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SIMPLE RANDOM WALKS ON SOME PARTIALLY DIRECTED PLANAR GRAPHS

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Introduction

This thesis is about simple random walks on some classes of lattice graphs in two dimensions, with a set of directed edges whose orientation is chosen either deterministically or randomly. We are mainly concerned with the type problem, that is to determine whether the simple random walk on such graphs is recurrent or transient.

Our starting point is a paper of 2003 by Campanino and Petritis [1]: the authors were concerned with the study of type and speed of a simple random walk $(M_n)_{n\geq 0}$ on partially directed versions of the square grid lattice. More precisely, for any $y \in \mathbb{Z}$ they required all the horizontal edges leading out from vertexes $(x, y), x \in \mathbb{Z}$ to be oriented in the same direction, specified by a $\{-1, 1\}$-valued variable ϵ_y: to the right if $\epsilon_y = 1$ and to the left if $\epsilon_y = -1$ (alternatively, we say that "level y" is oriented, respectively, to the right and to the left). All the vertical edges remain unoriented. Their main result is that if $(\epsilon_y)_{y\in\mathbb{Z}}$ is a family of i.i.d. Rademacher random variables, then M is ϵ-a.s. transient, that is

$$\sum_{n=1}^{\infty} \mathbb{P}_0(M_n = (0,0) \mid \sigma\{\epsilon_y | y \in \mathbb{Z}\}) < \infty \quad \text{a.s.}$$

After the work of Campanino and Petritis, their model and related ones have been further investigated by different authors: just to cite some of the papers that have appeared in recent years, in [12] and [11] a functional central limit theorem for the random walk is established; in [3] the authors obtain a local limit theorem for the probability of return to the origin, while in [13] the range of the walk is analyzed. In [2] the type problem is solved for a class of orientations given by periodic functions, whereas in [22] the a.s. transience is generalized to a certain class of stationary orientations. In [4] the authors analyze a more general model, where the probability of staying on one oriented line is non-constant. One common feature of these papers is that they all deal with directed version of the square grid lattice.

In this thesis we start instead by considering a different model, where the underlying graph is a honeycomb lattice. We recall that a honeycomb lattice is a sub-graph of the square grid obtained by eliminating a periodic set of vertical edges (see figure 1).

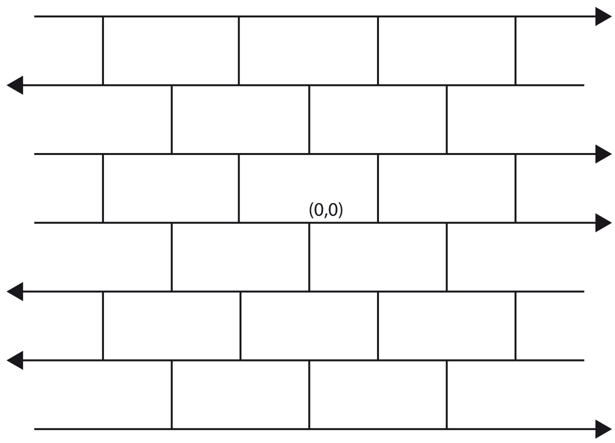


Figure 1: The randomly oriented lattice $\mathbf{H}_\epsilon$.

Precisely, the honeycomb lattice is $\mathbf{H} := (\mathbb{Z}^2, E)$, where $E = E_1 \backslash E_2$ with E_1 the set of nearest neighbor bonds in $\mathbb{Z}^2$ and

$$
\begin{aligned}
E_2 :=& \{((2j, 2k), (2j, 2k+1)) |\, j, k \in \mathbb{Z}\} \\
&\cup \{((2j, 2k+1), (2j, 2k)) |\, j, k \in \mathbb{Z}\} \\
&\cup \{((2j+1, 2k+1), (2j+1, 2k+2)) |\, j, k \in \mathbb{Z}\} \\
&\cup \{((2j+1, 2k+2), (2j+1, 2k+1)) |\, j, k \in \mathbb{Z}\}.
\end{aligned}
$$

Analogously to the square grid case, we let the horizontal levels be oriented by a family $(\epsilon_y)_{y \in \mathcal{Z}}$ of $\{-1, 1\}$-valued variables, while the vertical edges remain unoriented: we call the resulting randomly oriented graph $\mathbf{H}_\epsilon$. Our aim is to solve the type problem for the simple random walk on this graph, under different assumptions on the sequence ϵ.

One important tool in the analysis of the Campanino and Petritis model is the decomposition of the random walk into two components, a vertical motion and a horizontal one, such that on a certain sequence of random times it reproduces the law of the process. This decomposed walk encodes important information of the actual walk: in particular its recurrence behavior is the same as the actual random walk. However when dealing with the honeycomb lattice, the principal issue stems from the lack of decoupling between vertical and horizontal components of the walk that is instrumental in the aforementioned papers: here, the vertical skeleton of the walk is not -as in the square grid case- a one dimensional simple symmetric random walk, but a Markov process of order 2.

The first result of our work is theorem 1 in Chapter 1, which states that, if ϵ is a sequence of i.i.d. Rademacher random variables, then the random walk on $\mathbf{H}_\epsilon$ is ϵ-a.s. transient. This is obtained by using local limit estimates for Markov chains as established in [15], instead of local limit estimates for i.i.d. variables (as they have been used e.g. in [1]); and also a Tauberian theorem to generalize standard results for the simple random walk in $\mathbb{Z}$ to the vertical skeleton of $\mathbf{H}_\epsilon$, while some technical issues require to specialize the arguments of Campanino and Petritis to the new setting.

In Chapter 2 we consider a class of directed honeycomb lattices where a certain degree of periodicity in the orientations of the levels is assumed: first, in theorem 2 we show that if the sequence (ϵ_y) is deterministic and periodic, and such that $\sum_y \epsilon_y = 0$, then the random walk is recurrent; then, we introduce random perturbations around the origin that decay polynomially according to an exponent β and, depending of the value of β, we classify the type of the walk. Again, our results extend those obtained for the square grid oriented lattice (cfr. [2]). Here, mainly due to the lack of a reflection principle for the vertical skeleton, we develop a new technique to tackle the problem: the main idea is to exploit the periodicity of the orientations by means of a conditional local limit theorem for the occupation times of the vertical skeleton. [1]

Finally, we consider a model where the orientations are ergodic and non periodic,

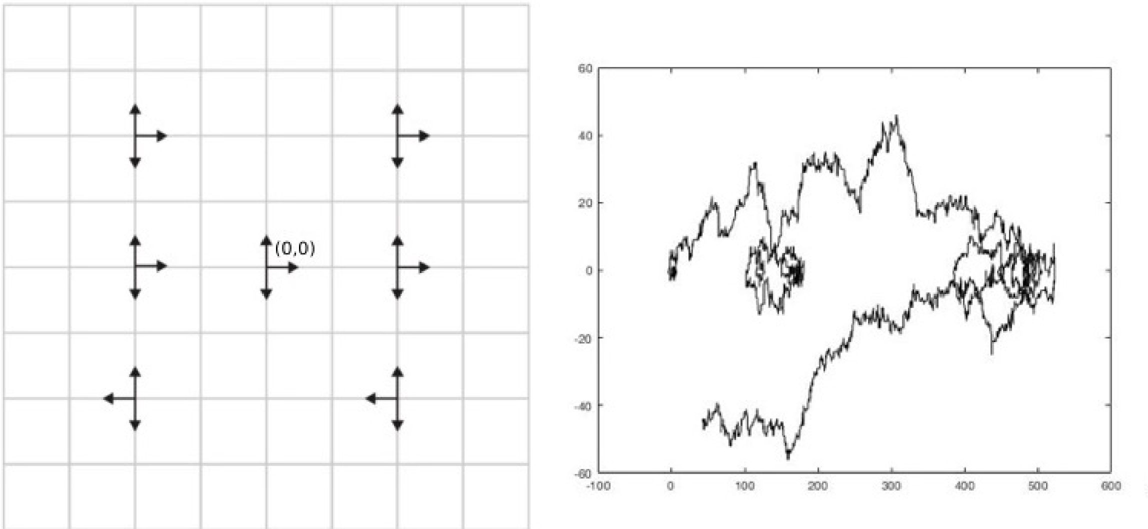
[1]These results have been accepted for publication in *Markov Processes & Related Fields*.

but the random walk is nonetheless a.s. recurrent: this shows in particular that the a.s. transience in the i.i.d. setting can't be generalized to the ergodic setting without assuming further hypothesis on the mixing rate of ϵ (cfr. [22]).

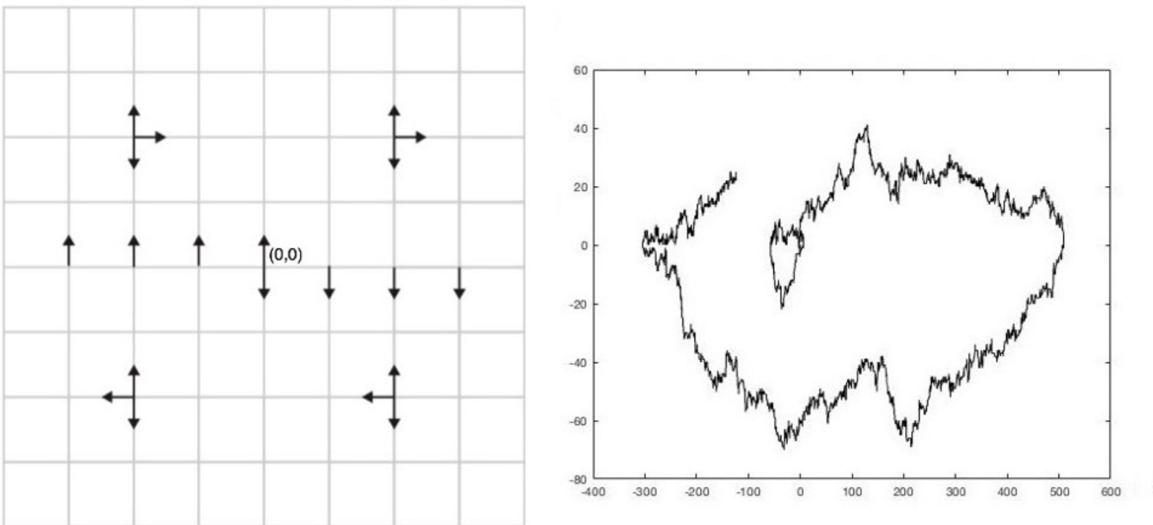
The results of Chapter 1 and 2 can be seen as a first step toward the generalization of the work of Campanino and Petritis to a larger class of partially directed graphs. Moreover, as we observe in appendix A, a simple reformulation of our theorems relates them to a conjecture (see [10]) involving the oriented square grid where both the vertical and horizontal levels are oriented. It turns out that, if the levels in one coordinate are randomly oriented by Rademacher random variables and in the other coordinate are alternate, the simple random walk is a.s. transient.

In this thesis we also consider the type problem for some so-called revolving random walks: this name comes from the fact that they are simple random walks on (deterministic) directed version of the two-dimensional square grid lattice that, because of the direction imposed by the oriented edges, are bound to revolve clockwise.

The first graph of this kind that we introduce, which we call $\mathbf{G}_1$, is in fact a square grid where the horizontal levels are oriented according to the deterministic function $\epsilon(y) = 1$ if $y \geq 0$, and $\epsilon(y) = -1$ otherwise, while the vertical edges stay unoriented (see figure 2(a)). The simple random walk on this graph was already known in the literature: it appeared for the first time in [1], where it is shown to be transient. Then, it has been studied again in [19] and [20]: the authors were mainly concerned about one-dimensional oscillating random walks and in fact they notice that the simple random walk on $\mathbf{G}_1$, seen at the times of consecutive returns to the x-axis, exhibits an oscillatory behavior. Interestingly, in the same work they introduce a new directed graph, that we shall call $\mathbf{G}_2$: precisely, $\mathbf{G}_2$ is obtained from $\mathbf{G}_1$ by redefining only the orientations of the edges leading out from x-axis, that is, $((v_1, 0), (w_1, w_2))$ with $v_1 = w_1$ and $w_2 = \pm 1$ is an edge if and only if $w_2 = -1$ and $v_1 = w_1 > 0$, or $w_2 = 1$ and $v_1 = w_1 < 0$, or $w_2 = \pm 1$ and $v_1 = w_1 = 0$ (see figure 2(b)). Although it is obtained by a simple modification of $\mathbf{G}_1$, the analysis of its type turns out to be more delicate and the authors conjectured that this revolving random walk is in fact recurrent.



(a) Graph $\mathbf{G}_1$



(b) Graph $\mathbf{G}_2$

Figure 2: The transient graph $\mathbf{G}_1$ is represented in figure (a), and the recurrent graph $\mathbf{G}_2$ in figure (b): the arrows indicate the orientation of the corresponding edges. At the right sides, a realization of 5000 steps of the corresponding random walks is shown.

In Chapter 3 we confirm their conjecture [2]: the idea of our proof is to use the

[2]All the results presented in Chapter 3 are from a joint work with Yuval Peres, Principal Researcher

Lyapunov function method (cfr.[20], Thm.2.5.2, p.53). To do so, we start considering a process W which is the continuous-time analogous of the simple random walk on $\mathbf{G}_2$, and show that W is recurrent: then, the analysis of this process leads us to find an appropriate Lyapunov function, whose expectation can be computed easily in the continuous case, and finally develop an approximation technique to estimate it in the discrete setting.

To complete the analysis, we obtain also a local limit theorem for the return probability to the origin of the random walk on $\mathbf{G}_1$, showing that it is asymptotic to a constant times $n^{-3/2}$ and giving, in particular, a new proof of transience.

at Microsoft Research, Redmond, and Yiping Hu, Ph.D. student from the University of Washington, Seattle. A preprint of our work is currently available on arXiv at `https://arxiv.org/abs/1807.03498`

Chapter 1

Random walks on randomly oriented honeycomb lattices

A *directed* (or equivalently, *oriented*) graph is a pair $\mathbf{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ is a denumerable set of vertexes and $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$ is a set of directed edges. Let $d_u^+ := |\{x \in \mathbb{V} | (x, u) \in \mathbb{E}\}|$ and $d_u^- := |\{x \in \mathbb{V} | (u, x) \in \mathbb{E}\}|$ be, respectively, the inwards degree and the outwards degree of $u \in \mathbb{V}$ and assume the graph is locally finite, i.e. $d_u^+, d_u^- < \infty$, $\forall u \in \mathbb{V}$.

Definition 1. The simple random walk on $\mathbf{G}$ started at $v_0 \in \mathbb{V}$ is a Markov chain $(M_n)_{n \in \mathbb{N}}$ with state space $\mathbb{V}$ and $M_0 = v_0$, and transition probabilities given by

$$\mathbb{P}(M_{n+1} = v | M_n = u) = \begin{cases} \frac{1}{d_u^-} & \text{if } (u, v) \in \mathbb{E}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2. We say that a state $x \in \mathbb{V}$ is recurrent if $\sum_{n=1}^{\infty} \mathbb{P}_x(M_n = x) = \infty$. Otherwise, the state is said to be transient.

Recurrence and transience are class properties (see e.g. [18], prop. 21.3), meaning in particular that for an irreducible Markov chain either all states are recurrent, or transient: we say, respectively, that the chain is recurrent, or transient.

Consider now the directed graph $\mathbf{H}_\epsilon$ which we defined in the introduction, where $\epsilon = (\epsilon_y)_{y \in \mathbb{Z}}$ is a sequence of $\{-1, 1\}$-valued variables. Note that for each vertex v of $\mathbf{H}_\epsilon$

we have $d_v := d_v^+ = d_v^- = 2$. Then the simple random walk M on $\mathbf{H}_\epsilon$ has the following transition probabilities

$$\mathbb{P}(M_{n+1} = v | M_n = u) = \begin{cases} \frac{1}{2} & \text{if } (u, v) \text{ is an allowed edge of } \mathbf{H}_\epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We shall assume $M_0 = (0, 0)$. Observe that M is necessarily a non reversible irreducible Markov chain.

Suppose $\epsilon = (\epsilon_y)_{y \in \mathbb{Z}}$ is a random sequence: in this case we call ϵ the random environment. We are interested in the type problem for the simple random walk on $\mathbf{H}_\epsilon$ under different hypothesis on ϵ.

We shall always assume all the random variables we use to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

The main result of this Chapter is the following:

Theorem 1. *Let $\epsilon = (\epsilon_y)_{y \in \mathbb{Z}}$ be a sequence of i.i.d. Rademacher random variables. Then the simple random walk M on $\mathbf{H}_\epsilon$ is ϵ-a.s. transient.*

1.1 Decomposition of the random walk

We want to decompose the random walk in two components, that encode respectively the vertical and the horizontal motion of the process, following the technique implemented in [1] in the case of a partially directed square grid lattice, but considering instead the honeycomb lattice.

Let ξ be a geometric random variable with values in $\{0, 1, 2, ...\}$ and success probability $\frac{1}{2}$; we want to interpret ξ as the number of consecutive horizontal steps that the random walk performs between two successive vertical steps. In particular, we notice that the two vertical steps share the same direction (both upward, or both downward) if and only if ξ has an odd outcome (cf. figure 1), which happens with probability

$$\mathbb{P}(\xi \text{ is odd}) = \sum_{m=0}^{\infty} \mathbb{P}(\xi = 2m + 1) = \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m+1} = \frac{1}{3}.$$

With this observation in mind, we want to define a discrete time process that records only the successive vertical movements of M.

Definition 3. The vertical skeleton of M is the process Y defined by

$$Y_k := \sum_{i=1}^{k} \nu_k, \tag{1.1}$$

for every $k \geq 0$, where $(\nu_k)_{k\geq 0}$ is a $\{+1,-1\}$-valued Markov chain with $v_0 = 1$ and transition matrix:

$$\pi_\nu := \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}, \text{ with } q = \frac{1}{3}.$$

Remark 1. Y is not a Markov chain. However, by a standard dilation of the state space into $\mathbb{Z} \times \{-1, 1\}$ we can turn it into a Markov chain $(Y_n, \nu_n)_{n\geq 0}$, defined by the following transition probabilities:

$$\begin{aligned} p_{(y,1),(y+1,1)} = p_{(y,-1),(y-1,-1)} &= \frac{1}{3} \\ p_{(y,1),(y-1,-1)} = p_{(y,-1)(y+1,1)} &= \frac{2}{3} \end{aligned}$$

for any $y \in \mathbb{Z}$, and $\mathbb{P}((Y_0, \nu_0) = (0, 1)) = 1$. Note that $(y, 1)$ corresponds to the state of Y being at y after an upward step, while $(y, -1)$ is the state of Y being at y after a downward step.

The next lemma 1 implies in particular the recurrence of the vertical skeleton: this result will be used extensively in our work.

Lemma 1. *We have, as $n \to \infty$*

$$\mathbb{P}_0(Y_{2n} = 0) \sim \frac{C}{\sqrt{n}},$$

where C is a positive constant.

Proof. The result follows directly by the application of a local limit theorem for Markov chains (th.3 in [15]) to the process ν. $\square$

Now we want to complete our decomposition by defining the process that represents the abscissa of the random walk $(M_n)_{n\geq 0}$ immediately after the n-th vertical movement has been performed.

Definition 4. Let $n \geq 0$. We define the *occupation measure* η_n at $y \in \mathbb{Z}$ by

$$\eta_n(y) := \sum_{k=0}^{n} \mathbf{1}_{\{Y_k = y\}}.$$

Definition 5. Let $(\xi_i^{(y)})_{i\geq 1,\, y\in\mathbb{Z}}$ be a family of i.i.d. geometric random variables with values in $\{0, 1, 2, ...\}$ parameter $p = \frac{1}{2}$. We call *embedded random walk* the process $(X_n)_{n\geq 0}$ defined by

$$X_n := \sum_{y\in\mathbb{Z}} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}, \tag{1.2}$$

with the convention that $\sum_i$ vanishes whenever $\eta_{n-1}(y) = 0$.

Let $n > 1$ and

$$T_n := n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$$

be the time just after the random walk M has performed its n-th vertical move. Then it's straightforward to see that

$$M_{T_n} = (X_n, Y_n).$$

(See figure 3 for an illustration of the decomposition)

Now denote by σ_n the sequence of consecutive returns to 0 of Y: note that by lemma 1 we have $\sigma_n < \infty$ almost surely $\forall n$. Obviously, $M_{T_{\sigma_n}} = (X_{\sigma_n}, 0)$.

Consider the following sigma-algebras

$$\begin{aligned} \mathcal{G} :=& \sigma(\epsilon_y, y \in \mathbb{Z}), \\ \mathcal{F}_n :=& \sigma(\nu_k, k \leq n) \end{aligned}$$

and $\mathcal{F} \vee \mathcal{G} = \sigma(\mathcal{F} \cup \mathcal{G})$, where $\mathcal{F} = \vee_n \mathcal{F}_n$. The next lemma 2 links the transience of $(X_{\sigma_n})_{n\geq 0}$ to the transience of the original random walk.

Lemma 2. *We have*

$$\sum_{n=0}^{\infty} \mathbb{P}_0(X_{\sigma_n} = 0 \mid \mathcal{F} \vee \mathcal{G}) < \infty \ a.s \implies \sum_{l=0}^{\infty} \mathbb{P}_0(M_l = (0,0) \mid \mathcal{F} \vee \mathcal{G}) < \infty \ a.s .$$

Proof. (See [1], lemma 2.3) □

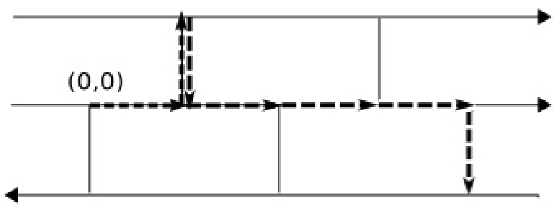


Figure 3: The first steps of a realization of the simple random walk on $\mathbf{H}_\epsilon$ started at $(0,0)$, assuming $\epsilon_0 = \epsilon_1 = 1$ and $\epsilon_{-1} = -1$. The decomposition into vertical skeleton Y and embedded random walk X gives here the following: $(X_0, Y_0) = (0,0), (X_1, Y_1) = (1,1)$, $(X_2, Y_2) = (1,0)$, $(X_3, Y_3) = (4,-1)$.

To conclude the section we derive the explicit expression for the characteristic function of the embedded random walk, a useful tool in the proof of theorem 1.

Definition 6. Let $n \in \mathbb{N}, y \in \mathbb{Z}$ and define

$$m_{n,o}^{(y)} := \sum_{k=0}^{n} \mathbf{1}_{\{Y_k = y, \nu_k = \nu_{k+1}\}},$$

and

$$m_{n,e}^{(y)} := \sum_{k=0}^{n} \mathbf{1}_{\{Y_k = y, \nu_k \neq \nu_{k+1}\}}.$$

Note that $\eta_n(y) = m_{n,o}^{(y)} + m_{n,e}^{(y)}$, and so we can split the embedded random walk X_n as follows

$$X_n = \sum_{y \in \mathbb{Z}} \epsilon_y \left(\sum_{i=1}^{m_{n-1,o}^{(y)}} \xi_{i,o}^{(y)} + \sum_{i=1}^{m_{n-1,e}^{(y)}} \xi_{i,e}^{(y)} \right),$$

where $\xi_{i,o}^{(y)}$ and $\xi_{i,e}^{(y)}$ are two independent families of i.i.d. random variables, that have the following laws:

$$\mathbb{P}(\xi_{i,o}^{(y)} = 2k+1) = \mathbb{P}(\xi = 2k+1 \mid \xi \text{ is odd}) = 3\mathbb{P}(\xi = 2k+1) = 3\left(\frac{1}{2}\right)^{2k+2} \tag{1.3}$$

for every $k \in \mathbb{N}$, and

$$\mathbb{P}(\xi_{i,e}^{(y)} = 2k) = \mathbb{P}(\xi = 2k \mid \xi \text{ is even}) = 3\left(\frac{1}{2}\right)^{2k+2}. \tag{1.4}$$

We shall call a random variable *odd geometric* if its law is given by (1.3), while (1.4) defines what we call an *even geometric* random variable.

Lemma 3. *We have*

$$\chi_o(\theta) := \mathbb{E}(\exp(i\theta\xi_{1,o}^{(y)})) = \frac{3e^{i\theta}}{4 - e^{2i\theta}}$$

and $\chi_e(\theta) := \mathbb{E}(\exp(i\theta\xi_{1,e}^{(y)})) = e^{-i\theta}\chi_o(\theta)$*, for every* $\theta \in \mathbb{R}$*.*

Proof. By (1.3) we have

$$\chi_o(\theta) := = \sum_{k\geq 0} e^{i\theta(2k+1)}\mathbb{P}(\xi_{1,o} = 2k+1) = \frac{3e^{i\theta}}{4}\sum_{k\geq 0} e^{i\theta(2k)}\left(\frac{1}{2}\right)^{2k} = \frac{3e^{i\theta}}{4 - e^{2i\theta}},$$

and similarly by (1.4), $\chi_e(\theta) = \frac{3}{4-e^{2i\theta}}$. □

Lemma 4. *The characteristic function of* X_n *is*

$$\mathbb{E}(\exp(i\theta X_n)) = \mathbb{E}\left(\prod_{y\in\mathbb{Z}} \chi_o(\theta\epsilon_y)^{m_{n-1,o}^{(y)}}\chi_e(\theta\epsilon_y)^{m_{n-1,e}^{(y)}}\right).$$

Proof. By lemma 3

$$\begin{aligned}
\mathbb{E}(\exp(i\theta X_n)) =& \mathbb{E}\left(\mathbb{E}\left(\exp(i\theta\sum_{y\in\mathbb{Z}}\epsilon_y\sum_{i=1}^{\eta_{n-1}(y)}\xi_i^{(y)}) \mid \mathcal{F}_n \vee \mathcal{G}\right)\right)\\
=& \mathbb{E}\left(\mathbb{E}\left(\prod_{y\in\mathbb{Z}}\exp(i\theta\epsilon_y\sum_{i=1}^{\eta_{n-1}(y)}\xi_i^{(y)}) \mid \mathcal{F}_n \vee \mathcal{G}\right)\right)\\
=& \mathbb{E}\left(\mathbb{E}\left(\prod_{y\in\mathbb{Z}}\exp(i\theta\epsilon_y(\sum_{i=1}^{m_{n-1,o}^{(y)}}\xi_{i,o}^{(y)} + \sum_{i=1}^{m_{n-1,e}^{(y)}}\xi_{i,e}^{(y)})) \mid \mathcal{F}_n \vee \mathcal{G}\right)\right)\\
=& \mathbb{E}\left(\mathbb{E}\left(\prod_{y\in\mathbb{Z}}\exp(i\theta\epsilon_y\sum_{i=1}^{m_{n-1,o}^{(y)}}\xi_{i,o}^{(y)})\exp(i\theta\epsilon_y\sum_{i=1}^{m_{n-1,e}^{(y)}}\xi_{i,e}^{(y)}) \mid \mathcal{F}_n \vee \mathcal{G}\right)\right)\\
=& \mathbb{E}\left(\prod_{y\in\mathbb{Z}}\chi_o(\theta\epsilon_y)^{m_{n-1,o}^{(y)}}\chi_e(\theta\epsilon_y)^{m_{n-1,e}^{(y)}}\right)
\end{aligned}$$

□

1.2 Proof of almost sure transience

Let m_o and m_e be, respectively, the mean of an odd geometric and of an even geometric random variable. We follow the strategy of [1] and define, for $n \geq 0$, the following families of events:

$$A_{n,1} := \{\max_{0 \leq k \leq 2n} |Y_k| < n^{\frac{1}{2}+\delta_1}\}, \ \delta_1 > 0$$
$$A_{n,2} := \{\max_{y \in \mathbb{Z}} \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2}\}, \ \delta_2 > 0$$
$$A_n := A_{n,1} \cap A_{n,2}$$
$$B_n := A_n \cap \{|\sum_{y \in \mathbb{Z}} \epsilon_y (m_o m^{(y)}_{2n-1,o} + m_e m^{(y)}_{2n-1,e})| > n^{\frac{1}{2}+\delta_3}\}, \ \delta_3 > 0$$

where δ_1, δ_2 and δ_3 are positive integers that will be chosen later. Observe that, for every n, $A_n \in \mathcal{F}_{2n}$ and $B_n \subset A_n, \ B_n \in \mathcal{F}_{2n} \vee \mathcal{G}$. Thus, we can write

$$p_n = p_{n,1} + p_{n,2} + p_{n,3},$$

where

$$p_n = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0)$$
$$p_{n,1} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, B_n)$$
$$p_{n,2} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, A_n / B_n)$$
$$p_{n,3} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, A_n^c).$$

In order to prove transience, we will provide estimates of, respectively, $p_{n,1}$, $p_{n,2}$ and $p_{n,3}$, from which we will deduce that p_n is summable. Then the result will follow at once thanks to the following lemma.

Lemma 5. *If $\sum_{n \geq 0} p_n < \infty$, then $(M_n)_{n \geq 0}$ is a.s. transient.*

Proof. From the trivial upper bound

$$\sum_{n \geq 0} \mathbb{P}(X_{\sigma_n} = 0) = \sum_{n \geq 0} \mathbb{P}(X_{\sigma_n} = 0, Y_{\sigma_n} = 0) \leq \sum_{n \geq 0} \mathbb{P}(X_{2n} = 0, Y_{2n} = 0),$$

we deduce that $\sum_{n\geq 0}\mathbb{P}(X_{\sigma_n}=0)<\infty$ and hence also

$$\sum_{n\geq 0}\mathbb{P}(X_{\sigma_n}=0\mid \mathcal{F}_n\vee\mathcal{G})<\infty \text{ a.s.}$$

By lemma 2, this implies the a.s. transience of $(M_n)_{n\geq 0}$. $\square$

1.2.1 Estimate of $p_{n,1}$

Define

$$N_o^+ := \sum_{k=1}^{2n}\mathbf{1}_{\{\epsilon_{Y_k}=1\}}\mathbf{1}_{\{\nu_k=\nu_{k+1}\}},$$

$$N_e^+ := \sum_{k=1}^{2n}\mathbf{1}_{\{\epsilon_{Y_k}=1\}}\mathbf{1}_{\{\nu_k\neq\nu_{k+1}\}},$$

$$N_o^- := \sum_{k=1}^{2n}\mathbf{1}_{\{\epsilon_{Y_k}=-1\}}\mathbf{1}_{\{\nu_k=\nu_{k+1}\}},$$

$$N_e^- := \sum_{k=1}^{2n}\mathbf{1}_{\{\epsilon_{Y_k}=-1\}}\mathbf{1}_{\{\nu_k\neq\nu_{k+1}\}},$$

and

$$\Delta_{n,o} := N_o^+ - N_o^-,$$

$$\Delta_{n,e} := N_e^+ - N_e^-,$$

$$\Sigma_{n,o} := N_o^+ + N_o^-,$$

$$\Sigma_{n,e} := N_e^+ + N_e^-,$$

Observe that

$$m_o\Delta_{n,o}+m_e\Delta_{n,e}=\sum_{y\in\mathbb{Z}}\epsilon_y(m_o m_{2n-1,o}^{(y)}+m_e m_{2n-1,e}^{(y)})$$

and $\Sigma_{n,o}+\Sigma_{n,e}=2n$.

Let $(\xi_k)_{k\geq 1}$ be a sequence of geometric random variables with values in $\{0,1,2,...\}$ and success probability $1/2$, and $(\xi_{k,o})_{k\geq 1}$ and $(\xi_{k,e})_{k\geq 1}$ be two families of, respectively, odd geometric and even geometric independent random variables (the families are independent also of each other). Moreover let ξ_o and ξ_e be respectively a odd geometric and a even geometric random variable, and define $s_o^2 := \sigma^2(\xi_o)$, $s_e^2 := \sigma^2(\xi_e)$.

Lemma 6. *We have*

$$\mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \exp\left(t(m_o\Delta_{n,o} + m_e\Delta_{n,e}) + \frac{t^2}{2}\left(s_o^2\Sigma_{n,o} + s_e^2\Sigma_{n,e}\right) + \mathcal{O}(t^3 n)\right).$$

Proof. Consider the generating function $\phi_o(t) = \mathbb{E}(\exp(t\xi_o)) = \frac{3e^t}{4-e^{2t}}$, defined in $t \in]-\infty, \ln 2[$, the largest domain in which $\phi_o(t) < \infty$. We have

$$\begin{aligned}\phi_o(t) = \sum_{k\geq 0}\mathbb{P}(\xi_o = k)e^{tk} = &\sum_{k\geq 0}\mathbb{P}(\xi_o = k)\left(1 + kt + \frac{(kt)^2}{2} + \mathcal{O}(t^3)\right)\\ =&1 + \mathbb{E}(t\xi_o) + \frac{1}{2}\mathbb{E}\left((t\xi_o)^2\right) + \mathcal{O}(t^3)\\ =&\exp\left(tm_o + t^2\frac{s_o^2}{2} + \mathcal{O}(t^3)\right) \end{aligned} \tag{1.5}$$

Analogously, we define $\phi_e(t)$ to be the generating function of ξ_e, and observe that

$$\phi_e(t) = \exp\left(tm_e + t^2\frac{s_e^2}{2} + \mathcal{O}(t^3)\right). \tag{1.6}$$

Also note that $\phi_e(t)$ is finite if and only if $t \in]-\infty, \ln 2[$. Finally, by (1.5) and (1.6) we have

$$\begin{aligned}\mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}) &= \mathbb{E}\left(\exp(t\sum_{k=1}^{2n}\mathbf{1}_{\{\epsilon_{Y_k}=1\}}\xi_k - t\sum_{k=1}^{2n}\mathbf{1}_{\{\epsilon_{Y_k}=-1\}}\xi_k) \mid \mathcal{F}_{2n} \vee \mathcal{G}\right)\\ &=\mathbb{E}\left(\prod_{k=1}^{N_o^+}\exp(t\xi_{k,o})\prod_{k=1}^{N_e^+}\exp(t\xi_{k,e})\prod_{k=N_o^++1}^{N_o^++N_o^-}\exp(-t\xi_{k,o})\prod_{k=N_e^++1}^{N_e^++N_e^-}\exp(-t\xi_{k,e}) \mid \mathcal{F}_{2n} \vee \mathcal{G}\right)\\ &=\phi_o(t)^{N_o^+}\phi_e(t)^{N_e^+}\phi_o(-t)^{N_o^-}\phi_e(-t)^{N_e^-}\\ &=\exp\big(t(m_o(N_o^+ - N_o^-) + m_e(N_e^+ - N_e^-))\\ &\quad +t^2\left(\frac{s_o^2}{2}(N_o^+ + N_o^-) + \frac{s_e^2}{2}(N_e^+ + N_e^-)\right) + \mathcal{O}(t^3 n)\big)\\ &=\exp\left(t(m_o\Delta_{n,o} + m_e\Delta_{n,e}) + \frac{t^2}{2}(s_o^2\Sigma_{n,o} + s_e^2\Sigma_{n,e}) + \mathcal{O}(t^3 n)\right).\end{aligned}$$

□

Proposition 1. *For large n, on the set B_n, we have*

$$\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathcal{O}(\exp(-n^{\delta'}))$$

for any $\delta' \in]0, 2\delta_3[$.

Proof. Using Markov inequality, we have for $t < 0$

$$\begin{aligned}\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq & \mathbb{P}(X_{2n} \leq 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \\ = & \mathbb{P}(tX_{2n} \geq 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \\ = & \mathbb{P}(\exp(tX_{2n}) \geq 1 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \\ \leq & \mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}).\end{aligned}$$

For $0 < t < \ln 2$, we obtain analogously the same bound

$$\begin{aligned}\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq & \mathbb{P}(X_{2n} \geq 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \\ \leq & \mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}).\end{aligned}$$

Then by lemma 6 we obtain

$$\begin{aligned}\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) & \leq \exp\left(t(m_o\Delta_{n,o} + m_e\Delta_{n,e}) + \frac{t^2}{2}\left(s_o^2\Sigma_{n,o} + s_e^2\Sigma_{n,e}\right) + \mathcal{O}(t^3 n)\right) \\ & \leq \exp\Big(t(m_o\Delta_{n,o} + m_e\Delta_{n,e}) \\ & \quad + \frac{t^2}{2}\max\{s_o^2, s_e^2\}(\Sigma_{n,o} + \Sigma_{n,e}) + \mathcal{O}(t^3 n)\Big) \\ & = \exp(t(m_o\Delta_{n,o} + m_e\Delta_{n,e}) + t^2 s^2 n + O(t^3 n)),\end{aligned}$$

where $s := \max\{s_o, s_e\}$. Then, for the case $m_o\Delta_{n,o} + m_e\Delta_{n,e} > n^{\frac{1}{2}+\delta_3}$ we choose $t = -\frac{n^{\delta_3 - \frac{1}{2}}}{2s^2}$ and get

$$\begin{aligned}\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) & \leq \exp\left(-\frac{n^{2\delta_3}}{2s^2} + \frac{n^{2\delta_3 - 1}s^2 n}{4s^4} + \mathcal{O}((n^{\delta_3 - \frac{1}{2}})^3 n)\right) \\ & = \exp\left(-\frac{n^{2\delta_3}}{4s^2} + \mathcal{O}(n^{3\delta_3 - \frac{1}{2}})\right).\end{aligned}$$

Finally, for the case $m_o\Delta_{n,o} + m_e\Delta_{n,e} < -n^{\frac{1}{2}+\delta_3}$ we choose $t = \frac{n^{\delta_3 - \frac{1}{2}}}{2s^2}$ and get exactly the same bound. $\square$

Corollary 1.

$$\sum_{n \in \mathbb{N}} p_{n,1} < \infty.$$

Proof. Observe that

$$\mathbb{P}(X_{2n}=0, Y_{2n}=0, B_n \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq \mathbf{1}_{B_n} \mathbb{P}(X_{2n}=0 \mid \mathcal{F}_{2n} \vee \mathcal{G}). \tag{1.7}$$

In proposition 1 we proved that, on B_n, $\mathbb{P}(X_{2n}=0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathcal{O}(\exp(-n^{\delta'}))$ for $\delta' \in]0, 2\delta_3[$. Then, taking expectations on both sides of (1.7) we obtain

$$p_{n,1} \leq \mathbb{E}(\mathcal{O}(\exp(-n^{\delta'}))\mathbf{1}_{B_n}) = \mathcal{O}(\exp(-n^{\delta'}))\mathbb{E}(\mathbf{1}_{B_n}) \leq \mathcal{O}(\exp(-n^{\delta'})).$$

Thus, $p_{n,1}$ is summable. □

1.2.2 Estimate of $p_{n,2}$

Lemma 7. *We have*

$$\mathbb{P}(X_{2n}=0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. In the proof of lemma 4 we saw that the conditional characteristic function of X_{2n} with respect to $\mathcal{F}_{2n} \vee \mathcal{G}$ takes on the following form:

$$\phi(\theta) := \mathbb{E}(\exp(i\theta X_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \prod_{y \in \mathbb{Z}} \chi_o(\theta \epsilon_y)^{m^{(y)}_{2n-1,o}} \chi_e(\theta \epsilon_y)^{m^{(y)}_{2n-1,e}}.$$

We have, by the inversion formula

$$\mathbb{P}(X_{2n}=0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta) d\theta.$$

By lemma 3 we have

$$r(\theta) := |\chi_e(\theta)| = |\chi_o(\theta)| = \frac{3}{\sqrt{17 - 8\cos(2\theta)}}.$$

Thus

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta) d\theta \leq & \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_{y \in \mathbb{Z}} |\chi_o(\theta \epsilon_y)|^{m^{(y)}_{2n-1,o}} |\chi_e(\theta \epsilon_y)|^{m^{(y)}_{2n-1,e}} d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_{y \in \mathbb{Z}} |\chi_o(\theta \epsilon_y)|^{\eta_{2n-1}(y)} d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\theta)^{\sum_{y \in \mathbb{Z}} \eta_{2n-1}(y)} d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\theta)^{2n} d\theta.
\end{aligned}$$

Now we use the parity of $r(\theta)$ and the fact that $r(\theta) < 1$ in $\theta \in]0, \pi[\cup]\pi, 2\pi[$ to bound with $K < 1$ the function $r(\theta)$ in the interval $[\frac{\pi}{4}, \frac{3}{4}\pi] \cup [-\frac{\pi}{4}, -\frac{3}{4}\pi]$. We obtain

$$\begin{aligned}\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq & \frac{1}{\pi}\int_0^{\frac{\pi}{4}} r(\theta)^{2n} d\theta + \frac{1}{\pi}\int_{\pi}^{\frac{5\pi}{4}} r(\theta)^{2n} d\theta + \mathcal{O}(K^{2n}) \\ = & \frac{2}{\pi}\int_0^{\frac{\pi}{4}} r(\theta)^{2n} d\theta + \mathcal{O}(K^{2n}).\end{aligned}$$

Now, we have $r(\theta) = 1 - \frac{8}{9}\theta^2 + \mathcal{O}(\theta^3)$ and so for large n

$$\int_0^{\frac{\pi}{4}} r(\theta)^{2n} d\theta \sim \int_0^{\frac{\pi}{4}} \left(e^{-\frac{8}{9}\theta^2}\right)^{2n} d\theta \sim \int_0^{\infty} \left(e^{-\frac{16}{9}n\theta^2}\right) d\theta \sim \frac{c}{\sqrt{n}},$$

with $c = \sqrt{\frac{9\pi}{16}}$. □

The next result is the analogous to proposition 4.6 in [1]; the proof doesn't require any modification when switching to our framework. However we include it due to a simplification that we achieve by applying the Cauchy-Schwarz inequality.

Proposition 2. *For large n, we have*

$$\mathbb{P}(A_n \backslash B_n \mid \mathcal{F}_{2n}) = \mathcal{O}(n^{-\frac{1}{4} + \frac{2\delta_3 + \delta_1}{2}}).$$

Proof. To simplify the notation, let $\hat{\eta}_{2n}(y) := m_o m_{2n-1,o}^{(y)} + m_e m_{2n-1,e}^{(y)}$. The required probability is an estimate, on the set A_n, of $\mathbb{P}(|\sum_{y\in\mathbb{Z}} \epsilon_y \hat{\eta}_{2n-1}(y)| \leq n^{\frac{1}{2}+\delta_3} \mid \mathcal{F}_{2n})$. Now let G be a centred Gaussian random variable with $\sigma = n^{\frac{1}{2}+\delta_3}$, independent (conditionally on $\mathcal{F}$) of $\epsilon_y \hat{\eta}_{2n-1}(y)$ for every $y \in \mathbb{Z}$. By lemma 3.2 in [2] we have the following inequality

$$\mathbb{P}\left(\left|\sum_{y\in\mathbb{Z}} \epsilon_y \hat{\eta}_{2n-1}(y)\right| \leq n^{\frac{1}{2}+\delta_3} \mid \mathcal{F}\right) \leq c\,\mathbb{P}\left(\left|\sum_{y\in\mathbb{Z}} \epsilon_y \hat{\eta}_{2n-1}(y) + G\right| \leq n^{\frac{1}{2}+\delta_3} \mid \mathcal{F}\right),$$

where the constant c is independent of n.

Let

$$\chi_2(t) := \mathbb{E}\left(\exp\left(it\sum_{y\in\mathbb{Z}} \epsilon_y \hat{\eta}_{2n-1}(y)\right) \mid \mathcal{F}_{2n}\right) = \prod_{y\in\mathbb{Z}} \cos(t\hat{\eta}_{2n-1}(y)),$$

where we used the fact that, conditionally on $\mathcal{F}_{2n}$, $\sum_{y\in\mathbb{Z}} \epsilon_y \hat{\eta}_{2n-1}(y)$ is the sum of i.i.d. symmetric Bernoulli random variables. Then, let

$$\chi_3(t) := \mathbb{E}\left(\exp(itG) \mid \mathcal{F}_{2n}\right) = \mathbb{E}\left(\exp(itG)\right) = \exp\left(-\frac{1}{2}t^2 n^{1+2\delta_3}\right).$$

Thus,

$$\mathbb{P}\left(\exp\left(it\sum_{y\in\mathbb{Z}}\epsilon_y\hat{\eta}_{2n-1}(y)+G\right)|\mathcal{F}_{2n}\right)=\chi_2(t)\chi_3(t),$$

because, conditionally on $\mathcal{F}_{2n}$, the variables $\sum_{y\in\mathbb{Z}}\epsilon_y\hat{\eta}_{2n-1}(y)$ and G are independent. Now we use the Plancherel formula (see e.g. [7]) and obtain

$$\mathbb{P}\left(\left|\sum_{y\in\mathbb{Z}}\epsilon_y\hat{\eta}_{2n-1}(y)\right|\le n^{\frac{1}{2}+\delta_3}\mid\mathcal{F}_{2n}\right)=\frac{n^{\frac{1}{2}+\delta_3}}{\pi}\int\frac{\sin(tn^{\frac{1}{2}+\delta_3})}{tn^{\frac{1}{2}+\delta_3}}\chi_2(t)\chi_3(t)\le Cn^{\frac{1}{2}+\delta_3}I,$$

where C is a positive constant and $I=\int\chi_2(t)\chi_3(t)dt$.

Fix $b_n=\frac{n^{\delta_4}}{n^{\frac{1}{2}+\delta_3}}$, for some $\delta_4>0$ and split the integral I into I_1+I_2, the first part for $|t|<b_n$, the second for $|t|>b_n$. We have

$$\begin{aligned}I_2=&\int_{|t|>b_n}\chi_2(t)\chi_3(t)dt\le C\int_{|t|>b_n}\exp\left(-\frac{t^2}{2}n^{1+2\delta_3}\right)\frac{dt}{\sqrt{2\pi}}\\=&\frac{C}{n^{\frac{1}{2}+\delta_3}}\int_{|s|>n^{\delta_4}}\exp\left(\frac{-s^2}{2}\right)\frac{ds}{\sqrt{2\pi}}\le 2\frac{C}{n^{\frac{1}{2}+\delta_3}}\frac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-n^{2\delta_4}}{2}\right)}{n^{\delta_4}},\end{aligned}$$

because for a standard normal distributed random variable X and for $x>0$ we have $\mathbb{P}(X\ge x)\le\frac{1}{x}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)$.

Now, we need to estimate I_1. Of course we have

$$I_1=\int_{|t|\le b_n}\chi_2(t)\chi_3(t)dt\le\int_{|t|\le b_n}\prod_{y\in\mathbb{Z}}|\cos\left(t\hat{\eta}_{2n-1}(y)\right)|dt.$$

Note that for $|x|\le\frac{\pi}{2}$ we have $|\cos(x)|\le\exp(\frac{-x^2}{2})$. By choosing $\delta_3>\delta_4+\delta_2$, we can ensure that $t\hat{\eta}_{2n-1}(y)<1<\frac{\pi}{2}$ for every y and sufficiently large n, and hence by Cauchy-Schwarz inequality

$$\begin{aligned}\int_{|t|\le b_n}\prod_{y\in\mathbb{Z}}|\cos\left(t\hat{\eta}_{2n-1}(y)\right)|dt\le&\int_{|t|\le b_n}\prod_{y\in\mathbb{Z}}\exp\left(-\frac{t^2\hat{\eta}^2_{2n-1}(y)}{2}\right)dt\\=&\int_{|t|\le b_n}\exp\left(-\frac{t^2}{2}\sum_{y=0}^{n^{\frac{1}{2}+\delta_1}}\hat{\eta}^2_{2n-1}(y)\right)dt\\\le&\int_{|t|\le b_n}\exp\left(-ct^2n^{\frac{3}{2}-\delta_1}\right)dt\\\le&\sqrt{\frac{\pi}{cn^{\frac{3}{2}-\delta_1}}}=\mathcal{O}(n^{-\frac{3}{4}+\frac{\delta_1}{2}}),\end{aligned}$$

where c is a positive constant. Finally, putting all together, we have

$$\mathcal{O}(n^{\frac{1}{2}+\delta_3}(I_1+I_2)) = \mathcal{O}(n^{\frac{1}{2}+\delta_3}I_1) = \mathcal{O}(n^{\frac{1}{2}+\delta_3}n^{-\frac{3}{4}+\frac{\delta_1}{2}})) = \mathcal{O}(n^{-\frac{1}{4}+\frac{2\delta_3+\delta_1}{2}}).$$

□

Corollary 2.

$$\sum_{n\in\mathbb{N}} p_{n,2} < \infty.$$

Proof. We have

$$\begin{aligned}
p_{n,2} =& \mathbb{P}(X_{2n}=0, Y_{2n}=0, A_n\backslash B_n) \\
=& \mathbb{E}(\mathbf{1}_{Y_{2n}=0}\mathbb{E}(\mathbf{1}_{A_n\backslash B_n}\mathbf{1}_{X_{2n}=0} \mid \mathcal{F}_{2n})) \\
=& \mathbb{E}(\mathbf{1}_{Y_{2n}=0}\mathbb{E}(\mathbb{E}(\mathbf{1}_{A_n\backslash B_n}\mathbf{1}_{X_{2n}=0} \mid \mathcal{F}_{2n}\vee\mathcal{G}) \mid \mathcal{F}_{2n})) \\
=& \mathbb{E}(\mathbf{1}_{Y_{2n}=0}\mathbb{E}(\mathbf{1}_{A_n\backslash B_n}\mathbb{P}(X_{2n}=0 \mid \mathcal{F}_{2n}\vee\mathcal{G}) \mid \mathcal{F}_{2n})) \\
=& \mathcal{O}\left(n^{-\frac{1}{2}}n^{-\frac{1}{2}}n^{-\frac{1}{4}+\frac{2\delta_3+\delta_1}{2}}\right) \\
=& \mathcal{O}(n^{-\frac{5}{4}+\frac{2\delta_3+\delta_1}{2}}).
\end{aligned}$$

Where we used the estimates of lemma 1, lemma 7 and proposition 2. Now it's enough to choose $2\delta_3+\delta_1<\frac{1}{2}$. □

1.2.3 Estimate of $p_{n,3}$

Notice that $A_n^c = A_{n,1}^c \cup A_{n,2}^c$. We are going to provide exponential estimates of both $\mathbb{P}(A_{n,1}^c \mid Y_{2n}=0)$ and $\mathbb{P}(A_{n,2}^c \mid Y_{2n}=0)$.

Lemma 8. *We have, for large n and for every $t>0$*

$$\mathbb{E}(e^{tY_{2n}}) \sim c\left(q\cosh t + \sqrt{q^2\cosh^2 t - (2q-1)}\right)^{2n},$$

with $c>0$.

Proof. We have, by the Markov property,

$$\mathbb{E}_{\nu_0}(e^{tY_{2n}}) = e^{t\nu_0}\int \pi_\nu(\nu_0,dy_1)e^{ty_1}\int \pi_\nu(y_1,dy_2)e^{ty_2}\cdots\int \pi_\nu(y_{2n-1},dy_{2n})e^{ty_{2n}}. \qquad (1.8)$$

It is now easy to see that can compute the quantity (1.8) by means of the $2n$-th power of the matrix

$$\pi_{\nu,t} := \begin{pmatrix} qe^t & (1-q)e^{-t} \\ (1-q)e^t & qe^{-t} \end{pmatrix}$$

with $q = \frac{1}{3}$. which has the following eigenvalues

$$\lambda_{1,2}(t) = q\cosh t \pm \sqrt{q^2\cosh^2 t - (2q-1)}.$$

By the spectral decomposition (e.g. see lemma 12.2 in [18]), we know that

$$(\pi_{\nu,t})^{2n} \sim \lambda_1^{2n}(t) h_1 h_1^T$$

for large n, where λ_1 is the largest eigenvalue and h_1 represents the (column) eigenvector associated with λ_1. Hence for large n

$$\mathbb{E}(e^{tY_{2n}}) = \sum_{y\in\{1,-1\}} (\pi_{\nu,t})^{2n}(\nu_0, y) \sim c\lambda_1^{2n}(t), \quad c > 0.$$

□

The following is an analogous to the classical reflection principle for simple symmetric random walk.

Lemma 9. *Let $n \in \mathbb{N}$, $y \in \mathbb{Z}$. We have*

$$\mathbb{P}(\max_{0\le k\le 2n} Y_k = y, Y_{2n} = 0) \le 2\mathbb{P}(Y_{2n} = 2y).$$

Proof. Consider a path in $\{\max_{0\le k\le 2n} Y_k = y, Y_{2n} = 0\}$ and reflect it when it visits y for the last time, say at time n_y, before returning to the origin (see figure 4). Note that the original path changes direction at time n_y, which happens with probability 2/3, while the reflected one keeps straight (in this case the probability is 1/3); this implies that the probability of the original path is 2 times the probability of the reflected one. Then the result follows by noting that the reflected path belongs to the event $\{Y_{2n} = 2y\}$. □

Proposition 3. *For large n, there exists $\delta > 0$ such that*

$$\mathbb{P}(A_{n,1}^c \mid Y_{2n} = 0) = \mathcal{O}\left(\exp\left(-n^\delta\right)\right).$$

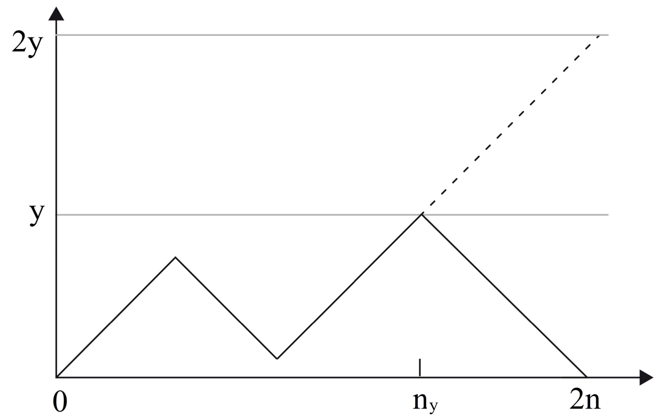


Figure 4: The reflection principle for the vertical skeleton Y_n. The probability of the path reflected at time n_y (after n_y the reflected path is represented by the dashed line) is half the probability of the original one.

Proof. Let $a_n = [n^{\frac{1}{2}+\delta_1}]$; we have

$$\mathbb{P}(\max_{0\leq k\leq 2n} Y_k \geq a_n \mid Y_{2n}=0) = \sum_{y\in\{a_n,a_{n+1},\dots,n\}} \frac{\mathbb{P}(\max_{0\leq k\leq 2n} Y_k = y, Y_{2n}=0)}{\mathbb{P}(Y_{2n}=0)}$$

The estimate for $\mathbb{P}(\min_{0\leq k\leq 2n} Y_k \leq -a_n \mid Y_{2n}=0)$ can be obtained by the same argument, so we shall omit it. By lemma 9

$$\begin{aligned}\sum_{y\in\{a_n,a_{n+1},\dots,n\}} \mathbb{P}(\max_{0\leq k\leq 2n} Y_k = y, Y_{2n}=0) \leq & 2\sum_{y\in\{a_n,a_{n+1},\dots,n\}} \mathbb{P}(Y_{2n}=2y)\\ =& 2\mathbb{P}(Y_{2n}\geq 2a_n)\\ =& 2\inf_{t>0}\mathbb{P}(\exp(tY_{2n})\geq \exp(2ta_n))\\ \leq & 2\inf_{t>0}\frac{\mathbb{E}(e^{tY_{2n}})}{e^{2ta_n}}.\end{aligned}$$

By lemma 8, we have that for large n

$$\mathbb{E}(e^{tY_{2n}}) \sim c\left(q\cosh t + \sqrt{q^2\cosh^2 t-(2q-1)}\right)^{2n}.$$

Now by the Taylor expansion at $t=0$, and substituting $q=\frac{1}{3}$, we have

$$q\cosh t + \sqrt{q^2\cosh^2 t-(2q-1)} = \frac{2+\sqrt{7}}{6} + st^2 + o(t^2) < 1 + st^2,$$

with $s = \frac{1}{6} + \frac{1}{3\sqrt{7}}$, where the inequality holds for $t \le t^*$ for sufficiently small t^*: whence $\mathbb{E}(e^{tY_{2n}}) < c(1 + 2nst^2) < c\exp(2nst^2)$. So for large n

$$\inf_{t>0} \frac{\mathbb{E}(e^{tY_{2n}})}{e^{2ta_n}} < c \inf_{t>0, t\le t^*} \exp(-2ta_n)\exp(2nst^2) = c\exp\left(-\frac{a_n^2}{2sn}\right) = c\exp\left(\frac{-n^{2\delta_1}}{2s}\right),$$

where in the first equality we used the fact that the minimum is attained at $t = \frac{a_n}{2ns}$. Then, putting all together and using lemma 1, we obtain

$$\mathbb{P}(A_{n,1}^c \mid Y_{2n} = 0) = \mathcal{O}\left(nn^{\frac{1}{2}}\exp\left(\frac{-n^{2\delta_1}}{2s}\right)\right).$$

□

Let $\sigma_{a,a}$ the time of first return to state $a \in \mathbb{Z} \times \{-1, 1\}$ of the Markov chain (Y_n, ν_n) starting at a.

Lemma 10. *We have*

$$\mathbb{E}(e^{-t\sigma_{a,a}}) \sim \exp(-c\sqrt{t}),$$

with $c > 0$*, i.e.*

$$\lim_{t\to 0} \frac{\mathbb{E}(e^{-t\sigma_{a,a}})}{\exp(-c\sqrt{t})} = 1.$$

Proof. By the local limit theorem for Markov chains ([15], lemma 14) we have

$$p_{a,a}^{(2n)} := \mathbb{P}_a((Y_{2n}, \nu_{2n}) = a) \sim \frac{C}{\sqrt{n}}$$

as $n \to \infty$ with $C > 0$. This implies, by the Tauberian theorem ([7], p.447, th.5), that there exists $C_1 > 0$ such that

$$G_{a,a}(s) := \sum_{k=0}^{\infty} p_{a,a}^{(k)} s^k \sim \frac{C_1}{\sqrt{1-s}}$$

as $s \to 1$. Then, using a standard result from the theory of Markov chains (see, for instance, [24], th.1.38), we see that as $s \to 1$

$$\mathbb{E}(s^{\sigma_{a,a}}) = 1 - \frac{1}{G_{a,a}(s)} \sim 1 - c\sqrt{1-s},$$

where $c = C_1^{-1}$. Finally, if we write $s = e^{-t}$, we have for $t \to 0$

$$\mathbb{E}(e^{-t\sigma_{a,a}}) \sim 1 - c\sqrt{1-e^{-t}} \sim 1 - c\sqrt{t} \sim e^{-c\sqrt{t}}.$$

□

Proposition 4. *There exist $\delta' > 0$ such that for large n*

$$\mathbb{P}(A^c_{n,2} \mid Y_{2n} = 0) = \mathcal{O}\left(\exp(-n^{\delta'})\right).$$

Proof. We have

$$\mathbb{P}(A^c_{n,2} \mid Y_{2n} = 0) = \mathbb{P}\left(\max_{y\in\mathbb{Z}} \eta_{2n-1}(y) \geq n^{\frac{1}{2}+\delta_2} \mid Y_{2n} = 0\right) \leq \sum_{y\in\mathbb{Z}} \frac{\mathbb{P}\left(\eta_{2n-1}(y) \geq n^{\frac{1}{2}+\delta_2}\right)}{\mathbb{P}(Y_{2n} = 0)}.$$

On the other hand we have

$$\mathbb{P}\left(\eta_{2n-1}(y) \geq a_n\right) \leq \mathbb{P}\left(\eta_{2n-1}(y,1) \geq \frac{a_n}{2}\right) + \mathbb{P}\left(\eta_{2n-1}(y,-1) \geq \frac{a_n}{2}\right). \tag{1.9}$$

Now let $\sigma^{(k)}_{a,a}$ be the time of k-th return to point a for the process $(Y_n, \nu_n)_{n\geq 0}$ starting at a. Observe that

$$\mathbb{P}\left(\eta_{2n-1}(a) \geq a_n\right) \leq \mathbb{P}_a\left(\sigma^{(\lfloor a_n \rfloor)}_{a,a} \leq 2n\right) \tag{1.10}$$

and consider the first term at the right hand side of (1.9). Notice that by lemma 10, $\mathbb{E}(e^{-t\sigma_{a,a}})^m \sim \exp(-cm\sqrt{t})$ for every $m \in \mathbb{N}$; then, for $C > 1$ there exists t^* s.t. for every $t < t^*$,

$$\mathbb{E}(e^{-t\sigma_{a,a}})^m \leq C\exp(-cm\sqrt{t}). \tag{1.11}$$

Hence, by (1.10) and (1.11), we have for sufficiently large n

$$\begin{aligned}
\mathbb{P}\left(\eta_{2n-1}(y,1) \geq \frac{a_n}{2}\right) &\leq \inf_{t>0} \mathbb{P}_y\left(\exp\left(-t\sigma^{([\frac{a_n}{2}])}_{(y,1),(y,1)}\right) \geq \exp(-2nt)\right)\\
&\leq \inf_{t>0} \exp(2nt)\left(\mathbb{E}\left(\exp\left(-t\sigma^{(1)}_{(y,1),(y,1)}\right)\right)\right)^{[\frac{a_n}{2}]}\\
&\leq C \inf_{t>0, t<t^*} \exp\left(2nt - \frac{a_n}{2}c\sqrt{t}\right)\\
&= C\exp\left(-\frac{c^2 a_n^2}{32n}\right)\\
&= C\exp\left(-c' n^{2\delta_2}\right)
\end{aligned}$$

with $c' = \frac{c^2}{32}$, where we used the fact that the minimum is attained at $t = \left(\frac{ca_n}{8n}\right)^2$.

Since we can provide, with the same procedure, an exponential estimate also for the rightmost term in (1.9), and using lemma 1, we finally obtain

$$\mathbb{P}(A^c_{n,2} \mid Y_{2n} = 0) \leq \sum_{y\in\mathbb{Z}} \frac{\mathbb{P}\left(\eta_{2n-1}(y) \geq n^{\frac{1}{2}+\delta_2}\right)}{\mathbb{P}(Y_{2n} = 0)} = \mathcal{O}\left(nn^{\frac{1}{2}} \exp(-cn^{\delta_2})\right).$$

□

Corollary 3.

$$\sum_{n\in\mathbb{N}} p_{n,3} < \infty.$$

Proof. Combining proposition 3 and 4, we know that for large n

$$\mathbb{P}(A_n^c \mid Y_{2n} = 0) = \mathcal{O}(\exp(-n^{\min\{\delta,\delta'\}})).$$

Then the result follows by the trivial majorization

$$p_{n,3} := \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, A_n^c) \leq \mathbb{P}(Y_{2n} = 0, A_n^c) \leq \mathbb{P}(A_n^c \mid Y_{2n} = 0).$$

□

The a.s. transience now follows from $p_n = p_{n,1} + p_{n,2} + p_{n,2}$ together with lemma 5.

Chapter 2

Periodic orientations with random perturbations

In the previous Chapter we showed that if the levels of the lattice are oriented according to i.i.d. Rademacher random variables then, due to the presence and level of fluctuations, the random walk exhibits almost sure transient behavior. Now we want to introduce a class of periodic orientations such that the fluctuations are absent and deduce the recurrence of the walk; later, we will also introduce random perturbations in the periodic orientations, and characterize the problem of type for a larger class of randomly oriented graphs.

2.1 Recurrence of the honeycomb lattice with periodic orientations

Let $Q > 1$ be an even integer and $f : \mathbb{Z} \longrightarrow \{-1, 1\}$ a Q-periodic function such that

$$\sum_{k=0}^{Q-1} f(k) = 0. \tag{2.1}$$

Accordingly, let $\mathbf{H}_f$ be the oriented honeycomb lattice whose vertical edges are unoriented and the horizontal ones are oriented according to the value of f: that is, if $f(y) = 1$ then the horizontal edges leading out from vertexes (x, y), $x \in \mathbb{Z}$, are right-directed, otherwise they are left-directed (cf. figure 5). Note that $\mathbf{H}_f$ is not a random graph. As before, we consider the simple random walk M on $\mathbf{H}_f$ and decompose it into the vertical skeleton Y and the embedded random walk X.

Theorem 2. *The simple random walk on $\mathbf{H}_f$ is recurrent.*

In order to take advantage of periodicity, we begin by analyzing the occupation times of the vertical skeleton restricted to $\mathbb{Z}_Q$.

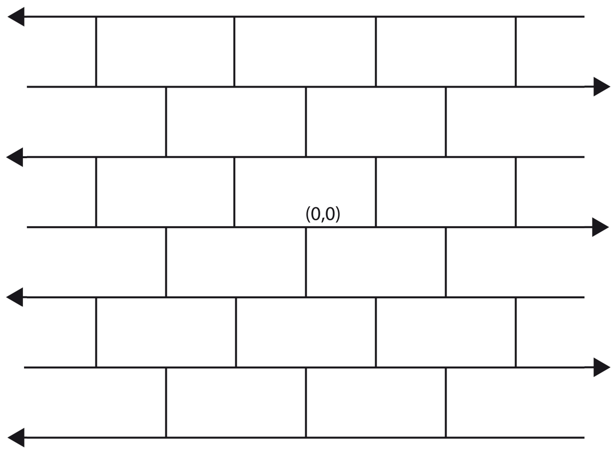


Figure 5: The lattice $\mathbf{H}_f$ with alternate orientations, i.e. $f(y) = (-1)^{|y|}$ and so the horizontal orientations have period $Q = 2$.

2.1.1 The vertical skeleton in the periodic case

Let $\mathbb{Z}_Q = \mathbb{Z}/Q$ and, for every $y \in \mathbb{Z}$, we write $\overline{y} = y \mod Q$. Define for every $n > 0$

$$W_n := (Y_{n-1}, \nu_{n-1}; Y_n, \nu_n)$$

and

$$\overline{W}_n := (\overline{Y}_{n-1}, \nu_{n-1}; \overline{Y}_n, \nu_n), \tag{2.2}$$

where $\overline{Y}_n = Y_n \mod Q$, and $W_0 := (-1, -1; 0, 1)$, $\overline{W}_0 := (\overline{-1}, -1; \overline{0}, 1)$.

Lemma 11. *The process $(\overline{W}_n)_{n\geq 0}$ is a one-class recurrent Markov chain with period 2. Its stationary distribution π is defined as follows*

$$\begin{cases} \pi(\overline{y}, \nu; \overline{y}', \nu') = \frac{2}{3}\frac{1}{2Q} & \text{if } \nu \neq \nu', \overline{y}' = \overline{y + \nu'} \\ \pi(\overline{y}, \nu; \overline{y}', \nu') = \frac{1}{3}\frac{1}{2Q} & \text{if } \nu = \nu', \overline{y}' = \overline{y + \nu'} \end{cases} \tag{2.3}$$

Proof. It is easy to verify that $(\overline{Y}_n, \nu_n)_{n\geq 0}$ is a Markov chain with $2Q$ elements and period 2, and that its stationary distribution is $\tilde{\pi}(\overline{y}, \nu) = \frac{1}{2Q}$, $\forall(\overline{y}, \nu) \in \mathbb{Z}_Q \times \{-1, 1\}$. Then $(\overline{Y}_n, \nu_n; \overline{Y}_{n+1}, \nu_{n+1})_{n\geq 0}$ is again a MC and its stationary distribution π can be derived from $\tilde{\pi}$; since

$$\sum_{(\overline{y},\nu)\in\mathbb{Z}_Q\times\{-1,1\}} \tilde{\pi}(\overline{y}, \nu) p_{(\overline{y},\nu),(\overline{y}',\nu')} = \tilde{\pi}(\overline{y}', \nu')$$

and

$$p_{(\overline{y},\nu;\overline{y}',\nu'),(\overline{y}',\nu',\overline{y}'',\nu'')} = p_{(\overline{y}',\nu'),(\overline{y}'',\nu'')}$$

then we shall define

$$\pi(\overline{y}, \nu; \overline{y}', \nu') := \tilde{\pi}(\overline{y}, \nu) p_{(\overline{y},\nu),(\overline{y}',\nu')}.$$

In fact we have

$$\begin{aligned} \pi(\overline{y}', \nu'; \overline{y}'', \nu'') &= \tilde{\pi}(\overline{y}', \nu') p_{(\overline{y}',\nu'),(\overline{y}'',\nu'')} \\ &= \sum_{(\overline{y},\nu)\in\mathbb{Z}_Q\times\{-1,1\}} \tilde{\pi}(\overline{y}, \nu) p_{(\overline{y},\nu),(\overline{y}',\nu')} p_{(\overline{y}',\nu'),(\overline{y}'',\nu'')} \\ &= \sum_{(\overline{y},\nu)\in\mathbb{Z}_Q\times\{-1,1\}} \pi(\overline{y}, \nu; \overline{y}', \nu') p_{(\overline{y},\nu;\overline{y}',\nu'),(\overline{y}',\nu',\overline{y}'',\nu'')} \end{aligned}$$

The other statements are also easy to verify. □

Note that W_n encloses the information of the last three movements of the vertical skeleton Y_n: the reason for considering such a process, and its analogous $\overline{W}_n$ in $(\mathbb{Z}_Q \times \{-1, 1\})^2$, is that we will need to control the number of times $(Y_n)_{n\geq 0}$ "changes direction" at a certain level before it returns to the origin. Then by taking advantage of the periodicity of the orientations, we will be able to bound the difference between the number of steps to the right and to the left of the embedded random walk X_n, distinguishing between the odd-valued and the even-valued steps, and to deduce that the probability of X_n returning to 0 is of order $n^{-1/2}$ for a set of paths with positive probability, which will imply the recurrence of M.

We begin by defining the following functionals of $\overline{W}$.

$$S_{n,e} := \sum_{i=1}^{2n} f_e(\overline{W}_i) := \sum_{i=1}^{2n} f(\overline{Y}_{i-1})\mathbf{1}_{\{\nu_{i-1}\neq\nu_i\}},$$

$$S_{n,o} := \sum_{i=1}^{2n} f_o(\overline{W}_i) := \sum_{i=1}^{2n} f(\overline{Y}_{i-1})\mathbf{1}_{\{\nu_{i-1}=\nu_i\}}.$$

Moreover for every $n \in \mathbb{N}$ define the event

$$\mathcal{Z}_n := \{W_{2n} = W_0\} = \{\overline{W}_{2n} = \overline{W}_0, Y_{2n} = 0\} \tag{2.4}$$

Proposition 5. *Let $C > 0$. We have, for sufficiently large n*

$$\mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq C\sqrt{n} \mid \mathcal{Z}_n) \geq \delta_C > 0.$$

Proof. To simplify our notation, we identify the states of $\overline{W}_n$ with the integers $\{1, 2, ..., 4Q\}$, with arbitrary order. Accordingly we define

$$\pi = (\pi_1, ..., \pi_{4Q})$$

to be the vector where the i-th component is the value that the stationary distribution takes at state i, and the occupation measure

$$\overline{\eta}_n = (\overline{\eta}_n(1), ..., \overline{\eta}_n(4Q)),$$

where $\overline{\eta}_n(i) := \sum_{k=0}^{n} \mathbf{1}_{\{\overline{W}_k=i\}}$, for $1 \leq i \leq 4Q$. By definition we have

$$S_{n,e} = \sum_{i=1}^{4Q} u_i\overline{\eta}_{2n}(i) = u\overline{\eta}_{2n}^T,$$

where $u \in \{-1, 0, 1\}^{4Q}$ is the vector such that u_i equals to the value that f_e takes on the i-th state. Analogously, let $v \in \{-1, 0, 1\}^{4Q}$ such that $S_{n,o} = v\overline{\eta}_{2n}^T$ and $w \in \{-1, 1\}^{4Q}$

such that $Y_{2n} = \sum_{i=1}^{2n} \nu_i = w\overline{\eta}_{2n}^T$. Note that u, v, w are linearly independent vectors and that we have

$$u(2n\pi)^T = v(2n\pi)^T = w(2n\pi)^T = 0, \tag{2.5}$$

by (2.3).

Let $c > 0$. By the multidimensional local limit theorem for the random vector $\overline{\eta}_{2n}$ (lemma 16 in [15]) we know that there exist a lattice $Z \subset \mathbb{Z}^{4Q}$ of dimension r, $r \geq 1$, and a constant $c' > 0$ dependent of c, such that

$$\mathbb{P}(\overline{\eta}_{2n} = x, \overline{W}_{2n} = \overline{W}_0) \geq \frac{c'}{n^{r/2}}, \tag{2.6}$$

for large n and for all $x \in Z$ such that $|x_i - \pi_i| \leq c\sqrt{n}$, $1 \leq i \leq 4Q$. Hence, by (2.6) and (2.5), and taking $c = \frac{C}{4Q}$, we have

$$\begin{aligned}
\mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq C\sqrt{n}; \mathcal{Z}_n) \geq & \mathbb{P}\left(|\overline{\eta}_{2n}(i) - \pi_i| \leq c\sqrt{n}, \forall i; \mathcal{Z}_n\right) \\
= & \sum_{\substack{x \in Z, wx^T = 0, \\ |x_i - \pi_i| \leq c\sqrt{n}, \forall i}} \mathbb{P}(\overline{\eta}_{2n} = x, \overline{W}_{2n} = \overline{W}_0) \\
\geq & |\{x \in Z, wx^T = 0, |x_i - \pi_i| \leq c\sqrt{n}, \forall i\}| \frac{c'}{n^{r/2}} \\
= & C' \frac{n^{(r-1)/2}}{n^{r/2}} \geq \frac{C'}{\sqrt{n}},
\end{aligned} \tag{2.7}$$

with $C' > 0$. Finally by (2.7) and lemma 1

$$\mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq C\sqrt{n} | \mathcal{Z}_n) \geq \frac{\mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq C\sqrt{n}, \mathcal{Z}_n)}{\mathbb{P}(Y_{2n} = 0)} \geq \delta_C > 0.$$

□

2.1.2 Proof of recurrence

In order to prove recurrence it is enough to show that $\sum_{k=1}^{\infty} \mathbb{P}(X_k = 0, Y_k = 0) = \infty$, since

$$\sum_{k=1}^{\infty} \mathbb{P}(M_k = (0,0)) \geq \sum_{k=1}^{\infty} \mathbb{P}(X_k = 0, Y_k = 0).$$

Define a set of constrained paths

$$\begin{aligned}\mathrm{Constr}(n,f) :=\Big\{&(\gamma,q) : \{-1,0,1,...,2n\} \longrightarrow \mathbb{Z}\times\{-1,1\} \text{ s.t. } \forall i,\ \gamma(i)=\gamma(i-1)\pm 1,\\ &(\gamma(-1),q(-1);\gamma(0),q(0)) = (\gamma(2n-1),q(2n-1);\gamma(2n),q(2n)) = W_0,\\ &\left|\sum_{i=1}^{2n} f(\overline{\gamma}_{i-1})\mathbf{1}_{\{q_{i-1}\neq q_i\}}\right| + \left|\sum_{i=1}^{2n} f(\overline{\gamma}_{i-1})\mathbf{1}_{\{q_{i-1}=q_i\}}\right| \le C\sqrt{n}\Big\}.\end{aligned}$$

If we prove that $\forall(\gamma,q)\in\mathrm{Constr}(n,f)$

$$\mathbb{P}(X_n=0 \mid (Y_i,\nu_i)=(\gamma(i),q(i))\,\forall i\le n) \ge \frac{c}{\sqrt{n}} \tag{2.8}$$

then the recurrence of the random walk will follow. In fact, if this is the case, thanks to (2.8) and to proposition 5 we would have for large n

$$\begin{aligned}\mathbb{P}(X_{2n}=0,Y_{2n}=0) \ge&\,\mathbb{P}\left(X_{2n}=0,\mathcal{Z}_n,|S_e(n)|+|S_o(n)|\le C\sqrt{n}\right)\\ =&\sum_{(\gamma,q)\in\mathrm{Constr}(n,f)} \mathbb{P}(X_{2n}=0 \mid (Y_i,\nu_i)=(\gamma(i),q(i))\,\forall i\le 2n)\\ &\times\mathbb{P}((Y_i,\nu_i)=(\gamma(i),q(i))\,\forall i\le 2n)\\ \ge&\frac{c}{\sqrt{n}}\sum_{(\gamma,q)\in\mathrm{Constr}(n,f)}\mathbb{P}((Y_i,\nu_i)=(\gamma(i),q(i))\,\forall i\le 2n)\\ =&\frac{c}{\sqrt{n}}\mathbb{P}(|S_e(n)|+|S_o(n)|\le C\sqrt{n},\mathcal{Z}_n)\\ \ge&\frac{c'}{n},\end{aligned}$$

with $c,c'>0$, and so

$$\sum_{n=1}^{\infty}\mathbb{P}(X_{2n}=0,Y_{2n}=0)\ge C''\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

To prove (2.8) we proceed as follows. Let N_e^+ and N_e^- be, respectively, the number of right (left) directed even steps of the embedded random walk up to time $2n$, and N_o^+, N_o^- the analogous quantities for the odd steps. Observe that $S_{n,e}=N_e^+-N_e^-$ and $S_{n,o}=N_o^+-N_o^-$. In particular, for all $(\gamma,q)\in\mathrm{Constr}(n,f)$ we have

$$|N_e^+-N_e^-|+|N_o^+-N_o^-|\le C\sqrt{n}. \tag{2.9}$$

Lemma 12. *There exist positive constants C_1, C_2, C_3, with $C_2 < C_3$, such that for every $(\gamma, q) \in \mathrm{Constr}(n, f)$ and $n \in \mathbb{N}$ we have*

$$\mathbb{E}(X_{2n}|(Y_i, \nu_i) = (\gamma(i), q(i))\, \forall i \leq 2n) \leq C_1\sqrt{n}$$

and

$$C_2 n \leq \sigma^2(X_{2n}|(Y_i, \nu_i) = (\gamma(i), q(i))\, \forall i \leq 2n) \leq C_3 n.$$

Proof. By (2.9)

$$\begin{aligned}\mathbb{E}(X_{2n}|(Y_i, \nu_i) = (\gamma(i), q(i))\, \forall i \leq 2n) =& \mathbb{E}(\xi_{1,e})(N_e^+ - N_e^-) + \mathbb{E}(\xi_{1,o})(N_o^+ - N_o^-) \\ \leq& \max\{\mathbb{E}(\xi_{1,o}), \mathbb{E}(\xi_{1,e})\}C\sqrt{n}.\end{aligned}$$

On the other hand, the conditional variance of X_{2n} is, by independence, the sum of the variances of the even and odd geometric random variables, and so since both of them have finite variance we obtain the result. □

Proposition 6. *There exists $c > 0$ such that, for every $(\gamma, q) \in \mathrm{Constr}(n, f)$ and sufficiently large n, we have*

$$\mathbb{P}(X_{2n} = 0|(Y_i, \nu_i) = (\gamma(i), q(i))\, \forall i \leq 2n) \geq \frac{c}{\sqrt{n}}.$$

Proof. Fix $(\gamma, q) \in \mathrm{Constr}(n, f)$; from now on every probability will be taken conditionally to $\{(Y_i, \nu_i) = (\gamma(i), q(i))\, \forall i \leq 2n\}$, although, in order to simplify the notation, we will sometimes omit to write it.

Let $(\xi_k)_{k\geq 1}$ be a sequence of random variables such that for every k, ξ_k represents the k-th step of the embedded random walk X_n. On $\{(Y_i, \nu_i) = (\gamma(i), q(i))\, \forall i \leq 2n\}$ we have

$$X_{2n} = \sum_{k=1}^{2n} \xi_k = \sum_{i=1}^{N_e^+} \xi_{i,e} + \sum_{i=1}^{N_o^+} \xi_{i,o} - \sum_{i=N_e^+ +1}^{N_e^+ + N_e^-} \xi_{i,e} - \sum_{i=N_o^+ +1}^{N_o^+ + N_o^-} \xi_{i,o},$$

and for every k let $a_k := \mathbb{E}(\xi_k)$, $b_k^2 := \sigma^2(\xi_k)$ and

$$A_n := \sum_{k=1}^{2n} a_k, \qquad B_n^2 := \sum_{k=1}^{2n} b_k^2.$$

First, we are going to show that

$$\left| B_n \mathbb{P}(X_{2n} = 0) - \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{A_n}{B_n}\right)^2} \right| \to 0, \tag{2.10}$$

as $n \to \infty$. Then, thanks to the lemma 12, we complete the proof.

To prove (2.10), we wish to generalize a classical approach due to Gnedenko (cfr [8]). Let $\phi_{\xi_k}(t) = \mathbb{E}(e^{it\xi_k})$, and $\phi_{X_{2n}}(t) = \mathbb{E}(e^{itX_{2n}}) = \phi_{\sum_{k=1}^{2n} \xi_k}(t) = \prod_{k=1}^{2n} \phi_{\xi_k}(t)$, and precisely

$$\phi_{X_{2n}}(t) = \chi_e(t)^{N_e^+} \chi_o(t)^{N_o^+} \chi_e(-t)^{N_e^-} \chi_o(-t)^{N_o^-},$$

where we recall that

$$\chi_o(t) = \frac{3e^{it}}{4 - e^{2it}}$$

and $\chi_e(t) = e^{-it}\chi_o(t)$. In particular note that $|\chi_o(t)| = |\chi_e(t)| = |\chi_o(-t)| = |\chi_e(-t)| = 1$ for $t = 0$ and $t = \pi$, and < 1 otherwise. Now, since $\sum_{k=-\infty}^{\infty} \mathbb{P}(X_{2n} = 2k)e^{i2kt} = \phi_{X_{2n}}(t)$, if we integrate both sides of this equation from $-\pi/2$ to $\pi/2$ we obtain $\pi\mathbb{P}(X_{2n} = 0) = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \phi_{X_{2n}}(x)dx$. Then

$$\pi\mathbb{P}(X_{2n} = 0) = \frac{1}{B_n} \int_{\frac{-\pi B_n}{2}}^{\frac{\pi B_n}{2}} \phi_{X_{2n}}(t/B_n)dt = \frac{1}{B_n} \int_{\frac{-\pi B_n}{2}}^{\frac{\pi B_n}{2}} e^{it\frac{A_n}{B_n}} \phi_{\frac{X_n - A_n}{B_n}}(t)dt.$$

The following equality is easily proved for every $z \in \mathbb{R}$.

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = \frac{1}{2\pi} \int e^{-itz - \frac{t^2}{2}} dt.$$

In particular, in our case, we take $z := -\frac{A_n}{B_n}$. We write

$$R_n := 2\pi \left[\frac{B_n}{2} \mathbb{P}(X_{2n} = 0) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{A_n}{B_n}\right)^2} \right] = J_1 + J_2 + J_3 + J_4, \tag{2.11}$$

where

$$J_1 = \int_{-A}^{A} e^{it\frac{A_n}{B_n}} \left[\phi_{\frac{X_{2n} - A_n}{B_n}}(t) - e^{-\frac{t^2}{2}} \right] dt$$

$$J_2 = -\int_{|t|>A} e^{it\frac{A_n}{B_n} - \frac{t^2}{2}} dt$$

$$J_3 = \int_{\epsilon B_n < |t| < \pi B_n/2} e^{it\frac{A_n}{B_n}} \phi_{\frac{X_{2n} - A_n}{B_n}}(t)dt$$

$$J_4 = \int_{A < |t| < \epsilon B_n} e^{it\frac{A_n}{B_n}} \phi_{\frac{X_{2n} - A_n}{B_n}}(t)dt$$

So to complete the proof we must show that these quantities tend to 0 as $n \to \infty$ and for sufficiently large A and small ϵ.

First, we show that the sequence $(\xi_k)_{k\geq 1}$ satisfies the Lyapunov condition with $\delta = 1$, that is

$$\lim_{n\to\infty} \frac{1}{B_n^{2+\delta}} \sum_{k=1}^{2n} \mathbb{E}|\xi_k - a_k|^{2+\delta} = 0.$$

In fact, by the previous lemma, $B_n^2 \sim Cn$ with $C > 0$ and the ξ_k's clearly have finite moment of the third order, so for appropriate $C' > 0$

$$\frac{1}{B_n^3} \sum_{k=1}^{2n} \mathbb{E}|\xi_k - a_k|^3 \sim \frac{1}{Cn^{3/2}} \sum_{k=1}^{2n} \mathbb{E}|\xi_k - a_k|^3 \leq \frac{C'n}{n^{3/2}} \sim \frac{C'}{n^{1/2}}.$$

Then by the central limit theorem we have that, as $n \to \infty$,

$$\phi_{\frac{X_{2n}-A_n}{B_n}}(t) \to e^{-\frac{t^2}{2}},$$

which implies $|J_1| \to 0$.

We have

$$|J_2| \leq \int_{|t|>A} |e^{-it\frac{A_n}{B_n}}||e^{-\frac{t^2}{2}}|dt = \int_{|t|>A} |e^{-\frac{t^2}{2}}| \leq \frac{2}{A}e^{-\frac{A^2}{2}}$$

and so by choosing a sufficiently large A we can make J_2 arbitrarily small.

For every k, $\phi_{\xi_k}(t)$ is either $\chi_e(t)$, $\chi_e(-t)$, $\chi_o(t)$ or $\chi_o(-t)$. Since for $\epsilon < |t| < \pi/2$ we have $|\phi_{\xi_k}(t)| < 1$, we can find $c > 0$ such that $|\phi_{\xi_k}(t)| \leq e^{-c} < 1$ for every k. Then, if $\epsilon B_n < |t| < \pi B_n/2$, we have

$$\begin{aligned} |\phi_{\frac{X_{2n}-A_n}{B_n}}(t)| &= \prod_{k=1}^{2n} |\phi_{\xi_k - a_k}(t/B_n)| = \prod_{k=1}^{2n} |e^{-ia_k t/B_n}||\phi_{\xi_k}(t/B_n)| \\ &= \prod_{k=1}^{2n} |\phi_{\xi_k}(t/B_n)| \leq \prod_{k=1}^{2n} e^{-c} = e^{-2cn}, \end{aligned}$$

which tends to 0 as $n \to \infty$. This implies $|J_3| \to 0$ as $n \to \infty$.

On the other hand, by Taylor expansion at $t = 0$

$$|\phi_{\frac{X_{2n}-A_n}{B_n}}(t)| = \prod_{k=1}^{2n} |\phi_{\xi_k - a_k}(t/B_n)| = \prod_{k=1}^{2n} \left|1 - \frac{\sigma_k^2 t^2}{2B_n^2} + o\left(\frac{t^2}{B_n^2}\right)\right|.$$

Now, if $|t| \leq \epsilon B_n$ for sufficiently small ϵ, we have

$$|\phi_{\frac{X_{2n}-A_n}{B_n}}(t)| < \prod_{k=1}^{2n} \left| 1 - \frac{\sigma_k^2 t^2}{4B_n^2} \right| < \prod_{k=1}^{2n} e^{-\frac{\sigma_k^2 t^2}{4B_n^2}} = e^{-t^2/4}.$$

Then

$$|J_4| \leq 2 \int_A^{\epsilon B_n} e^{-t^2/4} dt < 2 \int_A^{\infty} e^{-t^2/4} dt,$$

where the right hand side tends to 0 as $A \to \infty$. So we can make $|J_4|$ arbitrarily small. □

The proof of recurrence is now complete.

2.2 Periodic orientations with random perturbations

Let $f : \mathbb{Z} \to \{-1, 1\}$ be again a Q-periodic function such that $Q > 1$ is a even integer and f satisfies (2.1). In this section we consider the honeycomb lattice where the horizontal orientations are prescribed by a random perturbation of f, namely a sequence of random variables $\overline{\epsilon} = (\overline{\epsilon}_y)_{y \in \mathbb{Z}}$ defined by

$$\overline{\epsilon}_y := (1 - \lambda_y) f(y) + \lambda_y \epsilon_y, \tag{2.12}$$

where $\epsilon = (\epsilon_y)_{y \in \mathbb{Z}}$ is a sequence of i.i.d. Rademacher random variables, $\lambda = (\lambda_y)_{y \in \mathbb{Z}}$ is a $\{0, 1\}$-valued sequence of independent random variables, independent of ϵ, such that

$$\mathbb{P}(\lambda_y = 1) = \frac{c}{|y|^\beta} \tag{2.13}$$

for some constants c, β and large $|y|$.

Theorem 3. *If $\beta < 1$ the random walk M_n is (ϵ_y, λ_y)-a.s.transient.*

Theorem 4. *If $\beta > 1$ the random walk M_n is (ϵ_y, λ_y)-a.s.recurrent.*

Proof of theorem 3. To prove a.s. transience, we can follow the same technique we used for the case of an i.i.d. random environment. So we define, for $n \geq 0$, the following families of events:

$$A_{n,1} := \{\max_{0\le k\le 2n} |Y_k| < n^{\frac{1}{2}+\delta_1}\},\ \delta_1 > 0$$

$$A_{n,2} := \{\max_{y\in\mathbb{Z}} \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2}\},\ \delta_2 > 0$$

$$A_n := A_{n,1} \cap A_{n,2}$$

$$B_n := A_n \cap \{|\sum_{y\in\mathbb{Z}} \overline{\epsilon}_y(m_o m^{(y)}_{2n-1,o} + m_e m^{(y)}_{2n-1,e})| > n^{\frac{1}{2}+\delta_3}\},\ \delta_3 > 0$$

Now, it is clear that many of the estimates that we did in the proof of theorem 1 still hold: in fact, according to [2], we only need to provide an estimate on $A_n\backslash B_n$, conditionally to $\mathcal{F}_{2n}$. This estimate is given by the following result: [1]

Proposition 7 (Proposition 3.2, [2])**.** *For all $\beta < 1$, there exists a $\delta_\beta > 0$ such that, uniformly in $\mathcal{F}_{2n}$, for all large n*

$$\mathbb{P}(A_n\backslash B_n \mid \mathcal{F}_{2n}) = \mathcal{O}(n^{-\delta_\beta}).$$

□

Then, exactly as in the case of i.i.d. random environment, we show that $\mathbb{P}(X_n = 0, Y_n = 0)$ is summable and prove the a.s. transience.

Proof of theorem 4.

To prove a.s. recurrence we need to show that $\sum_{n\ge 0}\mathbb{P}(X_{2n} = 0, Y_{2n} = 0 \mid \mathcal{G}) = \infty$. We know from Borel-Cantelli lemma that

$$w := \max\{|y| \text{ such that } \lambda_y = 1\}$$

is $\overline{\epsilon}$-a.s. finite, i.e. there exists $L \in \mathbb{N}$ such that for almost every realization of $\overline{\epsilon}$ we have

$$L < \infty \quad \text{and} \quad wQ < L. \tag{2.14}$$

From now on we fix $\overline{\epsilon}_0$, one of such realizations, and consider all the probabilities as taken conditionally to $\overline{\epsilon}_0$, although for simplicity we will often omit to write it. Our strategy

[1] In [2] the graphs considered are partially directed square grids. However the same proof applies in the current framework without changes, and therefore we omit it.

is to control the time spent by the random walk within the strip $\{y \in \mathbb{Z} | |y| \leq L\}$, while we apply proposition 5 to control the fluctuations outside the strip, where the levels are periodically oriented.

Let

$$\overline{S}_{n,e}^{\leq L} := \sum_{i=1}^{2n} \mathbf{1}_{\{\nu_{i-1} \neq \nu_i, |Y_i| \leq L\}} \overline{\epsilon}_{Y_i},$$

$$\overline{S}_{n,e}^{\geq L} := \sum_{i=1}^{2n} \mathbf{1}_{\{\nu_{i-1} \neq \nu_i, |Y_i| \geq L\}} \overline{\epsilon}_{Y_i},$$

$$S_{n,e}^{\leq L} := \sum_{i=1}^{2n} \mathbf{1}_{\{\nu_{i-1} \neq \nu_i, |Y_i| \leq L\}} f(\overline{Y}_i),$$

$$S_{n,e}^{\geq L} := \sum_{i=1}^{2n} \mathbf{1}_{\{\nu_{i-1} \neq \nu_i, |Y_i| \geq L\}} f(\overline{Y}_i).$$

Note that $S_{n,e}^{\geq L} = \overline{S}_{n,e}^{\geq L}$. Moreover let

$$\overline{S}_{n,e} = \overline{S}_{n,e}^{\leq L} + \overline{S}_{n,e}^{\geq L},$$

$$S_{n,e} = S_{n,e}^{\leq L} + S_{n,e}^{\geq L}.$$

In a completely analogous way we define the quantities corresponding to the odd steps: $S_{n,o}^{\leq L}, S_{n,o}^{\geq L}, \overline{S}_{n,o}^{\leq L}, \overline{S}_{n,o}^{\geq L}, \overline{S}_{n,o}, S_{n,o}$.

Lemma 13. *We have*

$$|\overline{S}_{n,e}| \leq 2 \sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \leq L\}} + |S_{n,e}|,$$

$$|\overline{S}_{n,o}| \leq 2 \sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \leq L\}} + |S_{n,o}|.$$

Proof. We have

$$\begin{aligned}
|\overline{S}_{n,e}| =& |\overline{S}_{n,e}^{\leq L} + \overline{S}_{n,e}^{\geq L}| \\
=& |\overline{S}_{n,e}^{\leq L} - S_{n,e}^{\leq L} + S_{n,e}^{\leq L} + \overline{S}_{n,e}^{\geq L}| \\
=& |\overline{S}_{n,e}^{\leq L} - S_{n,e}^{\leq L} + S_{n,e}| \\
\leq& |\overline{S}_{n,e}^{\leq L} - S_{n,e}^{\leq L}| + |S_{n,e}| \\
\leq& 2 \sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \leq L\}} + |S_{n,e}|.
\end{aligned}$$

The same argument proves the analogous majorization for $\overline{S}_{n,o}$. □

We shall denote again by $\mathcal{Z}_n$ the event defined in (2.4).

Lemma 14. *There exists a constant $c' > 0$ such that for every $n \in \mathbb{N}$*

$$\mathbb{E}\left(\sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \le L\}} \mid \mathcal{Z}_n\right) \le c'\sqrt{n}.$$

Proof. We have

$$\mathbb{E}\left(\sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \le L\}} \mid \mathcal{Z}_n\right) = \sum_{i=1}^{2n} \mathbb{P}(|Y_i| \le L \mid \mathcal{Z}_n) = \sum_{k=-L}^{L} \sum_{i=1}^{2n} \mathbb{P}(|Y_i| = k \mid \mathcal{Z}_n).$$

By the local limit theorem (theorem 3 in [15]) applied to the Markov chain $(\nu_n)_{n \ge 0}$, we deduce that $\mathbb{P}_0(Y_i = k)$ is majorized by $\frac{c}{\sqrt{i}}$ for an appropriate constant $c > 0$ independent of k and for all sufficiently large i; Then we can find $c' > 0$ large enough such that $\mathbb{P}_0(Y_i = k) \le \frac{c'}{\sqrt{i}}$ for all $i > 0$. Hence

$$\begin{aligned}\sum_{i=1}^{2n} \mathbb{P}(Y_i = k \mid \mathcal{Z}_n) \le& \frac{\sum_{i=1}^{2n} \mathbb{P}_0(Y_i = k)\mathbb{P}_k(Y_{2n-i} = 0)}{\mathbb{P}_0(\mathcal{Z}_n)} \\ \le& C\sqrt{n} \int_{t=0}^{2n} \frac{1}{\sqrt{t(2n-t)}} dt = C\sqrt{n}\left[\arcsin\left(\frac{t-2n}{2n}\right)\right]_0^{2n} \le c'\sqrt{n}.\end{aligned}$$

□

Proposition 8. *We have*

$$\mathbb{P}(|\overline{S}_{n,e}| + |\overline{S}_{n,o}| \le C\sqrt{n} \mid \mathcal{Z}_n) \ge K_{C,L} > 0$$

with $C > 0$ and sufficiently large n.

Proof. By lemma 13 we have for large n

$$\begin{aligned}\mathbb{P}\left(\frac{|\overline{S}_{n,e}| + |\overline{S}_{n,o}|}{\sqrt{n}} \le C \mid \mathcal{Z}_n\right) \ge& \mathbb{P}\left(\frac{4\sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \le L\}} + |S_{n,e}| + |S_{n,o}|}{\sqrt{n}} \le C \mid \mathcal{Z}_n\right) \\ \ge& \mathbb{P}\left(\frac{|S_{n,e}| + |S_{n,o}|}{\sqrt{n}} \le C/2, \frac{\sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i| \le L\}}}{\sqrt{n}} \le C/2 \mid \mathcal{Z}_n\right).\end{aligned}$$

Now, by proposition 5

$$\mathbb{P}\left(\frac{|S_{n,e}|+|S_{n,o}|}{\sqrt{n}} \leq C/2 \mid \mathcal{Z}_n\right) \geq \delta_C > 0$$

and by the Markov inequality together with lemma 14

$$\mathbb{P}\left(\frac{\sum_{i=1}^{2n} \mathbf{1}_{\{|Y_i|\leq L\}}}{\sqrt{n}} \leq C/2 \mid \mathcal{Z}_n\right) \geq \delta'_{C,L} > 0,$$

where both δ_C and $\delta'_{C,L}$ tend to 1 as C grows to infinity. So if we take a sufficiently large C s.t. $\delta_{C',L} > 1-\delta_C$, the intersection between these two events will have positive probability. □

In analogy with the argument used in the periodic case, we define the following set of constrained paths

$$\begin{aligned}\mathrm{Constr}(n,f) := \{&(\gamma,q) : \{-1,0,1,...,2n\} \longrightarrow \mathbb{Z}\times\{-1,1\} \text{ s.t. } \forall i,\ \gamma(i)=\gamma(i-1)\pm 1,\\ &(\gamma(-1),q(-1);\gamma(0),q(0)) = (\gamma(2n-1),q(2n-1);\gamma(2n),q(2n)) = W_0,\\ &\left|\sum_{i=1}^{2n}\mathbf{1}_{\{q_{i-1}\neq q_i\}}\overline{\epsilon}_{\gamma_i}\right| + \left|\sum_{i=1}^{2n}\mathbf{1}_{\{q_{i-1}=q_i\}}\overline{\epsilon}_{\gamma_i}\right| \leq C\sqrt{n}\right\}.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{P}(X_{2n}=0,Y_{2n}=0) \geq &\mathbb{P}\left(X_{2n}=0,\mathcal{Z}_n,|\overline{S}_e(n)|+|\overline{S}_o(n)|\leq C\sqrt{n}\right)\\ = &\sum_{(\gamma,q)\in\mathrm{Constr}(n,f)} \mathbb{P}(X_{2n}=0 \mid (Y_i,\nu_i)=(\gamma(i),q(i))\,\forall i\leq 2n)\\ &\times \mathbb{P}((Y_i,\nu_i)=(\gamma(i),q(i))\,\forall i\leq 2n).\end{aligned}$$

Then by propositions 6 and 8, proceeding as in the proof of theorem 2, we show recurrence for the random walk conditionally to the realization $\overline{\epsilon}_0$ of the environment. But since the choice of $\overline{\epsilon}_0$ is arbitrary, with the only requirement that (2.14) is satisfied, and since this happens for a.e. realization, we proved a.s. recurrence.

2.3 Example of recurrence with non-periodic ergodic orientations

One question that one may ask, is whether it is possible to generalize the transience result obtained in theorem 1 to an ergodic sequence of random variables. In this section we shall construct a counterexample. In our model the sequence of orientations is in fact ergodic and non-periodic, but the simple random walk on the corresponding oriented lattice is nonetheless recurrent for almost every realization of the environment.

Remark 2. In [2] the authors give an example of a square lattice with non-periodic orientations such that the simple random walk is recurrent. The construction is deterministic and is done by starting with alternate orientations, and then recursively introducing some *defects*, i.e. levels where the orientation is opposite with respect to the one prescribed by $f(x) = (-1)^{|x|}$; the recurrence is thus proved by exploiting the sparsity of these defects together with the recurrence of the periodically oriented square grid. We will take inspiration from this approach to prove our result.

Theorem 5. *For every $n \in \mathbb{N}$ define the function $Z_n : \mathbb{Z} \longrightarrow \{-1, 1\}$ by*

$$Z_n(y) = \begin{cases} -1 & \text{if } y \in \{in | i \in \mathbb{Z}\}, \\ 1 & \text{otherwise.} \end{cases} \tag{2.15}$$

Let $T := (T_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables, with

$$\mathbb{P}(T_n = j) = \frac{1}{n} \tag{2.16}$$

for every n and $j \in \mathbb{N}$, $0 \leq j < n$.

Then there exists a strictly increasing sequence of prime numbers $a := (a_k)_{k \geq 1} \uparrow \infty$ with $a_1 > 2$, such that the honeycomb lattice where each horizontal level y is oriented randomly according to

$$\epsilon_a(y) := \begin{cases} (-1)^{|y|} \prod_{k=1}^{\infty} Z_{a_k}(y + T_{a_k}) & \text{if } Z_{a_k}(y + T_{a_k}) = 1 \text{ except for finitely many } k\text{'s} \\ 1 & \text{otherwise} \end{cases} \tag{2.17}$$

is recurrent for almost every realization of T. Moreover $\epsilon_a := (\epsilon_a(y))_{y\geq 0}$ is an ergodic sequence of random variables.

Remark 3. Note that we can choose the sequence a_n to grow sufficiently fast, so that the infinite product in (2.17) involves almost surely only a finite number of factors different from $+1$.

Let ϵ_0 be the alternate sequence, i.e. $\epsilon_a^0(y) := (-1)^{|y|}$ for every y. For $m \in \mathbb{Z}$, $m > 0$ define $\epsilon_a^m = (\epsilon_a^m(y))_{y\in\mathbb{Z}}$ by

$$\epsilon_a^m(y) := \epsilon_a^0(y) \prod_{k=1}^{m} Z_{a_k}(y + T_{a_k}) \tag{2.18}$$

for every y. Then, for almost every realization of T, $\epsilon_a(y) = \lim_{m\to\infty} \epsilon_a^m(y)$; based on this observation, in figure 6 we illustrate how to construct iteratively the sequence ϵ_a starting from ϵ_0.

We begin by showing a crucial property of the sequences ϵ_a^m.

Lemma 15. *If $a := (a_k)_{k\geq 1} \uparrow \infty$ is a strictly increasing sequence of prime numbers such that $a_1 > 2$, then for every $m \in \mathbb{N}$ and for every realization of T, ϵ_a^m is a sequence of period $2\prod_{i=1}^m a_i =: R_m$ and $\sum_{y=0}^{R_m-1} \epsilon_a^m(y) = 0$.*

Proof. The case $m = 0$ is trivial since the orientations are alternate.

Assume the result holds for $m > 0$ and fix a realization $\tilde{T}$ of T.

Consider the dynamical system on $\mathbb{Z}_{R_m}$ given by the map $\Phi(x) = x + a_{m+1} \mod R_m$. Let $O_{x_0}^{(m+1)} := \{x_0 + ja_{m+1} \mod R_m | j \in \mathbb{N}\}$ be the orbit of x_0. Since a_{m+1} and R_m are co-primes, we have

$$|O_{x_0}^{(m+1)}| = R_m \tag{2.19}$$

for every $x_0 \in \mathbb{Z}$, and so in particular (2.19) holds for $x_0 = -\tilde{T}_{a_{m+1}}$. Since by (2.18) the sequence ϵ_a^{m+1} is obtained from ϵ_a^m by changing the sign of $\epsilon_a^m(y)$ if and only if $y \in \{ja_{m+1} - \tilde{T}_{a_{m+1}} | j \in \mathbb{Z}\}$, then (2.19) implies by induction that the sequence ϵ_a^{m+1} has period $R_{m+1} = R_m a_{m+1}$, and satisfies

$$\sum_{y=0}^{R_{m+1}-1} \epsilon_a^{m+1}(y) = \sum_{y=0}^{R_{m+1}-1} \epsilon_a^m(y) - \sum_{y=0}^{R_m-1} \epsilon_a^m(y) = \sum_{y=0}^{R_{m+1}-1} \epsilon_a^m(y).$$

On the other hand, again by induction

$$\sum_{y=0}^{R_{m+1}-1} \epsilon_a^m(y) = \sum_{y=0}^{R_m a_{m+1}-1} \epsilon_a^m(y) = a_{m+1} \sum_{y=0}^{R_m-1} \epsilon_a^m(y) = 0,$$

which completes the proof.

□

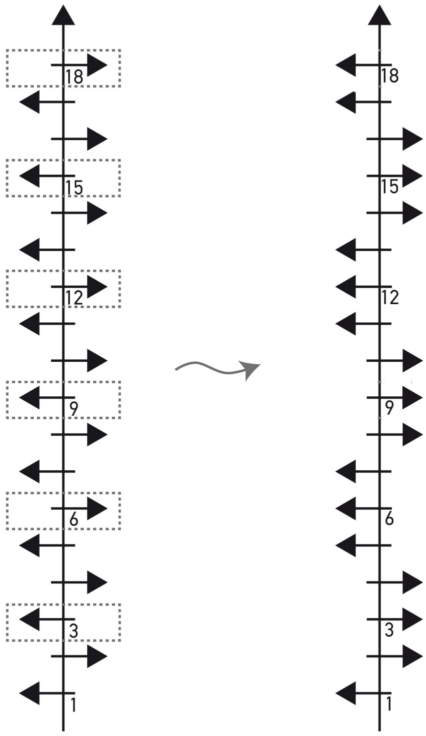


Figure 6: Construction of ϵ_a^1 starting from the alternate sequence ϵ_a^0, assuming $a_1 = 3$ and $T_1 = 0$: at the left hand side we consider the alternate sequence and select all the levels that are multiples of a_1; then we switch the orientation of the selected levels to obtain ϵ_a^1, shown at the right hand side. Note that the period of ϵ_a^1 is $2a_1 = 6$.

We are ready to prove theorem 5. We divide the proof into two parts: first we observe that the sequence is ergodic; then we show the recurrence of the walk.

Proof of ergodicity. Let $\Psi_n := \prod_{k=1}^n \mathbb{Z}_{a_k}$ and denote by G the group addition on Ψ_n, that is the transformation that adds $+1$ to each component of $x = (x_1, x_2, ..., x_n) \in$

Ψ_n, and let P be the product of n independent uniform distributions, i.e. $P = P_1 \otimes ... \otimes P_n$ where P_k is the uniform distribution on $\{0, 1, ..., a_k - 1\}$ for every k. Since $a = (a_k)_{k \in \mathbb{N}}$ is a sequence of primes, the orbit of any point $x \in \Psi_n$ covers all the space, whence

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} h(G^i(x)) = \mathbb{E}(h) \tag{2.20}$$

for all $x \in \Psi_n$ and for every measurable function h.

Now let $\Psi := \prod_{k=1}^{\infty} \mathbb{Z}_{a_k}$, and consider the system (Ψ, G, P), where G is the group addition on Ψ and $P = \otimes_{k=1}^{\infty} P_k$. Let $(X_k)_{k\geq 1}$ be a family of independent random variables such that for each k, X_k is distributed according to P_k, and define $X = (X_1, X_2, ...)$.

Suppose that there exists an invariant set I such that $0 < P(I) < 1$ and let $f := 1_I$. Clearly f is bounded in $L^1(\Psi)$. For each n define the function $f_n(X) := \mathbb{E}(f(X)|X_1, ..., X_n)$. It is easy to see that f_n is a bounded martingale in $L^1(\Psi)$ and by Doob theorem

$$\lim_{n\to\infty} f_n(X) = f(X) \tag{2.21}$$

a.s. and, by the dominated convergence theorem, in L^1. On the other hand, by (2.20) we have that for every n

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} f_n(G^i(X)) = \mathbb{E}(f_n(X)) \tag{2.22}$$

a.s. and, again by the dominated convergence theorem, in L^1. Then by (2.21) and (2.22) we get

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} f(G^i(X)) = \mathbb{E}(f(X)) \tag{2.23}$$

in $L^1(\Psi)$. Finally, by invariance of I, we have $\lim_{k\to\infty} \mathbb{E}|\frac{1}{k}\sum_{i=1}^{k} \mathbf{1}_I(G^i(X)) - P(I)| = \lim_{k\to\infty} \mathbb{E}|\mathbf{1}_{X\in I} - P(I)| \neq 0$ which is absurd.

Proof of recurrence. Let $(c_n)_{n\in\mathbb{N}} \uparrow \infty$ and $k \in \mathbb{N}$. By theorem 2 we know that the random walk on the lattice oriented according to ϵ_a^k is recurrent, and in particular $\exists L(\epsilon_a^k) < \infty$ such that on that lattice

$$\sum_{i=1}^{L} \mathbb{P}_0(M_i = 0) > c_k \tag{2.24}$$

for all $L \geq L(\epsilon_a^k)$.

Let $\delta < 1, x \in \mathbb{Z}$ and consider the following event:

$$E_{a,x} := \bigcap_{k=1}^{\infty} \{\{ja_k - T_{a_k} | j \in \mathbb{Z}\} \cap B_{x,\delta}^{(k)} = \emptyset\}, \tag{2.25}$$

where $B_{x,\delta}^{(k)} := \{x - a_k^\delta, x - a_k^\delta + 1, ..., x + a_k^\delta\}$. Let's show that $E_{a,x}$ occurs for some x, with probability 1. To this purpose note that

$$\mathbb{P}(E_{a,0}^c) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} \{\{ja_k - T_{a_k} | j \in \mathbb{Z}\} \cap B_{0,\delta}^{(k)} \neq \emptyset\}\right) \leq \sum_{k=1}^{\infty} \frac{2a_k^\delta}{a_k}. \tag{2.26}$$

By choosing an increasing sequence of primes $(a_k)_{k\in\mathbb{N}}$ that grows sufficiently fast, we can ensure that $\mathbb{P}(E_{a,0}^c) \leq c < 1$; whence $\mathbb{P}(E_{a,0}) > 0$, and by the ergodic theorem we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x=1}^{N} \mathbf{1}_{E_{a,x}}(\omega) = \lim_{N\to\infty} \frac{1}{N} \sum_{x=1}^{N} \mathbf{1}_{E_{a,0}}(S^x(\omega)) = \mathbb{P}(E_{a,0}) > 0,$$

a.s., where $S : \mathbb{Z} \to \mathbb{Z}$ is the shift map. Then with probability 1 we can find $x_0 \in \mathbb{Z}$ such that E_{a,x_0} occurs, and consider the simple random walk M started at $(0, x_0)$. For every k, the first a_k^δ levels around x_0 are oriented according to ϵ_a^{k-1}, by (2.25) and lemma 15; therefore, if

$$a_k^\delta \geq L(\epsilon_a^{k-1}), \tag{2.27}$$

we have by (2.24)

$$\sum_{i=1}^{\infty} \mathbb{P}_{(0,x_0)}(M_i = (0, x_0)) \geq \sum_{i=1}^{a_k^\delta} \mathbb{P}_{(0,x_0)}(M_i = (0, x_0)) > c_{k-1}.$$

Note that, possibly after modifying $(a_k)_{k\in\mathbb{N}}$ to a new sequence of primes $(a'_k)_{k\in\mathbb{N}}$, with $a'_k \geq a_k \ \forall k$, we can ensure that (2.27) is satisfied for every k, and since $(c_n)_{n\in\mathbb{N}} \uparrow \infty$ we deduce the recurrence of the random walk.

Chapter 3

Revolving random walks on oriented square grids

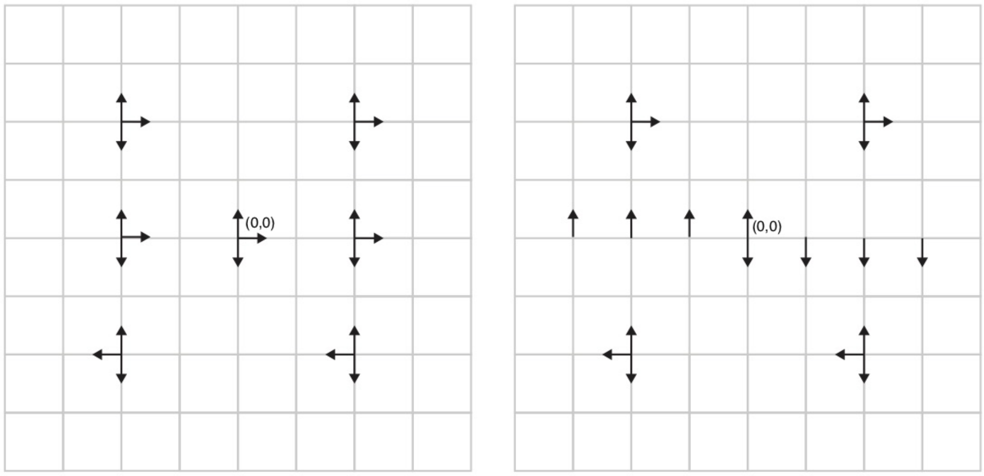
3.1 Introduction

In this Chapter we study the recurrence/transience behavior of the simple random walk on two partially directed versions of the two-dimensional square grid (see figure 7). [1]

The first graph is $\mathbf{G}_1 = (\mathbb{Z}^2, \mathbb{E}_1)$, with edge set $\mathbb{E}_1$ satisfying that a directed edge $(v, w) = ((v_1, v_2), (w_1, w_2)) \in \mathbb{E}_1$ if and only if $(w_1, w_2) = (v_1, v_2 \pm 1)$, or $(w_1, w_2) = (v_1 + 1, v_2)$ and $v_2 = w_2 \geq 0$, or $(w_1, w_2) = (v_1 - 1, v_2)$ and $v_2 = w_2 < 0$. The second graph we consider is $\mathbf{G}_2 = (\mathbb{Z}^2, \mathbb{E}_2)$, and can be obtained with a slight modification of $\mathbf{G}_1$ by redefining only the orientations of the edges leading out from x-axis, that is, $((v_1, 0), (w_1, w_2)) \in \mathbb{E}_2$ with $v_1 = w_1$ and $w_2 = \pm 1$ if and only if $w_2 = -1$ and $v_1 = w_1 > 0$, or $w_2 = 1$ and $v_1 = w_1 < 0$, or $w_2 = \pm 1$ and $v_1 = w_1 = 0$.

Graph $\mathbf{G}_1$ appeared for the first time in [1], where its transience was proved by computing the characteristic function of the corresponding embedded random walk. Then, more recently, graph $\mathbf{G}_1$ was reintroduced together with $\mathbf{G}_2$ in [19] [20]: the authors

[1] All the results presented in this Chapter are from a joint work with Yuval Peres, Principal Researcher at Microsoft Research, Redmond, and Yiping Hu, Ph.D. student from the University of Washington, Seattle. A preprint of our work is currently available on arXiv at `https://arxiv.org/abs/1807.03498`



(a) Graph $\mathbf{G}_1$ (b) Graph $\mathbf{G}_2$

Figure 7: The graph $\mathbf{G}_1$ in figure (a) is transient, whereas graph $\mathbf{G}_2$ in (b) is recurrent. The arrows indicate the orientation of the corresponding edges.

were mainly concerned with oscillating random walks on the line (and on simplexes of half lines), and they observe that the random walks on these two-dimensional graphs, at the times of successive returns to x-axis, have an oscillatory behavior. However, while they give a new proof of the transience of $\mathbf{G}_1$, graph $\mathbf{G}_2$ turns out to be more delicate, and they conjecture the recurrence of the graph. Our main result is a proof of their conjecture:

Theorem 6. *The simple random walk on graph* $\mathbf{G}_2$ *is recurrent.*

In order to prove the result, we shall start by considering a continuous analogue of the random walk which is recurrent, and apply the Lyapunov method together with an approximation technique to deduce the same result in the discrete framework. Here by Lyapunov method we mean the application of the following theorem (for a reference to it with several examples of its use we refer to [20].)

Theorem 7 (Lyapunov function recurrence criterion)**.** *An irreducible Markov chain* X_n *on a countable space* Σ *is recurrent if and only if there exists a function* $f : \Sigma \to \mathbb{R}_+$ *and* *a finite set $A \neq \emptyset$, $A \subset \Sigma$, such that $\forall x \in \Sigma \backslash A$*

$$\mathbb{E}[f(X_{n+1}) - f(X_n)|X_n = x] \leq 0 \tag{3.1}$$

and $f(x) \to \infty$ as $x \to \infty$.

Let's start by defining the aforementioned continuous analogue of the random walk on $\mathbf{G}_2$ and prove its recurrence.

Let $m \in \mathbb{R}^+$ and $(B_t^R)_{t\geq 0}$ be the one-dimensional standard Brownian motion starting at 0 with a reflecting barrier at 0 to stay in positive real line. We define a continuous-time process $(W_t)_{t\geq 0} := (W_t^{(1)}, W_t^{(2)})_{t\geq 0}$ on $\mathbb{R}^2$, together with a sequence of random times $(U_n)_{n\geq 0}$ in the following recursive manner: we set $U_0 := 0$ and $W_0 := (-m, 0)$ as the initial position; for every $n \geq 1$,

$$U_n := \min\{t > U_{n-1} + |W_{U_{n-1}}^{(1)}|; B_t^R = 0\}$$

and

$$W_t := \begin{cases} (t - U_{2n} + W_{U_{2n}}^{(1)}, B_t^R) & \text{if } t \in [U_{2n}, U_{2n+1}) \text{ for some } n \geq 0, \\ (-t + U_{2n-1} + W_{U_{2n-1}}^{(1)}, -B_t^R) & \text{if } t \in [U_{2n-1}, U_{2n}) \text{ for some } n > 0. \end{cases}$$

Clearly, $W_{U_n} = (W_{U_n}^{(1)}, 0)$ for every n and its x-coordinate $W_{U_n}^{(1)}$ changes sign alternately. It turns out that it suffices to keep track of $|W_{U_n}^{(1)}|$ at these returns to x-axis, since the recurrence of W_t would follow immediately from the recurrence of $|W_{U_n}^{(1)}|$. To this purpose, we define $H_n^B := |W_{U_n}^{(1)}|$ and call this discrete-time process $(H_n^B)_{n\geq 0}$ on $\mathbb{R}^+$ the *continuous ladder height process.* (cfr. figure 8)

The ladder height process is itself a Markov chain and has a nice representation as the product of i.i.d. random variables η_n's defined by

$$H_n^B := \eta_n H_{n-1}^B = m \prod_{i=1}^{n} \eta_i = \exp\left(\log m + \sum_{i=1}^{n} \log \eta_i\right) \tag{3.2}$$

for $n \geq 1$. One way to understand $\log \eta_i$'s is through the decomposition of each step of ladder height process into two parts, one from the starting point on x-axis to y-axis

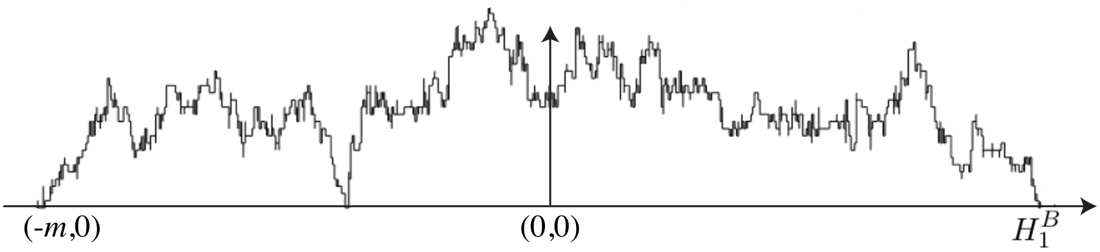


Figure 8: Illustration of the first step of the ladder height process.

and the other from y-axis back to x-axis. Let Z be a standard folded normal random variable, i.e. the absolute value of a standard normal random variable, and T_h a Lévy random variable independent of Z, i.e. the hitting time at 0 for a standard Brownian motion started at $h > 0$. Then by decomposing we have

$$H_1^B \overset{d}{=} T_{\sqrt{m}Z} \overset{d}{=} (\sqrt{m}Z)^2 T_1 = mZ^2 T_1 \tag{3.3}$$

and thus $\eta_1 \overset{d}{=} Z^2 T_1$ (see e.g. [6] p.170). On the other hand $T_1 =^d 1/|Z|^2$, since for $a > 0$

$$\mathbb{P}(T_1 > a) = \mathbb{P}(\max_{t \le a} B_t < 1) = \mathbb{P}(\sqrt{a}|Z| < 1) = \mathbb{P}(1/|Z|^2 > a)$$

by reflection principle.

It follows that $\log \eta_1$ is symmetric and, in particular, has mean zero: therefore by Chung-Fuchs theorem (see [5], th.4.2.7) $\sum_{i=1}^n \log \eta_i$ is recurrent. By (3.2) this implies the recurrence of the ladder height process and hence the recurrence of the continuous walk W_t.

The structure of the Chapter is as follows: in Section 3.2, we show how to adapt the argument above to the discrete setting and prove recurrence for the random walk on $\mathbf{G}_2$; then, in section 3.3 we analyze the simple random walk on $\mathbf{G}_1$, showing in particular its transience.

3.2 Recurrence of $\mathbf{G}_2$

Consider the simple random walk $(M_n)_{n\ge0} = (M_n^{(1)}, M_n^{(2)})_{n\ge0}$ on the graph $\mathbf{G}_2$. For simplicity we assume the random walk starts at $(M_0^{(1)}, M_0^{(2)}) = (-m, 0)$ for some fixed

$m \in \mathbb{Z}_+$ and use the notation $\mathbb{P}_m$ to make explicit the dependence on initial position. Sometimes we might want to start at $(M_0^{(1)}, M_0^{(2)}) = (0, h)$ for some $h \in \mathbb{Z}_+$, in which case we write $\mathbb{P}_h$. Analogous to the Brownian motion case, we can consider the discrete time *ladder height process* $(H_n)_{n\geq 0}$ with the state space $\mathbb{N}$. More rigorously, we define $H_n := |M_{\tau_n}^{(1)}|$ for $n \geq 0$, where $\{\tau_n\}_{n\geq 0}$ is a sequence of stopping time defined recursively as follows: $\tau_0 := 0$ and for $n \geq 1$,

$$\tau_n := \inf\{i > \tau_{n-1}; M_i^{(2)} = 0 \text{ and } M_i^{(1)} M_{\tau_{n-1}}^{(1)} \leq 0\}.$$

In the following, we will stick to the convention that $\log H_1 = 0$ when $H_1 = 0$ for simplicity. We also define for $n \geq 1$,

$$\sigma_n := \inf\{i > \tau_{n-1}; M_i^{(1)} = 0\} \tag{3.4}$$

and $V_n := |M_{\sigma_n}^{(2)}|$. Note that $\tau_{n-1} < \sigma_n \leq \tau_n$ for any $n \geq 1$. With this definition, in analogy with the decomposition (3.3) in the continuous setting, we can further decompose each step H_n of the ladder height process into two parts, one starting from $(M_{\tau_{n-1}}^{(1)}, 0)$ to the $(0, V_n)$ and the other from the $(0, V_n)$ to $(M_{\tau_n}^{(1)}, 0)$. Under $\mathbb{P}_m$, we should always consider H_1, H_1/m and V_1 as playing the same role as H_1^B, η_1 and $\sqrt{2m}Z$ in the continuous case respectively. Furthermore under $\mathbb{P}_h$, the correspondence is between H_1 and its continuous analogue $h^2 T_1/2$. Note that the extra constants $\sqrt{2}$ and $1/2$ come from the fact that the continuous analogue of the random walk in question should be constructed from a Brownian motion with twice the quadratic variation of a standard one.

It is not hard to see that the process H is a Markov chain in its own right starting at $H_0 = m$ and has the same recurrence property as the original chain M. The main difficulty in the combinatorial setting, however, is that the identity $\mathbb{E}_m \log(H_1/m) = 0$ only holds in the asymptotic sense. In fact, one can show that

$$\mathbb{E}_m \log(H_1/m) \to \mathbb{E}_m \log \eta_1 = 0$$

when $m \to \infty$, with the help of some Donsker-type arcsine laws (see [21], Prop.5.27, p.137). Unfortunately, this result is not sufficient to prove the recurrence. Hence instead of $\log H_1$, we consider a modified function $\sqrt{\log H_1}$ with the same convention at zero as above. Using the inequality $\sqrt{1+x} \le 1 + \frac{1}{2}x - \frac{1}{16}x^2$ for $x \in [-1,1]$, we obtain on the event $\{1 \le H_1 \le m^2\}$ that

$$\begin{aligned}
\sqrt{\log H_1} =& \sqrt{\log m + \log(H_1/m)} = \sqrt{\log m}\sqrt{1 + \frac{\log(H_1/m)}{\log m}} \\
\le& \sqrt{\log m}\left\{1 + \frac{\log(H_1/m)}{2\log m} - \frac{1}{16}\left[\frac{\log(H_1/m)}{\log m}\right]^2\right\} \\
\le& \sqrt{\log m} + \frac{\log H_1 - \log m}{2\sqrt{\log m}} - \frac{(\log H_1 - \log m)^2}{16(\log m)^{3/2}}.
\end{aligned}$$

Taking expectation, we get

$$\begin{aligned}
\mathbb{E}_m\sqrt{\log H_1} \le& \sqrt{\log m} + \frac{\mathbb{E}_m(\log H_1 - \log m)}{2\sqrt{\log m}} - \frac{\mathbb{E}_m(\log H_1 - \log m)^2}{16(\log m)^{3/2}} \\
&+ \frac{\mathbb{E}_m\left[(\log H_1 - \log m)^2; H_1 > m^2\right]}{16(\log m)^{3/2}} + \mathbb{E}_m\left[\sqrt{\log H_1}; H_1 > m^2\right] \\
\le& \sqrt{\log m} + \frac{\mathbb{E}_m(\log H_1 - \log m)}{2\sqrt{\log m}} - \frac{\mathbb{E}_m(\log H_1 - \log m)^2}{16(\log m)^{3/2}} \\
&+ 2\mathbb{E}_m\left[\log^2 H_1; H_1 > m^2\right] \\
=:& \sqrt{\log m} + \epsilon_1(m) - \epsilon_2(m) + \epsilon_3(m). \qquad (3.5)
\end{aligned}$$

Once we show that $\epsilon_1(m) + \epsilon_3(m) \ll \epsilon_2(m)$ for large enough m by giving reasonable bounds on their asymptotics, we can conclude $\sqrt{\log x}$ is a Lyapunov function for $(H_n)_{n\ge 0}$ and apply theorem 7 to prove the recurrence. Let us make a few comments about these errors before we proceed. The first order error $\epsilon_1(m)$ comes from the approximation of random walks with Brownian motions, the main difficulty we mentioned above, and its upper bound will be the main focus in this section. The approximation techniques we apply will be local central limit theorems and the Euler-Maclaurin formula. The second order term $\epsilon_2(m)$ is the reason behind our choice of function and it quantifies the amount we are able to exploit from using a concave function of $\log H_1$. Observe that $\mathbb{E}_m(\log H_1 - \log m)^2 \ge \mathrm{Var}_m(\log H_1)$, and we will show later in section 3.2.3 that the variance on the right hand side is uniformly bounded away from zero for all $m > 0$. For the truncation error $\epsilon_3(m)$, one should expect $\log H_1$ to be concentrated around $\log m$, and we will show in section 3.2.3 with Chernoff bounds that $\epsilon_3(m)$ decays polynomially and thus negligible compared to $\epsilon_2(m)$ when m goes to infinity.

In order to estimate $\epsilon_1(m)$, it is more convenient to consider the decomposition described by the random variables V_n's, see (3.4) for definition and the subsequent discussion of continuous analogues. Then

$$\begin{aligned}\mathbb{E}_m(\log H_1) =&\sum_{h=1}^{\infty}\mathbb{E}_m(\log H_1|V_1=h)\mathbb{P}_m(V_1=h)\\ =&\sum_{h=1}^{\infty}\left[2\log h+\mathbb{E}_h(\log H_1/h^2)\right]\mathbb{P}_m(V_1=h)\\ =&2\,\mathbb{E}_m(\log V_1)+\sum_{h=1}^{\infty}\left[\mathbb{E}_h(\log H_1)-2\log h\right]\mathbb{P}_m(V_1=h),\end{aligned}\tag{3.6}$$

so it suffices to estimate the corresponding approximation errors

$$\mathbb{E}_m(\log V_1)-\mathbb{E}\log(\sqrt{2m}Z)=\mathbb{E}_m(\log V_1)-(\log m)/2+\gamma/2\tag{3.7}$$

and

$$\mathbb{E}_h(\log H_1)-\mathbb{E}\log(h^2T_1/2)=\mathbb{E}_h(\log H_1)-2\log h-\gamma,\tag{3.8}$$

where γ is the Euler constant, Z and T_1 are defined in the paragraph above (3.3), and we use the result $\mathbb{E}\log T_1=-2\mathbb{E}(\log Z)=\gamma+\log 2$.

To this end, we define $p_{m,h}:=\mathbb{P}_m(V_1=h)$ to be the probability that the random walk starting from $(-m,0)$ hits the y-axis at point $(0,h)$ and $q_{h,l}:=\mathbb{P}_h(H_1=l)$ the probability that the random walk started at $(0,h)$ hits the x-axis at point $(l,0)$ for $m,h,l\in\mathbb{Z}_+$. Let $f_m(x):=\frac{\log(x)}{\sqrt{\pi m}}e^{-\frac{x^2}{4m}}$ and $g_h(x):=\log x\frac{h}{2\sqrt{\pi}x^{3/2}}e^{-\frac{h^2}{4x}}$ be functions define on $\mathbb{R}_+$. Then we can rewrite and decompose two errors as follows:

$$\begin{aligned}R_f(m):=&\sum_{h=1}^{\infty}\log h\,p_{m,h}-\int_0^{\infty}f_m(x)dx=\sum_{h=1}^{\infty}\left[\log h\,p_{m,h}-f_m(h)\right]+\sum_{h=m^{1/2+\delta}}^{\infty}f_m(h)\\ &+\left(\sum_{h=1}^{m^{1/2+\delta}}f_m(h)-\int_1^{m^{1/2+\delta}}f_m(x)dx\right)+\left(\int_1^{m^{1/2+\delta}}f_m(x)dx-\int_0^{\infty}f_m(x)dx\right)\\ =:&\,I_1+I_2+I_3+I_4\end{aligned}\tag{3.9}$$

and

$$R_g(h) := \sum_{l=1}^{\infty} \log l \, q_{h,l} - \int_0^{\infty} g_h(x)dx = \sum_{l=1}^{\infty} [\log l \, q_{h,l} - g_h(l)] + \sum_{l=1}^{h^{2-\delta}} g_h(l)$$
$$+ \left(\sum_{l=h^{2-\delta}}^{\infty} g_h(l) - \int_{h^{2-\delta}}^{\infty} g_h(x)dx \right) + \left(\int_{h^{2-\delta}}^{\infty} g_h(x)dx - \int_0^{\infty} g_h(x)dx \right)$$
$$=: J_1 + J_2 + J_3 + J_4, \tag{3.10}$$

where $\delta > 0$ is sufficiently small.

3.2.1 Local limit theorems

Throughout this section we shall denote the usual one-dimensional simple random walk on $\mathbb{Z}$ by S. We want to establish a local limit theorem for $p_{m,h}$ and $q_{h,l}$. First, we shall prove the following:

Lemma 16. *We have*

$$p_{m,h} := \mathbb{P}_m(V_1 = h) = \frac{1}{\sqrt{\pi m}} e^{-\frac{h^2}{4m}} + \mathcal{O}\left(\frac{1}{\sqrt{m}h^2} \wedge \frac{1}{m^{3/2}} + \frac{e^{-\frac{h^2}{8m}}}{m^{1-\delta}} \right).$$

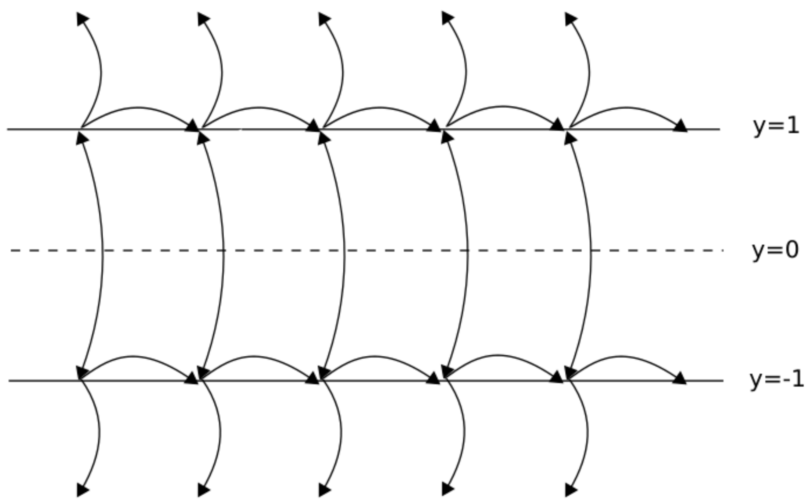


Figure 9: The modified graph $\mathbf{G}_2'$

Proof. Our approach is based on the fact that conditioned on the number of vertical steps before hitting y-axis, the vertical movement has the same law as S. To calculate the probability of n vertical steps, we hope to interpret the number of vertical steps before hitting y-axis as the sum of m i.i.d. geometric random variables, i.e. $G_{p,m} := \sum_{i=1}^{m} g_i$, with success probability $p = 1/3$ and values in $\{0, 1, 2, \dots\}$. The intuition is almost correct except that on graph $\mathbf{G}_2$, only vertical steps are allowed at ordinate zero. For this reason, we modify the transition probability of S by ignoring the origin as follows: $p(1, -1) = p(1, 2) = 1/2$ and $p(-1, 1) = p(-1, -2) = 1/2$, and write S' for the resulting random walk. We also consider a 2D modification, the random walk $(M_i'^{(1)}, M_i'^{(2)})_{i\geq 0}$ on an oriented graph $\mathbf{G}_2'$ where all the horizontal edges are to the right and all points on x-axis are ignored (see figure 9). Precisely, $\mathbf{G}_2' = (\mathbb{V}', \mathbb{E}_2')$ has vertex set $\mathbb{V}' = \mathbb{Z}^2 \backslash \mathbb{Z} \times \{0\}$, and $\mathbb{E}_2'$ consists of all edges leading to the nearest neighbors upward, downward and to the right. Then the intuition of geometric random variables holds for the random walk on $\mathbf{G}_2'$, noting that the conditional law of vertical movements has the same law as S'. For the process $(M_i^{(1)}, |M_i^{(2)}|)_{i\geq 0}$, define $p'_{m,h}$ analogously as the probability that the random walk started at $(-m, 1)$ hits the y-axis at point $(0, h)$ for $m, h \in \mathbb{Z}_+$. Then

$$\begin{aligned} p_{m,h} = p'_{m,h} &= \sum_{n=h}^{\infty} \left(\mathbb{P}_1(S'_n = -h) + \mathbb{P}_1(S'_n = h)\right) \mathbb{P}(G_{p,m} = n) \\ &= \sum_{n=h}^{\infty} \mathbb{P}_0(S_n = -h)\mathbb{P}(G_{p,m} = n) + \sum_{n=h}^{\infty} \mathbb{P}_0(S_n = h-1)\mathbb{P}(G_{p,m} = n) \\ &=: p_{m,h}^{(1)} + p_{m,h}^{(2)}. \end{aligned}$$

We will focus on $p_{m,h}^{(1)}$, as $p_{m,h}^{(2)}$ can be treated analogously. Let $\delta > 0$, we split the sum into two parts

$$p_{m,h}^{(1)} = \sum_{|n-2m|\leq m^{1/2+\delta}} \mathbb{P}_0(S_n = h)\mathbb{P}(G_{p,m} = n) + \mathcal{O}\left[\sum_{|n-2m|>m^{1/2+\delta}} \mathbb{P}(G_{p,m} = n)\right], \quad (3.11)$$

and notice that the second term in (3.11) decays exponentially fast by Chernoff bound.

Then, by applying the local limit theorem (see e.g. [17], p.36 [2]) to S we obtain

$$\begin{aligned}
p^{(1)}_{m,h} &= \sum_{|n-2m|\leq m^{1/2+\delta}} \left[\overline{p}_n(h) + \mathcal{O}\left(\frac{1}{m^{3/2}}\right)\right] \mathbb{P}(G_{p,m}=n) + \mathcal{O}(e^{-cm^{2\delta}}) \\
&= \left[\overline{p}_{2m}(h) + \mathcal{O}\left(\frac{1}{m^{3/2}} + \frac{e^{-\frac{h^2}{8m}}}{m^{1-\delta}}\right)\right] \sum_{|n-2m|\leq m^{1/2+\delta}} \mathbb{P}(G_{p,m}=n) + \mathcal{O}(e^{-cm^{2\delta}}) \\
&= \left[\overline{p}_{2m}(h) + \mathcal{O}\left(\frac{1}{m^{3/2}} + \frac{e^{-\frac{h^2}{8m}}}{m^{1-\delta}}\right)\right],
\end{aligned}$$

where we define $\overline{p}_n(h) := \frac{1}{\sqrt{2\pi n}} e^{-\frac{h^2}{2n}}$ and use the fact that if $|n-2m| \leq m^{1/2+\delta}$ then $\overline{p}_n(h) = \overline{p}_{2m}(h) + \mathcal{O}\left(\frac{e^{-\frac{h^2}{8m}}}{m^{1-\delta}}\right)$ by first order approximation. We conclude by noting that the same proof would go through if we apply instead the LLT in [17], eq. (2.4) on p.25. □

Now we consider the second part of our decomposition and prove a local limit theorem for $q_{h,l}$.

Lemma 17. *We have*

$$q_{h,l} := \mathbb{P}_h(H_1 = l) = \frac{h}{2\sqrt{\pi} l^{3/2}} e^{-\frac{h^2}{4l}} + \mathcal{O}\left(\frac{1}{l^{3/2}h} \wedge \frac{h}{l^{5/2}} + \frac{h}{l^{2-\delta}} e^{-\frac{h^2}{8l}}\right).$$

Proof. Let $G_{p,n} := \sum_{k=1}^{n} g_k$, with g_k's i.i.d. geometric random variables with success probability $p = 2/3$ and values in $\{0,1,2...\}$. Decomposing and conditioning on the number of vertical steps n, we have

$$\begin{aligned}
q_{h,l} &= \sum_{n=h}^{\infty} \mathbb{P}_0(S_n = h; S_k > 0, \forall 1 \leq k \leq n) \mathbb{P}(G_{p,n} = l) \\
&= \sum_{n=h}^{\infty} \frac{h}{n} \mathbb{P}_0(S_n = h) \mathbb{P}(G_{p,n} = l),
\end{aligned}$$

by the Ballot theorem (see e.g. [5], p.202 thm.4.3.2). Now, let $\delta > 0$ and split the sum into two parts as follows

$$\sum_{|n-2l|\leq l^{1/2+\delta}} \frac{h}{n} \mathbb{P}_0(S_n = h)\mathbb{P}(G_{p,n} = l) + \mathcal{O}\left[\sum_{|n-2l|>l^{1/2+\delta}} \mathbb{P}(G_{p,n} = l)\right]. \tag{3.12}$$

[2] This LLT and the following ones are stated for aperiodic random walks, but it is not difficult to deduce the analogue for bipartite walks, see e.g. pp. 26-27 of the cited book.

Notice that

$$\mathbb{P}(G_{p,n} = l) = \binom{n+l-1}{l} p^n (1-p)^l = \frac{n}{l}\mathbb{P}(G_{1-p,l} = n), \tag{3.13}$$

so for the second term of (3.12), we have

$$\begin{aligned}\sum_{|n-2l|>l^{1/2+\delta}} \mathbb{P}(G_{p,n} = l) = &\sum_{|n-2l|>l^{1/2+\delta}} \frac{n}{l}\mathbb{P}(G_{1-p,l} = n)\\ \leq &\mathbb{E}\left[G_{1-p,l}; |G_{1-p,l} - 2l| \geq l^{1/2+\delta}\right]\\ = &\mathcal{O}(e^{-cl^{2\delta}}),\end{aligned}$$

for appropriate $c > 0$ by the Chernoff bound. By (3.13) again, we can rewrite the first term of (3.12) as

$$\sum_{|n-2l|\leq l^{1/2+\delta}} \frac{h}{l}\mathbb{P}_0(S_n = h)\mathbb{P}(G_{1-p,l} = n)$$

and apply the local limit theorems and first order approximation as before. □

Thanks to lemma 16 and lemma 17 we can estimate the errors I_1 and J_1:

$$I_1 = \sum_{h=1}^{\infty} \log h\, \mathcal{O}\left(\frac{1}{\sqrt{m}h^2} + \frac{e^{-\frac{h^2}{8m}}}{m^{1-\delta}}\right) = \mathcal{O}\left(\frac{\log m}{m^{1/2-2\delta}}\right). \tag{3.14}$$

Here in the second term of the summation, we used a uniform bound for all $h \leq m^{1/2+\delta}$ and an integral to bound the sum for $h \geq m^{1/2+\delta}$, where the error is monotone in h. Similarly, we have

$$J_1 = \sum_{l=1}^{\infty} \log l\, \mathcal{O}\left(\frac{1}{l^{3/2}h} + \frac{h}{l^{2-\delta}}e^{-\frac{h^2}{8l}}\right) = \mathcal{O}\left(\frac{\log h}{h^{1-3\delta}}\right), \tag{3.15}$$

where in the second term of the (3.15) we used a uniform bound for all $l \geq h^{2-\delta}$ and an integral to bound the sum for $l \leq h^{2-\delta}$.

For errors I_2 and J_2, as in the case $h \geq m^{1/2+\delta}$ (and, respectively, $l \leq h^{2-\delta}$) mentioned above, it is straightforward to give exponential bounds with integrals:

$$I_2 = \mathcal{O}\left(e^{-cm^{2\delta}}\right) \tag{3.16}$$

and

$$J_2 := \mathcal{O}\left(e^{-ch^{\delta}}\right) \tag{3.17}$$

for some $c > 0$.

3.2.2 Euler-Maclaurin approximation

In this section we apply the Euler-Maclaurin formula to bound I_3 and J_3. We recall here the general formula (see e.g. [16]): let $n, p \geq 1$ be two integers, and $a, b \in \mathbb{R}$ with $a < b$. For any $\phi \in C^p[a,b]$ we have

$$\left[\sum_{i=a}^{b} \phi(i) - \int_a^b \phi(x)dx\right] = \sum_{j=1}^{p} \frac{B_j}{j!}\left[\phi^{(j-1)}(x)\right]_a^b + r(p,a,b), \tag{3.18}$$

where B_j are the Bernoulli coefficients and $r(p,a,b)$ is the remainder of order p. In our case we will need only a first order approximation, for which the following bound on the remainder is known

$$r(1,a,b) \leq C(b-a) \max_{a \leq x \leq b} |\phi'(x)|, \tag{3.19}$$

with $C > 0$ a positive constant. Moreover we recall that $B_1 = -1/2$.

Recall that $f_m(x) := \frac{\log(x)}{\sqrt{\pi m}} e^{-\frac{x^2}{4m}}$ and $f_m'(x) = \left(\frac{1}{x} - \frac{x \log x}{2m}\right)\frac{1}{\sqrt{\pi m}} e^{-\frac{x^2}{4m}}$. Hence, by the Euler-Maclaurin formula (3.18)

$$\begin{aligned} I_3 &\leq \sum_{k=0}^{(1/2+\delta)\log_2 m} \left[\sum_{h=2^k}^{2^{k+1}} f_m(h) - \int_{2^k}^{2^{k+1}} f_m(x)dx\right] \\ &\leq \sum_{k=0}^{(1/2+\delta)\log_2 m} \left[\frac{f_m(2^k) + f_m(2^{k+1})}{2} + r_k\right] = \mathcal{O}\left(\frac{\log m}{m^{1/2-2\delta}}\right), \end{aligned} \tag{3.20}$$

where $r_k := r(1, 2^k, 2^{k+1})$ and the last equality follows from (3.19), since

$$\begin{aligned} |r_k| \leq C2^k \max_{2^k \leq x \leq 2^{k+1}} |f_m'(x)| &\leq C2^k \max_{2^k \leq x \leq 2^{k+1}} \left(\frac{1}{x} + \frac{x \log x}{2m}\right)\frac{1}{\sqrt{\pi m}} e^{-\frac{x^2}{4m}} \\ &\leq C2^k \left(\frac{1}{2^k} + \frac{2^{k+1}(k+1)}{2m}\right)\frac{1}{\sqrt{\pi m}} \\ &= \mathcal{O}\left(\frac{1}{\sqrt{m}} + \frac{2^{2k} k}{m^{3/2}}\right). \end{aligned}$$

Let $g_h(x) := \log x \frac{h}{2\sqrt{\pi} x^{3/2}} e^{-\frac{h^2}{4x}}$ and $g_h'(x) = \left(1 - \frac{3 \log x}{2} + \frac{h^2 \log x}{4x}\right)\frac{h}{2\sqrt{\pi} x^{5/2}} e^{-\frac{h^2}{4x}}$. By the

Euler-Maclaurin formula (3.18),

$$
\begin{aligned}
J_3 &= \sum_{k=(2-\delta)\log_2 h}^{\infty} \left[\sum_{l=2^k}^{2^{k+1}} g_h(l) - \int_{2^k}^{2^{k+1}} g_h(x)dx \right] \\
&\leq \sum_{k=(2-\delta)\log_2 h}^{\infty} \left[\frac{g_h(2^k)+g_h(2^{k+1})}{2} + \tilde{r}_k \right] \\
&= \sum_{k=(2-\delta)\log_2 h}^{\infty} \mathcal{O}\left(\frac{hk}{2^{3k/2}} + \frac{h^3 k}{2^{5k/2}} \right) = \mathcal{O}\left(\frac{\log h}{h^{2-\frac{5\delta}{2}}} \right),
\end{aligned}
\tag{3.21}
$$

where we use the fact that

$$
\begin{aligned}
|\tilde{r}_k| \leq & C' 2^k \left(1 + \frac{3(k+1)}{2} + \frac{h^2(k+1)}{2^{k+2}} \right) \frac{h}{2\sqrt{\pi}2^{5k/2}} \\
= & \mathcal{O}\left(\frac{hk}{2^{3k/2}} + \frac{h^3 k}{2^{5k/2}} \right).
\end{aligned}
$$

We conclude this section by noting that the bounds on errors I_4 and J_4 follow from direct calculation:

$$
|I_4| = \int_0^1 f_m(x)dx + \int_{m^{1/2+\delta}}^{\infty} f_m(x)dx = \mathcal{O}\left(\frac{1}{\sqrt{m}} \right) \tag{3.22}
$$

and

$$
|J_4| = \int_0^{h^{2-\delta}} g_h(x)dx = \mathcal{O}\left(e^{-ch^{\delta}} \right). \tag{3.23}
$$

3.2.3 Proof of recurrence

In this section, we complete the proof of recurrence. By the formulas (3.7), (3.8), (3.9) and (3.10) and the estimates (3.14), (3.15), (3.16), (3.17), (3.20), (3.21), (3.22) and (3.23), we get

$$
R_f(m) := \mathbb{E}_m(\log V_1) - (\log m)/2 + \gamma/2 = \mathcal{O}\left(\frac{1}{m^{1/2-3\delta}} \right)
$$

and

$$
R_g(h) := \mathbb{E}_h(\log H_1) - 2\log h - \gamma = \mathcal{O}\left(\frac{1}{h^{1-3\delta}} \right),
$$

where γ is the Euler constant. Then, by (3.6) and lemma 16

$$\begin{aligned}\mathbb{E}_m(\log H_1) =& 2\,\mathbb{E}_m(\log V_1) + \sum_{h=1}^{\infty} [\mathbb{E}_h(\log H_1) - 2\log h]\,\mathbb{P}_m(V_1 = h) \\ =& \log m + \mathcal{O}\left(\frac{1}{m^{1/2-3\delta}}\right) + \sum_{h=1}^{\infty} \mathcal{O}\left(\frac{1}{h^{1-3\delta}}\right)\mathbb{P}_m(V_1 = h) \\ =& \log m + \mathcal{O}\left(\frac{1}{m^{1/2-3\delta}}\right),\end{aligned}$$

so we've shown

$$\epsilon_1(m) := \frac{\mathbb{E}_m\left(\log H_1 - \log m\right)}{2\sqrt{\log m}} = \mathcal{O}\left(\frac{1}{m^{1/2-4\delta}}\right). \tag{3.24}$$

For the truncation error $\epsilon_3(m)$, we have by lemma 16 and lemma 17

$$\begin{aligned}\epsilon_3(m) &= \sum_{l=m^2}^{\infty} \log^2 l\,\mathbb{P}_m(H_1 = l) = \sum_{l=m^2}^{\infty}\sum_{h=0}^{\infty} \log^2 l p_{m,h} q_{h,l} \\ &\le \sum_{l=m^2}^{\infty} \log^2 l \left[\sum_{h\le\sqrt{m}l^{\delta}} p_{m,h}q_{h,l} + \sum_{h>\sqrt{m}l^{\delta}} p_{m,h}\right] \\ &\le \sum_{l=m^2}^{\infty} \log^2 l \left[\sum_{h\le\sqrt{m}l^{\delta}} \mathcal{O}\left(\frac{1}{\sqrt{m}}\frac{h}{l^{3/2}}\right) + \mathcal{O}\left(e^{-cl^{2\delta}}\right)\right] \\ &= \sum_{l=m^2}^{\infty} \log^2 l\,\mathcal{O}\left(\frac{\sqrt{m}}{l^{3/2-2\delta}}\right) = \mathcal{O}\left(\frac{\log^2 m}{m^{1/2-4\delta}}\right),\end{aligned} \tag{3.25}$$

where for $h > \sqrt{m}l^{\delta}$, we apply Chernoff bounds by viewing $p_{m,h}$ as the sum of the absolute value of m many i.i.d random variables, each of which has the same law as the convolution of geometrically many Bernoulli distributions.

Finally, for the numerator in $\epsilon_2(m)$, we have

$$\mathbb{E}_m\left(\log H_1 - \log m\right)^2 \ge \mathrm{Var}_m(\log H_1) \ge \mathrm{Var}_m(\mathbb{E}[\log H_1 \mid V_1]). \tag{3.26}$$

To estimate the rightmost term, we notice that by lemma 16 for any $a > b > c > 0$, there exist $p_1, p_2 > 0$ such that $\mathbb{P}_m(V_1 \ge a\sqrt{m}) \ge p_1$ and $\mathbb{P}_m(b\sqrt{m} \ge V_1 \ge c\sqrt{m}) \ge p_2$ for large enough m. By the above estimate on $R_g(h)$, we obtain that on the event $\{V_1 \ge a\sqrt{m}\}$,

$$\mathbb{E}[\log H_1 \mid V_1] \ge 2\log a + \gamma + \log m + \mathcal{O}\left(\frac{1}{m^{(1-3\delta)/2}}\right). \tag{3.27}$$

Similarly on $\{b\sqrt{m} \geq V_1 \geq c\sqrt{m}\}$,

$$\mathbb{E}[\log H_1 \mid V_1] \leq 2\log b + \gamma + \log m + \mathcal{O}\left(\frac{1}{m^{(1-3\delta)/2}}\right). \tag{3.28}$$

Then, by the formula $\mathrm{Var}(X) = 1/2\mathbb{E}(X - X')^2$, where X' is an independent copy of X, and taking $X = \mathbb{E}[\log H_1 \mid V_1]$, we obtain thanks to (3.27) and (3.28)

$$\mathrm{Var}_m(\mathbb{E}[\log H_1 \mid V_1]) \geq 4p_1p_2(\log a - \log b)^2 + \mathcal{O}\left(\frac{1}{m^{(1-3\delta)/2}}\right),$$

for large enough m. Hence, we get

$$\epsilon_2(m) := \frac{\mathbb{E}_m\left(\log H_1 - \log m\right)^2}{16(\log m)^{3/2}} \geq \frac{4p_1p_2(\log a - \log b)^2 + \mathcal{O}\left(\frac{1}{m^{(1-3\delta)/2}}\right)}{16(\log(m))^{3/2}} \tag{3.29}$$

for large m.

We finish our proof of recurrence with (3.5), (3.24), (3.25) and (3.29). □

3.3 Transience of $\mathbf{G}_1$

Throughout this Section we denote by M the simple random walk on $\mathbf{G}_1$. Our main result is a local limit theorem for the return probabilities of M; a new proof of transience will be obtained as a corollary.

Let T_n be the time just after the n-th vertical step of M, and consider the usual decomposition into vertical skeleton and embedded random walk

$$M_{T_n} = (X_n, Y_n), \tag{3.30}$$

such that Y is the simple random walk on $\mathbb{Z}$ and $X_0 := 0$, $X_n := \sum_{i=0}^{n-1} \xi_i$ for $n > 0$, where ξ_i is the random variable representing the horizontal steps that M performs between the $i-$th and the $i+1$-th vertical step; note that $|\xi_i|$ is a geometric random variable with success probability $p = 2/3$, and $\mathrm{sgn}(\xi_i)$ is determined by Y_i. We shall prove the following.

Theorem 8. *We have for large n*

$$\mathbb{P}_0\left(M_{T_{2n}} = (0,0)\right) \sim \frac{1}{2\sqrt{\pi}n^{3/2}}.$$

Proof. Let σ be the first time that Y visits -1 and let $\alpha < 1$. We have

$$\mathbb{P}_0(\sigma > 2n^\alpha | Y_{2n} = 0) = \mathcal{O}\left(\frac{1}{n^{\frac{\alpha}{2}}}\right) \tag{3.31}$$

for large n; in fact by standard properties of the simple random walk in $\mathbb{Z}$

$$\begin{aligned}
\mathbb{P}_0(\sigma > 2n^\alpha, Y_{2n} = 0) &= \sum_{k=0}^{n^\alpha} \mathbb{P}_0(Y_1 \geq 0, Y_2 \geq 0, ..., Y_{2n^\alpha - 1} \geq 0, Y_{2n^\alpha} = 2k; Y_{2n} = 0) \\
&= \sum_{k=0}^{n^\alpha} \mathbb{P}_0(Y_{2n-2n^\alpha} = 2k)\mathbb{P}_0(Y_1 \geq 0, Y_2 \geq 0, ..., Y_{2n^\alpha - 1} \geq 0, Y_{2n^\alpha} = 2k) \\
&\leq \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \mathbb{P}_0(Y_1 \geq 0, Y_2 \geq 0, ..., Y_{2n^\alpha - 1} \geq 0) = \mathcal{O}\left(\frac{1}{n^{\frac{1+\alpha}{2}}}\right).
\end{aligned}$$

Define

$$A_n^+ := |\{0 \leq j < 2n | S_j \geq 0\}|$$

to be the amount of time spent by the vertical skeleton on the non-negative axis up to step $2n$. Let $1/2 < \delta < 1$. We have

$$\mathbb{P}_0(X_{2n} = 0, Y_{2n} = 0) = \sum_{m \geq 1} \sum_{|k-n| \leq n^{1/2+\delta}} \mathbb{P}_0(X_{2n} = 0, Y_{2n} = 0, A_n^+ = k, \sigma = 2m - 1) \tag{3.32}$$

$$+ \sum_{|k-n| > n^{1/2+\delta}} \mathbb{P}_0(X_{2n} = 0, Y_{2n} = 0, A_n^+ = k). \tag{3.33}$$

We begin by estimating (3.32); to this purpose we split again the sum into two parts, one for $1 \leq m \leq n^\alpha$ and the other for $m > n^\alpha$, and apply the modified Chung Feller theorem (see theorem 2.3.1 in [14]) to the vertical skeleton after the first time it goes below the origin.

We obtain

$$\sum_{m=1}^{n^\alpha} \sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}_0(X_{2n}=0, Y_{2n}=0, A_n^+=k, \sigma=2m-1)$$

$$=\sum_{m=1}^{n^\alpha} \sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}_0(X_{2n}=0 \mid Y_{2n}=0, A_n^+=k, \sigma=2m-1)$$

$$\times \mathbb{P}_0(A_n^+=k|\sigma=2m-1, Y_{2n}=0)\mathbb{P}_0(\sigma=2m-1 \mid Y_{2n}=0)\mathbb{P}_0(Y_{2n}=0)$$

$$\sim\frac{1}{2\sqrt{\pi}n^{3/2}}\mathbb{P}_0(\sigma\le 2n^\alpha \mid Y_{2n}=0) \sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}_0(X_{2n}=0 \mid Y_{2n}=0, A_n^+=k)$$

$$\sim\frac{1}{2\sqrt{\pi}n^{3/2}} \sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}(X_{2n,k}=0), \tag{3.34}$$

with $X_{2n,k} := \sum_{i=0}^{k-1}\xi_i - \sum_{i=k}^{2n-1}\xi_i$ for $1\le k\le 2n$, where $(\xi_i)_{i\ge 0}$ is a sequence of i.i.d. geometric random variables with success probability $p=2/3$ and values in $\{0,1,2,...\}$. Let $m_{n,k} := \mathbb{E}(X_{2n,k}) = k-n$ and $s_n := \sigma^2(X_{2n,k}) = 2n\sigma^2(\xi_1)$.

Then, by means of a local limit theorem for independent (not necessarily identically distributed) random variables (e.g. here we use [23], Chapter VII, theorem 5, p.197), we obtain

$$\sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}(X_{2n,k}=0) = \sum_{|k-n|\le n^{1/2+\delta}} \left[\overline{p}_n^{m_{n,k},s_n}(0) + \mathcal{O}\left(\frac{1}{n}\right)\right]$$

$$= \sum_{|j|\le n^{1/2+\delta}} \left[\overline{p}_n^{0,s_n}(j) + \mathcal{O}\left(\frac{1}{n}\right)\right]$$

$$=1+o(1)+\mathcal{O}\left(\frac{1}{n^{1/2-\delta}}\right), \tag{3.35}$$

where $\overline{p}_n^{m_{n,k},s_n}(x) = \frac{1}{\sqrt{2\pi s_n}}e^{-\frac{(x-m_{n,k})^2}{2s_n}}$.

Similarly

$$\sum_{m\ge n^\alpha+1} \sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}_0(X_{2n}=0, Y_{2n}=0, \sigma=2m-1, A_n^+=k)$$

$$=\mathcal{O}\left(\frac{1}{n^{3/2}}\right)\mathbb{P}_0(\sigma>2n^\alpha \mid Y_{2n}=0) \sum_{|k-n|\le n^{1/2+\delta}} \mathbb{P}(X_{2n,k}=0) = \mathcal{O}\left(\frac{1}{n^{\frac{3+\alpha}{2}}}\right), \tag{3.36}$$

by equations (3.31), (3.35) and because by the Chung Feller theorem we have that $P_0(A_n^+ = k|\sigma = 2m, Y_{2n} = 0) = \mathcal{O}\left(\frac{1}{n}\right)$ when $m \geq n^\alpha$ and $|k-n| \leq n^{1/2+\delta}$. Therefore, the term (3.32) is asymptotic to $\frac{1}{2\sqrt{\pi}n^{3/2}}$ for large n.

Finally, it remains to bound (3.33); to this purpose we use large deviation. Consider the trivial bound

$$\sum_{|k-n|>n^{1/2+\delta}} \mathbb{P}_0(X_{2n} = 0, Y_{2n} = 0, A_n^+ = k) \leq \sum_{|k-n|>n^{1/2+\delta}} \mathbb{P}(X_{2n,k} = 0),$$

and for $k \geq 0$ define $\hat{X}_{2n,k} := X_{2n,k} - m_{n,k}$. We have

$$\begin{aligned} \mathbb{P}\left(\hat{X}_{2n,k} \geq n^{1/2+\delta}\right) &= \inf_{t>0} \mathbb{P}(e^{t\hat{X}_{2n,k}} \geq e^{tn^{1/2+\delta}}) \\ &\leq \inf_{t>0} \frac{\mathbb{E}(e^{t\hat{X}_{2n,k}})}{e^{tn^{1/2+\delta}}} \\ &= \inf_{t>0} \frac{\left(\frac{2e^{-t/2}}{3-e^t}\right)^k \left(\frac{2e^{t/2}}{3-e^{-t}}\right)^{2n-k}}{e^{tn^{1/2+\delta}}} \\ &= \mathcal{O}\left(e^{-\frac{n^{2\delta}}{3}}\right), \end{aligned} \tag{3.37}$$

since, by Taylor expansion

$$\left(\frac{2e^{-t/2}}{3-e^t}\right)^k \left(\frac{2e^{t/2}}{3-e^{-t}}\right)^{2n-k} = 1 + \frac{3n}{4}t^2 + \mathcal{O}(nt^3),$$

and then we use the fact that the minimum is attained at $t^* = \frac{2}{3}n^{-1/2+\delta}$. Analogously we obtain

$$\mathbb{P}\left(\hat{X}_{2n,k} \leq -n^{1/2+\delta}\right) = \mathcal{O}\left(e^{-\frac{n^{2\delta}}{3}}\right). \tag{3.38}$$

Then, by (3.37) and (3.38)

$$\sum_{|k-n|>n^{1/2+\delta}} \mathbb{P}(X_{2n,k} = 0) = \sum_{|k-n|>n^{1/2+\delta}} \mathbb{P}\left(\hat{X}_{2n,k} = -(k-n)\right) = \mathcal{O}\left(ne^{-\frac{n^{2\delta}}{3}}\right).$$

This completes the proof. $\square$

Corollary 4. *The random walk on graph* $\mathbf{G}_1$ *is transient.*

Proof. By the transience of (X, Y), we can find $C > 0$ such that $\sum_n \mathbb{P}_0(X_n = x, Y_n = 0) \leq C < \infty$ for every $x \in \mathbb{Z}$. Whence

$$\sum_i \mathbb{P}_0(M_i = 0) = \sum_n \sum_{x \geq 0} \mathbb{P}_0(X_n = -x, Y_n = 0)(1/3)^x \leq C \sum_{x \geq 0} (1/3)^x < \infty.$$

□

Remark 4. By a slight modification of the above argument one can actually extend the limit theorem to most of the points $z = (z_1, 0) \in \mathbb{Z}^2$ with $|z_1| \leq n$ except for a subset of size $o(n)$, and obtain $\mathbb{P}_0\left(M_{T_{2n}} = z\right) \sim \frac{1}{2\sqrt{\pi}n^{3/2}}$. Similarly, for $|z_1| > n + n^{1/2+\delta}$ with $\delta > 0$, the probability is exponentially small.

Remark 5. With analogous but more involved calculations, a local limit theorem for the original chain M of the form $\mathbb{P}_0(M_{2n} = 0) \sim \frac{C}{n^{3/2}}$, $C > 0$ can be established.

Appendix A

The square grid with oriented horizontal and vertical levels

Consider a square grid lattice where all the lines, both horizontal and vertical, are randomly oriented. Precisely, let $(\epsilon_y)_{y\in\mathbb{Z}}, (\zeta_x)_{x\in\mathbb{Z}}$ be two independent families of $\{-1,1\}$-valued random variables, independent of each others: let the horizontal levels be oriented according to (ϵ_y), while the vertical ones according to (ζ_x). We denote this random graph by $\mathbf{F}_{\epsilon,\zeta}$.

Of course, the recurrence behaviour of $\mathbf{F}_{\epsilon,\zeta}$ depends of the distribution of (ϵ_y) and (ζ_x). If, for example, both sequences are deterministic and alternately oriented, we obtain the so-called Manhattan lattice, which is known to be recurrent (see [9]). However, if (ϵ_y) and (ζ_x) are chosen to be both Rademacher i.i.d. sequences, the type problem is still open. In fact, N-G. Plantard proposes the following conjecture (see [10]):

Conjecture 1. If (ϵ_y) and (ζ_x) are Rademacher i.i.d. sequences, independent of each other, then the simple random walk on $\mathbf{F}_{\epsilon,\zeta}$ is a.s. transient.

The goal of this section is to show that, by using the techniques developed in Chapters 1 and 2, we can determine the type for the simple random walk on a certain class of $\mathbf{F}_{\epsilon,\zeta}$. Precisely, we claim the following.

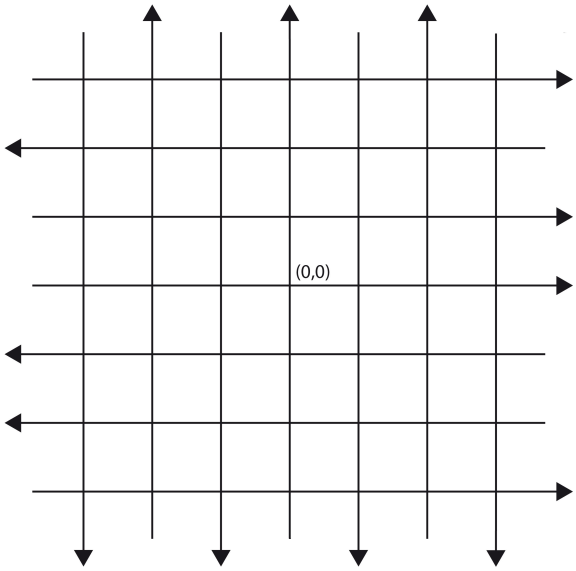


Figure 10: The random graph $\mathbf{F}_{\epsilon,\zeta}$, where we take $(\zeta_x)_{x\in\mathbb{Z}}$ to be the alternate sequence.

Theorem 9. *Let $\zeta := (\zeta_x)_{x\in\mathbb{Z}}$ be the deterministic alternate sequence, i.e. $\zeta_x = (-1)^{|x|}$ $\forall x \in \mathbb{Z}$.*

(i) If $\epsilon := (\epsilon_y)_{y\in\mathbb{Z}}$ is a sequence of i.i.d. Rademacher $\{-1,1\}$-valued random variables, then the simple random walk on $\mathbf{F}_{\epsilon,\zeta}$ is a.s. transient.

(ii) Let $Q > 1$ be an even integer. If $\epsilon := (\epsilon_y)_{y\in\mathbb{Z}}$ is a deterministic sequence with period Q and such that $\sum_{y=1}^{Q} \epsilon_y = 0$, then the simple random walk on $\mathbf{F}_{\epsilon,\zeta}$ is recurrent.

(iii) If $\epsilon^\beta := (\epsilon_{y,\beta})_{y\in\mathbb{Z}}$ is the sequence of random variables defined in (2.12), then the simple random walk on $\mathbf{F}_{\epsilon^\beta,\zeta}$ is a.s. transient if $\beta < 1$, and a.s. recurrent if $\beta > 1$.

Proof. (i) We want to apply the technique used in the proof of theorem 1, and so we start dividing the random walk into two components X and Y. Note that, because the vertical edges are alternate, in the present case the vertical skeleton takes the following form (cfr. figure 10 above)

$$Y_n := \sum_{i=1}^{n} \nu_i, \quad n \geq 1,$$

where $(\nu_i)_{i\geq 0}$ is a MC with transition matrix

$$\pi_\nu = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}, \text{ with } q = \frac{2}{3}.$$

(Observe that this is the same process we obtained in the honeycomb lattice, except that there we had $q = \frac{1}{3}$.)

The embedded random walk takes also the usual form (1.2), that is

$$X_n := \sum_{y\in\mathbb{Z}} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}.$$

Moreover, the dependencies between X and Y are the usual ones: when considering the embedded random walk conditioned to the vertical one, we need to distinguish between even and odd geometric random variables to track the horizontal steps.

Therefore, we can show a.s. transience by just repeating verbatim the proof of theorem 1.

(ii),(iii). Similarly, the results follow by repeating verbatim the proofs of theorems 2, 3 and 4 of Chapter 2.

□

Remark 6. Notice that the recurrence of the Manhattan lattice can be viewed as a particular case of theorem 9.(ii), when $Q = 2$.

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