ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

Dottorato di Ricerca in Matematica
Ciclo XXXI

Settore Concorsuale: 01/A3

Settore Scientifico Disciplinare: MAT/05

Harnack inequality in doubling quasi metric spaces and applications.

Presentata da: Chiara Guidi

Coordinatore Dottorato:
Chiar.ma Prof.ssa
Giovanna Citti

Supervisore:
Chiar.ma Prof.ssa
Annamaria Montanari

Esame finale anno 2019

To Lorenzo

Abstract

In this thesis we present an axiomatic approach to an invariant Harnack inequality for non homogeneous PDEs in the setting of doubling quasi-metric spaces. We adapt the abstract procedure developed by Di Fazio, Gutiérrez and Lanconelli, for homogeneous PDEs taking into account the right hand side of the equation. In particular we adapt the notions of double ball property and critical density property: these notions arise from Krylov-Safonov technique for uniformly elliptic operators and they imply Harnack inequality. Then we apply the axiomatic procedure to subelliptic equations in non divergence form involving Grushin vector fields and to X-elliptic operators in divergence form.

Introduction

Harnack inequality was first introduced in 1887 for non negative solutions of harmonic functions. Since then, it has been extended to non negative solutions of a huge variety of PDEs both in divergence and non divergence form and in Euclidean and non Euclidean setting. The importance of this type of inequality is widely recognized as it is a fundamental tool in the investigation of regularity of solutions of partial differential equations. Moser iteration technique ([39]) for elliptic operators in divergence form and Krylov-Safonov's measure theoretic approach ([32, 33, 40]) for linear equations in non divergence form and rough coefficients are cornerstones in the development of Harnack inequality procedures. The works [8] and [7] by Caffarelli and Caffarelli Cabré, where the Krylov-Safonov's technique is simplified and adapted to fully nonlinear operators, and the extension to the linearized Monge-Ampère equation in [9], enlightened the key role of the Alexandrov-Bakelman-Pucci maximum principle and the geometrical nature of the proof. It is indeed this last observation that inspired axiomatic procedures to Harnack inequality in the general setting of quasi metric spaces.

The term axiomatization of Harnack inequality refers to procedures that fix a quasi metric space and a set of real valued functions on this space and aim to find sufficient conditions on the set of functions considered in order to imply Harnack inequality over the balls defined by quasi distance of the underlying space. This kind of approach was established by Aimar, Forzani and Toledano in [2] for some families continuous functions, while Di Fazio, Gutiérrez and Lanconelli in [14] substituted the regularity assumption on the functions with a more natural requirement on the geometry of the underlying quasi metric space. Despite different assumptions, both results rely on covering lemmas and state that the Harnack inequality is a consequence of the power like decay of the distribution function of each element of the considered family. In both works [2, 14], the power decay property is a consequence of the validity of two other properties: the double ball and the critical density (see [14] for the definitions). These two notions, that are hidden in the works by Krylov and Safonov, arise from the analysis of the previously mentioned works by Caffarelli and Gutiérrez. Another type of abstract approach was established by Indratno, Maldonado and Silwal in [30]. Here the authors substitute the double ball property with an integral condition (see [30, equation (2.14)]) which makes the approach better suited for variational operators. These three axiomatic procedures have been used to prove Harnack inequality for non negative solutions to equations in divergence and non divergence form with underlying sub-Riemannian structures, see for example [1, 37, 41, 27].

Unfortunately none of the axiomatic approaches above mentioned can be directly applied to a PDE with non zero right hand side, indeed they can only handle functional sets closed under multiplication by positive constant, while the set of solutions of a linear non homogeneous PDE does not satisfy this requirement. However, a remark is in order. If a maximum principle holds true in a form that permits to establish pointwise-to-measure estimates for super solutions of the non homogeneous equation and if the existence of a solution for the Dirichlet problem for the corresponding homogeneous PDE is guaranteed, then it is possible to obtain the non homogeneous Harnack inequality from the homogeneous one by an elementary argument as in [25, Theorem 5.5]. However, when dealing with subelliptic PDEs in non divergence form, it is not known if a maximum principle holds true in a form that permits to establish pointwise-to-measure estimates for super solutions such as [24, Theorem 2.1.1]. For non divergence PDEs its proof depends in a crucial way upon the maximum principle of Aleksandrov, Bakelman and Pucci, see for example [22, Section 9.8]. This principle for some subelliptic PDEs has been recently studied, for example in [9, 6, 37, 13, 26, 3, 42]. However, nowadays this is still an interesting open problem for subelliptic PDEs in non divergence form and this lack precludes one from extending the method of [25, Theorem 5.5] to obtain a non homogeneous Harnack inequality in general subelliptic settings and motivates our interest in the study of an axiomatic approach.

Very recently in [23] we dealt with the problem of establishing an approach of this type in a form that permits to handle both homogeneous and non homogeneous equations in divergence and non divergence form in the general setting of quasi metric spaces. The purpose of this dissertation is to present the results of this investigation and their application to Grushin type and X-elliptic type operators, and some further developments.

The dissertation is organized in three main chapters followed by an Appendix. Here we give an outline of the thesis and briefly present our main results.

In Chapter 1 we present the abstract approach to non homogeneous scale invariant Harnack inequality obtained in [23, Section 2]. We recognize that the double ball and the critical density properties are again the right assumptions to make in order to obtain Harnack inequality. On the other hand, since the abstract approach has to be directly applicable to the case of PDEs with possibly non zero right hand side, we need to modify these two notions in order to take into account the non homogeneity. We will show how these two properties imply the power decay property, which in turn leads to a scale invariant Harnack inequality and consequently Hölder regularity estimates.

In Chapter 2 we apply the results of the previous chapter to the case of X-elliptic operators. These are second order operators in divergence form which are elliptic with respect to the vector fields generating the underlying metric space. In the proof of the critical density property we follow the ideas of [14] and we make use of some results by Uguzzoni [43]. In this last mentioned work the author proves a scale invariant Harnack inequality, via Moser iteration scheme, for non negative weak solutions to a class of X-elliptic operators that is more general than ours. The purpose of this chapter is to show that the abstract approach of Chapter 1 is well suited to operators in divergence form.

In Chapter 3 we extend, to the non homogeneous case, the Harnack inequality for non negative classical solutions to Grushin type equations proved by Montanari in [37]. The equations considered are of the type $Lu = x_1^2 f$ with L a subelliptic non divergence form operator with measurable coefficients and involving Grushin vector fields and the right hand side is such that $x_1 f \in L^2_{\text{loc}}$. When the right hand side is not of this type, Harnack inequality remains an interesting open problem, indeed the Alexandrov-Bakelman-Pucci estimates, which are a key tool towards the Harnack inequality, do not hold in general as shown in Theorem 3.7 and the subsequent Remark. The proof of the double ball and the critical density properties will require the construction of ad-hoc barriers in combination with a weighted Alexandrov-Bakelman-Pucci maximum principle by Montanari. From these two properties, again by means of the abstract procedure developed in Chapter 1, we deduce a non homogeneous scale invariant Harnack inequality and Hölder regularity estimates. This is, in fact, our main result in [23, Section 3]. After the preprint of this last mentioned work had been posted on arXiv:1709.03810 we learned that Diego Maldonado in [36] extended our example in [23, Section 3] to a larger class of non homogeneous PDEs with right hand side of the same type considered in this thesis. We will discuss his deep results at the end of Section 3.4. Then, we also prove Hölder regularity estimates for the X-gradient.

Finally in the Appendix we show how Grushin type operators are related to the prescribed Levi curvature equations in cylindrical coordinates. These equations are fully nonlinear subelliptic equations in non divergence form that arise from the problem of finding characterizing property of the domains of holomorphy in terms of a differential property of the boundary; they are therefore an important model of subelliptic operators related to the analysis in several complex variables.

Table of contents

Chapter 1

An axiomatic approach to Harnack inequality

In this chapter we develop an abstract theory to obtain Harnack inequality for non homogeneous PDEs in the setting of doubling quasi metric Hölder spaces. Our approach is a generalization of the one settled by Di Fazio, Gutiérrez and Lanconelli in [14], where an abstract theory to homogeneous Harnack inequality is established.

As usual in the axiomatic approaches to Harnack inequality, the idea is to consider a particular family of functions, and to prove that if this family satisfies certain properties (in our case the double ball and the critical density: Definitions 1.2.5 and 1.2.4), plus some structural conditions on the underlying space, then it satisfies the Harnack property (Definition 1.2.7). When the abstract theory machinary is applied to prove Harnack inequality for solutions to specific PDEs, the family of functions is chosen to be a subset of the set of non negative solution to the considered PDE (in [2] the solutions must be continuous while in [14] they just need to be measurable). In [2, 14, 30] they need to require the family to be closed under multiplication, precluding in this way the possibility to deal with non homogeneous PDEs. The novelty of our approach is that it permits to consider PDEs with non zero right hand side. Since we want to take into account the right hand side f we will need to introduce a function $\mathcal{S}_{\Omega,f}$ to maintain some kind of control on it. The role of this function will be made clearer in the next who chapters where two applications of the abstract theory are presented and the function $\mathcal{S}_{\Omega,f}$ will be chosen explicitly. For the moment it can be thought as some L^p norm of the right hand side.

At the beginning of this chapter some definitions and well known results about quasi metric spaces theory are recalled. Then we introduce the notions of critical density, double ball, power decay and Harnack property. With the aid of two Covering Theorems proved in [14] we show that under some structural conditions on the quasi metric space, the double ball and critical density properties imply the power decay property (Theorem 1.10). Finally in Theorem 1.11 we prove that the power decay property lead to the Harnack property, and consequently to Hölder regularity estimates. Results contained in this chapter have been published in [23, Section 2].

1.1 Definitions and preliminaries on quasi metric spaces

We recall some definitions and well known results about quasi metric spaces that will be extensively used in the sequel.

Definition 1.1.1. (Quasi distance) Let $Y \neq \emptyset$, we say that a function $d : Y \times Y \to [0,+\infty[$ is a quasi distance if

- $d(x,y) = d(y,x)$, for every $x,y \in Y$;
- for every $x,y \in Y$, $d(x,y) = 0$ if and only if $x = y$;
- (quasi triangle inequality) there exists a constant $K \geq 1$ such that

$$d(x,y) \leq K(d(x,z) + d(z,y)) \qquad \text{for every } x,y,z \in Y.$$

In this case the pair (Y,d) is called quasi metric space and the set

$$B_r(x) := \{y \in Y : d(x,y) < r\}$$

is called a d-ball of center $x \in Y$ and radius $r > 0$ (by abuse of notation, in the sequel, we will write "ball" instead of "d-ball").

Definition 1.1.2. ([11, p.66]) We say that the quasi metric space (Y,d) is of homogeneous type if the balls $B_r(y)$ are a basis of open neighborhoods and there exists a positive integer N such that for each x and $r > 0$, the ball $B_r(x)$ contains at most N points x_j such that $d(x_i,x_j) \geq r/2$ with $i \neq j$.

Definition 1.1.3. (Doubling space) Let (Y,d) be a quasi metric space and μ a positive measure on a σ-algebra of subsets of Y containing the d-balls. We say that the measure μ satisfies the doubling property if there exists a constant $C_D > 0$, called the doubling constant, such that

$$0 < \mu(B_{2r}(x)) \leq C_D \mu(B_r(x)), \qquad \text{for all } x \in Y \text{ and } r > 0.$$

In this case the triple (Y,d,μ) is called a doubling quasi metric space.

In the sequel we will assume $C_D > 2$ and we will use the notation $Q = \log_2 C_D$. We recall that every doubling quasi metric space is of homogeneous type. For a proof of this fact we refer the reader to [11]. In the sequel it will be useful the following lemma that contains a different version of the doubling property.

Lemma 1.1. *Let (Y,d,μ) be a doubling quasi metric space with doubling constant C_D and quasi triangle constant K. We define $Q = \log_2 C_D$, then we have*

$$\mu(B_{r_2}(x)) \le C_D \left(\frac{r_2}{r_1}\right)^Q \mu(B_{r_1}(x)), \quad \textit{for every } r_1 < r_2; \tag{1.1}$$

$$\mu(B_R(y)) \le C_D \left(\frac{2KR}{r}\right)^Q \mu(B_r(x)) \quad \textit{for every } B_r(x) \subset B_R(y). \tag{1.2}$$

Proof. Let $k \in \mathbb{N}$ be such that $2^{k-1} r_1 \le r_2 \le 2^k r_1$. Clearly we have

$$\left(\frac{r_2}{r_1}\right)^{\log_2 C_D} = \left(2^{\log_2(r_2/r_1)}\right)^{\log_2 C_D} = C_D^{\log_2 \frac{r_2}{r_1}} \ge C_D^{k-1}$$

and by repeatedly applying the doubling property we get

$$\mu(B_{r_2}(x)) \le \mu(B_{2^k r_1}(x)) \le C_D C_D^{k-1} \mu(B_{r_1}(x)) \le C_D \left(\frac{r_2}{r_1}\right)^Q \mu(B_{r_1}(x)).$$

This proves (1.1). Since $B_r(x) \subset B_R(y) \subset B_{2KR}(x)$, inequality (1.2) follows directly from (1.1) taking $r_2 = 2KR$, $r_1 = r$ and recalling that $\mu(B_R(y)) \le \mu(B_{2KR}(x))$. □

Definition 1.1.4. (Hölder quasi-distance) We say that the quasi distance d is Hölder continuous if there exist positive constants $\beta, \alpha \in]0,1]$ such that

$$|d(x,y) - d(x,z)| \le \beta d(y,z)^\alpha (d(x,y) + d(x,z))^{1-\alpha} \quad \text{for all } x,y,z \in Y. \tag{1.3}$$

In this case, the pair (Y,d) is said to be a Hölder quasi metric space.

Remark 1.2. *Every metric space is a Hölder quasi metric space. Indeed by the triangle inequality, for every $x,y,z \in Y$ we have*

$$d(x,z) - d(y,z) \le d(x,y) \le d(x,z) + d(y,z)$$

thus $|d(x,y) - d(x,z)| \le d(y,z)$ and (1.3) holds true with $\alpha = \beta = 1$.

The simplest example of Hölder doubling quasi metric space is the space $\mathbb{R}^n$ equipped with the Euclidean distance and the Lebesgue measure. More interesting examples are homogeneous

Lie group equipped with the Gauge distance and Lebesgue measure (see [14, Remark 2.5]). In the sequel we will always assume d to be a Hölder quasi-distance. This requirement on the regularity of d is not a restrictive assumption. Indeed, Marcías and Segovia [35, Theorem 2] proved that given a quasi distance d, it is always possible to construct an equivalent quasi distance d' which is Hölder continuous and such that d'-balls are open sets with respect the topology induced by d'. Moreover we recall that any Hölder doubling quasi metric space is separable (see [35]), consequently open sets are measurable as they are countable union of d-balls and μ is a Borel measure.

In the sequel we will also need two extra structural conditions on the quasi metric space (Y,d,μ). Let us fix $\Omega \subset Y$.

Definition 1.1.5. (Reverse doubling condition) We say that the doubling quasi metric space (Y,d,μ) satisfies the reverse doubling condition in Ω if there exists a constant $\delta \in]0,1[$ such that

$$\mu(B_r(x)) \leq \delta\mu(B_{2r}(x)),$$

for every $B_{2r}(x) \subset \Omega$.

Definition 1.1.6. (Ring condition) We say that the doubling quasi metric space (Y,d,μ) satisfies the ring condition if there exists a non negative function $\omega(\varepsilon)$ such that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and for every ball $B_r(x)$ and all $\varepsilon > 0$ sufficiently small we have

$$\mu\left(B_r(x) \setminus B_{(1-\varepsilon)r}(x)\right) \leq \omega(\varepsilon)\mu(B_r(x)).$$

Moreover we say that (Y,d,μ) satisfies the log-ring condition if it satisfies the ring condition with $\omega(\varepsilon) = o\left(\left(\log\frac{1}{\varepsilon}\right)^{-2}\right)$ as $\varepsilon \to 0^+$.

1.2 Double ball, critical density and power decay properties

Let us consider a Hölder doubling quasi metric space (Y,d,μ) and fix an open set $\Omega \subseteq Y$. As we have already said, we aim to build an axiomatic procedure that permits to prove non homogeneous Harnack inequality for non negative measurable solutions to a possibly non homogeneous PDE. For that reason we modify the notions of critical density, double ball, and power decay property given in [14] in order to take into account the non homogeneity of the PDE. We also need to consider families of functions that depend on the possibly non zero right hand side f. With these new definitions we prove that, under some additional hypotheses on the space (Y,d,μ), the double ball and the critical density properties imply the power decay property.

Let $\mathcal{B}(\Omega) = \{B_r(x) : B_r(x) \subset \Omega\}$ be the set of all the quasi metric balls contained in Ω partially ordered by inclusion. We fix a set

$$\mathcal{F}(\Omega) \subseteq \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable}\}$$

such that

- if $f \in \mathcal{F}(\Omega)$, then $\lambda f \in \mathcal{F}(\Omega)$ for every $\lambda \in \mathbb{R}$.

Also, we fix a function

$$\mathcal{S}_\Omega : \mathcal{B}(\Omega) \times \mathcal{F}(\Omega) \to [0, +\infty[$$

such that

- $\mathcal{S}_\Omega$ is order preserving in the first variable, that is $\mathcal{S}_\Omega(B_r(y), f) \le \mathcal{S}_\Omega(B_R(x), f)$ for every $B_r(y) \subseteq B_R(x)$ and for every $f \in \mathcal{F}(\Omega)$.
- $\mathcal{S}_\Omega$ is homogeneous in the second variable: for every $\lambda \in \mathbb{R}$, $f \in \mathcal{F}(\Omega)$ and $B_r(x) \subseteq \Omega$, we have $\mathcal{S}_\Omega(B_r(x), \lambda f) = |\lambda| \mathcal{S}_\Omega(B_r(x), f)$.

Definition 1.2.1. We define

$$\mathcal{L}(\Omega) = \{f \in \mathcal{F}(\Omega) :\ \mathcal{S}_\Omega(B_r(x), f) < +\infty \text{ for every } B_r(x) \subseteq \Omega\}.$$

Let us give a simple example of a space $\mathcal{L}(\Omega)$. If we consider $\mathbb{R}^n$ with the Lebesgue measure μ and the Euclidean distance, $\mathcal{F}(\Omega) = \{f : \Omega \to \mathbb{R},\ f \text{ measurable}\}$ and $\mathcal{S}_\Omega(B_r(x), f) = \left(\int_{B_r(x)} |f|^p \mathrm{d}\mu\right)^{1/p}$ we have that $\mathcal{L}(\Omega)$ is exactly the space $L^p_{loc}(\Omega)$.

Remark 1.3. *By the homogeneity of the function $\mathcal{S}_\Omega$ with respect to the second variable we have that if $f \in \mathcal{L}(\Omega)$, then $\lambda f \in \mathcal{L}(\Omega)$ for every $\lambda \in \mathbb{R}$.*

Definition 1.2.2. For $f \in \mathcal{L}(\Omega)$ we define $\mathbb{K}_{\Omega,f}$ a family of non negative measurable functions with domain contained in Ω

$$\mathbb{K}_{\Omega,f} \subset \{u : A \to \mathbb{R} \text{ such that } A \subset \Omega,\ u \ge 0 \text{ and } u \text{ is } \mu\text{-measurable}\}$$

such that the following two conditions hold:

- If $u \in \mathbb{K}_{\Omega,f}$ then $\lambda u \in \mathbb{K}_{\Omega,\lambda f}$ for all $\lambda \ge 0$.
- If $u \in \mathbb{K}_{\Omega,f}$ then for every $\lambda, \tau \ge 0$ such that $\tau - \lambda u \ge 0$ we have $\tau - \lambda u \in \mathbb{K}_{\Omega,-\lambda f}$.

In particular if $u \in \mathbb{K}_{\Omega,f}$ and its domain contains $A \subset \Omega$ we will write $u \in \mathbb{K}_{\Omega,f}(A)$.

In Chapters 2 and 3, where we will apply the abstract theory to specific operators, the definition of the family $\mathbb{K}_{\Omega,f}$ will be clarified. Roughly speaking, $\mathbb{K}_{\Omega,f}$ will contain all the non negative measurable solutions u with domain contained in Ω of an equation of the type $Lu = f$, where L is a second order partial differential operator.

As a convention, throughout this chapter, we do not specify the center of a ball if the center is a point $x_0 \in \Omega$, namely we write B_R instead of $B_R(x_0)$.

Definition 1.2.3. (Structural constant) We say that c is a structural constant if it is independent of each u belonging to the family $\mathbb{K}_{\Omega,f}$, of $f \in \mathcal{L}(\Omega)$ and of the balls defined by the quasi distance considered.

We are now ready to state the critical density, double ball and power decay property that will be crucial in the sequel.

Definition 1.2.4. (Critical density) Let $\nu \in]0,1[$. We say that $\mathbb{K}_{\Omega,f}$ satisfies the ν critical density property if there exist structural constants $\varepsilon_{CD}, c \in]0,1[$ depending on ν and a structural constant $\eta_{CD} > 1$ such that for every ball $B_{\eta_{CD}R} \subset \Omega$ and for every $u \in \mathbb{K}_{\Omega,f}(B_{\eta_{CD}R})$ with

$$\mu(\{x \in B_R : u(x) \geq 1\}) \geq \nu \mu(B_R),$$

we have

$$\inf_{B_{R/2}} u \geq c \quad \text{or} \quad \mathcal{S}_\Omega(B_{\eta_{CD}R}, f) \geq \varepsilon_{CD}.$$

In this case we say that the family $\mathbb{K}_{\Omega,f}$ satisfies the ν critical density $CD(\nu, c, \varepsilon_{CD}, \eta_{CD})$.

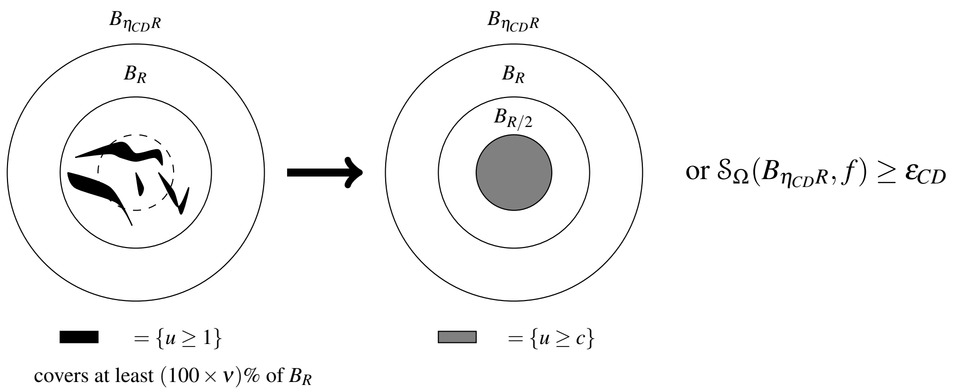

Fig. 1.1 Critical density property $CD(\nu, c, \varepsilon_{CD}, \eta_{CD})$

The critical density property $CD(\nu, c, \varepsilon_{CD}, \eta_{CD})$ illustrated in Figure 1.1 can be interpreted as follows: any $u \in \mathbb{K}_{\Omega,f}(B_{\eta_P R})$ is such that the information that the measure of the region of

B_R where $u \geq 1$ is at least the $(\nu \times 100)\%$ of the measure of B_R is enough to conclude that $\mathcal{S}_\Omega(B_{\eta_{CD}R}, f) \geq \varepsilon_{CD}$ or $u \geq c$ on the whole ball of halved radius.

Definition 1.2.5. (Double ball property) We say that $\mathbb{K}_{\Omega,f}$ satisfies the double ball property if there exist structural constants $\varepsilon_{DB}, \gamma \in]0,1[$ and $\eta_{DB} > 1$ such that for every $B_{\eta_{DB}R} \subset \Omega$ and for every $u \in \mathbb{K}_{\Omega,f}(B_{\eta_{DB}R})$ with

$$\inf_{B_{R/2}} u \geq 1 \quad \text{and} \quad \mathcal{S}_\Omega(B_{\eta_{DB}R}, f) < \varepsilon_{DB}$$

we have

$$\inf_{B_R} u \geq \gamma.$$

In this case we say that the family $\mathbb{K}_{\Omega,f}$ satisfies the double ball property $DB(\gamma, \varepsilon_{DB}, \eta_{DB})$.

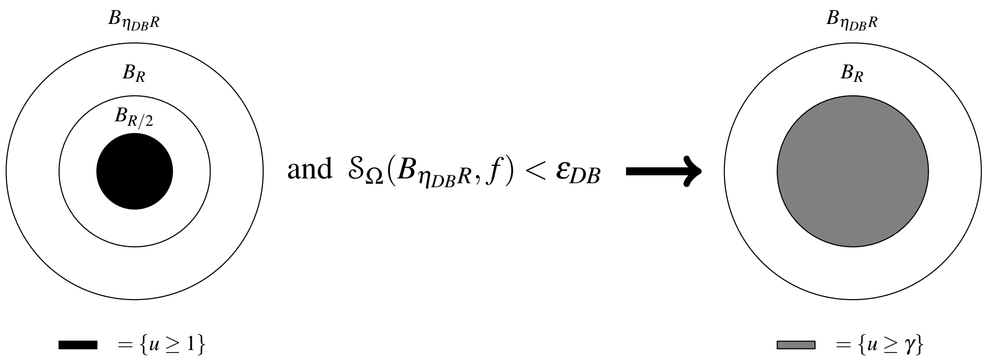

Fig. 1.2 Double Ball property $DB(\gamma, \varepsilon_{DB}, \eta_{DB})$

The double ball property $DB(\gamma, \varepsilon_{DB}, \eta_{DB})$ illustrated in Figure 1.2 can be interpreted as follows: if $\mathcal{S}_\Omega(B_{\eta_{DB}R}, f) < \varepsilon_{DB}$, for any $u \in \mathbb{K}_{\Omega,f}(B_{\eta_{DB}R})$ such that u is greater or equal 1 on the ball $B_{R/2}$, we have that u is grater than a positive constant γ on the whole ball of doubled radius B_R.

Definition 1.2.6. (Power decay) We say that the family of functions $\mathbb{K}_{\Omega,f}$ satisfies the power decay property if there exist structural constants $\gamma \in [0,1[$, $\varepsilon_P \in]0,1[$ and η_P, $M > 1$ such that for each $u \in \mathbb{K}_{\Omega,f}(B_{\eta_P R})$ with

$$\inf_{B_R} u \leq 1 \quad \text{and} \quad \mathcal{S}_\Omega(B_{\eta_P R}, f) < \varepsilon_P$$

we have

$$\mu(\{x \in B_{R/2} : u(x) > M^k\}) \leq \gamma^k \mu(B_{R/2}) \quad \text{for every } k \in \mathbb{N}.$$

In this case we say that the family $\mathbb{K}_{\Omega,f}$ satisfies the power decay property $PD(M,\gamma,\varepsilon_P,\eta_P)$.

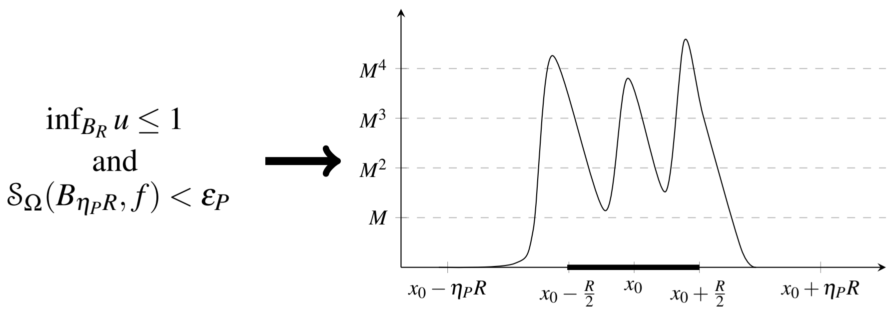

Fig. 1.3 Power Decay property $PD(M,\gamma,\varepsilon_P,\eta_P)$

The power decay property $PD(M,\gamma,\varepsilon_P,\eta_P)$ illustrated in Figure 1.3 can be interpreted as follows. If $\mathcal{S}_\Omega(B_{\eta_P R}, f) < \varepsilon_P$, for any $u \in \mathbb{K}_{\Omega,f}(B_{\eta_P R})$ such that u is smaller or equal 1 on the ball B_R, we have that the measure of the region inside the ball of halved radius $B_{R/2}$ where $u \geq M^k$ decays with rate γ^k as k increases.

Definition 1.2.7. We say that the family of functions $\mathbb{K}_{\Omega,f}$ satisfies the Harnack property if there exist structural constants $\eta > 1$ and $C > 0$ such that for each $u \in \mathbb{K}_{\Omega,f}(B_{\eta R})$ locally bounded we have

$$\sup_{B_R} u \leq C \left(\sup_{B_R} u + \mathcal{S}_\Omega(B_{\eta R}, f) \right).$$

In this case we say that $\mathbb{K}_{\Omega,f}$ satisfies the Harnack property $H(C,\eta)$.

It is not restrictive to require the critical density and the double ball properties to hold with the same constants $\eta_{CD} = \eta_{DB}$ and $\varepsilon_{CD} = \varepsilon_{DB}$, indeed

Remark 1.4. *If the family $\mathbb{K}_{\Omega,f}$ satisfies the ν critical density property $CD(\nu,c,\varepsilon_{CD},\eta_{CD})$ and the double ball property $DB(\gamma,\varepsilon_{DB},\eta_{DB})$ then it satisfies $CD(\nu,c,\varepsilon,\eta)$ and $DB(\gamma,\varepsilon,\eta)$ with $\varepsilon = \min\{\varepsilon_{DB},\varepsilon_{CD}\}$ and $\eta = \max\{\eta_{DB},\eta_{CD}\}$.*

The double ball and critical density properties are in general independent. However, the The following proposition shows that, for sufficiently small values of ν, the ν critical density property implies the double ball property.

Proposition 1.5. *Let C_D be the doubling constant, if $\mathbb{K}_{\Omega,f}$ satisfies the ν critical density property $CD(\nu,c,\varepsilon_{CD},\eta_{CD})$ for some $\nu \in]0,1/C_D^2[$, then $\mathbb{K}_{\Omega,f}$ satisfies the double ball property $DB(c,\varepsilon_{CD},2\eta_{DB})$.*

Proof. Suppose by contradiction that there exists $u \in \mathbb{K}_{\Omega,f}(B_{2\eta_{CD}R})$ with $B_{2\eta_{CD}R} \subset \Omega$, such that $\inf_{B_{R/2}} u \geq 1$ and $\mathcal{S}_\Omega(B_{2\eta_{CD}R}, f) < \varepsilon_{CD}$ but $\inf_{B_R} u < c$. Then by the ν critical density property we have

$$\mu(\{x \in B_{2R} : u(x) \geq 1\}) < \nu\mu(B_{2R}) \quad \text{or} \quad \mathcal{S}_\Omega(B_{2\eta_{CD}R}, f) \geq \varepsilon_{CD}.$$

If the second inequality holds we have an immediate contradiction. Otherwise if the first inequality holds, since $B_{R/2} \subseteq \{x \in B_{2R} : u(x) \geq 1\}$, we find

$$\mu(B_{R/2}) \leq \mu(\{x \in B_{2R} : u(x) \geq 1\}) \leq \nu\mu(B_{2R}) \leq \nu C_D^2 \mu(B_{R/2}) < \mu(B_{R/2})$$

a contradiction. □

Notice that it holds the following

Remark 1.6. *If the family $\mathbb{K}_{\Omega,f}$ satisfies the $\overline{\nu}$ critical density property $CD(\overline{\nu}, c, \varepsilon_{CD}, \eta_{CD})$, then it also satisfies $CD(\nu, c, \varepsilon_{CD}, \eta_{CD})$ for any $\nu > \overline{\nu}$, but, in general, it is not possible to prove the same for $\nu < \overline{\nu}$.*

Now we want to prove that in a Hölder doubling quasi metric space (Y, d, μ), if $\mathbb{K}_{\Omega,f}$ satisfies the double ball and critical density properties plus some other structural conditions on (Y, d, μ), then $\mathbb{K}_{\Omega,f}$ has also the power decay property. We will need few preliminary results and two covering theorems obtained by Di Fazio Gutiérrez and Lanconelli in [14]. These covering theorems will play a key role in the proof of the power decay property.

Covering Theorem 1. *Let (Y, d, μ) be a doubling quasi metric Hölder space satisfying the log-ring condition for all balls $B_t(x)$ and for all ε sufficiently small. Assume that there exist a ball $B_{R_0}(z)$ and a constant $\delta \in]0,1[$ such that $\mu(B_{R_0}(z)) < \delta\mu(B_{2R_0}(z))$. Then there exists a constant $c(\delta) \in]0,1[$ such that for any μ-measurable set $E \subset B_{R_0}(z)$ with $\mu(E) > 0$, there exists a family of balls $\{B_{r_j}(x_j)\}_{j=1}^{\infty}$ satisfying*

i) $r_j \leq 3KR_0$ for all $j \in \mathbb{N}$. Here K the constant in the quasi triangle inequality for d;

ii) all x_j are density points of E with respect to μ[1];

iii) $E \subset \bigcup_{j=1}^{\infty} B_{r_j}(x_j)$ a.e. in the measure μ;

iv) $\frac{\mu(B_{r_j}(x_j) \cap E)}{\mu(B_{r_j}(x_j))} = \delta$ for any $j \in \mathbb{N}$;

v) $\mu(E) \leq c(\delta)\mu\left(\bigcup_{j=1}^{\infty} B_{r_j}(x_j)\right)$.

[1] We recall that $x \in Y$ is a density point for $X \subset Y$ if $\frac{\mu(B_R(x) \cap X)}{\mu(B_r(x))} \to 1$ as $r \to 0^+$

Covering Theorem 2. *Let (Y,d,μ) be a doubling quasi metric Hölder space and suppose the function $r \mapsto \mu(B_r(x))$ is continuous. Moreover, we assume that there exist a ball $B_{R_0}(z)$ and a constant $\delta \in]0,1[$ such that $\mu(B_{R_0}(z)) < \delta\mu(B_{2R_0}(z))$. Then there exist two constants $c(\delta) \in]0,1[$ and $C_\delta > 0$ such that for any μ-measurable set $E \subset B_{R_0}(z)$ with $\mu(E) > 0$, there exists a family of balls $\{B_{r_j}(x_j)\}_{j=1}^{\infty}$ satisfying*

i) $r_j \le C_\delta R_0$ *for all* $j \in \mathbb{N}$*;*

ii) *all* x_j *are density points of* E *with respect to* μ*;*

iii) $E \subset \bigcup_{j=1}^{\infty} B_{r_j}(x_j)$ *a.e. in the measure* μ*;*

iv) $\frac{\delta}{C_D} \le \frac{\mu(B_{r_j}(x_j)\cap E)}{\mu(B_{r_j}(x_j))} < \delta$ *for any* $j \in \mathbb{N}$*;*

v) $\mu(E) \le c(\delta)\mu\left(\bigcup_{j=1}^{\infty} B_{r_j}(x_j)\right)$.

For the proof of Covering Theorem 1 and Covering Theorem 2 we refer the reader to [14, Theorems 3.3 and 3.4] respectively.

Proposition 1.7. *Suppose $\mathbb{K}_{\Omega,f}$ satisfies the double ball and the ν critical density properties $DB(\gamma,\varepsilon,\eta)$ and $CD(\nu,c,\varepsilon,\eta)$ for every $f \in \mathcal{L}(\Omega)$. Then there exists a structural constant $M_0 = \frac{1}{\gamma c} > 1$ such that for any positive constant α and for any $u \in \mathbb{K}_{\Omega,f}(B_{\eta R})$ with*

$$\mu(\{x \in B_R : u(x) \ge \alpha\}) \ge \nu\mu(B_R),$$

we have

$$\inf_{B_R} u \ge \frac{\alpha}{M_0} \quad or \quad \mathcal{S}_\Omega(B_{\eta R}, f) \ge \varepsilon\alpha c.$$

Proof. Since $u \in \mathbb{K}_{\Omega,f}(B_{\eta R})$ we have $\frac{u}{\alpha} \in \mathbb{K}_{\Omega,\frac{f}{\alpha}}(B_{\eta R})$. By the ν critical density property of $\mathbb{K}_{\Omega,\frac{f}{\alpha}}$ follows either $\inf_{B_{R/2}} \frac{u}{\alpha} \ge c$ or $\mathcal{S}_\Omega(B_{\eta R}, f) \ge \alpha\varepsilon \ge \alpha c\varepsilon$. In the second case we are done, otherwise $\frac{u}{\alpha c} \in \mathbb{K}_{\Omega,\frac{f}{\alpha c}}(B_{\eta R})$ and so, either $\mathcal{S}_\Omega(B_{\eta R}, f) \ge \alpha c\varepsilon$ or we can apply the double ball property to obtain $\inf_{B_R} \frac{u}{\alpha c} \ge \gamma$. Defining $\gamma c := \frac{1}{M_0}$ we conclude the proof. □

Lemma 1.8. *Suppose $\mathbb{K}_{\Omega,f}$ satisfies the double ball and the ν critical density properties $DB(\gamma,\varepsilon,\eta)$ and $CD(\nu,c,\varepsilon,\eta)$ for every $f \in \mathcal{L}(\Omega)$. Define $\theta := K(1+4\eta K) > 1$ with K the constant in the quasi triangle inequality. Let $u \in \mathbb{K}_{\Omega,f}(B_{\theta R})$,*

$$\inf_{B_R} u \le 1$$

and suppose there exist structural constants $\alpha > 0$, $\rho < 2KR$ and $y \in B_R$ such that

$$\mu(\{x \in B_\rho(y) : u(x) \ge \alpha\}) \ge \nu\mu(B_\rho(y)). \tag{1.4}$$

Then there exist positive structural constants σ, M_1 such that

$$\rho \leq \left(\frac{M_1}{\alpha}\right)^{\sigma} R \quad or \quad \mathcal{S}_{\Omega}(B_{\theta R}, f) \geq \frac{\varepsilon \alpha c \gamma^p}{M_0}. \tag{1.5}$$

Here $M_1 = (4K)^{1/\sigma} M_0$, M_0 is defined in Proposition 1.7, $\sigma = -\frac{\log 2}{\log \gamma}$ and $p \in \mathbb{N}$ is chosen so that $2^{p-1}\rho \leq 2KR \leq 2^p \rho$.

Proof. If the second inequality in (1.5) holds, the proof is completed. Thus we suppose $\mathcal{S}_{\Omega}(B_{\theta R}, f) < \frac{\varepsilon \alpha c \gamma^p}{M_0}$; this implies

$$\mathcal{S}_{\Omega}(B_r(\tilde{y}), f) < \frac{\varepsilon \alpha c \gamma^p}{M_0} \quad \text{for every } B_r(\tilde{y}) \subseteq B_{\theta R}. \tag{1.6}$$

Since $B_{\eta\rho}(y) \subset B_{\theta R}$ and inequality (1.4) holds, we apply Proposition 1.7 and taking into account (1.6) we deduce

$$\inf_{B_\rho(y)} u \geq \frac{\alpha}{M_0}. \tag{1.7}$$

Moreover if p is chosen as in the statement, since $y \in B_R$, it follows $B_{2^p \eta\rho}(y) \subseteq B_{\theta R}$ and so (1.6) implies

$$\mathcal{S}_{\Omega}(B_{2^{k+1}\eta\rho}(y), f) < \frac{\varepsilon \alpha c \gamma^p}{M_0} \leq \frac{\varepsilon \alpha c \gamma^k}{M_0} \quad \text{for every } 0 \leq k \leq p-1. \tag{1.8}$$

Hence we can repeatedly apply the double ball property to $\frac{u M_0}{\alpha \gamma^k}$ in $B_{2^{k+1}\eta\rho}(y)$, where $k = 0, \ldots, p-1$, obtaining

$$\inf_{B_{2^p\rho}(y)} u \geq \gamma^p \frac{\alpha}{M_0}. \tag{1.9}$$

Indeed by (1.7) we have $\inf_{B_\rho(y)} \frac{u M_0}{\alpha} \geq 1$, this and (1.8) with $k = 0$, allow us to use the double ball property of $\mathbb{K}_{\Omega, f\frac{M_0}{\alpha}}$ to get $\inf_{B_{2\rho}(y)} u \frac{M_0}{\alpha} \geq \gamma$. Now we have $\inf_{B_{2\rho}(y)} u \frac{M_0}{\alpha\gamma} \geq 1$. Again, by (1.8) with $k = 1$, and the double ball property, $\inf_{B_{4\rho}(y)} u \frac{M_0}{\alpha} \geq \gamma^2$. We repeat this procedure p times to find (1.9) and consequently

$$1 \geq \inf_{B_R} u \geq \inf_{B_{2^p\rho}(y)} u \geq \gamma^p \frac{\alpha}{M_0}.$$

From the first and the last inequality in the expression above we get $\gamma^p \leq \frac{M_0}{\alpha}$. Since $\gamma^\sigma = 1/2$, raising both side of the inequality to the power σ, it follows $2^{-p} \leq \left(\frac{M_0}{\alpha}\right)^\sigma$. Finally, multiplying both sides by $2^p \rho$ and keeping in mind the definition of p in the statement we get the thesis. □

Lemma 1.9. *Under the same hypotheses of Lemma 1.8 we have*

$$\rho \leq \left(\frac{M_1}{\alpha}\right)^{\sigma} R \quad \text{or} \quad \mathcal{S}_{\Omega}(B_{\theta R}, f) \geq \varepsilon c.$$

Proof. Let us define $\delta := \frac{\alpha \gamma^p}{M_0} \leq 1$. We shall prove that if $\mathcal{S}_{\Omega}(B_{\theta R}, f) < \varepsilon c$ then $\delta \leq 1$. Indeed, if $\mathcal{S}_{\Omega}(B_{\theta R}, f) < \varepsilon c \delta$ then by the proof of the previous lemma $\delta \leq 1$. On the other side, if $\varepsilon c \delta \leq \mathcal{S}_{\Omega}(B_{\theta R}, f) < \varepsilon c$, then obviously $\delta < 1$. Since in both cases $\delta \leq 1$, then $\rho \leq \left(\frac{M_1}{\alpha}\right)^{\sigma} R$. □

The next theorem shows that the double ball and the critical density properties combined, imply the power decay property. We will prove it under two different set of hypotheses on the Hölder doubling quasi-metric space. The proof follows the ideas in [14].

Theorem 1.10 (Power decay)**.** *Let (Y, d, μ) be a Hölder doubling quasi-metric space and consider $\Omega \subset Y$ open, $f \in \mathcal{L}(\Omega)$. Suppose there exists $\delta \in]0,1[$ such that $\mu(B_r(x)) \leq \delta \mu(B_{2r}(x))$ for every $B_{2r}(x) \subset \Omega$ and one of the following pairs of conditions holds*

(A1) $\mathbb{K}_{\Omega,f}$ *satisfies the double ball and the ν critical density properties $DB(\gamma, \varepsilon, \eta)$ and $CD(\nu, c, \varepsilon, \eta)$ for every $f \in \mathcal{L}(\Omega)$.*

(A2) (Y, d, μ) *satisfies the log-ring condition.*

or

(B1) $\mathbb{K}_{\Omega,f}$ *satisfies the ν critical density property $CD(\nu, c, \varepsilon, \eta)$ for a $\nu \in]0, 1/C_D^2[$ and for every $f \in \mathcal{L}(\Omega)$. Here C_D is the doubling constant.*

(B2) *The function $r \mapsto \mu(B_r(x))$ is continuous.*

Then the family $\mathbb{K}_{\Omega,f}$ satisfies the power decay property.

Proof. First of all we define $\tau := \max\{\nu, \delta\}$, by Remark 1.6, $\mathbb{K}_{\Omega,f}$ satisfies the critical density property $CD(\tau, c, \nu, \eta)$ for every $f \in \mathcal{L}(\Omega)$. That said, throughout the proof we will write ν instead of τ.
In order to prove the theorem under assumptions (A1), (A2) we consider $u \in \mathbb{K}_{\Omega,f}(B_{\eta P r})$ and set

$$E_k = \{x \in B_{\eta P r} : u(x) \geq M^k\}, \quad \text{for every } k \in \mathbb{N}.$$

Moreover, we suppose

$$B_{\eta P r} \subset \Omega, \quad \inf_{B_r} u \leq 1 \quad \text{and} \quad \mathcal{S}_{\Omega}(B_{\eta P r}, f) < \varepsilon_P \tag{1.10}$$

where $\eta_P, M > 1$ and $0 < \varepsilon_P < 1$ are structural constants that will be soon determined (see (1.15) and (1.18)). We shall prove that there exist structural constants $\omega \in]0,1[$ and $\tilde{M} > 1$ such that

$$\mu(\{x \in B_{r/2} : u(x) > \tilde{M}^t\}) \leq \omega^k \mu(B_{r/2}) \quad \text{for every } t \in \mathbb{N}.$$

Notice that the second inequality in (1.10) implies

$$\mathcal{S}_\Omega(B_\rho(x), f) < \varepsilon_P \quad \text{for every } B_\rho(x) \subseteq B_{\eta_P r}. \tag{1.11}$$

We claim it is possible to construct a family of balls B_k of radius t_k and concentric to $B_{\eta_P r}$ such that $r = t_0 > t_1 > t_2 > \cdots > r/2$ and

$$\mu(B_{k+1} \cap E_{k+2}) \leq c(\nu)\mu(B_k \cap E_{k+1}), \quad c(\nu) < 1, \quad k \in \mathbb{N}_0. \tag{1.12}$$

In particular we will construct this family by choosing $t_k = T_k r$ where T_k are defined by

$$\begin{cases} T_k = T_1 - \beta_1 q^3 \sum_{j=0}^{k-2} q^j, \; k > 2 \\ T_2 = 3/4 - \beta_1 q^3 \\ T_1 = 3/4 \end{cases} \quad \text{i.e.} \quad \begin{cases} T_{k+1} = T_k - \beta_1 q^{k+2}, \; k > 1 \\ T_1 = 3/4. \end{cases} \tag{1.13}$$

Here

$$q := 1/M^{\sigma\alpha}, \quad \beta_1 := (2K)^{1-\alpha} \beta M_1^{\sigma\alpha} (1 + M_1^\sigma)^{1-\alpha}, \tag{1.14}$$

σ, M_1 are defined in Lemma 1.8; α, β are the constants in Definition 1.1.4 and K is the quasi triangle inequality constant. Assuming the claim for a moment, from (1.12), we get

$$\begin{aligned} \mu(\{x \in B_{r/2} : u(x) > M^{k+2}\}) &\leq \mu(\{x \in B_{t_{k+1}} : u(x) > M^{k+2}\}) \\ &\leq (c(\nu))^{k+1} \mu(B_r) \\ &\leq (c(\nu))^{k+1} C_D \mu(B_{r/2}) \quad \text{for every } k \in \mathbb{N}_0. \end{aligned}$$

where C_D is the doubling constant. Consider a positive integer k_0 such that $(c(\nu))^{k_0} C_D < 1$, if we define $\tilde{M} = M^{k_0+2}$, from the last inequality, for any $t \in \mathbb{N}$, we have

$$\begin{aligned} \mu(\{x \in B_{r/2} : u(x) > \tilde{M}^t\}) &\leq \mu(\{x \in B_{r/2} : u(x) > M^{k_0+1+t}\}) \\ &\leq (c(\nu))^{k_0} C_D (c(\nu))^t \mu(B_{r/2}) \\ &\leq (c(\nu))^t \mu(B_{r/2}) \quad \text{for every } t \in \mathbb{N}. \end{aligned}$$

Hence $\mathbb{K}_{\Omega,f}$ has the power decay property $PD(\tilde{M}, c(\nu), \varepsilon_P, \eta_P)$.

We now explicitly define η_P and M (these specific choices will be motivated in the proof of the claim) such that

$$\begin{aligned} \eta_P &> \max\{K(3\eta K+1), \theta\}, \\ M &> \max\{M_0, \overline{M}\}, \end{aligned} \tag{1.15}$$

where $\overline{M}$ is a positive constant big enough to have

$$\max\left\{\beta_1 \frac{1}{\overline{M}^{2\sigma\alpha}}, \beta_1 \frac{1}{\overline{M}^{3\sigma\alpha}} \sum_{j\in\mathbb{N}_0} \frac{1}{\overline{M}^{j\sigma\alpha}}\right\} < \frac{1}{4}$$

and M_0 and θ are defined in Proposition 1.7 and Lemma 1.8 respectively. We remark that the definition of M and T_k implies

$$\frac{1}{2} < T_k \leq \frac{3}{4} \quad \text{for every } k \in \mathbb{N}. \tag{1.16}$$

To complete the proof of the Theorem 1.10, we are left with the proof of claim (1.12). We will explicitly show it in the case $k = 0$ and then for a generic $k \in \mathbb{N}$, for the sake of clarity we will subdivide each proof in five steps.
Proof of the claim for $k = 0$

Step I Consider $t_1 = T_1 r = 3/4r$. Since $B_1 \cap E_2 \subset B_r \subset B_{2r} \subset \Omega$, the Covering Theorem 1 by Di Fazio Gutierréz and Lanconelli ensures the existence of a level ν covering $\mathcal{F}_1 = \{B(x_h, r_h)\}$ of $B_1 \cap E_2$ where x_h are density points of $B_1 \cap E_2$, $r_h < 3Kr$ for every $h \in \mathbb{N}$ and

$$\nu = \frac{\mu(B_{r_h}(x_h) \cap B_1 \cap E_2)}{\mu(B_{r_h}(x_h))} \leq \frac{\mu(B_{r_h}(x_h) \cap E_2)}{\mu(B_{r_h}(x_h))}. \tag{1.17}$$

Step II We show that

$$B_{r_h}(x_h) \subset E_1, \quad \text{for every } h \in \mathbb{N}.$$

Since $\eta_P > K(3\eta K+1)$ and $r_h < 3Kr$ we have $B_{\eta r_h}(x_h) \subset B_{\eta_P r}$ so that $u \in \mathbb{K}_{\Omega,f}(B_{\eta r_h}(x_h))$. By (1.17) and Proposition 1.7 it follows $\inf_{B_{r_h}(x_h)} u \geq M^2/M_0 > M$ or $\mathcal{S}_\Omega(B_{\eta r_h}(x_h), f) \geq \varepsilon c M^2$. It suffices to choose

$$\varepsilon_P = \varepsilon c \tag{1.18}$$

in (1.10) and recall that, by definition we have $M > M_0 > 1$ to exclude the latter alternative and get $B_{r_h}(x_h) \subset E_1$.

Step III Now we prove $r_h < 2Kr$ for all $h \in \mathbb{N}$. Suppose by contradiction that there exists a $j \in \mathbb{N}$ such that $r_j > 2Kr$. Then $B_{2r}(x_j) \subset B_{2Kr}(x_j) \subset B_{r_j}(x_j)$. By Step II, $\inf_{B_{r_j}(x_j)} u \geq$

$M^2/M_0 > 1$ and so $\inf_{B_{2r}(x_j)} u > 1$. In addition, since x_j is a density point of $B_1 \cap E_2$, it follows that $x_j \in \overline{B_1}$. Finally, recalling that $t_1 < r$, we have $B_r \subset B_{2Kr}(x_j)$ and consequently $\inf_{B_r} u > 1$. This contradicts (1.10).

Step IV We next show that

$$B_{r_h}(x_h) \subset B_0, \quad \text{for every } h \in \mathbb{N}.$$

First of all notice that since $\eta_P > \theta$ we have $u \in \mathbb{K}_{\Omega,f}(B_{\theta r})$. This, (1.17) and Step III allow us to apply Lemma 1.9 with y, ρ and α replaced by x_h, r_h and M^2, obtaining

$$r_h \le \left(\frac{M_1}{M^2}\right)^{\sigma} r \quad \text{or} \quad \mathcal{S}_\Omega(B_{\theta r}, f) \ge \varepsilon c.$$

Since $\varepsilon_P = \varepsilon c$ and (1.11) holds, we conclude the first alternative take place. Now, if $z \in B_{r_h}(x_h)$, inequality (1.3) and the quasi triangle inequality imply

$$\begin{aligned} d(z,x_0) &\le d(x_h,x_0) + \beta (d(x_h,z))^{\alpha} (d(x_h,x_0) + d(x_0,z))^{1-\alpha} \\ &\le d(x_h,x_0) + (2K)^{1-\alpha} \beta (d(x_h,z))^{\alpha} (d(x_h,x_0) + d(x_h,z))^{1-\alpha} \\ &\le t_1 + (2K)^{1-\alpha} \beta \left(\frac{M_1}{M^2}\right)^{\sigma\alpha} r^{\alpha} \left(t_1 + \left(\frac{M_1}{M^2}\right)^{\sigma} r\right)^{1-\alpha}. \end{aligned}$$

keeping in mind $t_1 = T_1 r$ we get

$$\begin{aligned} d(z,x_0) &\le r\Big(T_1 + (2K)^{1-\alpha} \beta q^2 M_1^{\sigma\alpha} (T_1 + M_1^{\sigma} q^{2/\alpha})^{1-\alpha}\Big) \\ &\le r\Big(T_1 + (2K)^{1-\alpha} \beta q^2 M_1^{\sigma\alpha} (1 + M_1^{\sigma})^{1-\alpha}\Big) \\ &\le r(T_1 + \beta_1 q^2) \end{aligned}$$

where β_1 e q are the positive constants defined in (1.14). In virtue of our choice of t_1 and M we have $T_1 + \beta_1 q^2 < 1$ and so, $B_{r_h}(x_h) \subset B_r$ concluding the proof of Step IV.

Step V By Covering Theorem 1 (v) we have

$$\mu(B_1 \cap E_2) \le c(\nu)\mu\left(\bigcup_{h\in\mathbb{N}} B_{r_h}(x_h)\right),$$

on the other hand Step II and Step VI imply $B_{r_h}(x_h) \subset B_0 \cap E_1$ and hence

$$c(\nu)\mu\left(\bigcup_{h\in\mathbb{N}} B_{r_h}(x_h)\right) \le c(\nu)\mu(B_0 \cap E_1),$$

combining inequalities above we get (1.12) for $k = 0$.

Proof of the claim for a generic $k \in \mathbb{N}$

Step I Consider $t_{k+1} = T_{k+1} r$, where T_k is defined in (1.13). Since $B_{k+1} \cap E_{k+2} \subset B_r \subset B_{2r} \subset \Omega$, the Covering Theorem 1 by Di Fazio Gutiérrez and Lanconelli ensures the existence of a level ν covering $\mathcal{F}_{k+1} = \{B_{r_h}(x_h)\}$ of $B_{k+1} \cap E_{k+2}$ where x_h are density point of $B_{k+1} \cap E_{k+2}$, $r_h < 3Kr$ for every $h \in \mathbb{N}$ and

$$\nu = \frac{\mu(B_{r_h}(x_h) \cap B_{k+1} \cap E_{k+2})}{\mu(B_{r_h}(x_h))} \leq \frac{\mu(B_{r_h}(x_h) \cap E_{k+1})}{\mu(B_{r_h}(x_h))}. \tag{1.19}$$

Step II We show that

$$B_{r_h}(x_h) \subset E_{k+1}, \quad \text{for every } h \in \mathbb{N}.$$

Since $\eta_P > K(3\eta K + 1)$ and $r_h < 3Kr$ we have $B_{\eta r_h}(x_h) \subset B_{\eta_P r}$ so that $u \in \mathbb{K}_{\Omega,f}(B_{\eta r_h}(x_h))$. From (1.19) and Proposition 1.7 it follows that $\inf_{B_{r_h}(x_h)} u \geq M^{k+2}/M_0$ or $\mathcal{S}_\Omega(B_{\eta r_h}(x_h), f) \geq \varepsilon c M^{k+2}$. By (1.11), the definition of ε_P, and our choice of M we exclude the latter alternative and get $B_{r_h}(x_h) \subset E_{k+1}$.

Step III Now we prove that $r_h < 2Kr$ for all h. Suppose by contradiction that there exists a $j \in \mathbb{N}$ such that $r_j > 2Kr$. Then $B_{2r}(x_j) \subset B_{2Kr}(x_j) \subset B_{r_j}(x_j)$. By Step II $\inf_{B_{r_j}(x_j)} u \geq M^{k+2}/M_0 > 1$ so that $\inf_{B_{2r}(x_j)} u > 1$. In addition, since x_j is a density point of $B_{k+1} \cap E_{k+2}$, it follows that $x_j \in \overline{B_{k+1}}$. Finally, recalling that $t_{k+1} < r$, we have $B_r \subset B_{2Kr}(x_j)$ and consequently $\inf_{B_r} u > 1$, on the other hand (1.10) holds and we reach a contradiction.

Step IV We next show that

$$B_{r_h}(x_h) \subset B_k, \quad \text{for every } h \in \mathbb{N}.$$

First of all we notice that since $\eta_P > \theta$ we have $u \in \mathbb{K}_{\Omega,f}(B_{\theta r})$. This, the second inequality in (1.19) and Step III allow us to apply Lemma 1.9 with y, ρ and α replaced by x_h, r_h and M^{k+2} obtaining

$$r_h \leq \left(\frac{M_1}{M^{k+2}}\right)^\sigma r \quad \text{or} \quad \mathcal{S}_\Omega(B_{\theta r}, f) \geq \varepsilon c.$$

By (1.11) and the choice of ε_P we made, we can conclude that the first alternative take place. Now, if $z \in B_{r_h}(x_h)$, inequality (1.3) and the quasi triangle inequality imply

$$\begin{aligned} d(z,x_0) &\le d(x_h,x_0) + \beta(d(x_h,z))^{\alpha}(d(x_h,x_0)+d(x_0,z))^{1-\alpha} \\ &\le d(x_h,x_0) + (2K)^{1-\alpha}\beta(d(x_h,z))^{\alpha}(d(x_h,x_0)+d(x_h,z))^{1-\alpha} \\ &\le t_{k+1} + (2K)^{1-\alpha}\beta\left(\frac{M_1}{M^{k+2}}\right)^{\sigma\alpha} r^{\alpha}\left(t_{k+1}+\left(\frac{M_1}{M^{k+2}}\right)^{\sigma} r\right)^{1-\alpha}. \end{aligned}$$

Moreover, since $t_{k+1} = T_{k+1} r$ we have

$$\begin{aligned} d(z,x_0) &\le r\left(T_{k+1} + (2K)^{1-\alpha}\beta q^{k+2} M_1^{\sigma\alpha}\left(T_{k+1}+M_1^{\sigma} q^{(k+2)/\alpha}\right)^{1-\alpha}\right) \\ &\le r\left(T_{k+1} + (2K)^{1-\alpha}\beta q^{k+2} M_1^{\sigma\alpha}\left(1+M_1^{\sigma}\right)^{1-\alpha}\right) \\ &\le r(T_{k+1}+\beta_1 q^{k+2}) \end{aligned}$$

where β_1 e q are the positive constant defined in (1.14). Keeping in mind (1.13) and (1.16), we have $T_{k+1}+\beta_1 q^{k+2} = T_k < 1$ and so $B_{r_h}(x_h) \subset B_r$ concluding the proof of Step IV.

Step V One one hand by Covering Theorem 1 (v) we have

$$\mu(B_{k+1}\cap E_{k+2}) \le c(\nu)\mu\left(\bigcup_{h\in\mathbb{N}} B_{r_h}(x_h)\right),$$

on the other hand Step II and Step VI imply $B_{r_h}(x_h) \subset B_k \cap E_{k+1}$ and hence

$$c(\nu)\mu\left(\bigcup_{k\in\mathbb{N}} B_{r_h}(x_h)\right) \le c(\nu)\mu(B_k\cap E_{k+1}),$$

combining inequalities above we get (1.12) for general $k \in \mathbb{N}$.

This proves the claim and completes the proof Theorem 1.10 under assumptions (A1) and (A2).

Now suppose hypotheses (B1) and (B2) hold, the proof proceeds exactly as before with $\eta_P := 2\max\{K(3\eta K+1),\theta\}$, by using Covering Theorem 2 instead of Covering Theorem 1. Moreover (1.17) and (1.19) have to be replaced by

$$\frac{\nu}{C_D} \le \frac{\mu(B_{r_h}(x_h)\cap B_1\cap E_2)}{\mu(B_{r_h}(x_h))} \le \frac{\mu(B_{r_h}(x_h)\cap E_2)}{\mu(B_{r_h}(x_h))}$$

and
$$\frac{\nu}{C_D} \leq \frac{\mu(B_{r_h}(x_h) \cap B_{k+1} \cap E_{k+2})}{\mu(B_{r_h}(x_h))} \leq \frac{\mu(B_{r_h}(x_h) \cap E_{k+1})}{\mu(B_{r_h}(x_h))}$$
respectively. □

We explicitly remark that in theorem above the power decay property has been proved for every family $\mathbb{K}_{\Omega,f}$ with $f \in \mathcal{L}(\Omega)$.

1.3 Proof of the abstract Harnack inequality

This section is the core of our work, here we prove that the Harnack property is a consequence of the power decay property. We recall that our arguments are an adaptation to the ones given in [14]. In particular we will show that

Theorem 1.11 (Harnack inequality). *Let (Y,d,μ) be a doubling quasi metric Hölder space, suppose $\mathbb{K}_{\Omega,f}$ satisfies the power decay property $PD(M,\gamma,\varepsilon_P,\eta_P)$ for every $f \in \mathcal{L}(\Omega)$. Then, $\mathbb{K}_{\Omega,f}$ also satisfies the Harnack property for every $f \in \mathcal{L}(\Omega)$, that is for every $B_{\eta R}(x_0) \subset \Omega$, if $u \in \mathbb{K}_{\Omega,f}(B_{\eta R}(x_0))$ is locally bounded, there exists a positive structural constant C such that*
$$\sup_{B_R(x_0)} u \leq C\left(\inf_{B_R(x_0)} u + \mathcal{S}_\Omega(B_{\eta R}(x_0), f)\right).$$
Here $\eta = 2K(2K\eta_P + 1)$ and K is the constant in the quasi triangle inequality.

We remark that in virtue to [35, Theorem 2] it is not restrictive to require d to be a Hölder quasi distance, since if d do not satisfies this hypothesis, one can always consider equivalent Hölder quasi distance d'. We prove Harnack inequality using the following lemma and proposition.

Lemma 1.12. *Let (Y,d,μ) be a doubling quasi metric space, suppose $\mathbb{K}_{\Omega,f}$ satisfies the power decay property $PD(M,\gamma,\varepsilon_P,\eta_P)$ for every $f \in \mathcal{L}(\Omega)$. If $\mathcal{S}_\Omega(B_{2\eta_P R}(z_0), f) \leq \varepsilon_P$, $u \in \mathbb{K}_{\Omega,f}(B_{2\eta_P R}(z_0))$ is such that $\inf_{B_{2R}(z_0)} u \leq 1$, $u(x_0) \geq M^k$ and $B_{2\rho}(x_0) \subset B_R(z_0)$, then*
$$\sup_{B_\rho(x_0)} u \geq u(x_0)\left(1 + \frac{1}{M}\right). \tag{1.20}$$
Here $x_0 \in B_R(z_0)$, $k \geq 2$, $\rho = \frac{\gamma^{k/Q} R}{c_1}$, $Q = \log_2(C_D)$, $c_1 < \left(\frac{\gamma^{1/Q}(1-\gamma)^{1/Q}}{C_D^{1/Q} 4K\eta_P}\right)$, C_D is the doubling constant and K is the quasi triangle inequality constant.

Proof. We shall prove the statement by contradiction. Suppose (1.20) is not true and define

$$A_1 := \{x \in B_R(z_0) : u(x) \geq M^{k-1}\},$$
$$A_2 := \{x \in B_{\rho/(2\eta_P)}(x_0) : w(x) \geq M\}$$

where

$$w(x) := \frac{u(x_0)\left(1+\frac{1}{M}\right)-u(x)}{\frac{u(x_0)}{M}} = M+1-\frac{M}{u(x_0)}u(x) \in \mathbb{K}_{\Omega,-\frac{Mf}{u(x_0)}}(B_\rho(x_0)).$$

Since $w(x_0)=1$, $\inf_{B_{\rho/\eta_P}(x_0)} w(x) \leq 1$. Moreover $\mathcal{S}_\Omega(B_\rho(x_0), -\frac{M}{u(x_0)}f) = \mathcal{S}_\Omega(B_\rho(x_0), f)\frac{M}{u(x_0)} \leq \varepsilon_P M^{-k+1} \leq \varepsilon_P$, so the power decay property for $\mathbb{K}_{\Omega,-\frac{M}{u(x_0)}f}$ and $\mathbb{K}_{\Omega,f}$ implies respectively

$$\mu(A_2) \leq \gamma\mu(B_{\rho/(2\eta_P)}(x_0)) \quad \text{and} \quad \mu(A_1) \leq \gamma^{k-1}\mu(B_R(z_0)).$$

Recalling that $B_{\rho/(2\eta_P)}(x_0) \subset B_\rho(x_0) \subset B_r(z_0)$ we can show the inclusion $B_{\rho/(2\eta_P)} \subset A_1 \cup A_2$. Indeed if $x \in B_{\rho/(2\eta_P)}$ but $x \notin A_1$ then $u(x) < M^{k-1}$ so $w(x) \geq M$ and hence $x \in A_2$, vice versa if $x \in B_r(z_0)$ but $x \notin A_2$ then $w(x) < M$ so $u(x) > u(x_0) \geq M^k$ and hence $x \in A_1$. Consequently we estimate the measure of $B_{\rho/(2\eta_P)}(x_0)$ by

$$\mu(B_{\rho/(2\eta_P)}(x_0)) \leq \mu(A_1)+\mu(A_2) \leq \gamma^{k-1}\mu(B_R(z_0)) + \gamma\mu(B_{\rho/(2\eta_P)}(x_0)).$$

Now, since $B_{\rho/(2\eta_P)}(x_0) \subset B_R(z_0) \subset B_{2KR}(x_0)$, recalling Lemma 1.1, the last inequality becomes

$$\mu(B_{\rho/(2\eta_P)}(x_0)) \leq \left(\gamma^{k-1}C_D\left(\frac{4KR\eta_P}{\rho}\right)^Q + \gamma\right)\mu(B_{\rho/2\eta_P}(x_0))$$

where $Q = \log_2 C_D$. From the strict positiveness of the measure of $B_{\rho/(2\eta_P)}(x_0)$ and the definition of ρ and c_1 given in the statement, we get

$$1-\gamma \leq \gamma^{k-1}C_D\left(\frac{4KR\eta_P}{\rho}\right)^Q = C_D(4K\eta_P c_1)^Q\frac{\gamma^{k-1}}{\gamma^k} = \frac{C_D}{\gamma}(4K\eta_P c_1)^Q$$

that is equivalent to $c_0 := \frac{C_D}{\gamma(1-\gamma)}(4K\eta_P c_1)^Q \geq 1$. On the other hand, since $c_1 < \left(\frac{\gamma^{1/Q}(1-\gamma)^{1/Q}}{C_D^{1/Q}4K\eta_P}\right)$ we have $c_0 < 1$ reaching a contradiction. □

Proposition 1.13. *Let (Y,d,μ) be a doubling quasi metric space and suppose $\mathbb{K}_{\Omega,f}$ satisfies the power decay property $PD(M,\gamma,\varepsilon_P,\eta_P)$ for every $f \in \mathcal{L}(\Omega)$. Consider $B_{\eta R}(x_0) \subset \Omega$, and $u \in \mathbb{K}_{\Omega,f}(B_{\eta R}(x_0))$ and locally bounded. Then, if $\inf_{B_R(x_0)} u < 1$ and $\mathcal{S}_\Omega(B_{\eta R}(x_0), f) < \varepsilon_P$,* *there exists a positive structural constant C such that*

$$\sup_{B_R(x_0)} u \le C.$$

Proof. Consider $B_R(z)$ with $z \in B_R(x_0)$ and define

$$D := \sup_{x \in B_R(z)} u(x) g(x,R)$$

where

$$g(x,R) := \left(\frac{R - d(x,z)}{R} \right)^{\delta/\alpha},$$

δ is a structural constant that will be soon defined and α is the exponent in the Hölder property for d (see Definition 1.1.4). We claim that D is bounded from above by a structural constant C. Deferring the proof of the claim for a moment we have

$$u(x) \le C \left(\frac{R}{R - d(x,z)} \right)^{\delta/\alpha}, \quad \text{for all } x \in B_R(z) \text{ and for every } z \in B_R(x_0), \tag{1.21}$$

thus the thesis follows taking $x = z$ in (1.21).
Hence we are left with the proof of the claim, to this aim choose

$$\delta > 0 \quad \text{such that } \frac{1}{M} = \gamma^{\delta/Q},$$
$$\beta_* > 2(2K)^{1-\alpha} \beta \left(1 - \left(1 + \frac{1}{M} \right)^{-\alpha/\delta} \right)^{-1} > 2\beta (2K)^{1-\alpha},$$
$$k_0 \in \mathbb{N}, \quad k_0 > \frac{Q}{\log \gamma} \log \left(c_1 \left(2^{\frac{1}{1-\alpha}} - 1 \right) \right),$$

where M, ρ, γ and c_1 are defined in the statement of Lemma 1.12, α, β are as in Definition 1.1.4, and $Q = \log_2 C_D$. Notice that k_0 is a structural constant whose definition implies $\left(1 + \frac{\rho}{R}\right)^{1-\alpha} = \left(1 + \frac{\gamma^{k/Q}}{c_1}\right)^{1-\alpha} < 2$ for every $k \ge k_0$ and since u is non negative and locally bounded, $+\infty > D \ge 0$. If $D > 0$ pick $D^* \in (0,D)$, it suffices to show that D^* is bounded from above by a structural constant C to prove the claim. Since $u \ge 0$ is not identically zero, there exists $x_* \in B_R(z)$ such that $D^* < u(x_*) g(x_*,R)$, if $u(x_*) < 1$ we are done otherwise choose $k \in \mathbb{Z}$ such that $M^k \le u(x_*) < M^{k+1}$. For clarity sake we consider three different cases.

Case I If $k \le k_0$ then

$$D^* < M^{k+1} g(x_*, R) \le M^{k_0+1}.$$

Case II If $k > k_0$ and $\frac{D^*}{M} c_1^\delta < \beta_*^{\delta/\alpha}$, clearly D^* is bounded above by $\frac{M\beta_*^{\delta/\alpha}}{c_1^\delta}$.

Case III $k > k_0$ and $\frac{D^*}{M} c_1^\delta \ge \beta_*^{\delta/\alpha}$. We will show that this is never the case.

So let us assume $k > k_0$ and $\frac{D^*}{M} c_1^\delta \ge \beta_*^{\delta/\alpha}$. For $\rho = \frac{\gamma^{k/Q} R}{c_1}$ as in Lemma 1.20, we have

$$1 \ge g(x_*, R) > \frac{D^*}{M^{k+1}} = \frac{D^*}{M} (\gamma^k)^{\delta/Q} = \frac{D^*}{M} \left(c_1 \frac{\rho}{R} \right)^\delta,$$

hence, combining inequalities above and the definition of g we compute

$$d(x_*, z) < R - \beta_* R^{1-\alpha} \rho^\alpha. \tag{1.22}$$

Now, if $y \in B_\rho(x_*)$, by the Hölder property of the quasi distance and the quasi triangle inequality, the definition of k_0 and β_*, for every $k \ge k_0$ we have

$$\begin{aligned} d(y,z) &\le d(z,x_*) + (2K)^{1-\alpha} \beta \left(d(x_*,y) \right)^\alpha \left(d(x_*,y) + d(z,x_*) \right)^{1-\alpha} \\ &\le R - \beta_* R^{1-\alpha} \rho^\alpha + (2K)^{1-\alpha} \beta \rho^\alpha (\rho + R)^{1-\alpha} \\ &\le R - \beta_* R^{1-\alpha} \rho^\alpha + 2(2K)^{1-\alpha} \beta R^{1-\alpha} \rho^\alpha \\ &< R, \end{aligned}$$

hence

$$B_\rho(x^*) \subset B_R(z). \tag{1.23}$$

We can apply Lemma 1.12 with R, z_0 and x_0 replaced by KR, z and x_* respectively to obtain

$$\sup_{B_\rho(x_*)} u \ge u(x_*) \left(1 + \frac{1}{M} \right) > \frac{D^*}{g(x_*, R)} \left(1 + \frac{1}{M} \right). \tag{1.24}$$

Indeed since $B_{2KR\eta_P}(z) \subset B_{\eta R}(x_0)$, we have $\inf_{B_{2KR}(z)} u \le \inf_{B_R(x_0)} u$, and since $B_R(x_0) \subset B_{2KR}(z)$ we get $u \in \mathbb{K}_{\Omega,f}(B_{2KR\eta_P}(z))$; moreover $u(x_*) \ge M^k$ and $\mathcal{S}_\Omega(B_{2KR\eta_P}(z), f) < \varepsilon_P$. Thus all the hypotheses of Lemma 1.12 are satisfied.

By (1.23), for $y \in B_\rho(x_*)$

$$\sup_{B_\rho(x_*)} u \le D \sup_{y\in B_\rho(x_*)} \frac{1}{g(y,R)} = \frac{D}{g(x_*,R)} \sup_{y\in B_\rho(x_*)} \frac{g(x_*,R)}{g(y,R)}. \tag{1.25}$$

Moreover by the Hölder property, the quasi triangle inequality and (1.22) we have

$$\begin{aligned}\left(\frac{g(x_*,R)}{g(y,R)}\right)^{\alpha/\delta} &= \frac{R-d(z,x_*)}{R-d(y,z)} \\ &\le \frac{R-d(z,x_*)}{R-\left(d(z,x_*)+\beta(2K)^{1-\alpha}\rho^\alpha(d(z,x_*)+\rho)^{1-\alpha}\right)} \\ &\le \frac{1}{1-\frac{\beta(2K)^{1-\alpha}\rho^\alpha(R+\rho)^{1-\alpha}}{\beta_*\rho^\alpha R^{1-\alpha}}} \\ &\le \frac{1}{1-\frac{2\beta(2K)^{1-\alpha}}{\beta_*}}\end{aligned}$$

Combining (1.24), (1.25) and the inequality above we obtain

$$D^* < D\left(\frac{M}{1+M}\right)\left(\frac{\beta_*}{\beta_*-2\beta(2K)^{1-\alpha}}\right)^{\delta/\alpha}. \tag{1.26}$$

hence taking the limit for $D^* \to D$ in (1.26) we find $1 < \left(\frac{M}{1+M}\right)\left(\frac{\beta_*}{\beta_*-2\beta(2K)^{1-\alpha}}\right)^{\delta/\alpha}$ from which we get

$$\begin{aligned}\beta_*\left(\left(\frac{1+M}{M}\right)^{\alpha/\delta}-1\right)-2\beta(2K)^{1-\alpha}\left(\frac{1+M}{M}\right)^{\alpha/\delta} &< 0 \\ \beta_* < 2\beta(2K)^{1-\alpha}\left(1-\left(\frac{M+1}{M}\right)^{-\alpha/\delta}\right)^{-1}&\end{aligned}$$

which is in contrast with the previous choice of β_*. □

Proof of Theorem 1.11. It suffices to prove $\sup_{B_R(x_0)} u \le CM$ for every $M = \inf_{B_R(x_0)} u + \frac{\mathcal{S}_\Omega(B_{\eta R}(x_0),f)}{\varepsilon_P} + \delta$, with $\delta > 0$. Consider

$$\tilde{u} := \frac{u}{M} \in \mathbb{K}_{\Omega,\tilde{f}}, \quad \text{where } \tilde{f} := \frac{f}{M}$$

clearly $\inf_{B_R(x_0)} \tilde{u} \leq 1$ and $\mathcal{S}(B_{\eta R}(x_0), \tilde{f}) \leq \varepsilon_P$, so that by Proposition 1.13 we find $\sup_{B_R(x_0)} u \leq CM$, hence

$$\sup_{B_R(x_0)} u \leq \frac{C}{\varepsilon_P}\left(\inf_{B_R(x_0)} u + \mathcal{S}_\Omega(B_{\eta R}(x_0), f) + \delta\right).$$

By letting $\delta \to 0^+$ we get the thesis. □

We have just showed that in the setting of doubling quasi metric Hölder spaces with the reverse doubling property and satisfying the log-ring condition, for a family of function $\mathbb{K}_{\Omega,f}$ it is enough to satisfy the double ball and the critical density property to conclude that it has the Harnack property. We are going to prove that also the converse is true. Hence the Harnack property is equivalent to the double ball and the critical density properties jointly considered.

Theorem 1.14. *Suppose that $\mathbb{K}_{\Omega,f}$ satisfies the Harnack property $H(C,\eta)$. Then, the family $\mathbb{K}_{\Omega,f}$ satisfies the double ball and the ν critical density property $DB\left(\frac{k-1}{kC}, \frac{1}{kC}, 2\eta\right)$ and $CD(\nu, \frac{1}{kC}, \frac{k-1}{kC}, \eta)$ for any fixed $k > 1$ and for every u locally bounded.*

Proof. Since $\mathbb{K}_{\Omega,f}$ satisfies the Harnack property $H(C,\eta)$, there exist structural constants $C > 0$ and $\eta > 1$ such that for each $u \in \mathbb{K}_{\Omega,f}(B_{\eta r})$, locally bounded, the function u satisfies the inequality

$$\sup_{B_r} u \leq C\left(\inf_{B_r} u + \mathcal{S}(B_{\eta r}, f)\right).$$

We start proving the double ball property, so we assume

$$\sup_{B_r} u \geq 1 \qquad \text{and} \qquad \mathcal{S}_\Omega(B_{2\eta r}, f) < \frac{1}{kC} \quad \text{for a fixed } k > 1.$$

Then

$$1 \leq \inf_{B_r} u \leq \sup_{B_{2r}} u \leq C\left(\inf_{B_{2r}} u + \mathcal{S}_\Omega(B_{2\eta r}, f)\right) \leq C\left(\inf_{B_{2r}} u + \frac{1}{kC}\right)$$

from which we get the desired estimate for $\inf_{B_{2r}} u$,

$$\inf_{B_{2r}} u \geq \frac{k-1}{Ck}.$$

Now we want to prove the critical density property $CD(\nu, \frac{1}{kC}, \frac{k-1}{kC}, \eta)$ for each $k > 1$. Let us suppose that $\inf_{B_{r/2}} u < \frac{1}{kC}$ and $\mathcal{S}_\Omega(B_{\eta r}, f) < \frac{k-1}{kC}$, $k > 1$, then

$$\begin{aligned}\sup_{B_r} u &\leq C\left(\inf_{B_r} u + \mathcal{S}_\Omega(B_{\eta r}, f)\right)\\ &< C\left(\inf_{B_{r/2}} u + \frac{k-1}{kC}\right) < C\left(\frac{1}{kC} + \frac{k-1}{kC}\right) = 1.\end{aligned}$$

Hence $\mu(\{x \in B_r : u(x) < 1\}) \geq \mu(B_r) \geq \varepsilon\mu(B_r)$ for any $\varepsilon \in]0,1[$. Clearly, from this property we deduce that if there exists a constant $\nu \in]0,1[$ such that

$$\mu(\{x \in B_r : u(x) < 1\}) < (1-\nu)\mu(B_r)$$

i.e.

$$\mu(\{x \in B_r : u(x) \geq 1\}) \geq \nu\mu(B_r)$$

then we must have $\inf_{B_{r/2}} u \geq \frac{1}{kC}$ or $\mathcal{S}_\Omega(B_{\eta r}, f) \geq \frac{k-1}{kC}$, $k > 1$. □

1.4 Hölder regularity

We briefly discuss how to obtain Hölder regularity estimates in the abstract setting we have presented. It is well known that from the scale invariant Harnack inequality it is possible to obtain an oscillation inequality which in turns gives Hölder regularity estimates (see for example [22, Section 8.9]). We report the procedure with some minor modifications that are necessary to adapt classical arguments to our abstract setting. It will be needed the following Lemma.

Lemma 1.15 ([22] Lemma 8.23)**.** *Let ω and σ be two non decreasing functions on an interval $]0,R]$ satisfying the following inequality*

$$\omega(\tau\rho) \leq \gamma\omega(\rho) + \sigma(\rho)$$

for any $\rho \leq R$ and for some constants $0 < \gamma,\ \tau < 1$. Then, for any $0 < \mu < 1$ we have

$$\omega(\rho) \leq M\left(\left(\frac{\rho}{R}\right)^{\alpha}\omega(R) + \sigma(\rho^{\mu}R^{1-\mu})\right) \quad \textit{for any } \rho \leq R,$$

where $\alpha = (1-\mu)\frac{\log\gamma}{\log\tau}$ and $M = M(\gamma,\tau) > 0$.

In the next theorem will use the notation

$$M_r = \sup_{B_r(x_0)} u, \quad m_r = \inf_{B_r(x_0)} u, \quad \operatorname{osc}_{B_r(x_0)} u = M_r - m_r$$

and we define

$$\tilde{\mathbb{K}}_{\Omega,f} \subset \{u : A \to \mathbb{R} \text{ such that } A \subset \Omega,\ u \text{ is } \mu\text{-measurable}\}$$

such that the following three conditions hold:

- $\mathbb{K}_{\Omega,f} \subset \tilde{\mathbb{K}}_{\Omega,f}$
- If $u \in \tilde{\mathbb{K}}_{\Omega,f}$ then $\lambda u \in \tilde{\mathbb{K}}_{\Omega,\lambda f}$ for all $\lambda \geq 0$.
- If $u \in \tilde{\mathbb{K}}_{\Omega,f}$ then for every $\lambda, \tau \geq 0$ such that $\tau - \lambda u \geq 0$ we have $\tau - \lambda u \in \tilde{\mathbb{K}}_{\Omega,-\lambda f}$.

In other word $\tilde{\mathbb{K}}_{\Omega,f}$ is obtained from $\mathbb{K}_{\Omega,f}$ by removing the condition $u \geq 0$.

Theorem 1.16. *Suppose the family $\mathbb{K}_{\Omega,f}$ has the Harnack property $H(C,\eta)$ (Definition 1.2.7) and assume that for every ball $B_R(x_0) \subset \Omega$ and $u \in \tilde{\mathbb{K}}_{\Omega,f}(B_R(x_0))$, if $\lambda \in \mathbb{R}$ is such that $u - \lambda \geq 0$, we have $u - \lambda \in \mathbb{K}_{\Omega,f}(B_R(x_0))$. Then, there exists two positive structural constants c and $\alpha \in]0,1[$ such that*

$$\operatorname{osc}_{B_r(x_0)} u \leq c r^{\alpha} \left(R^{-\alpha} \sup_{B_R(x_0)} |u| + \frac{\mathcal{S}_{\Omega}(B_{r^{\mu}R^{1-\mu}}(x_0), f)}{r^{\alpha}} \right)$$

for every $u \in \tilde{\mathbb{K}}_{\Omega,f}(B_R(x_0))$ locally bounded, $r \in]0,R]$ and $\mu \in]0,1[$.

Proof. In the proof we denote positive structural constants by C (even if the value of the constant may change at each occurrence). The functions $M_r - u$ and $u - m_r$ are non negative so they belong to the family $\mathbb{K}_{\Omega,-f}(B_R(x_0))$ and $\mathbb{K}_{\Omega,f}(B_R(x_0))$, respectively. By the Harnack property we get

$$M_r - m_{r/\eta} \leq C\left(M_r - M_{r/\eta} + \mathcal{S}_{\Omega}(B_r(x_0), f)\right) \quad \text{for every } r \leq R$$
$$M_{r/\eta} - m_r \leq C\left(m_{r/\eta} - m_r + \mathcal{S}_{\Omega}(B_r(x_0), f)\right) \quad \text{for every } r \leq R.$$

Summing up the two inequalities above we find

$$\operatorname{osc}_{B_r(x_0)} u + \operatorname{osc}_{B_{r/\eta}(x_0)} u \leq C\left(\operatorname{osc}_{B_r(x_0)} u - \operatorname{osc}_{B_{r/\eta}(x_0)} u + 2\mathcal{S}_{\Omega}(B_r(x_0), f)\right)$$

from which we get the following oscillation inequality

$$\underset{B_{r/\eta}(x_0)}{\operatorname{osc}} u \le \frac{C-1}{C+1} \underset{B_r(x_0)}{\operatorname{osc}} u + 2\frac{C}{C+1} \mathcal{S}_\Omega(B_r(x_0), f) \quad \text{for every } r \le R.$$

Since $\underset{B_{r/\eta}(x_0)}{\operatorname{osc}} u$ and $\mathcal{S}_\Omega(B_r(x_0), f)$ are non decreasing with respect to r, for every $r \in]0,R]$ we can apply Lemma 1.15 with $\omega(r) \rightsquigarrow \underset{B_r(x_0)}{\operatorname{osc}} u$, $\sigma(r) \rightsquigarrow 2\frac{C}{C+1}\mathcal{S}_\Omega(B_r(x_0), f)$, $\tau \rightsquigarrow \frac{1}{\eta}$ and $\gamma \rightsquigarrow \frac{C-1}{C+1}$. We obtain

$$\underset{B_r(x_0)}{\operatorname{osc}} u \le c \left(\left(\frac{r}{R}\right)^\alpha \underset{B_R(x_0)}{\operatorname{osc}} u + \mathcal{S}_\Omega(B_{r^\mu R^{1-\mu}}(x_0), f) \right)$$

for every $r \in]0,R]$ and $\mu \in]0,1]$. The thesis follows recalling that $\underset{B_R(x_0)}{\operatorname{osc}} u \le 2 \sup_{B_R(x_0)} |u|$. $\square$

As a conclusion to this chapter we want to recall some literature regarding different axiomatic approach to Harnack inequality in the general setting of doubling quasi metric spaces. The first work we want to mention is [2] by Aimar Forzani and Toledano, where the authors prove Harnack inequality for some class of continuous functions as a consequence of the double ball and the power decay properties. Then, Di Fazio, Gutièrrez, and Lanconelli in [14] proved the right covering argument, necessary to relax the continuity assumption in [2]; under an additional assumption on the underlying quasi metric space, namely the ring condition, they are able to consider family of just measurable functions. Another type of approach was introduced by Indratno, Maldonado and Silwal in [30], where the authors replace the double ball property with an integral condition that makes their approach better suited for variational equations. In all these works the considered functional set is assumed to be closed under multiplication by positive constants, this assumption precludes one to directly apply the abstract procedures to family of solutions to PDE with non zero right hand side. In this chapter we have shown that it is possible to extend the approach in [14] making it well suited for non homogeneous equations.

Chapter 2

Application to X-elliptic operators

In this chapter we apply the abstract theory we have developed in Chapter 1 to a class of partial differential equations in divergence form related to a family of locally Lipschitz vector fields.

We first recall the notion of Carnot–Carathéodory distance, set the main assumptions on the metric space and we define the class of PDEs we want to deal with. Then we recall the definition of Sobolev spaces related to a family of vector fields and the notion of W^1 weak solutions. Finally, making use of some results obtained in [43, Section 2] we prove the ν critical density property for non negative measurable weak solutions to the considered equation for every $\nu \in]0,1[$. This property will allow us to set in motion the abstract theory machinery (Theorem 1.10 under hypotheses (B1)-(B2) and Theorem 1.11) to prove Harnack inequality and Hölder regularity estimate. We recall that Uguzzoni in [43] proved a Harnack inequality for a class of operator more general then ours by means of an adapted Moser iteration technique.

2.1 Definitions and main assumptions

In this section we recall the definition of Carnot–Carathéodory distance and set the main assumptions on the metric space and the class of operators considered.

On $\mathbb{R}^N$ we consider $X = \{X_1, \dots, X_m\}$ a family of vector fields with real valued locally Lipschitz coefficients d_{jk}

$$X_j = \sum_{k=1}^{N} d_{jk} \partial_{x_k}, \quad \text{for every } j = 1, \dots m.$$

We endow $\mathbb{R}^N$ with the Lebesgue measure $|\cdot|$.

Definition 2.1.1. Let $\gamma : [0;T] \to \mathbb{R}^N$ be a Lipschitz curve. We say that γ is subunit if there exist a vector valued function $\alpha : [0,T] \to \mathbb{R}^m$ such that

$$\dot{\gamma}(t) = \sum_{n=1}^{m} \alpha_n(t) X_n(\gamma(t)) \quad \text{almost everywhere in } [0,T]$$

and

$$\sup_{t \in [0,T]} \left(\sum_{n=1}^{m} |\alpha_n(t)|^2 \right)^{\frac{1}{2}} \leq 1.$$

The Carnot–Carathéodory distance or control distance associated to the family X is then defined as follows

Definition 2.1.2. (Carnot–Carathéodory distance) Given $x, y \in \mathbb{R}^N$, we define

$$d_X(x,y) = \inf\{T \in \mathbb{R} \text{ s.t. } \gamma : [0,T] \to \mathbb{R}^N \text{ is subunit, } \gamma(0) = x,\ \gamma(T) = y\}.$$

If the points x and y can not be connected through a subunit curve we set $d_X(x,y) = \infty$.

From now on we assume the family X to be such that

(C) The Carnot–Carathéodory distance d_X related to the family of vectors fields X is well defined (i.e. for every $x, y \in \mathbb{R}^N$ there exists a subunit curve joining x and y) and continuous with respect to the Euclidean topology.

We recall that in general on a bounded set of $\mathbb{R}^N$, the topology generated by the control distance is stronger than the Euclidean one. Indeed one has the following well known result (see for example [28, Proposition 11.2].)

Lemma 2.1. *Let $K \subset Y$ be compact. Then there exists a constant $C > 0$ depending on K and Y such that*

$$|x - y| \leq C d_X(x,y), \text{ for every } x, y \in K. \tag{2.1}$$

Hence, under assumption **(C)** we have that on an open bounded set the Carnot–Carathéodory topology is equivalent to the Euclidean one. We recall that **(C)** is true for a large class of vector fields satisfying the so called Hormander condition (See for example [5, p.191 assumption (H2)]) which includes Carnot groups and Grushin type vector fields. We will denote by $B_r(x)$ the ball of center $x \in \mathbb{R}^N$ and radius $r > 0$ defined by the distance d_X. We also assume the following "local" properties:

(D) (Doubling condition) For each compact set $K \subset \mathbb{R}^N$ there exist positive constants $C_D > 1$ and $R_0 > 0$ such that

$$0 < |B_{2r}(x)| \leq C_D |B_r(x)|$$

for every d_X-ball $B_r(x)$ with $x \in K$ and $r \le R_0$.

(P) (Poincaré inequality) For each compact set $K \subset \mathbb{R}^N$ there exists a positive constant C and $R_0 > 0$ such that

$$\fint_{B_r} |u - u_r|\, dx \le Cr \fint_{B_{2r}} |Xu|\, dx$$

for every C^1 function u and for every d-ball $B_r(x)$ with $x \in K$ and $r \le R_0$.

Here we have denoted the mean value of u by $u_r = \fint_{B_r} u\ \mathrm{d}x := \frac{1}{|B_r|}\int_{B_r} u\ \mathrm{d}x$ and the X-gradient of u and the norm of the X-gradient respectively by $Xu = (X_1u, \ldots, X_mu)$ and $|Xu| = \left(\sum_{j=1}^m |X_ju|^2\right)^{\frac{1}{2}}$. The number $Q = \log_2 C_D$ is called the local homogeneous dimension relative to the compact set K. Notice that, enlarging the doubling constant C_D in **(C)** if needed, we can always assume (and we do this) $Q > 2$.

From now on we let $K \subset \mathbb{R}^N$ be compact and $Y \subset K$ be open and connected. The space $(Y, d_X, |\cdot|)$ is then a doubling metric space for balls of small enough radius and, by Remark 1.2, d_X has the Hölder continuity property (1.3). Since we want to apply the abstract theory machinery presented in Chapter 1 we need to show that the structural hypotheses of Theorems 1.10 and 1.11 are satisfied. In particular we have to show that the function $r \mapsto |B_r(x)|$ is continuous and the reverse doubling property (Definition 1.1.5) holds true.

2.1.1 Reverse doubling property

Under the assumptions **(C)** and **(D)**, a local version of the reverse doubling property has been proven by Di Fazio, Gutiérrez and Lanconelli in [14, Theorem 2.10]. In particular they have proved the following

Proposition 2.2. *Suppose Y is open and connected and $(Y, d_X, |\cdot|)$ satisfies* **(C)** *and* **(D)**. *Then, for every $K \subset Y$, $K \neq Y$, there exists a positive constant R_0 depending on K and Y such that there exists a structural constant $\delta > 0$ and a radius $R_0 > 0$ depending on K and Y such that*

$$\delta |B_{2r}(x)| \ge |B_r(x)|$$

for every $B_{2r}(x) \subset K$ with $r \in]0; R_0[$.

They proved the proposition above as a special case of a more general theorem stated in the context of doubling quasi metric Hölder spaces.

Theorem 2.3 ([14], Theorem 2.9)**.** *Let* (Y,d,μ) *be a doubling quasi metric Hölder space and suppose there exist two constants* η,θ *such that* $1<\eta<2\theta<2$ *and the ring* $B_{2\theta r}(x)\setminus B_{\eta r}(x)$ *is non empty. Then, there exists a constant* $\delta=\delta(\eta,\theta,K,C_D,\alpha,\beta)\in]0,1[$ *such that*

$$\delta\mu(B_{2r}(x))\geq\mu(B_r(x))\quad \textit{for every } B_{2r}(x)\subset\Omega$$

Here α, β *are as in* (1.3)*.*

Proof. Let $y\in B_{2\theta r}(x)\setminus B_{\eta r}(x)$. First of all we want to show that there exists a constant $\sigma\in]0,1[$ depending on α and β such that $B_{\sigma r}(y)\subset B_{2r}(x)\setminus B_r(x)$. Indeed if $z\in B_{\sigma r}(y)$ we have

$$\begin{aligned} d(x,z) &\geq d(x,y)-\beta d(z,y)^{\alpha}\left(d(z,y)+d(x,y)\right)^{1-\alpha}\\ &\geq \eta r-\beta(\sigma r)^{\alpha}(\sigma r+2r)^{1-\alpha}\\ &=\left(\eta-\beta\sigma^{\alpha}(\sigma+2)^{1-\alpha}\right)r \end{aligned}$$

and

$$\begin{aligned} d(z,y) &\leq d(x,y)+\beta d(z,y)^{\alpha}\left(d(z,y)+d(x,y)\right)^{1-\alpha}\\ &\leq 2\theta r+\beta(\sigma r)^{\alpha}(\sigma r+2r)^{1-\alpha}\\ &=\left(2\theta+\beta\sigma^{\alpha}(\sigma+2)^{1-\alpha}\right)r. \end{aligned}$$

Hence we can choose $\sigma>0$ satisfying $\beta\sigma^{\alpha}(\sigma+2)^{1-\alpha}<\min\{2(1-\theta),\eta-1\}$ so that $B_{\sigma r}(y)\subset B_{2r}(x)\setminus B_r(x)$. Now Lemma 1.1 implies

$$\mu(B_{2r}(x))\ \geq\ \mu(B_r(x))\ +\ \mu(B_{\sigma r}(y))\ \geq\ \mu(B_r(x))\ +\ \left(C_D\left(\frac{4K}{\sigma}\right)^{\log_2 C_D}\right)^{-1}\mu(B_{2r}(x))$$

and the thesis follows for $\delta=1-\left(C_D\left(\frac{4K}{\sigma}\right)^{\log_2 C_D}\right)^{-1}$. □

Let us prove Proposition 2.2 by showing that the space $(Y,d_X,|\cdot|)$ satisfies the hypotheses of Theorem 2.3

Proof of Proposition 2.2. By Theorem 2.3 it is enough to prove that $\partial B_r(x)\neq\emptyset$ for every $x\in K$ and $r\in]0,R_0[$. So we let $y\in Y\setminus K$ and consider the compact set $K\cup\{y\}$. By Lemma 2.1 we know that $|x-y|\leq Cd_X(x,y)$ for a suitable constant $C=C(Y,K\cup\{y\})$ but on the other hand be can choose $R_0>0$ so that $|x-y|\geq CR_0$ for every $x\in K$, hence $d_X(x,y)\geq R_0$. Now let $\gamma\colon\ [0,1]\to Y$ be a continuous curve connecting x and y. Clearly the function $t\mapsto d_X(x,\gamma(t))$ is continuous and $d_X(x,\gamma(0)) = 0$ while $d_X(x,\gamma(1)) \geq R_0$. Thus, for every $r \in]0,R_0[$ there exists a $t \in]0,1[$ such that $d_X(x,\gamma(t)) = r$. This proves that $\partial B_r(x) \neq \emptyset$ as desired. □

2.1.2 Segment property

Another property that the space $(Y, d_x, |\cdot|)$ must satisfy in order to set in motion the abstract procedure is the property of continuity of the measure of the balls with respect to the radius. In [14], in the general setting of doubling metric spaces, this fact is proved to be a consequence of the following property.

Definition 2.1.3. (Segment property) We say that the metric space (Y,d) has the segment property if for every $x,y \in Y$ there exists a d-continuous curve $\gamma : [0,1] \to Y$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y), \quad \text{for every } t \in [0,1]. \tag{2.2}$$

More precisely we have

Lemma 2.4. *Let (Y,d,μ) be a doubling metric space. If (Y,d,μ) has the segment property, then for every $x \in Y$, the function $R \mapsto \mu(B_R(x))$ is continuous.*

Proof. Since

$$\lim_{r\to R^-} \mu(B_r(x)) = \mu(B_R(x)) \quad \text{and} \quad \lim_{r\to R^+} \mu(B_r(x)) = \mu(\overline{B_R(x)})$$

it suffices to prove that $\mu(\partial B_R) = 0$. For every $y \in \partial B_R(x)$ we consider a d-continuous path connecting y and x such that $d(x,y) = d(x,z) + d(z,y)$ for every $z \in \gamma$. The existence of such curve γ is guaranteed by the segment property. We fix $\tilde{z} \in \gamma$ so that $d(x,\tilde{z}) = \frac{r}{2}$, then $B_{\frac{r}{2}}(\tilde{z}) \subset B_R(x) \cap B_r(y)$ and $B_r(y) \subset B_{\frac{3}{2}r}(\tilde{z})$. Hence the doubling property (1.1) implies

$$\mu(B_R(x) \cap B_r(y)) \geq \mu(B_{\frac{r}{2}}(\tilde{z})) \geq C_D 3^{\log_2 C_D} \mu(B_{\frac{3}{2}r}(\tilde{z})) \geq C_D 3^{\log_2 C_D} \mu(B_r(y))$$

i.e.

$$\frac{\mu(B_R(x) \cap B_r(y))}{\mu(B_r(y))} \geq C_D 3^{\log_2 C_D}.$$

Since y is arbitrary, we deduce that $\partial B_R(x) \subset S$ where

$$S = \left\{ y \in Y \ : \quad \lim_{r\to 0} \fint_{B_r(y)} \chi_{B_R(x)} \, \mathrm{d}\mu \neq 0 \right\}. \tag{2.3}$$

On the other hand the Lebesgue differentiation Theorem [29, Theorem 1.8] implies

$$0 = \mu(S) \geq \mu(\partial B_R(x))$$

concluding the proof. □

We recall that in the particular setting of Carnot-Carathéodory spaces satisfying assumption **(C)** and **(D)**, the continuity of the function $r \mapsto |B_r(x)|$ for every for ball of small enough radius was proved in [12, Proof of Proposition 2.8]. We write the statement below for future reference.

Proposition 2.5. *Consider the space* $(Y, d_X, |\cdot|)$ *and assume* ***(C)*** *and* ***(D)*** *holds true. Then there exists a radius* $R_0 > 0$ *such that the function* $r \mapsto |B_r(x)|$ *is continuous for every* $r < R_0$.

2.2 Assumptions on the operator L

In this section we define the operator we want to deal with and set the assumption on it. Let us recall that K is a fixed compact set, $Y \subset K$ is open and we consider the space $(Y, d_X, |\cdot|)$ and assumptions **(C)**, **(D)** and **(P)**. In the open set $\Omega \subset Y$, we consider linear second order partial differential operators of the type

$$Lu = \sum_{i,j=1}^{N} \partial_i(b_{ij}\partial_j u) + \sum_{i=1}^{N} b_i \partial_i u \tag{2.4}$$

where b_{ij}, b_i are measurable functions and $B = \{b_{ij}\}_{i,j=1...N}$ is a symmetric matrix. Moreover we allow the coefficients of L to have degeneracy controlled by the family of vector fields X. More precisely we assume L to be uniformly X-elliptic in a bounded open set $\Omega \subset \mathbb{R}^N$:

Definition 2.2.1. (X-Elliptic operator) We say that the operator L is uniformly X-elliptic in an open subset $\Omega \subset \mathbb{R}^N$ if there exist positive constants $0 < \lambda \leq \Lambda$ and a non negative function γ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$

$$\lambda \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \leq \langle B(x)\xi, \xi \rangle \leq \Lambda \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2$$

$$\langle b(x), \xi \rangle^2 \leq \gamma^2(x) \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2.$$

Here we use the notation $b = (b_1, \ldots, b_N)$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$.

We recall that the definition of X-ellipticity was first explicitly introduced by Lanconelli and Kogoj in [34] where an homogeneous Harnack inequality for operators in principal form (i.e. of the type (2.4) with $b_i = 0$) is proved. X-elliptic operators have also been studied in [15, 25, 4, 43].

Applying the abstract theory developed in Chapter 1 we prove the non homogeneous Harnack inequality for non negative weak solution (in an appropriate sense that will be specified in the sequel) of the equation

$$Lu = f. \tag{2.5}$$

As in the elliptic case (see for example [22, Section 8.5]) we consider a right hand side f that include a divergence form term

$$f = g + \sum_{i=1}^{N} \partial_i h_i$$

In particular, since it is possible to write the equation $Lu = f$ as

$$Lu - f = \sum_{i,j=1}^{N} \partial_i (b_{ij} \partial_j u - h_i) + \sum_{i=1}^{N} b_i \partial_i u - g = 0$$

and L is X-elliptic, our assumption on divergence form term $h = (h_1, \ldots, h_N)$ are the following: h is a measurable function and there exists a non negative function γ_0 satisfying

$$\langle h(x), \xi \rangle^2 \leq \gamma_0^2(x) \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \text{ for every } x \in \Omega, \xi \in \mathbb{R}^N. \tag{2.6}$$

Also, we assume the following conditions on the low order terms b and on the right hand side f. For a suitable $p > \frac{Q}{2}$ it holds

(LT) $\gamma \in L^{2p}(\Omega)$.

(R) $\gamma_0 \in L^{2p}(\Omega) \quad \text{and} \quad g \in L^p(\Omega)$.

We remark that the operator L here considered does not contain first order terms, so it is a simplified version of the one studied by Gutiérrez, Lanconelli [25] and Uguzzoni [43] where they prove Harnack inequality using an adapted Moser's iteration technique. This restriction on the class of operators considered is motivated by the fact that in order to apply the results presented in Chapter 1 we need to require the family of solutions to be closed under sum of constants. Authors in [25] require dilation invariance of vector fields and a positivity condition on the operator ([25, equation (2.3)]), while in [43] these assumption are dropped. Since all our assumptions are of "local nature", the Harnack inequality will be obtained for balls with small radius and the constants appearing will depend on the compact set fixed. Very recently, in [4] Battaglia and Bonfiglioli obtained an invariant non homogeneus Harnack inequality for solutions to a class of sub-elliptic operators in divergence form under global doubling and Poincaré assumptions but no restrictions on the diameter of the set Ω.

2.3 W^1 weak solution for the operator L

In this section we recall the notions of Sobolev spaces modeled after a family of locally Lipschitz vector fields and of W^1 weak solution to the equation $Lu = f$.

First of all we recall that assumptions **(C)**, **(D)**, **(P)** imply the local Sobolev inequalities

$$\|u\|_{\frac{2Q}{Q-2}} \leq C(D)\|Xu\|_2 \quad \text{for every } u \in C_0^1(D) \tag{2.7}$$

for every open set D with sufficiently small diameter and closure contained in the interior of K and

$$\|u\|^*_{\frac{2Q}{Q-2}} \leq Cr\|Xu\|^*_2 \quad \text{for every } u \in C_0^1(B_r(x)) \tag{2.8}$$

$\bar{B}_r(x) \subset \Omega$, $2r < R_0$ For a deeper discussion on this result see [17, 20, 28].

For every D bounded domain supporting the Sobolev inequality (2.7) we define the space $W_0^1(D,X)$ to be the closure of $C_0^1(D)$ with respect to the norm $u \mapsto \|Xu\|_{L^2(D)}$.

Now we want to introduce the notion of weak X-gradient. For any $j = 1,\dots,m$ the formal adjoint of X_j is the unique operator X_j^* such that

$$\int_{\mathbb{R}^N} uX_jv \, \mathrm{d}x = \int_{\mathbb{R}^N} vX_j^*u \, \mathrm{d}x \quad \text{for all } u,v \in C_0^\infty(\mathbb{R}^N),$$

moreover given $u \in L^2(D)$ if there exists a function $\phi_j \in L^2(D)$ such that

$$\int_D \varphi\phi_j \, \mathrm{d}x = \int_D uX_j^*\varphi \, \mathrm{d}x \quad \text{for all } \varphi \in C_0^\infty(D)$$

we say that $X_ju = \phi_j$ exists in the weak sense. Hereafter Xu denotes the weak X-gradient of $u \in W_0^1(D,X)$: $Xu = (X_1u,\dots,X_mu)$. The Sobolev space $W^1(D,X)$, is the space

$$W^1(D,X) = \{u \in L^2(D) : \ X_ju \in L^2(D) \text{ for every } j = 1,\dots,m\}.$$

On $W^1(D,X)$ we consider the norm $\|u\|_{W^1} = \|u\|_{L^2(D)} + \|Xu\|_{L^2(D)}$ (see [20, Theorem A.2]). We adopt the usual notation

$$W^1_{\mathrm{loc}}(D,X) = \{u : \text{ for every } D'\text{open}, \ D' \subset\subset D \text{ we have } u \in W^1(D',X)\}.$$

Since we will deal with functions that are solutions to $Lu = f$ in a weak sense, the following Meyer-Serrin type result will be very useful in the sequel. The result was proved independently in [20, Theorem 1.13] and in [18, Theorem 1.2.3].

Theorem 2.6. *The set $C^\infty(D) \cap W^1(D,X)$ is dense in $W^1(D,X)$ with respect to the norm $\|\cdot\|_{W^1}$.*

By means of theorem above and an approximation argument it is possible to states assumption **(P)** for functions in $W^1(D,X)$ and the Sobolev inequality for functions in $W_0^1(D,X)$.

In order to introduce the notion of weak solution to the equation $Lu = h$ we consider the bilinear form

$$\mathfrak{L}(u,v) = \int_D \langle B\nabla u, \nabla v\rangle - \langle b, \nabla u\rangle v \, \mathrm{d}x \quad \text{for } u \in C^1(D),\ v \in C_0^1(D)$$

and the linear functional

$$\mathfrak{F}(v) = \int_D \langle h, \nabla v\rangle - gv \, dx \quad \text{for } v \in C_0^1(D).$$

Gutiérrez, Lanconelli in [25] and Uguzzoni in [43] showed respectively that $\mathfrak{L}$ can be extended continuously to $W^1(D,X) \cap L^r(D) \times W_0^1(D,X)$, where $\frac{1}{r} = \frac{1}{2} + \frac{1}{2p}$ and $\mathfrak{F}$ can be extended continuously to $W_0^1(D,X)$. Indeed, due to the positiveness of the matrix B, the X-ellipticity assumption, the Sobolev inequality and assumption **(LT)**, we have

$$\begin{aligned}
|\mathfrak{L}(u,v)| &\le \int_D \langle B\nabla u, \nabla u\rangle^{\frac{1}{2}} \langle B\nabla v, \nabla v\rangle^{\frac{1}{2}} + |\langle b, \nabla u\rangle||v| \, \mathrm{d}x \\
&\le \Lambda \int_D |Xu||Xv| \, \mathrm{d}x + \int_D |Xv||u|\gamma \, \mathrm{d}x \\
&\le \Lambda \|Xu\|_2 \|Xv\|_2 + \|Xv\|_2 \|u\|_r \|\gamma\|_{2p} \\
&\lesssim \|Xv\|_2 (\|Xu\|_2 + \|u\|_r \|\gamma\|_{2p}).
\end{aligned}$$

Moreover by assumption **(R)** and the Sobolev inequality we get

$$\begin{aligned}
|\mathfrak{F}(v)| &\le \int_D \gamma_0 |Xv| + |gv| \, \mathrm{d}x \\
&\le \|\gamma_0\|_2 \|Xv\|_2 + \|g\|_{\frac{2Q}{Q+2}} \|v\|_{\frac{2Q}{Q-2}} \\
&\lesssim \|Xv\|_2 (\|\gamma_0\|_{2p} + \|g\|_p).
\end{aligned}$$

Thus, by density, $\mathfrak{L}(u,v)$ and $\mathfrak{F}(v)$ can be uniquely prolonged to operators

$$\mathfrak{L}: \; W^1(D,X) \cap L^r(D) \times W_0^1(D,X) \to \mathbb{R},$$
$$\mathfrak{F}: \; W_0^1(D,X) \to \mathbb{R}.$$

Observations above give meaning to the following notion of weak solution

Definition 2.3.1. (Weak solution) We say that a function $u \in W^1_{loc}(\Omega,X)$ is a weak subsolution (resp. supersolution) to $Lu = f$ in Ω if, for every domain D, $D \subset\subset \Omega$ supporting the Sobolev inequality, we have

$$\mathfrak{L}(u,v) \le \mathfrak{F}(v) \quad (\text{resp. } \mathfrak{L}(u,v) \ge \mathfrak{F}(v)) \quad \text{for every } v \ge 0, \; v \in W_0^1(D,X).$$

We say that u is a solution if it is both a supersolution and a subsolution.

In the proof of the critical density property we will use of some arguments borrowed from the Moser iterative scheme, hence it will be crucial to have cut-off functions. In two independent works, Garofalo and Nhieu [21, Theorem 1.5] and Franchi, Serapioni e Serra Cassano [19] proved the existence of cut-off functions for Carnot-Carathéodory balls under assumption **(C)**:

Theorem 2.7. *Assume* **(C)**. *Let $B_R(x)$ be a bounded metric ball. Then, for every $0 < R_1 < R_2 < R$, there exists a d_X-Lipschitz continuous function $\phi \; : \mathbb{R}^N \to [0,\infty[$ such that ϕ is $W^1(B_{2R}(x),X)$ and*

- *$\phi \equiv 1$ on $B_{R_1}(x)$ and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B_{R_2}(x)$,*
- *$|X\phi| \le \frac{C}{R_2 - R_1}$ for almost every $x \in \mathbb{R}^N$.*

2.4 Critical density property for X-elliptic operators

In this section we show that non negative weak solutions to (2.5) have the ν-critical density property for every $\nu \in]0,1[$, thus the Harnack inequality follows from Theorem 1.10 under hypotheses (B) and Theorem 1.11. However, in order to set in motion the abstract machinery, all the structural hypotheses of Theorem 1.10 must be satisfied. Here we have considered and proved structural properties of "local nature" (**(D)**, **(C)**, Proposition 2.5 and Theorem 2.3), for this reason we need to require the following

(O) The d_X-diameter of Ω is small enough to have the double ball property, the reverse doubling, the continuity of the function $r \mapsto B_r(x)$, the Poincarè and the Sobolev inequality for every $B_R(x) \subset \Omega$.

Let us now reformulate the problem using the notation of the abstract approach. We set

$$\mathcal{F}(\Omega) = \left\{ f = g + \sum_{1=1}^{N} \partial_i h_i, \text{ satisfying (2.6) and } \textbf{(R)} \right\}$$

and for every $f \in \mathcal{L} := \mathcal{F}(\Omega)$, $B_R(x_0) \subseteq \Omega$, $R < r_0$, $\delta = 1 - \frac{Q}{2p}$ we define the function

$$\mathcal{S}_\Omega\left(B_R(x_0),\, f\right) = R^\delta \|\gamma_0\|_{L^{2p}(B_R(x_0))} + R^{2\delta} \|g\|_{L^p(B_R(x_0))}$$

and consider the family

$$\mathbb{K}_{\Omega,f} = \{u \in W^1_{loc}(\Omega, X) : u \geq 0 \text{ is a weak solution to (2.5)}\}.$$

It is easy to see that the definition of $\mathcal{S}_\Omega\left(B_R(x_0),\, f\right)$ is consistent with the abstract definition given in Chapter 1. Moreover, for every non negative $u \in \mathbb{K}_{\Omega,f}$ and $\lambda \geq 0$ we have $\lambda u \in \mathbb{K}_{\Omega,\lambda f}$ and if $\tau - \lambda u \geq 0$ then $\tau - \lambda u \in \mathbb{K}_{\Omega,-\lambda f}$, so, also the definition of $\mathbb{K}_{\Omega,f}$ is consistent with the abstract one (Definition 1.2.2). We will prove that $\mathbb{K}_{\Omega,f}$ has the critical density with the aid of three lemmas. The first lemma ensures the local boundedness of solutions to (2.5). This fact has been proved by Uguzzoni in [43] for a class of more general operators, indeed it is a fundamental step towards the proof Harnack inequality by means of the Moser iteration scheme. For the sake of completeness we report here the statement and the proof.

In the sequel we will abbreviate $B_R(x_0) = B_R$ for any $R > 0$ and the center x_0 to be understood.

Lemma 2.8 (Local boundedness)**.** *Let u be a weak solution to* (2.5) *in Ω, then u is locally bounded.*

Proof. Let us consider a ball B_{4R}, such that $\overline{B_{4R}} \subset \Omega$ and recall that we have assumed **(O)**. We define $\overline{u} = u^+ + \sigma$, for all $\sigma > 0$ and $H_n : [\sigma, +\infty) \to \mathbb{R}$

$$H_n(s) = \begin{cases} s^\beta & s \in [\sigma, n] \\ \beta n^{\beta-1}(s-n) + n^\beta & s > n \end{cases}, \qquad n \in \mathbb{N},\ \beta \geq 1.$$

Clearly H_n is non decreasing, C^1 and the sequence $(H_n)_{n\in\mathbb{N}}$ converges to s^β. Moreover we set

$$G(t) = \int_\sigma^t (H_n')^2\, \mathrm{d}x, \quad \text{and} \quad v = \eta^2 G(\overline{u})$$

where $\eta \in C_0^1(B_{4R})$ is a given cut-off function as in Theorem 2.7. Since u is a weak solution to (2.5) and $v \in W_0^1(B_{4R}, X)$, it holds $\mathfrak{L}(u,v) = \mathfrak{F}(v)$. Using an approximation argument we can assume u and v to be smooth, then

$$\begin{aligned}
0 = \mathfrak{F}(v) - \mathfrak{L}(u,v) &= -\int_{B_{4R}} \langle B\nabla u - h, \nabla v\rangle \, \mathrm{d}x + \int_{B_{4R}} (\langle b, \nabla u\rangle - g)\, v \, \mathrm{d}x \\
&\le -\int_{B_{4R}} \langle B\nabla u^+, \eta^2 G'(\overline{u})\nabla u^+\rangle \, \mathrm{d}x - \int_{B_{4R}} \langle B\nabla u, 2\eta G(\overline{u})\nabla\eta\rangle \, \mathrm{d}x + \\
&\quad + \int_{B_{4R}} \langle h, \eta^2 G'(\overline{u})\nabla u^+ + 2\eta G(\overline{u})\nabla \eta\rangle \, \mathrm{d}x + \int_{B_{4R}} (\langle b, \nabla u\rangle - g)\, \eta^2 G(\overline{u}) \, \mathrm{d}x.
\end{aligned}$$

Let $D = B_{4R} \cap \{u > 0\}$. Exploiting (2.6), the fact that L is X-elliptic (Definition 2.2.1) and moving the terms around we have

$$\begin{aligned}
\int_D \eta^2 G'(\overline{u})|Xu|^2 \, \mathrm{d}x \le 2\frac{\Lambda}{\lambda}\int_D \eta G(\overline{u})|Xu||X\eta| \, \mathrm{d}x + \frac{2}{\lambda}\int_D \eta G(\overline{u})|\gamma_0||X\eta| \, \mathrm{d}x + \\
+ \frac{1}{\lambda}\int_D \eta^2 G'(\overline{u})|\gamma_0||Xu| \, \mathrm{d}x + \\
+ \frac{1}{\lambda}\int_D (\gamma|Xu| + |g|)\, \eta^2 G(\overline{u}) \, \mathrm{d}x.
\end{aligned} \tag{2.9}$$

In order to estimate each of the four term in the right hand side of inequality above we set

$$a = \left(\frac{\gamma^2}{\lambda^2} + \frac{\gamma_0^2}{\lambda^2\sigma^2} + \frac{|g|}{\lambda\sigma}\right)^{\frac{1}{2}} \tag{2.10}$$

and we recall that

$$\begin{aligned}
H_n(s) \le sH_n'(s) \le \beta H_n(s), \quad G(t) \le tG'(t), \\
|X(H_n(\overline{u}))|^2 = G'(\overline{u})|Xu|^2\chi_{\{u>0\}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
2\frac{\Lambda}{\lambda}\int_D \eta G(\overline{u})|Xu||X\eta| \, \mathrm{d}x &\le \int_D \left(\frac{\Lambda^2}{\lambda^2\varepsilon}\overline{u}^2|X\eta|^2 + \eta^2|Xu|^2\varepsilon\right) G'(\overline{u}) \, \mathrm{d}x \\
&\le \frac{\Lambda^2}{\lambda^2\varepsilon}\int_D (\overline{u}H_n'(\overline{u}))^2|X\eta|^2 \, \mathrm{d}x + \varepsilon\int_D \eta^2|X(H_n(\overline{u}))|^2, \\
\frac{2}{\lambda}\int_D \eta G(\overline{u})|\gamma_0||X\eta| \, \mathrm{d}x \le \int_D G'(\overline{u})\overline{u}^2\left(a^2\eta^2 + |X\eta|^2\right) \, \mathrm{d}x &\le \int_D (\overline{u}H_n'(\overline{u}))^2\left(a^2\eta^2 + |X\eta|^2\right) \, \mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}\frac{1}{\lambda}\int_D \eta^2 G'(\overline{u})|\gamma_0||Xu|\,\mathrm{d}x &\le \frac{1}{2}\int_D \eta^2 G'(\overline{u})\left(|Xu|^2+a^2\overline{u}^2\right)\,\mathrm{d}x\\ &\le \frac{1}{2}\int_D \eta^2|X(H_n(\overline{u}))|^2\,\mathrm{d}x+\frac{1}{2}\int_D \eta^2\left(\overline{u}H_n'(\overline{u})\right)^2 a^2\,\mathrm{d}x,\\ \frac{1}{\lambda}\int_D (\gamma|Xu|+|g|)\,\eta^2 G(\bar{u})\,\mathrm{d}x &\le \int_D G'(\overline{u})\eta^2\left(|Xu|^2\varepsilon+\frac{1}{4\varepsilon}\overline{u}^2a^2+\overline{u}^2a^2\right)\,\mathrm{d}x\\ &\le \varepsilon\int_D \eta^2|X(H_n(\overline{u}))|^2\,\mathrm{d}x+\int_D \eta^2a^2(\overline{u}H_n'(\overline{u}))^2\left(1+\frac{1}{4\varepsilon}\right)\,\mathrm{d}x\end{aligned}$$

We plug estimates above in (2.9), then choosing ε small and moving terms around we get

$$\int_{B_{4R}} \eta^2|X(H_n(\overline{u}))|^2\,\mathrm{d}x \le C\beta^2\int_{B_{4R}} H_n(\overline{u})^2(|X\eta|^2+\eta^2a^2)\,\mathrm{d}x.$$

Here C is a positive structural constant, in the sequel a structural constant will be always denoted by C even if its value differs at each occurrence. Let us use the standard notation $\|u\|^*_{L^p}=\|u\|^*_{L^p(B_r)}=\left(\frac{1}{|B_r|}\int_{B_r}u^p\,\mathrm{d}x\right)^{\frac{1}{p}}$, from inequality above we deduce

$$\|\eta|X(H_n(\overline{u}))|\|^*_{L^2(B_{4R})}\le C\beta\left(\|H_n(\overline{u})|X\eta|\|^*_{L^2(B_{4R})}+\|H_n(\overline{u})\eta a\|^*_{L^2(B_{4R})}\right).$$

By the Sobolev inequality (2.8) we get

$$\|\eta H_n(\overline{u})\|^*_{L^q(B_{4R})}\le CR(\beta+1)\left(\|H_n(\overline{u})|X\eta|\|^*_{L^2(B_{4R})}+\|H_n(\overline{u})\eta a\|^*_{L^2(B_{4R})}\right) \qquad (2.11)$$

where $q=\frac{2Q}{Q-2}$. Now we focus on the term $\|H_n(\overline{u})\eta a\|^*_{L^2(B_{4R})}$. We use Hölder inequality and then the interpolation inequality $\|v\|^*_{L^\tau}\le\varepsilon\|v\|^*_{L^\omega}+\varepsilon^{\frac{\omega^{-1}-\tau^{-1}}{\tau^{-1}-\nu^{-1}}}\|v\|^*_{L^\nu}$ (valid for $\nu\le\tau\le\omega$) with $\tau=\frac{2p}{p-1}$, $\omega=q$, $\nu=2$ obtaining

$$\|H_n(\overline{u})\eta a\|^*_{L^2(B_{4R})}\le\|a\|^*_{L^{2p}(B_{4R})}\left(\varepsilon\|\eta H_n(\overline{u})\|^*_{L^q(B_{4R})}+\varepsilon^{\frac{Q}{Q-2p}}\|\eta H_n(\overline{u})\|^*_{L^2(B_{4R})}\right). \qquad (2.12)$$

We choose

$$\varepsilon=(2Ca^*(1+\beta))^{-1}\quad\text{with } a^*(r)=\sup_{\rho\le 4r}\left(\rho\|a\|^*_{L^{2p}(B_\rho)}\right).$$

Let us notice that we can choose σ so that a^* is controlled by a structural constant. Indeed if we set $\delta=1-\frac{Q}{2p}$ by the doubling condition (D) we have

$$a^*(r)\le Cr^\delta\left(\|\gamma\|^2_{L^{2p}(B_{4R})}+\sigma^{-2}\|\gamma_0\|^2_{L^{2p}(B_{4R})}+\sigma^{-1}\|g\|_{L^p(B_{4R})}\right)^{\frac{1}{2}}.$$

So that, if $\|g\|_{L^p(B_{4R})} = 0$ and $\|\gamma_0\|_{L^{2p}(B_{4R})} = 0$, a^* is bounded by a structural constant; otherwise one takes $\sigma = \mathcal{S}_\Omega(B_{4R}, f)$ obtaining

$$a^*(r) \le C\left(r^{2\delta}\|\gamma\|^2_{L^{2p}(B_{4R})} + 2\right)^{\frac{1}{2}} \le C\left(R_0^{2\delta}\|\gamma\|^2_{L^{2p}(B_{4R})} + 2\right)^{\frac{1}{2}} \tag{2.13}$$

as desired. Substituting (2.12) in (2.11) we find

$$\|\eta H_n(\overline{u})\|^*_{L^q(B_{4R})} \le C(\beta+1)^{1+\nu}\left(R\,\|H_n(\overline{u})|X\eta|\|^*_{L^2(B_{4R})} + \|H_n(\overline{u})\eta\|^*_{L^2(B_{4R})}\right)$$

with $\nu = \frac{Q}{2p-Q}$, then we let $n \to \infty$ to get

$$\left\|\eta\overline{u}^\beta\right\|^*_{L^q(B_{4R})} \le C(\beta+1)^{1+\nu}\left(R\left\|\overline{u}^\beta|X\eta|\right\|^*_{L^2(B_{4R})} + \left\|\overline{u}^\beta\eta\right\|^*_{L^2(B_{4R})}\right).$$

Inequality above holds true also for $\eta \in W_0^1(B_{4R})$, so we choose η to be a cut off function as in Theorem 2.7, with $R \le R_1 < R_2 \le 2R$. This choice of η and the doubling property **(D)** give

$$\left\|\overline{u}^\beta\right\|^*_{L^q(B_{R_1})} \le C(\beta+1)^{1+\nu}\left(1+\frac{R}{R_2-R_1}\right)\left\|\overline{u}^\beta\right\|^*_{L^2(B_{R_2})}. \tag{2.14}$$

It is convenient write inequality above using the following notation

$$\varphi(s,R) = \|\overline{u}\|^*_{L^s(B_R)}.$$

Then, (2.14) reads as

$$\varphi(\beta q, R_1) \le \left[C(\beta+1)^{1+\nu}\left(1+\frac{R}{R_2-R_1}\right)\right]^{\frac{1}{\beta}}\varphi(2\beta, R_2) \tag{2.15}$$

for any $\beta \ge 1$, and any R_1, R_2 such that $R \le R_1 < R_2 \le 2R$. Starting from this inequality we can set in motion the celebrate Moser iterative scheme to conclude the proof. To this aim let us choose $\tau \in]2, q[$ and, for any $n \in \mathbb{N}_0$ we set

$$\beta_n = \tau\left(\frac{q}{2}\right)^n, \quad r_n = R(1+2^{-n}).$$

At the n-th step of the iteration we consider inequality (2.15) with $(2\beta, R_1, R_2) = (\beta_n, r_{n+1}, r_n)$. So that, starting from inequality (2.15) with the choice $\beta = \tau$, $R_1 = r_1$, $R_2 = 2R$ (which corresponds to the case $n = 0$) it is possible to iteratively estimates the left most hand side obtaining

$$\varphi\left(\tau\left(\frac{q}{2}\right)^n\right) \le (Cq)^{2(1+\nu)(1+\sum_{k=1}^n k(2/q)^k)}\varphi(\tau,2R), \quad \text{for every } n \ge 1$$

here $C>1$ is a suitable structural constant. Recalling that $\lim_{s\to\infty}\varphi(s,R) = \sup_{B_R}\bar{u}$, and $q>2$, letting n to infinity we get

$$\sup_{B_R}\bar{u} \le C\|\bar{u}\|_{L^\tau(B_{2R})}$$

with $\bar{u} = u+\sigma$. In particular, letting $\tau \to 2$ we find

$$\sup_{B_R(x_0)} \overline{u} \le c\left(\fint_{B_{2R}(x_0)} \overline{u}^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \tag{2.16}$$

hence u^+ is locally bounded. The above argument applies also to u^-, indeed $-u$ solves $L(-u) = -f$. This concludes the proof. □

Remark 2.9. *We explicitly notice that in order to prove estimates* (2.16) *for* $\overline{u} = u^+ + \sigma$ *we have just used the fact that* u *is a weak subsolution to* (2.5).

Notice that in the proof it has not been used the fact that the solution u is non negative, this fact will be crucial in the proof of the next result instead.

Lemma 2.10 (Estimates for $\|X\log\overline{u}\|_{L^2(B_R)}$)**.** *Let* u *be a weak solution to* (2.5) *in* Ω, *then there exists a structural constant* $c>0$ *such that*

$$\fint_{B_R} |X\log\overline{u}|^2 \, \mathrm{d}x \le \frac{c}{R^2}$$

for every $\overline{B_{4R}} \subset \Omega$. *Here* $\overline{u} = u+\sigma$, *with* $\sigma = \mathcal{S}_\Omega(B_{4R}, f)$.

Proof. We exploit the non negativeness of u to define the test function $v = \eta^2\overline{u}^\beta$ with $\beta \ne 0$ and η as in Theorem 2.7 with $\mathrm{supp}(\eta) \subset B_{2R}$ and $\eta \equiv 1$ in B_R. By Lemma 2.8 we know that $u \in L^\infty_{\mathrm{loc}}(\Omega)$, hence $v \in W^1_0(B_{4R}, X)$. We proceed as in the proof of (2.9) obtaining

$$\begin{aligned}\int_{B_{4R}} \eta^2\overline{u}^{\beta-1}|Xu|^2 \, \mathrm{d}x \le \frac{2}{\lambda|\beta|}\int_{B_{4R}} \Lambda\eta\overline{u}^\beta|Xu||X\eta| \, \mathrm{d}x + \frac{2}{\lambda|\beta|}\int_{B_{4R}} \eta\overline{u}^\beta|\gamma_0||X\eta| \, \mathrm{d}x + \\ + \frac{1}{\lambda}\int_{B_{4R}} \eta^2\overline{u}^{\beta-1}|\gamma_0||Xu| \, \mathrm{d}x + \frac{1}{\lambda|\beta|}\int_{B_{4R}} (\gamma|Xu|+|g|)\,\eta^2\overline{u}^\beta \, \mathrm{d}x\end{aligned} \tag{2.17}$$

Let a be as in (2.10), then each of the four terms above can be estimated as follows

$$\frac{2}{\lambda|\beta|}\int_{B_{4R}}\Lambda\eta\overline{u}^{\beta}|Xu||X\eta|\,\mathrm{d}x \le \frac{1}{\lambda|\beta|}\int_{B_{4R}}\left(\eta^2\overline{u}^{\beta-1}|Xu|^2\lambda\varepsilon+\Lambda^2(\lambda\varepsilon)^{-1}\overline{u}^{\beta+1}|X\eta|^2\right)\,\mathrm{d}x$$
$$\le\frac{1}{6}\int_{B_{4R}}\eta^2\overline{u}^{\beta-1}|Xu|^2\,\mathrm{d}x+\frac{6\Lambda^2}{|\beta|^2\lambda^2}\int_{B_{4R}}\overline{u}^{\beta+1}|X\eta|^2\,\mathrm{d}x$$
$$\frac{2}{\lambda|\beta|}\int_{B_{4R}}\eta\overline{u}^{\beta}|\gamma_0||X\eta|\,\mathrm{d}x\le\frac{2}{|\beta|}\int_{B_{4R}}\eta\overline{u}^{\beta+1}a|X\eta|\,\mathrm{d}x\le\frac{1}{|\beta|}\int_{B_{4R}}\overline{u}^{\beta+1}\left(a^2\eta^2+|X\eta|^2\right)\,\mathrm{d}x$$
$$\frac{1}{\lambda}\int_{B_{4R}}\eta^2\overline{u}^{\beta-1}|\gamma_0||Xu|\,\mathrm{d}x\le\int_{B_{4R}}\eta^2\overline{u}^{\beta}a|Xu|\,\mathrm{d}x$$
$$\le\frac{1}{2}\int_{B_{4R}}\eta^2\overline{u}^{\beta-1}|Xu|^2\,\mathrm{d}x+\frac{1}{2}\int_{B_{4R}}a^2\eta^2\overline{u}^{\beta+1}$$
$$\frac{1}{\lambda|\beta|}\int_{B_{4R}}(\gamma|Xu|+|g|)\,\eta^2\overline{u}^{\beta}\,\mathrm{d}x\le\frac{1}{|\beta|}\int_{B_{4R}}\left(a|Xu|+a^2\overline{u}\right)\eta^2\overline{u}^{\beta}\,\mathrm{d}x$$
$$\le\frac{1}{|\beta|}\int_{B_{4R}}\frac{1}{2}\eta^2\left(|Xu|^2\overline{u}^{\beta-1}\varepsilon+a^2\overline{u}^{\beta+1}\varepsilon^{-1}\right)+a^2\eta^2\overline{u}^{\beta+1}\,\mathrm{d}x$$
$$\le\frac{1}{12}\int_{B_{4R}}\eta^2\overline{u}^{\beta-1}|Xu|^2\,\mathrm{d}x+\left(\frac{1}{|\beta|}+\frac{3}{|\beta|^2}\right)\int_{B_{4R}}a^2\eta^2\overline{u}^{\beta+1}\,\mathrm{d}x$$

Here we have chosen $\varepsilon=\frac{|\beta|}{6}$. Plugging those estimates in (2.17) and moving the terms around we get

$$\int_{B_{4R}}\eta^2\overline{u}^{\beta-1}|Xu|^2\,\mathrm{d}x\le\left(\min\left\{\frac{1}{6},\frac{|\beta|}{6}\right\}\right)^{-2}\int_{B_{4R}}\overline{u}^{\beta+1}F\,\mathrm{d}x$$

where $F=\left(1+6\frac{\Lambda^2}{\lambda^2}\right)|X\eta|^2+\eta^2a^2$. For $\beta=-1$ and a suitable structural constant C, equation above becomes

$$\int_{B_{4R}}|\eta X\log\overline{u}|^2\,\mathrm{d}x\le C\int_{B_{4R}}F\,\mathrm{d}x.$$

Hereafter we will denote a positive structural constant by C even if its value may change at each occurrence. This last estimate can be found in [43, pp 176]. Recalling the doubling condition **(D)** and the definition of η, inequality above becomes

$$\fint_{B_R}|X\log\overline{u}|^2\,\mathrm{d}x\le CC_D\fint_{B_{2R}}F\,\mathrm{d}x$$
$$\le C\fint_{B_{2R}}\frac{1}{R^2}+a^2\,\mathrm{d}x$$
$$=\frac{C}{R^2}\left(1+R^2\fint_{B_{2R}}a^2\,\mathrm{d}x\right).$$

Then, by using Hölder inequality, we estimate the right hand side by

$$\begin{aligned}\fint_{B_R} |X \log \overline{u}|^2 \, \mathrm{d}x &\leq \frac{C}{R^2}\left(R^2\left(\fint_{B_{2R}} a^{2p}\, dx\right)^{1/p} + 1\right)\\ &\leq \frac{C}{R^2}\big((a^*(R))^2 + 1\big).\end{aligned}$$

Since (2.13) shows that $a^*(R)$ is bounded from above by a structural constant we get the thesis. □

Lemma 2.11 (Fabes Lemma, [14] Lemma 7.4)**.** *Let $v \in W^1_{loc}(\Omega, X)$, $B_R \subset \Omega$ and define*

$$E = \{x \in B_R : v(x) = 0\}.$$

Then, if there exists $0 < \varepsilon \leq 1$ such that

$$|E| \geq \varepsilon |B_R|$$

we have

$$\fint_{B_R} |v|^2 \, dx \leq C R^2 \fint_{B_{2R}} |Xv|^2 dx$$

where $C \geq 0$ is a structural constant depending on ε.

Proof. Let us use the notation $v_E = \fint_E v \, \mathrm{d}x$ and $v_B = \fint_{B_R} v \, \mathrm{d}x$. Then, since $v \equiv 0$ on E, we have

$$\begin{aligned}|v(x)| &= |v(x) - v_E|\\ &\leq |v(x) - v_B| + |v_B - v_E|\\ &\leq |v(x) - v_B| + \frac{|B|}{|E|}\fint_{B_R} |v - v_B| \, \mathrm{d}x\\ &\leq |v(x) - v_B| + \frac{1}{\varepsilon}\fint_{B_R} |v - v_B| \, \mathrm{d}x\end{aligned}$$

Squaring both sides, taking the average over B_R and then using the Poincaré inequality [1] we find

$$\fint_{B_R} |v|^2 \, dx \leq \left(1 + \frac{2}{\varepsilon} + \frac{1}{\varepsilon^2}\right)\fint_{B_R} |v|^2 \, dx \leq C R^2 \fint_{B_{2R}} |Xv|^2 dx.$$

This concludes the proof. □

[1] here we use the the L^2 Poincaré inequality $\fint_{B_R} |v|^2 \, \mathrm{d}x \leq C R^2 \fint_{B_{2R}} |Xv|^2 \, \mathrm{d}x$ which is known to be implied by inequality **(P)** (see for example [20])

We recall that, by definition, a structural constant does not depend on $u \in \mathbb{K}_{\Omega,f}$ nor on the balls defined by the quasi distance. On the other hand it may depend on the ellipticity constants λ, Λ, the doubling constant C_D, the constant in the Poincaré inequality **(P)**, the Lipschitz constant of vector fields $\{X_i\}_{1,\dots,m}$ and $(\|\gamma\|_{2p}^2+1)^{1/2}$.

We are now ready to prove that the family $\mathbb{K}_{\Omega,f}$ satisfies the ν-critical density property for every $\nu \in]0,1[$.

Theorem 2.12. *Let $\overline{B_{8R}} \subset \Omega$, and suppose $u \in \mathbb{K}_{\Omega,f}(B_{8R})$. Then, there exist structural constants $\varepsilon, c \in]0,1[$, such that if u satisfies*

$$|\{x \in B_R : u(x) \geq 1\}| \geq \nu |B_R| \quad \textit{for some } \nu \in]0,1[$$

then we have

$$\inf_{B_{R/2}} u \geq c \quad \textit{or} \quad \mathcal{S}_\Omega(B_{8R}, f) \geq \varepsilon.$$

Here $c = c(\nu) \in]0,1[$.

Proof. Let $\sigma = \mathcal{S}_\Omega(B_{8R}, f)$ and define $h(\overline{u}) := \max\{-\log \overline{u}, 0\}$ with $\overline{u} = u + \sigma = u^+ + \sigma$, notice that $\sigma \geq 0$, in particular $\sigma = 0$ if and only if $g = 0$ and $h = 0$. We consider the function $w := h(\overline{u})$ and we observe that the quantity $\left(⨍_{B_R} w^2\, dx \right)^{\frac{1}{2}}$ is bounded from above by a structural constant. Indeed we have

$$\begin{aligned} |\{x \in B_R : w(x) = 0\}| &= |\{x \in B_R : \overline{u}(x) \geq 1\}| \\ &\geq |\{x \in B_R : u(x) \geq 1\}| \geq \nu |B_R|, \end{aligned}$$

hence, using the Fabes Lemma 2.11 and Lemma 2.10 we get

$$⨍_{B_R} |w(x)|^2 \, \mathrm{dx} \leq CR^2 ⨍_{B_{2R}} |Xw(x)|^2\, dx \leq CR^2 ⨍_{B_{2R}} |X \log(\overline{u}(x))|^2 \, \mathrm{dx} \leq C.$$

Now, if $\sigma = 0$, the function w is a weak subsolution to $Lu = 0$ with L the operator defined (2.4), otherwise, if $\sigma > 0$ we have $\inf_\Omega \bar{u} \geq \sigma$ and w is a weak subsolution to

$$\tilde{L}w = \sum_{i,j=1}^{N} \partial_i (b_{ij} \partial_j \omega) + \sum_{i=1}^{N} \tilde{b}_i \partial_i w = \tilde{f}$$

with

$$\tilde{b}_i = \begin{cases} b_i - \frac{h_i}{\overline{u}}, & \text{if } 0 < \overline{u} < 1 \\ b_i, & \text{otherwise} \end{cases} \quad , \qquad \tilde{f} = \begin{cases} -\frac{g + \sum_{i=1}^N \partial_i h_i}{\overline{u}}, & \text{if } 0 < \overline{u} < 1 \\ 0, & \text{otherwise} \end{cases} \quad .$$

In both cases the the assumptions **(LT)** and **(R)** are satisfied, so, recalling Remark 2.9, we can apply Lemma 2.8 obtaining

$$\sup_{B_{R/2}} \overline{w} \leq c \left(\fint_{B_R} \overline{w}^2 \, dx \right)^{\frac{1}{2}}$$

where we have defined $\sigma_w = \mathcal{S}_\Omega(B_{8R}, \tilde{f})$ and $\overline{w} = w + \sigma_w \leq w + 2$. So that

$$\sup_{B_{R/2}} w \leq \sup_{B_{R/2}} \overline{w} \leq c \left(\fint_{B_R} \overline{w}^2 \, dx \right)^{\frac{1}{2}} \leq c \left(\fint_{B_R} w^2 \, dx \right)^{\frac{1}{2}} + 4.$$

But the term $\left(\fint_{B_R} w^2 \, dx \right)^{\frac{1}{2}}$ is bounded from above by a structural constant, hence

$$\sup_{B_{R/2}} w \leq c$$

from which we get

$$\inf_{B_{R/2}} \overline{u}(x) \geq e^{-c} = c_0.$$

So that, if $\sigma < \varepsilon \leq \frac{c_0}{2}$ we find $\inf_{B_{R/2}} u(x) \geq c_0/2$. □

Structural constants in Theorem 2.12 are independent of the right hand side f, so families $\mathbb{K}_{\Omega,\lambda f}$ and $\mathbb{K}_{\Omega,-\lambda f}$ have the ν critical density property $CD(\nu, c, \varepsilon, \eta)$ for every fixed $0 < \nu < 1$, for every $\lambda \geq 0$ and $f \in \mathcal{L}(\Omega) = \mathcal{F}(\Omega)$ (here we are using the notation presented at the beginning of this section). Indeed if $u \in \mathbb{K}_{\Omega,f}$, then for every $\lambda > 0$, $u_\lambda = \lambda u \in \mathbb{K}_{\Omega,\lambda}$ and if $u_{-\lambda} := \tau - \lambda u \geq 0$, then $u_{-\lambda} \in \mathbb{K}_{\Omega,-\lambda f}$, so we can apply Theorem 2.12 to functions u_λ and $u_{-\lambda}$. Moreover hypothesis **(O)** guarantees the reverse doubling property and the continuity of the function $r \mapsto |B_r(x)|$. In addition, in Lemma 2.8 it has been proved that any weak solution u to (2.5), without sign assumption, belongs to L^∞_{loc}. Thus, recalling the definition of $\mathbb{K}_{\Omega,\lambda f}$ and $\mathcal{S}_\Omega$, we use Theorems 1.10 and 1.11 to get the following Harnack inequality.

Theorem 2.13. *Let $u \in W^1_{loc}(\Omega)$ be a non negative weak solution to $Lu = f$ in Ω. Then there exists a structural constant η such that for every $B_{\eta r} \subseteq \Omega$ we have*

$$\sup_{B_r} u \leq C \left(\inf_{B_r} u + r^\delta \|\gamma_0\|_{L^{2p}(B_{\eta r})} + r^{2\delta} \|g\|_{L^p(B_{\eta r})} \right).$$

Here C is a structural constant and $\delta = 1 - \frac{Q}{2p}$.

Furthermore, from Theorem 1.16 we get he following Hölder estimate

Theorem 2.14. *Let $u \in W^1_{loc}(\Omega)$ be a non negative weak solution to $Lu = f$ in Ω, then for every $\overline{B}_R(x_0) \subseteq \Omega$ we have*

$$\operatorname*{osc}_{B_r(x_0)} u \leq Cr^{\alpha} \left(R^{-\alpha} \sup_{B_R(x_0)} |u| + \|\gamma_0\|_{L^{2p}(B_R(x_0))} + \|g\|_{L^p(B_R(x_0))} \right)$$

for every $r \in]0, R]$.

Proof. It suffices to recall that $\mathcal{S}_\Omega(B_R(x_0), f) = R^{\delta} \|\gamma_0\|_{L^{2p}(B_R(x_0))} + R^{2\delta} \|g\|_{L^p(B_R(x_0))}$ with $\delta = 1 - \frac{Q}{2p}$ and to choose $\mu > \frac{\alpha}{\delta}$ in the statement of Theorem 1.16. □

Corollary 2.15. *Let $u \in W^1_{\text{loc}}(\Omega, X)$ be a non negative weak solution to $\mathcal{L}u = f$ in Ω. Then, there exists two structural constants $C > 0$ and $\alpha \in]0, 1[$ such that*

$$\sup_{x,y \in B_r} \frac{|u(x) - u(y)|}{d_X(x,y)^{\alpha}} \leq C 2^{\alpha} \left((4r)^{-\alpha} \sup_{B_{5r}} |u| + \|\gamma_0\|_{L^{2p}(\Omega)} + \|g\|_{L^p(\Omega)} \right)$$

for every ball $\bar{B}_{5r} \subset \Omega$.

Proof. Clearly it holds

$$\sup_{x,y \in B_r} \frac{|u(x) - u(y)|}{d_X(x,y)^{\alpha}} \leq \sup_{x,y \in B_r} \frac{\operatorname*{osc}_{B_{2d(x,y)}(x)} u}{d_X(x,y)^{\alpha}} \tag{2.18}$$

Moreover, keeping in mind that $B_{2d(x,y)}(x) \subset B_{4r}(x)$, by Theorem 2.14 we have

$$\operatorname*{osc}_{B_{2d(x,y)}(x)} u \leq C(2d(x,y))^{\alpha} \left((4r)^{-\alpha} \sup_{B_{4r}(x)} |u| + \|\gamma_0\|_{L^{2p}(\Omega)} + \|g\|_{L^p(\Omega)} \right).$$

Now it suffices to use estimates above in (2.18) and recall that $B_{4r}(x) \subset B_{5r}$ for every $x \in B_r$ to get the thesis. □

We explicitly notice that in the statement of Theorems 2.13 and 2.14 we do not make assumption on the radius because we have already made a stronger assumption on the diameter of Ω (assumption **(O)**). Moreover we want to recall again that the result contained in this chapter are not new as they have been proved by Uguzzoni in [43] for a class of more general operators (the author allows also the presence of first order term). What is new here is the method used to obtain the result. This shows that the approach presented in Chapter 1 is well adapted to divergence form operator.

Chapter 3

Application to Grushin type operators

In this chapter we apply the abstract procedure presented in Chapter 1 to Grushin-type operators L, a family operators that are elliptic with respect to the Grushin vector fields but they are not in divergence form. These type of operators arises from the geometric theory of several complex variables, more details about this fact will be given in the Appendix. Grushin type operators have been studied by Montanari in [37] where the author proves an invariant Harnack inequality for non negative classical solutions to the the homogeneous equation $Lu = 0$. In this chapter we improve the results obtained in [37] by considering the non homogeneous case; nevertheless we make use of many results and ideas developed here. Very recently this type of operators has been studied also by Maldonado in [36], we will give more details about his results at the end of Section 3.4.

We first introduce the family of operators we want to consider and summarize few well known facts about the Grushin metric. Then, we recall a weighted Alexandrov-Bakelman-Pucci type maximum principle proved by A. Montanari in [37]. Then, we prove the double ball and the critical density property by means of ad-hoc constructed barriers. Those properties will lead to an invariant Harnack inequality, via Theorem 1.10 under hypotheses (A1)-(A2) and Theorem 1.11. Finally, using the results in Chapter 2, we prove local Hölder estimates for the X-gradient of the solutions.

3.1 Some definitions and useful results

Here we recall some facts about Grushin metric and set the notation that will be used throughout the chapter. We start by defining Grushin vector fields and by introducing two quasi-distance functions that will be useful to prove the double ball and the critical density properties by means of ad-hoc barrier modeled after sublevel sets of these functions. We show that the log ring condition holds true in some of these sublevel sets and we present some useful ball-box theorems. Then, we define the class of operators we are concerned about and the family of functions on which we perform the abstract procedure of Chapter 1. Finally, we recall a weighted Alexandrov-Bakelman-Pucci type maximum principle proved by A. Montanari in [37] and we show that in general it is not possible to estimate the supremum of the negative part of $u \in W^{2,p}$ by the L^p norm of Lu with $0 < p < 3$.

3.1.1 Grushin metric and sublevel sets

Let us consider the space $(\mathbb{R}^2, d_X, |\cdot|)$, where $|\cdot|$ is the Lebesgue measure and d_X is the distance induced by Grushin vector fields

$$X_1 := \partial_{x_1}, \quad X_2 := x_1 \partial_{x_2}. \tag{3.1}$$

We recall that X_1, X_2 satisfy the well known Hörmander condition, so the control distance d_X (see Definition 2.1.2) is well defined. On the other hand X_1, X_2 are not left invariant with respect to any group law on $\mathbb{R}^2$. It is also known that the group of dilations $(\delta_t)_{t>0}$,

$$\delta_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad \delta_t(x) = (tx_1, t^2 x_2) \tag{3.2}$$

is such that the vector fields X_j are δ_t-homogeneous of degree one , i.e. for every $u \in C^1(\mathbb{R}^2)$, $x \in \mathbb{R}^2$ and $t > 0$

$$X_j(u \circ \delta_t)(x) = t(X_j u)(\delta_t(x)).$$

Using results proved by Franchi and Lanconelli in [16] it is possible to describe the structure of balls

$$B_{CC}(x,r) = \{y \in \mathbb{R}^2 : \, d_X(x,y) < r\}, \quad x \in \mathbb{R}^2, \, r > 0$$

by means of boxes, $x = (x_1, x_2)$,

$$\mathrm{Box}(x,r) :=]x_1 - r, x_1 + r[\times]x_2 - r(r + |x_1|), x_2 + r(r + |x_1|)[.$$

More precisely we have

Structure Theorem 1. *There exists a structural constant $\tilde{C} > 1$ such that*

$$\mathrm{Box}(x, \tilde{C}^{-1} r) \subset B_{CC}(x,r) \subset \mathrm{Box}(x, \tilde{C} r).$$

Remark 3.1. *We explicitly remark that the size of the Box depends on the first coordinate of the center and in particular we notice that the Lebesgue measure of balls with center* $(0,x_2)$ *and radius* r *is comparable to* r^3.

From Structure Theorem 1 we deduce the following chain of inequalities

$$\tilde{C}^{-2}|\mathrm{Box}(x,r)| \le |B_{CC}(x,r)| \le \tilde{C}^2|\mathrm{Box}(x,r)|$$

for every $x \in \mathbb{R}^2$, $r > 0$. This, makes the space $(\mathbb{R}^2, d_X, |\cdot|)$ a doubling metric space, but unfortunately it is not sufficient to conclude that the ring condition (see Definition 1.1.6 in Chapter 1) holds true. In [37] this problem is overcome by introducing a new Hölder quasi distance

$$\tilde{d}(x,y) := |x_1 - y_1| + \sqrt{x_1^2 + y_1^2 + 4|x_2 - y_2|} - \sqrt{x_1^2 + y_1^2} \tag{3.3}$$

on the measure space $(\mathbb{R}^2, |\cdot|)$. The triplet $(\mathbb{R}^2, \tilde{d}, |\cdot|)$ turns out to be a doubling quasi metric space satisfying the ring condition and the reverse doubling property. Indeed if we let

$$B(x,r) := \{y \in \mathbb{R}^2 : \tilde{d}(x,y) < r\}, \quad r > 0,\, x \in \mathbb{R}^2$$

be the quasi metric ball of center x and radius r, we have the following theorems.

Structure Theorem 2 ([37] Theorem 3.3)**.** *There exists a structural constant* $C_B > 1$ *such that*

$$\mathrm{Box}(x, C_B^{-1}r) \subset B(x,r) \subset \mathrm{Box}(x, C_B r) \quad \textit{for every } y \in \mathbb{R}^2,\ r > 0.$$

Hence

$$C_B^{-2}|B(x,r)| \le |\mathrm{Box}(x,r)| \le C_B^2|B(x,r)|.$$

Proof. Let us show the first inclusion. If $x \in B(y,r)$, we have

$$|x_1 - y_1| < r \quad \text{and} \quad \sqrt{x_1^2 + y_1^2 + 4|x_2 - y_2|} < r + \sqrt{x_1^2 + y_1^2}.$$

So that

$$\begin{aligned} 4|x_2 - y_2| &\le \left(r + \sqrt{x_1^2 + y_1^2}\right)^2 - (x_1^2 + y_1^2) = r\left(r + 2\sqrt{x_1^2 + y_1^2}\right) \\ &\le r\left(r + 2\sqrt{(r + |y_1|)^2 + y_1^2}\right) \le r\,(r + 2(2|y_1| + r)) \le 4r(r + |y_1|). \end{aligned}$$

Thus, $x \in \text{Box}(y,r)$. On the other hand, if $x \in \text{Box}(y,r)$, we have $|x_1 - y_1| < r$ and $|x_2 - y_2| < r(r + |y_1|)$. Then,

$$\begin{aligned} |x_1 - y_1| + \sqrt{x_1^2 + y_1^2 + 4|x_2 - y_2|} - \sqrt{x_1^2 + y_1^2} < \\ < r + \sqrt{x_1^2 + y_1^2 + 4r(r + |y_1|)} - \sqrt{x_1^2 + y_1^2} \\ = r + \sqrt{x_1^2 + (|y_1| + 2r)^2} - \sqrt{x_1^2 + y_1^2} \\ \leq r + |x_1| + |y_1| + 2r - \sqrt{x_1^2 + y_1^2} \leq 3r \end{aligned}$$

Hence $x \in B(y, 3r)$. Moreover

$$|\text{Box}(x,r)| \leq C_B^2 \frac{r}{C_B} \left(\frac{r}{C_B} + |x_1| \right) = C_B^2 |\text{Box}(x, C_B^{-1} r)| \leq C_B^2 |B(x,r)|$$

and

$$|\text{Box}(x,r)| \geq C_B^{-2} |\text{Box}(x, C_B r)| \geq C_B^{-2} |B(x,r)|.$$

This concludes the proof. □

We explicitly remark that theorem above implies the doubling doubling property in $(\mathbb{R}^2, \tilde{d}, |\cdot|)$, indeed we have

$$|B(x,2r)| \leq C_B^2 |\text{Box}(x,2r)| \leq (2C_B)^2 |\text{Box}(x,r)| \leq 2^2 C_B^4 |B(x,r)|. \tag{3.4}$$

Moreover the space $(\mathbb{R}^2, \tilde{d}, |\cdot|)$ has the reverse doubling property indeed we have the following remark

Remark 3.2. *For every $r > 0$ and $y \in \Omega$, the point $\left(\frac{3}{2} r + y_1, y_2\right)$ belongs to the ring $B(y, 5r/3) \setminus B(y, 4r/3)$ so hence the reverse doubling property follows from Theorem 2.3.*

The ring condition is proved in the following theorem.

Theorem 3.3 ([37], Theorem 3.4)**.** *There exists a non negative function ω, such that for every $r > 0$ and every sufficiently small value of $\varepsilon > 0$*

$$|B(x,r) \setminus B(x,(1-\varepsilon)r)| \leq \omega(\varepsilon) |B(x,r)| \tag{3.5}$$

with $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0+$.

Proof. We fix $y \in \mathbb{R}^2$ and define the function $f(r) = |B(y,r)|$. Using Fubini's theorem, it is possible to write f as

$$\begin{aligned} f(r) &= \int_{y_1-r}^{y_1+r} \left(\int_{4|x_2-y_2|<\left(r-|x_1-y_1|+\sqrt{x_1^2+y_1^2}\right)^2-(x_1^2+y_1^2)} \mathrm{d}x_2 \right) \mathrm{d}x_1 \\ &= \frac{1}{2}\int_{-r}^{r} (r-|t|)^2 + 2(r-|t|)\sqrt{(y_1+t)^2+y_1^2}\,\mathrm{d}t. \end{aligned}$$

Differentiating f we get

$$\begin{aligned} 0 \le f'(r) &= \int_{-r}^{r} (r-|t|) + \sqrt{(y_1+t)^2+y_1^2}\,\mathrm{d}t = r^2 + \int_{-r}^{r} \sqrt{(y_1+t)^2+y_1^2}\,\mathrm{d}t \\ &\le r^2 + r\left(\sqrt{(y_1+r)^2+y_1^2}\right) \le r^2 + r(2|y_1|+r) < 4r(r+|y1|). \end{aligned}$$

Now, the Lagrange mean value theorem ensures the existence of $\theta \in]1-\varepsilon, 1[$ such that

$$\begin{aligned} |B(x,r)\setminus B(x,(1-\varepsilon)r)| &= f(r) - f(r(1-\varepsilon)) = \varepsilon r f'(\theta r) \le 4\varepsilon r^2(r+|y_1|) \\ &= 4\varepsilon|\mathrm{Box}(x,r)| \le 4C_B^2|B(x,r)|. \end{aligned}$$

The last inequality follows from Structure Theorem 2. □

We now define suitable sublevel sets of particular functions g_r and h_r in which we are able to construct barriers that will be essential to prove the critical density and the double ball property. We construct these functions modifying the fundamental solution $\Gamma(x,0) = \left(x_1^4 + 4x_2^2\right)^{(2-Q)/4}$ with pole at the origin of the subelliptic Laplacian $X_1^2 + X_2^2$ where $Q = 3$ is the homogeneous dimension. As in [37], for every $r > 0$, $y = (y_1, y_2) \in \mathbb{R}^2$ we consider

$$\rho(x,y) = \left((x_1^2 - y_1^2)^2 + 4(x_2 - y_2)^2\right)^{1/4}, \tag{3.6}$$

and we define

$$\tilde{g}_r(x,y) = \begin{cases} \rho(x,y) & \text{if } |y_1| < r \\ \frac{1}{|y_1|}\rho^2(x,y) & \text{if } |y_1| \ge r. \end{cases}$$

We denote the sublevel sets of the function $\tilde{g}_r(\cdot, y)$ by

$$\tilde{G}(y,r) := \{x \in \mathbb{R}^2 : \tilde{g}_r(x,y) < r\}. \tag{3.7}$$

In order to avoid two zeros of $\tilde{g}_r$ in $\tilde{G}(y,r)$ we also define the function

$$g_r(x,y) = \begin{cases} \rho(x,y) & \text{if } |y_1| < r \\ \frac{1}{|y_1|}\rho^2(x,y) & \text{if } |y_1| \geq r \text{ and } x_1 y_1 \geq 0 \\ +\infty & \text{if } |y_1| \geq r \text{ and } x_1 y_1 < 0 \end{cases}$$

and consider its sublevel sets

$$G(y,r) := \{x \in \mathbb{R}^2 : g_r(x,y) < r\}.$$

Moreover, for every $r > 0$ and $y \in \mathbb{R}^2$, we define

$$\sigma(x,y) = \left((x_1^2 - y_1^2)^2 + 2y_1^2(x_1 - y_1)^2 + 4(x_2 - y_2)^2\right)^{1/4} \tag{3.8}$$

and

$$h_r(x,y) = \begin{cases} \sigma(x,y) & \text{if } |y_1| < r \\ \frac{1}{|y_1|}\sigma^2(x,y) & \text{if } |y_1| \geq r. \end{cases}$$

Sublevel sets of the function $h_r(\cdot,y)$ will be denoted by

$$H(y,r) := \{x \in \mathbb{R}^2 : h_r(x,y) < r\}. \tag{3.9}$$

Let us spend a few words to describe the main features of the sets defined above. The $\tilde{G}(y,r)$, are always symmetric with respect to the x_2 axis and they are connected for $|y_1| < r$, but when $|y_1| > r$ they have two connected components (see Figure 3.1).

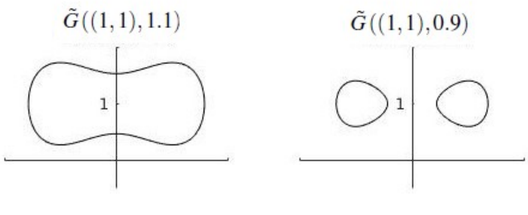

Fig. 3.1 Example of $\tilde{G}$ sublevel sets

Sets of type $G(y,r)$ coincide with $\tilde{G}(y,r)$ when $|y_1| < r$, and for $|y_1| > r$ they are the connected component of $\tilde{G}(y,r)$ that contains the center, so they are always connected (Figure 3.3).

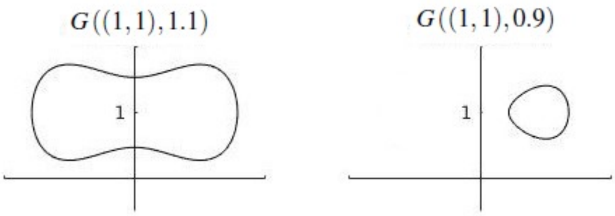

Fig. 3.2 Example of G sublevel sets

Sublevel sets $H(y,r)$ are connected but they are not symmetric with respect to the x_2 axis (see Figure 3.3).

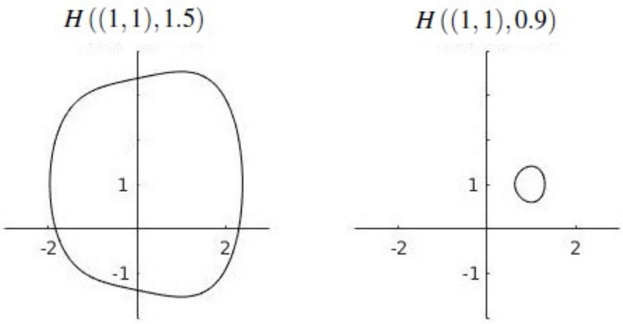

Fig. 3.3 Example of H sublevel sets

Theorems below compare sublevel sets $H(y,r)$ and $G(y,r)$ with boxes $\mathrm{Box}(y,r)$, those results are extensively used in the next section.

Structure Theorem 3. *There exists a structural constant $C_H > 1$ such that for every $y \in \mathbb{R}^2$ and $r > 0$*

$$\mathrm{Box}(y, C_H^{-1} r) \subset H(y,r) \subset \mathrm{Box}(y, C_H r).$$

Hence

$$C_H^{-2} |H(y,r)| \leq |\mathrm{Box}(y,r)| \leq C_H^2 |H(y,r)|.$$

Proof. To prove the inclusions we shall distinguish two case: $|y_1| < r$ and $|y_1| \geq r$. If we assume $|y_1| < r$ and $x \in \mathrm{Box}(y, r/4)$ then

$$|x_1 - y_1| < r/4, \quad |x_2 - y_2| < r/4(r/4 + |y_1|) < 1/4(1/4+1)r^2$$

and

$$\begin{aligned}
h_r(x,y)^4 &= \sigma^4(x,y) = (x_1-y_1)^2(x_1+y_1)^2 + 2y_1^2(x_1-y_1)^2 + 4(x_2-y_2)^2 \\
&\le (1/4)^2(|x_1|+|y_1|)^2 r^2 + 4(1/4)^2 r^4 (1/4+1)^2 + 2(1/4)^2 r^4 \\
&\le (1/4)^2(|x_1-y_1|+2|y_1|)^2 r^2 + 1/4 r^4 (1/4+1)^2 + 2(1/4)^2 r^4 \\
&\le (1/4)^2(1/4+2)^2 r^4 + 1/4(1/4+1)^2 r^4 + 2(1/4)^2 r^4 \\
&= ((1/4)^2(1/4+2)^2 + 1/4(1/4+1)^2 + 2(1/4)^2) r^4 \\
&= (213/256) r^4 < r^4.
\end{aligned}$$

Hence $x \in H(y,r)$. Moreover if $|y_1| < r$ and $x \in H(y,r)$ we have

$$|x_1^2 - y_1^2| < r^2, \quad 4|x_2-y_2|^2 < r^4$$

from which we get

$$|x_1-y_1|\,\big||x_1-y_1|-2|y_1|\big| < r^2, \quad |x_2-y_2| < \frac{r^2}{2} \le r(r+|y_1|).$$

Consequently if $|x_1-y_1| > 2|y_1|$, adding $|y_1|^2$ to both sides of the first inequality above and taking the square root gives

$$|x_1-y_1| - |y_1| < \sqrt{r^2+|y_1^2|} < 2r$$

from which we easily deduce $x \in \mathrm{Box}(y,3r)$. On the other hand if $|x_1-y_1| \le 2|y_1|$ then $x \in \mathrm{Box}(y,3r)$. Hence we have proved the thesis for $|y_1| < r$.
To prove the other case suppose $|y_1| \ge r$ and $x \in \mathrm{Box}(y, r/4)$ then

$$|x_1-y_1| < r/4, \quad |x_2-y_2| < r/4(r/4+|y_1|) < 1/4(1/4+1)|y_1|^2$$

and

$$\begin{aligned}
h_r(x,y)^2|y_1|^2 &= \sigma^4(x,y) = (x_1-y_1)^2(x_1+y_1)^2 + 2y_1^2(x_1-y_1)^2 + 4(x_2-y_2)^2 \\
&\le (1/4)^2(|x_1|+|y_1|)^2 r^2 + 4(1/4)^2 r^2 y_1^2 (1/4+1) + 2(1/4)^2 r^2 y_1^2 \\
&\le (1/4)^2(|x_1-y_1|+2|y_1|)^2 r^2 + 1/4 r^2 y_1^2 (1/4+1) + 2(1/4)^2 r^2 y_1^2 \\
&\le (1/4)^2(1/4+2)^2 r^2 y_1^2 + 1/4 r^2 y_1^2 (1/4+1) + 2(1/4)^2 r^2 y_1^2 \\
&= ((1/4)^2(1/4+2)^2 + 1/4(1/4+1)^2 + 2(1/4)^2) r^2 y_1^2 \\
&= (213/256) r^2 y_1^2 < r^2 y_1^2.
\end{aligned}$$

Hence $x \in H(y,r)$. Moreover if $x \in H(y,r)$

$$\sqrt{2}|y_1||x_1 - y_1| < r|y_1|, \quad 4|x_2 - y_2|^2 < r^2|y_1|^2$$

consequently

$$|x_1 - y_1| < r, \quad |x_2 - y_1| < \frac{r|y_1|}{2} < r(r + |y_1|),$$

hence $x \in \text{Box}(y,r)$. As in Structure Theorem 2, the chain of inclusions in the statement implies the chain of inequalities $C_H^{-2}|H(x,r)| \leq |\text{Box}(x,r)| \leq C_H^2|H(x,r)|$. □

The next ball-box theorem compares the measure of sublevel sets $G(x,r)$ with boxes $\text{Box}(x,r)$. The proof is similar to the one of Theorem 3.1.1, so it is omitted.

Structure Theorem 4 ([37], Theorem 3.6)**.** *There exists a structural constant $C_G > 1$ such that for every $y \in \mathbb{R}^2$ and $r > 0$*

$$\text{Box}(y, C_G^{-1}r) \subset G(y,r) \subset \text{Box}(y, C_G r).$$

Hence

$$C_G^{-2}|G(y,r)| \leq |\text{Box}(y,r)| \leq C_G^2|G(y,r)|.$$

Whenever $y = (y_1, 0) \in \mathbb{R}^2$, from the theorem above we get

$$c^{-1}r^2(r + |y_1|) \leq |G(y,r)| \leq cr^2(r + |y_1|) \tag{3.10}$$

$$\sup_{G(y,2r)} |x_1| \leq \tilde{c}(r + |y_1|) \tag{3.11}$$

$$C_M \max\{r, r(r + |y_1|)\} \geq \text{diam}(G(y,r)) \geq C_m \max\{r, r(r + |y_1|)\}. \tag{3.12}$$

with $c, \tilde{c}, C_M > 1$ and $C_m > 0$ structural constants. Hereafter we denote the Euclidean diameter of a set A by $\text{diam}(A)$

We now have introduced all the definitions and basic results concerning the Grushin plane that will be needed in the following sections. So we move to the definition of the Grushin operators.

3.1.2 Grushin type operators

Let $\Omega \subseteq \mathbb{R}^2$ be open and consider the Hölder doubling quasi metric space $(\Omega, \tilde{d}, |\cdot|)$ with $\tilde{d}$ defined in (3.3). We define the second order linear operator

$$L := a_{11}(x_1,x_2)X_1^2 + a_{22}(x_1,x_2)X_2^2 + 2a_{12}(x_1,x_2)X_1X_2, \tag{3.13}$$

where $a_{ij} : \Omega \to \mathbb{R}$ are measurable functions. We assume L is elliptic with respect to the family of vector fields $\{X_1, X_2\}$, i.e. there exist positive constants, called ellipticity constants, $0 < \lambda \le \Lambda$ such that for every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ we have

$$\lambda|\xi|^2 \le a_{11}\xi_1^2 + a_{22}\xi_2^2 + 2a_{12}\xi_1\xi_2 \le \Lambda|\xi|^2. \tag{3.14}$$

Notice that in the definition of the operator L the term $a_{21}X_2X_1$ is missing. Motivation to the study of these type of operators arises from the geometric theory of several complex variables. More precisely in the Appendix we show that, Levi curvature equations, which are fully nonlinear equation in non divergence form, in cylindrical coordinates are non divergence PDEs $Lu = f$ with the operator L structured as in (3.13). In this dissertation we focus on equations of the type

$$Lu = x_1^2 f$$

with

$$f \in \mathcal{F}(\Omega) = \{g : \Omega \to \mathbb{R}, \text{measurable and such that } x_1 g \in L^2_{\text{loc}}(\Omega)\}.$$

When the right hand side of the equation is not structured as above, Harnack inequality remains an interesting open problem. Indeed the Alexandrov-Bakelman-Pucci estimates, which are a key ingredient towards Harnack inequality, do not hold in general as shown in Theorem 3.7 and the subsequent Remark.

In accordance with the notation of Chapter 1, for every measurable set $A \subset \Omega$ we define

$$\mathcal{S}_\Omega(A, f) := \text{diam}(A)\|x_1 f\|_{L^2(A)},$$
$$\mathcal{L}(\Omega) = \{f \in \mathcal{F}(\Omega) : \mathcal{S}_\Omega(B(x,r), f) < +\infty, \text{ for every } B(x,r) \subseteq \Omega\}$$
$$\mathbb{K}_{\Omega,f} := \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : \ Lu = x_1^2 f, \ u \ge 0\}, \text{ for every } f \in \mathcal{L}(\Omega)$$

Here $\text{diam}(A)$ denotes the Euclidean diameter of A. Since the operator L is linear and with no zero-order term, if u is a classical solution to $Lu = x_1^2 f$, the function $\tau - \lambda u$ solves $L(\tau - \lambda u) = -\lambda x_1^2 f$, for every $\lambda, \ \tau \in \mathbb{R}$. So that the definition of $\mathbb{K}_{\Omega,f}$ is coherent with Definition 1.2.2.

3.1.3 An Alexandrov-Bakelman-Pucci maximum principle

For classical supersolution to (3.1.2), a weighted Alexandrov-Bakelman-Pucci maximum principle (ABP, henceforth) has been proved in [37, Theorem 2.5]; these kind of estimates are extremely useful as it provides a pointwise bound on solutions in terms of a measure theoretic quantity of the equation. We report the statement of the result for future references.

Theorem 3.4 (ABP estimates)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, $u \geq 0$ on $\partial\Omega$ is a classical solution to $Lu(x) \leq x_1^2 f(x)$ in Ω with $x_1 f \in L^2(\Omega)$. Then there exists a positive structural constant C such that*

$$\sup_{\Omega} u^- \leq C\mathrm{diam}(\Omega) \left(\int_{\Omega \cap \{u=\Gamma_u\}} (x_1 f^+)^2 \, dx \right)^{\frac{1}{2}}. \tag{3.15}$$

Here $\mathrm{diam}(\Omega)$ is the Euclidean diameter of Ω, $f^+(x) := \max\{f(x), 0\}$, Γ_u is the convex envelope[1] *of $-u^-(x) := -\max\{-u(x), 0\}$ in a Euclidean ball of radius $2\mathrm{diam}(\Omega)$ containing Ω and $u \equiv 0$ outside Ω.*

We remark that in [37, Theorem 2.5], the result is proved under the slightly more restrictive assumption on f, precisely it is required f to be bounded. The same proof remain valid when it is assumed $x_1 f \in L^2(\Omega)$ because, in this case, the right hand side of (3.15) is bounded. For the reader's convenience, we give a proof of Theorem 3.4. We will need the following well known results (see for example [24, Theorem 1.1.13, Exercise 1.1.14 and Theorem 1.4.5]).

Theorem 3.5. *Let Ω be open and $u \in C(\Omega)$. We denote by Du the sub-differential of u.*[2] *Then*

$$\mathcal{B} = \{E \subset \Omega : Du(E) \text{ is Lebesgue measurable}\}$$

is a Borel σ algebra. The function $\mu_u : \mathcal{B} \to \overline{R}$ defined by $\mu_u(E) = |Du(E)|$ is a Borel measure and it is finite on compact sets. Moreover, if $u \in C^2(\Omega)$ is a convex function on Ω, then

$$\mu_u(E) = \int_E \det(D^2 u)(x) dx$$

for any Borel set $E \subset \Omega$.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, assume $u \in C(\overline{\Omega})$, $u \geq 0$ on $\partial\Omega$ and define Γ_u is as in Theorem 3.4. Then there exists a positive structural constant c such that*

$$\sup_{\Omega} u^- \leq \frac{d}{c} \left(\mu_{\Gamma_u}(\{u = \Gamma_u\} \cap \Omega)\right)^{1/2}.$$

Proof of Theorem 3.4. We first prove the statement for u strictly convex on the contact set $\{u = \Gamma_u\}$. Under this assumption Du is a one-to-one map. Moreover, the contact set $\{u = \Gamma_u\}$

[1] We recall that the convex envelope of a continuous function u in a convex set Ω is the function Γ_u defined as follows: $\Gamma_u(x) = \sup\{w(x) : w \leq u, \, w \text{ is convex}\}$

[2] The sub-differential of $u \in C(\Omega)$, is the set valued function Du defined by $Du(x_0) = \{p \in \mathbb{R}^2 : u(x) \geq u(x_0) + p \cdot (x - x_0) \text{ for all } x \in \Omega\}$

do not intersects the line $\{x_1 = 0\}$, indeed $(a_{11}u_{11})(0,x_2) = Lu(0,x_2) \le 0$, and since a_{11} is positive then $u_{11}(0,x_2) \le 0$ while u is strictly convex in $\{u = \Gamma_u\}$. Notice that if $u \in C^2(\Omega)$ is convex then the symmetric matrix

$$X^2u = \begin{pmatrix} X_1^2u & X_2X_1u \\ X_2X_1u & X_2^2u \end{pmatrix} = \begin{pmatrix} u_{11} & x_1u_{12} \\ x_1u_{12} & x_1^2u_{22} \end{pmatrix} \tag{3.16}$$

is nonnegative definite. Here we have denoted by $u_{ij} = \partial_{x_i}\partial_{x_j}u$. Moreover, for u convex

$$\det(D^2u) = (u_{11}u_{22} - u_{12}^2) = \frac{\left(u_{11}x_1^2u_{22} - (x_1u_{12})^2\right)}{x_1^2} = \frac{\det(X^2u)}{x_1^2} \le \frac{\left(\operatorname{trace}(AX^2u)\right)^2}{4x_1^2\det A} \tag{3.17}$$

for every $A > 0$ and A symmetric. Let B_ε be a Euclidean ball of radius $2\operatorname{diam}(\Omega)$ containing Ω. The convex envelop Γ_u is a convex function so, for each $x_0 \in \overline{B}_\varepsilon \cap \{u = \Gamma_u\} \subset \Omega$, it has a supporting hyperplane at x_0. This hyperplane is also a supporting hyperplane for u at the same point. Thus $D\Gamma_u(x_0) \subset Du(x_0)$ for $x_0 \in \overline{B}_\varepsilon \cap \{u = \Gamma_u\} \subset \Omega$, and by Theorem 3.5 we have

$$\begin{aligned} |D\Gamma_u(\{u = \Gamma_u\} \cap \Omega)| &\le |Du(\{u = \Gamma_u\} \cap \Omega)| \\ &= \int_{\{u=\Gamma_u\}\cap\Omega} (\det D^2u)dx \\ &\le C\int_{\{u=\Gamma_u\}\cap\Omega} \frac{(Lu)^2}{x_1^2}dx \end{aligned} \tag{3.18}$$

where C is a positive constant depending on Λ, λ. The last inequality follows by applying (3.17), indeed u is convex on the set $\{u = \Gamma_u\} \cap \Omega$. Recalling that $Lu(x) \le f(x)x_1^2 = f^+(x)x_1^2$ on the contact set and by applying Theorem 3.6 and (3.18), we get the desired estimate.
Now assume u is only convex in the contact set an define $S_0 = \{x \in \Omega : \det D^2u = 0\}$. By Sard's Theorem we have $|Du(S_0)| = 0$. Since $E = \{u = \Gamma_u\} \cap \Omega$ is a Borel set, $E \cap S_0$ and $E \setminus S_0$ are also Borel sets. Hence

$$|Du(E)| = |Du(E \cap S_0)| + |Du(E \setminus S_0)| = |Du(E \setminus S_0)|$$

and by (3.18) we have

$$\begin{aligned} |D\Gamma_u(E)| \le |Du(E)| = |Du(E \setminus S_0)| &= \int_{E\setminus S_0} (\det D^2u)dx \\ \le C\int_{E\setminus S_0} \frac{(Lu)^2}{x_1^2}dx &\le C\int_{E\setminus S_0} f^2x_1^2dx \le C\int_E f^2x_1^2dx. \end{aligned}$$

This concludes the proof. □

Let us go back to the statement of Theorem 3.4. If we denote by $F = x_1^2 f$ the right hand side of the equation to which u is a supersolution, we can notice that, dissimilarly to what usually happens (see for example [7, Theorem 3.6]), $\sup_\Omega u^-$ is not estimated by a quantity that depends on the L^2 norm of F, but rather on the L^2 norm of F/x_1.
In the following we show that the weight on the right hand side of (3.15) is crucial. Precisely, we will show that ABP maximum principle of the type (3.15) can not hold in general for solutions u of $Lu \le f$ with $f \in L^{3-\varepsilon}$ and any $0 < \varepsilon < 3$. We will proceed exactly as in [13]. In that last mentioned article the authors consider a group of Heisenberg type of homogeneous dimension Q, they construct a non divergence form operator L_ε and show the impossibility of ABP type estimates for $W^{2,Q-\varepsilon}_{\text{loc}}$ solutions as a consequence of a non uniqueness result for solutions to the Dirichlet problem $L_\varepsilon u = 0$ in Ω, $u = 0$ on $\partial\Omega$; hereafter $W^{2,p}_{\text{loc}}(\Omega, X)$ indicates the Sobolev space of functions $u \in L^p_{\text{loc}}$ having weak derivatives $X_i X_j u \in L^p_{\text{loc}}$. Mimicking their approach and taking into account Remark 3.1 we get the following result

Theorem 3.7. *Let* $\rho(x,0) = \left(x_1^4 + 4x_2^2\right)^{1/4}$ *as in* (3.6)*. We define* $B_1(0) = \{x \in \mathbb{R}^2 : \rho^{1/4}(x,0) < 1\} \subset \mathbb{R}^2$ *be the ball of radius 1 and center the origin. Then, for every* $0 < \varepsilon < 3$*, there exists an operator* L_ε *of the type (3.13) with bounded measurable coefficients such that the following Dirichlet problem*

$$\begin{cases} L_\varepsilon u \le 0 & \text{in } B_1(0) \\ u = 0 & \text{on } \partial B_1(0). \end{cases} \tag{3.19}$$

has a non trivial solution $u_\varepsilon \in W^{2,3-\varepsilon}(B_1(0), X) \cap C(\overline{B_1(0)})$.

Proof. We use the notation $N(x_1, x_2) = \rho(x,0) = \left(x_1^4 + 4x_2^2\right)^{1/4}$. We choose the coefficients of the operator L_ε as follows

$$a_{11} = 1 + \gamma(\varepsilon)\frac{(X_1N)^2}{(X_1N)^2 + (X_2N)^2}$$
$$a_{22} = 1 + \gamma(\varepsilon)\frac{(X_2N)^2}{(X_1N)^2 + (X_2N)^2}$$
$$a_{12} = \gamma(\varepsilon)\frac{X_1NX_2N}{(X_1N)^2 + (X_2N)^2}$$

where $\gamma > 0$ is a constant depending on ε that will be chosen in the sequel. It will be convenient to explicitly write the coefficients of the operator L_ε by computing the derivatives of N with respect to the vector fields X_1 and X_2:

$$X_1 N = \frac{x_1^3}{N^3}, \qquad X_2 N = \frac{2x_1 x_2}{N^3},$$
$$X_1^2 N = \frac{12 x_1^2 x_2^2}{N^7}, \qquad X_2 X_1 N = -\frac{6x_1^4 x_2}{N^7}, \qquad X_2^2 N = \frac{2x_1^2}{N^7}(x_1^4 - 2x_2^2).$$

So we have

$$a_{11} = 1 + \gamma(\varepsilon)\frac{x_1^4}{N^4} \qquad a_{22} = 1 + \gamma(\varepsilon)\frac{4x_2^2}{N^4} \qquad a_{12} = 2\gamma(\varepsilon)\frac{x_1^2 x_2}{N^4}.$$

We look for a solution to (3.19) of the type $u_\varepsilon = F(N)$:

$$\begin{aligned}
L_\varepsilon F(N) &= a_{11} X_1^2 F(N) + 2a_{12} X_2 X_1 F(N) + a_{22} X_2^2 F(N) \\
&= F''(N)\left(a_{11}(X_1 N)^2 + 2a_{12} X_1 N X_2 N + a_{22}(X_2 N)^2\right) + F'(N) L_\varepsilon(N) \\
&= \frac{F''(N)}{N^{10}}\Big((x_1^4 + 4x_2^2 + \gamma x_1^4)x_1^6 + \gamma 8 x_1^6 x_2^2 + (x_1^4 + 4x_2^2 + \gamma 4x_2^2) 4x_1^2 x_2^2\Big) + \\
&\quad + \frac{F'(N)}{N^{11}}\Big((x_1^4 + 4x_2^2 + \gamma x_1^4) 12 x_1^2 x_2^2 - \gamma 24 x_1^6 x_2^2\Big) + \\
&\qquad + \frac{F'(N)}{N^{11}}(x_1^4 + 4x_2^2 + \gamma 4x_2^2) 2x_1^2 (x_1^4 - 2x_2^2) \\
&= \frac{F''(N)}{N^{10}} x_1^2 \left(\gamma(x_1^8 + 16x_2^4 + 8x_1^4 x_2^2) + N^4(x_1^4 + 4x_2^2)\right) + \frac{F'(N)}{N^{11}}\left(2x_1^2 N^8 - \gamma 4 x_1^2 x_2^2 N^4\right) \\
&= (1+\gamma)\frac{x_1^2}{N^2}\left(F''(N) + 2\frac{F'(N)}{(1+\gamma)N}\right) - \gamma 4 x_1^2 x_2^2 \frac{F'(N)}{N^7}.
\end{aligned}$$

In particular we choose $F(N) = N^{\lambda(\varepsilon)} - 1$ and we want to determine $\lambda > 0$ so that $L_\varepsilon N^{\lambda(\varepsilon)} \leq 0$, thus we require

$$\begin{aligned}
h_1 &= (1+\gamma)\frac{x_1^2}{N^2}\left(F''(N) + 2\frac{F'(N)}{(1+\gamma)N}\right) \\
&= x_1^2(1+\gamma)\frac{\lambda}{N^{-\lambda+4}}\left(\lambda - 1 + \frac{2}{(1+\gamma)}\right) = 0
\end{aligned} \tag{3.20}$$

and

$$h_2 = \gamma 4 x_1^2 x_2^2 \frac{F'(N)}{N^7} = \gamma\lambda 4 x_1^2 x_2^2 N^{\lambda - 8} > 0. \tag{3.21}$$

Equation (3.20) holds true for

$$\lambda = 1 - \frac{2}{1+\gamma} < 1 \quad \text{for } \gamma > 0. \tag{3.22}$$

Moreover for $\gamma > 1$ we have $\lambda > 0$, hence also inequality (3.21) is satisfied. As we have already said, for any $0 < \varepsilon < 3$ we want to construct a solution u_ε to $Lu_\varepsilon = 0$ belonging to $W^{2,3-\varepsilon}(B_1(0),X)$. Since the solution will be of the type $u = N^{\lambda(\varepsilon)} - 1$ it suffices to impose

$$(2-\lambda(\varepsilon))(3-\varepsilon) < 3 \quad \textit{i.e.} \quad \lambda(\varepsilon) > \max\left\{0, 1 - \frac{\varepsilon}{3-\varepsilon}\right\}$$

or equivalently

$$\gamma(\varepsilon) > \max\left\{1, \frac{6-3\varepsilon}{\varepsilon}\right\}. \tag{3.23}$$

Now, for any $0 < \varepsilon < 3$ we define $u_\varepsilon = N^{\lambda(\varepsilon)} - 1$. Clearly it satisfies the Dirichlet condition $u_\varepsilon|_{\partial B_1(0)} = 0$ and the choice of $\lambda(\varepsilon)$ we made ensures $Lu_\varepsilon = LN^{\lambda(\varepsilon)} \leq 0$ and $u_\varepsilon \in W^{2,3-\varepsilon}(B_1(0),X)$. This concludes the proof. □

From the above non uniqueness result we deduce the impossibility to obtain certain ABP estimates, more precisely we have the following remark.

Remark 3.8. *Let Ω be a bounded open set and L be structured as in (3.13) and suppose there exists $u \in W^{2,p}_{loc}(\Omega,X)$ a solution to $Lu \leq f$ with $f \in L^{3-\varepsilon}(\Omega)$. Then, in general, it is not possible to estimate the supremum of u^- on Ω by the L^p norm of the right hand side f with $p = 3-\varepsilon$. Indeed, if such an ABP maximum principle held true, using the result and the notations of the above theorem, we would have*

$$\sup_{B_1(0)} u_\varepsilon^- = \sup_{B_1(0)} (N^{\lambda(\varepsilon)} - 1)^- \leq 0$$

but $N^{\lambda(\varepsilon)}(0) = 0$ and $N^{\lambda(\varepsilon)} = \rho(0,x)^{\lambda(\varepsilon)}$ is an increasing function of $\rho(0,x)$, hence

$$\sup_{B_1(0)} (N^{\lambda(\varepsilon)} - 1)^- = \sup_{B_1(0)} \max\left\{0, 1 - N^{\lambda(\varepsilon)}\right\} = 1$$

reaching a contradiction.

Remark 3.9. *The counterexample in the previous remark does not apply to the ABP maximum principle (Theorem 3.4). Looking at the equations (3.20) and (3.21) we can notice that $L_\varepsilon F(N) = x_1^2 f$ with f that behaves like $N^{\lambda-4}$ for a suitable $\lambda \in]0,1[$. On the other hand the* *function f does not satisfy the hypotheses of Theorem 3.4 indeed $x_1 f \sim x_1 N^{\lambda-4} \in L^2(B_1(0))$ if and only if $\lambda > 3/2$ but, by (3.22) the counterexample works for $\lambda \in]0,1[$.*

3.2 Double ball property for Grushin type operators

In this section we prove double ball property for sublevel sets $H(y,r)$, and then extend it to balls $B(x,r)$ with the aid of Structure Theorems 2 and 3. The idea is to follow the procedure presented in [37]. In [37] sublevel sets of $\tilde{g}_r(\cdot,y)$ (see for the definition (3.7)) are considered, however in this thesis we prefer to use sublevel sets of $h_r(\cdot,y)$ (see for the definition (3.9)) in order to avoid the two zeros of the function $\tilde{g}$.

Lemma 3.10. *Let Λ, $\lambda > 0$ be the ellipticity constants and σ the function defined in (3.8). For $\alpha \le 4 - 10\Lambda/\lambda$, the function $\phi(x) = \sigma^\alpha$ is a classical solution to $L\phi \ge 0$ in $\{\sigma > 0\}$.*

Proof. We begin the proof computing first and second partial derivatives of σ:

$$\begin{aligned}
\sigma_{x_1} &= \sigma^{-3}\big((x_1^2 - y_1^2)x_1 + y_1^2(x_1 - y_1)\big) = \sigma^{-3}(x_1^3 - y_1^3) \\
\sigma_{x_2} &= 2\sigma^{-3}(x_2 - y_2) \\
\sigma_{x_1x_1} &= -3\sigma^{-7}\big(x_1^3 - y_1^3\big)^2 + 3\sigma^{-3}x_1^2 \\
\sigma_{x_1x_2} &= -3\sigma^{-7}\big(x_1^3 - y_1^3\big)(x_2 - y_2) \\
\sigma_{x_2x_2} &= -3\sigma^{-7}(x_2 - y_2)^2 + 2\sigma^{-3}.
\end{aligned}$$

Using calculation above it is straightforward to compute

$$\begin{aligned}
L\phi &= \alpha\sigma^{\alpha-2}\big((\alpha-1)(a_{11}\sigma_{x_1}^2 + 2x_1a_{12}\sigma_{x_2}^2 + x_1^2a_{22}\sigma_{x_2}^2) + \sigma L\sigma\big) \\
&= \alpha\sigma^{\alpha-2}\Big(a_{11}\big((\alpha-1)\sigma_{x_1}^2 + \sigma\sigma_{x_1x_1}\big) + 2x_1a_{12}\big((\alpha-1)\sigma_{x_1}\sigma_{x_2} + \sigma\sigma_{x_1x_2}\big)\Big) + \\
&\quad + \alpha\sigma^{\alpha-2}\Big(x_1^2a_{22}\big((\alpha-1)\sigma_{x_2}^2 + \sigma\sigma_{x_2x_2}\big)\Big).
\end{aligned}$$

Defining

$$\gamma := a_{11}(x_1^3 - y_1^3)^2 + 4a_{12}x_1(x_1^3 - y_1^3)(x_2 - y_2) + 4a_{22}x_1^2(x_2 - y_2)^2$$

we get

$$\begin{aligned} L\phi &= \alpha\sigma^{\alpha-2}\Big((\alpha-4)\sigma^{-6}\Big(a_{11}(x_1^3-y_1^3)^2+2a_{12}x_1(x_1^3-y_1^3)2(x_2-y_2)\Big) \\ &\qquad +(\alpha-4)\sigma^{-6}a_{22}x_1^2 4(x_2-y_2)^2+x_1^2\sigma^{-2}(3a_{11}+2a_{22})\Big) \\ &= \alpha\sigma^{\alpha-8}\big((\alpha-4)\gamma+x_1^2\sigma^4(3a_{11}+2a_{22})\big). \end{aligned}$$

From the assumption (3.14) on the coefficients of L, we deduce

$$\gamma \geq \lambda\Big((x_1^3-y_1^3)^2+4x_1^2(x_2-y_2)^2\Big) \quad \text{and} \quad 3a_{11}+2a_{22}\leq 5\Lambda,$$

so that

$$\begin{aligned} L\phi &\geq \alpha\sigma^{\alpha-8}\Big((\alpha-4)\lambda\Big((x_1-y_1)^2\big((x_1+y_1)x_1+y_1^2\big)^2+x_1^2 4(x_2-y_2)^2\Big)+5\Lambda x_1^2\sigma^4\Big) \\ &= \alpha\sigma^{\alpha-8}\Big(\big((\alpha-4)\lambda+5\Lambda\big)(x_1^2-y_1^2)^2x_1^2+4\big((\alpha-4)\lambda+5\Lambda\big)(x_2-y_2)^2x_1^2+ \\ &\qquad +y_1^2(x_1-y_1)^2\big(2\lambda(\alpha-4)(x_1+y_1)x_1+\lambda(\alpha-4)y_1^2+10\Lambda x_1^2\big)\Big) \\ &= \alpha\sigma^{\alpha-8}\Big(\big((\alpha-4)\lambda+5\Lambda\big)(x_1^2-y_1^2)^2x_1^2+4\big((\alpha-4)\lambda+5\Lambda\big)(x_2-y_2)^2x_1^2+ \\ &\qquad +y_1^2(x_1-y_1)^2\big(2(\lambda(\alpha-4)+5\Lambda)x_1^2+2\lambda(\alpha-4)x_1y_1+\lambda(\alpha-4)y_1^2\big)\Big) \\ &= \alpha\sigma^{\alpha-8}\Big(\big((\alpha-4)\lambda+5\Lambda\big)(x_1^2-y_1^2)^2x_1^2+4\big((\alpha-4)\lambda+5\Lambda\big)(x_2-y_2)^2x_1^2+ \\ &\qquad +y_1^2(x_1-y_1)^2\big((\lambda(\alpha-4)+10\Lambda)x_1^2+\lambda(\alpha-4)x_1^2+ \\ &\qquad\quad 2\lambda(\alpha-4)x_1y_1+\lambda(\alpha-4)y_1^2\big)\Big). \end{aligned}$$

Now, the definition of α, implies $\alpha(\lambda(\alpha-4)+5\Lambda)\geq 0$ and $\alpha(\lambda(\alpha-4)+10\Lambda)\geq 0$, thus

$$L\phi \geq \lambda\alpha(\alpha-4)\sigma^{\alpha-8}y_1^2(x_1-y_1)^2\big(x_1^2+2x_1y_1+y_1^2\big)\geq 0.$$

□

The next theorem uses lemma above to construct a barrier function Φ for the ring $R(y,r,3r):=H(y,3r)\setminus\overline{H(y,r)}$.

Theorem 3.11. *Let σ be the function defined in (3.8) and α as in Lemma 3.10. Then, there exist positive structural constants M_1, M_2 and $0<\gamma<1$ such that for every $y\in\mathbb{R}^2$ and $r>0$ the function $\Phi := M_2\sigma^\alpha - M_1$ satisfies*

- $\Phi\in C^2(R(y,r,3r))\cap C(\overline{R(y,r,3r)})$,
- $L\Phi\geq 0$ *in* $R(y,r,3r)$,
- $\Phi\mid_{\partial H(y,3r)}=0$,
- $\Phi\mid_{\partial H(y,r)}=1$,
- $\inf_{R(y,r,2r)}\Phi\geq\gamma$.

Proof. We choose M_1 and M_2 by imposing $\Phi\mid_{\partial H(y,3r)}=0$ and $\Phi\mid_{\partial H(y,r)}=1$, and we show that, with this choice, the constants M_1 and M_2 are positive. Since $H(y,r)$ are defined as sublevel set of the function $h_r(x,y)$ and the definition of this function changes in case $|y_1|<r$ or $|y_1|\geq r$, we have to distinguish four cases.

Case I $|y_1|<r$, we have
$M_1=\frac{3^\alpha}{1-3^\alpha}>0$, $M_2=\frac{1}{r^\alpha(1-3^\alpha)}>0$ and we define $M_3:=\Phi\mid_{\partial H(y,2r)}=\frac{2^\alpha-3^\alpha}{1-3^\alpha}>0$

Case II $3r\leq|y_1|$, we have
$M_1=\frac{3^{\alpha/2}}{1-3^{\alpha/2}}>0$, $M_2=\frac{1}{(r|y_1|)^{\alpha/2}(1-3^{\alpha/2})}>0$ and we define $M_3:=\Phi\mid_{\partial H(y,2r)}=\frac{2^{\alpha/2}-3^{\alpha/2}}{1-3^{\alpha/2}}>0$

Case III $r\leq|y_1|<2r$, we have
$M_1=\frac{3^\alpha}{(|y_1|/r)^{\alpha/2}-3^\alpha}>0$, $M_2=\frac{1}{(r|y_1|)^{\alpha/2}-(3r)^\alpha}>0$
and we define $M_3:=\Phi\mid_{\partial H(y,2r)}=\frac{2^\alpha-3^\alpha}{(|y_1|/r)^{\alpha/2}-3^\alpha}\geq\frac{2^\alpha-3^\alpha}{1-3^\alpha}>0$

Case IV $2r\leq|y_1|<3r$, we have
$M_1=\frac{3^\alpha}{(|y_1|/r)^{\alpha/2}-3^\alpha}>0$, $M_2=\frac{1}{(r|y_1|)^{\alpha/2}-(3r)^{\alpha/2}}>0$
and we define $M_3:=\Phi\mid_{\partial H(y,2r)}=\frac{(2|y_1|/r)^{\alpha/2}-3^\alpha}{(|y_1|/r)^{\alpha/2}-3^\alpha}\geq\frac{6^{\alpha/2}-3^\alpha}{2^{\alpha/2}-3^\alpha}>0$.

If we define $\gamma:=\min\left\{\frac{2^\alpha-3^\alpha}{1-3^\alpha},\frac{2^{\alpha/2}-3^{\alpha/2}}{1-3^{\alpha/2}},\frac{6^{\alpha/2}-3^\alpha}{3^{\alpha/2}-3^\alpha}\right\}$ we have $\Phi\mid_{\partial H(y,2r)}=M_3>\gamma$, and since Φ is constant on the sets $\partial H(y,\rho)$ and decreasing with respect to $\rho>0$ get $\Phi\mid_{R(y,r,2r)}>\gamma$. Finally, by Lemma 3.10, $L\Phi\geq 0$ on $R(y,r,3r)$, concluding the proof. □

We are now ready to prove the double ball property for sublevel sets H.

Theorem 3.12. *(Double ball property in $H(y,3r)$) Let C and γ be as in Theorem 3.4 and 3.11 respectively and define $\varepsilon < \frac{\gamma}{2C} < 1$. Then if $H(y,3r) \subset \Omega$ and u is a non negative classical solution to $Lu = x_1^2 f$ in $H(y,3r)$ satisfying*

$$\inf_{H(y,r)} u \geq 1 \quad \textit{and} \quad \operatorname{diam}(H(y,3r)) \|x_1 f\|_{L^2(H(y,3r))} < \varepsilon$$

we have

$$\inf_{H(y,2r)} u \geq \delta$$

where $\delta = \gamma/2$.

Proof. Let Φ be the barrier function defined in Theorem 3.11 and consider $\omega = u - \Phi$. Since $\omega \in C^2(R(y,r,3r)) \cap C(\overline{R(y,r,3r)})$, $\omega \geq 0$ on $\partial R(y,r,3r)$ and $L\omega \leq x_1^2 f$ in $R(y,r,3r)$ we can apply Theorem 3.4 to estimate ω^-:

$$\begin{aligned}
\sup_{R(y,r,3r)} ((\Phi - u)^+) = & \sup_{R(y,r,3r)} \omega^- \\
& \leq C \operatorname{diam}(R(y,r,3r)) \int_{R(y,r,3r)} (x_1 f^+)^2 \, dx \\
& \leq C\varepsilon
\end{aligned}$$

in particular

$$\sup_{R(y,r,2r)} ((\Phi - u)^+) \leq C\varepsilon.$$

Writing $-u = \Phi - u - \Phi$ and taking the supremum over $R(y,r,2r)$ we find

$$\sup_{R(y,r,2r)} (-u) \leq \sup_{R(y,r,2r)} (\Phi - u) + \sup_{R(y,r,2r)} (-\Phi)$$

i.e.

$$-\inf_{R(y,r,2r)} u \leq C\varepsilon - \inf_{R(y,r,2r)} \Phi.$$

It suffices to recall that $\inf_{R(y,r,2r)} \Phi = \gamma$ and the definition of ε to get

$$\inf_{R(y,r,2r)} u \geq \frac{\gamma}{2},$$

moreover, since by hypotheses $\inf_{H(y,r)} u \geq 1$ the thesis is proved. □

In the next theorem we repeatedly apply the double ball property on sublevel sets of h_r to get the same property for quasi metric balls $B(y,r)$.

Theorem 3.13. *(Double ball property in $B(y,\eta_{DB}r)$) There exist structural constants $0<\varepsilon_{DB},\gamma<1$ and $\eta_{DB}>2$ such that if u is a non negative classical solution to $Lu=x_1^2 f$ in $B(y,\eta_{DB}r)$ satisfying*

$$\inf_{B(y,r)} u \geq 1, \quad \textit{and} \quad \operatorname{diam}(B(y,\eta_{DB}r))\|x_1 f\|_{L^2(B(y,\eta_{DB}r))} < \varepsilon_{DB}$$

then

$$\inf_{B(y,2r)} u \geq \gamma.$$

More precisely we have $\eta_{DB}=12C^2$ with $C=C_H C_B$ and C_H, C_B defined Structure Theorems 2 and 3; $\varepsilon_{DB}=\varepsilon\delta^{p-1}$ and $\gamma=\delta^p$ where p is chosen so that $2^{p-1}(C_H C_B)^{-1}\leq 2C_H C_B \leq 2^p(C_H C_B)^{-1}$; ε and δ are defined in Theorem 3.12.

Proof. First of all we notice that the definition of η_{DB} and p imply the inclusion $H(y,2^pC^{-1}3r)\subset B(y,\eta_{DB}r)$, consequently for $0\leq k\leq p-1$ we have

$$\begin{aligned}\operatorname{diam}(H(y,2^kC^{-1}3r))\|x_1 f\|_{L^2(H(y,2^kC^{-1}3r)}&\leq \operatorname{diam}(B(y,\eta_{DB}r))\|x_1 f\|_{L^2(B(y,\eta_{DB}r))}\\&<\varepsilon\delta^{p-1}.\end{aligned} \tag{3.24}$$

Moreover, since $\inf_{B(y,r)} u\geq 1$, $H(y,C^{-1}r)\subset B(y,r)$ and (3.24) holds, we can use the double ball property in $H(y,3C^{-1}r)$ (Theorem 3.12) obtaining

$$\inf_{H(y,C^{-1}2r)} u \geq \delta.$$

In virtue of (3.24) we repeatedly apply ($p-1$ times) Theorem 3.12 to $\frac{u}{\delta^k}$ in $H(y,2^kC^{-1}3r)$ where $1\leq k\leq p-1$ and get

$$\inf_{H(y,C^{-1}2^pr)} u \geq \delta^p.$$

So, recalling that the definition of p, C and Structure Theorems 2 and 3 imply $B(y,2r)\subset H(y,C^{-1}2^pr)$, from the estimate above we deduce

$$\inf_{B(y,2r)} u \geq \delta^p.$$

This concludes the proof. □

3.3 Critical density property for Grushin type operators

In this section we prove critical density property for balls $B(y,r)$. Exactly as in [37] we obtain some rough estimates for the solutions on balls $G((y_1,0),r) \subset \Omega$ centered on the x_1 axis and then, by a dilation and translation argument, we refine the estimates and extend the result to every set of type $G(y,r)$. Once one has obtained the critical density property for every set $G(y,r) \subset \Omega$, Structure Theorems 4 and 2 give the property on balls $B(x,r) \subset \Omega$.

We start with the construction of a barrier on sets $\tilde{G}((y_1,0),2r)$.

Lemma 3.14 ([37], Lemma 4.2)**.** *There exist positive structural constants $\tilde{C} > 0$ and $M > 1$ such that for every $y = (y_1,0) \in \mathbb{R}^2$ and $r > 0$ there is a C^2 function $\tilde{\phi} : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\begin{aligned}
&\tilde{\phi} \geq 0 && \text{in } \mathbb{R}^2 \setminus \tilde{G}(y,2r),\\
&\tilde{\phi} \leq -2 && \text{in } \tilde{G}(y,r),\\
&\tilde{\phi} \geq -M && \text{in } \mathbb{R}^2,\\
&L\tilde{\phi}(x) \leq \tilde{C}\frac{x_1^2}{r^2(r+|y_1|)^2}\zeta(x) && \text{in } \mathbb{R}^2,
\end{aligned}$$

where $0 \leq \zeta \leq 1$ is a continuous function in $\mathbb{R}^2$ with $\operatorname{supp}\zeta \subset \overline{\tilde{G}(y,r)}$.

Proof. We just give a sketch of the proof. First of all it is considered the function $\rho^\alpha(x,y)$ where ρ is as in (3.6) with $\alpha \leq 2 - 3\frac{\Lambda}{\lambda}$, it is proved that ρ^α is a classical solution to $L(\rho^\alpha) \geq 0$ on the set $\{\rho > 0\}$. Then, the "radial" function $\varphi(x) = M_1 - M_2\rho^\alpha(x,y)$ is defined on $\mathbb{R}^2 \setminus \{\rho(x,y) = 0\}$ and the constants M_1 and M_2 are chosen by using an argument similar to the one given in Theorem 3.14. More precisely it is imposed

$$\varphi|_{\tilde{G}(y,2r)} = 0 \quad \text{and} \quad \varphi|_{\tilde{G}(y,r)} = -2 \tag{3.25}$$

and the value $-m = \varphi|_{\tilde{G}(y,r/2)}$ is considered. It is possible to show that M_1, M_2, m are positive; M_1 and m are uniformly bounded and M_2 is such that

$$\frac{K_1}{r^{\frac{\alpha}{2}}(r+|y_1|)^{\frac{\alpha}{2}}} \leq M_2 \leq \frac{K_2}{r^{\frac{\alpha}{2}}(r+|y_1|)^{\frac{\alpha}{2}}} \tag{3.26}$$

with K_1 and K_2 structural constants. Since the defining function of sets $\tilde{G}$ has two different definitions depending on whether $|y_1| < r$ or $|y_1| \geq r$, one has to impose (3.25) taking into account four different cases: $|y_1| \geq 2r$, $|y_1| < r/2$, $r/2 < |y_1| < r$, $r \leq |y_1| < 2r$.

Let $\beta \in \mathbb{N}$ and define

$$\psi(t) = \begin{cases} t, & \text{if } -m \leq t \\ -\int_{t+m}^{0} \frac{1}{1+s^{2\beta}} \, \mathrm{d}s - m, & \text{if } t > -m \end{cases}.$$

Since for $t < -m$ it holds

$$\psi'(t) = \frac{1}{1+(t+m)^{2\beta}} > 0, \quad \psi''(t) = \frac{2\beta(t+m)^{2\beta-1}}{(1+(t+m)^{2\beta})^2},$$

then $\psi \in C^2(\mathbb{R})$. The function

$$\tilde{\phi}(x) = \begin{cases} \psi(\tilde{\varphi}(x)), & \text{if } \rho(x,y) > 0, \\ -\int_{-\infty}^{0} \frac{1}{1+s^{2\beta}} \, \mathrm{d}s - m, & \text{if } \rho(x,y) = 0 \end{cases}$$

i.e.

$$\tilde{\phi}(x) = \begin{cases} \tilde{\varphi}(x), & \text{if } -m \leq \varphi(x) \text{ and } \rho(x,y) > 0, \\ -\int_{\varphi(x)+m}^{0} \frac{1}{1+s^{2\beta}} \, \mathrm{d}s - m, & \text{if } \varphi(x) < -m \text{ and } \rho(x,y) > 0 \\ -\int_{-\infty}^{0} \frac{1}{1+s^{2\beta}} \, \mathrm{d}s - m, & \text{if } \rho(x,y) = 0 \end{cases}$$

is constant on sets $\partial\tilde{G}(y,r)$ and increasing with respect to r, so that $\tilde{\phi} \geq 0$ in $\mathbb{R}^2 \setminus \tilde{G}(y,2r)$ and $\tilde{\phi} \leq -2$, in $\tilde{G}(y,r)$. Moreover, recalling that $L\varphi \leq 0$, in $\tilde{G}(y,r/2)$ it is possible to estimate $L\tilde{\phi}$ as follows

$$\begin{aligned} L\tilde{\phi} &= \psi''(\varphi)\left(a_{11}\varphi_{x_1}^2 + 2a_{12}x_1\varphi_{x_1}\varphi_{x_2} + x_1^2 a_{22}\varphi_{x_2}^2\right) + h'(\varphi)L\varphi \\ &\leq |\psi''(\varphi)|\Lambda\left((X_1\varphi)^2 + (X_2\varphi)^2\right) \\ &\leq |\psi''(\varphi)|\Lambda\left((-M_2\alpha x_1(x_1^2 - y_1^2)\rho^{\alpha-4})^2 + (-M_2\alpha x_2 x_1\rho^{\alpha-4})^2\right) \\ &\leq |\psi''(\varphi)|\Lambda M_2^2\alpha^2 x_1^2\rho^{2\alpha-8}\left((x_1^2 - y_1^2)^2 + 4x_2^2\right) = |\psi''(\varphi)|\Lambda M_2^2\alpha^2 x_1^2\rho^{2\alpha-4} \\ &\leq C_1\Lambda M_2^{1-2\beta} x_1^2\rho^{\alpha-4-2\alpha\beta}. \end{aligned}$$

Observe that $\rho < Cr^{\frac{1}{2}}(r+|y_1|)^{1/2}$ in $\tilde{G}(y,r/2)$ and chose $\beta \in \mathbb{N}$, $\beta > \max\{1, 1-4/\alpha\}$ so that $\alpha - 4 - 2\alpha\beta > 0$. Hence, recalling (3.26) , we get

$$L\tilde{\phi} \leq C_1\Lambda M_2^{1-2\beta} x_1^2\rho^{\alpha-4-2\alpha\beta} \leq \tilde{C}\frac{x_1^2}{r^2(r+|y_1|)^2}.$$

Eventually we recognize that since $L\rho^{\alpha} \geq 0$, in $\{\rho > 0\}$, we have $L\tilde{\phi} = L\varphi \leq 0$ in $\mathbb{R}^2 \setminus \tilde{G}(y,r/2)$ and $L\tilde{\phi} \leq C_2 \frac{x_1^2}{r^2(r+|y_1|^2)}$ in $\overline{\tilde{G}(y,r/2)}$. Now it suffices to choose a continuous function $0 \leq \zeta \leq 1$ such that $\zeta \equiv 1$ in $\overline{\tilde{G}(y,r/2)}$, $\zeta \equiv 0$ in $\mathbb{R}^2 \setminus \tilde{G}(y,2r/3)$, even in the first variable x_1 and satisfying $\tilde{\phi} \geq -M$ in $\mathbb{R}^2$, $L\tilde{\phi}(x) \leq \tilde{C}\frac{x_1^2}{r^2(r+|y_1|)^2}\zeta(x)$ in $\mathbb{R}^2$. □

Exploiting the barrier constructed in theorem above we prove some rough estimates for the solutions on balls centered on the x_1 axis.

Theorem 3.15. *Define $\tilde{\varepsilon} = (2C)^{-1}$ where $C > 0$ is the structural constant appearing in Theorem 3.4. Let $y = (y_1, 0)$, $r > 0$ and $u \in C^2(G(y,2r)) \cap C(\overline{G(y,2r)})$ be a non negative solution to $Lu \leq x_1^2 f$ in $G(y,2r)$ satisfying*

$$\inf_{G(y,r)} u \leq 1 \quad \textit{and} \quad \operatorname{diam}(G(y,2r))\|x_1 f\|_{L^2(G(y,2r))} < \tilde{\varepsilon}$$

Then there exist $0 < \nu < 1$, depending on $\tilde{\varepsilon}$ and $M > 1$ structural constant such that

$$|\{u \leq M\} \cap G(y,3r/2)| \geq \frac{\nu}{\max\{r+|y_1|, \frac{1}{r+|y_1|}\}} |G(y,3r/2)|. \tag{3.27}$$

Proof. The function $w := u + \tilde{\phi}$ with $\tilde{\phi}$ as in Lemma 3.14 satisfies $Lw \leq x_1^2\left(f + \zeta(x)\frac{\tilde{C}}{r^2(r+|y_1|)^2}\right)$ in $G(y,2r)$, $w \geq 0$ on $\partial G(y,2r)$, and

$$\inf_{G(y,r)} w \leq \inf_{G(y,r)} u - 2 \leq -1.$$

Thus it is straightforward to apply weighted ABP maximum principle Theorem 3.4 in $G(y,2r)$ and get

$$\begin{aligned} 1 &\leq \sup_{G(y,2r)} w^- \\ &\leq C\operatorname{diam}(G(y,2r))\left(\int_{\{w=\Gamma_w\}\cap G(y,2r)} \left(x_1 f + \zeta(x)\frac{\tilde{C}x_1}{r^2(r+|y_1|)^2}\right)^2 dx\right)^{1/2}. \end{aligned}$$

Let us set $U = \{w = \Gamma_w\} \cap G(y,2r)$. Taking into account (3.11) and (3.12) we further estimate the right hand side of inequality above by

$$\begin{aligned}
1 &\le C\mathrm{diam}(G(y,2r))\left(\|x_1 f\|_{L^2(G(y,2r))} + \frac{\tilde{C}}{r^2(r+|y_1|)^2}\left(\int_U \left(\zeta(x)x_1\right)^2 dx\right)^{\frac{1}{2}}\right) \\
&\le C\tilde{\varepsilon} + C\tilde{C}\frac{\mathrm{diam}(G(y,2r))}{r^2(r+|y_1|)^2}\left(\int_U \left(\zeta(x)x_1\right)^2 dx\right)^{\frac{1}{2}} \\
&\le C\tilde{\varepsilon} + C\tilde{C}C_M \max\{r, r(r+|y_1|)\}\frac{\tilde{c}(r+|y_1|)}{r^2(r+|y_1|)^2}\left(\int_U \zeta^2\, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover by Lemma 3.14, $0 \le \zeta \le 1$ and supp $\zeta \subset \overline{\tilde{G}(y,r)}$, so that keeping in mind (3.10), we have

$$\begin{aligned}
1 - C\tilde{\varepsilon} &\le K\frac{\max\{1, r+|y_1|\}}{r(r+|y_1|)}|\{w = \Gamma_w\} \cap G(y,2r) \cap \tilde{G}(y,r)|^{1/2} \\
&\le K\frac{\max\{1, r+|y_1|\}}{r(r+|y_1|)} r(r+|y_1|)^{\frac{1}{2}}\frac{|\{w = \Gamma_w\} \cap G(y,2r) \cap \tilde{G}(y,r)|^{\frac{1}{2}}}{|G(y,r)|^{\frac{1}{2}}} \\
&\le K\frac{\max\{1, r+|y_1|\}}{(r+|y_1|)^{\frac{1}{2}}}\frac{|\{w = \Gamma_w\} \cap G(y,2r) \cap \tilde{G}(y,r)|^{\frac{1}{2}}}{|G(y,r)|^{\frac{1}{2}}} \\
&\le K\max\{(r+|y_1|)^{-\frac{1}{2}}, (r+|y_1|)^{\frac{1}{2}}\}\frac{|\{u \le M\} \cap G(y,2r) \cap \tilde{G}(y,r)|^{\frac{1}{2}}}{|G(y,r)|^{\frac{1}{2}}}
\end{aligned}$$

where $K > 1$ is a structural constant. Since $w = \Gamma_w$ implies $w \le 0$ and consequently $u \le -\phi \le M$ we have

$$1 - C\tilde{\varepsilon} \le K\max\{(r+|y_1|)^{-\frac{1}{2}}, (r+|y_1|)^{\frac{1}{2}}\}\frac{|\{u \le M\} \cap G(y,2r) \cap \tilde{G}(y,r)|^{\frac{1}{2}}}{|G(y,r)|^{\frac{1}{2}}} \tag{3.28}$$

Now recalling definition of $\tilde{\varepsilon}$ and that by Structure Theorems 2 and 4 we have $|G(y,3r/2)| \le 9C_G^4/4|G(y,r)|$ we obtain

$$1 \le KC_G^4\frac{9}{2}\max\left\{(r+|y_1|)^{-\frac{1}{2}}, (r+|y_1|)^{\frac{1}{2}}\right\}\frac{|\{u \le M\} \cap G(y,2r) \cap \tilde{G}(y,3r/2)|^{\frac{1}{2}}}{|G(y,3r/2)|^{\frac{1}{2}}}. \tag{3.29}$$

At this stage we proceed as in [37, Theorem 5.1]. We need to consider three different cases. When $|y_1| \ge 2r$ the set $\tilde{G}(y,2r)$ is the disjoint union of the two set $\tilde{G}(y,2r)$ and $\tilde{G}(-y,2r)$ and $\tilde{G}(y,3r/2) \subset \tilde{G}(y,2r)$, while for $|y_1| < 3r/2$ we have $\tilde{G}(y,3r/2) = G(y,3r/2)$. In both cases we get $G(y_{2r}) \cap \tilde{G}(y,3r/2) = G(y,3r/2)$ and the thesis follows directly from the estimate above.

On the other hand if $3r/2 < |y_1| < 2r$ the set $\tilde{G}(y,3r/2)$ is the disjoint union of $G(-y,3r/2)$ and $G(y,3r/2)$ so estimate above is not enough to conclude. The idea is to get (3.28) for a smaller set so that conditions similar to the previous cases occur. We define $\rho = 2r/3$ and we proceed exactly as in estimate (3.28) but considering the smaller set $G(y,2\rho)$ instead of $G(y,2r)$. We remark that $u \in C^2(G(y,2\rho)) \cap C(\bar{(}G(y,2\rho)))$ is a non negative solution to $Lu \le x_1^2 f$ in $G(y,2\rho)$ and we can consider a function $\tilde{\tilde{\phi}}$ as in Lemma 3.14 so that $\tilde{w} = u + \tilde{\tilde{\phi}} \ge 0$ on $\partial G(y,2\rho)$. Let $\tau = \max\left\{ \left(\frac{3}{2}\rho + |y_1|\right)^{-\frac{1}{2}}, \left(\frac{3}{2}\rho + |y_1|\right)^{-\frac{1}{2}} \right\}$, then as in (3.28) we get

$$
\begin{aligned}
1 &\le 2K\tau \frac{|\{u \le M\} \cap G(y,2\rho) \cap \tilde{G}(y,3\rho/2)|^{-\frac{1}{2}}}{|G(y,3\rho/2)|^{1/2}} \\
&= 2K\tau \frac{|\{u \le M\} \cap G(y,3\rho/2)|^{1/2}}{|G(y,3\rho/2)|^{1/2}}
\end{aligned}
$$

where the last equality follows from the fact that $|y_1| > 3r/2 > 2\rho$. Relabeling $3\rho/2 = r$ we get

$$
\begin{aligned}
1 &\le 2K \max\{(r+|y_1|)^{-1/2}, (r+|y_1|)^{1/2}\} \frac{|\{u \le M\} \cap G(y,r)|^{1/2}}{|G(y,r2)|^{1/2}} \\
&\le KC_G^4 \frac{9}{2} \max\{(r+|y_1|)^{-1/2}, (r+|y_1|)^{1/2}\} \frac{|\{u \le M\} G(y,3r/2)|^{1/2}}{|G(y,3r/2)|^{1/2}}.
\end{aligned}
$$

from which we get the thesis. □

We use the same dilations and translations arguments performed in [37, Theorem 5.2], to extend Theorem 3.15 to every sublevel set $G(y,r)$ and improve constant in (3.27).

Theorem 3.16. *Define $\varepsilon_0 = \tilde{\varepsilon}\frac{C_m}{8C_M}$. Let $u \in C^2(G(y,2r)) \cap C(\overline{G(y,2r)})$ with $y \in \mathbb{R}^2$, $r > 0$ be a non negative solution to $Lu \le x_1^2 f$ in $G(y,2r)$ satisfying*

$$
\inf_{G(y,r)} u \le 1, \quad \textit{and} \quad \operatorname{diam}(G(y,2r)) \|x_1 f\|_{L^2(G(y,2r))} < \varepsilon_0
$$

then there exist structural constants $0 < \varepsilon < 1$, depending on ε_0 and $M > 1$ such that

$$
|\{u \le M\} \cap G(y,3r/2)| \ge \varepsilon |G(y,3r/2)|. \tag{3.30}
$$

Here C_m,C_M are the structural constants appearing in (3.12) and $\tilde{\varepsilon}$ is as in the statement of Lemma 3.15.

Proof. The proof is organized in four steps, at each step we prove the critical density property for a larger family of sets.

STEP I Fix $r=1$, $y_1\in[-1,1]$ and $y_2=0$. In this case we have $\max\{(r+|y_1|)^{1/2},(r+|y_1|)^{-1/2}\}<2$ and by applying Theorem 3.15, we find

$$|\{u\le M\}\cap G(y,3/2)|\ge\frac{\nu}{2}|G(y,3/2)|.$$

STEP II Fix $r\ge 0$, $|y_1|\le r$ and $y_2\in\mathbb{R}$. Keeping in mind (3.2) we introduce the change of variables

$$T(x)=T(x_1,x_2):=(rx_1,y_2+r^2x_2), \tag{3.31}$$

obtaining

$$X_i\tilde{u}(x)=r(X_iu)(T(x)),\quad \text{for } i=1,2;$$
$$X_2X_1\tilde{u}(x)=r^2(X_2X_1u)(T(x))$$

where $\tilde{u}(x):=u(T(x))$. Moreover $T(x)\in G(y,r)$ if and only if $x\in G((y_1/r,0),1)$. The new operator $\tilde{L}:=\tilde{a_{11}}X_1^2+2\tilde{a}_{12}X_2X_1+\tilde{a}_{22}X_2^2$, with $\tilde{a}_{i,j}(x):=a_{i,j}(T(x))$, $i\le j\in\{1,2\}$, is of the type (3.13) with ellipticity constants $\Lambda\ge\lambda>0$. We apply $\tilde{L}$ to $\tilde{u}$ obtaining

$$\tilde{L}\tilde{u}=r^2Lu(T(x))\le r^2(T(x))_1^2f(T(x))=r^4x_1^2f(T(x))=x_1^2\tilde{f}(x)$$

with $\tilde{f}(x):=r^4f(T(x))$. We claim that $\tilde{u}$ satisfies the hypotheses of Theorem 3.15 on $G((y_1/r,0),2)$. The only not obviously satisfied assumption is

$$\operatorname{diam}(G*)\|x_1\tilde{f}(x)\|_{L^2(G*)}<\tilde{\varepsilon} \tag{3.32}$$

where we have denoted $G*=G((y_1/r,0),2)$. Recalling (3.12) and changing coordinates ($x_1=\xi_1/r$, $x_2=(\xi_2-y_2)/r^2$) we have

$$\begin{aligned}\operatorname{diam}(G*)\|x_1\tilde{f}(x)\|_{L^2(G*)}&\le 4C_M\max\Big\{1,1+\frac{|y_1|}{r}\Big\}\Big(\int_{G*}(x_1\tilde{f}(x))^2\mathrm{d}x\Big)^{\frac12}\\&\le 4C_M\Big(1+\frac{|y_1|}{r}\Big)\Big(\int_{G(y,2r)}(r^3\xi_1f(\xi))^2\frac{1}{r^3}\mathrm{d}\xi\Big)^{\frac12}\\&\le 4C_M(r+|y_1|)r^{1/2}\|\xi_1f(\xi)\|_{L^2(G(y,2r))}\end{aligned} \tag{3.33}$$

on the other hand by hypothesis and (3.12)

$$\begin{aligned}\tilde{\varepsilon}\frac{C_m}{8C_M} &> \operatorname{diam}(G(y,2r))\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\\ &\geq C_m \max\{r, r(r+|y_1|)\}\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\end{aligned}$$

so that

$$\tilde{\varepsilon} \geq 8C_M r \max\{1, r+|y_1|\}\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}. \tag{3.34}$$

Now, if $r \geq 1$, it is straightforward to concatenate inequalities (3.33) and (3.34) to obtain (3.32). Otherwise if $r \leq 1$ and $|y_1| \leq r$, again combining (3.33) and (3.34) we find

$$\begin{aligned}\operatorname{diam}(G*)\|x_1\tilde{f}(x)\|_{L^2(G*)} &\overset{(3.33)}{\leq} 4C_M(r+|y_1|)r^{1/2}\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\\ &\leq 8C_M r^{3/2}\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\\ &\leq 8C_M r\max\{1, r+|y_1|\}\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\\ &\overset{(3.34)}{\leq} \tilde{\varepsilon}.\end{aligned}$$

Hence $\tilde{u}$ satisfies hypotheses of Theorem 3.15 with $y_1/r \in [-1,1]$, so that, by STEP I we get

$$|\{\tilde{u} \leq M\} \cap G((y_1/r,0),3/2)| \geq \frac{\nu}{2}|G((y_1/r,0),3/2)|$$

and consequently

$$\begin{aligned}|\{u \leq M\} \cap G(y,3r/2)| &= |T\left(\{\tilde{u} \leq M\} \cap G((y_1/r,0),3/2)\right)|\\ &= r^3|\{\tilde{u} \leq M\} \cap G((y_1/r,0),3/2)|\\ &\geq r^3\frac{\nu}{2}|G((y_1/r,0),3/2)|\\ &= \frac{\nu}{2}|G(y,3r/2)|.\end{aligned}$$

STEP III Fix $r > 0$, $|y_1| = 1$ and $y_2 = 0$. If $r \geq 1$ we apply STEP II, otherwise, in case $0 < r < 1$ we apply Theorem 3.15 and take into account that $\max\left\{r+|y_1|, \frac{1}{r+|y_1|}\right\} < 2$, so that

$$|\{u \leq M\} \cap G((y_1,0),3/2r)| > \frac{\nu}{2}|G((y_1,0),3/2r)|.$$

STEP IV Fix $r>0$, $y_1, y_2 \in \mathbb{R}$. If $|y_1| \le r$ we use STEP II. If $|y_1| > r$ apply the change of variable defined in the second step with r replaced by $|y_1|$ in (3.31). Again we want to make use of Theorem 3.15 and again the only need to check that

$$\operatorname{diam}(G^*)\|x_1\tilde{f}(x)\|_{L^2(G^*)} < \tilde{\varepsilon}, \tag{3.35}$$

where $\tilde{f}(x) := |y_1|^4 f(T(x))$ and $G^* = G((y_1/|y_1|,0), 2r/|y_1|)$.
Recalling (3.12) and changing coordinates ($x_1 = \xi_1/|y_1|$, $x_2 = (\xi_2 - y_2)/|y_1|^2$) we have

$$\begin{aligned}\operatorname{diam}(G^*)\|x_1\tilde{f}(x)\|_{L^2(G^*)} &\le 4C_M \max\left\{\frac{r}{|y_1|}, \frac{r}{|y_1|} + \frac{r^2}{|y_1|^2}\right\}\left(\int_{G^*}(x_1\tilde{f}(x))^2 \mathrm{d}x\right)^{1/2} \\ &\le \frac{4C_M r}{|y_1|^{1/2}} \max\{|y_1|, |y_1|+r\}\|\xi_1 f(\xi)\|_{L^2(G(y,2r))} \\ &\le \frac{4C_M r}{|y_1|^{1/2}}(|y_1|+r)\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\end{aligned} \tag{3.36}$$

If $|y_1| \ge 1$, it is straightforward to concatenate (3.36) and (3.34) obtaining (3.35). Otherwise if $0 < r < |y_1| < 1$, we estimate the right hand side of (3.36) as follows

$$\begin{aligned}\operatorname{diam}(G^*)\|x_1\tilde{f}(x)\|_{L^2(G^*)} &\le \frac{4C_M}{|y_1|^{1/2}} r(|y_1|+r)\|\xi_1 f(\xi)\|_{L^2(G(y,2r))} \\ &\le \frac{8C_M}{|y_1|^{1/2}} r|y_1|\|\xi_1 f(\xi)\|_{L^2(G(y,2r))}\end{aligned} \tag{3.37}$$

and then concatenate (3.37) with (3.34) and get the desired estimate.
Hence $\tilde{u}$ satisfies hypotheses of Theorem 3.15 and using the third step we find

$$\begin{aligned}|\{u \le M\} \cap G(y, 3r/2)| &= |T\left(\{\tilde{u} \le M\} \cap G((y_1/|y_1|,0), 3r/(2|y_1|))\right)| \\ &= |y_1|^3 |\{\tilde{u} \le M\} \cap G((y_1/|y_1|,0), 3r/(2|y_1|))| \\ &\ge \frac{\nu}{2}|y_1|^3 |G((y_1/|y_1|,0), 3r/(2|y_1|))| \\ &\ge \frac{\nu}{2}|G(y, 3r/2)|.\end{aligned}$$

Hence we have proved (3.30) with $\varepsilon = \frac{\nu}{2}$ and ν as in Theorem 3.15 □

Now we extend theorem above to quasi metric balls $B(y,r)$.

Theorem 3.17. *Define* ε_0 *as in Theorem 3.16. Let* $u \in C^2(B(y,2R)) \cap C(\overline{B(y,2R)})$ *with* $y \in \mathbb{R}^2$, $R > 0$ *be a non negative solution to* $Lu \leq x_1^2 f$ *in* $B(y,2R)$ *satisfying*

$$\inf_{B(y,R/2C^2)} u \leq 1, \quad \textit{and} \quad \operatorname{diam}(B(y,2R))\|x_1 f\|_{L^2(B(y,2R))} < \varepsilon_0.$$

Then there exist structural constants $0 < \nu < 1$*, depending on* ε_0 *and* $M > 1$ *such that*

$$|\{u \leq M\} \cap B(y,R)| \geq \nu |B(y,R)|.$$

Here ε, ε_0 *are defined in statement of Theorem 3.16;* $\nu = \varepsilon C^{-4}$ *with* $C = C_B C_G$ *with* C_G *and* C_B *the constants in Structure Theorem 4 and 2 respectively.*

Proof. First of all we define $r := 2R/3$, so $R > r$. Since inclusions $B(y,R/2C^2) \subset G(y,r/C)$ and $G(y,2r/C) \subset B(y,2R)$ imply respectively

$$\inf_{G(y,r/C)} u \leq \inf_{B(y,R/2C^2)} u \leq 1 \text{ and } \operatorname{diam}(G(y,2r/C))\|x_1 f\|L^2(G(y,2r/C)) < \varepsilon_0,$$

by Theorem 3.16 there exist structural constants $0 < \varepsilon < 1$ and $M > 1$ such that

$$|\{u \leq M\} \cap G(y,3r/(2C))| \geq \varepsilon |G(y,3r/(2C))|.$$

With the aid of Structure Theorems 3 and 2 we estimate from below the right hand side by

$$\varepsilon|G(y,3r/(2C))| \geq \varepsilon|\operatorname{Box}(x,RC_B/C^2)| \geq \varepsilon C^{-4}|\operatorname{Box}(x,RC_B)| \geq \varepsilon C^{-4}|B(y,R)|.$$

concluding the proof. □

Provided that we invert the relation of dependence between ν and ε_0, the negation of theorem above and the double ball property (Theorem 3.13) give the following critical density property

Theorem 3.18. *There exist structural constants* $\eta_{CD}, M > 1$ *and* $\nu, c, \varepsilon_{CD} \in]0,1[$ *such that if,* $u \in C^2(B(y,\eta_{CD}R)) \cap C(\overline{B(y,\eta_{CD}R)})$ *with* $y \in \mathbb{R}^2$, $R > 0$ *is a non negative solution to* $Lu \leq x_1^2 f$ *in* $B(y,\eta_{CD}R)$ *satisfying*

$$|\{u > M\} \cap B(y,R)| > (1-\nu)|B(y,R)|$$

then

$$\inf_{B(y,R)} u > c \quad \textit{or} \quad \operatorname{diam}(B(y,\eta_{CD}R))\|x_1 f\|_{L^2(B(y,\eta_{CD}R))} \geq \varepsilon_{CD}.$$

More precisely, $\eta_{CD} = 2\eta_{DB}$, $\varepsilon_{CD} = \min\{\gamma^p \varepsilon_{DB}, \varepsilon_0\}$, $c = \gamma^{p+1}$ *where* γ, $\varepsilon_{DB}, \eta_{DB}$ *are the constants defined in Theorem 3.13,* M, ν, ε_0 *are as in Theorem 3.17, and* $p \in \mathbb{N}$ *is chosen so that* $2^p > C^2 > 2^{p-1}$ *with* $C = C_G C_B$, C_G *and* C_B *are the constants in Structure Theorems 4 and 2 respectively.*

Proof. The negation of Theorem 3.17 says that if

$$|\{u \le M\} \cap B(y,R)| < \nu |B(y,R)|$$

i.e.

$$|\{u > M\} \cap B(y,R)| > (1-\nu)|B(y,R)|$$

then

$$\inf_{B(y,R/2C^2)} u > 1 \quad \text{or} \quad \operatorname{diam}(B(y,2R))\|x_1 f\|_{L^2(B(y,2R))} \ge \varepsilon_0.$$

If the second inequality holds or $\operatorname{diam}(B(y,\eta_{CD}R))\|x_1 f\|_{L^2(B(y,\eta_{CD}R))} \ge \varepsilon_{CD}$ there is nothing to prove. Otherwise since by the definition of p, $2^p/C^2 < 2$, for every $0 \le k \le p$ we have

$$\operatorname{diam}(B(y,R2^k\eta_{DB}/(2C^2))\|x_1 f\|_{L^2\left(B\left(y,R\frac{2^k\eta_{DB}}{2C^2}\right)\right)} \le \operatorname{diam}(B(y,\eta_{CD}R))\|x_1 f\|_{L^2(B(y,\eta_{CD}R))} < \varepsilon_{CD}$$

we can repeatedly apply the double ball property $p+1$ times (for $k = 0,\ldots,p$) in $B(y,R/2C^2)$ obtaining

$$\inf_{B(y,R)} u \ge \inf_{B(y,2^pR/C^2)} u \ge \gamma^{p+1}.$$

□

3.4 Harnack inequality

In this section we use the abstract approach of Chapter 1 to prove Harnack inequality for non negative classical solution to $Lu = x_1^2 f$ in quasi metric balls $B(x,r)$; then, Structure Theorem 1 and 2 imply the desired result on Carnot–Carathéodory metric balls $B_{CC}(x,r)$.

We recall that $\Omega \subset \mathbb{R}^2$ is open and the Hölder quasi metric space $(\Omega, \tilde{d}, |\cdot|)$ satisfies the doubling and the reverse doubling property (see (3.4) and Remark 3.2) and the log-ring condition ([37, Theorem 3.4]). At the end of subsection 3.1.2 we have observed that, due to the linearity of L and the absence of zero order terms in L, the family

$$\mathbb{K}_{\Omega,f} := \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : \; Lu = x_1^2 f, \; u \ge 0\}$$

with $f \in \mathcal{L}(\Omega) = \{g : \Omega \to \mathbb{R}$, measurable and $x_1 g \in L^2_{\text{loc}}(\Omega)\}$, satisfies

- $u \in \mathbb{K}_{\Omega,f} \Rightarrow \lambda u \in \mathbb{K}_{\Omega,\lambda f}$ for all $\lambda \geq 0$.
- $u \in \mathbb{K}_{\Omega,f}$, $\lambda, \tau \geq 0$ such that $\tau - \lambda u \geq 0 \Rightarrow \tau - \lambda u \in \mathbb{K}_{\Omega,-\lambda f}$.

It is also clear that the function $\mathcal{S}_\Omega(B(x,r), f) := \text{diam}(B(x,r))\|x_1 f\|_{L^2(B(x,r))}$ is increasing with respect to the first variable (here we think the family of all the quasi metric balls contained in Ω partially ordered by inclusion) and homogeneous in the second variable ($\mathcal{S}_\Omega(B(x,r), \lambda f) = |\lambda|\mathcal{S}_\Omega(B(x,r), f)$). Moreover the family $\mathbb{K}_{\Omega,f}$ satisfies the double ball and the critical density properties (Theorems 3.13 and 3.18). Thus we use Theorem 1.10 under hypotheses (A1)-(A2) and Theorem 1.11 obtaining the following invariant Harnack inequality on quasi metric balls B.

Theorem 3.19. *There exist structural constants $C, \eta > 1$ such that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a non negative solution to $Lu = x_1^2 f$ in Ω, then*

$$\sup_{B(y,r)} u \leq C \left(\inf_{B(y,r)} u + \text{diam}(B(y,\eta r))\|x_1 f\|_{L^2(B(y,\eta r))} \right)$$

for every $B(y,\eta r) \subset \Omega$.

Now, Structure Theorems 1 and 2 ensures the inclusions $B_{CC}(x,r) \subset B(x,Cr)$ and $B(x,C\eta r) \subset B_{CC}(x,C^2\eta r)$ with $C = C_B\tilde{C}$, so, if the ball $B_{CC}(x,C^2\eta r)$ is included in Ω, the invariant Harnack inequality for Carnot– Carathéodory metric ball $B_{CC}(x,r)$ follows by applying theorem above to quasi metric balls $B(x,Cr)$.

After the paper [23], which contains the proof of Theorem 3.19, had been posted on arXiv:1709.03810, we learned that D. Maldonado in [36] extended the example in this chapter to a larger class of PDEs. He studies certain degenerate elliptic PDEs modeled after suitable strictly convex function φ, of the type

$$\text{trace}(A(x)D^2u(x)) + \langle b, D^2\varphi^{-1/2}\nabla u\rangle + c(x)u(x) = f(x) \tag{3.38}$$

where $A(x)$ is a symmetric matrix such that for every $\xi \in \mathbb{R}^n$

$$\lambda\langle D^2\varphi^{-1}\xi,\xi\rangle \leq \langle A\xi,\xi\rangle \leq \Lambda\langle D^2\varphi^{-1}\xi,\xi\rangle, \quad \text{with } 0 < \lambda \leq \Lambda < +\infty \tag{3.39}$$

and φ is a strictly convex functions whose Monge-Ampère sections (see for example [24, Definition 3.1.1] for the definition of section) are quasi-metric balls and whose Monge-Ampère measure is doubling and satisfies the ring condition. Under some assumptions on φ and on the lower order terms coefficients b and c, the author obtains an invariant Harnack inequality for continuous $W^{2,n}_{\text{loc}}$ solutions to (3.38). Then, as an example he considers the class of PDEs defined by

$$x_1^2 \operatorname{trace}(A_\nu(x)D^2u(x)) + \langle \bar{b}, D^2\varphi^{-1/2}\nabla u\rangle + \bar{c}(x)u(x) = \bar{f}(x) \tag{3.40}$$

with A_ν and φ satisfying (3.39) in a open bounded set $\Omega \subset \mathbb{R}^2$, where

$$A_\nu = \begin{pmatrix} \frac{a_{11}}{x_1^{2\nu}} & \frac{a_{12}}{x_1^{\nu}} \\ \frac{a_{12}}{x_1^{\nu}} & a_{22} \end{pmatrix}, \qquad \varphi = \frac{x_1^{2\nu}}{(2\nu-2)(2\nu-1)} + \frac{x_2^2}{2}$$

and

$$\|\overline{f}x_1^{-\nu}\|_{L^2(S)}, \quad \|\overline{c}x_1^{-\nu}\|_{L^2(S)}, \quad \|\overline{b}x_1^{-\nu}\|_{L^2(S)} < \infty \tag{3.41}$$

for every Monge–Ampère section $S \subset \Omega$. This operators are more general then the ones studied in this Chapter where we have $\nu = 1$, $b = c = 0$. Notice that in case $\nu = 1$ it suffices to rename $\overline{f} = x_1^2 f$ to see that hypothesis (3.41) is equivalent to require $\|x_1 f\|_{L^2(S)} < \infty$ which is indeed our assumption. Maldonado's approach to Harnack inequality for solutions to (3.38) deeply relay on the ABP Maximum principle [22, Theorem Section 9.1 and Exercise 9.3.] and [10, Chapter 6], and with this tool the author proves the critical density, the double ball, the power decay properties and the Weak Harnack inequality on the quasi metric balls defined by the Monge-Ampére sections of φ. By taking into account the deep results in [36] it is easy to recognize that, in absence of zero order terms, the abstract approach presented in Chapter 1 also applies to solutions to the class of PDEs defined by (3.38). On the other hand, there are many subelliptic PDEs which are not contained in Maldonado's class and which satisfies our hypotheses. The easiest example is the Kohn Laplacian on the Heisenberg group $\mathbb{H}^1$, which is a X-elliptic operator in divergence form and which writes as

$$Lu(\xi) = \operatorname{trace}\left(A(\xi)D^2u(\xi)\right) = 0,$$

where D^2u is the Euclidean Hessian of u and

$$A(\xi) = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4(x^2+y^2) \end{pmatrix}$$

for all $\xi = (x,y,t) \in \mathbb{H}^1$. Obviously, the matrix $A(\xi)$ has minimum eigenvalue identically zero for all $\xi = (x,y,t) \in \mathbb{H}^1$ and therefore, as a quadratic form, it can not be estimated a.e. from below by the inverse of the Hessian matrix of a strictly convex function. More complicated examples of nondivergence subelliptic PDEs satisfying our conditions and which do not satisfy Maldonado's hypotheses could be find in [27, 41, 1].

3.5 A priori Hölder estimates for the X-gradient

It is well known that invariant Harnack inequalities give Hölder estimates for the solutions of PDEs (see Section 1.4). However, to study regularity properties of fully nonlinear PDEs one has to differentiate the equation and try to apply the theory to the equation solved by the derivatives of the solution.

In this section we prove local Hölder estimates for each component of the X-gradient of a solution to the equation $Lu = x_1 f$. The idea is to prove regularity for each component of the X-gradient by showing that it itself is a solution to an appropriate equation in divergence form. More precisely, $\partial_{x_2} u$ is a solution to the equation constructed by formal derivation of $Lu - x_1 f = 0$ with respect to ∂_{x_2}, and the regularity for $X_2 u$ is deduced from the regularity of $\partial_{x_2} u$. Analogously, we show that the first component of the X-gradient is a solution to the equation obtain by formal derivation of $Lu - x_1 f = 0$ with respect to X_1. The new equations we obtained by formal derivation, result to be X-elliptic so we refer to Chapter 2 to deduce Hö lder regularity. We remark that this procedure is a modification to the one presented in [22, Section 12.2] and deeply relay on the fact that we are working with an equation that depends only on two variables, in our case there is an added difficulty due to the non commutativeness of the vector fields X_2 and X_1. More precisely we will prove the following

Theorem 3.20. *Let Ω be a subset of $\mathbb{R}^2$ with small enough diameter (cf. **(O)**) $u \in C^3(\Omega)$ be a solution to $Lu = x_1 f$ with L the operator defined in (3.13). Moreover assume $f \in L^{2p}(\Omega)$ with $p > \frac{3}{2}$. Then there exist two positive structural constants C and $\alpha \in]0,1[$ such that for every $\overline{B_{CC}(x_0, 5r)} \subset \Omega$ it holds*

$$\sup_{x,y \in B_{CC}(x_0,r)} \frac{|\partial_{x_2} u(x) - \partial_{x_2} u(y)|}{d_X(x,y)^\alpha} \leq C \left((4r)^{-\alpha} \sup_{B_{CC}(x_0,5r)} |\partial_{x_2} u| + \left\| \frac{f}{a_{11}} \right\|_{L^{2p}(\Omega)} \right) \tag{3.42}$$

Moreover for every $\overline{B_{CC}(z,5\rho)} \subset B$, here $B = B_{CC}(x_0, r/2)$ we have

$$\sup_{x,y\in B_{CC}(z,\rho)} \frac{|X_1u(x) - X_1u(y)|}{d_X(x,y)^\alpha} \leq C\left(\frac{1}{4\rho^{-\alpha}} \sup_{B_{CC}(z,5\rho)} |X_1u| + \left\|\left|x_1\frac{f}{a_{11}}\right| + 2|\partial_{x_2}u|\right\|_{L^{2p}(B))}\right) \tag{3.43}$$

$$\sup_{x,y\in B_{CC}(z,\rho)} \frac{|X_2u(x) - X_2u(y)|}{d_X(x,y)^\alpha} \leq C\left(\frac{1}{4\rho^{-\alpha}} \sup_{B_{CC}(z,5\rho)} |X_2u| + \left\|\left|x_1\frac{f}{a_{11}}\right| + 2\left|\frac{a_{12}}{a_{22}}\partial_{x_2}u\right| + 2|\partial_{x_2}u|\right\|_{L^{2p}(B)}\right) \tag{3.44}$$

First of all we check that the setting is suitable for the purpose of applying results of Chapter 2. Let us consider the metric space $(\Omega, d_X, |\cdot|)$ where d_X is the control distance associated to X_1, X_2. Since $\{X_1, X_2\}$ satisfies the Hörmander condition, assumption **(C)** is satisfied and we have already proved that the doubling property holds true in $(\Omega, d_X, |\cdot|)$ (recall the Structure Theorem 1), so assumption **(D)** is satisfied too. Moreover D. Jerison in [31], proved the following Poincaré inequality on balls defined by the control distance associated with vector fields satisfying Hörmander condition and with small enough radius:

$$\int_{B_{CC}(x_0,r)} |u|^p \, \mathrm{d}x \leq Cr^p \int_{B_{CC}(x_0,\alpha r)} |Xu|^p \, \mathrm{d}x \tag{3.45}$$

with $u \in C^1(\overline{B_{CC}(x_0, \alpha r)})$, $\alpha > 1$, $1 \leq p < +\infty$. We recall that a Poincaré inequality for Grushin vector fields has been proved by Franchi and Lanconelli in [16]. Hence also **(P)** holds true and we are in position to apply the results of Chapter 2.

Proof of Theorem 3.20. We prove (3.42) by showing that $w = \partial_{x_2}u$ is a solution to an equation of the type (2.5). Let u be a C^3 solution to $Lu = x_1 f$ in Ω where L is the operator defined in (3.13). Since by (3.14) we know $a_{11} > 0$ we can consider the equation

$$X_1^2u + 2\frac{a_{12}}{a_{11}}X_2X_1u + \frac{a_{22}}{a_{11}}X_2^2u - x_1\frac{f}{a_{11}} = 0.$$

We formally differentiate equation above with respect to ∂_{x_2} and get

$$X_1^2w + \partial_{x_2}\left(2\frac{a_{12}}{a_{11}}x_1X_1w + \frac{a_{22}}{a_{11}}x_1X_2w - x_1\frac{f}{a_{11}}\right) = 0.$$

which is an equation in divergence form:

$$X_1(X_1w) + X_2\left(2\frac{a_{12}}{a_{11}}X_1w + \frac{a_{22}}{a_{11}}X_2w\right) = X_2\left(\frac{f}{a_{11}}\right). \tag{3.46}$$

It is not difficult to recognize that the operator is of X-elliptic type (Definition 2.2.1). Indeed if we let

$$B = \begin{pmatrix} 1 & 0 \\ 2x_1 \frac{a_{12}}{a_{11}} & x_1^2 \frac{a_{22}}{a_{11}} \end{pmatrix} \quad \text{and} \quad b_{ij} = (B)_{ij}, \quad b_i = d_i = c = 0$$

and we define

$$\mathcal{L}u = \sum_{i,j=1}^{2} \partial_{x_i} \left(b_{ij} \partial_{x_j} u \right) \tag{3.47}$$

equation (3.46) reads as

$$\mathcal{L}w = \partial_{x_2} h_2$$

where h is the vector $h = (h_1, h_2) = \left(0, x_1 \frac{f}{a_{11}}\right)$. Recalling (3.14), for every $\xi \in \mathbb{R}^2$ we have

$$\begin{aligned}\langle B\xi, \xi \rangle &\leq \frac{1}{\lambda} \left(a_{11}\xi_1^2 + 2a_{12}\xi_1 (x_1\xi_2) + a_{22}(x_1\xi_2)^2 \right) \leq \frac{\Lambda}{\lambda} \left(\xi_1^2 + (x_1\xi_2)^2 \right) \\ &= \frac{\Lambda}{\lambda} \left(\langle X_1, \xi \rangle^2 + \langle X_2, \xi \rangle^2 \right)\end{aligned}$$

and analogously

$$\langle B\xi, \xi \rangle \geq \frac{\lambda}{\Lambda} \left(\langle X_1, \xi \rangle^2 + \langle X_2, \xi \rangle^2 \right),$$

moreover

$$\langle h, \xi \rangle^2 = \left(\frac{f}{a_{11}} \right)^2 \langle X_2, \xi \rangle^2.$$

Since by hypotheses $f \in L^{2p}(\Omega)$ and from (3.14) we deduce $\lambda \leq a_{11} \leq \Lambda$, we have $\frac{f}{a_{11}} \in L^{2p}(\Omega)$. Then assumptions **(R)** and **(LT)** are satisfied (it suffices to take $\gamma_0 = |f/a_{11}|$ in (2.6)) so, by Corollary 2.15 we get (3.42). Consequently we can estimate $\|\partial_{x_2} u\|_{L^{2p}(B)}$, hereafter $B = B_{CC}(x_0, r/2)$. This fact will be used in the sequel.
In order to prove (3.44) it will be convenient to use the notation

$$u_1 = X_1 u \quad \text{and} \quad u_2 = X_2 u.$$

We notice that the function u_2 satisfies

$$\mathcal{L}u_2 = \partial_{x_1} \bar{h}_1 + \partial_{x_2} \bar{h}_2 \quad \text{in } B$$

with $\mathcal{L}$ the X-elliptic operator defined in (3.47) and $\bar{h} = (\bar{h}_1, \bar{h}_2) = \left(2w, x_1^2 \frac{f}{a_{11}} + 2x_1 w \frac{a_{12}}{a_{22}}\right)$. Moreover it holds

$$\begin{aligned} \langle \bar{h}, \xi \rangle^2 &\leq 2\,(2w)^2 \langle X_1, \xi \rangle^2 + 2\left(x_1 \frac{f}{a_{11}} + 2w \frac{a_{12}}{a_{22}}\right)^2 \langle X_2, \xi \rangle^2 \\ &\leq 2\left(2|w| + \left|x_1 \frac{f}{a_{11}}\right| + 2\left|w \frac{a_{12}}{a_{22}}\right|\right)^2 \left(\langle X_1, \xi \rangle^2 + \langle X_2, \xi \rangle^2\right). \end{aligned}$$

Again, since by hypotheses $f \in L^{2p}(\Omega)$ and we have just proved that $w = \partial_{x_2} u \in L^{2p}(B)$, we have $\gamma_0 = \left(2|w| + \left|x_1 \frac{f}{a_{11}}\right| + 2\left|w \frac{a_{12}}{a_{22}}\right|\right) \in L^{2p}(B)$, so **(R)** and **(LT)** are satisfied and Corollary 2.15 can be applied to u_2. This gives (3.44).

To get estimates for $u_1 = X_1 u$ we proceed in a similar way. We consider the equation

$$\frac{a_{11}}{a_{22}} X_1^2 u + 2\frac{a_{12}}{a_{22}} X_2 X_1 u + X_2^2 u - x_1 \frac{f}{a_{22}} = 0$$

which is well defined since by (3.14) we know that $a_{22} > 0$, and we rewrite it as follows

$$\frac{a_{11}}{a_{22}} X_1 u_1 + 2\frac{a_{12}}{a_{22}} X_2 u_1 + X_2 u_2 - x_1 \frac{f}{a_{22}} = 0.$$

Formally differentiating with respect to X_1 we find

$$X_1 \left(\frac{a_{11}}{a_{22}} X_1 u_1 + 2\frac{a_{12}}{a_{22}} X_2 u_1 - x_1 \frac{f}{a_{22}}\right) + X_1 X_2 u_2 = 0. \tag{3.48}$$

Now we notice that

$$X_1 X_2 u_2 = X_1 (x_1^2 \partial_{x_2}^2 u) = 2x_1 \partial_{x_2}^2 u + x_1^2 X_1 \partial_{x_2}^2 u = 2X_2 w + X_2^2 u_1.$$

Hence equation (3.48) is in divergence form

$$X_1 \left(\frac{a_{11}}{a_{22}} X_1 u_1 + 2\frac{a_{12}}{a_{22}} X_2 u_1\right) + X_2 (X_2 u_1) = X_1 \left(x_1 \frac{f}{a_{22}}\right) - 2X_2 w. \tag{3.49}$$

Again, if we let

$$\tilde{B} = \begin{pmatrix} \frac{a_{11}}{a_{22}} & 2x_1 \frac{a_{12}}{a_{22}} \\ 0 & x_1^2 \end{pmatrix}, \quad \tilde{b}_{ij} = (\tilde{B})_{ij} \quad \text{and} \quad \tilde{h} = (\tilde{h}_1, \tilde{h}_2) = \left(x_1 \frac{f}{a_{11}}, -2x_1 w\right)$$

equation (3.49) reads as

$$\sum_{1,j=1}^{2} \partial_{x_i}\left(\tilde{b}_{ij}\partial_{x_j}u_1\right) = \partial_{x_1}\tilde{h}_1 + \partial_{x_2}\tilde{h}_2$$

and, for every $\xi \in \mathbb{R}^2$ we have $\frac{\Lambda}{\lambda}\left(\xi_1^2 + (x_1\xi_2)^2\right) = \frac{\Lambda}{\lambda}\left(\langle X_1,\xi\rangle^2 + \langle X_2,\xi\rangle^2\right)$ so that, recalling (3.14) we find

$$\frac{\lambda}{\Lambda}\left(\langle X_1,\xi\rangle^2 + \langle X_2,\xi\rangle^2\right) \leq \langle B\xi,\xi\rangle \leq \frac{\Lambda}{\lambda}\left(\langle X_1,\xi\rangle^2 + \langle X_2,\xi\rangle^2\right).$$

Moreover it holds

$$\begin{aligned}\langle\tilde{h},\xi\rangle^2 &\leq 2\left(x_1\frac{f}{a_{11}}\right)^2\langle X_1,\xi\rangle^2 + 2\,(2w)^2\,\langle X_2,\xi\rangle^2\\ &\leq 2\left(\left|x_1\frac{f}{a_{11}}\right| + 2|w|\right)^2\left(\langle X_1,\xi\rangle^2 + \langle X_2,\xi\rangle^2\right)\end{aligned}$$

and the function $\gamma_0 = \left|x_1\frac{f}{a_{11}}\right| + 2|w|$ belongs to $L^{2p}(B)$. Now Corollary 2.15 gives Hölder estimate for X_1u concluding the proof. □

The prescribed Levi curvature equation in cylindrical coordinates

Here we want to show that Grushin type equations, considered in Chapter 3, arises from the prescribed Levi curvature equation in cylindrical coordinates. This fact was first recognized by Gutiérrez Lanconelli and Montanari and motivated the investigation carried out in [37]. Since the scope of this appendix is just to give a motivation to the study of Grushin type operators we do not aim to make it self complete and we refer the reader to [38] for precise definitions and further details.

Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded domain with boundary $\partial\Omega$. We consider $f \in C^2(\mathbb{C}^{n+1})$ a defining function for $\Omega = \{\zeta = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : f(\zeta) < 0\}$ such that $\partial f|_{\partial\Omega} \neq 0$. Here we have used the notations

$$f_i = \frac{\partial f}{\partial z_i}, \qquad f_{\bar{i}} = \frac{\partial f}{\partial \bar{z}_i}, \qquad \partial f = (f_1, \dots, f_{n+1}).$$

Analogous notations will be used for second order derivatives. Then, the complex Hessian matrix of the function f at a point p is

$$H_p(f) = \Big(f_{j,\bar{k}}(p)\Big)_{j,k=1,\dots,n}$$

and the complex tangent space to $\partial\Omega$ at a point p is

$$T_p^{\mathbb{C}}(\partial\Omega) = \{h \in \mathbb{C}^{n+1} : \langle h, \overline{\partial}_p f \rangle = 0\}$$

where we have denoted the standard Hermitian product in $\mathbb{C}^{n+1}$ by $\langle \cdot, \cdot \rangle$ and $\bar{\partial}_p f = \big(f_{\bar{1}}, \dots, f_{\overline{n+1}}\big)$. The Levi form of the function f at a point p is the restriction of the Hermitian form

$$\zeta \mapsto L_p(f, \zeta) = \langle H_p(f)\zeta, \zeta \rangle$$

to the complex tangent space $T_p^{\mathbb{C}}(\partial\Omega)$. We recall that a domain Ω is said to be strictly Levi pseudoconvex if the Levi form is strictly positive definite at each point of $\partial\Omega$. We also need to introduce the notion of B-normalized Levi matrix, that is the $n \times n$ Hermitian matrix

$$L_p(f,B) = \left(\frac{1}{|\partial_p f|} \langle H_p^T(f) u_j, u_k \rangle \right)_{j,k=1,\dots,n}$$

with $B = \{u_1, \dots, u_n\}$ an orthonormal basis for $T_p^{\mathbb{C}}(\partial\Omega)$. We remind that the eigenvalues of $L_p(f,B)$ only depend on the domain Ω. Then the total Levi curvature is

$$K_p^{(n)}(\partial\Omega) = \det(L_p(f,B)).$$

In the work [38, Section 2] the authors proved that the eigenvalues of the normalized Levi form for the boundary $\partial\Omega$ are the eigenvalues of the matrix

$$\left(I_n - \frac{\bar{\alpha} \otimes \alpha}{1 + |\alpha|^2} \right) A(f) \tag{A.1}$$

where

$$(\bar{\alpha} \otimes \alpha)_{jk} = \bar{\alpha}_j \alpha_k, \quad \alpha_j := \frac{f_j}{f_{n+1}}, \quad j,k = 1,\dots,n \tag{A.2}$$

and

$$A(f) = \left(A_{j,\bar{k}}(f) \right)_{j,k=1,\dots,n}$$

with

$$A_{j,\bar{k}}(f) = -\frac{1}{|f_{n+1}|^2} \det \begin{pmatrix} 0 & f_{\bar{k}} & f_{\overline{n+1}} \\ f_j & f_{j,\bar{k}} & f_{j,\overline{n+1}} \\ f_{n+1} & f_{n+1,\bar{k}} & f_{n+1,\overline{n+1}} \end{pmatrix}.$$

So we can compute

$$\begin{aligned} A_{j,\bar{k}}(f) &= -\det \begin{pmatrix} 0 & \frac{f_{\bar{k}}}{f_{\overline{n+1}}} & 1 \\ \frac{f_j}{f_{n+1}} & f_{j,\bar{k}} & f_{j,\overline{n+1}} \\ 1 & f_{n+1,\bar{k}} & f_{n+1,\overline{n+1}} \end{pmatrix} \\ &= -\det \begin{pmatrix} 0 & 0 & 1 \\ \frac{f_j}{f_{n+1}} & f_{j,\bar{k}} - \frac{f_{\bar{k}}}{f_{\overline{n+1}}} f_{j,\overline{n+1}} & f_{j,\overline{n+1}} \\ 1 & f_{n+1,\bar{k}} - \frac{f_{\bar{k}}}{f_{\overline{n+1}}} f_{n+1,\overline{n+1}} & f_{n+1,\overline{n+1}} \end{pmatrix} \end{aligned} \tag{A.3}$$

that is

$$A_{j,\bar{k}}(f) = f_{j,\bar{k}} - \frac{f_{\bar{k}}}{f_{\overline{n+1}}} f_{j,\overline{n+1}} - \frac{f_j}{f_{n+1}} f_{n+1,\bar{k}} + \frac{f_j f_{\bar{k}}}{|f_{n+1}|^2} f_{n+1,\overline{n+1}}. \tag{A.4}$$

With this formula in hands it is very easy to compute the total Levi curvature because it can be expressed in terms of the matrix $A(f)$. We have

$$K^{(n)}(\partial\Omega) = \frac{1}{|\partial f|^n} \det\left(\left(I_n - \frac{\bar{\alpha}\otimes\alpha}{1+|\alpha|^2}\right)A(f)\right) = \frac{|f_{n+1}|^2}{|\partial f|^{n+2}} \det A(f). \tag{A.5}$$

Let $\zeta = (z, z_{n+1})$ with $z = (z_1, \cdots, z_n)$. We introduce cylindrical coordinates

$$z_{n+1} = t + i\tau, \qquad r = |z|$$

and we consider f in the form $f(z, z_{n+1}) = u(|z|, t) - \tau$. We have

$$f_j = u_r \frac{\bar{z}_j}{2r}; \quad f_{\bar{j}} = u_r \frac{z_j}{2r}, \quad 1 \le j \le n.$$

$$f_{n+1} = \frac{1}{2}(u_t + i), \quad f_{\overline{n+1}} = \frac{1}{2}(u_t - i).$$

$$f_{j\,\bar{k}} = u_{rr}\frac{\bar{z}_j z_k}{4r^2} + u_r \frac{2\delta_{jk} r^2 - \bar{z}_j z_k}{4r^3}, \quad 1 \le j,k \le n.$$

$$f_{j\,\overline{n+1}} = u_{rt}\frac{\bar{z}_j}{4r}, \quad f_{n+1\,\bar{j}} = u_{rt}\frac{z_j}{4r}, \quad 1 \le j \le n.$$

$$f_{n+1\,\overline{n+1}} = \frac{1}{4}u_{tt}.$$

We can write (A.4) as

$$A(u-\tau) = \frac{u_r}{2r} I_n + \frac{\gamma}{4r^2}\bar{z}\otimes z$$

with

$$\gamma = \left(u_{rr} - \frac{u_r}{r} - 2\frac{u_r u_t}{1+u_t^2}u_{rt} + \frac{u_r^2}{1+u_t^2}u_{tt}\right).$$

Then

$$\det A = \left(\frac{u_r}{2r}\right)^{n-1}\left(\frac{u_r}{2r} + \frac{\gamma}{4}\right).$$

Moreover

$$\left(I_n - \frac{\bar{\alpha}\otimes\alpha}{1+|\alpha|^2}\right) = \left(I_n - \frac{u_r^2}{1+u_r^2+u_t^2}\frac{1}{r^2}\bar{z}\otimes z\right)$$

and we rewrite (A.5) as

$$K^{(n)}(\partial\Omega) = 2^n \frac{u_t^2+1}{\left(u_r^2+u_t^2+1\right)^{(n+2)/2}} \left(\frac{u_r}{2r}\right)^{n-1} \left(\frac{u_r}{2r}+\frac{\gamma}{4}\right). \tag{A.6}$$

The matrix in (A.1) can be written in cylindrical coordinates as

$$\begin{aligned}\left(I_n - \frac{\bar\alpha\otimes\alpha}{1+|\alpha|^2}\right)A(f) &= \left(\frac{u_r}{2r}I_n + \frac{\gamma}{4r^2}\bar z\otimes z\right)\left(I_n - \frac{u_r^2}{r^2(1+u_r^2+u_t^2)}\bar z\otimes z\right)\\ &= \frac{u_r}{2r}I_n + \left(\frac{\gamma}{4r^2}\frac{(1+u_t^2)}{(1+u_r^2+u_t^2)} - \frac{u_r^3}{2r^3(1+u_r^2+u_t^2)}\right)\bar z\otimes z\\ &= \frac{u_r}{2r}I_n + \beta\,\bar z\otimes z.\end{aligned}$$

Since $\det(\bar z\otimes z) = 0$ we have that

$$\det\left(\left(I_n - \frac{\bar\alpha\otimes\alpha}{1+|\alpha|^2}\right)A(f) - \frac{u_r}{2r}I_n\right) = \beta^n\det(\bar z\otimes z) = 0$$

and this implies that $\frac{u_r}{2r}$ is an eigenvalue of the Levi form with multiplicity $n-1$. So, if we denote by λ the other eigenvalue of the Levi form, we have

$$\operatorname{trace}\left(\left(I_n - \frac{\bar\alpha\otimes\alpha}{1+|\alpha|^2}\right)A(f) - \frac{u_r}{2r}I_n\right) = \lambda + (n-1)\frac{u_r}{2r}.$$

On the other hand we can compute

$$\operatorname{trace}\left(\left(I_n - \frac{\bar\alpha\otimes\alpha}{1+|\alpha|^2}\right)A(f) - \frac{u_r}{2r}I_n\right) = n\frac{u_r}{2r} + \beta r^2$$

and consequently $\lambda = \frac{u_r}{2r} + \beta r^2 = \left(\frac{u_r}{2r}+\frac{\gamma}{4}\right)\frac{1+u_t^2}{1+u_t^2+u_r^2}$ is an eigenvalue of the Levi form. We summarize the above observations in the following theorem

Theorem A.1. *The set* $\Omega = \{(r,t,\tau)\in\mathbb{R}_0^+\times\mathbb{R}^2 : u(r,t)-\tau<0\}$ *with* $u\in C^2$ *is strictly pseudoconvex if and only if*

$$\frac{u_r}{r} > 0$$

and

$$(1+u_t^2)\left(\frac{u_r}{r}+u_{rr}\right) - 2u_tu_ru_{rt} + u_r^2u_{tt} > 0.$$

In this situation the total Levi curvature of $\partial\Omega$ is by (A.6)

$$K^{(n)}(\partial\Omega) = \frac{1}{2}\frac{\left(\frac{u_r}{r}\right)^{n-1}}{\left(u_r^2+u_t^2+1\right)^{(n+2)/2}}\left((1+u_t^2)\left(\frac{u_r}{r}+u_{rr}\right)-2u_tu_ru_{rt}+u_r^2u_{tt}\right). \tag{A.7}$$

Now we use the notation

$$v=\frac{u_r}{r}, \qquad q=u_t, \qquad K=K^{(n)}(\partial\Omega)$$

to rewrite (A.7) in the equivalent form

$$(1+q^2)u_{rr}-2vqru_{rt}+v^2r^2u_{tt}=-(1+q^2)v+2K\frac{(1+v^2r^2+q^2)^{\frac{n+2}{2}}}{v^{n-1}}.$$

So, if we define $X_1=\partial_r$ and $X_2=r\partial_t$, the equation above becomes

$$\operatorname{trace}\left(\begin{pmatrix}1+q^2 & -vq\\ -vq & v^2\end{pmatrix}\begin{pmatrix}X_1^2u & X_2X_1u\\ X_2X_1u & X_2^2u\end{pmatrix}\right)=f(q,v,K)$$

that is

$$Lu=f(q,v,K) \tag{A.8}$$

with $f=-(1+q^2)v+2K\frac{(1+v^2r^2+q^2)^{\frac{n+2}{2}}}{v^{n-1}}$ and $L=(1+q^2)X_1^2-2vqX_2X_1+v^2X_2^2$. In particular, if Ω is strictly pseudoconvex, by Theorem A.1 we have $v>0$ and $\mathcal{A}=\begin{pmatrix}1+q^2 & -vq\\ -vq & v^2\end{pmatrix}$ is strictly positive definite. If we further assume q and v to be bounded, then the operator L in (A.8) is structured as in (3.13).

List of Symbols

Here is a brief list of notations frequently used in this thesis.

$B_r(x)$	quasi metric ball of center x and radius $r > 0$ (p. 2).
B_r	quasi metric ball of center an understood and fixed point x_0 and radius $r > 0$ (p. 2)
C_D	doubling constant (p. 2).
K	quasi triangle inequality constant (p. 2).
$\mathrm{diam}(\Omega)$	Euclidean diameter of the set Ω (p. 55).
$\mathrm{supp}\, u$	support of the function u (p. 41).
$\mathcal{B}(\Omega)$	set of all the quasi metric balls contained in Ω and ordered by inclusion (p. 5).
$\mathcal{F}(\Omega)$	particular subset of the real valued measurable functions defined on Ω (p. 5).
$\mathcal{S}_\Omega$	non negative function on $\mathcal{B}(\Omega) \times \mathcal{F}(\Omega)$, order preserving with respect to the first variable and homogeneous in the second variable (p. 5).
$\mathcal{L}(\Omega)$	subset of $\mathcal{F}(\Omega)$ containing all the functions f such that $\mathcal{S}_\Omega(\cdot, f)$ is finite at each ball in $\mathcal{B}(\Omega)$ (p. 5).
$\mathbb{K}_{\Omega,f}$	particular subset of the non negative real valued measurable functions with domain contained in Ω (p. 5).
$\tilde{\mathbb{K}}_{\Omega,f}$	particular subset of the measurable functions with domain contained in Ω which in particular contains $\mathbb{K}_{\Omega,f}$ (p. 25).
$\underset{B_r(x)}{\mathrm{osc}}\, u$	oscillation of the function u over the ball $B_r(x)$, that is $\sup_{B_r(x)} u - \inf_{B_r(x)} u$ (p. 25).
X	family of m vector fields $X = \{X_1, \dots, X_m\}$ on $\mathbb{R}^N$ (p. 27).
d_X	Carnot–Carathéodory distance associated to the family of vector fields X (p. 28).
Xu	X-gradient of the function u, that is $= (X_1 u, \dots, X_m u)$ (p. 29).

$B_{CC}(x,r)$	ball of center $x \in \mathbb{R}^2$ and radius $r > 0$ defined by the Carnot–Carathéodory distance associated to Grushin vector fields (p. 48).
$\mathrm{Box}(x,r)$	rectangle in $\mathbb{R}^2$ of center $x = (x_1, x_2)$ defined as $\mathrm{Box}(x,r) =]x_1 - r, x_1 + r[\times]x_2 - r(r + \lvert x_1 \rvert), x_2 + r(r + \lvert x_1 \rvert)[$ (p. 48).
$\tilde{d}$	particular Hölder quasi distance equivalent to the Carnot–Carathéodory distance associated to Grushin vector fields (p. 49).
$B(x,r)$	quasi metric ball of center $x \in \mathbb{R}^2$ and radius $r > 0$, defined by the quasi distance $\tilde{d}$ (p. 49).
$\tilde{G}(x,r)$	sublevel set of the function $\tilde{g}_r(x,\cdot)$, it might have two connected component (p. 51).
$G(x,r)$	sublevel set of the function $g_r(x,\cdot)$ (p. 52).
$H(x,r)$	sublevel set of the function $h_r(x,\cdot)$ (p. 52).

$\tilde{C}$	structural constant in the Ball-Box theorem for Carnot–Carathéodory balls B_{CC} . (p. 48).
C_B	structural constant in the Ball-Box theorem for balls B. (p. 49).
C_H	structural constant in the Ball-Box theorem for sublevel sets H. (p. 53).
C_G	structural constant in the Ball-Box theorem for sublevel sets G. (p. 55).

References

[1] F. Abedin, C. E. Gutiérrez, and G. Tralli, *Harnack's inequality for a class of non-divergent equations in the Heisenberg group*, Communications in Partial Differential Equations, 42 (2017), pp. 1644–1658.

[2] H. Aimar, L. Forzani, and R. Toledano, *Hölder regularity of solutions of PDE's: a geometrical view*, Communications in Partial Differential Equations, 26 (2001), pp. 1145–1173.

[3] Z. M. Balogh, A. Calogero, and A. Kristály, *Sharp comparison and maximum principles via horizontal normal mapping in the Heisenberg group*, Journal of Functional Analysis, 269 (2015), pp. 2669 – 2708.

[4] E. Battaglia and A. Bonfiglioli, *An invariant Harnack inequality for a class of subelliptic operators under global doubling and Poincaré assumptions, and applications*, Journal of Mathematical Analysis and Applications, 460 (2018), pp. 302 – 320.

[5] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics, Springer Berlin Heidelberg, 2007.

[6] X. Cabré, *Nondivergent elliptic equations on manifolds with nonnegative curvature*, Communications on Pure and Applied Mathematics, 50 (1997), pp. 623–665.

[7] L. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, no. v. 43 in American Mathematical Society: Colloquium publications, American Mathematical Society, 1995.

[8] L. A. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Annals of Mathematics, 130 (1989), pp. 189–213.

[9] L. A. Caffarelli and C. E. Gutiérrez, *Properties of the solutions of the linearized Monge-Ampère equation*, American Journal of Mathematics, 119 (1997), pp. 423–465.

[10] Y. Chen and L. Wu, *Second Order Elliptic Equations and Elliptic Systems*, Translations of mathematical monographs, American Mathematical Society, 1998.

[11] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, vol. 242 of Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 1 ed., 1971.

[12] D. Danielli, N. Garofalo, and D. M. Nhieu, *Trace inequalities for Carnot-Carathéodory spaces and applications*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 27 (1998), pp. 195–252.

[13] D. Danielli, N. Garofalo, and D.-M. Nhieu, *On the best possible character of the L^Q norm in some a priori estimates for non-divergence form equations in Carnot groups*, Proceedings of the American Mathematical Society, 131 (2003), pp. 3487–3498.

[14] G. Di Fazio, C. E. Gutièrrez, and E. Lanconelli, *Covering theorems, inequalities on metric spaces and applications to PDE's*, Mathematische Annalen, 341 (2008), pp. 255–291.

[15] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 10 (1983), pp. 523–541.

[16] ——, *Une métrique associée à une classe d'opérateurs elliptiques dégénérés*, in Conference on linear partial and pseudodifferential operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 1983, pp. 105–114.

[17] B. Franchi, G. Lu, and R. L. Wheeden, *A relationship between Poincaré-type inequalities and representation formulas in spaces of homogeneous type*, International Mathematics Research Notices, 1996 (1996), pp. 1–14.

[18] B. Franchi, R. Serapioni, and F. Serra Cassano, *Meyer-Serrin type theorem and relaxation of variational integrals depending on vector fields*, Houston J. of Math, (1996), pp. 859–888.

[19] ——, *Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields*, Bollettino della Unione Matematica Italiana-B, (1997), pp. 83–118.

[20] N. Garofalo and D. M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Communications on Pure and Applied Mathematics, 49 (1996), pp. 1081–1144.

[21] ——, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces*, Journal d'Analyse Mathématique, 74 (1998), pp. 67–97.

[22] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, 2001.

[23] C. Guidi and A. Montanari, *Abstract approach to non homogeneous Harnack inequality in doubling quasi metric spaces*, Nonlinear Analysis, 169 (2018), pp. 130 – 162.

[24] C. E. Gutiérrez, *The Monge-Ampère Equation*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser Boston, second ed., 2016.

[25] C. E. GUTIÉRREZ AND E. LANCONELLI, *Maximum principle, nonhomogeneous Harnack inequality, and Liouville theorems for X-elliptic operators*, Communications in Partial Differential Equations, 28 (2003), pp. 1833–1862.

[26] C. E. GUTIÉRREZ AND A. MONTANARI, *Maximum and comparison principles for convex functions on the Heisenberg group*, Communications in Partial Differential Equations, 29 (2005), pp. 1305–1334.

[27] C. E. GUTIÉRREZ AND F. TOURNIER, *Harnack inequality for a degenerate elliptic equation*, Communications in Partial Differential Equations, 36 (2011), pp. 2103–2116.

[28] P. HAJLASZ AND P. KOSKELA, *Sobolev met Poincaré*, vol. 145 (688) of Memoirs of the American Mathematical Society,, American Mathematical Society, Providence, RI, 2000.

[29] J. HEINONEN, *Lectures on Analysis on Metric Spaces*, Universitext, Springer New York, 2001.

[30] S. INDRATNO, D. MALDONADO, AND S. SILWAL, *On the axiomatic approach to Harnack's inequality in doubling quasi-metric spaces*, Journal of Differential Equations, 254 (2013), pp. 3369 – 3394.

[31] D. JERISON, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J., 53 (1986), pp. 503–523.

[32] N. KRYLOV AND M. SAFONOV, *An estimate on the probability that a diffusion hits a set of positive measure*, Soviet Math, (1979), pp. 253–256.

[33] ——, *A property of the solutions of parabolic equations with measurable coefficients*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya, 44 (1980), pp. 161–175.

[34] E. LANCONELLI AND A. E. KOGOJ, *X-elliptic operators and X-control distances*, Ricerche di Matematica, 49 (2000), pp. 223–244.

[35] R. A. MACÍAS AND C. SEGOVIA, *Lipschitz functions on spaces of homogeneous type*, Advances in Mathematics, 33 (1979), pp. 257 – 270.

[36] D. MALDONADO, *On certain degenerate and singular elliptic PDEs I: Nondivergence form operators with unbounded drifts and applications to subelliptic equations*, Journal of Differential Equations, 264 (2018), pp. 624 – 678.

[37] A. MONTANARI, *Harnack inequality for a subelliptic PDE in nondivergence form*, Nonlinear Analysis: Theory, Methods & Applications, 109 (2014), pp. 285 – 300.

[38] A. MONTANARI AND E. LANCONELLI, *Pseudoconvex fully nonlinear partial differential operators strong comparison theorems*, Journal of Differential Equations, 202 (2004), pp. 306 – 331.

[39] J. MOSER, *On a pointwise estimate for parabolic differential equations*, Communications on Pure and Applied Mathematics, 24 (1971), pp. 727–740.

[40] M. V. SAFONOV, *The Harnack inequality for elliptic equations and Hölder continuity of their solutions*, Zapiski Nauchnykh Seminarov LOMI, 96 (1980), pp. 272 – 287.

[41] G. TRALLI, *A certain critical density property for invariant Harnack inequalities in H-type groups*, Journal of Differential Equations, 256 (2014), pp. 461 – 474.

[42] N. S. TRUDINGER AND W. ZHANG, *Hessian measures on the Heisenberg group*, Journal of Functional Analysis, 264 (2013), pp. 2335 – 2355.

[43] F. UGUZZONI, *Estimates of the Green function for X-elliptic operators*, Mathematische Annalen, 361 (2015), pp. 169–190.

Acknowledgements

First of all I want to sincerely thank my advisor Prof. Annamaria Montanari, for her teachings, patience, time and precious advices. She has been a wise guide in the last five years, since my bachelor degree. I have learned a lot from her and I am really grateful for all the help and support she gives me.

I wish to thank Prof. Giovanna Citti and Prof. Alberto Parmeggiani for the help and the special attention they dedicate to us, young students, who are moving our first steps on the field of research.

I also wish to thank all the people I met during my visit at the Massachussets Institute of Technology, Boston. In particular I want to thank Prof. David Jerison not only for the useful and inspiring discussions but also for his contagious enthusiasm for the subject and his warm hospitality.

Then I want to thank Prof. Vittorio Martino, he is an amazing teacher and I really feel lucky for having the opportunity to learn from him.

Finally I want to thank my colleagues, who shared with me long days at the office, and Giulio Tralli for the discussion and references he suggested me.