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Numerical invariants and volume rigidity for hyperbolic lattices

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Abstract

Let $m \geq p \geq 3$ be positive integers. Given the fundamental group Γ of a finite-volume complete hyperbolic p-manifold M, it is possible to associate to any representation $\rho : \Gamma \to PO(m, 1)$ a numerical invariant called volume. This invariant is bounded by the hyperbolic volume of M and satisfies a well-known rigidity condition: if the volume of ρ is maximal, then ρ must be discrete and faithful. In this dissertation we prove a generalization of this rigidity result by showing that if a sequence of representations $\rho_n : \Gamma \to PO(m, 1)$ satisfies $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$, then there must exist a sequence of elements $g_n \in PO(m, 1)$ such that the representations $g_n \circ \rho_n \circ g_n^{-1}$ converge to a representation $\rho_\infty : \Gamma \to PO(m, 1)$ which preserves a totally geodesic copy of $\mathbb{H}^p$ in $\mathbb{H}^m$ and such that its $\mathbb{H}^p$-component is conjugated to the standard lattice embedding $i : \Gamma \to PO(p, 1) < PO(m, 1)$. We call ridigity at infinity this property of the volume function. The rigidity at infinity implies that if the representations ρ_n converge to an ideal point of the character variety, then the sequence of volumes must stay away from the maximum.

Let Γ be a non-uniform lattice in $PU(p, 1)$ (or $PSp(p, 1)$) without torsion. Assuming $m \geq p \geq 2$, we introduce the volume function for representations $\rho : \Gamma \to PU(m, 1)$ (or $\rho : \Gamma \to PSp(m, 1)$, respectively) and we prove that rigidity at infinity holds also for both the complex case and the quaternionic one.

Finally, if Γ is the fundamental group of a complete hyperbolic 3-manifold M with toric cusps, we define the ω-Borel invariant $\beta_n^\omega(\rho_\omega)$ associated to a representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$, where $\mathbb{C}_\omega$ is a field which can be constructed as a quotient of a suitable subset of $\mathbb{C}^{\mathbb{N}}$ with the data of a non-principal ultrafilter ω on $\mathbb{N}$ and a real divergent sequence λ_l such that $\lambda_l \geq 1$.

Since a sequence of ω-bounded representations ρ_l into $SL(n, \mathbb{C})$ determines a representation ρ_ω into $SL(n, \mathbb{C}_\omega)$, for $n = 2$ we study the relation between the invariant $\beta_2^\omega(\rho_\omega)$ and the sequence of Borel invariants $\beta_2(\rho_l)$. In particular we study the relation between the reducibility of the action induced on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ by a representation $\rho_\omega : \Gamma \to SL(2, \mathbb{C}_\omega)$ and the vanishing of the invariant $\beta_2^\omega(\rho_\omega)$.

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Introduction

Let G be a Lie group. We say that Γ is a lattice if Γ is a discrete subgroup of G such that $\overline{\mu}_{\mathcal{H}}(\Gamma\backslash G) < \infty$, where $\overline{\mu}_{\mathcal{H}}$ is the measure induced by the left Haar measure on the quotient $\Gamma\backslash G$. Assume that G admits X as Riemannian symmetric space. Any lattice Γ without torsion acts freely and properly discontinuously on X and hence the quotient $M = \Gamma\backslash X$ admits a natural structure of complete manifold which is locally isometric to X. Moreover M has finite volume. If M is compact we say that Γ is uniform, otherwise we refer to Γ as a non-uniform lattice.

Let Γ be a non-uniform lattice of G without torsion. The lattice Γ is said strongly rigid if for any other lattice Γ' of another Lie group G', every isomorphism $\varphi : \Gamma \to \Gamma'$ can be uniquely extended to an isomorphism $\Phi : G \to G'$ of the ambient Lie groups. The strong rigidity property has been widely studied so far, for instance in [Mos68] and in [Pra73]. Mostow proved in [Mos73] that any irreducible lattice in a connected semisimple Lie group $G \ncong PSL(2,\mathbb{R})$ with trivial center and no compact factor is strongly rigid. If we restrict our attention to lattices of the same Lie group G, Mostow strong rigidity theorem implies that if Γ and Γ' are isomorphic lattices there must exist an element $g \in G$ such that $g\Gamma g^{-1} = \Gamma'$.

If we assume $\Gamma < PSO(p,1)$ and we look at representations $\rho : \Gamma \to PO(m,1)$, the previous result may be strengthened by introducing the notion of volume for representations $\rho : \Gamma \to PO(m,1)$, where $m \geq p \geq 3$. When $m = p = 3$ the volume can be thought of as the integral of the pullback of the volume form on $\mathbb{H}^3$ along any pseudo-developing map $D : \mathbb{H}^3 \to \mathbb{H}^3$, as written both in [Dun99] and in [Fra04]. Since the volume is independent of the choice of the pseudeveloping map D, when D is a straight map this notion is a generalization of the volume of a solution for the gluing equations associated to a triangulation of M, given for instance in [NZ85]. In the more general context of $\Gamma < PSO(p,1)$ another way to define the volume of a representation $\rho : \Gamma \to PSO(p,1) = \mathrm{Isom}^+(\mathbb{H}^p)$ is based on the properties of the bounded cohomology of the group $\mathrm{Isom}^+(\mathbb{H}^p)$. In [BBI13] the authors prove that the volume class ω_p is a generator of the cohomology group $H^p_{cb}(\mathrm{Isom}^+(\mathbb{H}^p))$, hence, starting from it, we can construct a class in $H^p_b(\Gamma)$ by pulling back ω_p along ρ^*_b and then evaluate this class with a relative fundamental class $[N, \partial N] \in H^p(N, \partial N)$ via the Kronecker pairing. Here N is any compact core of $M = \Gamma\backslash\mathbb{H}^p$. For the case $p = 3$, the equivalence between the two different definitions it is shown for example in [Kim16]. To extend the notion of volume to the more general case of representations into the whole group of the isometries $PO(m,1)$, where $m \geq p$, the approach of [FK06] is to consider the infimum all over the volumes $\mathrm{Vol}(D)$, where $D : \mathbb{H}^p \to \mathbb{H}^m$ is a properly ending smooth ρ-equivariant map.

Since the volume is invariant under conjugation by an element of $PO(m,1)$, there exists a well-defined volume function on the character variety $X(\Gamma, PO(m,1))$ which is continuous with respect to the topology of the pointwise convergence. Moreover, this function satisfies a well-known rigidity condition. As written in both [Fra04] and [FK06], for any representation ρ we have $\mathrm{Vol}(\rho) \leq \mathrm{Vol}(M)$ and if equality holds ρ preserves a totally geodesic copy of $\mathbb{H}^p$ and its $\mathbb{H}^p$-component is conjugated to the standard lattice embedding $i : \Gamma \to PO(p,1) < PO(m,1)$. Beyond its intrinsic interest, this result has important consequences for example in the study of the AJ-conjecture for hyperbolic knot manifolds, as written in [LZ17].

So far we have described the notion and the properties of the volume function for representations of non-uniform lattices. The same results are true for uniform lattices, but for these ones the $PO(m,1)$-character variety may be degenerate. For instance, when Γ is a uniform lattice of $PSO(3,1)$, the hyperbolic component of the character variety $X(\Gamma, PO(3,1))$ is zero dimensional by [NZ85]. This reason leads us to care only about the non-uniform case.

Inspired by the work of Thurston about the compactification of the Teichmüller space for a closed surface of genus g exposed in [Th88] and generalizing the constructions for algebraic curves appeared in [CS83], in [MS84] J. Morgan and P. Shalen proposed a new way to compactify a generic algebraic variety V given a generating set $\mathcal{F}$ for the algebra of regular functions $\mathbb{C}[V]$. This particular method applied to the character variety $X(\Gamma, SL(2,\mathbb{C}))$ allows to interpret the ideal points of the compactification as projective length functions of isometric Γ-actions on real trees which are constructed as Bass–Serre trees associated to $SL(2,\mathbb{K}_v)$, where $\mathbb{K}_v$ is a suitable valued field (see [Ser80]). Lately Morgan extended the compactification to the variety $X(\Gamma, PO(m,1))$ in [Mor86], again by seeing the ideal points as projective lenght functions of isometric Γ-actions on real trees. We call this compactification the Morgan–Shalen compactification of $X(\Gamma, PO(m,1))$. It seems quite natural to ask if there exists a way to extend continuously the volume function to this compactification and which are the possible values attained at any ideal point. For instance, one could ask if it is possible to extend the ridigity of the volume function also at ideal points.

One of the main goal of this dissertation is to prove the following

Theorem 1 *Let Γ be a non-uniform lattice of $PSO(p,1)$ without torsion. Assume $m \geq p \geq 3$. Let $\rho_n : \Gamma \to PO(m,1)$ be a sequence of representations such that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$. Then there must exist a sequence of elements $g_n \in PO(m,1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to a reducible representation ρ_∞ which preserves a totally geodesic copy of $\mathbb{H}^p$ and whose $\mathbb{H}^p$-component is conjugated to the standard lattice embedding $i : \Gamma \to PO(p,1) < PO(m,1)$.*

An important consequence of this theorem will be

Corollary 2 *Let Γ be a non-uniform lattice of $PSO(p,1)$ without torsion. Assume $p \geq 3$. Suppose $\rho_n : \Gamma \to PO(m,1)$ is a sequence of representations with $m \geq p$. If the sequence is converging to any ideal point of the Morgan–Shalen compactification of $X(\Gamma, PO(m,1))$, then the sequence of volumes $\mathrm{Vol}(\rho_n)$ must be bounded from above by $\mathrm{Vol}(M) - \varepsilon$ with $\varepsilon > 0$.*

We are going to call rigidity at infinity the property of the volume function stated in Theorem 1. The proof of this theorem will be based essentially on the so-called BCG–natural map associated to a non-elementary representation $\rho : \Gamma \to PO(m,1)$, described in [BCG95], [BCG96] and [BCG99]. Given such a representation there exists a map $F : \mathbb{H}^p \to \mathbb{H}^m$ which is equivariant with respect to ρ, smooth and satisfies $Jac_p(F) \leq 1$ for every $x \in \mathbb{H}^p$, where $Jac_p(F)$ is the p-Jacobian of the map F. Moreover, the equality holds if and only if D_xF is an isometry, and we will exploit the fact that this claim can be made ε-accurate if $Jac_p(F) > 1 - \varepsilon$. These properties make the natural map F a powerful tool in the study of volume rigidity (see [BCS05] for this kind of applications).

However the construction of the BCG-natural map is much more general. Let Γ be a non-uniform lattice of G_p without torsion, with either $G_p = PU(p,1)$ or $G_p = PSp(p,1)$. We say that the lattice Γ is complex in the former case, quaternionic in the latter. Given a representation of $\rho : \Gamma \to G_m$, where $G_m = PU(m,1)$ if Γ is complex or $G_m = PSp(m,1)$ if Γ is quaternionic, we can adapt the procedure described by both [BCG99] and [Fra09] to obtain a natural map which satisfies the same properties listed previously.

The will to extend the strong rigidity at infinity in this more general context leads us to the introduction of the notion of volume for representations $\rho : \Gamma \to G_m$, with $m \geq p$. For uniform complex lattices the definition of volume for representations $\rho : \Gamma \to PU(m,1)$ is given both by [BCG99] and by [BCG07], whereas for non-uniform complex lattices we refer to [BI] and to [KM08]. Another interesting approach is exposed in [KK12], where the authors use the pairing between bounded cohomology and l^1-Lipschitz homology to define the volume of a representation. However, here we give a different version of it to adapt this notion to the non compact case, also for quaternionic lattices. Thanks to this definition we get

Theorem 3 *Let Γ be a non-uniform lattice of $PU(p,1)$ without torsion. Assume $p \geq 2$. Let $\rho_n : \Gamma \to PU(m,1)$ be a sequence of representations with $m \geq p$. If $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$, then there must exist a sequence of elements $g_n \in PU(m,1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to a reducible representation ρ_∞ which preserves a totally geodesic copy of $\mathbb{H}^p_{\mathbb{C}}$ and whose $\mathbb{H}^p_{\mathbb{C}}$-component is conjugated to the standard lattice embedding $i : \Gamma \to PU(p,1) < PU(m,1)$.*

And in the same way

Theorem 4 *Let Γ be a non-uniform lattice of $PSp(p,1)$ without torsion. Assume $p \geq 2$. Let $\rho_n : \Gamma \to PSp(m,1)$ be a sequence of representations with $m \geq p$. If $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$, then there must exist a sequence of elements $g_n \in PSp(m,1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to a reducible representation ρ_∞ which preserves a totally geodesic copy of $\mathbb{H}^p_{\mathbb{Q}}$ and whose $\mathbb{H}^p_{\mathbb{Q}}$-component is conjugated to the standard lattice embedding $i : \Gamma \to PSp(p,1) < PSp(m,1)$.*

Let now Γ be again a non-uniform lattice in $PSO(3,1)$ without torsion. By looking at representations $\rho : \Gamma \to PSL(n,\mathbb{C})$ it is possible to attach to every equivalence class of such a representation a suitable invariant called Borel invariant. Indeed, in [BBI] the authors prove that the Borel class $\beta(n)$, already introduced and studied into [Gon93], is a generator for the cohomology group $H^3_{cb}(PSL(n,\mathbb{C}))$. Thus, given a representation $\rho : \Gamma \to PSL(n,\mathbb{C})$, we can construct a class into $H^3_b(\Gamma)$ by pulling back $\beta(n)$ along ρ^*_b and then evaluate this new class on a fundamental class $[N, \partial N] \in H^3(N, \partial N)$, as done previously in the case of volume. Here N is still any compact core of $M = \Gamma \backslash \mathbb{H}^3$. When $n = 2$ this invariant is exactly the volume of the representation. The Borel invariant of a representation $\rho : \Gamma \to SL(n,\mathbb{C})$

will be the Borel invariant of the induced representation into $PSL(n,\mathbb{C})$. Moreover, since this invariant remains unchanged under conjugation, as before we have a well-defined function on the character variety $X(\Gamma, SL(n,\mathbb{C}))$, called Borel function, which is continuous with respect to the topology of the pointwise convergence.

In [Par12] Parreau extended the Morgan–Shalen interpretation of ideal points to the more general case of $X(\Gamma, SL(n,\mathbb{C}))$ by viewing an ideal point as a projective vectorial length function relative to an isometric action, this time on a Euclidean building of type A_{n-1}. The method suggested by [Par12] to obtain the Euclidean building and its isometric Γ-action is based on asymptotic cones and it reminds the ones already exposed both in [Bes88] and in [Pau88].

By following the same attitude assumed previously for the volume function, one could naturally ask if it is possible to extend continuously the Borel function to the ideal points of the Parreau compactification of $X(\Gamma, SL(n,\mathbb{C}))$. Going further, one could be interested in studying the possible values attained at ideal points and trying to formulate a rigidity result, which would generalize [BBI, Theorem 1]. This problem has already been conjectured in [Gui16, Conjecture 1].

In order to make a small step towards this direction we define a numerical invariant, the ω-Borel invariant, associated to a representation $\rho_\omega : \Gamma \to SL(n,\mathbb{C}_\omega)$, where $\mathbb{C}_\omega$ is a field obtained as a quotient of a suitable subset of $\mathbb{C}^{\mathbb{N}}$ by an equivalence relation which depends on a non-principal ultrafilter ω on $\mathbb{N}$ and a real divergent sequence λ_l with $\lambda_l \geq 1$. The motivation of this definition relies on the interpretation of the limit action of Γ on the Euclidean bulding of type A_{n-1} as a representation $\rho_\omega : \Gamma \to SL(n,\mathbb{C}_\omega)$, as proved in [Par12, Theorem 5.2].

The structure of the dissertation is the following. The first chapter is dedicated to preliminary definitions. We start by recalling the notion of bounded cohomology of locally compact groups. We describe the functorial approach to bounded cohomology and we exhibit an easy computation of some bounded cohomology groups, e.g. for $SL(2,\mathbb{C})$. Successively we describe the construction of the BCG-natural map. Let G_p be a rank-one Lie group of non-compact type and denote by X^p the symmetric space associated to G_p. Fix Γ a non-uniform lattice of G_p. Given the notion of barycentre of a positive Borel measure on $\partial_\infty X^p$, we explain the construction of the natural map F associated to a non-elementary representation $\rho : \Gamma \to G_m$. We introduce the definition of volume for representations $\rho : \Gamma \to G_m$ and we compare it with the volume of the ε-natural maps F^ε. These maps are smooth, ρ-equivariant and converge to F with respect to the C^1-topology (see [FK06]). We conclude the chapter with a quick overview about the Parreau compactification of the character variety $X(\Gamma, SL(n,\mathbb{C}))$ of any finitely generated group Γ. We follow [Par12] to describe the construction of the field $\mathbb{C}_\omega$. It follows a brief exposition about real trees and Euclidean buildings and how they can be associated to the ideal points of the compactification cited above.

The second chapter is devoted to the proof of the main theorems. We start by focusing our attention to real hyperbolic lattices and we underline some consequences of this result for the extendability of the volume function to the Morgan-Shalen compactification of $X(\Gamma, PO(m,1))$. Finally, we extend the rigidity theorem to the complex case and the quaternionic case.

In the third chapter we give the definition of the ω-Borel cohomology class $\beta^\omega(n)$ which will be an element of $H^3_b(SL^\delta(n,\mathbb{C}_\omega))$. We define the ω-Borel invariant $\beta^\omega_n(\rho_\omega)$

for a representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$ and we describe some of its properties. In particular we focus our attention on the case $n = 2$. We show that if a sequence of representations $\rho_l : \Gamma \to SL(2, \mathbb{C})$ induces a representation $\rho_\omega : \Gamma \to SL(2, \mathbb{C}_\omega)$ which determines a reducible action on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ with non-trivial length function, then it holds $\beta_2^\omega(\rho_\omega) = 0$.

The fourth chapter links the degeneration of natural maps to the vanishing of the invariant $\beta_2^\omega(\rho_\omega)$. Let $\rho_l : \Gamma \to SL(2, \mathbb{C})$ be a sequence of non-elementary representations diverging to an ideal point. Let $F_l : \mathbb{H}^3 \to \mathbb{H}^3$ be the sequence of natural maps associated to ρ_l, and let D_l be their measurable extensions to the boundary at infinity. If $\beta_{l,x} = (D_l)_*(\mu_x)$ is converging to the sum of two Dirac measures, we prove that $\beta_2^\omega(\rho_\omega) = 0$, where $\rho_\omega : \Gamma \to SL(2, \mathbb{C})$ is the representation associated to the sequence ρ_l.

We conclude the dissertation with some remarks and a short list of open problems regarding all these themes.

Chapter 1

Preliminary definitions

1.1 Continuous bounded cohomology

1.1.1 Definitions and topological interpretation

From now until the end of this section we denote by G a locally compact group. Before giving the definition of the bounded cohomology groups of G we need to give the following

Definition 1.1.1. A *Banach G-module* is a pair (π, E), where E is a Banach space and $\pi : G \to \mathrm{Isom}(E)$ is a representation which determines an action of G on E via linear isometries. We say that a Banach G-module (π, E) is *continuous* if the representation π is continuous, that is the action map $\theta_\pi : G \times E \to E$ defined by $\theta_\pi(g, v) := \pi(g)v$ is continuous with respect to the product topology on $G \times E$. The *maximal continuous submodule* of E is defined as

$$\mathcal{C}E := \{v \in E | \lim_{g \to e} ||g.v - v||_E = 0\}.$$

The module $\mathcal{C}E$ is the largest submodule of E on which the action map is continuous (to see that the condition above is equivalent to the notion of continuity for Banach G-module, we refer to [Mon01, Lemma 1.1.1.]).

A *morphism* between two Banach G-modules is a linear map between two Banach G-modules. If the map is also G-equivariant we call it a *G-morphism.*

We will usually refer to a Banach G-module (π, E) by writing only E, omitting the representation π. The Banach G-module of E-valued bounded continuous functions on G in degree n is given by

$$C^n_{cb}(G, E) := C_{cb}(G^{n+1}, E) = \{f : G^{n+1} \to E | f \text{ is continuous and } ||f||_\infty < \infty\}$$

where the supremum norm is defined as

$$||f||_\infty := \sup_{g_0,\dots,g_n \in G} ||f(g_0, \dots, g_n)||_E$$

and $C^n_{cb}(G, E)$ is endowed with the following G-module structure

$$(g.f)(g_0, \dots, g_n) := g.f(g^{-1}g_0, \dots, g^{-1}g_n)$$

for every element $g \in G$ and every function $f \in C^n_{cb}(G,E)$ (here the notation $g.f$ stands for the action of the element g on f). We denote by δ^n the homogeneous boundary operator of degree n, namely

$$\delta^n : C^n_{cb}(G,E) \to C^{n+1}_{cb}(G,E), \quad \delta^n f(g_0,\dots,g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0,\dots,\hat{g}_i,\dots g_{n+1}),$$

where the notation $\hat{g}_i$ indicates that the element g_i has been omitted. There is a natural embedding of E into $C^0_{cb}(G,E)$ given by the constant functions on G. This allows us to consider the following chain complex of G-modules

$$0 \longrightarrow E \longrightarrow C^0_{cb}(G,E) \xrightarrow{\delta^0} C^1_{cb}(G,E) \xrightarrow{\delta^1} \dots$$

and thanks to the compatibility of δ^n with respect to the G-action, we can consider the submodules of G-invariant vectors

$$0 \longrightarrow C^0_{cb}(G,E)^G \xrightarrow{\delta^0} C^1_{cb}(G,E)^G \xrightarrow{\delta^1} C^2_{cb}(G,E)^G \xrightarrow{\delta^2} \dots$$

Like in any other chain complex, we define the set of the n^{th}-*bounded continuous cocycles* as

$$Z^n_{cb}(G,E)^G := \ker\Big(\delta^n : C^n_{cb}(G,E)^G \to C^{n+1}_{cb}(G,E)^G\Big)$$

and the set of the n^{th}-*bounded continuous coboundaries*

$$B^n_{cb}(G,E)^G := \mathrm{im}\Big(\delta^{n-1} : C^{n-1}_{cb}(G,E)^G \to C^n_{cb}(G,E)^G\Big), \text{ and } B^0_{cb}(G,E) := 0.$$

Definition 1.1.2. The *continuous bounded cohomology* in degree n of G with coefficients in E is the space

$$H^n_{cb}(G,E) = \frac{Z^n_{cb}(G,E)^G}{B^n_{cb}(G,E)^G},$$

with the quotient seminorm

$$||[f]||_\infty := \inf ||f||_\infty,$$

where the infimum is taken over all the possible representatives of $[f]$.

By dropping the condition of boundedness, we can repeat the same construction above. More precisely, if we consider the G-module of continuous functions on G in degree n with values in E, namely

$$C^n_c(G,E) := C_c(G^{n+1},E) = \{f : G^{n+1} \to E | f \text{ is continuous}\}$$

and if we keep denoting by δ^n the homogeneous boundary operator introduced before, we define the space of homogeneous n^{th}-*continuous cocycles*

$$Z^n_c(G,E)^G := \ker\Big(\delta^n : C^n_c(G,E)^G \to C^{n+1}_c(G,E)^G\Big)$$

and the space of n^{th}-*continuous coboundaries*

$$B_c^n(G,E)^G := \mathrm{im}\Big(\delta^{n-1} : C_c^{n-1}(G,E)^G \to C_c^n(G,E)^G\Big), \text{ and } B_c^0(G,E) := 0.$$

Definition 1.1.3. The *continuous cohomology* in degree n of G with coefficients in E is the space

$$H_c^n(G,E) = \frac{Z_c^n(G,E)^G}{B_c^n(G,E)^G}.$$

For every $n \geq 0$ there exists an obvious map $C_{cb}^n(G,E) \to C_c^n(G,E)$ which simply forgets the boundedness of any function in $C_{cb}^n(G,E)$. This map is clearly a G-morphism and commutes with the homogeneous boundary operator, hence it induces a well-defined map

$$c : H_{cb}^n(G,E) \to H_c^n(G,E)$$

for every $n \geq 0$.

Definition 1.1.4. The map

$$c : H_{cb}^n(G,E) \to H_c^n(G,E)$$

is called *comparison map.*

Any continuous morphism $\varphi : G_1 \to G_2$ of locally compact groups determines in a natural way a sequence of maps

$$\varphi^* : C_{cb}^n(G_2,E) \to C_{cb}^n(G_1,E)$$

and

$$\varphi^* : C_c^n(G_2,E) \to C_c^n(G_1,E)$$

defined in a natural way by considering the pullback of cocycles

$$\varphi^*(f)(g_0,\ldots,g_n) := f(\varphi(g_0),\ldots,\varphi(g_n)), \;\; g_i \in G_1,$$

where $f \in C_{cb}^n(G_2)$ or $f \in C_c^n(G_2)$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} H_{cb}^\bullet(G_2,E) \arrow[r, "\varphi^*"] \arrow[d, "c"'] & H_{cb}^\bullet(G_1,E) \arrow[d, "c"] \\ H_c^\bullet(G_2,E) \arrow[r, "\varphi^*"] & H_c^\bullet(G_1,E). \end{tikzcd}$$

The definitions given so far have a clear interpretation when G is the fundamental group of a CW–complex X, G is endowed with the discrete topology and $E = \mathbb{R}$ considered as a trivial Banach G-module, where the norm is the standard Euclidean one. Indeed, if X admits a contractible universal cover $\tilde{X}$, then X is an

Eilenberg–MacLane space, that is $X = K(G, 1)$. In particular, the homotopy type of X depends only on G by Whitehead theorem. In this case it can be shown that

$$H_c^n(G, \mathbb{R}) \cong H^n(X, \mathbb{R}).$$

In analogous way we can consider the notion of singular bounded cohomology of X by restricting our attention only to bounded cochains. This leads to the definition of the singular bounded cohomology groups $H_b^n(X, \mathbb{R})$, firstly introduced by Gromov in [Gro83]. It is still true that

$$H_{cb}^n(G, \mathbb{R}) \cong H_b^n(X, \mathbb{R}),$$

but it can be shown even more. In fact, the bounded cohomology of G is canonically isometric isomorphic to the bounded cohomology of any countable CW–complex X such that $\pi_1(X) = G$ (see [Gro83] or [Iva87]).

1.1.2 Functorial approach to continuous bounded cohomology

The notion of continuous bounded cohomology for locally compact group can be given by following the so-called functorial approach. In order to do this, we need to introduce some machinery that we are going to use lately.

Definition 1.1.5. A *complex* $(E^\bullet, \partial^\bullet)$ of Banach G-modules is a sequence

$$\dots \longrightarrow E^{n-1} \xrightarrow{\partial^{n-1}} E^n \xrightarrow{\partial^n} E^{n+1} \xrightarrow{\partial^{n+1}} \dots$$

of Banach G-modules and G-morphisms such that $\partial^{n+1} \circ \partial^n = 0$ for every $n \in \mathbb{Z}$. A complex is said *continuous* if each E^n is a continuous Banach G-module. We usually refer to the complex by considering only $E^\bullet$ and omitting $\partial^\bullet$.

We denote by $\mathcal{C}E^\bullet$ the *maximal continuous subcomplex* of $E^\bullet$ defined as

$$\dots \longrightarrow \mathcal{C}E^{n-1} \xrightarrow{\partial^{n-1}} \mathcal{C}E^n \xrightarrow{\partial^n} \mathcal{C}E^{n+1} \xrightarrow{\partial^{n+1}} \dots$$

where ∂^n is obtained by restricting the boundary operator of the complex $E^\bullet$.

A *morphism* $\alpha^\bullet : E^\bullet \to F^\bullet$ of complexes of G-modules is a sequence of morphisms $\alpha^n : E^n \to F^n$ such that the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & E^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[d, "\alpha_{n-1}"'] & E^n \arrow[r, "\partial^n"] \arrow[d, "\alpha^n"'] & E^{n+1} \arrow[r, "\partial^{n+1}"] \arrow[d, "\alpha^{n+1}"'] & \dots \\
\dots \arrow[r] & F^{n-1} \arrow[r, "\partial^{n-1}"] & F^n \arrow[r, "\partial^n"] & F^{n+1} \arrow[r, "\partial^{n+1}"] & \dots
\end{tikzcd}$$

commutes. If for each $n \in \mathbb{Z}$ the map α^n is a G-morphism we call $\alpha^\bullet$ a *G-morphism* of complexes.

Definition 1.1.6. Given any two morphisms of complexes $\alpha^\bullet, \beta^\bullet : E^\bullet \to F^\bullet$, a *homotopy* $h^\bullet$ from $\alpha^\bullet$ to $\beta^\bullet$ is a sequence of morphisms $h^n : E^n \to F^{n-1}$ such that

$$h^{n+1} \circ \partial^n + \partial^{n-1} \circ h^n = \beta^n - \alpha^n,$$

for every $n \in \mathbb{Z}$. The previous condition is equivalent to require the commutativity of the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[d, bend right, "\alpha^{n-1}"'] \arrow[d, bend left, "\beta^{n-1}"] & E^{n} \arrow[r, "\partial^{n}"] \arrow[dl, "h^{n}"'] \arrow[d, bend right, "\alpha^{n}"'] \arrow[d, bend left, "\beta^{n}"] & E^{n+1} \arrow[r, "\partial^{n+1}"] \arrow[dl, "h^{n+1}"'] \arrow[d, bend right, "\alpha^{n+1}"'] \arrow[d, bend left, "\beta^{n+1}"] & \cdots \\
\cdots \arrow[r] & F^{n-1} \arrow[r, "\partial^{n-1}"] & F^{n} \arrow[r, "\partial^{n}"] & F^{n+1} \arrow[r, "\partial^{n+1}"] & \cdots
\end{tikzcd}$$

A morphism of complexes $\alpha^\bullet : E^\bullet \to F^\bullet$ is a *homotopy equivalence* if there exists a morphism $\beta^\bullet : F^\bullet \to E^\bullet$ such that the composition $\alpha^\bullet \circ \beta^\bullet$ is homotopic to the identity id_F whereas the composition $\beta^\bullet \circ \alpha^\bullet$ is homotopic to id_E.

A complex $E^\bullet$ of Banach G-modules admits a *contracting homotopy* $h^\bullet$ if there exists a homotopy between the identity id_E and the zero morphism of $E^\bullet$. More precisely, there must exist a sequence of maps $h^n : E^n \to E^{n-1}$ such that

$$h^{n+1} \circ \partial^n + \partial^{n-1} \circ h^n = \mathrm{id}_{E^n}$$

and $||h_n|| \leq 1$ for all $n \in \mathbb{Z}$. A complex $E^\bullet$ is *strong* if its maximal continuous subcomplex $\mathcal{C}E^\bullet$ admits a contracting homotopy.

Remark 1.1.7. Any definition which appears in Definition 1.1.6 can be strengthened by adding the requirement of G-equivariance. For instance, a G-homotopy will be homotopy where each h^n is a G-morphism. The same will hold for G-homotopic G-morphisms and so on.

Definition 1.1.8. The *cohomology* of the complex $E^\bullet$ is the collection of Banach G-modules defined as

$$H^n(E^\bullet) := \ker(\partial^n)/\mathrm{im}(\partial^{n-1}),$$

where each $H^n(E^\bullet)$ is equipped with the quotient seminorm.

Definition 1.1.9. A morphism $\varphi : E \to F$ of Banach spaces is *admissible* if there exists a morphism $\sigma : F \to E$ such that $||\sigma|| \leq 1$ and $\varphi \circ \sigma \circ \varphi = \sigma$. A Banach G-module E is *relatively injective* if for every injective admissible G-morphism $i : F \to H$ of continuous Banach G-modules and every G-morphism $\alpha : F \to E$ there is a G-morphism $\beta : H \to E$ satisfying $\beta \circ i = \alpha$ and $||\beta|| \leq ||\alpha||$.

Definition 1.1.10. Let E be a Banach G-module. A *resolution* of E is an exact complex $(E^\bullet, \partial^\bullet)$ of Banach G-modules such that $E^0 = E$ and $E^n = 0$ for every $n \leq -1$.

$$0 \longrightarrow E \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots$$

We say that $(E^\bullet, \partial^\bullet)$ is a *strong resolution* if the complex $(E^\bullet, \partial^\bullet)$ is strong.

Definition 1.1.11. Let E be a G-module and let $E^\bullet$ be a resolution of E which admits a contracting homotopy. The *complex of G-invariants* is the complex

$$0 \longrightarrow (E^0)^G \xrightarrow{\partial^0} (E^1)^G \xrightarrow{\partial^1} (E^2)^G \longrightarrow \dots$$

where $(E^i)^G := \{v \in E^i | g.v = v \text{ for every } g \in G\}$.

After the introduction of all this machinery, we are now ready to use it in our context to study the notion of continuous bounded cohomology of locally compact groups. The key point is that we can compute the continuous bounded cohomology of a locally compact group G with coefficients in E by using the cohomology of G-invariants of any strong resolution of E by relatively injective G-modules. More precisely, as shown in [Mon01, Theorem 7.2.1], it holds the following

Theorem 1.1.12 *Let G be a locally compact group and let E be any Banach G-module. Then, E admits a resolution by relatively injective Banach G-modules. Moreover, for any strong resolution $(E^\bullet, \partial^\bullet)$ by relatively injective Banach G-modules, there exists an isomorphism of topological vector space*

$$H^n((E^\bullet)^G) \cong H^n_{cb}(G, E)$$

for all $n \geq 0$.

We should not be surprised that the augmentation of the complex $(C^\bullet_{cb}(G, E), \delta^\bullet)$ is a particular case of strong resolution of E by relatively injective Banach G-modules. Hence the definition of continuous bounded cohomology of G with coefficients in E given in the previous section is nothing more that the cohomology of a particular resolution of E by the relatively injective Banach G-modules of continuous bounded functions.

It is worth noticing that the isomorphism written above is not a priori isometric. Even if this fact may result disappointing, it is always true that the isomorphism does not increase the norm. Moreover, there exist particular resolutions of E for which the isomorphism is actually isometric. For instance, consider the following resolution of E by relatively injective Banach G-modules. Let X be a locally compact space on which G acts continuously and properly. Suppose that $G \backslash X^{n+1}$ is paracompact for every $n \geq 0$. This happens for example when X is the symmetric space associated to a Lie group G of non-compact type. Define the complex

$$0 \longrightarrow E \xrightarrow{\epsilon} C^0_{cb}(X, E) \xrightarrow{\delta^0} C^1_{cb}(X, E) \xrightarrow{\delta^1} \dots$$

where $C^n_{cb}(X, E)$ is the set of continuous bounded function on $(n+1)$-tuples of points in X and δ^n is the homogeneous boundary operator. This complex is a strong resolution of E by relatively injective Banach G-modules. In particular the cohomology of the G-invariants $C^\bullet_{cb}(X, E)^G$ is isomorphic to the continuous bounded cohomology of G. Moreover this isomorphism is isometric (see [Mon01, Theorem 7.4.5]).

Even if we have only a strong resolution of E without the condition of relative injectivity of modules, it is possible to gain information about the bounded cohomology of G. If $(E^\bullet, \partial^\bullet)$ is a strong resolution of E, by [BM02, Proposition 1.5.2], there exists a canonical map

$$\mathfrak{c}^n : H^n((E^\bullet)^G) \to H^n_{cb}(G, E)$$

for all $n \geq 0$. For example, we are going now to construct a strong resolution of $\mathbb{R}$, seen as a trivial Banach G-modules, by studying suitable spaces on which G acts. More precisely, let X be a measurable space on which G acts measurably, that is the action map $\theta : G \times X \to X$ is measurable (G is equipped with the σ-algebra of the Haar measurable sets). We set

$$\mathcal{B}^\infty(X^n, \mathbb{R}) := \{f : X^n \to \mathbb{R} | f \text{ is measurable and } \sup_{x \in X^n} |f(x)| < \infty\},$$

and we endow it with the structure of Banach G-module given by

$$(g.f)(x_1, \ldots, x_n) := f(g^{-1}.x_1, \ldots, g^{-1}.x_n), \quad ||f||_\infty = \sup_{x \in X^n} |f(x)|$$

for every $g \in G$ and every $f \in \mathcal{B}^\infty(X^n, \mathbb{R})$. If $\delta^n : \mathcal{B}^\infty(X^n, \mathbb{R}) \to \mathcal{B}^\infty(X^{n+1}, \mathbb{R})$ is the standard homogeneous coboundary operator, for $n \geq 1$ and $\delta^0 : \mathbb{R} \to \mathcal{B}^\infty(X, \mathbb{R})$ is the inclusion given by constant functions, we get a cochain complex $(\mathcal{B}^\infty(X^\bullet, \mathbb{R}), \delta^\bullet)$. We denote by $\mathcal{B}^\infty_{\text{alt}}(X^{n+1}, \mathbb{R})$ the Banach G-submodule of alternating cochains, that is the set of elements satisfying

$$f(x_{\sigma(0)}, \ldots, x_{\sigma(n)}) = \text{sgn}(\sigma) f(x_0, \ldots, x_n),$$

for every permutation $\sigma \in S_{n+1}$.

In [BI02, Proposition 2.1] the authors prove that the complex $(\mathcal{B}^\infty(X^\bullet, \mathbb{R}), \delta^\bullet)$ is a strong resolution of $\mathbb{R}$. In particular it follows

Proposition 1.1.13 *There exists a canonical map*

$$\mathfrak{c}^\bullet : H^\bullet(\mathcal{B}^\infty(X^{\bullet+1}, \mathbb{R})^G) \to H^\bullet_{cb}(G, \mathbb{R}).$$

More precisely, every bounded measurable G-invariant cocycle $f : X^{n+1} \to \mathbb{R}$ determines canonically a class $\mathfrak{c}^n[f] \in H^n_{cb}(G, \mathbb{R})$. The same result holds for the subcomplex $(\mathcal{B}^\infty_{\text{alt}}(X^\bullet, \mathbb{R}), \delta^\bullet)$ of alternating cochains.

1.1.3 Examples and computations

We are ready to exhibit some elementary examples of computation of continuous bounded cohomology groups for a locally compact group G. We start giving the following

Definition 1.1.14. Let G be a locally compact group. Let μ be the left Haar measure on G and consider $\mathcal{F}(G) \subset L^\infty(G, \mu)$ a closed subspace containing the constant functions. A *mean* on $\mathcal{F}(G)$ is a continuous linear form $m : \mathcal{F}(G) \to \mathbb{R}$ such that

1. $m(f) \geq 0$ for every $f \in \mathcal{F}(G)$ which satisfies $f \geq 0$,

2. $m(\mathbf{1}) = 1$,

where **1** denotes the constant function equal to 1. A mean on $\mathcal{F}(G)$ is *G-invariant* if $m(g.f) = m(f)$ for every $f \in \mathcal{F}(G)$ and every $g \in G$.

A locally compact group G is *amenable* if the space $C_b(G, \mathbb{R})$ admits a G-invariant mean.

Examples of amenable groups are compact groups and solvable groups. In particular, the Borel subgroups of a Lie group are amenable. The computation of the continuous bounded cohomology of any amenable group is particularly easy. Indeed, it holds the following

Proposition 1.1.15 *Let G be an amenable topological group. Then*

$$H^n_{cb}(G, \mathbb{R}) = 0$$

for all integers $n \geq 1$.

A proof of the previous proposition can be found in [Mon01].

For our purposes we are going to compute the continuous bounded cohomology of the group $SL(2, \mathbb{C})$ in degree 3. We will start by computing the continuous cohomology in the same degree. When G is a Lie group there is a useful way to compute its continuous cohomology by studying the G-invariant differential forms on the associated symmetric space. More precisely, let G be a Lie group of non-compact type and let K be the maximal compact subgroup of G. Denote by X the symmetric space associated to G, that is $X = G/K$. Let $\Omega^n(X)$ be the space of n-th differential forms on X and let $\Omega^n(X)^G$ be the subspace of G-invariant forms.

Theorem 1.1.16 *There exists a natural isomorphism of groups, called Van Est isomorphism*

$$i_{VE} : \Omega^n(X)^G \to H^n_c(G, \mathbb{R})$$

for all $n \geq 0$.

We will give a short description of the previous isomorphism by following [Dup76]. Given a $(n+1)$-tuple of points $(x_0, \ldots, x_n)$ in X, we denote by $\tau(x_0, \ldots, x_n)$ the geodesic simplex whose vertices are the points $\{x_0, \ldots, x_n\}$ (recall that a geodesic simplex is defined inductively on the number of vertices). Fix a point $x \in X$ in the corresponding symmetric space. If $\omega \in \Omega^n(X)^G$, at the cochains level the isomorphism i_{VE} is given by

$$i_{VE}(\omega)(g_0, \ldots, g_n) = \int_{\tau(g_0 x, \ldots, g_n x)} \omega.$$

A priori the map defined above may depend on the choice of the basepoint x, but is can be proved that different choices of the basepoint lead to cohomologous cocycles. Hence, we get a well-defined element of $H^n_c(G, \mathbb{R})$. This offers us a way to compute the continuous cohomology of the group $SL(2, \mathbb{C})$. Indeed, since all the 3-forms of $\mathbb{H}^3$ which are invariant under the action of $SL(2, \mathbb{C})$ are parametrized by $\mathbb{R}$, it is straightforward to prove that the continuous cohomology group $H^3_{cb}(SL(2, \mathbb{C}), \mathbb{R})$ is isomorphic to $\mathbb{R}$. Moreover we can choose as preferred generator the standard volume form on $\mathbb{H}^3$. Additionally, since there exists an upper bound on the volume of hyperbolic simplices, this class is actually bounded and it determines a non-trivial class in the group $H^3_{cb}(SL(2,\mathbb{C}),\mathbb{R})$. The latter group can be computed in several ways. For example, by studying the cohomology of the G-invariant associated to the strong resolution of $\mathbb{R}$ given by relatively injective Banach $SL(2,\mathbb{C})$-modules $(L^\infty((\partial_\infty\mathbb{H}^3)^\bullet),\delta^\bullet)$ (see [Mon01, Theorem 7.5.3]). Here $L^\infty((\partial_\infty\mathbb{H}^3)^n)$ is the set of real bounded measurable functions on n-tuples of points in $\partial_\infty\mathbb{H}^3$ endowed with its natural structure of Banach $SL(2,\mathbb{C})$-module and δ^n is the homogeneous boundary operator. As consequence of [Blo00, Theorem 7.4.4], the cohomology group $H^3_{cb}(SL(2,\mathbb{C}),\mathbb{R})$ is one dimensional and it is generated by the volume function. More precisely, we give the following

Definition 1.1.17. The *Bloch–Wigner function* is defined as

$$D_2 : \mathbb{C}\setminus\{0,1\} \to \mathbb{R},\ \ D_2(z) := \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$$

where Li_2 is the dilogarithm function.

This function is continuous and it naturally extends on $\mathbb{P}^1(\mathbb{C})$ by zero. It reaches its maximum value at $z = (1+i\sqrt{3})/2$ and it corresponds to the hyperbolic volume of a regular ideal hyperbolic tetrahedron. By post-composing D_2 with the cross ratio cr we get a map

$$\mathrm{Vol} : \mathbb{P}^1(\mathbb{C})^4 \to \mathbb{R},\ \ \mathrm{Vol}(x_0,\ldots,x_3) := D_2(cr(x_0,\ldots,x_3))$$

which can be interpreted as the hyperbolic volume of the ideal tetrahedron whose vertices are given by $x_0,\ldots,x_3$. Thanks to the invariance of the cross ratio with respect to the diagonal action of $SL(2,\mathbb{C})$, it is clear that Vol is $SL(2,\mathbb{C})$-invariant. Moreover it is measurable with respect to the standard spherical measure on $\mathbb{P}^1(\mathbb{C})$ since it is obtained by composition of measurable functions.

Proposition 1.1.18 *The set of measurable functions $f : \mathbb{P}^1(\mathbb{C})^4 \to \mathbb{R}$ which are invariant under the natural action of $SL(2,\mathbb{C})$ and satisfy the cocycle condition forms a one-dimensional real vector space generated by the volume function* Vol.

From which we deduce

Proposition 1.1.19 *The comparison map*

$$c : H^3_{cb}(SL(2,\mathbb{C}),\mathbb{R}) \to H^3_c(SL(2,\mathbb{C}),\mathbb{R})$$

is an isomorphism. Both groups are isomorphic to $\mathbb{R}$ and they are generated by the volume class.

1.2 BCG-natural maps

For more details about the following definitions and constructions we strongly recomend the reader to see [BCG95], [BCG99] and [Fra09]. Denote by G_p a Lie group of rank-one and of non-compact type, namely $G_p = PO(p,1), PU(p,1)$ or $PSp(p,1)$ and let Γ be a discrete group such that $\Gamma\backslash G$ has finite Haar measure. In this section we are going to recall the definition of the so-called natural map associated to a non-elementary representation $\rho : \Gamma \to G_m$. Before doing this we need to recall the notion of barycentre of a positive Borel measure μ on $\partial_\infty X^p$, where X^p is the symmetric space associated to the group G_p.

1.2.1 Barycentre of positive Borel measure

We start by fixing some notation. Let G_p be either $PO(p,1)$, $PU(p,1)$ or $PSp(p,1)$. Denote by $\mathfrak{g}_p = T_e G_p$ the tangent space to G_p at the neutral element. If we endow $\mathfrak{g}_p$ with its natural structure of Lie algebra, we recall that $\mathfrak{g}_p$ admits an involution $\Theta : \mathfrak{g}_p \to \mathfrak{g}_p$ which allows us to decompose $\mathfrak{g}_p = \mathfrak{l} \oplus \mathfrak{p}$, where $\mathfrak{l}$ and $\mathfrak{p}$ are the eigenspaces with respect to 1 and -1 of the involution Θ. Moreover $\mathfrak{p}$ is naturally identified to any tangent space of the symmetric space X^p associated to G_p and since the restriction of the Killing form to $\mathfrak{p}$ is positive definite, this induces in a canonical way a Riemannian metric on X^p. If $G_p = PO(p,1)$ the associated symmetric space X^p is the real hyperbolic space of order p, that is $\mathbb{H}^p_{\mathbb{R}}$ (we will refer to the real hyperbolic space by omitting the subscript $\mathbb{R}$). In the same way, if $G_p = PU(p,1)$ we identify X^p with the complex hyperbolic space of order p, namely $\mathbb{H}^p_{\mathbb{C}}$. Finally if $G_p = PSp(p,1)$ the symmetric space X^p coincides with the quaternionic hyperbolic space $\mathbb{H}^p_{\mathbb{Q}}$ of order p. In the real case the sectional curvature is constant and equal to -1, whereas in the other two cases the sectional curvature of these spaces lies between -4 and -1. In particular, since X^p is always negatively curved, we can talk about the visual boundary of X^p and we denote it by $\partial_\infty X^p$.

Suppose to fix a point $x \in X^p$ as basepoint.

Definition 1.2.1. The Busemann function of X^p normalized at x, is the function

$$B_x : X^p \times \partial_\infty X^p \to \mathbb{R}, \quad B_x(y,\theta) = \lim_{t\to\infty} d(y,c(t)) - t,$$

where c is the geodesic ray starting at $x = c(0)$ and ending at θ.

From now until the end of the section we are going to fix a point in X^p as basepoint and we are going to denote it by O. Moreover, we will use the same letter O to denote basepoints in symmetric spaces of different dimension. Let $B_P(x,\theta)$ be the Busemann function of X^p normalized at O. Recall that, if we fix $\theta \in \partial_\infty X^p$, the Busemann function becomes a convex function with respect to the variable $x \in X^p$. Consider the function B_P and let β be a positive probability measure on $\partial_\infty X^p$. We define the map

$$\varphi_\beta : X^p \to \mathbb{R}, \quad \varphi_\beta(y) := \int_{\partial_\infty X^p} B_P(y,\theta) d\beta(\theta).$$

Thanks to the convexity of the Busemann function B_P the map φ_β is stricly convex, if we assume that β is not the sum of two Dirac measures. Additionally, if the measure β does not contain any atom of mass greater than or equal to $1/2$, the following condition holds

$$\lim_{y\to\partial_\infty X^p} \varphi_\beta(y) = \infty.$$

This implies that φ_β admits a unique minimum in X^p (see [BCG95, Appendix A]). On the other hand, if β contains an atom of mass at least $1/2$, then it is easy to check that the minimum of φ_β is $-\infty$ and it is attained when y coincides with the atom. In both cases it is worth noticing that the point at which φ_β attains its minimum does not depend on the choice of the basepoint O used to normalize the

Busemann function, since a different choice of basepoint would have modified the function φ_β by an additive constant.

Definition 1.2.2. Let β be any positive probability measure on the visual boundary $\partial_\infty X^p$ which is not the sum of two Dirac masses with the same weight. If β contains an atom x of mass greater than or equal to $1/2$ then we define its *barycentre* as

$$\mathrm{bar}_{\mathcal{B}}(\beta) = x,$$

otherwise we define it as the point

$$\mathrm{bar}_{\mathcal{B}}(\beta) = \mathrm{argmin}(\varphi_\beta).$$

The letter $\mathcal{B}$ wants to underline the dependence of the construction on the Busemann functions. The barycentre of β will be a point in $\overline{X}^p$ which satisfies the following properties:

- it is continuous with respect to the weak-* topology on the set of probability measures on $\partial_\infty X^p$, that is if $\beta_n \to \beta$ in the weak-* topology (and no measure is the sum of two atoms with equal weight) it holds

$$\lim_{n\to\infty} \mathrm{bar}_{\mathcal{B}}(\beta_n) = \mathrm{bar}_{\mathcal{B}}(\beta)$$

- it is G_p-equivariant, indeed for every $g \in G_p$ (if β is not the sum of two equal atoms) we have

$$\mathrm{bar}_{\mathcal{B}}(g_*\beta) = g(\mathrm{bar}_{\mathcal{B}}(\beta)),$$

- when β does not contain any atom of weight greater than or equal to $1/2$, it is characterized by the following equation

$$\int_{\partial_\infty X^p} dB_P|_{(\mathrm{bar}_{\mathcal{B}}(\beta),y)}(\cdot)d\beta(y) = 0. \tag{1.1}$$

1.2.2 The BCG–natural map

As before G_p is a rank-one Lie group. We still denote by X^p the symmetric space associated to the group G_p. Before starting, fix $k = p$ if $G_p = PO(p,1)$, $k = 2p$ if $G_p = PU(p,1)$ and $k = 4p$ if $G_p = PSp(p,1)$. The value k is simply the real dimension of the symmetric space X^p associated to G_p.

Definition 1.2.3. A *lattice* Γ in a Lie group G is a discrete subgroup such that $\bar{\mu}_{\mathcal{H}}(\Gamma\backslash G) < \infty$ where $\bar{\mu}_{\mathcal{H}}$ is the measure induced on the quotient $\Gamma\backslash G$ by the left Haar measure.

Let Γ be a lattice of G_p. We say that Γ is *real* if $G_p = PO(p,1)$, *complex* if $G_p = PU(p,1)$ or *quaternionic* if $G_p = PSp(p,1)$. If Γ is a lattice of G_p without torsion, then the action via isometries of Γ on X^p is free and properly discontinuous. In particular the quotient $M = \Gamma\backslash X^p$ admits a natural structure of Riemannian manifold which is locally isometric to X^p and has finite volume. We say that the lattice Γ is *uniform* if M is compact, otherwise we refer to Γ as a *non-uniform* lattice.

Definition 1.2.4. The *critical exponent* $\delta(\Gamma)$ associated to the group Γ is the infimum over all the possible positive real numbers for which the Poincaré series converges, that is

$$\delta(\Gamma) := \inf\{s \in [0,\infty] | \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} < \infty\}$$

where x is any point of X^p. The definition does not depend on the choice of the basepoint x used to compute $\delta(\Gamma)$.

When Γ is a non-uniform lattice of G_p, the critical exponent is always finite and by [Alb97, Theorem 2] we have that $\delta(\Gamma) = k + d - 2$. The number d is the real dimension of the algebra on which the hyperbolic space X^p is defined. Thus $d = 1$ if Γ is real, $d = 2$ if Γ is complex and $d = 4$ if Γ is quaternionic. Moreover, we remind that for $s = \delta(\Gamma)$ the series diverges by [Alb99, Proposition D], that is

$$\sum_{\gamma\in\Gamma} e^{-\delta(\Gamma)d(x,\gamma x)} = +\infty.$$

and for this reason we may refer to Γ as a lattice of *divergence type.*

Definition 1.2.5. Let $\mathcal{M}^1(Y)$ be the set of positive probability measures on a space Y. The *family of Patterson-Sullivan measures associated to a non-uniform lattice* Γ is a family of measures $\{\mu_x\} \in \mathcal{M}^1(\partial_\infty X^p)$, where $x \in X^p$, which satisfies the following properties

- the family is Γ-equivariant, that is $\mu_{\gamma x} = \gamma_*(\mu_x)$ for every $\gamma \in \Gamma$ and every $x \in X^p$,
- For every $x, y \in X^p$ it holds

$$d\mu_x(\theta) = e^{-\delta(\Gamma)B_y(x,\theta)} d\mu_y(\theta)$$

where $B_y(x,\theta)$ is the Busemann function normalized at y.

Remark 1.2.6. The construction of the family of Patterson-Sullivan measures has been generalized by [Alb97, Alb99] to any lattice of a Lie group G of non-compact type. The support of the measures μ_x coincides with the Furstenberg boundary $\partial_{\mathcal{F}} X$ of the symmetric space X, which can be thought of as the G-orbit of a regular point $\xi \in \partial_\infty X$. Since we are considering rank-one Lie group and

$$\mathrm{codim}_{\partial_\infty X} \partial_{\mathcal{F}} X = \mathrm{rank}(X) - 1$$

we have that $\partial_\infty X = \partial_{\mathcal{F}} X$ in our context.

Let $\{\mu_x\}$ be the family of Patterson-Sullivan measures associated to Γ and let $\rho : \Gamma \to G_m$ be a non-elementary representation. Set $\mu = \mu_O$. Recall that the action of Γ on $(\partial_\infty X^p \times \partial_\infty X^p, \mu \times \mu)$ is ergodic by [Nic89, Yue96, BM96, Rob00], for instance. Hence by both [Fra09] and [BM96, Corollary 3.2] there exists a ρ-equivariant measurable map

$$D : \partial_\infty X^p \to \partial_\infty X^m$$

and two different maps of this type must agree on a full μ-measure set.

Consider the space $\partial_\infty X^p \times \partial_\infty X^m$, with projections π_P and π_M over its factors. For any measure $\eta \in \mathcal{M}^1(\partial_\infty X^p \times \partial_\infty X^m)$ which satisfies $(\pi_P)_*\eta = \mu$, there exists a family of positive probability measures $\{\alpha_z\}_{z \in \partial_\infty X^m}$ on $\partial_\infty X^m$ such that

$$\int_{\partial_\infty X^p \times \partial_\infty X^m} \varphi(z,y) d\eta(z,y) = \int_{\partial_\infty X^p} \Big(\int_{\partial_\infty X^m} \varphi(z,y) d\alpha_z(y) \Big) d\mu(z)$$

for every $\varphi \in C^\infty(\partial_\infty X^p \times \partial_\infty X^m)$. In this case we say that the measure η *disintegrates* and we write

$$\eta = \mu \times \{\alpha_z\}.$$

For any $\eta \in \mathcal{M}^1(\partial_\infty X^p \times \partial_\infty X^m)$ as above we set

$$\eta_x = \mu_x \times \{\alpha_z\}$$

and we define

$$\beta_x := (\pi_M)_*(\eta_x).$$

This procedure determines a measure β_x which lives in $\mathcal{M}^1(\partial_\infty X^m)$ for every x. We want to emphasize that starting from a point $x \in X^p$ we end up with a measure $\beta_x \in \mathcal{M}^1(\partial_\infty X^m)$. In our context, since the representation $\rho : \Gamma \to G_m$ is non-elementary, we are allowed to define η via the measurable map D by setting $\alpha_z := \delta_{D(z)}$. In this way we can recognize β_x as $D_*(\mu_x)$. Hence the construction above is a generalization of the push-foward of measures and η is often called transport measure.

Since we have a non-elementary representation, β_x does not contain any atom of mass greater than or equal to 1/2. Indeed it holds

Lemma 1.2.7 *Let $\rho : \Gamma \to G_m$ be a non-elementary representation and let $D : \partial_\infty X^p \to \partial_\infty X^m$ be a ρ-equivariant measurable map. Then $D(x) \neq D(y)$ for almost every $(x,y) \in \partial_\infty X^p \times \partial_\infty X^p$.*

Proof. Define the set $A := \{(x,y) \in \partial_\infty X^p \times \partial_\infty X^p | D(x) = D(y)\}$. Since the map D is ρ-equivariant, A is a Γ-invariant measurable subset of $\partial_\infty X^p \times \partial_\infty X^p$. Recall that Γ acts ergodically on $\partial_\infty X^p \times \partial_\infty X^p$ with respect to the measure $\mu \times \mu$. In particular, the set A must have either null measure or full measure. By contradiction, suppose that A has full measure. This implies that for almost all x, the slice $A(x) := \{y \in \partial_\infty X^p | D(x) = D(y)\}$ has full measure in $\partial_\infty X^p$. The G_m-action preserves the class of μ, in particular, for any $\gamma \in \Gamma$, if $A(x)$ has full measure then so does $\gamma A(x)$. Since

Γ is countable, this implies that for almost all x, the set $A_\Gamma(x) := \cap_{\gamma\in\Gamma}\gamma^{-1}A(x)$ has full measure in $\partial_\infty X^p$. Fix now a point $y \in A_\Gamma(x)$. For any $\gamma \in \Gamma$ we have $(x, \gamma y) \in A$. In particular[1]

$$D(y) = D(x) = D(\gamma y) = \rho(\gamma)D(y)$$

for every $\gamma \in \Gamma$, but this would imply that ρ is elementary, which is a contradiction. $\square$

By the previous lemma, for all $x \in X^p$, we can define

$$F(x) := \mathrm{bar}_{\mathcal{B}}(\beta_x)$$

and this point will lie in X^m. In this way we get a map $F : X^p \to X^m$ (see Figure 1.1).

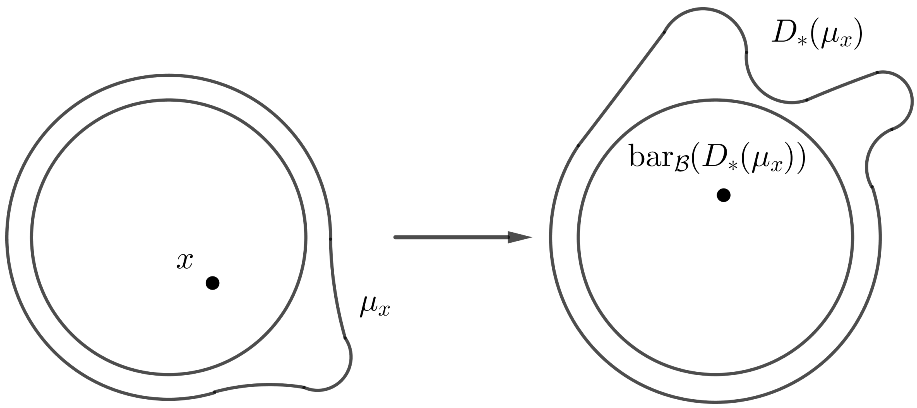

Figure 1.1: Construction of the natural map F.

Definition 1.2.8. The map $F : X^p \to X^m$ is called *natural map* for the representation $\rho : \Gamma \to G_m$.

Equation (1.1) becomes

$$\int_{\partial_\infty X^m} dB_M|_{(F(x),y)}(\cdot)d\beta_x(y) = 0. \tag{1.2}$$

and since $\beta_x = D_*(\mu_x)$, it can be rewritten as

$$\int_{\partial_\infty X^p} dB_M|_{(F(x),D(z))}(\cdot)d\mu_x(z) = 0. \tag{1.3}$$

The natural map is smooth and satisfies the following properties:

- Recall that we denoted by k the real dimension of the symmetric space X^p. Define the k-Jacobian of F as

$$Jac_k(F)(x) := \max_{u_1,\dots,u_k\in T_xX^p} ||D_xF(u_1)\wedge\ldots\wedge D_xF(u_k)||_{X^m}$$

[1]We use $\gamma = id$ in the first equality and the last follows by equivariance of D.

where $\{u_1, \ldots, u_k\}$ is an orthonormal frame of the tangent space $T_x X^p$ with respect to the standard metric induced by g_{X^p} and the norm $|| \cdot ||_{X^m}$ is the norm on $T_{F_n(x)} X^m$ induced by g_{X^m}.[2] We have $Jac_k(F) \leq 1$ and the equality holds at x if and only if $D_x F : T_x X^p \to T_{F(x)} X^m$ is an isometry.

- The map F is ρ-equivariant, that is $F(\gamma x) = \rho(\gamma) F(x)$.
- By differentiating (1.3), one gets that for all $x \in X^p$, $u \in T_x X^p$, $v \in T_{F(x)} X^m$ it holds

$$\int_{\partial_\infty X^p} \nabla dB_M|_{(F(x), D(z))}(D_x F(u), v) d\mu_x(z) =$$
$$\delta(\Gamma) \int_{\partial_\infty X^p} dB_M|_{(F(x), D(z))}(v) dB_P|_{(x,z)}(u) d\mu_x(z)$$

where ∇ is the Levi–Civita connection on X^m.

1.2.3 Volume of representations and ε-natural maps

Let Γ be a non-uniform lattice of G_p without torsion. If we denote by $M = \Gamma \backslash X^p$ we obtain a complete manifold of finite volume which is locally symmetric X^p and not compact. Moreover, as a consequence of Margulis lemma, it admits a decomposition

$$M = N \cup \bigcup_{i=1}^{h} C_i$$

where N is a compact core of finite volume and each C_i is a cuspidal neighborhood which can be seen as $N_i \times (0, \infty)$ where $\pi_1(N_i)$ is a discrete nilpotent parabolic subgroup of G_p (see [BGS85] or [Bow95]).

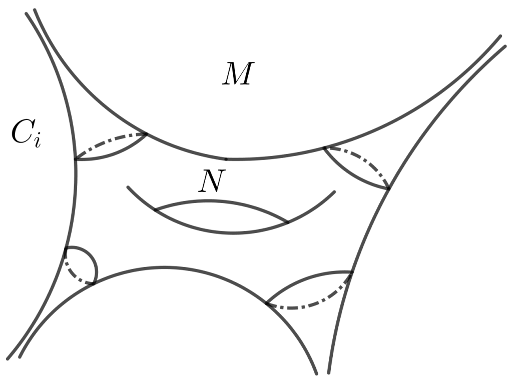

Figure 1.2: Decomposition of the manifold M.

[2] Actually, since here we used the maximal dimension k to define it, the k-Jacobian does not depend on the orthonormal frame used to compute it, since for two different orthonormal k-frames we have $u_1' \wedge \ldots \wedge u_k' = \pm u_1 \wedge \ldots \wedge u_k$.

As before, denote by k the real dimension of X^p. Let $\rho : \Gamma \to G_m$ be a representation, with $m \geq p$, and let $D : X^p \to X^m$ be a smooth ρ-equivariant map. By following the definition of [FK06] we want to define the volume $\mathrm{Vol}(D)$. Let g_{X^m} be the natural metric on X^m. The pullback of g_{X^m} along D defines in a natural way a pseudo-metric on X^p, which can be possibly degenerate, and hence it defines a natural k-form given by $\tilde{\omega}_D = \sqrt{|\det D^* g_{X^m}|}$. We remark that, in general, $\tilde{\omega}_D$ has only C^0-regularity. The equivariance of D with respect to ρ implies that the form $\tilde{\omega}_D$ is Γ-invariant and hence it determines a k-form on M. Denote this form by ω_D.

Definition 1.2.9. Let $\rho : \Gamma \to G_m$ be a representation and let $D : X^p \to X^m$ be any smooth ρ-equivariant map. The *volume* of D is defined as

$$\mathrm{Vol}(D) := \int_M \omega_D$$

We keep denoting by $D : X^p \to X^m$ a generic smooth ρ-equivariant map. For each cuspidal neighborhood $C_i = N_i \times (0, \infty)$, we know that $\pi_1(N_i)$ is parabolic, so it fixes a unique point in $\partial_\infty X^p$. Suppose $c_i = \mathrm{Fix}(\pi_1 N_i)$ and let $r(t)$ be a geodesic ray ending at c_i. We say that D is a *properly ending map* if all the limit points of $D(r(t))$ lie either in $\mathrm{Fix}(\rho(\pi_1 N_i))$ or in a finite union of $\rho(\pi_1 N_i)$-invariant geodesics.

Definition 1.2.10. Given a representation $\rho : \Gamma \to G_m$, we define its *volume* as

$$\mathrm{Vol}(\rho) := \inf\{\mathrm{Vol}(D)|\ D \text{ is smooth, } \rho\text{-equivariant and properly ending}\}.$$

When ρ is non-elementary, a priori the BCG–natural map $F : X^p \to X^m$ associated to ρ is not a properly ending map, hence we cannot compare its volume with the volume of representation ρ. However, by adapting the proofs contained in [FK06], for any $\varepsilon > 0$ it is possible to construct a family of smooth functions $F^\varepsilon : X^p \to X^m$ that C^1-converges to F as $\varepsilon \to 0$ and such that F^ε is a properly ending map for every $\varepsilon > 0$.

Definition 1.2.11. For any $\varepsilon > 0$ there exists a map $F^\varepsilon : X^p \to X^m$ called *ε-natural map* associated to ρ which satisfies the following properties

- F^ε is smooth and ρ-equivariant,
- at every point of X^p we have $Jac_k(F^\varepsilon) \leq 1 + \varepsilon$,
- for every $x \in X^p$ it holds $\lim_{\varepsilon \to 0} F^\varepsilon(x) = F(x)$ and $\lim_{\varepsilon \to 0} D_x F^\varepsilon = D_x F$,
- F^ε is a properly ending map.

Remark 1.2.12. The properly ending property of F^ε is guaranteed by the fact that $\pi_1(N_i)$ is parabolic and stabilizes each horosphere through the fixed point c_i. We want to underline that all the properties of F^ε and F descends directly from the properties of the Busemann functions. Moreover, since F^ε is a properly ending map, it holds trivially

$$\mathrm{Vol}(\rho) \leq \int_M \sqrt{|\det((F^\varepsilon)^* g_{X^m})|}.$$

We are going to use the previous estimate lately.

The following theorem states the rigidity of volume function.

Theorem 1.2.13 *Let Γ be a non-uniform lattice in G_p without torsion and let $\rho : \Gamma \to G_m$ be a representation, where $m \geq p$. Then* $\mathrm{Vol}(\rho) \leq \mathrm{Vol}(M)$ *and equality holds if and only if the representation ρ is a discrete and faithful representation of Γ into G_m. For real lattices we need to assume $p \geq 3$, whereas for both complex and quaternionic lattices we fix $p \geq 2$.*

For real lattices, the previous result corresponds to [FK06, Theorem 1.1], whereas for complex lattices we refer either to [BI] or to [KK12]. The statement regarding the quaternionic case can be considered as a different version of the rigidity result exposed in [Cor92].

1.3 Compactification of character varieties

1.3.1 Introduction to $\mathbb{R}$-trees and Euclidean buildings

This section is devoted to briefly recall some elementary definitions of the theory about real trees and Euclidean buildings. For a more detailed description see [Chi01] and [KL97].

Definition 1.3.1. A *real tree* $(\mathcal{T}, d)$ is a metric space which satisfies the following properties:

- Any two points $x, y \in \mathcal{T}$ are the endpoints of a unique closed segment, *i.e.* of an isometric embedding of a closed interval; we will denote this segment by $[x, y]$.
- If two segments have a common endpoint their intersection is a segment.
- If the intersection of two segments $[x, y] \cap [y, z]$ is the only point $\{y\}$, then $[x, y] \cup [y, z]$ is exactly the segment $[x, z]$.

An *isometry* of $\mathcal{T}$ is a bijection $g : \mathcal{T} \to \mathcal{T}$ such that $d(gx, gy) = d(x, y)$ for every pair of points $x, y \in \mathcal{T}$.

It is possible to subdivide isometries of a real tree into two categories: elliptic isometries and hyperbolic isometries. In order to do this, let g be a generic isometry of $\mathcal{T}$. Given a point $x \in \mathcal{T}$, the intersection $[x, gx] \cap [gx, g^2x]$ is still a segment thanks to the properties of $\mathcal{T}$. We denote this segment by $[gy, gx]$, where y is a suitable point in $[x, gx]$. We will classify g comparing the two distances $d(y, x)$ and $d(x, gx)$. We are going to distinguish three cases. Suppose $2d(x, y) \geq d(x, gx)$. If $2d(y, x) = d(x, gx)$, then y is the midpoint of $[x, gx]$ and it is fixed by g. Otherwise $2d(x, y) > d(x, gx)$ and the segment $[y, gy]$ is mapped to itself but g reverses the order. Moreover g fixes the midpoint of the segment $[x, gx]$.

Finally assume $2d(x, y) < d(x, gx)$. In this case y and gy are distinct, y lies in the segment $[x, gy]$ and gy lies in $[y, gx]$, respectively. The closed segments $[y, gy]$ and $[gy, g^2y]$ intersect only at gy, so their union is exactly $[y, g^2y]$. By applying an inductive reasoning, we consider $A := \bigcup_{n \in \mathbb{Z}} [g^n y, g^{n+1} y]$ which is an isometric embedding of a line and we observe that g acts on A as a translation of length $d(y, gy)$.

Definition 1.3.2. In the first two cases, that means $2d(x, y) \geq d(x, gx)$, we say that g is an *elliptic isometry*, otherwise we call g an *hyperbolic isometry* and we call A the *axis* of g.

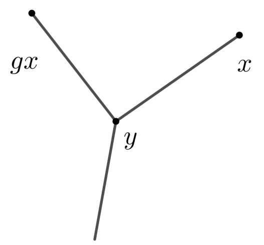

Figure 1.3: Elliptic isometry.

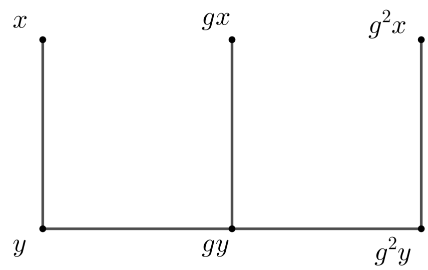

Figure 1.4: Hyperbolic isometry.

Definition 1.3.3. Given an isometry g of $\mathcal{T}$, the *length* of g is given by

$$l(g) := \min_{x \in \mathcal{T}} d(x, gx).$$

We define the *minimal locus* associated to g as

$$\mathrm{Min}(g) = \{x \in \mathcal{T} | l(g) = d(x, gx)\}.$$

The minimum which appears in the definition above always exists since we are studying real trees. For more general trees, called Λ-trees, we should have substituted the minimum with an infimum (see [Chi01, Chapter 3]).

If g is an elliptic isometry the length is trivial and $l(g) = 0$. The minimal locus of g coincide with the set of points fixed by g. If g is an hyperbolic isometry, the length is strictly positive. In this case the minimal locus of g coincides with its axis. However, in both cases, for any $x \in \mathcal{T}$ it holds

$$d(x, gx) = l(g) + 2d(x, \mathrm{Min}(g)).$$

We are going now to analyze isometric actions of a finitely generated group Γ on a real tree $\mathcal{T}$. Recall that since $\mathcal{T}$ is a real tree, it is also a $CAT(0)$–space. Hence it makes sense to refer to the boundary at infinity of $\mathcal{T}$. In this particular case the boundary $\partial_\infty \mathcal{T}$ can be seen as the limit of the inverse sistem $\pi_0(\mathcal{T} \setminus K)$ where K ranges all over the possible closed and bounded sets of $\mathcal{T}$. The elements of the boundary $\partial_\infty \mathcal{T}$ are called *ends*.

Definition 1.3.4. Let Γ be a group acting on a real tree $\mathcal{T}$. We say that the action of Γ is *reducible* if one of the following holds

- every elements of Γ fixes a point $x \in \mathcal{T}$;
- there exists an end $\varepsilon \in \partial_\infty \mathcal{T}$ fixed by Γ;

- there is an invariant line L for the action of Γ.

We call an action *semisimple* if it is *irreducible* or it fixes a point or it admits an invariant line. We say that the action is *minimal* if any subtree $\mathcal{T}' \subset \mathcal{T}$ stabilized by Γ is equal to $\mathcal{T}$.

Fix Γ a finitely generated group and suppose that Γ acts via isometries on a real tree $\mathcal{T}$. By applying the notion of length of an isometry to each element of Γ we get the notion of length of an isometric action.

Definition 1.3.5. Let Γ be a finitely generated group which acts on a real tree $\mathcal{T}$ by isometries. The *length function* of this action is given by

$$\mathfrak{L} : \Gamma \to \mathbb{R}_+, \quad \mathfrak{L}(\gamma) := l(\gamma).$$

Remark 1.3.6. The length function $\mathfrak{L}$ is a class function, that means it is invariant under conjugation by elements of Γ.

A first application of the notion of length function is the possibility to recognize isometric actions on $\mathcal{T}$ which admits a global fixed point. More precisely it holds

Proposition 1.3.7 *Let Γ be a finitely generated group which acts isometrically on a real tree $\mathcal{T}$. The action admits a global fixed point if and only if the associated translation length is trivial, that is $\mathfrak{L}(\gamma) = 0$ for every $\gamma \in \Gamma$.*

A proof of the previous proposition can be found for instance in [Kap01, Corollary 10.6].

Length functions are useful also to recognize reducible action. Indeed, we can distinguish reducible actions into *abelian actions*, which satisfy

$$\mathfrak{L}(\gamma_1\gamma_2) \leq \mathfrak{L}(\gamma_1) + \mathfrak{L}(\gamma_2)$$

for every $\gamma_1, \gamma_2 \in \Gamma$, and *dihedral actions*, for which the previous condition is satisfied only by hyperbolic isometries. For abelian actions the inequality above implies that for every $\gamma_1, \gamma_2 \in \Gamma$, their minimal loci intersect, that is

$$\mathrm{Min}(\gamma_1) \cap \mathrm{Min}(\gamma_2) \neq \varnothing.$$

The same happens also for dihedral actions if we restrict only to hyperbolic isometries.

On the other hand, if we restrict our attention to irreducible actions, then we are able to characterize them by studying their associated length functions. Recall that every semi-simple action without a global fixed point admit a minimal subtree (it is simply the union of the minimal loci associated to the hyperbolic elements of Γ).

Proposition 1.3.8 *Suppose to have two different semi-simple actions of Γ on two trees $\mathcal{T}, \mathcal{T}'$. If these actions have the same length functions, that is $\mathfrak{L}(\gamma) = \mathfrak{L}'(\gamma)$ for every $\gamma \in \Gamma$, then there is a Γ-equivariant isometry between their respective minimal subtrees $\mathcal{T}_{min}$ and $\mathcal{T}'_{min}$.*

See [CM87, Theorem 3.7] for a proof of the previous proposition.

A generalization of real trees is given by Euclidean buildings of type A_n. We are going to expose their definition and their properties following the work of [KL97]. Before doing this we recall that the symmetric group S_{n+1} acts naturally on $\mathbb{R}^{n+1}$ by permuting coordinates. If we denote by

$$\Pi := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 0\},$$

then Π is isomorphic to $\mathbb{R}^n$ and the action of S_{n+1} will still permute coordinates. We call the set

$$\overline{\mathfrak{C}} := \{(x_0, \ldots, x_n) \in \Pi | x_0 \geq x_1 \geq \ldots \geq x_n\}$$

a *closed Weyl chamber*. Any other closed Weyl chamber in Π is obtained by the action of S_{n+1}.

Definition 1.3.9. Endow $\mathbb{R}^n$ with the Euclidean metric. Let X be a metric space and let $\mathscr{A}$ be a family of isometric embeddings of $\mathbb{R}^n$ into X called *marked apartments*. We call *apartments* the family of images through the embeddings of $\mathscr{A}$. We call *Weyl chambers* of X the collection of images through the marked apartments of $\mathscr{A}$ into X of the Weyl chambers of $\mathbb{R}^n$. The set $\mathscr{A}$ defines a structure of *Euclidean building of type* A_n if the following properties are satisfied

- The system $\mathscr{A}$ is invariant under precomposition of an element of $\hat{W}_n = \mathbb{R}^n \rtimes S_{n+1}$ seen as a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. More precisely, given an injection $i : \mathbb{R}^n \to X$ with $i \in \mathscr{A}$ and an element $g \in \hat{W}_n$, it must hold $i \circ g \in \mathscr{A}$.

- Let $i, i' \in \mathscr{A}$. The set $I = i^{-1}(i'(\mathbb{R}^n))$ is a closed convex subset of $\mathbb{R}^n$ and the restriction of $(i')^{-1} \circ i$ to I coincides with the restriction of an element $g \in \hat{W}_n$.

- Given two points $x, x' \in X$, there exists at least an apartment passing through them.

- If C_1 and C_2 are two Weyl chambers of X, there exists an apartment A such that the intersections $A \cap C_1$ and $A \cap C_2$ still contain Weyl chambers.

- If A_1, A_2 and A_3 are apartments which intersect pairwise in half spaces, then $A_1 \cap A_2 \cap A_3 \neq \varnothing$ (see Figure 1.5).

An *automorphism* of X is an isometry $g : X \to X$ which preserves the set $\mathscr{A}$ of marked apartments. As done previously, we can distinguish different types of isometries.

Definition 1.3.10. Let X be a Euclidean building and let g be an automorphism. We define the *translation length* of g as

$$l(g) = \inf_{x \in X} d_X(x, gx).$$

If the infimum is attained we distinguish two cases. When the length is equal to zero, we say that g is an *elliptic isometry*, otherwise the translation length is

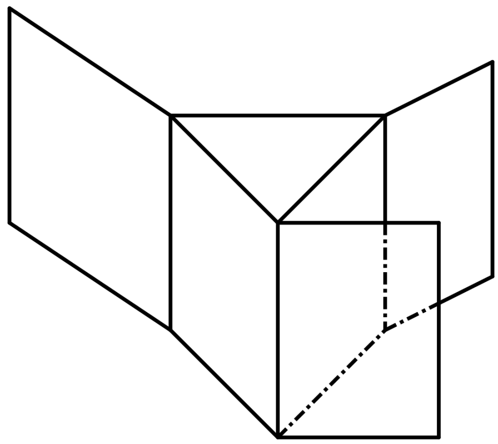

Figure 1.5: Configuration not allowed in a Euclidean building.

strictly positive and we call g a *hyperbolic isometry*. If the infimum is not attained we say that g is a *parabolic isometry*. The *minimal locus* associated to an isometry g is defined as

$$\mathrm{Min}(g) = \{x \in X | l(g) = d_X(x, gx)\}.$$

Both for elliptic and hyperbolic isometries the minimal locus is not-empty, whereas for parabolic ones is empty.

As in the case of real trees, isometric actions of a finitely generated group Γ on a Euclidean building of type A_n admit the notion of translation length function associated to them. However, there exists also a suitable generalization called vectorial length function. We are going to describe briefly this concept. Let X be a complete Euclidean building of type A_n. As before, we keep denoting by

$$\overline{\mathfrak{C}} := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 0, x_0 \geq x_1 \geq \ldots \geq x_n\}$$

the fundamental closed Weyl chamber. Since the action of the symmetric group is transitive on the Weyl chambers of $\mathbb{R}^n$, given a vector $v \in \Pi$ we can always find a vector $\Theta(v)$ such that $\Theta(v) \in \overline{\mathfrak{C}}$ and $\sigma(\Theta(v)) = v$, where σ is a suitable element of the symmetric group S_{n+1}. We say that $\Theta(v)$ is the *type* of the vector v into $\overline{\mathfrak{C}}$ (see Figure 1.6).

Now recall that given any pair of points $x, y \in X$, by the axioms we know that there exists an apartment A containing both of them. Let $i : \mathbb{R}^n \to X$ be the embedding whose image is A.

Definition 1.3.11. Let x, y be two points in A. The *vectorial type* $\delta(x, y)$ of the segment $[x, y]$ is the type of the vector whose endpoints are the points $a, b \in \mathbb{R}^n$ such that $i(a) = x$ and $i(b) = y$. Since the type of a segment is preserved by automorphisms, this notion is independent of the choice of the apartment A containing both x and y and hence it is well-defined.

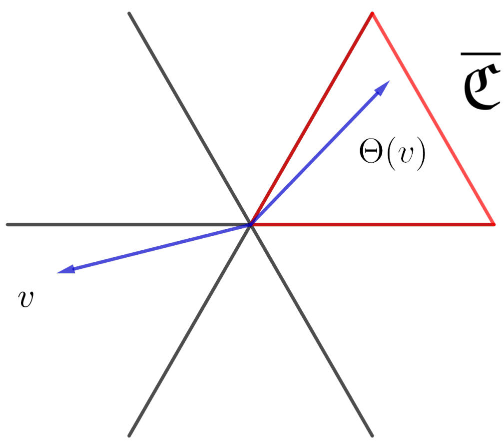

Figure 1.6: Type of a vector v.

The *translation vector* of an isometry g is the vectorial type of the segment $[x, gx]$ for any point $x \in \mathrm{Min}(g)$, that is

$$v(g) := \delta(x, gx), \quad x \in \mathrm{Min}(g).$$

The definition of $v(g)$ is independent of the choice of $x \in \mathrm{Min}(g)$. Given a finitely generated group acting by automorphisms on X, the *vectorial length function* associated to the action is defined as

$$\mathscr{V}_X : \Gamma \to \overline{\mathfrak{C}}, \quad \mathscr{V}_X(\gamma) := v(\gamma).$$

Euclidean buildings appear in several contexts, such as in the study of sequences of metric spaces. For instance, a way to obtain them is via the asymptotic cone construction. We are going to briefly recall this notion. We first introduce the concept of ultrafilter.

Definition 1.3.12. An *ultrafilter* ω on a set X is a family of subsets of X which satisfies:

- The empty set is not contained in ω, that is $\varnothing \notin \omega$.
- If $A \subset B$ and $A \in \omega$, then $B \in \omega$.
- Given a collection $A_1, \dots A_n$ such that $A_i \in \omega$ for every $i = 1, \dots, n$, then $A_1 \cap \dots \cap A_n \in \omega$.
- Given $A_1, \dots A_n$ such that $A_1 \sqcup \dots \sqcup A_n = X$, there exists exactly one $i_0 \in \{1, \dots, n\}$ so that $A_{i_0} \in \omega$.

An ultrafilter is *principal* and centered at $x \in X$ if for every set $A \in \omega$ it holds $x \in A$. Otherwise we say that the ultrafilter is *non-principal.*

The importance of ultrafilters relies on their power to force convergence of sequences of points in a topological space X by selecting a suitable limit point. For the sake of clarity we first need to introduce the following

Definition 1.3.13. Let X be a topological space and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of points in X. Fix an ultrafilter ω on the set of natural numbers $\mathbb{N}$. We say that the sequence ω-*converges* to x_0 if for every open neighborhood U of x_0 we have $\{k \in \mathbb{N} : x_k \in U\} \in \omega$.

A priori a sequence may admit no limit or several limits if the topology of the space X does not have good properties. To guarantee the existence and the uniqueness of the limit we need a compact Hausdorff space. Indeed, it holds

Proposition 1.3.14 *Let X be a topological space which is compact and Hausdorff. Then, for any ultrafilter ω on $\mathbb{N}$ and any sequence $(x_k)_{k\in\mathbb{N}}$ of points in X, there exists a unique point $x_0 \in X$ such that*

$$\omega\text{-}\lim_{k\to\infty} x_k = x_0.$$

Another remarkable property of ultrafilters is the compatibility with continuous functions between topological spaces.

Proposition 1.3.15 *Let $f : X \to Y$ be a continuous function between two compact Hausdorff spaces. Let ω be an ultrafilter on $\mathbb{N}$. For any sequence $(x_k)_{k\in\mathbb{N}}$ of points in X we have*

$$\omega\text{-}\lim_{k\to\infty} f(x_k) = f(\omega\text{-}\lim_{k\to\infty} x_k).$$

We are now ready to define the asymptotic cone of a metric space. Let (X, d) be a metric space. Fix an ultrafilter ω on $\mathbb{N}$ and a real divergent sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \geq 1$. Additionally, let $(*_k)_{k\in\mathbb{N}}$ be a sequence of basepoints in X. We consider

$$C_\omega(X, d/\lambda_k, *_k) := \{(x_k) \in X^{\mathbb{N}} | \exists C > 0, d(x_k, *_k) < C\lambda_k\}/\sim_\omega$$

where two sequences $(x_k)_{k\in\mathbb{N}}$ and $(x'_k)_{k\in\mathbb{N}}$ are identified by the relation $\sim_\omega$ if and only if $\omega\text{-}\lim_{k\to\infty} d(x_k, x'_k)/\lambda_k = 0$. If we denote by x_ω the equivalence class of the sequence $(x_k)_{k\in\mathbb{N}}$, we can endow the quotient space with a metric structure given by

$$d_\omega(x_\omega, x'_\omega) = \omega\text{-}\lim_{k\to\infty} d(x_k, x'_k)/\lambda_k$$

for every $x_\omega, x'_\omega \in C_\omega(X, d/\lambda_k, *_k)$.

Definition 1.3.16. The metric space $(C_\omega(X, d/\lambda_k, *_k), d_\omega)$ is the *asymptotic cone* of the metric space (X, d) with respect to the ultrafilter ω, the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the sequence of basepoints $(*_k)_{k\in\mathbb{N}}$.

Euclidean buildings appear as asymptotic cones of the symmetric spaces associated to $SL(n, \mathbb{C})$, by virtue of the following construction. Let X_n be the symmetric space of non-compact type associated to the group $SL(n, \mathbb{C})$. Let $(*_k)_{k\in\mathbb{N}}$ be a sequence of basepoints of X_n, let ω be a non-principal ultrafilter on $\mathbb{N}$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a real divergent sequence. Fix a maximal flat in X_n, which will be a copy of $\mathbb{R}^{n-1}$, the extended Weyl group $\hat{W}_n$ and the closed Weyl chamber previously introduced. A sequence of embeddings $i_k : \mathbb{R}^{n-1} \to X_n$ is ω-bounded if for every $k \in \mathbb{N}$ it holds $d(*_k, i_k(0)) < C\lambda_k$. In this case we have a map

$$i_\omega : \mathbb{R}^{n-1} \to C_\omega(X_n, d/\lambda_k, *_k), \quad i_\omega(v) = [i_k(\lambda_k v)]$$

which is an isometric embedding of $\mathbb{R}^{n-1}$. Denote by $\mathscr{A}_\omega$ the set of all the embeddings i_ω obtained in this way. As a consequence of [KL97, Theorem 5.2.1.] we have

Theorem 1.3.17 *Let X_n be the symmetric space associated to $SL(n,\mathbb{C})$. For any choice of the sequence $(*_k)_{k\in\mathbb{N}}$ of basepoints, any diverging sequence $(\lambda_k)_{k\in N}$ and any non-principal ultrafilter ω on $\mathbb{N}$, the asymptotic cone $C_\omega(X_n, d/\lambda_k, *_k)$ with the set of embeddings $\mathscr{A}_\omega$ is a complete Euclidean building of type A_{n-1}.*

1.3.2 The field $\mathbb{C}_\omega$

For more details regarding the definitions and the results contained in this section we refer to [Par12, Section 3.3]. As in the previous section let ω be a non-principal ultrafilter on $\mathbb{N}$ and let $(\lambda_k)_{k\in\mathbb{N}}$ be a real sequence that diverges to infinity and such that $\lambda_k \geq 1$ for every k. We define

$$\mathbb{C}_\omega = \{(a_k) \in \mathbb{C}^{\mathbb{N}} | \exists C > 0, \forall k \ |a_k|^{\frac{1}{\lambda_k}} < C\}/\sim_\omega$$

where $(a_k)_{k\in\mathbb{N}} \sim_\omega (b_k)_{k\in\mathbb{N}}$ if and only if $\omega\text{-}\lim_{k\to\infty} |a_k - b_k|^{\frac{1}{\lambda_k}} = 0$. It is easy to verify that the operations of pointwise sum and pointwise multiplication defined over $\mathbb{C}^{\mathbb{N}}$ are compatible with the equivalence relation $\sim_\omega$. Thus they define two operations of sum and multiplication over $\mathbb{C}_\omega$, which make $\mathbb{C}_\omega$ a field. There is a natural field embedding of $\mathbb{C}$ into $\mathbb{C}_\omega$ given by the constant sequences.

If we denote by a_ω the equivalence class $[(a_k)]$ of the sequence $(a_k)_{k\in\mathbb{N}}$, the function

$$|a_\omega|^\omega := \omega\text{-}\lim_{k\to\infty} |a_k|^{\frac{1}{\lambda_k}}$$

is an ultrametric absolute value on $\mathbb{C}_\omega$, that is it satisfies

$$|a_\omega + b_\omega|^\omega \leq \max\{|a_\omega|^\omega, |b_\omega|^\omega\}$$

for every pair $a_\omega, b_\omega \in \mathbb{C}_\omega$. It is worth noticing the elements of $\mathbb{C}$ different from 0, seen as the subfield of constant sequences, have all norm equal to 1.

Definition 1.3.18. The ultrametric field $(\mathbb{C}_\omega, |\cdot|^\omega)$ is called the *asymptotic cone* of $(\mathbb{C}, |\cdot|)$ with respect to the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the ultrafilter ω.

If we consider the distance induced by the absolute value $|\cdot|^\omega$ and we endow $\mathbb{C}_\omega$ with the metric topology, we obtain a topological field which is complete (see [Par12, Remark 3.10]), but it is not locally compact.

Proposition 1.3.19 *The field $\mathbb{C}_\omega$ is not locally compact with respect to the metric topology induced by the absolute value $|\cdot|^\omega$.*

Proof. Since $\mathbb{C}_\omega$ is a normed space, local compactness can be checked by verifying the compactness of the unit closed ball. Hence, it suffices to show that the closed ball

$$\overline{B}_1(0) := \{a_\omega \in \mathbb{C}_\omega | |a_\omega|^\omega \leq 1\}$$

is not compact. We are going to show that it is not sequentially compact. Consider the sequence $(n)_{n\in\mathbb{N}}$ where each element n has to be thought of as an element of $\mathbb{C}_\omega$ thanks to the standard embedding given by constant sequences. Given two different elements n and m it is clear that their distance in $\mathbb{C}_\omega$ is always equal to 1, indeed

$$|n-m|^\omega = \omega\text{-}\lim_{k\to\infty} |n-m|^{\frac{1}{\lambda_k}} = 1.$$

Hence it cannot exist a subsequence of $(n)_{n\in\mathbb{N}}$ which converges, as desired. □

The construction exposed above can be repeated, rather than for a field, for every m-dimensional normed vector space $(V, ||\cdot||)$ over $\mathbb{C}$. More precisely, we define

$$V_\omega := \{(v_k) \in V^{\mathbb{N}} | \exists C > 0, \forall k \ ||v_k||^{\frac{1}{\lambda_k}} < C\}/\sim_\omega,$$

where $(v_k)_{k\in\mathbb{N}}$ and $(u_k)_{k\in\mathbb{N}}$ are equivalent if and only if $\omega\text{-}\lim_{k\to\infty} ||u_k - v_k||^{\frac{1}{\lambda_k}} = 0$. Let v_ω be the equivalence class determined by $(v_k)_{k\in\mathbb{N}}$. It is possible to endow V_ω with a structure of m-dimensional $\mathbb{C}_\omega$-vector space by considering the operations induced by pointwise sum and by pointwise scalar multiplication. As before, we have a well-defined norm $||\cdot||^\omega$ given by

$$||v_\omega||^\omega := \omega\text{-}\lim_{k\to\infty} ||v_k||^{\frac{1}{\lambda_k}}.$$

Definition 1.3.20. The $\mathbb{C}_\omega$-vector space $(V_\omega, ||\cdot||^\omega)$ is the *asymptotic cone* of the vector space $(V, ||\cdot||)$ with respect to the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the ultrafilter ω.

We now focus our attention on the set of complex square matrices of order n, namely $M(n,\mathbb{C})$. If we endow $\mathbb{C}^n$ with the standard Hermitian stucture, we can choose as norm over $M(n,\mathbb{C})$ the norm which comes from thinking of a matrix as an operator between Hermitian spaces. Hence we can apply the construction above to the normed vector space $(M(n,\mathbb{C}), ||\cdot||)$. In this particular case we are able to enrich the structure of $M(n,\mathbb{C})_\omega$ by considering a multiplication. Indeed, the classic multiplication rows-by-columns is compatible with $\sim_\omega$ and hence it defines a structure of $\mathbb{C}_\omega$-algebra on $M(n,\mathbb{C})_\omega$.

Definition 1.3.21. The normed algebra $(M(n,\mathbb{C})_\omega, ||\cdot||^\omega)$ is called the *asymptotic cone* of the algebra $(M(n,\mathbb{C}), ||\cdot||)$ with respect to the scaling sequence $(\lambda_k)_{k\in\mathbb{N}}$ and the ultrafilter ω.

Definition 1.3.22. A sequence $(g_k) \in GL(n,\mathbb{C})^{\mathbb{N}}$ is ω-*bounded* if

$$\exists C > 0 : \forall k \ ||g_k||^{\frac{1}{\lambda_k}}, ||g_k^{-1}||^{\frac{1}{\lambda_k}} < C.$$

The previous condition implies that the sequence $(g_k)_{k\in\mathbb{N}}$ defines an element of $M(n,\mathbb{C})_\omega$ which admits a multiplicative inverse. We denote by $GL(n,\mathbb{C})_\omega$ the set of all the invertible elements of $M(n,\mathbb{C})_\omega$. This is a group with respect to the multiplication rows-by-columns. We denote by $SL(n,\mathbb{C})_\omega$ the subgroup

$$SL(n,\mathbb{C})_\omega := \{g_\omega \in GL(n,\mathbb{C})_\omega | \exists (g_k)_{k\in\mathbb{N}} \in g_\omega : \forall k \ \ \det(g_k) = 1\}.$$

Since we can also consider the normed algebra $(M(n,\mathbb{C}_\omega), ||\cdot||_\infty)$, where $||\cdot||_\infty$ is the standard supremum norm with respect to $|\cdot|^\omega$, it is natural to ask whether this algebra is isomorphic to $M(n,\mathbb{C})_\omega$ as normed algebra. The answer is given by [Par12, Corollary 3.18], which states that there is a natural isomorphism as normed $\mathbb{C}_\omega$-algebras between $M(n,\mathbb{C})_\omega$ and $M(n,\mathbb{C}_\omega)$. Moreover this isomorphism induces an isomorphism of groups between $SL(n,\mathbb{C})_\omega$ and $SL(n,\mathbb{C}_\omega)$.

Remark 1.3.23. Let X_n be the symmetric space associated to $SL(n,\mathbb{C})$. Fix a point O of X_n as basepoint. Let $(g_k)_{k\in\mathbb{N}}$ be any representative of an element $g_\omega \in SL(n,\mathbb{C})_\omega$. If for every $k \in \mathbb{N}$ there exists $C > 0$ such that $||g_k||, ||g_k^{-1}|| < C^{\lambda_k}$, then it follows that $d(g_k(O), O) < C\lambda_k$ for every $k \in \mathbb{N}$. This allows to define a natural action of $SL(n,\mathbb{C})_\omega$ via automorphisms on the asymptotic cone $C_\omega(X_n, d/\lambda_k, O)$, which is well-defined by [Par12, Proposition 3.20].

We conclude this section by introducing the space $\mathbb{P}^1(\mathbb{C})_\omega$. Denote by O the origin of the Poincaré disk model for $\mathbb{H}^3$. It should be clear that there exists a natural surjection

$$\pi : \mathbb{P}^1(\mathbb{C})^{\mathbb{N}} \to \partial_\infty C_\omega(\mathbb{H}^3, d/\lambda_k, O)$$

defined as it follows. Thinking of $\mathbb{P}^1(\mathbb{C})$ as the boundary at infinity of $\mathbb{H}^3$, a sequence of points $(\xi_k) \in \mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$ determines in a unique way a sequence of geodesic rays $(c_k)_{k\in\mathbb{N}}$ starting from O and ending at $(\xi_k)_{k\in\mathbb{N}}$. These rays allows us to define a geodesic ray $c_\omega : [0,\infty) \to C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ given by $c_\omega(t) := [c_k(\lambda_k t)]$. Hence, we can define $\pi((\xi_k)_{k\in\mathbb{N}}) := c_\omega(\infty)$. The space $\mathbb{P}^1(\mathbb{C})_\omega$ will be the quotient of $\mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$ by the equivalence relation induced by the surjection π. In this way $\mathbb{P}^1(\mathbb{C})_\omega$ is clearly identified with boundary at infinity of $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ and hence inherits in a natural way an action of $SL(2,\mathbb{C})_\omega$ given by $[h_k].[\xi_k] := [h_k.\xi_k]$. This action is well-defined because the action of $SL(2,\mathbb{C})_\omega$ on $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ is well-defined (see Remark 1.3.23). Moreover, it is possible to identify the space $\mathbb{P}^1(\mathbb{C})_\omega$ with the projective line $\mathbb{P}^1(\mathbb{C}_\omega)$. Since the absolute value $|\cdot|^\omega$ is ultrametric, the field $\mathbb{C}_\omega$ can be endowed with a natural valuation v^ω defined by

$$v^\omega : \mathbb{C}_\omega \to \mathbb{R}, \quad v^\omega(a_\omega) := -\log |a_\omega|^\omega.$$

In particular it makes sense to refer to the Bass–Serre tree $\Delta^{BS}(SL(2,\mathbb{C}_\omega))$ associated to the group $SL(2,\mathbb{C}_\omega)$. We refer to [Ota15, Section 5] for a detailed description of the construction of the tree $\Delta^{BS}(SL(2,\mathbb{C}_\omega))$.

Proposition 1.3.24 *There exists a natural isomorphism*

$$\Phi : C_\omega(\mathbb{H}^3, d/\lambda_k, O) \to \Delta^{BS}(SL(2,\mathbb{C}_\omega))$$

which is isometric and equivariant with respect to the actions of $SL(2,\mathbb{C})_\omega$ and $SL(2,\mathbb{C}_\omega)$, respectively.

For a proof of the previous proposition see [Par12, Proposition 3.21]. Recalling that the boundary at infinity of $\Delta^{BS}(SL(2,\mathbb{C}_\omega))$ coincides with $\mathbb{P}^1(\mathbb{C}_\omega)$, thanks to Proposition 1.3.24 the space $\mathbb{P}^1(\mathbb{C})_\omega$ can be indentified also with $\mathbb{P}^1(\mathbb{C}_\omega)$ and this identification is compatible with the actions of $SL(2,\mathbb{C})_\omega$ and $SL(2,\mathbb{C}_\omega)$, respectively.

Remark 1.3.25. Let V be a n-dimensional vector space over $\mathbb{C}_\omega$. Denote by $||\cdot||$ a norm on V. A basis $\mathcal{B}=\{v^1_\omega,\dots,v^n_\omega\}$ is *adapted* to the norm $||\cdot||$ if the following holds

$$||\sum_{i=1}^n a^i_\omega v^i_\omega|| = \max_{i=1,\dots,n} |a^i_\omega|^\omega ||v^i_\omega||, \text{ for every } a^1_\omega,\dots,a^n_\omega \in \mathbb{C}_\omega.$$

A norm which admits an adapted basis is called *adaptable*. A *good norm* is an adaptable norm which is ultrametric. The *volume* of a good norm is given by $\prod_{i=1}^n ||v^i_\omega||$ for any adapted basis $\mathcal{B}=\{v^1_\omega,\dots,v^n_\omega\}$ (see [GI63]). The *Goldman–Iwahori space* $\mathscr{N}^1(V)$ is the space of all the possible good norms of volume 1. This space has a natural structure of affine Euclidean building of type A_{n-1}, as proved in [Par00]. Moreover, the group $SL(n,\mathbb{C}_\omega)$ acts isometrically on it by $g_\omega.||v_\omega|| := ||g_\omega^{-1}v_\omega||$, for every $g_\omega\in SL(n,\mathbb{C}_\omega)$ and every $v_\omega\in V$.

We keep denoting by X_n the symmetric space associated to $SL(n,\mathbb{C})$ and we fix a basepoint $O\in X_n$. In [Par12, Proposition 3.21] the author proves that there exists an isometric isomorphism between the asymptotic cone $C_\omega(X_n, d/\lambda_k, O)$ and the Goldman–Iwahori space $\mathscr{N}^1(\mathbb{C}^n_\omega)$ and the isomorphism is equivariant with respect to the actions of $SL(n,\mathbb{C})_\omega$ and $SL(n,\mathbb{C}_\omega)$, respectively.

1.3.3 The Morgan–Shalen/Parreau compactification

Let Γ be a finitely generated group and let $S=\{\gamma_1,\dots,\gamma_s\}$ be a generating set. The representation variety is the algebraic variety defined by

$$\mathrm{Hom}(\Gamma, SL(n,\mathbb{C})) = \{\rho:\Gamma\to SL(n,\mathbb{C}) | \rho \text{ is a morphism of groups}\} \subset \mathbb{C}^{sn^2}.$$

Definition 1.3.26. The *character variety* $X(\Gamma, SL(n,\mathbb{C}))$ is the *GIT*-quotient of the representation variety $\mathrm{Hom}(\Gamma, SL(n,\mathbb{C}))$ by the conjugation action of $SL(n,\mathbb{C})$, that is

$$X(\Gamma, SL(n,\mathbb{C})) = \mathit{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, SL(n,\mathbb{C}))]^{SL(n,\mathbb{C})}),$$

where $\mathbb{C}[\mathrm{Hom}(\Gamma, SL(n,\mathbb{C}))]$ denotes the algebra of regular functions on $\mathrm{Hom}(\Gamma, SL(n,\mathbb{C}))$.

Given any affine algebraic variety V, J.W. Morgan and P.B. Shalen proposed in [MS84] a new way to compactify V based on the choice of a finite or countable family $\mathcal{F}$ of generating functions for the algebra of regular functions $\mathbb{C}[V]$. More precisely, consider $\mathcal{F}$ as above. Set

$$\mathbb{P}(\mathbb{R}^{\mathcal{F}}_+) = ([0,\infty)^{\mathcal{F}} \setminus \{0\})/(0,\infty)$$

where two distinct element $(s_f)_{f\in\mathcal{F}}$ and $(t_f)_{f\in\mathcal{F}}$ are identified if and only if there exists $\alpha\in(0,\infty)$ such that $s_f=\alpha t_f$ for every $f\in\mathcal{F}$. We define the following map

$$\varphi_{\mathcal{F}} : V \to \mathbb{P}(\mathbb{R}_+^{\mathcal{F}}), \quad \varphi_{\mathcal{F}}(x) = [\log(|f(x)| + 2)]_{f \in \mathcal{F}},$$

whose image into $\mathbb{P}(\mathbb{R}_+^{\mathcal{F}})$ is relatively compact by [MS84, Proposition I.3.1]. If we denote by V^+ the one-point compactification of V, by considering the closure of the image of the function

$$\Phi_{\mathcal{F}} : V \to V^+ \times \mathbb{P}(\mathbb{R}_+^{\mathcal{F}}), \quad \Phi_{\mathcal{F}}(x) = (x, \varphi_{\mathcal{F}}(x)),$$

we obtain a compactification of V which depends only the choice of the generating family $\mathcal{F}$.

Definition 1.3.27. The *$\mathcal{F}$-compactification* of the variety V is given by

$$\overline{V}^{\mathcal{F}} = \overline{\operatorname{im}(\Phi_{\mathcal{F}})}^{V^+ \times \mathbb{P}(\mathbb{R}_+^{\mathcal{F}})}.$$

The set of *$\mathcal{F}$-ideal points* of the compactification is defined as

$$\mathcal{B}_{\mathcal{F}}(V) := \overline{V}^{\mathcal{F}} \setminus V.$$

Now we focus our attention on $V = X(\Gamma, SL(2,\mathbb{C}))$. Denote by $\nu(\Gamma)$ the set of conjugacy classes of Γ and set $\mathscr{C} = \{\mathrm{Tr}_\gamma\}_{\gamma \in \nu(\Gamma)}$. The regular function Tr_γ is defined as

$$\mathrm{Tr}_\gamma : X(\Gamma, SL(2,\mathbb{C})) \to \mathbb{C}, \quad \mathrm{Tr}_\gamma(\rho) := \mathrm{Tr}(\rho(\gamma)).$$

Since $\mathscr{C}$ is a countable generating family for the algebra $\mathbb{C}[X(\Gamma, SL(2,\mathbb{C}))]$, as shown in [CS83], we can formulate the following

Definition 1.3.28. The *Morgan–Shalen compactification* of the character variety $X(\Gamma, SL(2,\mathbb{C}))$ is its $\mathscr{C}$-compactification, that is

$$\overline{X(\Gamma, SL(2,\mathbb{C}))}^{MS} := \overline{X(\Gamma, SL(2,\mathbb{C}))}^{\mathscr{C}}.$$

The interest in studying the Morgan–Shalen compactification of the character variety $X(\Gamma, SL(2,\mathbb{C}))$ relies on the possibilty to interpret each ideal point of the set $B_{\mathscr{C}}(X(\Gamma, SL(2,\mathbb{C})))$ as an isometric Γ-action on a suitable real tree. More precisely, let $S = \{\gamma_1, \dots, \gamma_s\}$ be a generating set for Γ. Given a representation $\rho : \Gamma \to SL(2,\mathbb{C})$, define the function

$$d_\rho : \mathbb{H}^3 \to \mathbb{R}, \quad d_\rho(x) := \sqrt{\sum_{i=1}^{s} d_{\mathbb{H}^3}(\rho(\gamma_i)x, x)^2}$$

and set

$$\lambda(\rho) := \inf_{x \in \mathbb{H}^3} d_\rho(x).$$

We call $\lambda(\rho)$ the *minimal displacement* associated to ρ. The importance of the minimal displacement is due to the following fact. Given a sequence of representations $\rho_k : \Gamma \to SL(2,\mathbb{C})$ define $\lambda_k := \lambda(\rho_k)$. If $\lambda_k \to \infty$, then the sequence ρ_k is diverging in the character variety $X(\Gamma, SL(2,\mathbb{C}))$.

Theorem 1.3.29 *Let Γ be a finitely generated group and suppose that a sequence of representations $\rho_k : \Gamma \to SL(2,\mathbb{C})$ satisfies $\lambda_k \to \infty$. Hence there exists a real tree $\mathcal{T}$ on which Γ acts isometrically and such that*

$$\lim_{k\to\infty} \frac{\log(|\mathrm{Tr}_\gamma(\rho_k)|+2)}{\lambda_k} = \mathfrak{L}_{\mathcal{T}}(\gamma)$$

for every element $\gamma \in \Gamma$. Here $\mathfrak{L}_{\mathcal{T}}$ is the length function associated to the action of Γ on $\mathcal{T}$. Moreover the action does not admit a global fixed point, that is the length $\mathfrak{L}$ is not identically zero.

If the previous equation holds for every $\gamma \in \Gamma$, we say that the sequence ρ_k *converges projectively* to the action of Γ on $\mathcal{T}$. This convergence can be interpreted as a convergence of translation length functions. More precisely, given a representation $\rho : \Gamma \to SL(2,\mathbb{C})$, we define the *translation length* of an element $\gamma \in \Gamma$ as

$$l(\rho(\gamma)) := \inf_{x\in\mathbb{H}^3} d_{\mathbb{H}^3}(\rho(\gamma)x, x)$$

which allows us to define a function

$$\mathfrak{L}_\rho : \Gamma \to \mathbb{R}, \quad \mathfrak{L}_\rho(\gamma) := l(\rho(\gamma)).$$

The function $\mathfrak{L}_\rho$ is the *translation length function* associated to ρ. Since when $|\mathrm{Tr}(\rho(\gamma))| \geq 1$ we have that

$$|2\log|\mathrm{Tr}(\rho(\gamma))| - l(\rho(\gamma))| \leq 2$$

we can substitute the limit which appears in Theorem 1.3.29 with the following expression

$$\lim_{k\to\infty} \frac{\mathfrak{L}_{\rho_k}(\gamma)}{\lambda_k} = \mathfrak{L}_{\mathcal{T}}(\gamma).$$

Hence the projective convergence of a sequence ρ_k to an isometric action of Γ on a real tree $\mathcal{T}$ can be reformulated as the projective convergence of the translation length functions $\mathfrak{L}_{\rho_k}$ to the translation lenght function $\mathfrak{L}_{\mathcal{T}}$.

In [MS84] the tree $\mathcal{T}$ is obtained by following the Bass–Serre construction for $SL(2,\mathbb{K}_v)$, where $\mathbb{K}_v$ is a valued field. Lately both [Bes88] and [Pau88] described a more geometrich approach to get the tree $\mathcal{T}$ based on Gromov–Hausdorff convergence. This procedure inspired Parreau to generalize the compactification to the case of the character variety $X(\Gamma, SL(n,\mathbb{C}))$ (see [Par12]). For this compactification the ideal points will be interpreted as vectorial length function associated to isometric actions of Γ on Euclidean buildings of type A_{n-1}.

In order to generalize the compactification of Morgan–Shalen to $X(\Gamma, SL(n,\mathbb{C}))$ we first need to introduce the notion of vectorial length function associated to a representation $\rho : \Gamma \to SL(n,\mathbb{C})$. Recall the definition of the closed Weyl chamber

$$\overline{\mathfrak{C}} := \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n | \sum_{i=1}^n \lambda_i = 0, \lambda_1 \geq \ldots \geq \lambda_n\}.$$

Additionally, thanks to the Iwasawa decomposition (see [Hel78, Chap. IX,Theorem 1.3]) every element $g \in SL(n,\mathbb{C})$ can be written in a unique way as $g = ehp$ where e

is an element of $SU(n)$, h is a diagonal matrix with determinant equal to 1 and positive real eigenvalues, and p is unipotent. We denote by $h(g)$ the diagonal component of the previous decomposition.

Definition 1.3.30. Given an element $g \in SL(n, \mathbb{C})$ we define the *Jordan projection of* g associated to $\overline{\mathfrak{C}}$ as

$$v(g) := (\log(\lambda_1(g)), \dots, \log(\lambda_n(g))) \in \overline{\mathfrak{C}}$$

where $\lambda_i(g)$ are the eigenvalues of the hyperbolic component $h(g)$ of g. Here we assume $\lambda_1(g) \geq \lambda_2(g) \geq \dots \geq \lambda_n(g)$. The *vectorial length function* of a representation $\rho : \Gamma \to SL(n, \mathbb{C})$ is given by

$$\mathscr{V}_\rho : \Gamma \to \overline{\mathfrak{C}}, \quad \mathscr{V}_\rho(\gamma) := v(\rho(\gamma)).$$

The previous notion generalizes the notion of translation length function for representations into $SL(2, \mathbb{C})$ and it will be one of the fundamental tool to define correctly the convergence of a sequence of representations to an ideal points of the Parreau compactification of $X(\Gamma, SL(n, \mathbb{C}))$. Define the function

$$\mathscr{V} : X(\Gamma, SL(n, \mathbb{C})) \to \overline{\mathfrak{C}}^{\mathscr{C}}, \quad \mathscr{V}([\rho]) := \mathscr{V}_\rho$$

where $\mathscr{V}_\rho([\gamma]) = v(\rho(\gamma))$. This function is well-defined since it is invariant by conjugation. Consider the space $\mathbb{P}\overline{\mathfrak{C}}^{\mathscr{C}}$ defined by

$$\mathbb{P}\overline{\mathfrak{C}}^{\mathscr{C}} := (\overline{\mathfrak{C}}^{\mathscr{C}} \setminus \{0\})/(0, \infty)$$

where two elements $v_{c \in \mathscr{C}}, w_{c \in \mathscr{C}} \in \overline{\mathfrak{C}}^{\mathscr{C}}$ are identified if and only if there exists an element $\alpha \in (0, \infty)$ such that $v_c = \alpha w_c$ for every $c \in \mathscr{C}$. If we set $X_0 := \mathscr{V}^{-1}(0)$, it is possible to show that X_0 is relatively compact. Moreover, the application

$$\mathbb{P}\mathscr{V} : X(\Gamma, SL(n, \mathbb{C})) \setminus X_0 \to \mathbb{P}\overline{\mathfrak{C}}^{\mathscr{C}}$$

has relatively compact image at infinity, i.e. there exists a compact K containing X_0 such that $\mathbb{P}\mathscr{V}(X(\Gamma, SL(n, \mathbb{C})) \setminus K)$ is relatively compact (see [Par12, Theorem 5.2]). This induces a natural compactification of $X(\Gamma, SL(n, \mathbb{C}))$ as follows. Define

$$\Phi_{\mathscr{C}} : X(n, SL(n, \mathbb{C})) \setminus X_0 \to X(\Gamma, SL(n, \mathbb{C}))^+ \times \mathbb{P}\overline{\mathfrak{C}}^{\mathscr{C}}, \quad \Phi_{\mathscr{C}}(x) = (x, \mathbb{P}\mathscr{V}(x)),$$

where $X(\Gamma, SL(n, \mathbb{C}))^+$ is the one-point compactification of $X(\Gamma, SL(n, \mathbb{C}))$. Since the image of $\mathbb{P}\mathscr{V}$ is relatively compact at infinity, the closure of $\operatorname{im}(\Phi_{\mathscr{C}})$ into the product $X(\Gamma, SL(n, \mathbb{C}))^+ \times \mathbb{P}\overline{\mathfrak{C}}^{\mathscr{C}}$ defines a compactification of $X(\Gamma, SL(n, \mathbb{C})) \setminus X_0$ and hence of $X(\Gamma, SL(n, \mathbb{C}))$.

Definition 1.3.31. The compactification of $X(\Gamma, SL(n, \mathbb{C}))$ obtained above is called *Parreau compactification* and we denote it by $\overline{X(\Gamma, SL(n, \mathbb{C}))}^P$. The set of ideal points of the compactification will be denoted by

$$B_P(X(\Gamma, SL(n, \mathbb{C}))) := \overline{X(\Gamma, SL(n, \mathbb{C}))}^P \setminus X(\Gamma, SL(n, \mathbb{C}))$$

By applying the same ideas of [MS84] also to the Parreau compactification, it is possible to interpret the ideal points as projective vectorial length functions of isometric actions Γ on Euclidean buildings. Moreover the construction of the building can be realized by taking the asymptotic cone of the symmetric space X_n associated to $SL(n, \mathbb{C})$.

More precisely, let ρ be a representation of Γ into $SL(n, \mathbb{C})$. As before, for a fixed generating set $S = \{\gamma_1, \dots, \gamma_s\}$ we define the function

$$d_\rho : X_n \to \mathbb{R}, \quad d_\rho(x) := \sqrt{\sum_{i=1}^{n} d_{X_n}(\rho(\gamma_i)x, x)^2}$$

and

$$\lambda(\rho) := \inf_{x \in X_n} d_\rho(x)$$

is the *minimal displacement* associated to ρ. A sequence of representations $\rho_k : \Gamma \to SL(n, \mathbb{C})$ diverging to an ideal point in the Parreau compactification will satisty $\lambda_k \to \infty$. Fix a sequence of basepoints x_k in X_n such that

$$d_{\rho_k}(x_k) \le \lambda_k + 1/k.$$

Up to conjugating ρ_k we can suppose that for every $k \in \mathbb{N}$ the basepoints $x_k = x_0$. In this setting we have

Theorem 1.3.32 *Let $\rho_k : \Gamma \to SL(n, \mathbb{C})$ be a sequence of representations such that $\lambda_k \to \infty$. Let $\flat \in B_P(X(\Gamma, SL(n, \mathbb{C})))$ be the ideal point to which the sequence ρ_k is converging.*

- *For any non-principal ultrafilter ω on $\mathbb{N}$ there exists an isometric action via automorphisms of Γ on the asymptotic cone $C_\omega(X_n, d/\lambda_k, x_0)$.*
- *Since $C_\omega(X_n, d/\lambda_k, x_0)$ is isometrically isomorphic to the Goldman–Iwahori space $\mathscr{N}^1(\mathbb{C}^n_\omega)$ of good norms with volume equal to 1, this action can be thought of as a representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$.*
- *The class $\flat \in \mathbb{P}\overline{\mathfrak{C}}^{\mathscr{C}}$ admits as representative the vectorial length function $\mathscr{V}_\omega$ associated to ρ_ω. Moreover the sequence of vectorial length functions $\mathscr{V}_{\rho_k}$ projectively converges to $\mathscr{V}_\omega$, that is for every $\gamma \in \Gamma$ it holds*

$$\omega\text{-}\lim_{k\to\infty} \frac{\mathscr{V}_{\rho_k}(\gamma)}{\lambda_k} = \mathscr{V}_\omega(\gamma).$$

Chapter 2

Volume and rigidity of hyperbolic lattices

2.1 Volume rigidity for lattices in $PO(3,1)$

In this chapter we are going to prove the rigidity at infinity for non-uniform lattices in rank-one Lie groups of non-compact type. The following section is devoted entirely to real lattices. Lately we will discuss complex and quaternionic lattices.

We are start working in $\mathbb{H}^3$. We fix the following setting.

- A group $\Gamma < PSO(3,1)$ so that $M = \Gamma \backslash \mathbb{H}^3$ is a (non-compact) complete hyperbolic manifold of finite volume.
- A base-point $O \in \mathbb{H}^3$ used to normalize the Busemann function $B(x,\theta)$, with $x \in \mathbb{H}^3$ and $\theta \in \partial_\infty \mathbb{H}^3$.
- The family $\{\mu_x\}$ of Patterson-Sullivan probability measures. Set $\mu = \mu_O$.
- A sequence of representations $\rho_n : \Gamma \to PO(3,1)$ such that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$.

Lemma 2.1.1 *The condition $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$ implies that, up to passing to a subsequence, we can suppose that no ρ_n is elementary.*

Proof. Elementary representations have zero volume and $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$, which is stricly positive. □

With an abuse of notation we still denote the subsequence of the previous lemma by ρ_n. Since no ρ_n is elementary we can consider the sequence of ρ_n-equivariant measurable maps $D_n : \partial_\infty \mathbb{H}^3 \to \partial_\infty \mathbb{H}^3$ and the corresponding sequence of BCG–natural maps $F_n : \mathbb{H}^3 \to \mathbb{H}^3$.

Lemma 2.1.2 *Up to conjugating ρ_n by a suitable element $g_n \in PO(3,1)$, we can suppose $F_n(O) = O$.*

Proof. Conjugating ρ_n by g reflects in post-composing F_n with g. We can choose g_n such $g_n(F_n(O)) = O$. □

The choice to fix the origin of $\mathbb{H}^3$ as the image of $F_n(O)$ is made to avoid pathological behaviours. For instance consider a sequence of loxodromic elements $g_n \in PO(3,1)$ which is divergent and define the representations $\rho_n := g_n \circ i \circ g_n^{-1}$, where $i : \Gamma \to PO(3,1)$ is the standard lattice embedding. Clearly this sequence of representations satisfies $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$ since for every $n \in \mathbb{N}$ we have $\mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$. However, there does not exist any subsequence of ρ_n converging to the holonomy of the manifold M.

Definition 2.1.3. For any $n \in \mathbb{N}$ and every $x \in \mathbb{H}^3$ we can define the following quadratic forms on $T_{F_n(x)}\mathbb{H}^3$:

$$\langle K_n|_{F_n(x)}u, u\rangle = \int_{\partial_\infty \mathbb{H}^3} \nabla dB|_{(F_n(x),D_n(\theta))}(u,u)d\mu_x(\theta)$$

$$\langle H_n|_{F_n(x)}u, u\rangle = \int_{\partial_\infty \mathbb{H}^3} (dB|_{(F_n(x),D_n(\theta))}(u))^2 d\mu_x(\theta)$$

for any $u \in T_{F_n(x)}\mathbb{H}^3$. The notation $\langle \cdot, \cdot \rangle$ stands for the scalar product on $T_{F_n(x)}\mathbb{H}^3$ induced by the hyperbolic metric on $\mathbb{H}^3$.

For sake of simplicity we are going to drop the subscripts in K_n and H_n. Recall that, since both the domain and the target have the same dimension, the 3-Jacobian $Jac_3(F_n)$ coincides the modulus of the jacobian determinant $\det(D_x F_n)$. As stated in [BCG96, Lemma 5.4], the following inequality holds for every $x \in \mathbb{H}^3$

$$|\det(D_x F_n)| \leq \left(\frac{4}{3}\right)^{\frac{3}{2}} \frac{\det(H_n)^{\frac{1}{2}}}{\det(K_n)}.$$

Lemma 2.1.4 *Suppose* $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$. *Then we have that* $|\det(D_x F_n)|$ *converges to* 1 *almost everywhere in* $\mathbb{H}^3$ *with respect to the measure induced by the standard metric.*

Proof. Denote by $F_n^\varepsilon : \mathbb{H}^3 \to \mathbb{H}^3$ the ε-natural maps introduced in Section 1.2.3. Recall that we have the following estimate

$$\mathrm{Vol}(\rho_n) \leq \int_M |\det(D_x F_n^\varepsilon)| d\mathrm{vol}_{\mathbb{H}^3}(x) = \mathrm{Vol}(F_n^\varepsilon)$$

and since $|\det(D_x F_n^\varepsilon)| \leq 1 + \varepsilon$ and $\lim_{\varepsilon\to 0} D_x F_n^\varepsilon = D_x F_n$, by the theorem of dominated convergence we get

$$\mathrm{Vol}(\rho_n) \leq \int_M |\det(D_x F_n)| d\mathrm{vol}_{\mathbb{H}^3}(x) \leq \mathrm{Vol}(M)$$

from which the statement follows. □

If $\mathcal{N}$ is the set of zero measure outside of which $|\det(D_x F_n)|$ is converging, for every $x \in \mathbb{H}^3 \setminus \mathcal{N}$ and fixed $\varepsilon > 0$ there must exist $n_0 = n_0(\varepsilon, x)$ such that $|\det(D_x F_n)| \geq 1 - \varepsilon$ for every $n > n_0$. Thus it holds

$$\left(\frac{4}{3}\right)^{\frac{3}{2}} \frac{\det(H_n)^{\frac{1}{2}}}{\det(K_n)} > 1 - \varepsilon$$

from which we can deduce

$$\frac{\det(H_n)}{(\det(K_n))^2} > \left(\frac{3}{4}\right)^3 (1-\varepsilon)^2 > \left(\frac{3}{4}\right)^3 (1-2\varepsilon).$$

Moreover, since $\mathbb{H}^3$ has costant sectional curvature equal to -1, we have $K_n = I - H_n$ (see [BCG95]). Here I stands for the identity on $T_{F_n(x)}\mathbb{H}^3$. Hence, by substituting the expression of K_n in the previous inequality, we get

$$\frac{\det(H_n)}{(\det(I-H_n))^2} > \left(\frac{3}{4}\right)^3 (1-2\varepsilon).$$

Consider now the set of positive definite symmetric matrices of order 3 with real entries and trace equal to 1, namely

$$Sym_1^+(3,\mathbb{R}) := \{H \in Sym(3,\mathbb{R}) | H > 0, \mathrm{Tr}(H) = 1\}.$$

Once we have fixed a basis of $T_{F_n(x)}\mathbb{H}^3$, we can identify H_n and K_n with the matrices representing these bilinear forms with respect to the fixed basis. Under this assumption, recall that $H_n \in Sym_1^+(3,\mathbb{R})$ for every $n \in \mathbb{N}$, as shown in [BCG96]. If we define

$$\psi : Sym_1^+(3,\mathbb{R}) \to \mathbb{R}, \quad \psi(H) = \frac{\det(H)}{(\det(I-H))^2},$$

we know that

$$\psi(H) \leq \left(\frac{3}{4}\right)^3$$

and the equality holds if and only if $H = I/3$ (see [BCG95, Appendix B]). It is worth noticing that the space $Sym_1^+(3,\mathbb{R})$ is not compact and a priori there could exist a diverging sequence of elements $H_n \in Sym_1^+(3,\mathbb{R})$ such that

$$\lim_{n\to\infty} \psi(H_n) = \left(\frac{3}{4}\right)^3.$$

We are going to show that this is impossible.

Proposition 2.1.5 *Suppose to have a sequence $H_n \in Sym_1^+(3,\mathbb{R})$ such that*

$$\lim_{n\to\infty} \psi(H_n) = \left(\frac{3}{4}\right)^3.$$

Then the sequence H_n must converge to $I/3$.

Proof. We start by observing that the function ψ is invariant by conjugation for an element $g \in GL(3,\mathbb{R})$. Indeed, $\psi(H)$ can be expressed as $\psi(H) = p_H(0)/(p_H(1))^2$, where p_H is the characteristic polynomial of H. Hence the claim follows. In particular, we have an induced function

$$\tilde{\psi} : O(3,\mathbb{R})\backslash Sym_1^+(3,\mathbb{R}) \to \mathbb{R}, \quad \tilde{\psi}(\bar{H}) = \psi(H),$$

where $\bar{H}$ denotes the equivalence class of the matrix H and the orthogonal group $O(3,\mathbb{R})$ acts on $Sym_1^+(3,\mathbb{R})$ by conjugation. We can think of the space $O(3,\mathbb{R})\backslash Sym_1^+(3,\mathbb{R})$ as the interior $\mathring{\mathbf{\Delta}}_2$ of the standard 2-simplex quotiented by the action of the symmetric group S_3 which permutes the coordinate of an element $(\lambda_1,\lambda_2,\lambda_3)\in\mathring{\mathbf{\Delta}}_2$. An explicit homeomorphism between the two spaces is given by

$$\Lambda : O(3,\mathbb{R})\backslash Sym_1^+(3,\mathbb{R}) \to S_3\backslash\mathring{\mathbf{\Delta}}_2, \quad \Lambda(\bar{H}) := [\lambda_1(H),\lambda_2(H),\lambda_3(H)],$$

where $\lambda_i(H)$ for $i=1,2,3$ are the eigenvalues of H. By defining $\Psi=\psi\circ\Lambda^{-1}$, we can express this function as

$$\Psi : S_3\backslash\mathring{\mathbf{\Delta}}_2 \to \mathbb{R}, \quad \Psi([a,b,c]) = \frac{abc}{((1-a)(1-b)(1-c))^2}.$$

We are going to think of Ψ as defined on $\mathring{\mathbf{\Delta}}_2$ and we are going to estimate this function on the boundary of $\mathbf{\Delta}_2$. Since $a+b+c=1$, with an abuse of notation we will write

$$\Psi(a,b) = \frac{ab(1-a-b)}{((1-a)(1-b)(a+b))^2},$$

identifying $\mathring{\mathbf{\Delta}}_2$ with the interior of the triangle τ in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ and $(0,1)$. If a sequence of points is converging to a boundary point of $\mathbf{\Delta}_2$, then we have a sequence (a_n,b_n) of points converging to a boundary point of τ. If the limit point is not a vertex of τ then $\lim_{n\to\infty}\Psi(a_n,b_n)=0$. For instance, suppose $\lim_{n\to\infty}(a_n,b_n)=(\alpha,0)$ with $\alpha\neq 0,1$. Hence

$$\lim_{n\to\infty}\Psi(a_n,b_n) = \lim_{n\to\infty}\frac{a_nb_n(1-a_n-b_n)}{((1-a_n)(1-b_n)(a_n+b_n))^2} = 0$$

as claimed. For the other boundary points which are not vertices, the computation is the same. The delicate points are given by the vertices $(0,0)$, $(1,0)$ and $(0,1)$. On these points the function Ψ cannot be continuously extended. However we can uniformly bound the possible limit values. Suppose to have a sequence (a_n,b_n) such that $\lim_{n\to\infty}(a_n,b_n)=(0,0)$. We have

$$\Psi(a_n,b_n) = \frac{a_nb_n(1-a_n-b_n)}{((1-a_n)(1-b_n)(a_n+b_n))^2} \sim \frac{a_nb_n}{(a_n+b_n)^2} \leq \frac{1}{4},$$

where the symbol $\sim$ denotes that the sequence on the left has the same behaviour of the sequence of the right in a neighborhood of $(0,0)$. Analogously, if $\lim_{n\to\infty}(a_n,b_n)=(1,0)$ then

$$\Psi(a_n,b_n) = \frac{a_nb_n(1-a_n-b_n)}{(1-a_n)(1-b_n)(a_n+b_n)} \sim \frac{b_n}{1-a_n}\left(1-\frac{b_n}{1-a_n}\right) \leq \frac{1}{4}.$$

and the same for $\lim_{n\to\infty}(a_n,b_n)=(0,1)$. The previous computation proves that Ψ is uniformly bounded by $1/4$ on the boundary of τ, hence on the boundary of $\mathbf{\Delta}_2$. Equivalently ψ is bounded by $1/4$ in a suitable neighborhood at infinity of $Sym_1^+(3,\mathbb{R})$, from which the statement follows. $\square$

Remark 2.1.6. In the following picture we can find the plot of the graph of the function Ψ. It is clear that the function converges to zero for a point of any edge of the triangle τ which is not a vertex. Moreover the function can be bounded by $1/4$ at the vertices of τ, hence Ψ achieves its maximum only at the point $(a,b) = (1/3, 1/3)$.

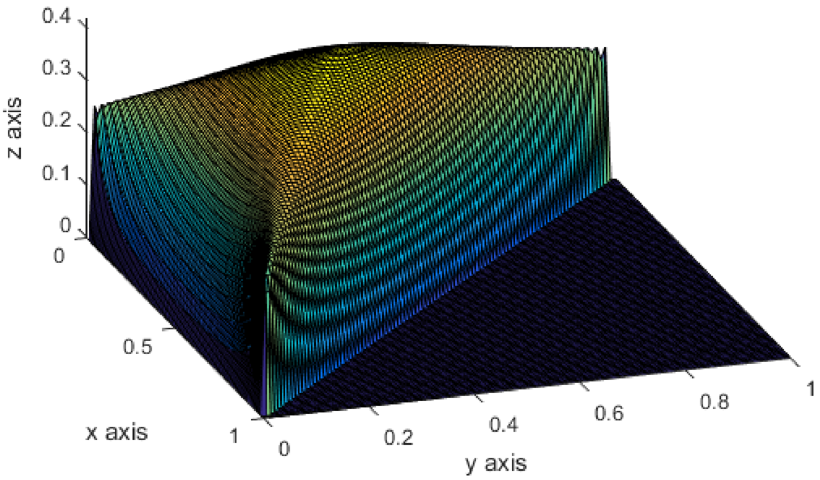

Figure 2.1: Graph of the function Ψ

We know that in our context we have

$$\left(\frac{3}{4}\right)^3 (1-2\varepsilon) \leq \psi(H_n) \leq \left(\frac{3}{4}\right)^3$$

for $n \geq n_0$. As a consequence of Proposition 2.1.5, the sequence H_n must converge to $I/3$. Hence H_n converges to $I/3$ almost-everywhere on $\mathbb{H}^3$. We are going to prove that this implies the uniform convergence of H_n to $I/3$ on compact sets. Before doing this we recall these two lemmas which can be found in [BCG95, Section 7].

Lemma 2.1.7 *Let $x, x' \in \mathbb{H}^3$ be such that the maximum eigenvalue of H_n satisfies $\lambda_n \leq 2/3$ at every point of the geodesic joining x to x'. Then there exists a positive constant C_1 such that*

$$d(F_n(x), F_n(x')) \leq C_1 d(x, x').$$

Lemma 2.1.8 *Let $x, x' \in \mathbb{H}^3$. Let P be the parallel transport from $F_n(x)$ to $F_n(x')$ along the geodesic which joins the two points. Denote by $H_n(x)$ the bilinear form defined on $T_{F_n(x)}\mathbb{H}^3$. Then there exists a positive constant C_2 such that*

$$||H_n(x) - H_n(x') \circ P|| \leq C_2(d(x,x') + d(F_n(x), F_n(x'))).$$

Proposition 2.1.9 *Suppose the sequence H_n converges almost everywhere to $I/3$. Thus it converges uniformly to $I/3$ on every compact set of $\mathbb{H}^3$.*

Proof. We will follow the same proof of [BCG95, Lemma 7.5]. Without loss of generality we may reduce ourselves to the case of a closed ball $\overline{B_r(O)}$ around the origin of the Poincaré model of $\mathbb{H}^3$. Since H_n is converging almost everywhere to $I/3$ on $\mathbb{H}^3$, hence in particular on $\overline{B_r(O)}$, by Egorov theorem, given a fixed $\eta > 0$ there will exist a compact set K and $N \in \mathbb{N}$ such that $\mathrm{Vol}(\overline{B_r(O)} \setminus K) < \eta$ and

$$||H_n(x) - I/3|| < \epsilon$$

for every $n \geq N$ and every $x \in K$. Moreover we can assume that the set $\overline{B_r(O)} \setminus K$ is sufficiently small not to contain any ball of radius ϵ, for $\epsilon > 0$. This assumption implies that for every $x \in \overline{B_r(O)}$ we must have $d(x, K) < \epsilon$. Fix now ϵ, K and a suitable value $n \geq N$ so that

$$||H_n(x) - I/3|| < \epsilon$$

for every $x \in K$. As in Lemma 2.1.8 we will write $H_n(x)$ to denote the bilinear form H_n defined on $T_{F_n(x)}\mathbb{H}^3$. By contradiction, suppose the statement is false. There must exist two points $x'_n \in \overline{B_r(O)}$ and $x_n \in K$ so that $d(x_n, x'_n) < \epsilon$ and $||H_n(x'_n) - I/3|| > C_3\epsilon$, where we can assume

$$\frac{1}{3\epsilon} \geq C_3 \geq C_2(C_1 + 1) + 1$$

and C_1 and C_2 are the constants introduced in the previous lemmas.

The continuity of the function $x \to H_n(x)$ implies the existence of a point x''_n contained in the geodesic segment $[x_n, x'_n]$ such that $||H_n(x''_n) - I/3|| = C_3\epsilon$. This implies that the maximum eigenvalue of H_n satisfies $\lambda_n \leq 2/3$ at every point of the geodesic segment $[x_n, x''_n]$. By applying Lemma 2.1.7 and Lemma 2.1.8 we get that

$$||H_n(x_n) - H_n(x''_n) \circ P|| \leq C_2(C_1 + 1)\epsilon,$$

where P is the parallel transport from $F_n(x_n)$ to $F_n(x''_n)$ along the geodesic segment joining them. Since $||H_n(x_n) - I/3|| < \epsilon$ we get a contradiction. $\square$

Thus, if we consider a closed ball $\overline{B_r(O)}$ with $r > 0$, there exists $n_1 = n_1(\varepsilon, r)$ such that for $n > n_1$ we have the following estimates

$$|2/3\langle D_xF_n(v), u\rangle| - \varepsilon < |\langle K_n \circ D_xF_n(v), u\rangle|, \quad \langle H_n u, u\rangle^{\frac{1}{2}} < ||u||/\sqrt{3} + \varepsilon.$$

As a consequence of the Cauchy–Schwarz inequality, we can write

$$|\langle K_n \circ D_xF_n(v), u\rangle| \leq 2(\langle H_n(u), u\rangle)^{\frac{1}{2}} \left(\int_{\partial_\infty \mathbb{H}^3} (dB|_{(x,\theta)}(v))^2 d\mu_x(\theta)\right)^{\frac{1}{2}},$$

for every $v \in T_x\mathbb{H}^3$ and $u \in T_{F_n(x)}\mathbb{H}^3$. Hence by taking $n > n_1$ we get

$$|2/3\langle D_xF_n(v),u\rangle| - \varepsilon \leq 2(||u||/\sqrt{3}+\varepsilon)(\int_{\partial_\infty \mathbb{H}^3} (dB|_{(x,\theta)}(v))^2 d\mu_x(\theta))^{\frac{1}{2}}.$$

Recall that $||dB||^2 = 1$. By considering on both sides the supremum on all the vectors u of norm equal to 1 we get

$$||D_xF_n(v)|| < \sqrt{3}||v|| + 3\varepsilon(||v|| + 1/2)$$

Again, by taking the supremum on all the vectors $||v|| = 1$ we get

$$||D_xF_n|| < \sqrt{3} + 9/2\varepsilon$$

hence $||D_xF_n||$ is uniformly bounded on $\overline{B_r(O)}$ for any $n > n_1$ and for any choice of $r > 0$. We are now ready to prove Theorem 1 when $m = p = 3$, that is

Theorem 2.1.10 *Let Γ be a non-uniform lattice of $PSO(3,1)$ without torsion. Let $\rho_n : \Gamma \to PO(3,1)$ be a sequence of representations such that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$. Then there must exist a sequence of elements $g_n \in PO(3,1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to the standard lattice embedding $i : \Gamma \to PO(3,1)$.*

Proof. Since we know that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$, the previous computation shows that $||D_xF_n||$ must be eventually uniformly bounded on every compact set of $\mathbb{H}^3$. Let $x \in \mathbb{H}^3$ be any point and let $\gamma \in \Gamma$. Let c be the geodesic joining x to γx. Denote by $L = d(x, \gamma x)$ so that the interval $[0, L]$ parametrizes the curve c. Consider a closed ball $\overline{B_r(O)}$ sufficiently large to contain in its interior both x and γx. On this ball there must exist a constant C such that $||D_xF_n|| < C$ for n bigger than a suitable value n_0. Thus, it holds

$$d(F_n(x), F_n(\gamma x)) \leq \int_0^L ||D_{c(t)}F_n(\dot{c}(t))||dt \leq \int_0^L ||D_{c(t)}F_n||dt \leq Cd(x, \gamma x).$$

Recall that given an element $g \in PO(3,1)$ its translation length is defined as $\mathfrak{L}_{\mathbb{H}^3}(g) := \inf_{y\in\mathbb{H}^3} d(gy, y)$. The previous estimate implies that the translation length of the element $\rho_n(\gamma)$ can be bounded by

$$\mathfrak{L}_{\mathbb{H}^3}(\rho_n(\gamma)) \leq d(\rho_n(\gamma)F_n(x), F_n(x)) \leq Cd(\gamma x, x)$$

and hence the sequence ρ_n is bounded in the character variety $X(\Gamma, PO(3,1))$. In particular, we can extract a subsequence of ρ_n which pointwise converges in $X(\Gamma, PO(3,1))$. Moreover, the choice made before to fix $F_n(O) = O$ guarantees that the subsequence must converge to a true representation ρ_∞. [1] By the continuity of the volume with respect to the pointwise convergence, we get

$$\mathrm{Vol}(\rho_\infty) = \lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M).$$

By the rigidity of volume function we know that ρ_∞ must be conjugated to i. As a consequence any convergent subsequence of ρ_n converges to a representation conjugated to i and the theorem is proved. □

[1] Recall the example of the sequence $g_n \circ i \circ g_n^{-1}$, where $(g_n)_{n\in\mathbb{N}}$ is a divergent sequence of loxodromic elements $g_n \in PO(3,1)$ and $i : \Gamma \to PO(3,1)$ is the standard lattice embedding.

2.2 Consequences and generalizations

In this section we are going to state some consequences of Theorem 2.1.10 regarding the Morgan–Shalen compactification of $X(\Gamma, PO(3,1))$. We also discuss generalizations of Theorem 2.1.10 to higher dimensional cases. We begin with the proof of

Corollary 2.2.1 *Suppose $\rho_n : \Gamma \to PO(3,1)$ is a sequence of representations converging to any ideal point of the Morgan–Shalen compactification of $X(\Gamma, PO(3,1))$. Then the sequence of volumes $\mathrm{Vol}(\rho_n)$ must be bounded from above by $\mathrm{Vol}(M) - \varepsilon$ with $\varepsilon > 0$.*

Proof. If there did not exist such an ε, we should have $\mathrm{Vol}(\rho_n) \to \mathrm{Vol}(M)$, but this contradicts Theorem 2.1.10. Indeed the sequence ρ_n should converge to a representation conjugated to the standard lattice embedding $i : \Gamma \to PO(3,1)$ and it could not converge to an ideal point. □

The previous result has a clear consequence in the study of the volume function on the character variety $X(\Gamma) = X(\Gamma, PO(3,1))$. Let $\overline{X(\Gamma)}^{MS}$ be the Morgan–Shalen compactification of the character variety X. The previous corollary can be restated as follows

Corollary 2.2.2 *Let $\mathrm{Vol} : X(\Gamma) \to \mathbb{R}$ be the volume function. Let $\mathcal{N}(i)$ be a small neighborhood in $X(\Gamma)$ of the class containing the standard lattice embedding i with respect to the topology of the pointwise convergence. Suppose that there exists a continuous extension $\overline{\mathrm{Vol}} : \overline{X(\Gamma)}^{MS} \to \mathbb{R}$. Then we can bound uniformly the restriction*

$$\overline{\mathrm{Vol}}|_{\overline{X(\Gamma)}^{MS} \setminus \mathcal{N}(i)} < \mathrm{Vol}(M) - \varepsilon$$

with a suitable value of $\varepsilon > 0$.

In particular, the previous corollary proves [Gui16, Conjecture 1] and hence [Gui16, Theorem 1.2] for representations into $PSL(2, \mathbb{C})$.

Now we prove a generalization of Theorem 2.1.10 when M is a p-manifold and ρ_n takes values in $PO(p,1)$ (for $p > 3$).

More precisely, let Γ be the fundamental group of a complete hyperbolic p-dimensional manifold M with finite volume. We show that, given a sequence of representations $\rho_n : \Gamma \to PO(p,1)$ such that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$, it is possible to find a sequence of elements $g_n \in PO(p,1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to the standard lattice embedding $i : \Gamma \to PO(p,1)$. The key point of the proof in the case $p = 3$ is given by Proposition 2.1.5, which is still valid in dimension bigger or equal than 4. Indeed, following what we have done before, consider the space

$$Sym_1^+(p, \mathbb{R}) := \{H \in Sym(p, \mathbb{R}) | H > 0, \mathrm{Tr}(H) = 1\}$$

of real symmetric matrices of order p with trace equal to 1 which are positive definite. The function

$$\psi : Sym_1^+(p,\mathbb{R}) \to \mathbb{R}, \quad \psi(H) := \frac{\det(H)}{(\det(H-I))^2}$$

induces a function $\tilde{\psi}$ on the quotient $O(p,\mathbb{R})\backslash Sym_1^+(p,\mathbb{R})$, where the orthogonal group acts by conjugation. As in the case of $p=3$, denote by $\mathring{\mathbf{\Delta}}_{p-1}$ the interior of the standard $(p-1)$-simplex and consider the action of S_p by permutation of coordinates. We can read the function ψ on the space $S_p\backslash\mathring{\mathbf{\Delta}}_{p-1}$ by considering

$$\Psi(a_1,\dots,a_p) := \prod_{i=1}^{p} \frac{a_i}{(1-a_i)^2}.$$

Indeed the space $O(p,\mathbb{R})\backslash Sym_1^+(p,\mathbb{R})$ is homeomorphic to the space $S_p\backslash\mathring{\mathbf{\Delta}}_{p-1}$ and the homeomorphism is realized by sending the class of a symmetric matrix H to the non-ordered p-tuple of its eigenvalues. We are now interested in extending the function Ψ to the space $S_p\backslash\mathbf{\Delta}_{p-1}$ and to do this we are going to consider the function as defined on $\mathring{\mathbf{\Delta}}_{p-1}$. Moreover, since $\sum_{i=1}^p a_i = 1$, with an abuse of notation, we are going to rewrite Ψ as

$$\Psi(a_1,\dots,a_{p-1}) = \frac{a_1\dots a_{p-1}(1-\sum_{i=1}^{p-1} a_i)}{(1-a_1)^2\dots(1-a_{p-1})^2(\sum_{i=1}^{p-1} a_i)^2}.$$

At every point of the boundary $\partial\mathbf{\Delta}_{p-1}$ which is not a vertex, the function clearly extends with zero. The same holds for the vertex $(1,0,\dots,0)$ corresponding to the $(p-1)$-tuple $(0,0,\dots,0)$. Indeed, near $(0,0,\dots,0)$ we have

$$\Psi(a_1,\dots,a_{p-1}) \sim \frac{a_1\dots a_{p-1}}{(\sum_{i=1}^{p-1} a_i)^2} \leq \frac{(\sum_{i=1}^{p-1} a_i)^{p-3}}{(p-1)^{p-1}}$$

where the symbol $\sim$ denotes that Ψ has the same behaviour of the expression on the right. For $p \geq 4$ the right-hand side is a function which converges to zero as $(a_1,\dots,a_{p-1}) \to (0,\dots,0)$. Moreover, since the function Ψ is invariant under the action of S_p on $\mathring{\mathbf{\Delta}}_{p-1}$ we have that its continuous extension must satisfy

$$\Psi(1,0,\dots,0) = \Psi(0,1,\dots,0) = \Psi(0,0,\dots,1)$$

and so the function can be extended to zero at any vertex. In particular given a sequence of matrices H_n such that $\lim_{n\to\infty}\psi(H_n) = (p/(p-1)^2)^p$ we have that the sequence H_n must converge to I/p, where I is the identity matrix of order p. From the previous considerations and following the same strategy of the case $p=3$, it is straightforward to prove

Theorem 2.2.3 *Let Γ be a non-uniform lattice of $PSO(p,1)$ without torsion. Let $\rho_n : \Gamma \to PO(p,1)$ be a sequence of representations such that $\lim_{n\to\infty}\mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$. It is possible to find a sequence of elements $g_n \in PO(p,1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to the standard lattice embedding $i : \Gamma \to PO(p,1)$.*

From which we deduce

Corollary 2.2.4 *Suppose $\rho_n : \Gamma \to PO(p,1)$ is a sequence of representations converging to any ideal point of the Morgan–Shalen compactification of $X(\Gamma, PO(p,1))$.*

Then the sequence of volumes $\mathrm{Vol}(\rho_n)$ *must be bounded from above by* $\mathrm{Vol}(M) - \varepsilon$ *with* $\varepsilon > 0$.

We conclude by discussing the generalization of Theorem 2.1.10 to the case where M is a p-manifold and ρ_n takes values in $PO(m, 1)$ (with $m > p \geq 3$).

More precisely, let $\rho_n : \Gamma \to PO(m, 1)$ be a sequence of representations such that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$. We show that there exists a sequence $g_n \in PO(m, 1)$ such that the sequence $g_n \circ \rho_n \circ g_n^{-1}$ converges to a representation ρ_∞ which preserves a totally geodesic copy of $\mathbb{H}^p$ and whose $\mathbb{H}^p$-component is conjugated to the standard lattice embedding $i : \Gamma \to PO(p, 1) < PO(m, 1)$.

The proof in this general case follows the line of the case $p = m$ but it needs some additional care. We do not rewrite the whole proof but we only concentrate on the subtleties which differ from the previous case.

Let $F_n : \mathbb{H}^p \to \mathbb{H}^m$ be the natural map associated to the representation ρ_n. We are going to follow [BCG99] for the notation. Recall that B_M denotes the Busemann function relative to the hyperbolic space of dimension m centered at the origin O. Similarly to what we have done before, for any $n \in \mathbb{N}$ and every $x \in \mathbb{H}^p$ we define the following quadratic forms on $T_{F_n(x)}\mathbb{H}^m$:

$$\langle K_n|_{F_n(x)} u, u\rangle = \int_{\partial_\infty \mathbb{H}^p} \nabla dB_M|_{(F_n(x), D_n(\theta))}(u, u) d\mu_x(\theta)$$

$$\langle H_n|_{F_n(x)} u, u\rangle = \int_{\partial_\infty \mathbb{H}^p} (dB_M|_{(F_n(x), D_n(\theta))}(u))^2 d\mu_x(\theta)$$

for any $u \in T_{F_n(x)}\mathbb{H}^m$. Since the dimension m is bigger than p, we will need to define another quadratic form, this time on $T_x\mathbb{H}^p$. For any $v \in T_x\mathbb{H}^p$, we define

$$\langle H_n'|_x v, v\rangle = \int_{\partial_\infty \mathbb{H}^k} (dB_P|_{(x,\theta)}(v))^2 d\mu_x(\theta).$$

For any quadratic form we are going to drop the subscript which refers to the tangent space on which the form is defined. As a consequence of the Cauchy–Schwarz inequality we get

$$\langle K_n \circ D_x F_n(v), u\rangle \leq (p - 1)(\langle H_n(u), u\rangle)^{\frac{1}{2}} (\langle H_n'(v), v\rangle)^{\frac{1}{2}}$$

for every $v \in T_x\mathbb{H}^p$ and every $u \in T_{F_n(x)}\mathbb{H}^m$.

By applying the same strategy of the proof of Lemma 2.1.4, we get that the condition $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$ implies that the p-Jacobian $Jac_p(F_n)$ of the natural maps F_n converges to 1 almost everywhere with respect the measure induced by the standard hyperbolic metric on $\mathbb{H}^p$. Since

$$Jac_p(F_n)(x) := \max_{u_1,\ldots,u_p \in T_x\mathbb{H}^p} ||D_x F_n(u_1) \wedge \ldots \wedge D_x F_n(u_p)||_{g_{\mathbb{H}^m}},$$

let $\{u_1, \ldots, u_p\}$ be the frame which realizes the maximum and denote by U_x the subspace $U_x := \mathrm{span}_{\mathbb{R}}\{u_1, \ldots, u_p\}$ of $T_x\mathbb{H}^p$ (in fact the subspace U_x coincides with $T_x\mathbb{H}^p$, but we prefer to mantain the same notation of [BCG99]). Set $V_{F_n(x)} := D_x F_n(U_x)$. We denote by $K_n(x)^V$, $H_n(x)^V$ and $H_n'(x)^U$ the restrictions of the forms

$K_n|_{F_n(x)}$, $H_n|_{F_n(x)}$ and $H'_n|_x$ to the subspace $V_{F_n(x)}$, $V_{F_n(x)}$ and U_x, respectively. As a consequence of the Cauchy–Schwarz inequality, as in [BCG99] it results

$$\begin{aligned} & \det(K_n(x)^V) Jac_p(F_n)(x) \\ \leq\ & (p-1)^p (\det(H_n^V(x)))^{\frac{1}{2}} (\det(H_n'^U(x)))^{\frac{1}{2}} \\ \leq\ & p^{-\frac{p}{2}} (p-1)^p (\det(H_n^V(x)))^{\frac{1}{2}} \end{aligned}$$

and since $K_n^V(x) = I - H_n^V(x)$ we get the estimate

$$Jac_p F_n(x) \leq \frac{(p-1)^p}{p^{\frac{p}{2}}} \frac{(\det(H_n^V(x)))^{\frac{1}{2}}}{\det(I - H_n^V(x))}.$$

In this way we can apply the same strategy followed for the case $p = m$ and hence it is straightforward to prove Theorem 1 and Corollary 2.

2.3 Volume for the figure eight knot

In this section we are going to describe some experimental tests which can be considered as a numerical evidence of the validity of Corollary 2. From now until the end of the section fix $M = \mathbb{S}^3 \setminus 4_1$, where 4_1 denotes the figure eight knot. It is well-known that the complement of the knot 4_1 in the 3-sphere admits a complete hyperbolic structure of finite volume. The value of the volume is given by $\mathrm{Vol}(M) = 2.0299$. The manifold M can be ideally triangulated with only two ideal regular tetrahedra with faces identified with respect to the scheme reported in Figure 2.2. The identifications have to respect the color and the orientation of each edge. Hence, for instance, we need to flip the face labelled with A' to reverse the orientation along the red edge before identifying it with the face labelled with A.

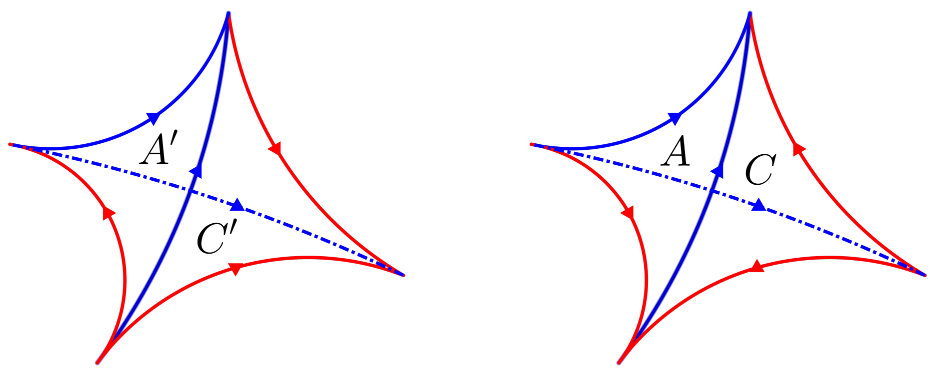

Figure 2.2: Ideal triangulation of the manifold M.

We are now interested in the study of the volume near the ideal points of the character variety $X(M, PSL(2,\mathbb{C}))$. Since we are focusing our attention on a knot manifold, we have $H^2(\pi_1(M)) \cong H^2(M) = 0$, hence all the possible representations into $PSL(2,\mathbb{C})$ can be lifted to representations into $SL(2,\mathbb{C})$. Indeed, the obstruction class to lift a generic representation $\rho : \pi_1(M) \to PSL(2,\mathbb{C})$ naturally lives in $H^2(\pi_1(M))$ by both [Cul86] and [GM93], which is trivial in our context. This allows us to study the volume function of the character variety $X(M, SL(2,\mathbb{C})) := X(\pi_1(M), SL(2,\mathbb{C}))$. We are going to compute an equation of this variety. Consider the following presentation of $\Gamma := \pi_1(M)$

$$\Gamma = \langle u, v | wv = uw, w = v^{-1}uvu^{-1} \rangle$$

which can be also rewritten as

$$\Gamma = \langle t, a, b | t^{-1}at = ab, t^{-1}bt = bab \rangle$$

where we set $t = u^{-1}, a = w$ and $b = vu^{-1}$. Following the notation of the previous chapter, given any word $\gamma \in \Gamma$, denote by Tr_γ the trace function

$$\mathrm{Tr}_\gamma : X(M, SL(2,\mathbb{C})) \to \mathbb{C}, \quad \mathrm{Tr}_\gamma(\rho) = \mathrm{Tr}(\rho(\gamma)).$$

If we now set

$$x = \mathrm{Tr}_u = \mathrm{Tr}_v, y = \mathrm{Tr}_{uv}, x_1 = \mathrm{Tr}_a, x_2 = \mathrm{Tr}_b$$

we get

$$x_2 = \mathrm{Tr}_{uv^{-1}} = \mathrm{Tr}_u \mathrm{Tr}_{v^{-1}} - \mathrm{Tr}_{uv} = x^2 - y \tag{2.1}$$

and in the same way

$$x_1 = \mathrm{Tr}_{uvu^{-1}v^{-1}} = 2x^2 + y^2 - x^2y - 2. \tag{2.2}$$

Observe that it holds

$$x_2 = \mathrm{Tr}_{t^{-1}bt} = \mathrm{Tr}_{bab} = \mathrm{Tr}_{abb} = \mathrm{Tr}_{ab}\mathrm{Tr}_b - \mathrm{Tr}_a = x_1x_2 - x_1.$$

or equivalently $x_1 + x_2 = x_1x_2$. Thus, by exploiting equations 2.1 and 2.2, we obtain

$$(x^2 - y - 2)(2x^2 + y^2 - x^2y - y - 1) = 0.$$

Hence, by considering the coordinates $x = \mathrm{Tr}_u, y = \mathrm{Tr}_{uv}$ we can write

$$X(M, SL(2,\mathbb{C})) = \{(x,y) \in \mathbb{C}^2 | (x^2 - y - 2)(2x^2 + y^2 - x^2y - y - 1) = 0\}.$$[2]

In this way we immediately understand that the variety has two irreducible components. The first one, given by equation $x^2 - y - 2 = 0$ corresponds to the classes of reducible representations. The second one instead contains the class of the holonomy of the complete hyperbolic structure of M. For this reason we are going to call this component the *hyperbolic component*. To determine the points of intersections between the two components, we easily solve the following system

[2]Compare the equation we get for the character variety $X(M, SL(2,\mathbb{C}))$ with the equation obtained in [GM93, Section 6].

$$\begin{cases} x^2 - y - 2 = 0 \\ 2x^2 + y^2 - x^2y - y - 1 = 0 \end{cases}$$

from which we get the points of coordinates $(\pm\sqrt{5}, 3)$. Consider now the embedding $i_0 : \mathbb{C}^2 \to \mathbb{P}^2(\mathbb{C})$ given by $i_0(x, y) = [1 : x : y]$. If we consider the projective closure of the image of $X(M, SL(2, \mathbb{C}))$ through the map i_0, we get two ideal points. The first point, of coordinates $[0 : 1 : 0]$ is common to both the components. The second point has coordinates $[1 : 0 : 0]$ and is an ideal point of the hyperbolic component.

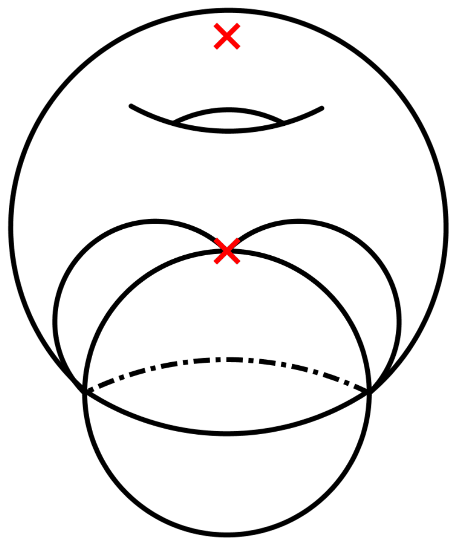

Figure 2.3: Character variety $X(M, SL(2, \mathbb{C}))$ and ideal points.

We want now to study the hyperbolic volume function around the ideal points of the hyperbolic component. Given the coordinate y_0 of a point lying in the hyperbolic component, we first need to determine a representation ρ_0 which is a representative of that point. The representation ρ_0 will be completely determined by the image of the generators u, v. Moreover, since it is an element of the hyperbolic component, we can assume the irreducibility of ρ_0 and hence we can fix as images

$$U := \rho_0(u) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \qquad V := \rho_0(v) = \begin{pmatrix} t & 0 \\ r & t^{-1} \end{pmatrix}$$

We solve the equation $(2 - y_0)x^2 + (y_0^2 - y_0 - 1) = 0$ in order to determine the possible values of x. If we choose a specific solution x_0 of the previous equation, we are able to find r, s, t by solving the following system

$$\begin{cases} s = t \\ x_0 = s + s^{-1} \\ y_0 = st + r + (st)^{-1}. \end{cases}$$

We are ready to compute the hyperbolic volume of the representation ρ_0 obtained by applying the following method. Recall that the Dirichlet domain of Γ in the Poincarè model of $\mathbb{H}^3$ has 5 ideal points corresponding to the 5-tuple of points $0, 1, \zeta_3, \zeta_3^2, \infty \in \mathbb{P}^1(\mathbb{C})$ on the Riemann sphere. Here ζ_3 corresponds to the third root of unity $(1 + i\sqrt{3})/2$. If we fix 0 as our preferred point, the other 4 can be obtained from 0 by acting with suitable elements of Γ. More precisely, it holds

$$(w^{-1}u).0 = 1, \quad (w^{-1}).0 = \infty, \quad (v^{-1}u).0 = \zeta_3, \quad (v^{-1}).0 = \zeta_3^2.$$

Consider the peripheral group of M, that is the fundamental group of its unique toric cusp. This group is generated by the meridian μ and the longitude λ of the knot 4_1. Moreover, we have

$$\mu = u^{-1}, \quad \lambda = uvu^{-1}v^{-1}u^2v^{-1}u^{-1}vu^{-1}$$

where we obtained the previous equations thanks to function `peripheral_curves(` `)` of the program `SnapPy`. Set $\mathcal{M} := \rho_0(\mu)$ and $\mathcal{L} := \rho_0(\lambda)$. Notice that, since we are interested in the study of a neighborhood of the ideal points, we can suppose that $\mathcal{M}$ is a hyperbolic matrix so that it admits two distinct fixed points on $\mathbb{P}^1(\mathbb{C})$. Thanks to the choice of the particular representative for (x_0, y_0) we can fix $0 \in \mathbb{P}^1(\mathbb{C})$ as one of the two possible points fixed by $\mathcal{M}$. If we define

$$z_1 = \rho_0(w^{-1}u).0, \quad z_\infty = \rho_0(w^{-1}).0, \quad z_x = \rho_0(v^{-1}u).0, \quad z_{xy} = \rho_0(v^{-1}).0$$

we call this choice a *decoration* for ρ_0. The decoration can be used to compute the hyperbolic volume of ρ_0. Indeed, if we denote by

$$\flat_1 = cr(0, z_1, z_\infty, z_x), \quad \flat_2 = cr(0, z_x, z_\infty, z_{xy})$$

the the hyperbolic volume $\mathrm{Vol}(\rho_0)$ can be expressed as

$$\mathrm{Vol}(\rho_0) = D_2(\flat_1) + D_2(\flat_2),$$

where D_2 is the Bloch–Wigner function introduced in the first chapter. We resume all the necessary steps to compute $\mathrm{Vol}(\rho_0)$ in **Algorithm 1**. Using this procedure to compute the volume, we can analyze the behaviour of the function around ideal points. Recall that the hyperbolic component has two ideal points of homogeneous coordinates $[1 : 0 : 0]$ and $[0 : 1 : 0]$ in $\mathbb{P}^2(\mathbb{C})$, respectively. By formally writing

$$x = \pm\sqrt{\frac{y^2 - y - 1}{2 - y}}$$

we understand that the point of coordinates $[1 : 0 : 0]$ is obtained when $y \to 2$, whereas the point $[0 : 1 : 0]$ is obtained when $y \to \infty$. Thus, if we choose any sequence $(y_n)_{n\in N}$ converging either to 2 or to ∞, we can first determine the sequence $(\rho_n)_{n\in\mathbb{N}}$. This allows us to study the evolution of the decorations associated to ρ_n and compute the associated volume $\mathrm{Vol}(\rho_n)$. In both cases, the experimental results suggest that the volume function is decresing to zero as the sequence of representations diverges to an ideal points. Indeed we conjecture that the volume function on $X(M, SL(2,\mathbb{C}))$ can be continuously extended by zero on the Morgan–Shalen compactification (which is simply the desingularization at infinity of the projective closure of $X(M, SL(2,\mathbb{C}))$). In Figure 2.4 we report an example of the evolution of a sequence of decorations as $y_n \to \infty$.

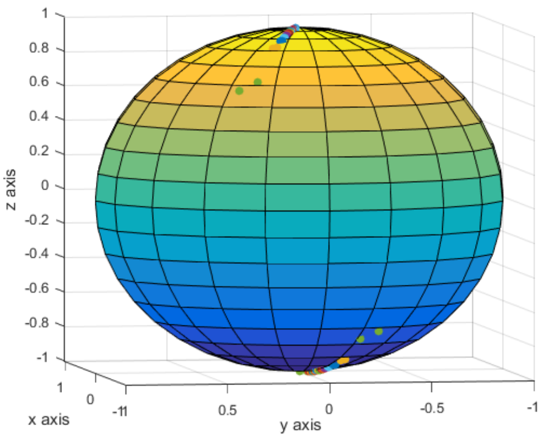

Figure 2.4: Evolution of the decoration near the ideal point $[0:1:0]$.

input : $y_0 \in \mathbb{C}$
output: $\rho_0 \in X(M, SL(2,\mathbb{C}))$, $\mathrm{Vol}(\rho_0) \in \mathbb{R}$

Solve $(2-y_0)x^2 + (y_0^2 - y_0 - 1) = 0$

Choose a solution x_0

Solve $s + s^{-1} = x_0$

Set $t = s$

Solve $st + r + (st)^{-1} = y_0$

Set

$$U = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \qquad V = \begin{pmatrix} t & 0 \\ r & t^{-1} \end{pmatrix}$$

Set $W = V^{-1}UVU^{-1}$, $X = V^{-1}U$, $Y = W^{-1}U$

Set $z_1 = \frac{Y_{12}}{Y_{22}}$, $z_\infty = \frac{(W^{-1})_{12}}{(W^{-1})_{22}}$, $z_x = \frac{X_{12}}{X_{22}}$, $z_{xy} = \frac{(Y^{-1})_{12}}{(Y^{-1})_{22}}$

Compute the cross ratios

$$\flat_1 = cr(0, z_1, z_\infty, z_x), \qquad \flat_2 = cr(0, z_x, z_\infty, z_{xy})$$

Compute the hyperbolic volume $\mathrm{Vol}(\rho_0) = D_2(\flat_1) + D_2(\flat_2)$

Algorithm 1: Hyperbolic volume of a point in $X(M, SL(2,\mathbb{C}))$ given y_0

2.4 Rigidity for complex and quaternionic lattices

In this section we will extend the ridigity results previously obtained for real lattices also to complex and quaternionic ones. We start by fixing the following setting.

- A lattice $\Gamma < G_p$ where $G_p = PU(p,1)$ or $G_p = PSp(p,1)$ so that $\Gamma\backslash X^p$ is a (non-compact) complete manifold of finite volume. Recall that X^p is the Riemannian symmetric space associated to G_p. Assume $p \geq 2$.
- A base-point $O \in X^p$ used to normalize the Busemann function $B_P(x,\theta)$, with $x \in X^p$ and $\theta \in \partial_\infty X^p$.
- The family $\{\mu_x\}$ of Patterson-Sullivan probability measures associated to Γ. Set $\mu = \mu_O$.
- A sequence of representations $\rho_n : \Gamma \to G_m$ such that $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$.

As before we easily see that the condition $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$ implies that, up to passing to a subsequence, we can suppose that no ρ_n is elementary. With an abuse of notation we still denote the subsequence of the previous lemma by ρ_n. Since no ρ_n is elementary we can consider the sequence of ρ_n-equivariant measurable maps $D_n : \partial_\infty X^p \to \partial_\infty X^m$ and the corresponding sequence of BCG–natural maps $F_n : X^p \to X^m$. Up to conjugating ρ_n by a suitable element $g_n \in G_m$, assume $F_n(O) = O$.

Definition 2.4.1. For any $n \in \mathbb{N}$ and every $x \in X^p$ we can define the following quadratic forms on $T_{F_n(x)}X^m$:

$$k_n|_{F_n(x)}(u,u) := \langle K_n|_{F_n(x)}u,u\rangle = \int_{\partial_\infty X^p} \nabla dB_M|_{(F_n(x),D_n(\theta))}(u,u)d\mu_x(\theta)$$

$$h_n|_{F_n(x)}(u,u) := \langle H_n|_{F_n(x)}u,u\rangle = \int_{\partial_\infty X^p} (dB_M|_{(F_n(x),D_n(\theta))}(u))^2 d\mu_x(\theta)$$

for any $u \in T_{F_n(x)}X^m$. The notation $\langle\cdot,\cdot\rangle$ stands for the scalar product on $T_{F_n(x)}X^m$ induced by the natural metric on X^m. Since the order m is bigger than p, we will need to define another quadratic form, this time on T_xX^p. For any $v \in T_xX^p$, we define

$$h'_n|_x(v,v) = \langle H'_n|_x v,v\rangle = \int_{\partial_\infty X^p} (dB_P|_{(x,\theta)}(v))^2 d\mu_x(\theta).$$

For any quadratic form we are going to drop the subscript which refers to the tangent space on which the form is defined. Since

$$Jac_k(F_n)(x) := \max_{u_1,\ldots,u_k \in T_xX^p} ||D_xF_n(u_1) \wedge \ldots \wedge D_xF_n(u_k)||_{X^m},$$

let $\{u_1, \ldots, u_k\}$ be the frame which realizes the maximum and denote by U_x the subspace $U_x := \mathrm{span}_{\mathbb{R}}\{u_1, \ldots, u_k\}$ of T_xX^p (since we are working with k-tuples, the subspace U_x coincides exactly with T_xX^p, but we prefer to mantain the same notation of [BCG99]). Set $V_{F_n(x)} := D_xF_n(U_x)$. We denote by $K_n^V(x)$, $H_n^V(x)$ and $H_n'^U(x)$ the restrictions of the forms $K_n|_{F_n(x)}$, $H_n|_{F_n(x)}$ and $H_n'|_x$ to the subspace $V_{F_n(x)}$, $V_{F_n(x)}$ and U_x, respectively. As consequence of the Cauchy–Schwarz inequality, as in [BCG99] it results

$$\begin{aligned} & \det(K_n(x)^V) Jac_k(F_n)(x) \\ \leq\; & (k+d-2)^k (\det(H_n^V(x)))^{\frac{1}{2}} (\det(H_n'^U(x)))^{\frac{1}{2}} \\ \leq\; & (k+d-2)^k (\det(H_n^V(x)))^{\frac{1}{2}} (\mathrm{Tr}(H_n'^U(x))/k)^{\frac{1}{2}} \\ \leq\; & k^{-\frac{k}{2}} (k+d-2)^k (\det(H_n^V(x)))^{\frac{1}{2}} \end{aligned}$$

Also in this case, the condition $\lim_{n\to\infty} \mathrm{Vol}(\rho_n) = \mathrm{Vol}(M)$ implies that $Jac_k(F_n) \to 1$ almost-everywhere on X^p with respect to the measure induced by the standard volume form. If $\mathcal{N}$ is the set of zero measure outside of which $Jac_k(F_n)$ is converging, for every $x \in X^p \setminus \mathcal{N}$ and fixed $\varepsilon > 0$ there must exist $n_0 = n_0(\varepsilon, x)$ such that $Jac_k(F_n) \geq 1 - \varepsilon$ for every $n > n_0$. Thus it holds

$$\left(\frac{(k+d-2)^2}{k} \right)^{\frac{k}{2}} \frac{\det(H_n^V)^{\frac{1}{2}}}{\det(K_n^V)} > 1 - \varepsilon$$

from which we can deduce

$$\frac{\det(H_n^V)}{(\det(K_n^V))^2} > \left(\frac{k}{(k+d-2)^2} \right)^k (1-\varepsilon)^2 > \left(\frac{k}{(k+d-2)^2} \right)^k (1-2\varepsilon).$$

This time X^p has sectional curvature which varies between -4 and -1, hence we can write $K_n^V = I - H_n^V - \sum_{i=1}^{d-1} J_i H_n^V J_i$, where $J_i(x)$ are orthogonal endomorphisms used to define the complex or the quaternionic structure on $T_{F_n(x)}X^m$ (see [BCG95]). Recall that $J_i^2 = -I$ at every point. Hence, by substituting the expression of K_n in the previous inequality, we get

$$\frac{\det(H_n^V)}{(\det(I - H_n^V - \sum_{i=1}^{d-1} J_i H_n^V J_i))^2} > \left(\frac{k}{(k+d-2)^2} \right)^k (1-2\varepsilon).$$

As done previously, once we have fixed a basis of $V_{F_n(x)}$, we can identify H_n^V, K_n^V and J_i with the matrices representing these endomorphisms with respect to the fixed basis. Under this assumption, recall that $H_n \in Sym_1^+(k, \mathbb{R})$ for every $n \in \mathbb{N}$, as shown in [BCG95]. If we define

$$\varphi : Sym_1^+(k, \mathbb{R}) \to \mathbb{R}, \quad \varphi(H) := \frac{\det(H)}{(\det(I - H - \sum_{i=1}^{d-1} J_i H J_i))^2},$$

we know that

$$\varphi(H) \leq \left(\frac{k}{(k+d-2)^2} \right)^k$$

and the equality holds if and only if $H = I/k$ (see [BCG95, Appendix B]).

Proposition 2.4.2 *Suppose to have a sequence $H_n \in Sym_1^+(k, \mathbb{R})$ such that*

$$\lim_{n\to\infty} \varphi(H_n) = \left(\frac{k}{(k+d-2)^2}\right)^k.$$

Then the sequence H_n must converge to I/k.

Proof. We are not going to work directly on the function φ but we will use the auxiliary function

$$\psi(H) := \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{\det(H)^{\frac{k-d}{k+d-2}}}{\det(I-H)^{\frac{2(k-1)}{k+d-2}}}.$$

By [BCG95, Lemme B.3], for every $H \in Sym_1^+(k, \mathbb{R})$ we have that $\varphi(H) \leq \psi(H)$. Moreover both functions attain the same maximum value

$$\max_{H \in Sym_1^+(k,\mathbb{R})} \varphi = \max_{H \in Sym_1^+(k,\mathbb{R})} \psi = \left(\frac{k}{(k+d-2)^2}\right)^k$$

at $H = I/k$.

We are going to study the properties of the function ψ. We start by observing that the function ψ is invariant by conjugation for an element $g \in GL(k, \mathbb{R})$. Indeed, $\psi(H)$ can be expressed as

$$\psi(H) = \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{p_H(0)^{\frac{k-d}{k+d-2}}}{p_H(1)^{\frac{2(k-1)}{k+d-2}}},$$

where p_H is the characteristic polynomial of H. Hence the claim follows. In particular, we have an induced function

$$\tilde{\psi} : O(k, \mathbb{R})\backslash Sym_1^+(k, \mathbb{R}) \to \mathbb{R}, \quad \tilde{\psi}(\bar{H}) = \psi(H),$$

where $\bar{H}$ denotes the equivalence class of the matrix H and the orthogonal group $O(k, \mathbb{R})$ acts on $Sym_1^+(k, \mathbb{R})$ by conjugation. As before, we can think of the space $O(k, \mathbb{R})\backslash Sym_1^+(k, \mathbb{R})$ as the interior $\mathring{\mathbf{\Delta}}_{k-1}$ of the standard $(k-1)$-simplex quotiented by the action of the symmetric group S_k which permutes the coordinates of an element $(\lambda_1, \ldots, \lambda_k) \in \mathring{\mathbf{\Delta}}_{k-1}$. By defining $\Psi = \psi \circ \Lambda^{-1}$, we can express this function as

$$\Psi : S_k\backslash\mathring{\mathbf{\Delta}}_{k-1} \to \mathbb{R}, \quad \Psi([a_1, \ldots, a_k]) = \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \prod_{i=1}^{k} \frac{(a_i)^{\frac{k-d}{k+d-2}}}{(1-a_i)^{\frac{2(k-1)}{k+d-2}}}.$$

We are going to think of Ψ as defined on $\mathring{\mathbf{\Delta}}_{k-1}$ and we are going to estimate this function on the boundary of $\mathbf{\Delta}_{k-1}$. Since $\sum_{i=1}^k a_i = 1$, with an abuse of notation we will write

$$\Psi(a_1,\dots,a_{k-1}) = \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{(a_1\dots a_{k-1}(1-\sum_{i=1}^{k-1} a_i))^{\frac{k-d}{k+d-2}}}{((1-a_1)\dots(1-a_{k-1})(\sum_{i=1}^{k-1} a_i))^{\frac{2(k-1)}{k+d-2}}}.$$

identifying $\mathring{\mathbf{\Delta}}_{k-1}$ with the interior of the simplex τ in $\mathbb{R}^{k-1}$ whose vertices are the origin $(0,0,\dots,0)$ and the vectors $e_i=(0,\dots,0,1,0,\dots,0)$ of the canonical basis, for $i=1,\dots,k-1$. If a sequence of points is converging to a boundary point of $\mathbf{\Delta}_{k-1}$, then we have a sequence $\{(a_1^{(n)},\dots,a_{k-1}^{(n)})\}_{n\in\mathbb{N}}$ of points in τ converging to a boundary point. If the limit point is not a vertex of τ then $\lim_{n\to\infty}\Psi(a_1^{(n)},\dots,a_{k-1}^{(n)})=0$. For instance, suppose

$$\lim_{n\to\infty}(a_1^{(n)},\dots,a_{k-1}^{(n)}) = (\alpha,0,\dots,0)$$

with $\alpha\neq 0,1$. Hence

$$\begin{aligned}&\lim_{n\to\infty}\Psi(a_1^{(n)},\dots,a_{k-1}^{(n)}) = \\ &\lim_{n\to\infty}\frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{(a_1^{(n)}\dots a_{k-1}^{(n)}(1-\sum_{i=1}^{k-1} a_i^{(n)}))^{\frac{k-d}{k+d-2}}}{((1-a_1^{(n)})\dots(1-a_{k-1}^{(n)})(\sum_{i=1}^{k-1} a_i^{(n)}))^{\frac{2(k-1)}{k+d-2}}} = 0\end{aligned}$$

as claimed. For the other boundary points which are not vertices, the computation is the same. The delicate points are given by the vertices of τ. On these points the function Ψ a priori cannot be continuously extended. Suppose to have a sequence $\{(a_1^{(n)},\dots,a_{k-1}^{(n)})\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}(a_1^{(n)},\dots,a_{k-1}^{(n)})=(0,0,\dots,0)$. We have

$$\Psi(a_1^{(n)},\dots,a_{k-1}^{(n)}) = \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{(a_1^{(n)}\dots a_{k-1}^{(n)}(1-\sum_{i=1}^{k-1} a_i^{(n)}))^{\frac{k-d}{k+d-2}}}{((1-a_1^{(n)})\dots(1-a_{k-1}^{(n)})(\sum_{i=1}^{k-1} a_i^{(n)}))^{\frac{2(k-1)}{k+d-2}}},$$

and since we are in a neighborhood of $(0,0,\dots,0)$ the sequence $\Psi(a_1^{(n)},\dots,a_{k-1}^{(n)})$ will have the same behaviour of the following sequence

$$\Psi(a_1^{(n)},\dots,a_{k-1}^{(n)}) \sim \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{(a_1^{(n)}\dots a_{k-1}^{(n)})^{\frac{k-d}{k+d-2}}}{(\sum_{i=1}^{k-1} a_i^{(n)})^{\frac{2(k-1)}{k+d-2}}}.$$

By looking carefully to the right-hand side of the inequality, we can estimate it as follows

$$\begin{aligned}&\frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{(a_1^{(n)}\dots a_{k-1}^{(n)})^{\frac{k-d}{k+d-2}}}{(\sum_{i=1}^{k-1} a_i^{(n)})^{\frac{2(k-1)}{k+d-2}}} \\ &\leq \frac{(k-1)^{\frac{2k(k-1)}{k+d-2}}}{(k+d-2)^{2k}} \frac{1}{(\sum_{i=1}^{k-1} a_i^{(n)})^{\frac{2(k-1)}{k+d-2}}} \left(\frac{(\sum_{i=1}^{k-1} a_i^{(n)})^{k-1}}{(k-1)^{k-1}}\right)^{\frac{k-d}{k+d-2}} \\ &= \frac{(k-1)^{\frac{(k-1)(k+d)}{k+d-2}}}{(k+d-2)^{2k}} (\sum_{i=1}^{k-1} a_i^{(n)})^{\frac{(k-1)(k-d-2)}{k+d-2}}.\end{aligned}$$

The last term which appears in the inequality above depends on the exponent $k-d-2$. More precisely, by the assumption $p \geq 2$ we already know that $k \geq d+2$, but we need to distinguish the case $k = d+2$ from the case $k > d+2$. Since we assumed either $G_p = PU(p,1)$ or $G_p = PSp(p,1)$, we can have either $d = 2$ or $d = 4$. Thus, if $k = d+2$, we should have $k = 4$ or $k = 6$. The cases $k = 6$ is not possible because the dimension of the tangent space of a quaternionic hyperbolic space is a multiple of 4, so we are going to analyze only the case $k = 4$. When $k = 4$, the space X^p becomes the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ and we get the estimate

$$\Psi(0,\dots,0) \leq \frac{3^{\frac{9}{2}}}{4^8}$$

which is stricly less then the maximum of Ψ. When $k > d+2$ the right-hand side of the inequality becomes a function which is continuous at $(0,\dots,0)$ and it converges to 0. Hence, in all the possible cases, we can bound $\Psi(a_1,\dots,a_{k-1})$ away from its maximum in a suitable neighborhood of the boundary $\partial\mathbf{\Delta}_{k-1}$. Moreover, since $\Psi(a_1,a_2,\dots,a_k)$ is a function which is invariant under the action of the group S_k, we get

$$\Psi(1,0,0,\dots,0) = \Psi(0,1,0,\dots,0) = \dots = \Psi(0,0,\dots,0,1),$$

and the claim follows because $\varphi(H) \leq \psi(H)$ for every $H \in Sym_1^+(k,\mathbb{R})$. $\square$

We know that in our context we have

$$\left(\frac{k}{(k+d-2)^2}\right)^k (1-2\varepsilon) \leq \varphi(H_n^V) \leq \left(\frac{k}{(k+d-2)^2}\right)^k$$

for $n \geq n_0$. As a consequence of Proposition 2.4.2, the sequence H_n^V must converge to I/k. Hence H_n^V converges to I/k almost-everywhere on X^p. By following the same proof of Proposition 2.1.9 we have

Proposition 2.4.3 *Suppose the sequence H_n^V converges almost everywhere to I/k. Thus it converges uniformly to I/k on every compact set of X^p.*

As a consequence of the Cauchy–Schwarz inequality, we can write

$$|k_n(v, D_xF(u))| \leq (k+d-2)h_n(v,v)^{\frac{1}{2}}h_n'(u,u)^{\frac{1}{2}},$$

for every $u \in T_xX^p$ and $v \in T_{F_n(x)}X^m$. Fix $r > 0$ and consider $\overline{B_r(O)}$ as compact set of X^p. By Proposition 2.4.3, we have that $\lim_{n\to\infty} H_n^V(x) = I/k$ for every x uniformly on $\overline{B_r(O)}$. This implies that

$$\lim_{n\to\infty} K_n^V(x) = \frac{k+d-2}{k}I, \qquad \lim_{n\to\infty} H_n^{'U}(x) = \frac{1}{k}I.$$

Hence by taking $n > n_1$, $u \in U_x$ and $v = D_xF_n(u)$ we get

$$(k+d-2)/k||D_xF_n(u)||^2_{X^m} - \varepsilon \leq (k+d-2)(||D_xF_n(u)||_{X^m}/\sqrt{k}+\varepsilon)(||u||_{X^p}/\sqrt{k}+\varepsilon).$$

By considering on both sides the supremum on all the vectors u of norm equal to 1 we get

$$||D_xF_n||^2 < k(||D_xF_n||/\sqrt{k} + \varepsilon)(1/\sqrt{k} + \varepsilon)$$

hence $||D_xF_n||$ is uniformly bounded on $\overline{B_r(O)}$ for any $n > n_1$ and for any choice of $r > 0$. Hence, by following the same strategy of the proof of Theorem 1, it is straightforward to prove both Theorem 3 and Theorem 4.

Chapter 3

The ω-Borel invariant

3.1 The cocycle Vol^ω

Fix an ultrafilter ω on $\mathbb{N}$ and a real divergent sequence $(\lambda_l)_{l\in\mathbb{N}}$. Recall that these data allow us to construct the field $\mathbb{C}_\omega$. This chapter is devoted to the introduction of the ω-Borel invariant for representations $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$, where Γ is a non-uniform torsion free lattice of $PSL(2,\mathbb{C})$.

From now until the end of the chapter we will consider the spaces $\mathbb{P}^1(\mathbb{C})_\omega$ and $\mathbb{P}^1(\mathbb{C}_\omega)$ identified, hence we will refer to any of these two as they were the same space. The same will be done also for the groups $SL(n,\mathbb{C})_\omega$ and $SL(n,\mathbb{C}_\omega)$. Moreover, to avoid a heavy notation we are going to refer to any sequence $(x_l)_{l\in\mathbb{N}}$ by dropping the parenthesis every time that we are considering the sequence itself instead of any of its single term.

In this section we are going to construct a generalization of the hyperbolic volume function which will live on $\mathbb{P}^1(\mathbb{C}_\omega)^4$. This generalization will reveal the fundamental tool to define the ω-Borel cocycle.

Before starting, we want to underline a delicate point. Since we want to exploit the properties of the standard Borel cocycle, one could try to define the new function Vol^ω simply by taking the ω-limit of the volumes, that is $\mathrm{Vol}^\omega(x^0_\omega,\ldots,x^3_\omega) = \omega\text{-}\lim_{l\to\infty}\mathrm{Vol}(x^0_l,\ldots,x^3_l)$, where x^i_l is any representative of x^i_ω. Unfortunately this definition is not correct. Indeed, if we suppose to have 3 points that coincide, say $x^0_\omega = x^1_\omega = x^2_\omega$, different choices of representatives lead to different values of the ω-limit of their volumes. Hence, we need to be careful.

Let $\mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$ be the space of 4-tuples of distinct points on $\mathbb{P}^1(\mathbb{C}_\omega)$. As in the standard case, there is a natural cross ratio function

$$cr_\omega : \mathbb{P}^1(\mathbb{C}_\omega)^{(4)} \to \mathbb{C}_\omega \setminus \{0,1\}, \quad cr_\omega(x^0_\omega,x^1_\omega,x^2_\omega,x^3_\omega) = \frac{(x^0_\omega - x^2_\omega)(x^1_\omega - x^3_\omega)}{(x^0_\omega - x^3_\omega)(x^1_\omega - x^2_\omega)},$$

which is well defined by its purely algebraic nature. Every x^i_ω may be considered in $\mathbb{C}_\omega$ or equal to ∞. Recall the definition of the Bloch–Wigner function by

$$D_2 : \mathbb{C} \to \mathbb{R}, \quad D_2(z) := \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|,$$

see Definition 1.1.17. By still denoting D_2 its continuous extension on $\mathbb{P}^1(\mathbb{C})$, we can formulate the following

Definition 3.1.1. The *ω-Bloch–Wigner function* is given by

$$D_2^\omega : \mathbb{C}_\omega \cup \{\infty\} \to \mathbb{R}, \quad D_2^\omega(x_\omega) := \omega\text{-}\lim_{l\to\infty} D_2(x_l) \text{ for } x_\omega \in \mathbb{C}_\omega \text{ and } D_2^\omega(\infty) := 0.$$

where x_l is any representative of the equivalence class x_ω.

Lemma 3.1.2 *If x_l and y_l are two sequences representing the same element in $\mathbb{C}_\omega$, then*

$$\omega\text{-}\lim_{l\to\infty} D_2(x_l) = \omega\text{-}\lim_{l\to\infty} D_2(y_l).$$

Proof. Since $\mathbb{P}^1(\mathbb{C})$ is compact and $\omega\text{-}\lim_{l\to\infty} |x_l - y_l|^{\frac{1}{\lambda_l}} = 0$, both sequences x_l and y_l will converge to the same limit in $\mathbb{C} \cup \{\infty\}$. Denote by ξ this point. As a consequence of Proposition 1.3.15 and by the continuity of D_2 we have

$$\omega\text{-}\lim_{l\to\infty} D_2(x_l) = D_2(\omega\text{-}\lim_{l\to\infty} x_l) = D_2(\xi) = D_2(\omega\text{-}\lim_{l\to\infty} y_l) = \omega\text{-}\lim_{l\to\infty} D_2(y_l),$$

as claimed. □

The previous lemma guarantees that the definition of the ω-Bloch–Wigner function is correct since it does not depend on the choice of the representative of the class x_ω.

Definition 3.1.3. The *ω-volume function* for a 4-tuple of points $(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^4$ is defined as

$$\mathrm{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) = \begin{cases} D_2^\omega(cr_\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3)) \text{ if } (x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^{(4)}, \\ 0 \text{ otherwise.} \end{cases}$$

Remark 3.1.4. Mantaining the notation of Chapter 1, we denote by Vol the composition $D_2 \circ cr$, where D_2 is the standard Bloch–Wigner function and cr is the cross ratio on $\mathbb{P}^1(\mathbb{C})$. Fix a 4-tuple $(x_\omega^0, \dots, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^4$ of distinct points. Thanks to the natural identification between $\mathbb{P}^1(\mathbb{C}_\omega)$ and $\mathbb{P}^1(\mathbb{C})_\omega$, we can think of each x_ω^i as the class of a sequence x_l^i of points in $\mathbb{P}^1(\mathbb{C})$. Now, it easy to see that

$$cr_\omega(x_\omega^0, \dots, x_\omega^3) = [cr(x_l^0, \dots, x_l^3)]$$

in $\mathbb{C}_\omega$ (if the x_ω^i are all distinct, also the terms of the sequences x_l^i are distinct ω-almost every $l \in \mathbb{N}$). By exploiting the previous identity, we can rewrite the definition of Vol^ω as follows

$$\begin{aligned}\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) &= D_2^\omega(cr_\omega(x_\omega^0, \dots, x_\omega^3)) = \omega\text{-}\lim_{l\to\infty} D_2(cr(x_l^0, \dots, x_l^3)) \\ &= \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3),\end{aligned}$$

and this is completely independent of the choice of representatives $x_l^0, \dots, x_l^3$. Hence Vol^ω coincides with the ω-limit of the standard volumes $\mathrm{Vol}(x_l^0, \dots, x_l^3)$ on a 4-tuple $(x_\omega^0, \dots, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$, where x_l^i is any representative for x_ω^i. Even though we have already underlined that this is not true on the whole space $\mathbb{P}^1(\mathbb{C}_\omega)^4$, we can always choose a suitable representative for each x_ω^i such that

$$\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3).$$

Proposition 3.1.5 *The function* Vol^ω *is a bounded, alternating,* $GL(2,\mathbb{C}_\omega)$*-invariant cocycle.*

Proof. Most of the properties we stated follow directly from the properties of the standard volume function Vol. We start by showing the $GL(2,\mathbb{C}_\omega)$-invariance. From now until the end of the proof we are going to pick suitable representative sequences for points in $\mathbb{P}^1(\mathbb{C}_\omega)$ such that

$$\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3).$$

Let $g_\omega \in GL(2,\mathbb{C}_\omega)$. We want to show that $g_\omega.\mathrm{Vol}^\omega = \mathrm{Vol}^\omega$.

$$g_\omega.\mathrm{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) = \mathrm{Vol}^\omega(g_\omega^{-1}.x_\omega^0, \dots, g_\omega^{-1}.x_\omega^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(g_l^{-1}.x_l^0, \dots, g_l^{-1}.x_l^3)$$

and thanks to the equivariance of the classic volume function we get

$$\omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(g_l^{-1}.x_l^0, \dots, g_l^{-1}.x_l^3) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, x_l^3) = \mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^3),$$

as required. The strategy to prove the alternating property and the cocycle property of Vol^ω is the same as above. Let $\sigma \in S_3$. It holds

$$\sigma.\mathrm{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) = \mathrm{Vol}^\omega(x_\omega^{\sigma(0)}, x_\omega^{\sigma(1)}, x_\omega^{\sigma(2)}, x_\omega^{\sigma(3)}) = \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^{\sigma(0)}, x_l^{\sigma(1)}, x_l^{\sigma(2)}, x_l^{\sigma(3)})$$
$$= \omega\text{-}\lim_{l\to\infty} \mathrm{sgn}(\sigma)\mathrm{Vol}(x_l^0, x_l^1, x_l^2, x_l^3) = \mathrm{sgn}(\sigma)\mathrm{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3).$$

In an analogous way, we have

$$\delta\mathrm{Vol}^\omega(x_\omega^0, \dots, x_\omega^4) = \sum_{i=0}^4 (-1)^i \mathrm{Vol}^\omega(x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^4) =$$

$$\sum_{i=0}^4 (-1)^i \omega\text{-}\lim_{l\to\infty} \mathrm{Vol}(x_l^0, \dots, \hat{x}_l^i, \dots, x_l^4) = \omega\text{-}\lim_{l\to\infty} \sum_{i=0}^4 (-1)^i \mathrm{Vol}(x_l^0, \dots, \hat{x}_l^i, \dots, x_l^4) = 0.$$

Finally, the boundedness is obvious since the ω-Bloch–Wigner is nothing more than the ω-limit of a sequence of real values all bounded by ν_3 on $\mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$ and it coincides with 0 on the complementary. Here ν_3 is the volume of a regular ideal hyperbolic tetrahedron in $\mathbb{H}^3$.

□

3.2 The cocycle B_n^ω

In order to define the ω-Borel invariant for a representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$, we first need to define the ω-Borel cocycle. We are going to follow the same construction exposed in [BBI, Section 3]. Let $\mathfrak{S}_k^\omega(m)$ be the following space

$$\mathfrak{S}_k^\omega(m) := \{(x_\omega^0, \dots, x_\omega^k) \in (\mathbb{C}_\omega^m)^{k+1} | \langle x_\omega^0, \dots x_\omega^k \rangle = \mathbb{C}_\omega^m\}/GL(m, \mathbb{C}_\omega)$$

where $GL(m, \mathbb{C}_\omega)$ acts on $(k+1)$-tuples of vectors by the diagonal action and $\langle x_\omega^0, \dots x_\omega^k \rangle$ is the $\mathbb{C}_\omega$-linear space generated by $x_\omega^0, \dots, x_\omega^k$. It obvious that if $k < m-1$ the space defined above is empty. For every m-dimensional vector space V over $\mathbb{C}_\omega$ and any $(k+1)$-tuple of spanning vectors $(x_\omega^0, \dots, x_\omega^k) \in V^{k+1}$, we choose an isomorphism $V \to \mathbb{C}_\omega^m$. Since any two different choices of isomorphisms are related by an element $g_\omega \in GL(m, \mathbb{C}_\omega)$, we get a well defined element of $\mathfrak{S}_k^\omega(m)$ which will be denoted by $[V; (x_\omega^0, \dots, x_\omega^k)]$. For

$$\mathfrak{S}_k^\omega := \bigsqcup_{m \geq 0} \mathfrak{S}_k^\omega(m) = \mathfrak{S}_k^\omega(0) \sqcup \dots \sqcup \mathfrak{S}_k^\omega(k+1)$$

we have two different face maps $\varepsilon_i^{(k)}, \eta_i^{(k)} : \mathfrak{S}_k^\omega \to \mathfrak{S}_{k-1}^\omega$ given by

$$\begin{aligned} \varepsilon_i^{(k)}[\mathbb{C}_\omega^m; (x_\omega^0, \dots, x_\omega^k)] &:= [\langle x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k \rangle; (x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k)], \\ \eta_i^{(k)}[\mathbb{C}_\omega^m; (x_\omega^0, \dots, x_\omega^k)] &:= [\mathbb{C}_\omega^m / \langle x_\omega^i \rangle; (x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k)]. \end{aligned}$$

As in [BBI], it is straightforward to prove

Lemma 3.2.1 *For all $0 \leq i < j \leq k$ the maps introduced above satisfy the following relations*

$$\begin{aligned} \varepsilon_j^{(k-1)} \varepsilon_i^{(k)} &= \varepsilon_i^{(k-1)} \varepsilon_{j+1}^{(k)}, \\ \eta_j^{(k-1)} \eta_i^{(k)} &= \eta_i^{(k-1)} \eta_{j+1}^{(k)}, \\ \eta_j^{(k-1)} \varepsilon_i^{(k)} &= \varepsilon_i^{(k-1)} \eta_{j+1}^{(k)}. \end{aligned}$$

We now define the operator

$$D_k : \mathbb{Z}[\mathfrak{S}_k^\omega] \to \mathbb{Z}[\mathfrak{S}_{k-1}^\omega], \quad D_k(\sigma) := \sum_{i=0}^k (-1)^i (\varepsilon_i^{(k)}(\sigma) - \eta_i^{(k)}(\sigma)),$$

where $\mathbb{Z}[\mathfrak{S}_k^\omega]$ is the free abelian group generated by $\mathfrak{S}_k^\omega$ and it is equal to 0 for $k \leq -1$. We still denote by $\varepsilon_i^{(k)}$ and $\eta_i^{(k)}$ the linear extensions of face maps to $\mathbb{Z}[\mathfrak{S}_k^\omega]$. As a consequence of Lemma 3.2.1 we get the condition $D_{k-1} \circ D_k = 0$. In this way we have constructed a chain complex $(\mathbb{Z}[\mathfrak{S}_\bullet^\omega], D_\bullet)$. With the purpose of dualizing this complex, we recall that we have a natural action of the symmetric group S_{k+1} on $\mathfrak{S}_k^\omega$, hence we can define

$$\mathbb{R}_{\mathrm{alt}}(\mathfrak{S}_k^\omega) := \{f : \mathfrak{S}_k^\omega \to \mathbb{R} | f \text{ is alternating with respect to the } S_{k+1}\text{-action}\}$$

and we can define D_k^* as the dual of $D_k \otimes id_{\mathbb{R}}$. The construction above produces a cochain complex $(\mathbb{R}_{\mathrm{alt}}(\mathfrak{S}^\omega_\bullet), D^*_\bullet)$.

We are going now to define a cocycle living in $\mathbb{R}_{\mathrm{alt}}(\mathfrak{S}^\omega_3)$ which will be used to construct the ω-Borel cocycle. Since the ω-volume function Vol^ω introduced in the previous section can be thought of as defined on $(\mathbb{C}^2_\omega \setminus \{0\})^4$, it is extendable to

$$\mathrm{Vol}^\omega : \mathfrak{S}^\omega_3 \to \mathbb{R}$$

where we set $\mathrm{Vol}^\omega|\mathfrak{S}^\omega_3(m)$ to be identically zero if $m \neq 2$ and

$$\mathrm{Vol}^\omega[\mathbb{C}^2_\omega; (v^0_\omega, \ldots, v^3_\omega)] := \begin{cases} \mathrm{Vol}^\omega(v^0_\omega, \ldots, v^3_\omega) & \text{if each } v^i_\omega \neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

By the compatibilty of the ω-limit with respect to finite sums, it should be clear that

Proposition 3.2.2 *The function* $\mathrm{Vol}^\omega \in \mathbb{R}_{\mathrm{alt}}(\mathfrak{S}^\omega_3)$ *is a cocycle, that is it holds* $D^*_4(\mathrm{Vol}^\omega) = 0$.

Since the proof of this proposition is the same as [BBI, Lemma 8, Lemma 9] we omit it. In order to define the ω-Borel cocyle we are going to introduce the spaces of affine flags in $\mathbb{C}^n_\omega$. A complete flag F_ω in $\mathbb{C}^n_\omega$ is a sequence of linear subspaces

$$F^0_\omega \subset F^1_\omega \subset \ldots \subset F^n_\omega$$

such that every F^i_ω has dimension i as $\mathbb{C}_\omega$-vector space. An affine flag (F_ω, v_ω) is a complete flag F_ω together with an n-tuple of vectors $v_\omega = (v^1_\omega, \ldots, v^n_\omega) \in (\mathbb{C}^n_\omega)^n$ such that

$$F^i_\omega = \mathbb{C}_\omega v^i_\omega + F^{i-1}_\omega, \quad i \geq 1.$$

It is clear that the group $GL(n, \mathbb{C}_\omega)$ acts naturally on the space of flags $\mathscr{F}(n, \mathbb{C}_\omega)$ and on the space of affine flags $\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)$ of $\mathbb{C}^n_\omega$. Let $\mathbb{Z}[\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1}]$ be the abelian group generated by $\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1}$ and let ∂_k be the standard boundary map induced by the face maps $\varepsilon^{(k)}_i : \mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1} \to \mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^k$ consisting in dropping the i^{th}-component for $1 \leq k \leq n-1$. Moreover set $\partial_0 : \mathbb{Z}[\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)] \to 0$. We are ready now to define

$$T_k : (\mathbb{Z}[\mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^k], \partial_k) \to (\mathbb{Z}[\mathfrak{S}^\omega_k], D_k)$$

which will enable us to construct a morphism between the dual of the complexes above (more precisely on their alternating versions). Given a multi-index $\mathbf{J} \in \{0, 1, \ldots, n-1\}^{k+1}$, we start by defining

$$\tau_{\mathbf{J}} : \mathscr{F}_{\mathrm{aff}}(n, \mathbb{C}_\omega)^{k+1} \to \mathfrak{S}^\omega_k$$

as the function

$$\tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \ldots, (F_{k,\omega}, v_{k,\omega})) := \left[\frac{\langle F^{j_0+1}_{0,\omega}, \ldots, F^{j_k+1}_{k,\omega} \rangle}{\langle F^{j_0}_{0,\omega}, \ldots, F^{j_k}_{k,\omega} \rangle}; (v^{j_0+1}_{0,\omega}, \ldots, v^{j_k+1}_{k,\omega}) \right]$$

and finally

$$T_k((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega})):=\sum_{\mathbf{J}\in\{0,\dots,n-1\}^{k+1}}\tau_{\mathbf{J}}((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega})).$$

Lemma 3.2.3 *For $k\geq 1$ we it holds*

- *If k is odd, then $T_{k-1}\circ\partial_k - D_k\circ T_k = 0$.*
- *If k is even, then $T_{k-1}\circ\partial_k - D_k\circ T_k = n^k[0;(0,\dots,0)]$ where $n^k[0;(0,\dots,0)]\in\mathbb{Z}[\mathfrak{S}^\omega_{k-1}(0)]$.*

Proof. We still denote by $\varepsilon_i^{(k)}$ and $\eta_i^{(k)}$ the face maps of Lemma 3.2.1. For every $0\leq i\leq k$ and every multi-index $\mathbf{J}\in\{0,\dots,n-1\}^{k+1}$ we have the following relations

(a) If $j_i\leq n-2$ then $\eta_i^{(k)}\circ\tau_{\mathbf{J}}=\varepsilon_i^{(k)}\circ\tau_{\mathbf{J}+\delta_i}$, where $\delta_i=(0,\dots,0,1,0,\dots,0)$ has the only entry equal to one in the i-th position.

(b) If $j_i=n-1$ then $\eta_i^{(k)}\circ\tau_{\mathbf{J}}((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega}))=[0;(0,\dots,0)]$.

(c) If $j_i=0$ then $\varepsilon_i^{(k)}\circ\tau_{\mathbf{J}}=T_{\mathbf{J}(i)}\circ\varepsilon_i^{(k)}$, where $\mathbf{J}(i)\in\{0,\dots,n-1\}^k$ is obtained from $\mathbf{J}$ by dropping j_i.

We can now evaluate

$$\begin{aligned}D_kT_k((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega}))=\sum_{i=0}^k(-1)^i\Big(&\sum_{\mathbf{J}}\varepsilon_i^{(k)}\tau_{\mathbf{J}}((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega}))\\&-\sum_{\mathbf{J}}\eta_i^{(k)}\tau_{\mathbf{J}}((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega}))\Big).\end{aligned}$$

We can now split the first inner sum into a first sum over all the possible multi-indices $\mathbf{J}\in\{0,\dots,n-1\}^{k+1}$ such that $j_i=0$ while the second sum will be over all the possible $\mathbf{J}\in\{0,\dots,n-1\}^{k+1}$ with $j_i\geq 1$. By exploiting relation (c) the first contribution results equal to $T_{k-1}\circ\varepsilon_i^{(k)}((F_{0,\omega},v_{0,\omega}),\dots,(F_{k,\omega},v_{k,\omega}))$. By applying relations (a) and (b), the second contribution together with the second inner sum gives us back $-n^k[0;(0,\dots,0)]$. $\square$

If we now recall that there exists a natural action of S_{k+1} on $\mathscr{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1}$ and dualize the complex considered so far, we get the cocomplex $(\mathbb{R}_{\mathrm{alt}}(\mathscr{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1}),\partial_k^*)$ of alternating cochains (∂_k^* is the dual of $\partial_k\otimes id_{\mathbb{R}}$). By denoting T_k^* the dual map of $T_k\otimes id_{\mathbb{R}}$, by Lemma 3.2.3 that T_k^* is a morphism a complexes taking values in $(\mathbb{R}_{\mathrm{alt}}(\mathscr{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^{k+1}))^{GL(n,\mathbb{C}_\omega)}$.

Definition 3.2.4. We define the ω-*Borel function* of degree n as

$$\begin{aligned}&B_n^\omega((F_{0,\omega},v_{0,\omega}),\dots,(F_{3,\omega},v_{3,\omega})):=T_3^*(\mathrm{Vol}^\omega)=\\&=\sum_{\mathbf{J}\in\{0,\dots,n-1\}^4}\mathrm{Vol}^\omega\left[\frac{\langle F_{0,\omega}^{j_0+1},\dots,F_{3,\omega}^{j_3+1}\rangle}{\langle F_{0,\omega}^{j_0},\dots,F_{3,\omega}^{j_3}\rangle};(v_{0,\omega}^{j_0+1},\dots,v_{3,\omega}^{j_3+1})\right].\end{aligned}$$

Using the same approach of [BBI] it is straghtfoward to prove that

Proposition 3.2.5 *The function B_n^ω is a bounded, alternating, strict $GL(n,\mathbb{C}_\omega)$-invariant cocycle on the space $\mathscr{F}_{\mathrm{aff}}(n,\mathbb{C}_\omega)^4$ of 4-tuples of affine flags which naturally descends to the space $\mathscr{F}(n,\mathbb{C}_\omega)^4$ of 4-tuples of flags. Moreover, for every 4-tuple of flags $(F_{0,\omega},\dots,F_{3,\omega}) \in \mathscr{F}(n,\mathbb{C}_\omega)^4$ we have the following bound*

$$|B_n^\omega(F_{0,\omega},\dots,F_{3,\omega})| \leq \frac{n(n^2-1)}{6}\nu_3.$$

We want now to use Proposition 1.1.13 in order to obtain the desired cohomology class. Before doing this we need to underline a delicate point in the discussion. By Proposition 1.3.19 the field $\mathbb{C}_\omega$ is not locally compact with respect to the topology induced by the ultrametric absolute value. In particular the group $SL(n,\mathbb{C}_\omega)$ cannot be locally compact with respect to the topology inherited by $M(n,\mathbb{C}_\omega)$ seen as $\mathbb{C}_\omega^{n^2}$. Hence it is meaningless to refer to the Haar measure or to the Haar σ-algebra for $SL(n,\mathbb{C}_\omega)$. In order to overcome these difficulties, we are going to consider $SL^\delta(n,\mathbb{C}_\omega)$, that is the group $SL(n,\mathbb{C}_\omega)$ endowed with the discrete topology. The same for $GL^\delta(n,\mathbb{C}_\omega)$. Moreover, in order to apply correctly Proposition 1.1.13, we are going to consider the discrete σ-algebra on both $\mathfrak{S}_k^\omega$ and $\mathscr{F}(n,\mathbb{C}_\omega)$.

Recall that $\mathfrak{S}_k^\omega(m)$ is a space on which the symmetric group S_{k+1} acts naturally. Let $\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_k^\omega)$ be the Banach space of bounded alternating Borel functions on $\mathfrak{S}_k^\omega$. The restriction of D_k^* gives us back a complex of Banach spaces $(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_\bullet^\omega), D_\bullet^*)$. By restricting the map T_k^* to the subcomplexes of bounded Borel functions and by applying Proposition 1.1.13 to $(\mathcal{B}^\infty_{\mathrm{alt}}(\mathscr{F}(n,\mathbb{C}_\omega)^{\bullet+1}), \partial_\bullet)$, we get a map

$$S^k_\omega(n) : H^k(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_\bullet^\omega)) \to H^k_b(GL^\delta(n,\mathbb{C}_\omega)).$$

Definition 3.2.6. With the notation above, we define the *ω-Borel cohomology class of degree n* as

$$\beta^\omega(n) := S^3_\omega(n)(\mathrm{Vol}^\omega) = \mathfrak{c}^3[B_n^\omega],$$

where $\mathfrak{c}^3 : H^3(\mathcal{B}^\infty_{\mathrm{alt}}(\mathscr{F}(n,\mathbb{C}_\omega)^{\bullet+1}))^{GL(n,\mathbb{C}_\omega)} \to H^3_b(GL^\delta(n,\mathbb{C}_\omega))$ is the canonical map of Proposition 1.1.13.

Remark 3.2.7. For every $k \in \mathbb{N}$ the groups $H^k_b(SL^\delta(n,\mathbb{C}_\omega))$ and $H^k_b(GL^\delta(n,\mathbb{C}_\omega))$ are isomorphic. Hence we can think of $\beta^\omega(n)$ as an element of both $H^3_b(SL^\delta(n,\mathbb{C}_\omega))$ and $H^3_b(GL^\delta(n,\mathbb{C}_\omega))$.

In fact, we have the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}_\omega^\times \arrow[r] & GL(n,\mathbb{C}_\omega) \arrow[r] & PGL(n,\mathbb{C}_\omega) \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[u] & SL(n,\mathbb{C}_\omega) \arrow[r] \arrow[u] & PSL(n,\mathbb{C}_\omega) \arrow[r] & 1
\end{tikzcd}$$

where $\mathbb{C}_\omega^\times$ is the group of invertible elements of $\mathbb{C}_\omega$ and μ_n is the group of the n-th roots of unity. By functoriality of bounded cohomology and since both $\mathbb{C}_\omega^\times$ and μ_n are amenable groups, we conclude that $H^k_b(GL^\delta(n,\mathbb{C}_\omega)) \cong H^k_b(SL^\delta(n,\mathbb{C}_\omega))$, as claimed.

3.3 The ω-Borel invariant for a representation ρ_ω

Let Γ be the fundamental group of a complete hyperbolic 3-manifold M with toric cusps. Recall that M can be decomposed as $M = N \cup \bigcup_{i=1}^{h} C_i$, where N is any compact core of M and for $i = 1, \ldots, h$ the component C_i is a cuspidal neighborhood diffeomorphic to $T_i \times (0, \infty)$, where T_i is a torus whose fundamental group corresponds to a suitable abelian parabolic subgroup of $PSL(2, \mathbb{C})$ (see Section 1.2.3). Our aim is to define a numerical invariant associated to any representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$. We start by fixing the notation. Let $i : (M, \varnothing) \to (M, M \setminus N)$ be the natural inclusion map. Since the fundamental group of the boundary ∂N is abelian, hence amenable, it can be proved that the maps $i_b^* : H_b^k(M, M \setminus N) \to H_b^k(M)$ induced at the level of bounded cohomology groups are isometric isomorphisms for $k \geq 2$ (see [BBF14]). Moreover, it holds $H_b^k(M, M \setminus N) \cong H_b^k(N, \partial N)$ by homotopy invariance of bounded cohomology. If we denote by c the canonical comparison map $c : H_b^k(N, \partial N) \to H^k(N, \partial N)$, we can consider the composition

$$H_b^3(SL^\delta(n, \mathbb{C}_\omega)) \xrightarrow{(\rho_\omega)_b^*} H_b^3(\Gamma) \cong H_b^3(M) \xrightarrow{(i_b^*)^{-1}} H_b^3(N, \partial N) \xrightarrow{c} H^3(N, \partial N),$$

where the isomorphism that appears in this composition holds since M is aspherical.

By choosing a fundamental class $[N, \partial N]$ for $H_3(N, \partial N)$ we are ready to give the following

Definition 3.3.1. The ω-*Borel invariant* associated to a representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$ is given by

$$\beta_n^\omega(\rho_\omega) := \langle (c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*) \beta^\omega(n), [N, \partial N] \rangle,$$

where the brackets $\langle \cdot, \cdot \rangle$ indicate the Kronecker pairing.

Remark 3.3.2. The previous definition is indipendent of the choice of the compact core N. Moreover, it can be easily extended to any lattice of $PSL(2, \mathbb{C})$.

We are going to generalize some of the classic results valid for the standard Borel invariant. The proofs are identical to the ones exposed in [BBI]. Before starting, we recall the existence of natural transfer maps

$$H_b^\bullet(\Gamma) \xrightarrow{\mathrm{trans}_\Gamma} H_{cb}^\bullet(PSL(2, \mathbb{C})) \qquad H^\bullet(N, \partial N) \xrightarrow{\tau_{DR}} H_c^\bullet(PSL(2, \mathbb{C})),$$

where $H_c^\bullet(PSL(2, \mathbb{C}))$ denotes the continuous cohomology groups of $PSL(2, \mathbb{C})$.

The transfer maps are defined as it follows. Let V_k be the set $C_b((\mathbb{H}^3)^{k+1}, \mathbb{R})$ of real bounded continuous functions on $(k+1)$-tuples of points of $\mathbb{H}^3$. With the standard homogeneous boundary operators and the structure of Banach $PSL(2, \mathbb{C})$-module given by

$$(g.f)(x^0, \ldots, x^n) := f(g^{-1}x^0, \ldots, g^{-1}x^n), \quad ||f||_\infty = \sup_{x^0, \ldots, x^n \in \mathbb{H}^3} |f(x^0, \ldots, x^n)|$$

for every $f \in C_b((\mathbb{H}^3)^{n+1}, \mathbb{R})$ and $g \in PSL(2,\mathbb{C})$, we get a complex $V_\bullet = C_b((\mathbb{H}^3)^{\bullet+1})$ of Banach $PSL(2,\mathbb{C})$-modules. Recall that this complex allows us to compute the continuous bounded cohomology of $PSL(2,\mathbb{C})$, indeed it holds

$$H^k(V_\bullet^{PSL(2,\mathbb{C})}) \cong H^k_{cb}(PSL(2,\mathbb{C}))$$

for every $k \geq 0$. Moreover, by substituting $PSL(2,\mathbb{C})$ with Γ, we have in an analogous way that

$$H^k(V_\bullet^\Gamma) \cong H^k_b(\Gamma)$$

for every $k \geq 0$. The previous considerations allow us to define the map

$$\mathrm{trans}_\Gamma : V_k^\Gamma \to V_k^{PSL(2,\mathbb{C})}, \quad \mathrm{trans}_\Gamma(c)(x_0, \dots, x_n) := \int_{\Gamma \backslash PSL(2,\mathbb{C})} c(\bar{g}x_0, \dots, \bar{g}x_n) d\mu(\bar{g}),$$

where c is any Γ-invariant element of V_k and μ is any invariant probability measure on $\Gamma\backslash PSL(2,\mathbb{C})$. Here $\bar{g}$ stands for the equivalence class of g into $\Gamma\backslash PSL(2,\mathbb{C})$.

Since $\mathrm{trans}_\Gamma(c)$ is $PSL(2,\mathbb{C})$-equivariant and trans_Γ commutes with the coboundary operator, we get a well-defined map

$$\mathrm{trans}_\Gamma : H^\bullet_b(\Gamma) \to H^\bullet_{cb}(PSL(2,\mathbb{C})).$$

We now pass to the description of the map τ_{DR}. If $\pi : \mathbb{H}^3 \to M = \Gamma\backslash\mathbb{H}^3$ is the natural covering projection, we set $U := \pi^{-1}(M \setminus N)$. Recall that the relative cohomology group $H^k(N, \partial N)$ is isomorphic to the cohomology group $H^k(\Omega^\bullet(\mathbb{H}^3, U)^\Gamma)$ of the Γ-invariant differential forms on $\mathbb{H}^3$ which vanish on U. Since, by Theorem 1.1.16 we have that $H^k_c(PSL(2,\mathbb{C}), \mathbb{R}) \cong \Omega^k(\mathbb{H}^3)^{PSL(2,\mathbb{C})}$, we define

$$\tau_{DR} : \Omega^k(\mathbb{H}^3, U)^\Gamma \to \Omega^k(\mathbb{H}^3)^{PSL(2,\mathbb{C})}, \quad \tau_{DR}(\alpha) := \int_{\Gamma\backslash PSL(2,\mathbb{C})} \bar{g}^* \alpha d\mu(\bar{g}),$$

where μ and $\bar{g}$ are the same as before. The map τ_{DR} commutes with the coboundary operators inducing a map

$$\tau_{DR} : H^k(N, \partial N) \cong H^k(\Omega^\bullet(\mathbb{H}^3, U)^\Gamma) \to H^k(\Omega^\bullet(\mathbb{H}^3)^{PSL(2,\mathbb{C})}) \cong H^k_c(PSL(2,\mathbb{C})).$$

For a more detailed description of the above maps we suggest to the reader to check [BBI13, Section 3.2].

Proposition 3.3.3 *For $k \geq 2$ the diagram*

$$\begin{tikzcd} H^k(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}^\omega_\bullet)) \arrow[r, "S^k_\omega(n+1)"] \arrow[dr, "S^k_\omega(n)"'] & H^k_b(GL^\delta(n+1, \mathbb{C}_\omega)) \arrow[d] \\ & H^k_b(GL^\delta(n, \mathbb{C}_\omega)) \end{tikzcd}$$

commutes. The vertical arrow is induced by the left corner injection $GL(n,\mathbb{C}_\omega) \to GL(n+1,\mathbb{C}_\omega)$. In particular we have that $\beta^\omega(n+1)$ restricts to $\beta^\omega(n)$.

Proof. Let $i_n : \mathbb{C}^n_\omega \to \mathbb{C}^{n+1}_\omega$ be the injection $i_n(x^1_\omega, \ldots, x^n_\omega) := (x^1_\omega, \ldots, x^n_\omega, 0)$. By an abuse of notation we define

$$i_n : \mathscr{F}_{\text{aff}}(n, \mathbb{C}_\omega) \to \mathscr{F}_{\text{aff}}(n+1, \mathbb{C}_\omega)$$

as $i_n((F_\omega, v_\omega)) = (\tilde{F}_\omega, \tilde{v}_\omega)$ where for $0 \leq j \leq n$ we have $\tilde{F}^j_\omega = i_n(F^j_\omega)$, $\tilde{v}^j_\omega = i_n(v^j_\omega)$ and $\tilde{v}^{n+1}_\omega = e_{n+1}$. If we set $\mathbf{J} \in \{0, \ldots n\}^{k+1}$ and $I = \{i : 0 \leq i \leq k \text{ such that } j_i = n\}$, it is easy to verify that if $I = \varnothing$ this implies $\mathbf{J} \in \{0, \ldots, n-1\}^{k+1}$ and

$$\tau_{\mathbf{J}}(i_n(F_{0,\omega}, v_{0,\omega}), \ldots, i_n(F_{k,\omega}, v_{k,\omega})) = \tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \ldots, (F_{k,\omega}, v_{k,\omega}))$$

while if $I \neq \varnothing$, then

$$\tau_{\mathbf{J}}(i_n(F_{0,\omega}, v_{0,\omega}), \ldots, i_n(F_{k,\omega}, v_{k,\omega})) = [\mathbb{C}_\omega; (\delta^I_0, \ldots, \delta^I_k)],$$

where $\delta^I_i = [e_{n+1}]$ if $i \in I$ and 0 otherwise. The previous considerations imply that i_n induces a commutative diagram of complexes

$$\begin{tikzcd}
\mathcal{B}^\infty_{\text{alt}}(\mathfrak{S}^\omega_k) \arrow[r, "T^*_k"] \arrow[dr, "T^*_k"'] & \mathcal{B}^\infty_{\text{alt}}(\mathscr{F}_{\text{aff}}(n+1, \mathbb{C}_\omega)^{k+1}) \arrow[d, "i^*_n"] \\
& \mathcal{B}^\infty_{\text{alt}}(\mathscr{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1})
\end{tikzcd}$$

and since the map i^*_n implements the restriction in bounded cohomology, the commutativity of the diagram which appears in the statement follows. In particular, by focusing our attention on the case of $k = 3$ we get

$$i^*_n(B^\omega_{n+1}) = i^*_n \circ T^*_3(\mathrm{Vol}^\omega) = T^*_3(\mathrm{Vol}^\omega) = B^\omega_n$$

as claimed. $\square$

Proposition 3.3.4 *For any representation $\rho_\omega : \Gamma \to SL(n, \mathbb{C}_\omega)$ the composition*

$$H^3_b(SL^\delta(n, \mathbb{C}_\omega)) \longrightarrow H^3_b(\Gamma) \xrightarrow{\text{trans}_\Gamma} H^3_{cb}(PSL(2, \mathbb{C}))$$

maps $\beta^\omega(n)$ to $\frac{\beta^\omega_n(\rho_\omega)}{\mathrm{Vol}(M)}\beta(2)$. In particular, it holds the following bound

$$|\beta^\omega_n(\rho_\omega)| \leq \frac{n(n^2-1)}{6}\mathrm{Vol}(M),$$

as in the classic case.

Proof. Recall that we have the following commutative diagram

$$\begin{tikzcd}
H_b^3(SL^\delta(n,\mathbb{C}_\omega)) \arrow[d, "(\rho_\omega)_b^*"] & \\
H_b^3(\Gamma) \arrow[d, "\cong"] \arrow[dr, "\mathrm{trans}_\Gamma"] & \\
H_b^3(N,\partial N) \arrow[d, "c"] & H_{cb}^3(PSL(2,\mathbb{C})) \arrow[d, "c"] \\
H^3(N,\partial N) \arrow[r, "\tau_{DR}"] & H_c^3(PSL(2,\mathbb{C})).
\end{tikzcd}$$

Since $H_{cb}^3(PSL(2,\mathbb{C})) \cong \mathbb{R}$ as a consequence of Proposition 1.1.19, there exists a suitable $\lambda \in \mathbb{R}$ such that

$$\mathrm{trans}_\Gamma \circ (\rho_\omega)_b^*(\beta^\omega(n)) = \lambda\beta(2).$$

Hence by composing both sides with the comparison map c, we obtain

$$c \circ \mathrm{trans}_\Gamma \circ (\rho_\omega)_b^*(\beta^\omega(n)) = c(\lambda\beta(2)) = \lambda(c\beta(2)) = \lambda\beta(2).$$

If we pick up $\omega_{N,\partial N} \in H^3(N,\partial N)$ such that its evaluation on the fundamental class $[N,\partial N]$ gives us back $\mathrm{Vol}(M)$, we have that $\tau_{DR}(\omega_{N,\partial N}) = \beta(2)$. In particular

$$\tau_{DR}(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*(\beta^\omega(n))) = \lambda\tau_{DR}(\omega_{N,\partial N})$$

and by injectivity of the map τ_{DR} in top degree we get

$$(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)(\beta^\omega(n)) = \lambda\omega_{N,\partial N}.$$

If we evaluate both sides on the fundamental class, we obtain

$$\beta_n^\omega(\rho_\omega) = \langle (c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)(\beta^\omega(n)), [N,\partial N]\rangle = \langle \lambda\omega_{N,\partial N}, [N,\partial N]\rangle = \lambda\mathrm{Vol}(M).$$

At the same time it holds

$$|\lambda| = \frac{||\mathrm{trans}_\Gamma \circ (\rho_\omega)_b^*\beta^\omega(n)||}{||\beta(2)||} \leq \frac{n(n^2-1)}{6},$$

from which it follows

$$|\beta_n^\omega(\rho_\omega)| \leq \frac{n(n^2-1)}{6}\mathrm{Vol}(M),$$

as claimed. □

Recall that there is a natural inclusion of fields of $\mathbb{C}$ into $\mathbb{C}_\omega$ given by constant sequences. In particular we have natural embeddings of $\mathbb{C}^m$ into $\mathbb{C}_\omega^m$ and of $SL(n,\mathbb{C})$ into $SL(n,\mathbb{C}_\omega)$. Since every representation $\rho : \Gamma \to SL(n,\mathbb{C})$ determines a representation $\hat{\rho}$ into $SL(n,\mathbb{C}_\omega)$ by composing it with the previous embedding, it is quite natural to ask which is the relation between $\beta_n^\omega(\hat{\rho})$ and $\beta_n(\rho)$. We have the following

Proposition 3.3.5 *Let $\rho : \Gamma \to SL(n, \mathbb{C})$ be a representation. If we denote by $\hat{\rho} : \Gamma \to SL(n, \mathbb{C}_\omega)$ the representation obtained by composing ρ with the natural embedding of $SL(n, \mathbb{C})$ into $SL(n, \mathbb{C}_\omega)$, we have*

$$\beta_n^\omega(\hat{\rho}) = \beta_n(\rho).$$

Proof. We are going to prove that the cohomology class $\beta^\omega(n)$ restricts naturally to the class $\beta(n)$. Let $j : SL(n, \mathbb{C}) \to SL(n, \mathbb{C}_\omega)$ be the natural embedding. By endowing both spaces with the discrete topology, we have a continuous morphism of groups that induces a map

$$j_b^* : H_b^3(SL^\delta(n, \mathbb{C}_\omega)) \to H_b^3(SL^\delta(n, \mathbb{C})).$$

We want to prove that $j_b^*(\beta^\omega(n)) = \beta(n)$. From this it will follow

$$\begin{aligned}\beta_n^\omega(\hat{\rho}) &= \langle (c \circ (i_b^*)^{-1} \circ \hat{\rho}_b^*)\beta^\omega(n), [N, \partial N]\rangle = \langle (c \circ (i_b^*)^{-1} \circ (j \circ \rho)_b^*)\beta^\omega(n), [N, \partial N]\rangle \\ &= \langle (c \circ (i_b^*)^{-1} \circ \rho_b^* \circ j_b^*)\beta^\omega(n), [N, \partial N]\rangle = \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)\beta(n), [N, \partial N]\rangle = \beta_n(\rho).\end{aligned}$$

Similarly to what we have done for the field $\mathbb{C}_\omega$, we define the configuration space

$$\mathfrak{S}_k(m) := \{(x^0, \dots, x^k) \in (\mathbb{C}^m)^{k+1} | \langle x^0, \dots, x^k\rangle = \mathbb{C}^m\}/GL(m, \mathbb{C}).$$

for every $k \geq m-1$. This family of spaces is exactly the family introduced by [BBI]. There exists a natural family of maps given by

$$\hat{j}_k(m) : \mathfrak{S}_k(m) \to \mathfrak{S}_k^\omega(m), \quad \hat{j}_k(m)[\mathbb{C}^m; (v^0, \dots, v^k)] := [\mathbb{C}_\omega^m; (v^0, \dots, v^k)],$$

where each vector v^i which appears on the right-hand side of the equation is thought of as an element of $\mathbb{C}_\omega^m$. This function is well-defined because $v^0, \dots, v^k$ are generators also for $\mathbb{C}_\omega^m$ as a $\mathbb{C}_\omega$-vector space and the identifications induced via conjugation by $GL(m, \mathbb{C})$ are respected. By denoting

$$\hat{j}_k := \hat{j}_k(0) \sqcup \hat{j}_k(1) \sqcup \dots \sqcup \hat{j}_k(k+1),$$

we get the following commutative diagram

$$\begin{tikzcd} H^3(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}^\omega_\bullet)) \arrow[r, "S^3_\omega(n)"] \arrow[d, "H^3(\hat{j}^*_\bullet)"'] & H^3_b(SL^\delta(n, \mathbb{C}_\omega)) \arrow[d, "j^*_b"] \\ H^3(\mathcal{B}^\infty_{\mathrm{alt}}(\mathfrak{S}_\bullet)) \arrow[r, "S^3(n)"] & H^3_b(SL^\delta(n, \mathbb{C})), \end{tikzcd}$$

where $\hat{j}^*_\bullet$ are the maps induced by $\hat{j}_\bullet$ on the Borel cochains. We will prove that $\mathrm{Vol} = \mathrm{Vol}^\omega \circ \hat{j}_3$, that is $H^3(\hat{j}^*_\bullet)[\mathrm{Vol}^\omega] = [\mathrm{Vol}]$. Let $m \in \{0, \dots, 4\}$. It is clear that $\mathrm{Vol} = \mathrm{Vol}^\omega \circ \hat{j}_3(m)$ for $m \neq 2$ because both sides are equal to zero. Let us now consider $[\mathbb{C}^2; (v^0, \dots, v^3)] \in \mathfrak{S}_3(2)$. If any of these vectors is 0 both functions evaluated on the 4-tuple give us back 0. Hence, we can suppose that each v^i is different from 0. If the vectors $v^0,\dots,v^3$ are in general position into $\mathbb{C}^2$, they still remain in general position into $\mathbb{C}^2_\omega$. Thus

$$\begin{aligned}&\mathrm{Vol}^\omega \circ \hat{j}_3(2)[\mathbb{C}^2;(v^0,\dots,v^3)] = \mathrm{Vol}^\omega[\mathbb{C}^2_\omega;(v^0,\dots,v^3)] = \omega\text{-}\lim_{l\to\infty}\mathrm{Vol}(v^0,\dots,v^3)\\ &= \mathrm{Vol}(v^0,\dots,v^3) = \mathrm{Vol}[\mathbb{C}^2;(v^0,\dots,v^3)].\end{aligned}$$

In the same way if $(v^0,\dots,v^3)$ are not in general position into $\mathbb{C}^2$, they will not be in general position in $\mathbb{C}^2_\omega$ either, so both $\mathrm{Vol}^\omega \circ \hat{j}_3(2)$ and Vol will evaluate to be zero, as desired. □

We want now to express $\beta^\omega_n(\rho_\omega)$ in terms of boundary maps. Recall that we can decompose $M = N \cup \bigcup_{i=1}^h C_i$, where N is the compact core we fixed before and each C_i is a cuspidal neighborhood, for $i = 1,\dots,h$. Since the fundamental group $H_i = \pi_1(C_i)$ is an abelian parabolic subgroup of $PSL(2,\mathbb{C})$, it has a unique fixed point ξ_i in $\mathbb{P}^1(\mathbb{C})$. We define the set

$$\mathscr{C}(\Gamma) := \bigcup_{i=1}^h \Gamma.\xi_i.$$

Definition 3.3.6. If $\Gamma = \pi_1(M)$ as above, given a representation $\rho_\omega : \Gamma \to SL(n,\mathbb{C}_\omega)$, a *decoration* for ρ_ω is a map

$$\varphi_\omega : \mathscr{C}(\Gamma) \to \mathscr{F}(n,\mathbb{C}_\omega)$$

that is equivariant with respect to ρ_ω.

Recall now that the cocycle B^ω_n is a strict cocycle, as in the standard case. Hence the class $(c \circ (i^*_b)^{-1} \circ (\rho_\omega)^*_b)\beta^\omega(n)$ can be represented in $H^3_b(\Gamma)$ by $\varphi^*_\omega(B^\omega_n)$, where φ_ω is a decoration for ρ_ω (we refer to [BI02, Corollary 2.7] for this result about the pullback of strict cocycles along boundary maps).

Let N be a fixed compact core of M. In order to realize the corresponding cocycle in $H^3_b(N,\partial N)$, we identify the universal cover $\tilde{N}$ of N with $\mathbb{H}^3$ minus a set of Γ-equivariant horoballs, each one centered at an element $\xi \in \mathscr{C}(\Gamma)$. We define a map $p : \tilde{N} \to \mathscr{C}(\Gamma)$ in two steps. We first send each horospherical section to the corresponding element. Then, for the interior of $\tilde{N}$, we map a fundamental domain to a chosen $\xi_0 \in \mathscr{C}(\Gamma)$ and we extend equivariantly. In this way, any bounded Γ-invariant cocycle $c : \mathscr{C}(\Gamma) \to \mathbb{R}$ determines a relative cocycle on $(N,\partial N)$ as it follows

$$\{\sigma : \Delta^3 \to \tilde{N}\} \mapsto c(p(\sigma(e_0)),\dots,p(\sigma(e_3))).$$

If τ is a relative triangulation of $(N,\partial N)$ and $\tilde{\tau}$ is the lifted triangulation of a fundamental domain in $(\tilde{N},\partial\tilde{N})$, the ω-Borel invariant $\beta^\omega_n(\rho_\omega)$ can be computed by the following formula

$$\beta^\omega_n(\rho_\omega) = \sum_{\tilde{\sigma}\in\tilde{\tau}} B^\omega_n(\varphi_\omega(p(\tilde{\sigma}(e_0))),\varphi_\omega(p(\tilde{\sigma}(e_1))),\varphi_\omega(p(\tilde{\sigma}(e_2))),\varphi_\omega(p(\tilde{\sigma}(e_3))))$$

where $\tilde{\sigma}$ is a lifted copy of the simplex $\sigma \in \tau$.

3.4 The case $n = 2$ and properties of the invariant $\beta_2^\omega(\rho_\omega)$

In this section we are going to focus our attention on the case of representations into $SL(2,\mathbb{C}_\omega)$. Suppose to have a sequence of representations $\rho_l : \Gamma \to SL(2,\mathbb{C})$ that determines a representation $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$. A sequence of decorations φ_l for ρ_l produces in a natural way a decoration φ_ω. Indeed it suffices to compose the standard projection $\pi : \mathbb{P}^1(\mathbb{C})^{\mathbb{N}} \to \mathbb{P}^1(\mathbb{C}_\omega)$ with the map $\prod \varphi_l : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$. We say that a decoration is *non-degenerate* if for every $\xi_0, \ldots, \xi_3 \in \mathscr{C}(\Gamma)$ such that $\xi_i \neq \xi_j$ for $i \neq j$, we have that the 4-tuple $(\varphi_\omega(\xi_0), \ldots, \varphi_\omega(\xi_3))$ contains at least 3 distinct points. If the decoration φ_ω is non-degenerate we have

$$\begin{aligned}\beta_2^\omega(\rho_\omega) &= \sum_{\tilde\sigma \in \tilde\tau} B_2^\omega(\varphi_\omega(p(\tilde\sigma(e_0))), \varphi_\omega(p(\tilde\sigma(e_1))), \varphi_\omega(p(\tilde\sigma(e_2))), \varphi_\omega(p(\tilde\sigma(e_3)))) = \\ &= \omega\text{-}\lim_{l\to\infty} \sum_{\tilde\sigma \in \tilde\tau} B_2(\varphi_l(p(\tilde\sigma(e_0))), \varphi_l(p(\tilde\sigma(e_1))), \varphi_l(p(\tilde\sigma(e_2))), \varphi_l(p(\tilde\sigma(e_3)))) \\ &= \omega\text{-}\lim_{l\to\infty} \beta_2(\rho_l),\end{aligned}$$

where the last equality is obtained by applying Corollary 2.7 of [BI02] to express the Borel invariant $\beta_2(\rho_l)$ in terms of the boundary maps φ_l. The second equality exploits the non-degenerancy of the decoration φ_ω, which allows us to pass from the evaluation of the ω-Borel cocycle on φ_ω to the ω-limit of evaluations of the standard Borel cocycle on φ_l. Hence we get

Proposition 3.4.1 *Let $\rho_l : \Gamma \to SL(2,\mathbb{C})$ be a sequence of representations with decorations φ_l. Let $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ be the representation associated to the sequence ρ_l. If the decoration φ_ω produced by the sequence φ_l is non-degenerate, we have*

$$\beta_2^\omega(\rho_\omega) = \omega\text{-}\lim_{l\to\infty} \beta_2(\rho_l).$$

Corollary 3.4.2 *Let $\rho_l : \Gamma \to SL(2,\mathbb{C})$ be a sequence of representations with decorations φ_l. Let $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ be the representation associated to the sequence ρ_l. Suppose $\beta_2^\omega(\rho_\omega) = \mathrm{Vol}(M)$. If the decoration φ_ω produced by the sequence φ_l is non-degenerate, there must exist a sequence $g_l \in SL(2,\mathbb{C})$ and a representation $\rho_\infty : \Gamma \to SL(2,\mathbb{C})$ such that*

$$\omega\text{-}\lim_{l\to\infty} g_l \rho_l(\gamma) g_l^{-1} = \rho_\infty(\gamma).$$

Proof. Thanks to the assumption of non-degenerancy, by applying Proposition 3.4.1 we desume that $\omega\text{-}\lim_{l\to\infty} \beta_2(\rho_l) = \mathrm{Vol}(M)$. The statement now follows directly by Theorem 1. $\square$

Assume that a sequence of representations $\rho_l : \Gamma \to SL(2,\mathbb{C})$ diverges to a ideal point of the character variety $X(\Gamma, SL(2,\mathbb{C}))$ and let $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ be the representation associated to the sequence. Recall that the identification between $SL(2,\mathbb{C}_\omega)$ and $SL(2,\mathbb{C})_\omega$ implies that the representation ρ_ω produces in a natural way an isometric action of Γ on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$. We are going to restrict our attention to reducible actions with non-trivial length function.

Proposition 3.4.3 *Let $\rho_l : \Gamma \to SL(2,\mathbb{C})$ be a sequence of representations and suppose it determines a representation $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ such that the isometric action induced by ρ_ω on $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ has non-trivial length function. If the action is reducible then $\beta_2^\omega(\rho_\omega) = 0$.*

Proof. Since the length function associated to the action induced by ρ_ω is non-trivial then the action does not admit a global fixed point. Moreover, since the action is reducible, it must admit either a fixed end or an invariant line. Suppose that there exists an end fixed by Γ. By Proposition 1.3.24 the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ is naturally identified with the Bass–Serre tree $\Delta^{BS}(SL(2,\mathbb{C}_\omega))$ associated to $SL(2,\mathbb{C}_\omega)$. Hence, there must exist an end of $\Delta^{BS}(SL(2,\mathbb{C}_\omega))$ fixed by the representation ρ_ω. Thus the image $\rho_\omega(\Gamma)$ is a subgroup of a suitable Borel subgroup N_ω of $SL(2,\mathbb{C}_\omega)$ and hence it is solvable, so amenable by [Zim84, Corollary 4.1.7]. This implies that the map $(\rho_\omega)_b^* = 0$ from which we conclude $\beta_2^\omega(\rho_\omega) = 0$.

Suppose now that the action of Γ admits an invariant line. This time the image $\rho_\omega(\Gamma)$ will be isomorphic to a subgroup of $\mathrm{Isom}(\mathbb{R})$. Being $\mathrm{Isom}(\mathbb{R})$ the semidirect group of the two amenable groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{R}$, it will be amenable by [Zim84, Proposition 4.1.6]. As before we will have $(\rho_\omega)_b^* = 0$, hence $\beta_2^\omega(\rho_\omega) = 0$. □

Remark 3.4.4. Another way to prove Proposition 3.4.3 is by using decorations. Indeed, if the action determined by ρ_ω admits a fixed end $\varepsilon_\omega \in \partial_\infty \Delta^{BS}(SL(2,\mathbb{C}_\omega))$ and since the boundary at infinity can be identified with $\mathbb{P}^1(\mathbb{C}_\omega)$, then the map $\varphi_\omega(\xi) = \varepsilon_\omega$ for $\xi \in \mathscr{C}(\Gamma)$ is a decoration and trivially it results $\beta_2^\omega(\rho_\omega) = 0$.

In the same way if the action admits an invariant line L_ω, we denote by ε_ω^1 and ε_ω^2 the ends of the line L_ω. For every $\xi \in \mathscr{C}(\Gamma)$ we can choose either ε_ω^1 or ε_ω^2 as the image of ξ for the decoration φ_ω. This implies that every possible choice produces a decoration for ρ_ω such that it results $\beta_2^\omega(\rho_\omega) = 0$.

Let $S = \{\gamma_1, \dots, \gamma_s\}$ be a generating set for the group Γ. Recall that if a sequence of representations $\rho_l : \Gamma \to SL(2,\mathbb{C})$ diverges in the character variety $X(\Gamma, SL(2,\mathbb{C}))$ to an ideal point of the Morgan–Shalen compactification, then the real sequence

$$\lambda_l := \inf_{x \in \mathbb{H}^3} \sqrt{\sum_{i=1}^{s} d(\rho_l(\gamma_i)x, x)}$$

is positive and divergent. As written in Theorem 1.3.32, for any non-principal ultrafilter ω on $\mathbb{N}$, by fixing $(\lambda_l)_{l\in N}$ as scaling sequence, we are able to construct in a natural way a representation $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ via the representations ρ_l.

Corollary 3.4.5 *Let $\rho_l : \Gamma \to SL(2,\mathbb{C})$ be a sequence of representations diverging to an ideal point of the Morgan–Shalen compactification of the character variety $X(\Gamma, SL(2,\mathbb{C}))$. Let $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ be the natural representation determined by the sequence $(\rho_l)_{l\in\mathbb{N}}$. If the representation is reducible, then $\beta_2^\omega(\rho_\omega) = 0$.*

Proof. It follows directly from Proposition 3.4.3 by obsverving that the ρ_ω has non-trivial length function since it is associated to a diverging sequence of representations. □

Chapter 4

Natural maps and abelian length functions

4.1 Sequences of natural maps

We ended the previous chapter by stating that if a sequence of representations $\rho_l : \Gamma \to SL(2,\mathbb{C})$ diverges to an ideal point of the character variety and it determines a reducible action via ρ_ω then $\beta_2^\omega(\rho_\omega)$ must vanish. In this chapter we are going to expose a criterion to get a reducible action based on the study of the sequence of boundary maps associated to the representations. Before starting we need to fix the following setting

- A group $\Gamma < PSL(2,\mathbb{C})$ so that $\Gamma\backslash\mathbb{H}^3$ is a complete hyperbolic manifold of finite volume.
- A base-point $O \in \mathbb{H}^3$ used to normalize the Busemann function $B(x,\theta)$, with $x \in \mathbb{H}^3$ and $\theta \in \partial_\infty\mathbb{H}^3$.
- The family $\{\mu_x\}$ of Patterson-Sullivan probability measures. Set $\mu = \mu_O$.
- A sequence of non-elementary representations $\rho_l : \Gamma \to SL(2,\mathbb{C})$. Let $\bar{\rho}_l$ be the induced representations into $PSL(2,\mathbb{C})$.
- A non-principal ultrafilter ω on $\mathbb{N}$.
- The sequence of measurable boundary maps $D_l : \partial_\infty\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$.
- The resulting sequence of BCG–natural maps $F_l : \mathbb{H}^3 \to \mathbb{H}^3$.

We first study the relation between the convergence of the natural map F_l and the convergence of the representations ρ_l in the character variety. We start recalling the notion of quasi-constant map.

Definition 4.1.1. A *quasi-constant map* $a_b : \partial_\infty\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$ is defined as

$$a_b(x) := \begin{cases} a & \text{if } x \neq b, \\ b & \text{if } x = b. \end{cases}$$

The same definition can be given in term of $\overline{\mathbb{H}}^3$ rather than $\partial_\infty\mathbb{H}^3$. The group $SL(2,\mathbb{C})$ admits a natural compactification whose ideal points are given by quasi-constant maps. Indeed any sequence of elements $g_l \in SL(2,\mathbb{C})$ converges up to the choice of a subsequence either to an element $g_\infty \in SL(2,\mathbb{C})$ or to a quasi-constant map a_b, with $a, b \in \partial_\infty\mathbb{H}^3$ (see [Kap01, Section 3.6]).

Proposition 4.1.2 *Let $\rho_l : \Gamma \to SL(2,\mathbb{C})$ be a sequence of representations. Suppose that the sequence of natural maps F_l associated to ρ_l converges pointwise to a map $F : \mathbb{H}^3 \to \mathbb{H}^3$. Then the sequence of representations ρ_l converges to a representations ρ_∞, up to passing to a subsequence.*

Proof. Let $S = \{\gamma_1, \dots, \gamma_s\}$ be a finite set of generators for the group Γ. It suffices to show that the limit of the sequence $\rho_l(\gamma_i)$ admits a subsequence converging to an element $g_i \in SL(2,\mathbb{C})$, for $i = 1, \dots, s$.

By contradiction, suppose that $\rho_l(\gamma_i)$ converges to a quasi-constant map a_b, with $a, b \in \partial_\infty\mathbb{H}^3$. This means that, if we endow $\overline{\mathbb{H}}^3$ with the euclidean metric, for every compact set $K \subset \overline{\mathbb{H}}^3 \setminus \{b\}$ we have a uniform convergence $\lim_{l\to\infty} \rho_l(\gamma_i)(z) = a$ for every $z \in K$. Moreover, since the natural maps F_l converges pointwise to a map F, for every $\varepsilon > 0$ and every $x \in \mathbb{H}^3$ there must exist a suitable $l_0 \in \mathbb{N}$ such that $d(F_l(x), F(x)) < \varepsilon$ for every $l > l_0$.

Denote by $\overline{B}_\varepsilon(F(x))$ the hyperbolic closed ball of radius ε around $F(x)$. Fix this ball as compact set and consider a small neighborhood U of a in $\overline{\mathbb{H}}^3$. The uniform convergence on $\overline{B}_\varepsilon(F(x))$ of the sequence $\rho_l(\gamma_i)$ to a implies that there exists $l_1 \in \mathbb{N}$ such that $\rho_l(\gamma_i)(\overline{B}_\varepsilon(F(x))) \subset U$ for every $l > l_1$.

As a consequence the sequence $\rho_l(\gamma_i)F(x)$ will eventually lie in U and hence the sequence $\rho_l(\gamma_i)F_l(x) = F_l(\gamma_i x)$ will eventually lie in U because

$$d(\rho_l(\gamma_i)F(x), \rho_l(\gamma_i)F_l(x)) = d(F(x), F_l(x))$$

and the right-hand side is less than ε, if $l > \max\{l_0, l_1\}$. In particular the limit of the sequence $F_l(\gamma_i x)$, that is $F(\gamma_i x)$, must lie in U, but this is a contradiction by the arbitrary choice of U and the claim is proved. $\square$

The previous proposition suggests that if a sequence $\rho_l : \Gamma \to SL(2,\mathbb{C})$ is diverging to an ideal point of the character variety, then the sequence of natural maps cannot converge pointwise to a map $F : \mathbb{H}^3 \to \mathbb{H}^3$. We are going to study more accurately this phenomenon. We keep adopting the setting fixed at the beginning of the chapter. Recall that up to conjugating ρ_l by a suitable element $g_l \in SL(2,\mathbb{C})$, we can suppose $F_l(O) = O$.

Let δ_p be the Dirac measure concentrated on p.

Definition 4.1.3. For any $l \in \mathbb{N}$ and every $z \in \partial_\infty\mathbb{H}^3$ we set

$$\alpha_{l,z} = \delta_{D_l(z)} \text{ and } \eta_l = \mu \times \{\alpha_{l,z}\}$$

Both $\{\alpha_{l,z}\}$ and η_l are probability measures.

Given a point $x \in \mathbb{H}^3$ we have $\eta_{l,x} := \mu_x \times \{\alpha_{l,z}\}$. Let π_1 and π_2 be the projections respectively on the first and on the second factor of the product $\partial_\infty\mathbb{H}^3 \times \partial_\infty\mathbb{H}^3$.

Definition 4.1.4. For any $l \in \mathbb{N}$ and $x \in \mathbb{H}^3$ we set

$$\beta_{l,x} = (D_l)_*(\mu_x) = (\pi_2)_*(\eta_{l,x}).$$

In particular we will have $F_l(x) = \mathrm{bar}_{\mathcal{B}}(\beta_{l,x})$.

Lemma 4.1.5 *The sequence η_l ω-converges to a positive probability measure η_∞ in the weak-* topology. Moreover, there exists a family $\{\alpha_{\infty,z}\}_{z\in\partial_\infty\mathbb{H}^3}$ of probability measures such that η_∞ disintegrates as*

$$\eta_\infty = \mu \times \{\alpha_{\infty,z}\}.$$

Proof. The weak-* limit exists by a compactness argument. Indeed, since $\partial_\infty\mathbb{H}^3 \times \partial_\infty\mathbb{H}^3$ is compact the set of probability measures is compact and Hausdorff with respect to the weak-* topology. Hence by Proposition 1.3.14 the ω-limit exists and it is unique. Since $(\pi_1)_*(\eta_l) = \mu$, by weak-* continuity of the pushfoward we deduce that $(\pi_1)_*(\eta_\infty) = \mu$. Therefore, we can apply the theorem of disintegration of measures (see [AFP00, Theorem 2.28]).

Hence, there exists a family $\{\alpha_{\infty,z}\}$ of positive probability measures such that $\eta_\infty = \mu \times \{\alpha_{\infty,z}\}$. □

Lemma 4.1.6 *For any x the measure $\eta_{l,x}$ has ω-weak-* limit $\eta_{\infty,x}$. Moreover, $\eta_{\infty,x}$ disintegrates as*

$$\eta_{\infty,x} = \mu_x \times \{\alpha_{\infty,z}\}.$$

Proof. We know that $d\mu_x(z) = e^{-2B(x,z)}d\mu(z)$ and $\mu = \mu_O$, whence

$$d\eta_{l,x}(\theta, z) = e^{-2B(x,z)}d\eta_l(\theta, z)$$

where the factor $e^{-2B(x,z)}$ does not depend on l. From Lemma 4.1.5 we deduce that

$$\omega\text{-}\lim_{l\to\infty} \eta_{l,x} = e^{-2B(x,z)}\eta_\infty$$

in the weak-* topology.

This implies

$$\eta_{\infty,x} = e^{-2B(x,z)}\eta_\infty = e^{-2B(x,z)}(\mu \times \{\alpha_{\infty,z}\}) = (e^{-2B(x,z)}\mu) \times \{\alpha_{\infty,z}\} = \mu_x \times \{\alpha_{\infty,z}\}$$

and the claim is proved. □

Lemma 4.1.7 *For any x the measure $\beta_{l,x}$ ω-weakly-* converges to a probability measure $\beta_{\infty,x} = (\pi_2)_*(\eta_{\infty,x})$. Moreover, the density class of $\beta_{\infty,x}$ does not depend on x.*

Proof. Since the pushfoward is weak-* continuous, by Proposition 1.3.15 we have that

$$\omega\text{-}\lim_{l\to\infty} (\pi_2)_*(\eta_{l,x}) = (\pi_2)_*(\eta_{\infty,x})$$

which proves the first claim. Now, let φ be a smooth positive function such that $\int_{\partial_\infty \mathbb{H}^3} \varphi(\theta) d\beta_{\infty,x}(\theta) = 0$. We have

$$
\begin{aligned}
0 = \int_{\partial_\infty \mathbb{H}^3} \varphi(\theta) d\beta_{\infty,x}(\theta) &= \int_{\partial_\infty \mathbb{H}^3 \times \partial_\infty \mathbb{H}^3} \varphi(\theta) d\eta_{\infty,x}(\theta, z) = \omega\text{-}\lim_{l\to\infty} \int_{\partial_\infty \mathbb{H}^3} \left(\int_{\partial_\infty \mathbb{H}^3} \varphi(\theta) d\alpha_{l,z}(\theta) \right) d\mu_x(z) \\
&= \omega\text{-}\lim_{l\to\infty} \int_{\partial_\infty \mathbb{H}^3} \left(\int_{\partial_\infty \mathbb{H}^3} \varphi(\theta) d\alpha_{l,z}(\theta) \right) e^{-2B(x,z)} d\mu_O(z) = \int_{\partial_\infty \mathbb{H}^3 \times \partial_\infty \mathbb{H}^3} \varphi(\theta) e^{-2B(x,z)} d\eta_{\infty,O}(z)
\end{aligned}
$$

and since $e^{-2B(x,z)}$ is stricly positive and $\eta_{\infty,O}$ is a positive measure, this implies

$$\int_{\partial_\infty \mathbb{H}^3 \times \partial_\infty \mathbb{H}^3} \varphi d\eta_{\infty,O} = 0$$

whence $\int_{\partial_\infty \mathbb{H}^3} \varphi \beta_{\infty,O} = 0$. The same argument show that if $\int_{\partial_\infty \mathbb{H}^3} \varphi d\beta_{\infty,O} = 0$ then $\int_{\partial_\infty \mathbb{H}^3} \varphi d\beta_{\infty,x} = 0$. In conclusion, for every x the measure $\beta_{\infty,x}$ is in the same density class of $\beta_{\infty,O}$. $\square$

The previous proposition gives us an important point to investigate. Indeed, if a sequence ρ_l diverges to an ideal point of the character variety $X(\Gamma, SL(2,\mathbb{C}))$ in the Morgan–Shalen compactification, then the limit measure $\beta_{\infty,O}$ must contain an atom of mass at least $1/2$, otherwise there will exist a limit map F which would be the pointwise limit of the maps F_l by the properties of the barycentre.

4.2 Abelian limit actions

In the previous section we have discovered that if the sequence of representations $\rho_l : \Gamma \to SL(2,\mathbb{C})$ is diverging to an ideal then the measures $\beta_{\infty,x}$ have an atom of mass greater than or equal to $1/2$. In this section we are going to prove that if the measures $\beta_{\infty,x}$ have as support two points, then the action on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ must be abelian. We will mantain the same notation of the previous section.

Since the representations ρ_l are diverging to an ideal point, if Γ is generated by $S = \{\gamma_1, \ldots, \gamma_s\}$, the sequence of minimal displacements

$$\lambda_l = \inf_{x \in \mathbb{H}^3} \sqrt{\sum_{i=1}^{s} d(\rho_l(\gamma_i)x, x)^2}$$

is diverging. Hence, the hypothesis allow us to define a representation $\rho_\omega : \Gamma \to SL(2, \mathbb{C}_\omega)$. By Proposition 1.3.24 the previous representation realizes an isometric action of Γ on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$.

Assume the support of the measure $\beta_{\infty,O}$ is a set of two points, say

$$\operatorname{supp}(\beta_{\infty,O}) = \{p, q\}.$$

By Lemma 4.1.7, for every $\gamma \in \Gamma$ we desume $\beta_{\infty,\gamma O} = \{p, q\}$.

Lemma 4.2.1 *Given $\gamma \in \Gamma$, suppose that*

$$\beta_{\infty,O} = P_1\delta_p + Q_1\delta_q, \quad \beta_{\infty,\gamma^{-1}O} = P_2\delta_p + Q_2\delta_q, \quad \beta_{\infty,\gamma O} = P_3\delta_p + Q_3\delta_q$$

where $P_i, Q_i \in \mathbb{R}, P_i + Q_i = 1$ and $P_i, Q_i > 0$. If the sequence $\rho_l(\gamma)$ diverges to a quasi-constant map x_y then $\{x, y\} = \{p, q\}$. Otherwise, if the sequence $\rho_l(\gamma)$ is bounded, then it admits a subsequence converging to an element $g \in SL(2, \mathbb{C})$ such that $g(\{p, q\}) = \{p, q\}$.

Proof. Assume $\rho_l(\gamma)$ diverges to a quasi-constant map x_y. We will prove the first part of the statement by contradiction. We start supposing that $\{x, y\} \cap \{p, q\} = \varnothing$. Given a $\delta > 0$, let $B_\delta(p)$ and $B_\delta(q)$ be two balls of radius δ with respect to the standard round metric on $\partial_\infty\mathbb{H}^3$. Consider in the same way two balls of radius δ around x and y, namely $B_\delta(x)$ and $B_\delta(y)$. Since $\beta_{l,O} \overset{*}{\rightharpoonup} \beta_{\infty,O}$ and $x \notin \{p, q\}$, there exists $l_0 = l_0(\delta, \varepsilon)$ such that for each $l > l_0$ it holds $\beta_{l,O}(B_\delta(x)) < \varepsilon$. The same argument applied to the sequence $\beta_{l,\gamma O}$ tells us that exists $l_1 = l_1(\delta, \varepsilon)$ such that for every $l > l_1$ we have $\beta_{l,\gamma^{-1}O}(B_\delta(p)) \geq P_2 - \varepsilon$. We recall that the family of measures $\{\beta_{l,x}\}$ is equivariant with respect the representation ρ_l, that is

$$\rho_l(\gamma)_*(\beta_{l,O}) = \beta_{l,\gamma O},$$

hence, for this reason, we will have

$$\beta_{l,\gamma^{-1}O}(B_\delta(p)) = \beta_{l,O}(\rho_l(\gamma)(B_\delta(p))) \text{ and } \beta_{l,\gamma^{-1}O}(B_\delta(q)) = \beta_{l,O}(\rho_l(\gamma)(B_\delta(q))).$$

Since $\rho_l(\gamma) \to x_y$ and $p \notin \{x, y\}$, there exists a suitable integer l_2 such that for $l > l_2$ the ball $B_\delta(p)$ is contained in the ball $B_\delta(x)$. Hence, by taking $l > \max\{l_0, l_1, l_2\}$ we get

$$P_2 - \varepsilon < \beta_{l,\gamma^{-1}O}(B_\delta(p)) = \beta_{l,O}(\rho_l(\gamma)(B_\delta(p))) \leq \beta_{l,O}(B_\delta(x)) < \varepsilon$$

leading us to a contradiction. Hence $\{x, y\} \cap \{p, q\} \neq \varnothing$. Without loss of generality we can suppose that $x = p$. By applying the same argument to $\rho_n(\gamma)^{-1}$, which diverges to y_x, we get $y = q$, as claimed.

If the sequence $\rho_l(\gamma)$ is bounded and converges to $g \in SL(2, \mathbb{C})$, the equivariance of the measures $\beta_{l,O}$ implies

$$g_*(\beta_{\infty,O}) = \beta_{\infty,\gamma O}.$$

from which the second part of the statement follows. □

Lemma 4.2.2 *Let $g_l \in SL(2, \mathbb{C})$ be a sequence of elements diverging to x_y, with $x \neq y$. Then the elements g_l are eventually loxodromic and if we denote by* $\text{Fix}(g_l) = \{x_l, y_l\}$ *the set of points fixed by g_l, up to relabelling the points x_l and y_l, we have $x_l \to x$ and $y_l \to y$.*

Proof. For $\delta > 0$ we fix neighborhoods $B_\delta(x)$ and $B_\delta(y)$ of x and y, respectively. The sequence $g_l \to x_y$, thus there exists $l_0 = l_0(\delta)$ such that for each $l > l_0$ we have that $g_l(\partial_\infty \mathbb{H}^3 \setminus B_\delta(y)) \subseteq B_\delta(x)$. In particular, we can suppose $g_l(B_\delta(x)) \subseteq B_\delta(x)$ thanks to the assumption $x \neq y$. By the Brower fixed point theorem, g_l admits a fixed point in $B_\delta(x)$, which we denote by x_l. By noticing that $g_l^{-1} \to y_x$, the previous reasoning applied to g_l^{-1} shows that there is a point $y_l \in B_\delta(y)$ fixed by g_l^{-1}, hence by g_l. Since we can consider δ small enough so that $B_\delta(x)$ and $B_\delta(y)$ do not intersect, we know that $x_l \neq y_l$. To sum up, we have found two sequences x_l, y_l of points fixed by g_l, such that

$$\forall \delta > 0, \exists l_0 : x_l \in B_\delta(x), y_l \in B_\delta(y), \ \ \forall l > l_0,$$

and the statement is proved. □

Proposition 4.2.3 *If* $\mathrm{supp}(\beta_{\infty,O}) = \{p, q\}$, *then it holds*

$$d_\omega(O_\omega, \mathrm{Min}(\rho_\omega(\gamma))) = 0,$$

that is the basepoint $O_\omega = [O]_\omega$ *lies in the minimal locus* $\mathrm{Min}(\rho_\omega(\gamma))$ *for every* $\gamma \in \Gamma$.

Proof. We first need to show that there exists a positive constant C such

$$d(\rho_l(\gamma)O, O) < C\lambda_l$$

where λ_l is the sequence of the minimal displacements fixed at the beginning of this section. We need to do this because the point O does not coincide with the point minimally displaced by ρ_l. If the sequence of $\rho_l(\gamma)$ is bounded we are done. Otherwise, the sequence $\rho_l(\gamma)$ is diverging to a quasi-constant map. Hence, by Lemma 4.2.2 they are eventually loxodromic. Thus we can write

$$d(\rho_l(\gamma)O, O) = \mathfrak{L}_{\mathbb{H}^3}(\rho_l(\gamma)) + 2d(O, \mathrm{Min}(\rho_l(\gamma))).$$

By the choice of the sequence λ_l we already know that $\mathfrak{L}_{\mathbb{H}^3}(\rho_l(\gamma)) < C_0\lambda_l$, where C_0 is a suitable constant. Indeed, we know that there exists a sequence x_l of points in $\mathbb{H}^3$ such that

$$d_{\rho_l}(x_l) = \sqrt{\sum_{i=1}^{s} d(\rho_l(\gamma_i)x_l, x_l)} \leq \lambda_l + 1/l.$$

Hence, by an easy computation, it follows

$$\mathfrak{L}_{\mathbb{H}^3}(\rho_l(\gamma)) = \inf_{x \in \mathbb{H}^3} d(\rho_l(\gamma)x, x) \leq d(\rho_l(\gamma)x_l, x_l) \leq ||\gamma||_S d_{\rho_l}(x_l) \leq C_0\lambda_l$$

as claimed. We are going to prove that $d(O, \mathrm{Min}(\rho_l(\gamma)))$ is a bounded sequence.

Let p_0 be the orthogonal projection of O on the geodesic determined by $\{p, q\}$. We denote by p_l the orthogonal projection of p_0 on the geodesic whose endpoints are $\{x_l, y_l\} = \mathrm{Fix}(\rho_l(\gamma))$.
By Lemma 4.2.1, if the sequence $\rho_l(\gamma)$ is diverging we have that $\omega\text{-}\lim_{l\to\infty}\rho_l(\gamma) = p_q$

either $\omega\text{-lim}_{l\to\infty}\rho_l(\gamma) = q_p$. We claim that $\omega\text{-lim}_{l\to\infty}p_l = p_0$. Let δ be a positive real number. Since $\omega\text{-lim}_{l\to\infty}x_l = p$ and $\omega\text{-lim}_{l\to\infty}y_l = q$, we know that the set

$$\{l : \angle_{p_0}(x_l, y_l) \geq \pi - \varepsilon\} \in \omega$$

for a suitable choice of $\varepsilon > 0$, by the continuity of the angle function $\angle_{p_0}(\cdot,\cdot)$. This guarantees that

$$\{l : d(p_0, \mathrm{Min}(\rho_l(\gamma))) = d(p_0, p_l) < \delta\} \in \omega.$$

By the arbitrary choice of δ we get $\omega\text{-lim}_{l\to\infty}d(p_0, p_l) = 0$, that is $\omega\text{-lim}_{l\to\infty}p_l = p_0$, as desired. It must hold

$$\omega\text{-}\lim_{l\to\infty}\frac{d(O, p_l)}{\lambda_l} = 0$$

because $\omega\text{-lim}_{l\to\infty}p_l = p_0$ and the distance $d(O, p_0)$ is bounded.

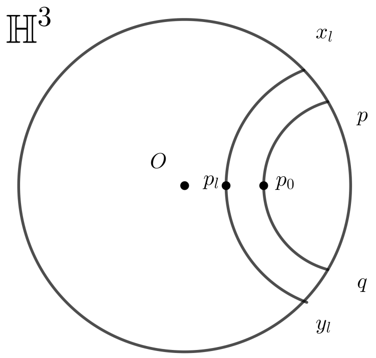

Figure 4.1: The sequence $(p_l)_{l\in\mathbb{N}}$ ω-converges to p_0.

From the estimate

$$d(O, \mathrm{Min}(\rho_l(\gamma))) \leq d(O, p_l)$$

we argue that there exists a positive constant C such that

$$d(\rho_l(\gamma)O, O) < C\lambda_l$$

for every $\gamma \in \Gamma$. Thus we have a well-defined isometric action of Γ on $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$.

The previous computation proves that the point O_ω lies in $\mathrm{Min}(\rho_\omega(\gamma))$ for every element $\rho_\omega(\gamma)$. The claim is clear if $\rho_l(\gamma)$ is bounded in $SL(2,\mathbb{C})$. Indeed since the sequence $d(\rho_l(\gamma)O, O)$ is bounded we have that

$$d_\omega(\rho_\omega(\gamma)O_\omega, O_\omega) = \omega\text{-}\lim_{l\to\infty}\frac{d(\rho_l(\gamma)O, O)}{\lambda_l} = 0$$

and so the isometry ρ_ω has translation length equal to zero and the point O_ω is fixed, that is $O_\omega \in \mathrm{Min}(\rho_\omega(\gamma))$.

Otherwise, considering the same notation as above, the sequence p_l defines a point p_ω in $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ and $d_\omega(O_\omega, p_\omega) = 0$. Moreover p_ω lies on $\mathrm{Min}(\rho_\omega(\gamma))$, indeed

$$d_\omega(p_\omega, \rho_\omega(\gamma)p_\omega) = \omega\text{-}\lim_{l\to\infty} \frac{d(p_l, \rho_l(\gamma)p_l)}{\lambda_l} = \omega\text{-}\lim_{l\to\infty} \frac{\mathfrak{L}_{\mathbb{H}^3}(\rho_l(\gamma))}{\lambda_l} = \mathfrak{L}_\omega(\rho_\omega(\gamma))$$

and this implies that $O_\omega \in \mathrm{Min}(\rho_\omega(\gamma))$. We denoted by $\mathfrak{L}_{\mathbb{H}^3}$ and by $\mathfrak{L}_\omega$ the translation length functions on $\mathbb{H}^3$ and on $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$, respectively. Hence, we have shown for every $\gamma \in \Gamma$ the isometry $\rho_\omega(\gamma)$ of the asymptotic cone has minimal locus passing through the basepoint O_ω. $\square$

By summarizing what we have shown so far, we get the following

Proposition 4.2.4 *Let Γ a non-uniform lattice of $PSL(2,\mathbb{C})$ without torsion and let ω be a non-principal ultrafilter on $\mathbb{N}$. Let $\rho_l : \Gamma \to SL(2,\mathbb{C})$ be a diverging sequence of non-elementary representations. Denote by $D_l : \partial_\infty\mathbb{H}^3 \to \partial_\infty\mathbb{H}^3$ the unique measurable map associated to ρ_l. If $\beta_{\infty,O} = \omega\text{-}\lim_{l\to\infty}(D_l)_*(\mu_x)$ is supported on two points, then the representation $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$ associated to the sequence $(\rho_l)_{l\in\mathbb{N}}$ determines an abelian action on $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$.*

Chapter 5

Open problems and final remarks

We want to conclude this dissertation with some remarks and a list of open problems related to the notions exposed so far. We start with some comments about the proof of the main rigidity theorems, that is Theorem 1, Theorem 3 and Theorem 4. A key point to show the rigidity at infinity of the volume function for representations of lattices in rank-one Lie groups is given by the sharpness of the estimate of the Jacobian of natural maps.

In both [CF03a] and [CF03b] the authors generalize the construction of natural maps to lattices in Lie groups of any rank by still obtaining an estimate on the Jacobian. The estimate is sharp for lattices in products of rank-one Lie groups, but this fails dramatically for Lie groups which cannot be written as products of rank-one Lie groups. A sharp estimate would be a fundamental ingredient to solve the minimal entropy rigidity conjecture, which is still an open problem (see [BCG96, Question 5]). However, the sharpness for lattices in products of rank one Lie groups suggests us that it should be possible to extend the strong rigidity at infinity at least in this more general context.

In the same way, it would be nice to have a rigidity result for the ω-Borel invariant in order to generalize [BBI, Theorem 1]. Given a non-uniform torsion free lattice Γ of $PSL(2,\mathbb{C})$ and a representation $\rho_\omega : \Gamma \to SL(2,\mathbb{C}_\omega)$, Corollary 3.4.2 gives us a result of weak rigidity for $\beta_2^\omega(\rho_\omega)$. Indeed we need to assume the non-degenerancy of the decoration φ_ω associated to ρ_ω to apply correctly Theorem 1 and conclude. The main difficulty in dropping the non-degenerancy hypothesis relies on the fact that a priori we do not know if the condition $\beta_2^\omega(\rho_\omega) = \mathrm{Vol}(M)$ implies automatically that $\beta_2^\omega(\rho_\omega) = \omega\text{-}\lim_{l\to\infty} \beta_2(\rho_l)$. The problem is even more complicated if we move to representations $\rho : \Gamma \to SL(n,\mathbb{C}_\omega)$. Indeed we do not know which conditions we have to assume about ρ_ω in order to get $\beta_n^\omega(\rho_\omega) = \omega\text{-}\lim_{l\to\infty} \beta_n(\rho_l)$. Finally, another interesting aspect would be the possibility to relate the vanishing of $\beta_n^\omega(\rho_\omega)$ with information about the action induced on the asymptotic cone $C_\omega(X_n, d/\lambda_l, O)$, similarly to what we have done for real trees (see Proposition 3.4.3).

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