Alma Mater Studiorum
Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA
Ciclo XXXI

Settore Concorsuale: 01/A2
Settore Scientifico Disciplinare: MAT 02

Super Jordan triple systems and Kantor triple systems

Presentata da: Antonio Ricciardo

Coordinatore Dottorato:	**Supervisore:**
Chiar.ma Prof.ssa	**Chiar.ma Prof.ssa**
Giovanna Citti	**Nicoletta Cantarini**

Esame finale anno 2019

Contents

Chapter 1

Introduction

Let $(A, \cdot)$ be an associative algebra. Then the commutative product

$$a \circ b = \frac{1}{2}(a \cdot b + b \cdot a)$$

and the skew-commutative product

$$[a, b] = \frac{1}{2}(a \cdot b - b \cdot a)$$

define on A a Jordan algebra and a Lie algebra structure, respectively. A deep relationship between these two kinds of algebras is given by the so-called Tits-Kantor-Koecher construction (TKK) [30, 35, 56], which establishes a bijection between isomorphism classes of Jordan algebras and isomorphism classes of Lie algebras endowed with a short grading induced by an $\mathfrak{sl}(2)$-triple.

Motivated by the work of M. Koecher on bounded symmetric domains [36], K. Meyberg extended the TKK correspondence to Jordan triple systems [42].

Definition 1.1. [25] A *Jordan triple system* (JTS) is a vector space V endowed with a trilinear map $(\cdot, \cdot, \cdot) : \otimes^3 V \to V$ satisfying the following axioms:

$$(u, v, (x, y, z)) = ((u, v, x), y, z) - (x, (v, u, y), z) + (x, y, (u, v, z)) \quad \text{principal identity}$$

$$(x, y, z) = (z, y, x) \quad \text{commutativity}$$

These are in fact particular examples of the so-called Kantor triple systems:

Definition 1.2. [31] A *Kantor triple system* (shortly, KTS) is a vector space V with a trilinear product $(\, , \, , \,)$ such that, in addition to the principal identity, it satisfies

$$K_{K_{u,v}(x),y} = K_{(yxu),v} - K_{(yxv),u} \quad \text{Kantor identity}$$

where $u, v, x, y, z \in V$ and $K_{x,y} : V \to V$ is defined by $K_{x,y}(z) = (xzy) - (yzx)$.

Kantor triple systems are also known as generalized Jordan triple systems of the second kind or $(-1,1)$-Freudenthal-Kantor triple systems. We will refer to $K_{x,y}$ as the "Kantor tensor" associated to $x, y \in V$. Note that a Jordan triple system is precisely a KTS all of whose associated Kantor tensors vanish.

Another class of triple systems, closely related to Jordan triple systems, is that of N=6 3-algbras. They were introduced in J. Bagger and N. Lambert's [7] as a model for a particular case of superconformal Chern-Simons theories in three dimensions and J. Palmkvist in [49] extended the TKK to N=6 3-algebras.

Definition 1.3. [7]An *N=6 3-algebra* is a vector space with a trilinear product $(\,,\,,\,)$ which satisfies the principal identity and

$$(x,y,z) = -(z,y,x) \quad \text{anti-commutativity}$$

In this work we will deal with linearly compact systems which may have infinite-dimension. In this case we also assume that triple products are continuous.

Simple finite-dimensional Jordan triple systems over an algebraically closed field were classified by O. Loos [39]. The first aim of this work is to systematically address the case where not all Kantor tensors are trivial. On the other hand, simple finite-dimensional N=6 3-algebras over an algebraically closed field were classified by J. Palmkvist in [49], however, the classification for the infinite-dimensional linearly-compact ones was obtained only later by N. Cantarini and V. Kac in [14]. The second aim of this thesis is to embed JTS and N=6 3-algebras inside a new class of super-symmetric triple systems, namely ϵ-super symmetric Jordan triple systems, in order to unify the TKK construction for JTS and that for N=6 3-algebras and get a more general version for sJTS.

Definition 1.4. Let $\epsilon \in \mathbb{Z}_2$. An *$\epsilon$-super Jordan triple system* (ϵ-sJTS) is a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a trilinear product compatible with the $\mathbb{Z}_2$-grading, i.e. $(V_{\bar{i}}, V_{\bar{j}}, V_{\bar{k}}) \subseteq V_{\bar{i}+\bar{j}+\bar{k}}$, and satisfying the following identities:

$$\begin{aligned} &(u,v,(x,y,z)) = -(-1)^{\alpha(u,v,x)+\epsilon|u|}(x,(v,u,y),z)+ \\ &+((u,v,x),y,z) + (-1)^{\beta(u,v,x,y)}(x,y,(u,v,z)) \qquad \text{super-principal identity} \\ &(x,y,z) = (-1)^{\alpha(x,y,z)}(z,y,x) \qquad \text{super-commutativity} \end{aligned} \tag{1.1}$$

where $|x| = \bar{i}$ if $x \in V_{\bar{i}}$, $\alpha(u,v,x) = |u||v| + |v||x| + |u||x|$, $\beta(u,v,x,y) = (|u|+|v|)(|x|+|y|)$.

S. Okubo and N. Kamiya in [45] introduced a notion of δ *Jordan-super triple systems* which satisfies identities similar to our super-principal identity and super-commutativity, anyway, their notion coincide with ours only in the case of $\delta = 1$ and $\epsilon = \bar{0}$.

Definition 1.5. A subspace $I \subset V$ of a (super)triple system V is called:

(i) an *ideal* if $(VVI) + (VIV) + (IVV) \subset I$,

(ii) a K-*ideal* if $(VVI) + (IVV) \subset I$,

(iii) a *left-ideal* if $(VVI) \subset I$.

If V is linearly compact, we also assume that I is closed in V.

We say that V is *simple* (resp. K-*simple*, resp. *irreducible*) if it has no non-trivial ideals (resp. K-ideals, resp. left-ideals).

We remark that such a triple system is either K-simple or it is polarized, i.e., it has a direct sum decomposition $V = V^+ \oplus V^-$ satisfying $(V^\pm V^\mp V^\pm) \subset V^\pm$ and $(V^\pm V^\pm V) = 0$ (see, e.g., [2]). The classification problem is thus reduced to the study of K-simple linearly compact KTS and ϵ-sJTS , that is the primary object of study of our work.

Definition 1.6. Let V and W be two triple systems. A bijective linear map $\varphi : V \to W$ is called:

(i) an *isomorphism* if $\varphi(x, y, z) = (\varphi(x)\varphi(y)\varphi(z))$ for all $x, y, z \in V$,

(ii) a *weak-isomorphism* if there exists another bijective linear map $\varphi' : V \to W$ such that $\varphi(x, y, z) = (\varphi(x)\varphi'(y)\varphi(z))$ for all $x, y, z \in V$.

If V and W are linearly compact, we also assume that φ and φ' are continuous.

Starting from a (finite-dimensional) Kantor triple system V, I. Kantor constructed a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ with $\mathfrak{g}_{-1} \simeq V$ and endowed with a $\mathbb{C}$-linear grade-reversing involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ (see Section 4 for the definition). The Lie algebra $\mathfrak{g}$ is defined as an appropriate quotient of a subalgebra of the (infinite-dimensional) universal graded Lie algebra generated by V (see also e.g. [29, §3]) and if V is a Jordan triple system then $\mathfrak{g}_{-2} = \mathfrak{g}_2 = 0$. Kantor then used this correspondence to classify the K-simple finite-dimensional KTS, up to weak-isomorphisms [31]. Dealing with weak-isomorphisms instead of isomorphisms amounts to the fact that triple systems associated to the same grading but with *different involutions* are actually regarded as equivalent. The structure theory of Kantor triple systems (up to weak-isomorphisms) has been the subject of recent investigations, see [2, 21, 48, 19].

The classification problem of K-simple KTS up to isomorphisms has been completely solved only in the *real classical* case by H. Asano and S. Kaneyuki [6, 29]. The exceptional case is more intricate and an interesting class of models has been constructed in the compact real case by D. Mondoc in [44, 43], making use of the structure theory of tensor products of composition algebras pioneered by B. Allison in [3]. Upon complexification, the finite-dimensional K-simple KTS obtained in [43] correspond exactly to the class of KTS of *extended Poincaré type* that we introduce in terms of spinors and Clifford algebras.

Although various examples of K-simple KTS are available in literature, a complete list is still missing. Our work aims to fill this gap and it provides the classification of *linearly compact complex K-simple KTS up to isomorphisms*.

On the other hand, V. Kac in [26] extended Kantor ideas to Jordan superalgebras, the super generalization of Jordan algebras. He obtained a bijection between finite-dimensional simple $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an $\mathfrak{sl}(2)$-triple and finite-dimensional simple Jordan superalgebras. The classification was later extended to infinite-dimensional linearly-compact Jordan superalgebras in N. Cantarini and V. Kac's [11]. The direct continuation of this work is given by the TKK for N=6 3-algebras which gives an equivalence between simple N=6 3-algebras and simple $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with the constraint of *consistency* between the $\mathbb{Z}$ and the $\mathbb{Z}_2$-grading: $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{2k}$, and a *graded conjugation*. We follow these lines to achieve the TKK for ϵ-sJTS .

In Chapter 3 we give the main examples of triple systems which will appear in the final classification. Section 3.1 is devoted to the K-simple KTS which are associated to the classical Lie algebras. They are the special linear KTS, series $\mathfrak{Ksl}$, with their orthogonal and symplectic subsystems $\mathfrak{Kso}$, $\mathfrak{Ksp}$ and the peculiar case of anti-reflexive matrices, $\mathfrak{Kar}$. We point out that some of them have different non-equivalent triple products. In Section 3.2 we give the examples of finite-dimensional ϵ-sJTS which appear in the classification of the K-simple finite-dimensional ones. A broad class of examples is obtained starting from a superspace of rectangular matrices and considering the supercommutative product

$$(x, y, z) = x\phi(y)z + (-1)^{\alpha}(x, y, z)z\phi(y)x\,,$$

where ϕ generalizes the notion of superinvolution or superantinvolution (cf. Remark 3.8). We also construct 3 different types of subsystems of the previous one. The other examples of ϵ-sJTS consists of: a vectorial triple systems, which are a generalization of the N=6 3-algebras $C^3(2n)$ (see e.g. [14]); a family of 4-dimensional ϵ-sJTS depending on a complex parameter α which are associated to the exceptional Lie superalgebra $D(2, 1; \alpha)$; a 10-dimensonal ϵ-sJTS associated to the exceptional Lie superalgebra $F(4)$.

In Section 4.1 we extend the TKK for N=6 3-algebras to the case of a non-consistent grading and relax the requirement of graded conjugation. We thus obtain the TKK for ϵ-sJTS , which associates bijectively K-simple ϵ-sJTS and simple 3-graded Lie superalgebras with a grade-reversing ϵ-involution (see 4.3 for its definition). In Section 4.2 we give a simplified version of Kantor's original correspondence $V \Leftrightarrow (\mathfrak{g}, \sigma)$, which makes use of N. Tanaka's approach to transitive Lie algebras of vector fields [54, 55] and can easily be adapted to linearly compact KTS. We reduce the problem of classifying KTS to the problem of classifying such pairs. Furthermore, in Section 5.2 we develop a structure theory of grade-reversing involutions which holds for all finite-dimensional simple $\mathbb{Z}$-graded complex Lie algebras and establish in this way an intimate relation with real forms (see Theorem 5.8).

The isomorphism classes of finite-dimensional K-simple KTS can be deduced by an analysis of the Satake diagrams, which is carried out in Section 5.3 and summarized in Corollary 5.12. In Theorem 5.13 we show that also the Lie algebra of derivations of any finite-dimensional K-simple KTS can be easily read off from the associated Satake diagram as well.

It is worth pointing out that the results contained in Section 5.3 hold for gradings of finite-dimensional simple Lie algebras of *any* depth and therefore provide an abstract classification of all the so-called generalized Jordan triple systems of any kind $\nu \geqslant 1$ ($\nu = 1$ are the Jordan triple systems, $\nu = 2$ the Kantor triple systems).

A complete list of simple, linearly compact, infinite-dimensional Lie algebras consists, up to isomorphisms, of the four simple Cartan algebras, namely, $W(m)$, $S(m)$, $H(m)$ and $K(m)$, which are respectively the Lie algebra of all formal vector fields in m indeterminates and its subalgebras of divergence free vector fields, of vector fields annihilating a symplectic form (for m even), and of vector fields multiplying a contact form by a function (for m odd) [16, 22]. It can be easily shown using [26, 28] that none of these algebras admits a non-trivial $\mathbb{Z}$-grading of finite length and hence, by the TKK construction for linearly compact KTS (Theorem 4.14 and Theorem 4.16), we immediately arrive at the following result.

Theorem 1.7. *Any K-simple linearly compact Kantor triple system has finite-dimension.*

Note that, consistently, a similar statement holds for simple linearly compact Jordan algebras, see [52].

In the finite-dimensional case a complete list, up to isomorphisms, of K-simple KTS consists of eight infinite series, corresponding to classical Lie algebras, and 23 exceptional cases, corresponding to exceptional Lie algebras.

The classical KTS are described in Section 3.1 and they are the complexifications of the compact K-simple KTS classified in [29] (see Theorem 6.1).

The KTS of exceptional type can be divided into three main classes, depending on the graded component $\mathfrak{g}_{-2}$ of the associated Tits-Kantor-Koecher algebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$. (Some authors refer to this Lie algebra simply as the Kantor algebra.) We say that V is:

(i) of *contact type* if $\dim \mathfrak{g}_{-2} = 1$;

(ii) of *extended Poincaré type* if $\mathfrak{g}_{-2} = U$ and $\mathfrak{g}_0$ is the Lie algebra direct sum of $\mathfrak{so}(U)$, of the grading element $\mathbb{C}E$ and of a reductive subalgebra acting trivially on $\mathfrak{g}_{-2}$;

(iii) of *special type* otherwise.

We determine these KTS in Sections 6.2, 6.3 and 6.4 respectively; we start with those of extended Poincaré type as this requires some preliminaries on Clifford algebras, which turn out to be useful for some KTS of contact type too. Our description of the products of extended Poincaré type in terms of spinors gives an alternative realization of the KTS studied by D. Mondoc and it is inspired by the appearance of triple systems in connection with different kinds of symmetries in supergravity theories (see e.g. [23, 33]). In particular, we use some results from [4, 5] and rely on Fierz-like identities which are deduced from the Lie bracket in the exceptional Lie algebras. The KTS of contact type are all associated to the unique contact grading of a simple complex Lie algebra and they are supported over $S^3\mathbb{C}^2$, $\Lambda^3_0\mathbb{C}^6$, $\Lambda^3\mathbb{C}^6$, the semispinor module $\mathbb{S}^+$ in dimension 12 and the 56-dimensional representation $\mathfrak{F}$ of E_7. We shall stress again that a contact grading has usually more than one associated KTS; for instance $\mathfrak{F}$ admits two different products, with algebras of derivations $E_6 \oplus \mathbb{C}$ and $\mathfrak{sl}(8, \mathbb{C})$, respectively. Finally, there are two KTS of special type, which are supported over $V = \Lambda^2(\mathbb{C}^5)^* \otimes \mathbb{C}^2$ and $V = \Lambda^3(\mathbb{C}^7)^*$, and associated to the Lie algebras E_6 and E_7, respectively.

All details on products, including their explicit expressions, are contained in the main results ranging from Theorem 6.9 in Subsection 6.2.1 to Theorem 6.31 in Subsection 6.4.2.

In Chapter 7 we recall the simple classical (see [27]) Lie superalgebras and study their 3-gradings, starting from their classification given in [14]. We point out that for the "strange" series $\mathfrak{p}(n)$ we changed its presentation slightly in order to get a consistent embedding of all the 3-graded classical Lie superalgebras, except for the ones which give rise to the vectorial ϵ-sJTS, in the same 3-grading of $\mathfrak{sl}(m, n)$.

The classical 3-graded simple Lie superalgebras corresponding to the vectorial ϵ-sJTS , on the other hand, can be viewed as degenerations of series of 5-graded Lie superalgebras, this is pointed out more precisely in Remark 7.6.

Finally, in Chapter 8 we carry out the classification of the grade-reversing ϵ-involutions of the classical simple 3-graded Lie superalgebras introduced in Chapter 7, this amount to all the finite-dimensional cases except the Lie superalgebra of Cartan type, $H(0,n)$. To achieve our goal we focus on the action of the automorphisms on the even part of the Lie superalgebras. Depending on the irreducibility of the odd part, in many cases this is enough to obtain the classification. In particular, in the exceptional cases we start from the action on the even part and we reconstruct the action on the whole Lie superalgebra.

We point out that the explicit description of the triple products of the ϵ-sJTS corresponding to the exceptional Lie superalgebras, namely $D(2,1;\alpha)$ and $F(4)$, are carried out in detail. The classification of the infinite-dimensional linearly-compact, as well as the case of $H(0,n)$, is still in progress and, for completeness reasons, it will not appear in this thesis but will be the subject of further studies by the author.

Chapter 2

Preliminary definitions

We begin with the classical algebraic structures and introduce their supersymmetric generalization which will be studied in the rest of the work.

Definition 2.1. A *Jordan triple system* (JTS) is a vector space with a trilinear product $(\, , \, , \,)$ such that:

$$(u, v, (x, y, z)) = ((u, v, x), y, z) - (x, (v, u, y), z) + (x, y, (u, v, z)) \quad \text{principal identity}$$

$$(x, y, z) = (z, y, x) \quad \text{commutativity}$$

Example 2.2. An example of JTS is given by the space $M_n(\mathbb{C})$ with product

$$(abc) = ab^t c + cb^t a$$

where x^t the usual transpose. We denote this JTS by $J(n, t)$.

Definition 2.3. Let V be a vector space with a trilinear product $(\, , \, , \,)$ and let $K_{u,v}(x) = (u, x, v) - (v, x, u)$. We say that V is a *Kantor triple system*, or simply KTS, if the following identities hold:

$$K_{K_{u,v}(x),y} = K_{(y,x,u),v} - K_{(y,x,v),u} \quad \text{auxiliar identity}$$

$$(u, v, (x, y, z)) = ((u, v, x), y, z) - (x, (v, u, y), z) + (x, y, (u, v, z)) \quad \text{principal identity}$$

The linear map $K_{u,v}$ is called the *Kantor tensor*.

Remark 2.4. It follows at once that a JTS is a KTS for which the Kantor tensor is trivial, i.e., $K_{u,v} = 0$. Moreover, it is possible to view JTS's and KTS's as special cases of a more general type of triple systems introduced by I. L. Kantor in [32] and called *generalized Jordan triple systems of the ν-th kind* (ν-GJTS), with $\nu \in \mathbb{N}$. In particular, JTS are 1-GJTS, while KTS are 2-GJTS.

Definition 2.5. An *N=6 3-algebra* is a vector space with a trilinear product $(\, , \, , \,)$ such that:

$$(u, v, (x, y, z)) = ((u, v, x), y, z) - (x, (v, u, y), z) + (x, y, (u, v, z)) \qquad \text{principal identity}$$

$$(x, y, z) = -(z, y, x) \qquad \text{anti-commutativity}$$

Example 2.6. We denote by $N(n, t)$ the space $M_n(\mathbb{C})$ with the N=6 triple-product

$$(abc) = ab^t c - cb^t a\,.$$

Definition 2.7. A *super vector space* is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If $v \in V_{\bar{i}}$ then v is said to be homogeneous of parity $\bar{i}$ and its parity is denoted by $|v| \in \mathbb{Z}_2$. A homogeneous element is said to be even if $v \in V_{\bar{0}}$ and odd if $v \in V_{\bar{1}}$. We will write $\dim(V) = (m|n)$ if $\dim(V_{\bar{0}}) = m$ and $\dim(V_{\bar{1}}) = n$. A *subspace* of a super vector space is a subspace of V as a vector space with the induced $\mathbb{Z}_2$-grading. We will say that a multilinear product defined on a super vector space V is compatible with the $\mathbb{Z}_2$-grading whenever $(V_{i_1}, \dots, V_{i_k}) \subset V_{i_1 + \dots + i_k}$. A super vector space with trivial $\mathbb{Z}_2$-grading, i.e. $V = V_{\bar{0}}$, is simply a vector space.

Moreover, we will write $V = \langle v_1, \dots, v_n | w_1, \dots, w_n \rangle$ to denote the span of the elements $v_1, \dots, v_n, w_1, \dots, w_n$, with $|v_i| = \bar{0}$ and $|w_j| = \bar{1}$.

Definition 2.8. A super vector space V is said *linearly-compact* if V is a topological super vector space which admits a fundamental system of neighbourhoods of zero consisting of subspaces of finite codimension and V is complete.

We refer to [11], Chapter 2, for the details on linearly-compact spaces. Here we simply recall that any finite-dimensional (super) vector space with the discrete topology is linearly-compact.

In the following we will implicitly suppose linearly-compactness of all the super vector spaces and if $\phi : V \to W$ is a map of linearly-compact vector spaces we suppose that ϕ is continuous.

Definition 2.9. Let $\epsilon \in \mathbb{Z}_2$. An *$\epsilon$-super Jordan triple system* (ϵ-sJTS) is a super vector space J with a trilinear product compatible with the $\mathbb{Z}_2$-grading and satisfying the following identities:

$$\begin{aligned} &(u, v, (x, y, z)) = -(-1)^{\alpha(u,v,x) + \epsilon |u|}(x, (v, u, y), z) + \\ &+((u, v, x), y, z) + (-1)^{\beta(u,v,x,y)}(x, y, (u, v, z)) \qquad \text{super-principal identity} \end{aligned} \tag{2.1}$$

$$(x, y, z) = (-1)^{\alpha(x,y,z)}(z, y, x) \qquad \text{super-commutativity}$$

where $\alpha(u, v, x) = |u||v| + |v||x| + |u||x|$, $\beta(u, v, x, y) = (|u| + |v|)(|x| + |y|)$.

The reason why ϵ is introduced is clarified by the following remark.

Remark 2.10. Let J be an ϵ-sJTS. Then $J_{\bar{0}}$ is a JTS. On the other side, if $\epsilon = 1$, $J_{\bar{1}}$ is an N=6 3-algebra. This follows form a simple check on the sign functions of Definition 2.9 when restricted to $J_{\bar{0}}$ or $J_{\bar{1}}$.

Definition 2.11. Let J, J' be a triple system. A *homomorphism* $\phi : J \to J'$ is an even linear map of superspaces such that

$$\phi(x, y, z) = (\phi(x), \phi(y), \phi(z))'$$

where $(\, , \, , \,)$ and $(\, , \, , \,)'$ are the triple products of J and J' respectively. The concepts of endomorphism and isomorphism are the usual ones.

A closed subspace $I \subset J$ is

- an *ideal* if $(J, J, I) + (J, I, J) + (I, J, J) \subseteq I$;
- a *K-ideal* if $(J, J, I) + (I, J, J) \subseteq I$;

We say that J is *simple* (resp. *K-simple*) if it has no non-trivial ideal (resp. K-ideal). Note that an ideal is a K-ideal, hence if J is K-simple then it is also simple.

We call *center* of J its subspace $Z(J)$ defined as follows

$$Z(J) = \{x \in J \,|\, (a, x, b) = 0, \ \forall a, b \in J\}.$$

Remark 2.12. Note that the center is an ideal. Indeed, if we denote the center by Z, by definition Z is a closed subspace, $(J, Z, J) = 0 \subseteq Z$ and, by the super-principal identity, $(J, (Z, J, J), J) = (J, Z, (J, J, J)) + ((J, Z, J), J, J) + (J, J, (J, Z, J)) = 0$ hence $(Z, J, J) \subseteq Z$.

Proposition 2.13. *Let J be a simple triple system. Then J is either K-simple or polarized, i.e. there exist two K-ideals I^+ and I^- of J such that $J = I^+ \oplus I^-$, $(I^{\pm}, I^{\pm}, J) = 0$.*

Proof. Let I be a nontrivial K-ideal which is not an ideal. Let $I^+ := I$ and $I^- := (J, I^+, J) \nsubseteq I^+$. I^- is a K-ideal: indeed $(J, J, I^-) = (J, J, (J, I^+, J)) = ((J, J, J), I^+, J) + (J, (J, J, I^+), J) + (J, I^+, (J, J, J)) \subseteq I^-$ and similarly it can be shown that $(I^-, J, J) \subseteq I^-$. We have that $V = I^+ + I^-$ is an ideal. In fact a sum of two K-ideals is still a K-ideal and $(J, V, J) = (J, I^+ + I^-, J) = (J, I^+, J) + (J, I^-, J)$, the first summand is by definition I^-, while the second summand is $(J, (J, I^+, J), J) = (I^+, J, (J, J, J)) + ((I^+, J, J), J, J) + (J, J, (I^+, J, J)) \subseteq I^+$. Since $V \neq 0$ is an ideal it must be $V = J$. We have that $V' = I^+ \cap I^-$ is an ideal of J too. Since V' is a K-ideal we just need to show that $(J, V', J) \subseteq V'$. We have that $(J, V', J) \subseteq (J, I^+, J) = I^-$ and $(J, V', J) \subseteq (J, I^-, J) \subseteq I^+$ simultaneously. It must be $V' = 0$ since $V' \subset I^+ \neq J$ is a proper ideal. We also have $(I^{\pm}, I^{\pm}, J) \subseteq I^+ \cap I^- = 0$. ☐

Chapter 3

Examples of triple systems

In this chapter we give the main examples of ϵ-sJTS and KTS. Before giving the explicit examples of KTS and ϵ-sJTS we fix some notation which will be useful later on. We denote by S_p the square matrix of order p with $(S_p)_{i,j} = \delta_{i,p+1-j}$ and we set $J_{2p} = \begin{pmatrix} 0 & S_p \\ -S_p & 0 \end{pmatrix}$. If $x \in M_{m,n}(\mathbb{C})$ we denote by $x^R = S_n x^t S_m$ the reflex of x . We set $S_{p,q} = \begin{pmatrix} 0 & 0 & S_p \\ 0 & Id_{q-2p} & 0 \\ S_p & 0 & 0 \end{pmatrix}$, $I_{p,q} = Diag(Id_p, -Id_q)$, $\hat{I}_{2p,2q} = Diag(I_{p,q}, -I_{q,p})$ and $H^{\pm}_{4p} = \begin{pmatrix} 0 & I_{p,p} \\ \pm I_{p,p} & 0 \end{pmatrix}$.

3.1 Examples of KTS

Example 3.1. Let A be an associative algebra and $^* : A \to A$ an antinvolution of A. Then $A \oplus A$ with triple product

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)_K = \begin{pmatrix} x_1 y_1^* z_1 + z_1 y_1^* x_1 - y_2^* x_2 z_1 \\ x_2 y_2^* z_2 + z_2 y_2^* x_2 - z_2 x_1 y_1^* \end{pmatrix} \tag{3.1}$$

is a KTS.

For instance, if $A = M_n(\mathbb{C})$ and * is the usual transposition, then $A \oplus A$ with the triple product (3.1) is a KTS. The same holds if $n = 2k$ is even and * is the symplectic transposition

$$^{st} : x \mapsto J_{2k} x^t J_{2k}^{-1} . \tag{3.2}$$

More generally, one can consider $M_{m,n}(\mathbb{C}) \oplus M_{r,m}(\mathbb{C})$ and an invertible linear map

$$^* : M_{m,n}(\mathbb{C}) \oplus M_{r,m}(\mathbb{C}) \to M_{n,m}(\mathbb{C}) \oplus M_{m,r}(\mathbb{C})$$

which is compatible with the two direct sum decompositions and satisfies the properties:

$$\begin{aligned} (x_1 y_1^* z_1)^* &= z_1^* y_1 x_1^* \ , \quad (x_2 y_2^* z_2)^* = z_2^* y_2 x_2^* \ , \\ (x_2^* y_2 z_1)^* &= z_1^* y_2^* x_2 \ , \quad (z_2 y_1 x_1^*)^* = x_1 y_1^* z_2^* \ , \end{aligned} \tag{3.3}$$

for $x_1, y_1, z_1 \in M_{m,n}(\mathbb{C})$, $x_2, y_2, z_2 \in M_{r,m}(\mathbb{C})$. Then $M_{m,n}(\mathbb{C}) \oplus M_{r,m}(\mathbb{C})$ with triple product (3.1) is a KTS. Note that if $m = n = r$, then property (3.3) precisely says that * is an antinvolution.

We introduce the triple systems $\mathfrak{Ksl}(m, n, r; t)$, given by the vector spaces $M_{m,n}(\mathbb{C}) \oplus M_{r,m}(\mathbb{C})$ with triple product (3.1) and

$$* : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^t \\ x_2^t \end{pmatrix} .$$

Likewise, if $m = 2h,\ n = 2k$ and $r = 2l$ are even, it is easy to see that the map

$$* : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} J_{2k}\, x_1^t\, J_{2h}^{-1} \\ J_{2h}\, x_2^t\, J_{2l}^{-1} \end{pmatrix} ,$$

satisfies (3.3) and we denote the corresponding KTS by $\mathfrak{Ksl}(m, n, r; st)$.

Example 3.2. Let $\varphi : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ and $\psi : M_{n,m}(\mathbb{C}) \to M_{n,m}(\mathbb{C})$ be any invertible linear maps of the form

$$\varphi(x) = BxA \ , \qquad \psi(x) = AxB \ ,$$

for some $A \in GL_n(\mathbb{C})$ and $B \in GL_m(\mathbb{C})$ which satisfy $B^2 = \pm \mathrm{Id}$. Then $M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C})$ with the triple product

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)_K = \begin{pmatrix} x_1\psi(y_2)z_1 + z_1\psi(y_2)x_1 - \varphi(y_1)x_2 z_1 \\ x_2\varphi(y_1)z_2 + z_2\varphi(y_1)x_2 - z_2 x_1 \psi(y_2) \end{pmatrix}$$

is a KTS.

In particular, we will use the notation $\mathfrak{Ksl}(m, n; k)$ for the KTS associated to $A = \mathrm{Id}$ and $B = \mathrm{Diag}(-\mathrm{Id}_k, \mathrm{Id}_{m-k})$.

Example 3.3. Let $A \in GL_n(\mathbb{C})$, $B \in GL_m(\mathbb{C})$ be such that $B^2 = \epsilon \mathrm{Id}$, $\epsilon = \pm 1$, and let us consider the associated KTS structure on $M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C})$ described in Example 3.2.

For any $x \in M_{m,n}(\mathbb{C})$ we define

$$x' = S_n x^t S_m \ , \tag{3.4}$$

and set

$$M = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C}) \mid x_2 = x_1' \} .$$

One can check that M is a subsystem of $M_{m,n}(\mathbb{C}) \oplus M_{n,m}(\mathbb{C})$ whenever $A' = \epsilon A$, $B' = \epsilon B$. (Note that $A' = A$ means that A is reflexive while $A' = -A$ anti-reflexive.) If this is the case, projecting onto the first component yields a KTS structure on $M_{m,n}(\mathbb{C})$ with the triple product

$$(x, y, z) = xAy'Bz + zAy'Bx - ByAx'z ,$$

for all $x, y, z \in M_{m,n}(\mathbb{C})$.

The reflexive matrices

$$A = \mathrm{Id} , \qquad B = \begin{pmatrix} 0 & 0 & S_k \\ 0 & \mathrm{Id}_{m-2k} & 0 \\ S_k & 0 & 0 \end{pmatrix} ,$$

give rise to a family of KTS over $\mathbb{C}$ which we denote by $\mathfrak{Kso}(m, n; k)$. Another natural class is obtained when $n = 2j$, $m = 2l$ are both even and A, B anti-reflexive; we let

$$A = J_{2j} S_{2j} , \qquad B = iJ_{2l} S_{2l} ,$$

and denote the associated KTS by $\mathfrak{Kso}(m, n; JS)$.

Example 3.4. Let $A \in GL_n(\mathbb{C})$, $B \in GL_{2m}(\mathbb{C})$ be such that $B^2 = \epsilon \mathrm{Id}$, $\epsilon = \pm 1$, and consider the associated KTS structure on $M_{2m,n}(\mathbb{C}) \oplus M_{n,2m}(\mathbb{C})$ of Example 3.2.

We define $\xi(x) = S_n x^t J_{2m}^{-1}$ for all matrices $x \in M_{2m,n}(\mathbb{C})$ and set

$$M = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in M_{2m,n}(\mathbb{C}) \oplus M_{n,2m}(\mathbb{C}) \mid x_2 = \xi(x_1) \} .$$

If $A' = -\epsilon A$ and $B^{st} = \epsilon B$ (recall (3.2), (3.4)), then M is a subsystem of $M_{2m,n}(\mathbb{C}) \oplus M_{n,2m}(\mathbb{C})$ and upon projecting onto the first factor $M_{2m,n}(\mathbb{C})$ one gets a KTS with triple product

$$(x, y, z) = xA\xi(y)Bz + zA\xi(y)Bx - ByA\xi(x)z ,$$

for all $x, y, z \in M_{2m,n}(\mathbb{C})$.

We note that the matrices $A = \mathrm{Id}$ and $B = J_{2m}$ satisfy the required conditions with $\epsilon = -1$ and denote the corresponding KTS over $\mathbb{C} = \mathbb{C}$ by $\mathfrak{Ksp}(2m, n; J)$.

If $n = 2l$ is even, we may also consider $A = J_{2l} S_{2l}$ and $B = \mathrm{Diag}(-\mathrm{Id}_k, \mathrm{Id}_{2m-2k}, -\mathrm{Id}_k)$, as they satisfy the conditions with $\epsilon = 1$. The associated KTS is denoted by $\mathfrak{Ksp}(2m, n; k)$.

Example 3.5. We let $\mathfrak{Kar}(n) = \mathbb{C}^n \oplus Aref_n(\mathbb{C})$ be the KTS with triple product

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} x_1y_1^tz_1 + z_1y_1^tx_1 - y_2^tx_2z_1 \\ x_2y_2^tz_2 + z_2y_2^tx_2 - z_2x_1y_1^t - (y_1')^t x_1' z_2 \end{pmatrix}$$

where $Aref_n(\mathbb{C})$ is the space of complex anti-reflexive $n \times n$ matrices (see (3.4)) .

3.2 Examples of ϵ-sJTS

Definition 3.6. Let V, W be super vector spaces. The space of linear maps from W to V, $Hom(W, V)$ inherits a grading from the gradings of V and W in the following way: $Hom(W, V)_{\bar{i}} = \{\phi \in Hom(W, V) \mid \phi(W_{\bar{j}}) \subseteq V_{\bar{j}+\bar{i}}\}$. If $dim(V) = (m_1|m_2)$ and $dim(W) = (n_1|n_2)$ we identify $Hom(W, V)$ with $M_{m_1+m_2,n_1+n_2}(\mathbb{C})$ and we denote this superspace by $M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C})$. When dealing with block matrices we will use the following notation to denote the dimension of blocks. When we write $\begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left(\begin{array}{c|c} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right) \end{array}$ we mean that a_{ij} is an $m_i \times n_j$ block, $i, j = 1, 2$, and the horizontal and vertical lines separate even elements from odd ones.

The $\mathbb{Z}_2$-grading of $M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C})$ is given by

$$(M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C}))_{\bar{0}} = \{X \in M_{(m_1+m_2,n_1+n_2)}(\mathbb{C}) | X = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left(\begin{array}{c|c} x & 0 \\ \hline 0 & y \end{array}\right) \end{array}\}$$

$$(M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C}))_{\bar{1}} = \{X \in M_{(m_1+m_2,n_1+n_2)}(\mathbb{C}) | X = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \left(\begin{array}{c|c} 0 & z \\ \hline w & 0 \end{array}\right) \end{array}\}.$$

The particular case $V = W$ gives rise to $Hom(V, V) := End(V)$ and if $dim(V) = (m|n)$ we identify $End(V)$ with $M_{(m|n),(m|n)}(\mathbb{C}) := M_{(m|n)}(\mathbb{C})$. The superspace $End(V)$ with composition is an *associative superalgebra*, where the condition of associativity in the super case is the usual one.

Proposition 3.7. *Let $\epsilon \in \mathbb{Z}_2$, let A be an associative superalgebra and let ϕ be an automorphism of A satisfying one of the following conditions*

$$\phi((x\phi(y)z)) = (-1)^{\epsilon|y|}\phi(x)y\phi(z)\,, \tag{3.5}$$

$$\phi((x\phi(y)z)) = (-1)^{\alpha(x,y,z)+\epsilon|y|}\phi(z)y\phi(x)\,. \tag{3.6}$$

Then A with triple product

$$(x, y, z) = x\phi(y)z + (-1)^{\alpha(x,y,z)}z\phi(y)x \tag{3.7}$$

is an ϵ*-sJTS.*

Proof. Let x, y, z, u, v be homogeneous elements of A. Expanding the triple product and using the definition of the sign functions α and β one obtains

$$\begin{aligned}
&(u, v, (x, y, z)) - ((u, v, x), y, z) - (-1)^{\beta(u,v,x,y)}(x, y, (u, v, z))+\\
&+(-1)^{\alpha(u,v,x)+\epsilon|u|}(x, (v, u, y), z) =\\
&= -(-1)^{\beta(u,v,x,y)}x\phi(y)u\phi(v)z - (-1)^{\alpha(u,v,x)}x\phi(v)u\phi(y)z-\\
&+(-1)^{\alpha(u,v,x)+\epsilon|u|}x\phi(v\phi(u)y)z + (-1)^{(\alpha(u,v,x)+\epsilon|u|+\alpha(v,u,y))}x\phi(y\phi(u)v)z-\\
&-(-1)^{\alpha((u,v,x),y,z)}z\phi(y)u\phi(v)x - (-1)^{(\beta(u,v,x,y)+\alpha(u,v,z)+\alpha(x,y,(u,v,z)))}z\phi(v)u\phi(y)x+\\
&+(-1)^{(\alpha(u,v,x)+\epsilon|u|+\alpha(x,(v,u,y),z))}z\phi(v\phi(u)y)x+\\
&+(-1)^{(\alpha(u,v,x)+\epsilon|u|+\alpha(v,u,y)+\alpha(x,(v,u,y),z))}z\phi(y\phi(u)v)x
\end{aligned} \tag{3.8}$$

By equation (3.8), the triple product satisfies the principal identity if the following system holds

$$\begin{cases}
(-1)^{\beta(u,v,x,y)}\phi(y)u\phi(v) + (-1)^{\alpha(u,v,x)}\phi(v)u\phi(y) =\\
= (-1)^{\alpha(u,v,x)+\epsilon|u|}\phi(v\phi(u)y) + (-1)^{(\alpha(u,v,x)+\epsilon|u|+\alpha(v,u,y))}\phi(y\phi(u)v)-\\
(-1)^{\alpha((u,v,x),y,z)}\phi(y)u\phi(v) + (-1)^{(\beta(u,v,x,y)+\alpha(u,v,z)+\alpha(x,y,(u,v,z)))}\phi(v)u\phi(y) =\\
= (-1)^{(\alpha(u,v,x)+\epsilon|u|+\alpha(x,(v,u,y),z))}\phi(v\phi(u)y)+\\
+(-1)^{(\alpha(u,v,x)+\epsilon|u|+\alpha(v,u,y)+\alpha(x,(v,u,y),z))}\phi(y\phi(u)v)
\end{cases}$$

A direct check shows that if ϕ satisfies either Equation (3.5) or (3.6) the system is satisfied. □

Remark 3.8. Recall that a superinvolution (resp. superantinvolution) of an associative superalgebra is an automorphism ϕ such that $\phi(ab) = \phi(a)\phi(b)$ (resp. $\phi(ab) = (-1)^{|a||b|}\phi(b)\phi(a)$) and $\phi^2(a) = a$. Condition (3.5) (resp. (3.6)) extends the concept of superinvolution (resp. superantinvolution). Indeed, if ϕ is a superinvolution (resp. superantinvolution) then it satisfies Equation (3.5) (resp. Equation (3.6)) with $\epsilon = 0$.

Remark 3.9. Even though the space of homomorphisms is not an associative algebra it is possible to use Proposition 3.7 to define on $\mathrm{Hom}(W,V)$ (and $\mathrm{Hom}(V,W)$) a structure of sJTS.

Let us embed $\mathrm{Hom}(W,V)$ (resp. $\mathrm{Hom}(V,W)$) into $\mathrm{End}(W\oplus V)$ by

$$\mathrm{Hom}(W,V)\ni f\to\bar f\in\mathrm{End}(W\oplus V)\quad \bar f(w,v)=(0,f(w))$$
$$(\text{resp. } \mathrm{Hom}(V,W)\ni g\to\bar g\in\mathrm{End}(W\oplus V)\quad \bar g(w,v)=(g(v),0)\ )$$

Let Φ be an ϵ-superinvolution or an ϵ-superantinvolution of the associative superalgebra $\mathrm{End}(W\oplus V)$ and suppose that $\phi:=\Phi_{|\mathrm{Hom}(W,V)}$ maps $\mathrm{Hom}(W,V)$ to $\mathrm{Hom}(V,W)$. The composition of elements of $\mathrm{Hom}(W,V)$ with those of $\mathrm{Hom}(V,W)$ is well defined and the subspace $\mathrm{Hom}(W,V)$ is closed under the triple product (3.7),
hence $\mathrm{Hom}(W,V)$ is a subsystem of $\mathrm{End}(W\oplus V)$. In the finite-dimensional case this yields a structure of ϵ-sJTS over $M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C})$ where the associative product is replaced with the usual product between (possibly rectangular) matrices.

It is also possible to prove an analogue of Proposition 3.7 for $\mathrm{Hom}(W,V)$ but this would result rather technical.

Example 3.10. Let $M=M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C})$ and $M'=M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C})$. We introduce the following maps from M to M':

$$\tau:\quad \left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)^{\tau}=\left(\begin{array}{c|c} a^t & -c^t\\ \hline b^t & d^t\end{array}\right);$$

$$s\tau:\quad \left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)^{s\tau}=\begin{pmatrix} J_{m_1} & 0\\ 0 & J_{m_2}\end{pmatrix}\left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)^{\tau}\begin{pmatrix} J_{n_1} & 0\\ 0 & J_{n_2}\end{pmatrix}^{-1},$$
if m_1,m_2,n_1,n_2 even;

$$os\tau:\quad \left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)^{os\tau}=\begin{pmatrix} S_{m_1} & 0\\ 0 & J_{m_2}\end{pmatrix}\left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)^{\tau}\begin{pmatrix} S_{n_1} & 0\\ 0 & J_{n_2}\end{pmatrix}^{-1},\ \text{if } m_2,n_2 \text{ even};$$

$$\iota:\quad \left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)\to\left(\begin{array}{c|c} a & -b\\ \hline c & -d\end{array}\right),\ \text{if } m_1=n_1,\ m_2=n_2;$$

$$\Pi:\quad \left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)\to\left(\begin{array}{c|c} d & c\\ \hline b & a\end{array}\right),\ \text{if } m_1=n_1=m_2=n_2;$$

$$\Pi\tau:\quad \left(\begin{array}{c|c} a & b\\ \hline c & d\end{array}\right)\to\left(\begin{array}{c|c} d^t & -b^t\\ \hline c^t & a^t\end{array}\right),\ \text{if } m_1=n_1=m_2=n_2.$$

Let δ_i be the map satisfying $(\delta_i)_{|M_{\bar 0}}=\mathrm{Id}_{M_{\bar 0}}$ and $(\delta_i)_{|M_{\bar 1}}=i\mathrm{Id}_{M_{\bar 1}}$, with i the imaginary unit. We denote by (M,ϕ) the space M with triple product defined by

$$(x,y,z)=x\phi(y)z+(-1)^{\alpha(x,y,z)}z\phi(y)x \tag{3.9}$$

where ϕ is either one of the maps just introduced or the identity map. We list which one of them is an $\bar{0}$-sJTS and which is an $\bar{1}$-sJTS:

$$\begin{aligned} \epsilon = \bar{0}: \quad & (M_{(m|n)}(\mathbb{C}), \mathrm{Id}),\ (M_{(m_1|m_2),(2n_1|2n_2)}(\mathbb{C}), os\tau), \\ & (M_{(m|m)}(\mathbb{C}), \Pi),\ (M_{(0|n)(n|0)}(\mathbb{C}), \mathrm{Id}),\ (M_{(m|m)}(\mathbb{C}), \Pi\tau)\,. \\ \epsilon = \bar{1}: \quad & (M_{(m|n)}(\mathbb{C}), \iota),\ (M_{(m_1|m_2),(n_1|n_2)}(\mathbb{C}), \tau), \\ & (M_{(2m_1|2m_2),(2n_1|2n_2)}(\mathbb{C}), s\tau),\ (M_{(m|m)}(\mathbb{C}), \Pi\tau \circ \delta_i)\,. \end{aligned} \tag{3.10}$$

We shall see in the following sections that these are all K-simple non-isomorphic ϵ-sJTS and that they are all related to the Lie superalgebra $\mathfrak{psl}$.

Example 3.11. Let $V_{m|2n} := M_{(m|2n)(1|0)}(\mathbb{C})$ and ϕ be one of the maps from $M_{(m|2n)(1|0)}(\mathbb{C})$ to $M_{(1|0)(m|2n)}(\mathbb{C})$

$$S\hat{I}_{p,m,q,n}: \begin{pmatrix} a \\ \hline b \end{pmatrix} \to \begin{pmatrix} a \\ \hline b \end{pmatrix}^{os\tau} \left(\begin{array}{c|c} S_{p,m} & 0 \\ \hline 0 & \hat{I}_{q,2n-2q} \end{array}\right);$$

$$SJ_{p,m,n}: \begin{pmatrix} a \\ \hline b \end{pmatrix} \to \begin{pmatrix} a \\ \hline b \end{pmatrix}^{os\tau} \left(\begin{array}{c|c} S_{p,m} & 0 \\ \hline 0 & J_{2n} \end{array}\right);$$

with $0 < p < [\frac{m}{2}], 0 < q < [\frac{n}{2}]$. Then $V_{m,2n}$ with triple product

$$(x, y, z) = x\phi(y)z + (-1)^{\alpha(x,y,z)} z\phi(y)x - (x\phi(y))^{os\tau} z\,. \tag{3.11}$$

is an ϵ-sJTS , denoted $(V_{m,2n}, \phi)$. The super-principal identity and super-commutativity can be checked directly using the properties of $\phi, os\tau$ and noting that the first two summands in the product satisfy already the axioms of ϵ-sJTS since they give the product of Example 3.10. We instead will derive this fact from Remark 7.6 and Chapter 8. We get the following $\bar{0}$ and $\bar{1}$-sJTS's

$$\begin{aligned} \epsilon = \bar{0}: \quad & (V_{m,2n}, S\hat{I}_{p,m,q,n}),\ 0 < p < [\tfrac{m}{2}], 0 < q < [\tfrac{n}{2}]; \\ \epsilon = \bar{1}: \quad & (V_{m,2n}, SJ_{p,m,n}),\ 0 < p < [\tfrac{m}{2}]. \end{aligned} \tag{3.12}$$

Different choices of p, q yield K-simple non-isomorphic ϵ-sJTS related to a particular 3-grading of the Lie superalgebra $\mathfrak{osp}(m, 2n)$.

Example 3.12. Let $M = M_{(m|n)}(\mathbb{C})$ and consider the following subspaces of M:

$$N_{m,n} = \{x \in M | x = \left(\begin{array}{c|c} a & b \\ \hline -b^R & d \end{array}\right), a = -a^R, d = d^R\},$$

$$N'_{m,n} = \{x \in M | x = \left(\begin{array}{c|c} a & b \\ \hline b^R & d \end{array}\right), a = a^R, d = d^R\}.$$

Notice that these are the subspaces of matrices in M fixed by the maps $f : x \to Ax^{\tau}B$ and $f' : x \to Bx^{\tau}A$ respectively, with $A = \mathrm{Diag}(S_m, -S_n)$, $B = \mathrm{Diag}(S_m, S_n)$. Suppose that $\phi : M \to M$ is an automorphism which exchanges $N_{m,n}$ and $N'_{m,n}$ and satisfying condition (3.6) or (3.5) and let $(\ ,\ ,\ )$ be the corresponding 3-product of (M, ϕ), cf. Example 3.10. Using the fact that τ is a superantinvolution and that $A^2 = B^2 = \mathrm{Id}$ one gets

$$\begin{aligned} f(x\phi(y)z) &= (-1)^{\alpha(x,y,z)}(Az^{\tau}\phi(y)^{\tau}x^{\tau}B) = (-1)^{\alpha(x,y,z)}(Az^{\tau}BB\phi(y)^{\tau}AAx^{\tau}B) = \\ &= (-1)^{\alpha(x,y,z)}(f(z)f'(\phi(y))f(x))\,. \end{aligned}$$

It follows that $f(x,y,z) = (x,y,z)$, hence $N_{m,n}$ is a sub ϵ-sJTS of M with product $(\ ,\ ,\ )$. We shall denote this ϵ-sJTS by $(N_{m,n}, \phi)$.

Notice that each of the following maps exchanges $N_{m,n}$ and $N'_{m,n}$:

$$\begin{aligned}
SI &: \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \to \mathrm{AdDiag}(S_m, I_{q,q})\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \text{ if } n = 2q; \\
IJ &: \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \to \mathrm{AdDiag}(I_{l,l}, S)\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \text{ if } m = 2l; \\
SJ &: \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \to \mathrm{AdDiag}(S_m, S_n)\left(\begin{array}{c|c} a & -b \\ \hline c & -d \end{array}\right); \\
II &: \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \to \mathrm{AdDiag}(I_{l,l}, I_{q,q})\left(\begin{array}{c|c} -a & b \\ \hline -c & d \end{array}\right), \text{ if } m = 2l, n = 2q.
\end{aligned}$$

Hence $(N_{m,n}, \phi)$ is an ϵ-sJTS . In particular, we get the following $\bar{0}$ and $\bar{1}$-sJTS's:

$$\begin{aligned} \epsilon = \bar{0} &: \quad (N_{m,2n}, SI),\ (N_{2m,n}, IJ); \\ \epsilon = \bar{1} &: \quad (N_{m,n}, SJ),\ (N_{2m,2n}, II). \end{aligned} \tag{3.13}$$

We will see that this are all K-simple non-isomorphic ϵ-sJTS related to a particular 3-grading of the Lie superalgebra $\mathfrak{osp}(2m, 2n)$.

Example 3.13. Let $M = M_{(m|n)}(\mathbb{C})$, $Q_{m,n} = \{x \in M | x = \left(\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right)\}$ and, if $m = n$, $P_n = \{x \in M | x = \left(\begin{array}{c|c} a & b \\ \hline c & -a^R \end{array}\right), b = b^R, c = -c^R\}$. We have that $Q_{m,n}$ (resp. P_n) are the spaces of matrices which are fixed by Π (resp. $\mathrm{AdDiag}(S_n, S_n) \circ \Pi\tau$). With the same arguments as in the previous example, it can be shown that P_n and $Q_{m,n}$ are sub ϵ-sJTS of (M, ϕ) if ϕ is one of the following maps:

$$\begin{aligned} P_n &: \quad \mathrm{Id}\,;\ \delta_i\,. \\ Q_{m,n} &: \quad \mathrm{Id}\,,\ m = n;\ \tau \circ \delta_i\,;\ s\tau \circ \delta_i\,, m, n \in 2\mathbb{N}. \end{aligned}$$

We denote such ϵ-sJTS by (P_n, ϕ) and $(Q_{m,n}, \phi)$ respectively. It follows that

$$\begin{aligned} \epsilon = \bar{0}: &\quad (P_n, \mathrm{Id}),\ (Q_{n,n}, \mathrm{Id}); \\ \epsilon = \bar{1}: &\quad (P_n, \delta_i),\ (Q_{m,n}, \tau \circ \delta_i),\ (Q_{2m,2n}, s\tau \circ \delta_i). \end{aligned} \tag{3.14}$$

These are all K-simple non-isomorphic ϵ-sJTS related to the Lie superalgebras $p(n)$ and $q(n)$ respectively.

Chapter 4

TKK construction

This chapter is devoted to the construction which relates ϵ-sJTS to Lie superalgebras and KTS to Lie algebras.

Definition 4.1. A Lie superalgebra $\mathfrak{g}$ is a super vector space with a bilinear product, $[,]$, compatible with the $\mathbb{Z}_2$-grading which satisfies the following identities

$$\begin{aligned} &[x, y] = (-1)^{|x||y|}[y, x] && \text{super anti-commutative} \\ &[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] && \text{super Jacobi identity} \end{aligned} \tag{4.1}$$

A Lie superalgebra is said $\mathbb{Z}$-graded if $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ and the bracket is compatible with the $\mathbb{Z}$-grading. Moreover, a $\mathbb{Z}$-graded Lie superalgebra is said

- *transitive* if for $a \in \mathfrak{g}_i$, $i \geqslant 0$, $[a, \mathfrak{g}_{-1}] = 0$ implies $a = 0$;
- *fundamental* if $\mathfrak{g}_{-k} = \mathfrak{g}_{-1}^k$;
- $(2k + 1)$-*graded* if $\mathfrak{g}_i = 0$ for $|i| > k$;
- *consistent* if $\mathfrak{g}_{\bar{0}} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_{2i}$ and, consequently, $\mathfrak{g}_{\bar{1}} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_{2i+1}$;
- *admissible* if $\dim(\mathfrak{g}_i) = \dim(\mathfrak{g}_{-i})$ when the $\mathfrak{g}_i$'s are finite-dimensional or if $\mathfrak{g}_i$ has same growth and size of $\mathfrak{g}_{-i}$ whenever they are infinite-dimensional;
- of *finite depth* d if $\mathfrak{g}_p = 0$ for all $p < -d$ and $\mathfrak{g}_{-d} \neq 0$, $d \in \mathbb{N}$.

The negatively graded part is $\mathfrak{g}_- = \bigoplus_{i<0} \mathfrak{g}_i$.

We refer to [10] for the definitions of growth and size.

Definition 4.2. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra and $E \in \mathfrak{g}$. If

$$[E, x] = ix, \forall x \in \mathfrak{g}_i, \forall i \in \mathbb{Z} \tag{4.2}$$

we call E the *grading element* of $\mathfrak{g}$.

Definition 4.3. Let $\epsilon \in \mathbb{Z}_2$ and let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra. An automorphism ϕ of $\mathfrak{g}$ is said:

- *grade-reversing* if $\phi(\mathfrak{g}_i) \subseteq \mathfrak{g}_{-i}$;
- *grade-preserving* if $\phi(\mathfrak{g}_i) \subseteq \mathfrak{g}_i$;
- an ϵ-*involution* if $\phi^2(x) = (-1)^{\epsilon|x|}x$.

Remark 4.4. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra. A necessary condition for the existence of a grade-reversing ϵ-involution is the grading being admissible.

Proposition 4.5. *Let $\mathfrak{g}$ be a 3-graded Lie superalgebra. $\mathfrak{g}$ is simple if and only if the following conditions are satisfied:*

- *$\mathfrak{g}_0$ acts irreducibly on $\mathfrak{g}_{-1}$;*
- *$\mathfrak{g}$ is transitive;*
- *$[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$ and $[a, \mathfrak{g}_1] = 0$ with $a \in \mathfrak{g}_0$ implies $a = 0$.*

Proof. For a proof of a more general version see [27], Proposition 1.2.8. □

We recall from [13] the definition of universal Lie superalgebra associated to a superspace:

Definition 4.6. Let V be a superspace. We define $W_{-1}(V) = V$ and for $k \geqslant 0$ we let $W_k(V)$ be the space of $(k+1)$-linear supersymmetric functions on V, i.e. $f \in W_k(V)$ if $f : V^{\otimes(k+1)} \to V$ satisfies the following property

$$f(\cdots, x, y, \cdots) = (-1)^{|x||y|} f(\cdots, y, x, \cdots).$$

Let $W(V) = \bigoplus_{k\geqslant -1} W_k(V)$ and define on $W(V)$ the following Lie bracket: for $f \in W_p(V), g \in W_q(V)$,

$$[f, g] = f\Box g - (-1)^{|f||g|} g\Box f \in W_{p+q}(V)\,,$$

where the square product is defined by:

$$f\Box g(x_0, \dots, x_{p+q}) = \sum_{\substack{i_0<\cdots<i_q \\ i_{q+1}<\cdots<i_{p+q}}} s(\tau) f(g(x_{i_0}, \dots, x_{i_q}), x_{i_{q+1}}, \dots, x_{i_{q+p}}),$$

with τ the permutation $\tau(j) = i_j$ and $s(\tau) = (-1)^{N(\tau)}$, N being the number of interchanges of indices of odd x_i's in the permutation τ.

Note that for linear functions the box product is the usual composition of functions and for any $v \in V$ it is $v\Box f = 0$ for all $f \in W(V)$. Furthermore $W(V)$ is transitive, indeed if $f \in W_p(V), p \geqslant 0,\ [f, v] = 0,\ \forall v \in W_{-1}(V)$ then $f(v, w_1, \dots, w_p) = 0$ for all $w_i \in V$ which implies $f = 0$.

4.1 TKK construction for ϵ-sJTS

Proposition 4.7. *Let* $(\mathfrak{g}, \sigma)$ *be a pair consisting of a 3-graded Lie superalgebra* $\mathfrak{g}$ *with a grade-reversing* ϵ*-involution* σ*. Then* $J(\mathfrak{g}, \sigma) := \mathfrak{g}_{-1}$ *with triple product*

$$(x, y, z) = [\,[x, \sigma(y)], z\,]$$

is an ϵ*-sJTS.*

Proof. Due to the super Jacobi identity, the super anticommutativity and the the fact that $\mathfrak{g}$ is 3-graded we have, for $u, v, x, y, z \in \mathfrak{g}_{-1}$:

$$\begin{aligned}(x, y, z) = &\ [[x, \sigma(y)], z] = [x, [\sigma(y), z]] - (-1)^{|x||y|}[\sigma(y), [x, z]] = \\ = &\ -(-1)^{|x|(|y|+|z|)}[[\sigma(y), z], x] = (-1)^{\alpha(x,y,z)}(z, y, x),\end{aligned}$$

and

$$\begin{aligned}(u, v, (x, y, z)) = &\ [[u, \sigma(v)], [[x, \sigma(y)], z]] = [[[u, \sigma(v)], [x, \sigma(y)]], z]+ \\ &+(-1)^{\beta(u,v,x,y)}[[[x, \sigma(y)], [[u, \sigma(v)], z]] = \\ = &\ [[[\,u, \sigma(v)\,],\ x\,], \sigma(y)\,], z] + (-1)^{|x|(|u|+|v|)}[[x, [[u, \sigma(v)], \sigma(y)]], z] \\ &+(-1)^{\beta(u,v,x,y)}(x, y, (u, v, z)) = \\ = &\ ((u, v, x), y, z) - (-1)^{\alpha(u,v,x)}[[x, [[\sigma(v), u], \sigma(y)]], z] \\ &+(-1)^{\beta(u,v,x,y)}(x, y, (u, v, z)) = \\ = &\ ((u, v, x), y, z) - (-1)^{\alpha(u,v,x)+\epsilon|u|}(x, (v, u, y), z) \\ &+(-1)^{\beta(u,v,x,y)}(x, y, (u, v, z))\end{aligned}$$

□

Definition 4.8. Let $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}', \sigma')$ be pairs consisting of a 3-graded Lie superalgebra and a grade-reversing ϵ-involution. If there exists a grade-preserving isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi \circ \sigma = \sigma' \circ \phi$ we say that $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}', \sigma')$ are *equivalent* pairs and if, moreover, $\mathfrak{g} = \mathfrak{g}'$ we say that σ and σ' are equivalent involutions.

Remark 4.9. It is straightforward to see that $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}', \sigma')$ are *equivalent* pairs if and only if $J(\mathfrak{g}, \sigma) \cong J(\mathfrak{g}', \sigma')$.

Proposition 4.10. *Let* J *be a centerless* ϵ*-sJTS. We define the following operators:*

$$L_{x,y}(z) = (x, y, z), \quad \varphi_x(y, z) = -(-1)^{|x||y|}(y, x, z) \tag{4.3}$$

and set $|L_{x,y}| = |x| + |y|$ *and* $|\varphi_x| = |x|$.

Then the 3-graded superspace

$$\begin{array}{ccccccc} & \mathrm{Lie}(J)_{-1} & & \mathrm{Lie}(J)_0 & & \mathrm{Lie}(J)_1 \\ \mathrm{Lie}(J) = & J & \oplus & \langle L_{x,y} | x, y \in J \rangle & \oplus & \langle \varphi_x | x \in J \rangle \end{array} \tag{4.4}$$

with product

$$\begin{aligned} &[x, y] = 0\,, \quad [L_{x,y}, z] = (x, y, z)\,, \quad [\varphi_x, y] = -(-1)^{|x||y|} L_{y,x} \\ &[L_{u,v}, L_{x,y}] = L_{(u,v,x),y} - (-1)^{\beta(u,v,x,y)+\epsilon|u|} L_{x,(y,u,v)}\,, \\ &[L_{x,y}, \varphi_z] = -(-1)^{|x||y|+\epsilon|x|} \varphi_{(y,x,z)}\,, \quad [\varphi_x, \varphi_y] = 0\,, \end{aligned} \tag{4.5}$$

is a subsuperalgebra of $W(J)$. Moreover, the following map is a grade-reversing ϵ-involution of $\mathrm{Lie}(J)$*:*

$$\sigma(x) = -\varphi_x\,, \ \sigma(L_{x,y}) = -(-1)^{|x||y|+\epsilon|y|} L_{y,x}\,, \ \sigma(\varphi_x) = -(-1)^{\epsilon|x|} x$$

Proof. The map $\varphi_x \in W_1(J)$: of course by supercommutativity of the triple product we have $\varphi_x(y, z) = -(-1)^{|x||y|}(y, x, z) = -(-1)^{|x||z|+|z||y|}(z, x, y) = (-1)^{|y||z|}\varphi_x(z, y)$. The fact that the product (4.5) is the one induced by $W(J)$ is a direct consequence of the defining identities of ϵ-sJTS, for example

$$\begin{aligned} [L_{u,v}, \varphi_y](x, z) = &\; L_{u,v} \square \varphi_y(x, z) - (-1)^{|y|(|u|+|v|)} \varphi_y \square L_{u,v}(x, z) = \\ = &\; L_{u,v}(\varphi_y(x.z)) - (-1)^{|y|(|u|+|v|)} \varphi_y(L_{u,v}(x), z) - \\ &\; -(-1)^{|y|(|u|+|v|)+|x||z|} \varphi_y(L_{u,v}(z), x) = \\ = &\; -(-1)^{|x||y|}(\,(u, v, (x, y, z)) - ((u, v, x), y, z) \\ &\; -(-1)^{\beta(u,v,x,y)}(x, y, (u, v, z))\,) = \\ = &\; -(-1)^{|x||y|+\alpha(u,v,x)+\epsilon|u|}(x, (v, u, y), z) = \\ = &\; -(-1)^{|u||v|+\epsilon|u|} \varphi_{(v,u,y)}(x, z) \end{aligned}$$

$$\begin{aligned} [L_{u,v}, L_{x,y}](z) = &\; L_{u,v}(L_{x,y}(z)) - (-1)^{\beta(u,v,x,y)} L_{x,y}(L_{u,v}(z)) = \\ = &\; (u, v, (x, y, z)) - (-1)^{\beta(u,v,x,y)}(x, y, (u, v, z)) = \\ = &\; ((u, v, x), y, z) - (-1)^{\alpha(u,v,x)+\epsilon|u|}(x, (v, u, y), z) = \\ = &\; (L_{(u,v,x),y} - (-1)^{\beta(u,v,x,y)+\epsilon|u|} L_{x,(y,u,v)})(z) \end{aligned}$$

$$\begin{aligned} [[\varphi_x, \varphi_y], z] = &\; [\varphi_x, [\varphi_y, z]] + (-1)^{|z||y|}[[\varphi_x, z], \varphi_y] = -(-1)^{|z||y|}[\varphi_x, L_{z,y}] - \\ &\; -(-1)^{|z||y|+|x||z|}[L_{z,x}, \varphi_y] = \\ = &\; -(-1)^{|x||z|+|x||y|+\epsilon|z|} \varphi_{(y,z,x)} + (-1)^{|z||y|+\epsilon|z|} \varphi_{(x,z,y)} = 0 \end{aligned}$$

The fact that σ is a morphism is a straightforward check. Besides, σ is injective: indeed if $\sigma(x) = \varphi_x = 0$ then $[[y, \varphi_x], z] = (y, x, z) = 0$ for any $y, z \in J$ and this means x is in the center of J, hence it must be 0, and similarly if $\sigma(L_{x,y}) = \pm L_{y,x} = 0$ implies $L_{y,x}(u, v, x) = (u, v, (L_{y,x}(z))) = (L_{y,x}(u), v, z) = 0$ for all $u, v, z \in J$, hence $(u, L_{x,y}(v), z) = 0$ implies $L_{x,y} = 0$. Surjectivity of σ follows from the definition of $\mathrm{Lie}(J)$ at once. □

Theorem 4.11. *Let* J *be a centerless linearly-compact* ϵ*-sJTS and* $\mathrm{Lie}(J)$ *its associated Lie superalgebra.*

(a) $\mathrm{Lie}(J)$ *is a 3-graded transitive Lie superalgebra and the product on* J *is given by* $(x, y, z) = [[x, \sigma(y)], z]$*;*

(b) $[\mathrm{Lie}(J)_{-1}, \mathrm{Lie}(J)_1] = \mathrm{Lie}(J)_0$ *and* $a \in \mathrm{Lie}(J)_0, [a, \mathrm{Lie}(J)_1] = 0$ *implies* $a = 0$*;*

(c) J *is K-simple (resp. finite-dimensional or linearly compact) if and only if* $\mathrm{Lie}(J)$ *is simple (resp. finite-dimensional or linearly compact);*

(d) two centerless ϵ*-sJTS* J, J' *are isomorphic if and only if* $\mathrm{Lie}(J)$ *and* $\mathrm{Lie}(J')$*, with their relative grade-reversing involutions, are isomorphic pairs.*

Proof. (a) $\mathrm{Lie}(J)$ is transitive since is a subalgebra of $W(J)$ and $\mathrm{Lie}(J)_{-1} = W(J)_{-1} = J$. The second part is clear.

(b) The identity $[\mathrm{Lie}(J)_{-1}, \mathrm{Lie}(J)_1] = \mathrm{Lie}(J)_0$ is immediate. Suppose $a \in \mathrm{Lie}(J)_0$ and $[a, \mathrm{Lie}(J)_1] = 0$. If we apply the grade reversing involution σ we get $[\sigma(a), \mathrm{Lie}(J)_{-1}] = 0$ and by transitivity $\sigma(a) = 0$ hence $a = 0$.

(c) Let $\mathrm{Lie}(J)$ be simple and $I \neq 0$ a K-ideal of J. We have $(J, J, I) \subseteq I$ which in terms of the Lie product is $[[\mathrm{Lie}(J)_{-1}, \mathrm{Lie}(J)_1], I] = [\mathrm{Lie}(J)_0, I] \subseteq I$. Since $\mathrm{Lie}(J)_0$ acts irreducibly on $\mathrm{Lie}(J)_{-1}$ and $I \neq 0$, by Proposition 4.5, it necessarily is $I = J$.
Conversely, suppose J is K-simple and $I \neq 0$ is a reducible $\mathrm{Lie}(J)_0$-submodule of $\mathrm{Lie}(J)_{-1} = J$. Then $(J, J, I) = [[\mathrm{Lie}(J)_{-1}, \sigma(\mathrm{Lie}(J)_{-1})], I] = [\mathrm{Lie}(J)_0, I] \subseteq I$ and by commutativity $(I, J, J) \subseteq I$. This shows that I is a non-zero K-ideal of J, hence $I = J$. The fact that finite-dimensionality and linearly-compactness are preserved follows from the fact that $\mathrm{Lie}(J)$ is a subalgebra of $W(J)$, cf. [11].

(d) If J, J' are isomorphic ϵ-sJTS and $\phi : J \to J'$ is an isomorphism then $\Phi : \mathrm{Lie}(J) \to \mathrm{Lie}(J')$, with $\Phi(x) = \phi(x)$, $\Phi(L_{x,y}) = L_{\phi(x),\phi(y)}$, $\Phi(\varphi_x) = \varphi_{\phi(x)}$, is an isomorphism of pairs.
On the other hand, if Φ is an isomorphism of pairs $\Phi : \mathrm{Lie}(J) \to \mathrm{Lie}(J')$, then

the restriction of $\Phi_{|J} := \phi : J \to J'$ (Φ is grade preserving and even) is an isomorphism of ϵ-sJTS. In fact $\phi((x,y,z)_J) = \phi([[x,\sigma(y)],z]) = [[\phi(x),\phi(\sigma(y))],\phi(z)] = [[\phi(x),\sigma'(\phi(y))],\phi(z)] = (\phi(x),\phi(y),\phi(z))_{J'}$.

□

4.2 TKK construction for KTS

Proposition 4.12. *Let* $(\mathfrak{g},\sigma)$ *be a pair consisting of a 5-graded Lie algebra* $\mathfrak{g}$ *with grade-reversing involution* σ*. Then* $K(\mathfrak{g},\sigma) := \mathfrak{g}_{-1}$ *with triple product*

$$(x,y,z) = [\,[x,\sigma(y)],z\,]$$

is a KTS.

Proof. Let $K = K(\mathfrak{g},\sigma)$. The proof that K satisfies the principal identity (2.1) is the same as the one for Proposition 4.7 with elements only in $\mathfrak{g}_{\bar{0}}$.

Notice that in K we have the following relation

$$\begin{aligned} K_{x,y}(z) &= (x,z,y) - (y,z,x) = [[x,\sigma(z)],y] - [[y,\sigma(z)],x] \\ &= [[x,\sigma(z)],y] + [x,[y,\sigma(z)]] \\ &= [[x,y],\sigma(z)]\ . \end{aligned}$$

As a consequence we get

$$\begin{aligned} K_{K_{u,v}(x),y}(z) &= [[[[u,v],\sigma(x)],y],\sigma(z)] = [[[u,v],[\sigma(x),y]],\sigma(z)] + [[[[u,v],y],\sigma(x)],\sigma(z)] \\ &= [[[u,v],[\sigma(x),y]],\sigma(z)] = [[u,[v,[\sigma(x),y]]],\sigma(z)] + [[[u,[\sigma(x),y]],v],\sigma(z)] \\ &= -[[[[y,\sigma(x)],v],u],\sigma(z)] + [[[[y,\sigma(x)],u],v],\sigma(z)] \\ &= K_{(y,x,u),v}(z) - K_{(y,x,v),u}(z)\ . \end{aligned}$$

□

Remark 4.13. As for ϵ-sJTS, if $(\mathfrak{g},\sigma)$ and $(\mathfrak{g}',\sigma')$ are pairs consisting of 5-graded Lie algebras with grade-reversing involution $K(\mathfrak{g},\sigma) \cong K(\mathfrak{g}',\sigma')$ if and only if there exists a grade-preserving isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\phi \circ \mathfrak{g} = \mathfrak{g}' \circ \phi$.

We recall that the *maximal transitive prolongation* (in the sense of N. Tanaka) of a negatively graded fundamental Lie algebra $\mathfrak{m} = \bigoplus_{i<0} \mathfrak{m}_i$ of finite depth is a $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{g}^\infty = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i^\infty \tag{4.6}$$

such that:

(i) $\mathfrak{g}_{-}^{\infty} = \mathfrak{m}$ as $\mathbb{Z}$-graded Lie algebras;

(ii) $\mathfrak{g}^{\infty}$ is transitive;

(iii) $\mathfrak{g}^{\infty}$ is maximal with these properties, i.e., if $\mathfrak{g}$ is another $\mathbb{Z}$-graded Lie algebra which satisfies (i) and (ii), then $\mathfrak{g} \subset \mathfrak{g}^{\infty}$ as a $\mathbb{Z}$-graded subalgebra.

The existence and uniqueness of $\mathfrak{g}^{\infty}$ is proved in [54] (the proof is for finite-dimensional Lie algebras but it extends verbatim to the infinite-dimensional case).

The maximal transitive prolongation (4.6) can be described as follows. First $\mathfrak{g}_0^{\infty} = \mathfrak{der}_0(\mathfrak{m})$ is the Lie algebra of all 0-degree derivations of $\mathfrak{m}$ and $[D, x] := Dx$ for all $D \in \mathfrak{g}_0^{\infty}$ and $x \in \mathfrak{m}$. The spaces $\mathfrak{g}_i^{\infty}$ for all $i > 0$ are defined inductively: the component

$$\mathfrak{g}_i^{\infty} = \big\{ D : \mathfrak{m} \to (\mathfrak{m} \oplus \mathfrak{g}_0^{\infty} \oplus \cdots \oplus \mathfrak{g}_{i-1}^{\infty}) \text{ s.t. } (i)\ D[x, y] = [Dx, y] + [y, Dx]\ , \forall\, x, y \in \mathfrak{m} \\ (ii)\ D(\mathfrak{m}_q) \subset \mathfrak{g}_{j+i}^{\infty} \text{ for all } j < 0 \big\} \tag{4.7}$$

is the space of i-degree derivations of $\mathfrak{m}$ with values in the $\mathfrak{m}$-module $\mathfrak{m} \oplus \mathfrak{g}_0^{\infty} \oplus \cdots \oplus \mathfrak{g}_{i-1}^{\infty}$. Again $[D, x] := Dx$, for all $D \in \mathfrak{g}_i^{\infty}$, $i > 0$, and $x \in \mathfrak{m}$. The brackets between non-negative elements of (4.6) are determined uniquely by transitivity; for more details and their explicit expression, we refer to the original source [54, §5].

In the infinite-dimensional linearly compact case the prolongation can be constructed in complete analogy, provided we take continuous derivations.

We shall now associate to a centerless Kantor triple system V, a 5-graded Lie algebra

$$\mathfrak{g} = \mathrm{Lie}(V) = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$$

with an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathcal{K}(\mathfrak{g}, \sigma) = V$.

We start with the negatively graded $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ defined by

$$\mathfrak{m}_{-1} = V\ , \qquad \mathfrak{m}_{-2} = \langle K_{x,y} \mid x, y \in V \rangle\ , \tag{4.8}$$

where the only non-trivial bracket is $[x, y] = K_{x,y}$, for all $x, y \in \mathfrak{m}_{-1}$ (if V is a Jordan triple system, then $\mathfrak{m}_{-2} = 0$ and $\mathfrak{m} = \mathfrak{m}_{-1}$ is trivially fundamental). The Lie algebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ we are interested is a subalgebra of the maximal prolongation $\mathfrak{g}^{\infty}$ of $\mathfrak{m}$.

We set $\mathfrak{g}_i = 0$ for all $|i| > 2$, $\mathfrak{g}_i = \mathfrak{m}_i$ for $i = -1, -2$, introduce linear maps $L_{x,y} : \mathfrak{m} \to \mathfrak{m}$, $\varphi_x : \mathfrak{m} \to \mathfrak{m}_{-1} \oplus \mathfrak{g}_0^{\infty}$ and $D_{x,y} : \mathfrak{m} \to \mathfrak{g}_0^{\infty} \oplus \mathfrak{g}_1^{\infty}$ given by

$$\begin{aligned} &[L_{x,y}, z] = (x, y, z)\ , && [\varphi_x, z] = L_{z,x}\ , && [D_{x,y}, z] = -\varphi_{K_{x,y}(z)}\ , \\ &[L_{x,y}, K_{u,v}] = K_{K_{u,v}(y),x}\ , && [\varphi_x, K_{u,v}] = K_{u,v}(x)\ , && [D_{x,y}, K_{u,v}] = L_{K_{u,v}(y)x} - L_{K_{u,v}(x)y}\ , \end{aligned} \tag{4.9}$$

where $u, v, x, y, z \in V$. Here $\mathfrak{g}_0 = \langle L_{x,y} \,|\, x, y \in V\rangle$, $\mathfrak{g}_1 = \langle \varphi_x \,|\, x \in V\rangle$ and $\mathfrak{g}_2 = \langle D_{x,y} \,|\, x, y \in V\rangle$.

Proposition 4.14. *The vector space $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ defined above is a fundamental transitive subalgebra of $\mathfrak{g}^\infty$. It is finite-dimensional (resp. linearly compact) if and only if V is finite-dimensional (resp. linearly compact).*

Proof. By axiom (ii) of Definition 1.2, we have

$$\begin{aligned}[L_{x,y}, [u, v]] &= [L_{x,y}, K_{u,v}] = K_{K_{u,v}(y)x} = K_{(x,y,u),v} - K_{(x,y,v),u} \\ &= [[L_{x,y}, u], v] + [u, [L_{x,y}, v]] \ ,\end{aligned}$$

hence $L_{x,y} \in \mathfrak{g}_0^\infty$. We also note that, by axiom (i) of Definition 1.2,

$$\begin{aligned}[[L_{u,v}, L_{x,y}], z] &= [L_{u,v}, (x, y, z)] - [L_{x,y}, (u, v, z)] = (u, v, (x, y, z)) - (x, y, (u, v, z)) \\ &= ((u, v, x), y, z) - (x, (v, u, y), z) = [L_{(u,v,x),y} - L_{x,(v,u,y)}, z]\end{aligned}$$

for all $z \in V$, hence, by transitivity,

$$[L_{u,v}, L_{x,y}] = L_{(u,v,x),y} - L_{x,(v,u,y)} \ , \tag{4.10}$$

for all $u, v, x, y \in V$ and $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}_0^\infty$. In a similar way

$$\begin{aligned}[\varphi_x, [u, v]] &= [\varphi_x, K_{u,v}] = K_{u,v}(x) = (u, x, v) - (v, x, u) \\ &= [[\varphi_x, u], v] + [u, [\varphi_x, v]]\end{aligned}$$

and $\varphi_x \in \mathfrak{g}_1^\infty$; we note that the inclusion $[\mathfrak{g}_1, \mathfrak{g}_{-2}] \subset \mathfrak{g}_{-1}$ and the equality $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$ hold by construction. Furthermore

$$\begin{aligned}[[L_{u,v}, \varphi_x], z] &= [L_{u,v}, L_{z,x}] - [\varphi_x, (u, v, z)] = L_{(u,v,z),x} - L_{z,(v,u,x)} - L_{(u,v,z),x} \\ &= -L_{z,(v,u,x)} = -[\varphi_{(v,u,x)}, z]\end{aligned}$$

for all $z \in V$, hence

$$[L_{u,v}, \varphi_x] = -\varphi_{(v,u,x)} \ , \tag{4.11}$$

for all $u, v, x \in V$ and $[\mathfrak{g}_1, \mathfrak{g}_0] \subset \mathfrak{g}_1$ as well.

To prove $D_{x,y} \in \mathfrak{g}_2^\infty$, it is first convenient to observe that $[\varphi_x, \varphi_y] \in \mathfrak{g}_2^\infty$ and then compute

$$\begin{aligned}[[\varphi_x, \varphi_y], z] &= [\varphi_x, [\varphi_y, z]] + [[\varphi_x, z], \varphi_y] = [\varphi_x, L_{z,y}] + [L_{z,x}, \varphi_y] \\ &= \varphi_{y,z,x} - \varphi_{x,z,y} = -\varphi_{K_{x,y}(z)}\end{aligned}$$

and

$$[[\varphi_x, \varphi_y], K_{u,v}] = [\varphi_x, [\varphi_y, K_{u,v}]] + [[\varphi_x, K_{u,v}], \varphi_y] = [\varphi_x, K_{u,v}(y)] + [K_{u,v}(x), \varphi_y]$$
$$= L_{K_{u,v}(y),x} - L_{K_{u,v}(x),y} \ ,$$

where $u, v, x, y, z \in V$. In other words

$$[\varphi_x, \varphi_y] = D_{x,y} \ , \tag{4.12}$$

hence $D_{x,y} \in \mathfrak{g}_2^\infty$ and $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$. We also note that

$$\begin{aligned} [L_{u,v}, D_{x,y}] &= [L_{u,v}, [\varphi_x, \varphi_y]] = [[L_{u,v}, \varphi_x], \varphi_y] + [\varphi_x, [L_{u,v}, \varphi_y]] \\ &= -[\varphi_{(v,u,x)}, \varphi_y] - [\varphi_x, \varphi_{(v,u,y)}] = -D_{(v,u,x),y} - D_{x,(v,u,y)} \end{aligned} \tag{4.13}$$

for all $u, v, x, y \in V$; in particular $[\mathfrak{g}_0, \mathfrak{g}_2] \subset \mathfrak{g}_2$.

Finally, we consider $[\varphi_u, D_{x,y}] \in \mathfrak{g}_3^\infty$ and compute

$$[[\varphi_u, D_{x,y}], z] = [\varphi_u, [D_{x,y}, z]] + [[\varphi_u, z], D_{x,y}] = [\varphi_{K_{x,y}(z)}, \varphi_u] + [L_{z,u}, D_{x,y}]$$
$$= D_{K_{x,y}(z),u} - D_{(u,z,x),y} - D_{x,(u,z,y)} \ ,$$

for all $u, x, y, z \in V$. Applying both sides to $v \in V$ immediately yields $[[[\varphi_u, D_{x,y}], z], v] = \varphi_w$, where

$$\begin{aligned} w &= -K_{K_{x,y}(z),u}(v) + K_{(u,z,x),y}(v) - K_{(u,z,y),x}(v) \\ &= 0 \,, \end{aligned}$$

by axiom (ii) of Definition 1.2. In summary $[\varphi_u, D_{x,y}] = 0$ by a repeated application of transitivity, $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$ and $[\mathfrak{g}_2, \mathfrak{g}_2] = [\mathfrak{g}_2, [\mathfrak{g}_1, \mathfrak{g}_1]] = 0$. The first claim of the proposition is proved.

The second claim is straightforward. We only note here that if V is linearly compact then $\mathfrak{g}$ is linearly compact too. This can be shown by the same arguments as [12, Lemma 2.1]. □

Now we define $\sigma : \mathfrak{g} \to \mathfrak{g}$ by

$$\begin{aligned} &\sigma(K_{x,y}) = D_{x,y} \ , \qquad \sigma(x) = -\varphi_x \ , \\ &\sigma(L_{x,y}) = -L_{y,x} \qquad \text{and} \\ &\sigma(\varphi_x) = -x \ , \qquad \sigma(D_{x,y}) = K_{x,y} \ , \end{aligned} \tag{4.14}$$

where $x, y \in V$. It can be easily checked that it is a well defined map as the center of V is zero; using (4.8)-(4.9) one gets the following.

Lemma 4.15. σ *is a grade reversing involution of* $\mathfrak{g}$.

Proposition 4.16. *Two centerless KTS,* V, V' *are isomorphic if and only if* $\mathtt{Lie}(V)$ *and* $\mathtt{Lie}(V')$, *with their relative grade-reversing involutions, are isomorphic pairs.*

Proof. It is proved with the same arguments used in Theorem 4.11. □

Chapter 5

$\mathbb{Z}$-graded Lie algebras and grade-reversing involutions

In this chapter we study the $\mathbb{Z}$-gradings of simple Lie algebras and their grade-reversing involutions. This will allow us to obtain the classification of KTS and will also be used in the classification of ϵ-sJTS.

5.1 Preliminaries on real and complex $\mathbb{Z}$-graded Lie algebras

We recall the following relevant result, see e.g. [55, Lemma 1.5].

Proposition 5.1. *A finite-dimensional simple $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ over $\mathbb{F}$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$) always admits a grade-reversing Cartan involution.*

In other words, there exists a grade-reversing involution $\theta : \mathfrak{g} \to \mathfrak{g}$ such that the form

$$B_\theta(x, y) = -B(x, \theta y) , \qquad x, y \in \mathfrak{g} , \tag{5.1}$$

is positive-definite symmetric ($\mathbb{F} = \mathbb{R}$) or positive-definite Hermitian ($\mathbb{F} = \mathbb{C}$), where B is the Killing form of $\mathfrak{g}$. Note that $\theta : \mathfrak{g} \to \mathfrak{g}$ is antilinear when $\mathbb{F} = \mathbb{C}$ — for uniformity of exposition in this section, we will still refer to any $\mathbb{R}$-linear involutive automorphism of a complex Lie algebra as an "involution".

Due to the results of Section 4.2, our interest is in a related but slightly different problem, i.e., *classifying* grade-reversing $\mathbb{C}$-*linear* involutions of $\mathbb{Z}$-graded complex simple Lie algebras, up to *zero-degree* automorphisms. We will shortly see that the solution of this problem is tightly related to suitable real forms of complex Lie algebras.

We begin with an auxiliary result – the $\mathbb{Z}$-graded counterpart of a known fact in the classification of real forms. Here and in the following section, we denote by $(\mathfrak{g}, \sigma)$ a pair consisting of a (finite-dimensional) $\mathbb{Z}$-graded simple Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and a grade-reversing involution $\sigma : \mathfrak{g} \to \mathfrak{g}$. We let G_0 be the connected component of the group of inner automorphisms of $\mathfrak{g}$ of degree zero.

Proposition 5.2. *For any pair $(\mathfrak{g}, \sigma)$ and a grade-reversing Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$, there exists $\phi \in G_0$ such that $\phi \circ \theta \circ \phi^{-1}$ commutes with σ.*

This result is proved in a similar way as in [34, Lemma 6.15, Theorem 6.16] and we omit details for the sake of brevity. The following corollary is proved as in [34, Corollary 6.19].

Corollary 5.3. *Any two grade-reversing Cartan involutions of a simple $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ are conjugate by some $\phi \in G_0$.*

5.2 Grade-reversing involutions and aligned pairs

We now introduce the main source of grade-reversing $\mathbb{C}$-linear involutions on complex simple Lie algebras. Let

$$\mathfrak{g}^o = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^o_p \tag{5.2}$$

be a *real* absolutely simple $\mathbb{Z}$-graded Lie algebra and $\theta : \mathfrak{g}^o \to \mathfrak{g}^o$ a grade-reversing Cartan involution. The complexification $\mathfrak{g}$ of $\mathfrak{g}^o$ is a simple $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ and the $\mathbb{C}$-linear extension of θ a grade-reversing involution $\sigma : \mathfrak{g} \to \mathfrak{g}$.

Definition 5.4. The pair $(\mathfrak{g}, \sigma)$ is the (complexified) *pair aligned* to $\mathfrak{g}^o$ and θ.

Definition 5.5. Two aligned pairs $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}', \sigma')$ are *isomorphic* if there is a zero-degree Lie algebra isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi \circ \sigma = \sigma' \circ \phi$.

Proposition 5.7 below says that the isomorphism class of an aligned pair does not depend on the choice of the Cartan involution but only on the real Lie algebra. To prove that, we first need a technical but useful result.

Lemma 5.6. *Let $\mathfrak{g}^o$ and $\widetilde{\mathfrak{g}}^o$ be $\mathbb{Z}$-graded real forms of a $\mathbb{Z}$-graded simple complex Lie algebra $\mathfrak{g}$. If the grade-reversing Cartan involutions*

$$\theta : \mathfrak{g}^o \to \mathfrak{g}^o , \qquad \widetilde{\theta} : \widetilde{\mathfrak{g}}^o \to \widetilde{\mathfrak{g}}^o , \tag{5.3}$$

have equal $\mathbb{C}$-linear extensions $\sigma, \widetilde{\sigma} : \mathfrak{g} \to \mathfrak{g}$ then $\mathfrak{g}^o$ and $\widetilde{\mathfrak{g}}^o$ are isomorphic as $\mathbb{Z}$-graded Lie algebras.

Proof. Let $\mathfrak{g}^o = \mathfrak{k} \oplus \mathfrak{p}$ and $\widetilde{\mathfrak{g}}^o = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{p}}$ be the Cartan decompositions associated to (5.3). In other words, the real forms $\mathfrak{u}^o = \mathfrak{k} \oplus i\mathfrak{p}$ and $\widetilde{\mathfrak{u}}^o = \widetilde{\mathfrak{k}} \oplus i\widetilde{\mathfrak{p}}$ are compact and

$$\begin{aligned} \vartheta_{\mathfrak{g}^o} &= \vartheta_{\mathfrak{u}^o} \circ \sigma = \sigma \circ \vartheta_{\mathfrak{u}^o} , \\ \vartheta_{\widetilde{\mathfrak{g}}^o} &= \vartheta_{\widetilde{\mathfrak{u}}^o} \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \vartheta_{\widetilde{\mathfrak{u}}^o} , \end{aligned} \tag{5.4}$$

where $\vartheta_{\mathfrak{g}^o}, \dots, \vartheta_{\widetilde{\mathfrak{u}}^o}$ are the antilinear involutions of $\mathfrak{g}$ associated to the real forms $\mathfrak{g}^o, \dots, \widetilde{\mathfrak{u}}^o$. It follows that $\vartheta_{\mathfrak{u}^o}$ and $\vartheta_{\widetilde{\mathfrak{u}}^o}$ are grade-reversing Cartan involutions of $\mathfrak{g}$ and $\vartheta_{\widetilde{\mathfrak{u}}^o} = \phi \circ \vartheta_{\mathfrak{u}^o} \circ \phi^{-1}$ for some zero-degree automorphism ϕ of $\mathfrak{g}$, by Corollary 5.3.

We replace $\widetilde{\mathfrak{g}}^o$ with the isomorphic $\mathbb{Z}$-graded real Lie algebra

$$\widehat{\mathfrak{g}}^o := \phi^{-1}(\widetilde{\mathfrak{g}}^o)$$

with grade-reversing Cartan involution $\widehat{\theta} = \phi^{-1} \circ \widetilde{\theta} \circ \phi : \widehat{\mathfrak{g}}^o \to \widehat{\mathfrak{g}}^o$. By construction the compact real form $\widehat{\mathfrak{u}}^o = \phi^{-1}(\widetilde{\mathfrak{u}}^o)$ coincides with $\mathfrak{u}^o$ and it is therefore stable under the action of both σ and the $\mathbb{C}$-linear extension $\widehat{\sigma}$ of $\widehat{\theta}$. We also note that $\widehat{\sigma} = \phi^{-1} \circ \widetilde{\sigma} \circ \phi = \phi^{-1} \circ \sigma \circ \phi$, where the last identity follows from our hypothesis $\widetilde{\sigma} = \sigma$.

The Lie algebra $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ of the connected component G_0 of the group of zero-degree inner automorphisms of $\mathfrak{g}$ decomposes into

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{u}^o) \oplus (\mathfrak{g}_0 \cap i\mathfrak{u}^o) ,$$

as $\vartheta_{\mathfrak{u}^o}(\mathfrak{g}_0) = \mathfrak{g}_0$. We denote by U the analytic subgroup of G_0 with Lie algebra $\mathrm{Lie}(U) = \mathfrak{g}_0 \cap \mathfrak{u}^o$ and note that the mapping

$$\Phi : (\mathfrak{g}_0 \cap i\mathfrak{u}^o) \times U \to G_0 , \qquad \Phi(X, u) = (\exp X)u ,$$

where $X \in \mathfrak{g}_0 \cap i\mathfrak{u}^o$, $u \in U$, is a diffeomorphism, cf. [24, Theorem 1.1, p. 252] and [34, Theorem 6.31]. In particular $\phi^{-1} = p \circ u$, for some $u \in U$ and an element $p \in \exp(\mathfrak{g}_0 \cap i\mathfrak{u}^o)$ which commutes with $u \circ \sigma \circ u^{-1}$, cf. a standard argument in the proof of [24, Proposition 1.4, p. 442]. It follows that

$$\widehat{\sigma} = u \circ \sigma \circ u^{-1}$$

and $u|_{\mathfrak{g}^o} : \mathfrak{g}^o \to \widehat{\mathfrak{g}}^o$ is the required isomorphism of $\mathbb{Z}$-graded real Lie algebras. □

We now deal with the isomorphism classes of aligned pairs.

Proposition 5.7. *Let $(\mathfrak{g} = \mathfrak{g}^o \otimes \mathbb{C}, \sigma)$ and $(\mathfrak{g}' = \mathfrak{g}'^o \otimes \mathbb{C}, \sigma')$ be aligned pairs with underlying real Lie algebras*

$$\mathfrak{g}^o = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^o_p , \qquad \mathfrak{g}'^o = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}'^o_p . \tag{5.5}$$

Then $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}', \sigma')$ are isomorphic if and only if $\mathfrak{g}^o$ and $\mathfrak{g}'^o$ are isomorphic as $\mathbb{Z}$-graded Lie algebras.

Proof. ($\Longleftarrow$) If the $\mathbb{Z}$-graded real Lie algebras $\mathfrak{g}^o$ and $\mathfrak{g}'^o$ are isomorphic, then, without any loss of generality, we may assume that they coincide. Hence $\sigma, \sigma' : \mathfrak{g} \to \mathfrak{g}$ are the $\mathbb{C}$-linear extensions of grade-reversing Cartan involutions $\theta, \theta' : \mathfrak{g}^o \to \mathfrak{g}^o$ and, by Corollary 5.3, there exists a zero-degree automorphism $\psi : \mathfrak{g}^o \to \mathfrak{g}^o$ such that $\psi \circ \theta = \theta' \circ \psi$. The $\mathbb{C}$-linear extension $\phi : \mathfrak{g} \to \mathfrak{g}$ of ψ is the required isomorphism of aligned pairs.

($\Longrightarrow$) Let $\phi : \mathfrak{g} \to \mathfrak{g}'$ be a zero-degree Lie algebra isomorphism such that $\phi \circ \sigma = \sigma' \circ \phi$. We replace $\mathfrak{g}'^o$ by the isomorphic $\mathbb{Z}$-graded real Lie algebra $\widetilde{\mathfrak{g}}^o := \phi^{-1}(\mathfrak{g}'^o)$ with grade-reversing Cartan involution $\widetilde{\theta} = \phi^{-1} \circ \theta' \circ \phi|_{\widetilde{\mathfrak{g}}^o} : \widetilde{\mathfrak{g}}^o \to \widetilde{\mathfrak{g}}^o$. The $\mathbb{C}$-linear extension of $\widetilde{\theta}$ is $\sigma = \phi^{-1} \circ \sigma' \circ \phi : \mathfrak{g} \to \mathfrak{g}$, hence Lemma 5.6 applies and $\mathfrak{g}^o$ and $\widetilde{\mathfrak{g}}^o$ are isomorphic. □

In summary, we have proved most of the following.

Theorem 5.8. *Let $(\mathfrak{g}, \sigma)$ be a pair consisting of a $\mathbb{Z}$-graded simple complex Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ and a grade-reversing involution $\sigma : \mathfrak{g} \to \mathfrak{g}$. Then $(\mathfrak{g}, \sigma)$ is the aligned pair associated to a $\mathbb{Z}$-graded real form $\mathfrak{g}^o = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^o_p$ of $\mathfrak{g}$. Isomorphic pairs correspond exactly to isomorphic $\mathbb{Z}$-graded real forms.*

Proof. In view of Proposition 5.7, it remains only to show that $(\mathfrak{g}, \sigma)$ is an aligned pair. By Proposition 5.2, there exists a grade-reversing Cartan involution $\vartheta : \mathfrak{g} \to \mathfrak{g}$ which commutes with σ, we denote by $\mathfrak{u}^o$ the associated compact real form of $\mathfrak{g}$. We note that $\sigma(\mathfrak{u}^o) = \mathfrak{u}^o$ and let $\mathfrak{u}^o = \mathfrak{k} \oplus i\mathfrak{p}$ be the ± 1-eigenspace decomposition of $\sigma|_{\mathfrak{u}^o}$.

Let $\mathfrak{g}^o = \mathfrak{k} \oplus \mathfrak{p}$ be the real form of $\mathfrak{g}$ with Cartan involution $\theta = \sigma|_{\mathfrak{g}^o} : \mathfrak{g}^o \to \mathfrak{g}^o$ and E the grading element of $\mathfrak{g}$, that is, the unique element $E \in \mathfrak{g}$ satisfying $[E, x] = px$ for all $x \in \mathfrak{g}_p$. We have

$$\sigma E = -E \Longrightarrow E \in \mathfrak{p} \oplus i\mathfrak{p},$$
$$\vartheta E = -E \Longrightarrow E \in i\mathfrak{k} \oplus \mathfrak{p},$$

hence $E \in \mathfrak{p} \subset \mathfrak{g}^o$. It follows that $\mathfrak{g}^o$ is a $\mathbb{Z}$-graded real form of $\mathfrak{g}$ and σ is the $\mathbb{C}$-linear extension of the grade-reversing Cartan involution $\theta : \mathfrak{g}^o \to \mathfrak{g}^o$. □

5.3 Classification of finite-dimensional Kantor triple systems

In this section, we describe all fundamental 5-gradings $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ of finite-dimensional complex simple Lie algebras and their real forms. When $\dim \mathfrak{g}_{\pm 2} = 1$, such gradings are usually called of *contact type*; in the general case we call them *admissible*. Admissible real gradings are in one-to-one correspondence with finite-dimensional K-simple complex KTS, cf. Theorems 4.14, 4.16, 5.8.

We first recall the description of $\mathbb{Z}$-gradings of complex simple Lie algebras (see e.g. [46]). Let $\mathfrak{g}$ be a complex simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system and by

$$\mathfrak{g}^\alpha = \left\{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ \text{ for all } \ H \in \mathfrak{h} \right\}$$

the associated root space of $\alpha \in \Delta$. Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ be the real subspace where all the roots are real valued; any element $\lambda \in (\mathfrak{h}_\mathbb{R}^*)^* \simeq \mathfrak{h}_\mathbb{R}$ with $\lambda(\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$ defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ on $\mathfrak{g}$ by setting:

$$\mathfrak{g}_0 = \ \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta \\ \lambda(\alpha) = 0}} \mathfrak{g}^\alpha \ , \qquad \mathfrak{g}_p = \bigoplus_{\substack{\alpha \in \Delta \\ \lambda(\alpha) = p}} \mathfrak{g}^\alpha \ , \quad \text{for all } \ p \in \mathbb{Z}^\times ,$$

and all possible gradings of $\mathfrak{g}$ are of this form, for some choice of $\mathfrak{h}$ and λ. We refer to $\lambda(\alpha)$ as the *degree* of the root α.

There exists a set of positive roots $\Delta^+ \subset \Delta$ such that λ is dominant, i.e., $\lambda(\alpha) \geqslant 0$ for all $\alpha \in \Delta^+$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the set of positive simple roots, which we identify with the nodes of the Dynkin diagram. The depth of $\mathfrak{g}$ is the degree $d = \lambda(\alpha_{\max})$ of the maximal root $\alpha_{\max} = \sum_{i=1}^{\ell} m_i \alpha_i$. A grading is fundamental if and only if $\lambda(\alpha) \in \{0, 1\}$ for all simple roots $\alpha \in \Pi$. Fundamental gradings on $\mathfrak{g}$ are denoted by marking with a cross the nodes of the Dynkin diagram of $\mathfrak{g}$ corresponding to simple roots α with $\lambda(\alpha) = 1$.

The previous arguments prove to the following result:

Proposition 5.9. *There exists a bijection between isomorphic simple Lie algebras with a fundamental* $(2k+1)$*-grading and Dynkin diagrams with marked nodes. In particular*

(i) 3-gradings correspond to Dynkin diagrams with one marked node α_i *with label* 1*;*

(ii) 5*-gradings correspond to Dynkin diagrams with either one marked node* α_i *with label* 2 *or two marked nodes* α_i, α_j *with label* 1*.*

The Lie subalgebra $\mathfrak{g}_0$ is reductive; the Dynkin diagram of its semisimple ideal is obtained from the Dynkin diagram of $\mathfrak{g}$ by removing all crossed nodes, and any line issuing from them.

We now recall the description of $\mathbb{Z}$-gradings of real simple Lie algebras, cf. [18]. Let $\mathfrak{g}^o$ be a real simple Lie algebra. Fix a Cartan decomposition $\mathfrak{g}^o = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace $\mathfrak{h}_\circ \subset \mathfrak{p}$ and a maximal torus $\mathfrak{h}_\bullet$ in the centralizer of $\mathfrak{h}_\circ$ in $\mathfrak{k}$. Then $\mathfrak{h}^o = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}^o$.

Denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g} = \mathfrak{g}^o \otimes \mathbb{C}$ with respect to $\mathfrak{h} = \mathfrak{h}^o \otimes \mathbb{C}$ and by $\mathfrak{h}_\mathbb{R} = i\mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ \subset \mathfrak{h}$ the real subspace where all the roots have real values. Conjugation $\vartheta : \mathfrak{g} \longrightarrow \mathfrak{g}$ of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}^o$ leaves $\mathfrak{h}$ invariant and induces an involution $\alpha \mapsto \bar\alpha$ on $\mathfrak{h}^*_\mathbb{R}$, transforming roots into roots. We say that a root α is compact if $\bar\alpha = -\alpha$ and denote by $\Delta_\bullet$ the set of compact roots. There exists a set of positive roots $\Delta^+ \subset \Delta$, with corresponding system of simple roots Π, and an involutive automorphism $\varepsilon \colon \Pi \to \Pi$ of the Dynkin diagram of $\mathfrak{g}$ such that $\bar\alpha = -\alpha$ for all $\alpha \in \Pi \cap \Delta_\bullet$, $\varepsilon(\Pi \setminus \Delta_\bullet) \subseteq \Pi \setminus \Delta_\bullet$ and

$$\bar\alpha = \varepsilon(\alpha) + \sum_{\beta \in \Pi \cap \Delta_\bullet} b_{\alpha,\beta}\,\beta \quad \text{for all } \alpha \in \Pi \setminus \Delta_\bullet \,.$$

The Satake diagram of $\mathfrak{g}^o$ is the Dynkin diagram of $\mathfrak{g}$ with the following additional data:

1. nodes in $\Pi \cap \Delta_\bullet$ are painted black;

2. if $\alpha \in \Pi \setminus \Delta_\bullet$ and $\varepsilon(\alpha) \neq \alpha$ then α and $\varepsilon(\alpha)$ are joined by a curved arrow.

A list of Satake diagrams can be found in e.g. [46].

Let $\lambda \in (\mathfrak{h}^*_\mathbb{R})^* \simeq \mathfrak{h}_\mathbb{R}$ be an element such that the induced grading on $\mathfrak{g}$ is fundamental. Then the grading on $\mathfrak{g}$ induces a grading on $\mathfrak{g}^o$ if and only if $\bar\lambda = \lambda$ [18, Theorem 3], or equivalently the following two conditions on the set
$\Phi = \left\{\alpha \in \Pi \mid \lambda(\alpha) = 1\right\}$ are satisfied:

$$1)\ \Phi \cap \Delta_\bullet = \emptyset\,; \qquad\qquad 2)\ \text{if } \alpha \in \Phi \quad \text{then} \quad \varepsilon(\alpha) \in \Phi\,.$$

There exists a bijection from the isomorphism classes of fundamental $\mathbb{Z}$-gradings of real simple Lie algebras and the isomorphism classes of marked Satake diagrams. In the real case too, the Lie subalgebra $\mathfrak{g}_0$ is reductive and the Satake diagram of its semisimple ideal is the Satake diagram of $\mathfrak{g}^o$ with all crossed nodes and any line issuing from them removed. A grading of $\mathfrak{g}^o$ is admissible if and only if the induced grading on $\mathfrak{g}$ is admissible.

We have just proved the following result:

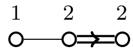
Theorem 5.10. *Let $\mathfrak{g}$ be a $(2k+1)$-graded Lie algebra and let $\Pi_1 = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ be the simple roots of degree 1. There is a bijection between grade-reversing involutions of $\mathfrak{g}$, up to equivalence, and Satake diagrams for which*

- *the nodes corresponding to simple roots in Π_1 are not black;*

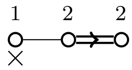
- *if* $\alpha \in \Pi_1$ *and there is an arrow between* α *and* β*, then* $\beta \in \Pi_1$.

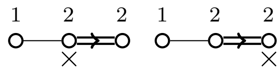
When in the situation of the previous proposition we will call the grade-reversing involution with the name of the real form corresponding to the Satake diagram. The list of Satake diagrams can be found in [47].

To better fix the ideas we give an explicit example .

Example 5.11. Let $\mathfrak{g} = so(7, \mathbb{C})$. The Dynkin diagram of $\mathfrak{g}$ is 

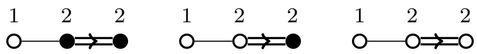
. By Proposition 5.9 the following $(2k+1)$-gradings are defined on $\mathfrak{g}$:

$k = 1$ 

$k = 2$ 

The Satake diagrams corresponding to the real forms of $\mathfrak{g}$ are

$so(1,6)$ $so(2,5)$ $so(3,4)$



By Proposition 5.10 we can conclude that the 3-grading of $\mathfrak{g}$ has 3 non-equivalent grade-reversing involutions corresponding to the real forms $so(p, 7-p)$, $p = 1, 2, 3$ while the 5-grading of $\mathfrak{g}$ given by marking the second node (resp. third node), has the 2 involutions corresponding to $so(2,5)$ and $so(3,4)$ (resp. the involution corresponding to $so(3,4)$).

Theorem 5.10 and a simple computation using the tables of Satake diagrams gives the enumeration of the finite-dimensional KTS with given Tits-Kantor-Koecher Lie algebra.

Corollary 5.12. *Let* $\mathfrak{g}$ *be a complex simple Lie algebra. Then the number* $K(\mathfrak{g})$ *of the K-simple KTS up to isomorphism with associated Tits-Kantor-Koecher Lie algebra* $\mathfrak{g}$ *is:*

(1) $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$ *with* $\ell \geqslant 1$*:*

- *if* $\ell = 2m+1$ *is odd with* m *even then* $K(\mathfrak{g}) = \frac{7m^2+10m}{4}$*;*
- *if* $\ell = 2m+1$ *is odd with* m *odd then* $K(\mathfrak{g}) = \frac{7m^2+10m-1}{4}$*;*
- *if* $\ell = 2m$ *is even then* $K(\mathfrak{g}) = \frac{3m^2+m}{2}$*;*

(2) $\mathfrak{g} = \mathfrak{so}(2\ell+1, \mathbb{C})$ *with* $\ell \geqslant 2$ *then* $K(\mathfrak{g}) = \frac{\ell(\ell-1)}{2}$*;*

(3) $\mathfrak{g} = \mathfrak{sp}(\ell, \mathbb{C})$ *with* $\ell \geqslant 3$*:*

- *if* $\ell = 2m + 1$ *is odd then* $K(\mathfrak{g}) = \frac{m^2+5m}{2}$*;*
- *if* $\ell = 2m$ *is even then* $K(\mathfrak{g}) = \frac{m^2+5m-4}{2}$*;*

(4) $\mathfrak{g} = \mathfrak{so}(2\ell, \mathbb{C})$ *with* $\ell \geqslant 4$*:*

- *if* $\ell = 2m + 1$ *is odd then* $K(\mathfrak{g}) = 2m^2 + 2m$*;*
- *if* $\ell = 4$ *then* $K(\mathfrak{g}) = 5$*;*
- *if* $\ell = 2m$ *is even with* $m > 2$ *then* $K(\mathfrak{g}) = 2m^2 - 1$*;*

(5) *if* $\mathfrak{g} = G_2$ *then* $K(\mathfrak{g}) = 1$*;*

(6) *if* $\mathfrak{g} = F_4$ *then* $K(\mathfrak{g}) = 3$*;*

(7) *if* $\mathfrak{g} = E_6$ *then* $K(\mathfrak{g}) = 8$*;*

(8) *if* $\mathfrak{g} = E_7$ *then* $K(\mathfrak{g}) = 7$*;*

(9) *if* $\mathfrak{g} = E_8$ *then* $K(\mathfrak{g}) = 4$*.*

5.4 The algebra of derivations

In this section we study the Lie algebra

$$\mathfrak{der}(V) = \{\delta : V \to V \mid \delta(xyz) = ((\delta x)yz) + (x(\delta y)z) + (xy(\delta z))\}$$

of derivations of a KTS V. This is an important invariant of a K-simple KTS and, as we will shortly prove, it can easily be described a priori by means of Theorem 5.8.

In Chapter 4 we saw that isomorphisms of V correspond to those of $(\mathfrak{g}, \sigma)$. It follows that $\mathfrak{der}(V)$ consists of (the restriction to $V = \mathfrak{g}_{-1}$ of) the 0-degree derivations of $\mathfrak{g}$ commuting with σ. If V is K-simple, then $\mathfrak{g}$ is simple, any derivation is inner and $\mathfrak{der}(V) = \{D \in \mathfrak{g}_0 \mid \sigma(D) = D\}$. Now note that $\mathfrak{g}_0$ is σ-stable reductive with the center which is at most 2-dimensional. When the grading corresponds to a positive subalgebra that is maximal parabolic, the center of $\mathfrak{g}_0$ is 1-dimensional and generated by the grading element E. Clearly $\sigma(E) = -E$. Otherwise, the marked Dynkin diagram of $\mathfrak{g}$ has two crossed nodes, this happens for all admissible gradings of $\mathfrak{sl}(\ell + 1, \mathbb{C})$, two of $\mathfrak{so}(2\ell, \mathbb{C})$ and one of E_6.

In summary, both the center and the semisimple part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$ decompose into the direct sum of ± 1-eigenspaces of σ and

$$\mathfrak{der}(V) = \{D \in \mathfrak{g}_0^{ss} \mid \sigma(D) = D\} \ ,$$

possibly up to a 1-dimensional central subalgebra.

Let $\mathfrak{g}^o$ be a real form of $\mathfrak{g}$ compatible with the grading and $(\mathfrak{g}_0^o)^{ss} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the Cartan decomposition of the semisimple part of $\mathfrak{g}_0^o$. The following result holds since $\sigma : \mathfrak{g} \to \mathfrak{g}$ is the $\mathbb{C}$-linear extension of the Cartan involution $\theta : \mathfrak{g}^o \to \mathfrak{g}^o$ of $\mathfrak{g}^o$, whose restriction

$$\theta|_{(\mathfrak{g}_0^o)^{ss}} : (\mathfrak{g}_0^o)^{ss} \to (\mathfrak{g}_0^o)^{ss}$$

to $(\mathfrak{g}_0^o)^{ss}$ is still a Cartan involution, see e.g. [8, Lemma 1.5].

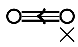
Theorem 5.13. *Let V be a K-simple KTS and $(\mathfrak{g} = \mathfrak{g}^o \otimes \mathbb{C}, \sigma)$ the associated aligned pair. Then*

$$\mathfrak{der}(V) = \mathfrak{k}_0 \otimes \mathbb{C}$$

possibly up to a 1-dimensional central subalgebra.

The list of Cartan decompositions of real semisimple Lie algebras can be found in [46]. Theorem 5.13 and an a priori identification of the algebra of derivations as the following example shows will be crucial ingredients to classify the KTS of exceptional type.

Example 5.14. By Corollary 5.12, there is just one KTS V with associated Tits-Kantor-Koecher algebra $\mathfrak{g} = G_2$. The aligned real form $\mathfrak{g}^o$ is the split form with marked Satake diagram



From the list of Dynkin diagrams and Satake diagrams one obtains that $(\mathfrak{g}_0^o)^{ss} \simeq \mathfrak{sl}(2, \mathbb{R})$, the maximal compact subalgebra $\mathfrak{k}_0 \simeq \mathfrak{so}(2, \mathbb{R})$, hence $\mathfrak{der}(V) \simeq \mathfrak{so}(2, \mathbb{C})$.

Chapter 6

The classification of KTS

6.1 The classification of classical Kantor triple systems

The list, up to isomorphism, of all *K-simple classical KTS over* $\mathbb{R}$ is due to Kaneyuki and Asano, see [29, 6]. We give here the classification over $\mathbb{C}$, written accordingly to the examples and conventions of Section 3.1.

Theorem 6.1. *A K-simple KTS over* $\mathbb{C}$ *with classical Tits-Kantor-Koecher Lie algebra is isomorphic to one of the following list:*

- $\mathfrak{Ksl}(r, m, n-m-r; t)$, $\quad n \geqslant 3,\ 1 \leqslant m \leqslant [(n-1)/2],\ 1 \leqslant r \leqslant n-m-1;$
- $\mathfrak{Ksl}(2r, 2m, 2(n-m-r); st)$, $\quad n \geqslant 3,\ 1 \leqslant m \leqslant [(n-1)/2],\ 1 \leqslant r \leqslant n-m-1;$
- $\mathfrak{Ksl}(n-2m, m; k)$, $\quad n \geqslant 3;\ 1 \leqslant m \leqslant [(n-1)/2],\ 0 \leqslant k \leqslant [(n-2m)/2];$
- $\mathfrak{Kso}(n-2m, m; k)$, $\quad n \geqslant 7,\ 2 \leqslant m \leqslant [(n-1)/2],\ 0 \leqslant k \leqslant [(n-2m)/2];$
- $\mathfrak{Kso}(2(n-2m), 2m; JS)$, $\quad n \geqslant 4;\ 2 \leqslant 2m \leqslant n-1;$
- $\mathfrak{Ksp}(2(n-m), m; J)$, $\quad n \geqslant 3,\ 1 \leqslant m \leqslant n-1;$
- $\mathfrak{Ksp}(2(n-2m), 2m; k)$, $\quad n \geqslant 3,\ 2 \leqslant 2m \leqslant n-1,\ 0 \leqslant k \leqslant [(n-2m)/2];$
- $\mathfrak{Kar}(n)$, $\quad n \geqslant 4\ .$

The ranges of natural numbers are chosen so that there are no isomorphic KTS in the above list.

Proof. By Theorem 5.10 specialized to 5-gradings, complexification of the classical KTS of compact type classified in [29] exactly gives the classification of all classical KTS over $\mathbb{C}$. □

Remark 6.2. We remark that the Lie algebras associated to the KTS $\mathfrak{Ksl}(m,n,r;t)$, $\mathfrak{Ksl}(m,n,r;st)$ and $\mathfrak{Ksl}(m,n;k)$ are all isomorphic to $\mathfrak{sl}(N,\mathbb{C})$, for some N, but with different gradings and involutions. In particular the involution is outer in the first two cases and inner in the last one. We also point out that Lie algebras associated to the KTS's of type $\mathfrak{Kso}$, resp. $\mathfrak{Ksp}$, are all orthogonal, respectively, symplectic.

On the other hand, the Lie algebra associated to the triple system $\mathfrak{Kar}(n)$ is $\mathfrak{so}(2n+2,\mathbb{C})$ with the grading given by marking the first and last node. The grade-reversing involution is the Chevalley involution.

Remark 6.3. From the description of classical KTS in Section 3.1, there are natural embeddings of triple systems

$$\begin{aligned} \mathfrak{Kso}(m,n) &\subset \mathfrak{Ksl}(m,n)\,, \\ \mathfrak{Ksp}(2m,n) &\subset \mathfrak{Ksl}(2m,n)\,. \end{aligned}$$

The system $\mathfrak{Kar}(n)$ has a different interesting embedding, which is dealt with in the final remark of this section.

We recall that the so-called generalized Jordan triple systems of the ν-th kind are those systems with associated Tits-Kantor-Koecher Lie algebra that is $(2\nu+1)$-graded. In particular those of the 1-st and 2-nd kind are the usual Jordan and, respectively, Kantor triple systems.

An interesting phenomenon happens for some JTS. The 3-graded Lie algebras associated to the JTS $4, 5, 11, 13$ of Table III, of [29, p. 110], i.e., the orthogonal Lie algebras with only the first node marked, are in a natural way graded subalgebras of some $\mathfrak{sl}(N,\mathbb{C})$ with a 5-grading. This gives rise to an embedding of the associated JTS in a special KTS. In other words, simple KTS admit suitable subsystems which are of the 1-st kind and yet simple.

A similar fact holds for $\mathfrak{Kar}(n)$, which admits a natural embedding in a simple generalized Jordan triple system of higher kind.

Remark 6.4. Let $A = M_{n_2,n_1}(\mathbb{C}) \oplus M_{n_3,n_2}(\mathbb{C}) \oplus M_{n_4,n_3}(\mathbb{C})$ with the 3-product

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right) = \begin{pmatrix} x_1 y_1^t z_1 + z_1 y_1^t x_1 - y_2^t x_2 z_1 \\ x_2 y_2^t z_2 + z_2 y_2^t x_2 - z_2 x_1 y_1^t - y_3^t x_3 z_2 \\ x_3 y_3^t z_3 + z_3 y_3^t x_3 - z_3 x_2 y_2^t \end{pmatrix}.$$

It can be shown that the product satisfies only condition (i) of Definition 1.2 and that it is a generalized JTS of the 3-rd kind. We are interested in the case

$$n_1 = n_4 = 1\,, \qquad n_2 = n_3 = n\,, \tag{6.1}$$

for which the associated Tits-Kantor-Koecher Lie algebra is $\mathfrak{sl}(2n+2,\mathbb{C})$ with the 7-grading given by marking the nodes $\{1, n+1, 2n+1\}$.

Let $n_1, \dots, n_4$ as above and consider the subspace of A given by

$$M = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in A \mid x_3 = -x_1',\ x_2' = -x_2 \right\} .$$

The space M is a simple subsystem of A of the 2-nd kind and it is isomorphic to $\mathfrak{Kar}(n)$.

To see this, it is sufficient to consider the subalgebra $\mathfrak{so}(2n+2,\mathbb{C})$ of anti-reflexive matrices of $\mathfrak{sl}(2n+2,\mathbb{C})$ and note that $\mathfrak{so}(2n+2,\mathbb{C})$ inherits a grading from $\mathfrak{sl}(2n+2,\mathbb{C})$ that is actually a 5-grading, the one mentioned in Example 3.5. Finally, the Chevalley involution of $\mathfrak{sl}(2n+2,\mathbb{C})$ restricts to the Chevalley involution of $\mathfrak{so}(2n+2,\mathbb{C})$ and our claim follows.

6.2 The exceptional Kantor triple systems of extended Poincaré type

This section relies on the so-called gradings of *extended Poincaré type*, see [5, Theorem 3.1], which have been extensively investigated by third author both in the complex and real case. We recall here only the facts that we need and refer to [4, 5] for more details.

Let (U, η) be a finite-dimensional complex vector space U endowed with a non-degenerate symmetric bilinear form η and $C\ell(U) = C\ell(U)_{\bar{0}} \oplus C\ell(U)_{\bar{1}}$ the associated Clifford algebra with its natural parity decomposition. We will make use of the notation of [38] and also of [1]; in particular, we adopt the following conventions:

- the product in $C\ell(U)$ satisfies $uv + vu = -2\eta(u, v)1$ for all $u, v \in U$.
- the symbol $\mathbb{S}$ denotes the complex spinor representation, i.e., an irreducible complex $C\ell(U)$-module, and Clifford multiplication on $\mathbb{S}$ is denoted by "$\circ$".
- the cover with fiber $\mathbb{Z}_4 = \{\pm 1, \pm i\}$ of the orthogonal group $O(U)$ is the Pin group $\mathrm{Pin}(U)$, with covering map given by the twisted adjoint action $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(U) \to O(U)$.
- if $\dim U$ is even, then $\mathbb{S}$ is $\mathfrak{so}(U)$-reducible and we denote by $\mathbb{S}^+$ and $\mathbb{S}^-$ its irreducible $\mathfrak{so}(U)$-submodules (the semispinor representations).

- we identify $\Lambda^\bullet U$ with $C\ell(U)$ (as vector spaces) via the isomorphism

$$u_1 \wedge \cdots \wedge u_k \mapsto \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \operatorname{sgn}(\pi) u_{\pi(1)} \cdots u_{\pi(k)}$$

  and let any $\alpha \in \Lambda^\bullet U$ act on $\mathbb{S}$ via Clifford multiplication, i.e. $\alpha \circ s$ for all $s \in \mathbb{S}$.

- we identify $\mathfrak{so}(U)$ with $\Lambda^2 U$ via $(u \wedge v)(w) = \eta(u, w)v - \eta(v, w)u$, where $u, v, w \in U$. We recall that the spin representation of $\mathfrak{so}(U)$ on $\mathbb{S}$ is half its action as a 2-form:

$$(u \wedge v) \cdot s = \frac{1}{2}(u \wedge v) \circ s$$

  for all $s \in \mathbb{S}$.

We will sometimes need the usual concrete realization of the representation of the Clifford algebra on the spinor module in terms of Kronecker products of matrices. Let us consider the 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and set

$$\alpha(j) = \begin{cases} 1 \text{ if } j \text{ is odd;} \\ 2 \text{ if } j \text{ is even.} \end{cases}$$

If $\dim U = 2k$ is even then a Clifford representation is given by the action of

$$C\ell(U) \simeq \underbrace{M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})}_{k-times} = \operatorname{End}(\mathbb{S})$$

on $\mathbb{S} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k-times}$ with the elements of a fixed orthonormal basis $(e_j)_{j=1}^{2k}$ of U realized as

$$e_j \mapsto E \otimes \cdots \otimes E \otimes g_{\alpha(j)} \otimes \underbrace{T \otimes \cdots \otimes T}_{[\frac{j-1}{2}]-times}, \tag{6.2}$$

for all $j = 1, \ldots, 2k$. Note that the basis elements satisfy $e_j^2 = -1$ as $\eta(e_i, e_j) = \delta_{ij}$ for all i, j. It follows that the volume $\mathrm{vol} = e_1 \cdots e_{2k}$ squares to $(-1)^k$ so that $\mathbb{S}^\pm$ are the ± 1-eigenspaces of the involution $i^k \mathrm{vol}$. In this paper, we will not need a concrete realization for $\dim U$ odd.

Definition 6.5. [1] A nondegenerate bilinear form $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ is called *admissible* if there exist $\tau, \sigma \in \{\pm 1\}$ such that $\beta(u \circ s, t) = \tau\beta(s, u \circ t)$ and $\beta(s, t) = \sigma\beta(t, s)$ for all $u \in U$, $s, t \in \mathbb{S}$.

Admissible forms are automatically $\mathfrak{so}(U)$-equivariant. If $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, we also require $\mathbb{S}^+$ and $\mathbb{S}^-$ to be either isotropic or mutually orthogonal w.r.t. β (in the former case we set $\iota = -1$, in the latter $\iota = 1$). The numbers (τ, σ), or (τ, σ, ι), are the *invariants associated with* β.

Set $\mathfrak{m}_{-2} = U$, $\mathfrak{m}_{-1} = \mathbb{S}$, $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$. If $\tau\sigma = -1$ we define on $\mathfrak{m}$ a structure of graded Lie algebra with Lie bracket given by the so-called "Dirac current"

$$\eta([s, t], u) = \beta(u \circ s, t) ,$$

for $s, t \in \mathbb{S}$, $u \in U$.

Definition 6.6. Any graded Lie algebra as defined above is called an *extended translation algebra*.

The main result of [4, 5] is the classification of maximal transitive prolongations

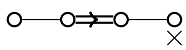
$$\mathfrak{g}^\infty = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$$

of extended translation algebras. The following result is also important.

Proposition 6.7. *[5, Theorem 2.3] If* $\dim U \geqslant 3$ *then for all* $p > 0$ *and* $X \in \mathfrak{g}_p$*, if* $[X, \mathfrak{g}_{-2}] = 0$ *then* $X = 0$*. In other words, elements of* $\mathfrak{g}_p$*,* $p > 0$*, are uniquely determined by their action on* $\mathfrak{g}_{-2}$*.*

6.2.1 The case $\mathfrak{g} = F_4$

There are 3 inequivalent KTS with Tits-Kantor-Koecher pair $(\mathfrak{g} = F_4, \sigma)$, cf. Corollary 5.12. In particular the grading of F_4 with marked Dynkin diagram



has two grade-reversing involutions, corresponding to real forms FI and FII, cf. Table 9 of [47, p.312-317].

In this case (U, η) is 7-dimensional and $\mathbb{S}$ is the 8-dimensional spinor module. (There are two such modules up to isomorphism, and they are equivalent as $\mathfrak{so}(V)$-representations. We have chosen the module for which the action of the volume element $\mathrm{vol} \in C\ell(U)$ is $\mathrm{vol} \circ s = s$ for all $s \in S$.)

More explicitly the graded components of the 5-grading of $\mathfrak{g} = F_4$ are $\mathfrak{g}_0 \simeq \mathfrak{so}(U) \oplus \mathbb{C}E$ and $\mathfrak{g}_{\pm 2} \simeq U$, $\mathfrak{g}_{\pm 1} \simeq \mathbb{S}$ as $\mathfrak{so}(U)$-modules. The negatively graded part $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$

is an extended translation algebra w.r.t. a bilinear form $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ with the invariants $(\tau, \sigma) = (-1, 1)$.

For any $s \in \mathbb{S}$ we introduce a linear map $\widehat{s} : \mathfrak{m} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ of degree 1 by

$$\begin{aligned} [\widehat{s}, t] &= -\Gamma^{(2)}(s,t) + \frac{1}{2}\beta(s,t)E \,, \\ [\widehat{s}, u] &= u \circ s \,, \end{aligned} \tag{6.3}$$

where $t \in \mathbb{S}$, $u \in U$ and

$$\Gamma^{(2)} : \mathbb{S} \otimes \mathbb{S} \to \mathfrak{so}(U) \,,$$

$$\eta(\Gamma^{(2)}(s,t)u, v) = \beta(u \wedge v \circ s, t) \,.$$

Using Proposition 6.7 (see also [5, Lemma 2.5]) one can directly check that $\mathfrak{g}_1 = \langle \widehat{s} \,|\, s \in \mathbb{S} \rangle$. Similarly we define a map $\widehat{u} : \mathfrak{m} \to \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of degree 2 for any $u \in U$ by

$$\begin{aligned} [\widehat{u}, s] &= -\widehat{u \circ s} \,, \\ [\widehat{u}, v] &= \lambda(u \wedge v) + \mu\eta(u,v)E \,, \end{aligned} \tag{6.4}$$

where $s \in \mathbb{S}$, $v \in U$ and where $\lambda, \mu \in \mathbb{C}$ are constants to be determined.

Lemma 6.8. *If* $\lambda = 2$ *and* $\mu = -1$ *then* $\widehat{u} \in \mathfrak{g}_2$ *for all* $u \in U$.

Proof. By $\mathfrak{so}(U)$-equivariance and transitivity, the action of $\mathfrak{g}_2$ on $\mathfrak{m}$ is necessarily of the above form for some λ, μ. We then compute

$$\begin{aligned} 0 = [\widehat{u}, [s, v]] &= [[\widehat{u}, s], v] + [s, [\widehat{u}, v]] \\ &= -[\widehat{u \circ s}, v] - \lambda[u \wedge v, s] + \mu\eta(u,v)s \\ &= (-1 + \frac{\lambda}{4} - \frac{\mu}{2})v \circ u \circ s - (\frac{\lambda}{4} + \frac{\mu}{2})u \circ v \circ s \,, \end{aligned}$$

for $u, v \in U$, $s \in \mathbb{S}$. The claim follows from the fact that the two terms vanish separately. □

The Lie brackets between elements of non-negative degrees are directly computed using transitivity and (6.3)-(6.4). They are given by the natural structure of Lie algebra of $\mathfrak{g}_0$ and

$$\begin{aligned} &[A, \widehat{s}] = \widehat{As} \,, \quad [A, \widehat{u}] = \widehat{Au} \,, \\ &[E, \widehat{s}] = \widehat{s} \,, \quad\; [E, \widehat{u}] = 2\widehat{u} \,, \quad [\widehat{s}, \widehat{t}] = \widehat{[s,t]} \,, \end{aligned} \tag{6.5}$$

where $A \in \mathfrak{so}(U)$, $\widehat{s}, \widehat{t} \in \mathfrak{g}_1$ and $\widehat{u} \in \mathfrak{g}_2$. We give details only for the third and last bracket. First compute

$$\begin{aligned} [[A, \widehat{u}], v] &= [A, [\widehat{u}, v]] - [\widehat{u}, Av] \\ &= 2[A, u \wedge v] - [\widehat{u}, Av] \\ &= 2Au \wedge v - \eta(Au, v)E \\ &= [\widehat{Au}, v] \end{aligned}$$

and

$$
\begin{aligned}
[[\widehat{s},\widehat{t}],u] &= [\widehat{s},u\circ t]-[\widehat{t},u\circ s] \\
&\equiv \frac{1}{2}\beta(s,u\circ t)E-\frac{1}{2}\beta(t,u\circ s)E \mod \mathfrak{so}(U) \\
&\equiv \beta(s,u\circ t)E = -\eta([s,t],u)E \mod \mathfrak{so}(U) \\
&\equiv [\widehat{[s,t]},u] \mod \mathfrak{so}(U)\ ,
\end{aligned}
$$

for all $A \in \mathfrak{so}(U)$, $s,t \in \mathbb{S}$, $u,v \in U$. The brackets follow from these identities, Proposition 6.7 and the fact that the action of $\widehat{u}$ on $\mathfrak{g}_{-2}$ is fully determined by its component on E, cf. (6.4).

The algebras of derivations of the two KTS on $\mathbb{S}$ are the complexifications of the maximal compact subalgebras of $\mathfrak{so}(7,\mathbb{R})$ and $\mathfrak{so}(3,4)$, respectively (see Table 9 of [47, p.312-317] and Theorem 5.13):

$$
\begin{aligned}
\mathfrak{der}(\mathbb{S}) &\simeq \mathfrak{so}(7,\mathbb{C}) \ \text{ for } \ FII\ , \\
\mathfrak{der}(\mathbb{S}) &\simeq \mathfrak{so}(3,\mathbb{C})\oplus\mathfrak{so}(4,\mathbb{C}) \ \text{ for } \ FI\ .
\end{aligned}
$$

The following main result describes both triple systems in a uniform fashion.

Theorem 6.9. *Let U be a 7-dimensional complex vector space with a non-degenerate symmetric bilinear form η and $\mathbb{S}$ the associated 8-dimensional spinor $C\ell(U)$-module. Let $\beta : \mathbb{S}\otimes\mathbb{S} \to \mathbb{C}$ be the unique (up to constant) admissible bilinear form on $\mathbb{S}$ and $\Gamma^{(2)} : \mathbb{S}\otimes\mathbb{S} \to \mathfrak{so}(U)$ the operator given by*

$$
\eta(\Gamma^{(2)}(s,t)u,v) = \beta(u\wedge v\circ s,t)\ ,
$$

where $s,t \in \mathbb{S}$ and $u,v \in U$. Fix an orthogonal decomposition

$$
U = W \oplus W^{\perp}
$$

of U with $\dim W = 7$ (i.e. with $W^{\perp}=0$, $W=U$) or $\dim W = 3$ and let $I = \mathrm{vol}_W \in C\ell(U)$ be the volume of W. (I acts on $\mathbb{S}$ as the identity if $\dim W = 7$ and as a paracomplex structure if $\dim W = 3$).

Then $\mathbb{S}$ with the triple product

$$
(rst) = -\Gamma^{(2)}(r, I\circ s)\cdot t + \frac{1}{2}\beta(r, I\circ s)t\ , \qquad r,s,t \in \mathbb{S}\ , \tag{6.6}
$$

is a K-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra $\mathfrak{g} = F_4$ and derivation algebra

$$
\mathfrak{der}(\mathbb{S}) = \mathfrak{stab}_{\mathfrak{so}(U)}(I) = \begin{cases} \mathfrak{so}(U) \ \text{if } \dim W = 7\ , \\ \mathfrak{so}(W)\oplus\mathfrak{so}(W^{\perp}) \ \text{if } \dim W = 3\ . \end{cases}
$$

Proof. It is immediate to see that $I \in \mathrm{Pin}(U)$ covers the opposite of the orthogonal reflection $r_W : U \to U$ across W, that is $\widetilde{\mathrm{Ad}}_I = -r_W$, and that $\beta(I \circ s, t) = \beta(s, I \circ t)$ for all $s, t \in \mathbb{S}$.

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the grade-reversing map defined by

$$\begin{aligned} \sigma(u) &= \widehat{r_W u} , & \sigma(s) &= \widehat{I \circ s} , \\ \sigma(A) &= r_W A r_W , & \sigma(E) &= -E , \\ \sigma(\widehat{s}) &= I \circ s , & \sigma(\widehat{u}) &= r_W u , \end{aligned} \tag{6.7}$$

where $u \in U$, $s \in \mathbb{S}$ and $A \in \mathfrak{so}(U)$. Clearly $\sigma^2 = 1$ and we now show that σ is a Lie algebra morphism. First note that $\sigma[A, s] = \widehat{I \circ As}$ and $[\sigma(A), \sigma(s)] = \widehat{t}$,

$$\begin{aligned} t &= [r_W A r_W, I \circ s] \\ &= I \circ A(I \circ I \circ s) \\ &= I \circ As , \end{aligned}$$

hence $\sigma[A, s] = [\sigma(A), \sigma(s)]$ for all $A \in \mathfrak{so}(U)$, $s \in \mathbb{S}$. Identity $\sigma[A, \widehat{s}] = [\sigma(A), \sigma(\widehat{s})]$ is analogous. Similarly $\sigma[\widehat{u}, s] = -\sigma(\widehat{u \circ s}) = -I \circ u \circ s$ while

$$\begin{aligned} [\sigma(\widehat{u}), \sigma(s)] &= -[\widehat{I \circ s}, r_W u] \\ &= -(r_W u) \circ I \circ s = \widetilde{\mathrm{Ad}}_I(u) \circ I \circ s \\ &= -I \circ u \circ s , \end{aligned}$$

proving $\sigma[\widehat{u}, s] = [\sigma(\widehat{u}), \sigma(s)]$ and, in the same way, $\sigma[u, \widehat{s}] = [\sigma(u), \sigma(\widehat{s})]$ for all $u \in U$, $s \in \mathbb{S}$. The remaining identities are straightforward, except for $\sigma[\widehat{s}, t] = [\sigma(\widehat{s}), \sigma(t)]$, $s, t \in \mathbb{S}$, which we now show. We have

$$\begin{aligned} \eta(\sigma[\widehat{s}, t]u, v) &= -\eta(\sigma(\Gamma^{(2)}(s, t))u, v) + \beta(s, t)\eta(u, v) \\ &= -\eta(\Gamma^{(2)}(s, t) r_W u, r_W v) + \beta(s, t)\eta(u, v) \\ &= -\beta((r_W u) \wedge (r_W v) \circ s, t) + \beta(s, t)\eta(u, v) \end{aligned}$$

and

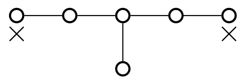
$$\begin{aligned} \eta([\sigma(\widehat{s}), \sigma(t)]u, v) &= -\eta([\widehat{I \circ t}, I \circ s]u, v) \\ &= +\eta(\Gamma^{(2)}(I \circ t, I \circ s)u, v) + \beta(I \circ t, I \circ s)\eta(u, v) \\ &= \beta(I \circ u \wedge v \circ I \circ t, s) + \beta(s, t)\eta(u, v) \\ &= \beta((r_W u) \wedge (r_W v) \circ t, s) + \beta(s, t)\eta(u, v) \\ &= \eta(\sigma[\widehat{s}, t]u, v) \end{aligned}$$

for all $u, v \in U$ so that $\sigma[\widehat{s}, t] = [\sigma(\widehat{s}), \sigma(t)]$.

The triple product (6.6) is the usual formula $(rst) = [[r, \sigma(s)], t]$ and the rest is clear. □

6.2.2 The case $\mathfrak{g} = E_6$

The grading of $\mathfrak{g} = E_6$ associated to



is of extended Poincaré type and all the 4 real forms of E_6 are compatible with this grading, leading to 4 non-isomorphic KTS with algebra of derivations equal to

$$\mathfrak{der}(\mathbb{S}) \cong \begin{cases} \mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}) & \text{for} \quad EI\,, \\ \mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C}) & \text{for} \quad EII\,, \\ \mathfrak{so}(7,\mathbb{C}) & \text{for} \quad EIII\,, \\ \mathfrak{so}(8,\mathbb{C}) & \text{for} \quad EIV\,, \end{cases} \tag{6.8}$$

possibly up to a 1-dimensional center. We note that this is the unique 5-grading of a simple exceptional Lie algebra whose associated parabolic subalgebra is not maximal. The fact that the center of $\mathfrak{g}_0$ is 2-dimensional makes our previous (and usual) approach to describe KTS practically inconvenient. In this subsection, we will use another approach and exploit the root space decomposition of E_6 to describe the KTS associated to EIV. The remaining products will be reduced to this case by a direct argument.

Let U be an 8-dimensional complex vector space, η a non-degenerate symmetric bilinear form on U and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ the spinor module, where each semispinor representation $\mathbb{S}^\pm$ is 8-dimensional. Let E be the grading element and F the operator acting as the volume $\mathrm{vol} \in C\ell(U)$ on $\mathbb{S}$ and trivially on U. It clearly commutes with E and $\mathfrak{so}(U)$. The grading of $\mathfrak{g} = E_6$ is then given by

$$\mathfrak{g}_{\pm 1} = \mathbb{S}\,, \quad \mathfrak{g}_0 = \mathfrak{so}(U) \oplus \mathbb{C}E \oplus \mathbb{C}F\,, \quad \mathfrak{g}_{\pm 2} = U\,.$$

We shall fix an isotropic decomposition of $U = W \oplus W^*$ and consider the natural $\mathfrak{so}(U)$-equivariant isomorphisms $\mathbb{S}^+ \cong \Lambda^{\mathrm{even}} W^*$ and $\mathbb{S}^- \cong \Lambda^{\mathrm{odd}} W^*$. With these conventions, the action of the Clifford algebra on $\mathbb{S}$ reads as

$$\begin{aligned} w \circ \alpha &= -2\imath_w \alpha\,, \\ w^* \circ \alpha &= w^* \wedge \alpha\,, \end{aligned} \tag{6.9}$$

for all $w \in W$, $w^* \in W^*$ and $\alpha \in \mathbb{S}$. There is an admissible bilinear form on $\mathbb{S}$ with the invariants $(\tau, \sigma, \iota) = (+,+,+)$; to avoid confusion with the elements of $\mathbb{S}$ we simply denote it by

$$\alpha \bullet \beta = (-1)^{[\frac{1}{2}(\deg(\alpha)+1)]} i_\omega(\alpha \wedge \beta)\,,$$

where $\alpha, \beta \in \mathbb{S}$, $[\cdot]$ is the "ceiling" of a rational number, i.e., its upper integer part, $\deg(\alpha)$ the degree of α as a differential form and $\omega \in \Lambda^4 W$ a fixed volume.

We depart with the KTS of type EIV. Throughout the section, we denote the unit constant in $\mathbb{S}$ by $\mathbb{1}$ and consider a basis $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^4})$ of W, with associated dual basis $(dx^1, \ldots, dx^4)$ of W^* and induced Hodge star operator $\star : \mathbb{S} \to \mathbb{S}$.

Theorem 6.10. *Let* U *be an* 8*-dimensional complex vector space with a non-degenerate symmetric bilinear form* η *and fix an isotropic decomposition* $U = W \oplus W^*$. *Then the associated* 16*-dimensional spinor representation* $\mathbb{S} \cong \Lambda^\bullet W^*$ *is a* K*-simple Kantor triple system with the triple product given by*

$$\begin{aligned} &(\mathbb{S}^\pm, \mathbb{S}^\mp, \mathbb{S}) = 0 \,, \\ &(\mathbb{1}, dx^{1234}, dx^i) = (dx^{1234}, \mathbb{1}, dx^{jkl}) = (dx^{jkl}, dx^i, dx^{1234}) = (dx^i, dx^{jkl}, \mathbb{1}) = 0 \,, \end{aligned} \tag{6.10}$$

and, for all other homogeneous differential forms, by

$$(\alpha, \beta, \gamma) = \begin{cases} (\alpha \bullet \beta)\gamma + (\beta \bullet \gamma)\alpha - (\alpha \bullet \gamma)\beta & \text{if} \quad \alpha, \beta, \gamma \in \mathbb{S}^\pm \\ (-1)^{[\frac{1}{2}(\deg(\alpha)+2)]} \star (\star(\alpha \wedge \gamma) \wedge \star\beta) & \text{if} \quad \alpha, \beta \in \mathbb{S}^\pm, \ \gamma \in \mathbb{S}^\mp, \\ & \qquad \deg(\alpha) + \deg(\gamma) \leqslant 4, \\ (-1)^{[\frac{1}{2}(\deg(\alpha)+1)]} \star (\star\alpha \wedge \star\gamma) \wedge \beta & \text{if} \quad \alpha, \beta \in \mathbb{S}^\pm, \ \gamma \in \mathbb{S}^\mp, \\ & \qquad \deg(\alpha) + \deg(\gamma) > 4. \end{cases} \tag{6.11}$$

The product is $\mathfrak{so}(U)$*-equivariant and its associated Tits-Kantor-Koecher Lie algebra is* E_6.

Proof. Identities (i) and (ii) of Definition 1.2 can be checked by tedious but straightforward computations, using the root space decomposition of E_6 and $\mathfrak{so}(U)$-equivariance of (6.10)-(6.11). We here simply record that $\mathfrak{so}(U)$ is generated by the two abelian parabolic subalgebras which exchange W and W^* so that $\mathfrak{so}(U)$-equivariance follows from the equivariance under the action of these subalgebras. The latter can be directly checked using (6.9). □

To proceed further, we first note that any KTS of type EI, EII, EIII is a *modification*, in Asano's sense [6], of the KTS of type EIV. In other words, we consider new triple products of the form

$$(\alpha, \beta, \gamma)_\Phi = (\alpha, \Phi(\beta), \gamma) \,, \tag{6.12}$$

where the product on the r.h.s. is the EIV product described in Theorem 6.10 and $\Phi : \mathbb{S} \to \mathbb{S}$ an involutive automorphism of it. It is known that a modification of a KTS is still a KTS.

Note that Φ has to be $\mathfrak{der}(\mathbb{S})$-equivariant, where $\mathfrak{der}(\mathbb{S})$ is detailed in (6.8) for all cases. Furthermore, since any endomorphism of $\mathbb{S}$ is realized by the action of some element in the Clifford algebra, we are led to consider $\mathfrak{der}(\mathbb{S})$-equivariant elements in $C\ell(U)$, which are also involutive automorphisms of the triple product of type EIV.

We introduce the required maps

$$\Phi(\alpha) = \begin{cases} \widetilde{\star}\alpha & \text{for} \quad \mathrm{EI}\,, \\ i(dx^1 \wedge (\widetilde{\star}\alpha) - \iota_{\frac{\partial}{\partial x^1}}(\widetilde{\star}\alpha)) & \text{for} \quad \mathrm{EII}\,, \\ i(dx^1 \wedge \alpha - \iota_{\frac{\partial}{\partial x^1}}\alpha) & \text{for} \quad \mathrm{EIII}\,, \end{cases} \tag{6.13}$$

where $\widetilde{\star}$ is the modified Hodge star operator given by $\widetilde{\star}\alpha = \epsilon(\alpha) \star \alpha$ with $\epsilon(\alpha) = -\alpha$ when $\deg(\alpha) = 1, 2$ and $\epsilon(\alpha) = \alpha$ when $\deg(\alpha) = 0, 3, 4$. It is easy to see that any Φ is the volume, up to some constant, of a non-degenerate subspace of U of appropriate dimension and hence realized as the action of an involutive and $\mathfrak{der}(\mathbb{S})$-equivariant element of the Pin group. As an aside, we note that conjugation by Φ in case EIII is nothing but the outer automorphism of $\mathfrak{so}(U)$ associated to the symmetry of the Dynkin diagram exchanging $\mathbb{S}^+$ and $\mathbb{S}^-$.

Theorem 6.11. *Let U be an 8-dimensional complex vector space with a non-degenerate symmetric bilinear form η and fix an isotropic decomposition $U = W \oplus W^*$. Then the associated 16-dimensional spinor representation $\mathbb{S} \cong \Lambda^\bullet W^*$ is a K-simple Kantor triple system with the modification*

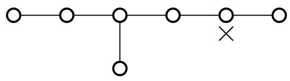
$$(\alpha, \beta, \gamma)_\Phi = (\alpha, \Phi(\beta), \gamma)\,, \qquad \alpha, \beta, \gamma \in \mathbb{S}\,, \tag{6.14}$$

of the triple product described in Theorem 6.10 by any of the three endomorphisms (6.13) of $\mathbb{S}$. Each product is $\mathfrak{der}(\mathbb{S})$-equivariant, see (6.8), and its associated Tits-Kantor-Koecher Lie algebra is E_6.

Proof. The KTS described in Theorem 6.10 is $\mathfrak{so}(U)$-equivariant, hence equivariant under the action of the connected component of the identity of $\mathrm{Pin}(U)$. It immediately follows that the map Φ in case EI is an automorphism of the KTS. A similar statement is directly checked for EIII and, consequently, for EII too. Hence, general properties of modifications apply. □

6.2.3 The case $\mathfrak{g} = E_7$

The grading



of $\mathfrak{g} = E_7$ is of extended Poincaré type. Let (U, η) be a 10-dimensional complex vector space with a non-degenerate symmetric bilinear form and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ the decomposition into semispinors of the 32-dimensional spinor $\mathfrak{so}(U)$-module $\mathbb{S}$. We have $\mathfrak{g}_0 \simeq \mathfrak{so}(U) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}E$, where E is the grading element, and

$$\mathfrak{g}_{-1} = \mathbb{S}^+ \boxtimes \mathbb{C}^2, \qquad \mathfrak{g}_1 = \mathbb{S}^- \boxtimes \mathbb{C}^2 ,$$
$$\mathfrak{g}_{-2} = U , \qquad \mathfrak{g}_2 = U ,$$

with their natural structure of $\mathfrak{g}_0$-modules. There are 3 compatible real forms EV, EVI, EVII, and the derivation algebras of the corresponding triple systems are respectively given by $\mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$, $\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{so}(9, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$.

We recall that in our convention (6.2) the Clifford algebra $C\ell(U)$ acts on

$$\mathbb{S} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k-\text{times}} , \qquad k = 5 ,$$

and that $\mathbb{S}^\pm$ are the ± 1-eigenspaces of the involution $i\,\mathrm{vol} = T \otimes \cdots \otimes T$. There is a unique (up to constant) admissible bilinear form $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ on $\mathbb{S}$ with invariants $(\tau, \sigma, \iota) = (-1, -1, -1)$. It is given by

$$\beta(x_1 \otimes \cdots \otimes x_5, y_1 \otimes \cdots \otimes y_5) = \omega(x_1, y_1) < x_2, y_2 > \cdots \omega(x_5, y_5) , \tag{6.15}$$

where $< -, - >$ (resp. ω) is the standard $\mathbb{C}$-linear product (resp. symplectic form) on $\mathbb{C}^2$ and $x_i, y_i \in \mathbb{C}^2$, $i = 1, \ldots, 5$. We note that semispinors $\mathbb{S}^\pm$ are isotropic and $(\mathbb{S}^\pm)^* \simeq \mathbb{S}^\mp$.

We consider $\mathfrak{so}(U)$-equivariant operators

$$\begin{aligned} &\Gamma : \mathbb{S} \otimes \mathbb{S} \to U , \\ &\eta(\Gamma(s,t), u) = \beta(u \circ s, t) , \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} &\Gamma^{(2)} : \mathbb{S} \otimes \mathbb{S} \to \mathfrak{so}(U) , \\ &\eta(\Gamma^{(2)}(s,t)u, v) = \beta(u \wedge v \circ s, t) , \end{aligned} \tag{6.17}$$

where $s, t \in \mathbb{S}$, $u, v \in U$. They are symmetric and satisfy $\Gamma(\mathbb{S}^\pm, \mathbb{S}^\mp) = \Gamma^{(2)}(\mathbb{S}^\pm, \mathbb{S}^\pm) = 0$.

The structure of graded Lie algebra on $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a mild variation of the usual Dirac current:

$$[s \otimes c, t \otimes d] = \Gamma(s,t)\omega(c,d) ,$$

where $s, t \in \mathbb{S}^+$, $c, d \in \mathbb{C}^2$. The Lie algebra $\mathfrak{g}_0$ acts naturally on $\mathfrak{m}$ by 0-degree derivations and we now in turn describe $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The general form of the adjoint action on $\mathfrak{m}$ of positive degree elements of $\mathfrak{g}$ is constrained due to $\mathfrak{g}_0$-equivariance: for any $r \in \mathbb{S}^-$, $c \in \mathbb{C}^2$ and $u \in U$, we have that $r \otimes c \in \mathfrak{g}_1$ and $\widehat{u} \in \mathfrak{g}_2$ act as

$$\begin{aligned}
[r \otimes c, t \otimes d] &= c_1 \underbrace{\Gamma^{(2)}(r,t)\omega(c,d)}_{\text{element of } \mathfrak{so}(U)} + c_2 \underbrace{\beta(r,t)\omega(c,d)E}_{\text{element of } \mathbb{C}E} + c_3 \underbrace{\beta(r,t)c \odot d}_{\text{element of } \mathfrak{sl}(2,\mathbb{C})} , \\
[r \otimes c, v] &= v \circ r \otimes c ,
\end{aligned} \tag{6.18}$$

and respectively

$$\begin{aligned}
[\widehat{u}, t \otimes d] &= -u \circ t \otimes d , \\
[\widehat{u}, v] &= c_4 \underbrace{u \wedge v}_{\text{element of } \mathfrak{so}(U)} + c_5 \underbrace{\eta(u,v)E}_{\text{element of } \mathbb{C}E} ,
\end{aligned} \tag{6.19}$$

for all $t \in \mathbb{S}^+$, $d \in \mathbb{C}^2$ and $v \in U$. The values of the constants $c_1, \ldots, c_5$ are now determined.

Proposition 6.12. $c_1 = -1$, $c_2 = \frac{1}{2}$, $c_3 = -1$ *and* $c_4 = 2$, $c_5 = -1$.

Proof. We let $r \in \mathbb{S}^-$, $t \in \mathbb{S}^+$, $u \in U$, $c, d \in \mathbb{C}^2$ as above and depart with

$$\begin{aligned}
0 &= [r \otimes c, [t \otimes d, u]] \\
&= [[r \otimes c, t \otimes d], u] + [t \otimes d, [r \otimes c, u]] \\
&= \omega(c,d)(c_1 \Gamma^{(2)}(r,t)u - 2c_2 \beta(r,t)u - \Gamma(t, u \circ r)) .
\end{aligned}$$

Abstracting $\omega(c,d)$ and taking the inner product with any $v \in U$ yields

$$\begin{aligned}
0 &= c_1 \beta(u \wedge v \circ r, t) - 2c_2 \beta(r,t)\eta(u,v) + \beta(u \circ v \circ t, r) \\
&= (c_1 + 1)\beta(u \wedge v \circ r, t) + (-2c_2 + 1)\beta(r,t)\eta(u,v) ,
\end{aligned}$$

where we used $u \circ v = u \wedge v - \eta(u,v)$. Since $\mathbb{S}^+ \simeq (\mathbb{S}^-)^*$ and $\mathrm{End}(\mathbb{S}^-) \simeq \Lambda^4 U \oplus \Lambda^2 U \oplus \Lambda^0 U$ acts faithfully on $\mathbb{S}^-$, the two terms vanish separately and the values of c_1 and c_2 follow.

The proof of $c_3 = -1$ relies on the explicit realization (6.2) of the Clifford algebra. Let $r \in \mathbb{S}^-$, $t \in \mathbb{S}^+$ and choose $c, d \in \mathbb{C}^2$ satisfying $\omega(c,d) = 1$, then

$$[r \otimes d, [t \otimes c, t \otimes d]] = \Gamma(t,t) \circ r \otimes d$$

and

$$\begin{aligned}
[r \otimes d, [t \otimes c, t \otimes d]] &= [[r \otimes d, t \otimes c], t \otimes d] + [t \otimes c, [r \otimes d, t \otimes d]] \\
&= -c_1 \Gamma^{(2)}(r,t) \cdot t \otimes d + c_2 \beta(r,t) t \otimes d + 3c_3 \beta(r,t) t \otimes d .
\end{aligned}$$

Abstracting d, substituting the values of c_1 and c_2 already determined and rearranging terms, we are left with the Fierz-like identity

$$\Gamma(t,t)\circ r = \Gamma^{(2)}(r,t)\cdot t + \beta(r,t)(\tfrac{1}{2}+3c_3)t\ . \tag{6.20}$$

We now choose suitable spinors

$$t = \begin{pmatrix}1\\+i\end{pmatrix}^{\otimes 5}\ ,\qquad r = \begin{pmatrix}1\\-i\end{pmatrix}^{\otimes 5}$$

and use (6.2) and (6.15) to get $\beta(r,t) = 32i$ and $\Gamma(t,t) = 0$. A similar but longer computation says that

$$\begin{aligned}\Gamma^{(2)}(r,t) &= -\frac{1}{2}\sum_{l,m}\beta(r, e_l\wedge e_m\circ t)e_l\wedge e_m\\ &= -\sum_{l\ \mathrm{odd}}\beta(r, e_l\wedge e_{l+1}\circ t)e_l\wedge e_{l+1}\\ &= i\sum_{l\ \mathrm{odd}}\beta(r,t)e_l\wedge e_{l+1}\\ &= -32\sum_{l\ \mathrm{odd}} e_l\wedge e_{l+1}\ ,\end{aligned}$$

from which $\Gamma^{(2)}(r,t)\cdot t = i80t$. In summary, identity (6.20) turns into $0 = i80t + i32(\frac{1}{2}+3c_3)t$, so that $c_3 = -1$. The values of the constants c_4, c_5 relative to the adjoint action of $\mathfrak{g}_2$ on $\mathfrak{m}$ are determined by similar but easier computations, which we omit. □

The description of the Lie brackets of $\mathfrak{g} = E_7$ is completed, with the exception of the Lie bracket of two elements $r\otimes c, s\otimes d \in \mathfrak{g}_1 = \mathbb{S}^-\boxtimes\mathbb{C}^2$, which is

$$[r\otimes c, s\otimes d] = \omega(c,d)\widehat{\Gamma(r,s)}\ .$$

To prove this identity, it is sufficient to note that elements of $\mathfrak{g}_2$ are fully determined by their adjoint action on $\mathfrak{g}_{-2}$ by (6.19) and check that the l.h.s. and r.h.s. yield the same result there.

Theorem 6.13 gives 3 different KTS associated with the graded Lie algebra just described.

Theorem 6.13. *Let U be a 10-dimensional complex vector space with a non-degenerate symmetric bilinear form η and $\mathbb{S} = \mathbb{S}^+\oplus\mathbb{S}^-$ the associated 32-dimensional spinor module. Let $\beta : \mathbb{S}\otimes\mathbb{S}\to\mathbb{C}$ be the admissible bilinear form on $\mathbb{S}$ with invariants $(\tau,\sigma,\iota) = (-1,-1,-1)$ and*

$$\Gamma : \mathbb{S}\otimes\mathbb{S}\to U\ ,\qquad \Gamma^{(2)} : \mathbb{S}\otimes\mathbb{S}\to\mathfrak{so}(U)\ ,$$

the $\mathfrak{so}(U)$*-equivariant operators defined in* (6.16)-(6.17)*. Fix an orthogonal decomposition*

$$U = W \oplus W^{\perp}$$

of U *with* $\dim W = 1$*,* $\dim W = 3$ *or* $\dim W = 5$ *and let* $I = \mathrm{vol}_W \in C\ell(U)$ *be the volume of* W*. (*I *acts on* $\mathbb{S}$ *as a complex structure if* $\dim W = 1, 5$ *and as a paracomplex structure when* $\dim W = 3$*.) Let* $J : \mathbb{C}^2 \to \mathbb{C}^2$ *be the standard complex structure on* $\mathbb{C}^2$*, except for* $\dim W = 3$ *where we set* $J := \mathrm{Id}$*.*

Then $\mathbb{S}^+ \otimes \mathbb{C}^2$ *with the triple product*

$$\begin{aligned}(xyz) = (-\Gamma^{(2)}(r, I \circ s) \cdot t + \tfrac{1}{2}\beta(r, I \circ s)t) \otimes \omega(b, Jc)d \\ - \beta(r, I \circ s)t \otimes (\omega(b, d)Jc + \omega(Jc, d)b)\ ,\end{aligned} \tag{6.21}$$

for all elements $x = r \otimes b$*,* $y = s \otimes c$ *and* $z = t \otimes d$ *of* $\mathbb{S}^+ \otimes \mathbb{C}^2$*, is a* $\mathbb{K}$*-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra* $\mathfrak{g} = E_7$ *and derivation algebra*

$$\mathfrak{der}(\mathbb{S}^+ \otimes \mathbb{C}^2) = \mathfrak{stab}_{\mathfrak{g}_0}(I, J) \simeq \begin{cases} \mathfrak{so}(9, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C}) \ \textit{if} \ \dim W = 1\ , \\ \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \ \textit{if} \ \dim W = 3\ , \\ \mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C}) \ \textit{if} \ \dim W = 5\ . \end{cases}$$

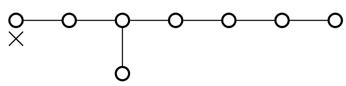
Proof. It follows from the fact that the involution

$$\begin{aligned} &\sigma(\widehat{u}) = r_W u\ , && \sigma(r \otimes d) = I \circ r \otimes Jd\ , \\ &\sigma(A) = \begin{cases} r_W A r_W \text{ if } A \in \mathfrak{so}(U) \\ JAJ^{-1} \text{ if } A \in \mathfrak{sl}(2, \mathbb{C}) \end{cases}, && \sigma(E) = -E\ , \\ &\sigma(s \otimes c) = I \circ s \otimes Jd\ , && \sigma(u) = \widehat{r_W u}\ , \end{aligned} \tag{6.22}$$

where $u \in U$, $r \otimes d \in \mathbb{S}^- \boxtimes \mathbb{C}^2$, $s \otimes c \in \mathbb{S}^+ \boxtimes \mathbb{C}^2$, is a Lie algebra morphism. The proof is analogous to that of Theorem 6.9, making use of the explicit expressions of the Lie brackets of $\mathfrak{g} = E_7$, the fact that $I \in \mathrm{Pin}(U)$ covers the opposite of the orthogonal reflection $r_W : U \to U$ across W and the identity $\beta(I \circ s, I \circ t) = \beta(s, t)$ for all $s, t \in \mathbb{S}$. □

6.2.4 The case $\mathfrak{g} = E_8$

The grading



of $\mathfrak{g} = E_8$ is of extended Poincaré type, with $\mathfrak{g}_0 = \mathfrak{so}(U) \oplus \mathbb{C}E$ and the other graded components given by $\mathfrak{g}_{\pm 1} = \mathbb{S}^{\mp}$, $\mathfrak{g}_{\pm 2} = U$. Here (U, η) is a 14-dimensional complex vector space with a non-degenerate symmetric bilinear form and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ the decomposition into semispinors of the corresponding 128-dimensional spinor $\mathfrak{so}(U)$-module $\mathbb{S}$.

There is an admissible bilinear form $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ with invariants $(\tau, \sigma, \iota) = (-1, +1, -1)$ and usual operators

$$\begin{aligned} &\Gamma : \mathbb{S} \otimes \mathbb{S} \to U\ , \\ &\eta(\Gamma(s,t), u) = \beta(u \circ s, t)\ , \end{aligned} \tag{6.23}$$

and

$$\begin{aligned} &\Gamma^{(2)} : \mathbb{S} \otimes \mathbb{S} \to \mathfrak{so}(U)\ , \\ &\eta(\Gamma^{(2)}(s,t)u, v) = \beta(u \wedge v \circ s, t)\ , \end{aligned} \tag{6.24}$$

$s, t \in \mathbb{S}$, $u, v \in U$. We note that $(\mathbb{S}^{\pm})^* \simeq \mathbb{S}^{\mp}$ and that (6.23)-(6.24) are both $\mathfrak{so}(U)$-equivariant and skewsymmetric. They also satisfy $\Gamma(\mathbb{S}^{\pm}, \mathbb{S}^{\mp}) = \Gamma^{(2)}(\mathbb{S}^{\pm}, \mathbb{S}^{\pm}) = 0$.

The arguments used in Subsection 6.2.1 for F_4 extend almost verbatim to E_8. In particular the non-trivial Lie brackets of $\mathfrak{g} = E_8$ are given by the natural action of $\mathfrak{g}_0$ on each graded component and

$$\begin{aligned} &[s_1, s_2] = \Gamma(s_1, s_2)\ , && [t_1, v] = v \circ t_1\ , && [t_1, s_1] = -\Gamma^{(2)}(t_1, s_1) + \tfrac{1}{2}\beta(t_1, s_1)E\ , \\ &[\widehat{u}, v] = 2u \wedge v - \eta(u, v)E\ , && [\widehat{u}, s_1] = -u \circ s_1\ , && [t_1, t_2] = \widehat{\Gamma(t_1, t_2)}\ , \end{aligned}$$

for all $s_1, s_2 \in \mathfrak{g}_{-1} \simeq \mathbb{S}^+$, $t_1, t_2 \in \mathfrak{g}_{+1} \simeq \mathbb{S}^-$, $v \in \mathfrak{g}_{-2} \simeq U$ and $\widehat{u} \in \mathfrak{g}_2 \simeq U$. The proof of the following result is similar to the proof of Theorem 6.9 and therefore omitted.

Theorem 6.14. *Let U be a 14-dimensional complex vector space with a non-degenerate symmetric bilinear form η and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ the associated 128-dimensional spinor module. Let $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ be the unique (up to constant) admissible bilinear form on $\mathbb{S}$ with invariants $(\tau, \sigma, \iota) = (-1, +1, -1)$ and (6.23)-(6.24) the naturally associated operators. Fix an orthogonal decomposition*

$$U = W \oplus W^{\perp}$$

of U with $\dim W = 3$ or $\dim W = 7$ and let $I = \mathrm{vol}_W \in C\ell(U)$ be the volume of W. (I acts on $\mathbb{S}$ as a paracomplex structure.) Then $\mathbb{S}^+$ with the triple product

$$(rst) = -\Gamma^{(2)}(r, I \circ s) \cdot t + \frac{1}{2}\beta(r, I \circ s)t\ , \qquad r, s, t \in \mathbb{S}^+\ , \tag{6.25}$$

is a K-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra $\mathfrak{g} = E_8$ and derivation algebra

$$\mathfrak{der}(\mathbb{S}^+) = \mathfrak{stab}_{\mathfrak{so}(U)}(I) \simeq \begin{cases} \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(11, \mathbb{C}) \text{ if } \dim W = 3\ , \\ \mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{so}(7, \mathbb{C}) \text{ if } \dim W = 7\ . \end{cases}$$

6.3 The exceptional Kantor triple systems of contact type

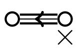
6.3.1 The case $\mathfrak{g} = G_2$

We now determine the Kantor triple system V with Tits-Kantor-Koecher pair $(\mathfrak{g} = G_2, \sigma)$, see also Example 5.14. To this aim, we consider a symplectic form ω on $\mathbb{C}^2$ and a compatible complex structure $J : \mathbb{C}^2 \to \mathbb{C}^2$, i.e., such that $\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in \mathbb{C}^2$.

Theorem 6.15. *The vector space $S^3\mathbb{C}^2$ with the triple product given by*

$$(x^3y^3z^3) = -\frac{3}{2}(\omega(x, Jy))^2(\omega(x, z)Jy \odot z^2 + \omega(Jy, z)x \odot z^2) + \frac{1}{2}(\omega(x, Jy))^3z^3$$

for all $x, y, z \in \mathbb{C}^2$ is a K-simple Kantor triple system. Its associated Tits-Kantor-Koecher pair $(\mathfrak{g}, \sigma)$ is the unique pair with $\mathfrak{g} = G_2$ and its derivation algebra is $\mathfrak{der}(S^3\mathbb{C}^2) = \mathfrak{so}(2, \mathbb{C})$.

The proof of Theorem 6.15 will occupy the remaining part of the section. We first recall that the unique fundamental 5-grading of G_2 is associated with the marked Dynkin diagram , with $\mathfrak{g}_0 \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}E$, where E is the grading element, and

$$\mathfrak{g}_{\pm 2} \simeq \mathbb{C} , \qquad \mathfrak{g}_{\pm 1} \simeq S^3\mathbb{C}^2 ,$$

as $\mathfrak{sl}(2, \mathbb{C})$-modules, by a routine examination of the roots of G_2. We fix a basis $\mathbb{1}$ of $\mathfrak{g}_{-2}$ and denote decomposable elements of $\mathfrak{g}_{-1}$ by x^3, y^3, where $x, y \in \mathbb{C}^2$. The negatively graded part $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$ has non-trivial Lie brackets

$$[x^3, y^3] = (\omega(x, y))^3\mathbb{1} , \tag{6.26}$$

and $\mathfrak{g}_0$ acts naturally on $\mathfrak{m}$ by 0-degree derivations. In particular $\mathfrak{m}$ can be extended to the non-positively $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}_{\leqslant 0} = \mathfrak{m} \oplus \mathfrak{g}_0$ and it is well known that G_2 is precisely the maximal transitive prolongation of $\mathfrak{g}_{\leqslant 0}$, see e.g. [57].

For any $x^3 \in S^3\mathbb{C}^2$ we introduce a linear map $\widehat{x}^3 : \mathfrak{m} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ of degree 1 by

$$\begin{aligned} &[\widehat{x}^3, y^3] = \tfrac{1}{2}(\omega(x, y))^2 \underbrace{(x \otimes \omega(y, -) + y \otimes \omega(x, -))}_{\text{element of } \mathfrak{sl}(2,\mathbb{C})} - \tfrac{1}{2}(\omega(x, y))^3E , \\ &[\widehat{x}^3, \mathbb{1}] = x^3 , \end{aligned} \tag{6.27}$$

where $y^3 \in \mathfrak{g}_{-1}$. A straightforward computation tells us that $\widehat{x}^3$ is an element of the first prolongation $\mathfrak{g}_1$ of $\mathfrak{g}_{\leqslant 0}$ for all $x^3 \in S^3\mathbb{C}^2$, therefore $\mathfrak{g}_1 = \langle \widehat{x}^3 \,|\, x^3 \in S^3\mathbb{C}^2 \rangle$. Similarly $\mathfrak{g}_2 = \mathbb{C}\widehat{\mathbb{1}}$, where $\widehat{\mathbb{1}} : \mathfrak{m} \to \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is given by

$$[\widehat{\mathbb{1}}, y^3] = \widehat{y}^3 , \qquad [\widehat{\mathbb{1}}, \mathbb{1}] = -E , \tag{6.28}$$

where $y^3 \in \mathfrak{g}_{-1}$. The remaining Lie brackets of G_2 can be directly computed using transitivity and (6.27)-(6.28). They are given by the natural structure of Lie algebra of $\mathfrak{g}_0$ and

$$\begin{aligned} &[A, \widehat{y}^3] = \widehat{[A, y^3]}\ , \quad [E, \widehat{y}^3] = \widehat{y}^3\ , \quad [\widehat{x}^3, \widehat{y}^3] = (\omega(x,y))^3 \widehat{\mathbb{1}}\ , \\ &[A, \widehat{\mathbb{1}}] = 0\ , \qquad\quad\ \ [E, \widehat{\mathbb{1}}] = 2\widehat{\mathbb{1}}\ , \end{aligned} \tag{6.29}$$

where $A \in \mathfrak{sl}(2, \mathbb{C})$ and $\widehat{x}^3, \widehat{y}^3 \in \mathfrak{g}_1$.

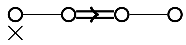
Now we define a linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ by

$$\begin{aligned} &\sigma(\mathbb{1}) = \widehat{\mathbb{1}}\ , &&\sigma(x^3) = \widehat{Jx^3}\ , \\ &\sigma(A) = -A^t\ , &&\sigma(E) = -E\ , \\ &\sigma(\widehat{x}^3) = -Jx^3\ , &&\sigma(\widehat{\mathbb{1}}) = \mathbb{1}\ , \end{aligned} \tag{6.30}$$

where we denoted the natural extension of J to a complex structure on $S^3\mathbb{C}^2$ by the same symbol. Using (6.26)-(6.29), one sees that (6.30) is a Lie algebra morphism, therefore a grade-reversing involution of G_2.

The associated triple product is given by the usual formula and it is invariant under the action of the stabilizer $\mathfrak{stab}_{\mathfrak{sl}(2,\mathbb{C})}(J) = \mathfrak{so}(2, \mathbb{C})$ of J in $\mathfrak{sl}(2, \mathbb{C})$ – this is indeed the associated algebra of derivations, see Example 5.14.

6.3.2 The case $\mathfrak{g} = F_4$

The grading with marked Dynkin diagram  admits one grade-reversing involution, the complexification of the Cartan involution of the split real form, see Table 9 of [47, p.312-317]. By Theorem 5.13, the derivation algebra of the associated KTS is the complexification of the maximal compact subalgebra $\mathfrak{u}(3)$ of $\mathfrak{sp}(6, \mathbb{R})$, i.e. $\mathfrak{gl}(3, \mathbb{C})$.

To describe the triple system, we consider a symplectic form ω on $\mathbb{C}^6$ and set $V = \Lambda^3_0\mathbb{C}^6$ to be the space of primitive 3-forms (kernel of natural contraction $\iota_\omega : \Lambda^3\mathbb{C}^6 \to \mathbb{C}^6$ by ω). Let $J : \mathbb{C}^6 \to \mathbb{C}^6$ be a complex structure on $\mathbb{C}^6$ compatible with ω and denote the natural extension to a complex structure on V by the same symbol.

We will tacitly identify $S^2\mathbb{C}^6$ with $\mathfrak{sp}(6, \mathbb{C})$ by means of the symplectic form

$$x \odot y : z \mapsto \omega(x,z)y + \omega(y,z)x\ ,$$

where $x, y, z \in \mathbb{C}^6$, and introduce $\mathfrak{sp}(6, \mathbb{C})$-equivariant operators

$$\bullet : \Lambda^2 V \to \mathbb{C}\ , \tag{6.31}$$

$$\vee : S^2 V \to \mathfrak{sp}(6, \mathbb{C})\ , \tag{6.32}$$

on V. The first operator is the non-degenerate skewsymmetric bilinear form on V given by

$$\alpha \wedge \beta = (\alpha \bullet \beta)\,\mathrm{vol}\ ,$$

where $\alpha, \beta \in V$ and $\mathrm{vol} = \frac{1}{3!}\omega^3 \in \Lambda^6\mathbb{C}^6$ is the normalized volume. The second operator is symmetric on V and given by (a multiple of) the projection to $S^2\mathbb{C}^6$ w.r.t. the decomposition

$$S^2V \simeq (V \circledcirc V) \oplus S^2\mathbb{C}^6$$

into $\mathfrak{sp}(6,\mathbb{C})$-irreducible submodules of S^2V. In our conventions this operator is normalized so that

$$\alpha \vee \beta = \sum_{i=1}^{3} p_i \odot q_i$$

when $\alpha = p_1 \wedge p_2 \wedge p_3$ and $\beta = q_1 \wedge q_2 \wedge q_3$, where $\{p_i, q_i \mid i = 1,2,3\}$ is a fixed basis of $\mathbb{C}^6$ satisfying $\omega(p_i,p_j) = \omega(q_i,q_j) = 0$, $\omega(p_i,q_j) = \delta_{ij}$.

Theorem 6.16. *The vector space* $V = \Lambda^3_0\mathbb{C}^6$ *with the triple product given by*

$$(\alpha\beta\gamma) = \frac{1}{2}(\alpha \bullet J\beta)\gamma + \frac{1}{2}(\alpha \vee J\beta)\cdot\gamma$$

for all $\alpha, \beta, \gamma \in V$ *is a* K*-simple Kantor triple system with derivation algebra* $\mathfrak{der}(V) = \mathfrak{stab}_{\mathfrak{sp}(6,\mathbb{C})}(J) \simeq \mathfrak{gl}(3,\mathbb{C})$ *and Tits-Kantor-Koecher Lie algebra* $\mathfrak{g} = F_4$.

Proof. The graded components of the 5-grading of $\mathfrak{g} = F_4$ are $\mathfrak{g}_0 \simeq \mathfrak{sp}(6,\mathbb{C}) \oplus \mathbb{C}E$ and $\mathfrak{g}_{\pm 2} \simeq \mathbb{C}$, $\mathfrak{g}_{\pm 1} \simeq V$ as $\mathfrak{sp}(6,\mathbb{C})$-modules. The negatively graded part $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$ has non-trivial Lie brackets

$$[\alpha, \beta] = (\alpha \bullet \beta)\mathbb{1}\ , \tag{6.33}$$

where $\mathbb{1}$ is a fixed basis of $\mathfrak{g}_{-2}$. In particular $\mathfrak{m}$ can be extended to the non-positively $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}_{\leqslant 0} = \mathfrak{m} \oplus \mathfrak{g}_0$ and F_4 is the maximal transitive prolongation of $\mathfrak{g}_{\leqslant 0}$ [57].

For any $\alpha \in V$ we introduce a linear map $\widehat{\alpha} : \mathfrak{m} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ by

$$\begin{aligned} [\widehat{\alpha}, \beta] &= \lambda\alpha \vee \beta + \mu(\alpha \bullet \beta)E\ , \\ [\widehat{\alpha}, \mathbb{1}] &= \alpha\ , \end{aligned} \tag{6.34}$$

for all $\beta \in \mathfrak{g}_{-1}$, where $\lambda, \mu \in \mathbb{C}$ are constants to be determined imposing $\widehat{\alpha} \in \mathfrak{g}_1$:

$$\begin{aligned} 0 = [\widehat{\alpha}, [\beta, \mathbb{1}]] &= [[\widehat{\alpha}, \beta], \mathbb{1}] + [\beta, [\widehat{\alpha}, \mathbb{1}]] \\ &= \mu(\alpha \bullet \beta)[E, \mathbb{1}] + [\beta, \alpha] \\ &= (-2\mu - 1)(\alpha \bullet \beta)\mathbb{1}\ . \end{aligned}$$

Hence $\mu = -\frac{1}{2}$ and a similar computation with $\widehat{\alpha}$ acting on $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$ yields $\lambda = -\frac{1}{2}$. In other words we just showed that $\mathfrak{g}_1 = \langle \widehat{\alpha} \,|\, \alpha \in V \rangle$ when $\mu = \lambda = -\frac{1}{2}$; similarly $\mathfrak{g}_2 = \mathbb{C}\widehat{\mathbb{1}}$, where $\widehat{\mathbb{1}} : \mathfrak{m} \to \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is given by

$$[\widehat{\mathbb{1}}, \beta] = \widehat{\beta} \ , \qquad [\widehat{\mathbb{1}}, \mathbb{1}] = -E \ , \tag{6.35}$$

for all $\beta \in \mathfrak{g}_{-1}$. The remaining Lie brackets of F_4 can be directly computed using transitivity and (6.34)-(6.35). They are given by the natural structure of Lie algebra of $\mathfrak{g}_0$ and

$$\begin{aligned} &[A, \widehat{\alpha}] = \widehat{[A, \alpha]} \ , \quad [E, \widehat{\alpha}] = \widehat{\alpha} \ , \quad [\widehat{\alpha}, \widehat{\beta}] = (\alpha \bullet \beta)\widehat{\mathbb{1}} \ , \\ &[A, \widehat{\mathbb{1}}] = 0 \ , \qquad\quad [E, \widehat{\mathbb{1}}] = 2\widehat{\mathbb{1}} \ , \end{aligned} \tag{6.36}$$

where $A \in \mathfrak{sp}(6, \mathbb{C})$ and $\widehat{\alpha}, \widehat{\beta} \in \mathfrak{g}_1$.

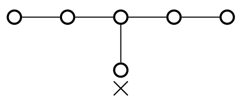
Using (6.33)-(6.36), one can directly check that the map $\sigma : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\begin{aligned} &\sigma(\mathbb{1}) = \widehat{\mathbb{1}} \ , \qquad \sigma(\alpha) = \widehat{J\alpha} \ , \\ &\sigma(A) = -A^t \ , \quad \sigma(E) = -E \ , \\ &\sigma(\widehat{\alpha}) = -J\alpha \ , \quad \sigma(\widehat{\mathbb{1}}) = \mathbb{1} \ , \end{aligned} \tag{6.37}$$

is a grade-reversing involution of F_4. □

6.3.3 The case $\mathfrak{g} = E_6$

The KTS with Tits-Kantor-Koecher pair $(\mathfrak{g} = E_6, \sigma)$ are 8, associated with 3 gradings. We already saw 4 of them in Subsection 6.2.2 and we study here those associated with the grading of contact type



There are 3 non-equivalent grade-reversing involutions, related to the real forms EI, EII and EIII, with derivation algebras respectively $\mathfrak{so}(6, \mathbb{C})$, $(\mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})) \oplus \mathbb{C}$ and $\mathfrak{sl}(5, \mathbb{C}) \oplus \mathbb{C}$.

Let $V = \Lambda^3\mathbb{C}^6$ and $\mathrm{vol} \in \Lambda^6\mathbb{C}^6$ a fixed volume. As for the case F_4 in Subsection 6.3.2 we introduce the $\mathfrak{sl}(6, \mathbb{C})$-equivariant operators

$$\begin{aligned} \bullet \quad &: \Lambda^2 V \to \mathbb{C} \ , \\ \vee \quad &: S^2 V \to \mathfrak{sl}(6, \mathbb{C}) \end{aligned}$$

with the $\bullet$ defined via wedge product as in Subsection 6.3.2 and $\vee$ defined by

$$(\alpha \vee \beta)(x) = i_{\alpha\bullet}(\beta \wedge x) + i_{\beta\bullet}(\alpha \wedge x),$$

for all $\alpha, \beta \in V$, where $i_{\alpha\bullet}$ is the contraction by $\alpha\bullet \in V^* \cong \Lambda^3(\mathbb{C}^6)^*$. We denote by M_1, M_2, the endomorphism of $\mathbb{C}^6$ represented by the matrices

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.38}$$

and by η the scalar product on $\mathbb{C}^6$ of signature $(+-+-+-)$. We let $\star : \Lambda^p\mathbb{C}^6 \to \Lambda^{6-p}\mathbb{C}^6$ be the corresponding Hodge star operator.

Theorem 6.17. *The vector space $V = \Lambda^3\mathbb{C}^6$ with triple product*

$$(\alpha\beta\gamma) = \frac{1}{2}(\alpha \bullet \star\beta)\gamma - \frac{1}{2}(\alpha \vee \star\beta)\cdot\gamma$$

for all $\alpha, \beta, \gamma \in V$ is a K-simple Kantor triple system with derivation algebra $\mathfrak{der}(V) = \mathfrak{stab}_{\mathfrak{sl}(6,\mathbb{C})}(\eta) \simeq \mathfrak{so}(6,\mathbb{C})$ and Tits-Kantor-Koecher Lie algebra $\mathfrak{g} = E_6$.

Proof. The graded components of $\mathfrak{g}$ are $\mathfrak{g}_0 \simeq \mathfrak{sl}(6,\mathbb{C}) \oplus \mathbb{C}E$, $\mathfrak{g}_{\pm1} \simeq V$ and $\mathfrak{g}_{\pm2} \simeq \mathbb{C}$. Direct arguments similar to those made for the contact grading of F_4 show that the Lie brackets are given by the natural action of the Lie algebra $\mathfrak{g}_0$ and

$$\begin{aligned} &[\alpha,\beta] = (\alpha\bullet\beta)\mathbb{1}\,, && [\widehat{\alpha},\widehat{\beta}] = (\alpha\bullet\beta)\widehat{\mathbb{1}}\,,\\ &[\widehat{\alpha},\beta] = \tfrac{1}{2}\alpha\vee\beta - \tfrac{1}{2}(\alpha\bullet\beta)E\,, && [\widehat{\alpha},\mathbb{1}] = \alpha\,,\\ &[\widehat{\mathbb{1}},\beta] = \widehat{\beta}\,, && [\widehat{\mathbb{1}},\mathbb{1}] = -E\,,\\ &[A,\widehat{\alpha}] = \widehat{[A,\alpha]}\,, && [A,\widehat{\mathbb{1}}] = 0\,, \end{aligned}$$

with $\alpha, \beta \in \mathfrak{g}_{-1}$, $\widehat{\alpha}, \widehat{\beta} \in \mathfrak{g}_1$, $\mathbb{1} \in \mathfrak{g}_{-2}$, $\widehat{\mathbb{1}} \in \mathfrak{g}_2$ and $A \in \mathfrak{sl}(6,\mathbb{C})$. The grade reversing involution is given by

$$\begin{aligned} &\sigma(\mathbb{1}) = \widehat{\mathbb{1}}\,, && \sigma(\alpha) = \widehat{\star\alpha}\,,\\ &\sigma(A) = -A^t\,, && \sigma(E) = -E\,,\\ &\sigma(\widehat{\alpha}) = \star\alpha\,, && \sigma(\widehat{\mathbb{1}}) = \mathbb{1}\,. \end{aligned} \tag{6.39}$$

and the triple product follows at once. □

Theorem 6.18. *The vector space $V = \Lambda^3\mathbb{C}^6$ with triple product*

$$(\alpha\beta\gamma) = \frac{1}{2}(\alpha \bullet M\beta)\gamma - \frac{1}{2}(\alpha \vee M\beta)\cdot\gamma$$

for all $\alpha, \beta, \gamma \in V$ and $M = M_1$ or M_2 as in (6.38) is a K-simple Kantor triple system with

$$\mathfrak{der}(V) = \mathfrak{stab}_{\mathfrak{sl}(6,\mathbb{C})}(M) \simeq \begin{cases} \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathbb{C} & \text{if } M = M_1 \\ \mathfrak{sl}(5,\mathbb{C}) \oplus \mathbb{C} & \text{if } M = M_2 \end{cases}$$

and Tits-Kantor-Koecher Lie algebra $\mathfrak{g} = E_6$.

Proof. The restriction of the involution σ to $\mathfrak{sl}(6,\mathbb{C})$ is either the complexification of the Cartan involution of $\mathfrak{su}(3,3)$ or $\mathfrak{su}(1,5)$. By [46], one has that $\sigma(A) = MAM^{-1}$ for any $A \in \mathfrak{sl}(6,\mathbb{C})$, where $M \in GL(6,\mathbb{C})$ is conjugated to either $\mathrm{Diag}(\mathrm{Id}_3, -\mathrm{Id}_3)$ or $\mathrm{Diag}(\mathrm{Id}_5, -1)$. In the first case we set $M = M_1$ and in the second $M = M_2$.

If we extend the natural action of M on $\mathfrak{g}_{\pm 1}$ to a grade-reversing involution of $\mathfrak{g}$ then necessarily

$$\begin{aligned}\sigma[\widehat{\alpha}, \beta] = [\sigma(\widehat{\alpha}), \sigma(\beta)] &= [M\alpha, \widehat{M\beta}] = -\tfrac{1}{2}(M\alpha \bullet M\beta)E - \tfrac{1}{2}(M\alpha \vee M\beta)\\ &= \tfrac{1}{2}(\alpha \bullet \beta)E + \tfrac{1}{2}M(\alpha \vee \beta)M\ ,\end{aligned}$$

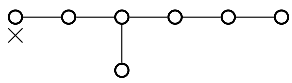
where we used that $(M\alpha \bullet M\beta) = -(\alpha \bullet \beta)$ and $(M\alpha \vee M\beta) = -M(\alpha \vee \beta)M$. In particular $\sigma(\alpha \vee \beta) = M(\alpha \vee \beta)M^{-1}$, hence $\sigma(A) = MAM^{-1}$ for all $A \in \mathfrak{sl}(6,\mathbb{C})$ and we arrive at the involution

$$\begin{aligned} &\sigma(\mathbb{1}) = -\widehat{\mathbb{1}}\ , &&\sigma(\alpha) = \widehat{M\alpha}\ ,\\ &\sigma(A) = MAM^{-1}\ , &&\sigma(E) = -E\ ,\\ &\sigma(\widehat{\alpha}) = M\alpha\ , &&\sigma(\widehat{\mathbb{1}}) = -\mathbb{1}\ , \end{aligned} \tag{6.40}$$

The claim on the algebra of derivations follows from $\mathfrak{stab}_{\mathfrak{sl}(6,\mathbb{C})}(M) \simeq \mathfrak{sl}(k,\mathbb{C}) \oplus \mathfrak{sl}(6-k,\mathbb{C}) \oplus \mathbb{C}$ where $k = 3, 1$ for M is conjugate to M_1 and M_2, respectively. □

6.3.4 The case $\mathfrak{g} = E_7$

The Kantor triple systems with Tits-Kantor-Koecher pair $(\mathfrak{g} = E_7, \sigma)$ are 7, associated to 3 different 5-gradings. The contact grading



of E_7 admits 3 grade-reversing involutions, corresponding to the real forms EV, EVI and EVII, with derivation algebras respectively $\mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{so}(6,\mathbb{C})$, $\mathfrak{gl}(6,\mathbb{C})$ and $\mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C})$.

The corresponding systems are close to those of extended Poincaré type studied in Section 6.2. Indeed $\mathfrak{g}_0 \simeq \mathfrak{so}(U) \oplus \mathbb{C}E$ and $\mathfrak{g}_{\pm 2} \simeq \mathbb{C}$, $\mathfrak{g}_{\pm 1} \simeq \mathbb{S}^+$ as $\mathfrak{so}(U)$-modules. Here (U, η) is a 12-dimensional complex vector space U with a non-degenerate symmetric bilinear form η and

$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$$

the decomposition into semispinors of the 64-dimensional spinor $\mathfrak{so}(U)$-module $\mathbb{S}$.

In this section, we will make extensive use of the explicit realization (6.2) of the Clifford algebra $C\ell(U)$ as endomorphisms of

$$\mathbb{S} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k-\text{times}} , \qquad k = 6 .$$

We note that in our conventions $\mathbb{S}^{\pm}$ are the ± 1-eigenspaces of the involution

$$i^k \,\mathrm{vol} = T \otimes \cdots \otimes T ,$$

that is the opposite of vol, as $k = 6$. There exists an admissible bilinear form $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ with the invariants $(\tau, \sigma, \iota) = (-1, -1, 1)$. It can be easily described using (6.2): if $< -, - >$ (resp. ω) is the standard $\mathbb{C}$-linear product (resp. symplectic form) on $\mathbb{C}^2$, then

$$\beta(x_1 \otimes \cdots \otimes x_6, y_1 \otimes \cdots \otimes y_6) = < x_1, y_1 > \omega(x_2, y_2) < x_3, y_3 > \cdots \omega(x_6, y_6) , \tag{6.41}$$

where $x_i, y_i \in \mathbb{C}^2$, $i = 1, \ldots, 6$. In particular $\mathbb{S}^{\pm}$ are orthogonal. As in Subsection 6.2.1, there exists an operator

$$\begin{aligned} &\Gamma^{(2)} : \mathbb{S} \otimes \mathbb{S} \to \mathfrak{so}(U) , \\ &\eta(\Gamma^{(2)}(s, t)u, v) = \beta(u \wedge v \circ s, t) , \end{aligned}$$

where $s, t \in \mathbb{S}$ and $u, v \in U$. It is a symmetric $\mathfrak{so}(U)$-equivariant operator.

We fix a basis $\mathbb{1}$ of $\mathfrak{g}_{-2}$ and write the graded Lie algebra structure on $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as

$$[s, t] = \beta(s, t)\mathbb{1} ,$$

for all $s, t \in \mathbb{S}^+$. We now turn to describe the positively-graded elements of $\mathfrak{g}$. For any $s \in \mathbb{S}^+$ we introduce the linear map $\widehat{s} : \mathfrak{m} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ of degree 1 by

$$\begin{aligned} &[\widehat{s}, t] = \lambda\Gamma^{(2)}(s, t) + \mu\beta(s, t)E , \\ &[\widehat{s}, \mathbb{1}] = s , \end{aligned} \tag{6.42}$$

where $t \in \mathbb{S}^+$ and $\lambda, \mu \in \mathbb{C}$ are constants to be determined. We note that the equations (6.42) are not identical to those encountered in the extended Poincaré case, see e.g. (6.3), as here there is no Clifford multiplication of elements from $\mathfrak{g}_{-2}$ with $\mathfrak{g}_{-1}$.

Proposition 6.19. *The map $\widehat{s} \in \mathfrak{g}_1$ for all $s \in \mathbb{S}^+$ if and only if $\lambda = \frac{1}{2}$, $\mu = -\frac{1}{2}$.*

Proof. We compute

$$\begin{aligned} 0 &= [\widehat{s}, [t, \mathbb{1}]] = [[\widehat{s}, t], \mathbb{1}] + [t, [\widehat{s}, \mathbb{1}]] \\ &= -2\mu\beta(s, t)\mathbb{1} + \beta(t, s)\mathbb{1} \end{aligned}$$

for all $s, t \in \mathbb{S}^+$ and directly infer $\mu = -\frac{1}{2}$. The proof of $\lambda = \frac{1}{2}$ is more involved and it relies on the explicit realization (6.2). We depart with

$$[\widehat{s}, [t, r]] = \beta(t, r)s$$

and note that

$$\begin{aligned}[\widehat{s}, [t, r]] &= [[\widehat{s}, t], r] + [t, [\widehat{s}, r]] \\ &= \lambda\Gamma^{(2)}(s, t) \cdot r - \lambda\Gamma^{(2)}(s, r) \cdot t \\ &\quad + \tfrac{1}{2}\beta(s, t)r - \tfrac{1}{2}\beta(s, r)t\end{aligned}$$

for all $s, t, r \in \mathbb{S}^+$. We now choose suitable spinors

$$s = r = \begin{pmatrix} 1 \\ +i \end{pmatrix}^{\otimes 6}, \qquad t = \begin{pmatrix} 1 \\ -i \end{pmatrix}^{\otimes 6}$$

and use (6.41) to get $\beta(s, r) = 0$, $\beta(s, t) = 64i$. Longer but straightforward computations similar to those of Proposition 6.12 yield

$$\Gamma^{(2)}(s, t) = 64 \sum_{l \text{ odd}} e_l \wedge e_{l+1}$$

and $\Gamma^{(2)}(s, s) = 0$.

We are therefore left with the identity

$$\begin{aligned}-64is &= \lambda\Gamma^{(2)}(s, t) \cdot s + 32is \\ &= 64\lambda \sum_{l \text{ odd}} e_l \wedge e_{l+1} \cdot s + 32is \\ &= -\frac{6i}{2} 64\lambda s + 32is\ ,\end{aligned}$$

from which $\lambda = \frac{1}{2}$ follows. □

Finally we consider the generator $\widehat{\mathbb{1}} : \mathfrak{m} \to \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the 1-dimensional component $\mathfrak{g}_2$ of $\mathfrak{g}$ defined by

$$[\widehat{\mathbb{1}}, t] = \widehat{t}\ , \qquad [\widehat{\mathbb{1}}, \mathbb{1}] = -E\ , \tag{6.43}$$

and note that $[\widehat{s}, \widehat{t}] = \beta(s, t)\widehat{\mathbb{1}}$ for all $s, t \in \mathbb{S}^+$. This completes the description of Lie brackets of $\mathfrak{g} = E_7$. We now turn to the KTS associated with the 3 different grade-reversing involutions.

The proof of Theorem 6.20 is similar but not identical to that of Theorem 6.9 and we therefore outline its main steps.

Theorem 6.20. *Let* U *be a* 12*-dimensional complex vector space with a non-degenerate symmetric bilinear form* η *and* $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ *the associated* 64*-dimensional spinor module. Let* $\beta : \mathbb{S} \otimes \mathbb{S} \to \mathbb{C}$ *be the admissible bilinear form on* $\mathbb{S}$ *with invariants* $(\tau, \sigma, \iota) = (-1, -1, 1)$ *and* $\Gamma^{(2)} : \mathbb{S} \otimes \mathbb{S} \to \mathfrak{so}(\mathrm{U})$ *the operator given by*

$$\eta(\Gamma^{(2)}(s,t)u, v) = \beta(u \wedge v \circ s, t) \ ,$$

where $s, t \in \mathbb{S}$ *and* $u, v \in \mathrm{U}$*.*

(1) Fix an orthogonal decomposition

$$\mathrm{U} = W \oplus W^{\perp}$$

of U *with* $\dim W = 6$ *or* $\dim W = 2$ *and let* $\mathrm{I} = i\,\mathrm{vol}_W$*, where* $\mathrm{vol}_W \in \mathrm{C}\ell(\mathrm{U})$ *is the volume of* W*. (*I *acts on* $\mathbb{S}$ *as a paracomplex structure in both cases.) Then the semispinor module* $\mathbb{S}^+$ *with the triple product*

$$(rst) = -\frac{1}{2}\Gamma^{(2)}(r, \mathrm{I} \circ s) \cdot t + \frac{1}{2}\beta(r, \mathrm{I} \circ s)t \ , \qquad r, s, t \in \mathbb{S}^+ \ , \tag{6.44}$$

is a K*-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra* $\mathfrak{g} = \mathrm{E}_7$ *and*

$$\mathfrak{der}(\mathbb{S}^+) = \mathfrak{stab}_{\mathfrak{so}(\mathrm{U})}(\mathrm{I}) \simeq \begin{cases} \mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{so}(6,\mathbb{C}) \ \textit{if} \ \dim W = 6 \ , \\ \mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C}) \ \textit{if} \ \dim W = 2 \ . \end{cases}$$

(2) Fix a split decomposition $\mathrm{U} = W \oplus W^*$ *of* U *into the direct sum of two isotropic* 6*-dimensional subspaces and let*

$$\mathrm{I}_t = \exp(tX) \in \mathrm{Spin}(\mathrm{U}) \ , \qquad X = \begin{pmatrix} \mathrm{Id}_W & 0 \\ 0 & -\mathrm{Id}_{W^*} \end{pmatrix} \ ,$$

be the complex 1*-parameter subgroup of the spin group generated by* $X \in \mathfrak{so}(\mathrm{U}) \simeq \mathfrak{spin}(\mathrm{U})$*. If we set* $\mathrm{I} = \mathrm{I}_t$ *for* $t = i\frac{\pi}{2}$ *then the semispinor module* $\mathbb{S}^+$ *with the triple product*

$$(rst) = -\frac{i}{2}\Gamma^{(2)}(r, \mathrm{I} \circ s) \cdot t + \frac{i}{2}\beta(r, \mathrm{I} \circ s)t \ , \qquad r, s, t \in \mathbb{S}^+ \ , \tag{6.45}$$

is a K*-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra* $\mathfrak{g} = \mathrm{E}_7$ *and*

$$\mathfrak{der}(\mathbb{S}^+) = \mathfrak{stab}_{\mathfrak{so}(\mathrm{U})}(\mathrm{I}) \simeq \mathfrak{gl}(W) \ .$$

Proof. (1) Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the grade-reversing map defined by

$$\begin{aligned} \sigma(\mathbb{1}) &= -\widehat{\mathbb{1}} \ , & \sigma(s) &= \widehat{\mathrm{I} \circ s} \ , \\ \sigma(A) &= r_W A r_W \ , & \sigma(E) &= -E \ , \\ \sigma(\widehat{s}) &= \mathrm{I} \circ s \ , & \sigma(\widehat{\mathbb{1}}) &= -\mathbb{1} \ , \end{aligned} \tag{6.46}$$

where $s \in \mathbb{S}^+$ and $A \in \mathfrak{so}(U)$. It is a Lie algebra involution, as $I \in \mathrm{Spin}(U)$ covers the opposite of the orthogonal reflection $r_W : U \to U$ across W, i.e., $\widetilde{\mathrm{Ad}}_I = -r_W$ and $\beta(I \circ s, t) = -\beta(s, I \circ t)$ for all $s, t \in \mathbb{S}$.

(2) First note that I acts as a complex structure on U via the spin cover $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(U) \to \mathrm{SO}(U)$. In particular $I^2 \in \mathrm{Spin}(U)$ sits in the twisted center

$$\mathrm{twcen}\, C\ell(U) = \{a \in C\ell(U) \mid a \circ v = -v \circ a \ \text{ for all } \ v \in V\}$$

of $C\ell(U)$, which is known to be generated by vol, as $\dim U$ is even. It follows that $I^2 = \alpha\,\mathrm{vol}$ for some fourth root of unity α and a direct computation with exponentials reveals that $I^2 \circ s = -s$ for the positive-chirality spinor

$$s = \begin{pmatrix} 1 \\ +i \end{pmatrix}^{\otimes 6} \in \mathbb{S}^+ \ .$$

In other words, $I^2 = \mathrm{vol}$ in the Clifford algebra (recall that $\mathbb{S}^+ = +1$-eigenspace of $-\,\mathrm{vol}$) and I acts as a complex structure both on U and $\mathbb{S}^+$.

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the grade-reversing map defined by

$$\begin{aligned} \sigma(\mathbb{1}) &= -\widehat{\mathbb{1}} \ , & \sigma(s) &= i\widehat{I \circ s} \ , \\ \sigma(A) &= IAI^{-1} \ , & \sigma(E) &= -E \ , \\ \sigma(\widehat{s}) &= iI \circ s \ , & \sigma(\widehat{\mathbb{1}}) &= -\mathbb{1} \ , \end{aligned} \tag{6.47}$$

and note that $\sigma^2 = 1$ by the above arguments. Using the expressions of the Lie brackets of $\mathfrak{g} = E_7$ and the equivariance properties

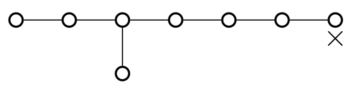
$$\beta(I \circ s, I \circ t) = \beta(s, t) \ , \qquad \Gamma^{(2)}(I \circ s, I \circ t) = I\Gamma^{(2)}(s, t)I^{-1}$$

for all $s, t \in \mathbb{S}$, we get that σ is a Lie algebra involution, with associated KTS (6.45).

□

6.3.5 The case $\mathfrak{g} = E_8$

Let



(6.48)

be the contact grading of $\mathfrak{g} = E_8$, with even graded components $\mathfrak{g}_0 = \mathbb{C}E \oplus E_7$, $\mathfrak{g}_{\pm 2} = \mathbb{C}$ and where $\mathfrak{g}_{\pm 1}$ is given by the defining 56-dimensional representation of E_7.

There are two inequivalent associated KTS, which correspond to the split real form EVIII and the real form EIX. The first has derivation algebra the (complexification of the) maximal compact subalgebra $\mathfrak{su}(8)$ of EV, the second the maximal compact subalgebra of EVII:

$$\mathfrak{der}(V) \simeq \begin{cases} \mathfrak{sl}(8, \mathbb{C}) \text{ or} \\ E_6 \oplus \mathbb{C} . \end{cases} \tag{6.49}$$

We anticipate that the KTS with $\mathfrak{der}(V) = E_6 \oplus \mathbb{C}$ is a variation of the triple system historically first introduced by Freudenthal to describe the defining representation of E_7 [20]. We will introduce a paracomplex structure $I : V \to V$ which relates the triple system of Freudenthal with ours, making evident that the KTS product is *not* equivariant under the whole E_7 (as for the Freudenthal product).

We note that $\mathfrak{g} = E_8$ with the grading (6.48) is the only 5-graded simple Lie algebra with $\mathfrak{g}_0$ of exceptional type. Due to this, instead of looking for a uniform description of $\mathfrak{g}$ and in particular of the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$, we shall use two different presentations, for each one of the two KTS. We begin with some introductory background on Jordan algebras.

Preliminaries on cubic Jordan algebras and associated structures

We recall that a Jordan algebra J is a complex vector space equipped with a bilinear product satisfying

$$A \circ B = B \circ A \,, \qquad (A \circ B) \circ A^2 = A \circ (B \circ A^2) \,,$$

for all $A, B \in J$. An important class of Jordan algebras is given by the *cubic* Jordan algebras developed in [53, 40]; we here sketch their construction, following the presentation of [41].

Definition 6.21. A *cubic norm* on a complex vector space V is a homogeneous map of degree three $N : V \to \mathbb{C}$ such that its full symmetrization

$$N(A, B, C) := \frac{1}{6}(N(A+B+C)-N(A+B)-N(A+C)-N(B+C)+N(A)+N(B)+N(C))$$

is trilinear.

We say that $c \in V$ is a basepoint if $N(c) = 1$. If V is a vector space equipped with a cubic norm and a fixed base point, one can define the following three maps:

1. the trace form,

$$\mathrm{Tr}(A) = 3N(c, c, A), \tag{6.50a}$$

2. a bilinear map,

$$S(A, B) = 6N(A, B, c), \tag{6.50b}$$

3. a trace bilinear form,

$$(A, B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B) - S(A, B). \tag{6.50c}$$

A Jordan algebra J with multiplicative identity $1 = c$ may be derived from any such vector space V if N is *Jordan*.

Definition 6.22. A cubic norm is *Jordan* if

1. the trace bilinear form (6.50c) is non-degenerate;
2. the quadratic adjoint map $\sharp\colon \mathfrak{J} \to \mathfrak{J}$ defined by

$$(A^\sharp, B) = 3N(A, A, B)$$

   satisfies $(A^\sharp)^\sharp = N(A)A$ for all $A \in \mathfrak{J}$.

We define the linearization of the adjoint map by

$$A \times B := (A + B)^\sharp - A^\sharp - B^\sharp \tag{6.51}$$

and note that $A \times A = 2A^\sharp$. (We remark that other authors define $A \times B$ with an additional factor $\frac{1}{2}$, in their conventions $A \times A = A^\sharp$.) Every vector space with a Jordan cubic norm gives rise to a Jordan algebra with unit $1 = c$ and Jordan product

$$A \circ B := \frac{1}{2}(A \times B + \mathrm{Tr}(A)B + \mathrm{Tr}(B)A - S(A, B)1). \tag{6.52}$$

We conclude with the definition of reduced structure group; for more details on cubic Jordan algebras, we refer the reader to the original sources [53, 40, 41], see also e.g. [37, §2].

Definition 6.23. The *reduced structure group* of a cubic Jordan algebra J with product (6.52) is the group

$$\mathrm{Str}_0(J) = \{\tau : J \to J \mid N(\tau A) = N(A) \text{ for all } A \in J\}$$

of linear invertible transformations of J preserving the cubic norm.

It is well known that over the field of complex numbers, there is a unique simple finite-dimensional exceptional Jordan algebra, called Albert algebra. It is a cubic Jordan algebra of dimension 27. The underlying vector space V is the space $\mathfrak{H}_3(\mathfrak{U})$ of 3×3 Hermitian matrices over the complex Cayley algebra $\mathfrak{U}$ (=complex octonions) and the cubic norm $N : \mathfrak{H}_3(\mathfrak{U}) \to \mathbb{C}$ the usual determinant of a matrix $A \in \mathfrak{H}_3(\mathfrak{U})$, the only proviso being that the order of the factors and the position of the brackets in multiplying elements from $\mathfrak{U}$ is important. The interested reader may find the explicit expression of N in e.g. [37, eq. (16)]. The trace forms (6.50a) and (6.50c) associated to the identity matrix as basepoint coincide in this case with the regular trace,

$$(A, B) = \frac{1}{2} \mathrm{Tr}(AB + BA) ,$$

whereas the adjoint $A^\sharp$ of $A \in \mathfrak{H}_3(\mathfrak{U})$ is the transpose of the cofactor matrix [37, pag. 934].

The reduced structure group $\mathrm{Str}_0(\mathfrak{H}_3(\mathfrak{U}))$ of $J = \mathfrak{H}_3(\mathfrak{U})$ is a simply connected simple Lie group of type E_6 and the natural action on J its 27-dimensional defining representation. Elements $\tau \in \mathrm{Str}_0(\mathfrak{H}_3(\mathfrak{U}))$ can be equally characterized by the following identity

$$\tau(A) \times \tau(B) = (\tau^*)^{-1}(A \times B) ,$$

for all $A, B \in \mathfrak{H}_3(\mathfrak{U})$, where τ^* is the transposed of τ relative to the trace bilinear form (6.50c). We recall that the defining representation J of E_6 and its dual J^* are *not* equivalent; in our conventions J^* is still represented by the set J but the action of $\mathrm{Str}_0(\mathfrak{H}_3(\mathfrak{U}))$ is $\tau \mapsto (\tau^*)^{-1}$. We will not distinguish between J and J^* if the action of $\mathrm{Str}_0(\mathfrak{H}_3(\mathfrak{U}))$ is clear from the context.

We now turn to recall Freudenthal's construction of the 56-dimensional defining representation of the Lie algebra E_7 from the 27-dimensional Albert algebra [20]. It is a special case of a more general construction by Brown in [9] which departs from any cubic Jordan algebra, but for our purposes it is enough to assume $J = \mathfrak{H}_3(\mathfrak{U})$ from now on.

We consider a vector space $\mathfrak{F} = \mathfrak{F}(J)$ constructed from J in the following way

$$\mathfrak{F}(J) = \mathbb{C} \oplus \mathbb{C} \oplus J \oplus J^*$$

and write an arbitrary element $x \in \mathfrak{F}$ as a "2×2 matrix"

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} , \quad \text{where } \alpha, \beta \in \mathbb{C} \quad \text{and} \quad A \in J, B \in J^* . \tag{6.53}$$

We note that the Lie algebra E_7 admits a 3-grading of the form

$$\begin{gathered} E_7 = J \oplus (E_6 \oplus \mathbb{C}G) \oplus J^* , \\ \deg(J) = -1 , \quad \deg(E_6 \oplus \mathbb{C}G) = 0 , \quad \deg(J^*) = 1 , \end{gathered} \tag{6.54}$$

where G is the corresponding grading element, and following [51] we will use the shortcut

$$\Phi = \Phi(\phi, X, Y, \nu) \in E_7$$
$$(\phi \in E_6,\ X \in J,\ Y \in J^*,\ \nu \in \mathbb{C}) ,$$

to denote elements of E_7. The action of E_7 on $\mathfrak{F}$ is as follows, see [20] and also [51, §2.1].

Proposition 6.24. *The representation of the Lie algebra E_7 on $\mathfrak{F}$ is given by*

$$\Phi(\phi, X, Y, \nu)\begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha\nu + (X, B) & \phi A - \frac{1}{3}\nu A + Y \times B + \beta X \\ -\phi^* B + \frac{1}{3}\nu B + X \times A + \alpha Y & -\beta\nu + (Y, A) \end{pmatrix} .$$

In particular the grading element of (6.54) is given by $G = \Phi(0, 0, 0, -\frac{3}{2})$.

Decomposing the tensor product $\mathfrak{F} \otimes \mathfrak{F}$ into irreducible representation of E_7, one readily sees that there exist unique (up to multiples) E_7-equivariant maps

$$\{\cdot, \cdot\} : \mathfrak{F} \otimes \mathfrak{F} \to \mathbb{C}$$

and

$$\times : \mathfrak{F} \otimes \mathfrak{F} \to E_7.$$

The first map is the standard symplectic form

$$\{x, y\} = \alpha\delta - \beta\gamma + (A, D) - (B, C) , \tag{6.55}$$

where

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} , \qquad y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix} , \tag{6.56}$$

are elements of $\mathfrak{F}$. The second is the so-called *Freudenthal product*, an appropriate extension of the operation (6.51) on J to the whole $\mathfrak{F}$, see e.g. [51, §2].

Proposition 6.25. *For $x, y \in \mathfrak{F}$ as in (6.56), the Freudenthal product is given by $x \times y := \Phi(\phi, X, Y, \nu)$, where*

$$\begin{aligned} \phi &= -\tfrac{1}{2}(A \vee D + C \vee B) \\ X &= -\tfrac{1}{4}(B \times D - \alpha C - \gamma A) \\ Y &= \tfrac{1}{4}(A \times C - \beta D - \delta B) \\ \nu &= \tfrac{1}{8}((A, D) + (C, B) - 3(\alpha\delta + \beta\gamma)), \end{aligned} \tag{6.57}$$

and $A \vee B \in E_6$ is defined by $(A \vee B)C = \frac{1}{2}(B, C)A + \frac{1}{6}(A, B)C - \frac{1}{2}B \times (A \times C)$. The group of automorphism of the Freudenthal product is a connected simple Lie group of type E_7.

We are now ready to describe the KTS with derivation algebra $E_6 \oplus \mathbb{C}$.

The first product

In the following, we denote by $I : \mathfrak{F} \to \mathfrak{F}$ the paracomplex structure on $\mathfrak{F}$ defined by

$$I \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha & A \\ -B & -\beta \end{pmatrix} .$$

Note that I is invariant under the subalgebra $E_6 \oplus \mathbb{C}G$ of E_7 (6.54).

Theorem 6.26. *The vector space $\mathfrak{F}$ with the triple product given by*

$$(xyz) = -4x \times Iy(z) + \frac{1}{2}\{x, Iy\}z$$

for all $x, y, z \in \mathfrak{F}$ is a K-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra E_8 and derivation algebra $\mathfrak{der}(\mathfrak{F}) = E_6 \oplus \mathbb{C}G$.

Proof. The Lie algebra $\mathfrak{g} = E_8$ with the 5-grading (6.48) has negatively graded part

$$\begin{aligned} \mathfrak{m} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \\ &= \mathbb{C}\mathbb{1} \oplus \mathfrak{F} \end{aligned}$$

with Lie brackets $[x, y] = \{x, y\}\mathbb{1}$ for all $x, y \in \mathfrak{F}$. Clearly $\mathfrak{g}_0 = E_7 \oplus \mathbb{C}E$ acts on $\mathfrak{m}$ by 0-degree derivations and it is known that E_8 is the maximal transitive prolongation of $\mathfrak{g}_{\leqslant 0}$ [57].

We will denote elements of $\mathfrak{g}_1 \simeq \mathfrak{F}$ by $\widehat{x}, \widehat{y} : \mathfrak{m} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ and fix a generator $\widehat{\mathbb{1}} : \mathfrak{m} \to \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}_2$. By E_7-equivariance, the adjoint action on $\mathfrak{m}$ is necessarily of the following form:

$$[\widehat{x}, y] = c_1(x \times y) + c_2\{x, y\}E , \qquad [\widehat{x}, \mathbb{1}] = x ,$$

and

$$[\widehat{\mathbb{1}}, y] = \widehat{y} , \qquad [\widehat{\mathbb{1}}, \mathbb{1}] = c_3 E ,$$

for some constants c_1, c_2, c_3 to be determined. First of all

$$\begin{aligned} 0 &= [\widehat{\mathbb{1}}, [x, \mathbb{1}]] = [[\widehat{\mathbb{1}}, x], \mathbb{1}] + [x, [\widehat{\mathbb{1}}, \mathbb{1}]] \\ &= [\widehat{x}, \mathbb{1}] + c_3[x, E] = x + c_3 x , \end{aligned}$$

and similarly

$$\begin{aligned} 0 &= [\widehat{x}, [y, \mathbb{1}]] = [c_2\{x, y\}E, \mathbb{1}] - \{x, y\}\mathbb{1} \\ &= -(2c_2 + 1)\{x, y\}\mathbb{1} , \end{aligned}$$

for all $x, y \in \mathfrak{F}$, whence $c_3 = -1$, $c_2 = -\frac{1}{2}$. On the other hand

$$[\widehat{x}, [y, z]] = \{y, z\} x \tag{6.58}$$

and

$$\begin{aligned} [[\widehat{x}, y], z] + [y, [\widehat{x}, z]] &= [c_1(x \times y) + c_2\{x, y\} E, z] - [c_1(x \times z) + c_2\{x, z\} E, y] \\ &= c_1 x \times y(z) + \frac{1}{2}\{x, y\} z - c_1 x \times z(y) - \frac{1}{2}\{x, z\} y \,. \end{aligned} \tag{6.59}$$

Choose $x = y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that $x \times y = \Phi(0, 0, 0, -\frac{3}{4})$, $\{x, y\} = x \times z = 0$, $\{x, z\} = \{y, z\} = -2$ and get $c_1 = 4$ equating (6.58) with (6.59). The adjoint action of the positively-graded part of $\mathfrak{g}$ on $\mathfrak{m}$ has been described. The remaining non-trivial Lie brackets of $\mathfrak{g}$ are given by the natural adjoint action of $\mathfrak{g}_0$ and $[\widehat{x}, \widehat{y}] = \{x, y\} \widehat{\mathbb{1}}$ for all $x, y \in \mathfrak{F}$.

Using the explicit expressions of the Lie brackets, it is a straightforward task to check that

$$\begin{aligned} &\sigma(\mathbb{1}) = -\widehat{\mathbb{1}} \,, && \sigma(x) = \widehat{Ix} \,, \\ &\sigma(\Phi(\phi, A, B, \nu)) = \Phi(\phi, -A, -B, \nu) \,, && \sigma(E) = -E \,, \\ &\sigma(\widehat{x}) = Ix \,, && \sigma(\widehat{\mathbb{1}}) = -\mathbb{1} \,, \end{aligned}$$

is a grade-reversing involution of $\mathfrak{g}$ and compute the triple product. □

The second product

To get the second KTS associated to the contact grading of E_8, we employ a description of E_7 dating back to Cartan [17].

The Lie algebra E_7 admits a symmetric irreducible decomposition

$$E_7 = \mathfrak{sl}(8, \mathbb{C}) \oplus \Lambda^4(\mathbb{C}^8)^* \,,$$

where the Lie subalgebra $\mathfrak{sl}(8, \mathbb{C})$ acts in the natural way on $\Lambda^4(\mathbb{C}^8)^*$. To describe the brackets of two elements in $\Lambda^4(\mathbb{C}^8)^*$, it is convenient to fix a volume $vol \in \Lambda^8(\mathbb{C}^8)$, let $\sharp : \Lambda^k(\mathbb{C}^8)^* \to \Lambda^{8-k}(\mathbb{C}^8)$ be the map which sends any ξ to $i_\xi(vol)$ and $\flat : \Lambda^{8-k}(\mathbb{C}^8) \to \Lambda^k(\mathbb{C}^8)^*$ its inverse. The maps $\sharp$ and $\flat$ are $\mathfrak{sl}(8, \mathbb{C})$-equivariant and can be thought as the analogues of the usual musical isomorphisms when only a volume is assigned. Let also

$$\bullet : \Lambda^k(\mathbb{C}^8) \otimes \Lambda^k(\mathbb{C}^8)^* \to \mathfrak{sl}(8, \mathbb{C})$$

be the unique $\mathfrak{sl}(8, \mathbb{C})$-equivariant map for any $k = 1, \ldots, 7$, which in our conventions is normalized so that

$$\boldsymbol{e}_{1\cdots k} \bullet \boldsymbol{e}^{1\cdots k} = \frac{8-k}{8} \sum_{i=1}^{k} \boldsymbol{e}_i \otimes \boldsymbol{e}^i - \frac{k}{8} \sum_{j=k+1}^{8} \boldsymbol{e}_j \otimes \boldsymbol{e}^j \,,$$

where (e_i) is the standard basis of $\mathbb{C}^8$ and (e^i) the dual basis of $(\mathbb{C}^8)^*$. With this in mind, the Lie bracket of $\alpha, \beta \in \Lambda^4(\mathbb{C}^8)^*$ is the element

$$[\alpha, \beta] = \alpha^\sharp \bullet \beta$$

of $\mathfrak{sl}(8, \mathbb{C})$.

The contact grading of $\mathfrak{g} = E_8$ is given by $\mathfrak{g}_0 = E_7 \oplus \mathbb{C}E$, $\mathfrak{g}_{\pm 2} \simeq \mathbb{C}$ and

$$\mathfrak{g}_{\pm 1} \simeq \Lambda^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8)^* ,$$

where the negatively graded part $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$ has the Lie brackets

$$[X, Y] = (x^*(y) - y^*(x))\mathbb{1}$$

for all $X = (x, x^*)$, $Y = (y, y^*)$ in $\mathfrak{g}_{-1} = \Lambda^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8)^*$. The action of $\mathfrak{g}_0$ as derivations of $\mathfrak{m}$ is the natural one of $\mathfrak{sl}(8, \mathbb{C}) \oplus \mathbb{C}E$ together with

$$[\alpha, X] = ((\alpha \wedge x^*)^\sharp, i_x(\alpha)) ,$$

where $\alpha \in \Lambda^4(\mathbb{C}^8)^*$ and $X \in \mathfrak{g}_{-1}$.

The next result follows from the fact that $\mathfrak{g} = E_8$ is the maximal prolongation of $\mathfrak{m} \oplus \mathfrak{g}_0$ [57] and from direct computations using $\mathfrak{sl}(8, \mathbb{C})$-equivariance.

Proposition 6.27. *For all* $X = (x, x^*) \in \Lambda^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8)^*$, *the operator* $\widehat{X} : \mathfrak{m} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ *given by*

$$[\widehat{X}, Y] = \underbrace{(x \bullet y^* + y \bullet x^*)}_{\text{element of } \mathfrak{sl}(8,\mathbb{C})} + \underbrace{(x^* \wedge y^* - (x \wedge y)^\flat)}_{\text{element of } \Lambda^4(\mathbb{C}^8)^*} - \frac{1}{2}\underbrace{(x^*(y) - y^*(x))E}_{\text{element of } \mathbb{C}E} ,$$
$$[\widehat{X}, \mathbb{1}] = X ,$$

where $Y \in \mathfrak{g}_{-1}$, *is an element of the first prolongation* $\mathfrak{g}_1$. *Similarly* $\widehat{\mathbb{1}} : \mathfrak{m} \to \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *given by*

$$[\widehat{\mathbb{1}}, Y] = \widehat{Y} , \qquad [\widehat{\mathbb{1}}, \mathbb{1}] = -E ,$$

is a generator of $\mathfrak{g}_2$.

It is not difficult to see that the remaining Lie brackets of $\mathfrak{g}$ are given by the action of E and

$$[A, \widehat{X}] = \widehat{[A, X]} , \qquad [\alpha, \widehat{X}] = \widehat{[\alpha, X]} , \qquad [\widehat{X}, \widehat{Y}] = \widehat{[X, Y]} \tag{6.60}$$

where $A \in \mathfrak{sl}(8, \mathbb{C})$, $\alpha \in \Lambda^4(\mathbb{C}^8)^*$ and $X, Y \in \Lambda^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8)^*$.

Theorem 6.28. *The vector space* $\Lambda^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8)^*$ *with the triple product*

$$(XYZ) = i \begin{pmatrix} \iota_{z^*}(x \wedge y) + \frac{1}{2}(x^*(y) + y^*(x))z + (x \bullet y^* - y \bullet x^*) \cdot z + (x^* + \wedge y^* \wedge z^*)^\sharp \\ \iota_z(x^* \wedge y^*) + \frac{1}{2}(x^*(y) + y^*(x))z^* + (x \bullet y^* - y \bullet x^*) \cdot z^* + (x \wedge y \wedge z)^\flat \end{pmatrix}$$

for all $X = (x, x^*)$, $Y = (y, y^*)$, $Z = (z, z^*)$ *in* $\Lambda^2(\mathbb{C}^8) \oplus \Lambda^2(\mathbb{C}^8)^*$ *is a* K*-simple Kantor triple system with Tits-Kantor-Koecher Lie algebra* E_8 *and derivation algebra* $\mathfrak{sl}(8, \mathbb{C})$.

Proof. The map defined by

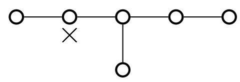
$$\begin{array}{lll} \sigma(\mathbb{1}) = \widehat{\mathbb{1}} \,, & \sigma(X) = \widehat{(ix, -ix^*)} \,, & \\ \sigma(A) = A \,, & \sigma(\alpha) = -\alpha \,, & \sigma(E) = -E \,, \\ \sigma(\widehat{X}) = (-ix, ix^*) \,, & \sigma(\widehat{\mathbb{1}}) = \mathbb{1} \,, & \end{array}$$

for all $X = (x, x^*)$, $A \in \mathfrak{sl}(8, \mathbb{C})$, $\alpha \in \Lambda^4(\mathbb{C}^8)^*$, is a grade-reversing involution of E_8. □

6.4 The exceptional Kantor triple systems of special type

6.4.1 The case $\mathfrak{g} = E_6$

The Lie algebra E_6 admits a special 5-grading which is not of contact or of extended Poincaré type. It is described by the crossed Dynkin diagram



and its graded components are $\mathfrak{g}_0 = \mathfrak{sl}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}E$, where E is the grading element, and

$$\mathfrak{g}_{-1} = \Lambda^2(\mathbb{C}^5)^* \boxtimes \mathbb{C}^2 \,, \qquad \mathfrak{g}_1 = \Lambda^2\mathbb{C}^5 \boxtimes \mathbb{C}^2 \,,$$
$$\mathfrak{g}_{-2} = \Lambda^4(\mathbb{C}^5)^* \,, \qquad \mathfrak{g}_2 = \Lambda^4\mathbb{C}^5 \,,$$

with their natural structure of $\mathfrak{g}_0$-modules. In the following, we denote forms in $\Lambda^2(\mathbb{C}^5)^*$ (resp. $\Lambda^4(\mathbb{C}^5)^*$) by α, β, γ (resp. ξ, ϕ, ψ) and polyvectors by $\widetilde{\alpha} \in \Lambda^2\mathbb{C}^5$, $\widetilde{\xi} \in \Lambda^4\mathbb{C}^5$, etc. Finally $a, b, c \in \mathbb{C}^2$ and we use the standard symplectic form ω on $\mathbb{C}^2$ to identify $\mathfrak{sl}(2, \mathbb{C})$ with $S^2\mathbb{C}^2$.

In order to describe the Lie brackets of E_6, we note that for $k = 1, \ldots, 4$ there exists a (unique up to constant) $\mathfrak{sl}(5, \mathbb{C})$-equivariant map

$$\bullet : \Lambda^k\mathbb{C}^5 \otimes \Lambda^k(\mathbb{C}^5)^* \to \mathfrak{sl}(5, \mathbb{C}) \,.$$

In our conventions, it is normalized so that

$$\boldsymbol{e}_{1\cdots k}\bullet\boldsymbol{e}^{1\cdots k}=\frac{5-k}{5}\sum_{i=1}^{k}\boldsymbol{e}_i\otimes\boldsymbol{e}^i-\frac{k}{5}\sum_{j=k+1}^{5}\boldsymbol{e}_j\otimes\boldsymbol{e}^j\ ,$$

where $(\boldsymbol{e}_i)$ is the standard basis of $\mathbb{C}^5$ and $(\boldsymbol{e}^i)$ the dual basis of $(\mathbb{C}^5)^*$. The structure of graded Lie algebra on $\mathfrak{m}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ is given by

$$[\alpha\otimes a,\beta\otimes b]=\omega(a,b)\alpha\wedge\beta\ ,$$

while the adjoint action on $\mathfrak{m}$ of positive-degree elements is of the form

$$\begin{aligned}&[\widetilde{\alpha}\otimes a,\beta\otimes b]=c_1\underbrace{\imath_{\widetilde{\alpha}}\beta\, a\odot b}_{\text{element of }\mathfrak{sl}(2,\mathbb{C})}+c_2\underbrace{\imath_{\widetilde{\alpha}}\beta\,\omega(a,b)\,E}_{\text{element of }\mathbb{C}E}+c_3\underbrace{\omega(a,b)\widetilde{\alpha}\bullet\beta}_{\text{element of }\mathfrak{sl}(5,\mathbb{C})}\ ,\\&[\widetilde{\alpha}\otimes a,\psi]=\imath_{\widetilde{\alpha}}\psi\otimes a\ ,\end{aligned}\tag{6.61}$$

and

$$\begin{aligned}&[\widetilde{\xi},\beta\otimes b]=-\imath_\beta\widetilde{\xi}\otimes b\ ,\\&[\widetilde{\xi},\psi]=c_4\underbrace{\imath_{\widetilde{\xi}}\psi\,E}_{\text{element of }\mathbb{C}E}+c_5\underbrace{\widetilde{\xi}\bullet\psi}_{\text{element of }\mathfrak{sl}(5,\mathbb{C})}\ ,\end{aligned}\tag{6.62}$$

for some constants $c_1,\ldots,c_5$ to be determined.

Proposition 6.29. *The constants in* (6.61)-(6.62) *are* $c_1=\frac{1}{2}$, $c_2=-\frac{3}{10}$, $c_3=-1$, $c_4=\frac{3}{5}$ *and* $c_5=1$.

Proof. As usual, elements of the first and second prolongation act as derivations on $\mathfrak{m}$.

First, let us choose $a,b\in\mathbb{C}^2$ such that $\omega(a,b)=1$ and compute

$$\begin{aligned}0&=[\widetilde{\alpha}\otimes a,[\beta\otimes b,\xi]]=[[\widetilde{\alpha}\otimes a,\beta\otimes b],\xi]+[\beta\otimes b,[\widetilde{\alpha}\otimes a,\xi]]\\&=-2c_2(\imath_{\widetilde{\alpha}}\beta)\xi+c_3(\widetilde{\alpha}\bullet\beta)\cdot\xi-\beta\wedge\imath_{\widetilde{\alpha}}\xi\end{aligned}$$

for all $\widetilde{\alpha}\in\Lambda^2\mathbb{C}^5$, $\beta\in\Lambda^2(\mathbb{C}^5)^*$, $\xi\in\Lambda^4(\mathbb{C}^5)^*$. Choosing suitable forms and polyvectors yields a regular non-homogeneous linear system in c_2, c_3. For instance if $\widetilde{\alpha}=\boldsymbol{e}_{12}$, $\beta=\boldsymbol{e}^{12}$ one gets

$$-2c_2-\tfrac{2}{5}c_3=1\ ,\qquad -2c_2+\tfrac{3}{5}c_3=0\ ,$$

taking $\xi=\boldsymbol{e}^{1234}$ and $\xi=\boldsymbol{e}^{2345}$, respectively. In other words $c_2=-\frac{3}{10}$, $c_3=-1$. The adjoint action of $\mathfrak{g}_1$ on $\mathfrak{g}_{-2}=[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]$ gives $c_1=\frac{1}{2}$ with an analogous computation.

We now note that

$$\begin{aligned}0=&[\widetilde{\xi},[\beta\otimes b,\psi]]=[[\widetilde{\xi},\beta\otimes b],\psi]+[\beta\otimes b,[\widetilde{\xi},\psi]]\\&=-[\imath_\beta\widetilde{\xi}\otimes b,\psi]+c_5[\beta\otimes b,\widetilde{\xi}\bullet\psi]+c_4(\imath_{\widetilde{\xi}}\psi)\beta\otimes b\\&=-\imath_{(\imath_\beta\widetilde{\xi})}\psi\otimes b-c_5(\widetilde{\xi}\bullet\psi)\cdot\beta\otimes b+c_4(\imath_{\widetilde{\xi}}\psi)\beta\otimes b\end{aligned}$$

holds for all $b \in \mathbb{C}^2$. Choosing $\widetilde{\xi} = \boldsymbol{e}_{1234}$, $\psi = \boldsymbol{e}^{1234}$ and, in turn, $\beta = \boldsymbol{e}^{12}$ and then $\beta = \boldsymbol{e}^{45}$, yields a system of linear equations

$$\tfrac{2}{5}c_5 + c_4 = 1 , \qquad -\tfrac{3}{5}c_5 + c_4 = 0 ,$$

whose unique solution is $c_4 = \frac{3}{5}$, $c_5 = 1$. □

The remaining Lie brackets follows easily, as $\mathfrak{g} = E_6$ is the maximal prolongation of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. They are given by the natural action of $\mathfrak{g}_0$ on $\mathfrak{g}_p$, $p \geqslant 0$, and by

$$[\widetilde{\alpha} \otimes a, \widetilde{\beta} \otimes b] = -\omega(a, b)\widetilde{\alpha} \wedge \widetilde{\beta}$$

for all $\widetilde{\alpha}, \widetilde{\beta} \in \Lambda^2\mathbb{C}^5$ and $a, b \in \mathbb{C}^2$.

By the results of Section 5.3 there is only one grade-reversing involution, namely the Chevalley involution with derivation algebra $\mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$. To describe the associated KTS, we denote by η the standard non-degenerate symmetric bilinear form on $\mathbb{C}^p$ for $p = 2, 5$ and extend the musical isomorphisms

$$\flat : \mathbb{C}^p \to (\mathbb{C}^p)^* , \qquad \natural : (\mathbb{C}^p)^* \to \mathbb{C}^p ,$$

to forms and polyvectors in the obvious way. When $p = 2$ we also consider the compatible complex structure $J : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\eta(a, b) = \omega(a, Jb)$ for all $a, b \in \mathbb{C}^2$.

Theorem 6.30. *The vector space $V = \Lambda^2(\mathbb{C}^5)^* \otimes \mathbb{C}^2$ with triple product*

$$\begin{aligned}((\alpha \otimes a)\ (\beta \otimes b)\ (\gamma \otimes c)) = &-\frac{1}{2}\eta(\alpha, \beta)(\omega(a, c)\gamma \otimes Jb + \omega(Jb, c)\gamma \otimes a) \\ &+ \frac{3}{10}\eta(\alpha, \beta)\eta(a, b)\gamma \otimes c - \eta(a, b)(\beta^\sharp \bullet \alpha) \cdot \gamma \otimes c\end{aligned}$$

for all $\alpha, \beta, \gamma \in \Lambda^2(\mathbb{C}^5)^*$, $a, b, c \in \mathbb{C}^2$, is a K-simple Kantor triple system with derivation algebra $\mathfrak{der}(V) = \mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$ and Tits-Kantor-Koecher Lie algebra $\mathfrak{g} = E_6$.

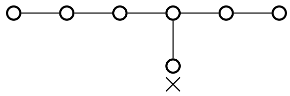
Proof. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the grade-reversing map defined by

$$\begin{aligned} \sigma(\xi) &= -\xi^\natural , & \sigma(\alpha \otimes a) &= \alpha^\natural \otimes Ja , \\ \sigma(A) &= -A^t , & \sigma(E) &= -E , \\ \sigma(\widetilde{\alpha} \otimes a) &= -\widetilde{\alpha}^\flat \otimes Ja , & \sigma(\widetilde{\xi}) &= -\widetilde{\xi}^\flat , \end{aligned} \tag{6.63}$$

for all $A \in \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C})$ and forms ξ, α, polyvectors $\widetilde{\xi}$, $\widetilde{\alpha}$, $a \in \mathbb{C}^2$. Clearly $\sigma^2 = 1$ and by the explicit expressions of the Lie brackets of E_6, one checks that σ is a Lie algebra morphism. □

6.4.2 The case $\mathfrak{g} = E_7$

The Lie algebra $\mathfrak{g} = E_7$ admits a special grading similar to that of E_6 in Subsection 6.4.1, i.e.,



with graded components $\mathfrak{g}_0 \simeq \mathfrak{sl}(7, \mathbb{C}) \oplus \mathbb{C}E$, $\mathfrak{g}_{-1} \simeq \Lambda^3(\mathbb{C}^7)^*$, $\mathfrak{g}_1 \simeq \Lambda^3\mathbb{C}^7$, $\mathfrak{g}_{-2} \simeq \Lambda^6(\mathbb{C}^7)^*$ and $\mathfrak{g}_2 \simeq \Lambda^6\mathbb{C}^7$. There is only one grade-reversing involution, i.e., the Chevalley involution. The symmetry algebra of the associated KTS is $\mathfrak{so}(7, \mathbb{C})$.

Let η be the standard non-degenerate symmetric bilinear form on $\mathbb{C}^7$, which we extend naturally to any $\Lambda^k\mathbb{C}^7$, $k \geqslant 0$. As in Subsection 6.4.1 we consider the $\mathfrak{sl}(7, \mathbb{C})$-equivariant projection

$$\bullet : \Lambda^k\mathbb{C}^7 \otimes \Lambda^k(\mathbb{C}^7)^* \to \mathfrak{sl}(7, \mathbb{C})$$

normalized so that

$$\boldsymbol{e}_{1\cdots k} \bullet \boldsymbol{e}^{1\cdots k} = \frac{7-k}{7}\sum_{i=1}^{k} \boldsymbol{e}_i \otimes \boldsymbol{e}^i - \frac{k}{7}\sum_{j=k+1}^{5} \boldsymbol{e}_j \otimes \boldsymbol{e}^j \ ,$$

where $(\boldsymbol{e}_i)$ is the standard basis of $\mathbb{C}^7$ and $(\boldsymbol{e}^i)$ the dual basis of $(\mathbb{C}^7)^*$. We denote by $\natural$ and $\flat$ the musical isomorphisms, inverse to each other, associated to η.

Theorem 6.31. *The vector space $V = \Lambda^3(\mathbb{C}^7)^*$ with triple product*

$$(\alpha\beta\gamma) = \frac{2}{7}\eta(\alpha, \beta)\gamma - (\beta^\sharp \bullet \alpha) \cdot \gamma$$

for all α, β, $\gamma \in V$ is a K-simple Kantor triple system with symmetry algebra $\mathfrak{der}(V) = \mathfrak{stab}_{\mathfrak{sl}(7,\mathbb{C})}(\eta) \simeq \mathfrak{so}(7, \mathbb{C})$ and Tits-Kantor-Koecher Lie algebra $\mathfrak{g} = E_7$.

Proof. The Lie brackets are given by the natural action of the grading element E, the standard action (resp. dual action) of the Lie algebra $\mathfrak{sl}(7, \mathbb{C})$ on $\Lambda^k\mathbb{C}^7$ (resp. $\Lambda^k(\mathbb{C}^7)^*$) for $k = 3, 6$ and

$$\begin{aligned}
&[\alpha, \beta] = \alpha \wedge \beta \ , && [\widetilde{\alpha}, \widetilde{\beta}] = -\widetilde{\alpha} \wedge \widetilde{\beta} \ ,\\
&[\widetilde{\alpha}, \beta] = -\tfrac{2}{7}\imath_{\widetilde{\alpha}}\beta\, E - (\widetilde{\alpha} \bullet \beta) \ , && [\widetilde{\alpha}, \xi] = i_{\widetilde{\alpha}}\xi \ ,\\
&[\widetilde{\xi}, \alpha] = i_\alpha \widetilde{\xi} \ , && [\widetilde{\xi}, \psi] = -\tfrac{4}{7}\imath_{\widetilde{\xi}}\psi\, E + (\widetilde{\xi} \bullet \psi) \ ,
\end{aligned}$$

with $\alpha, \beta \in \mathfrak{g}_{-1}$, $\widetilde{\alpha}, \widetilde{\beta} \in \mathfrak{g}_1$, $\xi, \psi \in \mathfrak{g}_{-2}$, $\widetilde{\xi}, \widetilde{\psi} \in \mathfrak{g}_2$. The grade reversing involution is

$$\begin{aligned}
\sigma(\xi) &= -\xi^\natural \ , & \sigma(\alpha) &= -\alpha^\natural \ ,\\
\sigma(A) &= -A^t \ , & \sigma(E) &= -E \ ,\\
\sigma(\widetilde{\alpha}) &= -\widetilde{\alpha}^\flat \ , & \sigma(\widetilde{\xi}) &= -\widetilde{\xi}^\flat
\end{aligned} \tag{6.64}$$

and the triple product follows as usual. $\square$

Chapter 7

3-graded Lie superalgebras

In this chapter we introduce the simple finite-dimensional Lie superalgebras and their 3-gradings.

Let $\mathfrak{g}$ be a classical simple Lie superalgebra, $\Delta = \Delta_{\bar{0}} + \Delta_{\bar{1}}$ its root system and $\mathfrak{h}$ a Cartan subalgebra. Any $\mathbb{Z}$-grading of $\mathfrak{g}$ is given by a degree function f on Δ and setting

$$\mathfrak{h} \subseteq \mathfrak{g}_0, e_\alpha \in \mathfrak{g}_j,\ e_{-\alpha} \in \mathfrak{g}_{-j} \text{ if } f(\alpha) = j, \tag{7.1}$$

where e_α is a root vector associated to the root α. Whenever $\mathfrak{g}$ is a matrix algebra $\mathfrak{h}$ will be a subspace of D, the set of diagonal matrices, and the roots will be expressed in terms of ϵ_i the elements of the standard basis of D^*.

When dealing with $\mathbb{Z}$-gradings of matrix superalgebras we will write

$$\left(\begin{array}{cc|cc} 0 & 1_{\bar{0}} & 0 & 1_{\bar{1}} \\ -1_{\bar{0}} & 0 & -1_{\bar{1}} & 0 \\ \hline 0 & 1_{\bar{1}} & 0 & 1_{\bar{0}} \\ -1_{\bar{1}} & & -1_{\bar{0}} & 0 \end{array}\right)$$

to denote that, for example, the elements of the form $\left(\begin{array}{cc|cc} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ are in $(\mathfrak{g}_1)_{\bar{0}}$, while the places left empty remark that there are no elements of $\mathfrak{g}$ with non-zero entries in those positions.

The classification of 3-graded simple linearly-compact Lie superalgebras $\mathfrak{g}$ with admissible grading is given in [14]. The notation for the simple Lie superalgebras follows [27].

Proposition 7.1 (N. Cantarini, V. G. Kac [14]). *A complete list of simple finite-dimensional 3-graded Lie superalgebras such that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have the same dimension, is, up to isomorphism, as follows:*

1. A_m, B_m, C_m, D_m, E_6, E_7 *with the* $\mathbb{Z}$*-gradings, defined for each* s *such that* $a_s = 1$ *by* $f(\alpha_s) = 1$, $f(\alpha_i) = 0$ *for all* $i \neq s$, *where* $\sum_i a_i \alpha_i$ *is the highest root;*

2. $psl(m, n)$ *with the* $\mathbb{Z}$*-gradings defined by:* $f(\epsilon_1) = \cdots = f(\epsilon_k) = 1$, $f(\epsilon_{k+1}) = \cdots = f(\epsilon_m) = 0$, $f(\delta_1) = \cdots = f(\delta_h) = 1$, $f(\delta_{h+1}) = \cdots = f(\delta_n) = 0$, *for each* $k = 1, \dots, m$ *and* $h = 0, \dots, n-1$;

3. $osp(2m+1, 2n)$ *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ *for all* $i \neq 1$, $f(\delta_j) = 0$ *for all* j;

4. $osp(2, 2n)$ *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_1) = 1$, $f(\delta_j) = 0$ *for all* j;

5. $osp(2, 2n)$ *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_1) = 1/2$, $f(\delta_j) = 1/2$ *for all* j;

6. $osp(2m, 2n)$, $m \geqslant 2$, *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ *for all* $i \neq 1$, $f(\delta_j) = 0$ *for all* j;

7. $osp(2m, 2n)$, $m \geqslant 2$, *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_i) = 1/2$, $f(\delta_j) = 1/2$ *for all* i, j;

8. $D(2, 1; \alpha)$ *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_1) = f(\epsilon_2) = 1/2$, $f(\epsilon_3) = 0$;

9. $F(4)$ *with the* $\mathbb{Z}$*-grading defined by:* $f(\epsilon_1) = 1$, $f(\epsilon_i) = 0$ *for all* $i \neq 1$, $f(\delta) = 1$;

10. $q(n)$ *with the gradings defined by:* $f(e_i) = f(\bar{e}_i) = -f(f_i) = -f(\bar{f}_i) = k_i$, *with* $k_s = 1$ *for some* s *and* $k_i = 0$ *for all* $i \neq s$;

11. $p(n)$ *with* $n = 2h \geqslant 2$ *and the grading defined by:* $\deg(e_h) = 1$ *on the even part and on the odd part the one induced by the even part;*

12. $H(0, n)$ *with the grading of type* $(|1, 0, \dots, 0, -1)$.

Remark 7.2. We point out that in the previous list all Lie superalgebras are classical except $H(0, n)$. The simple Lie superalgebra $H(0, n)$ is a special case of the infinite-dimensional Lie superalgebra of Cartan type $H(m, n)$. For this reason we will not deal with it in this work.

Definition 7.3. The Lie superalgebra $\mathfrak{gl}(m,n)$ is the Lie superalgebra of supermatrices, i.e.

$$\mathfrak{gl}(m,n) = \left\{ x \in M_{m+n}(\mathbb{C}) |\, x = \begin{array}{c} \\ m \\ n \end{array} \!\!\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \end{array} \right\},$$

the $\mathbb{Z}_2$-grading being

$$\mathfrak{gl}(m,n)_{\bar{0}} = \left\{ x \in \mathfrak{gl}(m,n) |\, x = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & d \end{array}\right) \right\},$$

$$\mathfrak{gl}(m,n)_{\bar{1}} = \left\{ x \in \mathfrak{gl}(m,n) |\, x = \left(\begin{array}{c|c} 0 & b \\ \hline c & 0 \end{array}\right) \right\}.$$

Here the bracket is given by the supercommutator $[x,y] = xy - (-1)^{|x||y|}yx$. We denote by $\mathfrak{sl}(m,n)$ the subsuperalgebra of $\mathfrak{gl}(m,n)$ consisting of supermatrices with zero supertrace, i.e.

$$\mathfrak{sl}(m,n) = \left\{ x \in \mathfrak{gl}(m,n) |\, x = \begin{array}{c} \\ m \\ n \end{array} \!\!\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \end{array},\ \operatorname{str}(x) := \operatorname{tr}(a) - \operatorname{tr}(d) = 0 \right\},$$

The simple Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(m,n)$ is defined by $\mathfrak{psl}(m,n) = \mathfrak{sl}(m,n)$, $m \neq n$ and $\mathfrak{psl}(n,n) = \mathfrak{sl}(n,n)/Id_{2n}$.

If $m \neq n$ (resp. $m = n$) the even part is isomorphic to $\mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}$ (resp. $\mathfrak{sl}(n,\mathbb{C}) \oplus \mathfrak{sl}(n,\mathbb{C})$) while the odd part is the sum of two irreducible representations of $\mathfrak{g}_{\bar{0}}$, namely $\mathfrak{g}_{\bar{1}} = V \oplus V^*$, $V \cong \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}$ (resp. $V \cong \mathbb{C}^n \otimes \mathbb{C}^n$) where $\mathbb{C}^k$ is the standard module for $\mathfrak{sl}(k,\mathbb{C})$. We denote by $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) the copy of $\mathfrak{sl}(m,\mathbb{C})$, resp. $\mathfrak{sl}(n,\mathbb{C})$, consisting of the elements of $\mathfrak{g}$ of the form $\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right)$ $\left(\text{resp. } \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & d \end{array}\right)\right)$.

The root system of $\mathfrak{psl}(m,n)$ is given in terms of linear functions $\epsilon_1, \ldots, \epsilon_m$ and $\delta_1 = \epsilon_{m+1}, \ldots, \delta_n = \epsilon_{m+n}$. The roots are

$$\Delta_{\bar{0}} = \{\epsilon_i - \epsilon_j,\ \delta_i - \delta_j,\ i \neq j\},\ \Delta_{\bar{1}} = \{\pm(\epsilon_i - \delta_j),\ i \neq j\}.$$

The grading of Proposition 7.1.2, denoted $\mathfrak{psl}(m,n)_{k,h}$, can be visualized with the following matrix

$$\begin{array}{c} \\ k \\ m-k \\ h \\ n-h \end{array} \begin{array}{c} \begin{array}{cccc} k & m-k & h & n-h \end{array} \\ \left(\begin{array}{cc|cc} 0 & 1_{\bar 0} & 0 & 1_{\bar 1} \\ -1_{\bar 0} & 0 & -1_{\bar 1} & 0 \\ \hline 0 & 1_{\bar 1} & 0 & 1_{\bar 0} \\ -1_{\bar 1} & 0 & -1_{\bar 0} & 0 \end{array}\right) \end{array}$$

If $m \neq n$ the grading element is $E = \mathrm{Diag}(x\mathrm{Id}_k, (x-1)\mathrm{Id}_{m-k}, x\mathrm{Id}_h, (x-1)\mathrm{Id}_{n-h})$, with $x = 1 - \frac{k-h}{m-n}$. If $m = n$ the element E with $E = \mathrm{Diag}(x\mathrm{Id}_k, (x-1)\mathrm{Id}_{m-k}, x\mathrm{Id}_h, (x-1)\mathrm{Id}_{n-h})$, and $x = 1 - \frac{k+h}{m+h}$ satisfies Equation (4.2), hence it acts as the grading element, and $\mathrm{str}(E) = (h-k)$, hence $E \notin \mathfrak{psl}(n,n)$, unless $k = h$ in which case we have $E \in \mathfrak{g}$. Note that, in general,

$$\begin{aligned} \dim \mathfrak{g}_{-1} = & \ (\ k(m-k) + h(n-h) \mid k(n-h) + h(m-k)\ ) \ = \\ = & \ \dim M_{(\,(m-k|n-h)\,,\,(k|h)\,)} . \end{aligned}$$

Remark 7.4. Notice that if $\mathfrak{g} = \mathfrak{psl}(m,n)_{k,h}$ and $x, z \in \mathfrak{g}_{-1}, y \in \mathfrak{g}_1$, then $[[x,y],z] = xyz + (-)^{\alpha(x,y,z)}zyx$. Indeed, we have $[[x,y],z] = xyz - (-1)^{|x||y|}yxz - (-1)^{|z|(|x|+|y|)}zxy + (-1)^{\alpha(x,y,z)}zyx$ and

$$\begin{aligned} yxz = & \left(\begin{array}{cc|cc} 0 & y_{11} & 0 & y_{12} \\ 0 & 0 & 0 & 0 \\ \hline 0 & y_{21} & 0 & y_{22} \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ x_{11} & 0 & x_{12} & 0 \\ \hline 0 & 0 & 0 & 0 \\ x_{21} & 0 & x_{22} & 0 \end{array}\right) z = \\ = & \left(\begin{array}{cc|cc} y_{11}x_{11} + y_{12}x_{21} & 0 & y_{11}x_{12} + y_{12}x_{22} & 0 \\ 0 & 0 & 0 & 0 \\ \hline y_{21}x_{11} + y_{22}x_{21} & 0 & y_{21}x_{12} + y_{22}x_{22} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ z_{11} & 0 & z_{12} & 0 \\ \hline 0 & 0 & 0 & 0 \\ z_{21} & 0 & z_{22} & 0 \end{array}\right) = 0\,. \end{aligned}$$

Similarly, it can be shown that $zxy = 0$.

Moreover, if we project $\mathfrak{g}_{-1}$ to $M_{\,(m-k|n-h)\,,\,(k|h)}(\mathbb{C})$, by

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ x_{11} & 0 & x_{12} & 0 \\ \hline 0 & 0 & 0 & 0 \\ x_{21} & 0 & x_{22} & 0 \end{array}\right) \to \left(\begin{array}{c|c} x_{11} & x_{12} \\ \hline x_{21} & x_{22} \end{array}\right)$$

and, in a similar way, $\mathfrak{g}_1$ to $M_{\,(k|h)\,,\,(m-k|n-h)}(\mathbb{C})$, we get an isomorphism of ϵ-sJTS between the one associated to $\mathfrak{g}$, with a grade-reversing ϵ-involution σ, and

$M_{(m-k|n-h),(k|h)}(\mathbb{C})$ with triple product defined by Equation (3.7), as explained in Remark 3.9, and with the map $\phi_\sigma = \sigma_{|\mathfrak{g}_{-1}}$. The fact that ϕ_σ satisfies the conditions given in Remark 3.9 follows from σ being a grade-reversing ϵ-involution.

Definition 7.5. Let V be a finite-dimensional super vector space and b an even non-degenerate bilinear superform on V, i.e. a non-degenerate bilinear form which is symmetric on $V_{\bar{0}}$, antisymmetric on $V_{\bar{1}}$ and such that $b(V_{\bar{0}}, V_{\bar{1}}) = b(V_{\bar{1}}, V_{\bar{0}}) = 0$. Notice that in this case $\dim(V_{\bar{1}})$ is necessarily even.

The orthosymplectic Lie superalgebra, denoted by $\mathfrak{osp}(b, V)$, is the Lie subalgebra consisting of endomorphisms which annihilate the superform b, $x \in \mathfrak{gl}(m, n)$ satisfying $b(x(v), w) = -(-1)^{|x||v|} b(v, x(w)),\ \forall v, w \in V$. If $\dim(V) = (m, 2n)$, then there is an obvious embedding of $\mathfrak{osp}(b, V)$ in $\mathfrak{sl}(m, n)$. In what follows we will consider b associated to the matrix $\begin{pmatrix} S_m & 0 \\ 0 & J_{2n} \end{pmatrix}$ and will denote this Lie superalgebra by $\mathfrak{osp}(m, 2n)$.

Let $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n),\ m > 0$, be the odd orthosymplectic Lie superalgebra. Then the elements of $\mathfrak{g}$ are of the form

$$X = \left(\begin{array}{ccc|cc} a & u & b & x_1 & x_2 \\ v & 0 & -u^R & y_1 & y_2 \\ c & -v^R & -a^R & z_1 & z_2 \\ \hline z_2^R & y_2^R & x_2^R & d & e \\ -z_1^R & -y_1^R & -x_1^R & f & -d^R \end{array}\right)$$

with $a, b, c \in M_m,\ b = -b^R,\ c = -c^R,\ u, v^t \in M_{m,1}(\mathbb{C}),\ d, e, f \in M_n,\ e = e^R,\ f = f^R$, $x_1, x_2, z_1, z_2 \in M_{m,n}(\mathbb{C}),\ y_1, y_2 \in M_{1,n}(\mathbb{C})$.

The even part of $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2m+1, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$ while the odd part is the irreducible $\mathfrak{g}_{\bar{0}}$-module $\mathbb{C}^{2m+1} \otimes \mathbb{C}^{2n}$, where $\mathbb{C}^{2m+1}$, resp. $\mathbb{C}^{2n}$, is the standard module of $\mathfrak{so}(2m+1)$, resp. $\mathfrak{sp}(2n, \mathbb{C})$. Its root system is given in term of linear functions $\epsilon_1, \ldots, \epsilon_m$ and $\delta_1 = \epsilon_{2m+1}, \ldots, \delta_n = \epsilon_{2m+n}$ and the roots are

$$\Delta_{\bar{0}} = \{\pm\epsilon_i \pm \epsilon_j,\ \pm\epsilon_i,\ \pm 2\delta_i,\ \pm\delta_i \pm \delta_j,\ i \neq j\},\ \Delta_{\bar{1}} = \{\pm\delta_i, \pm\epsilon_i \pm \delta_j\}.$$

The only 3-grading of $\mathfrak{g}$, up to isomorphism, is the one given by Proposition 7.1.3.

The grading can be represented by the following grading matrix:

$$
\begin{array}{c}
\\ 1 \\ 2m-1 \\ 1 \\ 2n
\end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2m-1 & 1 & 2n \end{array} \\
\left(\begin{array}{ccc|c}
0 & 1_{\bar 0} & & 1_{\bar 1} \\
-1_{\bar 0} & & 1_{\bar 0} & 0 \\
 & -1_{\bar 0} & 0 & -1_{\bar 1} \\ \hline
-1_{\bar 1} & 0 & 1_{\bar 1} & 0
\end{array}\right)
\end{array}
$$

The grading element is $E = \mathrm{Diag}(\,1\,,\,0_{2m-1}\,,\,-1\,,\,0_{2n}\,)$.

Let $\mathfrak{g} = osp(2, 2n)$ be the symplectic Lie superalgebra, whose elements are of the form

$$
X = \left(\begin{array}{cc|cc}
a & 0 & x_1 & x_2 \\
0 & -a & y_1 & y_2 \\ \hline
y_2^R & x_2^R & d & e \\
-y_1^R & -x_1^R & f & -d^R
\end{array}\right)
$$

with $a \in \mathbb{C},\ d, e, f \in M_n,\ e = e^R,\ f = f^R,\ x_1, x_2, y_1, y_2 \in M_{1,n}(\mathbb{C})$

The even part is isomorphic to $\mathbb{C} \oplus sp(2n, \mathbb{C})$ while the odd part is the sum of two irreducible $\mathfrak{g}_{\bar 0}$-modules $V(\omega_1) \oplus V(\omega_1)^*$, namely the first fundamental module and its dual. The root system is given in terms of linear functions ϵ_1 and $\delta_1 = \epsilon_3, \ldots, \delta_n = \epsilon_{n+1}$ and the roots are

$$
\Delta_{\bar 0} = \{\pm 2\delta_i,\ \pm\delta_i \pm \delta_j,\ i \neq j\},\ \Delta_{\bar 1} = \{\pm\epsilon_1 \pm \delta_j\}.
$$

The grading of Proposition 7.1.4 is of the form:

$$
\begin{array}{c}
\\ 1 \\ 1 \\ 2n
\end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 1 & 2n \end{array} \\
\left(\begin{array}{cc|c}
0 & & 1_{\bar 1} \\
 & 0 & -1_{\bar 1} \\ \hline
-1_{\bar 1} & 1_{\bar 1} & 0
\end{array}\right)
\end{array}
$$

and the grading element is $E = \frac{1}{2}\mathrm{Diag}(1, -1, 0_{2n})$.

The grading of Proposition 7.1.5 can be visualized as follows:

$$
\begin{array}{c}
\\ 1 \\ 1 \\ n-1 \\ n-1
\end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 1 & n-1 & n-1 \end{array} \\
\left(\begin{array}{cc|cc}
0 & & 0 & 1_{\bar 1} \\
 & 0 & -1_{\bar 1} & 0 \\ \hline
0 & 1_{\bar 1} & 0 & 1_{\bar 0} \\
-1_{\bar 1} & 0 & -1_{\bar 0} & 0
\end{array}\right)
\end{array}
$$

The grading element is $E = \frac{1}{2}\mathrm{Diag}(1, -1, \mathrm{Id}_n, -\mathrm{Id}_n)$.

Let $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ be the even orthosymplectic Lie superalgebra. We have that the elements of $\mathfrak{g}$ are of the form

$$X = \left(\begin{array}{cc|cc} a & b & x_1 & x_2 \\ c & -a^R & y_1 & y_2 \\ \hline y_2^R & x_2^R & d & e \\ -y_1^R & -x_1^R & f & -d^R \end{array}\right)$$

with $a, b, c \in M_m,\ b = -b^R,\ c = -c^R,\ d, e, f \in M_n,\ e = e^R,\ f = f^R,$ $x_1, x_2, y_1, y_2 \in M_{m,n}(\mathbb{C})$.

The even part is isomorphic to $\mathfrak{so}(2m, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$ while the odd part is the irreducible $\mathfrak{g}_{\bar{0}}$-module $\mathbb{C}^{2m} \otimes \mathbb{C}^{2n}$, where $\mathbb{C}^{2m}$ is the standard module of $\mathfrak{so}(2m, \mathbb{C})$ and $\mathbb{C}^{2n}$ is the standard module of $\mathfrak{sp}(2n, \mathbb{C})$. The root system is given in terms of linear functions $\epsilon_1, \dots, \epsilon_m$ and $\delta_1 = \epsilon_{2m+1}, \dots, \delta_n = \epsilon_{2m+n}$. The roots are

$$\Delta_{\bar{0}} = \{\pm\epsilon_i \pm \epsilon_j,\ \pm 2\delta_i,\ \pm\delta_i \pm \delta_j,\ i \neq j\},\ \Delta_{\bar{1}} = \{\pm\epsilon_i \pm \delta_j\}.$$

The 3-grading given in Proposition 7.1.6 can be treated in a similar way to the one of $\mathfrak{osp}(2m+1, 2n)$ and the grading element is the same without the $(m+1)$-th row and column.

The 3-grading of Proposition 7.1.7 is of the form

$$\begin{array}{c} \\ m \\ m \\ n \\ n \end{array} \begin{array}{c} \begin{array}{cccc} m & m & n & n \end{array} \\ \left(\begin{array}{cc|cc} 0 & 1_{\bar{0}} & 0 & 1_{\bar{1}} \\ -1_{\bar{0}} & 0 & -1_{\bar{1}} & 0 \\ \hline 0 & 1_{\bar{1}} & 0 & 1_{\bar{0}} \\ -1_{\bar{1}} & 0 & -1_{\bar{0}} & 0 \end{array}\right) \end{array}$$

and the grading element is $E = \frac{1}{2}\mathrm{Diag}(\mathrm{Id}_m, -\mathrm{Id}_m, \mathrm{Id}_n, -\mathrm{Id}_n)$.

Remark 7.6. Let $\mathfrak{g} = \mathfrak{osp}(m+2, 2n)$ with one of the 3-gradings of Proposition 7.1.3 , 4 , 6. If we identify $\mathfrak{g}_{-1}$ with $M_{(m|2n)(1|0)}(\mathbb{C})$ and $\mathfrak{g}_1$ with $M_{(1|0)(m|2n)}(\mathbb{C})$ by setting

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ 0 & -x_1^R & 0 & x_2' \\ \hline x_2 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 0 & y_1 & 0 & y_2 \\ 0 & 0 & -y_1^R & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & y_2'' & 0 \end{array}\right) \to \left(\begin{array}{c|c} y_1 & y_2 \end{array}\right)$$

with $x_2' = x_2^t J_{2n}, y_2'' = J_{2n} y_2^t$. By a straightforward calculation it follows that the triple product induced by $(\mathfrak{g}, \sigma)$ on $M_{(m|n)(1|0)}(\mathbb{C})$ is the one given in Example 3.11 and the triple product is

$$(x, y, z) = x\phi(y)z + (-1)^{\alpha(x,y,z)} z\phi(y)x - (x\phi(y))^{os\tau} z$$

where $\phi : M_{(m|2n)(1|0)}(\mathbb{C}) \to M_{(1|0)(m|2n)}(\mathbb{C})$ is determined by $\sigma_{|\mathfrak{g}_{-1}}$ and $os\tau$ is the map defined in Example 3.10.

It can be easily shown that this grading is induced in a natural way by the 5-grading of $sl(m+2, 2n)$ defined by $f(\epsilon_1) = -f(\epsilon_{m+2}) = 1$.

Definition 7.7. The simple Lie superalgebra $\mathfrak{g} = p(n)$, $n > 2$ is the Lie subalgebra of $sl(n, n)$ given by

$$p(n) = \{x \in psl(n, n), x = \left(\begin{array}{c|c} a & b \\ \hline c & -a^t \end{array}\right) \mid a \in sl(n, \mathbb{C}),\ b = b^t,\ c = -c^t\}$$

We have that $\mathfrak{g}_{\bar{0}} \cong sl(n, \mathbb{C})$ and $\mathfrak{g}_{\bar{1}}$ is the sum of the two irreducible $sl(n, \mathbb{C})$-modules $\Lambda^2(\mathbb{C}^n)^*$ and $S^2\mathbb{C}^n$.

The 3-grading of $p(n), n \in 2\mathbb{N}$, of Proposition 7.1.11 is not induced by $psl(n, n)_{k,h}$. It is convenient for our purposes to consider instead of $p(n)$ its image under the isomorphism $AdDiag(Id, S_n)$, which we still denote $p(n)$, with

$$p(n) = \{x \in psl(n, n), x = \left(\begin{array}{c|c} a & b \\ \hline c & -a^R \end{array}\right) \mid a \in sl(n, \mathbb{C}),\ b = b^R,\ c = -c^R\}.$$

With this identification, the 3-grading of $p(n)$ is the one induced by $psl(n, n)_{h,h}, n = 2h$.

Definition 7.8. The simple Lie superalgebra $\mathfrak{g} = q(n)$, $n > 2$ is the Lie subalgebra of $psl(n, n)$ given by

$$q(n) = \{x \in psl(n, n), x = \left(\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right) \mid tr(b) = 0\}$$

In $q(n)$ we consider the two copies of $sl(n, \mathbb{C})$ given by $q(n)_{\bar{0}}$ and $q(n)_{\bar{1}}$, and denote by e_α and $\bar{e}_\alpha$, respectively, the root vectors of the even $sl(n, \mathbb{C})$ and of the odd $sl(n, \mathbb{C})$.

The 3-grading of $q(n)$ of Proposition 7.1.10, denoted $q(n)_s$, is that induced by $psl(n, n)_{s,s}$ and the grading element is the same of $psl(n, n)_{s,s}$.

Remark 7.9. Notice that in all of the preceding cases, except for $so(m, 2n)$ with the 3-gradings of Proposition 7.1.3,4,6, if $x, z \in \mathfrak{g}_{-1}, y \in \mathfrak{g}_1$, then $[[x, y], z] = xyz +$

$(-)^{\alpha(x,y,z)}zyx$. This follows from Remark 7.4 and the fact that all the Lie superalgebras are subalgebras of $\mathfrak{psl}(m,n)$ with the induced 3-grading. As a consequence we can view $\mathfrak{g}_{-1}$ as a sub ϵ-sJTS of $M_{(m-k|n-h),(k|h)}(\mathbb{C})$. We list all the corresponding subspaces

$\mathfrak{g}$	$\mathfrak{g}_{-1}$
$\mathfrak{osp}(2m,2n)$	$\{x \in M_{(m\|n)}(\mathbb{C}) \| x = \left(\begin{array}{c\|c} a & b \\ \hline -b^R & d \end{array}\right), a = -a^R, d = d^R\}$
$\mathfrak{p}(n)$	$\{x \in M_{(n\|n)}(\mathbb{C}) \| x = \left(\begin{array}{c\|c} a & b \\ \hline c & -a^R \end{array}\right), b = b^R, c = -c^R\}$
$\mathfrak{q}(n)$	$\{x \in M_{(m\|n)}(\mathbb{C}) \| x = \left(\begin{array}{c\|c} a & b \\ \hline b & a \end{array}\right)\}$

In the first case the projection of $\mathfrak{g}_1$ is

$$\{x \in M_{(m|n)}(\mathbb{C})|x = \left(\begin{array}{c|c} a & b \\ \hline b^R & d \end{array}\right), a = -a^R, d = d^R\}$$

while in the other cases is equal to that of $\mathfrak{g}_{-1}$.

Definition 7.10. Let $\alpha \in \mathbb{C}\backslash\{0,-1\}$. The simple Lie superalgebra $D(\alpha) = D(2,1;\alpha)$ is the classical Lie superalgebra with Cartan matrix $\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$, see [27]. It has even part isomorphic to $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$, odd part isomorphic to the irreducible $\mathfrak{g}_{\bar{0}}$-module $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, where $\mathbb{C}^2$ is the standard $\mathfrak{sl}(2,\mathbb{C})$-module. We will denote by $\mathfrak{a}_i = \langle h_i, e_i, f_i\rangle, i = 1,2,3$, the i-th copy of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{g}_{\bar{0}}$ and we let $\mathbb{C}^2 = \langle w_+, w_-\rangle$ be the standard $\mathfrak{sl}(2,\mathbb{C})$-module, with $[h_i, w_\pm] = \pm w_\pm$. We will write $w_{(++)} \otimes w$ instead of $w_+ \otimes w_+ \otimes w \in \mathfrak{g}_1$, $w \in \mathbb{C}^2$.

The root system of $\mathfrak{g}$ is given by

$$\Delta_{\bar{0}} = \{\pm 2\epsilon_i\},\ \Delta_{\bar{1}} = \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\},$$

where $\pm 2\epsilon_i$ are the corresponding roots of the i-th copy of $\mathfrak{sl}(2,\mathbb{C})$. The product between odd elements is given by:

$$[u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3] = \sum_{\substack{\sigma \in S_3 \\ \sigma^3 = Id}} (u_{\sigma(1)}, v_{\sigma(1)})_J (u_{\sigma(2)}, v_{\sigma(2)})_J \Psi_{\sigma(3)}(u_{\sigma(3)}, v_{\sigma(3)}), \quad (7.2)$$

where $(,)_J$ is the standard symplectic form on $\mathbb{C}^2$ and Ψ_i is the map from $\mathbb{C}^2 \otimes \mathbb{C}^2$ to the i-th copy of $\mathfrak{sl}(2,\mathbb{C})$ defined by $\Psi_i(u_i, v_i)(w) = \lambda_i((v_i, w)_J u_i - (w, u_i)_J v_i)$ and

$\lambda_1 = \frac{\alpha}{2}, \lambda_2 = \frac{1}{2}, \lambda_3 = -\frac{1+\alpha}{2}$. Recall that $D(\alpha) \cong so(4,2)$ if $\alpha \in \{1, -2, -\frac{1}{2}\}$ and that $D(\alpha) \cong D(\beta)$ if $\alpha = \{(1+\beta)^{\pm 1}, \beta^{-1}, -(\frac{\beta}{1+\beta})^{\pm 1}\}$, see e.g. [27].

Let $\alpha \in \mathbb{C} - \{0, -1, 1, -2, -\frac{1}{2}\}$. The 3-grading of Proposition 7.1.8 is given by

$$D(\alpha)_i = \langle e_{i(2\epsilon_1)}, e_{i(2\epsilon_2)} | e_{i(\epsilon_1+\epsilon_2\pm\epsilon_3)} \rangle, \; i = \pm 1, \; D(\alpha)_0 = sl(1,2) \oplus \mathbb{C}E \,. \tag{7.3}$$

In particular, $D(\alpha)_{\bar{0}}$ consists of the direct sum of two copies of $sl(2,\mathbb{C})$ with the fundamental 3-grading and one with trivial grading. In terms of a_i we have

$$D(\alpha)_{\pm 1} = \langle e_{(1\pm)}, e_{(2\pm)} | w_{(\pm\pm)} \otimes w \rangle \,.$$

The purpose of the next remark is to calculate the triple commutator $[[\mathfrak{g}_{-1}, \mathfrak{g}_1], \mathfrak{g}_{-1}]$ of $\mathfrak{g} = D(\alpha)$.

Remark 7.11. Let $\alpha \in \mathbb{C} - \{0, -1, 1, -2, -\frac{1}{2}\}$ and let $\mathfrak{g}$ denote $D(\alpha)$, as introduced in Definition 7.10. Let $x, z \in \mathfrak{g}_{-1}$, and $y \in \mathfrak{g}_1$,

$$\begin{aligned} x &= a_x e_{(1-)} + b_x e_{(2-)} + w_{(--)} \otimes w_x \,, \\ y &= -a_y e_{(1+)} - b_y e_{(2+)} + w_{(++)} \otimes w_y \,, \\ z &= a_z e_{(1-)} + b_z e_{(2-)} + w_{(--)} \otimes w_z \,, \end{aligned}$$

with $a_x, a_y, a_z, b_x, b_y, b_z \in \mathbb{C}$ and $w_x, w_y, w_z \in \mathbb{C}^2$. We have

$$\begin{aligned} [x,y] = \; & [a_x e_{(1-)} + b_x e_{(2-)} + w_{(--)} \otimes w_x, -a_y e_{(1+)} - b_y e_{(2+)} + w_{(++)} \otimes w_y] = \\ = \; & a_x a_y h_1 + a_x w_{(-+)} \otimes w_y \\ & + b_x b_y h_2 + b_x w_{(+-)} \otimes w_y \\ & + a_y w_{(+-)} \otimes w_x + b_y w_{(-+)} \otimes w_x \\ & + (w_-, w_+)_J (w_-, w_+)_J \Psi_3(w_x, w_y) \\ & + (w_-, w_+)_J (w_x, w_y)_J (\Psi_1(w_-, w_+) + \Psi_2(w_-, w_+)) \\ = \; & a_x a_y h_1 + b_x b_y h_2 \\ & + (w_x, w_y)_J (\lambda_1 h_1 + \lambda_2 h_2) + \Psi_3(w_x, w_y) \\ & + w_{(-+)} \otimes (a_x w_y + b_y w_x) \\ & + w_{(+-)} \otimes (b_x w_y + a_y w_x) \end{aligned}$$

Hence, using the definitions of the bracket of $\mathfrak{g}$ and since $\lambda_1+\lambda_2+\lambda_3=0$, we get

$$\begin{aligned}
[[x,y],z] = \;& [[x,y], a_z e_{(1-)} + b_z e_{(2-)} + w_{(--)} \otimes w_z] \\
= \;& -2a_x a_y a_z e_{(1-)} - 2\lambda_1 a_z (w_x, w_y)_J e_{(1-)} \\
& - w_{(--)} \otimes a_z (b_x w_y + a_y w_x) \\
& -2b_x b_y b_z e_{(2-)} - 2\lambda_2 b_z (w_x, w_y)_J e_{(2-)} \\
& - w_{(--)} \otimes b_z (a_x w_y + b_y w_x) \\
& -(a_x a_y + b_x b_y) w_{(--)} \otimes w_z \\
& -(\lambda_1 + \lambda_2)(w_x, w_y)_J w_{(--)} \otimes w_z \\
& + w_{(--)} \otimes (\lambda_3((w_y, w_z)_J w_x - (w_z, w_x)_J w_y)) \\
& + (w_+, w_-)_J ((a_x w_y + b_y w_x), w_z)_J \Psi_1(w_-, w_-) \\
& + ((b_x w_y + a_y w_x), w_z)_J (w_+, w_-)_J \Psi_2(w_-, w_-) \\
= \;& -2(a_x a_y a_z + \lambda_1((w_x, w_y)_J a_z - (w_z, w_y)_J a_x + (w_x, w_z)_J b_y)) e_{(1-)} \\
& -2(b_x b_y b_z + \lambda_2((w_x, w_y)_J b_z - (w_z, w_y)_J b_x + (w_x, w_z)_J a_y)) e_{(2-)} \\
& - w_{(--)} \otimes ((a_x a_y + b_x b_y) w_z + (a_z a_y + b_z b_y) w_x + (a_x b_z + b_x a_z) w_y) \\
& - w_{(--)} \otimes (\lambda_1 + \lambda_2)((w_x, w_y)_J w_z - (w_z, w_y)_J w_x + (w_x, w_y)_J w_y) \,.
\end{aligned} \tag{7.4}$$

Definition 7.12. The exceptional simple Lie superalgebra $\mathfrak{g} = F(4)$ is the classical Lie superalgebra with Cartan matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$. The even part is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(7,\mathbb{C})$, the odd part is isomorphic to the irreducible module $\mathbb{C}^2 \otimes \mathrm{Spin}_7$, where Spin_7 is the highest-weight module with highest weight ω_3, the third fundamental weight of $\mathfrak{so}(7,\mathbb{C})$. The root system of $\mathfrak{g}$ is given by

$$\Delta_{\bar{0}} = \{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm\delta,\ i \neq j\},\ \Delta_{\bar{1}} = \{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)\},$$

where δ is the simple root of $\mathfrak{sl}(2,\mathbb{C})$ and $\epsilon_i,\ i = 1,2,3$ generate the root system of $\mathfrak{so}(7,\mathbb{C})$.

Let $(,)_A$ be the standard symplectic form on $\mathbb{C}^2$, $(,)_B$ be the, unique up to a scalar, invariant symmetric form on Spin_7, $\mathcal{B}$ a basis of $\mathfrak{so}(7,\mathbb{C})$ and

$$\begin{aligned}
\Psi_A : S^2\mathbb{C}^2 \to \mathfrak{sl}(2,\mathbb{C}) : \quad & \Psi_A(v_1, v_2)(v) = (v_2, v)v_1 - (v, v_1)v_2 \,, v \in \mathbb{C}^2 \\
\Psi_B : \Lambda^2 \mathrm{Spin}_7 \to \mathfrak{so}(7,\mathbb{C}) : \quad & \Psi_B(s_1, s_2) = \textstyle\sum_{x \in \mathcal{B}} (s_1, -x^t \cdot s_2)_B x \,.
\end{aligned}$$

The product of odd elements, $v_i \otimes s_i \in \mathbb{C}^2 \otimes \mathrm{Spin}_7$, is given by (cf. [50])

$$[v_1 \otimes s_1, v_2 \otimes s_2] = -\frac{4}{3}(v_1, v_2)_A \Psi_B(s_1, s_2) + (s_1, s_2)_B \Psi_A(v_1, v_2) \,,$$

The simple Lie superalgebra of type $F(4)$ has, up to equivalence, the 3-grading of Proposition 7.1.9. We have

$$\begin{array}{cc} \mathfrak{g}_i,\ i=\pm 1, & \mathfrak{g}_0 \\ \langle e_{i\delta}, e_{i\epsilon_1}, e_{i(\epsilon_1\pm\epsilon_2)}, e_{i(\epsilon_1\pm\epsilon_3)} | e_{\frac{i}{2}(\delta+\epsilon_1\pm\epsilon_2\pm\epsilon_3)}\rangle & osp(2,4)\oplus\mathbb{C}E. \end{array} \tag{7.5}$$

In particular, $\mathfrak{g}_{\bar{0}}$ is the direct sum of $sl(2,\mathbb{C})$ with the fundamental 3-grading and $so(7,\mathbb{C})$ with the 3-grading corresponding to the Dynkin diagram with the first node marked.

In the next remark we give an explicit realization of $\mathfrak{g} = F(4)$ and compute the triple commutator $[[\mathfrak{g}_{-1},\mathfrak{g}_1],\mathfrak{g}_{-1}]$ which will be needed in the next chapter.

Remark 7.13. Let U denote $\mathbb{C}^7$ with the bilinear form $(\cdot,\cdot)$ with matrix S_7, let $C\ell(U)$ denote the Clifford algebra, whose product we denote by juxtaposition, with defining relation $v^2 = (v,v)\mathbb{1}$, where $\mathbb{1}$ denotes the identity element of $C\ell(U)$. Let $\mathcal{C}$ be the basis of U consisting of $u_p = e_p, 1 \leqslant i \leqslant 4, u_p^* = e_{8-p}, p = 1,2,3,$ where $\{e_p\}$ is the standard basis of $\mathbb{C}^7$ and consider the isotropic decomposition of U given by $U = W \oplus \mathbb{C}u_4 \oplus W^*$, with $W = \langle u_i, i = 1,2,3\rangle, W^* = \langle u_i^*, i = 1,2,3\rangle$. We define the action of $u = w + cu_4 + w^* \in U$ on $s \in \Lambda^\bullet W$ by

$$u \circ s = \sqrt{2}(w \wedge s + i_{w^*}(s)) + (-1)^{\deg(s)} cs\,,$$

where $\deg(s)$ is the degree of s as an element of the exterior algebra and i_{w^*} denotes the interior product induced by the dual pairing between W and W^*. Since the action of U satisfies the defining relation of $C\ell(U)$, it extends to an action of $C\ell(U)$, thus we can identify $Spin_7$ with $\Lambda^\bullet W$ as $C\ell(U)$-modules. Moreover, we identify $so(7,\mathbb{C})$ with $\Lambda^2 U \subset C\ell(U), u\wedge v = \frac{1}{2}(uv - vu)$, by setting

$$(u\wedge v)(w) = (v,w)u - (w,u)v,\ u,v,w \in U,$$

and let $\Lambda^2 U$ acts on $\Lambda^\bullet W$ by

$$(u\wedge v)\cdot s = \frac{1}{2}u\wedge v\circ s = \frac{1}{4}(uv - vu)\circ s\,.$$

In this way we get the isomorphism $Spin_7 \cong \Lambda^\bullet W$ as $so(7,\mathbb{C})$-modules.

Let U' be the subspace of U with basis $\mathcal{C}' = \{u_2,u_3,u_4,u_3^*,u_2^*\}$, with the induced scalar product. Let $so(5,\mathbb{C})$ be embedded in $so(7,\mathbb{C})$ via the embedding induced by $\Lambda^2 U'$ in $\Lambda^2 U$. The restricted representation of $Spin_7 = \Lambda^\bullet W$ to $so(5,\mathbb{C})$ splits in two irreducible representations isomorphic to $Spin_5$, namely $Spin_7 = \mathbb{S}_1 \oplus \mathbb{S}_2,\ \mathbb{S}_1 = \Lambda^\bullet W', \mathbb{S}_2 = u_1 \wedge \Lambda^\bullet W'$, with $W' = \langle u_2,u_3\rangle \subset W$. The actions of $C\ell(U')$ on U' and $\mathbb{S}_1, \mathbb{S}_2$ are the restricted ones.

Let $\mathfrak{g}$ denote $F(4)$ with the 3-grading of Proposition 7.1.9, $f, h, e \in \mathfrak{g}_{\bar{0}}$ be the standard basis of $sl(2,\mathbb{C})$ and let $\mathbb{C}^2 = \langle v_+, v_- \rangle$ denote the standard $sl(2,\mathbb{C})$-module with $[h, v_\pm] = \pm v_\pm$. If we take as basis of the Cartan subalgebra of $so(7,\mathbb{C})$ the set $\{h_i = u_i \wedge u_i^* | i = 1,2,3\}$ we have that, for example, $u_1 \wedge u_2$ is a weight vector with weight $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. Indeed, $[h_i, u_1 \wedge u_2] = \frac{1}{2}(u_i \wedge u_i^*) \circ (u_1 \wedge u_2) = \frac{1}{4}(u_i u_i^* - u_i^* u_i) \circ (u_1 \wedge u_2) = \pm\frac{1}{2}(u_1 \wedge u_2)$, with the minus sign only if $k = 3$. Similar calculations yield the following identifications:

$$\begin{aligned}
(\mathfrak{g}_{-1})_{\bar{0}} &\cong \mathbb{C}f \oplus U' \wedge u_1^*\,, & (\mathfrak{g}_{-1})_{\bar{1}} &\cong \mathbb{C}v_- \otimes \mathbb{S}_1\,,\\
(\mathfrak{g}_0)_{\bar{0}} &\cong \mathbb{C}h \oplus \Lambda^2 U' \oplus \mathbb{C}E\,, & (\mathfrak{g}_0)_{\bar{1}} &\cong \mathbb{C}v_+ \otimes \mathbb{S}_1 \oplus \mathbb{C}v_- \otimes \mathbb{S}_2\,,\\
(\mathfrak{g}_1)_{\bar{0}} &\cong \mathbb{C}e \oplus u_1 \wedge U'\,, & (\mathfrak{g}_1)_{\bar{1}} &\cong \mathbb{C}v_+ \otimes \mathbb{S}_2\,,
\end{aligned}$$

with $E = \frac{1}{2}h + u_1 \wedge u_1^*$. Note that $\mathfrak{g}_0 \cong osp(2,4) \oplus \mathbb{C}E$ in a non-canonical way, due to the exceptional isomorphism $sp(4,\mathbb{C}) \cong so(5,\mathbb{C})$.

Let $x, z \in \mathfrak{g}_{-1},\ x = a_x f + u_x \wedge u_1^* + v_- \otimes s_x,\ z = a_z f + u_z \wedge u_1^* + v_- \otimes s_z$ and $y \in \mathfrak{g}_1,\ y = -a_y e - u_1 \wedge u_y + v_+ \otimes s_y$ with $a_x, a_y, a_z \in \mathbb{C}$, $u_x, u_y, u_z \in U'$, $s_x, s_y, s_z \in \Lambda^\bullet W'$. We have

$$\begin{aligned}
[x,y] = {} & [a_x f + u_x \wedge u_1^* + v_- \otimes s_x, -a_y e - u_1 \wedge u_y + v_+ \otimes u_1 \wedge s_y] = \\
= {} & a_x a_y h + a_x v_- \otimes u_1 \wedge s_y - [u_x \wedge u_1^*, u_1 \wedge u_y] + \\
& + v_+ \otimes (u_x \wedge u_1^* \cdot (u_1 \wedge s_y)) + a_y v_+ \otimes s_x + \\
& + v_- \otimes ((u_1 \wedge u_y) \cdot (s_x)) + [v_- \otimes s_x, v_+ \otimes u_1 \wedge s_y] = \\
= {} & a_x a_y h + a_x v_- \otimes u_1 \wedge s_y - u_x \wedge u_y + (u_x, u_y) u_1 \wedge u_1^* + \\
& + v_+ \otimes (\frac{1}{\sqrt{2}} u_x \circ s_y) + a_y v_+ \otimes s_x + v_- \otimes (\frac{1}{\sqrt{2}} u_1 \wedge (u_y \circ (s_x))) - \\
& + [v_- \otimes s_x, v_+ \otimes u_1 \wedge s_y]\,,
\end{aligned}$$

where we used $[v_1 \wedge v_2, v_3 \wedge v_4] = (v_2, v_3) v_1 \wedge v_4 - (v_4, v_1) v_3 \wedge v_2 - (v_1, v_3) v_2 \wedge v_4 + (v_2, v_4) v_1 \wedge v_3,\ v_i \in U, 1 \leqslant i \leqslant 4$.

Using the invariance of the form $(,)_B$, i.e. $(x \cdot s, t)_B = -(s, x \cdot t)_B, x \in \Lambda^2 U, s, t \in \Lambda^\bullet W$, if we fix $(\mathbb{1}, \omega)_B = 1,\ \omega = u_1 \wedge u_2 \wedge u_3 \in \Lambda^3 W$, we get

$$(s,t)_B = (-1)^{[\frac{\deg(s)+1}{2}]} i_{\omega^*}(s \wedge t)\,,$$

where the square bracket in the exponent denotes the integer part. We claim that $(,)_B$ is admissible with invariants $(-1, 1)$ (see Definition 6.5). Let s, t be elements of the standard basis of $\Lambda^\bullet W$ and $k = 1,2,3$. If $(u_k \circ s, t)_B = 0$ then $(s, u_k \circ t)_B = 0$, while if $(u_k \circ s, t)_B \neq 0$, we have

$$\begin{aligned}
(s, u_k \circ t)_B = {} & \sqrt{2}(-1)^{[\frac{\deg(s)+1}{2}]} i_{\omega^*}(s \wedge u_k \wedge t) \\
& \sqrt{2}(-1)^{[\frac{\deg(s)+1}{2}]+\deg(s)} i_{\omega^*}(u_k \wedge s \wedge t) \\
& (-1)^{[\frac{\deg(s)+2}{1}]+\deg(s)+[\frac{\deg(s)+1}{1}]}(u_k \circ s, t)\,.
\end{aligned}$$

A case by case check shows that $(-1)^{[\frac{\deg(s)+2}{1}]+\deg(s)+[\frac{\deg(s)+1}{1}]} = -1$. Similarly, the same relation can be proved for u_k^*, whereas for u_4 it is an immediate check.

In order to compute Ψ_B we consider the standard basis of $\Lambda^2 U$ (resp. of $\Lambda^2 U'$), denoted $\mathcal{B}$ (resp. $\mathcal{B}'$), consisting of the elements $v_i \wedge v_j$, where v_i, v_j are elements of $\mathcal{C}$ (resp. $\mathcal{C}'$). Note that, in particular, $u_1 \wedge u_1^* \in \mathcal{B}$ and $\mathcal{B} \supset \mathcal{B}'$. Moreover, in term of $\Lambda^2 U$ the map $x \to -x^t$ corresponds to $u \wedge v \to u^* \wedge v^*$, if x is identified with $u \wedge v$. Let $\Psi_{B'}(s,t)$ denote the map from $Spin_5 \otimes Spin_5$ to $\Lambda^2 U'$ given by

$$\Psi_{B'}(s,t) = \sum_{(u \wedge v) \in \mathcal{B}'} (s, (u^* \wedge v^*) \cdot (u_1 \wedge t))_B (u \wedge v)\,.$$

We have

$$\begin{aligned}
& [v_- \otimes s_x, v_+ \otimes u_1 \wedge s_y] = \\
= \; & -\tfrac{4}{3}\Psi_B(s_x, u_1 \wedge s_y) - (s_x, u_1 \wedge s_y)_B h = \\
= \; & -\tfrac{4}{3}\Psi_{B'}(s_x, u_1 \wedge s_y) + \tfrac{2}{3}(s_x, u_1 \wedge s_y) u_1 \wedge u_1^* \\
& -(s_x, u_1 \wedge s_y)_B h\,,
\end{aligned} \tag{7.6}$$

since $(s_x, (u^* \wedge v^*) \cdot (u_1 \wedge s_y)) = 0$, if $(u \wedge v) \in u_1 \wedge U' \oplus U' \wedge u_1^*$.

Summing up, we have

$$\begin{aligned}
[x,y] = \; & (a_x a_y - (s_x, u_1 \wedge s_y)_B)h - \\
& -u_x \wedge u_y - \tfrac{4}{3}\Psi_B(s_x, u_1 \wedge s_y) + \\
& +((u_x, u_y))u_1 \wedge u_1^* + \\
& +v_+ \otimes ((\frac{1}{\sqrt{2}} u_x \circ s_y) + a_y s_x) + \\
& +v_- \otimes ((\frac{1}{\sqrt{2}} u_1 \wedge (u_y \circ (s_x))) + a_x u_1 \wedge s_y)\,,
\end{aligned}$$

hence

$$
\begin{aligned}
[[x,y],z] = \; & -2a_z(a_x a_y - (s_x, u_1 \wedge s_y)_B)f + \\
& + \nu_- \otimes (-(a_x a_y - (s_x, u_1 \wedge s_y)_B)s_z) - \\
& -(u_y, u_z)u_x \wedge u_1^* + (u_x, u_z)u_y \wedge u_1^* - \\
& + \nu_- \otimes -\tfrac{1}{2}(u_x \wedge u_y) \circ s_z - \\
& -\tfrac{4}{3}[\Psi_B(s_x, u_1 \wedge s_y), u_z \wedge u_1^*] - \\
& + \nu_- \otimes (-\tfrac{4}{3}\Psi_B(s_x, u_1 \wedge s_y) \cdot s_z) + \\
& -((u_x, u_y))u_z \wedge u_1^* + \\
& + \nu_- \otimes (-\tfrac{1}{2}(u_x, u_y)s_z) - \\
& + \nu_- \otimes -a_z((\tfrac{1}{\sqrt{2}}u_x \circ s_y) + a_y s_x) + \\
& + \tfrac{4}{3}\Psi_B(((\tfrac{1}{\sqrt{2}}u_x \circ s_y) + a_y s_x), s_z) + \\
& + \nu_- \otimes (-(\tfrac{1}{2}u_z \circ (u_y \circ (s_x)) + \tfrac{1}{\sqrt{2}}a_x u_z \circ s_y)) - \\
& -2((\tfrac{1}{\sqrt{2}}u_1 \wedge (u_y \circ (s_x))) + a_x u_1 \wedge s_y, s_z)_B f = \qquad (7.7) \\
= \; & -2a_z a_x a_y f - \\
& -2(a_x s_z - a_z s_x, u_1 \wedge s_y)_B f - \\
& -\sqrt{2}(s_z, u_1 \wedge (u_y \circ (s_x)))_B f - \\
& + \tfrac{4}{3}a_y \Psi_B(s_x, s_z) + \\
& -((u_x, u_y)u_z + (u_z, u_y)u_x) \wedge u_1^* + \\
& + (u_x, u_z)u_y \wedge u_1^* + \\
& + \tfrac{2\sqrt{2}}{3}(\Psi_B(s_x, u_z \circ s_y) - \Psi_B(s_z, u_x \circ s_y)) + \\
& + \nu_- \otimes -(a_x a_y s_z + a_z a_y s_x) + \\
& + \nu_- \otimes (-\tfrac{1}{\sqrt{2}}(a_z u_x + a_x u_z) \circ s_y) + \\
& + \nu_- \otimes (-\tfrac{1}{2}((u_x u_y) \circ s_z + (u_z u_y) \circ s_x)) + \\
& + \nu_- \otimes (-\tfrac{4}{3}\Psi_{B'}(s_x, s_y) \cdot s_z + \tfrac{2}{3}(s_x, u_1 \wedge s_y)_B s_z)
\end{aligned}
$$

Note that in deriving Equation 7.7 we used the identities

$$
\begin{aligned}
(u_x, u_y)\mathbb{1} + (u_x \wedge u_y) &= (u_x, u_y)\mathbb{1} + \tfrac{1}{2}(u_x u_y - u_y u_x) \\
&= (u_x, u_y)\mathbb{1} + u_x u_y - (u_x, u_y)\mathbb{1} \\
&= (u_x u_y);
\end{aligned}
$$

$$
\begin{aligned}
-\tfrac{4}{3}[\Psi_B(s_x, u_1 \wedge s_y), u_z \wedge u_1^*] &= \tfrac{4}{3}[u_z \wedge u_1^*, \Psi_B(s_x, u_1 \wedge s_y)] \\
&= \tfrac{4}{3}\Psi_B(s_x, (u_z \wedge u_1^*) \cdot u_1 \wedge s_y) \\
&= \tfrac{2\sqrt{2}}{3}\Psi_B(s_x, u_z \circ s_y)
\end{aligned}
$$

Note that $[[x,y],z] = (-1)^{|x||y|+|y||z|+|x||z|}[[z,y],x]$ can be easily checked using Equation 7.7. In particular, in the last expression for the triple commutator the terms are rearranged such that the summands of each row satisfy super-commutativity on their own, e.g.

$$
\begin{aligned}
&-\sqrt{2}(s_z, u_1 \wedge (u_y \circ (s_x)))_B f = -2(s_z, (u_1 \wedge u_y)\cdot s_x))_B f = \\
&= 2((u_1 \wedge u_y)\cdot s_z, s_x)_B f = \sqrt{2}(u_1 \wedge (u_y \circ (s_z)), s_x)_B f = \\
&= \sqrt{2}(s_x, u_1 \wedge (u_y \circ (s_z)))_B f ,
\end{aligned}
$$

where we used the fact that $(,)_B$ is invariant and symmetric.

Let $s, t \in \Lambda^\bullet W'$. We define $(s,t)_{B'} = (s, u_1 \wedge t)_B$. We have that $(,)_{B'}$ is admissible with invariants $(1,-1)$. The fact that $(,)_B$ is admissible is transferred to the form $(,)_{B'}$ and to check the invariants is immediate. As a consequence we have that the form $(,)_{B'}$ is invariant with respect to the action of $\Lambda^2 U'$. Moreover, setting $\omega' = u_2 \wedge u_3 \in \Lambda^2 W'$, we have

$$
\begin{aligned}
(s,t)_{B'} = &\ (s, u_1 \wedge t) = (s,t)_B = (-1)^{[\frac{\deg(s)+1}{2}]} i_{\omega^*}(s \wedge u_1 \wedge t) \\
= &\ (-1)^{[\frac{\deg(s)+1}{2}]+\deg(s)} i_{(\omega')^*} i_{u_1^*}(u_1 \wedge s \wedge t) \\
= &\ (-1)^{[\frac{\deg(s)}{2}]} i_{(\omega')^*}(s \wedge t) ,
\end{aligned}
$$

where the identity $(-1)^{[\frac{\deg(s)}{2}]} = (-1)^{[\frac{\deg(s)+1}{2}]+\deg(s)}$ follows from a direct check.

Let $s, t \in \Lambda^\bullet W'$, then

$$
\begin{aligned}
\Psi_B(s,t) &= -\textstyle\sum_{u \in \mathcal{C}'} (s, (u_1 \wedge u^*)\cdot t)_B u \wedge u_1^* \\
&= -\textstyle\sum_{u \in \mathcal{C}'} (s, (u_1 \wedge u^*)\cdot t)_B u \wedge u_1^* \\
&= (-\tfrac{1}{\sqrt{2}} \textstyle\sum_{u \in \mathcal{C}'} (s, u_1 \wedge (u^* \circ t))_B u) \wedge u_1^* \\
&= (-\tfrac{1}{\sqrt{2}} \textstyle\sum_{u \in \mathcal{C}'} (s, u^* \circ t)_{B'} u) \wedge u_1^* .
\end{aligned}
$$

We set

$$
\Psi_{U'}(s,t) := -\frac{1}{\sqrt{2}} \sum_{u \in \mathcal{C}'} (s, u^* \circ t)_{B'} u .
$$

Note that $\sqrt{2}\Psi_{U'}$ satisfies the equation $(\sqrt{2}\Psi_{U'}(s,t), v) = (v \circ s, t)_{B'}$ (cf. Section 6.2). Furthermore, since u_1 anticommutes with the elements of $C\ell(U')$, the following identity holds

$$
\begin{aligned}
\Psi_{B'}(s,t) = &\ \textstyle\sum_{(u\wedge v) \in \mathcal{B}'} (s, (u^* \wedge v^*)\cdot(u_1 \wedge t))_B (u \wedge v) \\
= &\ \textstyle\sum_{(u\wedge v) \in \mathcal{B}'} (s, u_1 \wedge ((u^* \wedge v^*)\cdot t))_B (u \wedge v) \\
= &\ \textstyle\sum_{(u\wedge v) \in \mathcal{B}'} (s, (u^* \wedge v^*)\cdot t)_{B'} (u \wedge v) .
\end{aligned}
$$

We, thus, get the following expression for the triple commutator of Equation 7.7

$$
\begin{aligned}
[[x,y],z] = \quad & -2(a_x a_y a_z + (a_x s_z - a_z s_x, s_y)_{B'})f+ \\
& +\sqrt{2}(s_z, u_y \circ (s_x))_{B'} f+ \\
& +(\tfrac{4}{3} a_y \Psi_{U'}(s_x, s_z)) \wedge u_1^* - \\
& -(((u_x, u_y)u_z + (u_z, u_y)u_x - (u_x, u_z)u_y)) \wedge u_1^* + \\
& +(\tfrac{2\sqrt{2}}{3}(\Psi_{U'}(s_x, u_z \circ s_y) - \Psi_{U'}(s_z, u_x \circ s_y))) \wedge u_1^* + \\
& +\nu_- \otimes (-a_y(a_x s_z + a_z s_x) - \tfrac{1}{\sqrt{2}}(a_z u_x + a_x u_z) \circ s_y) + \\
& +\nu_- \otimes (-\tfrac{1}{2}((u_x u_y) \circ s_z + (u_z u_y) \circ s_x) + \\
& +\nu_- \otimes (-\tfrac{4}{3}\Psi_{B'}(s_x, s_y) \cdot s_z + \tfrac{2}{3}(s_x, s_y)_{B'}) s_z) \,.
\end{aligned}
\tag{7.8}
$$

Chapter 8

Involutions of Lie superalgebras

In this chapter we classify the grade-reversing ϵ-involutions of the classical simple 3-graded Lie superalgebras, we distinguish between the special and the exceptional case. To do this, we will make use of the description of the group of automorphisms of the simple Lie superalgebras obtained in [50]. Throughout the chapter we will make use of the notations introduced in Chapter 7.

Lemma 8.1. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra, σ an automorphism of $\mathfrak{g}$ and let $\sigma_{\bar{i}}$ denote the restriction of σ to $\mathfrak{g}_{\bar{i}}$. Suppose τ is another automorphism of $\mathfrak{g}$ such that $\sigma_{\bar{0}} = \tau_{\bar{0}}$. Then $\sigma = \tau \circ \delta_\lambda$, where δ_λ is the automorphism defined by $(\delta_\lambda)_{\bar{0}} = \mathrm{Id}_{\bar{0}}$ and $(\delta_\lambda)_{\bar{1}} = \lambda \mathrm{Id}_{\bar{1}}$, $\lambda = \pm 1$ if $\mathfrak{g}_{\bar{1}}$ is irreducible, while if $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ is a sum of two irreducible $\mathfrak{g}_{\bar{0}}$-modules then $(\delta_\lambda)_{\bar{0}} = \mathrm{Id}_{\bar{0}}$, $(\delta_\lambda)_{|\mathfrak{g}_+} = \lambda \mathrm{Id}_{\mathfrak{g}_+}$ and $(\delta_\lambda)_{|\mathfrak{g}_-} = \lambda^{-1}\mathrm{Id}_{\mathfrak{g}_-}$, $\lambda \in \mathbb{C}^\times$.*

Proof. Let $\mathfrak{g}, \sigma, \tau$ be as in the hypotheses. We have that $\delta = \tau^{-1} \circ \sigma$ is the identity on $\mathfrak{g}_{\bar{0}}$ and $\delta_{\bar{1}}$ is an isomorphism of the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$. Indeed $\delta([A, v]) = [A, \delta(v)]$, $\forall A \in \mathfrak{g}_{\bar{0}}$, $v \in \mathfrak{g}_{\bar{1}}$. By Schur's lemma δ acts as a scalar on each irreducible component. Suppose $\mathfrak{g}_{\bar{1}}$ is irreducible and $\delta(v) = \lambda v$ then $0 \neq [v, w] = \delta([v, w]) = \lambda^2 [v, w]$, $v, w \in \mathfrak{g}_{\bar{1}}$ thus $\lambda = \pm 1$. If $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_+$, with$\mathfrak{g}_\pm$ two irreducible representations of $\mathfrak{g}_{\bar{0}}$, then $\delta(v_\pm) = \lambda_\pm, v_\pm \in \mathfrak{g}_\pm$, for some $\lambda_\pm \in \mathbb{C}^\times$. In particular $0 \neq [v_+, v_-] = \delta([v_+, v_-]) = \lambda_- \lambda_+ [v_+, v_-]$ and the rest follows. □

Remark 8.2. Notice that if $(\mathfrak{g}, \sigma)$ gives rise to an ϵ-sJTS and σ commutes with δ_λ, then $\sigma \circ \delta_\lambda$ is an ϵ'-involution, $\epsilon' \in \mathbb{Z}_2$, if and only if $\lambda^4 = 1$ and if $\lambda = \pm i$ then $\epsilon \neq \epsilon'$ since $(\delta^2_{\pm i})_{\bar{1}} = -\mathrm{Id}_{\mathfrak{g}_{\bar{1}}}$.

Lemma 8.3. *Let $\mathfrak{g}$ be a 3-graded Lie superalgebra, σ an automorphism of $\mathfrak{g}$ and let E be the grading element. Then σ is grade-reversing (resp. preserving), if and only if $\sigma(E) = -E$ (resp. $\sigma(E) = E$).*

Proof. The proof for the grade-preserving case is similar to the grade-reversing one so we only treat the latter. Let σ be a grade-reversing automorphism. We have, for any $x \in \mathfrak{g}_i$,

$$-i\sigma(x) = [E, \sigma(x)] = \sigma([\sigma^{-1}(E), x]) \,,$$

which implies that $\sigma^{-1}(E) = -E$, hence $\sigma(E) = -E$.

Conversely, suppose $\sigma(E) = -E$. Then

$$[E, \sigma(x)] = \sigma[-E, x] = -i\sigma(x) \,,$$

which implies $\sigma(x) \in \mathfrak{g}_{-i}$. □

8.1 ϵ-involutions of special Lie superalgebras

In this subsection we give the grade-reversing ϵ-involutions of the special simple 3-graded Lie superalgebras, i.e. all the classical ones except for $D(\alpha)$ and $F(4)$.

Proposition 8.4. *Let $\mathfrak{g} = psl(m, n)_{k,h}$. If $m \neq n$, a complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}$ is the following:*

$$\iota_{\epsilon,k,h} := \mathrm{AdDiag}(\begin{pmatrix} 0 & \mathrm{Id}_k \\ \mathrm{Id}_k & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id}_h \\ (-1)^{\epsilon}\mathrm{Id}_h & 0 \end{pmatrix}) \,, \text{ if } m = 2k,\ n = 2h \,;$$

$$\epsilon = 0 : os\tau_{m,h,n} := \mathrm{AdDiag}(S_m, J_h, J_{n-h}) \circ \tau \,, \text{ if } h, n \in 2\mathbb{N} \,;$$

$$\epsilon = 1 : \tau;\ s\tau_{k,m,h,n} := \mathrm{AdDiag}(J_k, J_{m-k}, J_h, J_{n-h}) \,, \text{ if } k, m, h, n \in 2\mathbb{N}.$$

If $m = n$, a complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}$ consists of the previous ones together with the following:

$$\epsilon = 0 : \iota_{\bar{0},h,h} \circ \Pi \,, \text{ if } n = 2h = 2k \,;$$

$$\tau \circ \Pi \,, \text{ if } k = h \,;$$

$$\Pi \,, \text{ if } k = n,\ h = 0 \,;$$

$$\epsilon = 1 : \tau \circ \Pi \circ \delta_i \,, \text{ if } k = h \,.$$

Proof. Let $m \neq n$. An automorphism of $\mathfrak{g}$ is either of the form $\mathrm{Ad}(\mathrm{Diag}(A, B))$ or $\mathrm{Ad}(\mathrm{Diag}(A, B)) \circ \tau$, see [50], with

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{c|c} -a & -b \\ \hline c & -d \end{array}\right)^t = \left(\begin{array}{c|c} -a^t & c^t \\ \hline -b^t & -d^t \end{array}\right) ,$$

$A \in GL(m, \mathbb{C})$ and $B \in GL(n, \mathbb{C})$.

Let $\sigma = Ad(Diag(A, B))$ be a grade-reversing ϵ-involution. Then

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) = \left(\begin{array}{c|c} AaA^{-1} & AbB^{-1} \\ \hline BcA^{-1} & BdB^{-1} \end{array}\right). \tag{8.1}$$

In particular, if E is the grading element,

$$\sigma(E) = -E \iff \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) E = -E \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \iff$$

$$\iff \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \left(\begin{array}{c|c} \begin{pmatrix} xId_k & 0 \\ 0 & (x-1)Id_{m-k} \end{pmatrix} & 0 \\ \hline 0 & \begin{pmatrix} xId_h & 0 \\ 0 & (x-1)Id_{n-h} \end{pmatrix} \end{array}\right) =$$

$$= - \left(\begin{array}{c|c} \begin{pmatrix} xId_k & 0 \\ 0 & (x-1)Id_{m-k} \end{pmatrix} & 0 \\ \hline 0 & \begin{pmatrix} xId_h & 0 \\ 0 & (x-1)Id_{n-h} \end{pmatrix} \end{array}\right) \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

Thus, σ is grade-reversing if and only if $m = 2k, n = 2h, A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$, $A_1, A_2 \in GL(k, \mathbb{C}), B_1, B_2 \in GL(h, \mathbb{C})$. The condition $\sigma_{\bar{0}}^2 = Id$ implies $A_2 = \alpha A_1^{-1}$ and $B_2 = \beta B_1^{-1}$, $\alpha, \beta = \pm 1$. As a consequence one computes $\sigma_{\bar{1}}^2 = \delta_{\frac{\alpha}{\beta}}$. In particular, $\iota_{\epsilon,k,h}$ is a grade-reversing ϵ-involution of $\mathfrak{g}$.

The restriction of σ to $\mathfrak{a}_1$ (see Definition 7.3), resp. $\mathfrak{a}_2$, acts by the inner automorphism $Ad(A)$, resp. $Ad(B)$ and $\mathfrak{g}$ induces a 3-grading on $\mathfrak{a}_1$, resp. $\mathfrak{a}_2$, which is the one corresponding to the Dynkin diagram with the middle node marked, i.e. the k-th, resp. the h-th. By Proposition 5.10, the only possibility is that $Ad(A)$ and $Ad(B)$ are involutions corresponding to the real forms $\mathfrak{su}(k, k)$ and $\mathfrak{su}(h, h)$ respectively. We can conclude that $\iota_{\epsilon,k,h}$ is the only grade-reversing ϵ-involution up to equivalence of the form $AdDiag(A, B)$.

Now suppose $\sigma = Ad(Diag(A, B)) \circ \tau$. We have

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{c|c} -Aa^tA^{-1} & Ac^tB^{-1} \\ \hline -Bb^tA^{-1} & -Bd^tB^{-1} \end{array}\right). \tag{8.2}$$

In this case, the restrictions $\sigma_i = \sigma_{|\mathfrak{a}_i}, i = 1, 2$, act as the outer automorphisms $\sigma_1 = -Ad(A) \circ t$ of $\mathfrak{a}_1$ and $\sigma_2 = -Ad(B) \circ t$ of $\mathfrak{a}_2$ respectively, with t the usual transposition.

Since on $\mathfrak{a}_1$ we have the 3-grading given by marking the k-th node and the trivial grading if $k = m$ and on $\mathfrak{a}_2$ we have the 3-grading given by marking the h-th node or the trivial one if $h = 0$, by 5.10, one concludes that σ_1 can be either of type $\mathfrak{sl}(m, \mathbb{R})$ or $\mathfrak{sl}(m, \mathbb{H})$ if $m, k \in 2\mathbb{N}$ and σ_2 can be either of type $\mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{sl}(n, \mathbb{H})$ if $n, h \in 2\mathbb{N}$. As a consequence, see e.g. [47] for the classification of involutions, we have that the matrices A, B must be either symmetric or antisymmetric. Hence, we set $A^t = \alpha A, B^t = \beta B, \alpha, \beta = \pm 1$. Moreover, $\sigma(E) = -E$ is equivalent to $\mathrm{AdDiag}(A, B)$ being grade-preserving, since τ is grade-reversing. It follows from the description of E, that $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$, with $A_1 \in GL(k, \mathbb{C}), A_2 \in GL(m-k, \mathbb{C}), B_1 \in GL(h, \mathbb{C}), B_2 \in GL(n-h, \mathbb{C})$ with $A_i^t = \alpha A_i$ and $B_i^t = \beta B_i$, $i = 1, 2$. We calculate

$$\sigma^2 \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A(A^{-1})^t b B^t B^{-1} \\ -B(B^{-1})^t c A^t A^{-1} & 0 \end{pmatrix} = -\alpha\beta \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

The conditions derived for σ are satisfied if $\epsilon = \bar{0}$ and σ is $\mathrm{os}\tau_{m,h,n}$, or if $\epsilon = \bar{1}$ and σ is either τ or $s\tau_{k,m,h,n}$. Since they correspond to the real forms $\mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sl}(n, \mathbb{H})$ respectively, they cover all possible grade-reversing ϵ-involutions, up to equivalence.

Let $m = n$. In addition to the preceding cases, we also have the automorphisms of the form $\mathrm{Ad}(\mathrm{Diag}(A, B)) \circ \Pi$, $\mathrm{Ad}(\mathrm{Diag}(A, B)) \circ \Pi \circ \tau$ and their composition with $\delta_\lambda, \lambda \in \mathbb{C}^\times$, defined in Lemma 8.1.

Recall that unless $h = k$ there is no grading element, however we can still consider the element E introduced in Chapter 7. Since $E \in \mathfrak{gl}(n, n)$ the action of an automorphism of $\mathfrak{psl}(n, n)$ naturally extends to E and Lemma 8.3 still holds for E since it satisfies Equation (4.2). It is convenient to split E as the following sum: $E = E_1 + E_2 + E'$, with $E_1 = \mathrm{Diag}(a\mathrm{Id}_h, (a-1)\mathrm{Id}_{n-k}, 0_n)$, $E_2 = \mathrm{Diag}(0_n, b\mathrm{Id}_h, (b-1)\mathrm{Id}_{n-k})$, $E' = c(\mathrm{Id}_n, -\mathrm{Id}_n)$, $a = 1 - \frac{k}{n}$, $b = 1 - \frac{h}{n}$, $c = \frac{k-h}{2n}$. We have $E_1, E_2 \in \mathfrak{g}$ and $E' \notin \mathfrak{g}$ (note that E_1 and E_2 are the grading-elements for the induced 3-grading on $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively). Simple calculations show that $\mathrm{AdDiag}(A, B)(E) = \mathrm{AdDiag}(A, B)(E_1) + \mathrm{AdDiag}(A, B)(E_2) + E'$, $\tau(E) = -E$, $\Pi(E) = \Pi(E_1 + E_2) - E'$ and $\delta_\lambda(E) = E$.

Let σ be of the form $\mathrm{Ad}(\mathrm{Diag}(A, B))$. If σ is a grade-reversing ϵ-involution then $E' = 0$, since $\mathrm{AdDiag}(A, B)(E) = \mathrm{AdDiag}(E_1) + \mathrm{AdDiag}(E_2) + E' = -(E_1 + E_2 + E')$ and $h = k$. Arguments similar to the case $m \neq n$ let us conclude that $\iota_{\epsilon,k,k}$ is a grade-reversing ϵ-involution. Since there is only one equivalence class, this part of the statement is proved.

Similarly, for the case $\sigma = \mathrm{Ad}(\mathrm{Diag}(A, B)) \circ \tau$ one obtains the grade-reversing ϵ-involution of the case with $m \neq n$ and we only have to check if τ and $s\tau$ are equivalent under the action of the outer automorphisms one gets in addition to the case $m \neq n$. Let $h = k, h, n \in 2\mathbb{N}$. If ϕ is a grade-preserving automorphism of the form $\mathrm{AdDiag}(P, Q) \circ \Pi$ then $\phi(E) = E$ yields $P = \mathrm{Diag}(P_1, P_2), Q = \mathrm{Diag}(Q_1, Q_2), P_1, Q_1 \in GL(h, \mathbb{C}), P_2, Q_2 \in GL(n-h, \mathbb{C})$. We have

$$\begin{aligned}
\phi \circ s\tau \circ \phi^{-1} = \; & \mathrm{AdDiag}(P, Q) \circ \Pi \circ \mathrm{AdDiag}(J_h, J_{n-h}, J_h, J_{n-h}) \circ \\
& \circ \tau \circ \Pi \circ \mathrm{AdDiag}(P, Q)^{-1} = \\
= \; & \mathrm{AdDiag}(P, Q) \circ \mathrm{AdDiag}(J_h, J_{n-h}, J_h, J_{n-h}) \circ \\
& \circ (\Pi \circ \tau \circ \Pi) \circ \mathrm{AdDiag}(P, Q)^{-1} = \\
= \; & \mathrm{AdDiag}(P, Q) \circ \mathrm{AdDiag}(J_h, J_{n-h}, J_h, J_{n-h}) \circ \\
& \circ \tau \circ \delta_{-1} \circ \mathrm{AdDiag}(P, Q)^{-1} = \\
= \; & \mathrm{AdDiag}(P, Q) \circ \mathrm{AdDiag}(J_h, J_{n-h}, J_h, J_{n-h}) \circ \\
& \circ \mathrm{AdDiag}(P^t, Q^t) \circ \tau \circ \delta_{-1} = \\
= \; & \mathrm{AdDiag}(P_1 J_h P_1^t, P_2 J_{n-h} P_2^t, Q_1 J_h Q_1^t, Q_2 J_{n-h} Q_2^t) \circ \tau \circ \delta_{-1} .
\end{aligned}$$

The class of congruence for the matrix $J_i, i \in 2\mathbb{N}$ does not contain the identity, hence τ and $s\tau$ are not equivalent under ϕ. In the same way one checks that $\mathrm{AdDiag}(P, Q) \circ \tau \circ \Pi$ does not give an equivalence of τ and $s\tau$. The case $(k, h) = (n, 0)$ follows as well.

Let $\sigma = \mathrm{Ad}(\mathrm{Diag}(A, B)) \circ \Pi$ be a grade-reversing ϵ-involution. Then

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} AdA^{-1} & AcB^{-1} \\ BbA^{-1} & BaB^{-1} \end{pmatrix} \tag{8.3}$$

Since σ exchanges $\mathfrak{a}_1$ and $\mathfrak{a}_2$ they need to have the same grading, hence they are either both 3-graded and $h = k$ or trivially graded, case $(k, h) = (n, 0)$. We compute

$$\sigma^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ABa(AB)^{-1} & ABb(AB)^{-1} \\ BAc(BA)^{-1} & BAd(BA)^{-1} \end{pmatrix} .$$

It is clear that σ can only be a $\bar{0}$-involution and that $B = \pm A^{-1}$.
Suppose $h \neq 0$. Then $\Pi(E) = E$, hence $\mathrm{Ad}(A, \pm A^{-1})$ is grade-reversing, which implies $n = 2h$, $A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$. Let $\phi = \mathrm{Ad}(\mathrm{Diag}(P, Q))$ be a grade-preserving involution, $P = \mathrm{Diag}(P_1, P_2)$, $Q = \mathrm{Diag}(Q_1, Q_2)$ with $P_1, P_2, Q_1, Q_2 \in GL(h, \mathbb{C})$. Since $\Pi \circ \mathrm{Ad}(\mathrm{Diag}(P, Q)) = \mathrm{Ad}(\mathrm{Diag}(Q, P)) \circ \Pi$, we have

$$\begin{aligned}
\phi \circ \sigma \circ \phi^{-1} = \; & \mathrm{Ad}(\mathrm{Diag}(P, Q)) \circ \mathrm{Ad}(\mathrm{Diag}(A, \pm A^{-1})) \circ \mathrm{Ad}(\mathrm{Diag}(Q^{-1}, P^{-1})) \circ \Pi = \\
= \; & \mathrm{Ad}(\mathrm{Diag}(PAQ^{-1}, \pm QA^{-1}P^{-1})) \circ \Pi ;
\end{aligned}$$

By the generality of P, Q and the fact that $PAQ^{-1} = \begin{pmatrix} 0 & P_1 A_1 Q_2^{-1} \\ P_2 A_2 Q_1^{-1} & 0 \end{pmatrix}$ we can choose $P_i = A_i^{-1}$ and thus obtain that σ is equivalent to the involution $\iota_{\bar{0},h,h} \circ \Pi$. On the other hand, if $h = 0$, $\Pi(E) = -E$ and $Ad(A, \pm A^{-1})$ is grade-preserving. One gets that σ is equivalent to $AdDiag(PAQ^{-1}, QAP^{-1}) \circ \Pi$ for any invertible matrices P, Q. Setting $P = A^{-1}$, $Q = Id$ we get the $\bar{0}$-involution Π. We note also that $\Pi \circ \delta_\lambda = \delta_{\lambda^{-1}} \circ \Pi$ and as a result $(\sigma \circ \delta_\lambda)^2 = \sigma^2$.

Let $\sigma = Ad(Diag(A, B)) \circ \Pi \circ \tau$ be a grade-reversing ϵ-involution. Similarly to the previous case one derives that only the case $k = h$ occurs, that $AdDiag(A, B)$ is grade-preserving which implies $B = (A^{-1})^t$ and $A = Diag(A_1, A_2)$ with $A_1 \in GL(h, \mathbb{C}), A_2 \in GL(n-h, \mathbb{C})$. Studying the equivalence class of σ we get that it is equivalent to any involution of the same form with $A_i = P_i A_i Q_i^{-1}, i = 1, 2, P_1, Q_1 \in GL(h, \mathbb{C}), P_2, Q_2 \in GL(n-h, \mathbb{C})$. We conclude that in this case the only grade-reversing ϵ-involution is $\tau \circ \Pi$. Finally, $\sigma \circ \delta_\lambda = \delta_\lambda \circ \sigma$, hence $\sigma \circ \delta_\lambda$ is a $\bar{1}$-involution, if $\lambda = \sqrt{-1}$. □

Proposition 8.5. *Let $\mathfrak{g} = osp(2m+1, 2n)$ with the 3-grading of Proposition 7.1.3. A complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}$ is the following:*

$$\epsilon = 0 : S\hat{I}_{p,2m+1,q,n} = AdDiag(S_{p,2m+1}, \hat{I}_{2q,2n}), 1 \leqslant p \leqslant m-1, \ 1 \leqslant q \leqslant [\frac{n}{2}];$$

$$\epsilon = 1 : SJ_{p,2m+1,n} = AdDiag(S_{p,2m+1}, J_{2n}), \ 1 \leqslant p \leqslant m-1.$$

Proof. The automorphisms of $\mathfrak{g}$ are all inner, hence $Aut(\mathfrak{g}) = SO(2m+1, \mathbb{C}) \times SP(2n, \mathbb{C})$. Let $\sigma = Ad(Diag(A, B))$ with $A \in SO(2m+1, \mathbb{C}), \ B \in SP(2n, \mathbb{C})$ be a grade-reversing ϵ-involution. Then $\sigma_{\bar{0}}$ induces grade-reversing involutions σ_o, σ_{sp} on $so(2n+1, \mathbb{C})$ and $sp(2n, \mathbb{C})$ respectively. Due to Proposition 5.10, σ_o can be of type $so(p, 2m+1-p), 1 < p \leqslant l$, since $so(2m+1, \mathbb{C})$ has the 3-grading with the first node marked and, since $sp(2n, \mathbb{C})$ is trivially graded, σ_{sp} can be of type $sp(2n, \mathbb{R})$ or $sp(2q, 2(n-q)), \ 1 \leqslant q \leqslant [\frac{n}{2}]$. Let $\phi = AdDiag(P, Q)$ be a grade-preserving homomorphism. Then $\phi \circ \tau \circ \phi^{-1} = AdDiag(PAP^{-1}, QBQ^{-1})$, hence matrices A, B with different spectrum yield inequivalent grade-reversing ϵ-involution.

Condition $\sigma(E) = -E$ implies that $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & A_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ with $A_1 \in SO(2m-1, \mathbb{C})$. If we set $A_1 = S_{k,2m-1}, 0 \leqslant k \leqslant m-1$, and $B = J_{2n}$ or $B = \hat{I}_{2k,2n}, \ 1 \leqslant k \leqslant [\frac{n}{2}]$, we get inequivalent grade-reversing ϵ-involution since the choices for A_1 and for B all have different spectrum and, by our argument on the real forms, these are also the only grade-reversing ϵ-involution. Since $\sigma(\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}) = \begin{pmatrix} 0 & AbB^{-1} \\ BcA^{-1} & 0 \end{pmatrix}$ we have that if $B = J_{2n}$ then σ is a $\bar{1}$-involution, while in all the other cases is a $\bar{0}$-involution. □

Proposition 8.6. *Let* $\mathfrak{g}$ *be* $\mathrm{osp}(2,2n)$ *with the 3-grading of Proposition 7.1.4 and let* $\mathfrak{g}'$ *be* $\mathrm{osp}(2,2n)$ *with the 3-grading of Proposition 7.1.5. Any grade-reversing* ϵ*-involution of* $\mathfrak{g}$ *is equivalent to one of the following*

$$\epsilon = 0 : S\hat{I}_{2,q,n} = \mathrm{AdDiag}(S_2, \hat{I}_{2q,2n})\,,\ 1 \leqslant q \leqslant [\frac{n}{2}]\,;$$

$$\epsilon = 1 : SJ_{2,n} = \mathrm{AdDiag}(S_2, J_{2n})\,.$$

Any grade-reversing ϵ*-involution of* $\mathfrak{g}'$ *is equivalent to one of the following*

$$\epsilon = 0 : SH^{+}_{2,l} = \mathrm{AdDiag}\,(S_2, H^{+}_{4l})\ ,\ n = 2l\,;$$

$$\epsilon = 1 : SJ_{2,n} = \mathrm{AdDiag}(S_2, J_{2n})\,.$$

Proof. The automorphism group of $\mathrm{osp}(2,2n)$ is the semidirect product of the inner automorphisms and the outer involution, i.e. $\mathrm{Aut}(\mathrm{osp}(2,2n)) = SO(2) \times SP(2n) \rtimes \mathrm{Ad}(\mathrm{Diag}(S_2, \mathrm{Id}_{2n}))$. We first consider the case of $\mathfrak{g}$. Let $\sigma = \mathrm{Ad}(\mathrm{Diag}(A,B))$ with $A \in SO(2,\mathbb{C}),\ B \in SP(2n,\mathbb{C})$. Since A is of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ we see immediately that $\sigma(\mathfrak{g}_1) \nsubseteq \mathfrak{g}_{-1}$. Suppose $\sigma = \mathrm{Ad}(\mathrm{Diag}(AS_2, B)), A \in SO(2,\mathbb{C}),\ B \in SP(2n,\mathbb{C})$ is a grade-reversing ϵ-involution. Similarly to the case $\mathrm{osp}(2m+1,2n)$ one gets that $\mathrm{Ad}(B)$ is an involution of $\mathrm{sp}(2n,\mathbb{C})$, which is trivially graded, hence it is of type $\mathrm{sp}(2n,\mathbb{R})$ or $\mathrm{sp}(2q, 2(n-q)),\ 1 \leqslant q \leqslant [\frac{n}{2}]$. Since $\mathfrak{g}_{\bar{1}}$ is the sum of two irreducible $\mathfrak{g}_{\bar{0}}$-modules, by Lemma 8.1, we also have to consider $\sigma \circ \delta_\mu$, $\mu^4 = 1$, however, since $\mathrm{Ad}(AS_2)$ exchanges the irreducible components of $\mathfrak{g}_{\bar{1}}$, $\sigma \circ \delta_\mu = \delta_{\mu^{-1}} \circ \sigma$, and $\delta_\lambda \circ \sigma \circ \delta_{\lambda^{-1}} = \sigma \circ \delta_\mu$, if $\lambda^2 = \mu$. If $A = \mathrm{Id}$ and $B = J_{2n}$ or $B = \hat{I}_{2k,2n},\ 1 \leqslant k \leqslant [\frac{n}{2}]$ we get inequivalent grade-reversing ϵ-involutions and by the real forms argument they are all, up to equivalence. In particular σ is a $\bar{0}$-involution if $B = \hat{I}_{2k,2n},\ 1 \leqslant k \leqslant [\frac{n}{2}]$ and a $\bar{1}$-involution if $B = J_{2n}$.

We consider the case of $\mathfrak{g}'$. As it happened in the case of $\mathfrak{g}$, the inner automorphisms are not grade-reversing, hence we only need to consider the case of a grade-reversing ϵ-involution$\sigma = \mathrm{Ad}(\mathrm{Diag}(AS_2, B)), A \in SO(2),\ B \in SP(2n)$. Since the copy of $\mathrm{sp}(2n,\mathbb{C})$ is 3-graded, the one corresponding with the last node marked in the Dynkin diagram, $\mathrm{Ad}(B)$ is an involution of type $\mathrm{sp}(2n,\mathbb{R})$ or $\mathrm{sp}(n,n)$ if $n \in 2\mathbb{N}$. These are the only possibilities, since σ and $\sigma \circ \delta_\lambda$ are equivalent. In particular, $\sigma(E) = -E$ implies $B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$, $B \in SP(2n,\mathbb{C})$ implies $B_2 = -(S_n B_1^t S_n)^{-1}$ and $B^2 = \pm \mathrm{Id}$ yield $B_1 = \mp S B_1^t S$. We conclude that $B = J_{2n}$ is the only $\bar{1}$-involution up to equivalence, resp. $B_1 = B_2 = \mathrm{Diag}(\mathrm{Id}_{\frac{n}{2}}, -\mathrm{Id}_{\frac{n}{2}})$ is the only $\bar{0}$-involution. □

Proposition 8.7. *Let $\mathfrak{g}$ be $\mathrm{osp}(2m, 2n)$ with the 3-grading of Proposition 7.1.6 and let $\mathfrak{g}'$ be $\mathrm{osp}(2m, 2n)$ with the 3-grading of Proposition 7.1.7. A complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}$ is the following:*

$$\epsilon = 0 : S\hat{I}_{p,2m,q,n} = \mathrm{AdDiag}(S_{p,2m}, \hat{I}_{2q,2n})\,, 1 \leqslant p \leqslant m-1\,,\ 1 \leqslant q \leqslant [\frac{n}{2}]\,;$$

$$\epsilon = 1 : SJ_{p,2m,n} = \mathrm{AdDiag}(S_{p,2m}, J_{2n})\,,\ 1 \leqslant p \leqslant m-1\,.$$

A complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}'$ is the following:

$$\epsilon = 0 : SH^{+}_{2m,l} = \mathrm{AdDiag}(S_{2m}, H^{+}_{4l})\,,\ n = 2l\,;$$

$$H^{-}J_{l,n} = \mathrm{AdDiag}(H^{-}_{4l}, J_{2n})\,,\ m = 2l\,;$$

$$\epsilon = 1 : SJ_{2m,n} = \mathrm{AdDiag}(S_{2m}, J_{2n})\,;$$

$$H^{-}H^{+}_{l,q} = \mathrm{AdDiag}(H^{-}_{4l}, H^{+}_{4q})\,,\ m = 2l, n = 2q\,.$$

Proof. The automorphism group of $\mathfrak{g}$ is the semidirect product of the inner automorphisms and the outer involution T, $\mathrm{Aut}(\mathfrak{g}) = SO(2m, \mathbb{C}) \times SP(2n, \mathbb{C}) \rtimes T$, or equivalently $O(2m, \mathbb{C}) \times SP(2n, \mathbb{C})$.

Case $\mathfrak{g}$: Let $\sigma = \mathrm{Ad}(\mathrm{Diag}(A, B))$ with $A \in O(2m, \mathbb{C})$, $B \in SP(2n, \mathbb{C})$. Since the copy of $\mathrm{so}(2m, \mathbb{C})$ in $\mathfrak{g}_{\bar{0}}$ has the 3-grading represented by the Dynkin diagram with the first node marked, the only involutions compatible are those of type $\mathrm{so}(k, 2m-k)\,, 1 < k \leqslant m$, and since $\mathrm{sp}(2n, \mathbb{C})$ is trivially graded any involution is compatible, hence $\mathrm{Ad}(B)$ can be either of type $\mathrm{sp}(2n, \mathbb{R})$ or $\mathrm{sp}(2q, 2(n-q))\,, 1 < q \leqslant [\frac{n}{2}]$. The same arguments made for $\mathrm{so}(2m+1, 2n)$ let us conclude this case.

Case $\mathfrak{g}'$: Let $\sigma = \mathrm{Ad}(\mathrm{Diag}(A, B))$ be a grade-reversing ϵ-involution of $\mathfrak{g}'$. In this case, since both $\mathrm{so}(2m, \mathbb{C})$ and $\mathrm{sp}(2n, \mathbb{C})$ are 3-graded, the 3-grading being in both case the one corresponding to the respective Dyinkin diagram with the last node marked, by Proposition 5.10, the only compatible involutions are those of type $\mathrm{so}(m, m)$ and, if $m = 2l$, $\mathrm{so}^*(2l)$ of $\mathrm{so}(2m, \mathbb{C})$ and $\mathrm{sp}(2n, \mathbb{R})$ and, if $n \in 2\mathbb{N}$, $\mathrm{sp}(n, n)$ of $\mathrm{sp}(2n, \mathbb{C})$. The same arguments used for $\mathrm{osp}(2, 2n)$ lead us to $B = J_{2n}$ or, if $n = 2l$, $B = H^{+}_{4l}$. We have that $\mathrm{Ad}(A)$ is a grade-reversing involution of $\mathrm{so}(2m, \mathbb{C})$ if $A = \begin{pmatrix} 0 & A_1 \\ \pm A_1^{-1} & 0 \end{pmatrix}$ with $A_1^R = \pm A_1$. Thus, $A = S_{2m}$ and, if $n = 2p$, $A = H^{-}_{4p}$ are two inequivalent grade-reversing involutions of $\mathrm{so}(2m, \mathbb{C})$, since they satisfy all the requirements. The rest follows by Lemma 8.1, together with the fact that $\mathfrak{g}_{\bar{1}}$ is irreducible, and by evaluating the action of σ^2 on $\mathfrak{g}_{\bar{1}}$ in each of the 4 cases we obtained. □

Proposition 8.8. *Let $\mathfrak{g}$ be $\mathfrak{p}(n)$ with the 3-grading of Proposition 7.1.11, $n = 2h$. A complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}$ is the following:*

$$\epsilon = 0 : \iota_{\bar{0},h,h} .$$

$$\epsilon = 1 : \iota_{\bar{0},h,h} \circ \delta_i .$$

Proof. The group of automorphisms of $\mathfrak{g}$ is $Aut(\mathfrak{g}) = GL(n, \mathbb{C}) \rtimes \mathbb{C}^*$ with $GL(n, \mathbb{C})$ generated by $Ad(Diag(A, A^R))$ and $\mathbb{C}^*$ generated by $\{\delta_\lambda\}/\{\delta_\lambda | \lambda^{2n} = 1\}$. Recall that in this case $\mathfrak{g}_{\bar{1}}$ is the direct sum of two irreducible $\mathfrak{g}_{\bar{0}}$ modules.

Let $\sigma = AdDiag(A, A^R)$ be a grade-reversing ϵ-involution. We have that $\sigma_{\bar{0}}$ acts on $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(n, \mathbb{C})$ by the inner automorphism $Ad(A, A^R)$. Since $\mathfrak{sl}(n, \mathbb{C})$ has 3-grading the one with the h-th node marked and $n = 2h$, we have that $Ad(A, A^R)$ is of type $\mathfrak{su}(n, n)$. We conclude that the map $\iota_{\bar{0},h,h}$ is a $\bar{0}$-involution of $\mathfrak{p}(n)$ and is also the only one, up to equivalence. Since δ_λ commutes with $\iota_{\bar{0},h,h}$, we have that $(\iota_{\bar{0},h,h} \circ \delta_i)^2 = \delta_{-1}$, i.e. $\iota_{\bar{0},h,h} \circ \delta_i$ is a grade-reversing $\bar{1}$-involution, and since there is only one equivalence class, the proof is complete. □

Proposition 8.9. *Let $\mathfrak{g} = \mathfrak{q}(n)_s$. A complete list, up to equivalence, of grade-reversing ϵ-involutions of $\mathfrak{g}$ is the following:*

$$\epsilon = 0 : \iota_{\bar{0},s,s} \, , \; n = 2s \, ;$$

$$\epsilon = 1 : \tau \circ \delta_i ; \; s\tau_{2s,2n,2s,2n} \circ \delta_i \, , \; s, n \in 2\mathbb{N} \, .$$

Proof. The automorphisms of $\mathfrak{g}$ are of the form $Ad(Diag(A, A))$ or $Ad(Diag(A, A)) \circ \tau \circ \delta_i$, $A \in GL(n, \mathbb{C})$. Let $\sigma = Ad(Diag(A, A))$. Since the $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(n, \mathbb{C})$ has the 3-grading given by the Dynkin diagram with the s-th node marked and $\sigma_{\bar{0}}$ is an inner grade-reversing involution of $\mathfrak{g}_{\bar{0}}$, we have that $\sigma_{\bar{0}}$ is of type $\mathfrak{su}(n, n)$ and $n = 2s$. Thus, the automorphism $\iota_{\bar{0},s,s}$ is the only $\bar{0}$-involution of $\mathfrak{g}$, up to equivalence.

Let $Ad(Diag(A, A)) \circ \tau \circ \delta_i$. In this case we get that the $\sigma_{\bar{0}}$ is an outer grade-reversing involution of $\mathfrak{sl}(n, \mathbb{C})$, hence either of type $\mathfrak{sl}(n, \mathbb{R})$ or, if $n \in 2\mathbb{N}$, $\mathfrak{sl}(n, \mathbb{H})$. Since, as it was shown previously, $(\tau \circ \delta_i)^2 = \delta_{-1}$, we have that $\sigma = \tau \circ \delta_i$ and $\sigma = s\tau_{2s,2n,2s,2n} \circ \delta_i$ are grade-reversing $\bar{1}$-involutions of $\mathfrak{q}(n)$. The rest follows. □

8.2 ϵ-involutions of exceptional Lie superalgebras

In this subsection we give the grade-reversing ϵ-involutions of the exceptional Lie superalgebras in terms of their action on the even part, whereas the action on the odd part is explicitly described in the proofs.

Proposition 8.10. *Let* $\alpha \in \mathbb{C} - \{0, -1, 1, -2, -\frac{1}{2}\}$ *and* $\mathfrak{g} = D(\alpha)$ *with the 3-grading of Proposition 7.1.8. A complete list, up to equivalence, of grade-reversing* ϵ*-involutions of* $\mathfrak{g}$ *is the following:*

$$\epsilon = 0 : JJId_{2,2,2} = Ad(J_2) \times Ad(J_2) \times Ad(Id_2)\,;$$
$$\epsilon = 1 : JJJ_{2,2,2} = Ad(J_2) \times Ad(J_2) \times Ad(J_2)\,.$$

Proof. Let $\alpha \in \mathbb{C} - \{0, -1, 1, -2, -\frac{1}{2}\}$ and suppose $\alpha \neq \frac{1}{2}(-1 + \sqrt{-3})$. The automorphism group of $\mathfrak{g}$ is $Aut(\mathfrak{g}) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ with the action on $\mathfrak{g}_{\bar{0}}$ given by the i-th $SL(2, \mathbb{C})$ acting on $\mathfrak{a}_i \cong \mathfrak{sl}(2, \mathbb{C}), i = 1, 2, 3$.

Let $\sigma_{\bar{0}} = Ad(A_1) \times Ad(A_2) \times Ad(A_3)$ be a grade-reversing involution of $\mathfrak{g}_{\bar{0}}$. Since $\mathfrak{a}_i\,, i = 1, 2$ is 3-graded, $Ad(A_1)$ and $Ad(A_2)$ are grade-reversing involutions of type $\mathfrak{sl}(2, \mathbb{R})$, while $Ad(A_3)$ can be either of type $\mathfrak{sl}(2, \mathbb{R})$ or the identity. Let $A_1 = A_2 = J_2$ and $A_3 = Id_2$ or $A_3 = J_2$.

We define the extension of $\sigma_{\bar{0}}$ to $\mathfrak{g}$ by setting

$$\sigma_{\bar{1}}(v_1 \otimes v_2 \otimes v_3) = A_1(v_1) \otimes A_2(v_2) \otimes A_3(v_3)\,.$$

We show that σ is, in fact, an automorphism of $\mathfrak{g}$. Let $a_i \in \mathfrak{a}_i, v_i, w_i \in V_i\,, i = 1, 2, 3$. Indeed we have $[\sigma(a_1 + a_2 + a_3), \sigma(v_1 \otimes v_2 \otimes v_3)] = \sigma([a_1 + a_2 + a_3, v_1 \otimes v_2 \otimes v_3])$. Note that each A_i is an isometry of the symplectic product $(,)_J$, since $A_i^t J_2 A_i = J_2$, and that

$$\begin{aligned}\Psi_i(A_i(v_i), A_i(w_i))(u) &= \lambda_i(\,(A_i(w_i), u)_J A_i(v_i) - (u, A_i(v_i))_J A_i(w_i)\,) \\ &= \lambda_i A_i(\,((w_i, A_i^{-1}(u))_J v_i - (A_i^{-1}(u), v_i)_J w_i\,) \\ &= A_i(\Psi_i(v_i, w_i))A_i^{-1}(u)\,.\end{aligned}$$

From this fact and the definition of the product of odd elements, e.g. Equation (7.2), it follows that $[\sigma_{\bar{1}}(v_1 \otimes v_2 \otimes v_3), \sigma_{\bar{1}}(w_1 \otimes w_2 \otimes w_3)] = \sigma_{\bar{0}}([v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_2 \otimes w_3]$. Finally, we have that $JJId_{2,2,2} = Ad(J_2) \times Ad(J_2) \times Ad(Id_2)$, resp. $JJJ_{2,2,2} = Ad(J_2) \times Ad(J_2) \times Ad(J_2)$, is a $\bar{0}$-involution, resp. $\bar{1}$-involution. They are also the only ones, up to equivalence, by the real form argument, Lemma 8.1 and the fact that $\mathfrak{g}_{\bar{1}}$ is irreducible.

If $\alpha = \frac{1}{2}(-1 + \sqrt{-3})$ we also have to consider the outer automorphisms generated by d_3 which acts as the permutation $(1\,2\,3)$ on the simple components of $\mathfrak{g}_{\bar{0}}$ (see [50]). Since d_3 is not grade-reversing nor grade-preserving the equivalence classes of the grade-reversing ϵ-involutions are left unchanged. □

Theorem 8.11. *Let* $V = V_{\bar{0}} \oplus V_{\bar{1}}$, *with* $V_{\bar{0}} = \mathbb{C} \oplus \mathbb{C}$, $V_{\bar{1}} = \mathbb{C}^2$ *and let* $\alpha \in \mathbb{C} - \{0, -1, 1, -2, -\frac{1}{2}\}$. *We define*

$$\sigma'_p : V \to V, \ p = 1, 2, \quad (\sigma'_p)_{\bar{0}} = -\mathrm{Id}, \ (\sigma'_p)_{\bar{1}}(w) = J_2^{p-1} w \,.$$

Let $x, y, z \in V$, $x = (a_x, b_x, w_x), y = (a_y, b_y, w_y), z = (a_z, b_z, w_z)$, *with* $a_x, a_y, a_z, b_x, b_y, b_z \in \mathbb{C}, \ w_x, w_y, w_z \in \mathbb{C}^2$ *and set* $w'_y = (\sigma'_p)_{\bar{1}}(w_y)$.

The super space V *with triple product*

$$\begin{aligned}(x, y, z)_p = \ (\quad & -2a_x a_y a_z - \alpha((w_x, w'_y)_J a_z - (w_z, w'_y)_J a_x + (w_x, w_z)_J b_y) \quad , \\ & -2b_x b_y b_z - ((w_x, w'_y)_J b_z - (w_z, w'_y)_J b_x + (w_x, w_z)_J a_y) \quad , \\ & -(a_x a_y + b_x b_y) w_z + (a_z a_y + b_z b_y) w_x + (a_x b_z + b_x a_z) w'_y - \\ & -\tfrac{1+\alpha}{2}((w_x, w'_y)_J w_z - (w_z, w'_y)_J w_x + (w_x, w'_y)_J w'_y) \quad)\end{aligned} \tag{8.4}$$

is a K-simple $\bar{0}$*-sJTS if* $p = 1$ *and* $\bar{1}$*-sJTS if* $p = 2$ *with associated Lie superalgebra* $D(\alpha)$.

Proof. Let $\sigma_p, p = 1, 2$, be the grade-reversing ϵ-involutions of $\mathfrak{g} = D(\alpha)$ defined in Proposition 8.10, namely $\sigma_1 = JJId_{2,2,2}$, $\sigma_2 = JJJ_{222}$. If we identify $\mathfrak{g}_{\pm 1}$ with V via

$$a e_{(1\pm)} + b e_{(2\pm)} + w_{(\pm\pm)} \otimes w \to (a, b, w) \,,$$

we have that, under this identification, σ_p acts on the elements of $\mathfrak{g}_{-1}$ by σ'_p. The triple product of Equation 8.4 is the triple commutator of $[[x, \sigma_p(y)], z]$, by our identification of $\mathfrak{g}_{-1}$ with V and Equation 7.4. Hence, by Theorem 4.11, the statement follows. □

Proposition 8.12. *Let* $\mathfrak{g} = F(4)$ *with the 3-grading of Proposition 7.1.9. A complete list, up to equivalence, of grade-reversing* ϵ*-involutions of* $\mathfrak{g}$ *is the following:*

$$\epsilon = 0 : JS_p = Ad(J_2) \times Ad(S_{p,7}) \,, p = 1, 2 \,;$$
$$\epsilon = 1 : JS_p = Ad(J_2) \times Ad(S_7) \ .$$

Proof. The automorphism group of $\mathfrak{g}$ is $Aut(\mathfrak{g}) = SL(2, \mathbb{C}) \times SO(7, \mathbb{C})$. Let $\sigma_{\bar{0}} = Ad(A) \times Ad(B)$ be a grade-reversing involution of $\mathfrak{g}_{\bar{0}}$. Since the simple subalgebras $sl(2, \mathbb{C})$ and $so(7, \mathbb{C})$ are both 3-graded (see Definition 7.12) , $Ad(A)$ is of type $sl(2, \mathbb{R})$, while $Ad(B)$ is of type $so(p, 7-p)$, $p = 1, 2, 3$. Thus, $A = J_2$ and $B = S_{p,7}, p = 1, 2, 3$, are all the inequivalent grade-reversing involutions of $\mathfrak{g}_{\bar{0}}$, up to equivalence in $\mathfrak{g}_{\bar{0}}$ (i.e. conjugation by grade-preserving automorphisms of $\mathfrak{g}_{\bar{0}}$), since the choices for B have different spectrum.

Let σ be the extension of $\sigma_{\bar{0}}$ to $\mathfrak{g}$ (cf. [27, Proposition 1.1.1]), which, by Lemma 8.1 and the fact that $\mathfrak{g}_{\bar{1}}$ is irreducible, is unique up to composition with $\delta_{\pm 1}$. Let E be the grading element of $\mathfrak{g}$ and consider the adjoint action of E to $\mathfrak{g}_{\bar{0}}$.It follows at once that E is also the grading element for the 3-grading of $\mathfrak{g}_{\bar{0}}$. As a consequence, $\sigma(E) = \sigma_{\bar{0}}(E) = -E$, which proves that σ is grade-reversing. Furthermore, $(\sigma^2)_{\bar{0}} = \sigma_{\bar{0}}^2 = \mathrm{Id}$ implies that $(\sigma^2)_{\bar{1}} = \sigma_{\bar{1}}^2$ is a $\mathfrak{g}_{\bar{0}}$-modules isomorphism. With the same arguments used in the proof of Lemma 8.1, we conclude that $\sigma_{\bar{1}}^2 = \pm\mathrm{Id}$. This proves that any extension σ of $\mathfrak{g}_{\bar{0}}$ is a grade-reversing ϵ-involution. Together with our arguments on the equivalence classes of $\sigma_{\bar{0}}$, these facts conclude the proof, since, in this case, equivalence in $\mathfrak{g}$ is completely determined by equivalence in $\mathfrak{g}_{\bar{0}}$, as previously discussed. The following part is devolved to the explicit description of the extensions of σ.

Let $\mathrm{vol}_p \in C\ell(U), \mathrm{vol}_p = \dfrac{1}{\sqrt{2}^p}(u_1 - u_1^*)\cdots(u_{p-1} - u_{p-1}^*)(u_p - u_p^*), p = 1, 2, 3$. We define the extension of $(\sigma_p)_{\bar{0}}$ to $\mathfrak{g}$ by setting

$$(\sigma_p)_{\bar{1}}(\nu \otimes s) = J_2\nu \otimes \mathrm{vol}_p \circ s\,, \quad \text{if} \;\; (\sigma_p)_{\bar{0}} = \mathrm{Ad}(J_2) \times \mathrm{Ad}(S_{p,7}), \tag{8.5}$$

where $\nu \otimes s \in \mathbb{C}^2 \otimes \mathrm{Spin}_7$.

We prove that $(\sigma_p)_{\bar{1}}([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]) = [(\sigma_p)_{\bar{0}}(\mathfrak{g}_{\bar{0}}), (\sigma_p)_{\bar{1}}(\mathfrak{g}_{\bar{1}})]$. Let $(X, Y) \in \mathfrak{g}_{\bar{0}}$ and $\nu \otimes s \in \mathbb{C}^2 \otimes \mathrm{Spin}_7$, then

$$(\sigma_p)_{\bar{1}}([(X, Y), \nu \otimes s]) = (\sigma_p)_{\bar{1}}(X\nu \otimes Y(s)) = J_2X\nu \otimes \mathrm{vol}_p \circ (Y(s))\,.$$

It follows immediately that $J_2X\nu = J_2XJ_2^{-1}J_2\nu = (\mathrm{Ad}(J_2)(X))J_2\nu$. Hence, we need to prove

$$\mathrm{vol}_p \circ (Y(s)) = \mathrm{Ad}(S_{p,7})(Y)(\mathrm{vol}_p \circ s).$$

Notice that the twisted adjoint of $\nu o l_p$ covers $S_{p,7}$. Indeed, if $k \leqslant p$, we have

$$\begin{aligned}
\widetilde{\mathrm{Ad}}_{\nu o l_p}(u_k) = \;& (-1)^p \,\mathrm{vol}_p\, u_k\, \mathrm{vol}_p^{-1} = \\
= \;& (-1)^{p+\frac{p(p+1)}{2}} \tfrac{1}{2^p}(u_1 - u_1^*)\cdots(u_k - u_k^*)\cdots(u_p - u_p^*)u_k \\
& (u_1 - u_1^*)\cdots(u_k - u_k^*)\cdots(u_p - u_p^*) = \\
= \;& (-1)^{\frac{p(p+3)}{2}} \tfrac{1}{2^p}(u_k - u_k^*)u_k(u_k - u_k^*) \\
& ((u_1 - u_1^*)\cdots(u_{k-1} - u_{k-1}^*)(u_{k+1} - u_{k+1}^*)\cdots(u_p - u_p^*))^2 = \\
= \;& (-1)^{\frac{p(p+3)}{2} + \frac{(p-1)p}{2}} \tfrac{1}{2}(u_k - u_k^*)u_k(u_k - u_k^*) = \\
= \;& (-1)^{2(\frac{p(p+1)}{2})} \tfrac{1}{2} u_k^* u_k u_k^* = \\
= \;& u_k^*,
\end{aligned}$$

where we used the fact that $(u_j - u_j^*)^2 = -2\mathbb{1}$ and $\mathrm{vol}_p^2 = (-1)^{\frac{p(p+1)}{2}}\mathbb{1}$. Similarly one gets that, if $k \leqslant p$, $\widetilde{\mathrm{Ad}}_{\nu o l_p}(u_k^*) = u_k$. If $k > p$, then $\widetilde{\mathrm{Ad}}_{\nu o l_p}(u_k) = u_k$ and $\widetilde{\mathrm{Ad}}_{\nu o l_p}(u_k^*) = u_k^*$, by anticommutativity. Note that via our identification of $\mathfrak{so}(7, \mathbb{C})$ with $\Lambda^2(U)$, the adjoint action of $S_{p,7}$ on $\Lambda^2(U)$ is given by $\mathrm{Ad}(S_{p,7})(x) = S_{p,7}(u_1) \wedge S_{p,7}(u_2)$, if

$x \in so(7, \mathbb{C})$ corresponds to $u_1 \wedge u_2 \in \Lambda^2(U)$. It follows that

$$\begin{aligned}(\sigma_p)_{\bar{1}}[u \wedge v, s] = & \operatorname{vol}_p \circ (\tfrac{1}{4}(uv - vu) \circ s) \\ = & \tfrac{1}{4}(\operatorname{vol}_p(uv - vu)) \circ s \\ = & \tfrac{1}{4}(((-1)^p \operatorname{vol}_p u \operatorname{vol}_p^{-1})((-1)^p \operatorname{vol}_p v \operatorname{vol}_p^{-1} \operatorname{vol}_p)) \circ s \\ & -\tfrac{1}{4}(((-1)^p \operatorname{vol}_p v \operatorname{vol}_p^{-1})((-1)^p \operatorname{vol}_p u \operatorname{vol}_p^{-1} \operatorname{vol}_p)) \circ s \\ = & \tfrac{1}{2}(S_{p,7}(u)) \wedge (S_{p,7}(v)) \circ \operatorname{vol}_p \circ s = \\ = & [(\sigma_p)_{\bar{0}}(u \wedge v), (\sigma_p)_{\bar{1}}(s)] .\end{aligned}$$

We turn our attention to $[(\sigma_p)_{\bar{1}}(\mathfrak{g}_{\bar{1}}), (\sigma_p)_{\bar{1}}(\mathfrak{g}_{\bar{1}})]$. Suppose vol_p is an isometry of $(,)_B$. Then

$$\begin{aligned}[(\sigma_p)_{\bar{1}}(v \otimes s), (\sigma_p)_{\bar{1}}(w \otimes t)] = & (\operatorname{vol}_p \circ s, \operatorname{vol}_p \circ t)_B \Psi_A(J_2 v, J_2 w) + \\ & -\tfrac{4}{3}(J_2 v, J_2 w)_A \Psi_B(\operatorname{vol}_p \circ s, \operatorname{vol}_p \circ t) = \\ = & \operatorname{Ad}(J_2)((s,t)_B \Psi_A(v, w)) - \tfrac{4}{3}(v, w)_A \Psi_B(\operatorname{vol}_p \circ s, \operatorname{vol}_p \circ t)\end{aligned}$$

and

$$\begin{aligned}\Psi_B(\operatorname{vol}_p \circ s, \operatorname{vol}_p \circ t) = & \textstyle\sum_{x \in \mathcal{B}} (\operatorname{vol}_p \circ s, -x^t \cdot (\operatorname{vol}_p \circ t))_B x = \\ = & \textstyle\sum_{x \in \mathcal{B}} (s, (\operatorname{vol}_p)^{-1} \circ (-x^t \cdot (\operatorname{vol}_p \circ t)))_B x = \\ = & \textstyle\sum_{x \in \mathcal{B}} (s, (\operatorname{Ad}(S_{p,7})(-x^t)).t)_B x = \\ = & \textstyle\sum_{y \in \mathcal{B}''} (s, -y^t \cdot t)_B \operatorname{Ad}(S_{p,7})(y) = \operatorname{Ad}(S_{p,7})(\Psi_B(s,t)) ,\end{aligned}$$

with $\mathcal{B}''$ the basis of $so(7, \mathbb{C})$ consisting of the images of elements of the basis $\mathcal{B}$ under the action of $\operatorname{Ad}(S_{p,7})$. Hence if vol_p is an isometry σ_p is a grade-reversing ϵ-involution. The fact that vol_p is an isometry follows since $()_B$ is an admissible form with invariants $(-1, 1)$ on $Spin_7$.

Since $\operatorname{vol}_p^2 = (-1)^{\frac{p(p+1)}{2}}$ and $J_2^2 = -1$ we obtain that (σ_p) is a $\bar{0}$-involution if $p = 1, 2$ and a $\bar{1}$-involution if $p = 3$. □

Theorem 8.13. *Let* U' *denote* $\mathbb{C}^5$ *with the scalar product with matrix* S_5, $\mathbb{S}$ *denote* $\Lambda^\bullet W'$ *and let* $V = V_{\bar{0}} \oplus V_{\bar{1}}$, *with* $V_{\bar{0}} = \mathbb{C} \oplus U'$, $V_{\bar{1}} = \mathbb{S}$, $\operatorname{vol}_1' = \mathbb{1}, \operatorname{vol}_2' = \frac{(u_2 - u_2^*)}{\sqrt{2}}, \operatorname{vol}_3' = \frac{(u_2 - u_2^*)}{\sqrt{2}} \frac{(u_3 - u_3^*)}{\sqrt{2}}, \operatorname{vol}_p' \in C\ell(U'), p = 1, 2, 3,$ *with the notation introduced in Remark 7.13. Let* $x, y, z \in V$, $x = (a_x, u_x, s_x), y = (a_y, u_y, s_y), z = (a_z, u_z, s_z)$, $a_x, a_y a_{,z} \in \mathbb{C}, u_x, u_y, u_z \in U', s_x, s_y, s_z \in \mathbb{S}$ *and let*

$$\sigma_p'(y) = (-a_y, -S_{p-1,5}(u_y), \operatorname{vol}_p' \circ s_y) .$$

The super space V *with triple product*

$$
\begin{aligned}
(x,y,z)_p = \Big(& -2(a_x a_y a_z + (a_x s_z - a_z s_x, (\mathrm{vol}'_p \circ s_y))_{B'}) + \\
& +\sqrt{2}(s_z, (S_{p-1,5}(u_y)) \circ (s_x))_{B'}, \\
& -((u_x, (S_{p-1,5}(u_y)))u_z + (u_z, (S_{p-1,5}(u_y)))u_x - \\
& -(u_x, u_z)(S_{p-1,5}(u_y))) + \\
& +\tfrac{4}{3} a_y \Psi_{U'}(s_x, s_z) + \\
& +\tfrac{2\sqrt{2}}{3}(\Psi_{U'}(s_x, u_z \circ (\mathrm{vol}'_p \circ s_y)) - \Psi_{U'}(s_z, u_x \circ (\mathrm{vol}'_p \circ s_y))), \\
& -a_y(a_x s_z + a_z s_x) - \tfrac{1}{\sqrt{2}}(a_z u_x + a_x u_z) \circ (\mathrm{vol}'_p \circ s_y) - \\
& -\tfrac{1}{2}((u_x(S_{p-1,5}(u_y))) \circ s_z + (u_z(S_{p-1,5}(u_y))) \circ s_x) - \\
& -\tfrac{4}{3}\Psi_{B'}(s_x, (\mathrm{vol}'_p \circ s_y)) \cdot s_z + \tfrac{2}{3}(s_x, (\mathrm{vol}'_p \circ s_y))_{B'} s_z \Big)
\end{aligned}
\tag{8.6}
$$

is a K-simple $\bar{0}$*-sJTS if* $p = 1, 2$ *and* $\bar{1}$*-sJTS if* $p = 3$ *with associated Lie superalgebra* $F(4)$*.*

Proof. It follows from the realization of $\mathfrak{g} = F(4)$ given in Remark 7.13 and the fact that the grade-reversing ϵ-involutions of $F(4)$ defined in Proposition 8.12, namely $\sigma_p, p = 1, 2, 3$, act on the generic element $y \in \mathfrak{g}_{-1}, y = a_y f + u_y \wedge u_1^* + v_- \otimes s_y$, by

$$
\begin{aligned}
\mathfrak{g}_1 \ni \sigma_p(y) = \; & \mathrm{Ad}(J_2)(a_y f) + \mathrm{Ad}(S_{p,7})(u_y \wedge u_1^*) + J_2(v_-) \otimes \mathrm{vol}_p \circ s_y \\
= \; & -a_y e - u_1 \wedge S_{p-1,5}(u_y) + v_+ \otimes u_1 \wedge \mathrm{vol}'_p \circ s_y \, .
\end{aligned}
$$

If we identify $\mathfrak{g}_{\pm 1}$ with $\mathbb{C} \oplus U' \oplus \Lambda^\bullet W'$, via $af + u \wedge u_1^* + v_- \otimes s \to (a, u, s)$ and $ae + u_1 \wedge u + v_+ \otimes u_1 \wedge s \to (a, u, s)$, then, in these new coordinates, σ_p corresponds to σ'_p. Exchanging y with $\sigma'_p(y)$ in Equation 7.8 we obtain the triple product of Equation 8.6 and, due to Theorem 4.11, the statement follows. □

Bibliography

[1] Dmitri V. Alekseevsky and Vicente Cortés. Classification of N-(super)-extended Poincaré algebras and bilinear invariants of the spinor representation of $\mathrm{Spin}(p, q)$. *Comm. Math. Phys.*, 183(3):477–510, 1997.

[2] Bruce Allison, John Faulkner, and Oleg Smirnov. Weyl images of Kantor pairs. *Canad. J. Math.*, 69(4):721–766, 2017.

[3] Bruce N. Allison. Tensor products of composition algebras, Albert forms and some exceptional simple Lie algebras. *Trans. Amer. Math. Soc.*, 306(2):667–695, 1988.

[4] Andrea Altomani and Andrea Santi. Classification of maximal transitive prolongations of super-Poincaré algebras. *Adv. Math.*, 265:60–96, 2014.

[5] Andrea Altomani and Andrea Santi. Tanaka structures modeled on extended Poincaré algebras. *Indiana Univ. Math. J.*, 63(1):91–117, 2014.

[6] Hiroshi Asano. Classification of noncompact real simple generalized Jordan triple systems of the second kind. *Hiroshima Math. J.*, 21(3):463–489, 1991.

[7] Jonathan Bagger and Neil Lambert. Three-Algebras and N=6 Chern-Simons Gauge Theories. *Phys. Rev.*, D79:025002, 2009.

[8] Armand Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. of Math. (2)*, 75:485–535, 1962.

[9] Robert B. Brown. Groups of type E_7. *J. Reine Angew. Math.*, 236:79–102, 1969.

[10] Nicoletta Cantarini and Victor G. Kac. Infinite-dimensional primitive linearly compact Lie superalgebras. *Adv. Math.*, 207(1):328–419, 2006.

[11] Nicoletta Cantarini and Victor G. Kac. Classification of linearly compact simple Jordan and generalized Poisson superalgebras. *J. Algebra*, 313(1):100–124, 2007.

[12] Nicoletta Cantarini and Victor G. Kac. Classification of linearly compact simple Jordan and generalized Poisson superalgebras. *J. Algebra*, 313(1):100–124, 2007.

[13] Nicoletta Cantarini and Victor G. Kac. Classification of simple linearly compact n-Lie superalgebras. *Comm. Math. Phys.*, 298(3):833–853, 2010.

[14] Nicoletta Cantarini and Victor G. Kac. Classification of linearly compact simple algebraic $N = 6$ 3-algebras. *Transform. Groups*, 16(3):649–671, 2011.

[15] Nicoletta Cantarini, Antonio Ricciardo, and Andrea Santi. Classification of simple linearly compact kantor triple systems over the complex numbers. *Journal of Algebra*, 514:468 – 535, 2018.

[16] Elie Cartan. Les groupes de transformations continus, infinis, simples. *Ann. Sci. École Norm. Sup.*, 3(26):93–161, 1909.

[17] Elie Cartan. *Œuvres complètes. Partie I. Groupes de Lie*. Gauthier-Villars, Paris, 1952.

[18] Dragomir Ž. Djoković. Classification of **Z**-graded real semisimple Lie algebras. *J. Algebra*, 76(2):367–382, 1982.

[19] Alberto Elduque, Noriaki Kamiya, and Susumu Okubo. Left unital Kantor triple systems and structurable algebras. *Linear Multilinear Algebra*, 62(10):1293–1313, 2014.

[20] Hans Freudenthal. Beziehungen der E_7 und E_8 zur Oktavenebene. I. *Nederl. Akad. Wetensch. Proc. Ser. A.* **57** = *Indagationes Math.*, 16:218–230, 1954.

[21] Esther García, Miguel Gómez Lozano, and Erhard Neher. Nondegeneracy for Lie triple systems and Kantor pairs. *Canad. Math. Bull.*, 54(3):442–455, 2011.

[22] Victor Guillemin. Infinite dimensional primitive Lie algebras. *J. Differential Geometry*, 4:257–282, 1970.

[23] M. Günaydin, K. Koepsell, and H. Nicolai. Conformal and quasiconformal realizations of exceptional Lie groups. *Comm. Math. Phys.*, 221(1):57–76, 2001.

[24] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[25] Nathan Jacobson. Lie and Jordan triple systems. *Amer. J. Math.*, 71:149–170, 1949.

[26] Victor G. Kac. Classification of simple z-graded lie superalgebras and simple jordan superalgebras. *Communications in Algebra*, 5(13):1375–1400, 1977.

[27] Victor G. Kac. Lie superalgebras. *Advances in Math.*, 26(1):8–96, 1977.

[28] Victor G. Kac. Classification of infinite-dimensional simple linearly compact Lie superalgebras. *Adv. Math.*, 139:1–55, 1998.

[29] Soji Kaneyuki and Hiroshi Asano. Graded Lie algebras and generalized Jordan triple systems. *Nagoya Math. J.*, 112:81–115, 1988.

[30] Issai L. Kantor. Transitive differential groups and invariant connections in homogeneous spaces. *Trudy Sem. Vektor. Tenzor. Anal.*, 13:310–398, 1966.

[31] Issai L. Kantor. Certain generalizations of Jordan algebras. *Trudy Sem. Vektor. Tenzor. Anal.*, 16:407–499, 1972.

[32] Issai L. Kantor. Certain generalizations of Jordan algebras. *Trudy Sem. Vektor. Tenzor. Anal.*, 16:407–499, 1972.

[33] Sung-Soo Kim and Jakob Palmkvist. $N = 5$ three-algebras and 5-graded Lie superalgebras. *J. Math. Phys.*, 52(8):083502, 9, 2011.

[34] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[35] Max Koecher. Imbedding of Jordan algebras into Lie algebras. I. *Amer. J. Math.*, 89:787–816, 1967.

[36] Max Koecher. *An elementary approach to bounded symmetric domains*. Rice University, Houston, Tex., 1969.

[37] Sergei Krutelevich. Jordan algebras, exceptional groups, and Bhargava composition. *J. Algebra*, 314(2):924–977, 2007.

[38] Blaine H. Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[39] Ottmar Loos. *Lectures on Jordan triples*. The University of British Columbia, Vancouver, B.C., 1971.

[40] Kevin McCrimmon. The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras. *Trans. Amer. Math. Soc.*, 139:495–510, 1969.

[41] Kevin McCrimmon. *A taste of Jordan algebras*. Universitext. Springer-Verlag, New York, 2004.

[42] Kurt Meyberg. *Lectures on algebras and triple systems*. The University of Virginia, Charlottesville, Va., 1972. Notes on a course of lectures given during the academic year 1971–1972.

[43] Daniel Mondoc. Compact exceptional simple Kantor triple systems defined on tensor products of composition algebras. *Comm. Algebra*, 35(11):3699–3712, 2007.

[44] Daniel Mondoc. Compact realifications of exceptional simple Kantor triple systems defined on tensor products of composition algebras. *J. Algebra*, 307(2):917–929, 2007.

[45] Susumu Okubo and Noriaki Kamiya. Quasi classical Lie-super algebra and Lie-super triple systems. *University of Rochester Report UR-1462*, 1996.

[46] Arkadij L. Onishchik and Ernest B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[47] Arkadij L. Onishchik and Ernest B. Vinberg. *Lie Groups and Algebraic Groups*. Springer-Verlag, Berlin Heidelberg, 1990.

[48] Jakob Palmkvist. A realization of the Lie algebra associated to a Kantor triple system. *J. Math. Phys.*, 47(2):9, 2006.

[49] Jakob Palmkvist. Three-algebras, triple systems and 3-graded Lie superalgebras. *J. Phys. A*, 43(1):015205, 15, 2010.

[50] Vera V. Serganova. Automorphisms of simple Lie superalgebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(3):585–598, 1984.

[51] Osamu Shukuzawa. Explicit classifications of orbits in Jordan algebra and Freudenthal vector space over the exceptional Lie groups. *Comm. Algebra*, 34(1):197–217, 2006.

[52] Arkadii Slinko. Linearly compact algebras and coalgebras. *New Zealand J. Math.*, 25(1):95–104, 1996.

[53] Tonny A. Springer. Characterization of a class of cubic forms. *Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math.*, 24:259–265, 1962.

[54] Noboru Tanaka. On differential systems, graded Lie algebras and pseudogroups. *J. Math. Kyoto Univ.*, 10:1–82, 1970.

[55] Noboru Tanaka. On the equivalence problems associated with simple graded Lie algebras. *Hokkaido Math. J.*, 8(1):23–84, 1979.

[56] Jacques Tits. Une classe d'algèbres de Lie en relation avec les algèbres de Jordan. *Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math.*, 24:530–535, 1962.

[57] Keizo Yamaguchi. Differential systems associated with simple graded Lie algebras. In *Progress in differential geometry*, volume 22 of *Adv. Stud. Pure Math.*, pages 413–494. Math. Soc. Japan, Tokyo, 1993.