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Maximum Principle, Mean Value Operators and Quasi Boundedness in non-Euclidean Settings

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Introduction

This work deals with some classes of linear second order partial differential operators with non-negative characteristic form and underlying non-Euclidean structures. These structures are determined by families of locally Lipschitz-continuous vector fields in $\mathbb{R}^N$, generating metric spaces of Carnot-Carathéodory type. The Carnot-Carathéodory metric related to a family $\{X_j\}_{j=1,\dots,m}$ is the control distance obtained by minimizing the time needed to go from two points along piecewise trajectories of vector fields. We are mainly interested in the causes in which a Sobolev-type inequality holds with respect to the X-gradient, and/or the X-control distance is Doubling with respect to the Lebesgue measure in $\mathbb{R}^N$. This study is divided into three parts (each corresponding to a chapter), and the subject of each one is a class of operators that includes the class of the subsequent one.

In the first chapter, after recalling "X-ellipticity" and related concepts introduced by Kogoj and Lanconelli in [KL00], we show a Maximum Principle for linear second order differential operators for which we only assume a Sobolev-type inequality together with a lower terms summability. Adding some crucial hypotheses on measure and on vector fields (Doubling property and Poincaré inequality), we will be able to obtain some Liouville-type results. This chapter is based on the paper [GL03] by Gutiérrez and Lanconelli.

In the second chapter we treat some ultraparabolic equations on Lie groups. In this case $\mathbb{R}^N$ is the support of a Lie group, and moreover we require that vector fields satisfy left invariance. After recalling some results of Cinti [Cin07] about this class of operators and associated potential theory, we prove a scalar convexity for mean-value operators of $\mathcal{L}$-subharmonic functions, where $\mathcal{L}$ is our differential operator.

In the third chapter we prove a necessary and sufficient condition of regularity, for boundary points, for Dirichlet problem on an open subset of $\mathbb{R}^N$ related to sub-Laplacian. On a Carnot group we give the essential background for this type of operator, and introduce the notion of "quasi-boundedness". Then we show the strict relationship between this notion, the fundamental solution of the given operator, and the regularity of the boundary points.

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Chapter 1

Maximum Principle, non-homogeneous Harnack inequality, and Liouville theorems for X-elliptic operators

The aim of this Chapter is to prove a Maximum Principle for uniformly X-elliptic operators under Sobolev inequality and regularity assumption on the lower order terms. With these results at hand we obtain a Harnack inequality for non-homogeneous equations. As a consequence we derive a Liouville inequality, for which we require in addition Doubling property, Poincaré inequality and the dilatation invariance property. When the vector fields are left invariant with respect to left translations on a Carnot group, and form a basis of the first layer of its Lie algebra, then the operator (called sub-Laplacian) satisfies Harnack inequality (Bonfiglioli and Lanconelli, [BL01]). These authors massively use ad hoc Green's representation formulas.
For uniformly subelliptic operators on a Lie group (particular X-operators) the homogeneous Harnack inequality was proved by Varopoulos ([Var87]),

Saloff-Coste ([Sc90]) and Saloff-Coste with Stroock ([SS91]).
In some contexts we want to underline the crucial role of Hörmander condition on dimension of generated Lie algebra from a family of vector fields, in that case Lancia and Marchi ([LM97]) and later Cancelier with Xu ([CX00]) and Baldi with Franchi and Lu ([BFL00]) proved Maximum Principle and homogeneous Harnack inequalities.
Many other similar results can be found in literature, at the beginning especially in Heisenberg group and other dilation invariant operators with smooth coefficients (Korányi and Stanton [KS85], Geller [Gel83]). Finally we want to recall the case of non-smooth vector fields discussed by Franchi and Lanconelli ([FL83], [FL85]). Without giving a complete report of the results in literature, we can summarise as follows:

- Sub-Laplacian on Carnot group $\Rightarrow$ Harnack;
- Uniformly subelliptic operator on Lie group $\Rightarrow$ Harnack;
- Operator with smooth vector fields + Hörmander condition $\Rightarrow$ Harnack;
- Operator with non-smooth vector fields + Ellipticity $\Rightarrow$ Harnack.

We prove, for uniformly X-elliptic operators, that

- Sobolev + Regularity of lower order terms + Positivity condition $\Rightarrow$ Maximum Principle;
- Sobolev + Regularity of lower order terms + Positivity condition + Doubling + Poincaré + Invariance $\Rightarrow$ Harnack and Liouville.

We finally observe that many results present in the past literature are indeed included in our work.

1.1 Introduction

Let $\{X_1, \dots, X_m\}$ be a family of vector fields in $\mathbb{R}^N$, with locally Lipschitz continuous coefficients in $\mathbb{R}^N$.
We recall that f is locally Lipschitz countinuous if

$$\forall K \Subset \mathbb{R}^N \ \exists c_{f,K} \ : \ |f(x) - f(y)| \le c_{f,K}|x - y|, \quad \forall x, y \in K.$$

We denote the vector field X_j with the first order differential operator

$$X_j = \sum_{k=1}^{N} a_{kj}(x)\partial_k, \quad \partial_k = \frac{\partial}{\partial x_k} \tag{1.1}$$

and $X_jI(x)$ represents the vector

$$X_jI(x) := \begin{pmatrix} a_{1j}(x) \\ a_{2j}(x) \\ \vdots \\ a_{Nj}(x) \end{pmatrix} \ : \ \mathbb{R}^N \to \mathbb{R}^N, \quad a_{kj}(x) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}).$$

Here we consider linear second order differential operators of the form

$$Lu = \sum_{i,j=1}^{N} \partial_i(b_{ij}\partial_j u + d_i u) + \sum_{i=1}^{N} b_i \partial_i u + cu, \tag{1.2}$$

where $b_{ij}(x) = b_{ji}(x), d_i, b_i$ and c are measurable functions. Set $B := (b_{ij}), \ d := (d_1, \dots, d_N)$ and $b := (b_1, \dots, b_N)$.

1.2 X-Elliptic operators

Definition 1.2.1 (X**-elliptic).** Let Ω be an open subset of $\mathbb{R}^N$, L be the differential operator of the form (1.2). The operator L is X-elliptic in Ω if following conditions are satisfied:

1. There exists a constant $\lambda \in \mathbb{R}$, $\lambda > 0$, such that

$$\lambda \sum_{j=1}^{m} \langle X_jI(x), \xi\rangle^2 \le \langle B(x)\xi, \xi\rangle, \qquad \forall \xi \in \mathbb{R}^N, \ \forall x \in \Omega, \tag{1.3}$$

where $\langle B(x)\xi, \xi\rangle$ is the characteristic form of L given by

$$\langle B(x)\xi, \xi\rangle = \sum_{i,j=1}^{N} b_{ij}(x)\xi_i\xi_j; \tag{1.4}$$

2. There exists a function $\gamma : \Omega \to \mathbb{R},\ \gamma \geq 0$, such that

$$\langle d(x), \xi\rangle^2 + \langle b(x), \xi\rangle^2 \leq \gamma(x)^2 \sum_{j=1}^{m} \langle X_j I(x), \xi\rangle^2, \qquad \forall \xi \in \mathbb{R}^N,\ x \in \Omega \tag{1.5}$$

Definition 1.2.2 (**Uniformly X-Elliptic**). Let L be an X-elliptic differential operator defined as above. We say that L is uniformly X-elliptic in Ω if there exists $\Lambda > 0$ such that in addition the following condition is satisfied:

$$\langle B(x)\xi, \xi\rangle \leq \Lambda \sum_{j=1}^{m} \langle X_j I(x), \xi\rangle^2, \qquad \forall \xi \in \mathbb{R}^N,\ x \in \Omega. \tag{1.6}$$

1.3 On vector fields and associated control distance

Definition 1.3.1 (**X-Gradient**). Let $X_1, \ldots, X_m$ be differential operators as defined in (1.1), u be a function in $C^1(\Omega, \mathbb{R})$. We call X-gradient of u the vector

$$Xu := (X_1 u, \ldots, X_m u). \tag{1.7}$$

If $u \notin C^1(\Omega, \mathbb{R})$, the partial derivatives shall be intented in distribution sense.

Definition 1.3.2 (**Absolute continuity**). Let I be an interval of $\mathbb{R}$. A function $f : I \to \mathbb{R}^N$ is absolutely continuous on I if for every $\varepsilon > 0$, there is an $\eta > 0$ small enough so that whenever a sequence of pairwise disjoint sub-intervals $[x_k, y_k]$ of I, $k = 1, \ldots, n$ satisfies

$$\sum_{k=1}^{n} |y_k - x_k| < \eta \quad \Rightarrow \quad \sum_{k=1}^{n} ||f(y_k) - f(x_k)|| < \varepsilon,$$

where $||\cdot||$ shall be intended as Euclidean norm. Absolute continuity could be defined in a generic metric space (whose definition is given later), but we are not interested to do it.

Definition 1.3.3 (**X-Trajectory**). Let $\gamma : [0,1] \to \mathbb{R}^N$ be an absolutely continuous path. γ is an X-trajectory if

$$\dot{\gamma} = \sum_{j=1}^{m} a_j(s) X_j\big(\gamma(s)\big) \qquad \text{a.e. in } [0,1],$$

with $a_j : [0,1] \to \mathbb{R}$, $j = 1, \ldots, m$, are measurable functions.

Definition 1.3.4 (**X-Connection** and $\mathcal{T}(\cdot,\cdot)$). We will call $\mathbb{R}^N$ X-connected if for any $x, y \in \mathbb{R}^N$ there exists an X-trajectory connecting x and y (i.e. $\gamma(0) = x,\ \gamma(1) = y$). By $\mathcal{T}(x,y)$ we shall denote the set of all X-trajectories connecting x and y.

Definition 1.3.5 (**Control distance**). Suppose $\mathbb{R}^N$ is X-connected and let γ be an X-trajectory. If we set

$$||\gamma||_X := \sup_{t\in[0,1]} \left(\sum_{j=1}^{m} a_j^2(t) \right)^{1/2},$$

we define

$$d_X(x,y) := \inf\{||\gamma||_X : \gamma \in \mathcal{T}(x,y)\},$$

called control distance.

Definition 1.3.6 (**Quasi metric and metric**). Given a non-empty set Υ, a function $d : \Upsilon \times \Upsilon \to [0,+\infty)$ is called quasi metric if it is symmetric, strictly positive out of $\{x = y\}$ and there exists a constant $T \geq 1$ such that

$$d(x,y) \leq T\big(d(x,z) + d(y,z)\big)$$

for all $x, y, z \in \Upsilon$. The pair (Υ, d) is called quasi metric space. If $T = 1$, then d is called metric and the pair is called metric space.

Proposition 1.3.7 (**d_X is a metric**). *If $\mathbb{R}^N$ is X-connected, then the function $(x,y) \mapsto d_X(x,y)$ is a metric on $\mathbb{R}^N$.*

Proof. See ([BLU07] Proposition 5.2.3).

$\square$

Definition 1.3.8 (**d-balls and Doubling property (D)**). Let (Υ, d) be a quasi metric space. The d-ball with center $x \in \Upsilon$ and radius $r > 0$ is given by

$$B(x,r) = B_r(x) := \{y \in \Upsilon \mid d(x,y) < r\}.$$

Let μ be a positive measure on a σ-algebra of subsets of Υ containing the d-balls, then we say that μ satisfies the doubling property if there exists a positive constant D such that

$$0 < \mu\big(B(x,2r)\big) \leq D\mu\big(B(x,r)\big), \qquad \forall x \in \Upsilon \text{ and } r > 0. \tag{1.8}$$

(Υ, d, μ) will be consequently called *doubling quasi metric space.*
We recall an alternative version of the doubling property (1.8):

$$\mu\big(B(x,r_2)\big) \leq D\Big(\frac{r_2}{r_1}\Big)^Q \mu\big(B(x,r_1)\big), \tag{1.9}$$

where $r_1 < r_2$ and $Q = \log_2 D$.

In a general metric space the ball measure is not necessarily continuous with respect to d. A sufficient condition to have the continuity is the following property.

Definition 1.3.9 (**Segment property**). The metric space (Υ, d) has the segment property, if for any $x, y \in \Upsilon$ there exists a d-continuous curve $\gamma : [0,1] \to \Upsilon$ such that $\gamma(0) = x, \gamma(1) = y$ and

$$d(x,y) = d\big(x,\gamma(s)\big) + d\big(\gamma(s),y\big), \qquad \forall s \in [0,1].$$

Lemma 1.3.10 (**Segment property and d-continuity**). *Let (Υ, d, μ) be a doubling metric space satisfying the segment property. Then, for each $x \in \Upsilon$, the function $r \mapsto \mu\big(B(x,r)\big)$ is continuous with respect to d.*

Proof. If we set $B_r^* := \{y \in \Upsilon \mid d(x,y) \leq r\}$ and $\partial^* B_r := B_r^* \setminus B_r = \{y \in \Upsilon \mid d(x,y) = r\}$ we have

$$\lim_{\varrho \to r^-} \mu(B_\varrho) = \mu(B_r), \quad \lim_{\varrho \to r^+} \mu(B_\varrho) = \mu(B_r^*),$$

and so to prove the lemma it is enough to show that $\mu(\partial^* B_r) = 0$. Suppose by contradiction that $\mu(\partial^* B_r) > 0$. By Lebesgue's differentiation theorem in a doubling metric space ([Hei00] Theorem 1.8), we have

$$\chi_{B_r}(y) = \lim_{R\to 0} \fint_{B_R(y)} \chi_{B_r}(x)\,\mathrm{d}\mu(x) \qquad \text{for } \mu\text{-a.e. } y \in \Upsilon.$$

In particular , if y belongs to $\partial^* B_r$ we have $\chi_{B_r}(y) = 0$, because of $\partial^* B_r \cap B_r = \emptyset$. Therefore if we show that

$$\frac{\mu\big(B_r \cap B_R(y)\big)}{\mu\big(B_R(y)\big)} \geq C > 0 \tag{1.10}$$

for all $y \in \partial^* B_r$ and for all R sufficiently small we get a contradiction. Fix $y \in \partial^* B_r$, let x_0 be the center of B_r, and by the segment property let γ be a d-continuous curve joining x_0 and y such that $d(x_0, y) = d(x_0, z) + d(z, y)$ for all $z \in \gamma$. Picking $z \in \gamma$ such that $d(z, y) = R/2$, by the segment property we have that

$$r = d(x_0, y) = d(x_0, z) + d(z, y) = d(x_0, z) + R/2 \quad \Rightarrow \quad d(x_0, z) = r - R/2$$

and so we obtain for every $R \in]0, r[$ and for every $\xi \in B_{R/2}(z)$ that

$$\begin{aligned} d(\xi, x_0) &\leq d(\xi, z) + d(z, x_0) < R/2 + r - R/2 = r, \\ d(\xi, y) &\leq d(\xi, z) + d(z, y) < R/2 + R/2 = R, \end{aligned}$$

so it follows that

$$B_{R/2}(z) \subset B_r \cap B_R(y). \tag{1.11}$$

Moreover for every $\xi \in B_R(y)$ we have

$$d(\xi, z) \leq d(\xi, y) + d(y, z) < R + \frac{R}{2} = \frac{3R}{2},$$

whence

$$B_R(y) \subset B_{3R/2}(z). \tag{1.12}$$

Finally

$$\mu\big(B_r \cap B_R(y)\big) \overset{(1.11)}{\geq} \mu\big(B_{R/2}(z)\big) \overset{(1.9)}{\geq} \frac{1}{C_D 3^Q}\mu\big(B_{3R/2}(z)\big) \overset{(1.12)}{\geq} \frac{1}{C_D 3^Q}\mu\big(B_R(y)\big),$$

and so (1.10) follows.

□

Proposition 1.3.11 (**d-continuity** w.r.t. **Euclidean topology**). *If in the metric space $(\mathbb{R}^N, d)$ the function $(x, y) \mapsto d(x, y)$ is continuous with respect to Euclidean topology, then it satisfies the segment property.*

Proof. See ([GDMN96], Lemma 3.7) and ([FL83]), where this property was established for the first time in a non-Euclidean space.

□

Hereafter the Lebesgue measure of a measurable subset of $\mathbb{R}^N$ will be denoted by $|\cdot|$.

1.4 Conditions on the operators and on the control distance

Following conditions carry out a crucial role in our theory, and among them there are well known relationships we describe later.

Definition 1.4.1 (**Sobolev inequality - (S)**). Be $\Omega \subseteq \mathbb{R}^N$ an open set, a differential operator X satisfies **(S)** condition if there exists $q = q(\Omega) > 2$ and $S = S(\Omega, X) > 0$ such that

$$||u||_{L^q(\Omega)} \leq S_{\Omega,X}||Xu||_{L^2(\Omega)} \qquad \forall u \in C_0^1(\Omega). \tag{1.13}$$

We also set $Q := \frac{2q}{q-2}$, so $Q > 2$ and $q = \frac{2Q}{Q-2}$.

Definition 1.4.2 (**On lower terms - (L)**). Be L a differential operator of the form (1.2), where c is a measurable function, and γ is the function in (1.14) that controls lower order terms behaviour. L satisfies **(L)** condition if there exists $p \in]Q/2, +\infty[$ such that

$$\gamma \in L^{2p}(\Omega), \tag{1.14}$$

and

$$c \in L^p(\Omega). \tag{1.15}$$

Definition 1.4.3 (**Poincaré inequality - (P)**)**.** Be Ω an bounded open subset of $\mathbb{R}^N$, where K is compact. A differential operator X satisfies **(P)** condition if for each compact set $K \subset \Omega$ there exist positive constants $R_0,\ C,\ \nu > 1$ such that

$$\fint_{B_r} |u - u_r| \,\mathrm{d}x \leq Cr \fint_{B_{\nu r}} |Xu| \,\mathrm{d}x, \qquad \forall u \in C^1(\overline{\Omega}), \tag{1.16}$$

for any d-ball $B_r(x)$ with $r \leq R_0$, $B_{\nu r} \subset K$ and $u_r := \fint_{B_r} u(x)\,\mathrm{d}x$. For simplicity we shall assume $\nu = 2$.

Definition 1.4.4 (**Dilatation invariance - (I)**)**.** Let $\alpha_1, \ldots, \alpha_N$ be positive integers, $Q := \alpha_1 + \ldots + \alpha_N$, and for $R > 0$ we set $\delta_R x = (R^{\alpha_1} x_1, \ldots, R^{\alpha_N} x_N)$. We say that the vector fields $\{X_j\}$, $j = 1, \ldots, m$, satisfy **(I)** condition w.r.t. δ_R if they verify the following homogeneity property:

$$X_j\big(u(\delta_R x)\big) = R(X_j u)(\delta_R x), \tag{1.17}$$

Following consequences are well proved:

- **(D)** + **(P)** + continuity of $d \Rightarrow$ **(S)** on Ω where q is function of doubling constant(i.e. $Q = \log_2 D$),
  see ([GDMN96] Theorem 1.5),([FLW96]), ([HK00]).
- **(D)** + **(P)** + **(I)** $\Rightarrow$ **(S)** on every bounded open subset of $\mathbb{R}^N$ ([GDMN96]).

We shall underline that there exist Lipschitz vector fields for which **(S)** holds but **(D)** does not (See [GL03] Par. 6.2).

1.5 Essential theory of $C^1_0(\Omega)$, $W^1_0(\Omega, X)$, $W^1(\Omega, X)$

In this section we describe underlying structure of $C^1_0(\Omega)$, $W^1_0(\Omega, X)$, $W^1(\Omega, X)$. Hereafter Ω is a bounded open subset of $\mathbb{R}^N$ on which the operator X verifies Sobolev condition **(S)** for some $q > 2$ (if not different stated). To simplify notations the Lebesgue measure and the variables functions will be omitted.

Theorem 1.5.1 ($||Xu||_{L^2(\Omega)}$ **is a norm**). *Let u be a function of $C^1_0(\Omega)$, Xu be its X-gradient satisfying* **(S)** *condition. Then the function $u \mapsto ||Xu||_{L^2(\Omega)}$ is a norm in $C^1_0(\Omega)$.*

Proof. Some steps are trivial like positivity and linearity, they are obtained directly from definition of seminorm in $L^2(\Omega)$ and of X-gradient. Triangle inequality is direct consequence of Minkowski inequality with $p = 2$ and linearity of X-gradient. The most important step is to show that this seminorm is a norm (i.e. $||Xu||_{L^2(\Omega)} = 0 \ \Leftrightarrow \ u \equiv 0$). The "only if" part is due to Sobolev condition for $q > 2$:

$$0 = ||Xu||_{L^2(\Omega)} \geq ||u||_{L^q(\Omega)} \ \Rightarrow u \equiv 0 \ \text{ a.e. in } \Omega \ \Rightarrow \ u \equiv 0 \text{ in } \Omega,$$

because u belongs to $C^1_0(\Omega)$.

□

Definition 1.5.2 ($W^1_0(\Omega, X)$). If $\{\varphi_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ are Cauchy sequences in $C^1_0(\Omega)$ with the norm introduced above we can define the following equivalence relation

$$\{\varphi_j\}_{j\in\mathbb{N}} \ \sim \ \{\psi_j\}_{j\in\mathbb{N}} \ \Leftrightarrow \ ||X\varphi_j - X\psi_j||_{L^2(\Omega)} \xrightarrow[j\to+\infty]{} 0,$$

and the space

$$\mathcal{W}^1_0(\Omega, X) := \left\{ \{\varphi_j\}_{j\in\mathbb{N}} \ \middle| \ \varphi_j \in C^1_0(\Omega) \ , \ ||X\varphi_n - X\varphi_m||_{L^2(\Omega)} \xrightarrow[n,m\to+\infty]{} 0 \right\}.$$

By consequence we call

$$W^1_0(\Omega, X) := \mathcal{W}^1_0(\Omega, X) \Big/ \sim .$$

Definition 1.5.3 ($|| \cdot ||_{W^1_0(\Omega,X)}$). Let $u := [\{\varphi_j\}_{j\in\mathbb{N}}] \in W^1_0(\Omega, X)$ be the equivalence class of its sequence. We define

$$||u||_{W^1_0(\Omega,X)} := \lim_{j\to+\infty} ||X\varphi_j||_{L^2(\Omega)}. \tag{1.18}$$

It is well posed (easily seen taking the limit of the difference of two sequences in the same class representing u), and the proof that it's a norm is almost identical to the one of Theorem 1.5.1.
Moreover if α,β are real numbers we can define also

$$\alpha[\{\varphi_j\}_{j\in\mathbb{N}}]+\beta[\{\psi_j\}_{j\in\mathbb{N}}]:=[\{\alpha\varphi_j+\beta\psi_j\}_{j\in\mathbb{N}}],$$

that gives to $W_0^1(\Omega,X)$ a vector space structure.

Lemma 1.5.4. *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence in $C_0^1(\Omega)$, $q\geq 2$. We recall that $\Omega\subset\mathbb{R}^N$ is an open bounded set. We have*

$$\varphi_j \xrightarrow[j\to+\infty]{} u \text{ in } L^q(\Omega) \quad \Rightarrow \quad \varphi_j \xrightarrow[j\to+\infty]{} u \textit{ in } L^2(\Omega).$$

Proof. In general for every $f\in L^p(\Omega)$

$$\begin{aligned}
||f||^p_{L^2(\Omega)} &= \int_\Omega |f|^p\cdot 1 = \int_\Omega |f|^{q\frac{p}{q}}\cdot 1^{1-\frac{p}{q}} \leq \\
&\overset{\text{Hölder}}{\leq} \Big(\int_\Omega |f|^q\Big)^{\frac{p}{q}}\cdot\Big(\int_\Omega 1\Big)^{1-\frac{p}{q}} = \\
&= ||f||^p_{L^q(\Omega)}\cdot|\Omega|^{1-\frac{p}{q}}, \qquad (1.19)
\end{aligned}$$

and so by boundedness of Ω we have proved in particular that

$$\exists C_{q,\Omega}\in\mathbb{R} \quad : \quad ||\varphi_j-u||_{L^2(\Omega)}\leq C_{q,\Omega}\,||\varphi_j-u||_{L^q(\Omega)} \quad \forall q\geq 2. \qquad (1.20)$$

$\square$

Theorem 1.5.5 **($W_0^1(\Omega,X)$ embedded in $L^p(\Omega)$).** *Let $q>2$ be the constant for which* **(S)** *is verified. Then $W_0^1(\Omega,X)$ is a Banach space and for every p such that $2\leq p\leq q$ we have*

$$W_0^1(\Omega,X)\ \hookrightarrow\ L^p(\Omega).$$

Proof. If $u=[\{\varphi_j\}_{j\in\mathbb{N}}]\in W_0^1(\Omega)$, from **(S)** there exists $q>2$ such that

$$||\varphi_n-\varphi_m||_{L^q(\Omega)}\leq S_{\Omega,X}||X\varphi_n-X\varphi_m||_{L^2(\Omega)},$$

but from definition of u this right-hand side tends to zero, so $\{\varphi_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega)$. By its completeness

$$\exists u^* \in L^q(\Omega) \ : \ \varphi_j \xrightarrow[n\to+\infty]{} u^* \quad \text{in } L^q(\Omega).$$

We consider the map

$$\mathcal{T} \ : \ W_0^1(\Omega, X) \to L^q(\Omega), \quad u \to \mathcal{T}u := u^*.$$

We want to prove that $\mathcal{T}$ is injective, and by its structure it is enough to show that $\ker(\mathcal{T}) = [0]$.
So with $\{\varphi_j\}_{j\in\mathbb{N}}$ converging to zero in $L^q(\Omega)$ let us prove that $\{X\varphi_j\}_{j\in\mathbb{N}}$ tends to zero in $L^2(\Omega)$.
$\{X\varphi_j\}_{j\in\mathbb{N}}$ has limit in $L^2(\Omega)$ because it is a Cauchy sequence in a complete space. If we take ψ in $C_0^\infty(\Omega)$ and set $h := \lim\limits_{j\to+\infty} X\varphi_j$, we have

$$\int_\Omega X\varphi_j \cdot \psi \ \xrightarrow[j\to+\infty]{} \ \int_\Omega h \cdot \psi$$

because of the just given definitions. Integrating by parts left-hand side (allowed because of locally Lipschitz coefficients of X and Rademacher's theorem) we get

$$\int_\Omega \varphi_j X^*\psi \ \xrightarrow[j\to+\infty]{} \ 0, \quad (X^* \text{ formal adjoint of } X)$$

that tends to zero accordingly to $\varphi_j \xrightarrow[j\to+\infty]{} 0$ in $L^q(\Omega)$ and that $X^*\psi$ belongs to $L^{\overline{q}}(\Omega)$, with $\overline{q}$ such that $1/q + 1/\overline{q} = 1$. Hence

$$\forall\psi \in C_0^\infty(\Omega) \quad \int_\Omega h \cdot \psi = 0 \qquad \Rightarrow \qquad h = 0 \ \text{ a.e. in } \Omega,$$

and so $X\varphi_j \xrightarrow[j\to+\infty]{} 0$ in $L^2(\Omega)$.
Now we identify $W_0^1(\Omega, X)$ with $\mathcal{T}(W_0^1(\Omega, X)) \subseteq L^q(\Omega)$, and through that we arrive to previous definition of $W_0^1(\Omega, X)$, i.e.

$$u \in W_0^1(\Omega, X) \quad \Leftrightarrow \quad \begin{cases} \exists\{\varphi_j\}_{j\in\mathbb{N}} & \text{with } \ \varphi_j \in C_0^1(\Omega) \\ X\varphi_j \xrightarrow[j\to+\infty]{} f \ \text{ in } \ L^2(\Omega). \end{cases}$$

By consequence of **(S)** and Lemma 1.5.4 we have also that

$$\varphi_j \xrightarrow[j\to+\infty]{} u \quad \text{in} \quad L^2(\Omega),$$

and setting

$$Xu := \lim_{j\to+\infty} X\varphi_j \quad \text{in} \quad L^2(\Omega)$$

we arrive to Definition 1.5.3 $||u||_{W_0^1(\Omega,X)} = ||Xu||_{L^2(\Omega)}$. If $2 \le p \le q$, let us consider the map

$$\mathcal{I} \ : \ W_0^1(\Omega, X) \to L^p(\Omega), \quad u \to u,$$

which is still well posed because of Lemma 1.5.4. Moreover

$$\begin{aligned} ||\mathcal{I}(u)||_{L^p(\Omega)} &= ||u||_{L^p(\Omega)} \le C_{p,q,\Omega}\, ||u||_{L^q(\Omega)} \le \\ &\overset{\textbf{(S)}}{\le} C_{p,q,\Omega} S_{\Omega,X} ||Xu||_{L^2(\Omega)} = C_{p,q,\Omega} S_{\Omega,X} ||u||_{W_0^1(\Omega,X)}, \end{aligned}$$

whence

$$W_0^1(\Omega, X) \hookrightarrow L^p(\Omega) \quad 2 \le p \le q.$$

Now we shall recall a crucial result, which has been the real reason of $W_0^1(\Omega, X)$ usage.

Theorem 1.5.6 ($W_0^1(\Omega, X)$ **is complete**)**.** $(W_0^1(\Omega, X), ||\cdot||_{W_0^1(\Omega,X)})$ *is a complete space.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W_0^1(\Omega, X)$, from **(S)** it follows that it's a Cauchy sequence in $L^2(\Omega)$ as well. If $u := \lim\limits_{n\to+\infty} u_n$ in $L^2(\Omega)$ we want to show that

$$u \in W_0^1(\Omega, X) \quad \text{and} \quad u_n \xrightarrow[n\to+\infty]{} u \quad \text{in} \quad W_0^1(\Omega, X). \tag{1.21}$$

We know by convergence in $L^2(\Omega)$ that

$$\forall\, k \in \mathbb{N} \quad \exists\, n(k) \in \mathbb{N} \quad : \quad ||u - u_n||_{L^2(\Omega)} < \frac{1}{k} \qquad \forall n \ge n(k),$$

and we obtain that

$$\forall\ n(k) \quad \exists\ j(k) \in \mathbb{N} \quad : \quad \begin{cases} \|u_{n(k)} - \varphi_{n(k),j}\|_{L^2(\Omega)} < \frac{1}{k} & \forall j \geq j(k), \\ \|Xu_{n(k)} - X\varphi_{n(k),j}\|_{L^2(\Omega)} < \frac{1}{k} & \forall j \geq j(k). \end{cases}$$

If we set $\psi_k := \varphi_{n(k),j(k)}$ $(\in C_0^1(\Omega))$ by easy computations

$$\|u - \psi_k\|_{L^2(\Omega)} \leq \|u - u_{n(k)}\|_{L^2(\Omega)} + \|u_{n(k)} - \psi_k\|_{L^2(\Omega)} < \frac{2}{k},$$

we get that $\psi_k \xrightarrow[k\to+\infty]{} u$ in $L^2(\Omega)$. Furthermore

$$\begin{aligned} \|X\psi_N - X\psi_M\|_{L^2(\Omega)} &= \|X\psi_{n(N),j(N)} - X\psi_{n(M),j(M)}\|_{L^2(\Omega)} \leq \\ &\leq \|X\psi_{n(N),j(N)} - Xu_{n(N)}\|_{L^2(\Omega)} + \\ &+ \|Xu_{n(N)} - Xu_{n(M)}\|_{L^2(\Omega)} + \\ &+ \|Xu_{n(M)} - X\psi_{n(M),j(M)}\|_{L^2(\Omega)} \leq \\ &\leq \frac{1}{N} + \frac{1}{M} + \|Xu_{n(N)} - Xu_{n(M)}\|_{L^2(\Omega)}, \end{aligned}$$

but if $n(N), n(M)$ are sufficiently great we get

$$\|Xu_{n(N)} - Xu_{n(M)}\|_{L^2(\Omega)} = \|u_{n(N)} - u_{n(M)}\|_{W_0^1(\Omega,X)} < \varepsilon.$$

Then we have obtained that $\{X\psi_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, and so we can set

$$h := \lim_{N\to+\infty} \psi_N \quad \text{ in } L^2(\Omega).$$

Finally $u \in W_0^1(\Omega, X)$ and

$$\exists \varphi_j \in C_0^1(\Omega) \quad : \quad \begin{cases} \varphi_j \xrightarrow[j\to+\infty]{} u \ \text{ in } \ L^2(\Omega) \\ X\varphi_j \xrightarrow[j\to+\infty]{} h \ \text{ in } \ L^2(\Omega). \end{cases}$$

We should prove the second part of (1.21) in the distribution sense, i.e. for every $\psi \in C_0^\infty(\Omega)$

$$\int_\Omega Xu \cdot \psi = \int_\Omega u \cdot X^*\psi.$$

Through an integration by parts (admitted by Rademacher's theorem, every a_k is locally Lipschitz) we have

$$\begin{aligned}\int_\Omega h\cdot\psi &= \lim_{j\to+\infty}\int_\Omega X\varphi_j\cdot\psi = \lim_{j\to+\infty}\int_\Omega\Big(\sum_k a_k\partial_k\varphi_j\Big)\cdot\psi = \\ &= \lim_{j\to+\infty}\left[-\sum_k\int_\Omega\varphi_j\cdot\partial(a_k\psi)\right] = -\sum_k\int_\Omega u\cdot\partial(a_k\psi) = \\ &\stackrel{\text{def}}{=} \int_\Omega u\cdot X^*\psi.\end{aligned}$$

and so the assertion is proved.

□

Lemma 1.5.7. *If $u \in W^1_0(\Omega, X)$, then X_ju exists in the sense of distributions and $X_ju \in L^2(\Omega)$, for $j = 1, \ldots, m$. Moreover*

$$\int_\Omega (X_ju)v = -\int_\Omega\Big(uX_jv + \sum_{k=1}^N\partial_k(a_{jk})uv\Big),$$

for every $v \in W^1_0(\Omega, X)$.

Proof. Proceeding similarly to last theorem proof (i.e. integrating by parts and by density) it is easy to show above formula.

□

Corollary 1.5.8 (**X-gradient in $W^1_0(\Omega, X)$**)**.** *The X-gradient is well defined for any $v \in W^1_0(\Omega, X)$.*

Definition 1.5.9 ($W^1(\Omega, X)$)**.** We define

$$W^1(\Omega, X) := \{u \in L^2(\Omega) \mid Xu \in L^2(\Omega)\}$$

Theorem 1.5.10 ($||\cdot||_{W^1(\Omega,X)}$)**.** *The function*

$$u \quad\mapsto\quad ||u||_{L^2(\Omega)} + ||Xu||_{L^2(\Omega)}$$

is a norm in $W^1(\Omega, X)$.

Theorem 1.5.11 (density in $W^1(\Omega, X)$). *We have*

$$W^1(\Omega, X) = \overline{\{u \in C^\infty(\Omega) \mid u, Xu \in L^2(\Omega)\}}^{||\cdot||_{W^1(\Omega,X)}}.$$

Two last cited theorems are well proved and discussed in [GDMN96].

Lemma 1.5.12 (norms in $W_0^1(\Omega, X)$). *In $W_0^1(\Omega, X)$ we have that*

$$||\cdot||_{W_0^1(\Omega,X)} \quad \sim \quad ||\cdot||_{W^1(\Omega,X)}.$$

Proof. Because of (1.20) and **(S)** we write

$$||Xu||_{L^2(\Omega)} \leq ||u||_{L^2(\Omega)} + ||Xu||_{L^2(\Omega)} \leq (1 + C_{q,\Omega} S_{\Omega,X})||Xu||_{L^2(\Omega)}.$$

□

Remark 1.5.13. Of course we have the following inclusions:

$$W_0^1(\Omega, X) \subset W^1(\Omega, X) \subset W_{\text{loc}}^1(\Omega, X).$$

Lemma 1.5.14 (Chain rule in $W^1(\Omega, X)$). *Let $f \in C^1(\mathbb{R})$ and $f' \in L^\infty(\mathbb{R})$ and $u \in W^1(\Omega, X)$. Then we have that $f(u) \in W^1(\Omega, X)$ and $X\big(f(u)\big) = f'(u)Xu$.*
If $u \in W_0^1(\Omega, X)$ and $f(0) = 0$, then $f(u) \in W_0^1(\Omega, X)$.

Proof. The proof follows the lines of the proof of Lemma 7.5 in [GT83].

□

Definition 1.5.15 (Positive and negative functions). We define

$$u^+ := \max\{u, 0\} \qquad u^- := \min\{u, 0\}.$$

Trivial deductions are that $u = u^+ + u^-$ and $|u| = u^+ - u^-$.

Corollary 1.5.16. *If $u \in W_0^1(\Omega, X)$ or $u \in W^1(\Omega, X)$, the same holds for $|u|$ and we have*

$$X|u| = Xu \cdot \text{sgn}(u) = \begin{cases} Xu & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -Xu & \text{if } u < 0. \end{cases}$$

In particular, if $u \in W_0^1(\Omega, X)$*, then* $u^+ \in W_0^1(\Omega, X)$ *and*

$$Xu^+ = Xu \cdot \operatorname{sgn}(u^+) = \begin{cases} Xu & \textit{if } u > 0 \\ 0 & \textit{if } u \leq 0. \end{cases}$$

Proof. The proof follows the same lines of the one of Lemma 7.6 in [GT83].

□

1.6 Associated bilinear form and weak solutions

In this section we will give the definition of weak solution to the equation $Lu = f$, where L is the operator defined in (1.2). The simplified notation will be kept.

Definition 1.6.1 ($\mathcal{B}$ **and** $\mathcal{B}_0$)**.** If $u \in C^1(\Omega)$ and $v \in C_0^1(\Omega)$, with **(L)** condition verified, we set following bilinear form

$$\mathcal{B}(u, v) := \int_\Omega \Big(\langle B\nabla u, \nabla v\rangle + \langle d, \nabla v\rangle u - \langle b, \nabla u\rangle v - cuv \Big),$$

where B, b, d, c are defined in (1.2). Moreover we can simplify notations if we introduce

$$\mathcal{B}_0(u, v) := \int_\Omega \Big(\langle B\nabla u, \nabla v\rangle - \langle d + b, \nabla u\rangle v \Big),$$

that is interesting on its own. So we can write

$$\mathcal{B}(u, v) = \mathcal{B}_0(u, v) + \int_\Omega \Big(\langle d, \nabla(uv)\rangle - cuv \Big).$$

Definition 1.6.2 (**Positivity condition (+)**)**.** We tell that d and c defined in (1.2) satisfy positivity condition **(+)** if

$$\int_\Omega \Big(\langle d, \nabla\varphi\rangle - c\varphi \Big) \geq 0, \quad \forall \varphi \in C_0^1(\Omega),\ \varphi \geq 0.$$

Remark 1.6.3. If **(+)** is verified by d and c then we have

$$\mathcal{B}(u,v) \geq \mathcal{B}_0(u,v),$$

for all v such that $uv \geq 0$.

Lemma 1.6.4 ($\mathcal{B}$ **properties**). *If we assume* **(+)** *,***(L)** *and X-ellipticity of L in Ω, then the bilinear form $\mathcal{B}$ is well defined on $\big(C^1(\Omega)\times C^1_0(\Omega)\big)$ and it can be extended continuously to $\big(W^1(\Omega,X)\cap L^r(\Omega)\big)\times W^1_0(\Omega,X)$.*

Proof. If $(u,v)\in\big(C^1(\Omega)\times C^1_0(\Omega)\big)$, because of Cauchy-Schwartz inequality and positivity of B we have

$$\begin{aligned}|\mathcal{B}(u,v)| \ \leq\ & \int_\Omega \Big(\langle B\nabla u,\nabla u\rangle^{\frac12}\langle B\nabla v,\nabla v\rangle^{\frac12} + \\ & +|\langle d,\nabla v\rangle||u| + |\langle b,\nabla u\rangle||v| + |c||u||v|\Big). \qquad (1.22)\end{aligned}$$

Furthermore L is uniformly X-elliptic in Ω, then from (1.6)

$$\langle B\nabla u,\nabla u\rangle^{\frac12} \leq \Lambda^{\frac12}\Big(\sum_{j=1}^m \langle X_j I,\nabla u\rangle^2\Big)^{\frac12} = \Lambda^{\frac12}\Big(\sum_{j=1}^m (X_j u)^2\Big)^{\frac12} = \Lambda^{\frac12}|Xu|,$$

that is valid for v too. From (1.14) it follows that

$$|\langle d,\nabla v\rangle||u| \leq |u|\Big(\gamma^2\sum_{j=1}^m (X_j I\nabla v)^2\Big)^{\frac12} = |u|\gamma|Xv|,$$

that holds simmetrically for u and v. So from (1.22) we get

$$|\mathcal{B}(u,v)| \leq \Lambda\int_\Omega |Xu||Xv| + \int_\Omega \Big(|Xu||v| + |Xv||u|\Big)\gamma + \int_\Omega |c||u||v|, \quad (1.23)$$

and so the bilinear form $\mathcal{B}$ is well defined. Now if we set $\frac{1}{r} := \frac12 - \frac{1}{2p} = \frac{2p}{p-1}$ (so $r\in]2,q[$ if $p>\frac{Q}{2}$),we obtain from (1.23), **(S)** and **(L)** conditions (using

Hölder inequality and keeping in mind that $p > \frac{Q}{2}$)

$$\begin{aligned}
|\mathcal{B}(u,v)| &\leq \Lambda||Xu||_{L^2(\Omega)}||Xv||_{L^2(\Omega)} + ||Xv||_{L^2(\Omega)}||u||_{L^r(\Omega)}||\gamma||_{L^{2p}(\Omega)} \\
&\quad +||Xu||_{L^2(\Omega)}||v||_{L^r(\Omega)}||\gamma||_{L^{2p}(\Omega)} + ||u||_{L^r(\Omega)}||v||_{L^r(\Omega)}||c||_{L^p(\Omega)} \leq \\
&\leq \Big(\Lambda||Xu||_{L^2(\Omega)} + ||\gamma||_{L^{2p}(\Omega)}||u||_{L^r(\Omega)}\Big)||Xv||_{L^2(\Omega)} \\
&\quad +\Big(||\gamma||_{L^{2p}(\Omega)}||Xu||_{L^2(\Omega)} + ||c||_{L^p(\Omega)}||u||_{L^r(\Omega)}\Big)||v||_{L^r(\Omega)} \leq \\
&\leq C_1\Big(||Xu||_{L^2(\Omega)} + ||u||_{L^r(\Omega)}\Big)||Xv||_{L^2(\Omega)} \\
&\quad +C_1\Big(||Xu||_{L^2(\Omega)} + ||u||_{L^r(\Omega)}\Big)|\Omega|^{C_5}S_{\Omega,X}||Xv||_{L^2(\Omega)} \leq \\
&\leq C\Big(||Xu||_{L^2(\Omega)} + ||u||_{L^r(\Omega)}\Big)||Xv||_{L^2(\Omega)},
\end{aligned}$$

where $C_1 = (\Lambda + ||\gamma||_{L^{2p}(\Omega)} + ||c||_{L^p(\Omega)})$, and so $C = C(\Omega, S_{\Omega,X}, C_1)$.
Whence

$$(u,v) \quad \mapsto \quad \mathcal{B}(u,v)$$

can be extended continuously to $(W^1(\Omega,X) \cap L^r(\Omega)) \times W_0^1(\Omega,X)$.

□

Remark 1.6.5 ($\mathcal{B}_0$ **properties**). Of course Lemma 1.6.4 holds for $\mathcal{B}_0$.

Definition 1.6.6 (**Weak solution**). Let L be a differential operator of the form (1.2), uniformly X-elliptic in Ω satisfying **(L)** and **(+)**, where X satisfies **(S)**. $u \in W^1(\Omega,X)$ is a weak solution to

$$Lu = f, \qquad f \in L^1_{\text{loc}}(\Omega),$$

if

$$\mathcal{B}(u,v) = -\int_\Omega fv, \qquad \forall v \in C_0^1(\Omega).$$

Remark 1.6.7. We want to underline the role of Theorem 1.5.11 in the definition of weak solution. If $u \in W^1(\Omega,X)$ and $v \in C_0^1(\Omega)$ we can take

$\{u_j\}_{j\in\mathbb{N}}$ in $C^\infty(\Omega)\cap W^1(\Omega,X)$ s.t. $u_j \xrightarrow[j\to+\infty]{} u$ in $W^1(\Omega,X)$.
So for $\{u_j\}_{j\in\mathbb{N}}$ we have

$$\begin{aligned}
|\mathcal{B}(u_j,v)-\mathcal{B}(u_i,v)| &\overset{\mathcal{B}\text{ bil}}{=} |\mathcal{B}(u_j-u_i,v)| \le \\
&\overset{(1.23)}{\le} \Lambda\int_\Omega |Xu_j-Xu_i||Xv| \\
&\quad +\int_\Omega \Big(|Xu_j-Xu_i||v|+|u_j-u_i||Xv|\Big)\gamma \\
&\quad +\int_\Omega |c||u_j-u_i||v| \xrightarrow[j,i\to+\infty]{} 0,
\end{aligned}$$

since each term tends to zero,

$$\Lambda\int_\Omega |Xu_j-Xu_i||Xv| \le ||Xu_j-Xu_i||_{L^2(\Omega)}||Xv||_{L^2(\Omega)} \xrightarrow[j,i\to+\infty]{Xu_j\to Xu \text{ in } L^2(\Omega),\ Xv\in L^2(\Omega)} 0,$$

$$\int_\Omega |Xu_j-Xu_i||v|\gamma \xrightarrow[j,i\to+\infty]{Xu_j\to Xu \text{ in } L^{(2p)'}(\Omega),\ v\in L^\infty(\Omega),\ \gamma\in L^{2p}(\Omega)} 0,$$

$$\int_\Omega |u_j-u_i||Xv|\gamma \xrightarrow[j,i\to+\infty]{u_j\to u \text{ in } L^{(2p)'}(\Omega),\ Xv\in L^\infty(\Omega),\ \gamma\in L^{2p}(\Omega)} 0,$$

as the last one,

$$\begin{aligned}
\int_\Omega |c||u_j-u_i||v| &\le (\max_\Omega v)||c||_{L^p(\Omega)}||u_jv-u_iv||_{L^{\frac{p}{p-1}}(\Omega)} \le \\
&\le (\max_\Omega v)C_{p,q,\Omega}||c||_{L^p(\Omega)}||u_jv-u_iv||_{L^q(\Omega)} \le \\
&\le C_1||X(u_jv)-X(u_iv)||_{L^2(\Omega)} = \\
&= C_1||(Xu_j)v+u_j(Xv)-(Xu_i)v-u_i(Xv)||_{L^2(\Omega)} \le \\
&\le C_1||(Xu_j-Xu_i)v||_{L^2(\Omega)}+||(u_j-u_i)Xv||_{L^2(\Omega)} \le \\
&\le C_1\Big(||Xu_j-Xu_i||_{L^2(\Omega)}||v||_{L^\infty(\Omega)} \\
&\qquad +||u_j-u_i||_{L^2(\Omega)}||Xv||_{L^\infty(\Omega)}\Big) \xrightarrow[i,j\to+\infty]{} 0,
\end{aligned}$$

for the same reasons as above ($Xu_j \xrightarrow[j\to+\infty]{} Xu$ and $u_j \xrightarrow[j\to+\infty]{} u$ both in $L^2(\Omega)$).
We have set $C_1 := (\max_\Omega v)C_{p,q,\Omega}S_{\Omega,X}$, and of course it's easy to verify that $(2p)'\le 2$, $\frac{p}{p-1}\le q$. We have finally shown that the definition is well posed through Theorem 1.5.11.

1.7 On the Dirichlet problem for L in principal form

Let us study the Dirichlet problem when L is in principal form.

Proposition 1.7.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, X verify* **(S)** *and L be in the form (1.2), uniformly X-elliptic in Ω, and in principal form ($c \equiv 0$, $d = b = (0, \dots, 0)$). If $\varphi \in W^1(\Omega, X)$, then the problem*

$$\begin{cases} Lu = 0 \text{ in } \Omega \\ u - \varphi \in W_0^1(\Omega, X) \end{cases}$$

has a solution $u \in W^1(\Omega, X)$, and where $\varphi \leq 0$ we have $u \leq 0$.

Proof. Let us consider $\{u_j\}_{j\in\mathbb{N}}$ in $C^\infty(\Omega) \cap W^1(\Omega, X)$, with $u_j \xrightarrow[j\to+\infty]{} u \in W^1(\Omega, X)$. Then

$$|\mathcal{B}(u_j, u_j)| \leq \Lambda \int_\Omega \sum_{k=1}^m \langle X_k I, \nabla u_j \rangle^2 = \Lambda ||Xu_j||^2_{L^2(\Omega)} < +\infty.$$

Moreover

$$\begin{aligned} |\mathcal{B}(u_j, u_j) - \mathcal{B}(u_i, u_i)| &\leq |\mathcal{B}(u_j, u_j - u_i)| + |\mathcal{B}(u_j - u_i, u_i)| \leq \\ &\leq \Lambda\Big(||Xu_j||_{L^2(\Omega)} ||Xu_j - Xu_i||_{L^2(\Omega)} \\ &\quad + ||Xu_j - Xu_i||_{L^2(\Omega)} ||Xu_i||_{L^2(\Omega)} \Big). \end{aligned}$$

This right-hand side inequality tends to zero if j, i go to infinity, so by density we can define the functional $J(u) := \mathcal{B}(u, u)$. This functional is coercive, i.e.

$$\exists C \in \mathbb{R} \quad : \quad |J(u)| \geq C ||u||^2_{L^2(\Omega)} \qquad \forall u \in W^1(\Omega, X),$$

indeed

$$\begin{aligned} |J(u)| &= \lim_{j\to+\infty} |\mathcal{B}(u_j, u_j)| \geq \lambda \lim_{j\to+\infty} \int_\Omega \sum_{k=1}^m \langle X_k I, \nabla u_j \rangle^2 = \\ &= \lambda \lim_{j\to+\infty} ||Xu_j||^2_{L^2(\Omega)} \overset{\textbf{(S)}}{\geq} \lambda S_{\Omega,X} \lim_{j\to+\infty} ||u_j||^2_{L^q(\Omega)} \geq \\ &\overset{(1.19)}{\geq} C \lim_{j\to+\infty} ||u_j||^2_{L^2(\Omega)} = C ||u||^2_{L^2(\Omega)}, \end{aligned}$$

where $C := \lambda S_{\Omega,X}|\Omega|^{1-\frac{2}{q}}$.
J reaches its minimum on the closed convex subset $K := \varphi + W_0^1(\Omega, X)$ of $W_0^1(\Omega, X)$ in $u = \varphi + v$, with $v \in W_0^1(\Omega, X)$. Then $\mathcal{B}(u, v) = 0$ for each $v \in W_0^1(\Omega, X)$, and solvability of the Dirichlet problem is a natural consequence. Suppose that $\varphi \le 0$. By consequence $u = v + \varphi$ with $v \in W_0^1(\Omega, X)$, so $u^+ \in W_0^1(\Omega, X)$. Then $\mathcal{B}(u, u^+) = 0$, thus $\mathcal{B}(u^+, u^+) = 0$. L is in principal form, so it follows from X-ellipticity that $||Xu^+|| = 0$, and so from **(S)** $u^+ = 0$.

□

1.8 A maximum principle for uniformly X-elliptic operators

In this section like previous one we work on a bounded open subset of $\mathbb{R}^N$, and we assume that **(S)** is verified by each vector field X_j for $j = 1, \dots, m$. L of the form (1.2) is uniformly X-elliptic in Ω. The functions γ defined in (1.14) and c defined in (1.2), satisfy **(L)** conditions.

Definition 1.8.1 ($\sup u^+$ **on** $\partial\Omega$)**.** If $u \in W^1(\Omega, X)$ and $l \in \mathbb{R}$, we say that

$$u \le l \text{ on } \partial\Omega \quad \Leftrightarrow \quad (u - l)^+ \in W_0^1(\Omega, X),$$

and we define

$$\sup_{\partial\Omega} u^+ := \inf \Big\{ \{ l \in \mathbb{R} \mid u^+ \le l \text{ on } \partial\Omega \} \cup \{+\infty\} \Big\}.$$

Now we can prove the main theorem.

Theorem 1.8.2 (**Maximum Principle on** $W^1(\Omega, X)$)**.** *Let $Q/2 < p < +\infty$, $f \in L^p(\Omega)$, be* **(+)** *verified. Then there exists a constant C, independent of f and $u \in W^1(\Omega, X)$, being u weak sub-solution of $Lu = f$, such that*

$$\sup_{\Omega} u^+ \le \sup_{\partial\Omega} u^+ + C||f||_{L^p(\Omega)}. \tag{1.24}$$

Proof. Our first step is to prove the inequality

$$\sup_{\Omega} u^+ \leq C_1\big(||u^+||_{L^2(\Omega)} + ||f||_{L^p(\Omega)}\big),$$

for every $u \in W^1(\Omega, X)$ such that $u \leq 0$ on $\partial\Omega$ and verifing

$$\mathcal{B}(u, v) \leq -\int_{\Omega} fv \,\mathrm{d}x, \tag{1.25}$$

for any $v \in W_0^1(\Omega, X)$ such that $v \geq 0$ and $uv \geq 0$. It means that if $u < 0$ then $v \equiv 0$, otherwise if $u \geq 0$ then $v \geq 0$. We can read this request under a "support" argument: (1.25) has to be true for every positive $v \in W_0^1(\Omega, X)$, with $\operatorname{supp} v \subseteq \operatorname{supp} u^+$. C_1 is independent of choice of u and f. For this step the **(+)** condition is not necessary.

Now we build some functions with trivial properties that we recall. Let k be a proper real number we will obtain later, $N \geq k$ and $\beta > 1$. We define

$$H(z) := \begin{cases} z^\beta - k^\beta & \text{if } k \leq z \leq N \\ \beta N^{\beta-1}(z - N) + N^\beta - k^\beta & \text{if } z > N. \end{cases}$$

We have $H \in C^1([k, +\infty[)$ and $H' \in L^\infty([k, +\infty[)$. If we set

$$G(t) := \begin{cases} \displaystyle\int_k^t (H'(s))^2 \,\mathrm{d}s & \text{if } t \geq k \\ (H'(k))^2(t - k) & \text{if } t < k, \end{cases}$$

we can observe that $G \in C^1(\mathbb{R})$, and $G' \in L^\infty(\mathbb{R})$. Let $w(x) := u^+(x) + k$, and

$$\varphi(x) := G\big(w(x)\big) = \int_k^{w(x)} \big(H'(s)\big)^2 \,\mathrm{d}s.$$

Since $w \geq k$ then $\varphi \geq 0$. If $u < 0$ then $w = k$, and by consequence $\varphi = 0$, so $\varphi u \geq 0$.

By hypotheses $u \leq 0$ on $\partial\Omega$, so if we consider $\psi(t) := G(t + k)$, we have that $\psi \in C^1(\mathbb{R})$ and $\psi' \in L^\infty(\mathbb{R})$, with $\psi(0) = 0$. Whence $\psi \in W_0^1(\Omega, X)$, because of Lemma 1.5.14. Using (1.25) with φ as particular v we obtain

$$\mathcal{B}(u, \varphi) \leq -\int_{\Omega} fG(w) \,\mathrm{d}x. \tag{1.26}$$

Using again Lemma 1.5.14 with vector fields we have $X_i(\varphi) = G'(w)X_i(w)$, where $X_i(w) = X_i(u^+ + k) = X_i(u^+) = 0$ if $u \leq 0$, by Corollary 1.5.16. From X-ellipticity of L and that $\nabla\varphi = G'(w)\nabla(u^+)$

$$
\begin{aligned}
\mathcal{B}(u,\varphi) &= \int_\Omega \Big(\langle B\nabla u, \nabla\varphi\rangle + \langle d, \nabla\varphi\rangle u - \langle b, \nabla u\rangle\varphi - cu\varphi \Big)\,\mathrm{d}x = \\
&= \int_{\Omega\cap\{u>0\}} \langle B\nabla u, \nabla u\rangle G'(w)\,\mathrm{d}x + \int_{\Omega\cap\{u>0\}} \langle d, \nabla u\rangle G'(w)u\,\mathrm{d}x \\
&\quad - \int_\Omega \langle b, \nabla u\rangle G(w)\,\mathrm{d}x - \int_\Omega cuG(w)\,\mathrm{d}x \geq \\
&\geq \int_{\Omega\cap\{u>0\}} \lambda|Xu|^2 G'(w)\,\mathrm{d}x + \int_\Omega \langle d, \nabla w\rangle G'(w)u^+\,\mathrm{d}x \\
&\quad - \int_\Omega \gamma|Xu|G(w) - \int_\Omega cuG(w)\,\mathrm{d}x \geq \\
&\geq \int_\Omega \lambda|Xw|^2 G'(w)\,\mathrm{d}x - \int_\Omega \gamma|Xw|G'(w)u^+\,\mathrm{d}x \\
&\quad - \int_\Omega \gamma|Xu|G(w)\,\mathrm{d}x - \int_\Omega cuG(w)\,\mathrm{d}x \geq \\
&\geq \int_\Omega \lambda|Xw|^2 G'(w)\,\mathrm{d}x - \int_\Omega \gamma|Xw|G'(w)w\,\mathrm{d}x \\
&\quad - \int_\Omega \gamma|Xw|G(w)\,\mathrm{d}x - \int_\Omega cuG(w)\,\mathrm{d}x. \qquad (1.27)
\end{aligned}
$$

By a direct computation it's easy to prove that $G(s) \leq sG'(s)$, so by (1.26)

$$
\mathcal{B}(u,\varphi) \leq -\int_\Omega fG(w)\,\mathrm{d}x \leq \int_\Omega |f|G(w)\,\mathrm{d}x \leq \int_\Omega |f|wG'(w)\,\mathrm{d}x,
$$

and together with (1.27) we obtain

$$
\begin{aligned}
\int_\Omega \lambda|Xw|^2 G'(w)\,\mathrm{d}x &\leq \int_\Omega |f|wG'(w)\,\mathrm{d}x + 2\int_\Omega \gamma|Xw|G'(w)w\,\mathrm{d}x \\
&\quad + \int_\Omega cuG(w)\,\mathrm{d}x =: \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (1.28)
\end{aligned}
$$

So let us estimate every right-hand side addendum. $w \geq k > 0$, so about the first one

$$
\mathrm{I} = \int_\Omega \frac{|f|}{w} w^2 G'(w)\,\mathrm{d}x \leq \int_\Omega \frac{|f|}{k} w^2 G'(w)\,\mathrm{d}x.
$$

Recalling that $ab \le 1/2(\varepsilon a^2 + b^2/\varepsilon)$ for every $a, b \ge 0$ and for every $\varepsilon > 0$, about the second term we have

$$\begin{aligned} \mathrm{II} &= 2\int_\Omega \Big(|Xw|\sqrt{G'(w)}\Big)\Big(\gamma w\sqrt{G'(w)}\Big)\,\mathrm{d}x \le \\ &\le \varepsilon\int_\Omega |Xw|^2 G'(w)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega w^2 G'(w)\gamma^2\,\mathrm{d}x. \end{aligned}$$

Now the third addendum.

$$\mathrm{III} = \int_{\Omega\cap\{u>0\}} cuG(w)\,\mathrm{d}x = \int_\Omega cu^+G(w)\,\mathrm{d}x \le \int_\Omega |c|wG(w)\,\mathrm{d}x \le \int_\Omega |c|w^2G'(w)\,\mathrm{d}x.$$

Choosing $\varepsilon = \lambda/2$ and plugging the last three estimates into (1.28) we have

$$\frac{\lambda}{2}\int_\Omega |Xw|^2 G'(w)\,\mathrm{d}x \le \int_\Omega \Big(\frac{|f|}{k} + \frac{2\gamma^2}{\lambda} + |c|\Big) w^2 G'(w)\,\mathrm{d}x,$$

whence

$$\int_\Omega |Xw|^2 G'(w)\,\mathrm{d}x \le \int_\Omega \Big(\frac{2|f|}{\lambda k} + \Big(\frac{2\gamma}{\lambda}\Big)^2 + \frac{2|c|}{\lambda}\Big) w^2 G'(w)\,\mathrm{d}x.$$

We know that $w \ge k$, and then $G'(w) = |H'(w)|^2$. By chain rule (1.5.14) $X\big(H(w)\big) = H'(w)X(w)$, so from above relationship we obtain

$$\int_\Omega |X\big(H(w)\big)|^2\,\mathrm{d}x \le \int_\Omega \Big(\frac{2|f|}{\lambda k} + \Big(\frac{2\gamma}{\lambda}\Big)^2 + \frac{2|c|}{\lambda}\Big) |wH'(w)|^2\,\mathrm{d}x.$$

From $u \le 0$ on $\partial\Omega$ and $H(k) = 0$ Once more because of the chain rule (1.5.14) it follows that $H(w) \in W_0^1(\Omega, X)$. We can use **(S)** together with last inequality:

$$\begin{aligned} \left(\int_\Omega |H(w)|^q\,\mathrm{d}x\right)^{\frac{2}{q}} &= \Big(||H(w)||_{L^q(\Omega)}\Big)^2 \le S^2_{\Omega,X}||X(H(w))||^2_{L^2(\Omega)} \le \\ &\le S^2_{\Omega,X}\int_\Omega \Big(\frac{2|f|}{\lambda k} + \Big(\frac{2\gamma(x)}{\lambda}\Big)^2 + \frac{2|c|}{\lambda}\Big)|wH'(w)|^2\,\mathrm{d}x \le \\ &\overset{\text{Hölder}}{\le} S^2_{\Omega,X}\left(\int_\Omega \Big(\frac{2|f|}{\lambda k} + \Big(\frac{2\gamma(x)}{\lambda}\Big)^2 + \frac{2|c|}{\lambda}\Big)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad \cdot\left(\int_\Omega |wH'(w)|^{2p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}. \end{aligned}$$

Let us estimate the second factor as follows:

$$\begin{aligned}
&\left(\int_\Omega \Big(\frac{2|f|}{\lambda k} + \Big(\frac{2\gamma}{\lambda}\Big)^2 + \frac{2|c|}{\lambda}\Big)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \\
&\leq \frac{2}{\lambda}\left(\int_\Omega \Big(\frac{|f|}{k}\Big)^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \frac{4}{\lambda^2}\left(\int_\Omega \gamma^{2p}\,\mathrm{d}x\right)^{\frac{1}{p}} + \frac{2}{\lambda}\left(\int_\Omega |c|^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \qquad (1.29)
\end{aligned}$$

If we pick $k = ||f||_{L^p(\Omega)}$ the right-hand side of (1.29) is equal to:

$$M = M(\gamma, p, \Omega, \lambda, c) := \frac{2}{\lambda} + \frac{4}{\lambda^2}\left(\int_\Omega \gamma^{2p}\,\mathrm{d}x\right)^{\frac{1}{p}} + \frac{2}{\lambda}\left(\int_\Omega |c|^p\,\mathrm{d}x\right)^{\frac{1}{p}}, \qquad (1.30)$$

which is independent of u and f.
Hence

$$\left(\int_\Omega |H(w)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \leq S_{\Omega,X} M^{\frac{1}{2}} \left(\int_\Omega |wH'(w)|^{2p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}}.$$

Now let N tend to infinity. By definition of H, $\{H_N(w)\}_{N\in\mathbb{N}}$ is increasing and $H'(z) = \beta z^{\beta-1}$ when $z \leq N$. So taking the limit we have

$$\left(\int_\Omega |w^\beta - k^\beta|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \leq S_{\Omega,X} M^{\frac{1}{2}}\beta \left(\int_\Omega |w|^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}}.$$

Whence by triangle inequality in $L^q(\Omega)$

$$\begin{aligned}
\left(\int_\Omega w^{\beta q}\,\mathrm{d}x\right)^{\frac{1}{q}} &\leq \left(\int_\Omega |w^\beta - k^\beta|^q\,\mathrm{d}x\right)^{\frac{1}{q}} + k^\beta |\Omega|^{\frac{1}{q}} \leq \\
&\leq S_{\Omega,X} M^{\frac{1}{2}}\beta \left(\int_\Omega |w|^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}} + \left(\frac{1}{|\Omega|}\int_\Omega k^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}} |\Omega|^{\frac{1}{q}} \leq \\
&\leq S_{\Omega,X} M^{\frac{1}{2}}\beta \left(\int_\Omega |w|^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}} + \left(\frac{1}{|\Omega|}\int_\Omega w^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}} |\Omega|^{\frac{1}{q}} \leq \\
&= \Big(S_{\Omega,X} M^{\frac{1}{2}}\beta + |\Omega|^{\frac{1}{q}-\frac{1}{2p'}}\Big)\left(\int_\Omega w^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}} \leq \\
&\overset{\beta\geq 2}{\leq} \beta\Big(S_{\Omega,X} M^{\frac{1}{2}} + |\Omega|^{\frac{1}{q}-\frac{1}{2p'}}\Big)\left(\int_\Omega w^{2\beta p'}\,\mathrm{d}x\right)^{\frac{1}{2p'}}.
\end{aligned}$$

If we set $R := \left(S_{\Omega,X}M^{\frac{1}{2}} + |\Omega|^{\frac{1}{q}-\frac{1}{2p'}}\right)$ we obtain for $\beta > 1$ the inequality

$$||w||_{L^{\beta q}(\Omega)} \leq R^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}||w||_{L^{2\beta p'}(\Omega)},$$

that, if we set in addition $\theta := \frac{q}{2p'}$, ($\theta = \frac{Q(p-1)}{p(Q-2)} > 1$ because $p > \frac{Q}{2}$), becomes

$$||w||_{L^{\beta p'\theta}(\Omega)} \leq (R\beta)^{\frac{1}{\beta}}||w||_{L^{2\beta p'}(\Omega)}.$$

Taking $\beta = \theta^m$ with $m = 1,\ 2, \ldots$ we have

$$||w||_{L^{2p'\theta^{m+1}}(\Omega)} \leq \left(\prod_{j=1}^{m}(R\theta^j)^{\frac{1}{\theta^j}}\right)||w||_{L^{2p'\theta}(\Omega)}. \tag{1.31}$$

$w = u^+ + k$ belongs to $L^2(\Omega)$ (u^+ belongs to $L^2(\Omega)$ and k belongs to $L^\infty(\Omega) \cap L^1(\Omega)$ since Ω is bounded), and now letting $m \to \infty$

$$||w||_{L^\infty(\Omega)} \leq R^{\sum_{j=1}^{\infty}\theta^{-j}}\theta^{\sum_{j=1}^{\infty}j\theta^{-j}}||w||_{L^{2p'\theta}(\Omega)} = R^{\frac{1}{\theta-1}}\theta^{\frac{\theta}{(\theta-1)^2}}||w||_{L^{2p'\theta}(\Omega)}. \tag{1.32}$$

So if $||w||_{L^{2p'\theta}(\Omega)} = +\infty$ (1.32) is trivial, if $||w||_{L^{2p'\theta}(\Omega)} < +\infty$ (1.32) is verified through (1.31) for every $m \geq 1$.
Now we shall recall a very useful interpolation inequality, due to Gilbarg and Trudinger [GT83] (7.10).
If $\widetilde{p} \leq \widetilde{r} \leq +\infty$, and $\frac{1}{\widetilde{q}} = \frac{\lambda}{\widetilde{p}} + \frac{1-\lambda}{\widetilde{r}}$, with $\lambda \in]0, 1[$ and we set

$$\mu := \left(\frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right)\left(\frac{1}{\widetilde{q}} - \frac{1}{\widetilde{r}}\right)^{-1}, \tag{1.33}$$

we have for every $\varepsilon > 0$

$$||u||_{L^{\widetilde{q}}(\Omega)} \leq \varepsilon||u||_{L^{\widetilde{r}}(\Omega)} + \varepsilon^{-\mu}||u||_{L^{\widetilde{p}}(\Omega)}. \tag{1.34}$$

Choosing $\widetilde{q} = 2p'\theta$ ($\in]2, +\infty[$), $\widetilde{p} = 2$, $\widetilde{r} = \infty$, the inequality (1.34) can estimate from above right-hand side of (1.32):

$$||w||_{L^\infty(\Omega)} \leq R^\sigma\theta^\tau(\varepsilon||w||_{L^\infty(\Omega)} + \varepsilon^{1-p'\theta}||w||_{L^2(\Omega)}),$$

(where $\sigma := \frac{1}{\theta-1}$ and $\tau := \frac{\theta}{(\theta-1)^2}$). It can be minimized by $\varepsilon = (p'\theta - 1)^{\frac{1}{p'\theta}}(||w||_{L^2(\Omega)}/||w||_{L^\infty(\Omega)})^{\frac{1}{p'\theta}}$, reaching the value

$$||w||_{L^\infty(\Omega)} \leq R^{\sigma p'\theta}\theta^{\tau p'\theta}\left((p'\theta-1)^{\frac{1}{p'\theta}} + (p'\theta-1)^{\frac{1-p'\theta}{p'\theta}}\right)^{p'\theta}||w||_{L^2(\Omega)}.$$

So if we set

$$C_0 := R^{\sigma p'\theta}\theta^{\tau p'\theta}\Big((p'\theta-1)^{\frac{1}{p'\theta}} + (p'\theta-1)^{\frac{1-p'\theta}{p'\theta}}\Big)^{p'\theta} = R^{\sigma p'\theta}\theta^{\tau p'\theta}(p'\theta)^{p'\theta}(p'\theta-1)^{1-p'\theta},$$

we obtain $||w||_{L^\infty(\Omega)} \le C_0||w||_{L^2(\Omega)}$, and since $w = u^+ + k$,

$$\begin{aligned}
||u^+||_{L^\infty(\Omega)} &\le ||u^+ + k||_{L^\infty(\Omega)} + ||k||_{L^\infty(\Omega)} = ||w||_{L^\infty(\Omega)} + ||f||_{L^p(\Omega)} \le \\
&\le C_0||w||_{L^2(\Omega)} + ||f||_{L^p(\Omega)} \le \\
&\le C_0\big(||u^+||_{L^2(\Omega)} + ||f||_{L^p(\Omega)}|\Omega|^{\frac{1}{2}}\big) + ||f||_{L^p(\Omega)} \le \\
&\le C_1\big(||u^+||_{L^2(\Omega)} + ||f||_{L^p(\Omega)}\big),
\end{aligned}$$

with $C_1 := \max\{C_0, C_0|\Omega|^{\frac{1}{2}} + 1\}$, and we get the required result.

Now let's approach the second step.

Let $l = \sup_{\partial\Omega} u^+$, which can be supposed finite, otherwise (1.24) is trivial and there's nothing to prove. If u is sub-solution, the function $u - l$ belongs to $W^1(\Omega, X)$, and so for every $v \ge 0$ in $C_0^1(\Omega)$

$$\begin{aligned}
\mathcal{B}(u-l, v) &= \int_\Omega \Big(\langle B\nabla u, \nabla v\rangle + \langle d, \nabla v\rangle(u-l) - \langle b, \nabla u\rangle v - c(u-l)v\Big)\,\mathrm{d}x = \\
&= \mathcal{B}(u, v) - \int_\Omega (\langle d, \nabla v\rangle l)\,\mathrm{d}x + \int_\Omega (cvl)\,\mathrm{d}x \le \\
&\le -\int_\Omega fv\,\mathrm{d}x - l\left(\int_\Omega \big(\langle d, \nabla v\rangle - cv\big)\,\mathrm{d}x\right) \le \\
&\overset{(+)}{\le} -\int_\Omega fv\,\mathrm{d}x \le -\int_\Omega (-|f|v)\,\mathrm{d}x.
\end{aligned}$$

Thus the function $u - l$ is sub-solution of $Lu = -|f|$.

Suppose we have proved the assertion for $l = 0$.

If $\sup_{\partial\Omega} u^+ = +\infty$ there's nothing to prove.

If $\sup_{\partial\Omega} u^+ < +\infty$ then by the above considerations $\psi := u - l$ is sub-solution of $Lu = -|f|$, hence $\psi \in W^1(\Omega, X)$. Furthermore

$$\sup_{\partial\Omega}\psi^+ = \sup_{\partial\Omega}(u-l)^+ = \sup_{\partial\Omega}(u - \sup_{\partial\Omega}u^+)^+ \le \sup_{\partial\Omega}(u^+ - \sup_{\partial\Omega}u^+)^+ = 0.$$

Now we are in $l = 0$ case, by consequence

$$\sup_\Omega(u - \sup_{\partial\Omega}u^+)^+ = \sup_\Omega \psi^+ \le \sup_{\partial\Omega}\psi^+ + C||(-|f|)||_{L^p(\Omega)} = C||f||_{L^p(\Omega)}. \tag{1.35}$$

Whence

$$\begin{aligned}\sup_{\Omega} u^+ &= \sup_{\Omega}(u - \sup_{\partial\Omega} u^+ + \sup_{\partial\Omega} u^+)^+ \leq \\ &\leq \sup_{\Omega}(u - \sup_{\partial\Omega} u^+)^+ - \sup_{\partial\Omega} u^+ \leq \\ &\overset{(1.35)}{\leq} C||f||_{L^p(\Omega)} + \sup_{\partial\Omega} u^+.\end{aligned}$$

So let us consider the case $l = 0$.
In such a case $\sup_{\partial\Omega} u^+ = 0$, so $u^+ \in W_0^1(\Omega, X)$. Let k be a constant to be determined later, $M := \sup_{\Omega} u^+$ and let's pick the test function $\varphi := \dfrac{u^+}{M + k - u^+}$. $\varphi = \psi(u^+)$ if

$$\psi(s) := \begin{cases} \dfrac{s}{M + k - s} & \text{if } s \leq M \\ -\dfrac{M^2}{k^2} + \dfrac{m + k}{k^2} s & \text{if } s > M, \end{cases}$$

and also ψ belongs to $W_0^1(\Omega, X)$ because of the chain rule. On the other hand,

$$X\varphi = \frac{M + k}{(M + k - u^+)^2} X(u^+), \tag{1.36}$$

where $\varphi \geq 0$ and $\varphi u \geq 0$. Substituting this in the definition of sub-solution of $Lu = f$ and keeping in mind **(+)**, we obtain

$$\mathcal{B}_0(u, \varphi) \leq \mathcal{B}(u, \varphi) \leq -\int_{\Omega} f \frac{u^+}{M + k - u^+} \,\mathrm{d}x \leq \int_{\Omega} |f| \frac{u^+}{M + k - u^+} \,\mathrm{d}x. \tag{1.37}$$

Moreover, from (1.36), X-ellipticity of L, and Corollary (1.5.16) we have

$$\mathcal{B}_0(u, \varphi) \geq \lambda \int_{\Omega} |Xu^+|^2 \frac{M + k}{(M + k - u^+)^2} \,\mathrm{d}x - 2\int_{\Omega} |Xu^+| \frac{u^+}{M + k - u^+} \gamma \,\mathrm{d}x.$$

If we observe that $\frac{u^+}{M+k} \leq 1$ and that $M + k - u^+ \geq k$, plugging (1.37) into the above one and dividing to $M + k$ we obtain

$$\begin{aligned}\lambda \int_{\Omega} \frac{|Xu^+|^2}{(M + k - u^+)^2} \,\mathrm{d}x &\leq \int_{\Omega} \frac{|f|}{k} \,\mathrm{d}x + 2\int_{\Omega} \frac{|Xu^+|}{M + k - u^+} \gamma \,\mathrm{d}x \leq \\ &\leq \int_{\Omega} \frac{|f|}{k} \,\mathrm{d}x + \varepsilon \int_{\Omega} \frac{|Xu^+|^2}{(M + k - u^+)^2} \,\mathrm{d}x + \frac{1}{\varepsilon} \int_{\Omega} \gamma \,\mathrm{d}x.\end{aligned}$$

If we set $k = ||f||_{L^p(\Omega)}$ and $\varepsilon = \lambda/2$, then by Hölder inequality we get

$$\frac{\lambda}{2}\int_\Omega \frac{|Xu^+|^2}{(M+k-u)^2}\,\mathrm{d}x \le |\Omega|^{\frac{1}{p'}} + \frac{2}{\lambda}\int_\Omega \gamma\,\mathrm{d}x.$$

But if $w = \ln \dfrac{M+k}{M+k-u^+}$, we obtain

$$\int_\Omega |Xw|^2\,\mathrm{d}x \le \frac{2}{\lambda}|\Omega|^{\frac{1}{p'}} + \frac{4}{\lambda^2}\int_\Omega \gamma\,\mathrm{d}x =: C_4. \tag{1.38}$$

Since u belongs to $W_0^1(\Omega, X)$, we have that w belongs to $W_0^1(\Omega, X)$ too, and by **(S)**

$$\begin{aligned}\int_\Omega |w|^2\,\mathrm{d}x &\overset{q>2}{\le} \widetilde{C}\int_\Omega |w|^q\,\mathrm{d}x \le \widetilde{C}S_{\Omega,X}||Xw||^2_{L^2(\Omega)} \le \\ &\le \widetilde{C}S_{\Omega,X}C_4 \le C_2 S_{\Omega,X}.\end{aligned} \tag{1.39}$$

We claim that w satisfies the hypotheses of step 1, with respect to $\mathcal{B}_0$ instead of $\mathcal{B}$ and $-|f|/k$ instead of f, i.e.

$$\mathcal{B}_0(w, v) \le \int_\Omega \frac{|f|}{k} v\,\mathrm{d}x, \tag{1.40}$$

for every $v \in W_0^1(\Omega, X)$ such that $v \ge 0$ and $uv \ge 0$, or equally with the hypotheses on supports. Of course we already know that $w^+ = w \in W_0^1(\Omega, X)$. By definition of w we have

$$\nabla w = \frac{\nabla u^+}{M+k-u^+}.$$

If we set

$$\varphi := \frac{v}{M+k-u^+},$$

its gradient is

$$\nabla\varphi = \frac{\nabla v}{M+k-u^+} + \frac{v}{(M+k-u^+)^2}\nabla u^+.$$

Since $v = 0$ where $u < 0$, then $v\nabla u^+ = v\nabla u$. Applying the definition of weak sub-solution u (to $Lu = f$ in Ω) to φ as test function

$$\begin{aligned} \mathcal{B}(u,\varphi) &= \int_\Omega \langle B\nabla u, \nabla v\rangle \frac{1}{M+k-u^+}\,\mathrm{d}x \\ &\quad + \int_\Omega \langle B\nabla u, \nabla u^+\rangle \frac{v}{(M+k-u^+)^2}\,\mathrm{d}x \\ &\quad - \int_\Omega \langle d+b, \nabla u\rangle \frac{v}{M+k-u^+}\,\mathrm{d}x \leq \\ &\leq -\int_\Omega f\frac{v}{M+k-u^+}\,\mathrm{d}x. \end{aligned}$$

Noticing that $\langle B\nabla u, \nabla v\rangle = \langle B\nabla u^+, \nabla v\rangle$ and by keeping in mind that $\operatorname{supp} v \subseteq \operatorname{supp} u^+$, the above inequality becomes

$$\begin{aligned} \mathcal{B}(u,\varphi) &= \int_\Omega \langle B\nabla u^+, \nabla v\rangle \frac{1}{M+k-u^+}\,\mathrm{d}x \\ &\quad + \int_\Omega \langle B\nabla u^+, \nabla u^+\rangle \frac{v}{(M+k-u^+)^2}\,\mathrm{d}x \\ &\quad - \int_\Omega \langle d+b, \nabla u^+\rangle \frac{v}{M+k-u^+}\,\mathrm{d}x \leq \\ &\leq -\int_\Omega f\frac{v}{M+k-u^+}\,\mathrm{d}x. \end{aligned}$$

But we have that

$$\mathcal{B}_0(w,v) = \int_\Omega \langle B\nabla u^+, \nabla v\rangle \frac{1}{M+k-u^+}\,\mathrm{d}x - \int_\Omega \langle d+b, \nabla u^+\rangle \frac{v}{M+k-u^+}\,\mathrm{d}x,$$
$$\int_\Omega \langle B\nabla u^+, \nabla u^+\rangle \frac{v}{(M+k-u^+)^2}\,\mathrm{d}x \geq 0,$$

and direct consequence is that

$$\mathcal{B}_0(w,v) \leq \mathcal{B}(u,\varphi) \leq -\int_\Omega f\frac{v}{M+k-u^+}\,\mathrm{d}x \leq \int_\Omega |f|\frac{v}{k}\,\mathrm{d}x.$$

Thus w verifies (1.40), and we get step 1 with $w = w^+$ and supp $u^+ =$ supp w^+, i.e.

$$\sup_\Omega w \leq C_1\Big(||w||_{L^2(\Omega)} + \Big|\Big|\frac{|f|}{k}\Big|\Big|_{L^p(\Omega)}\Big).$$

Doing the same usual choice of k, keeping in mind (1.39) we obtain

$$\sup_{\Omega} w \leq C_1\Big(S_{\Omega,X}^{\frac{1}{2}} C_2^{\frac{1}{2}} + 1\Big) =: C.$$

Hence

$$\ln \frac{M+k}{M+k-u^+} \leq C \quad \Leftrightarrow \quad e^C u^+ \leq (M+k)e^C - (M+k).$$

Taking the sup of both sides

$$Me^C \leq Me^C + ke^C - M - k \quad \Leftrightarrow \quad M \leq k(e^C - 1),$$

and we finally have

$$\sup_{\Omega} u^+ \leq (e^C - 1)||f||_{L^p(\Omega)},$$

and so theorem is proved.

□

1.8.1 Condition $p < Q/2$ is sharp for the Maximum Principle

In this section we find out a counterexample with to the Maximum Principle of Theorem 1.8.2 assuming $p = Q/2$. The argument is the classical $\ln|\ln|$ example adapted to our setting. We suppose $Q \geq 3$.

Lemma 1.8.3. *Let $g :]0, \frac{1}{2}] \to \mathbb{R}$ be a continuous function such that $r \mapsto r^{Q-1} g(r)$ belongs to $L^1((0, 1/2])$. Let Q be the homogeneous dimension defined in (1.4.4), $\varrho : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a positive C^∞ function homogeneous of degree one w.r.t. $(\delta_\lambda)_{\lambda>0}$. Then $x \mapsto g(\varrho(x))$ belongs to $L^1(\Omega)$ and*

$$\int_\Omega g(\varrho(x))\,\mathrm{d}x = \omega_{Q,\varrho} \int_0^{\frac{1}{2}} r^{Q-1} g(r)\,\mathrm{d}r.$$

Proof. We have

$$\begin{aligned}\int_\Omega g(\varrho(x))\,\mathrm{d}x &= \int_{\{\varrho(x)\leq\frac{1}{2}\}} g(\varrho(x))\,\mathrm{d}x \\ &\overset{\text{co-area}}{=} \int_0^{\frac{1}{2}} \mathrm{d}t \int_{\{\varrho(x)=t\}} g(t)\frac{\mathrm{d}\sigma(x)}{|\nabla\varrho(x)|}. \end{aligned} \tag{1.41}$$

Since

$$|\{\varrho(x) < r\}| = \int_{\{\varrho(x)\le r\}} \mathrm{d}x \overset{x=\delta_r y}{=} \int_{\{\varrho(\delta_r y)\le r\}} r^Q \,\mathrm{d}y = r^Q \int_{\{\varrho(y)\le 1\}} \mathrm{d}y = Cr^Q,$$

but at the same time

$$|\{\varrho(x) < r\}| = \int_{\{\varrho(x)\le r\}} \mathrm{d}x \overset{\text{co-area}}{=} \int_0^r \mathrm{d}t \int_{\varrho(x)=t} \frac{\mathrm{d}\sigma(x)}{|\nabla\varrho(x)|},$$

and so the two right-hand sides are equal. Differentiating with respect to r we obtain

$$QCr^{Q-1} = \int_{\{\varrho(x)=r\}} \frac{\mathrm{d}\sigma(x)}{|\nabla\varrho(x)|}.$$

Plugging it into (1.41) we have the thesis with $\omega_{Q,\varrho} = QC$.

□

Proposition 1.8.4 (**A counterexample with** $p = Q/2$). *Let* $\{X_j\}_{j=1,\dots,m}$ *be a family of smooth vector fields verifing* **(I)** *and* **(S)**. *Suppose that the adjoint* X_j^* *of* X_j *is equal to* $-X_j$ *(it happens if and only if* $\mathrm{div}(X_j I) = 0$*). Let* L *be the second order self-adjoint operator*

$$L = \sum_{j=1}^m X_j^2,$$

which is X*-elliptic. Suppose that Hörmander condition is verified by vector fields, i.e.*

$$\text{rank Lie}(X_1,\dots,X_m)(x) = N, \qquad \forall x \in \mathbb{R}^N.$$

Then there exists an unbounded function $u \in W_0^1(\Omega, X)$ *weak solution to* $Lu = h$, *with* $h \in L^{Q/2}(\Omega)$.

Proof. Let $\varrho : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a positive C^∞ function homogeneous of degree one w.r.t. $(\delta_\lambda)_{\lambda>0}$, e.g.

$$\varrho(x) = \left((x_1^2)^{\frac{s}{\alpha_1}} + \cdots + (x_N^2)^{\frac{s}{\alpha_N}}\right)^{\frac{1}{2s}},$$

where s is the least common multiple of $\alpha_1, \dots, \alpha_N$. Let Ω be the open set

$$\Omega := \{x \in \mathbb{R}^N | \varrho(x) < 1/2\}$$

and define, for $x \in \overline{\Omega} \setminus \{0\}$,

$$u(x) := f\big(\varrho(x)\big) - f(1/2),$$

with $f(s) = \ln|\ln s|$. We have that $u \in C^\infty(\overline{\Omega} \setminus \{0\})$, u is unbounded in Ω, $u = 0$ on $\partial\Omega$, $X_j u = f'(\varrho) X_j \varrho$. Moreover

$$Lu = f''(\varrho)|X\varrho|^2 + f'(\varrho)L\varrho =: h \quad \text{in } \Omega \setminus \{0\}. \tag{1.42}$$

Once

1. $u \in L^2(\Omega)$;
2. $X_j u \in L^2(\Omega),\ \forall j \in \{1, \dots, m\}$;
3. $h \in L^{\frac{Q}{2}}(\Omega)$;

are proved we reach our purpose.
Using Lemma 1.8.3, property 1 immediately follows from direct integration if we take $g = \ln^2|\ln(\cdot)|$.
Property 2 can be proved noticing that the functions $X_j\varrho$, for $j = 1, \dots, m$ are bounded since they are smooth away from the origin and homogeneous of degree zero w.r.t. δ_λ. By consequence

$$|X_j u| = |f'(\varrho)| \cdot |X_j\varrho| \le C|f'(\varrho)|,$$

and using one more time Lemma 1.8.3,

$$\begin{aligned} \int_\Omega |X_j u|^2 \,\mathrm{d}x \quad &\le \quad C\omega_{Q,\varrho} \int_0^{\frac{1}{2}} r^{Q-1} |f'(r)|^2 \,\mathrm{d}r = \int_0^{\frac{1}{2}} r^{Q-1} \left|\frac{1}{r \ln r}\right|^2 \mathrm{d}r = \\ &\overset{\ln r = t}{=} \int_{-\infty}^{\ln\frac{1}{2}} e^{t(Q-1)} \left|\frac{1}{te^t}\right|^2 e^t \,\mathrm{d}t = \int_{-\infty}^{\ln\frac{1}{2}} e^{t(Q-2)} \left|\frac{1}{t}\right|^2 \mathrm{d}t. \end{aligned}$$

The result immediately follows from the hypothesis $Q \ge 3$.
To prove Property 3 we preliminarly notice that functions $|X\varrho|^2$ and $\varrho L\varrho$ are

smooth away from the origin and homogenous of degree zero w.r.t. δ_λ for $\lambda > 0$. Whence

$$\begin{aligned} |h| &= |f''(\varrho)|X\varrho|^2 + f'(\varrho)L\varrho| \leq C(|f''(\varrho)| + |f'(\varrho)|/\varrho) = \\ &= C\left(\left|-\frac{1}{(\varrho\ln\varrho)^2} - \frac{1}{\varrho^2\ln\varrho}\right| + \left|\frac{1}{\varrho^2\ln\varrho}\right|\right) \leq C_1\left|\frac{1}{\varrho^2\ln\varrho}\right| \end{aligned}$$

for $\varrho < \frac{1}{2}$. Again from Lemma 1.8.3, and using co-area formula

$$\int_\Omega |h|^{\frac{q}{2}}\,\mathrm{d}x \leq C\int_0^{\frac{1}{2}} \frac{1}{r|\ln r|^{\frac{Q}{2}}}\,\mathrm{d}r < +\infty,$$

since $\frac{Q}{2} > 1$.

□

1.9 Harnack inequality under doubling measure and Poincaré conditions

In this section we prove an invariant Harnack inequality for non-negative solution to $Lu = 0$ in Ω, where Ω is a bounded open subset of $\mathbb{R}^N$. We suppose that **(D)** and **(P)** are satisfied. We know that these two hypotheses together with continuity of the control distance imply Sobolev inequality, i.e.

$$||u||_{L^q_*(B_r)} \leq Cr||Xu||_{L^2_*(B_r)}, \quad \forall u \in C^1_0(B_r), \tag{1.43}$$

for every d-ball B_r with center in a fixed compact set $\widetilde{K}$, containing the closure of Ω and radius $r \leq r_0(\widetilde{K})$. We introduce following notation:

$$||u||_{L^s_*(B_r)} = \left(\fint_{B_r} |u|^s\,\mathrm{d}x\right)^{\frac{1}{s}} = \frac{1}{|B_r|^{\frac{1}{s}}}||u||_{L^s(B_r)}.$$

The exponent q in Sobolev inequality is function of Doubling constant D of **(D)** , that is

$$q = \frac{2Q}{Q-2} \quad \text{where} \quad Q = \log_2 D.$$

We also use the notations

$$a := (\gamma^2 + |c|)^{\frac{1}{2}}, \quad a^* := \sup_r (r||a||_{L^{2p}_*(B_r)}),$$

where the supremum is calculated on the set of $r > 0$ such that $B \subseteq \Omega$. Notice that $a \in L^{2p}(\Omega)$.

Theorem 1.9.1 (**Harnack inequality**). *Let $\{X_j\}_{j=1,\dots,m}$ be a family of smooth vector fields, d be the associated control distance, L be the differential operator of the form (1.2). Suppose that L is uniformly X-elliptic in Ω and satisfies* **(+)** *in $\mathbb{R}^N$. Suppose that* **(D)** *holds on d, that* **(L)** *holds with $Q = \log_2 D$, and* **(P)** *holds too.*
If $u \in W^1_{\mathrm{loc}}(\Omega, X)$ is a non-negative weak solution to $Lu = 0$, and $r \leq r_0(\overline{\Omega})/4$, then for any d-ball $B_{4r} \subseteq \Omega$ we have

$$\sup_{B_r} u \leq C \inf_{B_r} u, \tag{1.44}$$

with C as (structural) constant. We mean that C depends only on λ, Λ (from X-elliptic conditions), a^*, D, constant in Poincaré inequality, Lipschitz constants of vector fields, and p in **(L)**.

Proof. We suppose u bounded-below away from zero. This is not restrictive. Indeed, if $\inf_\Omega u = 0$ it suffices to replace u by $u+\varepsilon$, where $\varepsilon > 0$, and let ε go to zero in the final estimates. We follow now the same approach in [GT83], Section 8.6.
For every $\beta \in \mathbb{R}$, $\beta \neq 0$, and $\eta \in C^1_0(\Omega)$, with $\eta \geq 0$, set

$$v := \eta^2 u^\beta. \tag{1.45}$$

If $\beta \leq 1$, keeping in mind that $\inf_\Omega u > 0$, by the chain rule $v \in W^1_0(\Omega, X)$, and so v is an admissible test function in the integral form

$$\mathcal{B}(u, v) = 0 \quad \text{in } \Omega.$$

The same holds when $\beta > 1$ if u is bounded above, as we can assume without losing of generality. This condition can be removed by replacing u^β in (1.45), when $\beta > 1$, with the sequence of functions

$$u_k^\beta := \begin{cases} u^\beta & \text{if } u \leq k \\ \beta k^{\beta-1}(u-k) + k^\beta & \text{otherwise.} \end{cases}$$

The needed result follows letting k tend to infinity.

Let us now split the remaining part of the proof in two steps.

Step 1. So we have

$$\begin{aligned} 0 &= \mathcal{B}(u,v) = \mathcal{B}(u,\eta^2 u^\beta) = \\ &= \int_\Omega \Big(\langle B\nabla u, \nabla(\eta^2 u^\beta)\rangle + \langle d, \nabla(\eta^2 u^\beta)\rangle u \Big)\,\mathrm{d}x \\ &\quad - \int_\Omega \Big(\langle b, \nabla u\rangle(\eta^2 u^\beta) - cu(\eta^2 u^\beta) \Big)\,\mathrm{d}x = \\ &= \int_\Omega \Big(2\eta u^\beta \langle B\nabla u, \nabla \eta\rangle + \eta^2 \beta u^{\beta-1}\langle B\nabla u, \nabla u\rangle \Big)\,\mathrm{d}x \\ &\quad + \int_\Omega \Big(2\eta u^{\beta+1}\langle d, \nabla\eta\rangle + \eta^2\beta u^\beta \langle d, \nabla \eta\rangle \Big)\,\mathrm{d}x \\ &\quad - \int_\Omega \Big(\eta^2 u^\beta \langle b, \nabla u\rangle - cu^{\beta+1}\eta^2 \Big)\,\mathrm{d}x, \end{aligned}$$

hence by X-ellipticity, by **(+)** and by the above equality

$$\begin{aligned} \int_\Omega \eta^2\beta u^{\beta-1}\lambda |Xu|^2\,\mathrm{d}x &\leq \int_\Omega \Big(\eta^2\beta u^{\beta-1}\langle B\nabla u, \nabla u\rangle \Big)\,\mathrm{d}x \leq \\ &\leq 2\int_\Omega \Big(u^\beta \Lambda^{\frac{1}{2}}|Xu|\Lambda^{\frac{1}{2}}|X\eta|\eta \Big)\,\mathrm{d}x + 2\int_\Omega \Big(\eta u^{\beta+1}\gamma|X\eta| \Big)\,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\eta^2 u^\beta \gamma |Xu|(1+\beta^{-1})|\beta| \Big)\,\mathrm{d}x + \int_\Omega \Big(|c| u^{\beta+1}\eta^2 \Big)\,\mathrm{d}x. \end{aligned}$$

We obtain

$$\begin{aligned} \int_\Omega \eta^2 u^{\beta-1}|Xu|^2\,\mathrm{d}x &\leq \frac{2\Lambda}{\lambda|\beta|}\int_\Omega \eta|Xu||X\eta|u^\beta\,\mathrm{d}x \\ &\quad + \frac{1}{\lambda|\beta|}\int_\Omega \Big((|\beta|+1)|\beta|\eta^2 u^\beta |Xu| + 2\eta u^{\beta+1}|X\eta| \Big)\gamma\,\mathrm{d}x \\ &\quad + \frac{1}{\lambda|\beta|}\int_\Omega |c|\eta^2 u^{\beta+1}\,\mathrm{d}x. \end{aligned}$$

If $w := u^{\frac{\beta+1}{2}}$ when $\beta \neq -1$, and $w := \ln u$ when $\beta = 1$, by similar computations as above we obtain

$$\int_\Omega |\eta X w|^2 \,\mathrm{d}x \leq C(\beta+1)^2 \int_\Omega \Big((a\eta)^2 + |X\eta|^2\Big) w^2 \,\mathrm{d}x \tag{1.46}$$

if $\beta \neq 1$, and

$$\int_\Omega |\eta X w|^2 \,\mathrm{d}x \leq C \int_\Omega \Big((a\eta)^2 + |X\eta|^2\Big) \,\mathrm{d}x \tag{1.47}$$

if $\beta = -1$. C is a constant strictly greater than zero, only depending on Λ, λ, and the quotient $\frac{1+|\beta|}{|\beta|}$.

If B_r is a d-ball of radius r such that the ball B_{4r} with same center is contained in Ω, and η is a function belonging to $C_0^1(B_{4r})$, combining (1.43) with (1.46) and keeping in mind the triangle inequality we obtain

$$\begin{aligned} ||\eta w||_{L^q_*(B_{4r})} &\leq C_1 \cdot 4r \cdot ||X(\eta w)||_{L^2_*(B_{4r})} = C_1 r ||\eta X w + w X\eta||_{L^2_*(B_{4r})} \leq \\ &\leq C_1 r(1+|\beta+1|)\big(||a\eta w||_{L^2_*(B_{4r})} + ||wX\eta||_{L^2_*(B_{4r})}\big), \end{aligned} \tag{1.48}$$

in the case $\beta \neq 1$. In particular, using interpolation inequality (1.34),

$$\begin{aligned} ||a\eta w||_{L^2_*(B_{4r})} &\leq ||a||_{L^{2p}_*(B_{4r})} ||\eta w||_{L^{2p/(p-1)}_*(B_{4r})} \leq \\ &\leq ||a||_{L^{2p}_*(B_{4r})} \big(\varepsilon ||\eta w||_{L^q_*(B_{4r})} + \varepsilon^{-\mu} ||\eta w||_{L^2_*(B_{4r})}\big). \end{aligned} \tag{1.49}$$

We shall recall that the above inequalities are possible because of $2 < \frac{2p}{p-1} < q = \frac{2Q}{Q-2}$, giving $p > \frac{Q}{2}$. ε is strictly greater than zero and μ is given by (1.33), in this case $\mu = \frac{Q}{2p-Q}$. We now choose $\varepsilon = \frac{1}{2C_1(1+|\beta+1|)a^*}$, and plugging (1.49) into (1.48) we obtain for $\beta \neq 1$

$$||\eta w||_{L^q_*(B_{4r})} \leq C(1+|\beta+1|)^{1+\mu} \Big(||\eta w||_{L^2_*(B_{4r})} + r||wX\eta||_{L^2_*(B_{4r})}\Big), \tag{1.50}$$

where we remark C depends only on structural constants and on $\frac{1+|\beta|}{|\beta|}$. This inequality can be extended to every cut-off function $\eta \in W_0^1(B_{4r})$.

Now let us take two radii r_1 and r_2 such that

$$r \leq r_1 < r_2 \leq 2r,$$

and a cut-off function η such that

- $\eta \in W_0^1(B_{r_2})$;
- $\eta(x) = 1, \ \forall x \in B_{r_1}$;
- $|X\eta| \leq C(r_2 - r_1)^{-1}, \ C > 0$.

The constant C only depends on Lipschitz constants of X_j in Ω. The existence of such cut-off function was proved in [FSSC98] and in [GDMN98]. Then, from (1.50) and keeping in mind relationship among radii we obtain for every $q > 2$

$$||w||_{L^q_*(B_{r_1})} \leq C(1 + |\beta + 1)^{1+\mu} \left(1 + \frac{r}{r_2 - r_1}\right) ||w||_{L^2_*(B_{r_2})}. \tag{1.51}$$

Now following almost the same technique of Gilbarg and Trudinger in [GT83] at page 197, we get the desired estimates of the supremum and the infimum of u in B_r. In particular, for first iteration of (1.51) we take

- $\beta + 1 = \theta^k s$, with $s > 1$, $\theta = q/2$, (so $\beta_k = (q/2)^k s - 1$);
- $r_k = r(1 + 2^- k)$, with $k \geq 0$, (so $\{r_k\}_{k\in\mathbb{N}}$ is decreasing, $r_0 = 2r$ and $\lim_{k\to+\infty} r_k = r$).

With $k = 0$

$$\begin{aligned} ||w||_{L^q_*(B_{\frac{3r}{2}})} &= \left(\fint_{B_{\frac{3r}{2}}} |w|^q \,\mathrm{d}x\right)^{\frac{1}{q}} = \left(\fint_{B_{\frac{3r}{2}}} |u|^{\theta s} \,\mathrm{d}x\right)^{\frac{1}{q}} \leq \\ &\leq C_0(1+s)^{1+\mu}\left(1 + \frac{1}{1-\frac{1}{2}}\right)\left(\fint_{B_{2r}} |u|^s \,\mathrm{d}x\right)^{\frac{1}{2}}, \end{aligned}$$

and powering both sides to q we get first step iteraction.
Therefore for every k we have

$$\fint_{B_{r_{k+1}}} |u|^{\theta^{k+1}s} \,\mathrm{d}x \leq C_k\Big(1 + \theta^k s\Big)^{2\theta(1+\mu)}\Big(1 + 2^{k+1}\Big)^{2\theta}\left(\fint_{B_{r_k}} |u|^{\theta^k s} \,\mathrm{d}x\right)^{\theta},$$

and substituting in the right-hand side step $k-1$ we obtain

$$\begin{aligned}\fint_{B_{r_{k+1}}} |u|^{\theta^{k+1}s}\,\mathrm{d}x \;&\le\; C_k\Big(1+\theta^k s\Big)^{2\theta(1+\mu)}\Big(1+2^{k+1}\Big)^{2\theta} \\ &\cdot\Bigg(C_{k-1}\Big(1+\theta^{k-1}s\Big)^{2\theta(1+\mu)}\Big(1+2^k\Big)^{2\theta} \\ &\cdot\bigg(\fint_{B_{r_{k-1}}} |u|^{\theta^{k-1}s}\,\mathrm{d}x\bigg)^{\theta}\Bigg)^{\theta},\end{aligned}$$

and so iterating until $k=0$ we get

$$\begin{aligned}\fint_{B_{r_{k+1}}} |u|^{\theta^{k+1}s}\,\mathrm{d}x \;&\le\; \prod_{j=0}^{k}\bigg(C_{k-j}^{\theta^j}\Big(1+\theta^{k-j}s\Big)^{2\theta^{j+1}(1+\mu)}\Big(1+2^{k-j+1}\Big)^{2\theta^{j+1}}\bigg) \\ &\cdot\bigg(\fint_{B_{2r}} |u|^{s}\,\mathrm{d}x\bigg)^{\theta^{k+1}}.\end{aligned} \tag{1.52}$$

Now we shall power both sides to $\frac{1}{\theta^{k+1}s}$. The right-hand side iterated product is bounded for every k, and to prove it we can proceed as follows, e.g. for the first factor

$$\begin{aligned}\bigg(\prod_{j=0}^{k} C_{k-j}^{\theta^j}\bigg)^{\frac{1}{\theta^{k+1}s}} \;&=\; \bigg(\exp\sum_{j=0}^{k}\Big(\theta^j\ln C_{k-j}\Big)\bigg)^{\frac{1}{\theta^{k+1}s}} = \\ &=\; \exp\left(\frac{\sum_{j=0}^{k}\Big(\theta^j\ln C_{k-j}\Big)}{\theta^{k+1}s}\right) \le \\ &\le\; \exp\left(\frac{1}{s}\right)\exp\left(\frac{\sum_{j=0}^{k}\Big(\theta^j|\ln C_{k-j}|\Big)}{\theta^{k+1}}\right) \le \\ &\overset{|C_k|\le C,\ \forall k\in\mathbb{N}}{\le}\; C\exp\left(\frac{\sum_{j=0}^{k}\theta^j}{\theta^{k+1}}\right) = \\ &=\; C\exp\left(\frac{\frac{1-\theta^{k+1}}{1-\theta}}{\theta^{k+1}}\right) \le \widetilde{C} \quad \forall k\in\mathbb{N},\end{aligned}$$

and almost identically for

$$\bigg(\prod_{j=0}^{k}\Big(1+\theta^{k-j}s\Big)^{2\theta^{j+1}(1+\mu)}\bigg)^{\frac{1}{\theta^{k+1}s}},\quad \bigg(\prod_{j=0}^{k}\Big(1+2^{k-j+1}\Big)^{2\theta^{j+1}}\bigg)^{\frac{1}{\theta^{k+1}s}}.$$

So (1.52) becomes

$$\left(\fint_{B_{r_{k+1}}} |u|^{\theta^{k+1}s}\,\mathrm{d}x\right)^{\frac{1}{\theta^{k+1}s}} \leq C_s\left(\fint_{B_{2r}} |u|^{s}\,\mathrm{d}x\right)^{\frac{1}{s}}. \tag{1.53}$$

Now if we consider the left-hand side of the last inequality we have

$$\begin{aligned}\left(\fint_{B_{r_{k+1}}} |u|^{\theta^{k+1}s}\,\mathrm{d}x\right)^{\frac{1}{\theta^{k+1}s}} &= \frac{1}{|B_{r_{k+1}}|^{\frac{1}{\theta^{k+1}s}}}\left(\int_{B_{r_{k+1}}} |u|^{\theta^{k+1}s}\,\mathrm{d}x\right)^{\frac{1}{\theta^{k+1}s}} \geq \\ &\overset{B_r\subset B_{r_{k+1}}}{\geq} \frac{1}{|B_{r_{k+1}}|^{\frac{1}{\theta^{k+1}s}}}\left(\int_{B_{r}} |u|^{\theta^{k+1}s}\,\mathrm{d}x\right)^{\frac{1}{\theta^{k+1}s}}.\end{aligned}$$

Now letting k tend to infinity we get

$$\frac{1}{|B_{r_{k+1}}|^{\frac{1}{\theta^{k+1}s}}}\left(\int_{B_{r}} |u|^{\theta^{k+1}s}\,\mathrm{d}x\right)^{\frac{1}{\theta^{k+1}s}} \xrightarrow[k\to+\infty]{} 1\cdot ||u||_{L^\infty(B_r)},$$

because u is bounded above, so the first crucial inequality we obtain is

$$\sup_{B_r} u \leq C_s\left(\fint_{B_{2r}} u^{s}\,\mathrm{d}x\right)^{\frac{1}{s}}. \tag{1.54}$$

The constant $C_s > 0$ depends only on s and structural constants.
In the second iteration of (1.51) we take

- $\beta + 1 = -p_0\theta^k$, with $p_0 > 0$ small enough, $\theta = \frac{q}{2}$, (so $\beta_k = -p_0\left(\frac{q}{2}\right)^k - 1$);

and the radius sequence will be kept the same. Acting as before we obtain

$$\inf_{B_r} u \geq C_{p_0}\left(\fint_{B_{3r}} u^{-p_0}\,\mathrm{d}x\right)^{-\frac{1}{p_0}}. \tag{1.55}$$

The structural constant C_{p_0} depends also on p_0.
The third iteration of (1.51) has to be done with

- $\beta + 1 = s\theta^{-k-1}$, with $s > 1$, $\theta = \frac{q}{2}$, (so $\beta_k = s\left(\frac{q}{2}\right)^{-k-1} - 1$);

so we obtain

$$\left(\fint_{B_{2r}} u^s \,\mathrm{d}x\right)^{\frac{1}{s}} \leq C_{s,p_0}\left(\fint_{B_{3r}} u^{-p_0}\,\mathrm{d}x\right)^{-\frac{1}{p_0}}, \tag{1.56}$$

of course in non-trivial case of $s > p_0$.

Step 2. Keeping the same notation we prove now the existence of $p_0 > 0$, small enough, such that

$$\left(\fint_{B_{3r}} u^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}} \leq C\left(\fint_{B_{3r}} u^{-p_0}\,\mathrm{d}x\right)^{-\frac{1}{p_0}} \quad \forall B_{4r} \subseteq \Omega, \tag{1.57}$$

where, $C > 0$ is, as it always is, a structural constant depending on p_0 (which is structural too). So let $w := \ln u$. For any d-ball B_ϱ such that $B_{2\varrho} \subseteq \Omega$ we pick a cut-off function η about which we require

- $\eta \in W_0^1(\Omega, X)$;
- $\eta(x) = 1 \quad \forall x \in B_\varrho$;
- $\eta(x) = 0 \quad \forall x \in \Omega \setminus B_{2\varrho}$;
- $|X\eta| \leq C/\varrho$.

So we get

$$\begin{aligned}
\int_{B_\varrho} |Xw|^2\,\mathrm{d}x &\overset{\eta\equiv 1}{=} \int_{B_\varrho} |\eta Xw|^2\,\mathrm{d}x \leq \int_\Omega |\eta Xw|^2\,\mathrm{d}x \overset{(1.47)}{\leq} C\int_\Omega \left(\left(a\eta\right)^2 + |X\eta|^2\right)\mathrm{d}x \leq \\
&\leq C\int_\Omega \left(a^2\eta^2 + \frac{1}{\varrho^2}\chi_{B_{2\varrho}}\right)\mathrm{d}x = C\int_\Omega \left(a^2\eta^2\right)\mathrm{d}x + \frac{C}{\varrho^2}|B_{2\varrho}| = \\
&= C\int_{B_{2\varrho}} \left(a^2\eta^2\right)\mathrm{d}x + \frac{C}{\varrho^2}|B_{2\varrho}| \leq C\int_{B_{2\varrho}} a^2\,\mathrm{d}x + \frac{C}{\varrho^2}|B_{2\varrho}|.
\end{aligned}$$

Dividing the above inequality to $|B_\varrho|$ we get

$$\begin{aligned}
\fint_{B_\varrho} |Xw|^2\,\mathrm{d}x &\leq C\frac{|B_{2\varrho}|}{|B_\varrho|}\fint_{B_{2\varrho}} a^2\,\mathrm{d}x + \frac{C|B_{2\varrho}|}{\varrho^2|B_\varrho|} \overset{\textbf{(D)}}{\leq} C\left(\fint_{B_{2\varrho}} a^2\,\mathrm{d}x + \frac{1}{\varrho^2}\right) = \\
&= \frac{C}{\varrho^2}\left(\varrho^2\fint_{B_{2\varrho}} a^2\,\mathrm{d}x + 1\right) \overset{\text{Hö}}{\leq} \frac{C}{\varrho^2}\left(\varrho^2\left(\fint_{B_{2\varrho}} a^{2p}\,\mathrm{d}x\right)^{\frac{1}{p}} + 1\right) = \\
&= \frac{C}{\varrho^2}\left(\frac{1}{4}\left(2\varrho\left(\fint_{B_{2\varrho}} a^{2p}\,\mathrm{d}x\right)^{\frac{1}{2p}}\right)^2 + 1\right) \leq \frac{C}{\varrho^2}\left(\left(a^*\right)^2 + 1\right).
\end{aligned}$$

Moreover d-balls support **(P)**, and so using last inequality and **(D)** we obtain

$$\begin{aligned}
\fint_{B_\varrho} |w - w_\varrho| \, \mathrm{d}x &\overset{\textbf{(P)}}{\leq} C\varrho \fint_{B_{2\varrho}} |Xw| \, \mathrm{d}x \leq C\varrho \Big(\fint_{B_{2\varrho}} |Xw|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\
&\leq C\varrho \Big(\frac{1}{\varrho^2} \big((a^*)^2 + 1\big) \Big)^{\frac{1}{2}} = C\big((a^*)^2 + 1\big)^{\frac{1}{2}} = C,
\end{aligned}$$

where C is a structural constant. Buckey [Buk98] proved that John-Nirenberg's estimate holds in Doubling spaces, i.e.

$$\fint_{B_{3\varrho}} \exp\big(p_0 |w - w_{3r}|\big) \, \mathrm{d}x \leq M, \quad w_{3r} = \fint_{B_{3r}} w \, \mathrm{d}x$$

holds for every d-ball B_r such that $B_{4r} \subseteq \Omega$, where M and p_0 are ad hoc positive structural constants. Whence

$$\begin{aligned}
&\Big(\fint_{B_{3r}} u^{-p_0} \, \mathrm{d}x \Big) \Big(\fint_{B_{3r}} u^{p_0} \, \mathrm{d}x \Big) = \\
&\quad \overset{w = \ln u}{=} \Big(\fint_{B_{3r}} \exp(-p_0 w) \, \mathrm{d}x \Big) \Big(\fint_{B_{3r}} \exp(p_0 w) \, \mathrm{d}x \Big) = \\
&\quad = \Big(\fint_{B_{3r}} \exp\big(-p_0(w - w_{3r})\big) \, \mathrm{d}x \Big) \Big(\fint_{B_{3r}} \exp\big(p_0(w - w_{3r})\big) \, \mathrm{d}x \Big) \leq \\
&\quad \leq \Big(\fint_{B_{3r}} \exp\big(p_0 |w - w_{3r}|\big) \, \mathrm{d}x \Big)^2 \leq M^2,
\end{aligned}$$

and this proves (1.57). Now

$$\sup_{B_r} u \overset{(1.54)}{\leq} C \Big(\fint_{B_{2r}} u^s \, \mathrm{d}x \Big)^{\frac{1}{s}} \overset{(1.56)}{\leq} C \Big(\fint_{B_{3r}} u^{p_0} \Big)^{\frac{1}{p_0}} \overset{(1.57)}{\leq} C \Big(\fint_{B_{3r}} u^{-p_0} \Big)^{-\frac{1}{p_0}} \overset{(1.55)}{\leq} C \inf_{B_r} u,$$

and so the theorem follows.

□

1.10 Application to homogeneous vector fields

In this section we prove a Maximum Principle that holds uniformly on d-rings when L is of the form (1.2), and that holds uniformly on balls when

L is in principal form. d is control distance defined through $\{X_j\}_{j=1}^m$, family of vector fields that satisfy **(I)**. This condition implies homogeinity of degree one of d (i.e. $d(\delta_R x, \delta_R y) = Rd(x,y)$), and that a_{kj}, coefficients of ∂_k in every X_j, are homogeneous of degree $\alpha_k - 1$. Of course in the whole chapter when we talk about homogeinity we refer to δ_R-homogeneity in **(I)**, and we suppose **(D)** and **(P)** and by consequence **(S)** verified, as recalled in Section 1.4.

Theorem 1.10.1 (**Maximum Principle on rings**). *Let $\alpha_1, \dots, \alpha_N \in \mathbb{N}$, $Q := \sum_{j=1}^N \alpha_j$, $\{X_j\}_{j=1}^m$ be a family of vector fields satisfying* **(I)**. *Suppose L is of the form (1.2) and it is uniformly X-elliptic and it satisfies* **(+)** *in $\mathbb{R}^N$. Let $p > Q/2$, and let u be a weak solution to $Lu = f$ in the ring*

$$A_R(a,b) := \{x \in \mathbb{R}^N \mid aR < d(x) < bR\},$$

where $d(x) := d(x,0)$ and $0 < a < 1 < b$. Moreover suppose that γ and c satisfy the following conditions:

$$⨍_{A_r} \gamma^{2p}\, \mathrm{d}x \le CR^{-2p}, \qquad ⨍_{A_r} |c|^p\, \mathrm{d}x \le CR^{-2p}, \tag{1.58}$$

uniformly in R, i.e. C does not depend on R (but it can depend on a and b for example).
Then

$$\sup_{A_R} u^+ \le \sup_{\partial A_R} u^+ + CR^{2-\frac{Q}{p}} ||f||_{L^p(A_R)}, \tag{1.59}$$

where C is a constant independent of R.
Moreover, if $\gamma = 0$ (so $b \equiv 0$ and $d \equiv 0$) and $c = 0$, and $Lu = f$ in the ball $B_R(0)$, then

$$\sup_{B_r(0)} u^+ \le \sup_{\partial B_r(0)} u^+ + CR^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))}, \tag{1.60}$$

where C is a constant independent of R. We would like to recall that in this second case L is called in principal form.

Proof. By homogeneity of d we have that

$$A_R(a,b) = \delta_R\{x \in \mathbb{R}^N \mid a < d(x) < b\}.$$

The aim is to show the Maximum Principle on the unit ring (ring with $R = 1$) independently of radius using Theorem 1.8.2 and (1.58), and afterwards using a rescaling argument through homogeneity we extend the result to any radius. Henceforth we prefere to show arguments of functions because of many variable substitutions. So

$$\begin{aligned}\mathcal{B}(u(x), v(x)) &= \int_{A_R} \Big(\langle B(x)\nabla u(x), \nabla v(x)\rangle + \langle d(x), \nabla v(x)\rangle u(x)\Big)\,\mathrm{d}x \\ &\quad - \int_{A_R} \Big(\langle b(x), \nabla u(x)\rangle v(x) - c(x)u(x)v(x)\Big)\,\mathrm{d}x = \\ &= -\int_{A_R} f(x)v(x)\,\mathrm{d}x\end{aligned}$$

for every v with $\operatorname{supp} v \subseteq A_R$. Multiplying the above identity by R^2 and doing a change of variable $x = \delta_R y$ we get

$$\begin{aligned}&\int_{A_1} \Big(\langle B_R(y)\nabla u_R(y), \nabla v_R(y)\rangle + \langle d_R(y), \nabla v_R(y)\rangle u_R(y)\Big)\,\mathrm{d}y \\ &- \int_{A_1} \Big(\langle b_R(y), \nabla u_R(y)\rangle v_R(y) - c_R(y)u_R(y)v_R(y)\Big)\,\mathrm{d}y = -\int_{A_1} f_R(y)v_R(y)\,\mathrm{d}y\end{aligned}$$

where notations are due to a changing of variable, i.e.

$$\begin{aligned}B_R(y) &= \Big(R^2 R^{-\alpha_i} R^{-\alpha_j} b_{ij}(\delta_R y)\Big)_{1\le i,j\le N}; \\ d_R(y) &= \Big(R^{2-\alpha_i} d_i(\delta_R y)\Big)_{1\le i\le N}; \\ b_R(y) &= \Big(R^{2-\alpha_i} b_i(\delta_R y)\Big)_{1\le i\le N}; \\ c_R(y) &= R^2 c(\delta_R y); \\ f_R(y) &= R^2 f(\delta_R y); \\ u_R(y) &= u(\delta_R y).\end{aligned}$$

Thus u_R is a weak solution to the equation $L_R u_R = f_R$ in the ring A_1, where L_R is a obvious notation for the operator of the same form of L but with rescaled coefficients instead. Whence L_R is uniformly X-elliptic with same constants. Indeed, switching to a matrix representation, if

$$D_{R^{-1}} := \begin{pmatrix} R^{-\alpha_1} & 0 & \cdots & 0 \\ 0 & R^{-\alpha_2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & R^{-\alpha_N} \end{pmatrix},$$

then

$$B_R(y) = R^2 \cdot D_{R^{-1}} \cdot B(\delta_R y) \cdot D_{R^{-1}}.$$

Hence first X-ellipticity condition for B_R with any $\xi \in \mathbb{R}^N$, and any $y \in A_1$, by the following computations is verified:

$$\begin{aligned} \langle B_R(y)\xi, \xi\rangle &= \langle R^2 D_{R^{-1}} B(\delta_R y) D_{R^{-1}}\xi, \xi\rangle = R^2 \langle B(\delta_R y) D_{R^{-1}}\xi, (D_{R^{-1}})^T \xi\rangle \geq \\ &\geq R^2 \lambda \sum_{j=1}^{N} \langle X_j I(\delta_R y), D_{R^{-1}}\xi\rangle^2 = R^2 \lambda \sum_{j=1}^{N} \langle R^{-1} D_R X_j I(y), D_{R^{-1}}\xi\rangle^2 = \\ &= \lambda \sum_{j=1}^{N} \langle X_j I(y), \xi\rangle^2. \end{aligned}$$

Acting in the same way it's easy to prove other X-ellipticity conditions, and which one with $d_R(y)$ and $b_R(y)$ becomes

$$\langle d_R(y), \xi\rangle^2 + \langle b_R(y), \xi\rangle^2 \leq \gamma_R(y)^2 \sum_{j=1}^{m} \langle X_j I(y), \xi\rangle^2, \qquad \forall \xi \in \mathbb{R}^N,\ y \in A_1.$$

where $\gamma_R(y) = R\gamma(\delta_R y)$. Whence we can apply Theorem 1.8.2 to the function u_R on A_1, and we obtain

$$\sup_{A_1} (u_R)^+ \leq \sup_{\partial A_1} (u_R)^+ C_R ||f_R||_{L^p(A_1)}. \tag{1.61}$$

C_R shall be estimated through estimating M in (1.30) and C_4 in (1.38).

$$\begin{aligned}
M &= \frac{2}{\lambda}+\frac{4}{\lambda^2}\left(\int_{A_1}\left(R^2\gamma(\delta_R y)^2\right)^p \mathrm{d}y\right)^{\frac{1}{p}}+\frac{2}{\lambda}\left(\int_{A_1} R^{2p}|c(\delta_R y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\\
&= \frac{2}{\lambda}+\frac{4R^{2-\frac{Q}{p}}}{\lambda^2}\left(\int_{A_R}\gamma(x)^{2p}\,\mathrm{d}x\right)^{\frac{1}{p}}+\frac{2R^{2-\frac{Q}{p}}}{\lambda}\left(\int_{A_R}|c(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&= \frac{2}{\lambda}+\frac{4R^{2-\frac{Q}{p}}}{\lambda^2}\left(R^Q|A_1|\fint_{A_R}\gamma(x)^{2p}\,\mathrm{d}x\right)^{\frac{1}{p}}+\frac{2R^{2-\frac{Q}{p}}}{\lambda}\left(R^Q|A_1|\fint_{A_R}|c(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&= \frac{2}{\lambda}+\frac{4R^2|A_1|^{\frac{1}{p}}}{\lambda^2}\left(\fint_{A_R}\gamma(x)^{2p}\,\mathrm{d}x\right)^{\frac{1}{p}}+\frac{2R^2|A_1|^{\frac{1}{p}}}{\lambda}\left(\fint_{A_R}|c(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\overset{(1.58)}{\leq} \frac{2}{\lambda}+\frac{4R^2|A_1|^{\frac{1}{p}}}{\lambda^2}\left(CR^{-2p}\right)^{\frac{1}{p}}+\frac{2R^2|A_1|^{\frac{1}{p}}}{\lambda}\left(CR^{-2p}\right)^{\frac{1}{p}}\leq C,
\end{aligned}$$

uniformly in R. Analogously with C_4

$$C_4=\frac{2}{\lambda}|A_1|^{\frac{1}{p'}}+\frac{4}{\lambda^2}\int_{A_1}R^2\gamma(\delta_R y)^2\,\mathrm{d}y\leq C.$$

Thus C_R is bounded and doesn't depend on R, and so we can rewrite (1.61) obtaining (1.59). The proof of (1.60) is almost identical, where the ball should be normalized to unit ball (i.e. $B_R(0)=\delta_R(B_1(0))$, and Theorem 1.8.2 should be applied on the set $B_1(0)$.

□

Using the rescaling argument as in previous proof we can prove following invariant (w.r.t. a,b,R) Harnack inequality on rings.

Theorem 1.10.2. *Suppose L is of the form (1.2) and it is uniformly X-elliptic and it satisfies* (+) *in $\mathbb{R}^N$ with $N\geq 2$. Let $p>Q/2$, $0<a'<a<1<b<b'<+\infty$ and suppose that γ and c satisfy the following condition:*

$$\fint_{B(z,Rr)}\left(\gamma(x)^{2p}+|c(x)|^p\right)\mathrm{d}x\leq C_0(Rr)^{-2p} \tag{1.62}$$

uniformly in $R>0$, $0<r<\frac{a'}{2}$ and $z\in A_R(a',b')$. Then there exists a constant $C>0$ independent of R such that

$$\sup_{A_R(a,b)} u\leq C\inf_{A_R(a,b)} u,$$

for any non-negative solution to $Lu = 0$ in $A_r(a', b')$.

Proof. $N \geq 2$ and so the ring $A_1(a', b')$ is connected with respect to the Euclidean topology. The closed ring $\overline{A}_1(a, b)$ is a compact subset of $A_1(a', b')$. Using the same notations of the last theorem by a rescaling argument we have

$$\fint_{B(z,r)} \left(\gamma_R(x)^{2p} + |c_R(x)|^p\right) \mathrm{d}x \leq C_0 r^{-2p}$$

uniformly in R for every ball $B(z, r)$ with $0 < r < \frac{a'}{2}$ and $z \in A_1(a', b')$. We can choice a finite sequence of these balls covering the ring $A_1(a', b')$ and therefore we obtain that $\gamma_R \in L^{2p}\big(A_1(a', b')\big)$ and $c_R \in L^p\big(A_1(a', b')\big)$. By consequence using Theorem 1.9.1 with ring $A_1(a', b')$ we get

$$\sup_{B(x,r)} v \leq C \inf_{B(x,r)} v$$

for any non-negative solution to $L_r v = 0$ in $A_1(a', b')$, and for every ball $B(x, r)$ such that $x \in A_1(a, b)$ and $0 < 4r < \min\{a - a', b' - b\}$. The same holds if $v = u_R$. Since d induce the same topology as the Euclidean one, from connectness and compactness of $\overline{A}_1(a, b)$, it follows by a standard covering argument that

$$\sup_{A_1(a,b)} u_R \leq C \inf_{A_1(a,b)} u_R,$$

where C does not depend on R. The result follows from a rescaling argument.

□

Remark 1.10.3. It's easy to see that (1.62) can be consequence of

$$d(x)\big(\gamma^2(x) + |c(x)|\big)^{\frac{1}{2}} \leq C, \quad \forall x \in \mathbb{R}^N.$$

Remark 1.10.4. In the case $N = 1$ the Theorem 1.62 is false. A counterexample can be found defining $u = \pi/2 + \arctan x$. Such function does not satisfy thesis of Theorem 1.62, however it is positive, bounded and non-constant global solution to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \left(\frac{2x}{(1 + x^2)}\right) \frac{\mathrm{d}u}{\mathrm{d}x} = 0.$$

If we deal with operators in principal form, combining Proposition 1.7.1, Theorem 1.9.1 and the just proved theorems, we can prove non-homogeneous Harnack inequality.

Theorem 1.10.5. *Suppose the operator L is uniformly X-elliptic and in principal form. If u is a global solution (i.e. on every bounded open subset of $\mathbb{R}^N$) to $Lu = f$, then we have the following non-homogeneous Harnack inequality on metric balls:*

$$\sup_{B_{\frac{R}{4}}} u \leq C \inf_{B_{\frac{R}{4}}} u + \widetilde{C} R^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))}, \quad p > \frac{Q}{2}, \tag{1.63}$$

where the constants C, $\widetilde{C}$ do not depend of R and u.

Proof. Because of Proposition 1.7.1 we know there exists v solution to $Lv = 0$ in $B_R(0)$ with $v - u \in W_0^1(B_R(0), X)$. But $u \geq 0$, and the same proposition proves that $v \geq 0$ in $B_R(0)$. For any $R > 0$ we pick $v_R(x) := v(\delta_R x)$, which is solution to the equation $L_R v_R = 0$ in $B_1(0)$, and L_R is an operator in principal form uniformly X-elliptic, with constants independent of R (same situation met in Theorem 1.10.1). Thus, by Theorem 1.9.1

$$\sup_{B_{\frac{1}{4}}(0)} v_R \leq C \inf_{B_{\frac{1}{4}}(0)} v_R,$$

and for v it becomes:

$$\sup_{B_{\frac{R}{4}}(0)} v \leq C \inf_{B_{\frac{R}{4}}(0)} v. \tag{1.64}$$

Of course C is always independent of R.
So we have $L(u - v) = 0$ in $B_R(0)$ and $u - v$=0 on $\partial B_r(0)$ in the sense of $W_0^1(B_R(0), X)$. This means that $(u - v)^+ = 0$ on $\partial B_r(0)$, and then by (1.60)

$$\sup_{B_r(0)} (u - v) \leq C R^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))}. \tag{1.65}$$

From (1.60), with $v = v - u + u$ we obtain

$$\begin{aligned} \inf_{B_{\frac{R}{4}}(0)} v &\leq \sup_{B_{\frac{R}{4}}(0)} (v - u) + \inf_{B_{\frac{R}{4}}(0)} u \leq \sup_{B_R(0)} (v - u) + \inf_{B_{\frac{R}{4}}(0)} u \\ &\leq C R^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))} + \inf_{B_{\frac{R}{4}}(0)} u. \end{aligned} \tag{1.66}$$

Instead, if we write $u = u - v + v$ we get

$$\begin{aligned}
\sup_{B_{\frac{R}{4}}(0)} u &\leq \sup_{B_{\frac{R}{4}}(0)} (u-v) + \sup_{B_{\frac{R}{4}}(0)} v \overset{(1.64)}{\leq} \sup_{B_R(0)} (u-v) + C \inf_{B_{\frac{R}{4}}(0)} v \\
&\overset{(1.65)}{\leq} CR^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))} + \inf_{B_{\frac{R}{4}}(0)} u \\
&\overset{(1.66)}{\leq} CR^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))} + C\Big(CR^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))} + \inf_{B_{\frac{R}{4}}(0)} u\Big) \\
&= \widetilde{C} R^{2-\frac{Q}{p}} ||f||_{L^p(B_R(0))} + C \inf_{B_{\frac{R}{4}}(0)} u,
\end{aligned}$$

so theorem is proved.

□

Remark 1.10.6. The inequality (1.63) is false for operators which are not in principal form. E.g. let us consider the operator in Proposition 1.8.4 and $u(x) = 1 + \varrho(x)^2$. From (1.42) it follows that u solves $Lu + cu = 0$, where $c = \dfrac{-h}{(1+\varrho^2)}$, $h = 2|X\varrho|^2 + 2\varrho L\varrho$. So h is bounded, but u doesn't satisfy (1.63).

By arguing as in the classical case from Theorem 1.10.5 we deduce a Liouville-type theorem for operators in principal form.

Corollary 1.10.7. *Suppose the operator L is uniformly X-elliptic and in principal form. If u is a global solution (i.e. on every bounded open subset of $\mathbb{R}^N$) to $Lu = 0$, and $u \geq 0$, then u is constant.*

Combining Theorem 1.8.2 with Theorem 1.10.1 we get another Liouville type theorem for X-elliptic operators that aren't in principal form.

Theorem 1.10.8. *If hypotheses of Theorem 1.10.2 are satisfied, if $d \equiv 0$ and $c \equiv 0$, if $v > 0$ is a solution to the equation $Lv = 0$ in $\mathbb{R}^N$ ($N \geq 2$) and if u is a non-negative global weak solution to $Lu = 0$, then*

$$u = C \cdot v \text{ in } \mathbb{R}^N,$$

where $C \geq 0$ is a constant.

Proof. For every $R > 0$ we set

$$a_R := \sup\{a \geq 0 \mid u(x) - av(x) \geq 0,\ \forall x \in \partial B_d(0, R)\}.$$

Because of linearity of L, $L(u - a_R v) = 0$ in the open set $\Omega_R := \{d(x) < R\}$, and

$$u - a_R v \geq 0 \text{ on } \partial\Omega_R \quad \Leftrightarrow \quad a_R v - u \leq 0 \text{ on } \partial\Omega_R \quad \Leftrightarrow \quad (a_R v - u)^+ \in W_0^1(\Omega_R, X).$$

Because of the definition of a_R there exists a sequence $\{a_n\}_{n\in\mathbb{N}} \nearrow a_R$ for $n \to +\infty$, where for each a_n we have $u - a_n \geq 0$ on $\partial B_d(0, R)$. By this $(a_n v - u)^+ \in W_0^1(B_d(0, R), X)$ and letting n tend to infinity we have $(a_R v - u)^+ \in W_0^1(\Omega_R, X)$.
Then by the Maximum Principle (Theorem 1.8.2) $u - a_r v \geq 0$ in Ω_R. In particular if $\widetilde{R} < R$, then $u - a_R v \geq 0$ in $\Omega_{\widetilde{R}}$, and so $a_{\widetilde{R}} \geq a_R$. Then we can define

$$a := \lim_{R\to+\infty} a_R = \inf_{R>0} a_R$$

By definition $0 \leq a < +\infty$ and $u - av \geq 0$ in $\mathbb{R}^N$. Since $L(u - av) = 0$, by the Harnack inequality on rings (Theorem 1.10.2) we obtain

$$\sup_{A_R(\frac{1}{2},2)} (u - av) \leq C \inf_{A_R(\frac{1}{2},2)} (u - av),$$

with C independent of R. Keeping in mind that $(u - av)^+ \equiv u - av$, by the Maximum Principle

$$\sup_{d(x)<2R} (u - av) = \sup_{d(x)=2R} (u - av).$$

Moreover, again by Harnack inequality on rings and the maximum principle,

$$\begin{aligned}
\inf_{A_R(\frac{1}{2},2)} (u - av) &\le \inf_{d(x)=2R} (u - av) = \inf_{d(x)=2R} (u - a_{2R}v + a_{2R}v - av) \le \\
&\le \inf_{d(x)=2R} (u - a_{2R}v) + \sup_{d(x)=2R} v(a_{2R} - a) = \\
&= \sup_{d(x)=2R} v(a_{2R} - a) \le \sup_{A_R(\frac{1}{2},2)} v(a_{2R} - a) \le \\
&\le C \inf_{A_R(\frac{1}{2},2)} v(a_{2R} - a) \le C \inf_{d(x)=2R} v(a_{2R} - a) \le \\
&\le C \inf_{d(x)<2R} v(a_{2R} - a).
\end{aligned}$$

By consequence

$$\sup_{d(x)<2R} (u - av) \le C^2 \inf_{d(x)\le 2R} v(a_{2R} - a),$$

Letting R tend to infinity we now deduce that

$$u - av \equiv 0 \text{ in } \mathbb{R}^N,$$

since $u - av \ge 0$, and so the Theorem is proved.

$\square$

Chapter 2

A notion of convexity related to sub-solutions and mean-value operators for ultraparabolic equations on Lie groups

In this chapter we prove a scalar convexity for mean-value operators applied to sub-solutions of ultraparabolic equations on Lie groups. We will use a potential theory approach, which has been already investigated by Cinti in [Cin07], where $\mathcal{L}$-subharmonic functions have been characterized in terms of mean-value operators and representation formulas. The class of operators we treat is contained in a wider class about which the related potential theory and an invariant Harnack inequality were singled out by Kogoj and Lanconelli in [KL04].

2.1 Introduction

Let $\{X_1, \ldots, X_m\}$ be a family of smooth vector fields in $\mathbb{R}^N$, i.e. with C^∞ coefficients in $\mathbb{R}^N$. So, using the same notation as in the previous chapter, we have that

$$X_j = \sum_{k=0}^{N} a_{kj}(x)\partial_k, \quad \partial_k = \frac{\partial}{\partial x_k}, \tag{2.1}$$

and denoting $z := (x,t)$ an element of $\mathbb{R}^{N+1}$ we define

$$Y := X_0 - \partial_t,$$

vector field in $\mathbb{R}^{N+1}$. With this notation we set

$$\mathcal{L} \;:=\; \sum_{j=1}^{m} X_j^2 + Y, \tag{2.2}$$

$$\mathcal{L}_0 \;:=\; \sum_{j=1}^{m} X_j^2 + X_0. \tag{2.3}$$

Our main assuptions are:

- **(Left invariance)**
  there exists a homogeneous Lie group $\mathbb{L} := (\mathbb{R}^{N+1}, \circ, \delta_\lambda)$ such that
  (i) $X_1, \dots, X_m, Y$ are left invariant on $\mathbb{L}$;
  (ii)$X_1, \dots, X_m$ are δ_λ-homogeneous of degree one and Y is δ_λ-homogeneous of degree two;

- **($\mathcal{L}$-admissible path)**
  for every (x,t), $(\xi,\tau) \in \mathbb{R}^{N+1}$ with $t > \tau$ there exists an $\mathcal{L}$-admissible path $\eta : [0,T] \to \mathbb{R}^{N+1}$ such that $\eta(0) = (x,t)$ and $\eta(T) = (\xi,\tau)$.

We recall that a vector field X is called left invariant if for every $\alpha \in \mathbb{L}$ we have

$$X\big(\varphi(\alpha \circ x)\big) = (X\varphi)(\alpha \circ x), \quad \forall \varphi \in C^\infty(\mathbb{R}^{N+1}, \mathbb{R}),$$

(where we consider our vector fields as vector fields in $\mathbb{R}^{N+1}$), and δ_λ-homogeneous of degree $\alpha \in \mathbb{N}$ if

$$X\Big(\varphi\big(\delta_\lambda(x)\big)\Big) = \lambda^\alpha \big(X\varphi\big)\big(\delta_\lambda(x)\big).$$

We also recall that an $\mathcal{L}$-admissible path is a strictly related concept to Definition (1.3.3), i.e. η is called $\mathcal{L}$-admissible if it is absolutely continuous and

$$\eta'(s) = \sum_{j=1}^{m} l_j(s) X_j\big(\eta(s)\big) + l_0(s) Y\big(\eta(s)\big), \qquad \text{a.e. in } [0,T],$$

where $l_0, \ldots, l_m$ are piecewise constant real functions with $l_0 \geq 0$.

2.2 Basic $\mathcal{L}$-potential theory and $\mathcal{L}$-subharmonic functions

First of all we shall recall some consequences of our main assumptions, all proved in [KL04].

- **Hörmander condition** is satisfied, i.e.

  $$\text{rank Lie}\{X_1, \ldots, X_m\}(z) = N+1, \quad \forall z \in \mathbb{R}^{N+1}$$

  whence $\mathcal{L}$ and $\mathcal{L}_0$ are hypoelliptic in $\mathbb{R}^{N+1}$ and in $\mathbb{R}^N$ respectively;

- **Composition law** is Euclidean in the last variable component,

  $$(x,t) \circ (\xi,\tau) = \big(S(x,t,\xi,\tau), t+\tau\big), \quad \forall (x,t), (\xi,\tau) \in \mathbb{R}^{N+1},$$

  with S smooth function;

- **Dilatation form** for all $\lambda > 0$ is the following one:

  $$\delta_\lambda(x,t) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N, \lambda^2 t), \quad \sigma_1, \ldots, \sigma_N \in \mathbb{N};$$

- **Homogeneous dimension** of $\mathbb{L}$ is

  $$Q = \sum_{k=1}^{N} \sigma_k + 2,$$

  and we suppose that $Q \geq 5$, and hence the homogeneous dimension of $\mathbb{R}^N$ (i.e. $Q-2$), with respect to dilatations of first N components of $\mathbb{R}^{N+1}$, will be greater than 3;

- δ_λ**-homogeneous norm** exists on $\mathbb{L}$ and it is of degree one i.e.:
  1. $|\cdot| : \mathbb{R}^{N+1} \to [0, +\infty[, \quad |\cdot| \in C^\infty(\mathbb{R}^{N+1} \setminus \{0,0\}) \cup C(\mathbb{R}^{N+1})$;
  2. $|\delta_\lambda(z)| = \lambda|z|$, $|z^{-1}| = |z|, \quad \forall z \in \mathbb{R}^{N+1}$;
  3. $|z| = 0 \quad \Leftrightarrow \quad z = 0$;
- **Global fundamental solution** (=: Γ) to $\mathcal{L}$ exists and it belongs to $C^\infty(\mathbb{R}^{N+1} \setminus \{0,0\})$, such that (by definition) $\mathcal{L}\Gamma = \delta$. Moreover $\Gamma(x,t) > 0$ if and only if $t > 0$. We can define $\Gamma(z,\zeta) := \Gamma(\zeta^{-1} \circ z)$, and because of the left translation invariance we have $\mathcal{L}\Gamma(\cdot,\zeta) = \delta_\zeta$ with $\zeta \in \mathbb{R}^{N+1}$.

We stress that global fundamental solution is δ_λ-homogeneous of degree $2 - Q$ (Proposition 2.8 in [KL04]).
Throughout whole chapter we will keep the just explained notations, and in addition Ω will denote an open subset of $\mathbb{R}^{N+1}$ if not different stated.

Definition 2.2.1 (**$\mathcal{L}$-harmonic function**). Let $u : \Omega \to \mathbb{R}$ a smooth function such that

$$\mathcal{L}u = 0.$$

Then u will be called $\mathcal{L}$-harmonic.

Definition 2.2.2 (**Linear space $\mathcal{H}^{\mathcal{L}}$**). We denote by $\mathcal{H}^{\mathcal{L}}(\Omega)$ the linear space of $\mathcal{L}$-harmonic functions in Ω.

Definition 2.2.3 (**$\mathcal{L}$-regular set**). A bounded open set $V \subset \mathbb{R}^{N+1}$ is said to be $\mathcal{L}$-regular if, for any $\varphi \in C(\partial V)$, there exists a (unique) function such that

$$H^V_\varphi(x) = \varphi(x_0), \quad \forall x_0 \in \partial V,$$

and $H^V_\varphi \geq 0$ whenever $\varphi \geq 0$.

Because of maximum principle (proved in [KL04], Proposition 2.1), we can consider the map

$$C(\partial V, \mathbb{R}) \ni \varphi \mapsto H^V_\varphi(x) \in \mathbb{R},$$

for every fixed $x \in V$, whenever V is $\mathcal{L}$-regular. This map defines a positive linear functional on $C(\partial V)$, thus following definition is a consequence.

Definition 2.2.4 (**$\mathcal{L}$-harmonic measure**)**.** The Radon measure μ_z^V supported in ∂V, such that

$$H_\varphi^V(z) = \int_{\partial V} \varphi(\zeta)\,\mathrm{d}\mu_z^V(\zeta), \quad \forall \varphi \in C(\partial V)$$

will be called $\mathcal{L}$-harmonic measure related to V and z.

Definition 2.2.5 (**$\mathcal{L}$-hypoharmonic functions**)**.** Let $u : \Omega \to [-\infty, +\infty[$ be a function. If

1. u is upper semi-continuous (u.s.c.);
2. $u(z) \leq \displaystyle\int_{\partial V} u(\zeta)\,\mathrm{d}\mu_z^V(\zeta), \quad \forall V$ such that $\overline{V} \subset \Omega$, V $\mathcal{L}$-regular;

then we say that the function u is $\mathcal{L}$-hypoharmonic in Ω.
We denote the set of all $\mathcal{L}$-hypoharmonic functions in Ω by $\underline{\underline{\mathcal{S}}}(\Omega)$.

Definition 2.2.6 (**$\mathcal{L}$-subharmonic functions**)**.** Let $u : \Omega \to [-\infty, +\infty[$ be a function. If

1. u is $\mathcal{L}$-hypoharmonic in Ω ($u \in \underline{\underline{\mathcal{S}}}(\Omega)$);
2. u is finite in a dense subset of Ω;

then we say that the function u is $\mathcal{L}$-subharmonic in Ω.
We denote the set of all $\mathcal{L}$-subharmonic functions in Ω by $\underline{\mathcal{S}}(\Omega)$.

Definition 2.2.7 (**$\mathcal{L}$-hyperharmonic** (**$\mathcal{L}$-superharmonic**) **functions**)**.** Let u a function such that $-u \in \underline{\underline{\mathcal{S}}}^{\mathcal{L}}(\Omega)$ $\big(-u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)\big)$. Then we shall call u $\mathcal{L}$-hyperharmonic (u $\mathcal{L}$-superharmonic). We denote the set of all $\mathcal{L}$-hyperharmonic ($\mathcal{L}$-superharmonic) functions in Ω by $\overline{\overline{\mathcal{S}}}(\Omega)$ $\big(\overline{\mathcal{S}}(\Omega)\big)$.

Proposition 2.2.8. *Let $u : \Omega \to [-\infty, +\infty[$ be an u.s.c. function. Then, if $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$, we have $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}u \geq 0$ in the distribution sense.*

This proposition can be proved following the same steps of [NS84] Theorem 1 proof. Because of this proposition we can give definition of $\mathcal{L}$-Riesz measure related to u.

Definition 2.2.9. If $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$, then there exists a Radon measure μ supported in Ω such that $\mathcal{L}u = \mu$. We will call μ the $\mathcal{L}$-Riesz measure related to u.

Remark 2.2.10. Of course we have $\mathcal{H}^{\mathcal{L}}(\Omega) = \underline{\mathcal{S}}^{\mathcal{L}}(\Omega) \cap \overline{\mathcal{S}}^{\mathcal{L}}(\Omega)$.

Remark 2.2.11. In the sense of abstract potential theory (see, e.g. [CC72]), the map

$$\mathbb{R}^{N+1} \ni \Omega \mapsto \mathcal{H}^{\mathcal{L}}(\Omega)$$

is a *harmonic sheaf* and $(\mathbb{R}^{N+1}, \mathcal{H}^{\mathcal{L}})$ is a $\mathfrak{B}$-*harmonic space*. We recall this second statement is due to the following properties:

- the $\mathcal{L}$-regular sets form a basis of Euclidean topology ([Bon69] Corollary 5.2);

- $\mathcal{H}^{\mathcal{L}}$ satisfies *Doob convergence property*, i.e. if the pointwise limit of any increasing sequence of $\mathcal{L}$-harmonic functions on any open set is finite in a dense set then this limit is $\mathcal{L}$-harmonic ([KL00] Proposition 7.4 for proof, [CC72] for theory about);

- for every fixed $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$, the functions

$$z \mapsto -\Gamma(\zeta^{-1} \circ z), \qquad (x,t) \mapsto -\int_0^{\infty} \Gamma(\xi^{-1} \circ x, t)\, \mathrm{d}t$$

  are $\mathcal{L}$-subharmonic in $\mathbb{R}^{N+1}$, and their images (with $\zeta \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^N$ respectively as variables) separate the points of $\mathbb{R}^{N+1}$.

2.3 Representation formulas, $\mathcal{L}$-harmonic and $\mathcal{L}$-subharmonic functions

Definition 2.3.1 (**$\mathcal{L}$-ball**). Let $z \in \mathbb{R}^{N+1}$ and $r > 0$, we define the $\mathcal{L}$-ball of center z and radius r as follows:

$$\Omega_r(z) := \left\{ \zeta \in \mathbb{R}^{N+1} \mid \Gamma(\zeta^{-1} \circ z) > \frac{1}{r^{Q-2}} \right\}.$$

Proposition 2.3.2. *For every $z \in \mathbb{R}^{N+1}$, the $\mathcal{L}$-balls centered in z have the following properties:*

1. *for every $r > 0$, $\Omega_r(z)$ is a bounded non-empty set;*
2. *$\Omega_r(z)$ shrinks to $\{z\}$ when $r \to 0$, or equivalenty $\bigcap_{r>0} \overline{\Omega}_r(z) = \{z\}$;*
3. *we have that*
$$\lim_{r \to 0^+} \frac{|\Omega_r(z)|}{r^{Q-2}} = 0;$$
4. *for almost every $r > 0$, $\partial\Omega_r(z)$ is a N-dimensional C^∞ manifold;*
5. *if z=(x,t), then $\bigcup_{r>0} \Omega_r(z) = \mathbb{R}^N \times]-\infty, t[$.*

This proposition describes geometry induced by the operator through the fundamental solution, that has been used to define the $\mathcal{L}$-ball.

Proposition 2.3.3 (**Representation formulas**). *Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an open set, $u \in C^2(\Omega, \mathbb{R})$. Let Q be the homogeneous dimension of $\mathbb{R}^{N+1}$. Then*

$$u(z) = \mathcal{M}_r(u)(z) - \mathcal{N}_r(\mathcal{L}u)(z), \quad \forall \overline{\Omega}_r(z) \subseteq z \circ \Omega; \tag{2.4}$$

$$u(z) = M_r(u)(z) - N_r(\mathcal{L}u)(z), \quad \forall \overline{\Omega}_r(z) \subseteq z \circ \Omega; \tag{2.5}$$

where

$$\begin{aligned}
\mathcal{M}_r(u)(z) &:= \int_{\partial\Omega_r(0)} \mathcal{K}(\zeta) u(z \circ \zeta)\,\mathrm{d}\sigma(\zeta),\\
M_r(u)(z) &:= \frac{1}{r^{Q-2}} \int_{\Omega_r(z)} K(\zeta^{-1} \circ z) u(\zeta)\,\mathrm{d}\zeta,\\
\mathcal{N}_r(w)(z) &:= \int_{\Omega_r(z)} \big(\Gamma(\zeta^{-1} \circ z) - r^{2-Q}\big) w(\zeta)\,\mathrm{d}\zeta,\\
N_r(w)(z) &:= \frac{Q-2}{r^{Q-2}} \int_0^r \varrho^{Q-3} \mathcal{N}_\varrho(\mathcal{L}u)(z)\,\mathrm{d}\varrho.
\end{aligned}$$

If we use the same notation as (1.7), and $\nabla := (\partial_1, \cdots, \partial_N)$*, we have*

$$\begin{aligned}
\mathcal{K}(\zeta) &:= \frac{|X\Gamma(0,\zeta)|^2}{|\nabla\Gamma(0,\zeta)|},\\
K(\zeta^{-1} \circ z) &:= \frac{|X\Gamma(z,\zeta)|^2}{\Gamma^2(z,\zeta)}.
\end{aligned}$$

Proof can be found in [Cin07].

Proposition 2.3.4 (The kernel K). *Let $z \in \mathbb{R}^{N+1}$ be a fixed point. Let us summarize properties of the kernel K:*

- *K is left invariant with respect to the left translations on $\mathbb{L}$ (unlike $\mathcal{K}$);*
- *K is δ_λ-homogeneous of degree -2;*
- *$K(z,\cdot) \geq 0$ in $\mathbb{R}^{N+1}$;*
- *$K(z,\cdot) \in C^\infty\big(\{(\xi,\tau) \in \mathbb{R}^{N+1} \mid \tau < t\}\big)$;*
- *$\Sigma := \{\zeta = (\xi,\tau) \in \mathbb{R}^{N+1} \mid \tau < t, K(z,\zeta) = 0\}$ has an empty interior.*

Corollary 2.3.5. *If $u \in C^2(\Omega,\mathbb{R})$ then*

$$M_r(u)(z) = \frac{Q-2}{r^{Q-2}} \int_0^r \varrho \mathcal{M}_\varrho(u)(z)\,\mathrm{d}\varrho, \quad \forall \overline{\Omega}_r(z) \subseteq \Omega.$$

Proof. It is enough to multiply both sides of (2.4) by ϱ^{Q-3} and integrate between 0 and r with respect to ϱ to obtain

$$u(z)\frac{r^{Q-2}}{Q-2} = \int_0^r \varrho^{Q-3}\mathcal{M}_\varrho(u)(z)\,\mathrm{d}\varrho - \int_0^r \varrho^{Q-3}\mathcal{N}_\varrho(\mathcal{L}u)(z)\,\mathrm{d}\varrho.$$

Comparing result with (2.5) we reach the purpose.

□

Corollary 2.3.6. *If $u \in \mathcal{H}^{\mathcal{L}}(\Omega)$, then for every $z \in \Omega$ and $r > 0$ such that $\overline{\Omega}_r(z) \subseteq \Omega$ we have*

$$u(z) = \mathcal{M}_r(u)(z) \quad \textit{and} \quad u(z) = M_r(u)(z).$$

Vice versa holds true, if $u \in C(\Omega, \mathbb{R})$.

Theorem 2.3.7 (**Koebe Theorem**). *Let $u \in C(\Omega, \mathbb{R})$ be such that*

$$u(z) = \mathcal{M}_r(u)(z) \quad \textit{or} \quad u(z) = M_r(u)(z), \quad \forall \overline{\Omega}_r(z) \subseteq \Omega.$$

Then

$$u \in \mathcal{H}^{\mathcal{L}}(\Omega).$$

Let us define the Friedrichs' ε-mollifier adapted to our setting, which can be used to prove the above theorem, through building families of $\mathcal{L}$-subharmonic smooth functions that tend to $\mathcal{L}$-subharmonic non-smooth ones. Proof of following lemma is the same of the classic case.

Lemma 2.3.8 (ε**-mollifier**). *Let $J \in C_0^\infty(\mathbb{R}^{N+1})$, $J \geq 0$ be such that* $\operatorname{supp} J \subseteq B(0,1)$ *and* $\int_{\mathbb{R}}^{N+1} J = 1$, $u \in L^1_{\mathrm{loc}}(\Omega)$. *For $\varepsilon > 0$, we define the ε-$\mathcal{L}$-mollified of u in Ω the function*

$$u_\varepsilon \ : \ D_\varepsilon^\Omega \to \mathbb{R} \quad z \mapsto \int_\Omega u(\zeta) J\big(d_{\varepsilon^{-1}}(z \circ \zeta^{-1}\big)\varepsilon^{-Q}\,\mathrm{d}\zeta,$$

where $D_\varepsilon^\Omega := \{\zeta \in \mathbb{R}^{N+1} \mid \overline{B}(\zeta^{-1}, \varepsilon) \subset \Omega^{-1}\}$. We have that

$$u_\varepsilon \in C^\infty(D_\varepsilon^\Omega), \qquad u_\varepsilon \xrightarrow[\varepsilon \to 0]{L^1_{\mathrm{loc}}(\Omega)} u.$$

Lemma 2.3.9. *Let* $u : \Omega \to [-\infty, +\infty[$ *be an u.s.c. function,* $u \in L^1_{\mathrm{loc}}(\Omega)$. *The following statement holds true:*

$$u(z) \leq M_r(u)(z), \quad \forall \overline{\Omega}_r(z) \subseteq \Omega \qquad \Rightarrow \qquad u_\varepsilon(z) \leq M_r(u_\varepsilon)(z), \quad \forall \overline{\Omega}_r(z) \subseteq D^\Omega_\varepsilon.$$

Proof of this lemma can be found in [Cin07], Lemma 3.3.

Corollary 2.3.10. *Let* $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$. *There exists a sequence of smooth* $\mathcal{L}$-*subharmonic functions which tends to* u *in* $L^1_{\mathrm{loc}}(\Omega)$.

2.4 A notion of convexity

Let us introduce a new notion of (scalar) convexity, because the Euclidean one we will recall is obviously unappropriate.
Indeed, let I be an interval of $\mathbb{R}$ and $\varphi : I \to \mathbb{R}$. We say that φ is convex if, for all $s_1, s_2 \in I$,

$$\varphi(s) \leq \frac{s_2 - s}{s_2 - s_1}\varphi(s_1) + \frac{s - s_1}{s_2 - s_1}\varphi(s_2), \quad \forall s \in [s_1, s_2].$$

So this usual notion has to be extended in the following sense:

Definition 2.4.1 (**ψ-convexity**)**.** Let $\psi \in C(I, \mathbb{R})$, be strictly monotone. A function $\varphi \in C(I, \mathbb{R})$ is ψ-convex if, for all $s_1, s_2 \in I$ we have

$$\varphi(s) \leq \frac{\psi(s_2) - \psi(s)}{\psi(s_2) - \psi(s_1)}\varphi(s_1) + \frac{\psi(s) - \psi(s_1)}{\psi(s_2) - \psi(s_1)}\varphi(s_2), \quad \forall s \in [s_1, s_2]. \tag{2.6}$$

We explicitly note that right-hand side of (2.6) is of the form $a\psi(s) + b$, where a e b are constants such that $\varphi(s_j) = a\psi(s_j) + b$, $(j = 1, 2)$. It follows that

$$a = \frac{\varphi(s_1) - \varphi(s_2)}{\psi(s_1) - \psi(s_2)}, \quad b = \frac{\varphi(s_2)\psi(s_1) - \varphi(s_1)\psi(s_2)}{\psi(s_1) - \psi(s_2)}.$$

The meaning of such definition is a natural generalisation of the Euclidean one: the graph of φ between s_1 and s_2 lies below the graph of ψ. To ensure that ψ pass through the same two points we need the contribute of a and b. The classic definition coincindes with this one when ψ is affine.

We need now a lemma that will play a crucial role in the following pages.

Lemma 2.4.2. *Let* $0 < r_1 < r_2,\ 1 < \alpha,\ f \in C^1(]r_1, r_2[, \mathbb{R})$. *If*

$$r^\alpha \left[\frac{\mathrm{d}}{\mathrm{d}r} f(r)\right] \text{ non-decreasing} \tag{2.7}$$

$$\Rightarrow \quad f \text{ is } (r^{1-\alpha})\text{-convex}, \quad r \in]r_1, r_2[. \tag{2.8}$$

Proof. If we define $V_\alpha(r) := r^{1-\alpha}$, then we have

$$\frac{\mathrm{d}V_\alpha}{\mathrm{d}r} = (1-\alpha) r^{-\alpha} \overset{f^{-1}}{\Rightarrow} r^\alpha = (1-\alpha)\frac{\mathrm{d}r}{\mathrm{d}V_\alpha},$$

that plugged into (2.7) gives

$$(1-\alpha)\frac{\mathrm{d}r}{\mathrm{d}V_\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}r} f(r)\right] \overset{\text{chain rule}}{=} (1-\alpha)\frac{\mathrm{d}}{\mathrm{d}V_\alpha} f(r).$$

Whence $\frac{\mathrm{d}}{\mathrm{d}V_\alpha} f(\cdot)$ is non-increasing with respect to r. By consequence $\frac{\mathrm{d}}{\mathrm{d}V_\alpha} f(\cdot)$ is non-decreasing with respect to V_α. Let $V_\alpha(r) \in]V_\alpha(r_2), V_\alpha(r_1)[$. Because of the Lagrange mean-value Theorem there exist ν_1 and ν_2 such that,

$$\frac{d}{dV_\alpha} f(\nu_1) = \frac{f(r_1) - f(r)}{V_\alpha(r_1) - V_\alpha(r)}, \quad \nu_1 \in [V_\alpha(r), V_\alpha(r_1)[; \tag{2.9}$$

$$\frac{d}{dV_\alpha} f(\nu_2) = \frac{f(r) - f(r_2)}{V_\alpha(r) - V_\alpha(r_2)}, \quad \nu_2 \in]V_\alpha(r_2), V_\alpha(r)]. \tag{2.10}$$

We have that $V_\alpha(r_2) < \nu_2 < V_\alpha(r) < \nu_1 < V_\alpha(r_1)$ and that f is non-decreasing with respect to V_α; we obtain that left-hand side of (2.9) is greater or equal to left-hand side of (2.10), so the same holds between right-hand sides:

$$\frac{f(r_1) - f(r)}{V_\alpha(r_1) - V_\alpha(r)} \quad \geq \quad \frac{f(r) - f(r_2)}{V_\alpha(r) - V_\alpha(r_2)},$$

i.e.

$$f(r) \quad \leq \quad \frac{V_\alpha(r_2) - V_\alpha(r)}{V_\alpha(r_2) - V_\alpha(r_1)} f(r_1) + \frac{V_\alpha(r) - V_\alpha(r_1)}{V_\alpha(r_2) - V_\alpha(r_1)} f(r_2).$$

$\square$

2.5 Mean-value operator convexity of $\mathcal{L}$-subharmonic functions

Definition 2.5.1. Let $z \in \Omega \subseteq \mathbb{R}^{N+1}$, $r_1, r_2 \in \mathbb{R}$ such that $r_2 > r_1 > 0$ and $\overline{\Omega}_{r_2}(z) \subset \Omega$. We define $\mathcal{L}$-ring of center z:

$$\begin{aligned}\mathcal{A}(x, r_1, r_2) &:= \{\zeta \in \mathbb{R}^{N+1} \mid r_2^{2-Q} < \Gamma(\zeta^{-1} \circ z) < r_1^{2-Q}\} \\ &= \Omega_{r_2}(z) \setminus \overline{\Omega}_{r_1}(z).\end{aligned}$$

Proposition 2.5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $u \in \underline{\mathcal{S}}(\Omega) \cap C^2(\Omega, \mathbb{R})$. If $z \in \Omega$, and if*

$$R(z) := \sup\{\varrho > 0 \mid \overline{\Omega}_\varrho(z) \subset \Omega\},$$

then

$$\begin{aligned} r &\mapsto M_r(u)(z) \quad \text{and} \\ r &\mapsto N_r(u)(z)\end{aligned}$$

belong to $C^2(]0, R(z)[, \mathbb{R})$. Analogously

$$\begin{aligned} r &\mapsto \mathcal{M}_r(u)(z) \quad \text{and} \\ r &\mapsto \mathcal{N}_r(u)(z)\end{aligned}$$

belong to $C^1(]0, R(z)[, \mathbb{R})$.

Proof. We have

$$\begin{aligned} M_r(u)(z) &= \frac{1}{r^{Q-2}} \int_{\Omega_r(z)} K(\zeta^{-1} \circ z) u(\zeta) \, \mathrm{d}\zeta \\ &\overset{\zeta = z \circ \delta_r(\eta)}{=} \frac{1}{r^{Q-2}} \int_{\Omega_1(0)} K\big(\delta_r(\eta^{-1})\big) u\big(z \circ \delta_r(\eta)\big) r^Q \, \mathrm{d}\eta \\ &\overset{2.3.4}{=} \int_{\Omega_1(0)} K(\eta^{-1}) u\big(z \circ \delta_r(\eta)\big) \, \mathrm{d}\eta. \end{aligned}$$

Thus

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} M_r(u)(z) = \int_{\Omega_1(0)} K(\eta^{-1}) \frac{\partial^2}{\partial r^2} u\big(z \circ \delta_r(\eta)\big) \, \mathrm{d}\eta,$$

and first statement follows from hypotheses and by (2.5). By Corollary 2.3.5

$$r^{Q-2} M_r(u)(z) = (Q-2)\int_0^r \varrho^{Q-3}\mathcal{M}_\varrho(u)(z)\,\mathrm{d}\varrho, \quad \forall r \in]0, R(z)[,$$

and differentiating both sides with respect to r we obtain

$$\begin{aligned} &(Q-2)r^{Q-3}M_r(u)(z) + r^{Q-2}\frac{\mathrm{d}}{\mathrm{d}r}M_r(u)(z) = (Q-2)r^{Q-3}\mathcal{M}_r(u)(z) \\ \Rightarrow \quad &\frac{\mathrm{d}}{\mathrm{d}r}M_r(u)(z) = \frac{Q-2}{r}\big(\mathcal{M}_r(u)(z) - M_r(u)(z)\big). \end{aligned} \tag{2.11}$$

By regularity of $M_r(u)(z)$ it follows that $\mathcal{M}_r(u)(z)$ belongs to $C^1\big(]0, R(z)[, \mathbb{R}\big)$. Finally from (2.4) we get same regularity for $\mathcal{N}_r(u)(z)$.

□

Proposition 2.5.3. *Let $z \in \Omega$, $u \in C^2(\Omega, \mathbb{R})$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(u)(z) = \frac{Q-2}{r^{Q-1}}\int_{\Omega_r(z)} \mathcal{L}u(\zeta)\,\mathrm{d}\zeta,$$

for every r such that $r \in]0, R(z)[$.

Proof. Because of Proposition 2.5.2 we can differentiate (2.4), with respect to r, then for every $r \in]0, R(z)[$ we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(u)(z) - \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{N}_r(\mathcal{L}u)(z).$$

Whence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(u)(z) &= \frac{\mathrm{d}}{\mathrm{d}r}\int_{\Omega_r(z)} (\Gamma(\zeta^{-1}\circ z) - r^{2-Q})\mathcal{L}u(\zeta)\,\mathrm{d}\zeta \\
&\overset{\text{co-area}}{=} \frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \left(\int_{\{\zeta:\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}=\varrho\}} \frac{\big(\Gamma(\zeta^{-1}\circ z) - r^{2-Q}\big)\mathcal{L}u(\zeta)}{|X\big(\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}\big)|}\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}\varrho \\
&\overset{1}{=} \int_0^r \frac{\partial}{\partial r}\left(\int_{\{\zeta:\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}=\varrho\}} \frac{\big(\Gamma(\zeta^{-1}\circ z) - r^{2-Q}\big)\mathcal{L}u(\zeta)}{|X\big(\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}\big)|}\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}\varrho \\
&= \int_0^r \left(\int_{\{\zeta:\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}=\varrho\}} \frac{\frac{\partial}{\partial r}\big(\Gamma(\zeta^{-1}\circ z) - r^{2-Q}\big)\mathcal{L}u(\zeta)}{|X\big(\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}\big)|}\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}\varrho \\
&= -\int_0^r \frac{2-Q}{r^{Q-1}}\left(\int_{\{\zeta:\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}=\varrho\}} \frac{\mathcal{L}u(\zeta)}{|X\big(\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}\big)|}\,\mathrm{d}\sigma(\zeta)\right)\mathrm{d}\varrho \\
&\overset{\text{co-area}}{=} \frac{Q-2}{r^{Q-1}}\int_{\{\zeta:\Gamma(\zeta^{-1}\circ z)^{\frac{1}{2-Q}}>r^{2-Q}\}} (\mathcal{L}u)(\zeta)\,\mathrm{d}\zeta = \frac{Q-2}{r^{Q-1}}\int_{\Omega_r(z)} \mathcal{L}u(\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

$\square$

And now the main result.

Theorem 2.5.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$. We have:*

1. $\forall z \in \Omega,\ 0 < r_1 < r_2\ :\ \overline{\mathcal{A}}(z, r_1, r_2) \subseteq \Omega,$

$$r \mapsto \mathcal{M}_r(u)(x) \text{ is a } r^{2-Q}\text{-convex function on }]r_1, r_2[;$$

2. $\forall z \in \Omega,\ 0 < r_1 < r_2\ :\ \overline{\mathcal{A}}(z, r_1, r_2) \subseteq \Omega,$

$$r \mapsto r^Q M_r(u)(x) \text{ is a } r^{Q-2}\text{-convex function on }]r_1, r_2[.$$

Proof. *1.* At first we will show that the theorem is true for every $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega) \cap C^2(\Omega, \mathbb{R})$, and after through Proposition 2.3.10 we will prove the assertion.
Let $u \in \underline{\mathcal{S}}(\Omega) \cap C^2(\Omega, \mathbb{R})$. $\mathcal{A}(x, r_1, r_2) \subseteq \Omega$, as a consequence u belongs to $\underline{\mathcal{S}}\big(\mathcal{A}(x, r_1, r_2)\big)$. Since $(Q-2) > 0$ and $\mathcal{L}u \geq 0$ in $\mathcal{A}(z, r_1, r_2)$ (Proposition 2.2.8), we get

$$\begin{aligned} 0 &\leq (Q-2)\int_{\mathcal{A}(x,r_1,r_2)} \mathcal{L}u(\zeta)\,\mathrm{d}\zeta \\ &= (Q-2)\left[\int_{\Omega_{r_2}(z)} \mathcal{L}u(\zeta)\,\mathrm{d}\zeta - \int_{\Omega_{r_1}(z)} \mathcal{L}u(\zeta)\,\mathrm{d}\zeta\right] \\ &\overset{2.5.3}{=} r_2^{Q-1}\left[\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(u)(z)\right]_{r=r_2} - r_1^{Q-1}\left[\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(u)(z)\right]_{r=r_1}. \end{aligned}$$

This proves that

$$r \mapsto r^{Q-1}\left[\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{M}_r(u)(x)\right]$$

is non-decreasing in $]r_1, r_2[$. Thus by Lemma 2.4.2 $\mathcal{M}_r(u)(z)$ is a convex function of $r^{1-(Q-1)} = r^{2-Q}$ in such interval.

[1] $\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)dt = v'(x)f(x,v(x)) - u'(x)f(x,u(x)) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x,t)dt.$

Let $u \in \underline{\mathcal{S}}(\Omega)$. By Proposition 2.3.10 there exists $(u_n)_{n\in\mathbb{N}}$ where $u_n \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega) \cap C^\infty(\Omega, \mathbb{R})$ such that $\lim_{n\to+\infty} u_n = u$ in $L^1_{\text{loc}}(\Omega)$. We have

$$\begin{aligned}\lim_{n\to+\infty} \mathcal{M}_r(u_n)(x) &= \lim_{n\to+\infty} \int_{\partial\Omega_r(0)} \mathcal{K}(\zeta) u_n(z \circ \zeta)\, \mathrm{d}\sigma(\zeta) \\ &= \lim_{n\to+\infty} \int_{\partial\Omega_r(0)} \mathcal{K}(\zeta) u(z \circ \zeta)\, \mathrm{d}\sigma(\zeta) \\ &= \mathcal{M}_r(u)(x).\end{aligned}$$

Moreover every u_n is a r^{2-Q}-convex function, then letting this inequality go to infinity with respect to n

$$\begin{aligned}\lim_{n\to+\infty} \mathcal{M}_r(u_n)(x) &\le \lim_{n\to+\infty} \left[\frac{r_2^{2-Q} - r^{2-Q}}{r_2^{2-Q} - r_1^{2-Q}} \mathcal{M}_{r_1}(u_n)(z) \right] \\ &\quad + \lim_{n\to+\infty} \left[\frac{r^{2-Q} - r_1^{2-Q}}{r_2^{2-Q} - r_1^{2-Q}} \mathcal{M}_{r_2}(u_n)(z) \right],\end{aligned}$$

we have the obvious result which completes this step.

2. We shall proceed like *1.*, considering $u \in \underline{\mathcal{S}}(\Omega) \cap C^2(\Omega, \mathbb{R})$, with $r \in]r_1, r_2[$. Beginning from (2.11),

$$\frac{\mathrm{d}}{\mathrm{d}r} M_r(u)(z) = \frac{Q-2}{r} \big(\mathcal{M}_r(u)(z) - M_r(u)(z) \big),$$

multiplying both sides by r^{Q-2} we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r} \Big(r^{Q-2} M_r(u)(z) \Big) = (Q-2) r^{Q-3} \mathcal{M}_r(u)(z),$$

equivalently

$$r^{3-Q} \frac{\mathrm{d}}{\mathrm{d}r} \Big(r^{Q-2} M_r(u)(z) \Big) = (Q-2) \mathcal{M}_r(u)(z). \tag{2.12}$$

Form Proposition 2.5.3

$$\frac{\mathrm{d}}{\mathrm{d}r} \mathcal{M}_r(u)(z) = (Q-2) r^{1-Q} \int_{\Omega_r(z)} \mathcal{L}u(\zeta)\, \mathrm{d}\zeta \overset{\mathcal{L}u \ge 0}{\ge} 0,$$

and so $\mathcal{M}_r(u)(z)$ is a non-decreasing function of r, and the same holds for left-hand side of (2.12). By Lemma 2.4.2 (and noticing that $r^{1-3+Q} = r^{Q-2}$) we get that

$$r^{Q-2} M_r(u)(z) \ \text{ is a } r^{Q-2}\text{-convex function.}$$

With same proceedure as before, we can extend the result to $u \in \underline{\mathcal{S}}(\Omega)$.

□

Chapter 3

Quasi-boundedness and $\mathcal{S}$-regularity for Dirichlet problem

The aim of this chapter is to extend to the sub-Laplacians on stratified Lie groups the following theorem by Ülkü Kuran [Ku79].

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. A point $x \in \partial\Omega$ is regular for the classical Dirichlet problem if and only if the function

$$y \mapsto \Gamma(y - x)$$

is *quasi-bounded* in Ω.

Here Γ denotes the fundamental solution with pole at the origin of the usual Laplace operator Δ. We recall that a point $x \in \partial\Omega$ is said to be regular for the classical Dirichlet problem if denoting by ${}^{\Delta}H^{\Omega}_{\varphi}$ the Perron-Wiener-Brelot solution to

$$\begin{cases} \Delta u = 0 \quad \text{in} \ \ \Omega, \\ u|_{\partial\Omega} = \varphi \end{cases}$$

one has

$$\lim_{\Omega \ni z \to x} {}^{\Delta}H^{\Omega}_{\varphi} = \varphi(x) \quad \forall \varphi \in C(\partial\Omega, \mathbb{R}).$$

We also recall that a non-negative harmonic function $h : \Omega \to \mathbb{R}$ is said to be quasi-bounded if it is the supremum of an increasing sequence of non-negative

bounded harmonic functions in Ω. Kuran explicitly avoided Boulingand regularity criterion in his proof: he used some deep properties of the réduite and the balayage of the fundamental solution.
Here we adopt a different and easier approach, relaying on some properties of the Green functions and the Perron-Wiener-Brelot solutions to the Dirichlet problem [1].
We underline that some notions of this chapter will be the same as previous ones, because of the harmonic space structure.

3.1 Introduction

Keeping in mind notation (2.1), we only consider m smooth vector fields on $\mathbb{R}^N$. Given a stratified Lie group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ we denote by $\mathcal{S}$ its sub-Laplacian, i.e.

$$\mathcal{S} := \sum_{j=1}^{m} X_j^2.$$

So $\mathcal{S}$ is contained in the classes studied in the previous chapters. For our purposes it is crucial to recall the existence ([Fol75], [Gal81]) of a homogeneous norm $|\cdot|$ on $\mathbb{G}$ such that

$$\Gamma(x,y) := |x^{-1} \circ y|^{2-Q}$$

is the *fundamental solution* for $\mathcal{S}$, where Q is the homogeneous dimension of $\mathbb{G}$ ($Q \geq 3$ in our paper). We also know that

- $\Gamma(x,y) = \Gamma(y,x) \quad \forall x,y \in \mathbb{R}^N$;
- $\Gamma(\cdot,\cdot) \in C^\infty(\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \neq y\}, \mathbb{R})$.

For every open set $\Omega \subseteq \mathbb{R}^N$ analogously we define

$$\mathcal{H}(\Omega) := \{u \in C^\infty(\Omega) \mid \mathcal{S}u = 0\}.$$

[1] Brelot in [Br1], had already observed that some steps of Kuran proof could be easier obtained from some results in [Br44].

In [BLU07] it is proved that $(\mathbb{G}, \mathcal{H})$ is a $\mathfrak{S}$-harmonic space in the sense of [CC72].
Every definition about regularity, $\mathcal{S}$-harmonic measure and so on has been inherited from Chapter 2 consequently. It holds that u is *$\mathcal{S}$-harmonic* if it is both $\mathcal{S}$-subharmonic and $\mathcal{S}$-superharmonic, then $\mathcal{H}(\Omega) = \underline{\mathcal{S}}(\Omega) \cap \overline{\mathcal{S}}(\Omega)$ (see e.g. [BL03] Theorem 3.1).
It is worth to recall the definition of Perron-Wiener-Brelot solution to the Dirichlet problem for $\mathcal{S}$ in Ω with boundary $\varphi : \partial\Omega \to [-\infty, +\infty]$.
The sets of *upperfunctions* and *lowerfunctions* of φ in Ω are defined respectively as follows:

$$\overline{\mathcal{U}}_{\varphi}^{\Omega} := \{u \mid u \in \overline{\overline{\mathcal{S}}}(\Omega),\ \liminf_{y\to x} u(y) \geq \varphi(x)\ \forall x \in \partial\Omega,\ \inf_{\Omega} u > -\infty\};$$

$$\underline{\mathcal{U}}_{\varphi}^{\Omega} := \{u \mid u \in \underline{\underline{\mathcal{S}}}(\Omega),\ \limsup_{y\to x} u(y) \leq \varphi(x)\ \forall x \in \partial\Omega,\ \sup_{\Omega} u < +\infty\}.$$

The real extended functions $\overline{H}_{\varphi}^{\Omega} := \inf \overline{\mathcal{U}}_{\varphi}^{\Omega}$ and $\underline{H}_{\varphi}^{\Omega} := \sup \underline{\mathcal{U}}_{\varphi}^{\Omega}$ are called *upper solution* and *lower solution* respectively to

$$\begin{cases} \mathcal{S}u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

If $\underline{H}_{\varphi}^{\Omega} = \overline{H}_{\varphi}^{\Omega}$ and are $\mathcal{S}$-harmonic then φ is called *resolutive*. In this case one denotes $H_{\varphi}^{\Omega} := \underline{H}_{\varphi}^{\Omega} = \overline{H}_{\varphi}^{\Omega}$. We know from Wiener resolutivity Theorem (see e.g. [BLU07] Theorem 6.10.4) that all the continuous functions are resolutive. Moreover, for every x in Ω the map,

$$C(\partial\Omega, \mathbb{R}) \ni \varphi \mapsto H_{\varphi}^{\Omega}(x) \in \mathbb{R}$$

is linear and positive so that there exists a unique *Radon measure* μ_x^{Ω} with support in $\partial\Omega$ such that

$$H_{\varphi}^{\Omega}(x) = \int_{\partial\Omega} \varphi \, \mathrm{d}\mu_x^{\Omega},$$

called *$\mathcal{S}$-harmonic measure* of Ω at x. When Ω is $\mathcal{S}$-regular this definition gives back the previous one.

Definition 3.1.1 (**$\mathcal{S}$-regular point**). A point $x \in \partial\Omega$ is called *$\mathcal{S}$-regular* for Ω if

$$\lim_{\Omega \ni z \to x} H_\psi^\Omega = \psi(x) \quad \forall \psi \in C(\partial\Omega, \mathbb{R}).$$

Definition 3.1.2 (**Quasi-boundedness**). A non-negative $\mathcal{S}$-harmonic function h in Ω is said to be quasi-bounded in Ω if there exists an increasing sequence $(h_n)_{n\in\mathbb{N}}$ of non-negative bounded *$\mathcal{S}$-harmonic* functions in Ω such that

$$h = \lim_{n \to +\infty} h_n \ \text{ in } \Omega.$$

Next section 3.2 contains some preliminary results with some intent in themselves.

3.2 Some preliminary results

To begin with we recall the definition of $\mathcal{S}$-Green function.

Definition 3.2.1. Let Ω be an open subset of $\mathbb{G}$. A $\mathcal{S}$-Green function for Ω is a lower semicontinuous function $G_\Omega : \Omega \times \Omega \to]-\infty, +\infty]$ s.t.

(1). $G_\Omega(x, \cdot) = \Gamma_x(\cdot) + k_x(\cdot) \quad \forall x \in \Omega, \quad k_x \in \mathcal{H}(\Omega)$;

(2). $G_\Omega(x, y) \geq 0 \quad \forall x, y \in \Omega$;

(3). If $x \in \Omega,\ v_x \in \overline{\mathcal{S}}^+(\Omega),\ v_x(y) = \Gamma_x(y) + l_x(y),\ l_x \in \overline{\mathcal{S}}(\Omega),$ then $v_x(y) \geq G_\Omega(x, y) \quad \forall y \in \Omega.$

Remark 3.2.2. Properties (1) and (3) imply the uniqueness of G_Ω. Moreover

$$G_\Omega(x, y) = \Gamma_x(y) - H_{\Gamma_x}^\Omega(y), \quad G_\Omega(x, y) = G_\Omega(y, x),$$

and $G_\Omega > 0$ if and only if x and y belong to the same connected component of Ω. See Chapter 9.2 of [BLU07].

Theorem 3.2.3. *Let G_Ω as above and $x \in \partial\Omega$. Then,*

$$x \text{ is } \mathcal{S}\text{-regular} \quad \Leftrightarrow \quad \lim_{\Omega \ni z \to x} G_\Omega(y, z) = 0 \quad \forall y \in \Omega.$$

Proof. Assume $\lim_{\Omega \ni z \to x} G_\Omega(y,z) = 0$ for every y in Ω. Let O be a connected component of Ω whose boundary contains x, and choose a point $y \in O$. Then

$$O \ni z \mapsto G_\Omega(y,z)$$

is a $\mathcal{S}$-barrier function for O at x. Thus, by Boulingand's Theorem (see e.g. [BLU07] Theorem 6.12.2) x is $\mathcal{S}$-regular for O. Since O is an arbitrary connected component of Ω with $\partial\Omega \ni x$, this proves the $\mathcal{S}$-regularity of x for Ω.

Vice versa, if x is $\mathcal{S}$-regular, for every fixed $y \in \Omega$ we have that

$$\lim_{\Omega \ni z \to x} G_\Omega(y,z) = \Gamma_y(x) - \lim_{\Omega \ni z \to x} H^\Omega_{\Gamma_y}(z) = 0,$$

because Γ_y is continuous on $\partial\Omega$.

□

Theorem 3.2.4. *Let $\Omega \in \mathbb{G}$ be a bounded open set and let $h \in \mathcal{H}(\Omega)$, $h \geq 0$. Suppose h*

$$\widetilde{h} : \partial\Omega \to [0,\infty], \quad \widetilde{h}(x) = \lim_{z \to x} h(z)$$

is well defined. Then

$$h \textit{ is quasi-bounded in } \Omega \quad \Leftrightarrow \quad \widetilde{h} \textit{ is resolutive and } \ h \equiv H^\Omega_{\widetilde{h}}.$$

Proof.

($\Rightarrow$) If h is quasi-bounded in Ω there exists an increasing sequence (h_n) of non-negative and bounded $\mathcal{S}$-harmonic functions such that $\lim_{n\to\infty} h_n = h$. Every h_n belongs to $\underline{\mathcal{U}}^\Omega_{\widetilde{h}}$, and h belongs to $\overline{\mathcal{U}}^\Omega_{\widetilde{h}}$. By consequence

$$h = \lim_{n \to +\infty} h_n \leq \underline{H}^\Omega_{\widetilde{h}} \leq \overline{H}^\Omega_{\widetilde{h}} \leq h$$

in Ω.

($\Leftarrow$) The function $h_n : \partial\Omega \to \mathbb{R}$, $\ h_n = \min\{\widetilde{h}(z), n\}$ is resolutive. Then, for every z in Ω, we have

$$H^\Omega_{h_n}(z) = \int_{\partial\Omega} h_n(y) d\mu^\Omega_z(y) \overset{\textit{Beppo-Levi}}{\nearrow} \int_{\partial\Omega} \widetilde{h}(y) d\mu^\Omega_z(y) = H^\Omega_{\widetilde{h}}(z) = h(z).$$

The family $(H_{h_n})_{n\in\mathbb{N}}$ is $\mathcal{S}$-harmonic and increasing in Ω since $(h_n)_{n\in\mathbb{N}}$ is increasing on $\partial\Omega$. Moreover, $H^{\Omega}_{h_n} \leq n$ in Ω since the constant functions are $\mathcal{S}$-harmonic in Ω and $h_n \leq n$ on $\partial\Omega$. This proves that h is quasi-bounded and completes the proof.

□

3.3 Quasi-boundedness and $\mathcal{S}$-regularity

We are ready now to prove our main result, that links the quasi-boundedness of fundamental solution for $\mathcal{S}$ and the $\mathcal{S}$-regularity of the boundary points.

Theorem 3.3.1. *Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a stratified Lie group, $\mathcal{S}$ its sub-laplacian and Γ_x the fundamental solution of $\mathcal{S}$ with pole x. If $\Omega \subseteq \mathbb{G}$ is a bounded open set and $x \in \partial\Omega$ then*

$$x \text{ is } \mathcal{S}\text{-regular} \quad \Leftrightarrow \quad \Gamma_x \text{ is quasi-bounded on } \Omega.$$

Proof. ($\Leftarrow$) From Theorem 3.2.4 we have that

$$\Gamma_x(y) = H^{\Omega}_{\Gamma_x}(y) \qquad \forall y \in \Omega.$$

Let $G_\Omega(\cdot,\cdot)$ be $\mathcal{S}$-Green function of Ω. Let $y \in \Omega$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Ω such that $x_n \to x$ as n goes to infinity. We have

$$\begin{aligned} 0 &\leq \liminf_{n\to+\infty} G_\Omega(y,x_n) \leq \limsup_{n\to+\infty} G_\Omega(y,x_n) \leq \\ &\leq \limsup_{n\to+\infty} \Gamma_{x_n}(y) - \liminf_{n\to+\infty} H^{\Omega}_{\Gamma_{x_n}}(y). \end{aligned}$$

We have

$$\limsup_{n\to+\infty} \Gamma_{x_n}(y) = \Gamma_x(y)$$

since $x \to \Gamma_x(y)$ is continuous, being $y \in \Omega$. Moreover

$$\begin{aligned} \liminf_{n\to+\infty} H^{\Omega}_{\Gamma_{x_n}}(y) &= \liminf_{n\to+\infty} \int_{\partial\Omega} \Gamma_{x_n}(z) d\mu^{\Omega}_y(z) \geq \\ &\geq \int_{\partial\Omega} \liminf_{n\to+\infty} \Gamma_{x_n}(z) d\mu^{\Omega}_y(z) = \\ &= \int_{\partial\Omega} \Gamma_x(z) d\mu^{\Omega}_y(z) = H^{\Omega}_{\Gamma_x}(y) = \Gamma_x(y). \end{aligned}$$

Then $\lim_{n\to\infty} G(y, x_n) = 0$. By Theorem 3.2.3 we get $\mathcal{S}$-regularity of x.
($\Rightarrow$). Suppose that Ω is connected, and let $y \in \Omega$ and $z \in \mathbb{R}^N$. Define

$$g(y, z) \quad := \quad \Gamma_z(y) - \int_{\partial\Omega} \Gamma_z(t) d\mu_y^\Omega(t).$$

If $z \in \mathbb{R}^N \setminus \overline{\Omega}$ then Γ_z is $\mathcal{S}$-harmonic in a neighborhood of $\overline{\Omega}$. As a consequence

$$\int_{\partial\Omega} \Gamma_z(t) d\mu_y^\Omega(t) = \Gamma_z(y),$$

and $g(y, z) = 0$.
If $z \in \Omega$ then $g(y, z) = G(z, y) = G(y, z) \geq 0$. At any $\mathcal{S}$-regular boundary point t by Theorem 3.2.3 we have

$$\lim_{\Omega \ni z \to t} g(y, z) = \lim_{\Omega \ni z \to t} G(y, z) = 0,$$

and hence

$$\lim_{\Omega \not\ni z \to t} g(y, z) = 0.$$

Now, let $\partial\Omega_1$ be the set of the $\mathcal{S}$-irregular boundary points of Ω. We know that $\partial\Omega_1$ is an $\mathcal{S}$-polar set ([BC05] Theorem 3.1). Then there exists an $\mathcal{S}$-subharmonic function p on $\mathbb{R}^N$ such that

$$\begin{aligned} p(z) &= -\infty \quad \text{if} \ \ z \in \partial\Omega_1, \\ p(z) &> -\infty \quad \text{otherwise.} \end{aligned}$$

It is not restrictive to assume p strictly negative on $\partial\Omega$. For every $\varepsilon > 0$ define

$$g_\varepsilon(z) := g(y, z) + \varepsilon p(z), \quad z \in \mathbb{R}^N.$$

Then, since $g(y, z) \leq \Gamma_z(y) = \Gamma_y(z)$ and Γ_y is smooth of $\{z\}$, we have

$$\limsup_{\partial\Omega \not\ni z \to t} g_\varepsilon(z) < 0 \quad \forall t \in \partial\Omega_1.$$

This inequality extends to all $\partial\Omega$ thanks to (3.1).
Thus for every $t \in \partial\Omega$ there exists a connected neighborhood B_t of t such that

$$g_\varepsilon(z) < 0 \quad \forall z \in B_t \setminus \partial\Omega.$$

Then, since $\partial\Omega$ is compact, there exists an open set $B \supseteq \partial\Omega$ such that

$$g_\varepsilon(z) < 0 \quad \forall z \in B \setminus \partial\Omega.$$

Since g_ε is $\mathcal{S}$-subharmonic in B the Strong Maximum Principle for $\mathcal{S}$-subharmonic functions ([BL03] Theorem 3.2) implies that we also have

$$g_\varepsilon(z) < 0 \quad \forall z \in \partial\Omega,$$

for otherwise g_ε would attain a non-negative maximum at an interior point of B and, as a consequence, it would be constant and non-negative in some B_t. In particular

$$g_\varepsilon(x) < 0, \quad \text{i.e.} \quad g(y,x) < -\varepsilon p(x).$$

Since ε is arbitrary we deduce that $g(y,x) \le 0$.
On the other hand $g(y,x) \ge 0$ since

$$\begin{aligned} 0 &= \limsup_{\Omega \ni z \to x} g(y,z) = \limsup_{\Omega \ni z \to x} \Big(\Gamma_z(y) - \int_{\partial\Omega} \Gamma_z(t)\, d\mu_y^\Omega(t) \Big) \le \\ &\le \Gamma_x(y) - \liminf_{\Omega \ni z \to x} \int_{\partial\Omega} \Gamma_z(t)\, d\mu_y^\Omega(t) \le \\ &\le \Gamma_x(y) - \int_{\partial\Omega} \Gamma_x(t)\, d\mu_y^\Omega(t) = g(y,x). \end{aligned}$$

Thus, we have proved that

$$0 = g(y,x) = \Gamma_x(y) - \int_{\partial\Omega} \Gamma_x(t)\, d\mu_y^\Omega(t), \quad \forall y \in \Omega.$$

Then $\partial\Omega \ni t \mapsto \Gamma_x(t) \in \,]0,\infty]$ is resolutive and

$$H^\Omega_{\Gamma_x}(y) = \Gamma_x(y) \quad \forall y \in \partial\Omega.$$

Since $\Gamma_x(t) = \lim_{\Omega \ni z \to t} \Gamma_x(z)$ and $z \mapsto \Gamma_x(z)$ is $\mathcal{S}$-harmonic and non-negative in Ω, by Theorem 3.2.4 this implies that Γ_x is quasi-bounded in Ω. The proof is complete.

$\square$

3.4 Some applications

In this section we will show some applications of Theorem 3.3.1 to Carnot Groups.

Lemma 3.4.1. *Let $\mathbb{G} = (\mathbb{R}^{N-1} \times \mathbb{R}, \circ)$ be a Carnot group, Q and $d_{\mathbb{G}}$ be its homogeneous dimension and its homogeneous norm respectively. Let $O = \{(z,\tau) \in \mathbb{G} \mid \tau < 0\}$. If for every $(z,\tau) \in O$*

$$f :]0, +\infty[\mapsto \mathbb{R},\ t \mapsto d_{\mathbb{G}}((0,t)^{-1} \circ (z,\tau))$$

is monotone increasing then $(0,0)$ is $\mathcal{S}$-regular for every Ω bounded open subset of O such that $(0,0) \in \partial\Omega$.

Proof. Notice that

$$\Gamma_{(0,\frac{1}{n})}(\cdot,*) = d_{\mathbb{G}}^{2-Q}((0,\frac{1}{n})^{-1} \circ (\cdot,*))$$

belongs to $\mathcal{H}(\Omega)$, moreover it is positive and bounded in Ω. It follows from monotonicity of f that $(\Gamma_{(0,\frac{1}{n})})_{n\in\mathbb{N}}$ is an increasing sequence of non-negative bounded harmonic functions in Ω which tends to $\Gamma_{(0,0)}$.
Hence $\Gamma_{(0,0)}$ in quasi-bounded in Ω and by Theorem 3.3.1 the result follows.

□

Example 3.1.

Theorem 3.4.2. *Let $\mathbb{H}^N = (\mathbb{R}^{2N} \times \mathbb{R}, \circ)$ be the Heisenberg group. Denoting by (z,τ) a point of $\mathbb{H}^N$, $z \in \mathbb{R}^{2N}$, $\tau \in \mathbb{R}$, we know that for some $c > 0$ $d_{\mathbb{H}^N}(z,\tau) := c\sqrt[4]{|z|^4 + \tau^2}$ is a homogeneous norm on $\mathbb{H}^N$ that satisfies (3.1). The function*

$$f_{\mathbb{H}^N} :]0, +\infty[\mapsto \mathbb{R},\ t \mapsto d_{\mathbb{H}^N}\big((0,t)^{-1} \circ (z,\tau)\big)$$

is monotone strictly increasing for every fixed $(z,\tau) \in \mathbb{R}^N \times] - \infty, 0[$.

Proof. By hypotheses

$$\begin{aligned} f^4_{\mathbb{H}^N}(t) &= d^4_{\mathbb{H}^N}\big((z,\tau)^{-1}\circ(0,t)\big) = \\ &= d^4_{\mathbb{H}^N}\big((-z,-\tau)\circ(0,t)\big) = \\ &= d^4_{\mathbb{H}^N}\big((-z,t-\tau)\big) = c^4\big(|z|^4+(t-\tau)^2\big) \end{aligned}$$

and the assertion follows.

□

As consequence the point $x := (0,0)$ is $\Delta_{\mathbb{H}^N}$-regular for the ball with radius r in $\mathbb{H}^N$ and center $(0,-\frac{r^2}{c^2})$, i.e. for

$$D_r = \{(z,\tau)\in\mathbb{H}^N \ : \ c^4(|z|^4+(\tau+\frac{r^2}{c^2})^2) < r^4\}.$$

Proof. The set $O := \{(z,\tau)\in\mathbb{H}^N \mid \tau < 0\}$ and $f := f_{\mathbb{H}^N}$ satisfy Lemma 3.4.1 so we get immediately the assertion.
Doing an opportune translation this proves also that the point $(0,\frac{r^2}{c^2})$ is regular for the ball in $\mathbb{H}^N$ with radius r and center at the origin.

□

Example 3.2.

Another example can be done with Carnot group of step two in [BT02], for which Balogh and Tyson have explicitly found a homogeneous norm.

Theorem 3.4.3. *Let* $\mathbb{G}_1 := (\mathbb{R}^4\times\mathbb{R},\circ)$*. For every* $x = (x_1,x_2,x_3,x_4,t)$*,* $y = (y_1,y_2,y_3,y_4,\tau)\in\mathbb{G}_1$ *let the composition and the dilatation be defined as follow:*

- $x\circ y := \big(x_1+y_1,\dots,x_4+y_4,t+\tau+\frac{1}{2}(x_2y_1-x_1y_2+2x_4y_3-2x_3y_4)\big)$,
- $\delta_\lambda(x) := (\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4,\lambda^2 t)$.

$\mathbb{G}_1$ *is a homogenous Carnot group of step two with homogeneous dimension* $Q=6$*. If* $a := (\frac{1}{2}x_1^2+\frac{1}{2}x_1^2)$ *and* $b := (\frac{1}{2}x_1^2+\frac{1}{2}x_1^2+x_3^2+x_4^2)$ *then a homogeneous* *norm on* $\mathbb{G}_1$ *is*

$$d_{\mathbb{G}_1}(x) = c\sqrt[8]{\frac{(b^2+t^2)(a^2+\sqrt{b^2+t^2})^3}{b+\sqrt{b^2+t^2}}},$$

such that satisfies (3.1) for some $c > 0$. *Moreover the function*

$$f_{\mathbb{G}_1} :]0,+\infty[\ \mapsto \mathbb{R}, \quad t \mapsto d_{\mathbb{G}_1}\big((y_1,y_2,y_3,y_4,\tau)^{-1} \circ (0,0,0,0,t)\big)$$

is monotone strictly increasing for every fixed $(y_1,y_2,y_3,y_4,\tau) \in \mathbb{R}^4 \times]-\infty,0[$.

Proof. Computing the derivative of $f^8_{\mathbb{G}_1}$ it's easy to show that it's always strictly positive for every $t > 0$.

□

Let $\mathbb{G}_1$ be the Carnot group and $d_{\mathbb{G}_1}$ its homogeneous norm in Theorem 3.4.3. So the point $x := (0,0,0,0,\frac{r^2}{c^2})$ is $\Delta_{\mathbb{G}_1}$-regular for the ball with radius r in $\mathbb{G}_1$, i.e. for

$$D_r = \{x \in \mathbb{G}_1 \ : \ d_{\mathbb{G}_1}(x) < r\}.$$

Proof. Traslating D_r of $(0,0,0,0,-\frac{r^2}{c^2})$ we can apply Lemma 3.4.1 with $O := \{(z_1,z_2,z_3,z_4,\tau) \in \mathbb{G}_1 \mid \tau < 0\}$ and $f := f_{\mathbb{G}_1}$.

□

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