Alma Mater Studiorum
Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA
Ciclo XXXII

Settore Concorsuale: 01/A2
Settore Scientifico Disciplinare: MAT 03

Automorphisms of O'Grady's sixfolds

Presentata da: Annalisa Grossi

Coordinatore Dottorato:
Chiar.mo Prof.
Fabrizio Caselli

Supervisore:
Chiar.mo Prof.
Giovanni Mongardi

Esame finale anno 2020

Abstract

We study automorphisms of irreducible holomorphic symplectic (IHS) manifolds deformation equivalent to the O'Grady's sixfold (briefly of OG_6 type). In recent years the geometry of O'Grady's manifolds has been studied in order to increase the knowledge of them, but nothing was known, before this thesis, about automorphisms of these deformation types.

In the first part we introduce the concept of induced automorphisms. O'Grady introduced this IHS manifold in dimension six as the resolution of singularities of the Albanese fiber of a moduli space of sheaves on an abelian surface. We show that if the Picard lattice contains the class of the exceptional divisor of the O'Grady's desingularization, the O'Grady's six type manifold has a geometric realization as a moduli space of sheaves on an abelian surface. Moreover we show that an automorphism is induced by the abelian surface, if some lattice theoretic conditions are satisfied. In particular, we need that two copies of the hyperbolic lattice are contained in the co-invariant sublattice with respect to the induced action on the second integral cohomology lattice. We investigate also another notion of induced, which is the notion of automorphisms induced at the quotient. There exists a birational model for O'Grady's six type manifolds realized as the quotient of a $K3^{[3]}$ type manifold by a birational symplectic involution. Hence, we find a criterion to say if an automorphism of prime order of the O'Grady's sixfold lifts to an automorphism of the $K3^{[3]}$ type manifold. We also provide an example of induced automorphism which is induced at the quotient but not induced.

In the second part we classify non-symplectic automorphisms of prime order of OG_6 type manifold. In particular we study automorphisms which induced a non-trivial action on the second integral cohomology, which has a lattice structure, as for the other IHS manifolds. We classify, up to isometries, invariant and co-invariant sublattices with respect to the induced action on the second integral lattice, and we produce a list of non-symplectic automorphisms of prime order of OG_6 type manifold. We treat also the symplectic case and we use a similar approach as the one used in the non-symplectic case. However, we do not find symplectic automorphisms, but just symplectic birational morphisms for reasons related to the presence of some classes of divisors, the wall divisors, in the co-invariant sublattices.

Contents

List of Figures

List of Tables

Introduction

This thesis deals with the study of automorphisms of a deformation class of irreducible holomorphic symplectic manifolds (briefly IHS), which was discovered by O'Grady [76] in 2000. These manifolds are sixfolds with second Betti number 8 and are known in literature as manifolds of OG_6 type. There are three types of "building blocks" in the Beauville-Bogomolov decomposition [10, Theorem 2] of Ricci flat compact Kähler manifolds, namely complex tori, Calabi-Yau varieties, and irreducible holomorphic symplectic manifolds. By Yau's proof of Calabi's conjecture, having a Ricci flat metric is equivalent to having trivial first Chern class. O'Grady's sixfolds are a deformation class in the last type of the Beauville-Bogomolov decomposition, i.e. they are simply-connected compact Kähler manifolds carrying a holomorphic symplectic form which spans $H^{2,0}$. Irreducible holomorphic symplectic manifolds have a double nature, in fact the holonomy of a Ricci flat Kähler metric is equal to $Sp(r)$, hence they are are hyperkähler manifolds (see [10]). For a while it was thought that K3 surfaces are the unique example of irreducible holomorphic symplectic manifolds. We needed to wait for Fujiki [36] and Beauville [10] to find examples in higher dimensions. However, the main issue about irreducible holomorphic symplectic manifolds is the scarcity of the stock of examples, especially if we think of the many examples of CalabiYau's. For quite some time, every known irreducible symplectic manifold is a deformation of one of the following varieties: the Hilbert scheme parametrizing zero-dimensional subschemes of a K3 of fixed length [10], and the generalized Kummer variety parametrizing zero-dimensional subschemes of a complex torus of fixed length and whose associated 0-cycle sums up to 0 [10]. The keystone in finding new examples is due to the discovery made by Mukai [66] of a symplectic form on moduli spaces of certain sheaves on symplectic surfaces. This fact led to the hope that new irreducible holomorphic symplectic manifolds could be found with these constructions and a good theory was developed by various mathematicians; for more detailed references about this, the reader can refer to [47]. However it has been proved that all non-singular IHS manifolds obtained in this way were a deformation of known examples and the singular ones had a resolution of singularities which is IHS only in two cases, namely in O'Grady's six dimensional manifold [76] and in O'Grady's ten dimensional manifold [77]. Briefly: all known examples are deformations of a moduli space of semistable sheaves on a surface with trivial canonical bundle or, as in the last two cases, of a symplectic desingularization of such a moduli space. This new class of IHS sixfolds are usually called OG_6 type manifolds. They are not equivalent by deformation to $K3^{[3]}$ type manifolds neither to $K_3(A)$ type manifolds, have second Betti number 8 and are the main subject of our work. Manifolds in this class are obtained in two ways. The first construction due to O'Grady is obtained by taking a generic abelian surface and a Mukai vector w of square 2. As we will see in Section 1.3.1, the moduli space of Gieseker semistable sheaves with Mukai vector $2w$ is a singular tenfold with rational singularities, whose Albanese fiber admits a crepant resolution that is an IHS manifold in the family we are dealing with. O'Grady does this construction with a specific Mukai vector, $w = (1, 0, -1)$, but later the contribution given by many authors, first by M. Lehn and Sorger and then by Perego and Rapagnetta, allows to conclude that under the above assumption on w, the blow up of the Albanese fiber of the moduli space along its singular locus always gives a crepant resolution and these crepant resolutions are deformation equivalent, along smooth projective deformations, to the original O'Grady's example (see [87] and [80] for many details).

For what concern manifolds of OG_6 type, interesting progress has been made recently by Mongardi and Rapagnetta in [61]. In particular they showed that Classical Bimeromorphic global Torelli theorem holds for OG_6 type manifolds, which means that a necessary and sufficient condition to have a bimeromorphism between two manifolds X and Y of OG_6 type is the isometry of the integral Hodge structures $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$. We remark that the Classical Bimeromorphic global Torelli theorem rather rarely happens to hold for deformation equivalence classes of known IHS manifolds: it only holds (among known IHS manifolds) for K3 surfaces and their Hilbert schemes of n points if $n - 1$ is a prime power. The theorem fails for Hilbert schemes on K3 surfaces if n is not a prime power ([52, Section 9]) and fails for O'Grady's ten dimensional manifolds (see the counter example contained in [59]). Moreover this result fails for generalized Kummer manifolds (refer to [69] to see the counter example), as replacing the abelian surface, used to construct the generalized Kummer manifold with its dual, does not change the second Hodge structure, but the two Kummer manifolds are not birational. Due to this counterexample and due to the role of an abelian surface in the construction of O'Grady's six dimensional manifolds, one could expect a similar failure of the global Torelli theorem for O'Grady's sixfolds. However, this is not the case, and an intuitive motivation of this, depends on the relation of OG_6 type manifolds with $A \times A^\vee$, where A is the abelian surface involved in the construction of the moduli space, and with the Kummer K3 surface $A/\pm 1$ (see [62]).

When we have a new class of manifolds, one of the main aim is to know its invariants, for this reason the study of automorphisms of irreducible holomorphic symplectic manifolds has become a very active research field in the last years. The global Torelli theorem for K3 surfaces, due to Šapiro and Šafarevič, allows to reconstruct automorphisms of a K3 surface S starting from Hodge isometries of $H^2(S, \mathbb{Z})$, which preserve the Kähler cone. This result, together with foundational papers of Nikulin, namely [71] and [72], provided the instruments to investigate finite groups of automorphisms on K3's. The interesting point is that Huybrechts, Markman and Verbitsky (see [45], [52] and [96]) formulated similar results of Torelli type for IHS manifolds, and unexpectedly, also in these examples of higher dimension, the most important role to study automorphisms is assumed by the second integral cohomology. To be more precise automorphisms of IHS manifolds can be classified studying the action of an automorphism on the second cohomology group with integer coefficients (which carries a lattice structure, provided by the Beauville-Bogomolov-Fujiki quadratic form).

The following is an overview of what we know about automorphisms of IHS manifolds: many things we know about automorphisms of $K3^{[2]}$ type manifolds. The symplectic case (i.e. automorphisms which preserve the symplectic form of the manifold) is covered in [23] and [56]; moreover, the study of non-symplectic automorphisms was started by Beauville [9] and was developed by many other authors. We find these contributes in [16] and [13]. Another relevant paper about automorphisms of $K3^{[2]}$ type manifolds is [25], and in [17] we find some interesting remarks about the fixed locus of these automorphisms. Finally another remarkable work about automorphisms of $K3^{[n]}$ type manifolds, where $n \geq 3$, is the joint work of Camere and Alberto Cattaneo that we find in [24]. Moreover we can find an analysis of automorphisms of generalized Kummer manifolds in [94] and in [**?**].

Far less is known about O'Grady's manifolds: automorphisms of OG_6 type manifolds are precisely the subject on which we focus in the thesis, whose core can be divided into two parts, dealing with distinct aspects: on one hand we have the intent to classify these automorphisms, on the other hand we define the concept of induced automorphisms for OG_6 manifolds and we find a criterion to characterize them.

In **Chapter 1** we provide the reader with an overview of the basic results about irreducible holomorphic symplectic manifolds and we introduce the main tools to approach the study of automorphisms, i.e. we give basic notions of lattice theory and we recall the properties of the second cohomology for an IHS manifold. Moreover we recall the construction of the O'Grady's sixfolds. Manifolds in this family are obtained in two ways. The first one, given in Section 1.3.1, is due to O'Grady and it is obtained as a symplectic resolution of the Albanese fiber of a moduli space of sheaves on an abelian surface. The second construction, described in Section 1.4.1, was obtained in [62], by considering a principally polarized abelian surface A and its Kummer K3 surface S. On a moduli space of sheaves on S, the authors construct a non regular involution, whose quotient is birational to a manifold of OG6 type. These two models are useful in the analysis of induced automorphisms.

We know that K3 surfaces are a toy model for IHS manifolds, and for this reason in **Chapter 2** we present a classification of non-symplectic automorphisms of order 8 on K3 surfaces, under some assumptions on the fixed locus. This study allows us to become familiar with classification techniques of automorphisms which will be useful in the case of OG_6 manifolds. Given the low dimension of the K3 manifolds, we obtain a classification just by studying the fixed locus of the automorphisms, as we can find in Theorem 2.1.12.

The aim of **Chapter 3** is to define the concept of induced automorphisms. As we have seen in the first chapter, there exist two model for OG_6 type manifolds and for this reason we distinguish between induced automorphisms, and automorphisms induced at the quotient. The easiest example of IHS manifolds which arises from a symplectic surface, is the Hilbert scheme of n points on a K3 surface, constructed by Beauville in [10]. This kind of construction allows us to produce several examples of automorphisms on irreducible symplectic manifolds, simply by taking a K3 surface with non-trivial automorphism group and considering the induced action on its Hilbert scheme. These kinds of automorphisms are called natural in literature, and were studied by Beauville [10], Boissière [12] and many others. A generalization of the notion of natural automorphisms for moduli spaces is provided in [64].

This notion appeared for the first time for moduli spaces of sheaves in [79], a work inspired by the construction in [78, Section 5]. In [64] the authors extend the ideas drastically using developments in the theory of stability conditions by Bridgeland [22], and by Bayer-Macrì (see [6][7]) and Yoshioka (see [100]). Inspired by these recent works we adapt this notion for OG_6 type manifolds. In reference to the construction of O'Grady, we introduce the notion of induced automorphisms in order to state a criterion to determine whether a given automorphism on a manifold of OG_6 type is, in fact, induced by an automorphism of the Abelian surface that we use to define the moduli space. First of all, in Proposition 3.1.4 we find a sufficient condition to know when a manifold of OG_6 type is the symplectic resolution of the Albanese fiber of a moduli space on an abelian surface, then we provide in Theorem 3.2.6 a numerical criterion for the recognition of induced automorphisms. This criterion is applied in Chapter 4, in the classification of non-symplectic automorphisms of OG_6 type manifolds, and we will see that this criterion depends on the lattice structure of the Neron–Severi group of the manifold, and of invariant and co-invariant sublattices. Moreover this criterion, to be applied in the symplectic case i.e. in Chapter 5, will have to be adapted since we will have to consider symplectic birational automorphisms. We provide a numerical criterion for the recognition of induced automorphisms.

Then, in Section 3.3 we refer to the other model of OG_6, the birational one. In Theorem 3.3.3 we prove that, if there exists a class $E \in \mathrm{NS}(X)$ such that $E^2 = -2$ and $div(E) = 2$, then there exists a K3 surfaces S such that X is birational to the quotient of $S^{[3]}$ by a birational symplectic involution i. After some auxiliary Lemma, in Theorem 3.3.15, we find a criterion to say when an automorphism of the OG_6 type manifold lifts to an automorphism of the the Hilbert scheme, $S^{[3]}$, on S.

In **Chapter 4** we have in aim to classify non-symplectic automorphisms of prime order on OG_6 type manifolds. We give classification results about families of deformations of the pair (X, G) where X is a manifold of OG_6 type and $G \subset \mathrm{Aut}(X)$ is a group of prime order and non-symplectic automorphisms of X. As we have already said above, the second integral cohomology group $H^2(X, \mathbb{Z})$ of an IHS manifold X, has a lattice structure, and it is an important tool to approach the study of automorphisms. Moreover this lattice encodes most of the geometry of the manifold, for more details we can refer to [43]. The strategy employed to classify automorphisms of O'Grady sixfolds is related to the following representation map,

$$\nu : \mathrm{Aut}(X) \to O(H^2(X, \mathbb{Z})).$$

From [65] we know that $Ker(\nu) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$ and this is a deformation invariant. One of the main questions about automorphisms is about the image of ν (see [43, chapter 9]). In Proposition 4.0.9 we give an answer in this direction in the non-symplectic and prime order case. We call *effective* an isometry which is the image of ν. The idea to classify automorphisms from the second integral cohomology, looks like the one used for manifolds of Kummer type (for $n = 2$ see [63]). In fact, also in this case, the lattice of the second integral cohomology is not unimodular as we can find in [81]. The strategy is to compute effective isometries to obtain a classification of automorphisms that act non-trivially on the second cohomology. The classification is resumed in Theorem 4.2.13.

The treatment of the symplectic case is more complicated and **Chapter 5** is devoted to this. We are able to give results about the image of the following representation map,

$$\nu : \mathrm{Bir}(X) \to O(H^2(X,\mathbb{Z})),$$

where $\mathrm{Bir}(X)$ is the group of birational automorphisms. An element in the image is called *birational effective*. To classify these birational maps we need to consider wall divisors and prime exceptional divisors which are, roughly speaking, the walls of the Kähler cone, and the walls of the birational Kähler cone, respectively (see [52] for more details). If the co-invariant sublattice $S_G(X)$ of $H^2(X,\mathbb{Z})$ with respect to a symplectic action contains wall divisors, then the isometry is not effective, if it contains prime exceptional divisors, it is not birational effective. From [61] we know that if X is of OG_6 type, wall divisors are classes of square -2 and divisibility 1; on the other hand, prime exceptional divisors are classes of square -2 and divisibility 2, and classes of square -4 and divisibility 2. We study the case in which the induced action on the discriminant group $A_X \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ is trivial, and we obtain in Theorem 5.1.6 a complete classification of birational effective isometries of the OG_6 lattice. We know from Theorem 5.0.8 that wall divisors are in the co-invariant lattice of an OG_6 manifold whenever the group is symplectic, and consequently there is no way to find automorphisms using this strategy. We only obtain birational automorphisms that cannot be extended to regular automorphisms. Finally, in **Chapter 6**, we provide an example of a birational symplectic automorphism which is induced at the quotient but not induced. In Theorem 6.0.4 we find the explicit construction of this birational automorphism, starting from the co-invariant sublattice with respect to the induced action on the second integral cohomology. In general it holds that an induced automorphism is also induced at the quotient. In fact, if an automorphism is induced we find the abelian surface A to define the moduli space. If we can consider the Kummer surface of A, this is a K3 surface S, and this is what we need to define the Hilbert scheme $S^{[3]}$, and to verify the condition that allow us to say if it is induced at the quotient.

Chapter 1

Preliminaries

1.1 Lattices

In this Section we give an overview of lattice theory and of finite quadratic forms, recalling the fundamental definitions and results which we will use throughout the thesis. The work [72] due to Nikulin, is the most important which we will refer to, but there are also other classical sources, such as [29] and [46, Chapter 14].

Definition 1.1.1. A lattice Λ is by definition a free $\mathbb{Z}$-module of finite rank together with a symmetric bilinear form

$$(\ . \ ) : \Lambda \times \Lambda \longrightarrow \mathbb{Z},$$

which we will always assume to be non-degenerate.

A lattice Λ is called *even* if

$$(x)^2 = (x.x) \in 2\mathbb{Z}$$

for all $x \in \Lambda$, otherwise Λ is called *odd*. The determinant of the intersection matrix with respect to an arbitrary basis (over $\mathbb{Z}$) is called the *discriminant*, $\mathrm{disc}(\Lambda)$. A lattice Λ and the $\mathbb{R}$-linear extension of its bilinear form $(\ . \ )$ give rise to the real vector space $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ endowed with a symmetric bilinear form. The latter can be diagonalized with only 1 and -1 on the diagonal, as we assumed that $(\ . \ )$ is non-degenerate. The *signature* of Λ is (n_+, n_-), where $n_\pm$ is the number of ± 1 on the diagonal, and its *index* is $\tau(\Lambda) := n_+ - n_-$. The lattice Λ is called definite if either $n_+ = 0$ or $n_- = 0$ or, equivalently, if $\tau(\Lambda) = \pm \operatorname{rk} \Lambda$. Otherwise, Λ is *indefinite*. Finally, Λ is *hyperbolic* if $n_+ = 1$. The divisibility $\mathrm{div}(l)$ of an element $l \in \Lambda$ is the positive generator of the ideal $\{(l.m) | m \in \Lambda\} \subset \mathbb{Z}$. If t is a non-zero integer, $\Lambda(t)$ denotes the lattice of rank $r = \mathrm{rk}(\Lambda)$ and whose bilinear form is the one of Λ multiplied by t.

If Λ, Λ' are two lattices, their orthogonal direct sum is denoted by $\Lambda \oplus \Lambda'$: it is the lattice of rank $\mathrm{rk}(\Lambda) + \mathrm{rk}(\Lambda')$ on the free abelian group $\Lambda \oplus \Lambda'$ such that if $l_1, l_2 \in \Lambda$ and $l_1', l_2' \in \Lambda'$ then $(l_1, l_2) + (l_1', l_2') := (l_1 + l_1', l_2 + l_2')$.

If Λ is a lattice, a sublattice T of Λ is a subgroup $T \subset \Lambda$ with the property that the restriction of the bilinear form of Λ to T remains non-degenerate. For $T \subset \Lambda$ a sublattice, we define $T^\perp := \{l \in \Lambda \mid (l,t) = 0 \ \forall \ t \in T\}$. It is easy to check that

$T^\perp$ is a sublattice of Λ, called the orthogonal sublattice of T. In particular, the sublattice $T \oplus T^\perp \subset \Lambda$ has maximal rank, i.e. $\mathrm{rk}(T) + \mathrm{rk}(T^\perp) = \mathrm{rk}(\Lambda)$. A sublattice $T \subset \Lambda$ is called *primitive* if the quotient Λ/T is a free abelian group. The orthogonal complement $T^\perp$ of any sublattice $T \subset \Lambda$ is primitive; moreover, $(T^\perp)^\perp \subset T$ is the primitive sublattice of Λ generated by T (also called the saturation of T in Λ). If Λ, Λ' are two lattices, a morphism of lattices $\varphi : \Lambda \longrightarrow \Lambda'$ is a morphism of free abelian groups such that $(l_1, l_2) = (\varphi(l_1), \varphi(l_2))$ for all $l_1, l_2 \in \Lambda$. Since the bilinear form of any lattice is assumed to be non-degenerate, all morphisms of lattices are injective. An isometry is a bijective morphism of lattices; we denote by $O(\Lambda)$ the group of isometries of a lattice Λ to itself. If two lattices Λ, Λ' are isometric, we write $\Lambda \cong \Lambda'$. We will often use the term embedding to refer to a morphism of lattices which is not necessarily surjective. An embedding $i : \Lambda \hookrightarrow \Lambda'$ is primitive if the image $i(\Lambda) \subset \Lambda'$ is a primitive sublattice. The dual lattice of Λ is $\Lambda^* := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$, which admits the following equivalent description:

$$\Lambda^* = \{u \in \Lambda \otimes \mathbb{Q} : (u, v) \in \mathbb{Z} \ \forall \ v \in \Lambda\}. \tag{1.1}$$

Clearly Λ is a subgroup of Λ^*. Notice that, with respect to a basis $\{e_i\}_i$ of Λ and the dual basis $\{e_i^* := (e_i. \ )\}_i$ of Λ^*, the matrix which represents the inclusion $\Lambda \hookrightarrow \Lambda^*$ is simply the intersection matrix of Λ, i.e. the intersection with respect to $\{e_i\}_i$. Since $\Lambda \subset \Lambda^*$ is a subgroup of maximal rank, the quotient $A_\Lambda := \Lambda^*/\Lambda$ is a finite group, called the *discriminant group* of Λ. We denote with $\det(\Lambda)$ the order of the discriminant group A_Λ, i.e. the index of $\Lambda \subset \Lambda^*$, which coincides with $|\operatorname{disc}(\Lambda)|$. Moreover the *length*, $l(A_\Lambda)$ is defined as the minimal number of generators of A_Λ. If $A_\Lambda = \{0\}$ the lattice Λ is said to be *unimodular*. If instead $A_\Lambda \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$ for a prime number p and a non-negative integer k, then the lattice Λ is said to be *p-elementary*, in this case $l(A_\Lambda) = k$. Notice that the dual lattice Λ^* is not actually a lattice (it is not endowed with an integer valued bilinear form), however, using the representation 1.1, we can extend the bilinear form on Λ by $\mathbb{Q}$ linearity to $(.,.) : \Lambda^* \times \Lambda^* \longrightarrow \mathbb{Q}$. In particular if we consider $x_1, x_2 \in \Lambda^*$ and $l_1, l_2 \in \Lambda$, we have:

$$(x_1 + l_1, x_2 + l_2) = (x_1, x_2) + (x_1, l_2) + (x_2, l_1) + (l_1, l_2) \equiv (x_1, x_2) \ (mod \ \mathbb{Z}) \tag{1.2}$$

We recall the following definition

Definition 1.1.2. A *finite bilinear form* is a symmetric bilinear form $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ where A is a finite abelian group. A *finite quadratic form* is a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ such that:

i) $q(ka) = k^2 q(a)$ for all $k \in \mathbb{Z}$ and $a \in A$;

ii) $q(a + a') - q(a) - q(a') = 2b(a, a')$ in $\mathbb{Q}/2\mathbb{Z}$, where $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ is a finite bilinear form (called the bilinear form associated to q).

A finite quadratic form $q : A \to \mathbb{Q}/2\mathbb{Z}$ is said to be non-degenerate if the associated finite bilinear form b is non-degenerate, and by using b we define the orthogonal complement $H^\perp \subset A$ for any subgroup $H \subset A$. The isometry group $O(A)$ is the group of isomorphisms of A which preserves the finite quadratic form

q. Looking at the expression 1.2 if Λ is a lattice, the bilinear form (with rational values) on Λ^* descends to a well-defined finite bilinear form $A_\Lambda \times A_\Lambda \to \mathbb{Q}/\mathbb{Z}$.

In the case when the lattice Λ is even, we can associate to the bilinear form on A_Λ a finite quadratic form on q_Λ, defined as

$$q_\Lambda : A_\Lambda \to \mathbb{Q}/2\mathbb{Z},\ q_\Lambda(x+\Lambda) := (x,x) \pmod{2\mathbb{Z}}.$$

Notice that, for any two lattices Λ, Λ' there exists a canonical isomorphism $A_{\Lambda\oplus\Lambda'} \cong A_\Lambda \oplus A_{\Lambda'}$, which is an isometry with respect to the finite quadratic forms $q_{\Lambda\oplus\Lambda'}$ and $q_\Lambda \oplus q_{\Lambda'}$.
We recall the following result concerning finite quadratic forms.

Proposition 1.1.3. *Let q be a finite quadratic form on an abelian group A and $H \subset A$ a subgroup. If q is non-degenerate, then $|A| = |H||H^\perp|$. Moreover if the restriction $q_{|H}$ is non-degenerate, then $A = H \oplus H^\perp$ and $q = q_{|H} \oplus q_{|H^\perp}$*

Proof. See [72, Proposition 1.2.1] and [72, Proposition 1.2.2]. □

We now provide a list of examples of lattices that we will use throughout the thesis.

Example 1.1.4. Let $l \in 2\mathbb{Z}$, $l \neq 0$, we denote by $\langle l\rangle$ the rank one lattice $L = \mathbb{Z}e$, with $(e,e) = l$. It is positive definite if $l > 0$, negative definite otherwise. The equivalence class $\frac{e}{l} \in L\otimes\mathbb{Q}$ modulo L is a generator of $A_L \cong \frac{\mathbb{Z}}{l\mathbb{Z}}$, with $q_L(\frac{e}{l}+L) = \frac{1}{l}$.

Example 1.1.5. The lattice U is unimodular, hyperbolic lattice of rank two defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example 1.1.6. The E_8-*lattice* is given by the intersection matrix

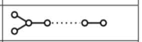
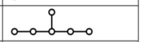
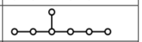
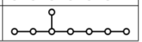
$$E_8 := \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

and it is, therefore, even, unimodular, positive definite (i.e. $n_- = 0$) of rank eight with discriminant equal to 1.

Lattices that are not unimodular and which are really important in this thesis are the lattices associated to the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8. Only the last one gives rise to a unimodular lattice as we have seen above.

To any graph Γ with simple edges the lattice $\Lambda(\Gamma)$ associated to Γ has a basis e_i corresponding to the vertices with an intersection matrix given by $(e_i.e_j) = 2$ if $i = j$, $(e_i.e_j) = 1$ if e_i and e_j are connected by an edge and $(e_i.e_j) = 0$ otherwise. So, for example, $A_1 = \langle 2\rangle$. In fact, the graphs of ADE type as drawn below are the only connected graphs for which the following holds: two vertices e_i, e_j of Γ are connected by at most one edge and the lattice $\Lambda(\Gamma)$ naturally associated with

Γ is positive definite. From a geometric point of view, lattices of ADE type occur as configurations of exceptional divisors of minimal resolutions of rational double points. Recall that rational double points are described explicitly by the following equations [46, Chapter 14]:

$A_{n\geq 1}$	$xy + z^{n+1}$	
$D_{n\geq 4}$	$x^2 + y(z^2 + y^{n-2})$	
E_6	$x^2 + y^3 + z^4$	
E_7	$x^2 + y(y^2 + z^3)$	
E_8	$x^2 + y^3 + z^5$	

The exceptional divisor of the minimal resolution of each of these singularities is a curve $\sum C_i$ with $C_i \simeq \mathbb{P}_1$, self-intersection $(C_i)^2 = -2$, and for $C_i \neq C_j$ one has $(C_i.C_j) = 0$ or$= 1$. The vertices of the dual graph correspond to the irreducible components C_i and vertices are connected by an edge if the corresponding curves C_i and C_j intersect. The dual graph is depicted in each of the cases in the last column. Alternatively, rational double points can be described as quotient singularities $\mathbb{C}^2/G$ by finite groups $G \subset SL(2,\mathbb{C})$. For instance, an A_n-singularity is isomorphic to the singularity of the quotient by the cyclic group of order n generated $\begin{pmatrix} \xi_n & 0 \\ 0 & \xi_n^{-1} \end{pmatrix}$ with ξ_n a primitive n-th root of unity, see [54, Chapter 4.6] for more details and references. We also record the discriminant groups of lattices of ADE type, see [35]:

Λ	A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
A_Λ	$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{0\}$

Example 1.1.7. For p prime, $p \equiv 1 (\bmod 4)$, let

$$H_p := \begin{pmatrix} (p-1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

It is a hyperbolic p-elementary lattice with $A_{H_p} \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$.

Example 1.1.8. For p prime, $p \equiv 3 (\bmod 4)$, let

$$K_p := \begin{pmatrix} -(p+1)/2 & 1 \\ 1 & -2 \end{pmatrix}.$$

It is a negative definite, p-elementary lattice with $A_{K_p} \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$. In particular $K_3 = A_2$.

Example 1.1.9. The *K3-lattice*

$$H^2(K3,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$$

is an even, unimodular lattice of signature $(3,19)$ and discriminant -1.

1.1.1 Orthogonal sublattices and primitive embeddings

Let Λ, Λ' be two even lattices, we say that they have *isomorphic discriminant forms* (and we write $q_\Lambda \cong q_{\Lambda'}$) if there exists a group isomorphism $\rho : A_\Lambda \to A_{\Lambda'}$ such that $q_\Lambda(x) = q_{\Lambda'}(\rho(x)) \in \mathbb{Q}/2\mathbb{Z}$ for all $x \in A_\Lambda$. By [72, Theorem 1.3.1.], Λ and Λ' have isomorphic discriminant forms if and only if there exist unimodular lattices V, V' such that $L \oplus V \cong L' \oplus V'$. Moreover by [72, Theorem 1.1.1(a)] the signature (v_+, v_-) of a unimodular lattice V satisfies $v_+ - v_- \equiv 0 \pmod 8$. Hence the following definition is well-posed.

Definition 1.1.10. The signature modulo 8 of a finite quadratic form q is

$$\operatorname{sign}(q) := n_+ - n_- \pmod 8$$

where (n_+, n_-) is the signature of an even lattice Λ such that $q_\Lambda = q$.

If $\Lambda_1 \hookrightarrow \Lambda$ has finite index, we have the following sequence of inclusions:

$$\Lambda_1 \hookrightarrow \Lambda \hookrightarrow \Lambda^* \hookrightarrow \Lambda_1^*,$$

such that the composition is just the canonical inclusion of Λ_1 in its dual Λ_1^*. Fix a basis $\{\alpha_i\}_i$ of Λ_1 and $\{\beta_i\}_i$ of Λ, and denote by G_{Λ_1} (respectively by G_Λ) the Gram matrix of the lattice Λ_1 (respectively of Λ) with respect to the chosen basis. If W is the matrix which represents the inclusion $\Lambda_1 \hookrightarrow \Lambda$, then the transposed matrix W^t represents $\Lambda^* \hookrightarrow \Lambda_1^*$, therefore we conclude that $G_{\Lambda_1} = W^t G_\Lambda W$. The determinant of W coincides with the index $[\Lambda : \Lambda_1]$, while the determinants of G_{Λ_1} and G_Λ are the discriminant of Λ_1 and Λ respectively, therefore

$$[\Lambda : \Lambda_1]^2 = \frac{\operatorname{discr}(\Lambda_1)}{\operatorname{discr}(\Lambda)} = \frac{|A_{\Lambda_1}|}{|A_\Lambda|}.$$

In a more general setting, if $\Lambda_1 \hookrightarrow \Lambda$ is a primitive sublattice of any rank, then $\Lambda_1 \oplus \Lambda_1^\perp \subset \Lambda$ has maximal rank, which implies

$$[\Lambda : (\Lambda_1 \oplus \Lambda_1^\perp)]^2 = \frac{\operatorname{discr}(\Lambda_1 \oplus \Lambda_1^\perp)}{\operatorname{discr} \Lambda} = \frac{|A_{\Lambda_1}||A^\perp_{\Lambda_1}|}{|A_\Lambda|}.$$

By the sequence of inclusions

$$\Lambda \oplus \Lambda_1^\perp \hookrightarrow \Lambda \hookrightarrow \Lambda^* \hookrightarrow (\Lambda \oplus \Lambda^\perp)^* \cong \Lambda_1^* \oplus (\Lambda_1^\perp)^*$$

the quotient $\Lambda/(\Lambda_1 \oplus \Lambda_1^*)$ is isomorphic to the subgroup $M \subset A_{\Lambda_1} \oplus A_{\Lambda_1^\perp}$, which is isotropic (i.e $(q_\Lambda \oplus q_{\Lambda^\perp})|_M = 0$), thus $M \subset M^\perp$ and $M^\perp/M \cong A_\Lambda$. In particular from the previous equality we have that $|A_{\Lambda_1}| \oplus |A_{\Lambda_1^\perp}| = |A_\Lambda||M|^2$. The projections

$$p_{\Lambda_1} : A_{\Lambda_1} \oplus A_{\Lambda_1^\perp} \to A_{\Lambda_1}, \quad p_{\Lambda_1^\perp} : A_{\Lambda_1} \oplus A_{\Lambda_1^\perp} \to A_{\Lambda_1^\perp},$$

are such that $M \cong M_{\Lambda_1} := p_{\Lambda_1}(M)$ and $M \cong M^\perp_{\Lambda_1} := p_{\Lambda_1^\perp}(M)$ as groups. Moreover, the composition

$$\gamma := p_{\Lambda^\perp} \circ (p_\Lambda)^{-1}|_{M_{\Lambda_1}} : M_{\Lambda_1} \to M_{\Lambda_1^\perp}$$

is an anti-isometry, i.e. an isomorphism of groups such that $q_{\Lambda_1}(x) = -q_{\Lambda_1^\perp}(\gamma(x))$ for all $x \in M_{\Lambda_1}$.

Lemma 1.1.11. *Let Λ be a unimodular lattice and $\Lambda_1 \subset \Lambda$ a primitive sublattice. Then, as groups,*

$$A_{\Lambda_1} \cong A_{\Lambda_1^\perp} \cong \frac{\Lambda}{\Lambda_1 \oplus \Lambda_1^\perp}.$$

Proof. Since $|A_\Lambda| = 1$, we have $|M|^2 = |A_{\Lambda_1}||A^\perp_{\Lambda_1}|$, hence $M_{\Lambda_1} = A_{\Lambda_1}$ and $M_{\Lambda_1^\perp} = M_{\Lambda_1^\perp}$. The projections, $p_{\Lambda_1}|_M$ and $p_{\Lambda_1^\perp}|_M$ are therefore isomorphisms of groups. □

We shall call two overlattices $S \hookrightarrow S'$ and $S \hookrightarrow S''$ *isomorphic* if there exists an automorphism of S extending to an automorphism of S' with S''. In order to formulate the next result, we observe that an isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ extends to a $\mathbb{Z}$-module isomorphisms $S_1^* \to S_2^*$ (denoted by φ^*) and determines an isomorphism $\overline{\varphi} : q_{S_1} \xrightarrow{\sim} q_{S_2}$ of their discriminant forms. In particular, there is an induced homomorphism $O(S) \to O(q_S)$ between the automorphism groups of S and q_S.

Proposition 1.1.12. *Two even overlattices $S \hookrightarrow S'$ and $S \hookrightarrow S''$ are isomorphic if and only if the isotropy subgroups $H_{S'} \subset A_S$ and $H_{S''} \subset A_S$ are conjugate under some automorphism of S.*

Proof. For a proof see [72, Proposition 1.4.2]. □

The following fundamental result, proved by Nikulin in [72, Proposition 1.15.1] describes primitive embeddings.

Theorem 1.1.13. *Let S be an even lattice of signature $(s_{(+)}, s_{(-)})$ and discriminant form q_S. Primitive embeddings $i : S \to L$, for L an even lattice of signature $(l_{(+)}, l_{(-)})$ and discriminant form q_L, are determined by quintuples $\theta_i := (H_S, H_L, \gamma, T, \gamma_T)$ such that:*

- *H_S is a subgroup of A_S,H_L is a subgroup of A_L and $\gamma : H_S \to H_L$ is an isomorphism $q_S|_{H_S} \cong q_L|_{H_L}$;*
- *T is a lattice of signature $(l_{(+)} - s_{(+)}, l_{(-)} - s_{(-)})$ and discriminant form $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, where $\Gamma \subseteq A_S \oplus A_L$ is the graph of γ and $\Gamma^\perp$ is its orthogonal complement in $A_S \oplus A_L$ with respect to the finite bilinear form associated to $(-q_S) \oplus q_L$;*
- *$\gamma_T \in O(A_T)$.*

The lattice T is isomorphic to the orthogonal complement of $i(S)$ in L. Moreover, two quintuples θ and θ' define isomorphic primitive sublattices if and only if $\overline{\mu}(H_S) = H'_S$ for some $\mu \in O(S)$ and there exists $\varphi \in O(A_L)$, $\nu : T \to T'$ isometries such that $\gamma' \circ \overline{\nu} = \varphi \circ \gamma$ and $\overline{\mu} \circ \gamma_T = \gamma'_{T'} \circ \overline{\nu}$.

Another crucial result due to Nikulin is the following (see [72, Proposition 1.5.1]).

Proposition 1.1.14. *A primitive embedding of an even lattice S into another even lattice, with discriminant form q and orthogonal complement isomorphic to K, is determined by a pair (H, γ), where $H \subset A_S$ is a subgroup and $\gamma : H \to A_K$ is a group monomorphism, while $q_K \circ \gamma = -q_S|H$ and*

$$(q_S \oplus q_K|(\Gamma_\gamma)^\perp)/\Gamma_\gamma \backsimeq q,$$

where Γ_γ *is the graph of* γ *in* $A_S \oplus A_K$. *Two such pairs* (H, γ) *and* (H', γ') *determine isomorphic primitive embeddings if and only if* $H = H'$, *and the injections* γ *and* γ' *are conjugate via some automorphism of* K, *and they determine primitive sublattice when there exist* $\varphi \in O(S)$ *and* $\psi \in O(K)$ *such that* $\gamma \circ \overline{\varphi} = \overline{\psi} \circ \gamma'$.

1.1.2 Existence and uniqueness

A fundamental invariant, in the theory of lattices, is given by the *genus*.

Definition 1.1.15. Two lattices, L and L' belong to same genus if $\mathrm{sign}(L) = \mathrm{sign}(L')$ and $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are isomorphic (as $\mathbb{Z}_p$-lattices) for all prime integers p.

Two lattices L and L' have the same genus if and only if $L \oplus U \cong L' \oplus U$ or equivalently if and only if they have the same signature and discriminant quadratic form:

Theorem 1.1.16. *The genus of an even lattice* L *is determined by the triple* $(l_{(+)}, l_{(-)}, q_L)$ *where* $(l_{(+)}, l_{-})$ *is the signature of the lattice and* q_L *is its discriminant quadratic form.*

Proof. See [72, Corollary 1.9.4]. □

Each genus contains only finitely-many isomorphism classes of lattices. It is an interesting problem to determine whether there exists an even lattice with given signature and discriminant quadratic form, and, if so, whether it is unique, up to isometries. The main results which we will need, regarding uniqueness of an indefinite lattice in its genus, are the following.

Theorem 1.1.17. *Let* L *be an even lattice with discriminant quadratic form* q_L *and signature* $(l_{(+)}, l_{(-)})$, *with* $l_{(+)} \geq 1$ *and* $l_{(-)} \geq 1$. *Up to isometries,* L *is the only lattice with invariants* $(l_{(+)}, l_{(-)}, q_L)$ *in all of the following cases:*

(i) $l_{(+)} + l_{(-)} \geq l(A_L) + 2$;

(ii) $l_{(+)} + l_{(-)} \geq 3$ *and* $discr(L) \leq 127$;

(iii) $l_{(+)} + l_{(-)} \geq 3$ *and* L *is* p*-elementary, with* p *odd;*

(iv) L *is* 2*-elementary.;*

Proof. The statement combines [72, Corollary 1.13.3], [29, Chapter 15, Corollary 22], [15, Theorem 2.2] and [32, Theorem 1.5.2].

□

We need to mention the following lemma:

Lemma 1.1.18. *Let* R *be a lattice, and let* $G \subset O(R)$, G *finite. Then the following hold:*

- $T_G(R)$ *contains* $\sum_{g \in G} gv$ *for all* $v \in R$.

- $S_G(R)$ *contains* $v - gv$ *for all* $v \in R$ *and all* $g \in G$.

- *$R/(T_G(R) \oplus S_G(R))$ is of $|G|$-torsion.*

where $T_G(R)$ is the invariant lattice, i.e. the eigenspace with respect to the eigenvalue 1, and $S_G(R)$ is the co-invariant sublattice i.e. the orthogonal complement of $T_G(R)$ in R.

Proof. It is obvious that $\sum_{g\in G} gv$ is G-invariant for all $v \in R$. For $w \in T_G(R)$ we have $(w, v) = (gw, gv) = (w, gv)$ for all $v \in R$ and all $g \in G$. Therefore $v - gv$ is orthogonal to all G-invariant vectors, hence it lies in $S_G(R)$. Let $t \in R$, we can write $|G|t = \sum_{g\in G} g(t) + \sum_{g\in G}(t - g(t))$, where the first term lies in $T_G(R)$ and the second in $S_G(R)$. □

The following Lemma gives some restrictions for a lattice to have an action of prime order:

Lemma 1.1.19. *Let L be a lattice and $G \subset O(L)$ a subgroup generated by φ of prime order p. Then*

$$m := \frac{rk(S_G(L))}{p-1}$$

is an integer and

$$\frac{L}{T_G(L)\oplus S_G(L)} \cong (\mathbb{Z}/p\mathbb{Z})^a.$$

Moreover, there are natural embeddings of $\frac{L}{T_G(L)\oplus S_G(L)}$ into the discriminant group $A_{T_G(L)}$ and $A_{S_G(L)}$, and $a \leq m$.

Proof. See [63, Lemma 1.8]. □

Since $T_G(\Lambda)$ and $S_G(\Lambda)$ are p-elementary lattices, we will use the following classification results to find them.

The first theorem deals with the case $p = 2$. For 2-elementary lattices, signature and length are not enough to determine the discriminant form. We need to introduce an additional invariant.

Definition 1.1.20. Let q be a quadratic form on a finite abelian group A. We define:

$$\delta(q) = \begin{cases} 0 \text{ if } q(x) \in \mathbb{Z}/2\mathbb{Z} \text{ for all } x \in A \\ 1 \text{ otherwise} \end{cases}$$

If L is an even lattice, we set $\delta(L) := \delta(q_L)$.

Theorem 1.1.21. *An even, 2-elementary lattice L of signature $(l_{(+)}, l_{(-)})$ is uniquely determined by the invariants $(l_{(+)}, l_{(-)}, l(A_L), \delta(L))$, up to isometries. There exists an even, 2-elementary lattice L with $sign(L) = (l_{(+)}, l_{(-)})$, $l(A_L) = a \geq 0$ and $\delta(L) = \delta \in \{0, 1\}$ if and only if $l_{(+)} \geq 0$, $l_{-} \geq 0$ and the following conditions are satisfied:*

$$\begin{cases} a \leq l_{(+)} + l_{(-)}; \\ l_{(+)} + l_{(-)} \equiv a \ (mod\ 2); \\ if\ \delta = 0\ then\ l_{(+)} - l_{(-)} \equiv 0 \ (mod\ 4); \\ if\ a = 0,\ then\ \delta = 0\ and\ l_{(+)} - l_{(-)} \equiv 0 \ (mod\ 8); \\ if\ a = 1,\ then\ l_{(+)} - l_{(-)} \equiv 1 \ (mod\ 8); \\ if\ a = 2,\ and\ l_{(+)} - l_{(-)} \equiv 4 \ (mod\ 8),\ then\ \delta = 0; \\ if\ \delta = 0,\ and\ l_{(+)} + l_{(-)} = a,\ then\ l_{(+)} - l_{(-)} \equiv 0 \ (mod\ 8). \end{cases}$$

Proof. See [32, Theorem 1.5.2]. □

And the second theorem deals with the case $p \neq 2$.

Theorem 1.1.22. *An even hyperbolic p-elementary lattice of rank r with $p \neq 2$ with invariants (r, a) exists if and only if the following conditions are satisfied:*

$$\begin{cases} a \leq r \\ r \equiv 0 \ (mod\ 2) \\ if\ a \equiv 0 \ (mod\ 2),\ then\ r \equiv 2 \ (mod\ 4) \\ if\ a \equiv 1 \ (mod\ 2),\ then\ p \equiv (-1)^{r/2-1} \ (mod\ 4) \\ if\ r \not\equiv 2 \ (mod\ 8),\ then\ r > a > 0 \end{cases}$$

Such a lattice is uniquely determined by the invariants (r, a) if $r \geq 3$.

Proof. See [83, Section 1]. □

Since the classification theorem of p-elementary lattices with $p \neq 2$ above deals only with hyperbolic lattices, we will need sometimes to split a lattice to study it. This is done by the following theorem:

Theorem 1.1.23. *Let L be an even lattice of rank r. If L is indefinite, and $r \geq 3 + l(A_L)$, then $L \simeq U \oplus L_0$ for some lattice L_0.*

Proof. See [72, Corollary 1.13.5]. □

The following theorems will also be useful along the classification:

Theorem 1.1.24. *Let L be an even lattice with signature (r_+, r_-) and discriminant form q_L. If L is indefinite and $l(A_L) \leq rk(L) - 2$, then L is the only lattice up to isometry with invariants (r_+, r_-, q_L).*

Proof. See [72, Theorem 1.13.3]. □

Proposition 1.1.25. *Let L be a lattice with a non-trivial action of order p, with rank $p - 1$, and discriminant d_L, then $\frac{d_L}{p^{p-2}}$ is a square in $\mathbb{Q}$.*

Proof. See [14, Section 4]. □

Let me recall this important result:

Theorem 1.1.26. *An even lattice with invariants (t_+, t_-, q) exists if and only if the following conditions are simultaneously satisfied:*

1. $t_+ - t_- \equiv sign\ q\ (mod\ 8)$

2. $t_+ \geq 0$, $t_- \geq 0$, $t_+ + t_- \geq l(A_q)$.

3. $(-1)^{t_-}|A_q| \equiv discrK(q_p)(mod(\mathbb{Z}_p^*)^2)$ *for all odd primes for which* $t_+ + t_- = l(A_{q_p})$.

4. $|A_q| \equiv \pm discrK(q_2)(mod(\mathbb{Z}_p^*)^2)$ *if* $t_+ + t_- = l(A_{q_2})$ *and* $q_2 \neq q_\theta^{(2)}(2) \oplus q_2'$.

Proof. See [74]. □

1.2 Irreducible holomorphic symplectic manifolds

1.2.1 Basic facts and examples

Definition 1.2.1. An *irreducible holomorphic symplectic* (IHS) manifold is a compact Kähler manifold X such that:

- $\pi_1(X) \cong \{1\}$
- $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$

where ω_X is an everywhere non-generate holomorphic two form.

The holomorphic form is referred to as the *symplectic form* of X. There are some properties that we can deduce from the definition of irreducible holomorphic symplectic manifolds (for more details, see for instance[43]).

Remark 1.2.2. In this remark we enumerate these properties:

(i) Since ω_X induces a symplectic form on the tangent space $T_x(X)$, for all $x \in X$, the complex dimension of X is even.

(ii) If dimX=2n, then $\chi(X, \mathscr{O}_X) = n + 1$, because for $k \in \{0, \ldots, 2n\}$ we have $H^0(X, \Omega_X^k) = \begin{cases} \mathbb{C}\omega_X^{k/2} & \text{if } k \equiv 0 \mod (2); \\ 0 & \text{if } k \equiv 1 \mod (2). \end{cases}$

(iii) Since ω_X^n generates $H^0(X, \Omega_X^{2n}) = H^0(X, K_X)$ and it is nowhere vanishing, it provides a trivialization of the canonical bundle, therefore $K_X \cong \mathscr{O}_X$.

(iv) The two-form ω_X defines an alternating homomorphism $TX \to \Omega_X^1$, which is bijective since ω_X is everywhere non-degenerate. As a consequence $TX \cong \Omega_X^1$ and thus $H^1(X, TX) = H^{1,1}(X)$.

(v) Since X is simply connected, we have $H_1(X, \mathbb{Z}) = 0$ and therefore (by the universal coefficient theorem) the second cohomology group with integer coefficients $H^2(X, \mathbb{Z})$ is torsion-free.

Since the canonical bundle is trivial, IHS manifolds have vanishing first Chern class: they constitute one of the building blocks of compact kähler manifolds Z such that $C_1(Z)_{\mathbb{R}} = 0$, as we find in the following theorem.

Theorem 1.2.3 (*Beauville–Bogomolov decomposition*)**.** *Let Z be a compact Kähler manifold with $c_1(Z)_{\mathbb{R}} = 0$. Z, up to an étale covering is isomorphic to*

$$T \times \prod_i V_i \times \prod_j X_j$$

where T is a complex torus, V_i are Calabi-Yau manifolds and X_j are IHS manifolds.

Proof. For a proof see [10, Theorem 2]. □

We recall the definition of a K3 surface.

Definition 1.2.4. A K3 surface is a compact complex smooth surface Σ such that $K_\Sigma \cong \mathscr{O}_\Sigma$ and $H^1(\Sigma, \mathscr{O}_\Sigma) = 0$.

We point out that every K3 surface is Kähler, even if this property is not explicitly requested in the definition (see [86]). All irreducible holomorphic symplectic manifolds of dimension two are K3. Notice that abelian surfaces (i.e. algebraic complex two-dimensional tori) are endowed with a non-degenerate holomorphic 2-form and therefore are holomorphic symplectic surfaces however they are not IHS because they are not simply connected.

Examples in higher dimensions are extremely hard to construct. E.g. complete intersections of dimension > 2 are never irreducible holomorphic symplectic. The following examples are the known ones.

Example 1.2.5. *Hilbert schemes of points on a K3 surface*
Let S be a K3 surface and $n \geq 1$ an integer. We will denote by $S^{[n]}$ the Hilbert scheme of n points on S, that is the scheme parametrizing zero-dimensional subschemes $(Z, \mathcal{O}_Z)$ of the surface S of length n (i.e. $\dim_{\mathbb{C}} \mathcal{O}_Z = n$). Notice that $S^{[n]}$ is, in general, just a complex space, but it is a scheme (even projective) if the K3 surface S is projective (see [10, §6]). The Hilbert scheme $S^{[n]}$ also arises as a minimal resolution of singularities of the n-symmetric product $S^{(n)}$, via the *Hilbert chow morphism*

$$\rho : S^{[n]} \to S^{(n)}$$

$$[(Z, \mathcal{O}_Z)] \longmapsto \sum_{p \in S} l(\mathcal{O}_{Z,p})p$$

where $l(\mathcal{O}_{Z,p})$ is the length of $\mathcal{O}_{Z,p}$, which is zero outside the (finite) set of points p in the support of Z. It was proved by Fogarty that ρ is the resolution of the singularities of $S^{(n)}$ and that $S^{[n]}$ is smooth; it is also Kähler because S is Kähler (see [95]). In his work [10], Beauville shows that $S^{[n]}$ is an IHS manifold of dimension $2n$, whose symplectic form comes from the symplectic form of the underlying K3 surface S. Any irreducible holomorphic symplectic manifold which is deformation equivalent to $S^{[n]}$, for some K3 surface S, is called a manifold of $K3^{[n]}$ type.

Example 1.2.6. *Generalized Kummer manifolds*
Let A be a complex two-dimensional torus and $n \geq 1$ an integer. The Hilbert

scheme $A^{[n+1]}$ is holomorphic symplectic, but it is not IHS since it is not simply connected. We consider the summation morphism

$$s : A^{[n+1]} \to A$$

$$[(Z, \mathcal{O}_Z)] \longmapsto \sum_{p \in A} l(\mathcal{O}_{Z,p})p$$

and we define $K_n(A) := s^{-1}(0)$ where 0 is the zero of the torus. The fiber $K_n(A)$ is now an IHS manifold of dimension $2n$, as proved by Beauville in [10], and we refer to these varieties as *generalized Kummer manifolds*. In particular, $K_1(A)$ is the Kummer K3 surface of the torus A, which is isomorphic to the blow up of the quotient $A/_{\pm 1}$. Irreducible holomorphic symplectic manifolds which are deformations of a generalized Kummer manifold are called IHS manifolds of *Kummer type*.

Hilbert schemes of points on a K3 surface and generalized Kummer manifolds provide two distinct ways to construct irreducible holomorphic symplectic manifolds in all even complex dimensions. Up to deformation, these are actually the only known examples of IHS manifolds, except in dimension six and in dimension ten, where we have two constructions (due to O'Grady) of irreducible holomorphic symplectic manifolds which are neither of $K3^{[n]}$ type, nor of Kummer type. They are called manifolds of OG_6 and OG_{10} type. The main topic of this thesis concern manifolds of OG_6 type and its group of automorphisms.

1.2.2 Cohomology of IHS manifolds

One of the main properties of IHS manifolds is that their second cohomology group with integer coefficients has a lattice structure since it is equipped with a non-degenerate symmetric bilinear form, which generalizes the intersection product. The following holds:

Theorem 1.2.7. *Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ and let ω be a symplectic form on X satisfying $\int_X (\omega \wedge \overline{\omega})^n = 1$. There exists a canonically defined pairing $(,)_X$ on $H^2(X, \mathbb{C})$, which is the Beauville-Bogomolov pairing, and a constant c_X called the Fujiki constant, such that the following holds:*

$$(\alpha, \alpha)_X = c_X \left(\frac{n}{2} \int_X (\omega \wedge \overline{\omega})^{n-1} \wedge \alpha^2 + (1-n) \left(\int_X \omega^{n-1} \wedge \overline{\omega}^n \wedge \alpha \right) \left(\int_X \omega^n \wedge \overline{\omega}^{n-1} \wedge \alpha \right) \right).$$

With respect to this intersection form, the signature of the lattice $H^2(X, \mathbb{Z})$ is $(3, b_2(X) - 3)$ by [10, Theorem 5]. These lattices have been studied by Beauville [10] for manifolds of $K3^{[n]}$ type and Kummer type, and by Rapagnetta for OG_6 and for OG_{10}, in [81] and [82] respectively.

The quadratic form associated to the Bauville-Bogomolov pairing is the Beauville-Bogomolov quadratic form which means that $q(\alpha) = (\alpha, \alpha)_X$. Moreover from the definition of the form we deduce that

$$q(\omega) = 0, \quad q(\omega + \overline{\omega}) > 0.$$

Remark 1.2.8. When we will use the Beauville-Bogomolov quadratic form q on $H^2(X,\mathbb{Z})$ for an IHS manifold, we will often write x^2 instead of $q(x)$, to denote the value of q on an element $x \in H^2(X,\mathbb{Z})$.

We stress the fact that the Fujiki constant c_X and the Beauville-Bogomolov quadratic form are deformation and birational invariants. As a consequence, for any IHS manifolds X' deformation equivalent to X we have a lattice isometry $H^2(X,\mathbb{Z}) \cong H^2(X,\mathbb{Z})$. Also notice that the Néron-Severi group

$$\mathrm{NS}(X) := H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

is a sublattice of $H^2(X,\mathbb{Z})$ which can be identified with

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap \omega^{\perp}$$

because $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$ inside $H^2(X,\mathbb{C})$.

Example 1.2.9. Let S be a K3 surface. Then the Beauville-Bogomolov quadratic form is just the intersection form on $H^2(S,\mathbb{Z})$. The lattice $H^2(S,\mathbb{Z})$ is unimodular and we have the following isometry:

$$(H^2(S,\mathbb{Z}), q) \cong L_{K3} := U^{\oplus 3} \oplus E_8^{\oplus 2}.$$

There exists a natural inclusion for any $n \geq 2$ (see [10, Proposition 6]):

$$i : H^2(S,\mathbb{Z}) \hookrightarrow H^2(S^{[n]},\mathbb{Z})$$

such that

$$H^2(S^{[n]},\mathbb{Z}) = i(H^2(S,\mathbb{Z})) \oplus \mathbb{Z}\delta$$

where 2δ is the class of the exceptional divisor E of the Hilbert–Chow morphism $\rho : S^{[n]} \to S^{(n)}$ (in particular, E is the locus in $S^{[n]}$ which parametrizes non-reduced zero-dimensional subscheme of length n). Another property that holds on algebraic classes is:

$$\mathrm{NS}(S^{[n]}) = i(\mathrm{NS}(S)) \oplus \mathbb{Z}\delta.$$

The class δ is such that $q(\delta) = -2(n-1)$, thus, for any manifold of $K3^{[n]}$ type, we have

$$(H^2(X,\mathbb{Z}), q) \cong L_n := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1) \rangle.$$

In particular $b_2(S^{[n]}) = rk(L_n) = 23$ and $\mathrm{sign}(L_n) = (3, 20)$. The Fujiki constant is $c = \frac{(2n)!}{n!2^n}$.

Using the Beauville-Bogomolov quadratic form, one can obtain the Euler characteristic of any divisor $D \in H^2(X,\mathbb{Z})$ (see [44, Example 23.19]):

$$\chi(X, D) = \binom{q(D)/2 + n + 1}{n}.$$

Example 1.2.10. Let X be an IHS manifold of Kummer type. Then the second integral cohomology is a lattice endowed with the Beauville-Bogomolov quadratic form and it hold that

$$(H^2(X,\mathbb{Z}), q) = U^{\oplus 3} \oplus \langle -2(n+1) \rangle$$

and the Fujiki constant is $c = \frac{(2n)!}{n!2^n}(n+1)$. In particular $b_2(X) = 7$.

Example 1.2.11. Let X be an IHS manifold deformation equivalent to an O'Grady sixfold. Then:

$$(H^2(X,\mathbb{Z}),q) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2},$$

and the Fujiki constant is $c = 60$. In particular, $b_2(X) = 8$.

Example 1.2.12. Let X be an IHS manifold deformation equivalent to an O'Grady tenfold. Then:

$$(H^2(X,\mathbb{Z}),q) \cong U^{\oplus 3} E_8^{\oplus 2} \oplus A_2,$$

and the Fujiki constant is $c = 945$. In particular, $b_2(X) = 24$, $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

It is interesting to notice that for the known IHS manifolds the second Betty number points out the deformation type.

As for K3 surfaces there exists a projectivity criterion for all IHS manifolds, which employs the Beauville-Bogomolov quadratic form.

Theorem 1.2.13. *Let X be an irreducible holomorphic symplectic manifold. Then X is projective if and only if there exists $l \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ such that $q(l) > 0$.*

Proof. See [44, Proposition 26.13]. □

An equivalent formulation of the projectivity criterion is that an IHS manifold X is projective if and only if the Néron-Severi sublattice, $\mathrm{NS}(X) \subset H^2(X,\mathbb{Z})$ is hyperbolic.

1.2.3 General results from deformation theory

Let X be a compact complex manifold. A *deformation* of X consists of a smooth proper morphism $\mathcal{X} \to S$, where $\mathcal{X}$ and S are connected complex spaces, and an isomorphism $X \simeq \mathcal{X}_0$, where $0 \in S$ is a distinguished point. Usually, only the germ of $(S,0)$ is considered. An *infinitesimal deformation* of X is a deformation with base space $S = Spec(\mathbb{C}[\varepsilon])/\varepsilon^2$.

Proposition 1.2.14. *The isomorphism classes of infinitesimal deformations of a compact complex manifold X are parametrized by elements in $H^1(X,TX)$.*

Proof. For a proof see [38, Proposition 22.1]. □

Definition 1.2.15. A deformation $\mathcal{X} \to (S,0)$ of X is called *universal* if any other deformation $\mathcal{X}' \to (S',0')$ is isomorphic to the pull-back under a uniquely determined morphism $\varphi : S' \to S$ with $\varphi(0') = 0$.

The universal deformation family is unique up to isomorphisms, it it exists. It will be denoted by $\mathcal{X} \to \mathrm{Def}(X)$, where $(\mathrm{Def}(X),0)$ is again considered as the germ of a complex space.
The main result in general deformation theory of complex manifolds is the following existence theorem which we will only state for manifolds without holomorphic vector fields.

Theorem 1.2.16 (Kuranishi)**.** *If X is compact complex manifold without global holomorphic vector fields, i.e. $H^0(X, TX) = 0$, then a universal deformation of X exists. Moreover, the universal deformation is universal for any of its fibers.*

The theorem readily applies to irreducible holomorphic symplectic manifolds. Indeed, since tangent and cotangent bundle are isomorphic, global holomorphic vector fields define global holomorphic one-forms, which do not exist on a simply connected compact Kähler manifold. Thus, an irreducible holomorphic symplectic manifold admits a universal deformation.
Moreover the following holds:

Proposition 1.2.17. *Let X be a compact Kähler manifold and let $\mathcal{X} \to S$ be any deformation of X.*

i) For $t \in S$ close to $0 \in S$, the fiber is a compact Kähler manifold.

ii) If K_X is trivial, then $K_{\mathcal{X}_t}$ is trivial for t close to 0 and the dimension of $H^1(\mathcal{X}_t, T\mathcal{X}_t)$ is independent of t.

Proof. For a proof see [38, Proposition 22.4]. □

Lemma 1.2.18. *Let $\mathcal{X} \to \operatorname{Def}(X)$ be the universal deformation of a compact complex manifold X with $H^0(X, TX) = 0$. For any $t \in \operatorname{Def}(X)$ close to $0 \in \operatorname{Def}(X)$ the Zarisky tangent space $T_t \operatorname{Def}(X)$ is naturally isomorphic to $H^1(\mathcal{X}_t, T\mathcal{X}_t)$.*

Proof. For a proof see [38, Lemma 22.5].

Definition 1.2.19. Let X be a compact complex manifold that admits a universal deformation $\mathcal{X} \to \operatorname{Def}(X)$. We say that the deformation of X are *unobstructed* if $\dim(T_0 \operatorname{Def}(X)) = \dim(\operatorname{Def}(X))$.

In other words the deformations of X are unobstructed if $\operatorname{Def}(X)$ is smooth, or, equivalently, if any infinitesimal deformation can be integrated over a small disk.
We know that for X an IHS manifold, $\dim(T_t(\operatorname{Def}(X))) = \dim(H^1(\mathcal{X}_t, T\mathcal{X}_t))$ is constant. If $\operatorname{Def}(X)$ is reduced this is enough to conclude that $\operatorname{Def}(X)$ is smooth, i.e. that X has unobstructed deformations. As the base space $\operatorname{Def}(X)$ of the universal deformation could, *a priori*, be non-reduced one has to argue more carefully to obtain:

Theorem 1.2.20. *If X is compact Hyperkähler manifold, the deformations of X are unobstructed.*

Proof. See [11]. □

Definition 1.2.21. Let X and X' be two IHS manifolds and let $G \subset \operatorname{Aut}(X)$, $G' \subset \operatorname{Aut}(X')$. Then (X, G) is deformation equivalent to (X', G') if $G \cong H \cong G'$ and there exists a flat family $\mathcal{X} \longrightarrow B$ and two maps $\{a\} \to B$, $\{b\} \to B$ such that $\mathcal{X}_a \cong X$ and $\mathcal{X}_b \cong X'$. Moreover we require that there exists a faithful action of the group H on $\mathcal{X}$ inducing fibrewise faithful actions of Hsuch that its restriction to $\mathcal{X}_a$ and $\mathcal{X}_b$ coincides with G and G'.

1.2.4 Moduli spaces and monodromy operators

Let X be an irreducible holomorphic symplectic manifold whose second cohomology lattice $H^2(X,\mathbb{Z})$ is isometric to a lattice L.

Definition 1.2.22. A *marking* of X is a choice of an isometry $\eta : H^2(X,\mathbb{Z}) \to L$. The pair (X,η) is called a *marked* irreducible holomorphic symplectic manifold. Two marked IHS manifolds (X,η), (X',η') are *isomorphic* if there exists a biregular (i.e. biholomorphic) isomorphism $f : X \to X'$ such that $\eta' = \eta \circ f^*$.

We can quotient the set of marked IHS pairs (X,η) with $H^2(X,\mathbb{Z}) \cong L$ by the isomorphism relation and we obtain:

$$\mathcal{M}_L := \{(X,\eta) | \eta : H^2(X,\mathbb{Z}) \to L \textit{ marking } \}/\cong .$$

The set $\mathcal{M}_L$ can be endowed with a structure of compact analytic complex space. We need to introduce the period map:

Definition 1.2.23. Let X be an irreducible holomorphic symplectic manifold and $\eta : H^2(X,\mathbb{Z}) \to L$ a marking. The *period domain* Ω_L is a complex space:

$$\Omega_L := \{k \in \mathbb{P}(L \otimes \mathbb{C}) | (k,k) = 0, (k,\overline{k}) > 0\}.$$

We know that ω satisfies, by definition of the Beauville-Bogomolov quadratic form, the two properties $(\omega,\omega) = 0$ and $(\omega,\overline{\omega}) > 0$. This implies that the choice of a marking η of X determines a point $\mathcal{P}(X,\eta) := \eta(H^{2,0}(X)) = \eta(\mathbb{C}\omega)$ in the period domain Ω_L. We can consider $p : \mathcal{X} \to I$ a flat deformation of the IHS manifold $X = p^{-1}(0)$. By Ehresmann's theorem (see [49, Theorem 2.6]) if $\eta : H^2(X,\mathbb{Z}) \to L$ is a marking of X , then there exists an open neighbourhood $J \subset I$ of the point 0 and a family of markings $F_t : \mathcal{X}_t \to L$ over J such that *eta* is the family of marking evaluated in 0. Then we define the map $\mathcal{P} : J \to \Omega_L$ as

$$\mathcal{P}(t) = F_t(H^{2,0}(\mathcal{X}_t)).$$

When considering the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$, the map $\mathcal{P} \to \Omega_L$ is called the (local) *period map.* We can now enunciate the following:

Theorem 1.2.24. *(Local Torelli theorem) Let (X,η) be a marked irreducible holomorphic symplectic manifold. The period map*

$$\mathcal{P} : \mathrm{Def}(X) \to \Omega_L$$

is a local isomorphism.

Proof. See [10, Theorem 5]. □

Using this local isomorphism, this universal deformations can be used as local charts for $\mathcal{M}_L$, which therefore is a compact non-Hausdorff complex space of dimension $h^{1,1}(X) = b_2(X) - 2$. There exists a holomorphic embedding $\mathrm{Def}(X) \hookrightarrow \mathcal{M}_L$, identifying $\mathrm{Def}(X)$ with an open neighbourhood of the point $(X,\eta) \in \mathcal{M}_L$. The maps $\mathcal{P} : \mathrm{Def}(X) \to \Omega_L$ can be glued together and we obtain a period map $\mathcal{P} : \mathcal{M}_L \to \Omega_L$ which is a local isomorphism by the Local Torelli theorem. It holds another meaningful result:

Theorem 1.2.25. *Let $\mathcal{M}_L^0$ be a connected component of the moduli space $\mathcal{M}_L$. Then the restriction of the period map $\mathcal{P}_0 : \mathcal{M}_L^0 \to \Omega_L$ is surjective.*

Proof. See [43, Theorem 8.1]. □

The natural following question is weather it holds a Global Torelli theorem for IHS manifolds as in the case of K3 surfaces. This is false in general, as Debarre shows in his counterexample (see [30]). However a weaker version of the global Torelli has been proved by Huybrechts, Markman and Verbitsky.

Theorem 1.2.26. *(Global Torelli theorem) Let $\mathcal{M}_L^0$ be a connected component of the moduli space $\mathcal{M}_L$. For each $\omega \in \Omega_L$, the fiber $P_0^{-1}(\omega)$ consists of pairwise inseparable points. If (X, η) and (X', η') are inseparable points of $\mathcal{M}_L^0$ then X and X' are bimeromorphic.*

Proof. See [52, Theorem 2.2]. □

We can formulate the Global Torelli also from a lattice-theoretic point of view. We will use principally this formulation in the following sections. In order to give this formulation we need to introduce the notion of **monodromy operator**.

Definition 1.2.27. Let X,Y be holomorphic symplectic manifolds. A lattice isometry $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is a *parallel transport operator* if there exists a smooth and proper family $\pi : \mathcal{X} \to B$ and a continuous path $\gamma : [0, 1] \to B$ such that $X \cong \mathcal{X}_{\gamma(0)}$ and $Y \cong \mathcal{X}_{\gamma(1)}$ and f is induced by parallel transport in the local system $R^2\pi_*\mathbb{Z}$ along γ.

A parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is called a *monodromy operator* of X.

The following is a necessary condition for an isometry $g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ to be a parallel transport operator. Denote by $\mathcal{C}_X \subset H^2(X, \mathbb{R})$ the cone

$$\{\alpha \in H^2(X, \mathbb{R}) : \ (\alpha, \alpha) > 0\}.$$

The $H^2(\mathcal{C}_X, \mathbb{Z}) \cong \mathbb{Z}$ and it comes with a canonical generator, which we call the *orientation class* on $\mathcal{C}_X$. Any isometry $g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ induces an isomorphism $\overline{g} : \mathcal{C}_X \to \mathcal{C}_Y$. The isometry g is said to be *orientation preserving* if $\overline{g}$ is. A parallel transport operator $g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is orientation preserving (see [52, Section 4]).

We denote by $Mon^2(X) \subset O(H^2(X, \mathbb{Z})$ the subgroup of monodromy operators, which is of finite index (see [52, Lemma 7.5]). In particular two marked pairs (X, η), (X', η') belong to the same connected component of $\mathcal{M}_L$ if and only if $\eta' \circ \eta^{-1}$ is a parallel transport operator. As a consequence, the number of connected components of $\mathcal{M}_L$ is $\pi_0(\mathcal{M}_L) = [O(H^2(X, \mathbb{Z}) : Mon^2(X)]$.

If X is an IHS manifold and $\eta : H^2(X, \mathbb{Z}) \to L$ is a marking, we can define

$$Mon^2(L) := \{\eta \circ \psi \circ \eta^{-1} | \psi \in Mon^2(X)\} \subset O(L).$$

The group $Mon^2(L) \subset O(L)$, whose elements are still called monodromy operators, is the same for any choice of a marked pair (X, η) in a connected component $\mathcal{M}_L^0 \subset \mathcal{M}_L$, but could a priori depend on $\mathcal{M}_L^0$. However, if the subgroup $Mon^2(X) \subset$

$O(H^2(X,\mathbb{Z}))$ is normal then $Mon^2(L)$ is independent on the choice of the connected component (see [52, Remark 7.2]).

The monodromy group has been studied and completely described for the most part of known IHS manifolds. In the case of K3 surfaces it was computed by Borcea (see [19]), in the case of manifolds of $K3^{[n]}$ type by Markman(see [52]), and in the case of manifolds of $K_n(A)$ type by Markman at first and by Mongardi. For O'Grady sixfolds Mongardi and Rapagnetta recently computed the group (see unpublished result). For O'Grady tenfolds very little is known, and the situation looks even more difficult by the lack of examples of monodromy operators. In the OG_6 case, the authors used a construction of OG_6 manifolds which relates these manifolds to manifolds of $K3^{[3]}$ type, so they could use Markman's results, but this methods does not works in the OG_{10} case. Moreover, Markman himself made a conjecture about their monodromy group, which was recently disproved by Mongardi; this counter-example sheds no light on the problem though.

Example 1.2.28. Let S be a K3 surface. We denote by $O^+(H^2(S,\mathbb{Z})$ the subgroup of $O(H^2(C,\mathbb{Z})$ of orientation preserving isometries: it is a normal subgroup of index two. Then $Mon^2(S) = O^+(H^2(S,\mathbb{Z})$ (see [19, Theorem A]), therefore the moduli space $\mathcal{M}_{K3}$ of marked K3 surfaces has two connected components, which corresponds to each other via the map $(S,\eta) \to (S,-\eta)$.

Example 1.2.29. Let X be an IHS manifold of $K3^{[n]}$ type, then Markman in [52] proved that $Mon^2(X) \subset O(H^2(X,\mathbb{Z})$ is a normal subgroup and provided several equivalent characterizations of monodromy operators of X, which we now recall.

Proposition 1.2.30. *Let X be a manifold of $K3^{[n]}$ type. An isometry $\psi \in O(H^2(X,\mathbb{Z})$ is a monodromy operator if and only if ψ is orientation preserving and it induces the action $\overline{\psi} = \pm id \in O(A_{H^2(X,\mathbb{Z})})$.*

In particular the index of $Mon^2(X)$ as a subgroup of $O^+(H^2(X,\mathbb{Z}))$ is 2^{r-1} where $r = \rho(n-1)$ is the number of distinct prime divisor of $n-1$. As a consequence, if $n = 2$ or $n-1$ is a prime power, then $Mon^2(X) = O^+(H^2(X,\mathbb{Z}))$, exactly as for K3 surfaces.

Example 1.2.31. Let X be an irreducible holomorphic symplectic manifold of generalized Kummer type. Let $O^+(H^2(X,\mathbb{Z}))$ be the group of orientation preserving isometries and define the subgroup

$$W(X) = \{g \in O^+(H^2(X,\mathbb{Z}))| g \text{ acts as } \pm id \text{ on } A_X\},$$

where A_X is the discriminant group. Denote by $\chi : W(X) \to \{\pm 1\}$ the corresponding character. The following is a characterization of $Mon^2(X)$ due to Mongardi (see [59, Theorem 2.3]).

Proposition 1.2.32. *Let X be an irreducible holomorphic symplectic manifold of $K_n(A)$ type, then*

$$Mon^2(X) = \{g \in W(X)|\ det(g)\chi(g) = 1\}.$$

Example 1.2.33. Let X be a manifold of OG_6 type. We know from [61, Theorem 5.4] that the Monodromy group for such a manifold is made by orientation preserving isometries, i.e. $Mon^2(X) = O^+(H^2(X, \mathbb{Z}))$, which means that

$$[O(H^2(X, \mathbb{Z})) : Mon^2(X)] = 2.$$

We can now state the Hodge-theoretic form of the global Torelli theorem.

Theorem 1.2.34. *Let X, Y be irreducible holomorphic symplectic manifolds. If there exists a parallel transport operator $\psi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ which is also an Hodge isometry, then X and Y are bimeromorphic. If, moreover, ψ maps a Kähler class to a Kähler class, then there exists a biregular isomorphism $f : Y \to X$ such that $f^* = \psi$.*

Proof. See [52, Theorem 1.3].

□

Since we know the Monodromy for manifolds of OG_6 type, we can apply the previous theorem and state the following result, which is a bimeromorphic global Torelli Theorem for manifolds of OG_6 type.

Theorem 1.2.35. *Bimeromorphic global Torelli (in the strongest form) holds for IHS manifolds of OG_6 type, i.e. two IHS manifolds X and X' of OG_6 type are bimeromorphic if and only if there exists a Hodge isometry between $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$.*

Proof. See [61, Theorem 5.4 (2)]. □

In the following we will denote by $Mon^2_{Hdg}(X) \subset Mon^2(X)$ the subgroup of monodromy operators which preserves the Hodge decomposition.

1.2.5 Kähler cone and wall divisors

Let X be an IHS manifold, the Beauville-Bogomolov quadratic form allows us to define several cones which are helpful in the study of automorphisms groups. These cones are contained in $H^{1,1}(X, \mathbb{R})$, or in its intersection with $H^2(X, \mathbb{Z})$. In general, for a IHS manifold X these cones are subcones of the positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$. The ample cone $\mathcal{A}_X$ and the movable cone $\mathrm{Mov}(X)$ are dual in $\mathcal{C}_X$ to wall divisors and stably prime exceptional divisors respectively. The aim of this section is to recall these cones and these divisors for manifolds of OG_6 type.

Definition 1.2.36. Let X be an irreducible holomorphic symplectic manifold. The *positive cone* $\mathcal{C}_X$ is the connected component of $\{x \in H^{1,1}(X, \mathbb{R}) | (x, x) > 0\}$ which contains the cone of Kähler classes, $\mathcal{K}_X$.

Recall that, in the case of K3 surfaces, the Kähler cone coincides with the set of real $(1, 1)$-classes which have positive intersection with all rational curves on the surface. Boucksom (see [20, Theorem 1.2]) generalizes the result for any IHS manifold X and we have:

$$\mathcal{K}_X = \{\alpha \in \mathcal{C}_X | \int_C \alpha > 0 \text{ for all rational curves } C \subset X\}.$$

Definition 1.2.37. A *prime* divisor on X is a reduced and irreducible effective divisor E, and it is *exceptional* if it is of negative Beauville-Bogomolov degree.

Definition 1.2.38. A *stably prime exceptional* divisor is a divisor D which is prime exceptional divisor in a generic deformation of the pair $(X, \mathcal{O}(D))$.

Prime exceptional divisors are stably prime exceptional divisors (see [52, Proposition 6.6 1]) but the converse does not hold in general. The easiest example of a stably prime exceptional divisor which is not prime exceptional is given by a reducible -2 curve on a K3 surface.

Definition 1.2.39. The *fundamental exceptional chamber* of X is the cone:

$$\mathcal{FE}_X = \{\alpha \in \mathcal{C}_X : (\alpha, E) > 0, \text{ for every stably prime exceptional divisor } E \subset X\}.$$

By [52, Proposition 5.6], $\mathcal{FE}_X$ is also the cone of classes $x \in \mathcal{C}_X$ such that $(x, D) > 0$ for any non-zero uniruled divisor $D \subset X$.

For this reason if S is a K3 surface, non-zero uniruled divisor are just rational curves $C \subset S$ and consequently $\mathcal{FE}_X = \mathcal{K}_X$. For IHS manifolds of higher dimensions, the Kähler cone is, in general, strictly contained in $\mathcal{FE}_X$. We will see in the following that the Kähler cone is actually a chamber of $\mathcal{FE}_X$, with respect to a suitable decomposition.

Definition 1.2.40. let X be an IHS manifold. The *birational Kähler cone* is defined in the following way:

$$\mathcal{BK}_X = \bigcup_{f: X \dashrightarrow X'} f^* \mathcal{K}_{X'}.$$

where $f : X \dashrightarrow X'$ runs through all birational maps $X \dashrightarrow X'$ from X to another IHS manifold X'.

The following results due to Markman hold (see [52, Corollary 5.7]):

Proposition 1.2.41. *Let X and Y be IHS manifolds, let $g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be a parallel transport operator, which is an isomorphism of Hodge structures. Let $\alpha_X \in \mathcal{FE}_X$, then $g(\alpha_X) \in \mathcal{FE}_Y$ if and only if there exists a birational map $f : Y \dashrightarrow X$ such that $g = f^*$.*

As a consequence of the following we have $\mathcal{BK}_X \subset \mathcal{FE}_X \subset \overline{\mathcal{BK}}_X$ (see [44, Theorem 4.3] [21]).

By [52, Proposition 5.6], we also have the inclusion $\mathcal{FE}_X \subset \overline{\mathcal{BK}}_X$; as a consequence, $\overline{\mathcal{FE}}_X = \overline{\mathcal{BK}}_X$. The positive cone is invariant under the action of $Mon^2_{Hdg}(X)$, and we can consider the following *chambers* in it.

Definition 1.2.42. Let X be an irreducible holomorphic symplectic manifold.

(i) An *exceptional chamber* of $\mathcal{C}_X$ is a subset of the form $g(\mathcal{FE}_X)$, for an isometry $g \in Mon^2_{Hdg}(X)$.

(ii) A *Kähler type chamber* of $\mathcal{C}_X$ is a subset of the form $g(f^*(\mathcal{K}_Y))$, for an isometry $g \in Mon^2_{Hdg}(X)$ and a birational map $f : X \dashrightarrow Y$

We know from [52, Theorem 6.18], that the Hodge monodromy operators act transitively on the set of exceptional chambers. Moreover, each exceptional chamber (and in particular $\mathcal{FE}_X$) is the interior of a fundamental domain for the action of a normal subgroup

$$W_{Exc} = \langle R_E | E \subset X \text{ prime exceptional divisor} \rangle \subset Mon^2_{Hdg}(X),$$

where R_E denotes the reflection with respect to the class E defined in the following way:

$$R_E(\alpha) := \alpha - 2\frac{(E,\alpha)}{(E,E)}E.$$

Remark 1.2.43. If E is a prime exceptional divisor, $R_E \in Mon^2_{Hdg}(X)$ by [52, Proposition 6.2].

Let $Mon^2_{Bir}(X) \subset Mon^2_{Hdg}(X)$ be the subgroup of monodromy operators induced by bimeromorphic maps from X to itself. From Proposition 1.2.41 we know that $Mon^2_{Bir} \subset Mon^2_{Hdg}$ is the stabilizer of the fundamental exceptional chamber. Moreover, from [52, Theorem 6.18] we know that the following equality holds:

$$Mon^2_{Hdg}(X) = W_{Exc} \rtimes Mon^2_{Bir}(X)$$

Remark 1.2.44. The Kähler type chambers are the translations of the Kähler cone, $\mathcal{K}_X$, by the Hodge monodromy operators. From Theorem 1.2.34 we know that distinct Kähler type chambers are disjoint, while the closure of two adjacent chambers intersect along a wall (since $\overline{\mathcal{BK}_X} = \overline{\mathcal{FE}_X}$).

By removing the orthogonal hyperplanes to stably prime exceptional divisors, the positive cone $\mathcal{C}_X$ is cut in a wall and chamber decomposition. One such chamber is the closure of the Birational Kähler cone (see [[52], Section 5.2]) and its algebraic part, i.e. $\overline{\mathcal{BK}_X} \cap H^{1,1}(X,\mathbb{R})$, is called the movable cone.

All the cones we have introduced so far live inside $H^{1,1}(X,\mathbb{R})$; we now want to study their intersections with the integral cohomology $H^2(X,\mathbb{Z})$. Recall that the Néron–Severi lattice is defined as $NS(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ and, since $H^1(X,\mathcal{O}(X)) = 0$, the first Chern class $c_1 : Pic(X) \to H^2(X,\mathbb{Z})$ provides an isomorphism $Pic(X) \cong NS(X)$.

Definition 1.2.45. A line bundle $L \in \mathrm{Pic}(X)$ is called *movable* if the codimension of the base locus of the linear system $|L|$ is at least two. The *movable cone* $\mathrm{Mov}(X) \subset NS(X)_{\mathbb{R}} := NS(X) \otimes \mathbb{R}$ is the cone generated by the classes of movable line bundles.

Proposition 1.2.46. *Let X be an IHS manifold. The interior of the movable cone coincides with $\mathcal{FE}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$. The group W_{Exc} acts faithfully on $\mathcal{C}_X \cap NS(X)_{\mathbb{R}}$ and there is a bijective correspondence between the set of exceptional chambers of X and the set of chambers of $\mathcal{C}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$ with respect to the action of W_{Exc}. In particular, $\overline{\mathrm{Mov}(X)} \subset \mathcal{C}_X$ is a fundamental domain for the action of W_{Exc} on $\mathcal{C}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$.*

Proof. See [52, Lemma 6.22]. □

If X is projective, the ample cone $\mathcal{A}_X$ (i.e. the cone in $\mathrm{NS}(X)_{\mathbb{R}}$ generated by ample classes) is contained inside $\mathrm{Mov}(X)$; more specifically, $\mathcal{A}_X = \mathcal{K}_X \cap \mathrm{NS}(X)_{\mathbb{R}}$.

The decomposition of the positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ into exceptional chambers can be adapted to the integral cohomology. In order to do so, we need to define wall-divisors.

Definition 1.2.47. A *wall-divisor* on an IHS manifold is a primitive divisor D with $D^2 < 0$, such that for every monodromy operator $g : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ that is an Hodge isometry,

$$g(D^{\perp}) \cap \mathcal{BK}_X = \emptyset.$$

By using the natural lattice embedding $H_2(X, \mathbb{Z}) \subset H^2(X, \mathbb{Q})$, induced by Bauville-Bogomolov-Fujiki form, a wall divisor is precisely a multiple of an extremal rational curve, up to the action of monodromy Hodge isometries, see [48, Poposition 2.3]. Also orthogonals to wall divisors give a wall and chamber decomposition of the positive cone (whence their name), and one of the open chambers is the Kähler cone. In particular, if we restrict this wall and chamber decomposition to the birational Kähler cone, we obtain the Kähler cones of all IHS birational models of X. Every stably prime exceptional divisor is a wall divisor (or more precisely, its primitive multiple is), but the converse does not hold and non-stably prime exceptional wall divisors are the wall which cause the non connectedness of the birational Kähler cone.

Notice that the movable cone $\mathrm{Mov}(X)$ is one of the connected components of

$$(\mathcal{C}_X \cap \mathrm{NS}(X)_{\mathbb{R}}) \setminus \bigcup_{E \in \mathcal{P}_{Ex}} E^{\perp}$$

where $\mathcal{P}_{Ex}$ s the set of prime exceptional divisors, while the ample cone $\mathcal{A}_X$ is one of the connected components of

$$(\mathcal{C}_X \cap \mathrm{NS}(X)_{\mathbb{R}}) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}$$

where $\Delta(X)$ is the set of wall-divisors.

As we already stated, wall-divisors are preserved under smooth deformations if their Hodge type does not change and they are invariants under parallel transport operator; in particular, we have the following result.

Theorem 1.2.48. *Let (X, η), (Y, μ) be marked irreducible holomorphic symplectic manifolds in the same connected component $\mathcal{M}_L^0$ of the moduli space $\mathcal{M}_L$. If $D \in \mathrm{NS}(X)$ is a wall-divisor (or a stably prime exceptional divisor) of X and $(\mu^{-1} \circ \eta)(D) \in \mathrm{NS}(Y)$, then $(\mu^{-1} \circ \eta)(D) \in \mathrm{NS}(Y)$ is a wall-divisor (respectively a stably prime exceptional divisor) of Y.*

Therefore it suffices to determine the classes of stably prime exceptional and wall divisors up to parallel transport.

Proof. See [58, Theorem 1.3].

□

Due to results of Mongardi and Rapagnetta we have a numerical characterization of wall divisors and prime exceptional divisors.

Lemma 1.2.49. *Let X be a manifold of OG_6 type. Let $D \in Div(X)$ and let $[D] \in \mathrm{Pic}(X)$ be its class. Then $[D]$ is the class of a multiple of a stably prime exceptional divisor if one of the following holds:*

- $[D]^2 = -4$ *and* $div(D) = 2$*,*
- $[D]^2 = -2$ *and* $div(D) = 2$*.*

Proof. See [61, Lemma 6.4]. □

Lemma 1.2.50. *Let X be a manifold of OG_6 type. Let $D \in Div(X)$ and let $[D] \in \mathrm{Pic}(X)$ be its class. Then $[D]$ is the class of a wall-divisor but not the class of a multiple of a stably prime exceptional divisor if $[D]^2 = -2$ and $div(D) = 1$.*

Proof. See [61, Lemma 6.6]. □

1.3 O'Grady's sixfolds

O'Grady's sixfolds are a deformation class of IHS manifolds which was firstly discovered by O'Grady [76]. Manifolds in this family are obtained in two known ways. The first construction, is obtained by taking a projective abelian surface and a Mukai vector w of square 2. The moduli space of Gieseker semistable sheaves with Mukai vector 2w is a singular tenfold with rational singularities, whose Albanese fiber admits a crepant resolution that is a IHS manifold in the family we are dealing with. This was proven by O'Grady [76] for a special Mukai vector. Later M. Lehn and Sorger [87] showed that, under our assumption on w, the blow up of the Albanese fiber of the moduli space along its singular locus always gives a crepant resolution and Perego and Rapagnetta proved [80] that these crepant resolutions are deformation equivalent, along smooth projective deformations, to the original O'Grady example.

A second construction was obtained in [62], by considering a principally polarized abelian surface A and its Kummer K3 surface S. On a moduli space of sheaves on S, the authors construct a non regular involution, whose quotient is birational to a manifold of OG_6 type. This last construction was used to compute the Hodge numbers of manifolds of OG_6 type.

1.3.1 O'Grady's construction

O'Grady discovered in 2000 a new example in dimension six of irreducible holomorphic symplectic manifold as the symplectic resolution of a certain subvariety of a moduli space of sheaves on an abelian surface A [76].
Now we refer to this construction and we take into consideration also the paper of Perego and Rapagnetta about deformations of the O'Grady's manifolds (see [80]). Let A be an abelian surface, an element $v \in \widetilde{H}(A,\mathbb{Z}) := H^{2*}(A,\mathbb{Z})$ will be written as $v = (v_0, v_1, v_2)$, where $v_i \in H^{2i}(A,\mathbb{Z})$, and $v_0, v_2 \in \mathbb{Z}$. If $v_0 \geq 0$ and $v_1 \in \mathrm{NS}(A)$,

then v is called Mukai vector. Recall that $\widetilde{H}(A,\mathbb{Z})$ has a pure weight-two Hodge structure defined as

$$\widetilde{H}^{2,0}(A) := H^{2,0}(A), \quad \widetilde{H}^{0,2}(A) := H^{0,2}(A),$$

$$\widetilde{H}^{1,1}(A) := H^0(A,\mathbb{C}) \oplus H^{1,1}(A) \oplus H^4(A,\mathbb{C}),$$

and a lattice structure with respect to the Mukai pairing $(.,.)$, which is defined in this way:

$$(r_1, l_1, s_1)(r_2, l_2, s_2) := l_1 l_2 - r_1 s_2 - r_2 s_1$$

In the following, we let $v^2 := (v, v)$ for every Mukai vector v; moreover, for every Mukai vector v define the sublattice

$$v^\perp := \{\alpha \in \widetilde{H}(A,\mathbb{Z}) | (\alpha, v) = 0\} \subseteq \widetilde{H}(A,\mathbb{Z}),$$

which inherits a pure weight-two Hodge structure from the one on $\widetilde{H}(A,\mathbb{Z})$. If $\mathscr{F}$ is a coherent sheaf on A, we define its *Mukai vector* to be

$$v(\mathscr{F}) := ch(\mathscr{F})\sqrt{td(A)} = (rk(\mathscr{F}), c_1(\mathscr{F}), ch_2(\mathscr{F})).$$

Let θ be an ample line bundle on A, i.e $\theta \in Amp(A)$, where $Amp(A) \subseteq \mathrm{NS}(A) \otimes \mathbb{R}$ is the ample cone of A. For every $n \in \mathbb{Z}$ and every coherent sheaf $\mathscr{F}$, let $\mathscr{F}(n\theta) := \mathscr{F} \otimes \mathscr{O}_A(n\theta)$. The Hilbert polynomial of $\mathscr{F}$ with respect to θ is $P_\theta(\mathscr{F})(n) := \chi(\mathscr{F}(n\theta))$, and the reduced Hilbert polynomial of $\mathscr{F}$ with respect to θ is

$$p_\theta(\mathscr{F}) := \frac{P_\theta(\mathscr{F})}{\alpha_\theta(\mathscr{F})},$$

where $\alpha_\theta(\mathscr{F})$ is the coefficient of the term of highest degree in $P_\theta(\mathscr{F})$.

We need to recall the definition of $v-$genericity of a polarization, where $v = (v_0, v_1, v_2)$ is a Mukai vector on A.

Definition 1.3.1. A polarization θ is $v-$generic if for every polystable sheaf $\mathscr{E}$ of Mukai vector v and every direct summand $\mathscr{F}$ of $\mathscr{E}$, we have $v(\mathscr{F}) \in \mathbb{Q} \cdot v$.

Let $\mathscr{E}$ be a $\theta-$semistable sheaf with Mukai vector v, and let $\mathscr{F} \subseteq \mathscr{E}$ a θ-destabilizing subsheaf with Mukai vector $u := (u_0, u_1, u_2)$.

Definition 1.3.2. The divisor associated to the pair $(\mathscr{E}, \mathscr{F})$ is defined as follows:

1. if $v_0 > 0$, it is the divisor $D := u_0 v_1 - v_0 u_1$;
2. if $v_0 = 0$, it is the divisor $D := u_2 v_1 - v_2 u_1$.

The set of the non-zero divisors associated to all the possible pairs is denoted $W_v(\theta)$.

The characterization of $v-$genericity is the following:

Lemma 1.3.3. *Let $v = (v_0, v_1, v_2)$ be a Mukai vector such that if $v_0 = 0$, then $v_2 \neq 0$. A polarization θ is $v-$generic if and only if $W_v(\theta) = \emptyset$.*

Proof. see [80, Lemma 2.3]. □

Remark 1.3.4. O'Grady (see [76, Introduction]) formulates a technical assumption which implies $v-$genericity in the case $v = (2, 0, -2)$ and it consists in

$$\text{There is no divisor } D \text{ on } A \text{ such that } C_1(D).\theta = 0 \text{ and } D.D = (-2). \tag{1.3}$$

In the following this will be useful to check $v-$genericity of an ample class θ in the case of O'Grady six dimensional manifolds.

We need also to mention the notion of v–walls and v–chambers: as these notions depends on v_0, we recall just the case $v_0 \geq 2$. If A is an abelian surface, let

$$|v| := \frac{v_0^2}{4}(v, v) + \frac{v_0^2}{2}.$$

Notice that $|v|$ depends only on (v, v) and v_0, and as $v_0 \geq 2$, then $|v| > 0$. Hence it makes sense to define

$$W_v = \{D \in \mathrm{NS}(A) | -|v| \leq D^2 < 0\}.$$

By Theorem 4.C.3 of [47], we have $W_v(\theta) \subseteq W_v$ for every $\theta \in Amp(A)$.

Definition 1.3.5. Let $D \in W_v$. The v–wall associated to D is

$$W^D := \{\alpha \in Amp(A) | D \cdot \alpha = 0\}.$$

Notice that the v–wall associated to $D \in W_v$ is an hyperplane in $Amp(A)$. By Theorem 4.C.2 of [47] the subset $\bigcup_{D \in W_v} W^D \subseteq Amp(A)$ is locally finite.

Definition 1.3.6. A connected component of $Amp(A) \setminus \bigcup_{D \in W_D} W^D$ is called a $v-$chamber.

A $v-$chamber is then an open connected subcone of $Amp(A)$. Now these $v-$chambers are important as if a polarization is in a $v-$chamber, then it is $v-$generic as shown in the following:

Lemma 1.3.7. *Let $v = (v_0, v_1, v_2)$ be a Mukai vector such that $v_0 \geq 2$, and let $\mathcal{C}$ be a $v-$chamber. If $\theta \in \mathcal{C}$, then θ is $v-$generic.*

If θ is a v–generic polarization, then it is not necessarily contained in some $v-$chamber. In general the moduli space $M_v(A, \theta)$ depends on the choice of θ. But we know that $M_v(A, \theta)$ does not change when θ is $v-$generic polarization moving in the closure of a $v-$chamber (see [80, Proposition 2.8]).

In the following we talk about stability conditions, where A is always a projective abelian surface and θ an ample divisor on A. A torsion-free sheaf $\mathscr{F}$ on A is *θ–semistable* if it is Gieseker semistable with respect to θ, i.e. for all proper subsheaf $\mathscr{E} \subset \mathscr{F}$ we have that

$$rk(\mathscr{F})\chi(\mathscr{E}(n\theta)) \leq rk(\mathscr{E})\chi(\mathscr{F}(n\theta)), \quad for\ al\ n \gg 0. \tag{1.4}$$

If there exists $\mathscr{E} \subset \mathscr{F}$ such that the inequality is an equality then $\mathscr{F}$ is strictly semistable, otherwise it is stable. There is also the notion of *slope–(semi)stability*: if for all $\mathscr{E} \subset \mathscr{F}$ with $0 < rk\mathscr{E} < rk\mathscr{F}$

$$\mu(\mathscr{E}) := \frac{1}{rk\mathscr{E}} c_1(\mathscr{E}) \cdot \theta^{k-1} \leq \frac{1}{rk\mathscr{F}} c_1(\mathscr{F}) \cdot \theta^{k-1} := \mu(\mathscr{F}), \quad k = \dim A,$$

$\mathscr{F}$ is θ–slope semistable. It is θ–slope stable if the inequality is always strict. Writing out the polynomials appearing in 1.4 one shows that θ–semistability implies θ–slope semistability, and θ–slope stability implies θ–stability. We recall that the moduli space of semistable torsion-free sheaves parametrize S–*equivalence* classes of such sheaves $[\mathscr{G}]$. To define S–equivalence one associates to a semistable sheaf $\mathscr{F}$ a direct sum of stable shaves $Gr(\mathscr{F})$, and then declares that $\mathscr{F}_1$ is S–equivalent to $\mathscr{F}_2$ if $Gr(\mathscr{F}_1) \cong Gr(\mathscr{F}_2)$. If rk($\mathscr{F}$)=2, we have $Gr(\mathscr{F}) = \mathscr{F}$, if $\mathscr{F}$ is stable and $Gr(\mathscr{F}) = \mathscr{L} \oplus (\mathscr{F}/\mathscr{L})$, if $\mathscr{F}$ is strictly semistable, and $\mathscr{L} \subset \mathscr{F}$ destabilizes. If $\mathscr{F}$ is a semistable sheaf we let $[\mathscr{F}]$ be its S–equivalence class.

Let θ be a $v-$generic polarization and v a Mukai vector on A. We write $M_v(A,\theta)$ (resp. $M_v^s(A,\theta)$) for the moduli space of θ-semistable (resp θ-stable) sheaves on A with Mukai vector v. In this setting we refer to the choice of Mukai vector due to O'Grady, $v = 2w$ where $w^2 = 2$, $w = (1, 0, -1)$ is a primitive Mukai vector on A. It is known that if $M_v^s \neq \emptyset$, then M_v^s is smooth, quasi-projective, of dimension v^2+2 and carries a symplectic form (see Mukai [66]). Since A is abelian, a further construction is necessary: choose $\mathscr{F}_0 \in M_v(A,\theta)$, and define $a_v : M_v(A,\theta) \longrightarrow A \times A^\vee$in the following way (see [98]): let $p_{A^\vee} : A \times A^\vee \longrightarrow A^\vee$ be the projection and $\mathscr{P}$ the Poincaré bundle on $A \times A^\vee$. For every $\mathscr{F} \in M_v(A,\theta)$, we let

$$a_v(\mathscr{F}) := (det(p_{A^\vee!}((\mathscr{F} - \mathscr{F}_0) \otimes (\mathscr{P} - \mathscr{O}_{A\times A^\vee})), det(\mathscr{F}) \otimes det(\mathscr{F}_0)^{-1}).$$

Moreover we define $K_v(A,\theta) := a_v^{-1}(0_A, \mathscr{O}_A)$, where 0_A is the zero of A.
We recall the following crucial result in the case v is a primitive Mukai vector:

Theorem 1.3.8 (Mukai, Yoshioka)**.** *Let A be an abelian surface, v a primitive Mukai vector and and θ a v-generic polarization. Then $M_v(A,\theta) = M_v^s(A,\theta)$. If $v^2 \geq 6$ then $K_v(A,\theta)$ is an irreducible symplectic variety of dimension $2n = v^2 - 2$, which is deformation equivalent to $K^n(A)$, the generalized Kummer variety of A, and there is a Hodge isometry between $v^\perp$ and $H^2(K_v,\mathbb{Z})$.*

If v is not primitive, which is the case we are interested in, then M_v can be singular: in view of this result we search for a moduli space containing points parametrizing strictly semistable sheaves, and singular at these points, admitting a symplectic desingularization, in the hope that the desingularization is a new irreducible symplectic variety. This is what was done to produce the new 10-dimensional O'Grady example [77], the moduli space being that of certain sheaves on K3. For the six-dimensional case we consider the moduli space of sheaves on an abelian surface, describe as follows. Let C be a smooth irreducible projective curve of genus two and $\mathcal{J} := Pic^0(C)$. We set $v := 2 - 2\eta\mathcal{J}$, where $\eta\mathcal{J} \in H^4(\mathcal{J};\mathbb{Z})$ is the orientation class of $\mathcal{J}$. Let M_v be the moduli space $M_v(\mathcal{J},\Theta)$, where Θ is a Theta divisor. Many of the results that we find in [77] for the moduli space M_v of torsion-free semistable rank-two sheaves on a K3 with $c_1 = 0$, $c_2 = 4$, remain valid for M_v, provided one makes the technical assumption established in equation 1.3

$$\text{There is no divisor } D \text{ on } \mathcal{J} \text{ such that } D\cdot\Theta = 0 \text{ and } D\cdot D = (-2).$$

One such result says that the singular locus of M_v coincides with the set of S-equivalence classes of strictly semistable sheaves, i.e. equivalent to $I_{p_1}\otimes\xi_1 \oplus I_{p_2}\otimes\xi_2$, where $p_i \in \mathcal{J}$ and $\xi_i \in \widehat{\mathcal{J}}$ ($\widehat{\mathcal{J}} :=$ Pic($\mathcal{J}$)). Most importantly, the procedure of [77] carries over to give a symplectic desingularization $\widetilde{\pi_v} : \widetilde{M_v} \to M_v$; we let $\widetilde{\omega_v}$ be the symplectic form on M_v. The variety M_v is of pure dimension 10 (see [76, Theorem 2.1.4]). It is not symplectically irreducible: consider the following map

$$a_v : M_v \to \mathcal{J} \times \widehat{\mathcal{J}}$$

$$[F] \mapsto (\sum c_2(F), [\det F]).$$

where $\sum c_2(F)$ (the Albanese map) is the sum of the points (with multiplicities) of any representative of $c_2(F) \in CH_0(\mathcal{J})$. Set $\widetilde{a_v} := a_v \circ \widetilde{\pi_v}$. As is easily checked $\widetilde{a_v}$ is surjective, hence M_v is not symplectically irreducible. Hence we consider the fiber

$$\widetilde{K}_v := \widetilde{a_v}^{-1}(0, \widehat{0}), \quad \widetilde{\omega} := \widetilde{\omega}_v|_{\widetilde{K}_v}.$$

The result of O'Grady is the following:

Theorem 1.3.9 (O'Grady)**.** *Keep assumptions as above, $\widetilde{K}_v$ is a six dimensional irreducible symplectic variety, i.e simply connected and with $H^{2,0}(\widetilde{K}_v)$ spanned by the symplectic form $\widetilde{\omega}$. furthermore $b_2(\widetilde{K}_v) = 8$. The deformation type of these manifolds is called OG_6.*

1.4 Moduli space of stable objects

We need to recall basic definitions and facts about moduli space of sheaves and Bridgeland stable objects on $K3$ and abelian surfaces. These results will be useful in Chapter 3. For many details we can refer to the work of Bridgeland [22].

Let S be a projective K3 surface. Mukai defined a lattice structure on $H^*(S, \mathbb{Z})$ by setting

$$(r_1, l_1, s_1)(r_2, l_2, s_2) = l_1 \cdot l_2 - r_1 s_2 - r_2 s_1,$$

where $r_i \in H^0$, $l_i \in H^2$ and $s_i \in H^4$. This lattice is referred to as the *Mukai lattice* and we call vectors $v \in H^*(S, \mathbb{Z})$ *Mukai vectors.* The Mukai lattice is isomorphic to Λ_{24}, the unique, up to isometry, even unimodular lattice of signature $(4, 20)$.

Furthermore we may introduce a weight-2 Hodge structure on $H^*(S, \mathbb{Z})$ by defining the $(1,1)$-part to be

$$H^{1,1}(S) \oplus H^0(S) \oplus H^4(S).$$

For an object $\mathcal{F} \in D^b(S)$, we define the *Mukai vector* of $\mathcal{F}$ by

$$v(\mathcal{F}) := \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_S} = (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), \mathrm{ch}_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F}))$$

It is of $(1,1)$-type and satisfies

(1) Either $r > 0$

(2) or r=0 and $l \neq 0$ effective

(3) or r=l=0 and $s > 0$.

Definition 1.4.1. A non-zero vector $v \in H^*(S, \mathbb{Z})$ satisfying $v^2 \geq 2$ and the conditions above is called a *positive* Mukai vector.

With this definition we can easily deduce the following lemma.

Lemma 1.4.2. *Let $v \in H^*(S,\mathbb{Z})$ be non-zero and of $(1,1)$-type satisfying $v^2 \geq 2$. Then either v or $-v$ is a positive Mukai vector.*

Let us now review some results on the birational geometry of moduli spaces of bridgeland stable objects on a K3 surface. Let S be a projective K3 surface and fix two classes $\beta, \omega \in \mathrm{NS}(S)_{\mathbb{R}}$. To these data, Bridgeland associates a stability condition $\tau := \tau_{\beta,\omega}$ on the derived category $D^b(S)$. The set of all such stability conditions $\tau_{\beta,\omega}$ is denoted by $\mathrm{Stab}(S)$. Next, we fix a primitive positive Mukai vector $v \in H^*(S,\mathbb{Z})$ and assume that τ is generic with respect to v. The coarse moduli space $M_\tau(v)$ of τ-stable objects of Mukai vector v is a projective manifold of $K3^{[n]}$ type [7, Theorem 3], and we have an isometry of weight-2 Hodge structures

$$H^2(M_\tau(v),\mathbb{Z}) \xrightarrow{\sim} v^\perp \subset H^*(S,\mathbb{Z}).$$

Bayer and Macrì studied the birational geometry of these moduli spaces: they introduced a chamber structure in $\mathrm{Stab}(S)$. This fact is summarized in the following:

Theorem 1.4.3. *(i) If τ and τ' are v-generic stability conditions then $M_\tau(v)$ and $M_{\tau'}(v)$ are birational.*

(ii) There is a surjective map

$$l : \mathrm{Stab}(S) \longrightarrow \mathrm{Mov}(M_\tau(v))$$

mapping every chamber of $\mathrm{Stab}(S)$ *onto a Kähler-type chamber such that for a generic τ' the moduli space $M_{\tau'}(v)$ is the birational model of $M_\tau(v)$ corresponding to the chamber containing $l(\tau')$.*

Note that, for every positive Mukai vector, at least one chamber in $\mathrm{Stab}(S)$ contains stability conditions $\tau_{\beta,\omega}$ whose stable objects are (up to a shift) stable sheaves in the sense of Gieseker.

Let $f : H^2(M_\tau(v),\mathbb{Z}) \to \Lambda$ be a marking, and denote by $\mathcal{P}$ the period map (restricted to a connected component of the moduli space of marked manifolds). The above theorem implies in particular, that every manifold in the fiber

$$\mathcal{P}^{-1}(\mathcal{P}(M_\tau(v),\mathbb{Z}))$$

is again a moduli space of stable objects on S with the same Mukai vector v. We even have the following stronger result.

Corollary 1.4.4. *Let X and X' be Hodge isometric manifolds of $K3^{[n]}$ type. Then X is a moduli space of stable objects on a K3 surface if and only if the same holds for X'.*

Proof. See [64, Corollary 2.30]. □

At least we need to recall the remark 2.31 of [64].

Remark 1.4.5. A very similar construction can be done in the case when A is an abelian surface. Again, $M_\tau(v)$ is a projective manifold, and the fiber $K_\tau(v)$ of the Albanese map $M_\tau(v) \to A \times A^\vee$ is of Kummer type [100, Theorem 1.9]. Again, the second cohomology of $K_\tau(v)$ is Hodge isometric to $v^\perp$, and we have an analogous result as in Theorem 1.4.3. The only important difference is the following: by [85, Lemma 3], for every 2-torus A there is a Hodge isometry g to its dual $A^\vee$. Thus the moduli space of marked 2-tori has four connected components (corresponding to (A, f), $(A, f \circ -Id)$, $(A^\vee, f \circ g)$ and $(A^\vee, f \circ g \circ -Id)$, where f is some marking of A). For every Mukai vector $v = (r, l, s) \in H^*(A, \mathbb{Z})$, we define its dual as $v^\vee := (r, g^*l, s)$. We see immediately that v is positive if and only if$v^\vee$ is positive, and the corresponding Albanese fibers $K_\tau(v)$ and $K_{\tau^\vee}(v^\vee)$ are Hodge isometric. (Here $\tau^\vee$ is the dual stability condition on $A^\vee$ defined in the obvious way). Note that in general $K_\tau(v)$ and $K_{\tau^\vee}(v^\vee)$ are not birational [69], but we, again, see that the moduli space of marked manifolds of Kummer type has (at least) four components. Summarizing, we can state that the above corollary holds also for manifolds of Kummer n type if n + 1 is a prime power.

1.4.1 A birational model for O'Grady's sixfolds

In this section we recall the construction of the birational model for manifolds of OG_6 type introduced in [62] to compute its Hodge numbers and its Chern numbers. The Hodge diamond of OG_6 type manifolds (see [62, Theorem 1.1]) is the following:

$$
\begin{array}{ccccccc}
 & & & H^{0,0} = 1 & & & \\
 & & H^{2,0} = 1 & H^{1,1} = 6 & H^{0,2} = 1 & & \\
 & H^{4,0} = 1 & H^{3,1} = 12 & H^{2,2} = 173 & H^{1,3} = 12 & H^{0,4} = 1 & \\
H^{6,0} = 1 & H^{5,1} = 6 & H^{4,2} = 173 & H^{3,3} = 1144 & H^{2,4} = 173 & H^{1,5} = 6 & H^{0,6} = 1 \\
 & H^{4,0} = 1 & H^{3,1} = 12 & H^{2,2} = 173 & H^{1,3} = 12 & H^{0,4} = 1 & \\
 & & H^{2,0} = 1 & H^{1,1} = 6 & H^{0,2} = 1 & & \\
 & & & H^{0,0} = 1 & & &
\end{array}
$$

When A is a general principally polarized abelian surface, the moduli space $M_v(A, \Theta)$ parametrizes pure 1-dimensional sheaves, and the IHS manifold $\widetilde{K}_v(A, \Theta)$ is the image of a surjective dominant degree 2 rational map whose domain is an IHS manifold of $K3^{[n]}$ type as we now recall. For more details we refer to [62].
Let $\Theta \in \mathrm{NS}(A)$ be a principal polarization on A. the Mukai vector $v_0 = (0, \Theta, 1)$ satisfies $v_0^2 = 2$, and hence, a natural support morphism $K_v \to |2\Theta| = \mathbb{P}^3$, realizing K_v as a Lagrangian fibration. By definition of K_v, the fiber over a smooth curve $C \in |2\Theta|$ is the kernel of the natural morphism $Pic^6(C) \to A$ (which is also the restriction of a_v to $Pic^6(C) \subset M_v(A, \Theta)$).It is well known that the morphism associated to the linear system $|2\Theta|$ is the quotient morphism $A \to A/\pm 1 \subset \mathbb{P}^3$ onto the singular Kummer surface of A. Let $S \to A/\pm 1$ be the minimal resolution of A. It is well known that S, the Kummer surface of A is a K3 surface. Notice that S comes

naturally equipped with the degree 4 nef line bundle D obtained by pulling back the hyperplane section of $A/\pm 1 \subset \mathbb{P}^3$. Consider the diagram

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\ a\ } & S \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{\ q\ } & A/\pm 1 \end{array}$$

where $\widetilde{A}$ is the blow up of A at its 16 2-torsion points or, equivalently, the ramified cover of S along the exceptional curves $E_1, \dots, E_{16}$ of p. Consider the moduli space $M_w(S)$ of sheaves on S with Mukai vector $w = (0, D, 1)$that are stable with respect to a choosen, sufficiently general polarization. This is an IHS manifold birational to the Hilbert cube of S and it has a natural morphism $M_w(S) \to |D| = \mathbb{P}^3$ realizing it as the relative compactified Jacobian of the linear system $|D|$ (also a Lagrangian fibration).

The morphisms in diagram 4 induce a rational generically $2:1$ map

$$b_* a^* = q^* p_* : M_w(S) \dashrightarrow M_w(A, \Theta).$$

Since $M_w(S)$ is simply connected, the image of this map lies in a fiber of a_v, giving a $2:1$ morphism $\Phi : M_w(S) \dashrightarrow K_v(A, \Theta)$. On the smooth fibers, this map restricts to the natural $2:1$ pull back morphism $Pic^3(C') \to Pic^6(C)$, whose image is precisely $\ker[Pic^6(C) \to A]$. Recall that $\sum_i E_i$ is divisible by two in $H^2(S, \mathbb{Z})$ and that the line bundle $\eta := \mathcal{O}_S(\frac{1}{2}\sum_i E_i)$ determines the double cover q. It follows that the involution on $M_w(S)$ corresponding to Φ is given by tensoring by η and $\widetilde{K}_v(A, \Theta)$ is a birational model of the "quotient" of $M_w(S)$ by the birational involution induced by tensorization by η.
What it is shown in [62] is that, for any abelian surface A and for an effective Mukai vector $v = 2v_0$ with $v_0^2 = 2$ on A, $\widetilde{K}_v(A, \Theta)$ admits a rational double cover from an IHS manifold $\underline{Y}_v(A, \Theta)$ of $K3^{[3]}$ type. Recall that the singular locus $\Sigma_v \subset K_v(A, \Theta)$ has codimension 2 and can be identified with $A \times A^\vee / \pm 1$ (see [62, section2]). Following [76], the symplectic resolution

$$\widetilde{K}_v(A, \Theta) \to K_v(A, \Theta)$$

can be obtained by two subsequent blow ups followed by a contraction: first one blows up the singular locus of Σ_v, then one blows up the proper transform of Σ_v itself (which is smooth); these two operations produce a manifold $\widehat{K}_v(A, \Theta)$ that has a holomorphic two form degenerating along the strict transform of the exceptional divisor of the first blow up; contracting this exceptional divisor finally gives the manifold $\widetilde{K}_v(A, \Theta)$ that has non-degenerate (hence symplectic) two form and a regular morphism $\widetilde{K}_v(A, \Theta) \to K_v(A, \Theta)$ which is, therefore, a symplectic resolution. The inverse image $\widetilde{\Sigma}$ of Σ_v in $\widetilde{K}_v(A, \Theta)$ is a smooth divisor, which is divisible by two in the integral cohomology by [81]. In [62] the authors show that the associated ramified double cover is a smooth manifold birational to an IHS manifold of $K3^{[3]}$ type, $\underline{Y}_v(A, \Theta)$, and which is equipped with a birational symplectic involution.

This enable us to reconstruct $\widetilde{K}_v(A, \Theta)$ starting from $\underline{Y}_v(A, \Theta)$ and its birational symplectic involution

$$\underline{\tau}_v : \underline{Y}_v(A, \Theta) \to \underline{Y}_v(A, \Theta).$$

More specifically, $\underline{Y}_v(A,\Theta)$ contains 256 $\mathbb{P}^3$s, the birational involution $\underline{\tau}_v$ is regular on the complement of these $\mathbb{P}^3$s, and, moreover, this involution lifts to a regular involution on the blow up $\overline{Y}_v(A,H)$ of $\underline{Y}_v(A,H)$ along the 256 $\mathbb{P}^3$'s. The fixed locus of the induced involution on $\overline{Y}_v(A,\Theta)$ is smooth and four dimensional, hence the blow up $\widehat{Y}_v(A,\Theta)$ of $\overline{Y}_v(A,\Theta)$ along this fixed locus carries an involution $\widehat{\tau}_v$ admitting a smooth quotient $\widehat{Y}_v(A,\Theta)/\widehat{\tau}_v$. This quotient is $\widehat{K}_v(A,\Theta)$ and $\widehat{Y}_v(A,\Theta)$ is its double cover branched over $\widehat{\Sigma}_v$. Finally $\widehat{K}_v(A,\Theta)$ is the blow up of $\widetilde{K}_v(A,\Theta)$ along 256 smooth 3–dimensional quadrics.

1.4.2 The local covering

If we fix a primitive Mukai vector $v_0 \in H^*_{alg}(A,\mathbb{Z})$ with $v_0^2 = 2$ and we set $v = 2v_0$, and we consider a v-generic ample line bundle H on A (see Section 2.1 of [80]). By [87, Theorem 1.1] the projective variety $K_v := K_v(A,H)$ admits a symplectic resolution $\widetilde{K}_v$ which is deformation equivalent to O'Grady's six dimensional example by [87, Theorem 1.6(2)]. In this section we recall the description of the singularity of K_v.

Since the singular locus Σ_v of K_v parametrizes polystable sheaves of the form $F_1 \oplus F_2$, with $F_i \in M_{v_0}(A,H)$, we have $\Sigma_v = K_v \cap Sym^2 M_{v_0}(A,H)$. Since $v_0^2 = 2$ the smooth moduli space M_{v_0} is isomorphic to $A \times A^\vee$ and, as the Albanese map alb is an isotrivial fibration, the singular locus Σ_v is isomorphic to $A \times A^\vee/\pm 1$. Since A has 16 two-torsion points, then the same holds for $A^\vee$, and consequently wa have $16 \cdot 16 = 256$ singular points in the quotient $A \times A^\vee/\pm 1$. For this reason that the singular locus Ω_v of Σ_v consists of 256 points representing sheaves of the form $F^{\oplus 2}$ with $F \in M_{v_0}(A,H)$.

The analytic type o the singularities appearing in K_v is completely known. If $p \in \Sigma_v \setminus \Omega_v$, i.e. p represents a polystable sheaf of the form $F_1 \oplus F_2$ where $F_1 \neq F_2$, there exists a neighborhood $U \subset K_v$ of p, in the classical topology, biholomorphic to a neighborhood of the origin in the hypersurface defined in $\mathbb{A}^7$ by the equation $\sum_{i=1}^3 x_i^2 = 0$ (see for example [1, Proposition 4.4] or [77, Proposition 1.4.1]), i.e. K_v has an A_1 singularity along $\Sigma_v \setminus \Omega_v$.

If $p \in \Omega_v$, the description of the analytic type of the singularity of K_v at p is due to Lehn and Sorger and it is contained in [87, Theorem 4.5]. To recall this description, let V be a four dimensional vector space, let σ be a symplectic form on V, and let $\mathfrak{sp}(V)$ be the symplectic Lie Algebra of (V,σ), i.e. the Lie algebra of the Lie group of the automorphisms of V preserving the symplectic form σ.

We let

$$Z := \{A \in \mathfrak{sp}(V) | A^2 = 0\}$$

be the subvariety of matrices of $\mathfrak{sp}(V)$ having square zero. It is known that Z is the closure of the nilpotent orbit of type $\mathfrak{o}(2,2)$, which parametrizes rank 2 square zero matrices. Moreover, by Criterion 2 of [41], Z is also a normal variety.

By [87, Theorem 4.5] if $p \in \Omega_v$, there exists an euclidean neighborhood of p in K_v, biholomorphic to a neighborhood of the origin in Z.

Let Σ be the singular locus of Z and let Ω be the singular locus of Σ. Let us recall that $\dim Z = 6$, $\dim \Sigma = 4$, $\dim \Omega = 0$ and, more precisely,

$$\Sigma = \{A \in Z | \operatorname{rk}(A) \leq 1\}, \quad \text{and} \quad \Omega = \{0\}.$$

Let $G \subset Gr(2,V) \subset \mathbb{P}(\wedge^2 V)$ be the Grassmanian of Lagrangian subspaces of V, notice that G is a smooth 3-dimensional quadric and set

$$\widetilde{Z} := \{(A,U) | A(U) = 0\} \subset Z \times G.$$

The restriction $\pi_G : \widetilde{Z} \to G$ of the second projection of $Z \times G$ makes $\widetilde{Z}$ the total space of a 3-dimensional vector bundle, the cotangent bundle of G. In particular $\widetilde{Z}$ is smooth and the restriction

$$f : \widetilde{Z} \to Z$$

of the first projection of $Z \times G$, which is an isomorphism when restricted to the locus of rank 2 matrices, is a resolution of the singularities. The fiber $f^{-1}(A)$ over a point $A \in \Sigma$, is a smooth $\mathbb{P}^1$ parametrizing Lagrangian subspaces contained in the 3-dimensional kernel of A and the central fiber $f^{-1}(0)$ is the whole G. As Z has an A_1 singularity along $\Sigma \setminus \Omega$ and G has dimension 3 it follows that $f : \widetilde{Z} \to Z$ is a symplectic resolution.

Now we are ready to recall Section 3 of [62] which is devoted to the local description of the double cover, branched along the singular locus, of O'Grady's singularity. It is known [26, Corollary 6.1.6] that the fundamental group of the open orbit $\mathfrak{o}(2,2)$ is isomorphic to $\mathbb{Z}/(2)$. We wish to extend this double cover to a ramified double cover of $\overline{\mathfrak{o}(2,2)} = Z$.

To this aim let

$$W := \{v \otimes w |\ \sigma(v,w) = 0\} \subset V \otimes V, \text{ and } \Delta_W = \{v \otimes v \text{ such that } v \in V\} \subset W$$

be the affine cover of the incidence subvariety

$$I := \{([v],[w]) |\ \sigma(v,w) = 0\} \subset \mathbb{P}(V) \times \mathbb{P}(V) \subset \mathbb{P}(V \times V).$$

Since I is smooth, the singular locus Γ of W consist only of the vertex $0 \in V \otimes V$. Moreover, since $I \subset \mathbb{P}(V \otimes V)$ is projectively normal, W is a normal variety.

Let

$$\tau : W \to W$$

be the involution induced by restricting the linear involution $\tau_{V \otimes V}$ on $V \otimes V$ that interchanges the two factors.

The following lemma exhibits W as the desired double cover of Z.

Lemma 1.4.6. *The morphism*

$$\varepsilon : W \longrightarrow Z$$

$$v \otimes w \longmapsto \sigma(v,.)w + \sigma(w,.)v$$

realizes Z as the quotient W/τ. In particular, ε is a finite $2:1$ morphism, the ramification locus of ε is Δ and the branch locus of ε is Σ.

Proof. See [62, Lemma 3.1].

□

1.5 Automorphisms

Let X be an IHS manifold, we denote by $\mathrm{Aut}(X)$ the group of automorphisms of X (biholomorphic maps from X to X) and by $\mathrm{Bir}(X)$ the group of bimeromorphic automorphisms. Clearly $\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$.

Theorem 1.5.1. *Let X be an IHS manifold and let $\eta : H^2(X, \mathbb{Z}) \to L$ be a marking. For a very general point $(X, \eta) \in \mathcal{M}_L$ we have* $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.

Proof. See [43, Proposition 9.2]. □

For all compact complex manifolds we have

$$\dim(\mathrm{Aut}(X)) = h^0(TX)$$

and, if X is an IHS manifold, $\dim(\mathrm{Aut}(X)) = h^{1,0}(X) = 0$, meaning that $\mathrm{Aut}(X)$ is a discrete group. We also to recall that, from [18, Theorem 2], we know that if X is projective, then $\mathrm{Bir}(X)$ is finitely generated. It is well defined the following homomorphism:

$$\nu : \mathrm{Bir}(X) \to O(H^2(X, \mathbb{Z}))$$
$$f \mapsto (f^*)^{-1}$$

where $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the pull-back of f. It preserves the Beauville-Bogomolov quadratic form. We know the following properties about ν (see [43, Proposition 9.1]):

(i) $\nu(\mathrm{Bir}(X)) = Mon^2_{Bir}(X) \subset Mon^2_{Hdg}(X)$;

(ii) $\nu(\mathrm{Aut}(X)) = \{g \in Mon^2_{Hdg}(X) | g(\mathcal{K}_X) \cap \mathcal{K}_X \neq \emptyset\}$;

(iii) $\nu^{-1}(\nu(\mathrm{Aut}(X))) = \mathrm{Aut}(X)$;

(iv) $\ker(\nu) \subset Aut(X)$ is finite.

These properties are consequences of Theorem 1.2.34.

By result of Hasset and Tschinkel, [39, theorem 2.1], the kernel of the homomorphism ν is invariant under smooth deformations of the manifold X and it has been computed for all known deformation types of IHS manifolds.

Theorem 1.5.2. *Let X be a manifold of OG_6 type. Then $Ker(\nu) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$.*

Proof. See [65, Thm 4.2]. □

In addition to the action on $H^2(X, \mathbb{Z})$, another important invariant of an automorphism $f \in Aut(X)$ is its action on $H^0(X, \Omega^2_X)$, i.e on the generator of the complex space $H^{2,0}(X)$, the symplectic form ω_X of the IHS manifold X. Since f^* is a Hodge isometry, and since $H^{2,0} \cong \mathbb{C}\omega_X$, i.e. it is a complex space of dimension 1, the action is forced to be $f^*(\omega_X) = \xi\omega_X$ where $\xi \in \mathbb{C}^*$; moreover, if f is of finite order m then $\xi^m = 1$.

Definition 1.5.3. An automorphism $f \in \mathrm{Aut}(X)$ is called *symplectic* if $f^*(\omega_X) = \omega_X$; otherwise f is called *non-symplectic.*

Remark 1.5.4. From [10, Proposition 6], if there exists a non-symplectic $f \in Aut(X)$, then the IHS manifold X is projective.

The following proposition is due to Nikulin:

Proposition 1.5.5. *Let X be an IHS manifold, suppose G is cyclic and generated by a non-symplectic element of maximal order m. Let ϕ be the Euler function. It holds that $\phi(m) \mid rk(T(X))$, i.e. $rk(T(X)) = \phi(m)n$ for some $n \in \mathbb{N} \setminus \{0\}$. In particular $\phi(m) \leq b_2(X) - \mathrm{rk}(NS(X))$.*

Definition 1.5.6. Let $f \in Aut(X)$ be an automorphism of finite order of an IHS manifold. The *invariant* lattice of f is

$$T_f(X) = H^2(X, \mathbb{Z})^{f^*} = \{u \in H^2(X, \mathbb{Z}) | f^*(u) = u\}$$

and the *co-invariant* lattice of f is

$$S_f(X) = (H^2(X, \mathbb{Z})^{f^*})^{\perp} \subset H^2(X, \mathbb{Z}).$$

Both $T_f(X)$ and $S_f(X)$ are primitive sublattices of $H^2(X, \mathbb{Z})$, since they can be expressed as kernels of lattice isometries: in particular if $m \in \mathbb{N}$ is the order of f, we have

$$T_{f(X)} = \ker(f^* - id), \qquad S_f(X) = \ker(id + f^* + \ldots + (f^*)^{m-1}) \tag{1.5}$$

We recall the following definitions of Neron-Severi and transcendental lattice.

Definition 1.5.7. The Neron-Severi lattice is the algebraic $(1,1)$-part of $H^2(X, \mathbb{C})$ i.e. $\mathrm{NS}(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$.
The transcendental lattice $\mathrm{T}(X)$ is the orthogonal complement of $\mathrm{NS}(X)$ in the second integral lattice, i.e. $\mathrm{T}(X) := \mathrm{NS}(X)^{\perp}$ and it is the smallest primitive sublattice of $H^2(X, \mathbb{Z})$ such that

$$H^{2,0}(X) \subset T(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$

It holds that

$$H^2(X, \mathbb{Z}) \otimes \mathbb{Q} = \mathrm{NS}(X) \otimes \mathbb{Q} \oplus \mathrm{T}(X) \otimes \mathbb{Q}.$$

In the following we explain the relative positions of the lattices $T_f(X)$, $S_f(X)$, with respect to $\mathrm{T}(X)$ and $\mathrm{NS}(X)$.

Proposition 1.5.8. *Let X be an IHS manifold and let $f \in Aut(X)$.*

(i) If f is symplectic , then $\mathrm{T}(X) \subset T_f(X)$ and $S_f(X) \subset \mathrm{NS}(X)$ and $S_f(X)$ is negative definite.

(ii) If f is non-symplectic, then $T_f(X) \subset \mathrm{NS}(X)$, $\mathrm{T}(X) \subset S_f(X)$ and $T_f(X)$ is hyperbolic.

Proof. We consider the two cases separately.

(i) If $x \in S_f(X)$, from equation 1.5, we have

$$(\sum_{i=0}^{m-1}(f^*)^i(x), \omega_X) = \sum_{i=0}^{m-1}((f^*)^i(x), \omega_X) = m(x, \omega_X),$$

where we use the fact that f^* is an isometry of $H^2(X, \mathbb{Z})$ and $f^*(\omega_X) = \omega_X$. As a consequence we have that $(x, \omega_X) = 0$ and for this reason $x \in \mathrm{NS}(X)$, since we know that $\mathrm{NS}(X) = \omega_X^\perp \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$; using the orthogonality we have $\mathrm{T}(X) \subset T_f(X)$. It is easy to see that there exists an invariant ample class, which is $\beta := \sum_{i=0}^{m-1}(f^*)^i(w)$ where $w \in \mathcal{K}_X$ is the Kähler class of X. Therefore we have

$$\mathbb{C}\omega_X \oplus \mathbb{C}\overline{\omega_X} \oplus \mathbb{C}\beta \subset T_f(X) \otimes \mathbb{C} \subset H^2(X, \mathbb{C}).$$

This means that there is a three-dimensional positive defined space in $T_f(X)$ and thus $S_f(X)$ is negative definite.

(ii) If f is non-symplectic and $f^*(\omega) = \xi\omega$, $\xi \neq 1$, we have that for any $x \in T_f(X)$

$$(x, \omega) = (f^*(x), f^*(\omega)) = (x, \xi\omega) = \xi(x, \omega) \Rightarrow (x, \omega) = 0.$$

This means that $T_f(X) \subset \mathrm{NS}(X)$ and taking the orthogonal we have $\mathrm{T}(X) \subset S_f(X)$. As we already remarked, the existence of a non-symplectic automorphism implies that X is projective, hence $\mathrm{NS}(X)$ is hyperbolic by Theorem 1.2.13. Since the positive class β, defined in the symplectic case, is still contained in $T_f(X) \otimes \mathbb{C}$, we conclude that $T_f(X) \subset \mathrm{NS}(X)$ is also hyperbolic.

□

In our work, we will be mainly interested in automorphisms of manifolds OG_6 type. In particular we will focus on automorphisms which act non-trivially on the second integral cohomology. Moreover we will exhibit some results about induced automorphisms, which means automorphisms that come from the Abelian surface that we use to built OG_6 as a moduli space (see Section 1.3.1 for more details) and automorphisms induced at the quotient, i.e. automorphisms which come from an automorphism of the $K3^{[3]}$ sixfolds that are used in the construction explained in Section 1.4.1.

To study automorphisms which act non-trivially on cohomology we need the following definitions:

Definition 1.5.9. Let ν be the homomorphism of groups:

$$\begin{aligned} \nu : Aut(X) &\longrightarrow O(H^2(X, \mathbb{Z})) \\ f &\longmapsto (f^*)^{-1} \end{aligned}$$

$\varphi \in O(H^2(X, \mathbb{Z}))$ is called **effective** if and only if $\varphi \in Im(\nu)$.

There is an analogous version of the Hodge theoretic Torelli theorem for abelian surfaces. It is well known, essentially already contained in [85], that

$$Mon^2(A) = SO^+(H^2(A, \mathbb{Z})),$$

where the lattice structure on $H^2(A, \mathbb{Z})$ is given by the cap product. We recall that the intersection form on $H^2(A, \mathbb{Z})$ is unimodular, even and of signature $(3, 3)$, hence, by the classification, there exists an isometry $H^2(A, \mathbb{Z}) \cong U^{\oplus 3}$ and we have an isomorphism

$$Mon^2(A) \cong SO^+(U^{\oplus 3}).$$

Theorem 1.5.10. *Let A be an abelian surface and let $\varphi \in O(H^2(A, \mathbb{Z}))$ be a monodromy operator which is an isometry of Hodge structures, then φ is effective, i.e. there exists an automorphism $\varphi \in \mathrm{Aut}(A)$ such that $\widetilde{\varphi}^* = \varphi$, if and only if a Kähler class is preserved by φ.*

Another main result due to Huybrechts is the following:

Theorem 1.5.11. *If X and X' are birational projective irreducible holomorphic symplectic manifolds, then:*

- *X and X' are diffeomorphic.*
- *For all k the weight-k Hodge structures of X and X' are isomorphic.*

Definition 1.5.12. Let ν be the omomorphism of groups:

$$\nu : Bir(X) \longrightarrow O(H^2(X, \mathbb{Z}))$$
$$f \longmapsto (f^*)^{-1}$$

$\varphi \in O(H^2(X, \mathbb{Z}))$ is called **birational effective** if and only if $\varphi \in Im(\nu)$.

We introduce the following convention: an automorphism of X is said to be an *automorphism of order n* if the induced action on $H^2(X, \mathbb{Z})$ has order n.

Remark 1.5.13. if X is a manifold of OG_6 type, then the second integral cohomology has a lattice structure and it holds that

$$H^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}.$$

Moreover the discriminant group associated to the second integral cohomology is

$$A_X = H^2(X, \mathbb{Z})^* / H^2(X, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}.$$

Lemma 1.5.14. *Let X be a manifold of OG_6 type, let $\varphi \in O(H^2(X), \mathbb{Z})$ be an isometry such that the induced action on A_X is trivial. Then there exists an embedding*

$$H^2(X, \mathbb{Z}) \hookrightarrow \Lambda = U^{\oplus 5}$$

and an isometry $\overline{\varphi} \in O(\Lambda)$ such that $\overline{\varphi}|_{H^2} = \varphi$.

Proof. Let $[v_1/2]$ and $[v_2/2]$ be two generators of A_X such that $v_1^2 = -2$ and $v_2^2 = -2$. We then have $\varphi([v_1/2]) = [v_1/2]$ and $\varphi([v_2/2]) = [v_2/2]$ i.e. $\varphi(v_1) = v_1 + 2w_1$ and $\varphi(v_2) = v_2 + 2w_2$. Consider now a lattice of rank 2 generated by two orthogonal elements x_1 and x_2 of square 2, its discriminant group is still $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and is generated by $[x_1/2]$ and $[x_2/2]$ with discriminant form given by $q(x_1/2) = 1/2$, $q(x_2/2) = 1/2$ and $(x_1, x_2) = 0$. Notice that $H^2 \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_2$ has an overlattice isometric to Λ which is generated by H^2, $\frac{x_1+v_1}{2}$ and $\frac{x_2+v_2}{2}$. We now extend φ on

$H^2 \oplus x_1 \oplus x_2$ by imposing $\varphi(x_1) = x_1$, $\varphi(x_2) = x_2$ and we thus obtain an extension $\overline{\varphi}$ of φ on Λ, defined by the $\mathbb{Q}$-linear extension, as follows:

- $\overline{\varphi}(e) = \varphi(e) \ \forall \ e \in H^2$,
- $\overline{\varphi}(x_1) = x_1$,
- $\overline{\varphi}(x_2) = x_2$,
- $\overline{\varphi}(\frac{x_1+v_1}{2}) = \frac{x_1+\varphi(v_1)}{2}$,
- $\overline{\varphi}(\frac{x_2+v_2}{2}) = \frac{x_2+\varphi(v_2)}{2}$.

□

Corollary 1.5.15. *Let X be a manifold of OG_6 type, and $G \subset O(H^2(X,\mathbb{Z}))$ a finite group of isometries such that the induced action on A_X is trivial. Then there exists a primitive embedding*

$$H^2(X,\mathbb{Z}) \hookrightarrow \Lambda \cong U^{\oplus 5}$$

such that G extends to a group of isometries of Λ and $S_G(X) = S_G(\Lambda)$ i.e. the induced action on $(H^2)^\perp \subset \Lambda$ is trivial.

Proof. Let x_1 and x_2 be two vectors of square 2 and v_1 and v_2 two vectors of square -2 in H^2 such that $(v_1, H^2) = 2\mathbb{Z}$ and $(v_2, H^2) = 2\mathbb{Z}$. Let Λ be the overlattice of $H^2 \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_2$ generated by H^2 and $\frac{x_1+v_1}{2}$ and $\frac{x_2+v_2}{2}$ and let us extend the action of G to Λ as in Lemma 1.5.14. A direct computation shows that $S_G(H^2) = S_G(\Lambda)$. □

Chapter 2

A motivating example on K3 surfaces

The main goal of the thesis is to classify automorphisms of O'Grady six dimensional manifolds. The study of non-symplectic automorphisms of prime order on irreducible holomorphic symplectic manifolds was completed by several authors: Nikulin in [74], Artebani, Sarti and Taki in [2], [4] and [92]. These papers are related to automorphisms of K3 surfaces. As we know, these are the examples of IHS manifolds in dimension 2. To classify automorphisms of IHS manifolds of higher dimensions we need methods used for the classification of automorphisms of K3 surfaces. On them, non-symplectic automorphisms of prime order are already classified. If the automorphism is not of prime order the setting is more complicated. Indeed, in this situation the "generic" case does not imply that the action of the automorphism is trivial on the Picard group [33, Section 11]. In the paper [93], Taki studies the case when the order of the automorphism is a prime power and the action is trivial on the Picard group. If we consider non-symplectic, non-trivial automorphisms of order 2^t, then by results of Nikulin we have $1 \leq t \leq 5$. We can find some other results about this in a paper by Schütt in the case of automorphisms of a 2-power order [84] and in a paper by Artebani and Sarti in the case of order 4 [4]. Recently in [91] Al Tabbaa, Sarti and Taki completed the study for purely non-symplectic automorphisms of order 16 and in [90] Al Tabbaa and Sarti studied the case of order 8 under the assumptions that the fourth power of the automorphism is the identity on the Picard lattice and, if we denote the automorphism by σ, $\mathrm{Fix}(\sigma^4)$ contains an elliptic curve.
To begin to become familiar with these classification techniques, in this chapter we classify non-symplectic automorphisms of order 8 on a K3 surface with certain hypotheses on the fixed locus. This will be an illuminating example and will be really useful to learn the method of classification. In particular, this chapter deals with purely non-symplectic automorphisms of order eight on K3 surfaces under the assumption that their fourth power σ^4 is the identity on the Picard lattice. This corresponds to the situation for the generic K3 surface in the moduli space of K3 surfaces with non-symplectic automorphism of order 8 and fixed action on the second cohomology with integer coefficients, see [33, Section 10]. The fixed locus $\mathrm{Fix}(\sigma)$ of such an automorphism σ is the disjoint union of smooth curves and points. We will

deal with the case in which $\mathrm{Fix}(\sigma^4)$ is not empty and contains only rational curves.

2.1 Non-symplectic automorphisms of order 8 on a K3 surface

2.1.1 Basic facts

For this part we will refer to [89]. Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of order 8. We assume that $\sigma^*(\omega_X) = \zeta_8\omega_X$ where ζ_8 is a primitive 8th root of unity. Such a σ is called *purely non-symplectic*, for simplicity we just call it *non-symplectic*, always meaning that the action is by a primitive 8th root of unity. We denote by k_σ the number of smooth rational curves fixed by σ and by N_σ the numbers of isolated points in $\mathrm{Fix}(\sigma)$. We denote by $r_{\sigma^j}, l_{\sigma^j}, m_{\sigma^j}$ and m_1 for $j = 1, 2, 4$ the rank of the eigenspace of $(\sigma^j)^*$ in $H^2(X, \mathbb{C})$ relative to the eigenvalues 1, -1, i and ζ_8 respectively (clearly $m_{\sigma^4} = 0$). We recall the invariant lattice:

$$T(\sigma^j) = \{x \in H^2(X, \mathbb{Z}) | (\sigma^j)^*(x) = x\},$$

and its orthogonal complement

$$S(\sigma^j) = T(\sigma^j)^\perp \cap H^2(X, \mathbb{Z}).$$

Since the automorphisms act purely non-symplectically, X is projective, see [71, Theorem 3.1], so that if we denote $\mathrm{rk}\, T(\sigma^j) = r_{\sigma^j}$, we have that $r_{\sigma^j} > 0$ for all $j = 1, 2, 4$ (one can always find an invariant ample class). On the other hand, one can easily show that $T(\sigma^j) \subseteq \mathrm{Pic}(X)$ for $j = 1, 2, 4$ so that the transcendental lattice satisfies $\mathrm{T}(X) \subseteq S(\sigma^j)$ for $j = 1, 2, 4$.

Remark 2.1.1. It is a straightforward computation that the invariants $r_{\sigma^j}, l_{\sigma^j}, m_{\sigma^j}$ and m_1 with $j = 1, 2, 4$ satisfy the following relations:

$$\begin{aligned} r_{\sigma^2} &= r_\sigma + l_\sigma; \\ l_{\sigma^2} &= 2m_\sigma; \\ 2m_{\sigma^2} &= 4m_1; \end{aligned} \qquad \begin{aligned} r_{\sigma^4} &= r_\sigma + l_\sigma + 2m_\sigma; \\ l_{\sigma^4} &= 4m_1; \end{aligned}$$

We remark that the invariants l_{σ^2} and m_{σ^2} are even numbers.

The moduli space for $K3$ surfaces carrying a non-symplectic automorphism of even order n, $n \neq 2$, with a given action on the $K3$ lattice is known to be a complex ball quotient of dimension $q - 1$ where q is the rank of the eigenspace V of σ^* in $H^2(X, \mathbb{C})$ relative to the eigenvalues $\zeta_n = e^{\frac{2\pi i}{n}}$, see [33, §11]. The complex ball is given by:

$$B = \{[w] \in \mathbb{P}(V) : (w, \overline{w}) > 0\}.$$

If n is even V is the ζ_n eigenspace of σ^* in $S(\sigma^{n/2}) \otimes \mathbb{C}$. This implies that the Picard group of a $K3$ surface corresponding to the generic point in the moduli space equals $T(\sigma^{n/2})$ see [33, Theorem 11.2].

2.1.2 The fixed locus

We denote by $\mathrm{Fix}(\sigma^j)$, $j = 1, 2, 4$ the fixed locus of the automorphism σ^j such that

$$\mathrm{Fix}(\sigma^j) = \{x \in X |\ \sigma^j(x) = x\}.$$

Clearly $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\sigma^2) \subseteq \mathrm{Fix}(\sigma^4)$. To describe the fixed locus of order 8 non-symplectic automorphisms we start recalling the following result about non-symplectic involutions, see [74, Theorem 4.2.2].

Theorem 2.1.2. *Let τ be a non-symplectic involution on a K3 surface X. The fixed locus of τ is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $g \geq 0$ and k smooth rational curves. Moreover, its fixed lattice $T(\tau) \subset \mathrm{Pic}(X)$ is a 2-elementary lattice with determinant 2^a such that:*

- $T(\tau) \cong U(2) \oplus E_8(2)$ *iff the fixed locus of τ is empty;*
- $T(\tau) \cong U \oplus E_8(2)$ *iff τ fixes two elliptic curves;*
- $2g = 22 - \mathrm{rk}\, T(\tau) - a$ *and* $2k = \mathrm{rk}\, T(\tau) - a$ *otherwise.*

Since $T(\tau)$ is 2-elementary its discriminant group $A_{T(\tau)} = T(\tau)^\vee / T(\tau) \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$, $a \in \mathbb{Z}_{>0}$. We introduce the invariant $\delta_{T(\tau)}$ of $T(\tau)$ by putting $\delta_{T(\tau)} = 0$ if $x^2 \in \mathbb{Z}$ for any $x \in A_{T(\tau)}$ and $\delta_{T(\tau)} = 1$ otherwise. By [72, Theorem 3.6.2], and [83, §1] $T(\tau)$ is uniquely determined by the invariant $\delta_{T(\tau)}$, rank, signature and the invariant a. The situation is resumed in Figure 2.1 from [75, §4].

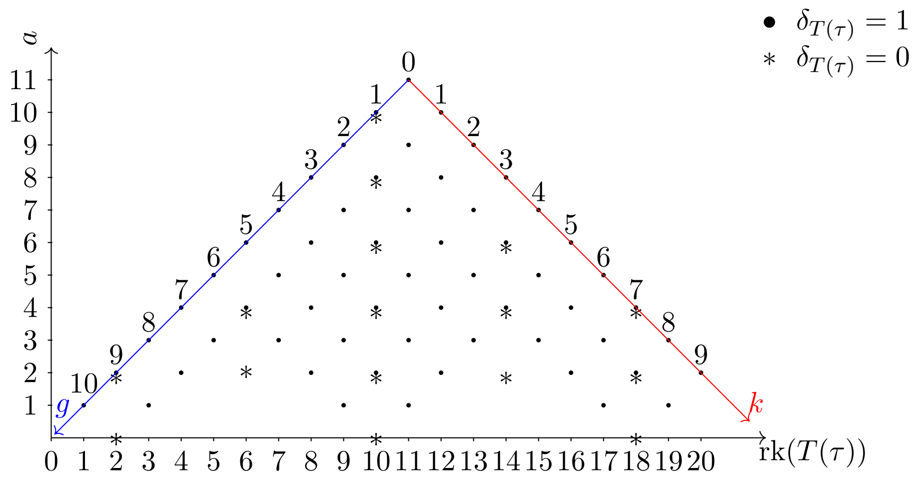


Figure 2.1: Order 2

We recall a result about non-symplectic automorphisms of order four on a K3 surface. This results are discussed in [3] and see also the Appendix of [91].

Theorem 2.1.3. *Let X be a K3 surface and σ be a purely non-symplectic automorphism of order four on it with $\mathrm{Pic}(X) = T(\sigma^2)$. If $\mathrm{Fix}(\sigma)$ contains a smooth rational curve and all curves fixed by σ^2 are rational, then the invariants associated to σ are* *as in Table 2.1. All cases in the table do exist (here m denotes the multiplicity of the eigenvalue i).*

m	r	l	n	k	a
4	10	4	6	1	0
3	13	3	8	2	0
	11	5	6	1	1
2	16	2	10	3	0
	14	4	8	2	1
	12	6	6	1	2
1	19	1	12	4	0
	13	7	6	1	3

Table 2.1: The case $g = 0$

From now on, σ will be an automorphism of order eight.

Remark 2.1.4. For each p_j, $j \in \{1, \ldots, N_\sigma\}$, fixed point for σ, there exists $i \in \{1, \ldots, t\}$ such that $p_j \in R_i'$, where R_i' is a fixed smooth rational curve for σ^4.

With the notation of the remark we have

Lemma 2.1.5. *The curve R_i' is σ-invariant for all $i \in \{1, \ldots, t\}$ and R_i' a smooth rational curve in* $\mathrm{Fix}(\sigma^4)$.

Proof. First of all we notice that since R_i' is fixed by σ^4 then also $\sigma(R_i')$ is fixed by σ^4. If R_i' is not σ-invariant this means that R_i' is sent to another rational curve $\sigma(R_i')$. We know by assumption that $x \in R_i'$ with x a fixed point, hence the intersection $R_i' \cap \sigma(R_i')$ is not empty. This is absurd in fact these two curves are fixed by σ^4 and so they can not intersect since the fixed locus of an involution is smooth by Theorem 2.1.2. □

We further denote by N_{σ^j}, k_{σ^j}, $j = 1, 2, 4$ the number of isolated points and smooth rational curves in $\mathrm{Fix}(\sigma^j)$. We observe that $N_{\sigma^4} = 0$ since σ^4 only fixes curves or is empty as explained in Theorem 2.1.2. Recall [90, Proposition 2.2] :

Proposition 2.1.6. *Let σ be a non-symplectic automorphism of order 8 acting on a K3 surface X. Then* $\mathrm{Fix}(\sigma)$ *is never empty nor it can be the union of two smooth elliptic curves. It is the disjoint union of smooth curves and $N_\sigma \geq 2$ isolated points. Moreover the following relations hold:*

$$n_{2,7} + n_{3,6} = 2 + 4\alpha, \quad n_{4,5} + n_{2,7} - n_{3,6} = 2 + 2\alpha, \quad N_\sigma = 2 + r_\sigma - l_\sigma - 2\alpha$$

where $n_{i,j}$ will be introduced in the following, after Remark 3.3.13.

Here we denote $\alpha = \sum_{K \subset Fix(\sigma)} (1 - g(K))$.

The fixed locus of an automorphism σ is then

$$\mathrm{Fix}(\sigma) = C \cup R_1 \cup \cdots \cup R_k \cup \{p_1, \ldots, p_N\} \tag{2.1}$$

where C is a smooth curve of genus $g \geq 0$ and R_i are rational curves.

We recall the following remarks and lemma which are important in the study of the fixed locus of σ.

Remark 2.1.7. A non-symplectic automorphism σ of order 8 acts on a set of smooth rational curves of X which are fixed by σ^4 either trivially, i.e. each smooth rational curve is σ-invariant or eventually pointwise fixed by σ, or it exchanges smooth rational curves two by two, or finally σ permutes four rational curves between them. In fact each curve in the set of four permuted smooth rational curves by σ has stabilizer group in $\langle\sigma\rangle$ of order 2, hence its σ-orbit has length four.

Lemma 2.1.8. *Four cyclic permuted smooth rational curves by a non-symplectic automorphism σ of order 8 on a K3 surface X, are either σ^4-invariant (not pointwise fixed), either pointwise fixed by σ^4.*

Proof. We can prove it simply as follows. Let C_i, $i \in \{1, \ldots, 4\}$ be four smooth rational curves such that $\sigma(C_i) = C_{i+1}$, $i = 1, 2, 3$ and $\sigma(C_4) = C_1$, and assume that C_1 is invariant by σ^4, then $\sigma^4(C_2) = \sigma^4(\sigma(C_1)) = \sigma(\sigma^4(C_1)) = \sigma(C_1) = C_2$. In particular if C_1 is pointwise fixed, then one proves in a similar way that C_2 is pointwise fixed. A similar proof holds also for C_3 and C_4, so we get the statement. □

We denote by $2a_\sigma$ the number of exchanged smooth rational curves by σ and fixed by σ^2, and by $4s_\sigma$ the number of smooth rational curves cyclic permuted by σ and pointwise fixed by σ^4 (and clearly they are interchanged by σ^2 two by two).

Remark 2.1.9. Let a_{σ^2} be the number of the pairs of rational curves interchanged by σ^2 and pointwisely fixed by σ^4, then $2a_{\sigma^2} = 4s$ and so $a_{\sigma^2} \in 2\mathbb{Z}$.

At a fixed point of σ the action can be linearized, see e.g. [71, Section 5]. We can find z_1 and z_2 local coordinates in a neighborhood of a fixed point x such that we can assume $x = (0, 0)$. We know that the symplectic form ω_X is an everywhere non-degenerate 2-form and this allow us to write it in local coordinates as $dz_1 \wedge dz_2$. We know that $\sigma^*(dz_1 \wedge dz_2) = \zeta_8(dz_1 \wedge dz_2)$ for this reason if we consider the local action diagonalized we need that the product of the eigenvalues with respect to z_1 and z_2 is equal to ζ_8. Since the automorphism is of finite order it can be locally diagonalized as follows (up to permutation of the coordinates but this doesn't play any role in the classification):

$$A_{1,0} = \begin{pmatrix} \zeta_8 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_{2,7} = \begin{pmatrix} i & 0 \\ 0 & \zeta_8^7 \end{pmatrix}, \qquad A_{3,6} = \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^6 \end{pmatrix}, \qquad A_{4,5} = \begin{pmatrix} -1 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}.$$

In the first case the point belongs to a smooth fixed curve, since we have an eigenvalue which is equal to 1, and this means that the second coordinate z_2 is totally preserved by the action of σ. In the other three cases it is an isolated fixed point. We say that an isolated point $x \in \mathrm{Fix}(\sigma)$ is of type (t, s) if the local action

at x is given by $A_{t,s}$. We denote by $n_{t,s}$ the number of isolated fixed points by σ with matrix $A_{t,s}$.
The following is a result about local actions of σ on a rational curve. With the same notation of Remark 2.1.4 if R'_i is σ-invariant (not pointwise fixed) each action of σ on R'_i has two fixed points. There are some restrictions about the possible actions of σ, in particular if we have an action on one of the fixed points then the action on the other point is determined. We state the following result which is inspired by [31].

Proposition 2.1.10. *Let σ be a non-symplectic automorphism of order 8 on a K3 surface X and suppose that the fixed locus of the involution σ^4 is the union of smooth rational curves R'_i. Then if p_1 is an isolated fixed point for σ and it is contained in R'_i, there exists another fixed point p_2 for σ on R'_i. If the local action in p_1 is of $(7,2)$-type then the local action in p_2 is of $(3,6)$-type and vice-versa. If the action in p_1 is of $(4,5)$-type then the action in p_2 is of $(5,4)$-type.*

Proof. We know that the action of a finite automorphism which is not the identity on a rational curve has two fixed points. Let p_1 and p_2 be the two fixed points on R'_i. The possible local actions for an automorphisms of order 8 are of type $(1,0)$, $(2,7)$, $(3,6)$ and $(4,5)$. The action of type $(1,0)$ doesn't happen since it means that the fixed point belongs to a smooth fixed curve, but the point is an isolated fixed point so we get a contradiction.
A morphism on a rational curve i.e. on $\mathbb{P}^1(\mathbb{C})$ has this form:

$$\sigma([z_0 : z_1]) = ([\gamma z_0 + \beta z_1 : \alpha z_0 + \delta z_1]),$$

where $\gamma\delta - \alpha\beta \neq 0$.
We can suppose that the two fixed points are 0 and ∞ which means that $p_1 = [1 : 0]$ in homogeneous coordinates $[z_0 : z_1]$ with $z_0 \neq 0$ and $p_2 = [0 : 1]$ in homogeneous coordinates $[z_0 : z_1]$ with $z_1 \neq 0$. Since the points p_1 and p_2 are points of the $K3$ surface which lie on the rational curve R'_i, the local coordinate in p_1, as a point of the rational curve, is $w := \frac{z_1}{z_0}$ and the local coordinate in p_2, as a point of the rational curve is $z := \frac{z_0}{z_1} = \frac{1}{w}$.
We can consider a local analytic neighborhood on the $K3$ surface of p_1 and we can call these local coordinates z_1 and z_2 where $z_2 = \frac{z_1}{z_0}$ since $p_1 \in \{z_0 \neq 0\}$. If the action is of $(7,2)$-type this means that

$$\sigma((z_1, z_2)) = (\zeta_8^7 z_1, \zeta_8^2 z_2).$$

In a local analytic neighborhood of p_2 we have coordinates (z'_1, z'_2) where we can choose $z'_1 = z_1$ and $z'_2 = \frac{z_0}{z_1}$ since $p_2 \in \{z_1 \neq 0\}$. Now we know that σ acts on the second component in this way:

$$z'_2 = 1/z_2 \mapsto 1/(\zeta_8^2 z_2) = \zeta_8^6 z'_2$$

and this implies that

$$z'_1 \mapsto \zeta_8^3 z'_1,$$

since we know which are the possible local actions of σ in a neighborhood of a fixed point. We can conclude that in p_2 we have an action of $(3,6)$-type i.e

$$\sigma((z'_1, z'_2)) = (\zeta_8^3 z'_1, \zeta_8^6 z'_2).$$

We obtain in a similar way that a $(4,5)$-type action in p_1 determines a $(5,4)$-type action in p_2. □

An important remark about the local behaviour of σ^2 in a neighborhood of a fixed point is the following:

Remark 2.1.11. The isolated fixed points by a non-symplectic automorphism σ of type $(2,7)$ and $(3,6)$, are also isolated fixed points in $\mathrm{Fix}(\sigma^2)$. The points of type $(4,5)$ in $\mathrm{Fix}(\sigma)$ are contained in a smooth fixed curve by σ^2. In fact the action of σ^2 at such points is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \zeta_8^2 \end{pmatrix}$ which implies that these points belong to a smooth curve in $\mathrm{Fix}(\sigma^2)$. For this reason we can say that if there exist points of $(4,5)$-type then $k_{\sigma^2} > k_\sigma$.

We denote by $2a$ the number of exchanged smooth rational curves by σ and fixed by σ^2, and by $4s$ the number of smooth rational curves cyclic permuted by σ and fixed by σ^4 (and clearly they are interchanged by σ^2 two by two).

2.1.3 The classification

Our goal is to give a classification of non-symplectic automorphisms of order eight under the assumption that the involution given by its fourth power fixes only rational curves. Let σ be such an automorphism, then

$$\mathrm{Fix}(\sigma^4) = R'_1 \cup \cdots \cup R'_t$$

where R'_i are smooth rational disjoint curves. This implies that:

$$\mathrm{Fix}(\sigma) = R_1 \cup \cdots \cup R_k \cup \{p_1, \ldots, p_N\}$$

which means that in the description of equation (2.1) we have $g(C) = 0$ and R_i are smooth disjoint rational curves.

Theorem 2.1.12. *Let σ be a non-symplectic automorphism of order 8 on a K3 surface X with $Pic(X) = T(\sigma^4)$. Suppose that* $\mathrm{Fix}(\sigma^4)$ *is not empty and it is the union of smooth rational curves. Then $k_\sigma \in \{0,1\}$ and the invariants of σ are as in Table 2.2*

Proof. To prove this result we consider two cases, $k_\sigma \geq 1$ and $k_\sigma = 0$.

$k_\sigma \geq 1$ Consider p_1 a fixed isolated point for σ. Since it is an isolated point it is not on a smooth rational curve of $\mathrm{Fix}(\sigma)$. From Remark 2.1.4 there exists a smooth rational curves in $\mathrm{Fix}(\sigma^4)$ such that $p_1 \in R'_i$. From Remark 2.1.5, R'_i is σ-invariant. Since a finite order automorphism of a rational curve has two fixed points, there exists another fixed point for σ on R'_i, we call it p_2. We deduce that N_σ is even. From what we know about the local behaviour of σ in a fixed point, we deduce that if a fixed point is of $(7,2)$-type then the other fixed point on the same rational curve R'_i is of $(6,3)$-type (recall that an action of (t,s)-type is equal to an action of (s,t)-type). If there is an action of

$(4,5)$-type on p_1 then there is an action of $(5,4)$-type on p_2. By these remarks we obtain:

$$n_{7,2} = n_{6,3}$$
$$n_{4,5} \in 2\mathbb{Z}.$$

Using Proposition 2.1.6 we obtain:

$$\alpha = k_\sigma \geq 1,$$

$$n_{3,6} = n_{7,2} \geq 3,$$
$$n_{4,5} \geq 4.$$

At this point we can consider the classification in Table 2.1. Using Remark 2.1.11 the number of curves in $\mathrm{Fix}(\sigma^2)$ is:

$$k_{\sigma^2} = k_\sigma + \frac{n_{4,5}}{2} + 2a_\sigma.$$

In this setting $k_{\sigma^2} \geq 3$ so the possible cases are $k_{\sigma^2} = 3$ and $k_{\sigma^2} = 4$. But from Remark 2.1.1 we know that m_{σ^2} has to be even so checking again in Table 2.1, the only case is $k_{\sigma^2} = 3$ and $m_{\sigma^2} = 2$. If $k_{\sigma^2} = 3$, since $n_{4,5} \geq 4$, the only possibility is $k_\sigma = 1$, $a_\sigma = 0$ and $n_{4,5} = 4$. Consequently $n_{2,7} = n_{3,6} = 3$. We can conclude that the number $N_\sigma = n_{4,5} + n_{3,6} + n_{2,7} = 10$.
We can use Remark 2.1.1 and Proposition 2.1.6 to conclude that $r_\sigma = 13$, $l_\sigma = 3$, $m_\sigma = 1$, $m_1 = 1$, $r_{\sigma^4} = 18$ and $l_{\sigma^4} = 4$. Moreover since $a_{\sigma^2} = 0$ this implies that $s = 0$ and for this reason $k_{\sigma^2} = 3$ and using Theorem 2.1.2 we conclude that $k_{\sigma^4} = 8$. Using Table 2.1 we compute the invariants for σ^2 which are $(r, m, l, N, k, a) = (16, 2, 2, 10, 3, 0)$.

$k_\sigma = 0$ By the same argument as in the previous case we deduce that $n_{3,6} = n_{2,7}$ and $n_{4,5}$ is even. Then we can use Proposition 2.1.6 and we obtain:

$$\alpha = k_\sigma = 0,$$

$$n_{3,6} = n_{7,2} = 1,$$
$$n_{4,5} = 2.$$

In this case $\mathrm{Fix}(\sigma)=\{p_1, p_2, p_3, p_4\}$, i.e. $N_\sigma = 4$. Observe that σ^2 is an automorphism of order 4 and it contains a rational curve in the fixed locus since two points of $(4,5)$-type in $\mathrm{Fix}(\sigma)$ become a fixed curve for σ^2. We can use the classification in Table 2.1 and as in the previous case, using Remark 2.1.11, the number of curves in $\mathrm{Fix}(\sigma^2)$ is:

$$k_{\sigma^2} = k_\sigma + \frac{n_{4,5}}{2} + 2a_\sigma.$$

In this setting $k_{\sigma^2} \geq 1$ and from Remark 2.1.1 we know that m_{σ^2} has to be even so using Table 2.1 we deduce that there are four possible cases.
If $\boldsymbol{m_{\sigma^2} = 4}$ then $\boldsymbol{k_\sigma^2 = 1}$. From the previous equation $a_\sigma = 0$. In this case the invariants for σ^2 are $(r, m, l, N, k, a) = (10, 4, 4, 6, 1, 0)$.
If $\boldsymbol{m_{\sigma^2} = 2}$ then $k_\sigma^2 \in \{3, 2, 1\}$. If $\boldsymbol{k_{\sigma^2} = 3}$ from the previous equation $a_\sigma = 1$.

In this case the invariants for σ^2 are $(r, m, l, N, k, a) = (16, 2, 2, 10, 3, 0)$.
If $\boldsymbol{k_{\sigma^2} = 2}$ from the previous equation $a_\sigma = \frac{1}{2}$, which is not possible.
If $\boldsymbol{k_{\sigma^2} = 1}$ from the previous equation $a_\sigma = 0$. In this case for σ^2 the invariants are $(r, m, l, N, k, a) = (12, 2, 6, 6, 1, 2)$.

m_1	m_σ	r_σ	l_σ	N_σ	k_σ	a_σ	Examples
1	1	13	3	10	1	0	$\star$
2	2	6	4	4	0	0	
1	1	9	7	4	0	1	$\star$
1	3	7	5	4	0	0	

Table 2.2: Invariants of the automorphism

□

2.1.4 Elliptic fibrations

Definition 2.1.13. Let X be a complex surface. An **elliptic fibration** is a holomorphic map $f : X \longrightarrow B$ to a smooth curve B such that the generic fiber is a smooth connected curve of genus one. A **jacobian elliptic fibration** is an elliptic fibration admitting a section $\pi : B \longrightarrow X$ such that $f \circ \pi = Id_B$. The surface X is called an **elliptic surface**. We denote by F_v the fiber $f^{-1}(v)$ over a point $v \in B$. The **Mordell-Weil group** is the group of sections of the elliptic fibration.

The **zero section** of an elliptic fibration is a chosen section $s : B \longrightarrow X$ and we identify the map s with the curve $s(B)$ on X. The point of intersection between the zero section and a fiber is the zero of the group law on the fiber.
For $K3$ surfaces we have that $B = \mathbb{P}^1$ (see [55]) and, if the fibration is jacobian, it admits a Weierstrass equation:

$$y^2 = x^3 + A(t)x + B(t), \tag{2.2}$$

where $A(t)$ and $B(t)$ are two polynomials with $t \in \mathbb{P}^1$ with complex coefficients such that deg($A(t)$)=8 and deg($B(t)$)=12. Here the zero section is $t \mapsto (0 : 1 : 0)$.
The discriminant of the fibration is a degree 24 polynomial:

$$\Delta(t) = 4A(t)^3 + 27B(t)^2. \tag{2.3}$$

The equation (2.2) is associated to an elliptic fibration if and only if $\Delta(t)$ does not vanish identically. Each zero of $\Delta(t)$ corresponds to a point v of the base $\mathbb{P}^1$ such that F_v is a singular fiber of the fibration. There are at most finitely many singular fibers. Let δ be the order of vanishing of Δ in the point corresponding to the singular fiber, by the *Kodaira classification* the possible singular fibers are recalled in Figure 2.2 where we denoted by Θ_0 the component of a fiber meeting the zero section. The first column in the Figure 2.2 contains the name of the reducible fiber according to Kodaira classification, the second the Dynkin diagram associated to the fiber, the last column contains the order of vanishing of Δ in the point corresponding to the singular fiber.

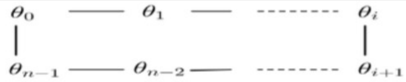
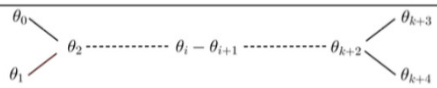
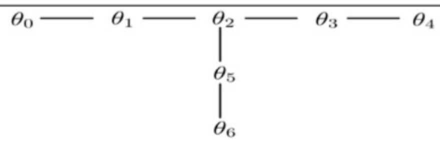
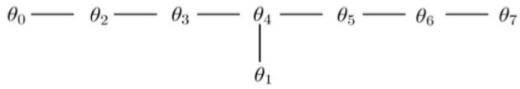
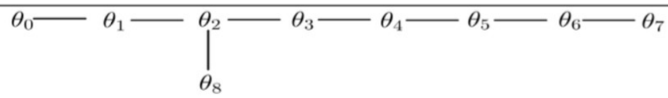
name	Dynkin diagrams	description	δ
II		a cuspidal rational curve	2
I_1		nodal rational curve	1
I_2	A_1	two rational curves meeting transversally at two points	2
$I_n, n \geq 3$	$\tilde{A}_n$		n
$I_n^*, n > 0$	$\tilde{D}_{k+4}$		$n+6$
III		two rational curves meeting in a point of order 2	3
IV		three rational curves all meeting at one point	4
IV^*	$\tilde{E}_6$		8
III^*	$\tilde{E}_7$		9
II^*	$\tilde{E}_8$		10

Figure 2.2: Kodaira classification

A **simple component** of a fiber is a component with multiplicity one. In Figure 2.3 we describe the singular fibers of an elliptic fibration with the multiplicities of the vertices of the extended Dynkin diagrams, and we list the components with their multiplicities.

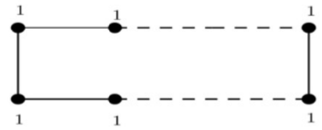
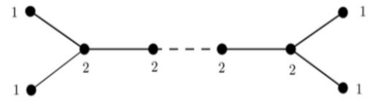
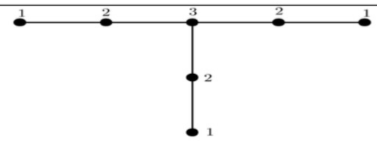
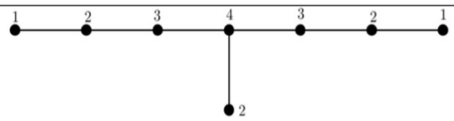
name	simple components	associated Dynkin diagram
$\tilde{A}_n$	$\Theta_i, \quad i=0,\dots,n-1$	
$\tilde{D}_{k+4}$	$\Theta_i, \quad i=0,1,k+3,k+4$	
$\tilde{E}_6$	$\Theta_i, \quad i=0,4,6$	
$\tilde{E}_7$	$\Theta_i, \quad i=0,7$	
$\tilde{E}_8$	Θ_7	

Figure 2.3: Dynkin diagrams with the multiplicities of the components

The Nèron–Severi group of a surface admitting an elliptic fibration contains the class of a fiber F (all the fibers are algebraic equivalent) and the class of the zero section s. Since the fibers are all algebraic equivalent $F \cdot F = 0$. The zero section intersect any fiber in one point, so that $F \cdot s = 1$. The sections of an elliptic fibration on $K3$ surfaces are rational curves and this implies that their self–intersection is -2. Moreover, if X is a $K3$ surface that admits an elliptic fibration, then there is an embedding of $\overline{U}$ in $\mathrm{NS}(X)$, where $\overline{U}$ is the two dimensional lattice

$$\overline{U} = \left\{ \mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \right\}.$$

Observe that the lattice $\overline{U}$ is isometric to the hyperbolic plane U, where U is the two dimensional lattice

$$U = \left\{ \mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

If $f : X \longrightarrow \mathbb{P}^1$ admits an n-torsion section s_n of order n in the Mordell Weil group then it induces an automorphism of the same order on X, which acts as the identity on the base of the fibration and as a translation by the section on each fiber hence this automorphism is a symplectic automorphism, see [46, chapter 15, Lemma 4.4]

2.1.5 Examples

In this section we give examples corresponding to the cases discussed in Theorem 2.1.12. They are constructed using elliptic fibrations on $K3$ surfaces.

Example $k_\sigma = 1$

The case $k_\sigma = 1$ in Theorem 2.1.12 occurs, this means that we can find a geometric example of a non-symplectic automorphism of order 8 on a K3 surface X such that its fixed locus is consists of a smooth rational curve and 10 isolated points and the fixed locus of σ^4 is made by three smooth rational disjoint curves. Consider the elliptic fibration X given as:

$$y^2 = x(x^2 + tp_6(t))$$

with $p_6(t) := (a_6t^6 + a_4t^4 + a_2t^2 + a_0) = (t^2 - \alpha_1)(t^2 - \alpha_2)(t^2 - \alpha_3)$, where $a_6, a_4, a_2, a_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, and the order 8 automorphism acting on it:

$$\sigma : (x, y, t) \mapsto (-ix, \zeta_8 y, -t).$$

By [50, section 3] a holomorphic 2-form can be written as $\omega_X = dt \wedge dx/2y$ so that one computes

$$\sigma(\omega_X) = \zeta_8 \omega_X$$

hence σ acts non-symplectically.

Moreover σ acts as an involution on the base $\mathbb{P}^1$ and it has order four on each fiber of the fibration.

The discriminant is $\Delta(t) = 4t^3(t^2 - \alpha_1)^3(t^2 - \alpha_2)^3(t^2 - \alpha_3)^3$. Recall that $t \in \mathbb{P}^1$ so if we consider the homogenization of the polynomial in coordinates $[t : u]$, we obtain $\Delta(t, u) = 4t^3(t^2 - \alpha_1)^3(t^2 - \alpha_2)^3(t^2 - \alpha_3)^3u^3$.
We take $\boldsymbol{\alpha_1 = 0}$ and $\boldsymbol{\alpha_2 = \alpha_3}$. In this hypothesis we have the elliptic fibration:

$$y^2 = x(x^2 + t(t^2 - \alpha_2)^2)$$

and $\Delta(t) = 4t^9(t^2 - \alpha_2)^6$ which is in homogeneous coordinates $[t : u]$ equal to $\Delta(t, u) = 4t^9(t^2 - \alpha_2)^6u^3$. For generic choice of the coefficient α_2 the fibration has 3 singular fibers which correspond to the three zeros of $\Delta(t, u)$.
To be more precise the fibration has a fiber of type III^* over 0, which corresponds to the $\tilde{E_7}$ Dynkin diagram, a fiber of type III over ∞, which consists in two rational curves meeting in a double point and two fibers of type I_0^* over $\pm\sqrt{\alpha_2}$. The action of σ on the base fixes two points, 0 and ∞ and so it preserves the fibers over these two points, III and III^*, and it exchanges the two fibers of type I_0^*. If $f : X \longrightarrow \mathbb{P}^1$ is the elliptic fibration then $\mathrm{Fix}(\sigma) \subseteq f^{-1}(0) \cup f^{-1}(\infty)$. Observe that the fibration has a two torsion section s_2, given by $t \mapsto (0 : 0 : 1) = (x : y : z)$ and the zero section s, given by $t \mapsto (0 : 1 : 0) = (x : y : z)$. These sections are preserved by the action of σ and they have two fixed points on it which are contained in the union of the fibers over 0 and ∞. These two sections are pointwise fixed on it by the action of σ^2.

The sections s_2 and s are simple sections which means that they meet one of the components of III^* of multiplicity one (not the same), they meet one of the two components of III (not the same) in a non-singular point and they meet the fibers of type I_0^* in one of the components of multiplicity 1 (not the same). We know that the components are not the same since the fixed loci are smooth.

Now we can see that the two fixed points on each section are contained in III^* and III. The fiber of type III consists in two tangent rational curves. Each of these two rational curves has a fixed point which is not the double one. Moreover, since a non-trivial finite order auto-morphism fixes two points on a rational curve, we can conclude that the double point of the fiber III is fixed by σ. On the fiber of type III^* we have the two fixed points given by the intersections of the two sections s ans s_2 with this fiber. Since the sections s and s_2 are not exchanged then all components in III^* are preserved. The only component of this fiber of multiplicity four is preserved by σ and it contains three fixed points so it is point-wise fixed. The component of multiplicity two which intersects the component of multiplicity four contains then a fixed point and we know that there is another fixed point on it. In conclusion $N_\sigma = 10$ and $k_\sigma = 1$ as we expect. The invariants are $(m_1, m_\sigma, r_\sigma, l_\sigma, N_\sigma, k_\sigma, a_\sigma) = (1, 1, 13, 3, 10, 1, 0)$, by Proposition 2.1.6.

The square of the automorphism $\sigma^2 : (x, y, t) \mapsto (-x, iy, t)$ preserves each fiber and acts as an automorphism of order four on it. Moreover σ^2 fixes two points on the generic smooth fiber, these two fixed points are contained in the two sections s and s_2. This gives that $k_{\sigma^2} \geq 2$.
Since $l_{\sigma^2} = 2m_\sigma = 2$ and $m_{\sigma^2} = 2m_1 = 2$ using Table 2.1 for the classification of non-symplectic automorphisms of order 4, we know that the invariants for σ^2 are $(r_{\sigma^2}, m_{\sigma^2}, l_{\sigma^2}, N_{\sigma^2}, k_{\sigma^2}, a_{\sigma_2}) = (16, 2, 2, 10, 3, 0)$.
The curves fixed by σ^2 are the curves fixed by σ and the two sections s and s_2. The points fixed by σ^2 are 10 but they are not the same fixed points by σ, in fact the 4 fixed points for σ on s and s_2 now lie on fixed curves (the curves s and s_2) and they do not give any contribution, but we add exactly 4 other points on the two fibers I_0^*. For this reason the number of fixed points remains the same.

Consider the curve defined by $y = 0$. From the equation of the elliptic fibration we obtain $x = 0$, which gives the zero section, and the curve C: $x^2 + t^2(t^2 - \alpha_2)^2 = 0$ which has a 2:1 morphism to $\mathbb{P}^1$ and has ramification points where $t^2(t^2 - \alpha_2)^2 = 0$. These points lie on the four singular fibers III, the two of type I_0^* and III^* (one over 0, one over $\sqrt{\alpha_2}$, one over $-\sqrt{\alpha_2}$ and one over ∞) and in fact C meets these fibers in points with double multiplicity. In particular C meets III in the double point, I_0^* in the two components of multiplicity one and III^* in the component of multiplicity two.

Recall that the Riemann-Hurwitz formula applied to a $2:1$ morphism from a curve C to the projective line $\mathbb{P}^1$ is given by:

$$2g(C) - 2 = 2(0 - 2) + \sum_{p \in C} (e_p - 1)$$

where the sum runs over the ramification points which are two points in this case, the point on the fiber over 0, i.e. on III^* and the point on the fiber over ∞ i.e on III and e_p is the ramification index at a ramification point p. In this case is $e_p = 2$. By using the formula we can compute in an easy way the genus of the curve $C \subseteq \mathrm{Fix}(\sigma^4)$ which is $g(C) = 0$ (i.e. the fixed curves by σ^4 are all rational) and then the curve C contains two fixed points, one of them of $(2,7)$-type and the other of $(6,3)$-type.

The rank of the invariant sublattice of the involution σ^4 is $r_{\sigma^4} = r_\sigma + l_\sigma + 2m_\sigma = 18$ and using the tabular of Proposition 2.1.2 we see that $2(k_{\sigma^4} - 1) = 18 - 4 = 14$ which means that $k_{\sigma^4} = 8$. In fact in this example we have the following fixed rational curves: three components of the fiber III^*, two sections s and s_2, the curve C and two other rational curves which are fixed on the two fibers of type I_0^*.

Example $k_\sigma = 0$

The case $k_\sigma = 0$ in Theorem 2.1.12 do exist when $k_{\sigma^2} = 3$. We can consider the same elliptic fibration of the previous example, and we fix $\alpha_1 = 0$ and $\alpha_2 = \alpha_3$ as before. As we have already observed the fibration has a two torsion section given by $t \mapsto (0:0:1) = (x:y:z)$. Denote by τ the symplectic involution associated to this two torsion section. As we have observed before, this involution is symplectic since it acts as a translation on each fiber and as the identity on the base of the fibration. The involution exchanges the zero section s and the two torsion section s_2. We cannot find fixed points for τ on the generic fiber since it acts as a translation, but we know (see [46]) that a symplectic involution has 8 fixed points on a K3 surface. Consequently the 8 fixed points for τ are on the singular fibers of the elliptic fibration, which are: the fiber over zero of type III^*, the two fibers over $\pm\sqrt{\alpha_2}$ of type I_0^* and the fiber over ∞, of type III (see the Kodaira classification in Figure 2.2). On each of the two fibers of type I_0^* we have two fixed points on the component of multiplicity two since the sections s and s_2 are exchanged. On the fiber of type III^* we have three fixed points, two of them are on the component of maximal multiplicity and the other is on the component of multiplicity two which intersects the component of maximal multiplicity. On the fiber of type III the two rational curves are exchanged and so we have a fixed point for τ which is the double point.
We consider $\sigma \circ \tau$ which is an automorphism of order eight. It is non-symplectic since in local coordinates we can write $\omega_X = \frac{dx \wedge dt}{2y}$. Since τ is symplectic, $\sigma \circ \tau(\omega_X) = \zeta_8 \omega_X$ which means that $\sigma \circ \tau$ is non-symplectic. Now the fixed point for τ on III is also a fixed point for σ. As a consequence it is a fixed point for $\sigma \circ \tau$. The fibers I_0^* are exchanged by σ so we cannot find fixed points for $\sigma \circ \tau$ on them and two of the three fixed points on III^* are on a fixed curve for σ so they contribute to $\mathrm{Fix}(\sigma \circ \tau)$. The last fixed point by τ on the component of multiplicity two is fixed also by σ so it is a fixed point for $\sigma \circ \tau$. Finally we conclude that $\mathrm{Fix}(\sigma \circ \tau) = \{p_1, p_2, p_3, p_4\}$ where one of them is the double point of III and it is of $(2,7)$-type, the other three are on the fiber III^*; two of them are of $(4,5)$-type and one of $(3,6)$-type, according with the result in Theorem 2.1.12.

Since σ, τ commute $(\sigma \circ \tau)^2 = \sigma^2$ and the behaviour of the order four automorphism is the same as it is described in the previous example.

Chapter 3

Induced automorphism groups

This chapter deals with several questions concerning irreducible holomorphic symplectic manifolds of OG_6 type and their automorphisms.

The easiest example of IHS manifolds that arises from a symplectic surface is the Hilbert scheme of n points on a K3 surface, constructed by Beauville in [10]. This kind of construction allows us to produce several examples of automorphisms on irreducible symplectic manifolds, simply by taking a K3 surface with non-trivial automorphism group and considering the induced action on its Hilbert scheme. These kinds of automorphisms are called *natural* in the literature, and were studied by Beauville [8], Boissière [12] and many others. Very few examples of non-natural automorphisms are known, such as those constructed in [79], and a numerical criterion to distinguish between natural and non-natural automorphisms is available only in special cases, as we can find in [18], and [57]. A generalization of the notion of natural automorphisms for moduli spaces of sheaves on a K3 surface is provided in [64]. This notion appeared the first time for moduli spaces of sheaves in the paper [79], a work inspired by the construction in [78, Section 5]. In [64] the authors extend the ideas drastically using developments in the theory of stability conditions by Bridgeland [22] and by Bayer-Macrì (see [6][7]) and Yoshioka (see [100]). Inspired by the recent works we re-adapt this notion for manifolds of OG_6 type and we show a criterion to determine weather a given automorphism is induced or induced at the quotient in the sense we are going to explain in greater detail in the following.

Moreover, as we have seen in section 1.4.1, there exists a birational model for OG_6 manifolds which has been introduced for the first time in [62]. This model is obtained considering a principally polarized abelian surface A and its Kummer $K3$ surface S. On the Hilbert scheme of three points on S, $S^{[3]}$ the authors of [62] construct a non-regular involution, whose quotient is birational to a manifold of OG_6 type.

Taking into consideration the birational model for OG_6 manifolds, the notion of induced automorphisms can be re-adapted also in this different contest. We introduce the notion of automorphisms induced at the quotient, in order to find a criterion to establish when an automorphisms of the birational model of an OG_6 manifolds lifts to an automorphism of the $S^{[3]}$ manifold, where S is the K3 surface appearing in the birational model. We will find a criterion for automorphisms of prime order and we will prove that in the non-symplectic case almost all the cases are covered, on the other hand this is not true in the case of symplectic automorphisms, as we will see in Chapter 5.

At the end of Chapter 5 we will show that all the induced automorphisms are induced at the quotient, but the opposite implication does not hold. In fact, in the last chapter of our work we will construct an example of an automorphism induced at the quotient but not induced.

3.1 A criterion for being a moduli space

This section is devoted to answering the following question: How can we determine if a given manifold of OG_6 type is the symplectic resolution of the Albanese fiber of a moduli space of stable objects on an abelian surface? We state a necessary and sufficient criterion entirely in terms of Hodge theory. In the following $\Lambda_8 := U^{\oplus 4}$ and $\Lambda_{10} := U^{\oplus 5}$.

Definition 3.1.1. Let X be a projective manifold of OG_6 type. Let $\sigma \in H^2(X,\mathbb{Z})$ be a class of square -2 and divisibility 2 and let i be a primitive lattice embedding of $\sigma^{\perp_{H^2(X,\mathbb{Z})}} \cong U^{\oplus 3} \oplus \langle -2 \rangle \hookrightarrow \Lambda_8$. It is possible to endow the latter with the (unique) Hodge structure, making i an embedding of Hodge structures such that the complement of the image of i is of type $(1,1)$. We call X a *numerical moduli space* if there exists $\sigma \in \mathrm{NS}(X)$ s.t. $\sigma^2 = -2$ and $div(\sigma) = 2$ and, through the previous embedding, $\Lambda_8^{1,1}$ contains a copy of the hyperbolic lattice U.

Lemma 3.1.2. *Let X be a projective manifold of OG_6 type. Consider the embedding $i : H^2(X,\mathbb{Z}) \hookrightarrow \Lambda_{10}$, and endow Λ_{10} with the (unique) Hodge structure such that the complement of the image of i is of $(1,1)$-type. X is a numerical moduli space if and only if there exists $\sigma \in \mathrm{NS}(X)$ such that $\sigma^2 = -2$ and $div(\sigma) = 2$ and there exists the following primitive embedding $U^{\oplus 2} \hookrightarrow \Lambda_{10}^{1,1}$.*

Proof. For the 'only if'-part we can consider those embeddings of Hodge structures $\sigma^\perp \cong U^{\oplus 3} \oplus \langle -2 \rangle \hookrightarrow \Lambda_8 \hookrightarrow \Lambda_{10}$ such that the complement of the images of these embeddings are of $(1,1)$-type. Since $U \cong (\Lambda_8)^\perp \subset \Lambda_{10}$, and since X is a numerical moduli space, we know that $U \hookrightarrow \Lambda_8^{1,1}$ and consequently $U^{\oplus 2} \hookrightarrow \Lambda_{10}^{1,1}$.
For the other direction we know that $H^2(X,\mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$, thus it holds that $\sigma^\perp \cong U^{\oplus 3} \oplus \langle -2 \rangle \hookrightarrow \Lambda_8 \hookrightarrow \Lambda_{10}$. Since $\Lambda_8 \hookrightarrow \Lambda_{10}$, we thus obtain $\Lambda_{10}^{1,1} \cong \Lambda_8^{1,1} \oplus U$. By our hypotheses we know that $U^{\oplus 2} \subset \Lambda_{10}^{1,1}$, therefore $U \subset \Lambda_8^{1,1}$ and we are done. □

Remark 3.1.3. In [64, Remark 5.6.], the conjectured condition on X to say that it is the desingularization of the Albanese fiber of a moduli space of stable objects on an abelian surface is different from what we state here, since the authors of [64] ask only two copies of U in $\Lambda_{10}^{1,1}$ as direct summand. In that case, since they start from $X \sim OG_6$ which is the desingularized Albanese fiber of a moduli space of stable objects on an abelian surface then there exists a class $\sigma \in \mathrm{NS}(X)$ s.t. $\sigma^2 = -2$ and $div(\sigma) = 2$. Actually this class is the exceptional divisor of the blow up of the fiber of the moduli space.

For the following proposition we refer to Section 1.4 when we consider moduli spaces of stable sheves in the derived category $D^b(X)$ where X is an abelian or a K3 surface.

Proposition 3.1.4. *Let X be a manifold of OG_6 type which is a numerical moduli space. Then there exists an abelian surface A s.t. X is birational to the desingularized Albanese fibre of a moduli space of stable objects of $D^b(A)$ for some stability condition $\theta \in \mathrm{Stab}(A)$.*
More precisely X is birational to $\widetilde{K}_u(A,\theta)$, where $\widetilde{K}_u(A,\theta) \longrightarrow K_u(A,\theta)$ is the symplectic resolution and $K_u(A,\theta) := alb^{-1}((0,0))$ where the Albanese map is

$$alb : M_u(A,\theta) \longrightarrow A \times A^\vee$$

$$F \mapsto (Alb(c_2(F)), \det(F) \otimes \det(F_0)^{-1}),$$

where $F_0 \in M_v(A,\theta)$ and $Alb : CH_0 \to A$ is the Albanese homomorphism.

Proof. Let $\sigma \in \mathrm{NS}(X)$ s.t. $\sigma^2 = -2$. We have an Hodge embedding $\sigma^{\perp_{H^2(X,\mathbb{Z})}} \cong U^{\oplus 3} \oplus \langle -2\rangle \hookrightarrow \Lambda_8$ where the complement is defined to be of $(1,1)$-type. Let w be the orthogonal complement of $\sigma^\perp$ in Λ_8, $w^2 = 2$. Notice that $\mathrm{sgn}(\sigma^\perp) = (3,4)$ and, since X is projective $\mathrm{sgn}(\mathrm{NS}(X)) = (1,*)$. Since $\sigma \in \mathrm{NS}(X)$ has negative square, the positive part of the signature of $(U^{\oplus 3} \oplus \langle -2\rangle)^{1,1}$ is the same of the positive part of the signature of $\mathrm{NS}(X)$. Thus we get $\Lambda_8^{1,1} \otimes \mathbb{Q} = ((U^{\oplus 3} \oplus \langle -2\rangle)^{1,1} \oplus \langle w\rangle) \otimes \mathbb{Q}$ and this implies that $\mathrm{sgn}(\Lambda_8^{1,1}) = (2,*)$. By hypotheses we know that X is a numerical moduli space, so $U \hookrightarrow \Lambda_8^{1,1}$. By a result of Shioda, [85, Theorem 2], there exists an abelian surface A such that $\Lambda_8^{1,1} \cong U \oplus \mathrm{NS}(A)$. Let θ be a w-generic stability condition, $u = 2w$ and $alb : M_u(A,\theta) \longrightarrow A \times A^\vee$, we define $K := K_u(A,\theta) = alb^{-1}(0,0)$. Using Remark 1.4.5 there exists an Hodge isometry

$$H^2(K_u(A,\theta),\mathbb{Z}) \xrightarrow{\cong} w^\perp \subset \Lambda_8. \tag{3.1}$$

Since there exists an isomorphism of Hodge structures $w^\perp \cong U^{\oplus 3} \oplus \langle -2\rangle$, we conclude that $H^2(K_u(A,\theta),\mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2\rangle$ [80, Theorem 1.7(2)]. The fiber $K_u(A,\theta)$ admits a symplectic resolution and $\widetilde{K}_u(A,\theta) \longrightarrow K_u(A,\theta)$ is such that the exceptional divisor of the blow up is E where $E^2 = -2$ and $E \in \mathrm{NS}(\widetilde{K}_u(A,\theta))$ (see [81, Corollary 3.5.13]). Furthermore $\widetilde{K}_u(A,\theta)$ is a manifold of OG_6 type, see Theorem 1.3.9, so we have $H^2(\widetilde{K}_u(A,\theta),\mathbb{Z}) \cong H^2(K_u(A),\mathbb{Z}) \oplus \mathbb{Z}\cdot E$. Thus we get

$$\varphi : H^2(\widetilde{K}_u(A,\theta),\mathbb{Z}) \longrightarrow U^{\oplus 3} \oplus \langle -2\rangle \oplus E \cong H^2(X,\mathbb{Z}).$$

The monodromy group for a manifold of OG_6 type is maximal, as we have seen in Example 1.2.33, which means that $Mon^2(OG_6) \cong O^+(H^2(X,\mathbb{Z})) \subset O(H^2(X,\mathbb{Z}))$, where $O^+(H^2(X,\mathbb{Z}))$ are the orientation preserving isometries. Since we have an Hodge isometry φ, we can say that $\pm\varphi$ is an orientation preserving Hodge isometry. For this reason we can conclude using Theorem 1.2.34 that X is birational to $\widetilde{K}_u(A,\theta)$. □

3.2 Automorphisms induced from an abelian surface

In this section we would like to know when an automorphism of a manifold of OG_6 type comes from an automorphism of the abelian surface. We need the following definition.

Definition 3.2.1. Let X be a manifold of OG_6 type and let $G \subset \mathrm{Aut}(X)$. We say that G is an *induced group of automorphisms* if there exists an abelian surface A with $G \subset \mathrm{Aut}(A)$, a G-invariant non-primitive Mukai vector $u = 2w$, $u \in H^*(A, \mathbb{Z})^G$ and a u-generic stability condition θ, which is G-invariant, such that X is birational to $\widetilde{K}_u(A, \theta)$, and the induced action on $\widetilde{K}_u(A, \theta)$ coincides with the given action of G on X.

Definition 3.2.2. Let X be a manifold of OG_6 type and let $G \subset \mathrm{Aut}(X)$. Let i be a primitive embedding of $H^2(X, \mathbb{Z})$ in Λ_{10}. Let Λ_{10} be endowed with the unique Hodge structure induced by $H^2(X, \mathbb{Z})$ in a way such that the orthogonal complement of the embedding is of $(1, 1)$ type. Then the group G is called *numerically induced* if the following hold:

(1) The group G acts trivially on the discriminant group A_X; the action can be extended to the lattice Λ_{10} with $S_G(\Lambda_{10}) \cong S_G(X)$.

(2) There exists $\sigma \in \mathrm{NS}(X)$, such that $\sigma^2 = -2$ and $div(\sigma) = 2$ and σ is G-invariant, such that $H^2(X, \mathbb{Z}) \hookrightarrow \Lambda_{10}$ is a Hodge embedding such that the $(1, 1)$-part of the lattice $T_G(\Lambda_{10})$ contains $U^{\oplus 2}$ as a direct summand.

Moreover we ask that for all $g \in G$, $\det(g^*) = 1$.

Remark 3.2.3. The last condition about the determinant of the isometry depends on the monodromy of abelian surfaces. As we have said in Section 1.5,

$$Mon^2(A) = SO^+(H^2(A, \mathbb{Z})),$$

where $H^2(A, \mathbb{Z}) \cong U^{\oplus 3}$.

Remark 3.2.4. The second condition in Definition 3.2.2 is equivalent to require that there exists $\sigma \in NS(X)$ such that $\sigma^2 = -2$ and $div(\sigma) = 2$. Moreover we ask that $\sigma^\perp \hookrightarrow \Lambda_8$ is a Hodge embedding such that the $(1, 1)$-part of the lattice $T_G(\Lambda_8)$ contains U as a direct summand.

Proposition 3.2.5. *Let X be a manifold of OG_6 type. Let $G \subset \mathrm{Aut}(X)$. If G is an induced group of automorphisms, then G is numerically induced.*

Proof. If $G \subset \mathrm{Aut}(X)$ is induced, by definition, there exists an abelian surface A such that X is the resolution of the Albanese fiber of the moduli spaces $M_u(A, \theta)$. For this reason we conclude that there exists in the Nèron-Severi group of X the class of the exceptional divisor, which corresponds to the resolution of the fiber, i.e. there exists $\sigma \in \mathrm{NS}(X)$, σ G-invariant, such that $\sigma^2 = -2$ and $div(\sigma) = 2$. Moreover it holds the following embedding

$$H^2(A, \mathbb{Z}) \hookrightarrow \Lambda_{10}.$$

We can extend the action of G on Λ_{10} trivially on the orthogonal complement of the embedding. By construction, the embedding is G-equivariant, which means that $S_G(A) \cong S_G(\Lambda_{10})$. Therefore, $S_G(\Lambda_{10}) \subset H^2(A, \mathbb{Z}) \cong U^{\oplus 3}$ and this prove that $U^{\oplus 2} \hookrightarrow T_G(\Lambda_{10})$.

We need to show that the action on $A_X := A_{H^2(X,\mathbb{Z})}$ (referring to the definition in Section 1.1) is trivial. To do this we recall the embedding in equation 3.2, which is an Hodge isometry

$$H^2(K_u(A,\theta),\mathbb{Z}) \xrightarrow{\cong} w^\perp \subset \Lambda_8. \tag{3.2}$$

Since by hypothesis the Mukai vector u, where $u = 2w$, is preserved by G, the action on A_w is trivial. Moreover $H^{2*}(A,\mathbb{Z}) \cong H^0(A,\mathbb{Z}) \oplus H^2(A,\mathbb{Z}) \oplus H^4(A,\mathbb{Z}) \cong \Lambda_8$ is a unimodular lattice, therefore the action on $A_{w^\perp}$ is trivial. The exceptional divisor σ such that $\sigma^2 = -2$ and $div(\sigma) = 2$, is invariant under the action of G and as a consequence the action is trivial also on the class of the discriminant group given by the σ divided by two and this show that the action on A_X is trivial. Moreover it holds (2) of the definition of numerically induced since, the orthogonal complement of $H^2(A,\mathbb{Z}) \cong U^{\oplus 3}$ in Λ_{10} is isomorphic to $U^{\oplus 2}$, and it is of $(1,1)$-type and in $T_G(\Lambda_{10})$ by construction. For this reason the condition (2) is verified. Finally since the group of automorphism is induced, by definition $G \subset \mathrm{Aut}(A)$ and this implies that the induced action of $g \in G$ on $H^2(A,\mathbb{Z})$ is a monodromy operator (see Theorem 1.2.34) and this means that $det(g^*) = 1$ (see Remark 3.2.4). □

Theorem 3.2.6. *Let X be a manifold of OG_6 type, and let $G \subset \mathrm{Aut}(X)$ be a numerically induced group of automorphisms. Then there exists a projective abelian surface A, with $G \subset \mathrm{Aut}(A)$, a G-invariant non-primitive Mukai vector $u = 2w$, and a u-generic stability condition θ such that X is birational to $\widetilde{K}_u(A,\theta)$ and G is induced.*

Proof. First of all let us consider the case G symplectic. Then we have $S_G(X) \subseteq \mathrm{NS}(X)$. Since G is numerically induced, we can write $T_G(\Lambda_{10}) = U^{\oplus 2} \oplus T$. We then have that $S_G(X)$ embeds in the second integral cohomology lattice of an abelian surface, i.e. in $H^2(A,\mathbb{Z}) \cong U^{\oplus 3}$, and its orthogonal is T, where the action of G is trivial. We give to this lattice the Hodge structure induced by Λ_{10}, and we denote by A the corresponding abelian surface. By proposition 3.1.4, X is the desingularized Albanese fiber of a moduli space of stable objects on A. We have that G acts on $H^2(A,\mathbb{Z})$ via Hodge isometries. We have that G is a group of orientation preserving Hodge isometries on A therefore $G \subset \mathrm{Aut}(A)$, and the induced action on $H^2(X,\mathbb{Z})$ is the action we started with. Now let us suppose that G is a non-symplectic group. This implies $T_G(X) \subset \mathrm{NS}(X)$. Without loss of generality, we can suppose $T_G(X) = \mathrm{NS}(X)$. We have $T_G(\Lambda_8) = U \oplus T$ and let A be the abelian surface associated to the Hodge structure on $U^\perp$ inside Λ_8. By proposition 3.1.4, X is birational to the desingularized Albanese fiber of the moduli space $M_u(A,\theta)$, in the previous notation $\widetilde{K}_u(A,\theta)$, and G is a group of Hodge isometries of A preserving $T = \mathrm{NS}(A)$. Therefore $G \subset \mathrm{Aut}(A)$ and its action on X coincides with the induced one.

□

Corollary 3.2.7. *If $X \sim OG_6$, $G = \langle \varphi \rangle \subset \mathrm{Aut}(X)$ is an induced group of prime order, automorphisms and $|G| = 2$, then $\mathrm{rk}(S_G(X))$ is even.*

3.3 Automorphisms induced at the quotient

Let $\mathcal{M}_{OG_6}$ be the moduli space of marked manifolds OG_6 type. We know from [62] that the odd Betty numbers are zero, and the Hodge diamond was presented in Section 1.4.1. In this case, since $h^{1,1}(X) = 6$, $\mathcal{M}_{OG_6}$ has dimension 6. We know that there exists a sublocus of $\mathcal{M}_{OG_6}$ which represents the manifolds of OG_6 type that can be realized as we have described in Section 1.4.1 as a quotient of an irreducible holomorphic symplectic manifold of $K3^{[3]}$ type by a birational symplectic involution $i : K_3^{[3]} \dashrightarrow K_3^{[3]}$. We will see later in the following the dimension of the locus of $\mathcal{M}_{OG_6}$ in which a manifold of OG_6 type is birational to this model built starting from a $K3^{[3]}$ type manifold. Moreover I will analyze the reciprocal behavior of this locus and of the locus in which an OG_6 type manifold is a numerical moduli space (see Section 3.1 for the definition).

Before starting with a crucial theorem, we need to make the following remark about the invariant and the co-invariant sublattice of $H^2(K3^{[3]}, \mathbb{Z})$ with respect to the symplectic birational involution.

Remark 3.3.1. We know from [8] that

$$H^2(\widetilde{K3^{[3]}/i}, \mathbb{Z})(2) \cong H^2(K3^{[3]}, \mathbb{Z})^i \oplus E,$$

where E is an exceptional divisor and $\widetilde{K3^{[3]}/i}$ is the desingularization of the quotient $K3^{[3]}/i$. Since we know $H^2(OG_6, \mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$ is a rank 8 lattice of signature $(3, 5)$, and E is an exceptional divisor such that $E^2 = -2$ and $div(E) = 2$, then $H^2(K3^{[3]}, \mathbb{Z})^i \cong T_i(K3^{[3]})$ is a rank 7 lattice of signature $(3, 4)$ which is isomorphic to $U^{\oplus 3} \oplus \langle -2 \rangle$.

Remark 3.3.2. As we know from Proposition 1.5.8, since i is symplectic, the co-invariant sublattice is negative definite. In this remark Λ is the Leech lattice, i.e. the unique (up to isometry) even unimodular negative definite lattice with no elements of square -2. We know that $\mathrm{rk}(\Lambda) = 24$ and that the induced action of i on $A_{S_G(K3^{[3]})}$ is trivial. This is true since $A_{S_i(K3^{[3]})}$ is 2-elementary and the action of i on this group is $-Id$ which is actually the identity on 2-elementary lattice (each element has order 2 hence it coincides with its opposite). For this reason it is possible to consider an i-equivariant primitive embedding

$$S_i(K3^{[3]}) \hookrightarrow \Lambda,$$

and to extend the action of i trivially on the orthogonal complement in a way such that $S_i(K3^{[3]}) = S_i(\Lambda)$. We can find in [60, Proposition A.13] a classification of the co-invariant sublattices of Λ with respect to involutions. Since we have a birational model of OG_6 as a quotient of $K3^{[3]}$ by the symplectic involution i, we know that the cohomology which survives in the quotient is the invariant part, $T_i(K3^{[3]})$, and this means that $\mathrm{rk}(T_i(K3^{[3]})) = b_2(\overline{OG_6}) = 7$, where $\overline{OG_6}$ is the O'Grady's sixfold before the blow up of the singular locus (see Remark 3.3.1). The second Betti number of a manifold of $K3^{[3]}$ type is 23 and hence $\mathrm{rk}(S_i(K3^{[3]})) = 23 - 7 = 16$.

Checking through the list of [60, Proposition A.13], we find only one co-invariant lattice of rank 16, and we have

$$S_i(K3^{[3]}) \cong BW_{16}(-1),$$

where BW_{16} is the Barnes-Wall lattice. It has the following Gram matrix that we take from [70]:

$$BW_{16} = \begin{pmatrix}
4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
-2 & 4 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & -2 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\
0 & 2 & 0 & 4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 0 & 2 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 1 & 2 & 2 & 1 \\
0 & -1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 4 & 0 & 2 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 2 & 0 & 4 & 2 & 2 \\
1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 2 & 4 & 2 \\
0 & 1 & 0 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 2 & 2 & 4
\end{pmatrix}$$

Consequently we compute the orthogonal complement:

$$T_i(K3^{[3]}) = S_i(K3^{[3]}, \mathbb{Z})^{\perp} \cong U(2)^{\oplus 3} \oplus \langle -4 \rangle.$$

Theorem 3.3.3. *Let X be a manifold of OG_6 type. If there exists $E \in \mathrm{NS}(X)$ such that $E^2 = -2$ and $div(E) = 2$, then there exists a $K3$ surface S, such that*

$$X \overset{bir}{\sim} Y.$$

Here Y is the resolution of singularities of $S^{[3]}/i$ i.e. the blow up of the singular locus of $S^{[3]}/i$, where $i\colon S^{[3]} \dashrightarrow S^{[3]}$ is a birational symplectic involution and $S^{[3]}$ is the Hilbert scheme of 3 points on S.

Proof. If $X \sim OG_6$, then $H^2(X, \mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$. By hypothesis we know that there exists $E \in \mathrm{NS}(X)$ such that $E^2 = -2$ and $div(E) = 2$. We can consider $H^2(X, \mathbb{Z}) \supset E^{\perp} =: L \cong U^{\oplus 3} \oplus \langle -2 \rangle$ with the induced weight two Hodge structure. From Remark 3.3.2 we know that $S_i(S^{[3]}) = BW_{16}(-1)$ and $T_i(S^{[3]}) = U(2)^{\oplus 3} \oplus \langle -4 \rangle$, respectively. Moreover we have $L(2) = U(2)^{\oplus 3} \oplus \langle -4 \rangle$. We can embed $BW_{16}(-1)$ in the (1,1) part of the second integral cohomology of a manifold of $K3^{[3]}$ provided with a weight two Hodge structure. Moreover we can embed $L(2)$ with its weight two Hodge structure in the orthogonal complement of $BW_{16}(-1)$ in $H^2(S^{[3]}, \mathbb{Z})$. Then we have $(U(2)^{\oplus 3} \oplus \langle -4 \rangle \oplus BW_{16}(-1)) \hookrightarrow H^2(S^{[3]}, \mathbb{Z})$. Due to this embedding it is possible to define on $H^2(S^{[3]}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -4 \rangle$

a compatible Hodge structure. For the surjectivity of the period map in [45], we know that there exists a manifold of $K3^{[3]}$ type with this Hodge structure in the second integral cohomology. Since we know that $BW_{16}(-1) \cong S_i(S^{[3]})$ and i is a birational symplectic involution, then $BW_{16}(-1)$ does not contain prime exceptional divisors. In this way the birational symplectic involution is well defined and we can consider the quotient $S^{[3]}/i$. The cohomolgy of the quotient is the invariant lattice with respect to the action of the involution, i.e. $T_i(S^{[3]})$, which means that $H^2(S^{[3]}/i, \mathbb{Z})(2) \cong T_i(S^{[3]}) \cong U(2)^{\oplus 3} \oplus \langle -4 \rangle$. The multiplication by a factor 2 depends on the fact that the involution is a 2:1 map. We can desingularize the quotient $S^{[3]}/i$, more precisely there exists a map

$$Y \longrightarrow S^{[3]}/i$$

such that the class of the exceptional divisor is a -2 class. This means that

$$H^2(Y, \mathbb{Z})(2) \cong H^2(S^{[3]}/i, \mathbb{Z}) \oplus \langle -2 \rangle \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2} \cong H^2(X, \mathbb{Z}).$$

The two varieties X and Y have the same Hodge structure, and since $Mon^2(Y) = O^+(H^2(Y, \mathbb{Z}))$, we can say that, if φ is an isomorphism between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$, then φ or $-\varphi$ is a Hodge parallel transport operator, which implies that X and Y are birational (see Theorem 1.2.34). □

Remark 3.3.4. Before starting the analysis about automorphisms induced at the quotient, it is important to remark that the birational models of sixfolds of OG_6 type taken into account can be defined in loci of codimension ≤ 2 in the moduli space of marked manifolds of $OG6$ type, and I would like to highlight if and when a manifold of OG_6 type admits both models, since it is relevant to the rest of the paper.

As we have noted above, if X is a manifold of OG_6 type, it is birational to the resolution of the fiber of the map *alb*, as we have described it in Proposition 3.1.4, if X is a numerical moduli space which means that X has to be projective and has to have a class of square -2 and divisibility 2 in the Nèron–Severi group. This implies that, in $\mathcal{M}_{OG_6}$, which is the moduli space of marked OG_6 type manifolds, the locus of the X birational to $\widetilde{K}_u(A, \theta)$ has codimension 2; if we consider the polarized manifold of OG_6 type, i.e. manifolds in which we have fixed a polarization, i.e. manifolds which are projective, they have codimension 1. On the other hand, if X is a manifold of OG_6 type, it admits as birational model the resolution of singularities of the quotient of a manifold of $K3^{[3]}$ type by a symplectic involution if it admits a class of square -2 and divisibility 2 in its Nèron–Severi group. This means that the codimension of the locus of manifolds of OG_6 type that admit this second birational model is 1. The codimension is equal to one, even If we consider the codimension in the locus of polarized manifolds of OG_6 type.

Moreover it holds the following inclusion

$$\{X \sim \ OG_6 \ s.t. \ X \overset{\text{bir}}{\simeq} \widetilde{K}_u(A, \theta)\} \subseteq \{X \sim \ OG_6 \ s.t. \ X \overset{\text{bir}}{\simeq} K3^{[3]}/i\},$$

in fact if we take $X \overset{\text{bir}}{\simeq} \widetilde{K}_u(A, \theta)$ we know that $\widetilde{K}_u(A, \theta)$ is a resolution of singularities of $K_u(A, \theta)$, and the exceptional divisor is a smooth divisor divisible by two in the integral cohomology by results of Rapagnetta [81]. There exists consequently an associated ramified double cover which is a smooth manifold birational to an IHS manifold of $K3^{[3]}$ type [62], which we denote by $\underline{Y}_v(A,\theta)$ which is equipped with a birational symplectic involution i. This manifold $\underline{Y}_v(A,\theta)$ is the Hilbert cube of S where S is the Kummer surface of A, and A is the one that we have used to define $\widetilde{K}_u(A,\theta)$. This means that $\underline{Y}_u(A,\theta) = S^{[3]}/i$. Up to deformation of the pair $(S^{[3]}, i)$, the quotient $S^{[3]}/i$ is birational to X.

Conjecture: If X is a polarized manifold, the previous inclusion is an equality.

This section is devoted to investigate automorphisms of manifolds of OG_6 type when this manifold is realized as a quotient of a manifold of $K3^{[3]}$ type. The main question that we would like to answer is about the existence of a criterion to determine when an automorphism of OG_6 manifold lifts to an automorphism of the manifold of $K3^{[3]}$ type that is involved in the birational model.

This definition will be useful in the following:

Definition 3.3.5. Let X be a manifold of OG_6 type and S a $K3$ surface such that $X \dashrightarrow S^{[3]}/i$ is a bimeromorphic map. Let $\varphi \in \mathrm{Aut}(X)$, φ is *induced at the quotient* if φ can be lifted to an automorphism $\tilde{\varphi} \in \mathrm{Aut}(S^{[3]})$ such that the induced action on the quotient coincides with φ.

For this part we refer to the construction made by Mongardi, Rapagnetta, Saccà in [62]. From now till the end of this section we will use the notation of section 3.3.3.

The objects that we use in this section depend on a non-primitive Mukai vector v, but we omit this dependence to avoid cumbersome notation. In $\underline{Y}_v(A,\theta)$ that we have mentioned above, there are 256 copies of $\mathbb{P}^3$ and every one of these involutions is not defined on al least one of these 256 copies of $\mathbb{P}^3$ [62]. We will briefly denote $\underline{Y}_v(A,\theta)$ by $\underline{Y}$ in the following. The only involution which is not defined on every one of the 256 copies of $\mathbb{P}^3$ is i, that is well defined on a contraction of $\underline{Y}$. The contraction is denoted by Y and it is such that the resolution of $Y/i = K$ is a manifold of OG_6 type.

Recall that $\mathcal{M}_{K3^{[3]}}$ is the marked moduli space of manifolds of $K3^{[3]}$ type which has dimension $21 = h^{1,1}(K3^{[3]})$. The manifold $\underline{Y}$ is of $K3^{[3]}$ type and i is a birational involution defined on it. This involution i is symplectic so $S_i(\underline{Y}) \subseteq \mathrm{NS}(\underline{Y})$.
Hence $\omega_{\underline{Y}} \in \mathbb{P}(T_i(K_3^{[3]}) \otimes \mathbb{C})$, which is a six dimensional space. Since $\omega_{\underline{Y}}$ is a period, which means that $\omega_{\underline{Y}}\overline{\omega_{\underline{Y}}} = 0$, we need to verify a quadratic equation in a space of dimension six, which means that

$$\{X \sim \ OG_6 \ s.t. \ X \overset{\mathrm{bir}}{\simeq} K3^{[3]}/i\} \subseteq \mathcal{M}_{OG_6}$$

is a subloci of codimension 1 in the moduli space of marked manifolds of OG_6 type, which is a six dimensional space.

Let $\underline{Y}$ and Y be as above. In [62] the authors show that i is well defined out of the 256 copies of $\mathbb{P}^3$, so we can consider Y which is a singular manifold made by con-

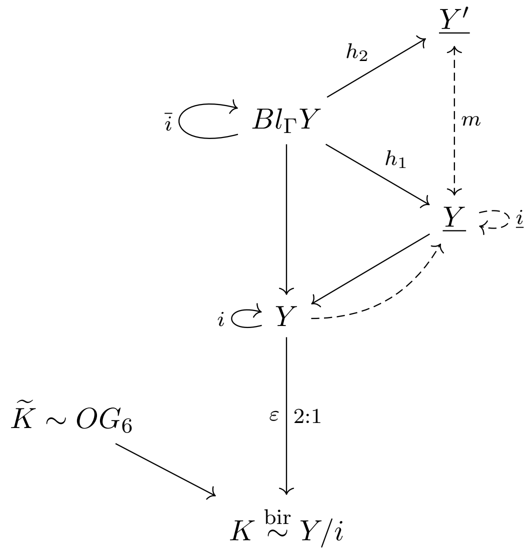


Figure 3.1: Main diagram

tracting the $\mathbb{P}^3$'s. We know that the $\mathbb{P}^3$'s are contractible using Nakano's contraction theorem [68]. This contraction is described in more details in [62, Proposition 5.3].

The second cohomology of $\underline{Y}$ is

$$H^2(\underline{Y}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -4 \rangle.$$

Since i is defined on Y, there is a regular morphism $Y \xrightarrow{2:1} Y/i$. We can consider the following diagram: Here Γ is the singular locus of Y which is composed by 256 points. K is a singular manifold of dimension 6 and the map

$$\tilde{K} \longrightarrow Y/i = K$$

is the blow up of the exceptional divisor E where $E^2 = -2$.
When we consider the contraction of the 256 copies of $\mathbb{P}^3$, we know by a straightforward computation about homology classes, that the second cohomology is modified. In particular the classes of lines in these copies of $\mathbb{P}^3$'s are the generators of the Barnes wall lattice, (see [29, Section 6.5, Proposition 11]), and we know that $BW_{16}(-1) \cong S_i(\underline{Y})$. Therefore in the contraction $\underline{Y} \to Y$, this lattice is contracted and hence it holds that [62, Lemma 6.5 (2)]

$$H^2(Y, \mathbb{Z}) = H^2(\underline{Y}, \mathbb{Z})^i \cong T_i(\underline{Y}) \cong \; H^2(K, \mathbb{Z})(2)$$
$$\cong \; (U^{\oplus 3} \oplus \langle -2 \rangle)(2) \cong U(2)^{\oplus 3} \oplus \langle -4 \rangle.$$

The manifold $\widetilde{K}$ in the previous diagram is of OG_6 type and it is the blow up of the singular locus of K. The class that we add in the second cohomology of $\widetilde{K}$ is the class of the exceptional divisor E, where $E^2 = -2$ and $div(E) = 2$.

If $\varphi \in Bir(\underline{Y})$ commutes with i, then φ preserves the locus on which i is not defined on $\underline{Y}$, and consequently φ preserves the singular locus of Y and the singular

locus of K, which consists of the exceptional divisor E. This assures that $E \in T_\varphi(K)$.
Then there is an embedding of finite index $T_i(\underline{Y}) \oplus S_i(\underline{Y}) \subseteq H^2(\underline{Y}, \mathbb{Z})$, where these are the invariant and co-invariant sublattices with respect to i on the second integral cohomology. Moreover

$$(S_\varphi(\underline{Y}) \cap T_i(\underline{Y})) \oplus (T_\varphi(\underline{Y}) \cap T_i(\underline{Y})) \subseteq T_i(\underline{Y}),$$

$$(S_\varphi(\underline{Y}) \cap S_i(\underline{Y})) \oplus (T_\varphi(\underline{Y}) \cap S_i(\underline{Y})) \subseteq S_i(\underline{Y}).$$

The same holds for Y.

Remark 3.3.6. Since $\widetilde{K}$ is a blow up, the exceptional divisor E is preserved by φ, i.e. $E \in T_\varphi(\widetilde{K})$, thus we have $H^2(\widetilde{K}, \mathbb{Z}) = H^2(K, \mathbb{Z}) \oplus E$. In addition $T_\varphi(Y) \cap T_i(Y) = T_\varphi(K)(2)$ and $T_\varphi(\widetilde{K}) = T_\varphi(K) \oplus E$.

Remark 3.3.7. When we consider the map $Y \xrightarrow{2:1} Y/i \cong K$, the second integral cohomology lattices behaves in this way $H^2(K, \mathbb{Z})(2) = \ H^2(Y, \mathbb{Z})^i \cong T_i(Y)$.

Starting from this, we will answer the question: which conditions are necessary for an automorphism of a manifold of OG_6 type to be induced at the quotient?

In the previous notation:

Proposition 3.3.8. *Let $\widetilde{K}$ be a manifold of OG_6 type and let $\widetilde{\varphi} \in O(H^2(\widetilde{K}, \mathbb{Z}))$ be an Hodge isometry such that $\widetilde{\varphi} \in Mon^2(\widetilde{K})$. Suppose $\widetilde{\varphi}$ preserves a Kähler class and there exists $E \in T_{\widetilde{\varphi}}(\widetilde{K})$ such that $E^2 = -2$ and $div(E) = 2$. Then there exists a contraction $\widetilde{K} \to K$, $\widetilde{\varphi}$ is effective (Definition 1.5.9) and there exists $\varphi : K \to K$ such that $\widetilde{\varphi}_{|K} = \varphi$.*

Proof. From Theorem 1.2.34, $\widetilde{\varphi}$ is effective. Since E is a prime exceptional divisor [61,] we can contract it. Hence we can define $\varphi_{K \setminus Sing(K)} = \widetilde{\varphi}_{\widetilde{K} \setminus E}$ □

In the notation of the commutative diagram 3.1 we have

Theorem 3.3.9. *Let $\widetilde{K}$ be a manifold of OG_6 type. Let $\widetilde{\varphi} \in Aut(\widetilde{K})$ such that there exists $E \in \mathrm{NS}(\widetilde{K})$ such that $\widetilde{\varphi}^*(E) = E$ with $E^2 = -2$ and $div(E) = 2$. Then $\widetilde{\varphi}$ lifts to an automorphism $\psi : Y \to Y$.*

Before starting to prove this theorem, let me recall the notation that we find in [62] to refer to these maps. The morphism $\varepsilon : Y \longrightarrow K$ is a finite 2:1 morphism, the ramification locus of ε is $\Delta \subseteq Y$ and the branch locus of ε is $\Sigma \subseteq K$.

Remark 3.3.10. The locus Σ coincide with the singular locus of K. The resolution of this locus is the exceptional divisor E in $\widetilde{K}$.

To prove Theorem 3.3.9 we need two results.

Lemma 3.3.11. *Let $\widetilde{K}$ be a manifold of OG_6 type. Let $\widetilde{\varphi} \in Aut(\widetilde{K})$ such that there exists $E \in NS(\widetilde{K}) \cap T_{\widetilde{\varphi}}(\widetilde{K})$ with $E^2 = -2$ and $div(E) = 2$, then $\widetilde{\varphi}$ lifts to an automorphism $\widetilde{\psi} : Y \setminus \Delta \longrightarrow Y \setminus \Delta$.*

Proof. Since $E \in \mathrm{NS}(\widetilde{K})$ we have from Theorem 3.3.3 that $\widetilde{K} \overset{\mathrm{bir}}{\sim} K3^{[3]}/i$. Since $E \in T_{\widetilde{\varphi}}(\widetilde{K})$, from Proposition 3.3.8 we have that $\varphi : K \longrightarrow K$ is well defined and $\widetilde{\varphi}_{|_K} = \varphi$. From [62, Remark 3.2, Theorem 4.2], we know the behaviour of the double cover $\varepsilon : Y \longrightarrow K$ and hence $\varepsilon^{-1}(K \setminus \Sigma) = Y \setminus \Delta$. Since the real codimension of Δ is greater than 2, then the map $\pi_1(Y \setminus \Delta) \twoheadrightarrow \pi_1(Y)$ is surjective. We have that $\pi_1(Y \setminus \Delta) = 0$ and $\varepsilon : Y \setminus \Delta \longrightarrow K \setminus \Sigma$ is an étale cover. We can consider the following diagram:

$$\begin{array}{ccc} Y \setminus \Delta & \xrightarrow{\widetilde{\psi}} & Y \setminus \Delta \\ {\scriptstyle\varepsilon}\big\downarrow{\scriptstyle 2:1} & & {\scriptstyle\varepsilon}\big\downarrow{\scriptstyle 2:1} \\ K \setminus \Sigma & \xrightarrow{\varphi} & K \setminus \Sigma \end{array}$$

From [40, Proposition 1.33] we know that if $\varphi(\varepsilon(\pi_1(Y \setminus \Delta))) \subseteq \varepsilon(\pi_1(Y \setminus \Delta))$ then φ lifts to an automorphism $\widetilde{\psi} : Y \setminus \Delta \longrightarrow Y \setminus \Delta$.

□

Now we would like to extend this $\widetilde{\psi} : Y \setminus \Delta \longrightarrow Y \setminus \Delta$ to an automorphism of Y. To do this we need the following result.

Lemma 3.3.12. *In the previous notations, let $\varepsilon : Y \longrightarrow K$ be a finite map and φ an automorphism of K. Suppose there exists an open subset U of K such that $\varphi_{|_U} : U \to U$ lifts to $\widetilde{\psi} : \varepsilon^{-1}(U) \to \varepsilon^{-1}(U)$, then $\widetilde{\psi}$ extends to a regular morphism $\psi : \varepsilon^{-1}(K) \longrightarrow \varepsilon^{-1}(K)$ such that $\psi_{|_{\varepsilon^{-1}(U)}} = \widetilde{\psi}$.*

$$\begin{array}{ccc} \varepsilon^{-1}(U) \subseteq Y & \xrightarrow{\widetilde{\psi}} & \varepsilon^{-1}(U) \subseteq Y \\ {\scriptstyle\varepsilon}\big\downarrow{\scriptstyle 2:1} & & {\scriptstyle\varepsilon}\big\downarrow{\scriptstyle 2:1} \\ U \subseteq K & \xrightarrow{\varphi} & U \subseteq K \end{array}$$

Proof. From hypothesis we know that $\varphi : K \longrightarrow K$ is regular.
If we denote $\Gamma_\varphi \subset K \times K$ the graph of the morphism, then it is well known that $p_1 : \Gamma_\varphi \xrightarrow{\cong} K$ is an isomorphism. For the same reason we have the graph

$$\Gamma_{\widetilde{\psi}} \subset \varepsilon^{-1}(U) \times \varepsilon^{-1}(U)$$

and the isomorphism $p_1 : \Gamma_{\widetilde{\psi}} \xrightarrow{\cong} \varepsilon^{-1}(U)$. We have that

$$\Gamma_{\widetilde{\psi}} \subset \varepsilon^{-1}(U) \times \varepsilon^{-1}(U) \subseteq Y \times Y$$

where the last is an inclusion in a compact. We can consider the Zariski closure of the graph, that we denote with $\overline{\Gamma_{\widetilde{\psi}}}$. The closure $\overline{\Gamma_{\widetilde{\psi}}}$ lies in a specific closed subset of $Y \times Y$, which is the fiber product over K. To be more precise the fiber product is $Y \times_{\varepsilon, \varphi\circ\varepsilon} Y \subset Y \times Y$.

$$\begin{array}{ccccc} \overline{\Gamma_{\widetilde{\psi}}} & \hookrightarrow & Y \times_{\varepsilon,\varphi\circ\varepsilon} Y & \hookrightarrow & Y \times Y \\ \vdots & \searrow^{\xi} & \big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\overline{\varepsilon}} \\ Y & \xrightarrow{\varepsilon} & \Gamma_\varphi \cong K & \hookrightarrow & K \times K \end{array}$$

In this commutative diagram $\overline{\varepsilon} := \varepsilon \times \varepsilon$ is finite, $\overline{\Gamma_{\widetilde{\psi}}}$ is a subset of $Y \times_{\varepsilon,\varphi\circ\varepsilon} Y$ and

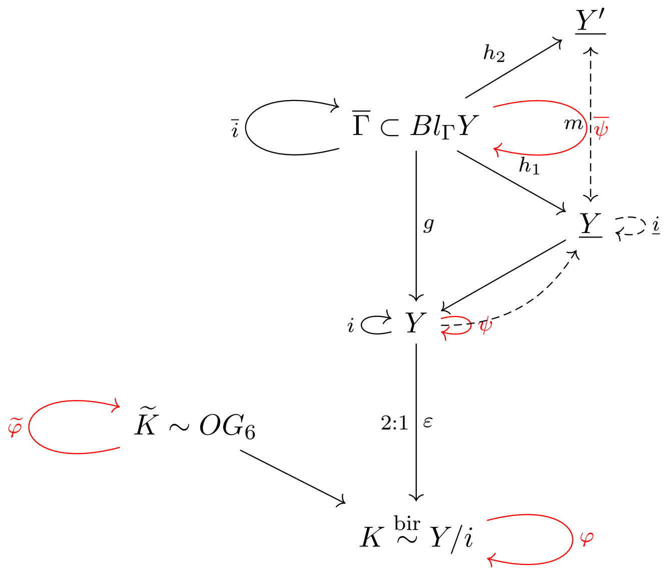
$$Y \times_{\varepsilon,\varphi\circ\varepsilon} Y \xrightarrow{\cong} \Gamma_{\varphi}$$

is an isomorphism by construction. For this reason $\xi : \overline{\Gamma_{\widetilde{\psi}}} \longrightarrow K$ is a finite morphism and consequently $\overline{\Gamma_{\widetilde{\psi}}} \longrightarrow Y$ is a finite morphism. Now by hypothesis we have that the previous map is injective on an open subset. Since Y is a normal variety we can conclude that $\overline{\Gamma_{\widetilde{\psi}}} \longrightarrow Y$ is an isomorphism, which implies that $\psi : Y \longrightarrow Y$ is a regular morphism, where ψ is such that $\psi_{|_{\varepsilon^{-1}(U)}} = \widetilde{\psi}$. □

Proof. (of Theorem 3.3.9)
Using Lemma 3.3.11 and Lemma 3.3.12, where $U = K \setminus \Sigma$, we can conclude.

□

So far we have shown under which conditions we can lift a morphism on $\widetilde{K} \sim OG_6$ to a regular morphism on Y, that is a singular manifold birational to an IHS manifold of $K3^{[3]}$ type [62, Lemma 5.2, Proposition 5.3]. Now we need to recall some results of [62] to know when this $\psi : Y \longrightarrow Y$, defined on a singular variety birational to an Hilbert scheme parametrizing 0-dimensional subscheme of lenght 3 on a $K3$ surface, lifts to a map on $\underline{Y}$ which is a smooth manifold of $K3^{[3]}$ type. The diagram is the following:



Here Γ is the singular locus of Y and it is consists of 256 singular points. We have that $\psi(\Gamma) = \Gamma$. In general this does not mean that each singular point is mapped in itself, there could be the possibility that these points are permuted. It is a classical result that the morphism $\psi : Y \longrightarrow Y$ extends in a direct way on the blow up of these singular points, which means that $\overline{\psi} : Bl_{\Gamma}Y \longrightarrow Bl_{\Gamma}Y$ is well defined. In fact one of these singular points is mapped in another singular points (for simplicity we assume that it is mapped in itself). So we are in the following hypothesis:

$$\text{Let } p \text{ be a singular point of } Y \text{ then } \psi(p) = p. \tag{3.3}$$

We already know the behaviour of ψ in a neighborhood of this point, and for this reason we know the behaviour of ψ on the normal bundle in this point. Hence we can define $\overline{\psi}$ on the blow up of Y. What we need to find is a sufficient condition to extend this automorphism on $\underline{Y}$, the manifold of $K3^{[3]}$ type. If we find this condition, we will be able to state when an automorphism of a manifold of OG_6 type is induced at the quotient (see Definition 3.3.5). As we have recalled in Section 1.4.2, $g^{-1}(\Gamma) = \overline{\Gamma}$ is the exceptional divisor of $Bl_\Gamma(Y)$ and consists of 256 copies of the incidence variety; every incidence variety is indicated by I_i and $I_i \subset \mathbb{P}(V) \times \mathbb{P}(V)^\vee$, where V is a 4 dimensional vector space, as we have described in Section 1.4.2. The incidence variety $I_i \subset \mathbb{P}(V) \times \mathbb{P}(V)^\vee$ has two natural $\mathbb{P}^2$ fibrations given by the projections onto $\mathbb{P}(V)$ and $\mathbb{P}(V)^\vee$. For any i, let $p_i : I_i \longrightarrow \mathbb{P}(V)$ be one of the two projections. We know that Y is locally analytically isomorphic to the cone W, the normal bundle of I_i in $Bl_\Gamma Y$ has degree -1 on the fibers of p_i. Therefore by applying Nakano's contraction Theorem ([68]), there exists a complex manifold $\underline{Y}$ and a morphism of complex manifolds $h : BL_\Gamma Y \longrightarrow \underline{Y}$ whose exceptional locus is $\overline{\Gamma}$ and is such that the image $J_i = h(I_i)$ of any component $\overline{\Gamma}$ is isomorphic to $\mathbb{P}^3$. Moreover the restriction of h to I_i equals p_i and h realizes $Bl_\Gamma Y$ as the blow up of $\underline{Y}$ along the disjoint union $J = h(\overline{\Gamma})$ of the $J_i's$. The complex manifold $\underline{Y}$ is a projective IHS manifold that is deformation equivalent to the Hilbert scheme parametrizing 0-dimensional subschemes of length 3 on a $K3$ surface. By construction $\underline{Y}$ has a regular birational morphism to Y contracting J to Γ which is made by 256 singular points. Since we would like to find a condition to extends the map $\overline{\psi}$ to a map $\underline{\psi} : \underline{Y} \longrightarrow \underline{Y}$, it is important to recall the Remark 5.4 in [62] which explains why the involution $\bar{\imath}$ can not be extended to a regular involution on $\underline{Y}$.

Remark 3.3.13. ([62, Remark 5.4]) Since the involution $\bar{\imath} : Bl_\Gamma(Y) \longrightarrow Bl_\Gamma(Y)$ sends the exceptional divisor of the blow up, $\overline{\Gamma}$, to itself, so for sure it descends to a rational involution $\underline{i} : \underline{Y} \longrightarrow \underline{Y}$ restricting to a regular involution on the complement $\underline{Y} \setminus J$ on the union of the projective spaces J_i in $\underline{Y}$. Since, by definition of i (this depends on the local structure), the involution $\bar{\imath}$ exchanges the two $\mathbb{P}^2$ fibrations on I_i, the indeterminacy locus of $\underline{i}$ is J. Finally, since $Bl_\Gamma(Y) \simeq Bl_J\underline{Y}$, the rational involution $\underline{i}$ may be described as the composition of a Mukai flop along J and an isomorphism outside of this locus.

In this setting it is obvious that $\underline{\psi}$ is well defined outside J which is made by the disjoint union of 256 copies of $\mathbb{P}^3$, but we would like to explain under which conditions it is possible to extend this map on these $\mathbb{P}^3$'s. To do this we need to focus on a fiber of a singular point p of Y, $g^{-1}(p) \simeq I_i$, which is a divisor of $Bl_\Gamma Y$. The preimage $g^{-1}(p) \simeq I_i$ is the incidence variety, and we know that this is a fibration with basis $\mathbb{P}^3$ and fiber $\mathbb{P}^2$, for this reason $I_i \simeq \mathbb{P}^5$ and by the local structure of this singularity described in [62], we obtain the following diagram.

$$\begin{array}{ccc} & & \mathbb{P}^3 \subset \underline{Y'} \\ & \overset{p_1}{\nearrow} & \uparrow \\ \mathbb{P}^3 \times \mathbb{P}^3 \supset I_1 \simeq I_2 \subset Bl_\Gamma Y & & \vdots \; Mukaiflop \\ & \underset{p_2}{\searrow} & \downarrow \\ & & \mathbb{P}^3 \subset \underline{Y} \end{array}$$

For the sake of notation, we call the incidence variety $I := I_1 \cong I_2$. Since $Bl_\Gamma Y \cong Bl_J \underline{Y}$, we have the following result.

Proposition 3.3.14. *$I \cong \mathbb{P}(\Omega_{\mathbb{P}^3})$ and $Pic(I) \cong Pic(\mathbb{P}^3 \times \mathbb{P}^3) \cong \langle H_1, H_2 \rangle$ where $H_1 = p_1^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $H_2 = p_2^*(\mathcal{O}_{\mathbb{P}^3}(1))$.*

Proof. $\underline{Y}$ is an IHS manifold of six dimension and $\mathbb{P}^3$ is a lagrangian subspace of $\underline{Y}$. The symplectic form $\sigma_{\underline{Y}}$ gives a duality between $\mathcal{T}_{\mathbb{P}^3}$ and $\Omega_{\mathbb{P}^3}$, but $\sigma_{\underline{Y}}$ on the tangent bundle is zero, this duality is the one that sends $\mathcal{N}_{\mathbb{P}^3 | \underline{Y}}$ to $\Omega_{\mathbb{P}^3}$ which are isomorphic. We know that the exceptional locus of this blow up is $I \cong \mathbb{P}(\mathcal{N}_{\mathbb{P}^3}) \cong \mathbb{P}(\Omega_{\mathbb{P}^3})$. We define $\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1) := p_1^*(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^3}(2))$. Since on I are defined two $\mathbb{P}^2$ fibrations, if we call $H_1 = p_1^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $H_2 = p_2^*(\mathcal{O}_{\mathbb{P}^3}(2))$ we can say that $Pic(\mathbb{P}^3 \times \mathbb{P}^3)$ is generated by $\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)$. By Lefschetz's Theorem for the Picard group, we know that $Pic(I) = Pic(\mathbb{P}^3 \times \mathbb{P}^3) = \langle H_1, H_2 \rangle$, where H_1 comes from the first fibration and H_2 comes from the second fibration. □

In the following theorem we find that a sufficient condition for an automorphism $\overline{\psi}$ defined on $Bl_\Gamma Y$ to descend to an automorphism on $\underline{Y} \sim K_3^{[3]}$ is that it doesn't exchange the fibers of the two $\mathbb{P}^2$ fibrations. In Remark 3.3.13 we find that the involution $\overline{i}$ defined on $Bl_\Gamma(Y)$ exchanges the fibers of the two fibrations and for this reason we can't extends $\overline{i}$ to an isometry on $\underline{Y}$, but we can define just a birational isometry $\underline{i}$ on it.

Theorem 3.3.15. *Let X be a manifold of OG_6 type and let Y be the 2:1 cover of X described above. Let $\varphi \in \mathrm{Aut}(X)$ an automorphism of prime order p, $p \neq 2$, such that $\mathrm{Sing}(Y) \subset \mathrm{Fix}(\varphi)$. Suppose there exists a class $E \in \mathrm{NS}(X) \cap T_\varphi(X)$ with $E^2 = -2$ and $div(E) = 2$. In these hypotheses φ is induced at the quotient.*

Remark 3.3.16. The request of Theorem 3.3.15 that $\Sigma \subset \mathrm{Fix}(\varphi)$ is true if we assume the condition expressed in 3.3.

Lemma 3.3.17. *Let f be an automorphism of $Bl_\Gamma Y$ that fixes the exceptional divisor, and f^* the induced action on $Pic(I) = \langle H_1, H_2 \rangle$. Then f^* is the identity or $f^*(H_1) = H_2$ and $f^*(H_2) = H_1$.*

Proof. From the hypothesis we know that H_1 and H_2 are hyperplane sections of the two copies of $\mathbb{P}^3$, that in this proof we denote by $\mathbb{P}^3$ and $(\mathbb{P}^3)^*$ to distinguish them. Recall that the pullback commutes with the intersection product, and for this reason, if we consider the product H_1^k, we can say that this is zero when $k \geq 4$

and the same holds true for H_2. We define $h_1 = \mathscr{O}_{\mathbb{P}^3}(1)$ and $h_2 = \mathscr{O}_{(\mathbb{P}^3)^*}(1)$, l is a line, then it holds

$$H_1^2 = p_1^*(h_1^2) = [p_1^{-1}(l)] = [(\mathbb{P}^3 \times l) \cap I],$$

where the class is in the Chow group. Moreover, for H_2 it holds the same:

$$H_2^3 = p_2^*(h_2^3) = [p_2^{-1}(*)] = [(* \times (\mathbb{P}^3)^* \cap I].$$

This is the fiber of the point $*$ and this is isomorphic to $\mathbb{P}^2$. The product $H_1^2 H_2^3$ is equal to 1, since this is an intersection of a line and a $\mathbb{P}^2$ in a generic position. With the same argument, but exchanging the role of H_1 and H_2 we obtain that $H_1^3 H_2^2$ is equal to 1.

Since the pullback operation commutes with the intersection form, from the initial remark we have that $f^*(H_1)^5 = f^*(H_1^5) = 0$. In general since the action of f^* preserves the Picard group of I, we can denote $f^*(H_1) = \alpha H_1 + \beta H_2$ and $f^*(H_2) = \gamma H_1 + \delta H_2$. With this notation we have:

$$(f^* H_1)^5 = \sum_{i=0}^{5} \binom{5}{i} \alpha^i \beta^{5-i} H_1^i H_2^{5-i} = 10\alpha^2\beta^3 H_1^2 H_2^3 + 10\alpha^3\beta^2 H_1^3 H_2^2 = 10\alpha^2\beta^3 + 10\alpha^2\beta^3.$$

Furthermore we have

$$\alpha^2\beta^2(\alpha+\beta) = 0.$$

In the same way for H_2 we obtain:

$$\gamma^2\delta^2(\gamma+\delta) = 0.$$

After some straightforward computation we obtain the following six cases:

$$\begin{cases} f^*(H_1) = H_1 \\ f^*(H_2) = H_2 \end{cases} \quad \begin{cases} f^*(H_1) = \pm(H_1 - H_2) \\ f^*(H_2) = H_2 \end{cases} \quad \begin{cases} f^*(H_1) = H_1 \\ f^*(H_2) = \pm(H_1 - H_2) \end{cases}$$

$$\begin{cases} f^*(H_1) = H_2 \\ f^*(H_2) = \pm(H_1 - H_2) \end{cases} \quad \begin{cases} f^*(H_1) = H_2 \\ f^*(H_2) = H_1 \end{cases} \quad \begin{cases} f^*(H_1) = \pm(H_1 - H_2) \\ f^*(H_2) = H_1 \end{cases}$$

We can notice that $f^*(H_1) = \pm(H_1 - H_2)$ is not allowed. In fact, let $l_1 \subset p_1^{-1}(p) \simeq \mathbb{P}^2$ and let $l_2 \subset p_2^{-1}(q) \simeq \mathbb{P}^2$ be two lines which lie in the two different fibrations. Since $f^*H_1.l_1 = f_*(f^*H_1.l_1) = H_1.f_*l_1$, we need to notice that f_*l_1 is a line which means $f_*l_1 \cong \mathbb{P}^1$ since f is an automorphism and for this reason $H_1.f_*l_1$ could be 1 or 0. If $H_1.f_*l_1 = 1$ and if it holds that $f^*(H_1) = H_1 - H_2$, then we have that $H_1 l_1 - H_2 l_1 = -1$ which is an absurd. This holds also in the other similar cases, choosing the right intersection with l_1 or l_2 and for this reason we can conclude that the two possible actions of f^* on $Pic(I)$ are the identity and the automorphism which exchanges H_1 and H_2. □

Proof. (of Theorem 3.3.15)
If $X \sim OG_6$ we know from Theorem 3.3.3 that X is birational to Y/i where Y is birational to an IHS manifold of $K3^{[3]}$ type. From Theorem 3.3.9 we know that φ lifts to ψ on Y. For the previous considerations we can say that ψ lifts to $\overline{\psi}$ in a

direct way, but we know that to descends to $\underline{Y}$ we need that the fibrations are not exchanged. From Lemma 3.3.17 we know the action of $\overline{\psi}$ on $\mathrm{Pic}(I)$, hence we deduce that if the order of the automorphism is prime $p > 2$, the action is the identity on $\mathrm{Pic}(I)$. The fibrations are not exchanged and we can define $\underline{\psi} : \underline{Y} \longrightarrow \underline{Y}$, which means that φ is induced at the quotient. □

Chapter 4

Non-symplectic automorphisms

This chapter is devoted to classify purely non-symplectic automorphisms of prime order on manifolds of OG_6 type. In doing this, we will be inspired by the techniques used in Chapter 2 to classify non-symplectic automorphisms on a K3 surfaces.

In this setting X is a manifold of OG_6 type, $G \subset \mathrm{Aut}(X)$ a finite group of non-symplectic automorphisms with $|G| = m$. We need the following remark and definitions. Let X, X_1 and X_2 be irreducible holomorphic symplectic manifolds.

Definition 4.0.1. An isomorphism $f\colon H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ is a *parallel transport operator* if there exists a smooth proper family $\pi\colon \mathcal{X} \to B$ of IHS manifolds over a base B, points $b_i \in B$, isomorphisms $\psi_i\colon X_i \to \mathcal{X}_{b_i}$, $i = 1, 2$ and a continuous path $\gamma\colon [0,1] \to B$ such that $\gamma(0) = b_1$ and $\gamma(1) = b_2$ and such that the parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ induces the homomorphism $\psi_{2_*} \circ f \circ \psi_1^*\colon H^*(\mathcal{X}_{b_1},\mathbb{Z}) \to H^*(\mathcal{X}_{b_2},\mathbb{Z})$. An isomorphism $g\colon H^k(X_1,\mathbb{Z}) \to H^k(X_2,\mathbb{Z})$ is said to be a *parallel-transport operator*, if it is the k-th graded summand of a parallel-transport operator f as above.

Definition 4.0.2. An automorphism $f\colon H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ is said to be a *monodromy operator* if it is a parallel-transport operator. The *monodromy group* $Mon(X)$ is the subgroup of $GL(H^*(X,\mathbb{Z}))$ consisting of all monodromy operators. We define $Mon^2(X)$ the image of $Mon(X)$ in $O(H^2(X,\mathbb{Z}))$.

Definition 4.0.3. Let $\varphi \in O(H^2(X,\mathbb{Z}))$ an isometry. We call φ an *Hodge operator* if the the $\mathbb{C}$-linearized action of φ is such that $\varphi(H^{2,0}(X)) \subseteq H^{2,0}(X)$ and $\varphi(H^{1,1}(X)) \subseteq H^{1,1}(X)$,

Remark 4.0.4. Let $G \subset \mathrm{Aut}(X)$ be a finite group and $\varphi \in G$. Then $\varphi^* \in O(H^2(X,\mathbb{Z}))$ is an Hodge-monodromy operator. This means that φ is an Hodge operator and that $\varphi^* \in Mon^2(X) \subset O(H^2(X,\mathbb{Z}))$.

Definition 4.0.5. Let X be an IHS manifold and ω_X be a generator of $H^{2,0}(X)$. Let $\varphi \in O(H^2(X,\mathbb{Z}))$ be an isometry of finite order m. The isometry is non-symplectic if the $\mathbb{C}$-linearized action of φ is such that $\varphi(\omega_X) = \alpha\,\omega_X$ with $\alpha \in \mathbb{C}$, $\alpha \neq 1$ and $\alpha^m = 1$.

Lemma 4.0.6. *Let X be an IHS manifold and let ω_X be a generator of $H^{2,0}(X)$. Let $G \subset \mathrm{Aut}(X)$ be a non-symplectic (Definition 1.5.3) and finite group of automorphisms. Then $\varphi^* \in O(H^2(X,\mathbb{Z}))$ is an isometry and let m be the order of φ^*. In this hypothesis the $\mathbb{C}$-linearized action of φ^* is such that $\varphi^*(\omega_X) = \zeta_m(\omega_X)$ where $\zeta_m \neq 1$ denotes a m-th root of unity.*

Proof. This is a consequence of Remark 4.0.4. □

From a result of Beauville (see [8]), we have the following proposition:

Proposition 4.0.7. *Let X be an IHS manifold. If $G \subset \mathrm{Aut}(X)$ is a finite group of purely non-symplectic automorphisms, then G is cyclic, which means that $G \cong \mathbb{Z}/m\mathbb{Z}$ where $|G| = m$.*

Now, inspired by the definitions of chapter 2, we need to introduce some notations: if $G \subset Aut(X)$, we define the invariant sublattice with respect the induced action of G on $H^2(X,\mathbb{Z})$

$$T_G(X) := H^2(X,\mathbb{Z})^G$$

The orthogonal complement of $T_G(X)$ in $H^2(X,\mathbb{Z})$ is the co-invariant sublattice

$$S_G(X) := T_G(X)^{\perp}.$$

It holds that

$$H^2(X,\mathbb{Z}) \otimes \mathbb{Q} = (T_G(X) \oplus S_G(X)) \otimes \mathbb{Q}.$$

Remark 4.0.8. Let X be an IHS manifold and $G \subset O(H^2(X,\mathbb{Z}))$ a finite subgroup of non-symplectic isometries. In the moduli space of pairs (X,G), where the action of G is fixed and the invariant sublattice $T_G(X)$ is fixed, the generic element is such that $T_G(X) = \mathrm{NS}(X)$ and consequently $T(X) = S_G(X)$. If the action is symplectic the generic point of the family of the pairs (X,G) is such that $T(X) = T_G(X)$ and consequently $S_G(X) = \mathrm{NS}(X)$. See [78] for a more detailed reference.

Proposition 4.0.9. *Let X be an IHS manifold of OG_6 type, suppose $G \cong \mathbb{Z}/p\mathbb{Z}$ is a non-symplectic group of automorphisms and p is a prime number. Then $p \in \{2,3,5,7\}$.*

Proof. From Proposition 1.5.5 there exists $n \in \mathbb{Z}$ such that $rk(T(X)) = \phi(p)n \leq 7$. Thus $\phi(p) = p - 1 \leq 7$ and this implies that $p \in \{2,3,5,7\}$. □

The main goal of our work is to classify automorphisms of manifolds of OG_6 type. If we classify effective isometries, i.e. we study $Im(\nu)$ for manifolds of OG_6 type, where

$$\nu : \mathrm{Aut}(X) \longrightarrow O(H^2(X,\mathbb{Z})),$$

we will obtain a classification, up to $\ker(\nu)$, of automorphisms, at least for the prime order and non-symplectic case, as it is shown in the next proposition.

Proposition 4.0.10. *If X is a IHS manifold of OG_6 type and $\varphi \in O(H^2(X,\mathbb{Z}))$ is a non-symplectic isometry of prime order $p \in \{3,5,7\}$ then φ is effective, if $p = 2$ then φ is effective up to a sign.*

Proof. To show this result we use Theorem 1.2.34. Since φ is non-symplectic, $\varphi(\omega_X) = \alpha\ \omega_X$ where $\alpha \neq 1$, $\alpha \in \mathbb{C}$ is a p-root of unity. Let $G = \langle\varphi\rangle \cong \mathbb{Z}/p\mathbb{Z}$ be the group of isometries generated by φ. With these hypothesis $T(X) \subseteq S_G(X)$. Since $T_G(X)$ and $S_G(X)$ are in direct sum and the same holds for $T(X)$ and $\mathrm{NS}(X)$, this implies that $T_G(X) \subseteq \mathrm{NS}(X)$. It is not restrictive to consider that $T_G(X) = \mathrm{NS}(X)$. Consequently $\varphi \in O(H^2(X,\mathbb{Z}))$ is an Hodge isometry. From [61, Theorem 5.4(1)] we know the monodromy of a manifold of OG_6 type, i.e. $Mon^2(X) = O^+(H^2(X,\mathbb{Z}))$, where $O^+(H^2(X,\mathbb{Z}))$ denotes the isometries which preserve the orientation of the positive cone of X, where the positive cone is the connected component of $\{x \in H^{1,1}(X,\mathbb{R}) | (x,x) > 0\}$ which contains the cone of Kähler classes, $\mathcal{K}_X$ (see Definition 1.2.36). Since φ is of prime order p, φ preserves the orientation of the positive cone if $p \neq 2$, otherwise $\pm\varphi$ preserves the orientation of the positive cone, which means that $\pm\varphi \in Mon^2(X)$.
The last thing that we need to check is that a Kähler class is sent to a Kähler class. In order to show this we need to show that in this hypothesis, if we consider the generic element od the pair (X, G), the manifold X is projective. Consider the class defined in this way $\omega_G := \sum_{\varphi\in G} \varphi(\omega)$, where ω is a Kähler class. By definition ω is positive and $\omega \in \mathrm{NS}(X)\otimes\mathbb{R}$, and since φ is an isometry, $\varphi(\omega)$ is positive. Since the positive classes lies in cone, ω_G is a positive class. In our assumption $T_G(X)\otimes\mathbb{R} = \mathrm{NS}(X)\otimes\mathbb{R}$, and by construction $\varphi(\omega_G) = \omega_G$, thus $\omega_G \in \mathrm{NS}(X) \otimes \mathbb{R}$. Consequently, since by definition $sgn(\mathrm{NS}(X)) = sgn(\mathrm{NS}(X) \otimes \mathbb{R})$, then $sgn(\mathrm{NS}(X)) = (1, *)$. For sure the positive part can not be greater than 1 since the orthogonal complement of $\mathrm{NS}(X)$ is $T(X) = S_G(X)$ and there are two positive classes in $T(X)$ which are the symplectic form and its conjugated. For the Huybrechts' projectivity criterion, since there is a positive class in the $\mathrm{NS}(X)$, then X is projective. Consequently, since $\mathrm{NS}(X) = T_G(X)$, there exists an invariant ample class. In this way we conclude that there exists a Kähler class which is preserved by φ and for this reason φ is effective.

□

4.0.1 How to classify automorphisms

From the previous proposition, if X is of OG_6 type and $\varphi \in O(H^2(X,\mathbb{Z}))$ is a prime order and non-symplectic isometry, φ is effective, i.e. it comes from an automorphism of X. Let $G = \langle\varphi\rangle$ be the group of non-symplectic isometries generated by φ. To classify effective isometries we can start from a decomposition of $H^2(X,\mathbb{Z})$ in sublattices T and S which are possible invariant and co-invariant sublattices.
By classifying we mean either one of the following:

(1) The first thing that we can do is to compute all the possible pairs of co-invariant and invariant sublattices of $H^2(X,\mathbb{Z})$, $(S_G(X), T_G(X))$. We can do it up to isometries of $T_G(X)$ and $S_G(X)$, this means that if in a table we classify using this criterion, two invariant sublattices with respect to the action of G which are not isometric, are in different pairs.

(2) After (1), another level of classification depends on the fact that each $S_G(X)$ (and in the same way $T_G(X)$) can be embedded in $H^2(X,\mathbb{Z})$ in ways that

are not isometric. This means that we can have $S_G(X)$ and $S'_G(X)$ which are isomorphic but such that do not exist an isometry of $H^2(X,\mathbb{Z})$ which sends $S_G(X)$ in $S'_G(X)$. This is equivalent to count the different primitive embeddings of $S_G(X)$ in $H^2(X,\mathbb{Z})$. To do the computation we need to use Theorem 1.1.13 due to Nikulin, or in some special cases that depend on the rank of $S_G(X)$, we can use Theorem 2.9 of [5]. The non-isometric primitive embeddings count the number of connected components of the period domain (see Definition 1.2.23), i.e. the different images of the pair (X,G) in the period domain, which correspond to the different choices of $S_G(X) = \mathrm{T}(X)$ that we can do.

(3) The third and best level of classification that we can do, concerns to count the different connected components of the moduli space of the pairs (X,G). We can have two manifolds of OG_6 type, X and Y, two pairs (X,G) and (Y,G) such that the lattices $(S_G(X),T_G(X))$ and $(S_G(Y),T_G(Y))$ are isometric to each others and such that there exists an isometry of $H^2(X,\mathbb{Z})$ such that $S_G(X)$ is sent to $S_G(Y)$, but such that there is no way to deform in a continuous way (X,G) to (Y,G) preserving the action of G. This means that (X,G) and (Y,G) are in different connected components of the moduli space of the pairs (X,G). The fact that there are more connected components in the moduli space of pairs (X,G) is related to the presence of wall divisors (see Lemma 1.2.50 for the characterization in the OG_6 case), which are not prime exceptional divisors (see Lemma 1.2.49 for a characterization in the OG_6 case) in the invariant lattice $T_G(X) = \mathrm{NS}(X)$.

The fact that the second integral cohomology of a manifold of OG_6 type is not unimodular often makes computations more complicated. It is therefore sometimes convenient to switch to a bigger unimodular lattice. We consider a primitive embedding $H^2(X,\mathbb{Z}) \hookrightarrow \Lambda := U^{\oplus 5}$. We observe that such an embedding is unique up to an isometry of Λ.

This embedding sends $U^{\oplus 3}$ identically into the first three copies of U and sends n_1 and n_2 which are the two generators of $(U^{\oplus 3})^\perp \subset H^2(X,\mathbb{Z})$ such that $n_1^2 = n_2^2 = -2$, to $e_4 - f_4$ and to $e_5 - f_5$, respectively, where e_4, f_4, e_5, f_5 form the usual basis of the last two copies of $U^{\oplus 2}$. The manifold X is of OG_6 type, and from what we know from Remark 1.5.13, the discriminant group of X is $A_X \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Let $[1,0]$ and $[0,1]$ be the two generators of A_X. Since G is a group of isometries, the induced action of G on A_X is either trivial or it exchanges $[1,0]$ with $[0,1]$.

Definition 4.0.11. Consider the primitive embedding of lattices

$$i\colon H^2(X,\mathbb{Z}) \hookrightarrow \Lambda,$$

and let φ be an isometry of $H^2(X,\mathbb{Z})$. We define the embedding φ-equivariant if there exists an isometry $\tilde{\varphi}$ on Λ such that $\tilde{\varphi}_{|i(H^2(X,\mathbb{Z}))} = \varphi$ and $\tilde{\varphi}$ is the identity on $H^2(X,\mathbb{Z})^{\perp_\Lambda}$.

Let $\varphi \in O(H^2(X,\mathbb{Z})$ and let $\tilde{\varphi}$ be the induced action on Λ. We know from Lemma 1.5.14 that if the induced action of φ on A_X is trivial, the embedding of $H^2(X,\mathbb{Z})$ in Λ is φ-equivariant.

In fact, if the action of $\varphi \in G$ is trivial on A_X then $\widetilde{\varphi}(e_4 + f_4) = e_4 + f_4$ and $\widetilde{\varphi}(e_5 + f_5) = e_5 + f_5$, otherwise if the action is not trivial on A_X then $\widetilde{\varphi}(e_4 + f_4) = e_5 + f_5$ and $\widetilde{\varphi}(e_5 + f_5) = e_4 + f_4$. As usual, we keep calling $T_G(\Lambda) := \Lambda^G$ and $S_G(\Lambda) := T_G(\Lambda)^{\perp}$. The advantage of this setting is that if G is of prime order p, then $T_G(\Lambda)$ and $S_G(\Lambda)$ are p–elementary lattices, by Lemma 1.1.18. This is by not true in general for $S_G(X)$ and $T_G(X)$. The idea is to define a list of possible $p-$elementary sublattices of Λ, (T', S'), where we know that $\langle 2\rangle^{\oplus 2} \subseteq T'$. These will be the possible invariant and co-invariant sublattices with respect to the action of G on Λ.
Starting from this list, it is possible to write down the list (T, S), where $T \cong (\langle 2\rangle^{\oplus 2})^{\perp} \subseteq T'$ and $S = S'$. These are the possible invariants and co-invariants sublattices of $H^2(X,\mathbb{Z})$ with respect to the action of a prime order isometry on it. We start from a list of $p-$elementary sublattices in an abstract way and to know which ones correspond to an action of an isometry on $H^2(X,\mathbb{Z})$ we can use results contained in [4], where the authors classify non-symplectic automorphisms of prime order on $K3$ surfaces. The trick is to find an embedding of S in $H^2(K3,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, which is a unimodular lattice of big rank. Using Lemma 1.5.14 we know it is possible to extend the action of G on the $K3-$lattice in a way such that it is trivial on $S^{\perp} \subseteq H^2(K3,\mathbb{Z})$. By construction $S \equiv S_G(K3)$ and so, if S is not in the list of possible co-invariant sublattices that we can find in the tables of classification in [4] we know that there is no isometry on $H^2(X,\mathbb{Z})$ such that the action has S as co-invariant sublattice. On the other hand if S is in the list we can conclude that there exists an isometry of $H^2(X,\mathbb{Z})$ such that S is the co-invariant sublattice and consequently T is the invariant one.

For $p = 2$ we know that the isometry which corresponds to the order-two action is $-Id_{S_G(\Lambda)}$; moreover since for the generic point of the moduli space of (X, G), $S_G(X) = T(X)$, the signature of $S_G(X) = (2, *)$, so we are sure that isometries of order two correspond to a non-symplectic involution. For $p \in \{3, 5, 7\}$, to know which co-invariant sublattices S, p-elementary that we have classified correspond to an isometry of order p we can consider, as we have explained above, the following $G-$equivariant embedding:

$$S_G(\Lambda) \hookrightarrow H^2(K3,\mathbb{Z}).$$

Since the induced action of G on A_X is trivial for $p \geq 3$, we know that $S_G(\Lambda) \cong S_G(K3)$. When the co-invariant lattice is isometric to the co-invariant lattice of a $K3$ surface with an automorphism of order p, we have a unique action on the lattice. Indeed in [4] it is proved that the connected component of the moduli space of $K3$ surfaces with an automorphism of order p is given by the isometry classes of the co-invariant lattice. For $p = 3$ all the 3-elementary lattices that we find in the list are possible co-invariant lattices with respect to an automorphism of order three since we can find them in the list in [2]. When $p = 5$ we can accept as co-invariant lattice only $S_G(\Lambda) = U \oplus H_5$ and when $p = 7$ we can accept the only one that we have found as 7-elementary sublattice of Λ i.e. $S_G(\Lambda) = U^{\oplus 2} \oplus K_7$.

Proposition 4.0.12. *Le X be an IHS manifold of OG_6type, suppose $f : H^2(X,\mathbb{Z}) \to H$ is a marking of X. Let $G \subset Mon^2(H)$ be a cyclic subgroup of isometries of order p. Assume that $T_G(H) \subset H$ has signature $(1, *)$ and suppose that $T_G(H) \subset NS(X)$.*

Then the group G therefore consists of parallel transport operators and the isometries are birational effective. If furthermore G preserves all wall divisors in $NS(X)$, then we actually construct automorphisms of X, which, in this case, can easily seen to be non-symplectic. Equivalently G is a group of effective isometries.

4.1 Classification for Λ

Suppose G is cyclic and generated by a non-symplectic element of maximal order p where p is prime, i.e. $G \cong \mathbb{Z}/p\mathbb{Z}$. For what we find in Proposition 4.0.9, $p \in \{2, 3, 5, 7\}$. Moreover, in this case $T_G(X)$ has signature $(1, rk(T_G(X)) - 1)$ and $S_G(X)$ has signature $(2, rk(S_G(X)) - 2)$, as we can find in Proposition 1.5.8. By the considerations of the previous section we obtain an embedding of $H^2(X, \mathbb{Z})$ into $\Lambda \cong U^{\oplus 5}$. Conversely $H^2(X, \mathbb{Z})$ can be considered as the orthogonal complement of a lattice $L \cong \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in Λ. We can call n_1 and n_2 the two orthogonal generators of square 2 in L. If the action of G on A_X is trivial, we have

$$S_G(X) \cong S_G(\Lambda) \ \ \textit{and} \ \ T_G(X) \cong \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{\perp} \subset T_G(\Lambda).$$

Thus $S_G(\Lambda)$ has signature $(2, *)$ and $T_G(\Lambda)$ has signature $(3, *)$.
If $p = 2$ and φ is an automorphism which exchanges $[1, 0]$ and $[0, 1] \in A_X$ we have:

$$T_G(X) \cong (n_1 + n_2)^{\perp} \subset T_G(\Lambda) \ \ \textit{and} \ \ S_G(X) \cong (n_1 - n_2)^{\perp} \subset S_G(\Lambda).$$

Thus $S_G(\Lambda)$ has signature $(3, *)$ and $T_G(\Lambda)$ has signature $(2, *)$. Since Λ is unimodular, then the lattices $S_G(\Lambda)$ and $T_G(\Lambda)$ are p-elementary and their discriminant group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^a$ for some integer $a \geq 0$. In order to classify non-symplectic automorphisms of manifolds of OG_6 type of prime order p, we first study p-elementary sublattices of $\Lambda \cong U^{\oplus 5}$ that might occur as invariant and co-invariant lattices of isometries of Λ of order p. In the case $p = 2$ we add a column $'\delta\,'$ to indicate whether the quadratic form of the discriminant group of the lattices at hand is integer valued ($\delta = 0$) or not ($\delta = 1$).

4.1.1 p=2 - trivial action of G on the discriminant group

Proposition 4.1.1. *The following is a complete list of co-invariant lattices $S_G(\Lambda)$ of signature $(2, *)$ and invariant lattices $T_G(\Lambda)$ of signature $(3, *)$ of order two isometries of Λ.*

Proof. Since $G \cong \mathbb{Z}/2\mathbb{Z}$ then $T_G(\Lambda)$ and $S_G(\Lambda)$ are 2-elementary lattices, which means that $T_G(\Lambda) \cong S_G(\Lambda) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$. For each value of $rk\ T_G(\Lambda)$ and consequently for each value of $rk\ S_G(\Lambda)$, we can establish an upper bound for a using Lemma 1.1.19. For each possible value of a we apply Theorem 1.1.21 and Theorem 1.1.23 for even hyperbolic 2-elementary lattices to find the lattices in the list above. □

No.	$S_G(\Lambda)$	$T_G(\Lambda)$	$rk\ (T_G(\Lambda))$	a	δ
1	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 3}$	$\langle 2\rangle^{\oplus 3}$	3	3	1
2	$U \oplus \langle -2\rangle^{\oplus 3} \oplus \langle 2\rangle$	$\langle 2\rangle^{\oplus 3} \oplus \langle -2\rangle$	4	4	1
3	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$U \oplus \langle 2\rangle^{\oplus 2}$	4	2	1
4	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 3}$	$\langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle^{\oplus 3}$	5	5	1
5	$U \oplus \langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle$	$U \oplus \langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle$	5	3	1
6	$U^{\oplus 2} \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus \langle 2\rangle$	5	1	1
7	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$U \oplus \langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	6	4	1
8	$U(2)^{\oplus 2}$	$U \oplus U(2)^{\oplus 2}$	6	4	0
9	$U \oplus \langle 2\rangle \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus \langle 2\rangle \oplus \langle -2\rangle$	6	2	1
10	$U \oplus U(2)$	$U^{\oplus 2} \oplus U(2)$	6	2	0
11	$U^{\oplus 2}$	$U^{\oplus 3}$	6	0	0
12	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle$	7	3	1
13	$U \oplus \langle 2\rangle$	$U^{\oplus 3} \oplus \langle -2\rangle$	7	1	1
14	$\langle 2\rangle^{\oplus 2}$	$U^{\oplus 3} \oplus \langle -2\rangle^{\oplus 2}$	8	2	1

Table 4.1: Order 2 trivial action on Λ

4.1.2 p=2 - non-trivial action of G

Proposition 4.1.2. *The following is a complete list of co-invariant lattices* $S_G(\Lambda)$ *of signature* $(3,*)$ *and invariant lattices* $T_G(\Lambda)$ *of signature* $(2,*)$ *of order two isometries of* Λ.

No.	$S_G(\Lambda)$	$T_G(\Lambda)$	$rk\ (T_G(\Lambda))$	a	δ
1	$U^{\oplus 3} \oplus \langle -2\rangle^{\oplus 2}$	$\langle 2\rangle^{\oplus 2}$	2	2	1
2	$U \oplus \langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	4	4	1
3	$U \oplus U(2)^{\oplus 2}$	$U(2)^{\oplus 2}$	4	4	0
4	$U^{\oplus 2} \oplus \langle 2\rangle \oplus \langle -2\rangle$	$U \oplus \langle 2\rangle \oplus \langle -2\rangle$	4	2	1
5	$U^{\oplus 2} \oplus U(2)$	$U \oplus U(2)$	4	2	0
6	$U^{\oplus 3}$	$U^{\oplus 2}$	4	0	0
7	$\langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle^{\oplus 3}$	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 3}$	5	5	1
8	$U \oplus \langle -2\rangle \oplus \langle 2\rangle^{\oplus 2}$	$U \oplus \langle 2\rangle \oplus \langle -2\rangle^{\oplus 2}$	5	3	1
9	$U^{\oplus 2} \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus \langle 2\rangle$	5	1	1
10	$U(2) \oplus \langle 2\rangle^{\oplus 2}$	$U \oplus U(2) \oplus \langle -2\rangle^{\oplus 2}$	6	4	1
11	$U \oplus \langle 2\rangle^{\oplus 2}$	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	6	2	1
12	$\langle 2\rangle^{\oplus 3}$	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 3}$	7	3	1

Table 4.2: Order 2 non-trivial action on Λ

Proof. This is a direct application of Theorem 1.1.21 and Theorem 1.1.23. □

4.1.3 p=3

Proposition 4.1.3. *The following is a complete list of coinvariant lattices* $S_G(\Lambda)$ *of signature* $(2,*)$ *and invariant lattices* $T_G(\Lambda)$ *of signature* $(3,*)$ *of order three isometries of* Λ.

No.	$S_G(\Lambda)$	$T_G(\Lambda)$	$rk\ (T_G(\Lambda))$	a
1	$U^{\oplus 2} \oplus A_2(-1)$	$U \oplus A_2$	4	1
2	$U \oplus A_2(-1) \oplus U(3)$	$U(3) \oplus A_2$	4	3
3	A_2	$U^{\oplus 3} \oplus A_2(-1)$	8	1

Table 4.3: Order 3 action on Λ

Proof. When $rk\ (T_G(\Lambda)) = 4$ we have $sign(T_G(\Lambda)) = (3,1)$ and $sign(S_G(\Lambda) = (2,4)$. We can find, by Lemma 1.1.19, that $a \in \{0,1,2,3\}$. We cannot have $a = 0$ because $S_G(\Lambda)$ cannot be unimodular by the fact that $2 \not\equiv 4\ (8)$. We cannot have $a = 2$ since we need that the discriminant of the lattice divided by p^{p-2} is a square in $\mathbb{Q}$, as we can find in Proposition 1.1.25. For this reason a must be odd.
If $a = 1$ then $S_G(\Lambda)$ is uniquely determined because we can write it as $U \oplus S'$ by Theorem 1.1.23 with S' hyperbolic of rank 4, which is unique by Theorem 1.1.22.
If $a = 2$ we can use Theorem 1.1.23 and Theorem 1.1.22 to conclude that this case is not allowed.
If $a = 3$ we can use Theorem 1.1.22 to conclude that there exists a lattice $T_G(\Lambda)$ with these properties. When $rk\ (T_G(\Lambda)) = 6$ we have $sign(T_G(\Lambda)) = (3,3)$ and $sign(S_G(\Lambda)) = (2,2)$. We can find, by Lemma 1.1.19, that $a \in \{0,1,2\}$, but a must be odd from Proposition 1.1.25, then $a = 1$. If $a = 1$ we can use Theorem 1.1.23 and Theorem 1.1.22 to conclude that this case is not allowed because we split $S_G(\Lambda) = U \oplus S'$ and $sign(S') = (1,1)$; so this is a hyperbolic lattice of rank $r = 2$. By Theorem 1.1.22 we conclude that this case is impossible because $p = 3 \not\equiv (-1)^{r/2-1} = 1\ (4)$.
When $rk\ (T_G(\Lambda)) = 8$ we have $sign(T_G(\Lambda)) = (3,5)$ and $sign(S_G(\Lambda)) = (2,0)$. We can find by Lemma 1.1.19, that $a \in \{0,1\}$. We cannot have $a = 0$ because $S_G(\Lambda)$ cannot be unimodular by the fact that $2 \not\equiv 0\ (8)$. If $a = 1$ then $T_G(\Lambda)$ is uniquely determined because we can write it as $U^{\oplus 2} \oplus T'$ by Theorem 1.1.23 applied two times. T' is hyperbolic of rank 4 and is unique by Theorem 1.1.22 and equal to $U \oplus A_2(-1)$. □

4.1.4 p=5

Proposition 4.1.4. *For $p = 5$ the rank of $S_G(\Lambda)$ must be equal to 4. The following is a complete classification with respect to the action of $G \cong \mathbb{Z}/5\mathbb{Z}$ on the unimodular lattice Λ.*

No.	$S_G(\Lambda)$	$T_G(\Lambda)$	$rk\ (T_G(\Lambda))$	a
1	$U \oplus H_5$	$U^{\oplus 2} \oplus H_5$	6	1

Table 4.4: Order 5 action on Λ

Proof. For $p = 5$ the rank of $S_G(\Lambda)$ must be equal to 4 because $rk(S_G(\Lambda)) = \alpha \cdot 4$ and $rk(s_G(\Lambda)) \leq 7$. From Lemma 1.1.19 we have $a \in \{0,1\}$. From Proposition 1.1.25 we know that a must be odd, this implies that $a = 1$. If $a = 1$ we can apply Theorem 1.1.23 to $S_G(\Lambda)$ and we obtain $S_G(\Lambda) = U \oplus S'$. S' is hyperbolic, 5-elementary of rank $r = 2$ and invariants $(r, a) = (2, 1)$. By Theorem 1.1.22 we obtain $S' = H_5$. In this case $S_G(\Lambda) = U \oplus H_5$ and $T_G(\Lambda) = U^{\oplus 2} \oplus H_5$. We recall that $H_5 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

□

4.1.5 p=7

Proposition 4.1.5. *For $p = 7$ there is only a possibility for the pair $(T_G(\Lambda), S_G(\Lambda))$:*

No.	$S_G(\Lambda)$	$T_G(\Lambda)$	$rk\ (T_G(\Lambda))$	a
1	$U^{\oplus 2} \oplus K_7$	$U \oplus K_7(-1)$	4	1

Table 4.5: Order 7 action on Λ

Proof. In this case we have $sign(T_G(\Lambda)) = (3, 1)$ and $sign(S_G(\Lambda)) = (2, 4)$. From Lemma 1.1.19 we obtain $a \in \{0, 1\}$. The case $a = 0$ is not allowed because $2 \not\equiv 4\ (8)$. If $a = 1$ we can apply Theorem 1.1.23 to $S_G(\Lambda)$ and we obtain $S_G(\Lambda) = U \oplus S'$. S' is hyperbolic, 7-elementary of rank $r = 4$, signature $(1, 3)$ and invariants $(r, a) = (4, 1)$. By Theorem 1.1.22 we obtain $S' = U \oplus K_7$. We recall that $K_7 = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$ □

4.2 Classification for $\mathbf{OG_6}$

Now we specialize to the case of manifolds of OG_6 type. As in the previous section we consider the primitive embedding

$$H^2(X, \mathbb{Z}) \hookrightarrow \Lambda := U^{\oplus 5}.$$

Lattice-theoretically this is done by choosing a sublattice $L = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in Λ. The cohomology $H^2(X, \mathbb{Z})$ will then be isometric to $L^{\perp}$. Keeping in mind that we can have different choices of embedding L in Λ related to the action of G, we will give a list of all configurations of lattices occurring as invariant and co-invariant lattices of prime order and non-symplectic automorphisms of manifolds of OG_6 type.

If L is embedded in the invariant lattice $T_G(\Lambda)$, the induced action on $H^2(X, \mathbb{Z})$ corresponds to a trivial action on A_X. In this situation the discriminant group of $S_G(X)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^a$ for some integer $a(S_G(X)) = a \geq 0$.

It is useful to recall when two lattices have the same genus.

Proposition 4.2.1. *L and T have the same genus $\iff$ $sgn(L) = sgn(T)$ and $q_{A_L} = q_{A_T}$.*

We show that if $p \geq 3$ we have that $T_G(X)$ is uniquely determined up to isometry regardless of the embedding of L in $T_G(\Lambda)$. We need to show that the discriminant group of $T_G(X)$, i.e. $A_{T_G(X)}$, is the same for each different embedding of $L =$

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in $T_G(\Lambda)$. If we have this result we can find an embedding of L in $T_G(\Lambda)$, we compute the orthogonal of L embedded in $T_G(\Lambda)$ and we find $T_G(X)$. If $T_G(X)$ is unique in its genus we obtain the unicity of the invariant lattice $T_G(X)$ of $H^2(X,\mathbb{Z})$.

First of all we can fix the notation, we have $L \oplus T_G(X) \subset T_G(\Lambda)$, which is an embedding of finite index and L and $T_G(X)$ are primitive sublattices of $T_G(\Lambda)$. We know that $A_{T_G(\Lambda)} = (\mathbb{Z}/p\mathbb{Z})^a$ for some a and $A_L = (\mathbb{Z}/2\mathbb{Z})^2$. Using Proposition 1.1.14, this embedding is given by two subgroups $H_L \subset A_L$ and $H_{T_G(X)} \subset A_{T_G(X)}$ and by an anti-isometry $\gamma : H_L \longrightarrow H_{T_G(X)}$. We can call $H_T = \Gamma_\gamma = \{(a,\gamma(a)), s.t.\ a \in H_L\}$ and we know that $A_{T_G(\Lambda)} = \Gamma_\gamma^\perp/\Gamma_\gamma = (H_T)^\perp/H_T$ where $H_T \subset A_L \oplus A_{T_G(X)}$ is an isotropic subgroup.

Proposition 4.2.2. *In the previous notation, if $p \neq 2$, we have:*

i) $H_L = A_L$ *and since γ is an anti-isometry,* $H_{T_G(X)} = A_L(-1)$

ii) $A_{T_G(\Lambda)} \subset A_{T_G(X)}$ *($A_{T_G(\Lambda)} \oplus A_L(-1) \subset A_{T_G(X)}$ and these are orthogonal complements with respect to the product in $A_{T_G(X)}$).*

iii) $A_{T_G(X)} = A_{T_G(\Lambda)} \oplus A_L(-1)$.

Proof. i) In this proof $N := T_G(X)$ and $T := T_G(\Lambda)$. Suppose $H_L \subsetneq A_L$. We would like to find $a \in A_L \setminus H_L$ such that $a \in A_T$. We will show that, if $a \in A_L \setminus H_L$ then $a \perp H_L$.

This can not happen because $A_T = (\mathbb{Z}/p\mathbb{Z})^a$ and $p \neq 2$, so it doesn't contain elements of order 2. For this reason we have to conclude that $H_L = A_L$. Let $a \neq 0 \in A_T = \Gamma_\gamma^\perp/\Gamma_\gamma$ which means that $a \in \Gamma_\gamma^\perp$ but $a \notin \Gamma_\gamma$. We need $(c,\gamma(c)) \in \Gamma_\gamma$ and we consider $e \in A_N$ such that $\gamma(e) = 0$, since γ is injective, $e = 0$ and in this way we are sure that $e \in H_N$. In these hypotesis we can write:

$$b_{A_{L\oplus N}}((a,e),(c,\gamma(c)) = b_{A_L}(a,c) + b_{A_N}(e,\gamma(c)) =$$
$$b_{A_L}(a,c) - b_{A_L}(\gamma^{-1}(0),\gamma^{-1}(\gamma(c))) = b_{A_L}(a - \gamma^{-1}(0), c) = b_{A_L}(a,c).$$

If we find a such that $b_{A_L}(a,c) = 0\ \forall\ c \in H_L$, we will find $(a,0) \in \Gamma_\gamma^\perp$ such that $(a,0) \notin \Gamma_\gamma$ i.e we will find $(a,0) \in A_T$. $A_L \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ so the proper subgroups H_L different from the trivial one are $\{[0,0],[0,1]\}, \{[0,0],[1,0]\}, \{[1,1],[0,0]\}$.

We can use the following trick. Let H_L be one of the non-trivial subgroups and let a be the $[0,0]$ class and b the other one. We have $H_T \subset A_L \oplus A_N$ and ${H_T}^\perp \subset A_L \oplus A_N$ and in particular $H_T = \{(a,\gamma(a)),(b,\gamma(b))\} \cong \mathbb{Z}/2\mathbb{Z}$. Now since $N = L^\perp \subset T$, we can say that $A_N \cong (\mathbb{Z}/2\mathbb{Z})^b \oplus (\mathbb{Z}/p\mathbb{Z})^a$, and for this reason we obtain $A_L \oplus A_N = (\mathbb{Z}/2\mathbb{Z})^{b+2} \oplus (\mathbb{Z}/p\mathbb{Z})^a$. We have $A_T = H_T^\perp/H_T$ and $q_{A_L \oplus A_N}$ is a non-degenerate quadratic form on $A_L \oplus A_N$, since we are taking the orthogonal of a group of order 2 which is H_T, we have $H_T^\perp = (\mathbb{Z}/2\mathbb{Z})^{b+1} \oplus (\mathbb{Z}/p\mathbb{Z})^a$. Moreover $A_T = H_T^\perp/H_T = (\mathbb{Z}/2\mathbb{Z})^b \oplus (\mathbb{Z}/p\mathbb{Z})^a$ and b has to be equal to zero since there are no elements of order 2 in A_T. If $b = 0$ $A_N = (\mathbb{Z}/p\mathbb{Z})^a$ which is false because we must have elements of order two in A_N. We can conclude that $H_L = A_L$ as we want.

ii) & iii) By the first step $H_N = A_L(-1) \subset A_N$, $H_N^{\perp} \subset A_N$. We would like to show that $H_N^{\perp} \cong A_T = H_T^{\perp}/H_T$.
The group A_L is $(\mathbb{Z}/2\mathbb{Z})^2$ and this implies that $H_N = A_L(-1) \cong (\mathbb{Z}/2\mathbb{Z})^2$ which means that $b = 2$.
$H_N = (\mathbb{Z}/2\mathbb{Z})^2 \subseteq A_N = (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/p\mathbb{Z})^a$ and this implies that $H_N^{\perp} \cong (\mathbb{Z}/p\mathbb{Z})^a$ since the last orthogonal complement is with respect to the product in A_N. Since $|H_N^{\perp}| = |A_T|$, the idea is to construct an isomorphism $g : H_N^{\perp} \to H_T^{\perp}/H_T$ which sends $\alpha \in H_N^{\perp}$, $\alpha \neq 0$ to $[g(\alpha)] = [(0, \alpha)]$. We need to verify that $g(\alpha) \in H_T^{\perp}$, in particular $(0, \alpha) \in A_L \oplus A_N$ and $\alpha \in H_N^{\perp} \subset A_N$, so if $(a, b) \in H_T$ we have $q_{A_L \oplus A_N}((0, \alpha), (a, b)) = q_{A_L}(0, a) + q_{A_N}(\alpha, b) = 0$ because q_{A_L} is non-degenerate, $\alpha \in H_N^{\perp}$ and $b \in H_N^{\perp}$. This allow us to conclude that $(0, \alpha) \in H_T^{\perp}$.
In this hypothesis g is well defined. Actually the last thing that we must prove is the injectivity of g. We have

$$[g(\alpha)] = g[(\beta)] \Leftrightarrow [(0, \alpha)] = [(0, \beta)],$$

$$(0, \alpha) = (0, \beta) + (c, \gamma(c)) \Rightarrow (0, \alpha - \beta) = (c, \gamma(c)) \Rightarrow (0, \alpha - \beta) \in H_T.$$

In this setting $\alpha - \beta = \gamma(0) = 0$, since γ is an anti-isometry. We can conclude that $\alpha = \beta$ an this forced g to be injective. Actually g is an isomorphism as we wanted to prove.
Moreover $H_N^{\perp} \cong A_T$, $A_L(-1) \cong H_N$, consequently $H_N \oplus H_N^{\perp} \subset A_N$, where the orthogonal complement of H_N is in A_N. Since q_{A_N} is non-degenerate, $H_N \oplus H_N^{\perp} = A_N$ i.e. $A_N = A_T \oplus A_L(-1)$. □

Remark 4.2.3. What we have done in point i) is exactly what we can not do in the case p=2. For this reason in that situation we have to use another strategy.

Remark 4.2.4. For what we have proved in the previous theorem we can say that the discriminant form of $T_G(X)$ depends only on $T_G(\Lambda)$ and $A_L(-1)$, so whatever the embedding of L in $T_G(\Lambda)$ is, the orthogonal complement will be in the same genus as $T_G(X)$. If we show that this lattice is unique in its genus, we will have the uniqueness of $T_G(X)$.

4.2.1 p=2, trivial action on the discriminant group

As in the notation of Proposition 4.2.2, we denote $N := T_G(X)$, $L := \langle 2 \rangle \oplus \langle 2 \rangle$ and $T := T_G(\Lambda)$. Since $p = 2$ we have $A_L \simeq (\mathbb{Z}/2\mathbb{Z})^2$, $A_T \simeq (\mathbb{Z}/2\mathbb{Z})^a$. Moreover it holds the following Lemmas.

Lemma 4.2.5. *Let L be any lattice and $\varphi \in O(L)$, an involution, then it holds that $S_\varphi(L) = T_{-\varphi}(L)$.*

Proof. It holds that

$$S_\varphi(L) := \ker(\varphi + 1) = \ker(-\varphi - 1) = \ker((-\varphi) - 1) = T_{-\varphi}(L).$$

□

Lemma 4.2.6. *Let T be a 2-elementary lattice, and $\varphi \in O(T)$ an involution, such that $\overline{\varphi} \in O(A_T)$ is trivial, then $T_\varphi(T) := T^\varphi$ is 2-elementary.*

Proof. Let $T \hookrightarrow \Lambda$ be a primitive embedding of T in the smallest unimodular lattice Λ, and let $L := T^{\perp_\Lambda}$. Let $\widetilde{\varphi} := \varphi \oplus -Id_L$ be an extension of φ to Λ. Suppose that $\mathrm{ord}(\varphi) = 2$. Then $-\widetilde{\varphi} \in O(\Lambda)$ is of order 2 and we have $T_{-\widetilde{\varphi}}(\Lambda) = S_{\widetilde{\varphi}}(\Lambda)$ from Lemma 4.2.5. Moreover $T_\varphi(T) = S_{-\varphi}(T) = S_{-\widetilde{\varphi}}(\Lambda)$, where the first equal holds from Lemma 4.2.5, and the second equal holds by construction. Thus $T_\varphi(T)$ is a co-invariant sublattice of a unimodular lattice with respect to an action of order 2 and this implies that $T_\varphi(T)$ is 2-elementary. □

Conjecture: The statement of Lemma 4.2.6 holds also for p-elementary lattices.

We can apply the previous result to our case taking $T_G(X)$ and G of order 2 as we want, using $T = T_G(\Lambda)$ and considering $T_G(X) \oplus \langle 2 \rangle^{\oplus 2} \subseteq T_G(\Lambda)$, a finite index embedding. We consider the extension of the action of G on $\langle 2 \rangle^{\oplus 2}$ as $-Id_L$ which is a non-trivial action, but trivial on the discriminant group. Using this setting $T_G(T_G(\Lambda)) = T_G(X)$ and from the previous result it is 2-elementary.

Consequently in our case N is 2-elementary, thus we have $A_N \simeq (\mathbb{Z}/2\mathbb{Z})^q$. If $A_T \simeq (\mathbb{Z}/2\mathbb{Z})^a$ q and a are related, as we will see later. In this case A_T, A_L, and A_N are all 2-elementary, so we need to use another strategy to obtain all the embeddings of L in $T_G(\Lambda)$ and consequently all possible $T_G(X)$. In this case the action on A_X is trivial, hence $S_G(X) = S_G(\Lambda)$ and $T_G(X)$ has signature $(1, *)$ and the latter is obtained as the orthogonal complement of $L \subset T_G(\Lambda)$.

As in the previous notation we have $L \oplus T_G(X) \subset T_G(\Lambda)$ which is an embedding of finite index, and L and $T_G(X)$ are primitive sublattices of $T_G(\Lambda)$. This embedding is given by two subgroups $H_L \subset A_L$ and $H_{T_G(X)} \subset A_{T_G(X)}$, and by an anti-isometry $\gamma : H_L \longrightarrow H_{T_G(X)}$. We can call $H_T = \Gamma_\gamma = \{(a, \gamma(a)), s.t.\ a \in H_L\}$ and we know that $A_{T_G(\Lambda)} = \Gamma_\gamma^\perp / \Gamma_\gamma = (H_T)^\perp / H_T$ where $H_T \subset A_L \oplus A_{T_G(X)}$ is an isotropic subgroup.

Proposition 4.2.7. *In the previous notation, $A_T \simeq (\mathbb{Z}/2\mathbb{Z})^a$ and $A_N \simeq (\mathbb{Z}/2\mathbb{Z})^q$, if the induced action on A_X is trivial, we have one of the following possibilities:*

i) $H_L = A_L \Rightarrow q = a + 2$.

ii) $H_L = 0 \Rightarrow q = a - 2$.

iii) $H_L \simeq \mathbb{Z}/2\mathbb{Z} \Rightarrow q = a$

Proof. For point $i)$ we refer to the Proposition 4.2.2 and we obtain $A_N = A_T \oplus A_L(-1)$, which means that $q = a + 2$.
For point $ii)$ we notice that $H_L = 0$ implies that $\gamma(H_L) = 0$ which means $H_T = 0$. In this case $H_T^\perp = A_L \oplus A_N$ and $A_T = (H_T)^\perp / H_T = A_L \oplus A_N$, so $q = a - 2$.
For point $iii)$ we have $(H_L, \gamma(H_L)) = (0, \gamma(0)), (1, \gamma(1)) \simeq \mathbb{Z}/2\mathbb{Z} = H_T$. $H_T^\perp \subset A_L \oplus A_N \simeq (\mathbb{Z}/2\mathbb{Z})^{q+2}$ and for this reason $H_T \simeq (\mathbb{Z}/2\mathbb{Z})^{q+1}$. Moreover $A_T = (H_T)^\perp / H_T = (\mathbb{Z}/2\mathbb{Z})^{q+1} / \mathbb{Z}/2\mathbb{Z} \simeq (\mathbb{Z}/2\mathbb{Z})^q$, and we conclude that $q = a$. □

This is a complete list of possible invariants and co-invariants lattices corresponding to the action of a group $G \cong \mathbb{Z}/2\mathbb{Z}$ on $H^2(X, \mathbb{Z})$ which acts trivially

on A_X. In this list $\mathrm{sgn}(S_G(X)) = (2,*)$ and $\mathrm{sgn}(T_G(X)) = (1,*)$. We denote $a(S_G(X))$ by a. In the column H_L we specify which is the corresponding value of $H_L \subseteq A_L \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
In the following table the last column is related to automorphisms which are induced and we are referring to chapter 3 for this notion. For sake of completeness we recall that the operative definition that we apply is Definition 3.2.2.

No.	$S_G(X)$	$T_G(X)$	a	$\delta(S_G(X)))$	H_L	*induced*
1	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 3}$	$\langle 2\rangle$	3	1	0	no
2.1	$U \oplus \langle -2\rangle^{\oplus 3} \oplus \langle 2\rangle$	$\langle 2\rangle \oplus \langle -2\rangle$	4	1	0	no
2.2	$U \oplus \langle -2\rangle^{\oplus 3} \oplus \langle 2\rangle$	$U(2)$	4	0	0	no
3.1	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	U	2	0	0	no
3.2	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$U(2)$	2	0	$\mathbb{Z}/2\mathbb{Z}$	no
3.3	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$\langle 2\rangle \oplus \langle -2\rangle$	2	1	$\mathbb{Z}/2\mathbb{Z}$	no
4	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 3}$	$\langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle$	5	1	0	no
5.1	$U \oplus \langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle$	$U \oplus \langle -2\rangle$	3	1	0	no
5.2	$U \oplus \langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle$	$\langle 2\rangle \oplus \langle -2\rangle^{\oplus 2}$	3	1	$\mathbb{Z}/2\mathbb{Z}$	no
6.1	$U^{\oplus 2} \oplus \langle -2\rangle$	$\langle -2\rangle^{\oplus 2} \oplus \langle 2\rangle$	1	1	A_L	no
6.2	$U^{\oplus 2} \oplus \langle -2\rangle$	$U \oplus \langle -2\rangle$	1	1	$\mathbb{Z}/2\mathbb{Z}$	no
7.1	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$U \oplus \langle -2\rangle^{\oplus 2}$	4	1	0	no
7.2	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	$U \oplus \langle 2\rangle \oplus \langle -2\rangle^{\oplus 3}$	4	1	$\mathbb{Z}/2\mathbb{Z}$	no
8	$U(2)^{\oplus 2}$	$\langle -2\rangle^{\oplus 2} \oplus U(2)$	4	1	$\mathbb{Z}/2\mathbb{Z}$	no
9.1	$U \oplus \langle 2\rangle \oplus \langle -2\rangle$	$\langle 2\rangle \oplus \langle -2\rangle^{\oplus 3}$	2	1	A_L	yes
9.2	$U \oplus \langle 2\rangle \oplus \langle -2\rangle$	$U \oplus \langle -2\rangle^{\oplus 2}$	2	1	$\mathbb{Z}/2\mathbb{Z}$	yes
10.1	$U \oplus U(2)$	$U(2) \oplus \langle -2\rangle^{\oplus 2}$	2	1	A_L	yes
10.2	$U \oplus U(2)$	$U \oplus \langle -2\rangle^{\oplus 2}$	2	1	$\mathbb{Z}/2\mathbb{Z}$	yes
11	$U^{\oplus 2}$	$U \oplus \langle -2\rangle^{\oplus 2}$	0	1	A_L	yes
12.1	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle$	$\langle -2\rangle^{\oplus 4} \oplus \langle 2\rangle$	3	1	A_L	no
12.2	$\langle 2\rangle^{\oplus 2} \oplus \langle -2\rangle$	$U \oplus \langle -2\rangle^{\oplus 3}$	3	1	$\mathbb{Z}/2\mathbb{Z}$	no
13	$U \oplus \langle 2\rangle$	$U \oplus \langle -2\rangle^{\oplus 3}$	1	1	A_L	no
14.1	$\langle 2\rangle^{\oplus 2}$	$U \oplus \langle -2\rangle^{\oplus 4}$	2	1	A_L	yes
14.2	$\langle 2\rangle^{\oplus 2}$	$U \oplus D_4(-1)$	2	0	$\mathbb{Z}/2\mathbb{Z}$	yes

Table 4.6: Order 2 trivial action on $H^2(X,\mathbb{Z})$

4.2.2 p=2, non-trivial action on the discriminant group

We know that $H^2(X,\mathbb{Z})$ is the orthogonal complement of $L \cong \langle 2\rangle^{\oplus 2}$ in $\Lambda \cong U^{\oplus 5}$ where n_1 and n_2 are the two vectors of square 2 in L. If the action of G on $A_X \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is non-trivial, we have

$$T_G(X) \cong (n_1+n_2)^{\perp} \subset T_G(\Lambda) \ \ and \ \ S_G(X) \cong (n_1-n_2)^{\perp} \subset S_G(\Lambda).$$

$S_G(\Lambda)$ has signature $(3,*)$ and $T_G(\Lambda)$ has signature $(2,*)$. It follows from the first section that all lattices $S_G(\Lambda)$ and $T_G(\Lambda)$ are 2-elementary, thus their discriminant group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^a$ for some integer $a \geq 0$.

In the following q is the Beauville-Bogomolov quadratic form introduced in Theorem 1.2.7. By an easy computation we obtain that $q(n_1+n_2) = 4$ and $q(n_1-n_2) = 4$.

We denote by $(.,.)$ the symmetric bilinear form associated to q. Let x, y be two elements in $H^2(X,\mathbb{Z})$. The bilinear from is defined in this way

$$(x,y) := \frac{q(x+y) - q(x) - q(y)}{2}.$$

Another request that we need for n_1 and n_2 is that $(n_1+n_2, n_1-n_2) = 0$. These two vectors are orthogonal, and $n_1 + n_2 + n_1 - n_2 = 2n_1$, i.e. the sum of them is twice a primitive vector. It holds the following lemma.

Lemma 4.2.8. *Let $G \subset O(\Lambda)$ be a finite group of isometries of Λ of order 2, such that the Let $T_G(\Lambda)$ and $S_G(\Lambda)$ be respectively the invariant and the co-invariant sublattice of Λ with respect to the action of G. The lattices $T_G(X)$ and $S_G(X)$ are the invariant and the co-invariant sublattices with respect to the action of G on $H^2(X,\mathbb{Z})$ such that the induced action of G on A_X is non-trivial. The lattices $T_G(X)$ and $S_G(X)$ exist if and only if there exist two vectors, $v_0 = n_1 - n_2 \in S_G(\Lambda)$ and $v_1 = n_1 + n_2 \in T_G(\Lambda)$, of square 4, orthogonal to each other and such that the sum is twice a primitive vector. In this situation*

$$T_G(X) := (n_1 + n_2)^{\perp} \subset T_G(\Lambda)$$

$$S_G(X) := (n_1 - n_2)^{\perp} \subset S_G(\Lambda).$$

Proof. For sure the "if" part is true since if we find the vectors n_1 and n_2 with the requested properties, we can compute $S_G(X)$ and $T_G(X)$. For the "only if" part we need to observe that if we have a non-trivial action of G on $A_X \cong \mathbb{Z}/2\mathbb{Z}$, then we know that L has to be embedded partially in $S_G(\Lambda)$ and partially in $T_G(\Lambda)$. Since the two elements $[1,0]$ and $[0,1]$ of are exchanged, for sure the sum of them is preserved and the difference is not preserved. □

Starting from the table in Proposition 4.1.2, we find that in the cases 4, 5, 6, 9 it is non possible to find two vectors with the properties explained above.

Example 4.2.9. In case 4, $S_G(\Lambda) \cong U^{\oplus 2} \oplus \langle 2\rangle \oplus \langle -2\rangle$ and $T_G(\Lambda) \cong U \oplus \langle 2\rangle \oplus \langle -2\rangle$. Since $T_G(\Lambda) \oplus S_G(\Lambda) \subset \Lambda$, we can define the two copies of U in $S_G(\Lambda)$ generated by $\{e_1, f_1, e_2, f_2\}$, the vector of square 2 is $e_3 + f_3$ in the embedding above and the vector of square -2 is $e_4 - f_4$. In the same way we can call $\{e_5, f_5\}$ the generators of U in $T_G(\Lambda)$, the vector of square 2 is $e_4 + f_4$ w.r.t. the choice which we have done before, and the vector of square -2 is $e_3 - f_3$. Doing this choice for the basis we can take $v_0 = e_1 + f_1 + e_3 + f_3$ or $v_0 = 2(e_3 + f_3) + e_1 - f_1 + e_2 - f_2$ and $v_1 = e_5 + f_5 + e_4 + f_4$ or $v_1 = 2(e_4 + f_4) + e_3 - f_3 + e_5 - f_5$. Since we have these possible choices the sum $v_0 + v_1$ is not equal to twice a primitive vector.

Example 4.2.10. In case 2 we can compute for instance $S_G(\Lambda) \cong U \oplus \langle 2\rangle^2 \oplus \langle -2\rangle^2$ and $T_G(\Lambda) \cong \langle 2\rangle^2 \oplus \langle -2\rangle^2$. Since $T_G(\Lambda) \oplus S_G(\Lambda) \subset \Lambda$, we can define the copy of U in $S_G(\Lambda)$ generated by $\{e_1, f_1\}$, the two vectors of square 2 are $e_2 + f_2$ and $e_3 + f_3$ in the embedding above and the two vectors of square -2 are $e_4 - f_4$ and $e_5 - f_5$. In the same way we can call the vectors of square 2 in $T_G(\Lambda)$ $e_4 + f_4$ and $e_5 + f_5$ w.r.t. the choice which we have done before, and the vectors of square -2 are $e_2 - f_2$ and $e_3 - f_3$.

We have for sure at least two possible choices for v_0 and v_1: the first one is $v_0 =$

$e_4 - f_4 + e_5 - f_5 + 2(e_1 + f_1)$ and $v_1 = e_4 + f_4 + e_5 + f_5$, the second one is $v_0 = e_4 - f_4 + e_5 - f_5 + 2(e_2 + f_2)$ and $v_1 = e_4 + f_4 + e_5 + f_5$. In the first case $S_G(X) \cong \langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle 4 \rangle$ and $T_G(X) \cong \langle -2 \rangle^{\oplus 2} \oplus \langle 4 \rangle$. In the second case $S_G(X) \cong U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -4 \rangle$ and $T_G(X) \cong \langle -2 \rangle^{\oplus 2} \oplus \langle 4 \rangle$. These two pairs of lattices are not isomorphic.

The same situation happens in cases 7 and 10. There is just one possible choice for v_0 and v_1 in cases 3, 5, 8, 11, 12 of Proposition 4.1.2.

We can find the list of these lattices in the following table.

Remark 4.2.11. The following is not a complete list of possible invariant and co-invariant sublattices in this case since we did not show that the choices that we have computed for v_0 and v_1 are the unique that we can do. The different cases that we can have depend on the possible embeddings of v_0 in $S_G(\Lambda)$ and of v_1 in $T_G(\Lambda)$.

No.	$S_G(X)$	$T_G(X)$	$a(S_G(X))$
1	$U^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	$\langle 4 \rangle$	3
2.1	$\langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	$\langle -2 \rangle^{\oplus 2} \oplus \langle 4 \rangle$	6
2.2	$U \oplus \langle -2 \rangle \oplus \langle 2 \rangle \oplus \langle -4 \rangle$	$\langle -2 \rangle^{\oplus 2} \oplus \langle 4 \rangle$	4
3	$U \oplus U(2) \oplus \langle -4 \rangle$	$U(2) \oplus \langle -4 \rangle$	4
7.1	$\langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle \oplus \langle -4 \rangle$	$\langle -2 \rangle^{\oplus 3} \oplus \langle 4 \rangle$	5
7.2	$\langle -2 \rangle^{\oplus 2} \oplus \langle 2 \rangle \oplus \langle 4 \rangle$	$\langle -2 \rangle^{\oplus 2} \oplus \langle 2 \rangle \oplus \langle -4 \rangle$	5
8	$U \oplus \langle -2 \rangle \oplus \langle 4 \rangle$	$\langle 2 \rangle \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	3
10.1	$\langle 2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	$U \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	4
10.2	$U(2) \oplus \langle 4 \rangle$	$U(2) \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	4
11	$U \oplus \langle 4 \rangle$	$U \oplus \langle -2 \rangle^{\oplus 2} \oplus \langle -4 \rangle$	2
12	$\langle 2 \rangle \oplus \langle 4 \rangle$	$U \oplus \langle -2 \rangle^{\oplus 3} \oplus \langle -4 \rangle$	3

Table 4.7: Order 2 non-trivial action on $H^2(X, \mathbb{Z})$

Remark 4.2.12. In the following the last two columns in the cases $p = 3, 5, 7$ are related to automorphisms which are induced (i.) and induced at the quotient (i.q.) and we are referring to chapter 3 for this notion. For sake of completeness we recall that the operative definition that we apply to check if an automorphism is induced is Definition 3.2.2, and the one that we apply to check if an automorphism is induced at the quotient is the result of Theorem 3.3.15.

4.2.3 p=3

For $p = 3$ the action on A_X is trivial, hence $S_G(X) = S_G(\Lambda)$ and $T_G(X)$ has signature $(1, *)$ and the latter is obtained as the orthogonal complement of the primitive sublattice $L \subset T_G(\Lambda)$. As we have seen in the previous section, all the lattices $S_G(\Lambda)$ are admissible, so we need to find a primitive embedding of L in the invariant lattice $T_G(\Lambda)$ and to compute the orthogonal complement to find $T_G(X)$. As first thing, in the case $S_G(\Lambda) = U \oplus U(3) \oplus A_2(-1)$, then $T_G(\Lambda) = U(3) \oplus A_2$. In this case $A_{T_G(\Lambda)} \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 3}$. We know that there exists a primitive embedding of L in a lattice isomorphic to $T_G(\Lambda)$ if there exists an isomorphism between a subgroup of $A_L \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and a subgroup of $A_{T_G(\Lambda)}$ (Theorem 1.1.13). We notice that the orders of the respective subgroups are co-prime, and for this reason the unique isomorphism between the subgroups that we can have is the one between the trivial subgroups. Consequently we get $A_L \oplus A_{T_G(\Lambda)} \cong A_{T_G(X)}$, i.e. $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus (\mathbb{Z}/3\mathbb{Z})^{\oplus 3} \cong (\mathbb{Z}/6\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}$, and this implies $l(A_{T_G(X)}) = 3$. This is a contradiction since the rank of $T_G(X) = 2$ and for this reason this case is not allowed (see Theorem 1.1.26).

No.	$S_G(X)$	$T_G(X)$	$a(S_G(X))$	*i.*	*i. at the q.*
1	$U^{\oplus 2} \oplus A_2(-1)$	$\langle -2 \rangle \oplus \langle 6 \rangle$	1	no	yes
2	A_2	$U \oplus A_2(-1) \oplus \langle -2 \rangle^{\oplus 2}$	1	yes	yes

Table 4.8: Order 3 action on $H^2(X, \mathbb{Z})$

In the first case of Table 4.8 we have $T_G(\Lambda) = U \oplus A_2$. An embedding consists in taking a vector of square two in U and another orthogonal vector of square two in A_2. With this primitive embedding the orthogonal complement is $T_G(X) = \langle -2 \rangle \oplus \langle 6 \rangle$. Now we notice, using Theorem 1.1.13, that to find a primitive embedding of L in $T_G(\Lambda)$ we need to define an isomorphism between a subgroup of $A_L \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and a subgroup of $A_{T_G(\Lambda)} \cong \mathbb{Z}/3\mathbb{Z}$. Since the orders of the subgroups of A_L are 1, 2, 4 and the orders of the subgroups of $A_{T_G(\Lambda)}$ are 1, 3 to have an isomorphism, the only possibility is to take the trivial subgroup. In this case $L^\perp \cong T_G(X) \subset T_G(\Lambda)$ and we can apply Theorem 1.1.13 to note that there are no others primitive embeddings, up to isometries, of L in $T_G(\Lambda)$ since the unique subgroup of A_L that we can choose is $H_L \cong Id$. In the second case of Table 4.8 we can embed the two generators of L in $U^{\oplus 2}$, and we obtain $T_G(X) = U \oplus \langle -2 \rangle^{\oplus 2} \oplus A_2(-1)$. In this case $l(A_{T_G(X)}) = 2$ since $A_{T_G(X)} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. We have that the hypothesis of Theorem 1.1.17 is verified since $rank(T_G(X)) \geq 2$. Also in this case we have uniqueness up to isometry of $T_G(X)$.

4.2.4 p=5

For $p = 5$ the action on A_X is trivial, hence $S_G(X) = S_G(\Lambda)$ and $T_G(X)$ has signature $(1, *)$ and the latter is obtained as the orthogonal of the sublattice $L \subset T_G(\Lambda)$. As we have seen in Table 4.4 the only admissible lattice for $T_G(\Lambda)$ is $U^{\oplus 2} \oplus H_5$.

No.	$S_G(X)$	$T_G(X)$	$a(S_G(X))$	*i.*	*i. at the q.*
1	$U \oplus H_5$	$\langle -2 \rangle \oplus \langle -10 \rangle \oplus U$	1	yes	yes

Table 4.9: Order 5 action on $H^2(X, \mathbb{Z})$

As in the second case for $p = 3$ we can notice that $rk(T_G(X)) = 4$ and $l(A_{T_G(X)}) = 2$, for this reason $T_G(X)$ is uniquely determined up to isometry of $T_G(X)$.

4.2.5 p=7

As in the second case for $p = 3$ we can notice that $rk(T_G(X)) = 4$ and $l(A_{T_G(X)}) = 2$, for this reason $T_G(X)$ is uniquely determined up to isometry of $T_G(X)$.

No.	$S_G(X)$	$T_G(X)$	$a(S_G(X))$	*i.*	*i. at the q.*
1	$U^{\oplus 2} \oplus K_7$	$\langle -2\rangle \oplus \langle 14\rangle$	1	no	yes

Table 4.10: Order 7 action on $H^2(X,\mathbb{Z})$

The following is a result that summarizes what has been said so far about non-symplectic and prime order automorphisms of manifolds of OG_6 type, with respect to the 3 levels of classification that we have described in Section 4.0.1

Theorem 4.2.13. *Let X be a manifold of OG_6 type. Let G be a non-symplectic group of automorphisms of order p. If $p = 5$ or $p = 7$ there exists a unique pair $(S_G(X), T_G(X))$, up to isometry, of the invariant and co-invariant sublattices of $H^2(X,\mathbb{Z})$. If $p = 3$ there are two pairs of $(S_G(X), T_G(X))$ up to isometry, if $p = 2$ there are more then ten pairs of $(S_G(X), T_G(X))$.*
If $p = 3$ we have two connected component of the period domain with respect to this action, i.e. there are two images of the pairs (X, G) in the period domain. If $p = 7$ we have we have a unique image of the pair (X, G) in the period domain.

Proof. The proof concerns the classification that we have done in this Chapter. In fact, for the first level of classification we can find the number of the pairs $(S_G(X), T_G(X))$ up to isometry in the tables above, where each $T_G(X)$ and $S_G(X)$ is computed up to isometry. For the second level of classification we can consider [5, Theorem 2.9], and using this result we find a unique embedding of $S_G(X)$ (or equivalently of $T_G(X)$) in $H^2(X,\mathbb{Z})$, up to isometry of $H^2(X,\mathbb{Z})$. This depends on the fact that for the two cases of $p = 3$ and in the case of $p = 7$, the rank of $S_G(X)$ or the rank of $T_G(X)$ is equal to 2 and there are three copies of U in $H^2(X,\mathbb{Z})$. □

Chapter 5

Symplectic birational automorphisms

In this section we would like to discuss and classify symplectic birational morphisms on manifolds of OG_6 type. In the following, unless otherwise stated, X is an irreducible holomorphic symplectic manifold of OG_6 type.

Definition 5.0.1. Let $G \subseteq \mathrm{Aut}(X)$ a group of automorphisms of X, then we say that G is a symplectic group if $\varphi^*(\omega_X) = \omega_X$ for each $\varphi \in G$, where ω_X is the symplectic form of X.

Let now G be a group of symplectic automorphisms, G generated by φ, where φ is of prime order p. Since there exists an action of φ on $H^2(X, \mathbb{Z})$ we can find the invariant lattice $T_G(X)$ and the co-invariant lattice $S_G(X)$. We know that there exists an integer m such that $\mathrm{rk}(S_G(X)) = m(p-1)$ and the discriminant group of $H^2(X, \mathbb{Z})$ which we have denoted throughout the thesis with A_X is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. If $p \neq 2$ the induced action of φ on A_X, is trivial.

If $p = 2$ the induced action of φ on A_X could be non trivial which means that it exchanges the generators $[1,0]$ and $[0,1]$ of the discriminant group (see Proposition 1.5.5).

Remark 5.0.2. Since by definition the action on $T_G(X)$ is trivial, φ acts in a trivial way also on $A_{T_G(X)}$.

Proposition 5.0.3. *Let G be a finite group of automorphisms, the action of φ on A_X is trivial $\iff$ the action of φ on $A_{S_G(X)}$ is trivial.*

Proof. We know that the finite index embedding $S_G(X) \oplus T_G(X) \subseteq H^2(X, \mathbb{Z})$ is given by an isotropic subgroup H such that $H \subseteq A_{S_G(X)} \oplus A_{T_G(X)}$ and $A_X = H^\perp/H$, so if there exists a non trivial action on an element of A_X this comes from a non trivial action on an element of $A_{S_G(X)}$. On the other hand, if the action of G is trivial on A_X, then we can consider a primitive embedding $H^2(X, \mathbb{Z}) \hookrightarrow \Lambda = U^{\oplus 5}$ and extend the action of G trivially on the orthogonal complement.

We know that $A_{S_G(\Lambda)} \cong A_{S_G(X)}$ and $A_{T_G(\Lambda)} \cong A_{S_G(\Lambda)}$ since Λ is unimodular and the embedding is G-equivariant. Obviously the action of φ on $A_{T_G(\Lambda)}$ is trivial and so we can conclude that the same holds for $A_{S_G(X)}$. $\square$

Proposition 5.0.4. *Let X be a manifold of OG_6 type and let $G \subset \mathrm{Bir}(X)$ be a finite group of birational symplectic maps. Then the following assertions are true:*

(i) $S_G(X)$ and $T_G(X)$ are non-degenerate and $S_G(X)$ is negative definite.

(ii) $T(X) \subset T_G(X)$ and $S_G(X) \subset NS(X)$.

(iii) $S_G(X)$ contains no prime exceptional divisors.

Proof. The proof of the first two items is taken from Lemma 3.5 of [60], we sketch it here for the reader's convenience. To prove that $S_G(X)$ and $T_G(X)$ are non-degenerate let $H^2(X, \mathbb{C}) = \oplus_\rho U_\rho$ be the decomposition in orthogonal representations of G, where U_ρ contains all irreducible representations of G of character ρ inside $H^2(X, \mathbb{C})$. Obviously $T_G(X) = U_{Id|\mathbb{Z}}$ and $S_G(X) = H^2(X, \mathbb{Z}) \cap \oplus_{\rho \neq Id} U_\rho$, which implies they are orthogonal and of trivial intersection. Hence they are both non-degenerate.

It is known that $\varphi(\omega_X) = \omega_X$. The transcendental lattice is the smallest sublattice of $H^2(X, \mathbb{Z})$ such that $T(X) \otimes \mathbb{C} \supseteq H^{2,0} \oplus H^{0,2}$. Recall that $(H^{0,2} \oplus H^{2,0}) \cap H^2(X, \mathbb{R})$ is positive definite. Here $T(X) \subset T_G(X)$ and if ω is a Kähler class, $\omega_G = \sum_{g \in G} g(\omega)$ is preserved by the action of φ, i.e. $\omega_G \in T_G(X) \otimes \mathbb{R}$. Since ω_G is a positive class, it holds $\mathrm{sgn}(T_G(X)) = (3, *)$, consequently $S_G(X)$ is negative definite. As a consequence we have that $S_G(X) \subset \mathrm{NS}(X)$.
For the last item assume on the contrary that we have an element $c \in S_G(X)$ which is a prime exceptional divisor. Then using Markman it is known that there exists $n \in \mathbb{Z}$, $n > 0$, such that either $\pm\, nc$ is represented by an effective divisor D on X. Let $D' = \sum_{\varphi \in G} \varphi(D)$ which is also an effective divisor on X, but $[D'] \in S_G(X) \cap T_G(X) = 0$. This implies D' is linearly equivalent to 0, which is impossible. □

As a consequence of the previous theorem we have a result that will be really useful in the part of classification of symplectic birational morphisms.

Corollary 5.0.5. *If X is a manifold of OG_6 type and $G \subseteq O(H^2(X, \mathbb{Z}))$ is a group of prime order p of symplectic isometries of X then $\mathrm{rk}(S_G(X)) \leq 5$ and $p = 2, 3, 5$.*

Proof. Let $|G| = p$, $S_G(X)$ is a negative definite sublattice of $H^2(X, \mathbb{Z})$ and the signature of the latter is $(3, 5)$ for a manifold of OG_6 type. We know that $rk(S_G(X)) = m(p-1)$ and for reasons that depends on the signature this rank should be ≤ 5 , so p is a prime number ≤ 6. □

Moreover, we know that:

Proposition 5.0.6. *Let X be an IHS manifolds and $G \subsetneq \mathrm{Aut}(X)$ a group of symplectic automorphisms, then $S_G(X)$ contains no wall divisors.*

Proof. If G is symplectic, then $S_G(X)$ is negative definite. Since $T_G(X) \otimes \mathbb{R}$ contains a Kähler class, its orthogonal can not contain wall divisors, from the definition of them (see Definition 1.2.47). □

Remark 5.0.7. Before the next Proposition we recall that the characterization of stably prime exceptional and wall divisors for OG_6 type manifolds is contained in Lemma 1.2.49 and Lemma 1.2.50.

Theorem 5.0.8. *Let X be an OG_6 type manifold and $G \subset O(H^2(X,\mathbb{Z}))$ a non-trivial subgroup of symplectic isometries with trivial action on A_X, then $S_G(X)$ contains wall divisors.*

Proof. As in the non-symplectic setting it is possible to define an embedding of $S_G(X)$ in the second integral cohomology of a K3 surface. In fact, we know that $\mathrm{rk}(S_G(X)) \leq 5$ and the action on $A_{S_G(X)}$ is trivial, which means that the action on the second K3 cohomology can be extended trivially out of $S_G(X)$, i.e. the embedding:

$$S_G(X) \hookrightarrow H^2(K3,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

is such that $S_G(X) = S_G(K3)$. If $S_G(X)$ does not contain classes of square -2 and divisibility 1, this means that $S_G(K3)$ does not contain wall divisors for a K3 surface (see [53, proposition 1.5]) and this has to be the co–invariant sublattice of a K3 surface with respect to a symplectic automorphism. But we know from [73] that the co–invariant sublattice of a K3 surface with respect to a symplectic action has rank ≥ 8 and this is a contradiction.

□

Corollary 5.0.9. *Let X be a manifold of OG_6 type, and $G \subset O(H^2(X,\mathbb{Z}))$ a non–trivial subgroup of symplectic isometries with trivial action on A_X, then G is not effective.*

Proof. This is a direct consequence of Theorem 5.0.8. □

As a consequence we have that in the symplectic case, if the induced action on A_X si trivial, we do not obtain symplectic automorphisms of the manifold of OG_6 type, starting from non trivial isometries of the second integral cohomology. What we can analyze is when the group of isometries is birational effective and in the remaining part of the chapter we will give a classification of birational symplectic automorphisms.

We always have a group of symplectic automorphisms acting trivially on the second cohomology: the kernel of the map $\nu : \mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z}))$ is deformation invariant and was determined by Mongardi and Wandel in [65]. It is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\times 8}$ which means that it is composed by 256 symplectic involutions.

Proposition 5.0.10. *If X is a manifold of OG_6 type, and G is a group of symplectic isometries on $H^2(X,\mathbb{Z})$ with a trivial action on A_X, such that one of the following holds:*

- $\mathrm{rk}(S_G(X)) \leq 4$,
- $\mathrm{rk}(S_G(X)) = 5$ *and* $l(A_{S_G(X)}) \leq 3$,

then there exists an embedding of $S_G(X)$ in $E_8(-1)$, which is the unique unimodular negative definite lattice of rank 8, up to isometries.

Proof. We can notice that $S_G(X)$ has to be negative definite and it is a sublattice of $H^2(X,\mathbb{Z})$, so $\mathrm{rk}(S_G(X)) \leq 5$. We can define a lattice $\overline{S_G(X)} := S_G(X)(-1)$ (which has the same rank, the same discriminant group and is positive definite, so the discriminant form is the same but with opposite sign).

Now we want to define a finite index embedding of $S_G(X) \oplus \overline{S_G(X)}$ in a unimodular lattice K. We need $S_G(X) = \overline{S_G(X)}^{\perp}$ and $\overline{S_G(X)} = S_G{}^{\perp}$. Since we have $\operatorname{sgn}(S_G(X)) = (0,*)$, we can say that $\operatorname{sgn}(\overline{S_G(X)}) = (*,0)$ and $\operatorname{sgn}(K) = (*,*)$. To do this we need a subgroup of $A_{S_G(X)}$, a subgroup of $A_{\overline{S_G(X)}}$ and an anti-isometry γ between them. In this case we can choose as these subgroups the discriminant groups themselves and since the discriminant form is the same but with opposite sign, $\gamma : A_{S_G(X)} \to A_{\overline{S_G(X)}}$ is automatically defined. Now it's easy to find that $K \cong U^{\oplus \operatorname{rk}(S_G(X))}$.
Since $\overline{S_G(X)}$ exists it has to satisfy the 4 conditions of Theorem 1.1.26. Let E be a unimodular, negative definite lattice which is the smallest in which we can embed $S_G(X)$, and let N be the orthogonal complement of $S_G(X)$ inside E. Now we want to show that actually N exists. It should be negative definite and with the same discriminant form of $\overline{S_G(X)}$. The idea is to fix the discriminant form, i.e. $q_{A_{\overline{S_G(X)}}}$.
We denote with (t_+, t_-) the signature of $\overline{S_G(X)}$. We need that $t_+ = 0$ in order to verify the two first conditions of Theorem 1.1.26. We need to control just the two first conditions since to check the third and the fourth conditions we need that there exists no prime numbers for which the rank of the lattice is equal to the length on the discriminant group.
The second condition is verified by our assumptions on $\operatorname{rk}(S_G(X))$ and $l(A_{S_G(X)})$ and the first implies that $t_+ - t_- \equiv \operatorname{sgn}\ q_{A_{\overline{S_G(X)}}}$ $(mod\ 8)$, where $\operatorname{sgn}(\overline{S_G(X)}) \in \{(1,0),(2,0),(3,0),(4,0),(5,0)\}$.
If $\operatorname{rk}(S_G(X)) \leq 4$, then $\operatorname{rk}(\overline{S_G(X)})) \leq 4$. We need to fit that $-(t_-) \equiv 4\ (mod\ 8)$, which is verified if $(t_-) = 4$. This implies that N is a negative definite lattice of rank 4. It exists from Theorem 1.1.26, since $l(A_{S_G(X)}) = l(A_N) \leq 4$, this means that E is a unimodular, negative definite lattice of rank 8, i.e. $E \cong E_8(-1)$. If $\operatorname{rk}(S_G(X)) < 4$, then $-(t_-) \in \{1,2,3\}$ and so there exists an embedding of $S_G(X)$ in $E_8(-1)$.
On the other hand, if $\operatorname{rk}(S_G(X)) = 5$, then $-(t_-) \equiv 5\ (mod\ 8)$ and this implies that $t_- \in \{3, 11, \ldots\}$. If $t_- = 3$ then $\operatorname{rk}(N) = 3$ and using Theorem 1.1.26 we have that N in this way exists if and only if $l(A_{S_G(X)}) = l(A_N) \leq 3$, but this holds by hypothesis. Also in this case this means that E is a unimodular, negative definite lattice of rank 8, i.e. $E \cong E_8(-1)$. □

Remark 5.0.11. The case $rk(S_G(X)) = 5$ with $l(A_{S_G(X)}) \in \{4,5\}$ must be treated in a different way. In fact in this case we cannot use the strategy of Proposition 5.0.10 since it is not possible to find a primitive embedding of $S_G(X)$ in a unimodular lattice of rank 8. For this reason the case $\operatorname{rk}(S_G(X)) = 5$ will be classified in the case of prime order action, using the support of SAGE [97].

As a direct consequence of Proposition 5.0.10 we have the following result.

Corollary 5.0.12. *Let X be a manifold of OG_6 type, and let $G \subset O(H^2(X,\mathbb{Z}))$ be a group of symplectic isometries such that one of the hypothesis of Proposition 5.0.10 is verified, then $G \hookrightarrow O(E_8)$ and if the induced action of G on the discriminant group of $H^2(X,\mathbb{Z})$ is trivial, then $S_G(X) \cong S_G(E_8)(-1)$.*

Corollary 5.0.13. *If $|G|$ is odd there exists an embedding of $S_G(X)$ in $E_8(-1)$.*

Proof. If $|G|$ is odd, then $\mathrm{rk}(S_G(X))$ is even. Thus $\mathrm{rk}(S_G(X)) \leq 4$ and Theorem 5.0.10 applies.

5.1 Birational symplectic automorphisms

The following is a fundamental result for the classification of birational effective isometries of manifolds pf O'Grady six type.

Theorem 5.1.1. [Thm. 1.5 [52]]

If X is an irreducible holomorphic symplectic manifold, $\mathcal{FE}_X$ is an open cone, which is the interior of a closed generalized convex polyhedron in $\mathcal{C}_X$. $\mathcal{BK}_X$ is a dense open subset of $\mathcal{FE}_X$.

Definition 5.1.2. Let X be an IHS manifold and ω_X be a generator of $H^{2,0}(X)$. Let $\varphi \in O(H^2(X,\mathbb{Z}))$ be an isometry of finite order. The isometry is symplectic if the $\mathbb{C}$-linearized action of φ is such that $\varphi(\omega_X) = \omega_X$.

Proposition 5.1.3. *If X is a projective manifold of OG_6 type, $G \subset O(H^2(X,\mathbb{Z}))$ is a finite group of symplectic isometries, then $\mathcal{BK}_X$ is preserved.*

Proof. If G is a finite symplectic group of isometries, in a generic point of the family of deformations of the pair (X, G), we have $S_G(X) = \mathrm{NS}(X)$ and $T(X) = T_G(X)$. Since G is symplectic, $S_G(X)$ contains no prime exceptional divisors, see Proposition 5.0.4, and we can not find divisors in $\mathrm{T}(X)$, we conclude that X has no prime exceptional divisors. From [52] we can deduce that $\mathcal{FE}_X$ has only a chamber which coincides with $\mathcal{C}_X$. It follows that

$$\overline{\mathcal{BK}}_X = \overline{\mathcal{C}}_X$$

and this implies that the birational Kähler cone is preserved, since the positive cone is preserved by an isometry.

□

Let me recall this crucial result of Markman:

Corollary 5.1.4. [[52], Corollary 5.7] *Let X_1 and X_2 be irreducible holomorphic symplectic manifolds, $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel transport operator, which is an isomorphism of Hodge structures, and $\alpha_1 \in \mathcal{FE}_{X_1}$ a very general class. Then $g(\alpha_1)$ belongs to $\mathcal{FE}_{X_2}$, if and only if there exists a bimeromorphic map $f : X_1 \dashrightarrow X_2$, such that $g = f_*$.*

The following is a birational version of Theorem 1.2.34 for manifolds of OG_6 type. For sake of completeness we give the proof in this case.

Theorem 5.1.5. *Let X be an IHS manifold of OG_6 type and let $\varphi \in O(H^2(X,\mathbb{Z}))$ be an isometry of finite order. The isometry φ is birational effective $\iff$*

- *φ is an Hodge isometry on $H^2(X,\mathbb{C})$*
- *$\varphi \in Mon^2(X) \subset O(H^2(X,\mathbb{Z}))$*
- *An element of $\mathcal{BK}_X$ is sent to an element of $\mathcal{BK}_X$.*

Proof. Let $\varphi \in O(H^2(X,\mathbb{Z}))$. If φ is non-symplectic, it holds Proposition 4.0.10 which assures that φ is effective and in particular birational effective. We can assume that φ is symplectic. Since φ si symplectic, we know that $\mathrm{sgn}(T_G(X)) = (3,*)$. For this reason the orientation of $\mathcal{C}_X$ is preserved (since this is equivalent to check that the orientation of a positive 3-space, which is contained in $T_G(X)$, is preserved, see [52, Section 4]). Consequently $\varphi \in Mon^2(X) = O^+(H^2(X,\mathbb{Z}))$. The isometry φ is an Hodge isometry by construction and since it is a birational effective isometry, it comes from a birational map of X and from Proposition 5.0.4 we know that $S_G(X)$ contains no prime exceptional divisors. For this reason $\mathcal{BK}_X \cap S_G(X) = \mathcal{C}_X \cap S_G(X)$ and the last one is preserved and so we have the third condition.
For the other direction we want to show that if $\varphi \in O(H^2(X,\mathbb{Z}))$ is an Hodge-Monodromy operator which sends a Kähler class to a Kähler class for another birational model, φ comes from a birational map of X via the representation map

$$\nu : \mathrm{Bir}(X) \to O(H^2(X,\mathbb{Z}))$$

Since we have in aim to apply Corollary 5.1.4, we need to know if a very general class of $\mathcal{FE}_X$ is sent to a class of $\mathcal{FE}_X$. We know that $\mathcal{BK}_X$ is an open subset and the very general classes are a dense subset of $\mathcal{BK}_X$. We know that an element of $\mathcal{BK}_X$ is sent to an element of $\mathcal{BK}_X$, so for regularity of φ there exists an open subset of $\mathcal{BK}_X$ which is sent to an open subset of $\mathcal{BK}_X$ and consequently a very general class is sent to an element of $\mathcal{FE}_X$. We can apply Corollary 5.1.4 and Theorem 5.1.1 in our setting, and we know that there exists a birational map f of X such that $\varphi = f^*$ which means that φ is birational effective.

□

The following is an operative result about how to find birational effective isometries:

Theorem 5.1.6. *Let $G \subset O(E_8)$ be a group of finite order isometries and let X be an IHS manifold of OG_6 type. Suppose there exists a primitive G-equivariant embedding of $S_G(E_8(-1))$ in $H^2(X,\mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$. Suppose that* $\mathrm{NS}(X) \cong S_G(E_8(-1))$ *under the above embedding. Suppose that $\nexists\ \sigma \in \mathrm{NS}(X)$ such that $\sigma^2 = -2$ and $div(\sigma) = 2$ and $\nexists\ \sigma \in \mathrm{NS}(X)$ such that $\sigma^2 = -4$ and $div(\sigma) = 2$ (i.e.* $\mathrm{NS}(X)$ *contains no prime exceptional divisor). Then G is a group of birational effective isometries and the corresponding birational maps are symplectic.*

Proof. In this setting since $S_G(E_8(-1))$ is negative definite since it is a sublattice of $E_8(-1)$ which is negative definite. Moreover since $E_8(-1)$ is unimodular, the action on $A_{E_8(-1)}$ is trivial and the action on $A_{S_G(E_8(-1))}$ is trivial because the embedding is G-equivariant; we can extend trivially the action outside $\mathrm{NS}(X)$, which means that $\mathrm{NS}(X) = S_G(X)$ and $T(X) = T_G(X)$. From hypothesis $\mathrm{NS}(X) = S_G(X)$ has no prime exceptional divisors and for this reason $\mathcal{C}_X = \mathcal{BK}_X$. The group G is a group of Hodge isometries by construction and if we take $\varphi \in G$, $\varphi \in Mon^2(OG_6)$ because $Mon^2(OG_6) = O^+(H^2(OG_6,\mathbb{Z}))$, i.e. $Mon^2(OG_6)$ is made by orientation preserving isometries (see Theorem 1.2.33). We know that $S_G(X)$ is negative definite, thus the positive cone is preserved and also the $\mathcal{BK}_X$ is preserved. We can apply the Theorem

5.1.5 and we can conclude that G is a group of birational effective isometries and the corresponding birational maps are symplectic.

□

From the last theorem it makes sense to analyze subgroups of $O(E_8)$ such that their covariant lattice can be embedded in $U^{\oplus 5} = \Lambda$ and afterwards to specialize to $U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$. Finally we will check the additional conditions of the Theorem 5.1.6 to check that $\mathrm{NS}(X)$ does not contain prime exceptional divisors.

5.2 Induced birational symplectic morphisms

For this part we will refer to Section 3.2, since we have in aim to obtain similar results in the symplectic case. Here we provide a classification for symplectic birational morphisms. As a consequence we can speak about induced birational symplectic morphisms on manifolds of OG_6 type; definitions and lattice theoretic results of Section 3.2 holds also in this case, but for sake of completeness we will give them again in this setting.

Definition 5.2.1. Let X be a manifold of OG_6 type and let $G \subset \mathrm{Bir}(X)$. We say that G is an *induced group of birational morphisms* if there exists an abelian surface A with $G \hookrightarrow \mathrm{Bir}(A)$, a G-invariant non-primitive Mukai vector $u = 2w$, $u \in H^*(A, \mathbb{Z})^G$ and a u-generic stability condition θ such that X is birational to $\tilde{K}_u(A)$, and the induced action on $\tilde{K}_u(A, \theta)$ is a birational action which coincides with the action of G on X.

The Definition 3.2.2 of numerically induced depends only on the lattice structure of the second cohomology of OG_6 hence it can be applied also to birational morphisms. For this reason Theorem 3.2.6 holds with this statement:

Theorem 5.2.2. *Let X be a manifold of OG_6 type and let $G \subset \mathrm{Bir}(X)$ be a numerically induced group of birational morphisms. Then there exists a projective abelian surface A with $G \hookrightarrow \mathrm{Aut}(A)$, a G-invariant non-primitive Mukai vector $u = 2w$ and a u-generic stability condition θ such that X is birational to $\tilde{K}_u(A, \theta)$ and G is an induced group of birational morphisms.*

Moreover, it holds the following result:

Proposition 5.2.3. *If X is a manifold of OG_6 type and $G \subset \mathrm{Bir}(X)$ is a finite symplectic induced group of birational morphisms then $T_G(X) \cap \mathrm{NS}(X) \neq \{0\}$ and the -2-class σ of divisibility 2 requested in the definition 3.2.2 is an invariant class i.e. $\sigma \in T_G(X)$.*

Proof. To show this result we need Proposition 5.0.4. We know that $S_G(X)$ contains no prime exceptional divisors and $S_G(X) \subset \mathrm{NS}(X)$. Since G is numerically induced, we know from Proposition 3.2.5 that there exists a class $\sigma \in \mathrm{NS}(X)$ such that $\sigma^2 = -2$ and $div(\sigma) = 2$, and such that σ is G-invariant and this implies that $\sigma \in T_G(X)$. In fact if the group is numerically induced this means that the group is induced and this implies that the automorphisms is defined on the singular moduli space and on the smooth one, and this means that the singular locus of the singular moduli space is preserved by the action of the induced automorphism, and this implies that σ is preserved. □

Theorem 5.2.4. *If A is an abelian surface, $\varphi \in O(H^2(A,\mathbb{Z}))$ is birational effective* $\iff$

- *φ is an Hodge isometry on $H^2(A,\mathbb{C})$*
- *$\varphi \in Mon^2(A) \subset O(H^2(A,\mathbb{Z}))$*
- *A Kähler class is sent to a Kähler class.*

Corollary 5.2.5. *If X is a manifold of OG_6 type, $G = \langle\varphi\rangle \subset \mathrm{Bir}(X)$ is a finite induced group of birational automorphisms, and $|G| = 2$ then $\mathrm{rk}(S_G(X))$ is even.*

What we have already described in Corollary 5.2.5 about the structure of $S_G(X)$ also applies in the symplectic case, but there is an additional result if the group is induced:

Remark 5.2.6. If X is a manifold of OG_6 type and $G = \langle\varphi\rangle \subset \mathrm{Bir}(X)$ is an induced group of birational automorphisms, then φ^* is birational effective for the abelian surface so it has to satisfy Theorem 5.2.4. Moreover, if $G = \langle\varphi\rangle$ is an induced group of birational morphisms, and the induced action of φ on $A_{S_G(X)}$ is trivial, then $S_G(X) \cong S_G(A)$ where A is the corresponding abelian surface of Theorem 3.2.6.

Proposition 5.2.7. *If $S_G(X)$ is the co-invariant lattice with respect to the action of a finite symplectic group of induced birational morphisms on a manifold of OG_6 type, then $\mathrm{rk}(S_G(X)) \leq 3$.*

Proof. First of all we know by Remark 5.2.6 that $S_G(X) = S_G(A)$. Hence $\mathrm{rk}(T_G(A)) \geq 3$ because it contains the symplectic form, its conjugated and the invariant Kähler class. Since the rank of $H^2(A,\mathbb{Z}) = 6$ then $\mathrm{rk}(S_G(X)) \leq 3$. □

5.3 Some remarks on strictly semistable sheaves

In the following we will explain what kind of obstructions do we have in the symplectic case to extend a birational induced map to an automorphism. Let A be an abelian surface and $G \subset \mathrm{Aut}(A)$ a finite group of symplectic automorphisms (we need that A is projective to define the moduli space). This implies that $\mathrm{NS}(A) \cap T_G(A) \neq \{0\}$. In this setting $T(A) \subseteq T_G(A)$ and $S_G(A) \subseteq \mathrm{NS}(A)$. We have a classification of the invariant and co-invarinat sublattices of $H^2(A,\mathbb{Z})$ with respect to a symplectic action (see [37]) and we know that $\mathrm{rk}(S_G(A)) \geq 2$ and since A is projective $\rho(A) \geq 3$.

We are interested in the case in which $G \subset \mathrm{Aut}(A)$ is a symplectic group of automorphisms. We need to induce the action of G on the moduli space $M_v(A,\theta)$ where $\theta \in \mathrm{NS}(A)$ is a stability condition. To have this, we ask that θ is G-invariant, i.e. $\theta \in T_G(A)$. Another request to induce the action on $M_v(A,\theta)$ is that $v \cap H^2(X,\mathbb{Z}) \in T_G(A)$. From [37] we know that in $S_G(A)$ there are divisors D such that $D.D = -2$ and $D.\theta = 0$. Using the Remark 1.3.4 we see that in this case θ is not v-generic. Consequently the subspace of $M_v(A,\theta)$ of strictly semistable shaves with respect to θ does not consist only in destabilizing sub sheaves described by O'Grady (see [76, Lemma 2.1.2]) but the locus of strictly semistable shaves is richer. In particular, let $[\mathscr{F}] \in M_v(A, \theta)$ and assume

$$0 \to \mathscr{L}_1 \to \mathscr{F} \to \mathscr{L}_2 \to 0$$

is a destabilizing sequence, we have the following theorem:

Theorem 5.3.1. *[76, Lemma 2.1.2] Keep notation as above, then $\mathscr{L}_i \cong I_{x_i} \otimes \xi_i$, where I_{x_i} is the ideal sheaf of a point $x_i \in A$ and $\xi_i \in A^\vee$. Conversely, if $\mathscr{F}$ fits into an exact sequence as above, with $\mathscr{L}_i$ of this form, then $\mathscr{F} \in M_v$ and $\mathscr{F}$ is strictly semistable.*

We need to recall that if A is an abelian surface, then $H^2(A, \mathbb{Z})$ has a lattice structure. i.e. a non-degenerate symmetric bilinear form with integer values. This lattice is know to be isomorphic to $U^{\oplus 3}$, where U is the hyperbolic plane, i.e. the rank 2 lattice with intersection form

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The group $H^{2,0}(A)$ is 1-dimensional and generated by a nowhere vanishing holomorphic 2-form ω. Let $\mathrm{Aut}(A)$ be the group of automorphisms of A.

Definition 5.3.2. We call $\sigma \in \mathrm{Aut}(A)$ *symplectic* if σ preserves the symplectic form and we call it *non-symplectic* otherwise.

In Section 2 of his paper, O'Grady describes the desingularization of these strictly semistable sheaves of the moduli space ([76]).

For this reason we know that O'Grady destabilizing sequences are of this form and if we consider the Mukai vectors associated to these sheaves we have that $v(\mathscr{L}_i) = (1, 0, -1)$, since $\mathrm{rk}(I_{x_i}) = 1$, $c_1(I_{x_i}) = 0$ and $\mathrm{ch}_2 = -c_2(I_{x_i}) = -1$, and $v(\mathscr{F}) = (2, 0, -2)$. In our setting we have:

Theorem 5.3.3. *Let $M_v(A, \theta)$ be a moduli space of stable sheaves on a projective abelian surface A, with respect to a stability condition $\theta \in \mathrm{NS}(A) \cap T_G(A)$, and Mukai vector $v = (2, 0, -2)$. Let $G \subset \mathrm{Aut}(A)$ be a group of symplectic automorphisms on A. Suppose $v \cap H^2(A, \mathbb{Z}) \in T_G(A)$ and there exists $D \in \mathrm{NS}(A) \cap S_G(A)$ such that $D^2 = -2$ and $D.\theta = 0$. Consider the sequence*

$$0 \to \mathscr{F}_1 \to \mathscr{F} \to \mathscr{F}_2 \to 0 \tag{5.1}$$

where $\mathscr{F} \in M_v$, $\mathscr{F}_1$ is a subsheaf of $\mathscr{F}$ of Mukai vector $(1, D, -1)$ and $\mathscr{F}_2$ is a subsheaf $\mathscr{F}$ of Mukai vector $(1, -D, -1)$. The moduli spaces $M_{(1,D,-1)}(A, \theta)$ and $M_{(1,-D,-1)}(A, \theta)$ are non-empty and 5.1 is a destabilizing sequence i.e. the sheaf $\mathscr{F}$ is strictly θ-semistable.

Proof. To prove this we use equation 1.4. We need to compute the Hilbert polynomials $P_\theta(\mathscr{F}_1)(n) = \chi(\mathscr{F}_1(n\theta))$ and $P_\theta(\mathscr{F})(n) = \chi(\mathscr{F}(n\theta))$ and to write down the following equality.

$$\mathrm{rk}(\mathscr{F})\chi(\mathscr{F}_1(n\theta)) = 2\int_A \mathrm{ch}(\mathscr{F}_1)\,\mathrm{ch}(\mathscr{O}(n)) = 2\left(\frac{n^2\theta^2}{2} + nD.\theta - 1\right) = n^2\theta^2 - 2,$$

$$\mathrm{rk}(\mathscr{F}_1)\chi(\mathscr{F}(n\theta)) = \int_A \mathrm{ch}(\mathscr{F})\,\mathrm{ch}(\mathscr{O}(n)) = n^2\theta^2 - 2.$$

This means that $\mathscr{F}$ is strictly θ-semistable.
The moduli space $M_{(1,\pm D,-1)}(A,\theta) \neq \emptyset$ since $(1,\pm D,-1)^2 = 1-2+1 = 0 \leq 0$ and $\mathrm{rk}(\mathscr{F}) = 1 > 0$ (see [99]). □

Remark 5.3.4. The sequence of Theorem 5.3.3 is a destabilizing sequence which is not considered by O'Grady in [76]. This happens since θ is not v-generic.

From the previous remark we deduce that these sheaves in $M_v(A,\theta)$ form a singular locus which remains singular even if we apply the desingularization described by O'Grady in [76]. Consider a sheaf described in 5.1, since $\mathrm{rk}(\mathscr{F})$=2 and $\mathscr{F}$ is strictly semistable we have $Gr(\mathscr{F}) = \mathscr{F}_1 \oplus (\mathscr{F}/\mathscr{F}_1) \cong \mathscr{F}_1 \oplus \mathscr{F}_2$. Since $(1,\pm D,1)^2 = 0$, $M_{(1,\pm D,-1)}(A,\theta)$ are abelian surfaces (see [67]), we call them A_1 and A_2. We deduce that

$$A_1 \times A_2 \subset \mathrm{Sing}(M_v(A,\theta)).$$

We consider the construction described by O'Grady in [76] and we deduce that, since the fibration $a_v : M_v(A,\theta) \longrightarrow A \times A^\vee$ is isotrivial, then for reasons related to the dimension of the objects involved, we have some singular points:

$$P_1 \dots P_l \in \mathrm{Sing}(K_v(A,\theta)),$$

and these points remain singular even if we consider the desigularization described by O'Grady in [76]. In fact, with this choice of strictly semi-stable sheaves, $\widetilde{K}_v(A,\theta)$ is no longer regular, but it is a six dimensional variety and $P_1,\dots,P_l$ are singular points on it.
We know that θ is on a v-wall in the description above since it is not v-generic. We know that the classes $D \in \mathrm{NS}(A)$ such that $D.D = -2$ and $D.\theta = 0$ are of finite number which depends on $\mathrm{NS}(A)$, let me call them $D_1,\dots,D_m$, so it makes sense to consider a little increase of θ in this way:

$$\widetilde{\theta}_1 := \theta + \sum_{i=1}^{m} \varepsilon_i D_i,$$

where $\varepsilon_i \in \mathbb{Q}$ are small positive coefficients, small enough to have $\widetilde{\theta}_1$ positive. In the same way but with different sign we define:

$$\widetilde{\theta}_2 := \theta - \sum_{i=1}^{m} \varepsilon_i D_i,$$

and as for the other case we have $\widetilde{\theta}_2$ positive. These two ample classes $\widetilde{\theta}_i$ $(i = 1,2)$ and we can choose ε_i in a way such that $\widetilde{\theta}_i$ are v-generic since $\nexists\, D \in \mathrm{NS}(A)$ such that $\widetilde{\theta}_i D = 0$. For this reason we can consider $K_v(A,\widetilde{\theta}_i)$ which is a singular variety of OG_6 type, where the singularities are given by the destabilizing sequence of O'Grady type (see [76]) and these are the only ones. So we know how to desingularize this space and we obtain $\tilde{K}_v(A,\widetilde{\theta}_i)$ which are two smooth IHS manifolds of OG_6 type.

The sequence of this form:

$$0 \to \mathscr{F}_1 \to \mathscr{F} \to \mathscr{F}_2 \to 0,$$

where $\mathscr{F}_1 \in M_{(1,D,-1)}(A, \widetilde{\theta}_1)$, $\mathscr{F}_2 \in M_{(1,-D,-1)}(A, \widetilde{\theta}_1)$ and $\mathscr{F} \in M_v(A, \widetilde{\theta}_1)$ is no more a destabilizing sequence with respect to the stability condition $\widetilde{\theta}_1$. Consequently the subspace of sub sheaves of $M_v(A, \widetilde{\theta}_1)$ which fits in this exact sequence is not made by S-equivalence classes but by the sheaves themselves, which means that we find $\mathbb{P}(\mathscr{E}xt^1(\mathscr{F}_2, \mathscr{F}_1))$ as a subspace of the smooth locus of $K_v(A, \widetilde{\theta}_1)$. There exists a generalized version of Hirzebruch-Riemann-Roch formula for arbitrary coherent sheaves. First of all define for $\mathscr{E}, \mathscr{F}$ coherent sheaves, the Euler pairing

$$\chi(\mathscr{E}, \mathscr{F}) := \sum (-1)^i \text{dim Ext}^i(\mathscr{E}, \mathscr{F}). \tag{5.2}$$

Serre duality implies $\chi(\mathscr{E}, \mathscr{F}) = \chi(\mathscr{F}, \mathscr{E})$ i.e. the Euler pairing is symmetric. Note that for $E = \mathscr{O}_X$ we find $\chi(\mathscr{O}_X, \mathscr{F}) = \chi(\mathscr{F})$ and more generally, for $\mathscr{E}$ locally free $\chi(\mathscr{E}, \mathscr{F}) = \chi(\mathscr{E}^* \otimes \mathscr{F})$. Then 5.2 generalizes to

$$\chi(\mathscr{E}, \mathscr{F}) = \int \text{ch}(\mathscr{E})^* \, \text{ch}(\mathscr{F}) \, \text{td}(A).$$

Recall that for a locally free sheaf $\mathscr{E}$ it holds $\text{ch}(\mathscr{E})^* = \text{ch}(\mathscr{E}^*)$ and then applying this to $\mathscr{F}_1$ and $\mathscr{F}_2$ and with stability condition $\widetilde{\theta}_1$ we obtain

$$\chi(\mathscr{F}_2, \mathscr{F}_1) = ext^0(\mathscr{F}_2, \mathscr{F}_1) - ext^1(\mathscr{F}_2, \mathscr{F}_1) + ext^2(\mathscr{F}_2, \mathscr{F}_1) = \int \text{ch}(\mathscr{F}_2)^* \, \text{ch}(\mathscr{F}_1) \, \text{td}(A).$$

By Poincaré duality $ext^0(\mathscr{F}_2, \mathscr{F}_1) = ext^2(\mathscr{F}_2, \mathscr{F}_1)$ and

$$\int \text{ch}(\mathscr{F}_2)^* \, \text{ch}(\mathscr{F}_1) \, \text{td}(A) = -4.$$

Then it follows:

$$2 \; ext^0(\mathscr{F}_2, \mathscr{F}_1) - ext^1(\mathscr{F}_2, \mathscr{F}_1) = -4.$$

By Proposition 1.2.7 of [47], since the reduced Hilbert polynomials $p(\mathscr{F}_1)$ and $p(\mathscr{F}_2)$ are such that $p(\mathscr{F}_2) > p(\mathscr{F}_1)$, then $Hom(\mathscr{F}_2, \mathscr{F}_1) = ext^0(\mathscr{F}_2, \mathscr{F}_1) = 0$. Finally we get $ext^1(\mathscr{F}_2, \mathscr{F}_1) = 4$ and

$$\mathbb{P}(\mathscr{E}xt^1(\mathscr{F}_2, \mathscr{F}_1)) \cong \mathbb{P}^3.$$

We have the following situation, l copies of $\mathbb{P}^3$ which we will call $\mathbb{P}^3_1, \dots, \mathbb{P}^3_l \in K_v(A, \widetilde{\theta}_1)$ and the same but with the dual for the other v-generic ample divisor $\widetilde{\theta}_2$, i.e. $(\mathbb{P}^3_1)^* \dots, (\mathbb{P}^3_l)^* \in K_v(A, \widetilde{\theta}_2)$. This is due to the fact that the roles of $\mathscr{F}_1$ and $\mathscr{F}_2$ are completely exchangeable and this means that if we consider the exact sequence

$$0 \to \mathscr{F}_2 \to \mathscr{F} \to \mathscr{F}_1 \to 0$$

with respect to $\widetilde{\theta}_2$ we have the dual copies of $\mathbb{P}^3$.

Now we can desingularize the two six dimensional varieties $K_v(A, \widetilde{\theta}_1)$ and $K_v(A, \widetilde{\theta}_2)$ using the O'Grady techniques (see [76, Section 2]) and we obtain two smooth manifolds of OG_6 type, $\tilde{K}_v(A, \widetilde{\theta}_1)$ and $\tilde{K}_v(A, \widetilde{\theta}_2)$ respectively. We can summarize saying that $\tilde{K}_v(A, \widetilde{\theta}_i)$ with $i = 1, 2$ is the resolution of singularities of $\tilde{K}_v(A, \theta)$; this resolution consists in replacing the points $P_1, \dots, P_l$ with the copies of $\mathbb{P}^3$ or with the dual copies $(\mathbb{P}^3)^*$. In fact as we can find in [1], if we have a moduli space with a fixed Mukai vector v and we change the stability condition, i.e. the ample class $\theta \in \mathrm{NS}(A)$, passing from a stability condition which is v-generic to another which is not v-generic, we know that the the generic one is a resolution of singularities of the non-generic. Moreover $\tilde{K}_v(A, \widetilde{\theta}_i))$ for $i = 1, 2$ is a smooth symplectic manifold of OG_6 type which dominates a singular variety $\tilde{K}_v(A, \theta)$ and for this reason $\tilde{K}_v(A, \widetilde{\theta}_1) \to \tilde{K}_v(A, \theta)$ and $\tilde{K}_v(A, \widetilde{\theta}_2) \to \tilde{K}_v(A, \theta)$ are resolution of singularities.

Since α_1 is a contraction of $\mathbb{P}^3$'s on the points $P_1, \dots, P_l$ and α_2 is a contraction of some $(\mathbb{P}^3)^*$'s on the same points, we can say that there exists a birational map

$$\tilde{K}_v(A, \widetilde{\theta}_1) \dashrightarrow \tilde{K}_v(A, \widetilde{\theta}_2)$$

$$\begin{array}{ccccc}
\mathbb{P}^3_1, \dots, \mathbb{P}^3_l \subset \tilde{K}_v(A, \widetilde{\theta}_1) & & \longrightarrow & & K_v(A, \widetilde{\theta}_1) \\
\Big\updownarrow {\scriptstyle Mukai flop} & \searrow^{\alpha_1} & & & \Big\downarrow {\scriptstyle \pi_1} \\
& & P_1, \dots, P_l \in \tilde{K}_v(A, \theta) & \longrightarrow & K_v(A, \theta) \\
& \nearrow_{\alpha_2} & & & \Big\uparrow {\scriptstyle \pi_2} \\
(\mathbb{P}^3_1)^*, \dots, (\mathbb{P}^3_l)^* \subset \tilde{K}_v(A, \widetilde{\theta}_2) & & \longrightarrow & & K_v(A, \widetilde{\theta}_2)
\end{array}$$

If we want that the morphism on $\tilde{K}_v(A, \widetilde{\theta}_i)$, shortly $\widetilde{K}$ from here to the end of this section, is induced from an automorphism of A then $S_G(\widetilde{K}) \cong S_G(A)$. The classes of divisors D in $S_G(A)$ of square -2 are also classes of $S_G(\widetilde{K})$. The copies of $\mathbb{P}^3$ that we find in $\widetilde{K}$ contain lines. The classes of these lines are in the fibers of an extremal contraction $\tilde{K}_v(A, \widetilde{\theta}_1) \to K_v(A, \theta)$ and are of negative self intersection so for [58, Lemma 1.3] they include wall divisors (see [58, Lemma 1.4 and Proposition 1.5]).

Let $G \subset \mathrm{Aut}(\widetilde{K})$. If we take an ample divisor L and we take the class $\overline{L} = \sum_{\varphi \in G}(\varphi^*(L))$ this is an invariant ample class and we can not find wall divisors in its orthogonal complement. In a similar way if $G \subset \mathrm{Bir}(\widetilde{K})$ it holds the same with movable divisors instead of ample and prime exceptional divisors instead of wall divisors (see [58]).

For this reason if $G \subset \mathrm{Bir}(\widetilde{K})$ we can find wall divisors in $S_G(\widetilde{K})$ which are these classes of lines of square -2 and divisibility 1 and, if we do not find prime exceptional divisors, then we conclude that G is an induced group of birational morphisms on $\tilde{K}_v(A, \widetilde{\theta}_i)$ which is an O'Grady six dimensional manifold.

5.4 A classification with trivial action on the discriminant group

5.4.1 The prime order case

Let X be a manifold of OG_6 type. In this section we would like to provide a classification of possible groups G of symplectic birational morphisms of prime order on manifolds of OG_6 type.

Since there are no results about the impossibility of a non-trivial action of G on A_X we need to admit also this case. This can happen only in the case $|G|$ divides $|A_X| = 4$, which means if $|G| = 2$ (G is a group of prime order).

In the following we will study only the case in which if $|G| = 2$, then the action of G on A_X is trivial. As in the non-symplectic case and using the Proposition 5.0.10 and the Corollary 5.0.5, we first analyze prime order subgroups of $O(E_8)$ such that their co-invariant lattice has rank ≤ 5, then we specialize to $U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$ and we check the additional conditions of Theorem 5.1.6. To obtain this classification we use some results of Nikulin, that we find in [72, Section 1.12], moreover we need [72, Theorem 1.10.1, Theorem 1.8.1], and [32, Theorem 1.5.2]. Some other useful results for classification of lattices are due to Gordon-Nipp and to the database of Gabriel Nebe and Neil Sloane.

Rank $S_G(X)$	G	$S_G(X) = S_G(E_8(-1))$
1	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)$
2	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 2}$
2	$\mathbb{Z}/3\mathbb{Z}$	$A_2(-1)$
3	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 3}$
4	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 4}$
4	$\mathbb{Z}/2\mathbb{Z}$	$D_4(-1)$
4	$\mathbb{Z}/3\mathbb{Z}$	$A_2(-1)^{\oplus 2}$
4	$\mathbb{Z}/5\mathbb{Z}$	$A_4(-1)$
5	$\mathbb{Z}/2\mathbb{Z}$	$D_4(-1) \oplus A_1(-1)$

Table 5.1: Co-invariant sublattices with respect to a symplectic and prime order action; case $\mathrm{rk}(S_G(X)) \leq 4$ or $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) \leq 3$

This is the list that we obtain in the case of Proposition 5.0.10, in fact we are in one of the two following conditions $\mathrm{rk}(S_G(X)) \leq 4$ or $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) \leq 3$. The two more cases that we need to consider to complete the classification in the prime order case for symplectic isometries, are the following:

- $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) = 4$,
- $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) = 5$.

These two cases are not treated in the classification obtained starting from $E_8(-1)$-root lattice since in these cases it is not possible to define a primitive embedding of $S_G(X)$ in the $E_8(-1)$-root lattice.

If $l(A_{S_G(X)}) = 4$ we have $\det(S_G(X)) = 2^4$. This is not allowed, since a 2-elementary lattice with these invariants does not exist by the classification of 2-elementary lattices (see Theorem 1.1.21).

In the second case we have $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) = 5$. Such a lattice exists by Theorem 1.1.21. Since $S_G(X)(1/2)$ is an odd unimodular lattice of signature $(0,5)$, then it is unique in its genus. In this way we get the uniqueness of $S_G(X) = A_1(-1)^{\oplus 5}$.

Rank $S_G(X)$	$l(A_{S_G(X)})$	G	$S_G(X)$
5	5	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 5}$

Table 5.2: Co-invariant sublattice with respect to a symplectic and prime order action; case $\mathrm{rk}(S_G(X)) = 5$

In order to obtain the classification of birational symplectic automorphisms, we can now proceed by checking which of these lattices admits an embedding in $U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$ without prime exceptional divisors. Clearly in this contest we are treating morphisms of manifolds of OG_6 type, for this reason prime exceptional divisors are referred to this deformation type and are the ones described in Lemma 1.2.49 that we recall for sake of completeness.

Lemma 5.4.1. *Let X be a manifold of OG_6 type. Let $D \in Div(X)$ and let $[D] \in$* Pic(X) *be its class. Then $[D]$ is the class of a multiple of a stably prime exceptional divisor if one of the following holds:*

- $[D]^2 = -4$ *and* $div(D) = 2$*,*
- $[D]^2 = -2$ *and* $div(D) = 2$*.*

We remark that, since all prime exceptional divisors have non-trivial divisibility, a sufficient condition is that all elements of these lattices are embedded with trivial divisibility. Moreover notice that, all lattices of rank at most 3 can be embedded into $U^{\oplus 3}$ and, checking that if $|G| = 2$ then $\mathrm{rk}(S_G(X))$ is even, using Corollary 5.2.5, we obtain induced automorphisms. If we want to establish when an automorphism φ of X, a manifold of OG_6 type, is induced in the sense of definition 3.2.1 we need to check that $\mathrm{rk}(S_G(X)) \leq 3$ and $U^{\oplus 2} \subset T_G(\Lambda_{10})$ in the embedding $H^2(X,\mathbb{Z}) \hookrightarrow \Lambda_{10}$. Moreover we need a class $\sigma \in \mathrm{NS}(X)$ s.t. $\sigma^2 = -2$ and $div(\sigma) = 2$. Using Theorem 5.2.3 we know that since the isometry is symplectic σ is in $T_G(X)$. The following is a classification with respect to the induced action of $G = \langle \varphi \rangle$ on Λ_{10}.

$S_G(\Lambda_{10})$	$T_G(\Lambda_{10})$
$A_1(-1)$	$A_1 \oplus U^{\oplus 4}$
$A_1(-1)^{\oplus 2}$	$A_1^{\oplus 2} \oplus U^{\oplus 3}$
$A_2(-1)$	$A_2 \oplus U^{\oplus 3}$
$A_1(-1)^{\oplus 3}$	$A_1^{\oplus 3} \oplus U^{\oplus 2}$
$A_1(-1)^{\oplus 4}$	$A_1^{\oplus 4} \oplus U$
$D_4(-1)$	$D_4 \oplus U$
$A_2(-1)^{\oplus 2}$	$A_2^{\oplus 2} \oplus U$
$A_4(-1)$	$A_4 \oplus U$
$D_4(-1) \oplus A_1(-1)$	$D_4 \oplus A_1$
$A_1(-1)^{\oplus 5}$	$A_1^{\oplus 5}$

Table 5.3: Embedding of the co-invariant sublattices in Λ_{10}

With the same techniques used in the non-symplectic case, we obtain the following list of groups of prime order that can act symplectically on O'Grady's sixfolds. In the following Table, *induced* refers to Definition 3.2.1, bir.eff. means *birational effective* and refers to Definition 1.5.12 and i.q. means *induced at the quotient* and refers to Definition 3.3.5.

Rank	G	$S_G(X)$	$T_G(X)$	induced	bir. eff.	i.q.
1	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)$	$A_1 \oplus U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$	Yes	Yes	
2	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 2}$	$A_1^{\oplus 2} \oplus U \oplus \langle -2\rangle^{\oplus 2}$	Yes	Yes	
2	$\mathbb{Z}/3\mathbb{Z}$	$A_2(-1)$	$A_2 \oplus U \oplus \langle -2\rangle^{\oplus 2}$	Yes	Yes	Yes
3	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 3}$	$A_1^{\oplus 3} \oplus \langle -2\rangle^{\oplus 2}$	Yes	Yes	
4	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 4}$	$\langle 2\rangle^{\oplus 3} \oplus \langle -2\rangle$	No	No	
4	$\mathbb{Z}/2\mathbb{Z}$	$D_4(-1)$	$\begin{pmatrix} 6 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \oplus \langle -2\rangle$	No	Yes	
4	$\mathbb{Z}/3\mathbb{Z}$	$A_2(-1)^{\oplus 2}$	$A_2 \oplus \langle 6\rangle \oplus \langle -2\rangle$	No	Yes	Yes
4	$\mathbb{Z}/5\mathbb{Z}$	$A_4(-1)$	$\begin{pmatrix} 6 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \oplus \langle -2\rangle$	No	Yes	Yes
5	$\mathbb{Z}/2\mathbb{Z}$	$D_4(-1) \oplus A_1(-1)$	$\begin{pmatrix} 6 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$	No	No	
5	$\mathbb{Z}/2\mathbb{Z}$	$A_1(-1)^{\oplus 5}$	$A_1^{\oplus 3}$	No	No	

Table 5.4: Prime order symplectic isometries

Remark 5.4.2. In the last column we list the automorphisms which are induced at the quotient.(i.q.). As we expect the automorphisms which are induced (in the sense of abelian surfaces) are induced at the quotient. We leave the last column incomplete since we don't know in general if involutions are induced at the quotient or not since we don't know if the action exchanges or not the fibers of the two different $\mathbb{P}^2$ fibrations as we have explained in Section 3.3

The previous table is not a complete classification since we just find, when it is possible, a suitable embedding of $S_G(X)$ in $H^2(X,\mathbb{Z})$ in a way such that there are no prime exceptional divisors. We do not compute all the possible, up to isometries, primitive embeddings of $S_G(X)$ in $H^2(X,\mathbb{Z})$.

Remark 5.4.3. If we find an embedding in which all the generators and all the linear combinations of generators have trivial divisibility, the embedding is primitive, on the other hand there are primitive embeddings in which there exists elements with divisibility greater than one.

There are three cases in the previous table in which it is not possible to find an embedding of $S_G(X)$ in $H^2(X,\mathbb{Z})$ without prime exceptional divisors:

(i) $S_G(X) = A_1(-1)^{\oplus 4}$,

(ii) $S_G(X) = D_4(-1) \oplus A_1(-1)$,

(iii) $S_G(X) = A_1(-1)^{\oplus 5}$.

If we show that all the possible primitive embeddings of $A_1(-1)^{\oplus 4}$ admit prime exceptional divisors we can conclude that also all primitive embeddings of $D_4(-1)\oplus A_1(-1)$ and $A_1(-1)^{\oplus 5}$ admit prime exceptional divisors. In fact it holds the following proposition:

Proposition 5.4.4. *If S is an even lattice such that $S \hookrightarrow L$ is a primitive embedding without prime exceptional divisors then all the primitive sublattices S' of S have the same property.*

Since there exists a primitive embedding of $A_1(-1)^{\oplus 4}$ in $D_4(-1) \oplus A_1(-1)$ then it does not admit a primitive embedding without prime exceptional divisors. A possible embedding is the following:

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \hookrightarrow \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \end{pmatrix}$$

If we call $\{a_1, a_2, a_3, a_4\}$ the generators of $A_1(-1)^{\oplus 4}$ and $\{b_1, b_2, b_3, b_4, b_5\}$ the generators of $A_1(-1) \oplus D_4(-1)$ we can consider the following primitive embedding:

$$\begin{aligned} a_1 &\longmapsto b_1 \\ a_2 &\longmapsto b_2 \\ a_3 &\longmapsto b_4 \\ a_4 &\longmapsto b_5 \end{aligned}$$

In the case $S_G(X) = A_1(-1)^{\oplus 5}$ it is easy to find the following primitive embedding:

$$A_1(-1)^{\oplus 4} \hookrightarrow A_1(-1)^{\oplus 5}.$$

So we need to compute all the possible primitive embeddings of $S := A_1(-1)^{\oplus 4}$ in $L := H^2(X, \mathbb{Z})$. We compute the discriminant group and we find $A_S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ and $A_L \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} = \{[0,0], [0,1], [1,0], [1,1]\}$, so using a result of Nikulin, [72, Theorem 1.15.1], we know that primitive embeddings are determined by quintuples $\Theta_i := (H_S, H_L, \gamma, T, \gamma_T)$. The possible subgroups H_S and H_L are

$$\{\{0\}, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}\}.$$

We have two ways up to isomorphism to choose $\mathbb{Z}/2\mathbb{Z}$ in A_L: $\{[0,0], [1,0]\}$ and $\{[0,0], [1,1]\}$. We know the values of the discriminant form q_L on the generators of the discriminant group, $q_L([1,0]) = q_L([0,1]) = -\frac{1}{2}$. In the same way we know the values of the discriminant form of S on the generators, which are in this set: $q_S([1,0,0,0]) = q_S([0,1,0,0]) = q_S([0,0,1,0]) = q_S([0,0,0,1]) = -\frac{1}{2}$. We have the following four case:

- $H_S = H_L = \{0\}$. The rank of the orthogonal complement T is 4 and the signature is $(3,1)$. In this case $\gamma : \{[0,0,0,0]\} \longrightarrow \{[0,0]\}$ and $\Gamma = \{[0,0,0,0],[0,0]\}$, so $\Gamma^\perp \cong A_S \oplus A_L$, and $\Gamma^\perp/\Gamma \cong A_S \oplus A_L$. The quadratic form is $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma} \cong ((-q_S) \oplus q_L)|_{A_S \oplus A_L}$. For this reason $l(A_T) = 6$. Since $l(A_T) > \mathrm{rk}(T)$, the lattice T does not exists (see Theorem 1.1.26).

- $H_S = H_L = \mathbb{Z}/2\mathbb{Z}$, in the case $\{[0,0],[1,0]\}$, which means that the discriminant form holds $-\frac{1}{2}$ on the generator of this subgroup. In this case $\gamma([1,0,0,0]) = ([1,0])$, $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$, $\Gamma^\perp \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 5}$ and $\Gamma^\perp/\Gamma \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$. We compute the discriminant form q_T on $\Gamma^\perp/\Gamma$ and it holds $\frac{1}{2}$ on three generators of the discriminant group of T and $-\frac{1}{2}$ on the fourth generator. The lattice T is indefinite and 2-elementary, so by Theorem 1.1.17, the lattice T is unique up to isometries, and it holds $T \cong \langle 2\rangle^{\oplus 3} \oplus \langle -2\rangle$:

$$T \cong \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

If we call $\{e_1, f_1, e_2, f_2, e_3, f_3, z_1, z_2\}$ the generators of $H^2(X,\mathbb{Z})$, the embedding of S in L is the following:

$$\begin{aligned} a_1 &\longmapsto e_1 - f_1 \\ a_2 &\longmapsto e_2 - f_2 \\ a_3 &\longmapsto e_3 - f_3 \\ a_4 &\longmapsto z_1 \end{aligned}$$

As we can see in this case a_4 is a prime exceptional divisor, an element of square -2 and divisibility 2.

- $H_S = H_L = \mathbb{Z}/2\mathbb{Z}$, in the case $\{[0,0],[1,1]\}$, which means that the discriminant form holds $-1 \equiv 1$ in $\mathbb{Q}/2\mathbb{Z}$ on the generator of this subgroup. In this case $\gamma([1,1,0,0]) = ([1,1])$, $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$, $\Gamma^\perp \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 5}$ and $\Gamma^\perp/\Gamma \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$. We compute the discriminant form q_T on $\Gamma^\perp/\Gamma$ and it holds $\frac{1}{2}$ on two generators of the discriminant group of T and 0 on the other two which are not orthogonal in fact the form holds $\frac{1}{2}$ between them. The lattice T is indefinite and 2-elementary, so by Theorem 1.1.17, the lattice T is unique up to isometries, $T \cong \langle 2\rangle^{\oplus 2} \oplus U(2)$:

$$T \cong \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Remark 5.4.5. Observe that T which we obtain in this last case is isomorphic to the previous one. In fact

$$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cong \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

If we call $\{e_1, e_2, e_3, e_4\}$ the generators of $U(2) \oplus \langle 2 \rangle^{\oplus 2}$ and $\{f_1, f_2, f_3, f_4\}$ the generators of $\langle 2 \rangle^{\oplus 3} \oplus \langle -2 \rangle$ we consider this map:

$$\begin{aligned} e_1 &\longmapsto f_3 - f_4 \\ e_2 &\longmapsto \tfrac{f_3+f_4}{2} \\ e_3 &\longmapsto f_1 \\ e_4 &\longmapsto f_2 \end{aligned}$$

and this is an isomorphism.

Since these lattices are isomorphic, also in this case S admits a prime exceptional divisor in the embedding.

- $H_S = H_L = \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$. In this case $\gamma([1,0,0,0]) = ([1,0]), \gamma([0,1,0,0]) = ([0,1])$, $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, $\Gamma^{\perp} \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$ and $\Gamma^{\perp}/\Gamma \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$. We compute the discriminant form q_T on $\Gamma^{\perp}/\Gamma$ and it holds $\frac{1}{2}$ on the two generators of the discriminant group of T. Since $4 \geq l(A_T) + 2$, by Theorem 1.1.17, the lattice T is unique up to isometries, $T \cong \langle 2 \rangle^{\oplus 2} \oplus U$:

$$T \cong \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The embedding of S in L is the following:

$$\begin{aligned} a_1 &\longmapsto e_2 - f_2 \\ a_2 &\longmapsto e_3 - f_3 \\ a_3 &\longmapsto z_1 \\ a_4 &\longmapsto z_2 \end{aligned}$$

As we can see in this case a_4 is a prime exceptional divisor, an element of square -2 and divisibility 2.

Example 5.4.6. We take into consideration $S_G(X) = D_4(-1)$ and we show how to build embeddings in Λ_{10}, to check if there are two copies of U in $T_G(\Lambda)$. Moreover we build embeddings of $S_G(X)$ in $H^2(X, \mathbb{Z})$ taking care to not have prime exceptional divisors.

First of all we call $\{v_1, v_2, v_3, v_4\}$ a basis for the lattice $D_4(-1)$ and $\{e_i\, f_i\}$ for $i = 1, \cdots 5$ a basis for Λ_{10}. It is well defined the following embedding:

$$\begin{aligned} v_1 &\longmapsto e_1 - f_1 + f_2 \\ v_2 &\longmapsto e_2 - f_2 \\ v_3 &\longmapsto e_3 - f_3 + f_2 \\ v_4 &\longmapsto e_4 - f_4 + f_2 \end{aligned}$$

The orthogonal complement is:

$$T_G(\Lambda) \cong U \oplus D_4$$

There is a unique copy of U in $T_G(\Lambda)$ and this is the reason why this birational map is not induced by a map of the abelian surface.
On the other hand, we have to show that there exists a way to find an embedding of $S_G(X) = S_G(\Lambda)$ in $H^2(X,\mathbb{Z})$ such that there are no prime exceptional divisors. This means that the the isometry of $H^2(X,\mathbb{Z})$ such that the corresponding invariant and co-invariant lattices are $T_G(X)$ and $S_G(X)$ is birational effective. This means, by definition, that there exists a birational map defined on a manifold of OG_6 type, such that the induced action on the cohomology is the action of this example.
In fact there exists an embedding of $S_G(X)$ in $H^2(X,\mathbb{Z})$ without prime exceptional divisors. Let $\{e_1, f_1 e_2, f_2, e_3, f_3, z_1, z_2\}$ be a basis for the O'Grady six lattice, the following embedding is without prime exceptional divisors:

$$\begin{aligned} v_1 &\longmapsto e_1 - f_1 + f_2 \\ v_2 &\longmapsto e_2 - f_2 \\ v_3 &\longmapsto e_3 - f_3 + f_2 \\ v_4 &\longmapsto e_1 + f_1 + z_1 + z_2 + f_2 \end{aligned}$$

Example 5.4.7. For the case $S_G(X) \cong A_2(-1)^{\oplus 2}$ we can consider the following embedding which is without prime exceptional divisors:

$$\begin{aligned} v_1 &\longmapsto e_1 - f_1 + e_2 \\ v_2 &\longmapsto f_2 - e_2 \\ v_3 &\longmapsto z_1 + e_3 \\ v_4 &\longmapsto z_2 + f_3 \end{aligned}$$

And we can compute the orthogonal complement which is:

$$T_G(X) \cong A_2 \oplus \begin{pmatrix} -2 & -4 \\ -4 & -2 \end{pmatrix}$$

We need to remark that $\begin{pmatrix} -2 & -4 \\ -4 & -2 \end{pmatrix}$ is isometric to $\begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}$, where the last is the one that you find in the table 5.4.

5.4.2 The finite case

In this section we would like to provide a classification of possible group G of symplectic birational morphisms of finite order on manifolds of OG_6 type. The E8-root lattice is the unique even, unimodular, positive-definite lattice of rank 8 and its isometries group is the corresponding Weyl group, $W(E_8)$. It follows from a Theorem of Steinberg (see [88, Theorem 1.5]) that the stabilizer of a sublattice of a root lattice inside the corresponding Weyl group is a reflection group. The conjugacy classes of reflections subgroups for $W(E_8)$ are known, (see [34, Table 5]).

Let me recall this result of Hohn and Mason which is fundamental to find the possible co-invariant sublattices of $E_8(-1)$ in order to apply Theorem 5.1.6 and classify symplectic isometries of manifolds of OG_6 type.

Theorem 5.4.8. *[42, Theorem 3.6] In its action on the E_8-root lattice, the Weyl group of type E_8 has 41 orbits of fixed-point sublattices. These are in bijective correspondence with the isomorphism types of full subgraphs of the Coxeter graph for E_8, the lattice-stabilizers being the Coxeter groups determined by these subgraphs. The coinvariant lattices are the corresponding root lattices.*

Using the previous theorem we can find this list of possible sublattices corresponding to co-invariant sublattices of $E_8(-1)$.

Rank $S_G(E_8(-1))$	G	$S_G(E_8(-1))$
3	S_4	$A_3(-1)$
3	$\mathbb{Z}/2\mathbb{Z} \oplus S_3$	$A_1(-1) \oplus A_2(-1)$
4	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus S_3$	$A_1(-1)^{\oplus 2} \oplus A_2(-1)$
4	$\mathbb{Z}/2\mathbb{Z} \oplus S_4$	$A_1(-1) \oplus A_3(-1)$
5	$\mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus S_3$	$A_1(-1)^{\oplus 3} \oplus A_2(-1)$
5	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus S_4$	$A_1(-1)^{\oplus 2} \oplus A_3(-1)$
5	$\mathbb{Z}/2\mathbb{Z} \oplus S_5$	$A_1(-1) \oplus A_4(-1)$
5	S_6	$A_5(-1)$
5	$S_4 \ltimes \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$	$D_5(-1)$
5	$S_3^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$	$A_2(-1)^{\oplus 2} \oplus A_1(-1)$
5	$S_3 \oplus S_4$	$A_2(-1) \oplus A_3(-1)$

Table 5.5: Co-invariant sublattices with respect to a symplectic and finite order action

Theorem 5.4.9. *Let X be a manifold of OG_6 type and $G \subset O(H^2(X,\mathbb{Z}))$ be a subgroup of symplectic isometries of finite order such that $\mathrm{rk}(S_G(X)) \leq 4$. The following is a list of birational effective isometries with respect to these actions. The prime order case action is computed in table 5.4.*

R	G	$S_G(X)$	$T_G(X)$	is induced	bir. eff.	i.q.
3	S_4	$A_3(-1)$	$A_3 \oplus \langle -2\rangle^{\oplus 2}$	Yes	Yes	Yes
3	$\mathbb{Z}/2\mathbb{Z} \oplus S_3$	$A_1(-1) \oplus A_2(-1)$	$A_1 \oplus A_2 \oplus \langle -2\rangle^{\oplus 2}$	Yes	Yes	Yes
4	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus S_3$	$A_1(-1)^{\oplus 2} \oplus A_2(-1)$	$A_1^{\oplus 2} \oplus \begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}$	No	Yes	Yes
4	$\mathbb{Z}/2\mathbb{Z} \oplus S_4$	$A_1(-1) \oplus A_3(-1)$	$A_1 \oplus \begin{pmatrix} 2 & -2 & 0 \\ -2 & -2 & 4 \\ 0 & 4 & -2 \end{pmatrix}$	No	Yes	Yes

Table 5.6: Finite order symplectic isometries with $\mathrm{rk}(S_G(X)) \leq 4$

We know that if $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) \leq 3$ then the classification starting from the $E_8(-1)$ lattice is complete and this is what we can find in the following table. The case $\mathrm{rk}(S_G(X)) = 5$ is incomplete since $l(A_{S_G}(X))$ could be also 4 or 5 and these cases are not classified here.

Theorem 5.4.10. *Let X be a manifold of OG_6 type and $G \subset O(H^2(X,\mathbb{Z}))$ be a subgroup of symplectic isometries of finite order such that $\mathrm{rk}(S_G(X)) = 5$ and $l(A_{S_G(X)}) \leq 3$. The following is a list of birational effective isometries with respect to these actions. The prime order case action is computed in table 5.4.*

R	G	$S_G(X)$	$T_G(X)$	is induced	bir. eff.	i.q.
5	$\mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus S_3$	$A_1(-1)^{\oplus 3} \oplus A_2(-1)$	$A_1 \oplus A_2$	No	No	No
5	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus S_4$	$A_1(-1)^{\oplus 2} \oplus A_3(-1)$	A_3	No	No	No
5	$\mathbb{Z}/2\mathbb{Z} \oplus S_5$	$A_1(-1) \oplus A_4(-1)$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$	No	Yes	No
5	S_6	$A_5(-1)$	$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 6 \end{pmatrix}$	No	Yes	No
5	$S_4 \ltimes \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$	$D_5(-1)$	$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$	No	Yes	No
5	$S_3^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$	$A_2(-1)^{\oplus 2} \oplus A_1(-1)$	$\langle 2 \rangle \oplus \langle 6 \rangle^{\oplus 2}$	No	Yes	No
5	$S_3 \oplus S_4$	$A_2(-1) \oplus A_3(-1)$	$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$	No	Yes	No

Table 5.7: Finite order symplectic isometries with $\mathrm{rk}(S_G(X)) = 5$

The cases that do not admit an embedding with only trivial divisibility elements in $H^2(OG_6,\mathbb{Z})$ are $A_1(-1)^{\oplus 3} \oplus A_2(-1)$ and $A_1(-1)^{\oplus 2} \oplus A_3(-1)$. Since $A_1(-1)^{\oplus 4}$ admits a primitive embedding in these two lattices, we can use Proposition 5.4.4. In this way we conclude that these two lattices do not admit a primitive embedding without prime exceptional divisors. An embedding of $A_1(-1)^{\oplus 4}$ in $A_1(-1)^{\oplus 3} \oplus A_2(-1)$ is the following:

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \hookrightarrow \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

If we call $\{a_1, a_2, a_3, a_4\}$ the generators of $A_1(-1)^{\oplus 4}$ and $\{b_1, b_2, b_3, b_4, b_5\}$ the generators of $A_1(-1)^{\oplus 3} \oplus A_2(-1)$ we can consider the following primitive embedding:

$$\begin{aligned} a_1 &\longmapsto b_1 \\ a_2 &\longmapsto b_2 \\ a_3 &\longmapsto b_3 \end{aligned}$$

$$a_4 \longmapsto b_4$$

The embedding of $A_1(-1)^{\oplus 4}$ in $A_1(-1)^{\oplus 2} \oplus A_3(-1)$ is the following:

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \hookrightarrow \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

If we call $\{a_1, a_2, a_3, a_4\}$ the generators of $A_1(-1)^{\oplus 4}$ and $\{b_1, b_2, b_3, b_4, b_5\}$ the generators of $A_1(-1)^{\oplus 3} \oplus A_2(-1)$ we can consider the following primitive embedding:

$$\begin{aligned} a_1 &\longmapsto b_1 \\ a_2 &\longmapsto b_2 \\ a_3 &\longmapsto b_3 \\ a_4 &\longmapsto b_5 \end{aligned}$$

Example 5.4.11. In this example, the co-invariant sublattice is $S_G(X) = A_1(-1) \oplus A_4(-1)$, and we can find a primitive embedding in $H^2(OG_6, \mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle^{\oplus 2}$ in this way:

$$\begin{aligned} v_1 &\longmapsto e_1 - f_1 \\ v_2 &\longmapsto z_1 + e_2 \\ v_3 &\longmapsto f_2 - e_2 \\ v_4 &\longmapsto e_2 + e_3 - f_3 \\ v_5 &\longmapsto z_2 - e_3 \end{aligned}$$

We compute the orthogonal complement, which is:

$$T_G(X) \cong \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}.$$

The classification that we have done in this chapter is not a compete classification of all possible invariant and co-invariant sublattices with respect to a prime order or finite order symplectic action. In this Chapter the unique thing that we have classified, for a given possible co-invariant sublattice, is that it admits a primitive embedding in $H^2(X, \mathbb{Z})$ without prime exceptional divisor. What we can say is only that co-invariant sublattices which admit an embedding without prime exceptional divisors for sure are the co-invariant sublattices with respect to birational symplectic automorphisms of a manifold of OG_6 type. To classify this birational symplectic morphisms we have use the E_8-root lattice.

Remark 5.4.12. We have to notice is that is the classification of this Chapter we have considered only the case in which the induced action on A_X is trivial. The classification in which the induced action on A_X is non-trivial is still to be calculated and there are possibilities that in such a case, the symplectic action on integral cohomology is effective and we can find a classification of symplectic automorphisms of manifolds of OG_6 type.

Chapter 6

An example of induced automorphism

We have already defined when an automorphism is induced at the quotient (see Definition 3.3.5). We need to notice that there are two ways to exhibit induced automorphisms. In the first case we need to know the conditions to lift an automorphism of a manifold of OG_6 type, which is birational to $K3^{[3]}/i$ to an automorphism of a $K3^{[3]}$. This point is discussed in Section 3.3.

Otherwise we can take a group of automorphisms G of the $K3^{[3]}$ such that $i \subseteq G$ where i is the Rapagnetta's involution and we can consider the induced action of G on $K3^{[3]}/i$ which is birational to OG_6. Since we have a classification for the automorphisms of manifold of OG_6 type, we can try to recognize in this list an automorphism which comes from the $K3^{[3]}$.

This second approach is the way we chose to find examples of automorphisms induced at the quotient. In [42] the authors determine the orbits of fixed-point sublattices of the Leech lattice with respect to the action of the Conway group Co_0. The Leech lattice Λ is the only positive-definite, even, unimodular lattice of rank 24 with no elements of square -2 [51] [28]. The group of automorphisms of Λ is the Conway's group Co_0 [27] and so $\mathrm{Aut}(\Lambda)/\pm 1 = Co_0/\pm 1 = Co_1$. Höhn and Mason describe in [42, Theorem 1.1] a classification of the 290 orbits on the set of fixed-point sublattices of Λ under the action of Co_0. The purpose of their note is not merely to enumerate the orbits of fixed- point sublattices, but to provide in addition a detailed analysis of their properties. In particular, this includes the stabilizers G, which are the (largest) subgroups of Co_0 that stabilize a given fixed-point sublattice pointwise. In this section, to provide this example we refer to the orbits of fixed-point lattices and their fixing groups which are given in Table 1 in Section 4 of [42]. The geometry of $K3$ surfaces and certain hyperkähler manifolds X, is controlled, using Torelli-type theorems, by lattices related to Λ. In this way, symmetry groups of X can be mapped into Co_0, and properties of the fixed-point lattices control which groups may appear. This is what is done in [56] for hyperkähler manifolds. The extension of Mukai's Theorem to more general contexts is currently an active research area, and it is widely expected that knowledge of the stabilizers G with $T_G(\Lambda) \geq 4$, where $T_G(\Lambda)$ is the invariant sublattice in the usual notation, will lead

to the classification of all finite symplectic automorphism groups of hyperkähler manifolds of $K3^{[n]}$ type [60].
Let X be a manifold of $K3^{[3]}$ type, we know that the second integral cohomology has a lattice structure: $H^2(X,\mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -4 \rangle$. The smallest unimodular lattice in which this $K3^{[3]}$ lattice embeds is the Mukai lattice $:= \Lambda_{24} = U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$, which is a lattice of rank 24 and signature $(4, 20)$. It's important to notice that if $v \in \Lambda_{24}$ is a Mukai vector, then $v^{\perp} \cong H^2(X,\mathbb{Z})$.

Recall the following result:

Theorem 6.0.1. *[60] Let X be an IHS manifold of $K3^{[n]}$ type and let G be a finite group of symplectic automorphisms of X. Then $G \subset Co_1$ and $S_G(X) = S_G(\Lambda)$ for some conjugacy class of G in Co_1.*

With $Aut_s(X)$ we refer to symplectic automorphisms. The following is a good starting point:

Theorem 6.0.2. *[60] Let $G \subset Co_0$ be a group of isometries such that $rk(S_G(\Lambda)) \leq 20$ and $rk(T_G(\Lambda)) > l(A_{T_G(\Lambda)})$. Then there exist an integer n and a manifold X of $K_3^{[n]}$ type such that $G \subset Aut_s(X)$ and $S_G(X) \cong S_G(\Lambda)$.*

The condition in the above theorem is actually equivalent to $rk(S_G(\Lambda)) + l(A_{S_G(\Lambda)}) < 24$. Now the point is to consider a group G of birational maps of X such that $\langle i \rangle \subseteq G$ where i is the Rapagnetta's involution. Since we know from [62, Remark 5.4] that this involution can not be extended to an automorphism of X, we need that the condition above is not verified which means $rk(S_G(\Lambda)) + l(A_{S_G(\Lambda)}) = 24$. We stress the fact that for any n, the Rapagnetta's involution does not extend (see [60, Proposition 4.6]). In particular $rk(S_G(\Lambda)) = 24 - rk(T_G(\Lambda))$, and $A_{S_G(\Lambda)} \cong A_{T_G(\Lambda)}$ since they are complement in a unimodular lattice, we have that $rk(T_G(\Lambda)) = l(A_{T_G(\Lambda)})$.

If G is a group of birational symplectic maps of X, a manifold of $K3^{[3]}$ type, such that $\langle i \rangle \subseteq G$, where i is the Rapagnetta's involution, we are sure that G induces an action on the quotient $K3^{[3]}/i$, which is a singular model of a manifold of OG_6 type, K in the previous notation. Since we classified the symplectic birational maps for manifolds of OG_6 type, the point is that we can try to find in the list of Table 1 in Section 4 of [42] a group which acts on a $K3^{[3]}$ type manifold and such that contains the Rapagnetta's involution. Now this group induces an action on $H^2(OG_6,\mathbb{Z})$ and we can find the structure of $S_G(OG_6)$ and $T_G(OG_6)$. If we find this sublattices in our list of symplectic actions on OG_6 we can conclude that this action is an action which comes from the $K3^{[3]}$ manifold, which means that G is induced at the quotient.

Proposition 6.0.3. *Let X be a manifold of $K3^{[3]}$ type, let G be a finite group of birational maps of X, such that $\langle i \rangle \subseteq G$, where i is the Rapagnetta's involution. Consider the maximal G with $S_G(X)$ as co-invariant sublattice. Let Y be the resolution of the quotient of X by i. Consider the following representation map:*

$$\nu : Aut(Y) \longrightarrow O(H^2(Y,\mathbb{Z})).$$

It holds that $Ker(\nu) \subseteq G$.

Proof. It is known from [65] that $Ker(\nu) \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 8}$. If $\langle i \rangle \subseteq G$ then $S_i(X) \subseteq S_G(X)$. From [65] we know that the co-invariant lattice $S_i(X)$ is $BW_{16}(-1)$, the Barnes-Wall lattice, an even lattice negative definite that we will show below here. There exists the following injective map:

$$\widetilde{O}(A_{BW_{16}(-1)}) \hookrightarrow O(A_{BW_{16}(-1)})$$

and $Ker(\nu) \subset \widetilde{O}(A_{BW_{16}(-1)}) \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 9}$ [65]. Let $\tau \in Ker(\nu)$ be an involution, τ acts trivially on $A_{BW_{16}(-1)}$, so τ extends in a trivial way outside $BW_{16}(-1) \subset H^2(K3^{[3]}, \mathbb{Z})$, consequently $S_\tau(K3^{[3]}) \subseteq BW_{16}(-1)$. Now we know that $BW_{16}(-1) \subseteq S_G(K3^{[3]})$ so $S_\tau(K3^{[3]}) \subseteq S_G(K3^{[3]})$ $\forall \tau \in Ker(\nu)$ and this implies that $\tau \in G$, which concludes the proof. □

Now we can consider the list in Table 1 in Section 4 of [42]. From the previous result since we want G such that $i \in G$ we have to consider G such that $Ker(\nu) \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 8}$ is contained in G. We recognize these groups because in the column "type" of the table we find the notation Mon_a or Mon_a^* and in the column G we find $[2^s].G'$ where $s \in \mathbb{Z}$ and $s \geq 9$. When we look at G as a group of involutions on $K3^{[3]}$ we can say that i is not in $Ker(\nu)$, since when we look at $Ker(\nu)$ on $K3^{[3]}/i$ we know that $Ker(\nu)$ is made by the involutions of OG_6 which acts trivially on the second cohomology. We have to stress that when we consider the quotient, $K3^{[3]}/i$, then i is the identity and not an involution on this manifold.

Since the Rapagnetta's involution is "not contained" in $Ker(\nu)$ then G contains $\mathbb{Z}/2\mathbb{Z}^{\oplus 9}$.

Theorem 6.0.4. *The case $S_G(OG_6) = A_4(-1)$ of the table 5.4 corresponds to a birational maps of OG_6 induced at the quotient, which means that this action on OG_6 comes from an action on the manifold of $K3^{[3]}$ type, where OG_6 is the blow up of the singular locus of $K3^{[3]}/i$.*

Proof. The Table 1 of Section 4 of [42] show a classification of the action of a group G on the Leech Lattice Λ. The second column is the rank of $T_G(\Lambda)$. If we consider the case 100 we find that $G \cong [2^9].A_5$, which means that the group which acts on the Leech Lattice is an extension of A_5, the alternating group on 5 elements, with a group of order 2^9, $rk(T_G(\Lambda)) = 4$ and consequently $rk(S_G(\Lambda)) = 20$. Now we have that $(S_G(\Lambda) \oplus T_G(\Lambda)) \otimes \mathbb{Q} = \Lambda \otimes \mathbb{Q}$ and they are orthogonal complement in a unimodular lattice, which means that $A_{T_G(\Lambda)} \cong A_{S_G(\Lambda)}$ and $q_{T_G(\Lambda)} = -q_{S_G(\Lambda)}$ [72]. The same is true for Λ_{24}, the smallest unimodular lattice which embeds in the lattice of a manifold X of $K3^{[3]}$ type. In fact if G acts on Λ_{24} then it holds $S_G(\Lambda_{24}) \oplus T_G(\Lambda_{24}) = \Lambda_{24}$ and for the same reason $A_{T_G(\Lambda_{24})} \cong A_{S_G(\Lambda_{24})}$ and $q_{T_G(\Lambda_{24})} = -q_{S_G(\Lambda_{24})}$. From [60, Lemma 3.5], the action of a symplectic automorphism on $A_{S_G(X)}$ is trivial, so through this embedding $H^2(X, \mathbb{Z}) \hookrightarrow \Lambda_{24}$, we can extends the action of G outside $S_G(X)$ in a trivial way, which means that $S_G(\Lambda_{24}) = S_G(X)$. From [60, Theorem 3.6] we have that $S_G(\Lambda) = S_G(X)$, so we have $A_{S_G(\Lambda)} \cong A_{S_G(\Lambda_{24})}$ and $q_{S_G(\Lambda)} = q_{S_G(\Lambda_{24})}$. It is known that the embedding $H^2(X, \mathbb{Z}) \hookrightarrow \Lambda_{24}$ is such that $H^2(X, \mathbb{Z})^{\perp_{\Lambda_{24}}} = v$ where v is the Mukai vector and $v^2 = 2n-2 = 4$ since here $n = 3$. Now we need to find the Mukai vector v in $T_G(\Lambda_{24})$ and $T_G(\Lambda_{24}) \cong T_G(X) \oplus \langle v \rangle$.

Since from the previous discussion $S_i(X) \subset S_G(X)$, it holds that $T_G(X) \subset T_i(X)$. From computations in Section 3.3, $T_G(X) \cap T_i(X) = 2\ T_G(\overline{OG_6})$ where $\overline{OG_6}$ is the O'Grady 6 singular manifold, and now $T_G(X) \cap T_i(X) = T_G(X) = 2\ T_G(\overline{OG_6})$. Now we can deduce $T_G(\Lambda_{24})$ from the table of [42] and we can compute $T_G(X)$ as orthogonal complement of a vector of square 4. We know that a rank 4 lattice of signature $(4, 0)$ and discriminant form $2_{II}^{-4}5^{-1}$, with respect to the notation oh Hohn and Mason [42], has discriminant group $\mathbb{Z}/2\mathbb{Z}^{\oplus 4} \oplus \mathbb{Z}/5\mathbb{Z}$. A representative lattice with this features is $T_G(\Lambda_{24}) = A_4(2)$. Now we can take a vector $v \in A_4(2)$ of square 4 and we compute the rank 3 lattice $T_G(X) = 2\ T_G(\overline{OG_6})$.

There exists the following embedding in the unimodular lattice $U^{\oplus 4}$:

$$H^2(\overline{OG_6}, \mathbb{Z}) \cong U^{\oplus 3} \oplus \langle -2 \rangle \hookrightarrow U^{\oplus 4}$$

and the orthogonal complement in this embedding is a vector of square 2 which correspond to the Mukai vector in the context of X. $T_G(\overline{OG_6}) \oplus \langle -2 \rangle \cong T_G(U^{\oplus 4}) = \frac{T_G(\Lambda_{24})}{2} = A_4$. Moreover since the action of a symplectic group on $A_{S_G(X)}$ is trivial, $S_G(OG_6) = S_G(\overline{OG_6}) = S_G(U^{\oplus 4})$. Now we know $T_G(U^{\oplus 4})$ and we can find $S_G(U^{\oplus 4})$ and consequently $S_G(OG_6)$ since they are orthogonal complement in a unimodular lattice. We check the list in the table of Section 5.4 and we discover that $S_G(OG_6) = A_4(-1)$ corresponds to this case. So we can say that this is an example of birational map of a manifold of OG_6 type, which is induced at the quotient. (i.e. it comes from an automorphism of $K3^{[3]}$). □

Remark 6.0.5. This example, which is the number 100 of the table of Höhn and Mason [42] corresponds to an automorphism which is induced at the quotient but not induced.

Conjecture: If X is a manifold of OG_6 type, and φ is an induced automorphism, then it is induced at the quotient.

Bibliography

[1] Enrico Arbarello and Giulia Saccà. Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties. *Advances in Mathematics*, 329:649–703, 2018.

[2] Michela Artebani and Alessandra Sarti. Non-symplectic automorphisms of order 3 on k3 surfaces. *Mathematische Annalen*, 342(4):903, 2008.

[3] Michela Artebani and Alessandra Sarti. Symmetries of order four on K3 surfaces. *J. Math. Soc. Japan*, 67(2):1–31, 2015.

[4] Michela Artebani, Alessandra Sarti, and Shingo Taki. K3 surfaces with non-symplectic automorphisms of prime order. *Mathematische Zeitschrift*, 268(1-2):507–533, 2011.

[5] W Barth, C Peters, and A Van de Ven. Compact complex surfaces, volume 4 of ergebnisse der mathematik und ihrer grenzgebiete (3)[results in mathematics and related areas (3)], 1984.

[6] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3's via wall-crossing: nef and movable cones, Lagrangian fibrations. *Inventiones mathematicae*, 198(3):505–590, 2014.

[7] Arend Bayer and Emanuele Macrì. Projectivity and birational geometry of Bridgeland moduli spaces. *Journal of the American Mathematical Society*, 27(3):707–752, 2014.

[8] Arnaud Beauville. Some remarks on Kähler manifolds with $c_1 = 0$. *Classification of algebraic and analytic manifolds, Katata, 1982*, 1983.

[9] Arnaud Beauville. Antisymplectic involutions of holomorphic symplectic manifolds. *Journal of Topology*, 4(2):300–304, 2011.

[10] Arnaud Beauville et al. Variétés kähleriennes dont la premiere classe de Chern est nulle. *Journal of Differential Geometry*, 18(4):755–782, 1983.

[11] Fedor Alekseevich Bogomolov. Hamiltonian Kähler manifolds. In *Doklady Akademii Nauk*, volume 243, pages 1101–1104. Russian Academy of Sciences, 1978.

[12] Samuel Boissière. Automorphismes naturels de l'espace de douady de points sur une surface. *Canadian Journal of Mathematics*, 64(1):3–23, 2012.

[13] Samuel Boissière, Chiara Camere, Giovanni Mongardi, and Alessandra Sarti. Isometries of ideal lattices and hyperkähler manifolds. *International Mathematics Research Notices*, 2016(4):963–977, 2016.

[14] Samuel Boissiere, Chiara Camere, Giovanni Mongardi, and Alessandra Sarti. Isometries of ideal lattices and hyperkahler manifolds. *International Mathematics Research Notices*, 2016(4):963–977, 2016.

[15] Samuel Boissière, Chiara Camere, and Alessandra Sarti. Complex ball quotients from manifolds of $K3^{[n]}$-type. *Journal of Pure and Applied Algebra*, 223(3):1123–1138, 2019.

[16] Samuel Boissière, Chiara Camere, Alessandra Sarti, et al. Classification of automorphisms on a deformation family of hyperkähler fourfolds by p-elementary lattices. *Kyoto Journal of Mathematics*, 56(3):465–499, 2016.

[17] Samuel Boissière, Marc Nieper-Wißkirchen, and Alessandra Sarti. Smith theory and irreducible holomorphic symplectic manifolds. *Journal of Topology*, 6(2):361–390, 2013.

[18] Samuel Boissière and Alessandra Sarti. A note on automorphisms and birational transformations of holomorphic symplectic manifolds. *Proceedings of the American Mathematical Society*, 140(12):4053–4062, 2012.

[19] Ciprian Borcea. Diffeomorphisms of a K3 surface. *Mathematische Annalen*, 275(1):1–4, 1986.

[20] Sébastien Boucksom. Le cône kählérien d'une variété hyperkählérienne. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 333(10):935–938, 2001.

[21] Sébastien Boucksom. Divisorial zariski decompositions on compact complex manifolds. In *Annales scientifiques de l'Ecole normale supérieure*, volume 37, pages 45–76, 2004.

[22] Tom Bridgeland et al. Stability conditions on K3 surfaces. *Duke Mathematical Journal*, 141(2):241–291, 2008.

[23] Chiara Camere. Symplectic involutions of holomorphic symplectic fourfolds. *Bulletin of the London Mathematical Society*, 44(4):687–702, 2012.

[24] Chiara Camere and Alberto Cattaneo. Non-symplectic automorphisms of odd prime order on manifolds of k3 [n]-type. *manuscripta mathematica*, pages 1–44.

[25] Chiara Camere, Grzegorz Kapustka, Michał Kapustka, and Giovanni Mongardi. Verra four-folds, twisted sheaves, and the last involution. *International Mathematics Research Notices*, 2019(21):6661–6710, 2018.

[26] David H Collingwood and William M McGovern. *Nilpotent orbits in semisimple Lie algebra: an introduction.* CRC Press, 1993.

[27] John Horton Conway. A perfect group of order 8,315,553,613,086,720,000 and the sporadic simple groups. *Proceedings of the National Academy of Sciences of the United States of America*, 61(2):398, 1968.

[28] John Horton Conway. A characterisation of Leech's lattice. *Inventiones mathematicae*, 7(2):137–142, 1969.

[29] John Horton Conway and Neil James Alexander Sloane. *Sphere packings, lattices and groups*, volume 290. Springer Science & Business Media, 2013.

[30] Olivier Debarre. Un countre-exemple au théorème de torelli pour les variétés symplectiques irréductibles. *CR Acad. Sci. Paris*, 299:681–684, 1984.

[31] Jimmy Dillies. Order 6 non-symplectic automorphisms of K3 surfaces. *arXiv preprint arXiv:0912.5228*, 2009.

[32] Igor Dolgachev. Integral quadratic forms: applications to algebraic geometry. *Séminaire Bourbaki*, 25:251–278, 1982.

[33] Igor Dolgachev and Shigeyuki Kondō. Moduli of K3 surfaces and complex ball quotients. In *Arithmetic and geometry around hypergeometric functions*, pages 43–100. Springer, 2007.

[34] James Matthew Douglass, Götz Pfeiffer, and Gerhard Röhrle. On reflection subgroups of finite coxeter groups. *Communications in Algebra*, 41(7):2574–2592, 2013.

[35] Wolfgang Ebeling. Lattices and codes, friedr. *Vieweg & Sohn*, 2002.

[36] Akira Fujiki. On primitively symplectic compact kähler v-manifolds of dimension 4. classification of algebraic and analytic manifolds. pm 39, 71–250, 1983.

[37] Akira Fujiki. Finite automorphism groups of complex tori of dimension two. *Publications of the Research Institute for Mathematical Sciences*, 24(1):1–97, 1988.

[38] Mark Gross, Daniel Huybrechts, and Dominic Joyce. *Calabi-Yau manifolds and related geometries: lectures at a summer school in Nordfjordeid, Norway, June 2001*. Springer Science & Business Media, 2012.

[39] Brendan Hassett and Yuri Tschinkel. Hodge theory and lagrangian planes on generalized kummer fourfolds. *arXiv preprint arXiv:1004.0046*, 2010.

[40] Allen Hatcher. *Algebraic topology*. 2005.

[41] Wim Hesselink. The normality of closures of orbits in a lie algebra. *Commentarii Mathematici Helvetici*, 54(1):105–110, 1979.

[42] Gerald Höhn and Geoffrey Mason. The 290 fixed-point sublattices of the leech lattice. *Journal of algebra*, 448:618–637, 2016.

[43] Daniel Huybrechts. Compact hyperkähler manifolds: basic results. *Inventiones mathematicae*, 135(1):63–113, 1999.

[44] Daniel Huybrechts. Compact hyperkähler manifolds. calabi-yau manifolds and related geometries (nordfjordeid, 2001), 161–225, universitext, 2003.

[45] Daniel Huybrechts. A global torelli theorem for hyperkähler manifolds (after verbitsky). *arXiv preprint arXiv:1106.5573*, 2011.

[46] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158. Cambridge University Press, 2016.

[47] Daniel Huybrechts and Manfred Lehn. The geometry of moduli spaces of sheaves, aspects of math. *E31, Friedr. Vieweg & Sohn, Braunschweig*, 1997.

[48] Andreas Knutsen, Margherita Lelli-Chiesa, and Giovanni Mongardi. Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds. *Transactions of the American Mathematical Society*, 371(2):1403–1438, 2019.

[49] Kunihiko Kodaira. *Complex manifolds and deformation of complex structures.* Springer, 2006.

[50] Shigeyuki Kondo. Automorphisms of algebraic K3 surfaces which act trivially on picard groups. *Journal of the Mathematical Society of Japan*, 44(1):75–98, 1992.

[51] John Leech. Notes on sphere packings. *Canadian Journal of Mathematics*, 19:251–267, 1967.

[52] Eyal Markman. A survey of torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, pages 257–322. Springer, 2011.

[53] Eyal Markman et al. Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections. *Kyoto Journal of Mathematics*, 53(2):345–403, 2013.

[54] Kenji Matsuki. *Introduction to the Mori program.* Springer Science & Business Media, 2013.

[55] Rick Miranda. *The basic theory of elliptic surfaces.* ETS, 1989.

[56] Giovanni Mongardi. Automorphisms of hyperkähler manifolds. *arXiv preprint arXiv:1303.4670*, 2013.

[57] Giovanni Mongardi. On natural deformations of symplectic automorphisms of manifolds of $K3^{[n]}$-type. *Comptes Rendus Mathematique*, 351(13-14):561–564, 2013.

[58] Giovanni Mongardi. A note on the kähler and mori cones of hyperkähler manifolds. *Asian Journal of Mathematics*, 19(4):583–592, 2015.

[59] Giovanni Mongardi. On the monodromy of irreducible symplectic manifolds. *Algebraic Geometry*, 3(3):385–391, 2016.

[60] Giovanni Mongardi. Towards a classification of symplectic automorphisms on manifolds of K3-n type. *Mathematische Zeitschrift*, 282(3-4):651–662, 2016.

[61] Giovanni Mongardi and Antonio Rapagnetta. Monodromy and birational geoemetry of O'Grady's sixfolds. *arXiv preprint arXiv:arXiv:1909.07173*, 2019.

[62] Giovanni Mongardi, Antonio Rapagnetta, and Giulia Saccà. The Hodge diamond of O'Grady's six-dimensional example. *Compos. Math.*, 154(5):984–1013, 2018.

[63] Giovanni Mongardi, Kévin Tari, and Malte Wandel. Prime order automorphisms of generalised kummer fourfolds. *manuscripta mathematica*, 155(3-4):449–469, 2018.

[64] Giovanni Mongardi and Malte Wandel. Induced automorphisms on irreducible symplectic manifolds. *Journal of the London Mathematical Society*, 92(1):123–143, 2015.

[65] Giovanni Mongardi and Malte Wandel. Automorphisms of O'Grady's manifolds acting trivially on cohomology. *Algebr. Geom.*, 4(1):104–119, 2017.

[66] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. *Inventiones mathematicae*, 77(1):101–116, 1984.

[67] Shigeru Mukai et al. Semi-homogeneous vector bundles on an abelian variety. *Journal of Mathematics of Kyoto University*, 18(2):239–272, 1978.

[68] Shigeo Nakano. On the inverse of monoidal transformation. *Publications of the Research Institute for Mathematical Sciences*, 6(3):483–502, 1971.

[69] Yoshinori Namikawa. Counter-example to global torelli problem for irreducible symplectic manifolds. *Mathematische Annalen*, 324(4):841–845, 2002.

[70] Gabriele Nebe and Neil J.A. Sloane. Nebe lattices. *web site:http://www.math.rwth-aachen.de/ Gabriele.Nebe/LATTICES/LAMBDA16.html.*

[71] Viacheslav Valentinovich Nikulin. Finite groups of automorphisms of Kählerian surfaces of type $K3$. *Uspehi Mat. Nauk*, 31(2(188)):223–224, 1976.

[72] Viacheslav Valentinovich Nikulin. Integral symmetric bilinear forms and some of their applications. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 43(1):111–177, 1979.

[73] Viacheslav Valentinovich Nikulin. Finite automorphism groups of Kähler K3 surfaces. *Trans. Moscow Math. Soc*, 38(2):71–135, 1980.

[74] Viacheslav Valentinovich Nikulin. Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. algebrogeometric applications. *Journal of Soviet Mathematics*, 22(4):1401–1475, 1983.

[75] Viacheslav Valentinovich Nikulin. Discrete reflection groups in lobachevsky spaces and algebraic surfaces. In *Proceedings of the international congress of mathematicians*, volume 1, pages 654–671. Citeseer, 1986.

[76] Kieran Gregory O'Grady. A new six dimensional irreducible symplectic variety. *J. Algebraic Geom. 12 (2003), 435-505*, 12.

[77] Kieran Gregory O'Grady. Desingularized moduli spaces of sheaves on a K3, i. *J. Reine. Angew. Math.*, 512:49–117, 1999.

[78] Keiji Oguiso and Stefan Schrer. Enriques manifolds. *Journal fr die reine und angewandte Mathematik (Crelles Journal)*, 2011(661):215–235, 2011.

[79] Hisanori Ohashi and Malte Wandel. Non-natural non-symplectic involutions on symplectic manifolds of $K3^{[2]}$-type. *arXiv preprint arXiv:1305.6353*, 2013.

[80] Arvid Perego and Antonio Rapagnetta. Deformation of the O'Grady moduli spaces. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013(678):1–34, 2013.

[81] Antonio Rapagnetta. Topological invariants of O'Grady's six dimensional irreducible symplectic variety. *Mathematische Zeitschrift*, 256(1):1–34, 2007.

[82] Antonio Rapagnetta. On the Beauville form of the known irreducible symplectic varieties. *Mathematische Annalen*, 340(1):77–95, 2008.

[83] Aleksei Nikolaevich Rudakov and Igor Rostislavovich Shafarevich. Surfaces of type k3 over fields of finite characteristic. *Itogi Nauki i Tekhniki. Seriya" Sovremennye Problemy Matematiki. Noveishie Dostizheniya"*, 18:115–207, 1981.

[84] Matthias Schütt. $K3$ surfaces with non-symplectic automorphisms of 2-power order. *J. Algebra*, 323(1):206–223, 2010.

[85] Tetsuji Shioda. The period map of abelian surfaces. *J. Fac. Sci. Univ. Tokyo*, 25:47–59, 1978.

[86] Yum-Tong Siu. Every K3 surface is Kähler. *Inventiones mathematicae*, 73(1):139–150, 1983.

[87] Christoph Sorger and Manfred Lehn. La singularité de O'Grady. *J. Algebraic Geom*, 15:753–770, 2006.

[88] Robert Steinberg. Differential equations invariant under finite reflection groups. *Transactions of the American Mathematical Society*, 112(3):392–400, 1964.

[89] Dima Al Tabbaa, Annalisa Grossi, and Alessandra Sarti. Symmetries of order eight on K3 surfaces without high genus curves in the fixed locus. *arXiv preprint arXiv:1906.02616*, 2019.

[90] Dima Al Tabbaa and Alessandra Sarti. Order eight non-symplectic automorphisms on elliptic k3 surfaces. *arXiv preprint arXiv:1612.01184*, 2016.

[91] Dima Al Tabbaa, Alessandra Sarti, and Shingo Taki. Classification of order sixteen non-symplectic automorphisms on K3 surfaces. *arXiv preprint arXiv:1409.5803*, 2014.

[92] Shingo Taki. Classification of non-symplectic automorphisms of order 3 on K3 surfaces. *Math. Nachr.*, 284:124–135, 2011.

[93] Shingo Taki. Classification of non-symplectic automorphisms on $K3$ surfaces which act trivially on the Néron-Severi lattice. *J. Algebra*, 358:16–26, 2012.

[94] Kévin Tari. *Automorphismes des variétés de Kummer généralisées.* PhD thesis, Poitiers, 2015.

[95] Jean Varouchas. Kähler spaces and proper open morphisms. *Mathematische Annalen*, 283(1):13–52, 1989.

[96] Misha Verbitsky et al. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Mathematical Journal*, 162(15):2929–2986, 2013.

[97] web site: https://www.sagemath.org. Sage.

[98] Kōta Yoshioka. Some examples of Mukai's reflections on K3 surfaces. *J. reine angew. Math.*, 515:97–123, 1999.

[99] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Mathematische Annalen*, 321(4):817–884, 2001.

[100] Kōta Yoshioka et al. Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface. In *Development of moduli theory—Kyoto 2013*, pages 473–537. Mathematical Society of Japan, 2016.