathcal{H}$ if there exists a function $f \in \mathcal{H}$ such that

$$\forall x \in X, f(x) = 0 \iff x \in \{x_i\}.$$

For the Hardy space, zero sequences are well known [45, Theorem 2.3] while for the Dirichlet space the characterization of zero sequences is a notorius difficult problem (see [26] [80] [72] [57] and [61]). Nonetheless the following theorem due to Shapiro and Shields holds

Theorem 2.3.5. *[80] Let $\{z_i\} \subset \mathbb{D}$ a sequence in the unit disc. If*

$$\sum_{i=1}^{\infty} \Big(\log \frac{e}{1-|z_i|^2}\Big)^{-1} < +\infty,$$

then $\{z_i\}$ is a zero sequence for the Dirichlet space.

This theorem is sharp only in the sense that any condition weaker than this which is sufficient for a sequence to be a zero sequence must depend not only on the modulus of $\{z_i\}$ but also on their argument [61, Theorem 4.4.2].

2.3.3 Regular unitarily invariant spaces

Next we introduce a general class of RKHS in the unit ball of $\mathbb{C}^d$ which includes the Hardy and Dirichlet spaces. It has the advantage that it encompasses many concrete examples but also it is narrow enough so interesting theorems can be proved.

A *regular unitarily invariant space* is a reproducing kernel Hilbert space $\mathcal{H}$ on $\mathbb{B}_d$ whose reproducing kernel is of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n, \tag{2.3.2}$$

where $a_0 = 1$, $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$. We think of the last condition as a regularity condition, as it is natural to assume that the power series defining K has radius of convergence 1, since $\mathcal{H}$ is a space of functions on the ball of radius 1. Under this assumption, the limit, if it exists, necessarily equals 1. We recover H^2 and $\mathcal{D}$ by choosing $d = 1$ and $a_n = 1$, respectively $a_n = \frac{1}{n+1}$, for all $n \in \mathbb{N}$. It can be proven that if $\mathcal{H}$ is a regular unitarily invariant space then the polynomials are automatically multipliers for the space.

More background on these spaces can be found in [19, 43, 51].

2.4 The complete Nevanlinna Pick property

The prototype for of all interpolation problems should probably be considered the Pick's interpolation problem. Suppose one is given $z_1, z_2, \dots z_N$ points in $\mathbb{D}$ and $w_1, w_2, \dots w_N$ complex number. What is a necessary and sufficient condition so that there exists $\varphi \in H^\infty(\mathbb{D})$, (a bounded analytic function of supremum norm at most 1 such that

$$\varphi(z_i) = w_i, \;\; i = 1, \dots N?$$

The key observation which will allow us to put the Pick problem in the framework of the theory of reproducing kernel Hilbert spaces is the fact that $\mathcal{M}(H^2(\mathbb{D})) = H^\infty$. That is because if $f \in H^2$ and $\varphi \in H^\infty$

$$\int_0^{2\pi} |\varphi(re^{it}) f(re^{it})|^2 dt \le \|\varphi\|^2_{H^\infty} \int_0^{2\pi} |f(re^{it})|^2 dt \le \|\varphi\|^2_{H^\infty} \|f\|^2_{H^2}.$$

Therefore Pick's problem can be seen as a problem of interpolation by multipliers of a RKHS. In this light one can formulate a general version of Pick's problem.

Suppose $\mathcal{H}$ is a RKHS on X and we are given a finite sequence of points $x_1, x_2, \dots x_N \in X$ and a bounded linear operators $W_1, W_2, \dots W_N \in$

$\mathcal{B}(\ell^2_\mu, \ell^2_\nu)$ what is a necessary and sufficient condition such that there exists $\Phi \in \mathcal{M}(\mathcal{H} \otimes \ell^2_\mu, \mathcal{H} \otimes \ell^2_\nu)$ of operator norm at most 1 which interpolates the data, i.e.,

$$\Phi(x_i) = W_i, \ \ i = 1, \dots N?$$

In this generality we can formulate a necessary condition.

Theorem 2.4.1. *Let $\mathcal{H}$ be a RKHS on X, let $x_1, x_2, \dots x_N \in X$ and let $W_1, W_2, \dots W_N \in \mathcal{B}(\ell^2_\mu, \ell^2_\nu)$. A necessary condition to be able to solve the corresponding Pick's interpolation problem is that the $\mathcal{B}(\ell^2_\nu)$-operator valued matrix*

$$\left[(I_\nu - W_i W_j^*) k(x_i, x_j) \right]_{i,j=1}^N \tag{2.4.1}$$

is positive semi-defnite.

Proof. Suppose that such a Φ exists. As usual this is equivalent to

$$I_{\mathcal{H}\otimes\ell^2_\nu} - M_\Phi M_\Phi^* \geq 0$$

on $\mathcal{H} \otimes \ell^2_\nu$. In particular if $v_1, v_2, \dots v_N \in \ell^2_\nu$,

$$\begin{aligned}
0 &\leq \Big\langle [I_{\mathcal{H}\otimes\ell^2_\nu} - M_\Phi M_\Phi^*]\big(\sum_{i=1}^N k_{x_i} \otimes v_i\big), \sum_{j=1}^N k_{x_j} \otimes v_j\big) \Big\rangle_{\mathcal{H}\otimes\ell^2_\nu} \\
&= \sum_{i,j=1}^N \Big[k(x_j, x_i) \langle v_i, v_j \rangle_{\ell^2_\nu} - \big\langle M_\Phi(k_{x_i} \otimes \Phi^*(x_i) v_i), k_{x_j} \otimes v_j \big\rangle_{\mathcal{H}\otimes\ell^2_\nu} \Big] \\
&= \sum_{i,j=1}^N \Big[k(x_j, x_i) \langle v_i, v_j \rangle_{\ell^2_\nu} - \big\langle k_{x_i} \otimes \Phi^*(x_i) v_i, k_{x_j} \otimes \Phi^*(x_j) v_j \big\rangle_{\mathcal{H}\otimes\ell^2_\nu} \Big] \\
&= \sum_{i,j=1}^N k(x_j, x_i) \langle (I_{\ell^2_\nu} - \Phi(x_j)\Phi^*(x_i)) v_i, v_j) \rangle_{\ell^2_\nu}.
\end{aligned}$$

□

The previous condition is not always sufficient. But the cases when it is are so important that they deserve a definition.

Definition 2.4.2. We say that a RKHS $\mathcal{H}$ with reproducing kernel k has the $\mu \times \nu$ Nevanlinna-Pick property if condition (2.4.1) is also sufficient to solve the interpolating problem. If a kernel has the $\mu \times \nu$ Nevanlinna-Pick property for all μ, ν we say that it is a *complete Nevanlinna-Pick Kernel.*

Let us now present mostly without proofs three different situations which are representative of the possible behaviours that one should expect.

Example (The Paley Wiener Space)**.** *We say that an entire function f is of exponential type A if there exists a positive constant C such that $|f(z)| \leq Ce^{A|z|}, z \in \mathbb{C}$. It can be shown that for such functions, if $f|_{\mathbb{R}} \in L^2(\mathbb{R})$, the Fourier transform is supported on the interval $[-\pi, \pi]$, hence we can define the norm*

$$\|f\|^2_{PW^2_A} := \int_{-A}^{A} |\widehat{f}(x)|^2 dx < +\infty.$$

It can be shown that that the space PW^2_A is a RKHS. We usually take $A = \pi$. Then the reproducing kernel is given by

$$\sigma_\pi(z, w) := \frac{\sin \pi(z - \overline{w})}{\pi(z - \overline{w})}, \quad z, w \in \mathbb{C}.$$

By Corollary 2.2.2 if $\varphi \in \mathcal{M}(PW^2_\pi)$, it must be a bounded entire function therefore it should be constant. Therefore the Paley-Wiener spaces has only trivial multipliers.

This in particular implies that the Paley Wiener space does not have the Pick Property.

Example (The Bergman Space)**.** *The Bergman space $A^2(\mathbb{D})$ is the space of analytic functions in the unit disc which are square integrable with respect to the normalized area measure. Now let $\varphi \in H^\infty(\mathbb{D})$ and $f \in A^2(\mathbb{D})$,*

$$\int_{\mathbb{D}} |\varphi(z)f(z)|^2 dA(z) \leq \|\varphi\|^2_{H^\infty} \|f\|^2_{A^2}.$$

Hence,

$$\mathcal{M}(A^2(\mathbb{D})) = H^\infty(\mathbb{D}).$$

Although the multiplier algebra of the Bergman space contains a lot of non trivial elements, it turns out that they are not enough for the space to have the Nevanlinna-Pick property.

Suppose we want to solve the scalar interpolation problem for two points, so take for convenience $z_1 = w_1 = 0$, then the Pick matrix for the Bergman kernel is positive semidefnite if and only if

$$|w_2| \leq |z_2|\sqrt{2 - |z_2|^2}$$

But we know that analytic functions in the unit ball of H^∞ reduce hyperbolic distance hence if such an interpolating function φ were to exist one should have

$$|w_2| = |\varphi(z_2)| \leq |z_2|,$$

which cleary it is not the case for all admissible choices of w_2.

Example (Regular Unitarily Invariant Spaces)**.** *The following theorem, essentially due to Agler and McCarth [2] (see also [71])d resolves the problem of when a regular unitarily invariant kernel satisfies the complete Nevanlinna Pick property.*

Theorem 2.4.3. *Suppose $\mathcal{H}$ is an irreducble regular unitarily invariant with reproducing kernel*

$$k(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n.$$

Let also the Taylor expansion of k^{-1} be

$$\frac{1}{\sum_{n=0}^{\infty} a_n t^n} = \sum_{n=0}^{\infty} c_n t^n.$$

Then, $\mathcal{H}$ has the CNP property if and only if

$$c_n \leq 0, \ \ \forall n \geq 1.$$

In particular the Hardy space has the CNP property. It is interesting to notice that the CNP property is only an isometric invariant and not an isomorphic invariant. This can be illustrated by the fact that the Dirichlet space satisfies the property with the $\|\cdot\|_{\mathcal{D},1}$ norm but not with $\|\cdot\|_{\mathcal{D}}$.

Spaces satisfying the complete Nevanlinna Pick property enjoy many useful properties. More precisely, such spaces have a rich multipliers space which allows for many problems to be transferred from the Hilbert space it self to $\mathcal{M}(\mathcal{H})$ and vice versa. In the rest of this section we are going to explore an instance of this phenomenon.

The first one will be an extremal problem. Suppose we are given a sequence of points $\{x_i\} \subset X$ and $x \in X \setminus \{x_i\}$. We would like to study the maximazation problems

$$\sigma_{\mathcal{H}} := \sup\{\operatorname{Re} f(x) : f|_{\{x_i\}} = 0, \|f\|_{\mathcal{H}} \leq 1\} \tag{2.4.2}$$

and

$$\sigma_{\mathcal{M}} := \sup\{\operatorname{Re} \varphi(x) : \varphi|_{\{x_i\}} = 0, \|\varphi\|_{\mathcal{M}} \leq 1\} \tag{2.4.3}$$

A priori these two problems could be completely unrelated.

Theorem 2.4.4. *[2, Theorem 9.33] Let $\mathcal{H}$ an irreducible RKHS with the scalar Pick property. Suppose that $\sigma_{\mathcal{H}} > 0$. Then the extremal problems* (2.4.2) *and* (2.4.3) *have unique extremal function f_0 and φ_0 respectively, which are related by the equation*

$$f_0 = \varphi_0 \frac{k_x}{\|k_x\|}.$$

Consequently,

$$\sigma_{\mathcal{H}} = \sigma_{\mathcal{M}} \|k_x\|.$$

2.5 Basics in basis theory of Hilbert spaces

It will be useful for later to establish some terminology and present some basic results from the theory of bases in Hilbert spaces. A standard reference for the material in this section is [53].

What are we going to talk about in this chapter makes sense in an arbitrary Hilbert space $\mathcal{H}$, even if a kernel structure is not specified.

Suppose we have a sequence of vectors $\{x_i\} \in \mathcal{H}$. For most of what is coming we assume that at least our sequence is *topologically free*, i.e.

$$x_i \notin \vee\{x_j : j \neq i\}.$$

Such systems always have what is called a dual system, that is a sequence $\{y_i\}$, with the property

$$\langle x_i, y_j \rangle_{\mathcal{H}} = \delta_{ij}, \forall i, j \in \mathbb{N}$$

To construct such a sequence just pick a nonzero vector in the space $\vee\{x_j\} \ominus \vee\{x_j : j \neq i\}$ and normalize it appropriately.

In fact as just shown it is possible to choose the vectors y_j to belong in $\vee\{x_j\}$. In this case the biorthogonal system is uniquely determined and is called the *minimal dual system.* A stronger requirement is to ask for a system to be *uniformly minimal* which means that there exists a dual system such that

$$\sup_i \|x_i\| \|y_i\| < +\infty.$$

We can formally define an *analysis* operator associated to the sequence which maps an element $h \in \mathcal{H}$ to the sequence of its Fourier coefficients

$$\mathcal{F} : \mathcal{H} \to \ell^2, \ \ h \mapsto \{\langle h, x_i \rangle_{\mathcal{H}}\}_i.$$

Proposition 2.5.1. *The analysis operator is densely defined and closed when $\{x_i\}$ is topologically free.*

Proof. Let $\{y_i\}$ the minimal dual system of $\{x_i\}$. The operator is densely defined on $\mathcal{H}$ because $\mathcal{H} = \vee\{y_i\} \oplus (\vee\{x_i\})^{\perp}$. It vanishes on the orthogonal component and is well defined on the span of y_i. The proof that the operator is closed is quite standard and we will omit it. □

The (formal) adjoint of this operator is called *synthesis* operator and is given by

$$\mathcal{F}^* : \ell^2 \to \mathcal{H}, \ \ \{\alpha_i\} \mapsto \sum_i \alpha_i x_i.$$

Definition 2.5.2. A sequence such that the associated synthesis operator is bounded it is called *Bessel* sequence. If $\mathcal{F}^*$ is also bounded below it is called *Riesz* sequence

The matrix of the operator $G := \mathcal{F}\mathcal{F}^*$ with respect to the standard orthonormal basis of ℓ^2 is called the Grammian of the sequence. More explicitly,

$$G_{ij} = \langle x_i, x_j \rangle_{\mathcal{H}}.$$

We therefore have the following proposition.

Proposition 2.5.3. *For a topologically free system the following are equivalent.*

(a) $\{x_i\}$ *is a Bessel system.*

(b) G *is a bounded matrix in* ℓ^2.

(c) The range of $\mathcal{F}$ *is contained in* ℓ^2.

Proof. Items (a) and (c) are equivalent by the closed graph theorem. While (b) and (c) because $\|\mathcal{F}\mathcal{F}^*\| = \|\mathcal{F}\|^2$. □

Before we proceed to our next proposition we shall need the following lemma.

Proposition 2.5.4. *Let* $A : \mathcal{H} \dashrightarrow \mathcal{K}$ *a linear densely defined closed and surjective operator. Then there exists a bounded right inverse to* A.

Proof. Let $D(A)$ be the domain of A. Then we can endow it with the graph norm, i.e.

$$\|x\|_{\mathcal{G}}^2 = \|x\|_{\mathcal{H}}^2 + \|Ax\|_{\mathcal{K}}^2.$$

With this norm $D(A)$ becomes a Hilbert space and $A : D(A) \mapsto \mathcal{K}$ becomes continuous with this norm. Therefore it has a bounded right inverse which remains continuous with the $\mathcal{H}$ norm. □

Proposition 2.5.5. *For a topologically free system the following are equivalent.*

(i) The minimal dual system is a Bessel sequence.

(ii) G *is bounded below in* ℓ^2.

(iii) The range of $\mathcal{F}$ *contains* ℓ^2.

Proof. Let $\{x_i\}$ a topologically free system, and $\{y_i\}$ its minimal dual. First we prove the equivalence of the first two elements in the list. Note, that since G is a positive operator, the requirement that G is bounded below is equivalent to the statement $G \geq \varepsilon \,\mathrm{id}$ for some $\varepsilon > 0$. Which in turn is equivalent to the inequality

$$\|\sum_{i=1}^{N} \alpha_i x_i\|^2 \geq \varepsilon \sum_{i=1}^{n} |\alpha_i|^2,$$

for all N tuples of complex numbers α_i. Suppose $\{y_i\}$ is and G_∂ is its Gramm matrix,

$$\|\sum_{i=1}^{N}\alpha_i x_i\|^2 \geq \sup\Big\{\Big|\langle\sum\alpha_i x_i, \sum\beta_j y_j\rangle_{\mathcal{H}}\Big| : \|\sum\beta_j y_j\| \leq 1\Big\}$$
$$\geq \sup\Big\{\Big|\sum\alpha_i\overline{\beta_i}\Big| : \Big(\sum|b_i|^2\Big)^{1/2} \leq \|G_\partial\|^{-1}\Big\}$$
$$= \frac{1}{\|G_\partial\|}\sum_{i=1}^{N}|\alpha_i|^2.$$

Note that this direction holds true even if $\{y_i\}$ is just **a** dual system of $\{x_i\}$. To see the other direction, the argument is the same with the role of x_i and y_i reversed noticing that we can reverse the inequalities because of minimality and the bounded below hypothesis.

Now we prove the equivalence of (i) and (ii). Let $\{\alpha_i\} \in \ell^2$, then

$$\mathcal{F}\Big(\sum\alpha_i y_i\Big) = \{\alpha_i\}.$$

In the other direction, Proposition 2.5.4 allow us to construct a bounded right inverse of $\mathcal{F}$ which maps e_i t y_i.

$$\mathcal{F}R = \mathrm{id}_{\ell^2}, \ \ R : \ell^2 \to \vee\{x_i\} \subset \mathcal{H}.$$

It follows that the Grammian of $\{y_i\}$ is bounded and hence they are a Bessel system.

□

One of the most extraordinary theorems in the theory of bases in Hilbert spaces is Feichtinger's Theorem [29], a consequence of the positive solution of the Kadison-Singer problem, by Marcus-Spielman-Srivastava [59]. We present it here without a proof.

Theorem 2.5.6 (Feichtinger's Theorem)**.** *Any Bessel sequence $\{x_i\}$ is a finite union of Riesz sequences.*

2.6 Interpolating sequences

Let now $\mathcal{H}$ a RKHS,

Definition 2.6.1. Let $\{x_i\}$ a sequence of points in X and $g_i := \frac{k_{x_i}}{\|k_{x_i}\|}$ (the sequence of the normalized kernel vectors).

- We say that $\{x_i\}$ is Universally Interpolating (UI), if $\{g_i\}$ is a Riesz system in $\mathcal{H}$,

- Simply Interpolating (SI) if the minimal dual system of $\{g_i\}$ is Bessel,
- Carleson sequence (C) if the system $\{g_i\}$ is a Bessel system.
- Weakly Separated (WS) if it separated with respect to the *Gleason metric* $d_{\mathcal{H}}(x_i, x_j) := \sqrt{1 - |\langle g_i, g_j\rangle_{\mathcal{H}}|^2}$.
- Strongly Separated (SS) if the system g_i is uniformly minimal.

This definition is sensible in view of Propositions 2.5.3 and 2.5.5. So what actually means that a sequence is Simply Intrpolating is that for any sequence of data $\{a_i\} \in \ell^2$ there exists an interpolating function $f \in \mathcal{H}$, i.e., $f(a_i) = \|k_{x_i}\|a_i$. While Universally Interpolating sequences guarantee also that an inequality of the form

$$\sum_{i=1}^{\infty} \frac{|f(x_i)|^2}{\|k_{x_i}\|^2} \leq C\|f\|^2_{\mathcal{H}},$$

for some $C > 0$. To put this inequality in context it helps consider the *discrete measure associated to the sequence*

$$\mu_{\{x_i\}} := \sum_{i=1}^{\infty} \frac{\delta_{x_i}}{\|k_{x_i}\|}.$$

Then to say that a sequence $\{x_i\}$ is Carleson is to say that the associate measure is a *Carleson measure for* $\mathcal{H}$, i.e.

$$\mathcal{H} \subset L^2(X, \mu_{\{x_i\}}).$$

In the jargon of interpolation theory the synthesis operator is usually called the *weighted restriction operator* associated to the sequence and denoted by $T_{\mathcal{H}}$ when the sequence of points $\{x_i\}$ is understood from the context.

2.6.1 Universally interpolating sequences

Our next goal is to prove the following theorem.

Theorem 2.6.2. *[4, Theorem 1.1] If $\mathcal{H}$ is a RKHS with the CNP property then a sequence $\{x_i\} \subset X$ is Universally Interpolating if and only if it is Weakly Separated and Carleson.*

Proof. First we prove the direct implication. For the converse we will need some more preparation. That a (UI) sequence is (C) is evident by defnition, to see the (WS) part just let $\lambda \in \mathbb{C}, |\lambda| = 1$, by the Riesz basis property property for $i \neq j$

$$\varepsilon \leq \|g_i - \lambda g_j\|^2 = 2(1 - \overline{\lambda}\,\mathrm{Re}\langle g_i, g_j\rangle_{\mathcal{H}}).$$

Taking infimum over all unimodular λ,

$$\frac{\varepsilon}{2} \leq 1 - |\langle g_i, g_j \rangle_{\mathcal{H}}|.$$

$\square$

The following theorem is of fundamental importance and justifies the time we spent on tensor multipliers.

Theorem 2.6.3. *[Agler, McCarthy, Theorem 9.46] Let k a CNP kernel and $\{x_i\}$ a sequence of points, let g_i the corresponding normalized kernel vectors and let e_i be the standard orthonormal basis of ℓ^2.*

(a) $\{x_i\}$ is simply Interpolating if and only if there exists a multiplier $\Psi \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ such that

$$\Psi(x_i) = e_i = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots \end{pmatrix}.$$

Proof. If the sequence is simply Interpolating by the discussion in the previous paragraph the associated Grammian G is bounded below, or equivalently there exists $\varepsilon > 0$ such that

$$G - \varepsilon I \geq 0.$$

Or to state it in a Pick matrix form,

$$[(1 - \varepsilon e_i \cdot e_j^*) k(x_i, x_j)] \geq 0.$$

Hence by the row Pick property there exists a multiplier $\widetilde{\Phi} \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ of norm at most one, such that $\widetilde{\Psi}(x_i) = \sqrt{\varepsilon} e_i$. Then $\Psi := \frac{\widetilde{\Psi}}{\sqrt{\varepsilon}}$ is the desired multiplier.

The converse follows by the same argument, because the existence of such a multiplier implies the positivity of the Pick matrix. $\square$

Next theorem is an intermediate step in the proof, although of independent interest.

Theorem 2.6.4. *Let k a kernel with the two point scalar pick property and a $\{x_i\}$ Carleson and Weakly Separated sequence. Then there exists a sequence of multipliers $\theta_i \in \mathcal{M}(\mathcal{H})$, $\|\theta_i\|_{\mathcal{M}(\mathcal{H})} \leq 1$ such that*

$$\theta_i(x_j) = \varepsilon \delta_{ij},$$

for some $\varepsilon > 0$.

Proof. Fix an $i \in \mathbb{N}$ and let $\phi_{ij} \in \mathcal{M}(\mathcal{H})$, $\|\varphi_{ij}\|_{\mathcal{M}(\mathcal{H})} \leq 1$ such that,

$$\phi_{ij}(x_i) = 0, \phi_{ij}(x_j) = d_{\mathcal{H}}(x_i, x_j).$$

Such a matrix exists by positivity of the correspondent Pick matrix and the two point scalar pick property. The consider the multiplier $\theta_i \in \mathcal{M}(\mathcal{H})$

$$\theta_i := \prod_{j \neq i} \phi_{ij}.$$

(Check that the infinite product converges to a multiplier). Then θ_i vanishes on all points except x_j where it takes the value

$$\theta_i(x_i) = \prod_{j \neq i} d_{\mathcal{H}}(x_i, x_j).$$

Each factor is bounded away from zero by the Weak Separation condition, and also

$$\sum_{j \neq i}(1 - d_{\mathcal{H}}(x_i, x_j)) \leq 2\sum_{j \neq i}(1 - d_{\mathcal{H}}(x_i, x_j)^2) = \sum_{j \neq i} |\langle g_i, g_j \rangle_{\mathcal{H}}|^2 \leq \|G\|^2_{\ell^2}.$$

Hence $\inf_{i \in \mathbb{N}} |\theta_i(z_i)| > 0$. □

Proof of converse in Theorem 2.6.2. Let $\{x_i\}$ a sequence which is Weakly Separated and Carleson. By defnition the system $\{g_i\}$ of normalized reproducing kernels forms a Bessel sequence, therefore by Feichtinger's Theorem it can be written as a finite union of Riesz systems or equivalently our sequence is a finite union of Universally Interpolating sequences. Therefore the claim will be proved if we show that the union of two (UI) sequences is (UI) if it is (WS).

We shall use the following notation. If $\{a_i\}, \{b_i\}$ are two infinite sequence we write $\{a_i\} \wedge \{b_i\}$ for the sequence

$$a_1, b_1, a_2, b_2, a_3, \ldots$$

Let $\{x_i^{(k)}\}, k = 1, 2$ be (UI) and $\{x_i\} := \{x_i^{(1)}\} \wedge \{x_i^{(2)}\}$ be (WS).

The union is also a Carleson sequence therefore there exist θ_i as in Theorem 2.6.4. Finally there exist multipliers $\Psi^{(1)}, \Psi^{(2)} \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ as in Theorem 2.6.3. define the Ψ by

$$\Psi(x) := \left(\Psi^{(1)}(x) \wedge \Psi^{(2)}(x)\right) \begin{pmatrix} \theta_1(x) & 0 & 0 & \ldots \\ 0 & \theta_2(x) & 0 & \\ 0 & 0 & \theta_3(x) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.6.1}$$

In fact $\Psi \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ and $\Psi(x_i) = e_i$. Hence, again by Theorem 2.6.3 the sequence is Universally Interpolating. □

In a recent preprint [52] Hartz has found an alternative proof of the fact that weakly separated Carleson sequences are universally interpolating which has the advantage that it does not invoke the Feichtinger Theorem. The core of the proof is the fact that every CNP space satisfies the so called column-row property which simply says that if $\Phi \in \mathcal{M}(\mathcal{H}, \mathcal{H} \otimes \ell^2)$ then $\Phi^t \in \mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ (Φ^t denotes the matrix transpose of Φ). This might look as an innocent statement but its proof is quite non trivial. In fact the converse statement, which looks superficially similiar, fails spectacularly for many spaces.

2.6.2 Simply interpolating sequences

Simply Interpolating sequences are not so well understood in the generality of CNP spaces. One first observation is the following.

Proposition 2.6.5. *If $\{x_i\} \subset X$ is a simply interpolating sequence for $\mathcal{H}$, then $\{x_i\}$ is strongly separated.*

Proof. We saw that the weighted restriction operator T has a bounded right inverse $R : \ell^2 \mapsto \mathcal{H}$ when the sequence $\{x_i\}$ is simply interpolating. Let now e_i the standard orthonormal basis of ℓ^2 and consider the functions $f_i := Re_i$. The identity $TR = \mathrm{id}$ implies that

$$f_i(z_j) = \delta_{ij} \|k_{x_i}\|,$$

and the boundedness of R that $\|f_i\| \leq \|R\|$. Hence f_i is a system dual to g_i which satisfies the hypotheses of uniform minimaility. □

For spaces with the scalar Pick property this naturally translates to a property of the multiplier algebra.

Corollary 2.6.6. *If $\{x_i\}$ is a simply interpolating sequence for a RKHS with the scalar Pick property, then there exist $\varphi_i \in \mathcal{M}(\mathcal{H})$, such that $\varphi_i(z_j) = \delta_{ij}$ and $\sup_i \|\varphi_i\|_{\mathcal{M}(\mathcal{H})} < +\infty$.*

Proof. This is immediate by an application of Theorem 2.4.4. □

Apart from this implication little is known about simply interpolating sequences. For the Hardy space already Carleson [28] showed that strongly separated sequences are automatically universally interpolating, and as a consequence universally and simply interpolating sequences coincide.

The situation in the Dirichlet space is quite different. In two unpublished preprints, Bishop [21] and Marshall & Sundberg [60] showed that there exist simply interpolating sequences for the Dirichlet space which are not universally interpolating.

In the same preprint, Bishop [21], proves the following theorem. Following Bishop we use the notation

$$d(z) = \log \frac{e}{1 - |z|^2}.$$

Theorem 2.6.7. *[21, Theorem 1.2] A sequence $\{z_i\}$ is simply interpolating for $\mathcal{D}$ iff for every $i \in \mathbb{N}$ there exists a function $f_i \in \mathcal{D}$ such that $\|f_i\| \leq Cd(z_i)^{-1}, \|f_i\|_{H^\infty} \leq C$ and $f_i(z_j) = \delta_{ij}$, for some positive constant $C > 0$.*

One can observe that the condition that Bishop gives is the strong separation condition plus a uniform bound on the H^∞ norms of the dual system. With the machinery of Pick spaces we can easily lift the condition $\|f_i\|_{H^\infty} \leq C$.

Corollary 2.6.8. *Strongly separated sequences and simply interpolating sequences coincide for the Dirichlet space.*

Proof. Let $\{z_i\}$ strongly separated. Since $\mathcal{D}$ is a CNP space, there exist $\varphi_i \in \mathcal{M}(\mathcal{D})$ as in Corollary 2.6.6. Then the functions

$$f_i := \varphi_i \frac{k_{z_i}}{d(z_i)}$$

satisfy the condition in Theorem 2.6.7. □

Later we will come back in this theorem and we will also provide a quantitative version of it.

It is therefore natural to pose the conjecture that in every space with the CNP property strongly separated and simply interpolating sequences coincide. This problem probably appeared first in [2, p. 145]. In the best of our knowledge this is still an open problem.

Even more, we would like to know in which spaces the simply interpolating sequences coincide with the universally interpolating ones (as in the Hardy space) and in which spaces there exist genuine simply interpolating sequences (as in the Dirichlet space). A possible conjecture would be that the spaces where the two notions coincide must have H^∞ as their multiplier algebra, but the evidence to support such a conjecture are rudimental.

2.7 The Whitney decomposition of the unit disc.

In this section we are going to introduce a tool of foremost importance in the modern approach to the Dirichlet space theory. The idea is to tile the hyperbolic disc in such a way such that the tiles are roughly hyperbolic discs of constant hyperbolic radius. Then the set of tiles carries a natural

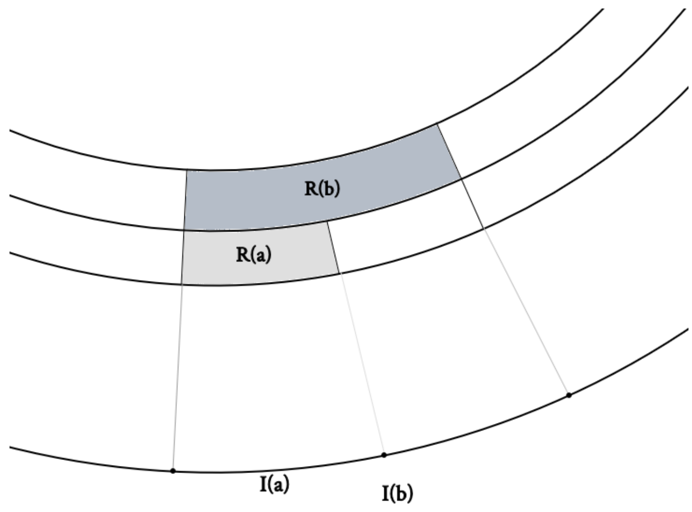

Figure 2.2: Two adjacent regions such that $a \preceq b$.

homogeneous tree structure on which one can define a discrete version of the Dirichlet space. Very often results about the discrete Dirichlet space can be transferred back to the usual Dirichlet space courtesy of the local oscillation estimates for Dirichlet functions.

We shall now formalize this idea.

Let $n \geq 0$ and $0 \leq k < 2^n$ we associate to this couple of indices the region

$$R(n,k) := \{z = re^{2\pi it} : 1 - 2^{-n} \leq r < 1 - 2^{-n-1},\ \frac{k}{2^n} \leq t < \frac{k+1}{2^n}\}.$$

The regions $R(n,k)$ form a partition of the unit disc and one can inscribe and circumscribe hyperbolic discs of comparable hyperbolic radius, uniformly over all allowable pairs of indices (n,k).

Let also $I(n,k) := \{z/|z| : z \in R(n,k)\}$, and let τ the set of allowable indices. We define a partial order on it in the following way. Let $\alpha = (n,k), \beta = (n',k') \in \tau$ we write

$$\alpha \preceq \beta \iff I(\alpha) \subset I(\beta).$$

The Hasse graph of this partial order relation is a dyadic tree with root vertex the pair $\omega := (0,0)$ which we will continue to denote with τ since no confusion arises. By tree we mean a connected graph without loops. Another way to think of τ is the following. For any allowable index (n,k) consider the center point $z(n,k)$ of the rectangle $R(n,k)$[1] then the collection of points

[1] Any point in $R(n,k)$ would serve the same purpose as long as it is defined in a canonical way, so that we do not have to invoke the axiom of countable choice.

$z(n,k)$ inherits the tree structure of τ. Often we prefer this realization of τ which is more concrete.

Let us introduce a piece of notation about trees. A geodesic $\{\alpha_i\}$ is a (finite or infinite) sequence of edges such that for every $\alpha_j \in \{\alpha_i\}$, $\{\alpha_0, \ldots, \alpha_j\}$ is the shortest walk between α_0 and α_j. Notice that for every edge α there exists a unique geodesic $\{\alpha_0 = \omega, \alpha_1, \ldots, \alpha_N = \alpha\} =: [\omega, \alpha]$ which starts at the root and ends at α. Here N is the level of α, which we denote by $d_\tau(\alpha)$.

Recalling the order relation $\preceq$ we define successor sets $S(\alpha) = \{\beta \in \tau : \beta \preceq \alpha\}$ and predecessor sets $P(\alpha) = \{\beta \in \tau : \alpha \preceq \beta\}$, as well as the set of sons of α, $s(\alpha) := \{\beta \in E(T) : \alpha \preceq \beta, |\beta| = \alpha + 1\}$. This set has two members $\sigma_-\alpha$ and $\sigma_+\alpha$. The parent of α is the unique $p(\alpha) \in \tau$ which satisfies $\alpha \in s(p(\alpha))$. We call T_α the subtree of T rooted at α which has as vertices the set $S(\alpha)$. The boundary $\partial\tau$ of a tree τ is defined as the set of infinite geodesics with starting point ω, and has a topology generated by the basis $\{\partial\tau_\alpha\}_{\alpha\in\tau}$ where $\partial\tau_\alpha$ is the set of infinite geodesics passing through α. It turns out that this space is metrizable and $\tau \cup \partial\tau$ is a compactification of τ with the edge counting metric. Note that the order relation extends naturally to an order on the set $\tau \cup \partial T$.

For a function $f : \tau \mapsto \mathbb{C}$ we define the gradient

$$\nabla f(\alpha) := \begin{cases} f(\omega), & \alpha = \omega, \\ f(\alpha) - f(p(\alpha)), & \alpha \neq \omega. \end{cases}$$

Naturally the Sobolev $H_2(\tau)$ space of the tree is the space of functions $f : \tau \mapsto \mathbb{C}$ such that

$$\|f\|^2_{H_2(\tau)} := \sum_{\alpha\in\tau} |\nabla f(\alpha)|^2 < +\infty.$$

2.7.1 The $T(1)$-Theorem for Dirichlet Carleson measures

To demonstrate the parallelism between the theories of the Dirichlet space and $H_2(\tau)$ we shall discuss a $T(1)$ type characterization of Carleson measures for the Dirichlet space. Let us call $CM(\mathcal{D})$ and $CM(H_2(\tau))$ the space of Carleson measures for the Dirichlet space in the disc and on the tree respectively.

Lemma 2.7.1. *[14, Theorem 9] Let μ a positive measure on $\mathbb{D}$. Then $\mu \in CM(\mathcal{D})$ if and only if the measure*

$$\widehat{\mu}(\alpha) := \mu(R(\alpha)), \forall\alpha \in \tau$$

belongs to $CM(H_2(\tau))$.

Therefore the question of characterizing Carleson measures reduces to the corresponding question for a discrete space. The next theorem shows that a $T(1)$ testing type condition characterizes the discrete Carleson measures and therefore the standard Carleson measures for the Dirichlet space.

Theorem 2.7.2. *A measure on τ is in $CM(H_2(\tau))$ if and only if*

$$\sum_{\beta \preceq \alpha} \mu(S(\beta))^2 \leq C(\mu)\mu(S(\alpha)),$$

for some $C(\mu) > 0$ depending possibly on μ. The smallest constant $C(\mu)$ such that this inequality holds is denoted by $[\mu]_{CM}$.

Another way to say that a measure is Carleson for $H_2(\tau)$ is to say that the *Hardy operator*

$$\mathcal{I} : \ell^2(\tau) \mapsto \ell^2(\mu) \tag{2.7.1}$$

$$\mathcal{I}f(\alpha) := \sum_{\beta \succ \alpha} f(\beta), \tag{2.7.2}$$

has finite norm.

There are various proofs of Theorem 2.7.2 ([9], [82], [10], [11], [8] and [58]). Here we shall give a new, simple proof.

Lemma 2.7.3. *Let H_1, H_2 be Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. Then*

$$\|T\|_{H_1 \to H_2} = \|T^*\|_{H_2 \to H_1} = \sqrt{\|TT^*\|_{H_1}}.$$

Lemma 2.7.4. *Let μ, ν positive measures in τ such that $\mu(S(\alpha)) \leq \nu(S(\alpha))$ for all $\alpha \in \tau$. Then if $f : \tau \to \mathbb{R}_{\geq 0}$ is decreasing, in the sense that $f(\alpha) \geq f(\beta)$ when $\alpha \preceq \beta$, we have*

$$\sum_{\alpha \in \tau} f(\alpha)\mu(\alpha) \leq \sum_{\alpha \in \tau} f(\alpha)\nu(\alpha).$$

Proof. Let $t > 0$, since f is decreasing the set $\{\alpha \in \tau : f(\alpha) \geq t\}$ is a stopping time, therefore it can be written as a disjoint union $\bigcup_{k=1}^{\infty} S(\alpha_k)$. Hence,

$$\mu(\alpha \in \tau : f(\alpha) > t) = \sum_{k=1}^{\infty} \mu(S(\alpha_k)) \leq \sum_{k=1}^{\infty} \nu(S(\alpha_k)) = \nu(\alpha \in \tau : f(\alpha) > t).$$

From the distributional formula we have

$$\sum_{\alpha \in \tau} f(\alpha)\mu(\alpha) \quad = \quad \int_0^{\infty} \mu(\alpha \in \tau : f(\alpha) \geq t)dt$$

$$
\begin{aligned}
&\leq \int_0^\infty \nu(\alpha \in \tau : f(\alpha) \geq t)dt \\
&= \sum_{\alpha\in\tau} f(\alpha)\nu(\alpha).
\end{aligned}
$$

□

Proof of Theorem 2.7.2. We start with the non-trivial part of the proof which is the sufficiency of the condition. Suppose our tree is finite, so that the Hardy operator has already some finite norm, and we will establish a norm estimate independent of the depth of the tree.

The adjoint of the Hardy operator is given by the formula

$$\mathcal{I}^* g(\alpha) = \sum_{\beta\preceq\alpha} g(\beta)\mu(\beta).$$

Therefore,

$$
\begin{aligned}
\mathcal{I}^*\mathcal{I}g(\alpha) &= \sum_{\beta\preceq\alpha}\sum_{\gamma\succ\beta} g(\gamma)\mu(\beta) \\
&= \sum_{\gamma\preceq\alpha} g(\gamma)\mu(S(\gamma)) + \mu(S(\alpha))\sum_{\gamma\succ\alpha} g(\gamma) \\
&= \sum_{\gamma\preceq\alpha} g(\gamma)\mu(S(\gamma)) + \mu(S(\alpha))\mathcal{I}g(\alpha) \\
&= T_1 g(\alpha) + T_2 g(\alpha).
\end{aligned}
$$

Therefore,

$$\|\mathcal{I}^*\mathcal{I}\|_{\ell^2(\tau)} \leq \|T_1\|_{\ell^2(\tau)} + \|T_2\|_{\ell^2(\tau)}.$$

The second norm can be computed in terms of the norm $\mathcal{I}$,

$$
\begin{aligned}
\|T_2 g\|^2_{\ell^2(\tau)} &= \sum_\alpha \mu(S(\alpha))^2 \left(\mathcal{I}g(\alpha)\right)^2 \\
&\leq [\mu]_{CM}\sum_\alpha \mu(\alpha)\left(\mathcal{I}g(\alpha)\right)^2 \leq [\mu]_{CM}\|g\|^2_{\ell^2(\tau)}\|\mathcal{I}\|^2_{\ell^2(\tau)\to\ell^2(\mu)},
\end{aligned}
$$

where the first inequality comes from Lemma 2.7.4 and the hypothesis on μ.

A standard calculation shows that $T_2^* = T_1$ (with respect to the inner product in $\ell^2(\tau)$), hence we have for free the estimate on the norm of T_1. Putting everything together we get

$$\|\mathcal{I}\|^2_{\ell^2(\tau)\to\ell^2(\mu)} = \|\mathcal{I}\mathcal{I}^*\|_{\ell^2(\tau)} \leq \sqrt{2}[\mu]^{1/2}_{CM}\|\mathcal{I}\|_{\ell^2(\tau)\to\ell^2(\mu)}.$$

Since $\mathcal{I}$ is bounded because our tree is finite we can divide both sides of the inequality with its norm to get

$$\|\mathcal{I}\|_{\ell^2(\tau)\to\ell^2(\mu)} \leq \sqrt{2}[\mu]^{1/2}_{CM}.$$

This concludes the proof of the one direction. To see why the condition is also necessary, suppose that $\mathcal{I}$ is bounded, then so is $\mathcal{I}^*$. In other words,

$$\sum_{\alpha\in\mathcal{T}} |\mathcal{I}^* f(\alpha)|^2 \leq \|\mathcal{I}\|^2 \sum_{\alpha\in\mathcal{T}} |f(\alpha)|^2 \mu(\alpha).$$

Putting $f = \chi_{S(\beta)}$ gives the desired inequality. □

2.8 Potential theory

2.8.1 Hardy Sobolev spaces in the unit ball of $\mathbb{C}^d$

Let $d \in \mathbb{N}$ we denote by $\mathbb{B}^d$ the unit ball of $\mathbb{C}^d$. We would like to introduce a family of regular unitarily invariant spaces which naturally generalize the Dirichlet and Hardy space from one dimension.

The function theory in the unit ball of $\mathbb{C}^d$ is in many respects similar to the one of the unit disc but it also exhibits radically different phenomena.

Let $z = (z_1, \ldots, z_d), w = (w_1, \ldots, w_d) \in \mathbb{C}^n$, we shall repeatedly use the standard innner product of $\mathbb{C}^d$,

$$\langle z, w\rangle := \sum_{i=1}^{d} z_i \overline{w_i}.$$

We will also use the multi index notation, i.e. for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d, z \in \mathbb{C}^d$ we define

$$\begin{aligned} z^\alpha &:= z_1^{\alpha_1} \ldots z_d^{\alpha_d} \\ |\alpha| &:= \alpha_1 + \cdots + \alpha_d \\ \alpha! &:= \alpha_1! \ldots \alpha_d!. \end{aligned}$$

Let also $\mathcal{U}(\mathbb{C}^d)$ the group of unitary $d \times d$ matrices, i.e. all square $d \times d$ matrices U with entries in $\mathbb{C}$ such that

$$U^*U = UU^* = \mathrm{id}\,.$$

Such matrices define a transitive action on $\mathbb{C}^d$ which preserves the inner product and in particular it leaves invariant the unit sphere $\partial\mathbb{B}_d$. It can be proven that there exists a unique positive Borel probability measure on $\partial\mathbb{B}_d$ which is invariant under the action of $\mathcal{U}(\mathbb{C}^d)$. We shall call this measure the surface measure on $\partial\mathbb{B}_d$ and we will denote it by σ.

It is quite out of scope to give a complete introduction to the function theory of the unit ball here so we will only concentrate on some aspects that are of particular interest for us. The interested reader can find much more material in [74].

Naturally the Hardy space $H^2(\mathbb{B}_d)$ is defined as the space of functions $f \in \mathcal{O}(\mathbb{B}_d)$ (holomorphic in the unit ball) such that

$$\sup_{0\le r<1} \int_{\partial\mathbb{B}_d} |f(r\zeta)|^2 d\sigma(\zeta) < +\infty.$$

The later quantity defines a norm which comes from an inner product. Much of the function theory of the Hardy space $H^2(\mathbb{D})$ can be transferred in this setting.

Let us now define a radial fractional derivation operator as follows. Fix a real parameter $s \in \mathbb{R}$ and let $f \in \mathcal{O}(\mathbb{B}_d)$. Then f has a an expansion of the form

$$f(z) = \sum_{n\ge 0} f_n(z)$$

where f_n is a homogeneous polynomial of degree n and the series converges locally uniformly. We define the formal operator $(\mathrm{id} + \mathcal{R})^s$ as follows

$$(\mathrm{id} + \mathcal{R})^s f(z) = \sum_{n\ge 0} (1+n)^s f_n(z).$$

Then the Hardy-Sobolev space $\mathcal{H}_s$ is the space of $f \in \mathcal{O}(\mathbb{B}_d)$ such that

$$(\mathrm{id} + \mathcal{R})^s f \in H^2(\mathbb{B}_d).$$

The corresponding norm is simply $\|f\|_s := \|(\mathrm{id} + \mathcal{R})^s f\|_{H^2}$. It can be proven that for $s > \frac{d}{2}$ every function in $\mathcal{H}_s$ has a continuous extension on the boundary, therefore many aspects the theory, such as exceptional sets for example, in this case are less interesting. The critical case $s = \frac{d}{2}$ corresponds to the Dirichlet space (when $d = 1$ this is the standard Dirichlet space). Another space which deserves special attention is the Drury Arveson space which corresponds to the parameter value $s = \frac{d-1}{2}$. This space is central in the study of operator inequalities of the von Neumann type [44].

Henceforth we will always implicitly assume that $s \le \frac{d}{2}$. For some equivalent norm on these spaces the reproducing kernel is given by the formula

$$K_s(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{d-2s}}, & s < \frac{d}{2}, \\ \log \frac{e}{1-\langle z,w\rangle}, & s = \frac{d}{2}. \end{cases} \tag{2.8.1}$$

This shows that the spaces $\mathcal{H}_s$ are regular unitarily invariant[2]and furthermore that $\mathcal{H}_s$ has the complete Nevanlinna Pick property when $s \in [\frac{d-1}{2}, \frac{d}{2}]$.

[2]More precisely there exists an equivalent norm on $\mathcal{H}_s$ with respect to which the spaces are regular unitarily invariant.

2.8.2 Non-isotropic Riesz potentials and capacities

In this section we introduce a way to measure subsets of $\partial\mathbb{B}_d$ which is adapted to the function theory in the spaces $\mathcal{H}_s$. First we must say a few words about the non isotropic geometry of the unit sphere $\partial\mathbb{B}_d$.

2.8.3 The Koranyi metric

For the purposes of holomorphic function theory the Euclidean metric restricted on $\partial\mathbb{B}_d$ is not a natural way to measure distances. The natural quantity is the so called *Koranyi metric.* Let $z, w \in \overline{\mathbb{B}_d}$ we define

$$d_{\mathcal{K}}(z,w) := |1 - \langle z, w\rangle|^{\frac{1}{2}}.$$

It can be proved [74, Proposition 5.1.2] that $d_{\mathcal{K}}$ defines a metric on $\partial\mathbb{B}_d$. If $\zeta \in \partial\mathbb{B}_d$ the corresponding *Koranyi ball* is the set

$$Q_\delta(\zeta) = \{\eta \in \partial\mathbb{B}_d : d_{\mathcal{K}}(\zeta, \eta) < \delta\}.$$

The Koranyi metric is also invariant under orthogonal transformations, i.e.

$$d_{\mathcal{K}}(z,w) = d_{\mathcal{K}}(Uz, Uw), \;\; \forall z, w \in \overline{\mathbb{B}_d}, \forall U \in \mathcal{U}(\mathbb{C}^d).$$

The next lemma is often useful in calculations.

Lemma 2.8.1. *[74, Proposition 5.1.2] There exists an absolute constant $c > 0$ such that for any $\zeta \in \partial\mathbb{B}_d$, $0 < \delta < \sqrt{2}$*

$$\frac{1}{c}\delta^{2d} \leq \sigma(Q_\delta(\zeta)) \leq c\delta^{2d}.$$

2.8.4 Non isotropic Bessel capacities a la Adams & Hedberg

We can now go ahead and define a non isotropic potential theory on $\partial\mathbb{B}_d$. When defining capacity for (compact) sets $E \subset \partial\mathbb{B}_d$ induced by the Hardy-Sobolev spaces $\mathcal{H}_s$, there are at least two possible approaches. Each one can be viewed as natural depending on the perspective. The two definitions turn out to be equivalent in the sense that the capacities defined are comparable with absolute constants. In particular, capacity zero sets coincide in both senses.

The first definition, introduced in [3, p. 489], is motivated by the fact that the spaces $\mathcal{H}_s$ can be understood as potential spaces and fits into the framework of the general potential theory of Adams and Hedberg [1]. Let $M^+(\partial\mathbb{B}_d)$ denote the set of positive regular Borel measures on $\partial\mathbb{B}_d$. We let σ be the normalized surface measure on $\partial\mathbb{B}_d$. If $E \subset \partial\mathbb{B}_d$ is compact, let

$M^+(E)$ be the set of all measures in $M^+(\partial\mathbb{B}_d)$ that are supported on E. For $0 \le s < d$, consider the kernel

$$k_s(z,w) = \frac{1}{|1-\langle z,w\rangle|^{d-s}} \quad (z,w \in \overline{\mathbb{B}_d})$$

and also set

$$k_d(z,w) = \log \frac{e}{|1-\langle z,w\rangle|} \quad (z,w \in \overline{\mathbb{B}_d}).$$

Definition 2.8.2. Let $0 \le s \le d$, let $\mu \in M^+(\partial\mathbb{B}_d)$ and let $E \subset \partial\mathbb{B}_d$ be compact.

(a) The *non-isotropic Riesz potential* of μ is

$$\mathcal{I}_s(\mu)(z) = \int_{\partial\mathbb{B}_d} k_s(z,w)d\mu(w) \quad (z \in \partial\mathbb{B}_d).$$

We extend the definition to non-negative measurable functions $f \in L^1(\partial\mathbb{B}_d, d\sigma)$ by letting $\mathcal{I}_s(f) = \mathcal{I}_s(f\,d\sigma)$.

(b) The *non-isotropic Bessel capacity* of E is defined by

$$C_{s,2}(E) = \inf\{\|f\|^2_{L^2(\partial\mathbb{B}_d,d\sigma)} : \mathcal{I}_s(f) \ge 1 \text{ on } E, f \ge 0\}.$$

(c) The quantity $\|\mathcal{I}_s(\mu)\|^2_{L^2(\partial\mathbb{B}_d,d\sigma)} \in [0,\infty]$ is called the *energy* of μ.

To see why this is a special case of the general potential theory introduced by Adams and Hedberg in [1] recall that to define a potential theory in their setting [1, Definition 2.3.1] one needs a reference measure space $(M,\mathcal{M},\nu)$ and a non negative kernel $g : \mathbb{R}^n \times M \mapsto [0,+\infty]$ such that for every $y \in M$, $g(\cdot,y)$ is lower semicontinuous on $\mathbb{R}^n$ and for every $x \in \mathbb{R}^n$, $g(x,\cdot)$ is $M-$measurable.

If we let $n = 2d$, M be the unit sphere $\partial\mathbb{B}_d$ equipped with the Borel $\sigma-$algebra, $\nu = \sigma$ and finally

$$g(x,y) := k_s(z,w)\chi_{\partial\mathbb{B}_d}(x),$$

where $z = (x_1 + ix_2, \dots, x_{2d-1} + ix_{2d}), w = (y_1 + iy_2, \dots, y_{2d-1} + iy_{2d})$, then we find ourselfs in the setting of Adams and Hedberg.

This observation allows us to freely use the potential theory of Adams and Hedberg. In particular, by [1, Theorem 2.5.1], we have the following "dual" expression for the capacity $C_{s,2}(\cdot)$,

$$C_{s,2}(E)^{1/2} = \sup\{\mu(E) : \mu \in M^+(E),\ \|\mathcal{I}_s(\mu)\|_{L^2(\partial\mathbb{B}_d,d\sigma)} \le 1\}, \qquad (2.8.2)$$

which holds for at least all compact sets $E \subset \mathbb{T}$. In particular, $C_{s,2}(E) > 0$ if and only if E supports a probability measure of finite energy.

2.8.5 An alternative approach

A different approach, which can be justified by regarding $\mathcal{H}_s$ as a reproducing kernel Hilbert space, is the following; cf. [46, Chapter 2].

Sometimes it is wise to distinguish between the space $\mathcal{H}_s$ with the norm induced by the Hardy space and the same space equipped with the equivalent norm which gives as reproducing kernel the functions in (2.8.1). When it is necessary to do so we write $\mathcal{D}_a$ for the later space with $a = d - 2s$.

Definition 2.8.3. Let $\frac{d-1}{2} < s \leq \frac{d}{2}$, let $a = d - 2s$, let $\mu \in M^+(\partial\mathbb{B}_d)$ and let $E \subset \partial\mathbb{B}_d$ be compact.

(a) The $\mathcal{D}_a$-*potential* of μ is

$$\mathcal{I}_{2s}(\mu)(z) = \int_{\partial\mathbb{B}_d} |K_s(z,w)|\, d\mu(w).$$

(b) The $\mathcal{D}_a$-*energy* of μ is defined by

$$\mathcal{E}(\mu, \mathcal{D}_a) = \int_{\partial\mathbb{B}_d} \int_{\partial\mathbb{B}_d} |K_s(z,w)| d\mu(z) d\mu(w).$$

(c) The $\mathcal{D}_a$-*capacity* of E is defined by

$$c_\alpha(E)^{1/2} = \sup\{\mu(E) : \mu \in M^+(E),\ \mathcal{E}(\mu, \mathcal{D}_a) \leq 1\}.$$

As for the Bessel capacity, $c_\alpha(E) > 0$ if and only if E supports a probability measure of finite energy. The capacities $c_\alpha(\cdot)$ fit into the framework of capacities on compact metric spaces developed in [46, Chapter 2]. When $d = 1$ and $\alpha = 0$ we usually refer to the capacity c_0 as *logarithmic capacity* and we usually write c instead of c_0 to denote it.

It appears to be well known to experts that the capacities c_α and $C_{s,2}(\cdot)$ are equivalent if $a = d - 2s$, the point being that the corresponding energies are comparable. A proof in the case $d = 1, s = \frac{1}{2}$ can be found in [32, Lemma 2.2]. In the case $s \neq \frac{d}{2}$, the crucial estimate is stated in [31, Remark 2.1] without proof. A proof of the estimate in one direction in this case is contained in [3, p.442-442]. For the sake of completeness, we provide an argument that applies to all cases under consideration. We adapt the proof in [32, Lemma 2.2] to the non-isotropic geometry of $\partial\mathbb{B}_d$.

Lemma 2.8.4. *Let* $\frac{d-1}{2} < s \leq \frac{d}{2}$ *and* $\mu \in M^+(\partial\mathbb{B}_d)$. *Then*

$$\mathcal{I}_s(\mathcal{I}_s(\mu)) \approx \mathcal{I}_{2s}(\mu),$$

where the implied constants only depend on s *and* d.

Proof. We will show that

$$\int_{\partial\mathbb{B}_d} k_s(z,\zeta)k_s(\zeta,w)d\sigma(\zeta) \approx k_{2s}(z,w) \quad (z,w\in\partial\mathbb{B}_d).$$

The statement then follows by integrating both sides with respect to μ and using Fubini's theorem.

Let $z,w\in\partial\mathbb{B}_d$ and set $\delta=\frac{d_{\mathcal{K}}(z,w)}{2}$. Then, in order to estimate the kernel

$$\int_{\partial\mathbb{B}_d} \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{d-s}|1-\langle\zeta,w\rangle|^{d-s}} = \int_{\partial\mathbb{B}_d} \frac{d\sigma(\zeta)}{d_{\mathcal{K}}(z,\zeta)^{2(d-s)}d_{\mathcal{K}}(\zeta,w)^{2(d-s)}},$$

we split the domain of integration $\partial\mathbb{B}_d$ as follows

$$\begin{aligned}\partial\mathbb{B}_d =& \big(\{d_{\mathcal{K}}(\zeta,z)\le d_{\mathcal{K}}(\zeta,w)\}\setminus Q_\delta(z)\big)\cup\big(\{d_{\mathcal{K}}(\zeta,w)\le d_{\mathcal{K}}(\zeta,z)\}\setminus Q_\delta(w)\big)\\ &\cup Q_\delta(z)\cup Q_\delta(w).\end{aligned}$$

We denote by I, I′, II, II′ the corresponding integrals. By the symmetry of the problem it suffices to estimate I and II.

For I, we note that if $\zeta\in Q_\delta(z)$, then $\delta\le d_{\mathcal{K}}(\zeta,w)\le 3\delta$ by the triangle inequality for d. Hence, integrating with the help of the distribution function, we find that

$$\begin{aligned}\mathrm{I} &\approx \frac{1}{\delta^{2(d-s)}}\int_{Q_\delta(z)}\frac{d\sigma(\zeta)}{d_{\mathcal{K}}(z,\zeta)^{2(d-s)}}\\ &= \frac{1}{\delta^{2(d-s)}}\int_0^\infty \sigma(\{\zeta\in Q_\delta(z): d_{\mathcal{K}}(z,\zeta)\le t^{\frac{-1}{2(d-s)}}\})dt\\ &= \frac{\sigma(Q_\delta(z))}{\delta^{4(d-s)}}+\frac{1}{\delta^{2(d-s)}}\int_{\delta^{-2(d-s)}}^\infty \sigma(\{\zeta\in\partial\mathbb{B}_d: d_{\mathcal{K}}(z,\zeta)\le t^{\frac{-1}{2(d-s)}}\})dt\\ &\approx \delta^{-2(d-2s)}+\frac{1}{\delta^{2(d-s)}}\int_{\delta^{-2(d-s)}}^\infty t^{\frac{-d}{d-s}}dt\\ &\approx \delta^{-2(d-2s)}.\end{aligned}$$

Next, using the fact that $d_{\mathcal{K}}(z,\zeta)\le\sqrt{2}$ for all $z,\zeta\in\partial\mathbb{B}_d$, we see that

$$\begin{aligned}\mathrm{II} &\le \int_{\partial\mathbb{B}_d\setminus Q_\delta(z)}\frac{d\sigma(\zeta)}{d_{\mathcal{K}}(z,\zeta)^{4(d-s)}}\\ &= \int_0^\infty \sigma(\{\zeta\in\partial\mathbb{B}_d:\delta< d_{\mathcal{K}}(z,\zeta)\le t^{\frac{-1}{4(d-s)}}\})dt\\ &\lesssim 1+\int_{2^{-2(d-s)}}^{\delta^{-4(d-s)}} t^{\frac{-d}{2(d-s)}}dt\\ &\lesssim \begin{cases}\delta^{-2(d-2s)}, & \text{if } s<\frac{d}{2},\\ \log(\delta^{-2}), & \text{if } s=\frac{d}{2}.\end{cases}\end{aligned}$$

Combining the estimates for I and II and recalling the definition of δ we see that

$$\int_{\partial\mathbb{B}_d} \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{d-s}|1-\langle \zeta,w\rangle|^{d-s}} \lesssim \begin{cases} \frac{1}{|1-\langle z,w\rangle|^{d-2s}}, & \text{if } s<\frac{d}{2} \\ \log\left(\frac{e}{|1-\langle z,w\rangle|}\right), & \text{if } s=\frac{d}{2}. \end{cases}$$

To establish the lower bound, it suffices to consider $z,w\in\partial\mathbb{B}_d$ for which $d_{\mathcal{K}}(z,w)$ is small. In the case $s<\frac{d}{2}$, the lower bound follows from the treatment of the integral I above. Let $s=\frac{d}{2}$. Notice that in the region $\mathcal{U}_{z,w}=\{\zeta\in\partial\mathbb{B}_d : d_{\mathcal{K}}(z,w)\le d_{\mathcal{K}}(w,\zeta)\}$, the triangle inequality yields $d_{\mathcal{K}}(\zeta,z)\le 2d_{\mathcal{K}}(\zeta,w)$. Hence integrating again with the distribution function and writing $\delta=d_{\mathcal{K}}(z,w)$, we estimate

$$\begin{aligned} \int_{\partial\mathbb{B}_d} \frac{d\sigma(\zeta)}{d_{\mathcal{K}}(\zeta,w)^d d_{\mathcal{K}}(\zeta,z)^d} &\gtrsim \int_{\mathcal{U}_{z,w}} \frac{d\sigma(\zeta)}{d_{\mathcal{K}}(\zeta,w)^{2d}} \\ &= \int_0^{\delta^{-2d}} \sigma(\{\zeta\in\partial\mathbb{B}_d : \delta\le d_{\mathcal{K}}(\zeta,w)\le t^{\frac{-1}{2d}}\})dt \\ &= \int_0^{\delta^{-2d}} \sigma(Q_{t^{-\frac{1}{2d}}}(w))dt - \delta^{-2d}\sigma(Q_\delta(w)) \\ &\ge c_0\log(\delta^{-1})-c_1, \end{aligned}$$

where $c_0,c_1>0$ are constants depending only on the dimension d. This shows the lower bound for small δ, which concludes the proof. ☐

From this lemma the equivalence of the capacities $C_{s,2}(\cdot)$ and c_α for $a=d-2s$ follows easily.

Corollary 2.8.5. *Let $\frac{d-1}{2}<s\le\frac{d}{2}$, let $a=d-2s$ and $\mu\in M^+(\partial\mathbb{B}_d)$. Then*

$$\|\mathcal{I}_s(\mu)\|^2_{L^2(\partial\mathbb{B}_d,d\sigma)} \approx \mathcal{E}(\mu,\mathcal{D}_a).$$

Hence $C_{s,2}(E)\approx c_\alpha(E)$ for compact subsets $E\subset\partial\mathbb{B}_d$. Here, all implied constants only depend on d and s.

Proof. For a measure $\mu\in M^+(\partial\mathbb{B}_d)$, we compute

$$\begin{aligned} \|\mathcal{I}_s(\mu)\|^2_{L^2(\partial\mathbb{B}_d,d\sigma)} &= \int_{\partial\mathbb{B}_d}\left(\int_{\partial\mathbb{B}_d}\frac{d\mu(z)}{|1-\langle z,\zeta\rangle|^{d-s}}\right)^2 d\sigma \\ &= \int_{\partial\mathbb{B}_d}\int_{\partial\mathbb{B}_d}\int_{\partial\mathbb{B}_d}\frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{d-s}|1-\langle w,\zeta\rangle|^{d-s}}d\mu(z)d\mu(w) \\ &= \int_{\partial\mathbb{B}_d}\mathcal{I}_s(\mathcal{I}_s(\mu))d\mu(w). \end{aligned}$$

Thus, Lemma 2.8.4 yields that

$$\|\mathcal{I}_s(\mu)\|^2_{L^2(\partial\mathbb{B}_d,d\sigma)} \approx \int_{\partial\mathbb{B}_d}\mathcal{I}_{2s}(\mu)(w)d\mu(w)=\mathcal{E}(\mu,\mathcal{D}_a).$$

Since the energies involved are comparable, so are the capacities by (2.8.2). ☐

2.8.6 Condensers in the plane

When working in the complex plane, there is another potential theoretic notion which will be important in the sequel, that of a condenser.

Definition 2.8.6. Let $B \subset \mathbb{C}$ a Jordan domain and $E, F \subset \overline{B}$ two compact sets. The triplet (B, E, F) is called a condenser with plates E, F and field B.

The capacity of a condenser (B, E, F) is defined as

$$\mathrm{Cap}_B(E, F) := \inf \int_B |\nabla u|^2 dA,$$

where the infimum is taken over all *admissible* functions u which are real valued, continuous in $\overline{B}$ locally Lipschitz continuous in B and equal to 1 on E and 0 on F.

We will be interested only in the case that $B = \mathbb{D}$. It should be noted then that capacity is invariant under the action of the Möbius transformations that preserve $\mathbb{D}$. Also we slightly extend the definition in the case that one of the plates is a countable disjoint union of compact sets in the obvious way. If

$$E = \bigsqcup E_n$$

then

$$\mathrm{Cap}_{\mathbb{D}}(E, F) := \lim_n \mathrm{Cap}_{\mathbb{D}}(\bigsqcup_{k \leq n} E_k, F).$$

The limit exists by the monotonicity of capacity.

Condensers are of interest because in some sense they provide a conformal invariant version of capacity. We shall try to make this clearer. The following observation is well known [7].

Lemma 2.8.7. *Let $\Delta_1(0)$ be the hyperbolic centered at 0 of radius 1 and $E \subset \mathbb{T}$ an at most countable union of closed arcs. Then*

$$C_{\frac{1}{2},2}(E) \approx \mathrm{Cap}_{\mathbb{D}}(\Delta_1(0), E),$$

where the implied constants are absolute.

Therefore, while in general for an automorphism of the unit disc φ, $\varphi(0) = a$, $C_{\frac{1}{2},2}(E) \neq C_{\frac{1}{2},2}(\varphi(E))$ we have that

$$C_{\frac{1}{2},2}(\varphi(E)) \approx \mathrm{Cap}_{\mathbb{D}}(\Delta_1(a), \varphi(E)).$$

Chapter 3

Simply interpolating sequences for the Dirichlet space

3.1 Introduction

In this chapter we revisit the question of simply interpolating sequences for the Dirichlet space.

Recall that in the Dirichlet space the norm of the kernel vectors is

$$\|k_z\|_{\mathcal{D}}^2 = \log \frac{e}{1-|z|^2} =: d(z).$$

Also $d_h(\cdot,\cdot)$ stands for the hyperbolic distance in the unit disc

$$d_h(r,0) = \log_2 \frac{1+r}{1-r}, \;\; 0 < r < 1,$$

and and one can see that

$$d(z) \approx d_h(0,z) + 1.$$

As it has been already discussed in Section 2.6 the first to study interpolation problems in the Dirichlet space have been Bishop [21] and Marshall & Sundberg [60]. Their work, unfortunately, remains unpublished but most of their results can be found also in other sources. As we already know universally interpolating sequences are characterized by the weak separation condition and the Carleson measure condition. In this concrete situation weak separation is equivalent to say [60] that there exists $\varepsilon > 0$ such that

$$d_h(z_i, z_j) \geq \varepsilon(d_h(z_i, 0) + 1), \;\; \forall i \neq j. \qquad (\mathrm{WS}_{\mathcal{D}})$$

The classical characterization of Carleson measures for the Dirichlet space in terms of logarithmic capacity given by Stegenga [81] is the following. We denote by c the logarithmic capacity of compact subsets of

$\mathbb{T}$. We also adopt the notation $S(z)$ for the Carleson box which closest point to the origin is z and $I_z := \overline{S(z)} \cap \mathbb{T}$. Then a measure is Carleson for the Dirichlet space if and only if it satisfies the following *sub-capacitary condition*, i.e. there exists $C_\mu > 0$ such that for any $z_1, \dots z_k \in \mathbb{D}$

$$\mu\Big(\bigcup_{i=1}^{k} S(z_i)\Big) \leq C_\mu c\Big(\bigcup_{i=1}^{k} I_{z_i}\Big). \tag{SC}$$

Already Bishop notes that if the measure associated to a weakly separated sequence $\mathcal{Z} = \{z_i\}$ satisfies the one box sub-capacitary condition ($k = 1$ in (SC)) the sequence is simply interpolating, and he constructs a sequence which is simply but not universally interpolating (soon after it became also clear that the same result is implicit in the work of Marshall and Sundberg).

A later contribution comes from the work of Arcozzi Rochberg and Sawyer [12], which can be found in a published form in [15]. They prove that not only the one box sub-capacitary condition together with weak separation is not necessary for simple interpolation but they even construct simply interpolating sequences $\mathcal{Z}$ such that the associated measure is *infinite*.

To see why this result is somewhat surprising, it helps consider the connection with zero sets in the Dirichlet space. Recall that a sequence $\mathcal{Z}$ is called a *zero set* for the Dirichlet space if there exists $f \in \mathcal{D}$ not identically zero, such that $f(z) = 0, \forall z \in \mathcal{Z}$. A characterization of zero sets is a difficult problem. Nonetheless, every simply interpolating sequence $\{z_i\}$ is automatically a zero set. That is because we can find an $f \in \mathcal{D}$ such that $f(z_0) = 1, f(z_i) = 0, \forall i \geq 1$, then the Dirichlet function $(z - z_0)f(z)$ is not identically zero and it vanishes on $\mathcal{Z}$.

As we saw in Section 2.3, one of the most general sufficient criteria for a sequence $\{z_i\}$ in order to be a zero sequence for the Dirichlet space is the convergence of the sequence

$$\sum_{i=1}^{\infty} \frac{1}{d(z_i)} < +\infty,$$

or in our language, that it has a finite associated measure [1]. Furthermore this result is sharp in the sense that any other sufficient criterion for a zero sequence must depend not only on $\{|z_i|\}$ but also on their argument [62]. So, if a sequence has infinite associate measure is not always clear if it is a zero set, let alone a simply interpolating sequence.

[1] In fact the analogous condition in the Hardy space H^2 is exactly the Blaschke condition which characterized completely the zero sets in the Hardy space

3.1.1 Main results

In the direction of a better understanding of simply interpolating sequences we prove a capacitary characterization of strongly separated sequences and therefore, in light of Corollary 2.6.8, simply interpolating sequences in the Dirichlet space, under the additional assumption that the associated measure is finite. Due to the aforementioned results of Arcozzi Rochberg & Sawyer [12] this does not constitute a full characterization of simply interpolating sequences.

Our result involves an interesting condenser capacitary condition which we will now discuss.

Suppose $z \in \mathbb{D} \setminus \{0\}$, as we saw earlier the Carleson boxes $S(z)$ fit well with the geometry of the Hardy space, but often for the geometry of the Dirichlet space one needs to modify them. Let $0 < \eta \leq 1$ and $z^* := z/|z|$ we define the *blow up* of the Carleson box

$$S^\eta(z) := \{w \in \mathbb{D} : w \in S(z^*(1-(1-|z|)^\eta)), |w| \geq |z|\}.$$

Consistently with our previous notation we write $I_z^\eta := \overline{S^\eta(z) \cap \mathbb{T}}$. Note that everything reduces to the standard situation when $\eta = 1$. We denote by $\Delta_r(z)$ the hyperbolic disc or hyperbolic radius r centered at z. See also Figure 3.1.1.

We can now formulate our condition. We say that a weakly separated sequence $\mathcal{Z} := \{z_i\}$ satisfies the *capacitary condition* if there exist constants $K > 0, \gamma < 1$, depending only on $\mathcal{Z}$ such that,

$$\operatorname{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \bigcup_{z \in S^\gamma(z_i) \cap \mathcal{Z}} I_z\right) \leq \frac{K}{d(z_i)}, \quad \forall z_i \in \mathcal{Z}. \tag{CC}$$

The meaning of this otherwise obscure inequality is physically quite simple. If one considers a condenser with one plate a hyperbolic disc of constant radius around a point of the sequence, and as second plate the union of the intervals I_z for all other points in the sequence in the "vicinity" of z_i, then an electric charge of one unit in one plate, creates an electric field of total energy which bounds asymptotically the hyperbolic distance of z_i to the origin.

The plates of the condenser are marked with grey and bold intervals in Figure 3.1.1.

Theorem 3.1.1. *Let $\{z_i\}$ be a sequence in the unit disc which has finite associated measure, i.e. $\sum_{i=1}^{\infty} 1/d(z_i) < +\infty$. Then, $\{z_i\}$ is simply interpolating for the Dirichlet space iff it is weakly separated and satisfies the capacitary condition.*

The proof of Theorem 3.1.1 is constructive in the sense that for given data we construct explicitly the interpolation operator (i.e. a bounded right

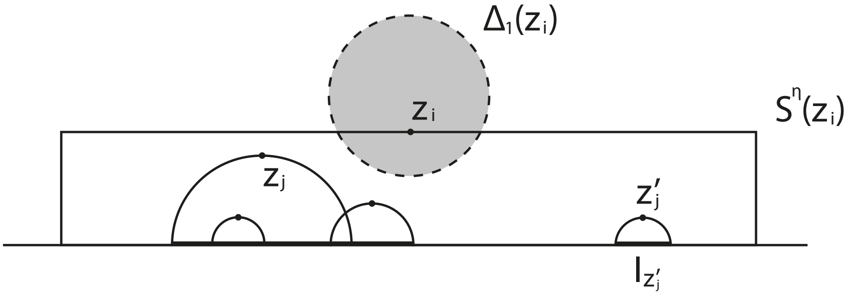

Figure 3.1: A possible configuration of points.

inverse of $T_{\mathcal{D}}$) . In the literature there are two main ways to construct Dirichlet functions which solve interpolation problems, either of universal or simple type. They are both based on some kind of building blocks but the constructions are quite different. The first one, initiated by Böe in [22] was used to solve the universal interpolation problem in Besov spaces, and later exploited further by Arcozzi Rochberg and Sawyer in [12] to give sufficient conditions for simple interpolation in the Dirichlet space. It should be mentioned that traces of Böe's construction can be found in the work of Marshall and Sundberg [60]. The second construction is due to Bishop [21], and makes use of conformal mappings. In this work we combine both approaches and we construct building blocks that have the best of both worlds, in the sense that the relevant error terms arising are easier to control. We should also mention that the abstract approach of Aleman Hartz McCarthy and Richter [4], does not seem to be able to give any advantage in this concrete situation.

Another feature of our construction is that we use an iterative scheme of interpolation which is based on a quantitative version of the Corollary 2.6.8;

$$(SS) \iff (SI).$$

To be more precise we should first quantify the conditions (SS) and (SI). Let $\mathcal{Z}$, as always, be a sequence in the unit disc, we define its *strong separation constant*, denoted by $StrongSep(\mathcal{Z})$ as the infimum of all $C_{\mathcal{Z}} > 0$ such that (SS) holds. Similarly for weak separation, we call weak separation constant the supremum over all ε such that $\mathrm{WS}_{\mathcal{D}}$ holds. If $\mathcal{Z}$ is simply interpolating an application of the closed graph theorem provides a constant $C_{\mathcal{Z}} > 0$ such that for an $\alpha \in \ell^2$ there exists $f \in \mathcal{D}$ such that

$$T_{\mathcal{D}} f = \alpha$$
$$\|f\|_{\mathcal{D}} \leq C_{\mathcal{Z}} \|\alpha\|_{\ell^2}.$$

Again the infimum over all such constants we call it the *simple interpolation constant* of $\mathcal{Z}$ and we write $Int(\mathcal{Z})$. We are justified therefore to call the next theorem a quantitative version of Bishop's Theorem.

Theorem 3.1.2. *Let $K_0 > 0$. If a sequence $\mathcal{Z}$ satisfies $StrongSep(\mathcal{Z}) \leq K_0$, then*

$$Int(\mathcal{Z}) \leq C_{K_0},$$

where C_{K_0} depends only on K_0 and not on $\mathcal{Z}$.

The proof of this theorem depends largely on the original proof of Bishop [21], but it requires a careful extraction of the relevant constants.

A second direction in which we pursue is that of understanding better the one-box subcapacitary condition of Bishop. To this end we introduce a kind of interpolating sequences which is intermediate between simply interpolating and universally interpolating. The definition applies to all RKHS.

Definition 3.1.3. Let $\mathcal{H}$ a RKHS on X with kernel k. Suppose $\{x_i\} \subset X$ is a sequence of points and let G be the associated Grammian. We shall say that $\{x_i\}$ is a $(2, \infty)$-interpolating Sequence if it is simply interpolating and

$$\|G\|^2_{(2,\infty)} := \sup_i \sum_j |G_{ij}|^2 < +\infty,$$

or equivalently that G defines a bounded linear operator from ℓ^2 to ℓ^∞.

It is a classical theorem of Carleson that $(2, \infty)$-interpolating sequences as defined above are the interpolating sequences for H^∞. Our formulation is a disguise of the original theorem.

One other remark, which was the motivation for exploring this kind of interpolation is that Universally interpolating sequences are $(2, \infty)-$ interpolating which are simply interpolating. Therefore they provide an intermediate situation between these two situations.

As we have already mentioned, in the Hardy space (UI) coincide with (SI) sequences, but this need not be the case in other RKHS. In the Dirichlet space there exist simply interpolating sequences with infinite associated measure therefore such sequences they cannot be even $(2, \infty)$-interpolating [13]. The situation in the Dirichlet space can be summarized as follows

$$(UI) \quad \underset{\not\Leftarrow}{\Rightarrow} \quad (2, \infty) - \text{Interpolation} \quad \underset{\not\Leftarrow}{\Rightarrow} \quad (SI).$$

The next theorem, which is suggested by Seip in [78, p. 32][2], provides a geometric characterization of $(2, \infty)$-interpolating sequences for the classical Dirichlet space.

Theorem 3.1.4. *Let $\mathcal{Z} = \{z_i\} \subset \mathbb{D}$. Then $\{z_i\}$ is $(2, \infty)$-interpolating for the Dirichlet space if and only if it is (WS) and the corresponding measure $\mu_{\mathcal{Z}}$ satisfies the single box condition, i.e.,*

$$\exists C > 0,\ \mu_{\mathcal{Z}}(S(I)) \leq C\big(\log \frac{1}{|I|}\big)^{-1},\ \forall\, arc\ I \subset \mathbb{T}. \tag{SB}$$

[2] Seip suggests that the equivalence follows by a calculus argument which is indeed the case. Since we were unable to spot a reference we include a proof for the purpose of completeness.

This theorem can be seen as a more precise version of a theorem of Bishop [21, Theorem 1.3] which essentially says that a sequence satisfying the (SB) condition is simply interpolating for the Dirichlet space.

Corollary 3.1.5. *In the Dirichlet space a sequence is* $(2,\infty)-$*interpolating if and only if it is (WS) and the associated Grammian is* $(2,\infty)$ *bounded.*

The statement above is vacuously true in the Hardy space as well simply because $(2,\infty)-$interpolating sequences coincide with (UI) sequences. Therefore a natural question is in which extend this situation generalizes to reproducing kernel Hilbert spaces with the complete Nevanlinna Pick property. We state our question as a conjecture.

Conjecture 3.1.6. In every RKHS with the complete Nevanlinna Pick property $(2,\infty)$-Interpolation is equivalent to (WS) and $(2,\infty)$ boundedness of the Grammian.

3.1.2 Connections with non analytic interpolation in $H_2(\mathbb{D})$

Next we consider the problem of simple interpolation in the Sobolev space $H_2(\mathbb{D})$, the space of $L^2(\mathbb{D})$ functions on the unit disc with weak partial derivatives of first order also in $L^2(\mathbb{D})$. In this space pointwise evaluations are not well defined, therefore the definition of interpolation has to be somewhat different. We shall say that a sequence $\{z_i\}$ is simply interpolating for $H_2(\mathbb{D})$ if there exists $\varepsilon > 0$ such that for any $\alpha = \{a_i\} \in \ell^2(\mathbb{N})$, there exists $u \in H_2(\mathbb{D})$ such that $u|_{\Delta_\varepsilon(z_i)} \equiv \sqrt{d(z_i)} \cdot a_i$. We choose the weights $d(z_i)$ in the definition of (SI) sequences for $H_2(\mathbb{D})$ in analogy with the holomorphic case.

In this case we have a complete characterization of simply interpolating sequences.

Theorem 3.1.7. *A sequence* $\{z_i\} \subset \mathbb{D}$ *is simply interpolating for* $H_2(\mathbb{D})$ *iff it is weakly separated and satisfies the capacitary condition.*

From this theorem we can derive a useful corollary.

Corollary 3.1.8. *Suppose that a sequence* $\{z_i\}$ *has finite associated measure and that for any* $\alpha \in \ell^2$ *we can find* $u \in H_2(\mathbb{D})$ *such that*

$$u|_{\Delta_\varepsilon}(z_i) \equiv \sqrt{d(z_i)}a_i.$$

Then we can find a holomorphic u *such that* $u(z_i) = \sqrt{d(z_i)}a_i$.

3.1.3 Other results about simple interpolation

In order to illustrate the power of our results we will prove a sufficient condition for simple interpolation which generalizes the so called weak simple condition[3] of Arcozzi Rochberg and Sawyer [12, Theorem A] which in its

[3]Not to be confused with the weak separation condition which in [12] is called just "separation".

turn generalizes the Bishop's one box subcapacitary condition.

In analogy with the weak simple condition of Arcozzi et al, given a sequence $\{z_i\}$ and $\gamma < 1$ we shall say that z_i has *γ-uninterrupted view* of z_j if there exists no other point $z_k \in S^\gamma(z_i)$ in the sequence such that $S^\gamma(z_k) \supseteq S(z_k)$. Therefore the next theorem implies [12, Theorem A] for $\gamma = 1$.

Theorem 3.1.9. *Let $\{z_i\}$ a weakly separated sequence with constant of weak separation $\varepsilon > 0$. Assume it also satisfies*

$$\sum \frac{1}{d(z_j)} \leq \frac{C}{d(z_i)},$$

where the sum is taken over all z_j in the sequence such that z_i has γ - uninterrupted view of z_j with $\gamma \in (1-\varepsilon, 1]$. Then it satisfies the capacitary condition.

In analogy with the case of universal interpolation, a natural property for the capacitary condition that one could ask is to respect unions. More precisely, if $\{z_i\}, \{w_i\}$ are universally interpolating sequences such that their union is weakly separated, then $\{z_i\} \cup \{w_i\}$ is also universally interpolating, simply because the sum of two Carleson measures is a Carleson measure. Furthermore the union of two zero sequences is a zero sequence, although this statement is somewhat more difficult to prove [46]. Here we show that not only this is not true in the case of simply interpolating sequences but we have the following more dramatic failure.

Theorem 3.1.10. *There exist sequences $\{z_i\}, \{w_i\}$ in $\mathbb{D}$ such that $\{z_i\}$ is universally interpolating and $\{w_i\}$ is simply interpolating for the Dirichlet space both with finite associated measures, their union is weakly separated but it is not a simply interpolating sequence.*

3.1.4 Organization of the material

Section 3.2 is a collection of definitions and known results together with some elementary estimates on capacities of condensers that will be used throughout. In Section 3.4 we give the proof of Theorem 3.1.7, that introduces some of the techniques that will be used later without involving the complications of analyticity. In Section 3.5 we present a proof of the quantitative version of Bishop's Theorem . In Section 3.6 using the quantitative version of Bishop's Theorem we provide the proof of Theorem 3.1.1. In Section 3.7 we characterize $(2, \infty)$-interpolating sequences, proving Theorem 3.1.4. Finally in Section 3.8 we give the proofs of Theorems 3.1.9 and 3.1.10.

3.2 Condensers and capacity.

In this section we take a close look to the capacitary condition that appears in Theorem 3.1.1. The basic idea is that the capacitary condition can be state equivalently in terms of logarithmic capacity. This is the content of Proposition 3.2.9.

Another tool we will develop in this chapter is a number of stability results. We would like to know that under certain operations on a condenser the capacity remains essentially the same.

For a set $E \subset \mathbb{T}$ we shall write $c(E)$ for its logarithmic capacity. Recall that from Lemma 2.8.7 we have

$$c(E) \approx \mathrm{Cap}_{\mathbb{D}}(\Delta_1(0), E).$$

In this chapter, when working with condensers, it will often be useful to think of logarithmic capacity approximately as the quantity on the right. Since we will be interested only in the asymptotic behaviour of sets of vanishing logarithmic capacity this is not a real issue.

We now turn to the condensers appearing in the capacitary condition and some variants. Suppose that we have a base point $z \in \mathbb{D}$ and a finite sequence of points $z_1, \dots z_N \in \mathbb{D}$. One can associate a number of condensers to this configuration of points. We are interested in three types of condensers

$$(\mathbb{D}, \Delta_1(z), \bigcup_{j=1}^{N} S(z_j)),$$
$$(\mathbb{D}, \Delta_1(z), \bigcup_{j=1}^{N} \Delta_1(z_j)),$$
$$(\mathbb{D}, \Delta_1(z), \bigcup_{j=1}^{N} I_{z_j}).$$

Some justification is necessary. First of all let us mention that although such type of condensers do not appear explicitly in the literature, one can trace this construction back in the work of Bishop [21, Theorem 1.2], although the condensers appearing there are of "analytic nature" meaning that the admissible functions are required to be analytic. On the other hand, in the work of Arcozzi Rochberg and Sawyer[12] the authors characterize simply interpolating sequences for a Dirichlet type space defined on a tree in terms of a discrete condenser capacity reminiscent of our definition (see the tree capacitary condition [12, p.6]).

Initially we will work with condensers of the third type, but as it turns out, under the separation hypothesis all condensers have comparable capacities.

3.2.1 Condensers and capacity blow up

We proceed now to the first of our stability results. Suppose we are given an arc $I \subset \mathbb{T}, \kappa > 0, 0 < \eta \leq 1$ then we define $\kappa \cdot I^\eta$ as the arc having the same midpoint with I and length $\kappa|I|^\eta$. Then, in general if $G \subset \mathbb{T}$ is an open set we define the "blow up" $\kappa \cdot G^\gamma$ naturally as

$$\kappa \cdot G^\eta := \bigcup_{I \subset G} \kappa \cdot I^\eta.$$

Note that when $\eta < 1$ the "exponential blow up" due to η has a far bigger effect that the "scalar blow up" due to κ. The following observation is due to Bishop [21]. Another proof of this fact using potential theory on trees exists implicitly in [13, Lemma 2.7].

Lemma 3.2.1. *Let $G \subset \mathbb{T}$ an open set $\kappa > 0$, $0 < \eta < 1$. There exists a constant $C_{\kappa,\eta} > 0$ such that*

$$c(\kappa \cdot G^\eta) \leq C_{\kappa,\eta} c(G).$$

In the next lemma $\omega(z, \cdot, \mathbb{D})$ stands for the harmonic measure at z.

Lemma 3.2.2. *Let $I \subset \mathbb{T}$, $z \in \mathbb{D}, |z| \geq 1/2$ and $0 < \eta < 1$. If $|I|^\delta \leq 1 - |z|$ for some $0 < \delta < \eta < 1$ then,*

$$\omega(z, I^\eta, \mathbb{D}) \leq C_{\delta,\eta}\omega(z, I, \mathbb{D})^\alpha,$$

for some $\alpha > 0$ which depends on δ and η but not on I, z. In fact the estimate is true if we choose $\alpha = \frac{\eta-\delta}{1-\delta}$.

Proof. Without loss of generality we can assume that $I = [0, \sigma] := \{e^{i2\pi\theta} : 0 \leq \theta \leq \sigma\}$. Since $I^\eta \subset [0, \sigma^\eta] \cup [\sigma - \sigma^\eta, \sigma] := I^\eta_+ \cup I^\eta_-$ it suffices to prove the inequality only for the interval I^η_+.

Now let $z = re^{i\theta}$ as in the statement. We can write $1 - r = \sigma^\rho$ for some $0 < \rho \leq \delta < \eta$, and $\theta = \sigma^x, x \geq 0$. The standard estimate for the harmonic measure of an arc gives

$$\omega(z, I^\eta, \mathbb{D}) \approx \int_{-\sigma^{x-\rho}}^{\sigma^{\eta-\rho}-\sigma^{x-\rho}} (1 + s^2)^{-1} ds. \tag{3.2.1}$$

And similarly

$$\omega(z, I, \mathbb{D}) \approx \int_{-\sigma^{x-\rho}}^{\sigma^{1-\rho}-\sigma^{x-\rho}} (1 + s^2)^{-1} ds. \tag{3.2.2}$$

We have to distinguish two cases. First consider the case $0 \leq x \leq \rho$. Since $\sigma^{1-\rho} - \sigma^{x-\rho} \leq 0$, estimate (3.2.1) becomes

$$\omega(z, I^\eta, \mathbb{D}) \lesssim \frac{\sigma^{\eta-\rho}}{1 + (\sigma^{\eta-\rho} - \sigma^{x-\rho})^2} \leq \sigma^{\eta+\rho-2x}.$$

In a similar fashion

$$\omega(z, I, \mathbb{D})^\alpha \gtrsim \frac{\sigma^{\alpha(1-\rho)}}{(1+\sigma^{2(x-\rho)})^\alpha} \gtrsim \sigma^{\alpha(1+\rho-2x)}.$$

The last quantity is always bigger than $\sigma^{\eta+\rho-2x}$ if $\alpha = \frac{\eta-\delta}{1-\delta} > 0$.

For the remaining case $x > \rho$, first we estimate $(1+s^2)^{-1}$ by 1 and we get $\omega(z, I_+^\eta, \mathbb{D}) \lesssim \sigma^{\eta-1}$. For the reverse estimate for $\omega(z, I, \mathbb{D})$ we estimate again in the simplest way, because in that case $[-\sigma^{x-\rho}, \sigma^{1-\rho} - \sigma^{x-\rho}] \subset [-1, 1]$, and since $(1+s^2)^{-1} \geq \frac{1}{2}$ on this interval

$$\omega(z, I, \mathbb{D})^\alpha \gtrsim \sigma^{\alpha(1-\eta)} \geq \sigma^{\eta-\rho}.$$

The last inequality is true for all $0 \leq \rho < \delta$ if $\alpha = \frac{\eta-\delta}{1-\delta}$.

□

Proposition 3.2.3. *Let $z, z_i \in \mathbb{D}$, $i \in \mathbb{N}, |z| \geq 1/2$ and suppose that there exist $\alpha > 1, 0 < \beta < 1$ such that $(1-|z_i|)^\beta \leq (1-|z|)^\alpha$. Then,*

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{\infty} I_{z_i}^\beta\Big) \leq C_{\alpha,\beta} \cdot \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{\infty} I_{z_i}\Big).$$

Proof. Let us denote by ϕ_z the disc automorphism which interchanges 0 and z. By lemma 3.2.2, there exist constants $C, \eta > 0$, depending only on α, β, such that $|\phi_z(I_{z_i}^\beta)| \leq C \cdot |\phi_z(I_{z_i})|^\eta$. Since $\phi_z(I_{z_i}) \subset \phi_z(I_{z_i}^\beta)$, we get that $\phi_z(I_{z_i}^\beta) \subset C \cdot \phi_z(I_{z_i})^\eta$.

In this case we can estimate as follows, for $N \in \mathbb{N}$ fixed.

$$\begin{aligned}
\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} I_{z_i}^\beta\Big) &= \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^{N} \phi_z(I_{z_i}^\beta)\Big) \\
&\leq \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^{N} C \cdot \phi_z(I_{z_i})^\eta\Big) \\
&\leq C\, \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^{N} \phi_z(I_{z_i})\Big) \\
&= C\, \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} I_{z_i}\Big).
\end{aligned}$$

The result follows by letting N go to infinity. □

3.2.2 Hyperbolic geometry in the disc and stability of Carleson boxes under automorphisms of the unit disc

One obstacle we will have to overcome when dealing with the capacitary condition is the fact that it involves an intrinsically conformally invariant quantity (the condenser capacity) and a geometric object (Carleson box) which is not defined in terms of hyperbolic geometry of the disc.

A manifestation of this phenomenon is in the following observation. Suppose we consider an arc $I_w \subset \mathbb{T}$ corresponding to a point w in the unit disc. Then in general, under a disc automorphism ϕ_w

$$\phi_z(I_w) \neq I_{\phi_z(w)}.$$

One way to get around this problem is to ask for the point z to be closer to the origin than w is. In such a case we can expect a stability of the geometry of Carleson boxes under a disc automorphism.

Lemma 3.2.4. *Let $z \in \mathbb{D}$ and*

$$G = \bigcup_i I_{w_i}$$

an open set. Suppose also that

$$|z| \leq |w_i|, \ \ \forall i.$$

Then, there exists an absolute constant $\kappa > 0$, such that

$$\frac{1}{\kappa} \cdot \phi_z(G) \subset \bigcup_i I_{\phi_z(w_i)} \subset \kappa \cdot \phi_z(G).$$

Proof. Notice that since an automorphism of the unit disc extends to a homeomorphism on the boundary it suffices to prove the claim when G is a single interval I_w.

It suffices to prove the claim when $|z| \geq 1/2$. Even more, it is always true that $\phi_z(I_w) \cap I_{\phi_z(w)} \neq \emptyset$, therefore our claim will follow if we prove that their lengths are comparable. To this end, consider the case $\zeta \in I_w$ and $|z^* - \zeta| \geq 2\pi(1 - |z|)$. Consequently,

$$\begin{aligned} |z^* - w^*| &\geq |z^* - \zeta| - |\zeta - w^*| \\ &\geq |z^* - \zeta| - \pi(1 - |w|) \\ &\geq |z^* - \zeta| - \pi(1 - |z|) \\ &\geq \pi|z^* - \zeta|/2. \end{aligned}$$

Hence, for $z, w \in \mathbb{D}$ and $\zeta \in I_w$ as before, we have that

$$|1 - z\overline{w}| \approx \max\{1 - |z|, 1 - |w|, |z^* - w^*|\} \gtrsim \max\{1 - |z|, |z^* - \zeta|\} \approx |1 - z\overline{\zeta}|.$$

But this last estimate remains true also in the case $|z^* - \zeta| \leq 2\pi(1-|z|)$.

As a matter of fact, the converse inequalities follows by a similar consideration, examining the cases $|z^*-w^*| \geq 2\pi(1-|w|)$ and $|z^*-w^*| \leq 2\pi(1-|w|)$.

Hence,

$$|\phi_z(I_w)| = \int_{I_w} \frac{1-|z|^2}{|1-\bar{\zeta}z|^2}|d\zeta| \approx \frac{(1-|z|)(1-|w|)}{|1-\overline{w}z|^2} \approx |I_{\phi_z(w)}|.$$

□

3.2.3 Stability of condenser capacity under perturbation of plates

In this point we introduce a tool which proves to be critical for our constructions. We introduce it here because it will come handy in the proof of the next lemmas, but we will use it again in section 3.5 as a building block for our interpolating functions.

First a bit of notation. For an interval $I \subset \mathbb{T}$ we write $S(I)$ for the Carleson box $S(w)$ such that $I_w = I$ and also if $G \subset \mathbb{T}$ is an open set on the circle we use the notation

$$S(G) := \bigcup_{I \subset G} S(I).$$

Let now $G \subset \mathbb{T}$ then there exists an equilibrium measure μ_G for G (as defined for example in [46, p.19]) and an associated holomorphic potential defined as

$$\varphi_G(z) := \int_{\mathbb{T}} \log \frac{e}{1-z\bar{\zeta}} d\mu_G(\zeta).$$

This function has some useful properties.

Proposition 3.2.5. *[32, Lemma 2.3] Let G and φ_G as before, then the following is true.*

1. $|\Im\varphi_G(z)| \leq \frac{\pi c(G)}{2}, \ \forall z \in \mathbb{D}$,
2. $0 \leq \mathrm{Re}(\varphi_G)(z) \leq 1, \ \forall z \in \mathbb{D}$,
3. $|\varphi_G| \leq \frac{\pi}{2}\mathrm{Re}(\varphi_G)$,
4. $|\varphi_G(z)| \geq \varepsilon, \ \forall z \in S(G)$,
5. $\|\varphi_G\|^2_{\mathcal{D}} \leq c_0 c(G)$,
6. *φ_G is univalent.*

The fact that it is univalent comes from the observation that it has a derivative with positive real part.

Lemma 3.2.6. *Suppose that* $(\mathbb{D}, E, F)$ *is a condenser and* $0 < a < b$. *Also* $u \in H_2(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ *such that* $u \leq a$ *in* E, $u \geq b$ *in* F. *Then*

$$\mathrm{Cap}_{\mathbb{D}}(E, F) \leq \frac{1}{(b-a)^2} \int_{\mathbb{D}} |\nabla u|^2 dA.$$

Proof. Define the function

$$g := \min\{\max\{\frac{u-a}{b-a}, 0\}, 1\}.$$

Then g is an admissible function for the condenser $(\mathbb{D}, E, F)$, hence,

$$\mathrm{Cap}_{\mathbb{D}}(E, F) \leq \int_{\mathbb{D}} |\nabla g|^2 dA \leq \frac{1}{(b-a)^2} \int_{\mathbb{D}} |\nabla u|^2 dA.$$

□

Lemma 3.2.7. *Suppose that* $w \in \mathbb{D}, d(w, 0) > 2$ *and* u *is the equilibrium potential for the condenser* $(\mathbb{D}, \Delta_1(0), \Delta_1(w))$. *Then* $u(z) \geq 1/2$ *for every* z *such that* $d(z, w) \leq d(z, 0)$, *where* d *is the hyperbolic distance in* $\mathbb{D}$.

Proof. First consider ϕ an automorphism of the unit disc such that $\phi(0) = -r, \phi(w) = r, r > 0$. By conformal invariance, $v := u \circ \phi$ is the equilibrium potential for the condenser $(\mathbb{D}, \Delta_1(-r), \Delta_1(r))$. Also by symmetry, $v(-x + iy) = 1 - v(x + iy)$, therefore $v(iy) = 1/2$. Suppose now that at some point $z_0 \in \mathbb{D}, \mathrm{Re}(z_0) > 0$, $v(z_0) < 1/2$. In that case the function h defined by

$$h(z) := \begin{cases} v(z), & \text{if } \mathrm{Re}(z) \leq 0, \\ \max\{\frac{1/2 + v(z_0)}{2}, v(z)\}, & \text{if } \mathrm{Re}(z) \geq 0, \end{cases}$$

is admissible for the condenser and has strictly smaller Dirichlet integral, which contradicts the fact that v is the minimizer. □

We can now prove the following.

Proposition 3.2.8. *Suppose that* $z \in \mathbb{D}$ *and* $z_1, \ldots, z_N \in \mathbb{D}$, *such that* $|z_i| \geq |z|$ *and* $d(z, z_i) \geq 2$. *Then,*

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} S(z_i)\Big) \approx \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} \Delta_1(z_i)\Big) \approx \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} I_{z_i}\Big).$$

Where the implied constants are absolute.

Before going into the proof, let us remark that the assumption $d(z, z_i) > 2$ is indeed necessary (although the particular constant is not essential) because otherwise even for $N = 1$ it might happen that the capacity of the first and second condenser goes to infinity while the capacity of the third one remains bounded. Since we are interested in comparability of the capacities only when they tend to zero, this is not a major issue for us.

Proof. Trivially the first capacity is bigger than the third one.

To prove the other estimates we first examine the case $z = 0$. Consider the set $E = \bigcup_{j=1}^N I_{z_j}$. Consider the equilibrium potential φ_E and apply to it lemma 3.2.6 in order to get

$$\begin{aligned}\operatorname{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^N S(z_i)\Big) &\leq \frac{1}{(c - C_0 c(E))^2}\int_{\mathbb{D}} |\nabla f_E(z)|^2 dA(z)\\ &\lesssim c(E)\\ &= \operatorname{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^N I_{z_i}\Big).\end{aligned}$$

Without loss of generality in the above estimate we assumed that $c(E)$ is sufficiently small, otherwise the estimate is trivial.

Before we proceed let us note that by Dirichlet's principle [60] the equilibrium potential u for the condenser $\Big(\Delta_1(z), \bigcup_{i=1}^N \Delta_1(z_i)\Big)$ is harmonic in the domain $\Omega = \mathbb{D} \setminus \{\overline{\Delta_1}(z_1), \overline{\Delta_1}(z_2) \dots, \overline{\Delta_1}(z_N)\}$.

For a fixed $i \in \{1, \dots, N\}$ let u_i be the equilibrium potential for the condenser $\operatorname{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \Delta_1(z_i)\Big)$. Then by the maximum principle

$$u \geq u_i \text{ on } \Omega.$$

By appealing to lemma 3.2.7, we get $u \geq u_i \geq 1/2$ on I_{z_i}. Hence,

$$\operatorname{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^N I_{z_i}\Big) \leq 4 \operatorname{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^N \Delta_1(z_i)\Big).$$

Suppose now that z is not necessarily zero. By the stability lemma 3.2.4 we can find $\kappa > 0$ such that

$$\phi_z(\bigcup_{j=1}^N S(z_j)) \subset \bigcup_{j=1}^N \kappa \cdot S(\phi_z(z_j))$$

Applying this we get

$$\begin{aligned}\operatorname{Cap}_{\mathbb{D}}(\Delta_1(z), \bigcup_{j=1}^N S(z_j)) &\leq \operatorname{Cap}_{\mathbb{D}}(\Delta_1(0), \bigcup_{j=1}^N \kappa \cdot S(\phi_z(z_j)))\\ &\lesssim \operatorname{Cap}_{\mathbb{D}}(\Delta_1(0), \bigcup_{j=1}^N \kappa \cdot I_{\phi_z(z_j)})\\ &\lesssim \operatorname{Cap}_{\mathbb{D}}(\Delta_1(0), \bigcup_{j=1}^N \phi_z(I_{z_j}))\\ &= \operatorname{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^N I_{z_i}\Big).\end{aligned}$$

The remaining estimates

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} I_{z_i}\Big) \lesssim \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} \Delta_1(z_i)\Big)$$
$$\lesssim \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{j=1}^{N} S(z_j)\Big).$$

can be handled in much the same way, therefore the proof will be omitted. □

The following proposition allows us to express the condenser capacity appearing in Theorem 3.1.1 in terms of logarithmic capacity.

Proposition 3.2.9. *Suppose that $z, z_1, \ldots z_N \in \mathbb{D}$ such that $1 - |z_i| \leq (1 - |z|)/2$. Then*

$$c\Big(\bigcup_{i=1}^{N} I_{\phi_z(z_i)}\Big) \approx \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N} I_{z_i}\Big),$$

where the implied constants are absolute.

Proof. We have that

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z), \bigcup_{i=1}^{N}\Big) = \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^{N} \phi_z(I_{z_i})\Big)$$
$$\approx \mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(0), \bigcup_{i=1}^{N} I_{\phi_z(z_i)}\Big)$$
$$= c\Big(\bigcup_{i=1}^{N} I_{\phi_z(z_i)}\Big).$$

The second estimate is justified with an argument identical to the one in the proof of Proposition 3.2.8. □

3.2.4 Strong separation and condensers capacity

At this point let us introduce the following notation. If $\mathcal{Z}$ is a sequence and $\gamma < 1$ we write

$$\mathcal{V}_\gamma(z) := \{z_j \in \mathcal{Z} : S^\gamma(z) \cap S^\gamma(z_j) \neq \emptyset, |z_j| \geq |z|\}.$$

This is a slightly bigger neighborhood of points than $S^\gamma(z) \cap \mathcal{Z}$ but nonetheless triangle inequality gives

$$\mathcal{V}_\gamma(z) \subset S^\gamma(2I_z)$$

As a consequence we get that the capacitary condition does not change substantially if we ask a bit more.

Lemma 3.2.10. *If a weakly separated sequence satisfies the capacitary condition with constants $K, 1 > \gamma > 0$ then excluding a finite number of points, it satisfies the condition*

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z_i), \bigcup_{z_j\in\mathcal{V}_\eta(z_i)} I_{z_j}\Big) \le \frac{K}{d(z_i)},$$

for $1 > \eta > \gamma$.

Proof. As already noted, excluding possibly a finite number of points close to the origin

$$\mathcal{V}_\eta(z_i) \subset S(2I^\eta_{z_i}) \subset S^\gamma(z_i).$$

□

The following is a simple lemma about the geometry of weakly separated sequences. A proof of it (actually a stronger statement) can be found [60]. We provide a proof of the part that we are going to use for completeness.

Lemma 3.2.11. *Suppose that $\{z_i\}$ is a weakly separated sequence, with separation constant ε. Then if $1 > \gamma > 1-\varepsilon$, with finite exceptions, if $z_j \in \mathcal{V}_\gamma(z_i)$*

$$(1-|z_j|)^\gamma \le \frac{1-|z_i|}{2}.$$

Proof. The proof of this lemma is nothing more than the simple geometric fact that for $z \in \mathbb{D}$ and $c_0 > 0$ the region

$$\{w \in \mathbb{D} : |w| \ge |z|, 2(1-|w|)^\gamma \le 1-|z|,\ \ |z^* - w^*| \le c_0(1-|z|)^\gamma\}$$

is contained eventually (for z close to the boundary) in any hyperbolic disc with center z and radius $(d_h(z,0)+1)/\varepsilon$, since $1/\varepsilon > 1/(1-\gamma)$. □

Proposition 3.2.12. *Suppose that $\{z_i\}$ is a $K-$ strongly separated sequence in the unit disc. If $\gamma < 1$ as in Lemma 3.2.11, there exists $C > 0$ depending only on K such that,*

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z_i), \bigcup_{z_j\in\mathcal{V}_\gamma(z_i)} I_{z_j}\Big) \le \frac{C_{K,\gamma}}{d(z_i)},$$

for all but finitely many z_i.

Proof. For a fixed point z_i then there exists a function $f_i \in \mathcal{D}$, such that $f_i(z_i) = 1, f_i(z_j) = 0$ for $i \ne j$ and $\|f_i\|^2_{\mathcal{D}} \le K/d(z_i)$. By the standard oscillation estimate for Dirichlet functions $|f(z)-f(w)| \le \|f\|_{\mathcal{D}} C_0\sqrt{d(z,w)}$, we have that

$$|f(w)-1| \le c_0/\sqrt{d(z_i,0)},\ \ w \in \Delta_1(z_i),$$

and

$$|f(w)| \leq c_0/\sqrt{d(z_i,0)},\ w \in \Delta_1(z_j),\ j \neq i.$$

Therefore by Lemma 3.2.6 applied to f_i

$$\mathrm{Cap}_{\mathbb{D}}(\Delta_1(z_i), \bigcup_{j\neq i} \Delta_1(z_j)) \leq \frac{1}{(1-2C_0/\sqrt{d(z_i)})^2}\int_{\mathbb{D}} |f_i'|^2 dA.$$

Which gives the estimate

$$\mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \bigcup_{j\neq i} \Delta_1(z_j)\right) \leq 2K/d(z_i),$$

for all but finite many z_i.

Now notice that if again we exclude again from the sequence a finite number of points, because of Lemma 3.2.11, it is true that $z_j \in \mathcal{V}_\gamma(z_i)$ implies that $(1-|z_j|) \leq (1-|z_i|)/2$. And therefore the result follows from Lemma 3.2.8. The remaining finite points can be included if we choose C large enough.

□

3.3 Constructing interpolating building blocks

As we have already mentioned our interpolation theorems are constructive in the sense that interpolating functions are constructed by pasting together functions that behave essentially as "bump" functions. If one requires these bump functions to be holomorphic is clear that they cannot vanish in any non discrete set, therefore our goal is to make them very small outside a certain region, in a sense to be made precise . Here we will preset two complementary constructions. We start with the first one due to Bøe [22, Lemma 4.1].

Lemma 3.3.1 (Böe [22])**.** *Suppose that $z \in \mathbb{D}$ and $\alpha < 1$. Then there exists a function $f = f_{z,\alpha}$, such that*

1. $f(z) = 1$,
2. $\|f\|_{\mathcal{D}}^2 \leq C_\alpha/d(z)$,
3. $\|f\|_\infty \leq C_\alpha$,
4. $|f(w)| \leq C_\alpha e^{-6d(z_i)}$ *and,*
5. $|f'(w)| \leq C_\alpha e^{-6d(z_i)}$ *for all* $w \notin S^\alpha(z)$.

We will refer to the function $f_{z,\alpha}$ as the Böe's function associated to z and $S^\alpha(z)$.

The second construction is essentially due to Bishop [21, Lemma 2.13]. Here we give a construction which is based on holomorphic potentials. Bishop's original idea was to use conformal mapping and extremal distance techniques based on a result of Marshall and Garnett [49, Theorem 4.1]. We shall do a very similar construction but based on logarithmic potentials instead. In many ways the two constructions are equivalent but our has some slight advantages. First, using the machinery developed in Section 3.2 we can prove a conformal invariant version of the construction and secondly we are able to control the dependence of our constants on the parameters which is crucial for proving the quantitative estimate that Theorem 3.1.7 requires.

We start with a simple lemma for harmonic measure.

Lemma 3.3.2. *Let I be a closed arc in $\mathbb{T}$. For $0 < \eta < 1$ we have the following elementary estimate*

$$\omega(z, \mathbb{T} \setminus I^\eta, \mathbb{D}) = \int_{\mathbb{T}\setminus I^\eta} \frac{1-|z|^2}{|\zeta - z|^2} |d\zeta| \le C_\eta \cdot c(I),$$

for $z \in S(I)$.

Proof. Without loss of generality $S(I) \subset \{\zeta = re^{i\theta} \in \mathbb{D} : 0 \le \theta \le |I|, 1-r \le |I|\}$. In this case we have

$$\omega(re^{it}, \mathbb{T}\setminus I^\eta, \mathbb{D}) = \frac{1}{2\pi}\int_{|I|^\eta}^{2\pi} \frac{1-r^2}{|e^{is} - re^{it}|^2} ds \le C \int_{|I|^\eta}^{2\pi} \frac{1-r}{|s-t|^2} ds.$$

This last integral can be seen to be of the right magnitude

$$\int_{|I|^\eta}^{2\pi} \frac{1-r}{|s-t|^2} ds \le C\frac{1-r}{|I|^\eta - t} \le C \frac{|I|}{|I|^\eta - |I|} \le C|I|^{1-\eta} \le Cc(I).$$

□

The following lemma is essentially the conformal invariant version of [21, Lemma 2.13].

Lemma 3.3.3. *Let $z, z_1, z_2, \dots z_N \in \mathbb{D}$, such that $1 - |z_i| \le (1-|z|)/2$, and*

$$\mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z), \bigcup_{i=1}^N I_{z_i}\right) \le K/d(z).$$

Where $K \le K_0$. Then there exists a constant C, depending only on K_0, and a function $g_z \in \mathcal{D}$ such that

1. $g_z(z) = 1$,
2. $\int_{\mathbb{D}} |g_z'|^2 dxdy \leq \frac{C}{d(z)}$,
3. $\|g_z\|_\infty \leq C$
4. $|g_z(w)| \leq Ce^{-6d(z)}$, $w \in S(F)$,
5. $\int_{S(F)} |g_z'|^2 dA \leq Ce^{-6d(z_i)}$.

Proof. Throughout the proof C denotes a positive constant depending only on K_0. If a constant depends on some other parameter we will denote it by an index.

Fix some arbitrary $\eta < 1$, and apply to our assumption first Proposition 3.2.9 and then Lemma 3.2.1 with parameter η to arrive at

$$c(\bigcup_{i=1}^{\infty} \phi_z(I_{z_i})^\eta) \leq \frac{C}{d(z)}.$$

Set $E := \bigcup_{i=1}^{N} \phi_z(I_{z_i})^\eta$. Let also φ_E the holomorphic equilibrium potential associated to E and $\psi_E := 1 - (1 - c(E))\varphi_E$. Our building block will be the function

$$f_E := e^{-\frac{A}{\psi_E}},$$

where $A > 0$ is a constant to be specified. By the properties of holomorphic potentials, it is clear that

$$\psi_E(z) \in [c(E), 1] \times [-\frac{\pi c(E)}{2}, \frac{\pi c(E)}{2}], \ \ \forall z \in \mathbb{D}.$$

At this point some elementary euclidean geometry on the image of ψ_E gives

$$|\psi_E| \leq \frac{\pi}{2} \operatorname{Re} \psi_E.$$

This is the crucial estimate that will allow us to derive the desired properties of f_E.

Clearly f_E is bounded by 1 because the real part of the exponent is positive.

The Dirichlet integral of f_E can be estimated as follows

$$\begin{aligned}
\int_{\mathbb{D}} |f_E'|^2 dxdy &= A^2 \int_{\mathbb{D}} \Big|\frac{\psi_E'}{\psi_E^2}\Big|^2 e^{-2A \operatorname{Re} \psi_E / |\psi_E|^2} dA \\
&\leq A^2 \int_{\mathbb{D}} \frac{e^{-\pi A/|\psi_E|}}{|\psi_E|^4} |\psi_E'|^2 dA \\
&\leq \Big(\frac{4}{\pi e}\Big)^4 \frac{1}{A^2} \int_{\mathcal{D}} |\psi_E'|^2 dA \\
&\leq c_0 \frac{c(E)}{A^2}.
\end{aligned}$$

With c_0 absolute positive constant.

On the other hand the value of f_E at the origin remains bounded below.

$$|f_E(0)| = e^{-\frac{A}{1-(1-c(E))c(E)}} \geq e^{-\frac{4A}{3}}.$$

Suppose now that $w \in \bigcup_{i=1}^N S(\phi_z(I_{z_i}))$. Let $J_i = \phi_z(I_{z_i})$. Then $w \in S(J_i)$ for some i, recalling Lemma, 3.3.2

$$\begin{aligned}
\operatorname{Re}\psi_E(w) &= \int_{\mathbb{T}} \frac{1-|w|^2}{|\zeta-w|^2} \operatorname{Re}\psi_E(\zeta)|d\zeta| \\
&\leq \int_{\mathbb{T}\setminus J_i^\eta} \frac{1-|w|^2}{|\zeta-w|^2}|d\zeta| + c(E)|J_i^\eta| \\
&= \omega(w, \mathbb{T}\setminus J_i^\eta, \mathbb{D}) + c(E)|J_i^\eta| \\
&\leq c_0 c(J_i) \\
&\leq c_0 c(E) \\
&\leq \frac{C}{d(z)}.
\end{aligned}$$

Hence, $|f_E(w)| \leq e^{-\frac{Ad(z)}{2C}}$.

Finally, using the conformality of logarithmic potentials,

$$\begin{aligned}
\int_{\bigcup_{i=1}^N J_i} |f_E'|^2 dA &= A^2 \int_{\bigcup_{i=1}^N J_i} \frac{|\psi_E'|^2}{|\psi_E|^4} e^{-2A\operatorname{Re}\psi_E/|\psi_E|^2} dA(z) \\
&\leq A^2 \int_{\bigcup_{i=1}^N \psi_E(J_i)} \frac{1}{|z|^4} e^{-2A\operatorname{Re}z/|z|^2} dA(z) \\
&\leq A^2 \int_0^{c(E)} \int_{c(E)}^{Cc(E)} \frac{e^{-A/x}}{x^4} dxdy \\
&\leq C_A e^{-\frac{Ad(z)}{C}}.
\end{aligned}$$

Set $A = 12C$.

We claim that $g_z = f_E \circ \phi_z / f(0)$ satisfies the required conditions. Properties (1)-(3) and (5) are invariant under Möbius transformations, we get property (4) after an application of Lemma 3.2.4. □

3.4 Simple Interpolation in $H_2(\mathbb{D})$

The idea of the proof is to interpolate every value separately with functions in $H_2(\mathbb{D})$ that have disjoint supports. The simple minded idea to take functions constant on $\Delta_1(z_i)$ that vanish outside a bigger hyporbolic disc does not work, so we have to construct the disjoint regions in a slightly more sophisticated way.

After a series of elementary lemmas we will proceed to the proof.

Let γ as in Lemma 3.2.11. Then let us define the regions S_i associated to the sequence

$$S_i := S^\gamma(z_i) \setminus \bigcup_{z_j \in \mathcal{V}_\gamma(z_i)} S^\gamma(z_j).$$

Lemma 3.4.1. *The regions S_i are pairwise disjoint and, for all but finitely many z_i, we that $\Delta_1(z_i) \subset S_i$ and $\Delta_1(z_j) \subset \mathbb{D} \setminus S_i$ for all $j \neq i$.*

Proof. Let $i \neq j$. Without loss of generality $|z_j| \geq |z_i|$. Hence, either $S^\gamma(z_i) \cap S^\gamma(z_j) = \emptyset$ or $z_j \in \mathcal{V}_\gamma(z_i)$. In both cases $S_i \cap S_j = \emptyset$.

From Lemma 3.2.11 it follows that $\Delta_1(z_i) \subset S_i$, if z_i is sufficiently close to the boundary. Since S_i are pairwise disjoint, $\Delta_1(z_j) \subset \mathbb{D} \setminus S_i$, for any $j \neq i$. □

Lemma 3.4.2. *A sequence $\{z_i\} \subset \mathbb{D}$ is simply interpolating for $H_2(\mathbb{D})$ if a cofinite subsequence of it is.*

Proof. It suffice to show that if $\{z_i\}_{i=1}^\infty$ is a simply interpolating sequence then $\{z_i\}_{i=0}^\infty$ is. Fix $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon}(z_i)$ are pairwise disjoint, and such that for all $\alpha = \{a_i\} \in \ell^2$ there exists $u \in H_2(\mathbb{D})$ such that $u|_{\Delta_\varepsilon(z_i)} \equiv \sqrt{d(z_i)}a_i, i \geq 1$. Let also $\varepsilon' > 0$ such that $\overline{\Delta_\varepsilon}(z_0) \subset \Delta_{\varepsilon'}(z_0)$ and $\overline{\Delta_{\varepsilon'}} \cap \bigcup_{i=1}^\infty \overline{\Delta_\varepsilon}(z_i) = \emptyset$. Then there exists $\xi \in C^\infty(\mathbb{D})$, such that $\xi \equiv 0$ on $\Delta_\varepsilon(z_0)$ and $\xi \equiv 1$ on $\mathbb{D} \setminus \Delta_{\varepsilon'}(z_0)$. Let $\alpha = \{a_i\}_{i=0}^\infty \in \ell^2$ and u as before. The function

$$v := \xi u + a_0(1 - \xi),$$

is the interpolating function for the sequence $\{z_i\}_{i=0}^\infty$ and the data $\{a_i\}_{i=0}^\infty$. □

Theorem 3.4.3. *A sequence $\{z_i\} \subset \mathbb{D}$ is simply interpolating for $H_2(\mathbb{D})$ iff it is weakly separated and satisfies the capacitary condition.*

Proof. We will start with the more involved direction which is the sufficiency of the conditions in the statement. Notice that if the capacitary condition is satisfies for some $\alpha < 1$, then it is satisfied for all $\gamma, \alpha < \gamma < 1$. Therefore we can assume that it is satisfied for γ as large as the one in Lemma 3.2.11. Assume without loss of generality that $\Delta_1(z_i) \cap \Delta_1(z_j) = \emptyset, i \neq j$. The estimates that we state next might fail for a finite number of points in our sequence but in that case Lemma 3.4.2 allows us to initially disregard any finite number of points.

Then suppose that $f_i := f_{z_i,\gamma}$ is Böe's function for z_i and $S^\gamma(z_i)$. There exists a constant $C_0 > 0$ such that $|f_i(z)| \leq C_0 e^{-6d(z_i)}$, for all $z \notin S^\gamma(z_i)$ and $|1 - f_i(z)| \leq C_0/d(z_i), z \in \Delta_1(z_i)$. Set

$$u_i := \min\{\max\{\frac{|f_i| - C_0 e^{-6d(z_i)}}{1 - C_0/d(z_i) - C_0 e^{-6d(z_i)}}, 0\}1\}.$$

The function just constructed satisfies $u_i|_{\Delta_1(z_i)} \equiv 1$, $u_i|_{\mathbb{D}\setminus S^\gamma(z_i)} \equiv 0$, $\|u\|^2_{H_2(\mathbb{D})} \le C/d(z_i)$ and $\|u_i\|_\infty \le C$.

Next we apply Lemma 3.2.8 to the stronger version of the capacitary condition in Lemma 3.2.10 and we arrive at

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z_i), \bigcup_{z_j\in\mathcal{V}_\gamma(z_i)} S^\gamma(z_j)\Big) \le \frac{K}{d(z_i)}$$

Hence by definition of condenser capacity there exists $v_i \in H_2(\mathbb{D})$ such that $v_i|_{\Delta_1(z_i)} \equiv 1$, $v_i|_{S^\gamma(z_j)} \equiv 0$ for all $z_j \in \mathcal{V}_\gamma(z_i)$ and $\|v_i\|^2_{H_2(\mathbb{D})} \le C/d(z_i)$. Without loss of generality we can also assume that $\|v_i\|_\infty \le 1$

Our interpolation building blocks will be the functions $w_i := u_i \cdot v_i$. Notice that by construction $\operatorname{supp} w_i \subset S_i$, hence, $w_i|_{\Delta_1(z_j)} \equiv \delta_{ij}$ by Lemma 3.4.1. Furthermore $\|w_i\|^2_{H_2(\mathbb{D})} \le C/d(z_i)$ and $\|w_i\|_\infty \le C$.

This observation clearly suggests that, if $\alpha = \{a_i\} \in \ell^2(\mathbb{N})$, the obvious candidate for interpolation is the function $F := \sum_{i=1}^\infty a_i\sqrt{d(z_i)}w_i$. It takes the right values on hyperbolic discs $\Delta_1(z_i)$. It remains to show that is is actually in $H_2(\mathbb{D})$.

Let $N \in \mathbb{N}$. $F_N := \sum_{i=1}^N a_i\sqrt{d(z_i)}w_i$. Then,

$$\begin{aligned}
&\int_{\mathbb{D}} |\nabla F_N(z)|^2 + |F_N|^2(z)dA(z) \\
&\quad\le \sum_{i=1}^N |a_i|^2 d(z_i)\int_{S_i} |\nabla w_i(z)|^2 dA(z) + \sum_{i=1}^N |a_i|^2 d(z_i)\int_{S_i} |w_i(z)|^2 dA(z) \\
&\quad\le C\sum_{i=1}^N |a_i|^2 d(z_i)\|w_i\|^2_{H_2(\mathbb{D})} + \sum_{i=1}^N |a_i|^2 d(z_i)\|w_i\|^2_\infty |S_i| \\
&\quad\le C\Big(\sum_{i=1}^N |a_i|^2 + \sum_{i=1}^N |a_i|^2 d(z_i)(1-|z_i|)^\gamma\Big) \\
&\quad\le C\sum_{i=1}^\infty |a_i|^2.
\end{aligned}$$

Now we turn to the necessity of the conditions. Without loss of generality $0 \in \{z_i\}$ and $\varepsilon = 1$. We should notice that also here there exists a weighted restriction operator, defined on the subspace of $H_2(\mathbb{D})$ of functions constant on hyperbolic discs $\Delta_1(z_i)$. Hence if a sequence is simply interpolating as in the Dirichlet space we can solve the interpolation problem with "norm control", meaning that we can find $u \in H_2(\mathbb{D})$ such that

$$u|_{\Delta_1(z_i)} \equiv \sqrt{d(z_i)}a_i$$
$$\|u\|^2_{H_2(\mathbb{D})} \le C\|a\|^2_{\ell^2}.$$

Considering such a u_j which interpolates $\{\delta_{ij}\}_i$, then u is also an admissible function for the condenser $\big(\Delta_1(z_i), \Delta_1(z_j)\big)$, for any $i \neq j$. It is immediate that

$$\mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_j), \Delta_1(z_i)\right) \leq \|u\|^2_{H_2(\mathbb{D})} \leq \frac{C}{d(z_j)}$$

Since also by conformal invariance of condenser capacity we have

$$\mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \Delta_1(z_j)\right) \approx \frac{1}{\log \frac{1}{|I_{\phi_{z_i}(z_j)}|}} \approx \frac{1}{d(z_i, z_j)}$$

the sequence is weakly separated. The capacitary condition then follows by the definition of condenser capacity and Lemma 3.2.8.

□

3.5 A quantitative version of Bishop's theorem

We are now in a position to prove Theorem 3.1.2. The only substantial difference with the proof of the non holomorphic case is that the function v_i which in the proof of Theorem 3.1.7 (that exist by our assumptions on the condenser capacity) they can no longer be used since they are not holomorphic. Their role will be played by the functions constructed in Lemma 3.3.3.

Proof of Theorem 3.1.2. We will denote by C a general constant depending only on K_0. Suppose that $\{z_i\}$ is $K-$ strongly separated. Let us exclude initially a finite of points from the sequence such that Lemmas 3.2.11 and Propositions 3.2.3, 3.2.12 apply. We can always add these points in the sequence in the end.w

In our construction we will need three types pf building blocks. The functions constructed by Bishop, Bøe and the sequences of functions guaranteed by the strong separation hypothesis. First we perform a simple trick so that the sequence of functions coming from strong separation are uniformly bounded in modulus. We know that there exist multipliers $m_i \in \mathcal{M}(\mathcal{D})$ such that $\|m_i\|_{\mathcal{M}(\mathcal{D})} \leq K$ and $m_i(z_j) = \delta_{ij}$. Consider the functions $f_i := \frac{m_i k_{z_i}}{d(z_i)}$. It is immediate that $\|f_i\|^2_{\mathcal{D}} \leq K/d(z_i), f_i(z_j) = \delta_{ij}$ and $\|f_i\|_\infty \leq C$.

The sequence is also weakly separated by a constant $\varepsilon = \varepsilon(K) > 0$. Let $\gamma = 1 - \varepsilon/4$ so that Lemma 3.2.11 applies. We know that $\mathcal{Z}$ satisfies the condition in Proposition 3.2.12 for γ as defined here, hence by applying Proposition 3.2.3 (for $\beta = \gamma$ and $\alpha = 1 + \varepsilon/4$) we get

$$\mathrm{Cap}_{\mathbb{D}}\Big(\Delta_1(z_i), \bigcup_{z_j \in \mathcal{V}_\gamma(z_i)} I^\gamma_{z_j}\Big) \leq \frac{C}{d(z_i)}.$$

Let now g_i be the functions that we get if we apply Lemma 3.3.3 to the condenser $\left(\Delta_1(z_i), \bigcup_{\mathcal{V}_\gamma(z_i)} I^\gamma_{z_j}\right)$. Finally, let h_i be Böe's function associated to z_i and $S^\gamma(z_i)$. Multiply these functions together to get $u_i := f_i g_i h_i$.

Suppose we are given a sequence $\alpha = \{a_i\} \in \ell^2(\mathbb{N})$. As in the non holomorphic case the obvious choice for the interpolating function would be $F := \sum_{i=1}^\infty a_i \sqrt{d(z_i)} u_i$. At least formally F assumes the correct values, the problem now being that u_i do not have disjoint supports. Nevertheless u_i are small outside the regions S_i, as defined in Lemma 3.4.1, in the following sense. Assume $z \notin S_i$, then $|u_i(z)| \le Ce^{-6d(z_i)}$, and if $S(E_i) := \bigcup_{z_j \in \mathcal{V}_\gamma(z_i)} S^\gamma(z_j)$

$$\begin{aligned}
\int_{S(E_i)} |u_i'|^2 dA &\le 3\int_{S(E_i)} |f_i' g_i h_i|^2 dA \\
&\quad + 3\int_{S(E_i)} |f_i g_i' h_i|^2 dA + 3\int_{S(E_i)} |f_i g_i h_i'|^2 dA \\
&\le Ce^{-6d(z_i)} \int_{S(E_i)} |f_i'|^2 dA \\
&\quad + C\int_{S(E_i)} |g_i'|^2 dA + Ce^{-6d(z_i)} \int_{S(E_i)} |h_i'|^2 dA \\
&\le Ce^{-6d(z_i)}.
\end{aligned}$$

In the same way

$$\int_{\mathbb{D}\setminus S^\gamma(z_i)} |u_i'|^2 dA \le Ce^{-6d(z_i)}.$$

Set $F_N = \sum_{i=1}^N a_i \sqrt{d(z_i)} u_i$. And estimate as follows

$$\begin{aligned}
\int_{\mathbb{D}} |F_N'|^2 dA &\le 2\sum_{i=1}^N |a_i|^2 d(z_i) \int_{S_i} |u_i'|^2 dA + 2\Big(\sum_{i=1}^N |a_i|\sqrt{d(z_i)}\Big[\int_{\mathbb{D}\setminus S_i} |u_i'|^2 dA\Big]^{\frac{1}{2}}\Big)^2 \\
&\le 2\sum_{i=1}^N |a_i|^2 d(z_i) \int_{\mathbb{D}} |u_i'|^2 dA + c\Big(\sum_{i=1}^N |a_i|\sqrt{d(z_i)} e^{-6d(z_i)}\Big)^2 \\
&\le C\sum_{i=1}^N |a_i|^2 + C\sum_{i=1}^N |a_i|^2 \sum_{i=1}^N d(z_i) e^{-12d(z_i)} \\
&\le C\sum_{i=1}^\infty |a_i|^2.
\end{aligned}$$

The constant C above is independent of N because every weakly separated sequence satisfies $\sum_{i=1}^\infty d(z_i)e^{-12d(z_i)} < +\infty$ (see for example [12]). Hence, $\|F_N\|_{\mathcal{D}} \le C\sum_{i=1}^\infty |a_i|^2$. By choosing a *weak*$-*$ cluster point of the sequence we have a function f which solves the interpolation problem. Since C depends only on K_0 the interpolation constant can be chosen uniformly, as in the statement. □

3.6 Simple interpolation in $\mathcal{D}$ for finite measure sequences

The necessity of the capacitary condition comes from Proposition 3.8.1 together with Lemma 3.2.9. The other direction follows from the next proposition.

Proposition 3.6.1. *If $\{z_i\}$ has finite associated measure, then,*

$$(WS) + (CC) \implies (SI).$$

Proof. Since this proof requires an inductive argument on finite subsets of the sequence we will be more careful with our constants.

Let $\{z_i\}$ be a sequence as in the statement. We can choose a constant $C_0 > 1$, which depends only on the sequence, that is large enough such that for all $z_i \in \{z_i\}$ there exists $f_i \in \mathcal{D}$ with $\|f_i\|^2_{\mathcal{D}} \leq C_0/d(z_i)$, $f_i(z_i) = 1$ and $|f_i(z_j)|^2 \leq C_0 e^{-d(z_i)}$ if $j \neq i$. The functions f_i are constructed by multiplying Böe's function $f_{z_i,\gamma}$ with the function g_{z_i} from Lemma 3.3.3. Also, by the quantitative version of Bishop's Theorem, there exists a constant $C_1 > 1$ such that for every sequence of points $\mathcal{E} \subset \mathbb{D}$ which is $K-$ strongly separated, $K \leq 4C_0 + 1$, satisfies $\mathrm{Int}(\mathcal{E}) \leq C_1$. By deleting a finite number of points in the sequence we can ensure that

$$e^{-d(z_i)} \leq \frac{(1 - 1/\sqrt{2})^2}{4C_1C_0^2} \frac{1}{d(z_i)}, \text{ for all } i \in \mathbb{N},$$

and

$$\sum_{j=1}^{\infty} \frac{1}{d(z_j)} \leq 1.$$

For some $N \in \mathbb{N}$ set

$$M_N := \sup_{\substack{\mathcal{E} \subset \{z_i\} \\ |\mathcal{E}| = N}} \mathrm{StrongSep}(\mathcal{E}), \quad A_N := \sup_{\substack{\mathcal{E} \subset \{z_i\} \\ |\mathcal{E}| = N}} \mathrm{Int}(\mathcal{E}). \tag{3.6.1}$$

Notice that $M_1 \leq 1$, but *a priori* we don't even know if $M_N < \infty$ for $N > 1$. We claim that $M_N \leq 4C_0 + 1$, for all $N \geq 1$. Suppose the statement is true for some $N \geq 1$. Consider any $\mathcal{E} \subset \{z_i\}$ such that $|\mathcal{E}| = N + 1$ and fix some $z_i \in \mathcal{E}$. Let f_i be the function as defined above. By the induction hypothesis we have $A_N \leq C_1$, hence, there exists $g_i \in \mathcal{D}$ such that $g_i(w) + f_i(w) = 0$, for all $w \in \mathcal{E} \setminus \{z_i\}$ with

$$\|g_i\|^2_{\mathcal{D}} \leq C_1 \sum_{w \in \mathcal{E} \setminus \{z_i\}} \frac{|f_i(w)|^2}{d(w)} \leq C_0C_1e^{-d(z_i)} \sum_{j=1}^{\infty} \frac{1}{d(z_j)} \leq \frac{1}{4d(z_i)}.$$

Furthermore,

$$\begin{aligned}|g_i(z_i)|^2 &\le \|g_i\|_{\mathcal{D}}^2\|k_{z_i}\|_{\mathcal{D}}^2 \\ &\le C_0C_1e^{-d(z_i)}C_0d(z_i) \\ &\le C_1C_0^2d(z_i)e^{-d(z_i)} \\ &\le (1-1/\sqrt{2})^2.\end{aligned}$$

Finally, consider the function $h_i := (f_i+g_i)/(f_i(z_i)+g_i(z_i))$. By definition $h_i(z_i) = 1$ and $h_i(w) = 0$ for all $w \in \mathcal{E} \setminus \{z_i\}$. Also,

$$\|h_i\|_{\mathcal{D}}^2 = \frac{\|f_i+g_i\|_{\mathcal{D}}^2}{|f_i(z_i)+g_i(z_i)|^2} \le 4\|f_i\|_{\mathcal{D}}^2 + 4\|g_i\|_{\mathcal{D}}^2 \le \frac{4C_0}{d(z_i)} + \frac{1}{d(z_i)}.$$

Since z_i was arbitrary by definition of M_{N+1}, we have that $M_{N+1} \le 4C_0+1$. The induction is complete and it gives that $\limsup_{N\to\infty} M_N \le 4C_0+1 < \infty$. Therefore $\{z_i\}$ is strongly separated.

□

3.7 $(2,\infty)$ Interpolating sequences

Proof of Theorem 3.1.4. The proof is nothing more than an elementary but careful computation. Let us start by proving that
$(WS) + (SB) \implies (2,\infty)-$Interpolation.

Recall that $\Gamma_\sigma(z^*)$ is the Stolz angle at $z^* = z/|z|$. [4]

Fix $z_i \in \mathcal{Z}$ and some $\eta \in (0,1)$. Then we have

$$\begin{aligned}\sum_{j=1}^{\infty}|G_{ij}|^2 &= \sum_{j=1}^{\infty}\frac{\left|\log\frac{e}{(1-z_i\overline{z_j})}\right|^2}{d(z_i)d(z_j)} \\ &\lesssim \sum_{j=1}^{\infty}\frac{\left(\log\frac{e}{|1-z_i\overline{z_j}|}\right)^2}{d(z_i)d(z_j)} \\ &= \sum_{z_j\in S(z_i)} * \;+ \sum_{z_j\in\Gamma_\sigma(z_i^*),z\notin S(z_i)} * \;+ \sum_{z_j\notin\Gamma_\sigma(z_i^*)\cup S(z_i)}\frac{\left(\log\frac{e}{|1-z_i\overline{z_j}|}\right)^2}{d(z_i)d(z_j)} \\ &= (I)+(II)+(III).\end{aligned}$$

Naturally we proceed to estimate quantities $(I),(II)$ and (III) separately. Starting with (I),

$$(I) \approx \sum_{z_j\in S(z_i)}\frac{d(z_i)}{d(z_j)} = d(z_i)\mu_{\mathcal{Z}}(S(z_i)) \overset{(SB)}{\lesssim} \frac{d(z_i)}{\log\frac{1}{|I_{z_i}|}} \lesssim 1.$$

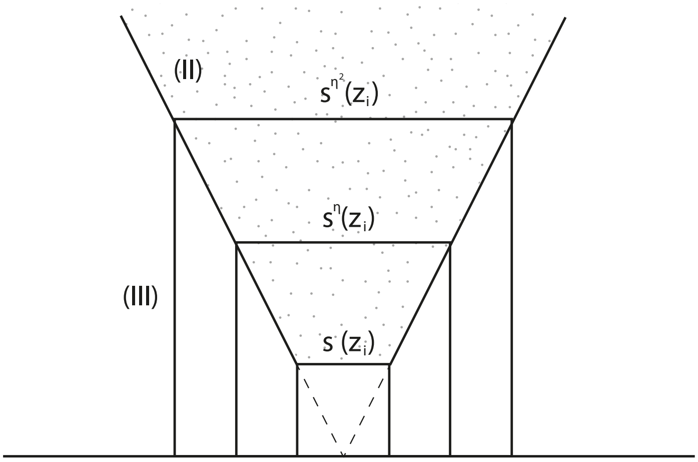

Figure 3.2: The splitting in the proof of Theorem 3.1.4.

Let us now estimate (III), for this fix $\eta \in (0,1)$, ($\eta = 1/2$ will do) and proceed as follows

$$
\begin{aligned}
(III) &\approx \sum_{z_j \notin \Gamma_\sigma(z_j^*) \cup S(z_i)} \frac{\left(\log \frac{e}{|z_i^*-z_j^*|}\right)^2}{d(z_i)d(z_j)} \\
&= \frac{1}{d(z_i)} \sum_{k=0}^{\infty} \sum_{\substack{z_j \notin \Gamma_\sigma(z_j^*)\cup S(z_i) \\ z_j \in S^{\eta^{k+1}}(z_i) \setminus S^{\eta^k}(z_i)}} \left(\log \frac{e}{|z_i^* - z_j^*|}\right)^2 d(z_j)^{-1} \\
&\lesssim \frac{1}{d(z_i)} \sum_{k=0}^{\infty} \sum \left(\log \frac{e}{(1-|z_i|)^{\eta^k}}\right)^2 d(z_j)^{-1} \\
&\lesssim \frac{1}{d(z_i)} \sum_{k=0}^{\infty} \sum_{z_j \in S^{\eta^{k+1}}(z_i) \setminus S^{\eta^k}(z_i)} \left[d(z_j)^{-1}\eta^{2k}\left(\log \frac{1}{1-|z_i|}\right)^2 + d(z_j)^{-1}\right] \\
&\approx d(z_i) \sum_{k=0}^{\infty} \eta^{2k} \mu_{\mathcal{Z}}(S^{\eta^{k+1}}(z_i)) + \mu_{\mathcal{Z}}(\mathbb{D}) \\
&\lesssim d(z_i) \sum_{k=0}^{\infty} \eta^{k-1}\left(\log \frac{1}{1-|z_i|}\right)^{-1} + \mu_{\mathcal{Z}}(\mathbb{D}) \\
&\lesssim 1.
\end{aligned}
$$

It remains a similar estimate for (II), where (WS) comes into play. That is because of weak separation each region $E_k := \Gamma_\sigma(z_i^*) \cap \left(S^{\eta^{k+1}}(z_i) \setminus S^{\eta^k}(z_i)\right)$

[4] σ is fixed and all constants depend silently on σ, only restriction we require in order to simplify our calculations is that it is large enough such that $\Gamma(z) \supseteq \{re^{i\theta} : r \leq |z|, e^{i\theta} \in I_z\}$

contains at most a bounded number of points of the sequence (see also Figure 3.2). Suppose this number is N_0. Then we have,

$$
\begin{aligned}
(II) &\approx \sum_{k=0}^{\infty} \sum_{z_j \in E_k} \frac{d(z_j)}{d(z_i)} \\
&\leq \frac{N_0}{d(z_i)} \sum_{k=0}^{\infty} \log \frac{1}{(1-|z_i|)^{\eta^k}} \\
&\lesssim 1.
\end{aligned}
$$

To prove the other direction just notice that

$$
\begin{aligned}
\sum_{j=1}^{\infty} |G_{ij}|^2 &\gtrsim \sum_{z_j \in S(z_i)} \frac{\left|\log \frac{e}{(1-z_i\overline{z_j})}\right|^2}{d(z_i)d(z_j)} \\
&\approx \sum_{z_j \in S(z_i)} \frac{d(z_i)}{d(z_j)} = d(z_i).
\end{aligned}
$$

The conclusion then follows by the hypothesis. □

3.8 Some remarks on the capacitary condition

3.8.1 A stronger condition implying the capacitary condition

Proof of Theorem 3.1.9. The idea of the proof is an old one, originally due to Shapiro and Shields, and it amounts to use Nevanlinna-Pick property in order to compensate for the lack of Blaschke products in the Dirichlet space. Let z_i a point in the sequence. Let $S_\circ^\gamma(z_i)$ the set of points z_j in the sequence such that z_i has γ-uninterrupted view of z_j. For each $z_j \in S_\circ^\gamma(z_i)$ consider the multiplier $\psi_{ij} \in \mathcal{M}(\mathcal{D})$ which vanishes at z_j, maximizes $\operatorname{Re}\psi_{ij}(z_i)$ and has multiplier norm ≤ 1. Due to the Nevanlinna Pick property of $\mathcal{D}$ (see [2, Theorem 9.43],

$$
\psi_{ij}(z_i) = 1 - \frac{|\langle k_{z_i}, k_{z_j}\rangle|^2}{\|k_{z_i}^2\|_{\mathcal{D}}\|k_{z_i}\|_{\mathcal{D}}^2}.
$$

Then consider a $weak - *$ cluster point of the sequence $\psi_{ij_1}^2\psi_{ij_2}^2\dots\psi_{ij_N}^2$, where $S_\circ^\gamma(z_i) = \{z_{j_1}, z_{j_2}, \dots\}$. Let's call this cluster point ψ_i. Obviously it vanishes on all points in $S_\circ^\gamma(z_i)$ and at z_i takes the value

$$
\psi_i(z_i) = \prod_{z_j \in S_\circ^\gamma(z_i)} 1 - \frac{|\langle k_{z_i}, k_{z_j}\rangle|^2}{\|k_{z_i}^2\|_{\mathcal{D}}\|k_{z_i}\|_{\mathcal{D}}^2}.
$$

To estimate this infinite product we use our hypothesis

$$\begin{aligned}\sum_{z_j\in S_\circ^\gamma(z_i)} \frac{|\langle k_{z_i},k_{z_j}\rangle|^2}{\|k_{z_i}^2\|_{\mathcal{D}}\|k_{z_i}\|_{\mathcal{D}}^2} &\lesssim \sum_{z_j\in S_\circ^\gamma(z_i)} \frac{\left(\log\frac{1}{|1-\overline{z_i}z_j|}\right)^2}{d(z_i)d(z_j)}\\ &\lesssim \sum_{z_j\in S_\circ^\gamma(z_i)} \frac{d(z_i)}{d(z_j)}\\ &\le C.\end{aligned}$$

Considering also that the sequence is weakly separated, hence every individual term of the infinite product is bounded away from zero in modulus we can conclude that $|\psi_i(z_i)|$ is bounded below by a constant independent of i. Consider the functions $f_i := k_{z_i}\psi_i/(\|k_{z_i}\|_{\mathcal{D}}^2|\psi_i(z_i)|)$. The same argument as in the proof of Lemma 3.2.12 applied to the functions f_i, shows that

$$\mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \bigcup_{z_j\in S_\circ^\gamma(z_i)} I_{z_j}\right) \le \frac{C}{d(z_i)}.$$

But then the estimate for the capacitary condition is immediate by Proposition 3.2.3 and Lemma 3.2.11.

$$\begin{aligned}\mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \bigcup_{z_j\in S^\gamma(z_i)} I_{z_j}\right) &\le \mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \bigcup_{z_j\in S_\circ^\gamma(z_i)} I_{z_j}^\gamma\right)\\ &\lesssim \mathrm{Cap}_{\mathbb{D}}\left(\Delta_1(z_i), \bigcup_{z_j\in S_\circ^\gamma(z_i)} I_{z_j}\right)\\ &\le C/d(z_i).\end{aligned}$$

□

3.8.2 A negative result

In order to construct the counter example in Theorem 3.1.10 we will exploit the Whitney decomposition of the unit disc and the corresponding analysis of the Dirichlet space on the tree, $H_2(\tau)$.

The tree model is convenient because it gives a necessary condition for interpolation in the Dirichlet space in terms of capacities defined on trees, which are highly more computable with respect to their continuous counterparts. In fact, in [12], Arcozzi Rochberg and Sawyer gave the following necessary condition for simple interpolation.

In what follows we think of the tree τ constructed in Section 2.7 as the collection of the centerpoints of the rectangles $R(n,k)$. Let $U \subset \tau$ and $\alpha \in \tau \setminus U$. We define the tree capacity of the condenser (U,α) as

$$\mathrm{Cap}_\tau(\alpha,U) := \inf \sum_{\beta\in\tau\setminus\{\omega\}} |\nabla f(\beta)|^2,$$

where the infimum is taken over all $f : \tau \mapsto \mathbb{R}$ such that $f(\alpha) = 1$ and $f(\gamma) = 0, \forall \gamma \in U$.

Proposition 3.8.1 (Tree Capacitary Condition)**.** *Suppose that $\{z_n\} \subset \tau$ is a simply interpolating sequence for the Dirichlet space. Then there exists a constant $C > 0$ such that*

$$\mathrm{Cap}_\tau(\alpha, \{z_i\} \setminus \{\alpha\}) \leq \frac{C}{d(\alpha)}$$

for all $\alpha \in \{z_i\}$.

Lemma 3.8.2. *$d_\tau(z) \approx d(z)$, $z \in \tau$. The constant of comparison is $\frac{2}{\log 2}$.*

Proof. Let $z = z(n, k)$, then $|z| = 1 - 2^{-n}$.Without loss of generality $n \neq 0$. Therefore $d(0, z) = \frac{1}{2}\log(2 + 2^{-n}) + \frac{n}{2}\log 2 \geq \frac{\log 2}{2}n$. On the other hand $d(0, z) \leq n + \frac{1}{2}\log\frac{3}{2} \leq 2n$. □

The tree capacitary condition is much easier to analyse, mainly because there exists a recursive formula for its computation [12, p. 32].

Given $\alpha, \beta \in \tau$, $\alpha \prec \beta$ and $U_\pm \subset S(\beta_\pm)$ we have

$$\mathrm{Cap}_\tau(\alpha, U_+ \cup U_-) = \frac{\mathrm{Cap}_\tau(\alpha, U_+) + \mathrm{Cap}_\tau(\alpha, U_-)}{1 + d_\tau(\alpha, \beta)[\mathrm{Cap}_\tau(\alpha, U_+) + \mathrm{Cap}_\tau(\alpha, U_-)]}.$$

Proof of Theorem 3.1.10. Let $z_0 \in \tau$, and set $N = d_\tau(z_0)$. Assume for simplicity that $\sqrt{N}$ is an integer. Consider also the points $\{w_i\}_{i=0}^{\sqrt{N}}$, where $w_0 = z$ and $w_{i+1} = \sigma_+ w_i$, and the points $z_i = \sigma_-^{(N)} w_i, 0 \leq i \leq \sqrt{N}$. Due to the shape of the representation of this configuration of points as a graph we shall write $comb(z_0) := \{z_1, \ldots, z_{\sqrt{N}}\}$. (See also Figure 3.3.)

Let us start with an estimate of $\mathrm{Cap}_\tau(z_0, comb(z_0))$. This can be done by applying the recursive formula 3.8.2. Let $c_i = \mathrm{Cap}_\tau(w_i, \{z_{i+1}, \ldots, z_{\sqrt{N}}\})$, $i < \sqrt{N}$ and $c_{\sqrt{N}} = 0$. Then the recursive formula gives

$$c_{i-1} = \frac{\frac{1}{N} + c_i}{1 + \frac{1}{N} + c_i} = \rho\begin{pmatrix} 1 & \frac{1}{N} \\ 1 & \frac{1}{N} + 1 \end{pmatrix}(c_i).$$

Where ρ is the map $\rho : M_2(\mathbb{C}) \mapsto$ Möb, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\rho}{\mapsto} \left(z \mapsto \frac{az+b}{cz+d}\right)$. Since ρ is a homomorphism we conclude that

$$\mathrm{Cap}_\tau(z_0, comb(z_0)) = c_0 = \rho\Big(\begin{pmatrix} 1 & \frac{1}{N} \\ 1 & \frac{1}{N} + 1 \end{pmatrix}^{\sqrt{N}}\Big)(0).$$

After diagonalizing the matrix we get

$$\begin{pmatrix} 1 & \frac{1}{N} \\ 1 & \frac{1}{N} + 1 \end{pmatrix}^{\sqrt{N}} = \begin{pmatrix} \delta_2 & \Delta \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (1-\Delta)^{\sqrt{N}} & 0 \\ 0 & (1-\delta_2)^{\sqrt{N}} \end{pmatrix}\begin{pmatrix} \delta_2 & \Delta \\ 1 & 1 \end{pmatrix}^{-1},$$

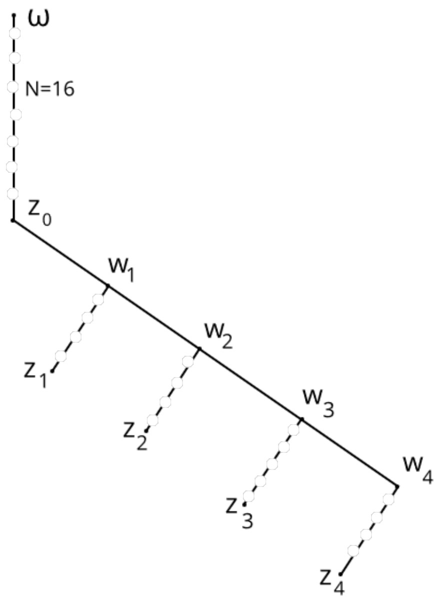

Figure 3.3: A comb with 4 teeth

where

$$\Delta = \frac{-\frac{1}{N} + \sqrt{(\frac{1}{N})^2 + 4\frac{1}{N}}}{2} \quad , \delta_2 = \frac{-\frac{1}{N} - \sqrt{(\frac{1}{N})^2 + 4\frac{1}{N}}}{2}.$$

A simple algebraic manipulation of the previous expression leads to

$$\frac{c_0}{1/\sqrt{N}} = \frac{c_0}{\Delta}\sqrt{N}\Delta = \frac{1 - \left(\frac{1-\Delta}{1-\delta_2}\right)^{\sqrt{N}}}{1 - \frac{\Delta}{\delta_2}\left(\frac{1-\Delta}{1-\delta_2}\right)^{\sqrt{N}}}\sqrt{N}\Delta \overset{N\to\infty}{\longrightarrow} \frac{e^2-1}{e^2+1} > 0.$$

Because $\Delta/\delta_2 \to -1$, $\left(\frac{1-\Delta}{1-\delta_2}\right)^{\sqrt{N}} \to e^{-2}$. Hence, for N sufficiently big $c_0 \geq \frac{1}{10\sqrt{N}}$.

Also we can calculate the total mass that each $comb(z_0)$ carries

$$\sum_{i=1}^{\sqrt{N}} \frac{1}{d(z_i)} \lesssim \sum_{i=1}^{\sqrt{N}} \frac{1}{d_\tau(z_i)} \lesssim \sum_{i=1}^{\sqrt{N}} \frac{1}{N} = \frac{1}{\sqrt{N}} \lesssim \frac{1}{\sqrt{d(z_0)}}.$$

A last remark is the following. There exists an $\eta < 1$, which can be chosen independently of N such that if $1 \leq i \neq j \leq \sqrt{N}$ then $S^\eta(z_i) \cap S^\eta(z_j) = \emptyset$.

Consider now a new sequence of points $\{\omega_i\}$ such that for any $\alpha \in comb(\omega_i), \beta \in comb(\omega_j)$ $S^\eta(\alpha) \cap S^\eta(\beta) = \emptyset$ for $i \neq j$ and some $0 < \eta < 1$, and also $\sum_{i=1}^{\infty} 1/\sqrt{d(\omega_i)} < \infty$. By a theorem of Axler [17] $\{\omega_i\}$ has a universally interpolating subsequence. We can assume without loss of generality that

$\{\omega_i\}$ itself is universally interpolating. Set $\{w_i\} = \bigcup_{i=1}^{\infty} comb(\omega_i)$. It is clear that the union of the two sequences it cannot be simply interpolating because it fails the tree capacitary condition

$$\mathrm{Cap}_\tau(z_j, \{z_i\}_{j\neq i} \cup \{w_i\}) \geq \mathrm{Cap}_\tau(z_j, comb(z_j)) \geq \frac{1}{10\sqrt{d_\tau(z_j)}}.$$

Nevertheless $\{w_i\}$ is simply interpolating by Theorem 3.1.1 because there exists $\eta < 1$ such that $S^\eta(w_i) \cap S^\eta(w_j) = \emptyset$, it has finite associated measure and it is weakly separated. □

Concluding remarks

If we have a sequence $\{z_i\} \subset \tau$, it would be interesting to know whether the tree capacitary condition implies the capacitary condition, for that would mean that the simply interpolating sequences for the tree coincide with the simply interpolating sequences for the Dirichlet space, at least for finite measure sequences, which are much easier to understand mainly due to the recursive relations for tree capacities.

Another question which remains open is if the characterization of simply interpolating sequences carries over to the case of infinite associated measure, something that is suggested by the analogous result for $H_2(\mathbb{D})$. In fact if one examines the proof of Theorem 3.1.1 can see that that would be true if $\ell^\infty(\mathbb{N}) \subset \{\{f(z_i)\} : f \in \mathcal{D}\}$ for every simply interpolating sequence. There are even some questions in $H_2(\mathbb{D})-$interpolation which remain open. For example in our definition we introduced a parameter ε and it is not clear at all how the interpolation constant depends on the parameter ε. Moreover our definition of $H_2(\mathbb{D})$ interpolation, although fit for our purposes, is only one of the natural definition that one could come up with for example, instead one could ask only that the interpolating function $u \in H_2(\mathbb{D})$ is only on average equal to the data, i.e.

$$\frac{1}{|\Delta_\varepsilon(z_i)|}\int_{\Delta_\varepsilon(z_i)} udA = \sqrt{d(z_i)}a_i.$$

Such questions have not been investigated, but it is the authors opinion that it would be very interesting to explore them further.

Chapter 4

Random interpolating sequences for Dirichlet type spaces

4.1 Introduction

In this chapter we consider a probabilistic version of the interpolation problems studied so far. In particular we consider random sequences of the following kind. Let $\mathcal{Z}(\omega) = \{z_n\}$ with $z_n = \rho_n e^{i\theta_n(\omega)}$ where $\theta_n(\omega)$ is a sequence of independent random variables, all uniformly distributed on $[0, 2\pi]$ (Steinhaus sequence), and $\rho_n \in [0, 1)$ is a sequence of *a priori* fixed radii. Depending on distribution conditions on (ρ_n) as will be discussed below, we ask about the probability that $\mathcal{Z}(\omega)$ is interpolating for the Dirichlet spaces $\mathcal{D}_\alpha$, $0 \le \alpha \le 1$. Recall that the weighted Dirichlet space $\mathcal{D}_\alpha$, $0 \le \alpha \le 1$, is the space of all analytic function f on the unit disc $\mathbb{D}$ such that

$$\|f\|_\alpha^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha \mathrm{d}A(z) < \infty.$$

For $f(z) = \sum_{n \ge 0} a_n z^n$, $z \in \mathbb{D}$, the above expression is equivalent to $\sum_{n \ge 0} (1+n)^{1-\alpha} |a_n|^2$. If $\alpha = 1$, $\mathcal{D}_1$ is the Hardy space H^2, and the classical Dirichlet space $\mathcal{D}$ corresponds to $\alpha = 0$.

The problems we would like to study in this chapter are inspired by results by Cochran [39] and Rudowicz [75] who considered random interpolation in the Hardy space. Since interpolation in this space is characterized by separation (in the pseudohyperbolic metric) and by the Carleson measure condition (note that the Hardy space was the pioneering space with a kernel satisfying the complete Pick property), those authors were interested in a 0-1 law for separation, see [39], and a condition for being almost surely a Carleson measure [75], which led to a 0-1 law for interpolation. It is thus natural to discuss separation, Carleson measure type conditions and interpolation in Dirichlet spaces.

We also would like to observe that generally, when the deterministic frame does not give a full answer to a problem, or if the deterministic conditions are not so easy to check, it is interesting to look at the random situation. In particular, it is interesting to ask for conditions ensuring that a sequence picked at random is or is not interpolating almost surely (i.e., which are in a sense "generic situations"?). In this context, it is also worth mentioning the huge existing literature around Gaussian analytic functions which investigates the zero distribution in classes of such functions [55].

Concerning separation in Dirichlet spaces $\mathcal{D}_\alpha$, $0 < \alpha < 1$, this turns out to be the same as in the Hardy spaces (see [78, p.22]), so that in that case Cochran's result perfectly characterizes the situation. The separation in the classical Dirichlet space, however, is much more delicate than in the Hardy space. We establish here a 0-1-type law for separation in $\mathcal{D}$. While our proof of this fact is inspired by Cochran's ideas, it requires a careful adaptation to the metric in that space.

Concerning Carleson measure type results in Dirichlet spaces, $0 < \alpha < 1$, we will first discuss the situation in the Hardy space and improve Rudowicz' result simplifying his proof. Our new proof carries over to the Dirichlet situation and allows, together with Stegenga's characterization of Carleson measures, to discuss the results on interpolation in $\mathcal{D}_\alpha$. As it turns out, we are able to exhibit a peculiar breakpoint in the behaviour of such interpolating sequences depending on the weight α: for $1/2 < \alpha \leq 1$, almost sure separation corresponds to almost sure interpolation, while for $0 \leq \alpha \leq 1/2$, almost sure zero sequences correspond to almost sure interpolating sequences. Observe that for the critical value $\alpha = 1/2$ the behaviour is the same as in the Dirichlet space. Commenting further on this breakpoint $\alpha = 1/2$, we would like to mention work by Newman and Shapiro [66] who show that if I is a non-constant singular inner function, then $I \notin \mathcal{D}_{1/2}$.

The interesting feature in connection with Stegenga's result is that switching to the random setting, it turns out that his condition for unions of intervals reduces to a one-box condition involving just one interval, and for which the capacity can be estimated.

Since zero sequences are of some importance as we have seen, another central ingredient of our discussion is a rather immediate adaption of Bogdan's result on almost sure zero sequence in the Dirichlet space to the case of weighted Dirichlet spaces which we add for completeness in an annex.

We will now discuss in details the results we have obtained.

4.1.1 Back to the Hardy space

As pointed out in the introduction, before considering the situation in the Dirichlet space, it seems appropriate to re-examine the situation in the Hardy space. Recall that Cochran established a 0-1 law for (pseudohyperbolic) separation (see Theorem 4.1.3 below) and Rudowicz showed that Cochran's condition for separation implies almost surely the Carleson measure condition. This implies that interpolation is characterized by the condition ensuring almost sure separation. As it turns out the situation in Dirichlet spaces is quite different. So, in order to get a better understanding we start stating an improvement of Rudowicz' results on random Carleson measures in the Hardy space which will help to better understand the case of Dirichlet spaces.

Recall that the measure $\mathrm{d}\mu_{\mathcal{Z}} = \sum_{z\in\mathcal{Z}}(1-|z|^2)\delta_z$ is a Carleson measure for H^2 if there is a constant C such that for every interval $I \subset \mathbb{T}$,

$$\mu_{\mathcal{Z}}(S(I)) \leq C|I|,$$

for all Carleson boxes $S(I)$. We will prove that a weaker condition than Rudowicz' leads to Carleson measures almost surely in the Hardy space. We first need to introduce a notation:

$$N_n = \#\{z \in \mathcal{Z}(\omega) : 1-2^{-n} \leq |z| \leq 1-2^{-n-1}\}, \quad n = 0, 1, 2 \ldots.$$

Theorem 4.1.1. *Let $\beta > 1$ and suppose*

$$\sum_{n\geq 1} 2^{-n} N_n^{\beta} < +\infty.$$

Then the measure $\mathrm{d}\mu_{\mathcal{Z}}$ is a Carleson measure almost surely in the Hardy space.

As a result, the Carleson measure condition alone is not sufficient to give a 0-1 law for interpolation in the Hardy space.

Note that Rudowicz [75] showed that the above condition with $\beta = 2$ is sufficient. We will construct an example showing that it is not possible to replace β by 1, so that (almost sure or not) zero sequences do not imply almost surely the Carleson measure condition. This makes the Hardy space a singular point in this respect within the scale of weighted Dirichlet spaces, $0 \leq \alpha \leq 1$.

4.1.2 Interpolation in Dirichlet spaces $\mathcal{D}_\alpha$, $0 < \alpha < 1$

In this subsection we mention the results connected to interpolation: zero sequences, separation, and Carleson measures.

Random zero sequences in Dirichlet spaces

A central role in our interpolation results will be played by random zero sequences. Indeed, for an interpolating sequence in the Dirichlet space it is necessary to be a zero sequence (interpolation implies that there are functions vanishing on the whole sequence except for one point z_0, and multiplying this function by $(z - z_0)$ yields a function in the Dirichlet space vanishing on the whole sequence). We recall some results on random zero set in Dirichlet spaces. Carleson proved in [25] that when

$$\sum_{z\in\mathcal{Z}} \|k_z\|_\alpha^{-2} < \infty \tag{4.1.1}$$

then the Blaschke product B associated to $\mathcal{Z}$ belongs to $\mathcal{D}_\alpha$, $0 < \alpha < 1$ (for $\alpha = 1$ this corresponds to the Blaschke condition for the Hardy space). When $\alpha = 0$, (4.1.1) is the condition of Shapiro–Shields that we have already encountered in Section 2.3 and is sufficient for $\mathcal{Z}$ to be a zero set for the classical Dirichlet space $\mathcal{D}_1$. Note that if $0 < \alpha \le 1$ then

$$\sum_{z\in\mathcal{Z}} \|k_z\|_\alpha^{-2} \asymp \sum_{z\in\mathcal{Z}} (1-|z|)^\alpha \asymp \sum_n 2^{-\alpha n} N_n$$

and if $\alpha = 0$ then

$$\sum_{z\in\mathcal{Z}} \|k_z\|_0^{-2} \asymp \sum_{z\in\mathcal{Z}} |\log(1-|z|)|^{-1} \asymp \sum_n n^{-1} N_n.$$

On the other hand, it was proved by Nagel–Shapiro–Shields in [62] that if $\{r_n\} \subset (0,1)$ does not satisfy (4.1.1), then there is $\{\theta_n\}$ such that $\{r_n e^{i\theta_n}\}$ is not a zero set for $\mathcal{D}_\alpha$. Bogdan [23, Theorem 2] gives a condition on the radii $|z_n|$ for the sequence $\mathcal{Z}(\omega)$ to be almost surely zeros sequence for $\mathcal{D}$:

$$P(\mathcal{Z}(\omega) \text{ is a zero set for } \mathcal{D}) = \begin{cases} 1 \\ 0 \end{cases} \text{iff } \sum_n n^{-1} N_n \begin{cases} < \infty \\ = \infty. \end{cases} \tag{4.1.2}$$

Bogdan's arguments carry over to $\mathcal{D}_\alpha$, $\alpha \in (0,1)$. For the sake of completeness, we will prove in the annex, Section 4.6, the following result on almost sure zero sequences.

Theorem 4.1.2. *Let $0 < \alpha \le 1$. Then*

$$P(\mathcal{Z}(\omega) \textit{ is a zero set for } \mathcal{D}_\alpha) = \begin{cases} 1 \\ 0 \end{cases} \textit{iff } \sum_n 2^{-\alpha n} N_n \begin{cases} < \infty \\ = \infty. \end{cases} \tag{4.1.3}$$

Interpolation in Dirichlet spaces $\mathcal{D}_\alpha$, $0 < \alpha < 1$

As pointed out earlier, interpolation is intimately related with separation conditions and Carleson measure type conditions. Recall that a sequence $\mathcal{Z}$

is called (pseudohyperbolically) separated if

$$\inf_{\substack{z,w\in\mathcal{Z}\\ z\neq w}} \rho(z,w) = \inf_{\substack{z,w\in\mathcal{Z}\\ z\neq w}} \frac{|z-w|}{|1-\overline{z}w|} \geq \delta_{\mathcal{Z}} > 0.$$

Since in Dirichlet spaces $\mathcal{D}_\alpha$, $0 < \alpha \leq 1$, the natural separation ($\mathcal{D}_\alpha$-separated sequence) is indeed pseudohyperbolic separation [78, p.22], we recall Cochran's separation result on pseudohyperbolic separation.

Theorem 4.1.3 (Cochran). *A sequence $\mathcal{Z}(\omega)$ is almost surely (pseudohyperbolically) separated if and only if*

$$\sum_n 2^{-n} N_n^2 < +\infty. \tag{4.1.4}$$

We should pause here to make a crucial observation. We have already mentioned that interpolating sequences are necessarily zero-sequences. Also separation is another necessary condition for interpolation. Now the condition for zero sequences (4.1.3) depends on α while the separation condition does not, and it follows that depending on α, it is one condition or the other which is dominating. From (4.1.3) and (4.1.4) it is not difficult to see that this breakpoint is exactly at $\alpha = 1/2$ (for $\alpha = 1/2$, (4.1.3) still implies (4.1.4)). This motivates already the necessary conditions of our central Theorem 4.1.5 below. Another ingredient of that result comes from Carleson measures which have been characterized by Stegenga using capacitory conditions. For these we have the following result which is in the spirit of Theorem 4.1.1 in the Hardy space.

Theorem 4.1.4. *Let $0 < \alpha < 1$. If*

$$\sum_n 2^{-\alpha n} N_n < \infty \tag{4.1.5}$$

then $\mu_{\mathcal{Z}} = \sum_{z\in\mathcal{Z}} (1-|z|^2)^\alpha \delta_z$ is almost surely a Carleson measure for $\mathcal{D}_\alpha$.

Observe that contrarily to the Hardy space, where he had to pick $\beta > 1$, here the exponent is exactly α. We obtain a similar result in the classical Dirichlet space. So, in view of Theorem 4.1.2, we can deduce that almost sure zero sequences give rise to almost sure Carleson measures in $\mathcal{D}_\alpha$, $0 \leq \alpha < 1$ (which in particular includes the classical Dirichlet space).

We can now state our first main result.

Theorem 4.1.5.

(i) Let $1/2 < \alpha < 1$, then

$$P\big(\mathcal{Z}(\omega) \text{ is interpolating for } \mathcal{D}_\alpha\big) = \begin{cases} 1 \\ 0 \end{cases} \text{iff } \sum_n 2^{-n} N_n^2 \begin{cases} < \infty \\ = \infty. \end{cases}$$

(ii) Let $0 < \alpha \leq 1/2$. *Then*

$$P\big(\mathcal{Z}(\omega) \textit{ is interpolating for } \mathcal{D}_\alpha\big) = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. \textit{ iff } \sum_n 2^{-\alpha n} N_n \left\{ \begin{array}{l} < \infty \\ = \infty. \end{array} \right.$$

An interesting reformulation of the above results connects random interpolation with random zero sequences and random separated sequences as stated in the following corollary.

Corollary 4.1.6. *The following statements hold:*

1. *Let* $1/2 < \alpha \leq 1$. *The sequence* $\mathcal{Z}(\omega)$ *is almost surely interpolating for* $\mathcal{D}_\alpha$ *if and only if it is almost surely separated.*

2. *Let* $0 < \alpha \leq 1/2$. *The sequence* $\mathcal{Z}(\omega)$ *is almost surely interpolating for* $\mathcal{D}_\alpha$ *if and only if it is almost surely a zero sequence.*

Observe that for $\alpha = 1/2$, the condition for being almost surely a zero sequence is strictly stronger than the condition for being almost surely separated so that in the limit case $\alpha = 1/2$ it is indeed the almost sure zero condition that drives the situation.

4.1.3 Interpolation in the classical Dirichlet space

For the classical Dirichlet space we will first establish a result on separation, and then use again the fact that in the random situation Stegenga's capacitory condition on unions of intervals reduces to a single interval.

Separation in the Dirichlet space

In the case $\alpha = 0$, the separation is given in a different way. Recall from Section 2.6 the definition of the Gleason metric

$$d_{\mathcal{D}}(z, w) = \sqrt{1 - \frac{|k_w(z)|^2}{k_z(z)k_w(w)}}, \qquad z, w \in \mathbb{D}.$$

A sequence $\mathcal{Z}$ is called $\mathcal{D}$–separated if

$$\inf_{\substack{z,w\in\mathcal{Z} \\ z\neq w}} d_{\mathcal{D}}(z, w) > \delta_{\mathcal{Z}} > 0$$

for some $\delta_{\mathcal{Z}} < 1$. This is equivalent to (see [78, p.23])

$$\frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2} \leq (1 - |z|^2)^{\delta_{\mathcal{Z}}^2}, \qquad z, w \in \mathcal{Z}. \tag{4.1.6}$$

For separation in the Dirichlet space $\mathcal{D}$ we obtain the following 0-1 law.

Theorem 4.1.7.

$$P(\mathcal{Z}(\omega) \text{ is } \mathcal{D}\text{–separated}) = \begin{cases} 1, & \text{if } \exists \gamma < 1 \text{ such that } \sum_n 2^{-\gamma n} N_n^2 < \infty, \\ 0, & \text{if } \forall \gamma < 1 \text{ holds that } \sum_n 2^{-\gamma n} N_n^2 = \infty. \end{cases}$$

We observe that in both conditions we can replace the sum by a supremum (this amounts to replacing γ by a slightly bigger or smaller value).

As we will see, this is a very mild condition in comparison to the almost sure zero- or Carleson-measure condition, and does not play a big role for the almost sure interpolation problem.

Interpolation in the Dirichlet space $\mathcal{D}$

Recall that Bogdan showed that $\mathcal{Z}(\omega)$ is almost surely a zero sequence for $\mathcal{D}$ if and only if $\sum_n n^{-1} N_n < +\infty$. This motivates already the necessary part of the following complete characterization of almost surely *universal* interpolating sequences for $\mathcal{D}$. The sufficiency comes essentially from the fact that the condition easily implies the (mild) separation condition, as well as the Carleson measure condition which will be shown in a similar fashion as in Theorem 4.1.4.

Theorem 4.1.8.

$$P\big(\mathcal{Z}(\omega) \text{ is universal interpolating for } \mathcal{D}\big) = \begin{cases} 1 \\ 0 \end{cases} \text{iff } \sum_n n^{-1} N_n \begin{cases} < \infty \\ = \infty. \end{cases}$$

We can reformulate the above result in the same spirit as Corollary 4.1.6

Corollary 4.1.9. *A sequence is almost surely interpolating for $\mathcal{D}$ if and only if it is almost surely a zero sequence for $\mathcal{D}$.*

4.1.4 Organization of the material

This material is organized as follows. In the next section we present the improved version of the Rudowicz result concerning random Carleson measures in Hardy spaces which is the guideline for the corresponding result in the Dirichlet space. Indeed, this largely clarifies and simplifies not only the situation in the Hardy space, but also indicates the direction of investigation for the Dirichlet space. In Section 3 we prove the sufficient condition for interpolation in $\mathcal{D}_\alpha$, $0 < \alpha < 1$. Here, we will also prove Corollary 4.1.5. In the following section we show the 0-1 law on separation in the classical Dirichlet space. This requires a subtle adaption of the Cochran discussion in the Hardy space to the much more intricate geometry in the Dirichlet space. The proofs of the results on interpolating sequences in the classical Dirichlet space are contained in Section 5. Actually, as in the Hardy space, the core of the proof being probabilistic, we are able to get rid of analytic functions. In the final Section 6, we give some indications to the 0-1 law on zero-sequences in weighted Dirichlet spaces based on Bogdan's proof in the classical Dirichlet space.

4.2 Carleson condition in the Hardy space

Before considering Carleson measure conditions in the Dirichlet space, we will discuss the situation in the Hardy space, in particular we will prove here Theorem 4.1.1. We will also construct an example showing that it is not possible to choose $\beta = 1$ in the statement of Theorem 4.1.1.

4.2.1 Proof of Theorem 4.1.1

We start introducing some notation. Let

$$I_{n,k} = \{\mathrm{e}^{2\pi i t} : t \in [k2^{-n}, (k+1)2^{-n})\} \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, 2^n$$

be dyadic intervals and $S_{n,k} = S(I_{n,k})$ the associated Carleson window. In order to check the Carleson measure condition for a positive Borel measure μ on $\mathbb{D}$ it is clearly sufficient to check the Carleson measure condition for windows $S_{n,k}$:

$$\mu(S_{n,k}) \leq C|I_{n,k}| = C2^{-n},$$

for some fixed $C > 0$ and every $n \in \mathbb{N}$, $k = 0, \ldots, 2^n$. Given n, k and $m \geq k$ let $X_{n,m,k}$ be the number of points of $\mathcal{Z}$ contained in $S_{n,k} \cap A_m$ (we stratify the Carleson window $S_{n,k}$ into a disjoint union of layers $S_{n,k} \cap A_m$). Since A_m contains N_m points and the (normalized) length of $S_{n,k}$ is 2^{-n}, we have $X_{n,m,k} \approx B(2^{-n}, N_m)$ (binomial law). In order to show that $d\mu_{\mathcal{Z}}$ is almost surely a Carleson measure we thus have to prove the existence of C such that

$$\mu_{\mathcal{Z}}(S_{n,k}) = \sum_{m \geq n} 2^{-m} X_{n,m,k} \leq C2^{-n}$$

almost surely, in other words we have to prove

$$\sup_{n,k} 2^n \sum_{m \geq n} 2^{-m} X_{n,m,k} \leq C$$

almost surely (in ω). The estimate above had already been investigated by Rudowicz [75]. Here we will proceed in a different way with respect to Rudowicz' argument to obtain an improved version of his result and which allows to better understand the Dirichlet space situation.

Proof of Theorem 4.1.1. In view of our preliminary remarks, we need to look at the random variable

$$Y_{n,k} = 2^n \sum_{m=n}^{+\infty} 2^{-m} X_{n,m,k},$$

where, as said above, $X_{n,m,k} \approx B(2^{-n}, N_m)$. Hence, saying that $Y_{n,k} \geq A$ means that there are Carleson windows for which the Carleson measure constant is at least A. Also denote by $G_{Y_{n,k}}$ the probability generating function of the random variable $Y_{n,k}$, i.e. $G_{Y_{n,k}}(s) = \mathbb{E}(s^{Y_{n,k}})$. It is well known that for a random variable X which follows a binomial distribution with parameters p, N we have that $G_X(s) = (1-p+ps)^N$.

By the hypothesis, for n sufficiently large, $N_n \leq 2^{(1-\epsilon)n}, \epsilon = 1 - 1/\beta$. Introduce now two parameters $A, s > 0$ to be specified later. By Markov's inequality we have that

$$\begin{aligned}
\log P(Y_{n,k} \geq A) &= \log P(s^{Y_{n,k}} \geq s^A) \\
&\leq \log\big(\frac{1}{s^A} G_{Y_{n,k}}(s)\big) \\
&= \sum_{m \geq n} N_m \log(1 - 2^{-n} + 2^{-n} s^{2^{n-m}}) - A\log(s) \\
&\leq 2^{-n} \sum_{m \geq n} N_m (s^{2^{n-m}} - 1) - A\log(s) \\
&= \sum_{m \geq 0} N_{n+m} 2^{-(n+m)} 2^m (s^{2^{-m}} - 1) - A\log(s).
\end{aligned}$$

At this point notice that $x(a^{1/x} - 1) \leq a$, for all $x \geq 1, a > 0$, which together with the hypothesis on N_n gives

$$\log P(Y_{n,k} \geq A) \leq \sum_{m \geq 0} 2^{-\epsilon(n+m)} s - A\log(s) = \frac{2^\epsilon}{2^\epsilon - 1} s 2^{-\epsilon n} - A\log(s).$$

Now set $s = 2^{\frac{\epsilon n}{2}}, A = \frac{4}{\epsilon}$ in the last inequality to get

$$\log P(Y_{n,k} \geq A) \leq \frac{2^\epsilon}{2^\epsilon - 1} 2^{-\frac{\epsilon n}{2}} - 2n\log(2).$$

Hence, $P(Y_{n,k} \geq \frac{4}{\epsilon}) \leq C(\epsilon) 2^{-2n}$.

In view of an application of the Borel-Cantelli Lemma we compute

$$\sum_{n \geq 0} \sum_{k=1}^{2^n} P(Y_{n,k} \geq A) \leq C(\epsilon) \sum_{n \geq 0} 2^n \times 2^{-2n} < \infty.$$

Hence, by the Borel-Cantelli Lemma, the event $Y_{n,k} \geq A$ can happen for at most a finite number of indices (n, k). In particular the Carleson measure constant of $d\mu_{\mathcal{Z}}$ is almost surely at most A except for a finite number of Carleson windows. □

4.2.2 An example

Here we construct an example showing that we cannot choose $\beta = 1$ in Theorem 4.1.1. For $\gamma > 1$, let

$$N_n = \frac{2^n}{n^\gamma}, \quad n \geq 1.$$

Clearly $\sum_{n\geq 1} 2^{-n} N_n < +\infty$. For $n \in \mathbb{N}^*$ define the event

$$A_{n,k}^N = \bigcup_{m=n}^{n+N} (X_{n,k,m} \geq 2^{m-n}).$$

In this case, we have $2^n \mu_{\mathcal{Z}}(S_{n,k}) \geq N+1$. Since $X_{n,k,m}$ are mutually independent for fixed n, k, we have

$$P(A_{n,k}^N) = \prod_{m=n}^{n+N} P(X_{n,k,m} \geq 2^{m-n}).$$

The following well known elementary lemma will be very useful (it is essentially approximation of the binomial law by the Poisson law). We refer for instance to [20] for the material on probability theory — essentially elementary — used in this chapter.

Lemma 4.2.1. *If X is a binomial random variable with parameters p, N, then for every $s = 0, 1, 2, \ldots$,*

$$\lim_{\substack{N\to\infty \\ pN\to 0}} \frac{P(X = s)}{(pN)^s} = \lim_{\substack{N\to\infty \\ pN\to 0}} \frac{P(X \geq s)}{(pN)^s} = \frac{1}{s!}.$$

Note that in view of the Lemma we can replace in $A_{n,k}^N$ the condition $X_{n,k,m} \geq 2^{m-n}$ by $X_{n,k,m} = 2^{n-m}$. Since in our situation $N = 2^m/m^\gamma \to \infty$ and $pN = N_m/2^n \leq 2^{m-n}/m^\gamma \to 0$ ($n \leq m \leq n+N$, N fixed), we get

$$\begin{aligned}
P(A_{n,k}^N) &= \prod_{m=n}^{n+N} P(X_{n,k,m} = 2^{m-n}) \simeq \prod_{m=n}^{n+N} \left(\frac{1}{(2^{m-n})!} \left(\frac{N_m}{2^n}\right)^{2^{m-n}} \right) \\
&= \prod_{m=n}^{n+N} \left(\frac{1}{(2^{m-n})!} \left(\frac{2^{m-n}}{m^\gamma}\right)^{2^{m-n}} \right) \\
&= \prod_{m=0}^{N} \left(\frac{1}{(2^m)!} \left(\frac{2^m}{(m+n)^\gamma}\right)^{2^m} \right) \\
&\simeq \frac{1}{n^{2^{N+1}\gamma}}.
\end{aligned} \tag{4.2.1}$$

The crucial observation here is that for every fixed N this probability goes polynomially to zero. In particular

$$\sum_{n\geq 0}\sum_{k=0}^{2^n-1} P(A_{n,k}^N) \approx \sum_{n\geq 0} \frac{2^n}{n^{2^{N+1}\gamma}} = +\infty.$$

We need to apply a reverse version of the Borel-Cantelli lemma. This works for events which are independent. However this is not the case for $A_{n,k}^N$ and $A_{i,j}^N$ when the corresponding dyadic annuli meet. Since $A_{n,k}^N$ intersects the annuli A_l, $n \leq l \leq n+N$ and $A_{i,j}^N$ intersects A_l, $i \leq l \leq i+N$, the events $A_{n,k}^N$ and $A_{i,j}^N$ are dependent when $[n, n+N] \cap [i, i+N] \neq \emptyset$, i.e. when $|n-i| \leq N+1$. We will appeal to a more general version of the Borel-Cantelli Lemma, see Lemma 4.6.1. This requires that

$$\liminf_{M\to+\infty} \frac{\sum_{i,j,k,n\leq M} P(A_{i,j}^N \cap A_{n,k}^N)}{\left(\sum_{k,n\leq M} P(A_{n,k}^N)\right)^2} \leq 1.$$

The difference here is of course made by the elements with $|n-i| \leq N+1$ since for $|n-i| > N+1$, we have $P(A_{i,j}^N \cap A_{n,k}^N) = P(A_{i,j}^N) \times P(A_{n,k}^N)$. In particular

$$\sum_{i,j,k,n\leq M} P(A_{i,j}^N \cap A_{n,k}^N) = \Big(\sum_{k,n\leq M} P(A_{n,k}^N)\Big)^2 + \sum_{\substack{|n-i|\leq N+1\\ i,j,k,n\leq M}} P(A_{i,j}^N \cap A_{n,k}^N) - \sum_{\substack{|n-i|\leq N+1\\ i,j,k,n\leq M}} P(A_{i,j}^N) \times P(A_{n,k}^N).$$

For fixed N we can assume i and n big so that $N \ll N_n, N_i$. Note that $P(A_{i,j}^N \cap A_{n,k}^N) = P(A_{i,j}^N) \times P(A_{n,k}^N | A_{i,j}^N)$, and we have to estimate $P(A_{n,k}^N | A_{i,j}^N)$. The idea is to observe that the conditional probability essentially behaves like the unconditional one, i.e. the knowledge of $A_{i,j}^N$ does not interfer too much on the probability of $A_{n,k}^N$.

More precisely, if $|n-i| \leq N+1$, the fact that $A_{i,j}^N$ has occured reduces the number of points available for $A_{n,k}^N$ (in the annulus A_m) by at most 2^N which we can again assume neglectible with respect to N_m, $n \leq m \leq n+N$.

One has to pay a little bit attention here. A priori, it could happen that $S_{i,j} \subset S_{n,k}$ or $S_{n,k} \subset S_{i,j}$ (in these cases the knowledge of $A_{i,j}^N$ has some indicidence to that of $A_{n,k}^N$ or vice versa), but this requires $|j-k| \leq 2^N$ which is uniformly bounded. The corresponding sum of probabilities is thus also uniformly bounded and dividing by the square of eventually divergent partial sums makes these terms neglectible. So we can assume that this pathological situation does not occur.

Then, in order to estimate the probability of $A_{n,k}^N$, under the condition $A_{i,j}^N$, we can run a similar computation as in (4.2.1) with N_m replaced by

$N_m - N = N_m(1 - N/N_m)$ (for those m for which the annulus A_m lies in both Carleson boxes; for the others we keep N_m), which yields a comparable probability:

$$P(A_{i,j}^N \cap A_{n,k}^N) = (1 + \varepsilon_{n,k,i,j}) P(A_{i,j}^N) \times P(A_{n,k}^N)$$

where $\varepsilon_{n,k,i,j} \to 0$ when i, j, n, k get big. Then

$$\sum_{\substack{|n-i|\le N+1\\ i,j,k,n\le M}} P(A_{i,j}^N \cap A_{n,k}^N) - \sum_{\substack{|n-i|\le N+1\\ i,j,k,n\le M}} P(A_{i,j}^N) \times P(A_{n,k}^N)$$
$$= \sum_{n,k\le M} P(A_{n,k}^M)(1 - P(A_{n,k}^M) + \sum_{\substack{0<|n-i|\le N+1\\ i,j,k,n\le M}} \varepsilon_{n,k,i,j} P(A_{i,j}^N) \times P(A_{n,k}^N),$$

which, when dividing through $\left(\sum_{k,n\le M} P(A_{n,k}^N)\right)^2 \to +\infty$, $M \to \infty$, goes to 0.

From Lemma 4.6.1 we conclude that for every N there exists infinitely many n, k such that $2^n \mu_{\mathcal{Z}}(S_{n,k}) \ge N + 1$, which concludes the example.

4.3 Proof of Theorem 4.1.5

A key ingredient in the proof of Theorem 4.1.5 is Theorem 4.1.4 which we recall here for convenience.

Theorem. *Let* $0 < \alpha < 1$. *If*

$$\sum_n 2^{-\alpha n} N_n < \infty \tag{4.3.1}$$

then $\mu_{\mathcal{Z}}$ *is almost surely a Carleson measure for* $\mathcal{D}_\alpha$.

Recall also that

$$\mu_{\mathcal{Z}} = \sum_{z\in\mathcal{Z}} \frac{1}{\|k_z\|_\alpha^2} \delta_z,$$

where

$$\|k_z\|_\alpha^2 = \begin{cases} \log \dfrac{e}{1-|z|^2}, & \alpha = 0, \\ \dfrac{1}{(1-|z|^2)^\alpha}, & 0 < \alpha \le 1. \end{cases}$$

The case $\alpha = 0$ will be useful later in the study of the classical Dirichlet space $\mathcal{D}$.

It shall be observed that the proof does not work for the Hardy space case $\alpha = 1$, for which we have seen that it is possible to construct sequences (r_n) satisfying the Blaschke condition, but the associated sequences $\mathcal{Z}(\omega)$

are not almost surely interpolating.

We will need Stegenga's characterization of Carleson measures for Dirichlet spaces which involves the capacities c_α defined in Section 2.8. In connection with this result we recall the following three facts. Once these facts collected, the proof is essentially the same as in the Hardy space.

The first fact we would like to recall are the following known estimates (see [78, p.19])

$$c_\alpha(I) \simeq \begin{cases} |I|^\alpha, & 0 < \alpha < 1, \\ \left(\log \dfrac{e}{|I|}\right)^{-1}, & \alpha = 0, \end{cases} \tag{4.3.2}$$

where I is an interval.

The second fact is Stegenga's result (see e.g. [78, p.19]).

Theorem 4.3.1. *A nonnegative Borel measure μ on $\mathbb{D}$ is a Carleson measure for $\mathcal{D}_\alpha$, $0 \le \alpha < 1$, if and only if there exists a constant $K > 0$ such that*

$$\sum_{j=1}^{n} \mu(S(I_j)) \le K c_\alpha \left(\bigcup_{j=1}^{n} I_j \right), \tag{4.3.3}$$

for each finite collection of disjoint subarcs $I_1, I_2, \ldots, I_n$ of the unit circle, and arbitrary n.

The last fact is the following observation. There exists a universal constant K such that for every finite collection I_1, I_2, ..., I_n of subsarcs of $\mathbb{T}$, and I an arc of $\mathbb{T}$, with $|I| = \sum_{j=1}^{n} |I_j|$, we have

$$c_\alpha(I) \le K c_\alpha \left(\bigcup_{j=1}^{n} I_j \right). \tag{4.3.4}$$

Concerning (4.3.4), we refer to [46, Theorem 2.4.5] for the case $\alpha = 0$. The general case $\alpha \in (0,1)$ is shown exactly in the same way as for $\alpha = 0$.

Proof. We can now essentially repeat the argument from the Hardy space case. In view of (4.3.4), in order to apply Theorem 4.3.1, it is sufficient to show that the condition of Theorem (4.1.4) implies almost surely

$$\sum_{j=1}^{n} \mu(S(I_j)) \le K c_\alpha(I),$$

where I is an interval of length $\sum_{j=1}^{n} |I_j|$. Since the distribution of points in $S(J)$ for an arc J does not depend on its position on $\mathbb{T}$, we can assume that I_j are adjacent intervals. Then, setting $I = \bigcup_{j=1}^n I_j$ we have

$$\bigcup_{j=1}^{n} S(I_j) \subset S(I),$$

and for the application of Stegenga's theorem it is enough to show that almost surely

$$\mu(S(I)) \leq Kc_\alpha(I). \tag{4.3.5}$$

In view of our discussion on $\mathcal{D}$ we should emphasize that the above discussion is true for every $\alpha \in (0,1]$.

Consider now the case $\alpha \in (0,1)$. Then (4.3.5) becomes

$$\sum_{z\in S(I)} (1-|z|^2)^\alpha \leq c|I|^\alpha.$$

As in the Hardy space, it is enough to discuss this inequality for dyadic intervals $I = I_{n,k}$, and covering again $S_{n,k}$ by dyadic arcs we obtain the following random variable

$$Y_{n,k}^\alpha = 2^{\alpha n} \sum_{m\geq n} 2^{-\alpha m} X_{n,m,k}.$$

We now get using again $x(a^{1/x}-1) \leq a$ for $x \geq 1$, $a > 0$,

$$\begin{aligned}
\log P(Y_{n,k}^\alpha \geq A) &= \log P(s^{Y_{n,k}} \geq s^A) \leq \log\big(\frac{1}{s^A} G_{Y_{n,k}}(s)\big) \\
&= \sum_{m\geq n} N_m \log(1 - 2^{-n} + 2^{-n} s^{2^{\alpha(n-m)}}) - A\log(s) \\
&\leq 2^{-n} \sum_{m\geq n} N_m (s^{2^{\alpha(n-m)}} - 1) - A\log(s) \\
&\leq 2^{-(1-\alpha)n} \sum_{m\geq n} N_m 2^{-\alpha m} \underbrace{2^{\alpha(m-n)}(s^{2^{\alpha(n-m)}} - 1)}_{\leq s} - A\log(s). \\
&\leq 2^{-(1-\alpha)n} s c_n - A\log(s),
\end{aligned}$$

where $c_n = \sum_{m\geq n} 2^{-\alpha m} N_m$ is the value of the remainder sum which tends to zero, and which we thus can assume less than 1. Now set $s = 2^{(1-\alpha)n}$, $A = \frac{4}{1-\alpha}$ in the last inequality to get

$$\log P(Y_{n,k} \geq A) \leq 1 - 2n\log(2).$$

Hence, $P(Y_{n,k} \geq \frac{4}{1-\alpha}) \leq C(\alpha)2^{-2n}$, and we conclude as in the Hardy space case. □

Let us give the proof of Theorem 4.1.5.

Proof of Theorem 4.1.5. (i) Let $1/2 < \alpha < 1$.

If $\mathcal{Z}$ is interpolating almost surely, then it is separated almost surely, which implies $\sum_n 2^{-n} N_n^2 < +\infty$.

If $\sum_n 2^{-n} N_n^2 < +\infty$, then $\mathcal{Z}$ is almost surely separated. Moreover, the condition implies that $N_n \leq c 2^{n/2}$ for some constant $c > 0$, and hence

$$\sum_n 2^{-\alpha n} N_n \leq c \sum_n 2^{-(\alpha - 1/2)n} < +\infty.$$

Theorem 4.1.4 implies that $\mu_{\mathcal{Z}}$ is almost surely a Carleson measure so that $\mathcal{Z}$ is almost surely interpolating.

The remaining cases of a divergent sum and interpolation with zero probability follows from the above and the Kolomogorov 0-1 law.

(ii) Consider the case $0 < \alpha \leq 1/2$.

If $\mathcal{Z}$ is interpolating almost surely, then it is a zero sequence almost surely, which implies $\sum_n 2^{-\alpha n} N_n < +\infty$ by Theorem 4.1.2.

Suppose $\sum_n 2^{-\alpha n} N_n < +\infty$. Again by Theorem 4.1.4 we that $\mu_{\mathcal{Z}}$ is almost surely a Carleson measure.

The condition clearly implies also that $\sum_n 2^{-n} N_n^2 < +\infty$, which further yields that the sequence is almost surely separated. Hence it is almost surely an interpolating sequence.

Agian, the remaining cases of a divergent sum and interpolation with zero probability follows from the above and the Kolomogorov 0-1 law.

□

4.4 Separated random sequences for the Dirichlet space

We will now prove the separation result in $\mathcal{D}$.

Proof of Theorem 4.1.7. Separation with probability 0. Assume that for all $\gamma \in (1/2, 1)$ we have $\sup_k 2^{-\gamma k} N_k^2 = \infty$. As it turns out, under the condition of the Theorem, separation already fails in dyadic annuli (without taking into account radial Dirichlet separation).

Assume now that $\gamma_l \to 1$ as $l \to \infty$ and $\sup_k 2^{-\gamma_l k} N_k^2 = \infty$ for every l. For each $k = 1, 2 \ldots$, let $I_k = [1 - 2^{-k+1}, 1 - 2^{-k})$. Define

$$\Omega_k^{(l)} = \{\omega : \exists (i,j),\ i \neq j \text{ with } \rho_i, \rho_j \in I_k \text{ and } |\theta_i(\omega) - \theta_j(\omega)| \leq \pi 2^{-\gamma_l k}\}.$$

In view of (4.1.6), if $\omega \in \Omega_k^{(l)}$, this means that in the dyadic annulus A_k there are at least two points close in the Dirichlet metric. To be more precise, if $\omega \in \Omega_k^{(l)}$, then there is a pair of distinct point $z_i(\omega)$ and $z_j(\omega)$ such that $|z_i|, |z_j| \in I_k$ and $|\arg z_i(\omega) - \arg z_j(\omega)| \leq \pi 2^{-\gamma_l k}$. Hence

$$\frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-\overline{z_i}z_j|^2} \geq c\frac{2^{-2k}}{2^{-2k}+\pi 2^{-2\gamma_l k}},$$

where the constant c is an absolute constant. Hence

$$\frac{(1-|z_i|^2)(1-|z_j|^2)}{|1-\overline{z_i}z_j|^2} \geq c\frac{1}{1+\pi 2^{k(1-\gamma_l)}} \geq c' 2^{-k(1-\gamma_l)} \geq c''(1-|z_i|)^{1-\gamma_l}.$$

Absorbing c'' into a suitable change of the power $\delta_l^2 := 1-\gamma_l$ into $\delta_l'^2$ (which can be taken by choosing for instance $2\delta_l > \delta_l' > \delta_l$ provided k is large enough), then by (4.1.6)

$$d_{\mathcal{D}}(z_i(\omega), z_j(\omega)) \leq \delta_l'.$$

Our aim is thus to show that for every $l \in \mathbb{N}$, we can find almost surely $z_i(\omega) \neq z_j(\omega)$ such that $d_{\mathcal{D}}(z_i(\omega), z_j(\omega)) \leq \delta_l'$, i.e. $P(\Omega_k^{(l)}) = 1$. (Note that $\delta_l' \to 0$ when $l \to +\infty$.)

Let us define a set $E := \{j : 2^{-\gamma_l j-1} N_j \leq 1\}$. Observe that when $k \notin E$, then at least two points are closer than $\pi 2^{-\gamma_l k}$ (this is completely deterministic), so that in that case $P(\Omega_k^{(l)}) = 1$. Hence if $E \subsetneq \mathbb{N}$, then we are done.

Consider now the case $E = \mathbb{N}$, and let $k \in E = \mathbb{N}$. We will use the Lemma on the probability of an uncrowded road [39, p. 740], which states

$$P(\Omega_k^{(l)}) = 1 - (1 - N_k 2^{-\gamma_l k-1})^{N_k-1}$$

(since $E = \mathbb{N}$ this is well defined).

We can assume that $N_k \geq 2$ (since obviously $\sum_{k:N_k<2} 2^{-\gamma_l k} N_k < \infty$). In particular $N_k^2/2 \leq N_k(N_k-1) \leq N_k^2$. Since $\log(1-x) \leq -x$, we get

$$\begin{aligned}
\sum_{k:N_k\geq 2} (N_k-1)\log(1-N_k 2^{-\gamma_l k-1}) &\leq -\sum_{k:M_k\geq 2} (N_k-1)N_k 2^{-\gamma_l k-1} \\
&\leq -\frac{1}{2}\sum_{k:N_k\geq 2} N_k^2 2^{-\gamma_l k-1} \\
&= -\infty
\end{aligned}$$

by assumption. Hence, taking exponentials in the previous estimate,

$$\prod_{k\in E, N_k\geq 2} (1-N_k 2^{-\gamma_l k-1})^{N_k-1} = 0,$$

which implies, by results on convergence on infinite products, that

$$\sum_k P(\Omega_k^{(l)}) = \infty.$$

Since the events $\Omega_k^{(l)}$ are independent, by the Borel–Cantelli Lemma,

$$P(\limsup \Omega_k^{(l)}) = 1,$$

where

$$\limsup \Omega_k^{(l)} = \bigcap_{n\geq 1} \bigcup_{k\geq n} \Omega_k^{(l)} = \{\omega : \omega \in \Omega_k^{(l)} \text{ for infinitely many } k\}.$$

In particular, since the probability of being in infinitely many $\Omega_k^{(l)}$ is one, there is at least one $\Omega_k^{(l)}$ which happens with probability one. So that again $P(\Omega_k^{(l)}) = 1$.

As a result, the probability that the sequence is δ_l'-separated in the Dirichlet metric is zero for every l. Since $\delta_l' \to 0$ when $l \to +\infty$, we deduce that

$$P(\omega : \{z(\omega)\} \text{ is separated for } \mathcal{D}) = 0.$$

Separation with probability 1. Now we assume that $\sum_k 2^{-\gamma k} N_k^2 < +\infty$ for some $\gamma \in (1/2, 1)$. Let us begin defining a neighborhood in the Dirichlet metric. For that, fix $\eta > 1$ and $\alpha \in (0,1)$. Given $z \in \mathcal{Z}$, so that for some k, $z \in A_k$. Consider

$$T_z^{\eta,\alpha} = \{z = re^{it} : (1-|z|)^\eta \leq 1-r \leq (1-|z|)^\eta, |\theta - t| \leq (1-r)^\alpha\}.$$

Figure 4.1 represents the situation.

Our aim is to prove that under the condition $\sum_k 2^{-\gamma k} N_k^2 < +\infty$, there exists $\eta > 1$ and $\alpha \in (0,1)$ such that $T_z^{\eta,\alpha}$ does not contain any other point of $\mathcal{Z}$ except z, and this is true for every $z \in \mathcal{Z}$ with probability one. For this we need to estimate

$$P(T_z^{\eta,\alpha} \cap \mathcal{Z} = \{z\}).$$

Let us cover

$$T_z^{\eta,\alpha} = \bigcup_{j=k/\eta}^{\eta k} (T_z^{\eta,\alpha} \cap A_j),$$

and we need that for every $j \in [k/\eta, \eta k] \setminus \{k\}$, $(T_z^{\eta,\alpha} \cap A_j) \cap \mathcal{Z} = \emptyset$ and $(T_z^{\eta,\alpha} \cap A_k) \cap \mathcal{Z} = \{z\}$. Note that $X_j = \#(T_z^{\eta,\alpha} \cap A_j \cap \mathcal{Z}) \approx B(N_j, 2^{-\alpha j})$,

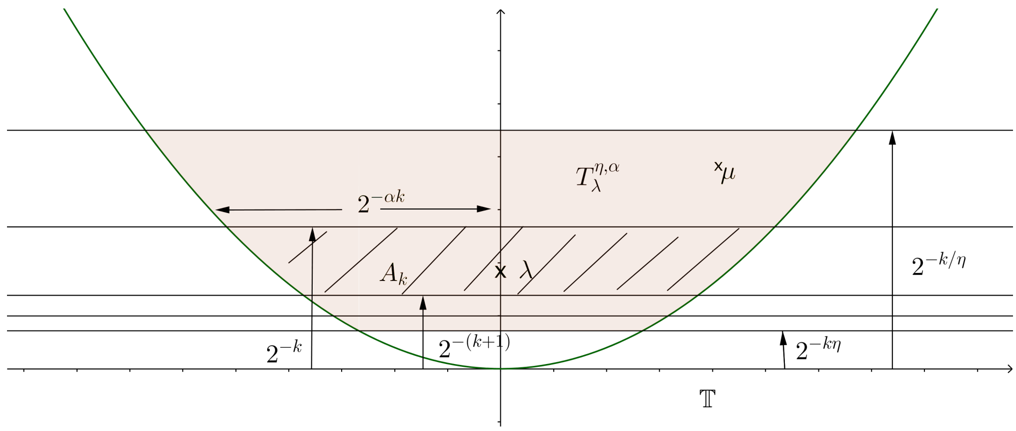

Figure 4.1: Dirichlet neighborhood.

$j \neq k$, and $X_k \approx B(N_k - 1, 2^{-\alpha k})$ (since we do not count z in the latter case). Hence, since the arguments of the points are independent, we have

$$\begin{aligned} P(T_z^{\eta,\alpha} \cap \mathcal{Z} = \{z\}) &= P\Big(\big(\bigcap_{j=k/\eta, j\neq k}^{\eta k} (X_j = 0)\big) \cap (X_k = 1)\Big) \\ &= \prod_{j=k/\eta, j\neq k}^{j=\eta k} \Big(P(X_j = 0)\Big) \times P(X_k = 1). \end{aligned}$$

From the binomial law we have $P(X_j = 0) = (1 - 2^{-\alpha j})^{N_j}$, for $j \in [k/\eta, \eta k] \setminus \{k\}$. Also, assuming $0 < \gamma < \alpha < 1$, we have $N_j 2^{-\alpha j} = o(j)$, so that

$$P(X_j = 0) = (1 - 2^{-\alpha j})^{N_j} \approx 1 - N_j 2^{-\alpha j}.$$

Moreover

$$P(X_k = 1) = N_k 2^{-\alpha k}(1 - 2^{-\alpha k})^{N_k - 1} \approx N_k 2^{-\alpha k}.$$

Hence we get

$$\begin{aligned} P(T_z^{\eta,\alpha} \cap \mathcal{Z} = \{z\}) &\approx \exp\Big(\sum_{j=k/\eta, j\neq k}^{j=\eta k} \ln(P(X_j = 0)\Big) \times N_k 2^{-\alpha k} \\ &\approx \left(1 - \sum_{j=k/\eta, j\neq k}^{j=\eta k} N_j 2^{-\alpha j}\right) \times N_k 2^{-\alpha k}. \end{aligned}$$

Again we use $\gamma < \alpha < 1$ to see now that the sum $\sum_{j=k/\eta, j\neq k}^{\eta k} N_j 2^{-\alpha j}$ is convergent and goes to zero when $k \to \infty$. This shows in particular that the fact of considering the event of having points in neighboring annuli of A_k containing z can be neglected. Hence

$$P(T_z^{\eta,\alpha} \cap \mathcal{Z} = \{z\}) \approx N_k 2^{-\alpha k}.$$

We now sum over all $z \in \mathcal{Z}$ by summing over all dyadic annuli A_k and the N_k points contained in each annuli:

$$\sum_{z\in\mathcal{Z}} P(T_z^{\eta,\alpha} \cap \mathcal{Z} = \{z\}) \approx \sum_{k\in\mathbb{N}} N_k \times N_k 2^{-\alpha k} = \sum_{k\in\mathbb{N}} N_k^2 2^{-\alpha k}.$$

For $\alpha > \gamma$, this sum converges by assumption. Using the Borel-Cantelli lemma we deduce that $T_z^{\eta,\alpha} \cap \mathcal{Z} = \{z\}$ for all but finitely many z with probability one. Obviously these finitely many neighborhoods $T_z^{\eta,\alpha}$ contain finitely many points between which a lower Dirichlet distance exists. This achieves the proof of the separation. □

It should be observed that the above proof only involves α but not η, so that it is the separation in the annuli which dominates the situation.

4.5 Proof of Theorem 4.1.8

In order to prove Theorem 4.1.8 we will use a method similar to the one for $\mathcal{D}_\alpha$.

Let us again observe that interpolation implies the zero sequence condition, so that by Bogdan's result, if $\mathcal{Z}$ is almost surely interpolating then $\sum_n N_n/n < +\infty$.

So suppose now $\sum_n N_n/n < +\infty$. As mentioned in the proof of Theorem 4.1.5, it is enough to check that we almost surely have (4.3.5), now with $\alpha = 1$. So the condition translates to

$$\sum_{z\in S(I)} \left(\log \frac{e}{1-|z|}\right)^{-1} \le c \left(\log \frac{1}{|I|}\right)^{-1},$$

which, using the usual dyadic discretization $I = I_{n,k}$, translates to

$$\sum_{m\ge n} \frac{1}{m} X_{n,m,k} \le C\frac{1}{n} \text{ almost surely.}$$

This leads to estimate the tail of the random variables

$$Y_{n,k} = \sum_{m\ge n} \left(\frac{n}{m}\right) X_{n,m,k}.$$

To do that, introduce again two positive parameters s, A. Using the formula for the generating function of a binomially distributed random variable and Markov's inequality we can estimate as follows

$$\begin{aligned}\log P(Y_{n,k} \geq A) &= \log P(s^{Y_{n,k}} > s^A) \\ &\leq \sum_{m\geq n} N_m \log(1 - 2^{-n} + 2^{-n} s^{(\frac{n}{m})}) - A\log(s) \\ &\leq 2^{-n} \sum_{m\geq n} \frac{N_m}{m} (\frac{m}{n})(s^{(\frac{n}{m})} - 1)n - A\log(s) \\ &\leq n2^{-n} s \sum_{m\geq n} \frac{N_m}{m} - A\log(s).\end{aligned}$$

Setting $s = 2^{n/2}$ and $A = 4$ the above calculation gives

$$P(Y_{n,k} > 4) \lesssim C2^{-2n}.$$

Again, an application of the Borel-Cantelli Lemma concludes the proof.

As already mentioned in preceding cases, the Kolmogorov 0-1 law allows to get also the cases of a divergent sum and interpolation with probability 0.

4.6 Annex : Proof of Theorem 4.1.2

Carleson proved in [25, Theorem 2.2] that, for $0 < \alpha < 1$, if

$$\sum_{z\in\mathcal{Z}} (1 - |z|)^\alpha < \infty$$

then the Blaschke product B associated to $\mathcal{Z}$ belongs to $\mathcal{D}_\alpha$. So the sufficiency part of Theorem 4.1.2 follows immediately from this result (and is moreover deterministically true).

For the proof of the converse we will need the following two lemmas. The first one is a version of the Borel-Cantelli Lemma [20, Theorem 6.3].

Lemma 4.6.1. *If $\{A_n\}$ is a sequence of measurable subsets in a probability space (X, P) such that $\sum P(A_n) = \infty$ and*

$$\liminf_{n\to\infty} \frac{\sum_{j,k\leq n} P(A_j \cap A_k)}{[\sum_{k\leq n} P(A_k)]^2} \leq 1, \tag{4.6.1}$$

then $P(\limsup_{n\to\infty} A_n) = 1$.

The second Lemma is due to Nagel, Rudin and Shapiro [62, 63] who discussed tangential approach regions of functions in $\mathcal{D}_\alpha$.

Lemma 4.6.2. *Let $f \in \mathcal{D}_\alpha$, $0 < \alpha < 1$. Then, for a.e. $\zeta \in \mathbb{T}$, we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each region*

$$|z - \zeta| < \kappa(1 - |z|)^\alpha, \qquad (\kappa > 1).$$

Proof of Theorem 4.1.2. In view of our preliminary observations, we are essentially interested in the converse implication. So suppose $\sum_n 2^{-\alpha n} N_n = +\infty$ or equivalently

$$\sum_n (1 - \rho_n)^\alpha = +\infty. \tag{4.6.2}$$

We have to show that $\mathcal{Z}$ is not a zero sequence almost surely. For this, introduce the intervals $I_\ell = (\mathrm{e}^{-i(1-\rho_\ell)^\alpha}, \mathrm{e}^{i(1-\rho_\ell)^\alpha})$ and let $F_\ell = \mathrm{e}^{i\theta_\ell} I_\ell$. Denoting by m normalized Lebesgue measure on $\mathbb{T}$, observe that

$$m(F_\ell) = m(I_\ell) = (1 - \rho_\ell)^\alpha.$$

We have for every $\varphi \in F_\ell$, $z_\ell \in \Omega_{\kappa,\varphi} = \{z \in \mathbb{D} : |z - \mathrm{e}^{i\varphi}| < \kappa(1 - |z|)^\alpha\}$. By Lemma 4.6.2 it suffices to prove that $|\limsup_\ell F_\ell| > 0$ a.s. (the latter condition means that there is a set of strictly positive measure on $\mathbb{T}$ to which $\mathcal{Z}$ accumulates in Dirichlet tangential approach regions according to Lemma 4.6.2, which is of course not possible for a zero sequence). Let E denote the expectation with respect to the Steinhaus sequence (θ_n). By Fubini's theorem we have $E[m(F_j \cap F_k)] = m(I_j)m(I_k)$, $j \neq k$, (the expected size of intersection of two intervals only depends on the product of the length of both intervals). By Fatou's Lemma and (4.6.2)

$$\begin{aligned}
E\Big[\liminf_{n\to\infty} \frac{\sum_{j,k\le n} m(F_j \cap F_k)}{[\sum_{k\le n} m(F_k)]^2}\Big] &\le \liminf_{n\to\infty} E\Big[\frac{\sum_{j,k\le n} m(F_j \cap F_k)}{[\sum_{k\le n} m(F_k)]^2}\Big] \\
&= \liminf_{n\to\infty} \frac{\sum_{j,k\le n} E[m(F_j \cap F_k)]}{[\sum_{k\le n} m(F_k)]^2} \\
&= \liminf_{n\to\infty} \frac{\sum_{j,k\le n, j\neq k} m(I_j)m(I_k) + \sum_{k\le n} m(I_k)}{[\sum_{k\le n} m(I_k)]^2} \\
&= \liminf_{n\to\infty} \left(1 + \frac{\sum_{k\le n} m(I_k)(1 - m(I_k))}{[\sum_{k\le n} m(I_k)]^2}\right).
\end{aligned}$$

Now, since $1 - m(I_k) \to 1$, and by (4.6.2), keeping in mind that $m(I_k) = (1 - \rho_k)^\alpha$, we have

$$\lim_{n\to\infty} \frac{\sum_{k\le n} m(I_k)(1 - m(I_k))}{[\sum_{k\le n} m(I_k)]^2} = 0.$$

This implies that (4.6.1) holds on a set B of positive probability and hence, by the zero-one law, on a set of probability one. From Lemma 4.6.1 we conclude $P(\limsup_{n\to\infty} F_n) = 1$ a.s., which is what we had to show. □

Chapter 5

Totally null and capacity zero sets for Dirichlet type spaces

5.1 Background

It is well known that H^2 can be identified with the closed subspace of all functions in $L^2(\partial\mathbb{D})$ whose negative Fourier coefficients vanish. Correspondingly, subsets of $\partial\mathbb{D}$ of linear Lebesgue measure zero frequently play the role of small or negligible sets in the theory of H^2 and related spaces. For instance, a classical theorem of Fatou shows that every function in H^2 has radial limits outside of a subset of $\partial\mathbb{D}$ of Lebesgue measure zero; see for instance [54, Chapter 3]. For the disc algebra

$$A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f\big|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D})\},$$

the Rudin–Carleson theorem shows that every compact set $E \subset \partial\mathbb{D}$ of Lebesgue measure zero is an interpolation set for $A(\mathbb{D})$, meaning that for each $g \in C(E)$, there exists $f \in A(\mathbb{D})$ with $f\big|_E = g$. In fact, one can achieve that $|f(z)| < \|g\|_\infty$ for $z \in \overline{\mathbb{D}} \setminus E$ (provided that g is not identically zero); this is called peak interpolation. In particular, there exists $f \in A(\mathbb{D})$ with $f\big|_E = 1$ and $|f(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus E$, meaning that E is peak set for $A(\mathbb{D})$. Conversely, every peak set and every interpolation set has Lebesgue measure zero. For background on this material, see [48, Chapter II].

In the theory of the classical Dirichlet space $\mathcal{D}$ a frequently used notion of smallness of subsets of $\partial\mathbb{D}$ is that of having logarithmic capacity zero; see Section 2.8 . This notion is particularly important in the potential theoretic approch to the Dirichlet space. A theorem of Beurling shows that every function in $\mathcal{D}$ has radial limits outside of a subset of $\partial\mathbb{D}$ of (outer) logarithmic capacity zero; see [46, Section 3.2]. In the context of boundary interpolation, Peller and Khrushchëv [69] showed that a compact set $E \subset$

$\partial\mathbb{D}$ is an interpolation set for $A(\mathbb{D}) \cap \mathcal{D}$ if and only if E has logarithmic capacity zero. Many of these considerations have been extended to standard weighted Dirichlet spaces and their associated capacities, and more generally to Hardy–Sobolev spaces on the Euclidean unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ by Cohn [40] and by Cohn and Verbitsky [42].

As mentioned already these spaces belong to the large family of regular unitarily invariant spaces. In studying regular unitarily invariant spaces $\mathcal{H}$ and especially their multipliers, a functional analytic smallness condition of subsets of $\partial\mathbb{B}_d$ has proved to be very useful in recent years. This smallness condition has its roots in the study of the ball algebra

$$A(\mathbb{B}_d) = \{f \in C(\overline{\mathbb{B}_d}) : f\big|_{\mathbb{B}_d} \in \mathcal{O}(\mathbb{B}_d)\}$$

as explained in Rudin's book [74, Chapter 10].

A complex regular Borel measure μ on $\partial\mathbb{B}_d$ is said to be $\mathcal{M}(\mathcal{H})$-*Henkin* if whenever (p_n) is a sequence of polynomials satisfying $\|p_n\|_{\mathcal{M}(\mathcal{H})} \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} p_n(z) = 0$ for all $z \in \mathbb{B}_d$, we have

$$\lim_{n\to\infty} \int_{\partial\mathbb{B}_d} p_n \, d\mu = 0.$$

A Borel subset $E \subset \partial\mathbb{B}_d$ is said to be $\mathcal{M}(\mathcal{H})$-*totally null* if $|\mu|(E) = 0$ for all $\mathcal{M}(\mathcal{H})$-Henkin measures μ. The Henkin condition can be rephrased in terms of a weak-$*$ continuity property; see Section 5.4 for this reformulation and for more background.

In the case of H^2, a measure is Henkin if and only if it is absolutely continuous with respect to Lebesgue measure on $\partial\mathbb{D}$. Hence the totally null sets are simply the sets of Lebesgue measure 0. Beyond the ball algebra, these notions were first studied by Clouâtre and Davidson for the Drury–Arveson space [37], and then for more general regular unitarily invariant spaces by Bickel, M^{c}Carthy and Hartz [19]. Just as in the case of the ball algebra, Henkin measures and totally null sets appear naturally when studying the dual space of algebras of multipliers [37, 43], ideals of multipliers [38, 43], functional calculi [19, 36], and peak interpolation problems for multipliers [37, 43].

5.2 Main results

In this article, we will compare the functional analytic notion of being totally null with the potential theoretic notion of having capacity zero. As was pointed out in [43], for the Dirichlet space $\mathcal{D}$, the energy characterization of logarithmic capacity easily implies that every compact subset of $\partial\mathbb{D}$ that is $\mathcal{M}(\mathcal{D})$-totally null necessarily has logarithmic capacity zero. We will show that for Hardy–Sobolev spaces spaces on the ball, including the Dirichlet space on the disc, the two notions of smallness in fact agree.

Our main result concerning the Hardy–Sobolev spaces $\mathcal{H}_s$ is the following.

Theorem 5.2.1. *Let $d \in \mathbb{N}$ and let $\frac{d-1}{2} < s \leq \frac{d}{2}$. A compact subset $E \subset \partial\mathbb{B}_d$ is $\mathcal{M}(\mathcal{H}_s)$-totally null if and only if $C_{s,2}(E) = 0$.*

In particular, taking $d = 1$ and $s = \frac{1}{2}$, we see that in the context of the classical Dirichlet space $\mathcal{D}$, a compact subset $E \subset \partial\mathbb{D}$ is $\mathcal{M}(\mathcal{D})$-totally null if and only if it has logarithmic capacity zero.

A direct proof of Theorem 5.2.1 will be provided in Section 5.5.

Moreover, we will prove an abstract result about totally null sets, which, in combination with work on exceptional sets by Ahern and Cohn [3] and by Cohn and Verbitsky [42], will yield a second proof of Theorem 5.2.1. This result applies to some spaces that are not covered by Theorem 5.2.1, such as the Drury–Arveson space.

It is possible to interpret the capacity zero condition as a condition involving the reproducing kernel Hilbert space $\mathcal{H}$ (cf. Proposition 5.5.2 below), whereas the totally null condition is a condition on the multiplier algebra $\mathcal{M}(\mathcal{H})$. Complete Pick spaces form a class of spaces in which it is frequently possible to go back and forth between $\mathcal{H}$ and $\mathcal{M}(\mathcal{H})$.

If $\mathcal{H}$ is a reproducing kernel Hilbert space on $\mathbb{B}_d$, let us say that a compact subset $E \subset \partial\mathbb{B}_d$ is an *unboundedness set for $\mathcal{H}$* if there exists $f \in \mathcal{H}$ so that $\lim_{r \nearrow 1} |f(r\zeta)|$ for all $\zeta \in E$. The following result covers the spaces in Theorem 5.2.1, but it also applies, for example, to the Drury–Arveson space, which corresponds to the end point $s = \frac{d-1}{2}$.

Theorem 5.2.2. *Let $\mathcal{H}$ be a regular unitarily invariant complete Pick space on $\mathbb{B}_d$. A compact set $E \subset \partial\mathbb{B}_d$ is an unboundedness set for $\mathcal{H}$ if and only if E is $\mathcal{M}(\mathcal{H})$-totally null.*

A refinement of this result will be proved in Section 5.6. The results of Ahern and Cohn [3] and of Cohn and Verbitsky [42] on exceptional sets show that in the case of the spaces $\mathcal{H}_s$ for $\frac{d-1}{2} < s \leq \frac{d}{2}$, a compact subset $E \subset \partial\mathbb{B}_d$ is an unboundedness set for $\mathcal{H}_s$ if and only if $C_{s,2}(E) = 0$. Indeed, the "only if" part follows from [3, Theorem B], the "if" part is contained in the construction on p. 443 of [3]; see also [42, p. 94]. Thus, we obtain another proof of Theorem 5.2.1.

5.3 Applications

We close the introduction by mentioning some applications of Theorem 5.2.1. The first application concerns peak interpolation. Extending the work of Peller and Khrushchëv [69] on boundary interpolation in the Dirichlet space, Cohn and Verbitsky [42, Theorem 3] showed that every compact subset $E \subset \partial\mathbb{B}_d$ with $C_{s,2}(E) = 0$ is a strong boundary interpolation set for $\mathcal{H}_s \cap A(\mathbb{B}_d)$.

This means that for every $g \in C(E)$, there exists $f \in \mathcal{H}_s \cap A(\mathbb{B}_d)$ with $f\big|_E = g$ and $\max(\|f\|_{\mathcal{H}_s}, \|f\|_{A(\mathbb{B}_d)}) \le \|f\|_{C(E)}$. Combining Theorem 5.2.1 with a peak interpolation result for totally null sets of Davidson and Hartz [43], we can strengthen the result of Cohn and Verbitsky in two ways. Firstly, we replace $\mathcal{H}_s \cap A(\mathbb{B}_d)$ with the smaller space $A(\mathcal{H}_s)$, which is defined to be the multiplier norm closure of the polynomials in $\mathcal{M}(\mathcal{H}_s)$. Thus,

$$A(\mathcal{H}_s) \subset \mathcal{M}(\mathcal{H}_s) \cap A(\mathbb{B}_d) \subset \mathcal{H}_s \cap A(\mathbb{B}_d)$$

with contractive inclusions. Secondly, we obtain a strict pointwise inequality off of E.

Theorem 5.3.1. *Let $d \in \mathbb{N}$, let $\frac{d-1}{2} < s \le \frac{d}{2}$ and let $E \subset \partial\mathbb{B}_d$ be compact with $C_{s,2}(E) = 0$. Then for each $g \in C(E) \setminus \{0\}$, there exists $f \in A(\mathcal{H}_s)$ so that*

1. *$f\big|_E = g$,*
2. *$|f(z)| < \|g\|_\infty$ for every $z \in \overline{\mathbb{B}_d} \setminus E$, and*
3. *$\|f\|_{\mathcal{M}(\mathcal{H})} = \|g\|_\infty$.*

Proof. According to [43, Theorem 1.4], the conclusion holds when $\mathcal{H}_s$ is replaced with any regular unitarily invariant space $\mathcal{H}$ and E is $\mathcal{M}(\mathcal{H})$-totally null. Combined with Theorem 5.2.1, the result follows. □

In fact, in the setting of Theorem 5.3.1, there exists an isometric linear operator $L : C(E) \to A(\mathcal{H}_s)$ of peak interpolation; see [43, Theorem 8.3]. In a similar fashion, one can now apply other results of [43] in the context of the spaces $\mathcal{H}_s$, replacing the totally null condition with the capacity zero condition. In particular, this yields a joint Pick and peak interpolation result (cf. [43, Theorem 1.5]) and a result about boundary interpolation in the context of interpolation sequences (cf. [43, Theorem 6.6]).

Our second application concerns cyclic functions. Recall that a function $f \in \mathcal{H}_s$ is said to be *cyclic* if the space of polynomial multiples of f in dense in $\mathcal{H}_s$. It is a theorem of Brown and Cohn [24] that if $E \subset \partial\mathbb{D}$ has logarithmic capacity zero, then there exists a function $f \in \mathcal{D} \cap A(\mathbb{D})$ that is cyclic for $\mathcal{D}$ so that $f\big|_E = 0$; see also [47] for an extension to other Dirichlet type spaces on the disc. The following result extends the theorem of Brown and Cohn to the spaces $\mathcal{H}_s$ on the ball, and moreover achieves that $f \in A(\mathcal{H}_s)$, so in particular, f is a multiplier.

Corollary 5.3.2. *Let $d \in \mathbb{N}$, let $\frac{d-1}{2} < s \le \frac{d}{2}$ and let $E \subset \partial\mathbb{B}_d$ be compact with $C_{s,2}(E) = 0$. Then there exists $f \in A(\mathcal{H}_s)$ that is cyclic for $\mathcal{H}_s$ so that $E = \{z \in \overline{\mathbb{B}_d} : f(z) = 0\}$.*

Proof. Applying Theorem 5.3.1 to the constant function $g = 1$, we find $h \in A(\mathcal{H}_s)$ so that $h\big|_E = 1, |h(z)| < 1$ for $z \in \overline{\mathbb{B}_d} \setminus E$ and $\|h\|_{\mathcal{M}(\mathcal{H}_s)} = 1$. Set $f = 1 - h$. Clearly, f vanishes precisely on E. The fact that $\|h\|_{\mathcal{M}(\mathcal{H}_s)} \leq 1$ easily implies that f is cyclic; see for instance [5, Lemma 2.3] and its proof. □

5.4 Regular unitarily invariant spaces and totally null sets

Let $\mathcal{H}$ be a regular unitarily invariant space. We let $\mathcal{M}(\mathcal{H})$ denote the multiplier algebra of $\mathcal{H}$. Identifying a multiplier φ with the corresponding multiplication operator on $\mathcal{H}$, we can regard $\mathcal{M}(\mathcal{H})$ as a WOT closed subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$. By trace duality, $\mathcal{M}(\mathcal{H})$ becomes a dual space in this way, and hence is equipped with a weak-$*$ topology. The density of the linear span of kernel functions in $\mathcal{H}$ implies that on bounded subsets of $\mathcal{M}(\mathcal{H})$, the weak-$*$ topology agrees with the topology of pointwise convergence on $\mathbb{B}_d$. In a few places, we will use the following basic and well known fact, which we state as a lemma for easier reference. For a proof, see for instance [43, Lemma 2.2].

Lemma 5.4.1. *Let $\mathcal{H}$ be a regular unitarily invariant space and let $\varphi \in \mathcal{M}(\mathcal{H})$. Let $\varphi_r(z) = \varphi(rz)$ for $0 \leq r \leq 1$ and $z \in \mathbb{B}_d$. Then $\|\varphi_r\|_{\mathcal{M}(\mathcal{H})} \leq \|\varphi\|_{\mathcal{M}(\mathcal{H})}$ for all $0 \leq r \leq 1$ and $\lim_{r \nearrow 1} \varphi_r = \varphi$ in the weak-$*$ topology of $\mathcal{M}(\mathcal{H})$.*

Let $M(\partial\mathbb{B}_d)$ be the space of complex regular Borel measures on $\partial\mathbb{B}_d$.

Definition 5.4.2. Let $\mathcal{H}$ be a regular unitarily invariant space.

(a) A measure $\mu \in M(\partial\mathbb{B}_d)$ is said to be $\mathcal{M}(\mathcal{H})$-Henkin if the functional

$$\mathcal{M}(\mathcal{H}) \supset \mathbb{C}[z_1, \ldots, z_d] \to \mathbb{C}, \quad p \mapsto \int_{\partial\mathbb{B}_d} p \, d\mu,$$

extends to a weak-$*$ continuous functional on $\mathcal{M}(\mathcal{H})$.

(b) A Borel subset $E \subset \partial\mathbb{B}_d$ is said to be $\mathcal{M}(\mathcal{H})$-totally null if $|\mu|(E) = 0$ for all $\mathcal{M}(\mathcal{H})$-Henkin measures μ.

By [19, Lemma 3.1], the definition of Henkin measure given here is equivalent to the one given in the introduction in terms of sequences of polynomials converging pointwise to zero. The set of $\mathcal{M}(\mathcal{H})$-Henkin measures forms a band (see [19, Lemma 3.3]), meaning in particular that μ is $\mathcal{M}(\mathcal{H})$-Henkin if and only if $|\mu|$ is Henkin. This band property implies that a compact set E is $\mathcal{M}(\mathcal{H})$-totally null if and only if $\mu(E) = 0$ for every positive $\mathcal{M}(\mathcal{H})$-Henkin measure μ that is supported on E; see [43, Lemma 2.5].

5.5 Direct proof of Theorem 5.2.1

To prove Theorem 5.2.1, we will make use of holomorphic potentials. Since several of our proofs involve reproducing kenrel arguments, it is slightly more convenient to work with the spaces $\mathcal{D}_a$ rather than with $\mathcal{H}_s$.

Definition 5.5.1. Let $0 \le a < 1$ and let $\mu \in M^+(\partial \mathbb{B}_d)$. The holomorphic potential of μ is the function

$$f_\mu : \mathbb{B}_d \to \mathbb{C}, \quad z \mapsto \int_{\partial \mathbb{B}_d} K_a(z,w) \, d\mu(w).$$

Let $A(\mathbb{B}_d)$ denote the ball algebra. If $\mu \in M^+(\partial \mathbb{B}_d)$, let

$$\rho_\mu : A(\mathbb{B}_d) \to \mathbb{C}, \quad f \mapsto \int_{\partial \mathbb{B}_d} f \, d\mu$$

denote the associated integration functional.

The following functional analytic interpretation of the holomorphic potential and of capacity will show that every totally null set has capacity zero. In the case of the Dirichlet space on the disc, it is closely related to the energy formula for logarithmic capacity in terms of the Fourier coefficients of a measure; see for instance [46, Theorem 2.4.4].

Proposition 5.5.2. *Let $\mu \in M^+(\partial \mathbb{B}_d)$ and let $0 \le a < 1$. The following assertions are equivalent:*

(i) $\mathcal{E}(\mu, \mathcal{D}_a) < \infty$,

(ii) *the densely defined functional ρ_μ is bounded on $\mathcal{D}_a$,*

(iii) $f_\mu \in \mathcal{D}_a$.

In this case,

$$\mathcal{E}(\mu, \mathcal{D}_a) \approx \|\rho_\mu\|^2_{(\mathcal{D}_a)^*} = \|f_\mu\|^2_{\mathcal{D}_a},$$

where the implied constants only depend on a and d, and

$$\rho_\mu(g) = \langle g, f_\mu \rangle_{\mathcal{D}_a}$$

for all $g \in \mathcal{D}_a$.

Proof. For ease of notation, we write $f = f_\mu$, $\rho = \rho_\mu$ and $k_w(z) = K_a(z,w)$. For $0 \le r < 1$, define $f_r(z) = f(rz)$ and $\rho_r(f) = \rho(f_r)$. Then each $f_r \in \mathcal{D}_a$ and each ρ_r is a bounded functional on $\mathcal{D}_a$. First, we connect f_r and ρ_r, which will be useful in all parts of the proof. By the reproducing property of the kernel, we find that

$$\langle k_z, f_r \rangle = \overline{f(rz)} = \int_{\partial \mathbb{B}_d} k_{rz}(w) \, d\mu(w) = \int_{\partial \mathbb{B}_d} k_z(rw) \, d\mu(w) = \rho_r(k_z)$$

for all $z \in \mathbb{B}_d$. Since finite linear combinations of kernels are dense in $\mathcal{D}_a$, it follows that

$$\rho_r(g) = \langle g, f_r \rangle \tag{5.5.1}$$

for all $g \in \mathcal{D}_a$ and hence $\|\rho_r\|_{(\mathcal{D}_a)^*} = \|f_r\|_{\mathcal{D}_a}$.

Next, we show the equivalence of (ii) and (iii). If $f \in \mathcal{D}_a$, then $\lim_{r \nearrow 1} f_r = f$ in $\mathcal{D}_a$ and hence for all $g \in A(\mathbb{B}_d) \cap \mathcal{D}_a$, Equation (5.5.1) shows that

$$\rho(g) = \lim_{r \nearrow 1} \rho_r(g) = \lim_{r \nearrow 1} \langle g, f_r \rangle = \langle g, f \rangle,$$

so ρ is bounded on $\mathcal{D}_a$. In this case, $\|\rho\|_{(\mathcal{D}_a)^*} = \|f\|_{\mathcal{D}_a}$, which establishes the final statement of the proposition holds as well. Conversely, if ρ is bounded on $\mathcal{D}_a$, then since $\|\rho_r\|_{(\mathcal{D}_a)^*} \le \|\rho\|_{(\mathcal{D}_a)^*}$, it follows that $\sup_{0 \le r < 1} \|f_r\|_{\mathcal{D}_a} \le \|\rho\|_{(\mathcal{D}_a)^*}$, hence $f \in \mathcal{D}_a$.

It remains to show the equivalence of (i) and (iii) and that $\mathcal{E}(\mu, \mathcal{D}_a) \approx \|f\|^2_{\mathcal{D}_a}$. With the help of Equation (5.5.1), we see that

$$\begin{aligned} \|f_r\|^2_{\mathcal{D}_a} = \langle f_r, f_r \rangle = \rho_r(f_r) &= \int f(r^2 z)\, d\mu(z) \\ &= \iint K_a(rz, rw)\, d\mu(w) d\mu(z), \end{aligned}$$

where all integrals are taken over $\partial \mathbb{B}_d$. Taking real parts and using the fact that $\operatorname{Re} K_a$ and $|K_a|$ are comparable, we find that

$$\|f_r\|^2_{\mathcal{D}_a} \approx \iint |K_a(rz, rw)| d\mu(z) d\mu(w).$$

Thus, if $f \in \mathcal{D}_a$, then Fatou's lemma shows that

$$\mathcal{E}(\mu, \mathcal{D}_a) = \iint |K_a(z, w)|\, d\mu(z) d\mu(w) \lesssim \|f\|^2_{\mathcal{D}_a}.$$

Conversely, if $\mathcal{E}(\mu, \mathcal{D}_a) < \infty$, we use the basic inequality

$$\Big| \frac{1}{1 - r^2 \langle z, w \rangle} \Big| \le 2 \Big| \frac{1}{1 - \langle z, w \rangle} \Big| \quad (z, w \in \mathbb{B}_d)$$

and the Lebesgue dominated convergence theorem to find that

$$\lim_{r \nearrow 1} \|f_r\|^2_{\mathcal{D}_a} \lesssim \mathcal{E}(\mu, \mathcal{D}_a),$$

so $f \in \mathcal{D}_a$ and $\|f\|^2_{\mathcal{D}} \lesssim \mathcal{E}(\mu, \mathcal{D}_a)$. □

With this proposition in hand, we can prove the "only if" part of Theorem 5.2.1, which we restate in equivalent form (see Corollary 2.8.5 for the equivalence). The idea is the same as that in the proof of [43, Proposition 2.6].

Proposition 5.5.3. *Let $0 \le a < 1$ and let $E \subset \partial\mathbb{B}_d$ be compact. If E is $\mathcal{M}(\mathcal{D}_a)$-totally null, then $c_\alpha(E) = 0$.*

Proof. Suppose that $c_\alpha(E) > 0$. Then E supports a probability measure μ of finite energy $\mathcal{E}(\mu, \mathcal{D}_a)$. By Proposition 5.5.2, we see that the integration functional ρ_μ is bounded on $\mathcal{D}_a$. In particular, it is weak-$*$ continuous on $\mathcal{M}(\mathcal{D}_a)$. Hence E is not $\mathcal{M}(\mathcal{D}_a)$-totally null. □

To prove the converse, we require the following fundamental properties of the holomorphic potential of a capacitary extremal measure of a compact subset $E \subset \mathbb{B}_d$, i.e. a measure for which the supremum in (2.8.2) is achieved. If $a > 0$, these properties are contained in the proof of [3, Theorem 2.10], see also [32, Lemma 2.3] for a proof in the case $d = 1$ and $a = 0$. An argument that directly works with the logarithmic capacity (in the case $d = 1$ and $a = 0$) can be found on pp. 40–41 of [46]. We briefly sketch the argument in general.

Lemma 5.5.4. *Let $E \subset \partial\mathbb{B}_d$ be a compact set with $c_\alpha(E) > 0$. There exists a positive measure μ supported on E so that the corresponding holomorphic potential f_μ satisfies*

(a) $f_\mu \in \mathcal{D}_a$ *with* $\|f_\mu\|^2_{\mathcal{D}_a} \lesssim c_\alpha(E)$.

(b) $\liminf_{r \nearrow 1} \operatorname{Re} f_\mu(r\zeta) \gtrsim 1$ *for all* $\zeta \in \operatorname{int}(E)$, *and*

(c) $|f_\mu(z)| \lesssim 1$ *for all* $z \in \mathbb{B}_d$.

Here, the implied constants only depend on a and d.

Proof. Let $s = \frac{d-a}{2}$, so that $\mathcal{H}_s = \mathcal{D}_a$ with equivalent norms. The general theory of Bessel capacities (see [1, Theorem 2.5.3]), combined with the maximum principle for the capacity $C_{s,2}(\cdot)$ [3, Lemma 1.15] implies that there exists a positive measure μ supported on E so that

1. $\mu(E) = \|\mathcal{I}_s(\mu)\|^2_{L^2(\partial\mathbb{B}_d, d\sigma)} = C_{s,2}(E)$;

2. $\mathcal{I}_s(\mathcal{I}_s(\mu)) \ge 1$ on $E \setminus F$, where F is a countable union of compact sets of $C_{s,2}$-capacity zero.

3. $\mathcal{I}_s(\mathcal{I}_s(\mu)) \lesssim 1$ on $\partial\mathbb{B}_d$.

(See also [46, Corollary 2.4.3] for an approach using the logarithmic capacity in the case $d = 1$ and $a = 0$, instead of the Bessel $C_{s,2}$ capacity.)

Item (1) and Corollary 2.8.5 show that $\mathcal{E}(\mu, \mathcal{D}_a) \approx \|\mathcal{I}_s(\mu)\|^2_{L^2(\partial\mathbb{B}_d,\sigma)} \approx c_\alpha(E)$, hence Proposition 5.5.2 yields that (a) holds.

Lemma 2.8.4 and Item (3) show that for $z \in \partial\mathbb{B}_d$, we have

$$\int_{\partial\mathbb{B}_d} |K_a(z, w)| d\mu(w) = \mathcal{I}_{2s}(\mu)(z) \approx \mathcal{I}_s(\mathcal{I}_s(\mu))(z) \lesssim 1,$$

so in combination with the basic inequality $|\frac{1}{1-r\langle z,w\rangle}| \le 2|\frac{1}{1-\langle z,w\rangle}|$ for $z, w \in \partial\mathbb{B}_d$ and $0 \le r < 1$, we see that (c) holds.

To establish (b), notice that (c) implies that $f_\mu \in H^\infty(\mathbb{B}_d)$, so f_μ has radial boundary limits f_μ^* almost everywhere with respect to σ, and $f_\mu = P[f_\mu^*]$, the Poisson integral of f_μ^*. Fatou's lemma and the fact that $\operatorname{Re} K_a$ and $|K_a|$ are comparable show that for σ-almost every $z \in \partial\mathbb{B}_d$, the estimate

$$\begin{aligned}\operatorname{Re} f_\mu^*(z) = \lim_{r\nearrow 1} \int_{\partial\mathbb{B}_d} \operatorname{Re} K_a(rz,w)d\mu(w) &\gtrsim \int_{\partial\mathbb{B}_d} |K_a(z,w)|\, d\mu(w) \\ &= \mathcal{I}_{2s}(\mu)(z).\end{aligned}$$

Now $C_{s,2}(K) = 0$ implies that $\sigma(K) = 0$ for compact sets $K \subset \partial\mathbb{B}_d$. (This is because $\sigma\big|_K$ has finite energy, which for instance follows from Proposition 5.5.2 since $\mathcal{D}_a$ is continuously contained in $H^2(\mathbb{B}_d)$.) Therefore, Item (2) and Lemma 2.8.4 imply that $\operatorname{Re} f_\mu^*(z) \gtrsim 1$ for σ-almost every $z \in E$. In combination with $\operatorname{Re} f_\mu = P[\operatorname{Re} f_\mu^*]$, this easily implies (b). □

In [30], Cascante, Fàbrega and Ortega showed that if $0 < a < 1$ and if the holomorphic potential f_μ is bounded in $\mathbb{B}_d$, then it is a multiplier of $\mathcal{D}_a$. They also proved an L^p-analogue of this statement. We will require an explicit estimate for the multiplier norm of f_μ. It seems likely that the arguments in [30] could be used to obtain such an estimate. Instead, we will provide a different argument in the Hilbert space setting, based on the following result of Aleman, M^c^Carthy, Richter and Hartz [6]. It also applies to the case $a = 0$ without changes. The function V_f below is called the *Sarason function* of f.

Theorem 5.5.5 ([6])**.** *Let $0 \le a < 1$, let $f \in \mathcal{D}_a$ and define*

$$V_f(z) = 2\langle f, K_a(\cdot,z)f\rangle - \|f\|^2.$$

If $\operatorname{Re} V_f$ is bounded in $\mathbb{B}_d$, then $f \in \mathcal{M}(\mathcal{D}_a)$ and

$$\|f\|_{\mathcal{M}(\mathcal{D}_a)} \lesssim \|\operatorname{Re} V_f\|_\infty^{1/2},$$

where the implied constant only depends on a and d.

Proof. In [6, Theorem 4.5], it is shown that if $\mathcal{H}$ is a normalized complete Pick space that admits an equivalent norm which is given by an L^2-norm of derivatives of order at most N, then boundedness of $\operatorname{Re} V_f$ implies that $f \in \mathcal{M}(\mathcal{H})$, and

$$\|f\|_{\mathcal{M}(\mathcal{H})} \lesssim (\|\operatorname{Re} V_f\|_\infty + 3)^{N+\frac{1}{2}}.$$

This applies in particular to the spaces $\mathcal{D}_a$. The improved bound on the multiplier norm of f follows from the scaling properties of both sides of the inequality. Indeed, if $t > 0$, then $V_{tf} = t^2 V_f$, so applying the above inequality to the function tf, we find that

$$\|f\|^2_{\mathcal{M}(\mathcal{D}_a)} \lesssim \frac{1}{t^2}(t^2 \|\operatorname{Re} V_f\|_\infty + 3)^{2N+1}$$

for all $t > 0$. If $\|\operatorname{Re} V_f\|_\infty = 0$, then taking $t \to \infty$ above yields $f = 0$. If $\|\operatorname{Re} V_f\|_\infty \neq 0$, then choosing $t = \|\operatorname{Re} V_f\|_\infty^{-1/2}$, we obtain the desired estimate. (The choice of t could be optimized to improve the implicit constants, but we will not do so here.) □

With the help of Theorem 5.5.5, we can establish the desired multiplier norm estimate of f_μ. It can be regarded as a quantitative version of the result of Cascante, Fàbrega and Ortega [30] in the Hilbert space setting.

Proposition 5.5.6. *Let $0 \le a < 1$ and let $\mu \in M^+(\partial \mathbb{B}_d)$. If f_μ is bounded in $\mathbb{B}_d$, then f_μ is a multiplier of $\mathcal{D}_a$, and*

$$\|f_\mu\|_{\mathcal{M}(\mathcal{D}_a)} \approx \|f_\mu\|_\infty,$$

where the implied constants only depend on a and d.

Proof. Since the multiplier norm dominates the supremum norm, we have to show the inequality "$\lesssim$". Let $f = f_\mu$ and

$$f_r(z) = f(rz) = \int_{\partial \mathbb{B}_d} K_a(rz, w)\, d\mu(w) = \int_{\partial \mathbb{B}_d} K_a(z, rw)\, d\mu(w). \tag{5.5.2}$$

We will show that $\|f_r\|_{\mathcal{M}} \lesssim \|f_r\|_\infty$ for all $0 < r < 1$, where the implied constant is independent of f and r. To simplify notation, write $k_w(z) = K_a(z, w)$. We will use Theorem 5.5.5 and instead show that

$$\sup_{z \in \mathbb{B}_d} \operatorname{Re}\langle f_r, k_z f_r \rangle \lesssim \|f_r\|^2_\infty.$$

Since the map

$$\partial \mathbb{B}_d \to \mathcal{D}_a, \quad w \mapsto k_{rw},$$

is continuous, the integral on the right-hand side of (5.5.2) converges in $\mathcal{H}$. Thus, by the reproducing property of the kernel,

$$\begin{aligned}\operatorname{Re}\langle f_r, k_z f_r \rangle &= \iint \langle k_{rw}, k_z k_{r\zeta} \rangle d\mu(\zeta) d\mu(w) \\ &= \iint \overline{k_{r\zeta}(rw) k_z(rw)}\, d\mu(\zeta) d\mu(w).\end{aligned}$$

Equation (5.5.2) shows that $\int k_{r\zeta}(rw)\, d\mu(\zeta) = f_r(rw)$, hence

$$\operatorname{Re}\langle f_r, k_z f_r \rangle = \int \overline{f_r(rw) k_z(rw)}\, d\mu(w)$$

and so

$$\operatorname{Re}\langle f_r, k_z f_r\rangle \le \|f_r\|_\infty \int |k_z(rw)|\, d\mu(w) \lesssim \|f_r\|_\infty \operatorname{Re} \int k_z(rw)\, d\mu(w)$$
$$= \|f_r\|_\infty \operatorname{Re} f_r(z) \le \|f_r\|_\infty^2. \qquad \square$$

We are ready to provide the direct proof of Theorem 5.2.1, which we restate in equivalent form.

Theorem 5.5.7. *Let $0 \le a < 1$ and let $E \subset \partial\mathbb{B}_d$ be compact. Then E is $\mathcal{M}(\mathcal{D}_a)$-totally null if and only if $c_\alpha(E) = 0$.*

Proof. The "only if" part was already established in Proposition 5.5.3. Conversely, suppose that $c_\alpha(E) = 0$. By upper semi-continuity of capacity, there exists a decreasing sequence (E_n) of compact neighborhoods of E so that $\lim_{n\to\infty} c_\alpha(E_n) = 0$; see [46, Theorem 2.1.6]. Let μ_n be a positive measure supported on E_n as in Lemma 5.5.4 and let $g^{(n)} = f_{\mu_n}$ be the corresponding holomorphic potential. We claim that

1. $\liminf_{r\nearrow 1} \operatorname{Re} g^{(n)}(r\zeta) \gtrsim 1$ for all $\zeta \in E$ and all $n \in \mathbb{N}$;
2. the sequence $(g^{(n)})$ converges to 0 in the weak-$*$ topology of $\mathcal{M}(\mathcal{D}_a)$.

Indeed, Part (1) is immediate from Lemma 5.5.4. To see (2), we first observe that Lemma 5.5.4 (c) and Proposition 5.5.6 imply that the sequence $(g^{(n)})$ is bounded in multiplier norm. Using Lemma 5.5.4 (a), we see that $\|g^{(n)}\|^2_{\mathcal{D}_a} \lesssim c_\alpha(E_n)$, so $(g^{(n)})$ converges to zero in the norm of $\mathcal{D}_a$ and in particular pointwise on $\mathbb{B}_d$, hence (2) holds.

Let now ν be a positive $\mathcal{M}(\mathcal{D}_a)$-Henkin measure that is supported on E. We will finish the proof by showing that $\nu(E) = 0$; see the discussion following Definition 5.4.2. Item (1) above and Fatou's lemma show that

$$\nu(E) = \int_E 1\, d\nu \lesssim \liminf_{r\nearrow 1} \int_{\partial\mathbb{B}_d} \operatorname{Re} g^{(n)}(r\zeta)\, d\nu(\zeta).$$

Since ν is $\mathcal{M}(\mathcal{D}_a)$-Henkin, the associated integration functional ρ_ν extends to a weak-$*$ continuous functional on $\mathcal{M}(\mathcal{D}_a)$, which we continue to denote by ρ_ν. Since $\lim_{r\nearrow 1} g_r^{(n)} = g^{(n)}$ in the weak-$*$ topology of $\mathcal{M}(\mathcal{D}_a)$ by Lemma 5.4.1, we find that for all $n \in \mathbb{N}$,

$$\lim_{r\nearrow 1} \int_{\partial\mathbb{B}_d} \operatorname{Re} g_n(r\zeta) d\nu(\zeta) = \operatorname{Re} \rho_\nu(g_n).$$

Thus,

$$\nu(E) \lesssim \operatorname{Re} \rho_\nu(g_n)$$

for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ and using Item (2), we see that $\nu(E) = 0$, as desired. $\square$

5.6 Proof of Theorem 5.2.2

In this section, we prove a refined version of Theorem 5.2.2. Let $\mathcal{H}$ be a regular unitarily invariant space on $\mathbb{B}_d$. Recall that a compact set $E \subset \mathbb{B}_d$ is said to be an *unboundedness set for* $\mathcal{H}$ if there exists $f \in \mathcal{H}$ with $\lim_{r \nearrow 1} |f(r\zeta)| = \infty$ for all $\zeta \in E$. We also say that E is a *weak unboundedness for* $\mathcal{H}$ if there exists a separable auxiliary Hilbert space $\mathcal{E}$ and $f \in \mathcal{H} \otimes \mathcal{E}$ so that $\lim_{r \nearrow 1} \|f(r\zeta)\| = \infty$ for all $\zeta \in E$.

Theorem 5.6.1. *Let $\mathcal{H}$ be a regular unitarily invariant complete Pick space on $\mathbb{B}_d$. The following assertions are equivalent for a compact set $E \subset \mathbb{B}_d$.*

(i) *E is $\mathcal{M}(\mathcal{H})$-totally null.*

(ii) *E is an unboundedness set for $\mathcal{H}$.*

(iii) *E is a weak unboundedness set for $\mathcal{H}$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that E is totally null. In the first step, we will show that for each $M > 1$, there exists $f \in \mathcal{H} \cap A(\mathbb{B}_d)$ satisfying

1. $f\big|_E = M$,
2. $\|f\|_{\mathcal{H}} \leq 1$, and
3. $\operatorname{Re} f \geq 0$ on $\overline{\mathbb{B}_d}$.

Let $\varepsilon = 1/M$. Since E is totally null, the simultaneous Pick and peak interpolation result [43, Theorem 1.5] shows that there exists $\eta \in A(\mathcal{H}) \subset \mathcal{H} \cap A(\mathbb{B}_d)$ satisfying $\eta\big|_E = (1-\varepsilon^2)^{1/2}$, $\eta(0) = 0$ and $\|\eta\|_{\mathcal{M}(\mathcal{H})} \leq 1$. It follows that the column multiplier

$$\begin{bmatrix} \varepsilon \\ (1-\varepsilon^2)^{1/2}\eta \end{bmatrix}$$

has multiplier norm at most one, so the implication (b) $\Rightarrow$ (a) of part (i) of [6, Theorem 1.1] implies that the function f defined by

$$f = \frac{\varepsilon}{1 - (1-\varepsilon^2)^{1/2}\eta}$$

belongs to the closed unit ball of $\mathcal{H}$. Moreover, since $\|\eta\|_{\mathcal{M}(\mathcal{H})} \leq 1$, we find that $|\eta(z)| \leq 1$ for all $\zeta \in \overline{\mathbb{B}_d}$, from which it follows that $f \in A(\mathbb{B}_d)$ and $\operatorname{Re} f \geq 0$. Clearly, $f\big|_E = \frac{1}{\varepsilon} = M$. This observation finishes the construction of f.

The above construction yields, for each $n \geq 1$, a function $f_n \in \mathcal{H} \cap A(\mathbb{B}_d)$ satisfying $f_n\big|_E = 1$, $\|f_n\|_{\mathcal{H}} \leq 2^{-n}$ and $\operatorname{Re} f_n \geq 0$. Define $f = \sum_{n=1}^\infty f_n \in \mathcal{H}$. Let $\zeta \in E$. Then for each $N \in \mathbb{N}$, we have that

$$\liminf_{r \nearrow 1} \operatorname{Re} f(r\zeta) \geq \sum_{n=1}^{N} \operatorname{Re} f_n(\zeta) = N.$$

Thus, $\lim_{r \nearrow 1} |f(r\zeta)| = \infty$ for all $\zeta \in E$, so E is an unboundedness set for $\mathcal{H}$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Suppose that E is a weak unboundedness set for $\mathcal{H}$ and let $f \in \mathcal{H} \otimes \mathcal{E}$ satisfy $\|f\| \le 1$ and $\lim_{r \nearrow 1} \|f(r\zeta)\| = \infty$ for all $\zeta \in E$. By the implication (a) $\Rightarrow$ (b) of part (i) of [6, Theorem 1.1], we may write $f = \frac{\Phi}{1-\psi}$, where $\Phi \in \mathcal{M}(\mathcal{H}, \mathcal{H} \otimes \mathcal{E}), \psi \in \mathcal{M}(\mathcal{H})$ have multiplier norm at most 1 and $|\psi(z)| < 1$ for all $z \in \mathbb{B}_d$. In particular, $\|\Phi(z)\| \le 1$ for all $z \in \mathbb{B}_d$, hence $\lim_{r \nearrow 1} \psi(r\zeta) = 1$ for all $\zeta \in E$.

Let now μ be a positive $\mathcal{M}(\mathcal{H})$-Henkin measure that is supported on E and let ρ_μ denote the associated weak-$*$ continuous integration functional on $\mathcal{M}(\mathcal{H})$. We have to show that $\mu(E) = 0$; see the discussion following Definition 5.4.2. To this end, we write $\psi_r(z) = \psi(rz)$ and let $n \in \mathbb{N}$. Applying the dominated convergence theorem and the fact that $\lim_{r \nearrow 1} \psi_r^n = \psi^n$ in the weak-$*$ topology of $\mathcal{M}(\mathcal{H})$ (see Lemma 5.4.1), we find that

$$\mu(E) = \lim_{r \nearrow 1} \int_E \psi_r^n \, d\mu = \lim_{r \nearrow 1} \int_{\partial \mathbb{B}_d} \psi_r^n \, d\mu = \rho_\mu(\psi^n).$$

Since ψ is a contractive multiplier satisfying $|\psi(z)| < 1$ for all $z \in \mathbb{B}_d$, it follows that ψ^n tends to zero in the weak-$*$ topology of $\mathcal{M}(\mathcal{H})$. So taking the limit $n \to \infty$ above, we conclude that $\mu(E) = 0$, as desired. $\square$

Let us briefly compare the direct proof of the implication "capacity 0 implies totally null" given in Section 5.5 with the proof via Theorem 5.2.2. If $E \subset \partial \mathbb{B}_d$ is a compact set with $C_{s,2}(E) = 0$, then the work of Ahern and Cohn [3] and of Cohn and Verbitsky [42] shows that E is unboundedness set for $\mathcal{H}_s$. To show this, they use holomorphic potentials and their fundamental properties (cf. Lemma 5.5.4) to construct a function $f \in \mathcal{H}_s$ satisfying $\lim_{r \nearrow 1} |f(r\zeta)| = 1$ for all $\zeta \in E$. Proceeding via Theorem 5.2.2, one then applies the factorization result [6, Theorem 1.1] to f to obtain a multiplier ψ of $\mathcal{H}$ of norm at most 1 satisfying $\lim_{r \nearrow 1} \psi(r\zeta) = 1$ for all $\zeta \in E$, from which the totally null property of E can be deduced.

The direct proof given in Section 5.5 uses holomorphic potentials as well, this time to construct a sequence of functions in $\mathcal{H}$, which, roughly speaking, have large radial limits on E compared to their norm. It is then shown that the holomorphic potentials themselves form a bounded sequence of multipliers, from which the totally null property of E can once again be deduced.

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