Alma Mater Studiorum - Universitá di Bologna

DOTTORATO DI RICERCA

Fisica

Ciclo XX

**Settore scientifico disciplinare di afferenza: FIS/02**

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# WORLDLINE APPROACH TO HIGHER SPIN FIELDS

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**Esame finale anno 2008**

# Introduction and motivations

The main object of this thesis is the analysis and the quantization of spinning particle models which employ extended "one dimensional supergravity" on the worldline, and their relation to the theory of higher spin fields (HS).

In the first part of this work we will describe the classical theory of massless spinning particles with an $SO(N)$ extended supergravity multiplet on the worldline, in flat and more generally in maximally symmetric backgrounds. These (non)linear sigma models describe, upon quantization, the dynamics of particles with spin $\frac{N}{2}$ [1] [2] (see also [3] [4] [5] for further analysis on the $N = 2$ case).

Then we will analyze carefully the quantization of spinning particles with $SO(N)$ extended supergravity on the worldline, for every $N$ and in every dimension $D$. The physical sector of the Hilbert space reveals an interesting geometrical structure: the generalized higher spin curvature (HSC). We will see, in particular, that these models of spinning particles describe a subclass of HS fields whose equations of motions are conformally invariant at the free level [6] [7] [8] [9] [10]; in $D = 4$ this subclass describes all massless representations of the Poincaré group.

In the third part of this work we will consider the one-loop quantization of $SO(N)$ spinning particle models by studying the corresponding partition function on the circle. After gauge fixing the supergravity multiplet, the partition function reduces to an integral over the corresponding moduli space which will be computed using orthogonal polynomial techniques.

Finally we will extend our canonical analysis, described previously for flat space, to maximally symmetric target spaces (i.e. $(A)dS$ background). The quantization of these models produce $(A)dS$ HSC as the physical states of the Hilbert space; we will then use an iterative procedure and Pochhammer functions to solve the differential Bianchi identity in maximally symmetric spaces.

In the last part of this work we will construct spinning particle models with $sp(2)$ $R$ symmetry, coupled to Hyper Kähler and Quaternionic Kähler (QK) backgrounds[1].

[1]Spinning particle models with $U(N)$ $R$ symmetry in Kähler background have been considered in [11].

Motivated by the correspondence between $SO(N)$ spinning particle models and HS gauge theory, and by the notorious difficulty one finds in constructing an interacting theory for fields with spin greater than two [12], we intend to employ these one dimensional supergravity models to study and extract informations on HS.
One advantage of using the "worldline" point of view, for studying HS fields, is the fact that this approach gives a quite interesting representation of the one-loop quantum field theory (**qft**) effective action in terms of a quantum mechanical model (**qm**) (see [13] [14] [15] and references therein). Let us briefly review the relation of first quantized particle to the formalism of **qft**.

**The worldline approach** to **qft** is a powerful technique to calculate effective actions in quantum field theories and study anomalies, amplitudes etc... . The main idea can be introduced as follows: consider a **qft** action principle and use a path integral on the fields $\phi(x)$ to construct the effective action $\Gamma_{qft}$ (second quantization); after that, one can convert $\Gamma_{qft}$ into a path integral over space-time coordinate $x$ (first quantized theory, or equivalently particle theory). These two kinds of approaches are equivalent, at least at the perturbative level.

As an example one may consider the simple case of a free scalar field $\phi$

$$S_{qft}[\phi] = \int d^4x \frac{1}{2}\partial^\mu\phi\partial_\mu\phi = \int d^4x \frac{1}{2}\phi(-\Box)\phi \ . \tag{1}$$

The partition function reads

$$\begin{aligned} Z_{qft} \equiv e^{-\Gamma_{qft}} = \int \mathcal{D}\phi e^{-S_{qft}[\phi]} &= \Big(\boldsymbol{det}(-\frac{1}{2}\Box)\Big)^{-\frac{1}{2}} \\ &= exp\Big(-\frac{1}{2}\boldsymbol{tr}(\boldsymbol{ln}(-\frac{1}{2}\Box)\Big) \ . \end{aligned} \tag{2}$$

Then the effective action $\Gamma_{qft}$ can be re-expressed as

$$\begin{aligned} \Gamma_{qft} &= \frac{1}{2}\boldsymbol{tr}\big(\boldsymbol{ln}(-\frac{1}{2}\Box)\big) \\ &= -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\boldsymbol{tr}\ e^{-\beta\left(-\frac{1}{2}\Box\right)} \\ &\sim -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int_{PBC} e^{-S_{qm}[x]} \end{aligned} \tag{3}$$

where $\beta$, the Schwinger parameter, will be recognized as the proper time of a relativistic particle and $PBC$ means that we take periodic boundary conditions $x(\beta) = x(0)$. In the second line of (3) there appears the operator $(-\frac{1}{2}\Box)$ which may be interpreted as the hamiltonian operator of a suitable **qm** model. The trace is then computed by using the path integral for the **qm** action $S_{qm}$ given by

$$S_{qm}[x] = \int_0^\beta d\tau\ \frac{1}{2}\dot{x}^2 \ . \tag{4}$$

Originally this **qm** model was considered to be "fictitious"; however, later, it has been realized that it arises from the quantization of the bosonic particle model

$$S_{qm}[x,e] = \int_0^1 d\tau \frac{\dot{x}^2}{2e} \tag{5}$$

with a suitable gauge fixing of the one dimensional einbein $e$[2]. The partition function on the circle ($PBC$), for the model (5) reads formally as

$$Z_{qm} = \int_{PBC} \frac{\mathcal{D}x\mathcal{D}e}{\mathrm{Vol(Gauge)}} e^{-S_{qm}[x,e]}; \tag{6}$$

the gauge fixing of the einbein to the proper time ($e = \beta$) produces an integral over the modulus $\beta$ [16]. The gauge fixed partition function is:

$$\begin{aligned} Z_{qm} &\equiv \int_0^\infty \frac{d\beta}{\beta} \int_{PBC} \mathcal{D}x e^{-S_{qm}[x,e=\beta]} \\ &= \int_0^\infty \frac{d\beta}{\beta} Z_{qm}(\beta) \end{aligned} \tag{7}$$

and reproduces (3). Concluding one has:

$$\Gamma_{qft} \sim \int_{PBC} \frac{\mathcal{D}x\mathcal{D}e}{\mathrm{Vol(Gauge)}} e^{-S_{qm}[x,e]} \; . \tag{8}$$

Thus we see that a **qft** object (in particular $\Gamma_{qft}$) can be calculated by using a **qm** model.

After having emphasized the relations between particle and fields (first and second quantization) let us focus our attention on **qm** path integrals.
Consider the action principle

$$S_{qm} = \frac{1}{\beta} \int_0^1 d\tau \Big[\frac{1}{2}\dot{x}^\alpha \dot{x}^\beta \delta_{\alpha\beta} + \beta^2 V(x)\Big] \tag{9}$$

where $V(x)$ is a scalar potential. To compute the path integral one can extracts the dependence on the zero modes $x_0^\alpha$

$$x^\alpha(\tau) = x_0^\alpha + y^\alpha; \tag{10}$$

$y^\alpha$ is the quantum fluctuation and we impose Dirichlet boundary conditions $y^\alpha(0) = y^\alpha(1) = 0$. This describes a loop with a fixed point.

[2]Note that a canonical analysis of (5) produces the massless Klein-Gordon equation of motion as a constraint of the physical sector of the Hilbert space.

At this point the computation is straightforward and the **qm** path integral can be evaluated in perturbative expansion:

$$\int_{PBC} Dx e^{-\frac{1}{\beta}\int_0^1 d\tau[\frac{1}{2}\dot{x}^2+\beta^2 V(x)]} = \frac{1}{(2\pi\beta)^2}\int d^D x_0 \Big(a_0 + \underbrace{a_1\beta + a_2\beta^2 + \cdots}_{=0 \text{ in the free case}}\Big) \tag{11}$$

where the coefficients $a_i$ are the so called Seeley-DeWitt coefficients [17] in the coincidence limit; they characterize the quantum theory, for example we can use them to identify the counterterms needed to renormalize the full one loop effective action. In the free case limit one finds that $a_i = 0 \quad \forall i \neq 0$ whiel $a_0$ counts the degrees of freedom $(Dof)$ propagated into the loop. In the example discussed above one finds, in fact, $a_0 = 1$, that coincide with the degrees of freedom associated to a scalar field.
At this point one can introduce more general backgrounds, for example electromagnetic background $(V(x,\dot{x}) = -igA(x)\cdot\dot{x})$ or gravitational background, $(\delta_{\alpha\beta} \longrightarrow g_{\mu\nu}(x)$ and $V(x) = \xi R(x))$. In this cases, in order to define properly the path integrals, one requires the introduction of a regularization scheme to make sense on the integration over paths, and the fixing of certain renormalization conditions, which makes sure that different regularization schemes will produce the same results. We are not going to describe all the details, since regularization is not the main object of our work (more details on path integral and anomalies could be found in [18]).

The previous considerations can be extended to the spinning case by introducing extra fermionic degrees of freedom $\psi^\alpha$, the supersymmetric partner of $x^\alpha$, on the worldline [19] [20] [21].
Note that supersymmetry on the worldline does not imply supersymmetry on space-time. In fact all the spinning particle models we will deal with describe the propagation of multiplets of the same statistical nature (either fermionic or bosonic).
In fact it is well known that the dynamics of a spin 1/2 particle in $D = 4$ Minkowsky background can be described by using a quantum mechanical model with $N = 1$ local supersymmetry on the worldline:

$$S^{(1)} = \int dt \big[p_\alpha \dot{x}^\alpha + \frac{i}{2}\psi^\alpha\dot{\psi}^\beta\eta_{\alpha\beta} - e\big(\frac{1}{2}p^\alpha p_\alpha\big) - i\chi(\psi^\alpha p_\alpha)\big] \ . \tag{12}$$

A canonical analysis of this model produces the massless Dirac equation as constraints for the physical sector of the Hilbert space. In (2.30) $e$ and $\chi$ are lagrangian multipliers introduced to implement into the action the first class constraints $H = p^2$ (i.e. the hamiltonian, the generator of worldline diffeomorphism) and $Q = \psi^\alpha p_\alpha$ (i.e. supercharge, the generator of supersymmetry).
More in general, in this work, we will study spinning particle models enjoying $N$ local worldline supersymmetries, $SO(N)$ gauge symmetry and worldline diffeomorphism: the $SO(N)$ spinning particle models.

**The $SO(N)$ spinning particle models** have been extensively studied in [1] [2] [3] [8] [22] [23] [24] [25] [26].

In **Chapter 1** we will start describing the model in flat Minkowsky target space and we will analyze in details its symmetries and the first class constraints algebra, namely the $SO(N)$ extended supersymmetry algebra. After that we will focus our attention on the possible coupling with gravitational background preserving one dimensional local supersymmetry. Howe et al. have analyzed this problem in [2] and have concluded that, with a given generalization of the worldline local supersymmetry transformation, $SO(N)$ spinning particle models can be consistently coupled to a gravitational background only when $N = 0, 1, 2$; however Kuzenko and Yarevskaya have relaxed this no go theorem introducing in [27] a coupling with maximally symmetric space (i.e. $(A)dS$) for every $N$, by extending in a clever way the local worldline supersymmetry transformation rules. In **Chapter 1** we will propose also an alternative approach to this problem; we will work in first order formalism and we will study the first class constraints algebra in curved space showing explicitly that in maximally symmetric space, turns out to be closed, yet non linear. Finally we will compare our results with the Kuzenko and Yarewskaya ones. In **Appendix B** we will also discuss the construction of the BRST charge associated to this non linear $SO(N)$ extended supersymmetry algebra.

The $SO(N)$ spinning particles, from our point of view, are very interesting because in $D = 4$ Minkowskian background, one finds, via canonical analysis, the well known Bargman Wigner equation of motion [28] describing the dynamics of spin $s = \frac{N}{2}$ fields (i.e. Higher Spin Field).

**Higher Spin Fields** and in particular HS gauge theory have attracted a great deal of attention in the search for generalizations of the known gauge theories of fields of spin 1 (Maxwell and Yang-Mills theory), spin 2 (Einstein general relativity) and spin $\frac{3}{2}$ (supergravity); see [29] [30] [31] [32] [33] and references therein.
Further motivations in studying HS arise from Superstring Theory; superstring theory is, in fact, the most important candidate for a unified theory of interactions. The key idea of string theory is very simple: extension of point particles (i.e. 0 dimensional object) to one dimensional object (i.e. string) with length $l_s \sim 10^{-33} cm$. The Regge slope $\alpha'$ and the string tension $T$ are defined as

$$l_s = \sqrt{2\alpha'} = \frac{1}{\sqrt{\pi T}} \ . \tag{13}$$

Vibrational string states generate an infinite number of states with mass $m_s$ and spin $s$ given by:

$$m_s^2 \sim \frac{1}{\alpha'}(s-1) \sim T(s-1) \ . \tag{14}$$

String theory naturally describes an infinite number of excited massive states, with increasing mass and spin. String tension is usually taken to be of the same order of the Plank mass ($M_{pl} \sim 10^{19} Gev$), for this reason at low energy just the massless modes are exited (with $s \leqslant 2$) and HS states are too heavy and cannot be not observed at low energy. In the high energy limit, all the HS string states may effectively become massless, so that one may be left with an infinite number of massless HS states, i.e. a

HS gauge theory [35].
For all this reason a better understanding of the dynamics of HS states is important for the analysis of quantum properties of String Theory.
The construction of an interacting HS theory is a main long standing problem (see [36] for recent developments in the spin 3 case); the general Coleman-Mandula and Haag-Lopuszanski-Sohnius theorem of the possible symmetries of the unitary S-matrix of the quantum field theory in $D = 4$ Minkowsky space [37] does not allow conserved currents associated with the symmetries of fields with spin greater than two to contribute to the S-matrix (see also [38]). This no-go theorem might be overcome if the higher spin symmetries would be spontaneously broken. There exists also another way out: one could construct the interacting HS field theory in a vacuum background with a non-zero cosmological constant (i.e. $(A)dS$), in which the S-matrix theorem does not apply. Positive results along these lines have been achieved and the most notorious is perhaps the Vasilievs interacting field equations, which involve an infinite number of fields with higher spin [12], but an action principle for them is still lacking; further and interesting analysis on HS fields in $(A)dS$ background can be found in [39] and [40] [41].

Relations between HS field theory and Superstring Theory suggest us that one has to study the HS dynamics in $D \neq 4$ and in particular in $D = 10$, that is the superstring critical dimension. Note now that in $D = 4$ symmetric tensors of rank $s$ (that we denote with $\phi_{\alpha_1 \ldots \alpha_s}$) describe all possible higher spin representation of the Poincaré group; in this case, in fact, all the irreducible massless representations of the Poincaré group are classified by using the group $SO(2)$ whose Young tableaux are single rows; this implies that massless HS fields could be represented by totally symmetric tensors. Something news happens in $D \neq 4$; in higher dimension, symmetric tensors do not describe all possible HS fields. For example in $D = 10$ the compact subgroup of the little group is $SO(8)$; the representations of $SO(8)$ are, in general, Young tableaux with mixed symmetry; see [42] for a recent review on this topics.

In **Appendix A** we will discuss different approaches to HS gauge theory. In particular we will focus our attention on the geometric approach to HS and *conformal HS gauge theory*. Conformal HS is an interesting and important subclass of HS fields whose (at least linearized) equations of motion are conformally invariant; it's important to observe that in this case, in every even $D > 4$ dimensions, conformal HS fields are not completely symmetric tensors, and the corresponding Young tableaux are rectangles with $s$ columns and $\frac{D}{2} - 1 = n - 1$ rows (i.e. $s \otimes (n-1)$ Young tableaux); fields with this kind of symmetries will be denoted with $\phi_{[n-1]_1 \cdots [n-1]_s}$ (further analysis and references on *conformal HS gauge theory* can be found in [43]).
In general, integer HS fields dynamics, both for completely symmetric or with the symmetries of a rectangular Young tableaux, is described by the Frosndal (for completely symmetric tensor) and Fronsdal-Labastida (that is the generalization for tensors with mixed symmetry) equation of motion:

$$G\phi_{\alpha_1 \cdots \alpha_s} = 0 \qquad \text{and} \qquad G\phi_{[n-1]_1 \cdots [n-1]_s} = 0 \tag{15}$$

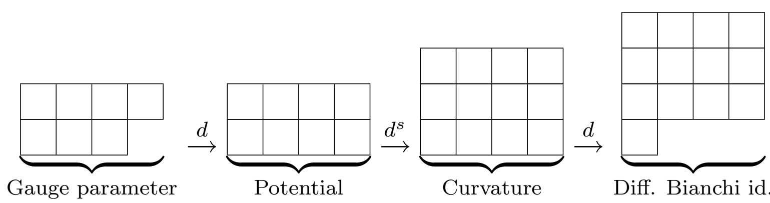
where $G$ is the Fronsdal-Labastidal kinetic operator. It is important to emphasize that this operator is a second order differential operator which guarantees the unitary of the theory.

**Generalized Higher Spin Curvatures** (HSC), or equivalently HS field strengths, are the key ingredients of the geometrical approach to HS gauge theory.
More in general gauge theories are usually presented in terms of a local symmetry of an action principle. Other ingredients, however, are gauge parameters, gauge fields equation of motion and Bianchi identities.

Let now $Y_p$ be a Young diagram such that the number of cells of $Y_p$ is $p$. We define $\Omega^p_{(Y)}(\mathbb{R}^D)$ to be the vector space of tensor fields of rank $p$ on $\mathbb{R}^D$ which have the Young symmetry type $Y_p$. We define now the differential operator

$$d = (-)^p \mathbf{Y}_{p+1} \partial : \Omega^p_{(Y)}(\mathbb{R}^D) \to \Omega^{p+1}_{(Y)}(\mathbb{R}^D). \qquad (16)$$

This operator acts as partial derivative of the tensor $T \in \Omega^p_{(Y)}(\mathbb{R}^D)$, then we have to apply the Young symmetrizer $\mathbf{Y}_{p+1}$ to obtain a tensor in $\Omega^{p+1}_{(Y)}$; note that in general $d^2 \neq 0$.
Let us restrict now ourselves to tensor fields in $\mathbb{R}^D$ with the symmetries of a Young tableaux with number of columns strictly smaller than $s+1$, and we consider gauge potential with the symmetry of a rectangular Young tableaux with $s$ columns[3]. Gauge theory for arbitrary spin $s$ in $\mathbb{R}^D$, by using the differential operator (16), can be packaged into a complex, that schematically one can writes as:




; (17)

the fact that (17) is a complex follows form the property $d^{s+1} = 0$ [44]; moreover the *generalized Poincaré Lemma* implies that the complex (17) is also an exact sequence. This assures that the the differential Bianchi identity can be solved by introducing the gauge potential, and this solution is unique (at least in $\mathbb{R}^D$); moreover exactness of (17) implies that the curvature is gauge invariant only with respect the transformation $\delta(\text{potential}) = d(\text{gauge parameter})$.
In particular, let now $d_i$ be the usual exterior derivative acting on the $i^{th}$ column so that:

$$d_i \prod_{j=1}^{s} d_j = 0 \qquad \forall\ i = 1, ..s\ . \qquad (18)$$

In virtue of the *generalized Poincaré Lemma* one has

$$d_i \mathcal{R} = 0 \ \ \forall \ \ i = 1, 2, ..., s \ \ \Rightarrow \ \ \mathcal{R} = \prod_{j=1}^{s} d_j \phi \qquad (19)$$

[3]Let us recall that in $D = 4$ gauge fields have the symmetry of a Young tableaux $s \otimes 1$ while in $D \neq 4$ and in particular in even dimension $D = 2n$ we can construct also representation of the Poincaré group with the symmetry of the Young tableaux $s \otimes (n-1)$.

where $\mathcal{R}$ (i.e. curvature) and $\phi$ (i.e. potential) are multiforms which components are irreducible tensor fields with the symmetry of the Young tableaux $s\otimes(m+1)$ and $s\otimes m$ respectively, for some $m$. More details and generalization to tensor with different kind of symmetries can be found in [45] [46].

Higher spin curvatures are constructed as the natural generalization of the spin 1 Maxwell field strength and of the linearized Riemann tensor in the case of spin 2. HS gauge potential can be introduced by solving the differential Bianchi identity (19). The main advantage one has by using this geometrical approach, is the fact that the theory is automatically gauge invariant; otherwise HS potential equation of motion (obtained from the traceless condition on the HSC) suffers for higher derivative problem, and this implies that the theory is not unitary. This problem can be cured as follows: in virtue of the *generalized Poincaré Lemma* one can introduce the so called compensator field and after having gauge fixed it to zero the HS equation of motion reduces to (15) (more details can be found in **Appendix A**).

In **Chapter 2** we will analyze the physical contents of spinning particle models with $SO(N)$ extended supergravity multiplet on the world line, in flat target space, and for every dimension $D$. We will focus our attention on the integer spin case (even $N$), and we will quantize the model "á la Dirac"; the phase space coordinates will be turned into operators and the Hamiltonian, supersymmetry and $SO(N)$ constraints are imposed as operatorial constraints on the Hilbert space states. We will compare our results with the *conformal HS gauge theory* and we will show that a canonical analysis produce "conformal" HSC as physical states of the Hilbert space.

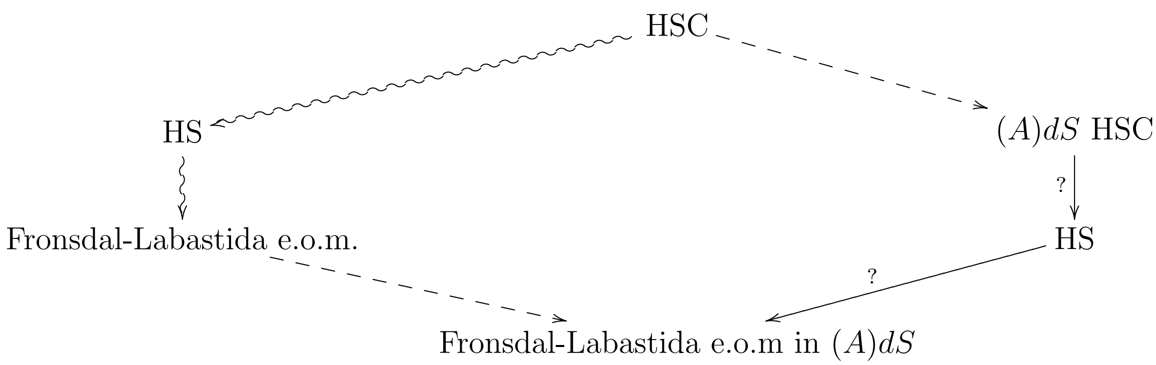
After having analyzed the spectrum of the $SO(N)$ spinning particle model, and after having understood its physical contents we proceed computing the path integral.

**One loop quantization** of spinning particles is the main objective of **Chapter 3**. First of all we will gauge fix the one dimensional supergravity multiplet on the circle and we will introduce the Faddeev-Popov determinant to extract the volume of the gauge group. We will calculate the path integral on the one-dimensional torus in order to obtain a compact formula which gives the number of physical degrees of freedom of the spinning particles for all $N$ in every dimensions $D$. Let us emphasize that the computation of the path integral allows us to obtain the correct measure on the moduli space of the supergravity multiplet on the circle; this is going to be extremely useful also to compute more general quantum corrections arising when couplings to background fields are introduced.

Spinning particles with $SO(N)$ supergravity multiplet on the worldline, can also propagate, in fact, in maximally symmetric space (i.e. $(A)dS$). In this work we intend to analyze the physical content of the spinning particle model with $SO(N)$ supergravity multiplet on the worldline, coupled to maximally symmetric space. In the literature the dynamics of HS in $(A)dS$ background has been extensively studied; in $(A)dS_4$, for example, integer HS gauge potential $\phi_{\alpha_1\cdots\alpha_s}$ equation of motion reads

$$G^{(A)dS}\phi_{\alpha_1\cdots\alpha_s}=0; \tag{20}$$

where $G^{(A)dS}$ is the Fronsdal-Labastida kinetic operator obtained starting from $G$ by minimal coupling plus terms introduced to restore gauge invariance. However it is not yet clear how to obtain this operator starting from a pure geometrical object (i.e. $(A)dS$ HSC). A possible explicative scheme is the following:




where:

$A \rightsquigarrow B$

- means that we go from $A$ to $B$ by using the *generalized Poincaré Lemma*

$A \dashrightarrow B$

- means that $B$ is obtained by minimal covariantization of $A$.

$A \xrightarrow{?} B$

- means that one should obtain $B$, starting from $A$ by using the *generalized Poincaré Lemma*; the interrogation point on the arrow emphasizes that an extension of this Lemma in $(A)dS$ space is still lacking, and we cannot use it as a guide line.

Recently some progresses in constructing $(A)dS$ HSC have been obtained by Enquist and Hohm in [47] where they have studied the dynamic of HS in $(A)dS$ in frame-like formulation[4], and by Manvelyan and Ruhl in [49].

**Conformal HSC in** $(A)dS_{2n}$ will be extensively studied in **Chapter 4**. A canonical analysis of $SO(N)$ spinning particles produces $(A)dS$ HSC as the physical sector of the Hilbert space. In order to find the relation between HS field strength and the gauge potential, and in particular construct the Fronsdal-Labastida operator in $(A)dS$, we cannot reproduce the procedure described for the flat case because the *generalized Poincaré Lemma* is the key ingredient one needs to extract the potential from the curvature (equivalently solve differential Bianchi identity) and construct gauge theories.

[4]Gravity (i.e. spin 2 theory) could be described using metric and curvature, or vielbein and spin connection. Frame-like approach is based on the generalization of vielbein and spin connection, instead of metric and curvature, to the HS case (see [48] for a pedagogical review and references therein).

For this reason we will propose a different approach: we start from the results obtained in the flat case limit, we will add suitable $(A)dS$ corrections and we will use supercharges constraints (i.e. differential Bianchi identity) to fix extra terms. In particular we will propose an iteration procedure that let us to solve differential Bianchi identity for every even dimension $D$ and every integer spin $s$. The analysis is technically complicated, for this reason we have decided to postpone part of it, regarding Pochhammer function, in **Appendix C**.

Recently spinning particle models with Hyper Kähler (HK) and Quaternionic-Kähler (QK) background have attracted a great deal of attention in the context of studying radial quantization of BPS black-hole [50].

**HK and QK** $N = 4$ **one dimensional supergravity** will be analyzed in **Chapter 5**.

HK and QK geometries in dimension $4n$ and signature $(2n, 2n)$ enjoy $sp(2n)$ and $sp(2) \otimes sp(2n)$ holonomy, respectively. Thus we will decompose $SO(2n, 2n)$ tangent space indices with respect to the $sp(2) \otimes sp(2n)$ subgroup and we will construct spinning particle models with $N = 4$ supersymmetry and $sp(2)$ "internal" symmetry: the $sp(2)$ spinning particle model.

In HK this model enjoys rigid worldline translation, $sp(2)$ symmetry and $N = 4$ supersymmetry. We will thus gauge these symmetries and we will study the first class constraints algebra.

Then we will analyze the model with QK background and we will show that in QK target space is no longer possible to maintain rigid supersymmetry, just a model with local supersymmetry is allowed. We will analyze the first class constraints algebra, that in this case becomes a non Lie algebra, and we will show that two possible interesting gauged model should be studied with:

- Rigid $sp(2)$ symmetry, local supersymmetry and worldline diffeomorphism.
- Local $sp(2)$ symmetry, local supersymmetry and worldline diffeomorphism.

We will focus our attention on the first one and we will construct the BRST charge. Moreover we will gauge fix the one dimensional supergravity multiplet on the circle and we will construct also the gauge fixed action.

# Contents

# Notations

During this work we will use the following notation:

| Notation | Meaning |
|---|---|
| $D$ | Space time dimension |
| $D = 2n$ | Even space time dimension |
| $\alpha, \beta, ...$ | Flat space time indices |
| $\eta_{\alpha\beta} \sim (-, +, ..., +)$ | Minkowskyan metric |
| $\delta_{\alpha\beta} \sim (+, +, ..., +)$ | Euclidean metric |
| $\mu, \nu, ...$ | Curved space time indices |
| $g_{\mu\nu}$ | Curved metric |
| $N$ | Number of one dimensional supercharges |
| $i, j, .. = 1, 2, ..., N$ | $SO(N)$ vector indices |
| $I, J, ... = 1, 2, ..., N/2$ | $SO(N)$ vector indices in complex base |
| $\cdot$ | Contraction over space time indices |
| $\circ$ | Contraction over $SO(N)$ vector indices |
| $\{\ ,\ \}_{pb}$ | Poisson and Dirac bracket |
| $[\ ,\ ]$ | Commutators or anticommutators |
| $[xy] = \frac{1}{2}(xy - yx)$ | Antisimmetrization |
| $[m]$ | m indices totally antisimmetrized |
| $\{xy\} = \frac{1}{2}(xy + yx)$ | symmetrization |
| HS | Higher Spin field |
| HSC | Generalized Higher Spin Curvature |
| $R$ | Background curvature |
| $\mathcal{R}$ | HSC |
| $\phi$ | Integer HS field |

# Chapter 1

# $SO(N)$ spinning particle model

In this Chapter we will describe in details the main object of our study: the spinning particle model with $SO(N)$ extended supergravity multiplet on the worldline. This model has attracted a great deal of attention in the context of studying the dynamics of higher spin field (HS). As was shown by Gershun and Tkach in [1], and later by Howe et al. in [2], the mechanics action of spinning particle with a gauged $N$-extended worldline supersymmetry and a local $SO(N)$ invariance, in $D = 4$ Minkowskian background, describes the dynamics of free massless HS. In Ref [1] the massive case was completely treated too. For even dimension it was thoroughly shown in [2] that upon quantization, the physical wave functions are subject to a relativistic conformally invariant equation for pure spin $\frac{N}{2}$. Let us postpone the canonical analysis to the next Chapter; in the following we shall focus our attention on the symmetries of the model and on the interaction with gravitational background preserving one dimensional supersymmetry.

These one dimensional supergravity models were thought for a while to be consistently extended to include coupling with an arbitrary gravitational background only for $N = 0, 1, 2$. When $N$ is bigger than 2, it was originally concluded in [2] that the only space compatible with "*standard*" local worldline supersymmetry transformations rules, is the flat one.
Nevertheless Kuzenko and Yarevskaya have shown in [27] that, with a suitable generalization of worldline supersymmetry transformation rules, this statement can be generalized, for every $N$, to space-time with constant non-zero curvature (i.e. maximally symmetric space):

$$R_{\mu\nu\rho\sigma} = b(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \qquad \rightarrow \qquad R = b\ d(d-1) \tag{1.1}$$

where $b$ is a parameter related to the cosmological $\Lambda$ by the relation

$$\Lambda = b\frac{(d-1)(d-2)}{2}\ . \tag{1.2}$$

The case $b > 0$ corresponds to $dS$ space, $b < 0$ to $AdS$.
In the following we intend to discuss and analyze the results obtained by Howe et al. in [2] and Kuzenko and Yeravskaya in [27]; we will rederive them by using Hamiltonian analysis that reveals an interesting non linear structure of the $SO(N)$ extended superalgebra.

The $SO(N)$ spinning particle action principle is characterized by a $N$ extended supergravity

multiplet on the worldline. The gauge fields $(e, \chi_i, a_{ij})$ of the $N$ supergravity contain in particular the einbein $e$ which gauges worldline translation, $N$ gravitinos $\chi_i$ which gauges the $N$ worldline supersymmetry, and the standard $SO(N)$ gauge fields. The einbein and the gravitinos correspond to constraints that eliminate negative norm state and make the particle model consistent with unitarity [1]. Let us start discussing the "rigid" model. The coordinate space is spanned by the real bosonic variable $x^\alpha(\tau)$, that represents a map from the worldline to the $D$-dimensional Minkowsky space time, and its supersymmetric partners $\psi_i^\beta$; $\alpha, \beta = 1, .., D$ are Lorentz indices, while $i, j = 1, ..., N$ are $SO(N)$ vector indices.
The dynamics is governed by the simple action principle

$$S = \int dt \big[\frac{1}{2}(\dot{x}^\alpha \dot{x}^\beta + i\psi_i^\alpha \psi_i^\beta)\eta_{\alpha\beta}\big] \ . \tag{1.3}$$

This model is invariant with respect rigid time translation, $SO(N)$ and rigid supersymmetry; the corresponding Noether charges are

$$\begin{aligned} H &= \frac{1}{2}\dot{x}^\alpha \dot{x}_\alpha \\ Q_i &= \dot{x}_\alpha \psi_i^\alpha \\ J_{ij} &= i\psi_{[i}^\alpha \psi_{j]}^\beta \eta_{\alpha\beta} \ . \end{aligned}$$

We extend now the model by gauging the symmetry generated by $H$, $Q_i$ and $J_{ij}$:

$$S = \int dt \Big[\frac{1}{2}e^{-1}\eta_{\alpha\beta}(\dot{x}^\alpha - i\chi_i\psi_i^\alpha)(\dot{x}^\beta - i\chi_i\psi_i^\beta) + \frac{i}{2}\eta_{\alpha\beta}\psi_i^\alpha \dot{\psi}_i^\beta - \frac{i}{2}a_{ij}\psi_i^\alpha \psi_j^\beta \eta_{\alpha\beta}\Big] \tag{1.4}$$

where $(e, \chi_i, a_{ij})$ is the one dimensional $SO(N)$ extended supergravity multiplet; thus (1.4) enjoys local symmetries

**Supersymmetry**

$$\begin{aligned} \delta x^\beta &= i\epsilon_i \psi_i^\beta \\ \delta\psi_i^\beta &= -\epsilon_i e^{-1}(\dot{x}^\beta - i\chi_j\psi_j^\beta) \\ \delta e &= 2i\chi_i\epsilon_i \\ \delta\chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j \\ \delta a_{ij} &= 0 \end{aligned} \tag{1.5}$$

**Local** $SO(N)$

$$\begin{aligned} \delta x^\beta &= 0 \\ \delta\psi_i^\beta &= \alpha_{ij}\psi_j^\beta \\ \delta e &= 0 \\ \delta\chi_i &= \alpha_{ij}\chi_j \\ \delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{ik}a_{kj} - \alpha_j^k a_{ki} \end{aligned} \tag{1.6}$$

**Worldline diffeomorphism**

$$\begin{aligned}
\delta x^\beta &= \xi \dot{x}^\beta \\
\delta \psi_i^\beta &= \xi \dot{\psi}_i^\beta \\
\delta e &= \partial_\tau(\xi e) \\
\delta \chi_i &= \partial_\tau(\xi \chi_i) \\
\delta a_{ij} &= \partial_\tau(\xi a_{ij}).
\end{aligned} \tag{1.7}$$

We introduce now $p_\alpha$, the conjugated momentum to $x^\alpha$; in first order formalism the action (1.4) becomes:

$$S^{(1)} = \int dt \Big[ p_\alpha \dot{x}^\alpha + \frac{i}{2}\psi_i^\alpha \dot{\psi}_i^\beta \eta_{\alpha\beta} - e(\frac{1}{2}p^\alpha p_\alpha) - i\chi_i(\psi_i^\alpha p_\alpha) - \frac{i}{2}a_{ij}(\psi_i^\alpha \psi_i^\beta \eta_{\alpha\beta}) \Big] \ . \tag{1.8}$$

In the previous expression we recognize the constraints

$$\begin{aligned}
Q_i &= p_\alpha \psi_i^\alpha \\
H &= \frac{1}{2}\eta_{\alpha\beta}p^\alpha p^\beta \\
J_{ij} &= i\psi_{[i}^\alpha \psi_{j]}^\beta \eta_{\alpha\beta} \ .
\end{aligned}$$

Note that these constraints are first class and the algebra reads:

$$\{Q_i, Q_j\}_{pb} = -2i\delta_{ij}H, \qquad \{J_{ij}, Q_k\}_{pb} = -2i\delta_{k[i}Q_{j]}, \qquad \{J_{ij}, J^{kl}\}_{pb} = -i\delta_{[i}^{[k} J_{j]}^{l]} \tag{1.9}$$

Let us recall to the reader that in the fermionic sector we have eliminated the second-class constraint

$$p_\psi^\alpha + \frac{i}{2}\psi^\alpha \approx 0 \tag{1.10}$$

and we use Dirac bracket

$$\{\psi_i^\alpha, \psi_j^\beta\}_{pb} = -i\delta_{ij}\eta^{\alpha\beta}. \tag{1.11}$$

We intend now to extend the analysis introducing a coupling with gravitational background. Our convention is the following: $\mu, \nu, ...$ are curved indices, we use the flat ones $\alpha, \beta, ..$ for worldline fermions and we introduce the vielbein $e_\alpha^\mu$ to raise and lower these indices:

$$e_\mu^\alpha e_\nu^\beta \eta_{\alpha\beta} = g_{\mu\nu} \qquad \psi_i^\alpha = \psi_i^\mu e_\mu^\alpha \ . \tag{1.12}$$

The action proposed in [2], as the natural extension of (1.4) to the curved background case, reads:

$$\begin{aligned}
S = \int dt \quad & \Big[ \frac{1}{2e} g_{\mu\nu}(\dot{x}^\mu - i\chi_i \psi_i^\alpha e_\alpha^\mu)(\dot{x}^\nu - i\chi_i \psi_i^\alpha e_\alpha^\nu) + \frac{i}{2}\psi_i^\alpha(\dot{\psi}_{i\alpha} - a_{ij}\psi_{j\alpha} + \dot{x}^\mu \omega_{\mu\alpha\beta}\psi_i^\beta) \\
& + \frac{1}{8} e \psi_i^\alpha \psi_i^\beta \psi_j^\gamma \psi_j^\delta R_{\alpha\beta\gamma\delta} \Big]
\end{aligned} \tag{1.13}$$

where $\omega_{\mu\alpha\beta}$ is the spin connection field, and $R_{\alpha\beta\gamma\delta}$ is the curvature tensor.
A suitable generalization of (1.5) to the curved target space case, reads[1]:

$$\begin{aligned}
\delta x^\mu &= i\epsilon_i\psi_i^\mu \\
\delta\psi_i^\alpha &= -\epsilon_i e^{-1}(\dot{x}^\mu e_\mu^\alpha - i\chi_j\psi_j^\alpha) + i\epsilon_j\psi_j^\beta\psi_i^\gamma e_\beta^\mu\omega_{\mu\gamma}{}^\alpha \\
\delta e &= 2i\chi_i\epsilon_i \\
\delta\chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j \\
\delta a_{ij} &= 0\ .
\end{aligned} \tag{1.14}$$

The Action (1.13) transforms under (1.14) as

$$\delta S = \int dt\big[\frac{i}{2}(\epsilon_i\chi_k - \frac{1}{2}\delta_{ik}\epsilon_l\chi_l)\psi_k^\alpha\psi_i^\beta\psi_j^\gamma\psi_j^\delta R_{\alpha\beta\gamma\delta} + \frac{i}{8}e\epsilon_k\psi_k^\mu\psi_i^\alpha\psi_i^\beta\psi_j^\gamma\psi_j^\delta\nabla_\mu R_{\alpha\beta\gamma\delta}\big]\ . \tag{1.15}$$

Howe, Penati, Pernici and Towsend have shown in [2] that this term vanishes only for $N = 0, 1$ or 2:

- $N = 1 \rightarrow \psi^\alpha\psi^\beta\psi^\gamma\psi^\delta R_{\alpha\beta\gamma\delta}$ vanishes because of the cyclic symmetry of Riemann tensor;
- $N = 2 \rightarrow \psi_k^\mu\psi_i^\alpha\psi_i^\beta\psi_j^\gamma\psi_j^\delta\nabla_\mu R_{\alpha\beta\gamma\delta}$ vanishes because of Bianchi identity, while the first one is zero because in this case the contraction of Riemann tensor with four Grassmann variables has to be proportional to $\delta_{ik}$.

Thus for $N = 0$, 1,2 there are no constraints on the background; if $N \geq 3$ equation (1.15) vanishes only if the space time is flat: $R_{\alpha\beta\gamma\delta} = 0$. This suggests that worldline supersymmetry (1.14) is not compatible with gravitational background.
Howe, Penati, Pernici and Towsend concluded that this result seems to be very "natural" and strictly related to the old problem of constructing interaction between massless higher spin field and gravitational background.

However there is still an opportunity to construct $SO(N)$ spinning particle models coupled to gravitational background: modify (1.14). We will discuss it in the next section.

## 1.1 Kuzenko and Yarevskaya construction

Kuzenko and Yeravskaya in [27] have shown how to introduce a coupling to constant non zero curvature space. The starting point is the ansatz proposed by Siegel in [51] in which he has constructed the $D$-dimensional action principle (1.4) starting from an explicitly conformal $O(D, 2)$ invariant mechanics action in $D$ space and 2 time dimensions; Siegel has also shown in [52] that all conformal wave equations, in all dimensions, can be derived in this way.
In the following will not focus our attention on the technique proposed in [27] to introduce a coupling with maximally symmetric space; more details can be found in the original paper.

[1]Note that this generalization in phase space coordinate corresponds with the minimal substitution of the momentum $p$ with covariant momentum $\pi$.

We limit ourselves in describing the results, and we will compare them with our Hamiltonian analysis in the next section.
The action principle proposed by Kuzenko and Yarevskaya reads:

$$S^{KY} = \int dt \Big[\frac{1}{2e} g_{\mu\nu}(\dot{x}^\mu - i\chi_i\psi_i^\alpha e_\alpha^\mu)(\dot{x}^\nu - i\chi_i\psi_i^\alpha e_\alpha^\nu) + \frac{i}{2}\dot{\psi}_i^\alpha(\psi_{i\alpha} - \tilde{a}_{ij}\psi_{j\alpha} + \dot{x}^\mu\omega_{\mu\alpha\beta}\psi_i^\beta)\Big], \quad (1.16)$$

where $g_{\mu\nu}$ is the $(A)dS$ metric.
This action enjoys local supersymmetry:

$$\begin{aligned}
\delta x^\mu &= i\epsilon_i\psi_i^\mu \\
\delta\psi_i^\alpha &= -\epsilon_i e^{-1}(\dot{x}^\mu e_\mu^\alpha - i\chi_j\psi_j^\alpha) + i\epsilon_j\psi_j^\beta\psi_i^\gamma e_\beta^\mu \omega_{\mu\gamma}{}^\alpha \\
\delta e &= 2i\chi_i\epsilon_i \\
\delta\chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j \\
\delta\tilde{a}_{ij} &= -ib\epsilon_{[i}\psi_{j]\alpha}\dot{x}^\mu e_\mu^\alpha \ .
\end{aligned} \quad (1.17)$$

Note now that in maximally symmetric space (i.e. $R_{\alpha\beta\gamma\delta} = b(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})$) the action principle (1.16) coincides with (1.13) except for the following redefinition of the $SO(N)$ gauge field.

$$\tilde{a}_{ij} = a_{ij} - \frac{i}{2}eb\psi_i^\alpha\psi_{j\alpha} \ . \quad (1.18)$$

Thus we can conclude that spinning particle models with $SO(N)$ extended supergravity on the worldline, can be coupled to $(A)dS$ background. In order to preserve local worldline supersymmetry, the $SO(N)$ gauge field has to transform under supersymmertry. In particular, from (1.17) and (1.18), one finds that (1.13) is invariant with respect to the following supersymmetry transformation rules:

$$\begin{aligned}
\delta x^\mu &= i\epsilon_i\psi_i^\mu \\
\delta\psi_i^\alpha &= -\epsilon_i e^{-1}(\dot{x}^\mu e_\mu^\alpha - i\chi_j\psi_j^\alpha) + i\epsilon_j\psi_j^\beta\psi_i^\gamma e_\beta^\mu \omega_{\mu\gamma}{}^\alpha \\
\delta e &= 2i\chi_i\epsilon_i \\
\delta\chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j \\
\delta a_{ij} &= -b\chi_k(\epsilon_k\psi_i^\alpha\psi_{j\alpha} + \psi_k^\alpha\epsilon_{[i}\psi_{j]\alpha}).
\end{aligned} \quad (1.19)$$

## 1.2 Hamiltonian analysis

We would like now to reanalyze, and extend just a little bit, the results discussed above by using Hamiltonian approach.
First of all let us recall some useful formula [53].
Le $X$ be matter fields both fermonic and bosonic and $P_X$ their conjugated momentum. We consider the action principle, written in first order formalism

$$S^{(1)} = \int dt[\dot{X}P_X - H_0] \ . \quad (1.20)$$

We introduce now a set of gauge fields $G_p$ to enforce into the action a set of constraints $V_p \approx 0$

$$S^{(1)} = \int dt[\dot{X}P_X - H_0 - G_p V^p] \ . \tag{1.21}$$

These constraints are first class and the algebra reads

$$\begin{aligned} \{V_p, V_q\}_{pb} &= C^t_{pq} V_t \\ \{H_0, V_p\}_{pb} &= K^q_p V_q \end{aligned} \tag{1.22}$$

where $C^t_{pq}$ and $K^q_p$ are, in general, functions of the phase space coordinates.
Action (1.21) is invariant with respect the following transformation:

$$\begin{aligned} \delta_\epsilon X &= \{X, \epsilon^p G_p\}_{pb} \\ \delta_\epsilon G^q &= \dot{\epsilon}^q + G^t \epsilon^p C^q_{pt} - \epsilon^p K^q_p \end{aligned} \tag{1.23}$$

where in an obvious notation $\epsilon^q$ is the gauge parameter associated to the gauge generator (or equivalentely first class constraint) $V^q$.

We use now relations (1.22) and (1.23) to analyze spinning particle model in $(A)dS$. Let us recall that the $SO(N)$ extended algebra in flat background reads

$$\{Q_i, Q_j\}_{pb} = -2i\delta_{ij}H, \qquad \{J_{ij}, Q_k\}_{pb} = -2i\delta_{k[i}Q_{j]}, \qquad \{J_{ij}, J^{kl}\}_{pb} = -i\delta^{[k}_{[i} J^{l]}_{j]} \ .$$

In a generic curved space one may attempt to covariantize it. Using tangent space flat indices instead of curved ones it is straightforward to generalize the $SO(N)$ generators, and using the vielbein $e^\mu_\alpha$, one obtains the covariantization of susy generators that thus reads

$$Q_i = \psi^\alpha_i \, e^\mu_\alpha \, \pi_\mu \tag{1.24}$$

where $\pi_\mu$ is the "covariant" momentum $\pi_\mu = p_\mu - \frac{i}{2}\omega_{\mu\alpha\beta}\psi^\alpha \circ \psi^\beta$,and $\psi^\alpha \circ \psi^\beta/2i$ the generators of the $SO(D)$ Lorentz group. In our notation $\circ$ means contraction over $SO(N)$ vector indices. The Hamiltonian constraint in $(A)dS$ background becomes

$$H = \underbrace{\frac{1}{2}\pi^\mu \pi^\nu g_{\mu\nu}}_{H_0} - \underbrace{\frac{1}{8}\psi^a \circ \psi^b \psi^c \circ \psi^d R_{abcd}}_{H_R} \ . \tag{1.25}$$

After a straightforward computation one obtains

$$\{Q_i, Q_j\}_{pb} = -ig^{\mu\nu}\pi_\mu \pi_\nu \, \delta_{ij} + \frac{i}{2} R_{\alpha\beta\gamma\delta} \, \psi^\alpha_{\{i}\psi^\beta_{j\}} \, \psi^\gamma \circ \psi^\delta \tag{1.26}$$

$$\{H, Q_i\}_{pb} = -\frac{1}{8}\nabla_\sigma R_{\alpha\beta\gamma\delta} \, \psi^\sigma_i \psi^\alpha \circ \psi^\beta \psi^\gamma \circ \psi^\delta \ . \tag{1.27}$$

The latter vanishes for locally (Riemannian) symmetric spaces. However, even in such a case the closure of (1.26) seems not to be guaranteed, at least not in a trivail way. Let us now restrict ourselves to manifolds equipped with the Riemann tensor

$$R_{\alpha\beta\gamma\delta} = f(x)(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}). \tag{1.28}$$

The algebra now reads

$$\begin{aligned}
\{Q_i, Q_j\}_{pb} &= -2iH_0\delta_{ij} + if(x)J_{ik}J_{jk} \\
&= -2iH\delta_{ij} + if(x)J_{ik}J_{jk} - i\frac{f(x)}{2}J_{lk}J_{lk}\delta_{ij} \\
\{Q_i, H\}_{pb} &= \frac{1}{4}\psi_i^{\mu}\partial_\mu f(x)J_{jk}J^{jk} \\
\{Q_i, J_{jk}\}_{pb} &= 2\delta_{k[i}Q_{j]} \\
\{J_{ij}, J_{lm}\}_{pb} &= SO(N) \text{ algebra} .
\end{aligned} \tag{1.29}$$

from which one can compute the transformation rules for the $SO(N)$ gauge fields:

$$\begin{aligned}
\delta^{susy}a_{ij} &= \frac{f(x)}{2}\psi_j^{\alpha}\psi_{k\alpha}(\chi_i\epsilon_k + \chi_k\epsilon_i) - \frac{f(x)}{2}\psi_i^{\alpha}\psi_{k\alpha}(\chi_j\epsilon_k + \chi_k\epsilon_j) \\
&\quad + f(x)\psi_i^{\alpha}\psi_{j\alpha}\chi_k\epsilon_k + \sum_{k\neq i,j}\frac{i}{2}\psi_k^{\mu}\psi_i^{\alpha}\psi_{j\alpha}\partial_\mu f(x)e\epsilon_k
\end{aligned} \tag{1.30}$$

$$\delta^{diff}a_{ij} = \partial_\tau(\xi a_{ij}) + \sum_{k\neq i,j}\frac{i\xi}{2}\psi_k^{\mu}\psi_i^{\alpha}\psi_{j\alpha}\partial_\mu f(x)\chi_k \ ; \tag{1.31}$$

when $f(x) = b$ one recovers the transformation rules (1.23).
Equation (1.23) and (1.30) are not exactly the same, but they differ for a trivial symmetry $\delta^{new}$, which vanishes on shell. In particular

$$\delta^{new}a_{ij} = \frac{1}{2}\psi_{[i}\psi_k(\chi_{[k}\epsilon_{j]]}) = -\frac{i}{2}\psi_p\psi_q(\frac{i}{2}\delta_{jp}\chi_{[i}\epsilon_{q]} + \frac{i}{2}\delta_{iq}\chi_{[j}\epsilon_{p]} + \frac{i}{2}\delta_{ip}\chi_{[q}\epsilon_{j]} + \frac{i}{2}\delta_{jq}\chi_{[p}\epsilon_{i]}) \tag{1.32}$$

that looks like $\frac{\delta S}{\delta a_{pq}}C_{ijpq}$ with $C_{ijpq} = -C_{pqij}$; note that the variation of the action with respect (1.32) vanishes, in fact, only for symmetry reason.

We will analyze now, for every dimension $D$, the space-time with non constant curvature we have used above (1.28):

$$R_{\alpha\beta\gamma\delta} = f(x)(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}) \tag{1.33}$$

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} = (D-1)f(x)\eta_{\beta\delta} \tag{1.34}$$

$$R = R^{\alpha}{}_{\alpha} = D(D-1)f(x). \tag{1.35}$$

We impose the Bianchi identity:

$$\nabla_\sigma R_{\alpha\beta\gamma\delta} + \nabla_c R_{\alpha\beta\delta\sigma} + \nabla_d R_{\alpha\beta\sigma\gamma} = 0. \tag{1.36}$$

Let us now contract the previous formula with $\eta^{\alpha\gamma}\eta^{\beta\delta}$

$$\nabla_\alpha(R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}R) = 0. \tag{1.37}$$

When we substitute (1.34) (1.35) in (1.37) we obtain

$$\frac{(D-1)(D-2)}{2}\partial_\beta f(x) = 0. \tag{1.38}$$

This condition is satisfied in every dimension when $f(x)$ is a constant ($(A)dS$ and Minkowsky) and just in dimension two for every $f(x)$.

Finally the $SO(N)$ extended algebra (1.29) in $(A)dS$ background becomes:

$$\boxed{\begin{array}{rcl} \{Q_i, Q_j\}_{pb} &=& -2iH\delta_{ij} + ibJ_{ik}J_{jk} - i\frac{b}{2}J_{lk}J_{lk}\delta_{ij} \\ \{Q_i, J_{jk}\}_{pb} &=& 2\delta_{k[i}Q_{j]} \\ \{J_{ij}, J^{kl}\}_{pb} &=& -i\delta^{[k}_{[i}J^{l]}_{j]} \end{array}} \tag{1.39}$$

In Appendix B we will use Koszul-Tate algorithm to the construct the BRST charge associated to this quadratic superalgebra.

## 1.3 Comments

In this Chapter we have constructed and analyzed spinning particles with $SO(N)$ extended supergravity on the worldline. In particular we have rederived the model coupled to maximally symmetric background, with a canonical analysis at the classical level. This approach has produced as a byproduct more general coupling in $D = 2$, and has revealed an interesting non linear structure of the first class constraints algebra. The action principle we started with is the one proposed by Kuzenko and Yarevskaya in [27]

$$S^{KY} = \int dt[\frac{1}{2e}g_{\mu\nu}(\dot{x}^\mu - i\chi_i\psi_i^\alpha e_\alpha^\mu)(\dot{x}^\nu - i\chi_i\psi_i^\alpha e_\alpha^\nu) + \frac{i}{2}\dot{\psi}_i^\alpha(\psi_{i\alpha} - \tilde{a}_{ij}\psi_{j\alpha} + \dot{x}^\mu\omega_{\mu\alpha\beta}\psi_i^\beta)].$$

Let us redefine now the $SO(N)$ gauge fields $\tilde{a}_{ij}$ as (this is obviously legal):

$$\tilde{a}_{ij} = a_{ij} + ibk\ e\ \psi_i^\alpha\psi_{j\alpha} \tag{1.40}$$

where $k$ is an arbitrary constant. The first class constraints become

$$\begin{array}{rcl} Q_i &=& \psi_i^\mu \pi_\mu \\ H &=& \frac{1}{2}\pi^2 + \frac{k}{4}\psi_i^a\psi_i^b\psi_j^c\psi_j^d R_{abcd} \\ J_{ij} &=& i\psi^a_{[i}\psi_{j]a} \end{array} \tag{1.41}$$

and the algebra reads:

$$\begin{array}{rcl} \{Q_i, Q_j\}_{pb} &=& -2iH\delta_{ij} + ib(\frac{1}{2} - k)J_{ik}J_{jk} + ikbJ_{lk}J_{lk}\delta_{ij} \\ \{Q_i, H\}_{pb} &=& = 2ib(\frac{1}{2} + k)Q_jJ_{ij}. \end{array} \tag{1.42}$$

During this work we prefer to use $k = -\frac{1}{2}$ because with this choice the gauge fixing procedure seems to be easier. In Appendix B we will discuss also the gauge fixing of the $SO(N)$ spinning particles in $(A)dS$ background and we will construct the gauge fixed action.

We resume now, for commodity, the results obtained in this chapter, regarding spinning particles models with $SO(N)$ extended supergravity multiplet on the worldline with $(A)dS$ background:

- **Action:**

$$
\begin{aligned}
S = \quad & \int dt \Big[ \frac{1}{2e} g_{\mu\nu}(\dot{x}^\mu - i\chi_i \psi_i^\alpha e_\alpha^\mu)(\dot{x}^\nu - i\chi_i \psi_i^\alpha e_\alpha^\nu) + \frac{i}{2}\dot{\psi}_i^\alpha(\psi_{i\alpha} - a_{ij}\psi_{j\alpha} + \dot{x}^\mu \omega_{\mu\alpha\beta}\psi_i^\beta) \\
& + \frac{1}{8} e \psi_i^\alpha \psi_i^\beta \psi_j^\gamma \psi_j^\delta R_{\alpha\beta\gamma\delta} \Big].
\end{aligned}
$$

- **Symmetries:**

**Supersymmetry**

$$
\begin{aligned}
\delta x^\mu &= i\epsilon_i \psi_i^\mu \\
\delta \psi_i^\alpha &= -\epsilon_i e^{-1}(\dot{x}^\mu e_\mu^\alpha - i\chi_j \psi_j^\alpha) + i\epsilon_j \psi_j^\beta \psi_i^\gamma e_\beta^\mu \omega_{\mu\gamma}{}^\alpha \\
\delta e &= 2i\chi_i \epsilon_i \\
\delta \chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j \\
\delta a_{ij} &= -b\chi_k(\epsilon_k \psi_i^\alpha \psi_{j\alpha} + \psi_k^\alpha \epsilon_{[i}\psi_{j]\alpha}).
\end{aligned}
$$

**Local** $SO(N)$

$$
\begin{aligned}
\delta x^\mu &= 0 \\
\delta \psi_i^\beta &= \alpha_{ij}\psi_j^\beta \\
\delta e &= 0 \\
\delta \chi_i &= \alpha_{ij}\chi_j \\
\delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{ik}a_{kj} - \alpha_j^k a_{ki}
\end{aligned}
$$

**Worldline diffeomorphism**

$$
\begin{aligned}
\delta x^\mu &= \xi \dot{x}^\mu \\
\delta \psi_i^\beta &= \xi \dot{\psi}_i^\beta \\
\delta e &= \partial_\tau(\xi e) \\
\delta \chi_i &= \partial_\tau(\xi \chi_i) \\
\delta a_{ij} &= \partial_\tau(\xi a_{ij}).
\end{aligned}
$$

- **Algebra:**

$$\begin{aligned}
\{Q_i, Q_j\}_{pb} &= -2iH\delta_{ij} + ibJ_{ik}J_{jk} - i\frac{b}{2}J_{lk}J_{lk}\delta_{ij} \\
\{Q_i, J_{jk}\}_{pb} &= 2\delta_{k[i}Q_{j]} \\
\{J_{ij}, J^{kl}\}_{pb} &= -i\delta^{[k}_{[i}J^{l]}_{j]}
\end{aligned}$$

# Chapter 2

# From spinning particle to HS

In this Chapter we give a detailed analysis of the quantization of the $SO(N)$ spinning particle models with flat background. We will show that spinning particles with $SO(N)$ extended local supersymmetries on the worldline, constructed and analyzed in the previous Chapter, describe the propagation of particles of spin $N/2$ in four dimension. A canonical analysis produces, in fact, the massless Bargmann-Wigner equations [28] as constraints for the physical sector of the Hilbert space, and these equations are known to describe massless free particles of arbitrary spin.

We proceed analyzing in details two very interesting cases, $N = 3$ and 4, corresponding to spin $\frac{3}{2}$ and 2, in order to derive explicitly the well known massless Rarita-Schwinger and graviton equation of motion in $D = 4$.

More generally, $SO(N)$ spinning particles are conformally invariant and describe all possible conformal free particles in arbitrary dimensions, as shown by Siegel in [7]. In the second part of this Chapter we will analyze carefully the even $N$ case (i.e. integer spin) in every even dimension $D = 2n$. In particular we will show that $SO(N)$ spinning particle models produce, upon quantization, conformal higher spin curvature (HSC) as physical states of the Hilbert space; more details about HSC and the geometrical approach to HS fields can be found in Appendix A.

Finally we will solve the differential Bianchi identity and we will derive the Fronsdal-Labastida kinetic operator, describing the dynamics of free HS fields with mixed symmetry.

## 2.1 Bargman Wigner equation of motion

It is well known that the Klein-Gordon equation for a spin 0 particle can be obtained by quantization of a relativistic particle model. Moreover a canonical analysis of spinning particles with $N = 1$ worldline supersymmetry produce the massless Dirac equation of motion. More in general the wave equations for arbitrary spin can be obtained by quantization of spinning particle models with $SO(N)$ extended local worldline supersymmetry [2]. Precisely, in $D = 4$, $SO(N)$ spinning particle action describes, upon quantization, a massless particle of spin $\frac{N}{2}$.

Let us rewrite for commodity, the one dimensional supergravity model we would like to analyze from a first quantized point of view:

$$S^{(1)} = \int dt[p_\alpha \dot{x}^\alpha + \frac{i}{2}\psi_i^\alpha \dot{\psi}_i^\beta \eta_{\alpha\beta} - eH - i\chi_i Q_i - \frac{i}{2}a_{ij}J_{ij}] \tag{2.1}$$

where $(e, \chi_i, a_{ij})$ is the one dimensional supergravity multiplet we introduce to enforce into the action the constraints

$$\boxed{H = \frac{1}{2}p^2 \approx 0} \tag{2.2}$$

$$\boxed{Q_i = \psi_i^\alpha p_\alpha \approx 0} \tag{2.3}$$

$$\boxed{J_{ij} = i\psi_i^\alpha \psi_{j\alpha} \approx 0} \; . \tag{2.4}$$

When one quantizes *"a la Dirac"* these models, the constraints $H$, $Q_i$ and $J_{ij}$ have to be imposed as operatorial conditions on the states on the Hilbert space. In principle these states are complex, but in what follows we shall suppose that they satisfy a reality condition. This is possible for arbitrary even $D$ when $N$ is even. The required reality condition takes the form of identification under worldline time reversal and was discussed in [54] for the spin $\frac{1}{2}$ particle. We shall now show that the equations obtained in this way are relativistic wave equations for spin $\frac{N}{2}$. For the sake of concreteness we shall discuss the $D = 4$ case, extension to $D \neq 4$ will be analyzed later.

To perform the quantization let us turn phase space variables into operators

$$O \to \boldsymbol{O} \; ; \tag{2.5}$$

Poisson and Dirac brackets are promoted to (anti)commutators (we use $\hbar = 1$)

$$\{\cdot, \cdot\}_{pb} \to i[\cdot, \cdot] \; . \tag{2.6}$$

Phase space coordinates, both bosonic and fermionic, satisfy:

$$[\boldsymbol{x}^\alpha, \boldsymbol{p}_\beta] = i\delta^\alpha_\beta \qquad\qquad [\boldsymbol{\psi}_i^\alpha, \boldsymbol{\psi}_j^\beta] = \delta_{ij}\eta^{\alpha\beta}; \tag{2.7}$$

and the $SO(N)$ extended supersymmetry algebra (1.9) becomes:

$$[\boldsymbol{Q}_i, \boldsymbol{Q}_j] = 2\delta_{ij}\boldsymbol{H} \qquad [\boldsymbol{J}_{ij}, \boldsymbol{Q}_k] = 2\delta_{k[i}\boldsymbol{Q}_{j]} \qquad [\boldsymbol{J}_{ij}, \boldsymbol{J}^{kl}] = \delta^{[k}_{[i}\boldsymbol{J}^{l]}_{j]} \; . \tag{2.8}$$

From the previous algebra one can observe that, after having imposed the condition $\boldsymbol{Q}_i|\phi\rangle = \boldsymbol{J}_{ij}|\phi\rangle = 0$ the constraint $\boldsymbol{H}|\phi\rangle = 0$ is automatically satisfied.
The wave function is a multispinor $\phi(x)_{a_1.....a_N}$, where $a_i$ are spinorial indices.
The Grassmann operators $\boldsymbol{\psi}_i^\alpha$ are the generators of Clifford algebra; they can be represented in terms of Dirac $\gamma$-matrices as follows[1]

$$\boldsymbol{\psi}_i^\alpha = \frac{1}{\sqrt{2}} \underbrace{\gamma_* \otimes ..... \otimes \gamma_*}_{i-1} \otimes \gamma^\alpha \otimes \underbrace{1 \otimes ... \otimes 1}_{N-i} \ . \tag{2.9}$$

In the bosonic sector the $\boldsymbol{x}$, $\boldsymbol{p}$ commutation relation is realized in the usual way by settings

$$\begin{aligned} \boldsymbol{x}^\alpha &= x^\alpha \\ \boldsymbol{p}^\alpha &= -i\frac{\partial}{\partial x^\alpha} \ . \end{aligned}$$

We start imposing the constraint (2.3); this implies that, with respect each spinorial index, $\phi_{a_1.....a_N}$ satisfies a Dirac-type equation:

$$\boldsymbol{Q_i}|\phi\rangle = 0 \qquad \Rightarrow \qquad (\partial\!\!\!/)^{ba_i}\phi_{a_1...a_i...a_N} = 0 \ . \tag{2.10}$$

Let us now use the operator $\boldsymbol{J_{ij}}$. It could be written in gamma matrices basis as

$$\boldsymbol{J_{ij}} = -\frac{i}{2} \underbrace{1 \otimes ..... \otimes 1}_{i-1} \otimes \gamma_* \gamma^\alpha \otimes \underbrace{\gamma_* \otimes \gamma_*}_{j-i-1} \otimes \gamma_\alpha \otimes \underbrace{1 \otimes ... \otimes 1}_{N-j} \ . \tag{2.11}$$

The relation $\boldsymbol{J_{ij}}|\phi\rangle = 0$ yields

$$(\gamma^\alpha)_{a_i}{}^{\tilde{a}_i}(\gamma^\alpha)_{a_j}{}^{\tilde{a}_j}\phi_{a_1...\tilde{a}_i...\tilde{a}_j...a_N} = 0. \tag{2.12}$$

We contract the previous expression with any element $\Gamma^{(m)}$ of the four-dimensional Clifford algebra basis

$$\Gamma^{(m)} = (1, \gamma^\alpha, \gamma^{\alpha\beta}, \gamma^*\gamma^\alpha, \gamma^*) \tag{2.13}$$

and we obtain a set of relations:

$$(\gamma^\alpha \Gamma^{(m)} \gamma_\alpha)^{\tilde{a}_i \tilde{a}_j}\phi_{a_1...\tilde{a}_i...\tilde{a}_j...a_N} = 0 \ . \tag{2.14}$$

Note that we use the charge conjugation matrix to lower and riser spinorial indices

$$(\Gamma^{(m)})_{ab} = (\Gamma^{(m)})_a{}^{\tilde{b}} C_{\tilde{b}b} \ . \tag{2.15}$$

We use now the identity

$$(\gamma^\alpha \Gamma^{(m)} \gamma_\alpha) = (-1)^n (D - 2m)\Gamma^{(m)} \tag{2.16}$$

[1]We work, now, in four dimension; this construction could be generalized to every even $D$. Let us comment that $\gamma^*$ plays a fundamental role to realize the anticommutation relation (2.7), this is one of the most relevant obstacle in generalizing this construction for odd $D$ where $\gamma^* = 1$.

and in $D=4$ equation (2.14) reduces to

$$(\Gamma^{(m)})^{\tilde{a}_i\tilde{a}_j}\phi_{a_1\ldots\tilde{a}_i\ldots\tilde{a}_j\ldots a_N}=0 \qquad \text{with} \qquad \Gamma^{(m)}\neq\gamma^{\alpha\beta}\ . \tag{2.17}$$

Since in $D=4$ $\gamma^{\alpha\beta}$ is symmetric in spinor indices, the previous relation implies that $\phi_{a_1\ldots\tilde{a}_i\ldots\tilde{a}_j\ldots a_N}$ is a completely symmetric multispinor. This could be easily understood by expanding $\phi$ on the Clifford algebra basis; relation (2.17) implies that just terms proportional to $\gamma^{\alpha\beta}$ survive. This conclude the proof since from (2.17) and (2.10) we obtain explicitly the Bargman-Wigner e.o.m.

$$\boxed{(\partial\!\!\!/)^{ba_i}\phi_{a_1\ldots a_i\ldots a_N}=0 \quad \text{where} \quad \phi_{a_1\ldots\ldots a_N} \quad \text{is a completely symmetric multispinor}} \tag{2.18}$$

## 2.2 Gravitino

In this section we will show that, by solving explicitly the equation (2.17) for $N=3$ in $D=4$, one obtains the massless Rarita-Schwinger equation of motion describing the dynamics of a spin $\frac{3}{2}$ field[2]:

$$(\gamma^{\alpha\beta\delta})_a{}^c\partial_\beta\Psi_{\delta|c}\equiv\gamma^{\alpha\beta\delta}\partial_\beta\Psi_\delta=0 \tag{2.19}$$

with

$$\gamma^{\alpha\beta\delta}=\gamma^\alpha\gamma^\beta\gamma^\delta-\eta^{\alpha\beta}\gamma^\delta-\gamma^{\beta\delta}\gamma^\alpha+\eta^{\alpha\delta}\gamma^\beta\ .$$

It is very useful to rewrite (2.19) by contracting it with $\gamma_\alpha$ and $\partial_\alpha$:

$$\gamma_\alpha\gamma^{\alpha\beta\delta}\partial_\beta\Psi_\delta=0 \qquad\Rightarrow\qquad \partial\!\!\!/(\gamma\cdot\Psi)-\partial\cdot\Psi=0 \tag{2.20}$$

$$\partial_\alpha\gamma^{\alpha\beta\delta}\partial_\beta\Psi_\delta=0 \qquad\Rightarrow\qquad \partial\!\!\!/\partial\!\!\!/(\gamma\cdot\Psi)-\partial^2(\gamma\cdot\Psi)=0\ . \tag{2.21}$$

We expand now the wave function on the gamma matrix basis, in this way:

$$\phi_{abc}=(\Gamma^{(n)})_{ab}\chi_{(n)|c}. \tag{2.22}$$

As was just discussed in the previous section, $\boldsymbol{J_{ij}}|\phi\rangle=0$, in four dimension implies:

[2]Where we don't write spinor indices $a,b,c,..$ it means that they are contracted.

$$(1)^{a_i a_j}\phi_{a_1 a_2 a_3} = 0 \tag{2.23}$$

$$(\gamma^\alpha)^{a_i a_j}\phi_{a_1 a_2 a_3} = 0. \tag{2.24}$$

From (2.22), (2.23) and (2.24) one finds that the wave function can be written as

$$\phi_{abc} = (\gamma^{\alpha\beta})_{ab}\chi_{\alpha\beta|c} \tag{2.25}$$

and $\chi_{\alpha\beta|c}$ has to satisfy the relation

$$(\gamma^\alpha)^{ac}\chi_{\alpha\beta|c} = 0\ . \tag{2.26}$$

Now we use the supercharges constraint (2.3) and we obtain:

$$(\gamma^\rho)_{\tilde{a}_i}{}^{a_i}\partial_\rho\phi_{a_1 a_2 a_3} = 0 \qquad i = 1,2,3\ . \tag{2.27}$$

This relation plays a key role; it implies that $\chi_{\alpha\beta|c}$ is a closed two form, so it could be written as

$$\chi_{\alpha\beta|c} = \partial_\alpha\Psi_{\beta|c} - \partial_\beta\Psi_{\alpha|c}\ . \tag{2.28}$$

We substitute now the previous expression in (2.26) and we obtain:

$$\partial\!\!\!/\Psi_\alpha - \partial_\alpha(\gamma\cdot\Psi) = 0 \tag{2.29}$$

Contraction of the previous relation with $\gamma^\alpha$ and $\gamma^\alpha\partial\!\!\!/$ produces

$$\gamma^\alpha(\partial\!\!\!/\Psi_\alpha - \partial_\alpha(\gamma\cdot\Psi)) = 0 \qquad \Rightarrow \qquad \partial\cdot\Psi - \partial\!\!\!/(\gamma\cdot\Psi) = 0 \tag{2.30}$$

$$\gamma^\alpha\partial\!\!\!/(\partial\!\!\!/\Psi_\alpha - \partial_\alpha(\gamma\cdot\Psi)) = 0 \qquad \Rightarrow \qquad \partial\!\!\!/\partial\!\!\!/(\gamma\cdot\Psi) - \partial^2(\gamma\cdot\Psi) = 0\ . \tag{2.31}$$

Note now that (2.30) and (2.31) coincide with (2.20) and (2.21) and this conclude our proof.

## 2.3 Graviton

In this section we will discuss the case $N = 4$ (i.e. spin 2) in $D = 4$. This example reveals to be very useful for future analysis; we will use a different way, with respect the one we have discussed in the previous section, to realize the algebra.
In particular we use complex combination of $SO(N)$ vector indices

$$(\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) \qquad \Rightarrow \qquad (\lambda_1^\alpha, \bar{\lambda}_1^\alpha, \lambda_2^\alpha, \bar{\lambda}_2^\alpha) \tag{2.32}$$

where

$$\lambda_1^\alpha = \frac{\psi_1^\alpha + i\psi_2^\alpha}{2} \qquad \bar\lambda_1^\alpha = \frac{\psi_1^\alpha - i\psi_2^\alpha}{2}$$

$$\lambda_2^\alpha = \frac{\psi_3^\alpha + i\psi_4^\alpha}{2} \qquad \bar\lambda_2^\alpha = \frac{\psi_3^\alpha - i\psi_4^\alpha}{2}.$$

Equations (2.2) (2.3) and (2.4) in this complex base become.

$$\boldsymbol{H} = \tfrac{1}{2}\boldsymbol{p}^2$$

$$\boldsymbol{Q}_I = \boldsymbol{\lambda}_I^\alpha \boldsymbol{p}_\alpha \qquad \boldsymbol{Q}^I = \bar{\boldsymbol{\lambda}}_I^\alpha \boldsymbol{p}_\alpha$$

$$\boldsymbol{J}_{IJ} = i\boldsymbol{\lambda}_I\boldsymbol{\lambda}_J \qquad \boldsymbol{J}^{IJ} = i\bar{\boldsymbol{\lambda}}_I\bar{\boldsymbol{\lambda}}_J$$

$$\boldsymbol{J}_I{}^J = \tfrac{i}{2}[\boldsymbol{\lambda}_I, \bar{\boldsymbol{\lambda}}_J].$$

where $I, J = 1, 2$. In our notation raised indices are complex indices

$$\boldsymbol{J}_{I\bar J} \equiv \boldsymbol{J}_I{}^J \qquad \boldsymbol{J}_{\bar I\bar J} \equiv \boldsymbol{J}^{IJ} \ .$$

In the following we prefer, sometimes, consider the operators $\boldsymbol{J}_I{}^J$ with $I \neq J$ and $I = J$ separately; to this aim we introduce the notation

$$\boldsymbol{J}_I{}^J = \begin{cases} \tilde{\boldsymbol{J}}_I{}^J & \text{when} \quad I \neq J \\ \boldsymbol{J}_I{}^I & \text{when} \quad I = J \end{cases} ; \tag{2.33}$$

note that this notation doesn't imply in $\boldsymbol{J}_I{}^I$, where is not explicitly indicated, a sum over complex $SO(N)$ vector indices.
In this new base Grassmann variables satisfy the anticommutation relation

$$[\bar{\boldsymbol{\lambda}}_I^\alpha, \boldsymbol{\lambda}_J^\beta] = \delta_{IJ}\eta^{\alpha\beta} \tag{2.34}$$

and the algebra is realized by setting

$$\boldsymbol{\lambda}_I^\alpha = \lambda_I^\alpha \qquad \bar{\boldsymbol{\lambda}}_I^\alpha = \frac{\partial}{\partial\lambda_I^\alpha}. \tag{2.35}$$

Note that in writing $J_I{}^I$ we have fixed the following ordering

$$\boldsymbol{J}_I{}^I \equiv \frac{i}{2}\Big(\lambda_I^\alpha \frac{\partial}{\partial\lambda_I^\alpha} - \frac{\partial}{\partial\lambda_I^\alpha}\lambda_I^\alpha\Big) = i\lambda_I^\alpha\frac{\partial}{\partial\lambda_I^\alpha} - 2i \ .$$

States of the full Hilbert space can be described as functions of the coordinates $x^\alpha$ and $\lambda_I^\alpha$. We denote, as usual, with $x^\alpha$ the eigenvalues of the operator $\boldsymbol{x}^\alpha$, while for the fermionic variables we use bra coherent states defined by

$$\langle\lambda_I^\alpha|\boldsymbol{\lambda}_I^\alpha = \langle\lambda_I^\alpha|\lambda_I^\alpha \ .$$

Any state $|\phi\rangle$ can then be described by the wave function

$$\phi(x,\lambda_I) = (\langle x| \otimes \langle \lambda_1| \otimes \langle \lambda_2|)|\phi\rangle. \tag{2.36}$$

Since $\lambda_I$ are anticommuting variables, the wave function has a finite expansion

$$\begin{aligned}\phi(x,\lambda_I) &= A(x) + A(x)^I_{\alpha_1}\lambda_I^{\alpha^1} + A(x)^{II}_{\alpha_1\alpha_2}\lambda_I^{\alpha_1}\lambda_I^{\alpha_2} + A(x)^{IJ}_{\alpha_1\beta_1}\lambda_I^{\alpha_1}\lambda_J^{\beta_1} + \\ &+ A(x)^{III}_{\alpha_1\alpha_2\alpha_3}\lambda_I^{\alpha_1}\lambda_I^{\alpha_2}\lambda_I^{\alpha_3} + A(x)^{IIJ}_{\alpha_1\alpha_2\beta_1}\lambda_I^{\alpha_1}\lambda_I^{\alpha_2}\lambda_J^{\beta_1} + ....\end{aligned} \tag{2.37}$$

We start solving the constraint $\boldsymbol{J}_I{}^I$. In particular we use the relation $\boldsymbol{J}_I{}^I|\phi\rangle = 0$ to select only one tensorial structure on the expression (2.37):

$$\boldsymbol{J}_I{}^I\phi(x,\lambda_I) = 0 \quad \Rightarrow \quad \phi(x,\lambda_I) = A(x)_{\alpha_1\alpha_2\beta_1\beta_2}\lambda_1^{\alpha_1}\lambda_1^{\alpha_2}\lambda_2^{\beta_1}\lambda_2^{\beta_2}. \tag{2.38}$$

From the other $SO(4)$ operator we learn that:

$$\boldsymbol{J}^{12}\phi = 0 \quad \Rightarrow \quad \eta^{\alpha_1\beta_1}A(x)_{\alpha_1\alpha_2\beta_1\beta_2} = 0 \tag{2.39}$$

$$\boldsymbol{J}_1{}^2\phi = 0 \quad \Rightarrow \quad A(x)_{[\alpha_1\alpha_2\beta_1]\beta_2} = 0. \tag{2.40}$$

It is not hard to show that at this point the constraint $\boldsymbol{J}_{12}|\mathcal{R}\rangle = 0$ is automatically satisfied. Let us now resume the informations we have obtained above:

- The wave function is proportional to a 4-rank tensor $\phi(x) \backsim A(x)_{\alpha_1\alpha_2\beta_1\beta_2}$.
- From (2.38) one can easily reads the symmetry property of $A$:

$$A(x)_{\alpha_1\alpha_2\beta_1\beta_2} = -A(x)_{\alpha_2\alpha_1\beta_1\beta_2} = A(x)_{\beta_1\beta_2\alpha_1\alpha_2} = -A(x)_{\alpha_1\alpha_2\beta_2\beta_1}$$

- The 4-rank tensor $A(x)_{\alpha_1\alpha_2\beta_1\beta_2}$ satisfies the algebraic Bianchi identity (2.40);
- Note also that $A_{\alpha_1\alpha_2\beta_1\beta_2}$ has to be traceless (2.57).

We use now supercharges and we obtain:

$$\boldsymbol{Q}_I\phi = 0 \quad \Rightarrow \quad \partial_{[\delta}A(x)_{\alpha_1\alpha_2]\beta_1\beta_2} = 0 \qquad A \text{ closed} \tag{2.41}$$

and

$$\boldsymbol{Q}^I\phi = 0 \quad \Rightarrow \quad \partial^\delta A(x)_{\delta\alpha_2\beta_1\beta_2} = 0 \qquad A \text{ co-closed.} \tag{2.42}$$

We proceed now analyzing carefully relations (2.39)-(2.42).
Note that equation (2.41) implies that $A(x)$ is a closed multiform, equivalently satisfies differential Bianchi identity with respect the first couple of indices $\alpha_1,\alpha_2$ or the second couple $\beta_1,\beta_2$. Thus we can use the *Poincaré Lemma* to introduce a rank-3 tensor $K_{\alpha_1\beta_1\beta_2}$ defined by:

$$A(x)_{\alpha_1\alpha_2\beta_1\beta_2} = \partial_{\alpha_1}K_{\alpha_2\beta_1\beta_2} - \partial_{\alpha_2}K_{\alpha_1\beta_1\beta_2}. \tag{2.43}$$

We substitute now (2.43) in (2.40), and we solve it as a function of $K$; explicitly we obtain:

$$\partial_{[\alpha_1}\tilde{K}_{\alpha_2\beta_1]\beta_2} = 0 \qquad \Rightarrow \qquad \tilde{K}_{\alpha_2\beta_1\beta_2} = \partial_{\alpha_2}\tilde{\chi}_{\beta_1\beta_2} - \partial_{\beta_1}\tilde{\chi}_{\alpha_2\beta_2} \tag{2.44}$$

where $\tilde{K}_{\alpha_2\beta_1\beta_2} = K_{[\alpha_2\beta_1]\beta_2}$; this tensor with mixed symmetry is strictly related to the Fierz-field that has been extensively studied by Novello and Neves in [55] as an alternative approach to describe gravity (known in literature as Teleparallel gravity [56]).
An iterative procedure let us write (2.44) in terms of $K$ and in particular one finds:

$$2K_{\alpha_2\beta_1\beta_2} = \partial_{\alpha_2}\tilde{\chi}_{[\beta_1\beta_2]} - \partial_{\beta_1}\chi_{\alpha_2\beta_2} + \partial_{\beta_2}\chi_{\alpha_2\beta_1} \tag{2.45}$$

where we have defined $\chi_{\alpha\beta} = \tilde{\chi}_{\{\alpha\beta\}}$.
We are ready now to write the rank-4 tensor $A(x)$ as a function of a rank-2 tensor $\chi$:

$$A(x)_{\alpha_1\alpha_2\beta_1\beta_2} = \partial_{\beta_1}\partial_{[\alpha_2}\chi_{\alpha_1]\beta_2} - \partial_{\beta_2}\partial_{[\alpha_2}\chi_{\alpha_1]\beta_1} \ . \tag{2.46}$$

Note that, at this point, condition (2.42) is automatically satisfied.
We substitute now (2.46) in (2.57), and we obtain the graviton e.o.m.

$$\partial_\alpha\partial_\beta\chi_{\alpha\delta} - \partial_\alpha\partial_\delta\chi_{\alpha\beta} - \Box\chi_{\delta\beta} - \partial_\beta\partial_\delta\chi = 0 \tag{2.47}$$

where $\chi = \eta^{\alpha\beta}\chi_{\alpha\beta}$. Finally it's not hard to prove that (2.47) enjoys the gauge symmetry:

$$\delta\chi_{\alpha\beta} = \partial_{\{\alpha}\xi_{\beta\}}. \tag{2.48}$$

## 2.4 From spinning particle to integer higher spin gauge theory

We use now the idea inherited from the $N = 4$ case. We consider even $N = 2s$ and we work in even dimension $D = 2n$. Let us forget, in the following, about the bold font, as its obvious that we are now dealing with quantum operators.
We use complex combinations of the $SO(N)$ vector indices

$$\psi_i^\alpha \quad \text{with} \quad i = 1, \ldots, N = 2s \ , \qquad \Rightarrow \qquad (\lambda_I^\alpha, \bar{\lambda}_I^\alpha) \quad \text{with} \quad I = 1, .., s \tag{2.49}$$

so that

$$[\lambda_I^\alpha, \bar{\lambda}^{\beta J}] = \eta^{\alpha\beta}\delta_I{}^J \tag{2.50}$$

with

$$\bar{\lambda}_{\beta I} = \frac{\partial}{\partial\lambda^{\beta I}} \ . \tag{2.51}$$

Explicitly the $SO(N)$ generators are:

$$J_{IJ} \equiv i\lambda_I^\alpha\lambda_{J\alpha} \tag{2.52}$$

$$J_I{}^J \equiv i\lambda_I^\alpha\frac{\partial}{\partial\lambda_J^\alpha} - in\delta_I^J \tag{2.53}$$

$$J^{IJ} \equiv i\frac{\partial}{\partial\lambda_I^\alpha}\frac{\partial}{\partial\lambda_{J\alpha}} \tag{2.54}$$

and the algebra is

$$[Q_I, Q^J] = 2\delta_I{}^J H$$

$$[J_{IJ}, Q^K] = i\delta_{[J}{}^K Q_{I]} \qquad [J_I{}^J, Q_K] = i\delta_K{}^J Q_I$$

$$[J^{IJ}, Q_K] = i\delta_K{}^{[J} Q^{I]} \qquad [J_I{}^J, Q^K] = -i\delta_I{}^K Q^J \ .$$

The wave function, $\mathcal{R}(x^\alpha, \lambda_I^\alpha) = (\langle x| \otimes \langle\lambda_I|)|\mathcal{R}\rangle$, depends on the coordinate $x^\alpha$ and on the fermionic degrees of freedom $\lambda_I^\alpha$; in analogy with the spin 2 case, it can be Taylor expanded in terms of $\lambda$.
We start imposing the constraint $J_I{}^I|\mathcal{R}\rangle = 0$ and we find that:

$$\boxed{J_I{}^I|\mathcal{R}\rangle = 0 \Rightarrow \mathcal{R}(x^\alpha, \lambda_I^\alpha) = \mathcal{R}_{\alpha_1..\alpha_n|...|\beta_1..\beta_n}(x)\, \lambda_1^{\alpha_1}..\lambda_1^{\alpha_n}...\lambda_s^{\beta_1}..\lambda_s^{\beta_n}} \ . \tag{2.55}$$

Note that this implies that in the wave function survives only a tensor structure that has $s$ blocks of $n$ antisymmetric indices. For commodity in the following we prefer to use the shortcut notation $\mathcal{R}_{[n]_1\ldots[n]_s}\lambda_1^{[n]_1}\cdots\lambda_s^{[n]_s}$.
We proceed now imposing the other constraints on the wave function (2.55).

The constraint $\tilde{J}_I{}^J$ remove a $\lambda_J^\sigma$ from the $J^{th}$ block and add a $\lambda_I^\sigma$ in the $I^{th}$ one; it produces the condition

$$\mathcal{R}_{[n]_1\cdots[n+1]_I\cdots[n-1]_J\ldots[n]_s} = 0 \ . \tag{2.56}$$

This is a constraints that guaranties the algebraic Bianchi identity.

We use now $J^{IJ}$. It removes two Grassmann variables, one from the $I^{th}$ and one from the $J^{th}$ block; the corresponding vector indices are contracted and the condition $J^{IJ}|\mathcal{R}\rangle = 0$ yields:

$$\eta^{\sigma\gamma}\mathcal{R}_{[n]_1,\cdots,\sigma[n-1]_I,\cdots,\gamma[n-1]_J,\cdots,[n]_s} = 0 \ . \tag{2.57}$$

Thus this constraint remove all the traces into $\mathcal{R}$.

One can notes that at this point the condition $J_{IJ}|\mathcal{R}\rangle = 0$ is automatically satisfied. It does not arise as a consequence of the algebra, but it is related to the hermiticity properties. One could view it as a consequence of a duality symmetry which takes the dual in the $I^{th}$ block of indices by the operation

$$\lambda_I \ \leftrightarrow \ \bar{\lambda}_I \tag{2.58}$$

which can be obtained with a discrete $O(N)$ transformation and in particular a reflection on one real coordinate. In particular $J_{IJ}|\mathcal{R}\rangle = 0$ reduces to a sum of traceless constraints that are just implemented into the condition $J^{IJ}|\mathcal{R}\rangle = 0$. Explicitly one has that the operator $J_{IJ}$ adds two Grassmann variables into the $I^{th}$ and $J^{th}$ block:

$$\mathcal{R}_{[n]_1\cdots[n]_s}\lambda_1^{[n]_1}..\lambda_I^{[n]_I}\lambda_I^{\sigma}..\lambda_J^{[n]_J}\lambda_{I\sigma}..\lambda_s^{[n]_s} = 0. \tag{2.59}$$

Note now that when vector indices into $[n]_I$ and $[n]_J$ assume the value $\sigma$ the previous relation vanishes identically because of the property of the Grassmann variables. Let us recall that $2n = D$, for this reason

$$\mathcal{R}_{[n]_1\cdots[n]_s}\lambda_1^{[n]_1}..\lambda_I^{[n]_I}\lambda_I^{c}..\lambda_J^{[n]_J}\lambda_{Ic}..\lambda_s^{[n]_s}$$

does not vanish identically only if at least one index into the $I^{th}$ block is contracted with one index into the $J^{th}$ block; it implies that the constraint $J_{IJ}|\mathcal{R}\rangle = 0$ produces a sum of traceless, double traceless and multiple traceless conditions

$$\sum_{k=1}^{n} c_k \boldsymbol{tr^k} R_{[n]_1\cdots[n]_s} = 0 \tag{2.60}$$

for some coefficients $c_k$.

We are ready now to attack the supercharges constraints. $Q_I$ adds a Grassmann variable into the $I^{th}$ block and its vector index is contracted with a partial derivative

$$\partial_\sigma \mathcal{R}_{[n]_1\cdots[n]_s}\lambda_1^{[n]_1}\cdots\lambda_I^{[n]_I}\lambda_I^{\sigma}\cdots\lambda_s^{[n]_s} = 0; \tag{2.61}$$

thus $Q_I$ produces the differential Bianchi identity (i.e. closing condition) as a constraint on the wave function.
In addition $Q^J$ removes a Grassmann variable from the $J^{th}$ block and substitute it with a partial derivative:

$$\partial^\sigma \mathcal{R}_{[n]_1\cdots,[n-1]_I\sigma,\cdots,[n]_s}\lambda_1^{[n]_1}\cdots\lambda_I^{[n-1]_I}\cdots\lambda_s^{[n]_s} = 0, \tag{2.62}$$

that is the co-closing condition on $\mathcal{R}$, that at this point it is automatically satisfied as a consequence of the algebra ($[J^{IJ}, Q_K] = i\delta_K{}^J Q^I - i\delta_K{}^I Q^J$).
Let us recall that $[Q_I, Q^I] = 2H$; it implies that, at this point, also the constraint $H$ is automatically satisfied.

Let us summarize and explain further what we have obtained before:

- The constraints $J_I{}^J|R\rangle = 0$ correspond to the subgroup $U(s) \subset SO(2s)$, which is manifestly realized in the complex basis. The curvature $\mathcal{R}$ that solves these $U(s)$ constraints has "$s$" symmetric blocks of "$n$" antisymmetric indices each, and satisfies the algebraic Bianchi identities . Antisymmetry in each block is manifest.
- Symmetry between blocks can be shown by using finite $SO(s) \subset U(s)$ rotations. For example, consider the rotation that exchanges $\lambda_I \to \lambda_J$ and $\lambda_J \to -\lambda_I$ for fixed $I$ and $J$. This proves symmetry under exchange of the block relative to the fermions $\lambda_I$ with the one relative to the fermion $\lambda_J$.

- Note that the fermionic Fock vacuum $|V\rangle \sim V(x)$ is not invariant under the subgroup $[U(1)]^s \subset U(s)$, as the generators $J_I{}^{I'}$ for $I = 1, .., s$ transform it by an infinitesimal phase ($J_I{}^{I'}|V\rangle = -i\frac{D}{2}|V\rangle$). It is the vector $|\mathcal{R}\rangle$ in eq. (2.55) that is left invariant.

This is collected in the following table:

$$
\begin{array}{rclcrcll}
J_I{}^I|\mathcal{R}\rangle & = & 0 & \Rightarrow & |\mathcal{R}\rangle \sim \mathcal{R}_{[n]_1\cdots[n]_s} & & & \textbf{HSC} \\
\tilde{J}_I{}^J|\mathcal{R}\rangle & = & 0 & \Rightarrow & \mathcal{R}_{[n+1]_1[n-1]_2\ldots[n]_s} & = & 0 & \textbf{Algebraic Bianchi identity} \\
Q_I|\mathcal{R}\rangle & = & 0 & \Rightarrow & \partial_{[\beta}\mathcal{R}_{\alpha_1..\alpha_n][n]_2\ldots[n]_s} & = & 0 & \mathcal{R}\ \textbf{closed} \\
J^{IJ}|\mathcal{R}\rangle & = & 0 & \Rightarrow & \mathcal{R}^{\sigma}{}_{[n-1]_1,\sigma[n-1]_2\ldots[n]_s} & = & 0 & \mathcal{R}\ \textbf{traceless}
\end{array}
$$

$$
\begin{array}{rclcrcll}
J_{IJ}|\mathcal{R}\rangle & = & 0 & \Rightarrow & \sum c_k \boldsymbol{tr^k} R_{[n]_1\cdots[n]_s} = 0 & = & 0 & \textbf{Sum of traceless conditions} \\
Q^I|\mathcal{R}\rangle & = & 0 & \Rightarrow & \partial^{\alpha_1}\mathcal{R}_{\alpha_1..\alpha_n[n]_2\ldots[n]_s} & = & 0 & \mathcal{R}\ \textbf{co-closed}
\end{array}
$$

These coincides with the geometrical equations of de Wit-Freedman [31] but in a different basis [29] [9].
Thus we can conclude that spinning particle models with $SO(N)$ extended supergravity on the worldline, produce, upon quantization, conformal HSC as the physical sector of the Hilbert space (more details could be found in Appendix A).

In Young tableaux language we have:

$$
|\mathcal{R}\rangle \sim \mathcal{R}_{a_1^1\cdots a_n^1|\ldots|a_1^s\cdots a_n^s} \equiv
\begin{array}{|c|c|c|c|}
\hline
a_1^1 & \cdot & \cdot & a_1^s \\ \hline
\cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot \\ \hline
a_n^1 & \cdot & \cdot & a_n^s \\ \hline
\end{array}
\tag{2.63}
$$

and

$$
\partial_{[a}\mathcal{R}_{a_1^1\cdots a_n^1]|\ldots|a_1^s\cdots a_n^s} \equiv
\begin{array}{|c|c|c|c|}
\hline
a_1^1 & \cdot & \cdot & a_1^s \\ \hline
\cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot \\ \hline
a_n^1 & \cdot & \cdot & a_n^s \\ \hline
\partial \\ \cline{1-1}
\end{array}
= 0
\tag{2.64}
$$

## 2.5 HS gauge potential

We would like now to solve the differential Bianchi identity (2.40) and introduce the gauge potential (i.e. HS field).
We define the operator

$$q_s \equiv Q_1 Q_2 .. Q_s \tag{2.65}$$

which satisfies $Q_I q_s = q_s Q_I = 0$ for any $I$. In fact, powers of the $Q_I$ may be non vanishing up to the $s$-th power, since then any additional application of any of the $Q_I$'s makes it vanish, because of the algebra $[Q_I, Q_I] = 0$ which in particular implies $Q_I^2 = 0$ at fixed $I$. Then the differential Bianchi identity (i.e. $Q_I|\mathcal{R}\rangle = 0$) can be solved by using the *generalized Poincaré Lemma*[3] [45] [46] as

$$|R\rangle = q_s|\phi\rangle \ . \tag{2.66}$$

We focus now our attention on the $U(S)$ constraints; we compute

$$J_I{}^J q_s|\phi\rangle = ([J_I{}^J, q_s] + q_s J_I{}^J)|\phi\rangle = q_s(i\delta_I{}^J + J_I{}^J)|\phi\rangle = 0 \tag{2.67}$$

which can be solved by requiring that

$$J_I{}^J|\phi\rangle = -i\delta_I{}^J|\phi\rangle \tag{2.68}$$

which says that $|\phi\rangle$ has the form

$$|\phi\rangle \sim \phi_{\alpha_1..\alpha_{n-1}|...|\alpha_1..\alpha_{n-1}}(x)\, \lambda_1^{\alpha_1}..\lambda_1^{\alpha_{n-1}}...\lambda_s^{\alpha_1}..\lambda_s^{\alpha_{n-1}} \tag{2.69}$$

and satisfies the algebraic Bianchi identities. In particular, the tensor $\phi$ is symmetric under block exchange. Thus equation (2.55), the algebraic and differential Bianchi identity are solved.
Note now that $\phi$ and is represented by the Young tableaux:

$$\phi_{\alpha_1..\alpha_{n-1}|...|\alpha_1..\alpha_{n-1}}(x) = \begin{array}{|c|c|c|c|} \hline \alpha_1^1 & \cdot & \cdot & \alpha_1^s \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \alpha_{n-1}^1 & \cdot & \cdot & \alpha_{n-1}^s \\ \hline \end{array} \tag{2.70}$$

One may now implement the traceless condition and then all the other constraints are solved automatically. We force now the generalized curvature to be traceless (in our mind this step

[3] This Lemma assures also that the solution we are going to propose is unique.

coincides with the computation of the e.o.m.):

$$\begin{aligned}
J^{12}\, q_s|\phi\rangle &= J^{12}\, Q_1Q_2Q_3...Q_s|\phi\rangle = \underbrace{Q_3...Q_s}_{q_{extra}}\, J^{12}\, Q_1Q_2|\phi\rangle \\
&= q_{extra}\Big[[J^{12},Q_1]Q_2 + Q_1[J^{12},Q_2] + Q_1Q_2J^{12}\Big]|\phi\rangle \\
&= q_{extra}\Big[-iQ^2Q_2 + iQ_1Q^1 + Q_1Q_2J^{12}\Big]|\phi\rangle \\
&= q_{extra}\Big[-2iH + i(Q_1Q^1+Q_2Q^2) + Q_1Q_2J^{12}\Big]|\phi\rangle \\
&= q_{extra}\Big[-2iH + iQ_IQ^I + \frac{1}{2}Q_IQ_JJ^{IJ}\Big]|\phi\rangle \\
&= q_{extra}iG|\phi\rangle
\end{aligned} \tag{2.71}$$

where we have defined the Fronsdal-Labastida operator

$$\boxed{G = -2H + Q_IQ^I + \frac{i}{2}Q_IQ_JJ^{JI}} \tag{2.72}$$

which is manifestly $U(s)$ invariant. In fact, one may compute $[J_I{}^J, G] = 0$. A similar expression holds for $J^{12} \to J^{IJ}$ so that imposing the traceless condition one obtains (in an obvious notation)

$$q_{extra\,IJ}\; iG|\phi\rangle = 0\ . \tag{2.73}$$

Before describing a solution to this equation let us discuss the gauge symmetries. Using an arbitrary vector field $V^\mu(x)$ we may define

$$\bar{V}^I = V^\mu \bar{\lambda}^I_\mu \tag{2.74}$$

and use it to describe the following gauge transformation

$$\delta|\phi\rangle = Q_K\bar{V}^K|\xi\rangle \tag{2.75}$$

which is a gauge symmetry of $|\mathcal{R}\rangle = q_s|\phi\rangle$. Since $[J_I{}^J, Q_K\bar{V}^K] = 0$, one requires that $J_I{}^J|\xi\rangle = -i\delta_I{}^J|\xi\rangle$ to guarantee that $|\phi\rangle$ and $\delta|\phi\rangle$ describe tensors with the same Young tableaux. Having familiarized with these techniques, let us describe the solution to eq. (2.73). Recalling that the product of $s+1$ $Q_I$'s must vanish, one obtains the following solution

$$G|\phi\rangle = Q_IQ_JQ_K\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle \tag{2.76}$$

which depends on an arbitrary vector field contained in $\bar{W}^I = W^\mu\bar{\lambda}^I_\mu$ and on $|\rho\rangle$ which satisfies $J_I{}^J|\rho\rangle = -i\delta_I{}^J|\rho\rangle$ (so that it belongs to the same space of $|\phi\rangle$ and $|\xi\rangle$, i.e. it has the same Young tableaux). These are the equation of motion for HS fields written using the so called *compensator fields* described by $\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle$ (see Appendix A, [29] and references therein). To study how gauge symmetries act on these equations, one may compute the gauge variation of $G|\phi\rangle$ using (2.75)

$$G\delta|\phi\rangle = \frac{i}{2}Q_IQ_JQ_K\bar{V}^K\bar{J}^{JI}|\xi\rangle\ . \tag{2.77}$$

This implies that the compensator field transforms as:

$$\delta(\bar{W}^K\bar{W}^J\bar{W}^I|\rho\rangle) = \frac{i}{2}\bar{V}^{[K}\bar{J}^{JI]}|\xi\rangle \tag{2.78}$$

With the choice $V^\mu = W^\mu$ and $J^{JI}|\xi >= -W^J W^I|\rho >$ we can gauge fix the compensator to zero.
Now the HS potential e.o.m. reads

$$G|\phi >= 0 \ . \tag{2.79}$$

The previous e.o.m. is gauge invariant

$$G\delta|\phi >= \frac{i}{2}Q_I Q_J Q_K J^{[JI}V^{K]}|\xi > \tag{2.80}$$

only if we force the gauge parameter to be traceless. Thus we have obtained the Fronsdal-Labastida equation of motion, with the correct traceless condition on the gauge parameter.
We apply now the operator

$$Q^I + \frac{i}{2}Q_J J^{JI} \tag{2.81}$$

on (A.38) one obtains the relation

$$\frac{1}{4}Q_J Q_K Q_M J^{KM}J^{JI}|\phi\rangle = Q_J Q_K Q_M(-iQ^P - iQ_N J^{NP})W^J W^K W^M + iQ^M W^J W^K W^P)|\rho\rangle \tag{2.82}$$

which implies that the gauge fixing of the compensator to zero forces the HS spin field to be double traceless.

This conclude our analysis; we have in fact solved explicitly (2.2) (2.3) and (2.4) for every even $N$ (i.e. integer spin) in every dimension $D = 2n$. In particular we have reobtained, by using our language and notations, the well known Fornsdal-Labastida e.o.m. describing the dynamics of free higher spin field with mixed symmetry. One advantage one finds by using the approach described above is the fact that equations are quite compact and do not depend on the space time dimension.

# Chapter 3

# One loop quantization and counting degrees of freedom

In this Chapter we study the one-loop quantization of the spinning particle model with a $SO(N)$ extended local supersymmetry on the worldline. We restrict our analysis to flat space, and we will calculate the path integral on the one-dimensional torus to obtain compact formulas which give the number of physical degrees of freedom of the spinning particles for all $N$ in every dimensions. We will obtain also the correct measure on the moduli space of the supergravity multiplet on the one-dimensional torus; this should be useful also to construct the quantum field theory effective action and to compute more general quantum corrections arising when couplings to background fields are introduced.

Our starting point is the Minkowskian action (1.4) or equivalently (1.8). In the following we prefer to use euclidean conventions, and perform a Wick rotation to euclidean time $t \to -i\tau$, accompanied by the Wick rotations of the $SO(N)$ gauge fields $a_{ij} \to ia_{ij}$, just as done in [4] for the $N = 2$ model. We obtain the euclidean action

$$S[X,G] = \int_0^1 d\tau \, \left[\frac{1}{2}e^{-1}(\dot{x}^\alpha - \chi_i\psi_i^\alpha)^2 + \frac{1}{2}\psi_i^\alpha(\delta_{ij}\partial_\tau - a_{ij})\psi_{j\alpha}\right] \tag{3.1}$$

where $X = (x^\alpha, \psi_i^\alpha)$ collectively describes the coordinates $x^\alpha$ and the extra fermionic degrees of freedom $\psi_i^\alpha$ of the spinning particle, and $G = (e, \chi_i, a_{ij})$ represents the set of gauge fields of the $SO(N)$ extended worldline supergravity. The euclidean model (3.1) enjoys the gauge symmetries on the supergravity multiplet given by

$$\begin{aligned} \delta e &= \dot{\xi} + 2\chi_i\epsilon_i \\ \delta\chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j + \alpha_{ij}\chi_j \\ \delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im}a_{mj} + \alpha_{jm}a_{im} \end{aligned} \tag{3.2}$$

where $(\xi, \epsilon_i, \alpha_{ij})$ are diffeomorphism, supersymmetry and $SO(N)$ gauge parameters; in the previous formula we have also Wick rotated the gauge parameters $\epsilon_i \to -i\epsilon_i$, $\xi \to -i\xi$.

In this chapter, in particular, we would like to study the partition function on the one-dimensional torus $T^1$

$$Z \sim \int_{T^1} \frac{\mathcal{D}X\mathcal{D}G}{\mathrm{Vol}\,(\mathrm{Gauge})}\, e^{-S[X,G]}\ . \tag{3.3}$$

Bosonic coordinates have periodic boundary conditions (PBC), while we will take fermions and gravitini with antiperiodic boundary conditions (ABC) on the one dimensional torus. This choice will reveals to be very useful soon.

Our strategy reads:

- Gauge fix the local symmetries (3.2). In particular we will choose a gauge which fixes completely the supergravity multiplet up to some moduli.
- Use the Faddeev-Popov method to extract the volume of the gauge group.
- Calculate the path integral.

## 3.1 Gauge fixing on the torus

We start now fixing the gauge freedom. In particular using (3.2) we choose the convenient gauge configuration

$$(e, \chi_i, a_{ij}) = (\beta, 0, \hat{a}_{ij}) \tag{3.4}$$

where $\beta$ and $\hat{a}_{ij}$ are constants. Let us now discuss in more details this gauge choice.

**EINBEIN:**

The gauge choice of the einbein is rather standard, and produces an integral over the proper time $\beta$ [16].

**GRAVITINI:**

The fermions and the gravitinis are taken with antiperiodic boundary conditions (ABC). This implies that the gravitino can be completely gauged away as there are no zero modes for the differential operator that relates the gauge parameters $\epsilon_i$ to the gravitinos, see eq. (3.2).

$SO(N)$ **GAUGE FIELDS:**

For the $SO(N)$ gauge fields, the gauge conditions $a_{ij} = \hat{a}_{ij}(\theta_k)$ can be chosen to depend on a set of constant angles $\theta_k$, with $k = 1, ..., r$, where $r$ is the rank of group $SO(N)$, taking values on the Cartan torus of the Lie algebra of $SO(N)$. These angles are the moduli of the gauge fields on the torus and must be integrated over a fundamental region. Let us now show explicitly how to reach this gauge configuration.
We parametrize the one-dimensional torus of the worldline by $\tau \in [0, 1]$ with periodic boundary conditions on $\tau$.

Let us start with the simpler $SO(2) = U(1)$ group. For this case the finite version of the gauge transformations (3.2) looks similar to the infinitesimal one

$$\begin{aligned} a' &= a + \dot{\alpha} \\ &= a + \frac{1}{i} g^{-1} \dot{g} \ , \qquad g = e^{i\alpha} \in U(1) \ . \end{aligned} \tag{3.5}$$

One could try to fix the gauge field to zero by solving

$$a + \dot{\alpha} = 0 \qquad \Rightarrow \qquad \alpha(\tau) = -\int_0^\tau dt\, a(t) \ , \tag{3.6}$$

but this would not be correct as the gauge transformation

$$\tilde{g}(\tau) \equiv e^{-i\int_0^\tau dt\, a(t)} \tag{3.7}$$

is not periodic on the torus, $\tilde{g}(0) \neq \tilde{g}(1)$. In general this gauge transformation is not admissible as it modifies the boundary conditions of the fermions. Thus one introduces the constant

$$\theta = \int_0^1 dt\, a(t) \tag{3.8}$$

and uses it to construct a periodic gauge transformation connected to the identity ("small" gauge transformation)

$$g(\tau) \equiv e^{-i\int_0^\tau dt\, a(t)}\, e^{i\theta\tau} \ . \tag{3.9}$$

This transformation brings the gauge field to a constant value on the torus

$$a'(\tau) = \theta \ . \tag{3.10}$$

Now "large" gauge transformations $e^{i\alpha(\tau)}$ with $\alpha(\tau) = 2\pi n\tau$ are periodic and allow to identify

$$\theta \sim \theta + 2\pi n \ , \qquad n \text{ integer} \ . \tag{3.11}$$

Therefore $\theta$ is an angle, and one can take $\theta \in [0, 2\pi]$ as the fundamental region of the moduli space for the $SO(2)$ gauge fields on the one-dimensional torus.

The general case of $SO(N)$ can be treated similarly, using path ordering prescriptions to take into account the non-commutative character of the group. Finite gauge transformations can be written as

$$a' = g^{-1} a g + \frac{1}{i} g^{-1} \dot{g} \ , \qquad g = e^{i\alpha} \ , \qquad \alpha \in \text{Lie}(SO(N)) \ . \tag{3.12}$$

One can define the gauge transformation

$$\tilde{g}(\tau) = \text{P} e^{-i\int_0^\tau dt\, a(t)} \tag{3.13}$$

where "P" stands for path ordering. This path ordered expression solves the equation

$$\partial_\tau \tilde{g}(\tau) = -i a(\tau) \tilde{g}(\tau) \tag{3.14}$$

and could be used to set $a'$ to zero, but it is not periodic on the torus, $\tilde{g}(0) \neq \tilde{g}(1)$, and thus is not admissible. Therefore one identifies the Lie algebra valued constant $A$ by

$$e^{-iA} = \mathrm{P}e^{-i\int_0^1 dt\, a(t)} \tag{3.15}$$

so that the gauge transformation given by

$$g(\tau) \equiv \mathrm{P}e^{-i\int_0^\tau dt\, a(t)}\, e^{iA\tau} \tag{3.16}$$

is periodic and brings the gauge potential equal to a constant

$$a'(\tau) \;=\; A\,. \tag{3.17}$$

Since the constant $A$ is Lie algebra valued, it is given in the vector representation by an antisymmetric $N \times N$ matrix, which can always be skew diagonalized by an orthogonal transformation. One can recognize that the parameters $\theta_i$ contained in the latter equations are angles, since one can use "large" $U(1)$ gauge transformation contained in $SO(N)$ to identify

$$\theta_i \sim \theta_i + 2\pi n_i\,, \qquad n_i \text{ integer}\,. \tag{3.18}$$

The range of these angles can be taken as $\theta_i \in [0, 2\pi]$ for $i = 1, \ldots, r$, with $r$ the rank of the group. Further identifications restricting the range to a fundamental region are discussed in the next sections.

## 3.2 Computing the partition function

We are ready now to write the gauge fixed partition function

$$\begin{aligned} Z \;=\; & -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int \frac{d^Dx}{(2\pi\beta)^{\frac{D}{2}}} \\ & \times \underbrace{K_N}_{(1)} \underbrace{\left[\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\right]}_{(2)} \underbrace{\Big(\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC}\Big)^{\frac{D}{2}-1}}_{(3)} \underbrace{\mathrm{Det}'\,(\partial_\tau - \hat{a}_{adj})_{PBC}}_{(4)} \end{aligned} \tag{3.19}$$

The previous formula contains the well-known proper time integral with the appropriate measure for one-loop amplitudes, and the spacetime volume integral with the standard free particle measure ($(2\pi\beta)^{-\frac{D}{2}}$). In addition we have:

**(1)** $K_N$ is a normalization factor that implements the reduction to a fundamental region of moduli space and will be discussed in the next section separately for even and odd $N$ .

**(2)** This contribution contains the integrals over the $SO(N)$ moduli $\theta_k$ and the determinants of the ghosts and of the remaining fermion fields.

**(3)** This is the determinants of the susy ghosts and of the Majorana fermions $\psi_i^\alpha$ which all have antiperiodic boundary conditions (ABC) and transform in the vector representation of $SO(N)$.

**(4)** This determinant is due to the ghosts for the $SO(N)$ gauge symmetry. They transform in the adjoint representation and have periodic boundary conditions (PBC), so they have zero modes (corresponding to the moduli directions) which are excluded from the determinant (this is indicated by the prime on Det′).

The whole second line (1) + (2) + (3) + (4) computes the number of physical degrees of freedom, normalized to one for a real scalar field,

$$Dof(D,N) = K_N \left[ \prod_{k=1}^{r} \int_0^{2\pi} \frac{d\theta_k}{2\pi} \right] \Big( \text{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC} \Big)^{\frac{D}{2}-1} \text{Det}'\,(\partial_\tau - \hat{a}_{adj})_{PBC} \tag{3.20}$$

In fact, for $N=0$ there are neither gravitinos nor gauge fields, $K_0 = 1$, and all other terms in the formula are absent [19], so that

$$Dof(D,0) = 1 \tag{3.21}$$

as it should, since the $N=0$ model describes a real scalar field in target spacetime. We now present separate discussions for even $N$ and odd $N$, as typical for the orthogonal groups, and explicitate further the previous general formula.

### 3.2.1 Even case: $N = 2r$

To get a flavor of the general formula let us briefly review the $N=2$ case treated in [4]. We have a $SO(2) = U(1)$ gauge field $a_{ij}$ which can be gauge fixed to the constant value

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \tag{3.22}$$

where $\theta$ is an angle that corresponds to the $SO(2)$ modulus. A fundamental region of gauge inequivalent configurations is given by $\theta \in [0, 2\pi]$ with identified boundary values: it corresponds to a one-dimensional torus. The factor $K_2 = 1$ because there are no further identifications on moduli space, and the formula reads

$$\begin{aligned} Dof(D,2) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \Big( \underbrace{\text{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC}}_{(2\cos\frac{\theta}{2})^2} \Big)^{\frac{D}{2}-1} \underbrace{\text{Det}'\,(\partial_\tau)_{PBC}}_{1} \\ &= \begin{cases} \frac{(D-2)!}{[(\frac{D}{2}-1)!]^2} & \text{even } D \\ 0 & \text{odd } D \end{cases} . \end{aligned} \tag{3.23}$$

This formula correctly reproduces the number of physical degrees of freedoms of a gauge $(\frac{D}{2}-1)$–form in even dimensions $D$. Instead, for odd $D$, the above integral vanishes and one has no degrees of freedom left. This may be interpreted as due to the anomalous behavior of an odd number of Majorana fermions under large gauge transformations [57]. In this formula the first determinant is due to the $D$ Majorana fermions, responsible for a power $\frac{D}{2}$

of the first determinant, and to the bosonic susy ghosts, i.e. the Faddeev–Popov determinant for local susy, responsible for the power $-1$ of the first determinant. This determinant is more easily computed using the $U(1)$ basis which diagonalizes the gauge field in (3.22). The second determinant is due the $SO(2)$ ghosts which of course do not couple to the gauge field in the abelian case. A zero mode is present since these ghosts have periodic boundary conditions and is excluded from the determinant. This last determinant does not contribute to the $SO(2)$ modular measure.

In the general case, the rank of $SO(N)$ is $r = \frac{N}{2}$ for even $N$, and by constant gauge transformations one can always put a constant field $a_{ij}$ in a skew diagonal form

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 & . & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & . & 0 & 0 \\ 0 & 0 & -\theta_2 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & \theta_r \\ 0 & 0 & 0 & 0 & . & -\theta_r & 0 \end{pmatrix} . \tag{3.24}$$

The $\theta_k$ are angles since large gauge transformations can be used to identify $\theta_k \sim \theta_k + 2\pi n$ with integer $n$. The determinants are easily computed pairing up coordinates into complex variables that diagonalize the matrix (B.15). Then

$$\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec}) = \prod_{k=1}^{r} \mathrm{Det}\,(\partial_\tau + i\theta_r)\,\mathrm{Det}\,(\partial_\tau - i\theta_r) \tag{3.25}$$

and thus

$$\Big(\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC}\Big)^{\frac{D}{2}-1} = \prod_{k=1}^{r} \Big(2\cos\frac{\theta_k}{2}\Big)^{D-2} . \tag{3.26}$$

Similarly

$$\begin{aligned} \mathrm{Det}'\,(\partial_\tau - \hat{a}_{adj})_{PBC} &= \prod_{k=1}^{r} \mathrm{Det}'\,(\partial_\tau) \\ &\times \prod_{k<l} \mathrm{Det}\,(\partial_\tau + i(\theta_k+\theta_l))\,\mathrm{Det}\,(\partial_\tau - i(\theta_k+\theta_l)) \\ &\times \prod_{k<l} \mathrm{Det}\,(\partial_\tau + i(\theta_k-\theta_l))\,\mathrm{Det}\,(\partial_\tau - i(\theta_k-\theta_l)) \\ &= \prod_{k<l} \Big(2\sin\frac{\theta_k+\theta_l}{2}\Big)^2 \Big(2\sin\frac{\theta_k-\theta_l}{2}\Big)^2 . \end{aligned} \tag{3.27}$$

Thus, with the normalization factor $K_N = \frac{2}{2^r r!}$ one obtains the final formula

$$\begin{aligned} Dof(D,N) &= \frac{2}{2^r r!}\Bigg[\prod_{k=1}^{r} \int_0^{2\pi} \frac{d\theta_k}{2\pi} \Big(2\cos\frac{\theta_k}{2}\Big)^{D-2}\Bigg] \\ &\times \prod_{k<l} \Big(2\sin\frac{\theta_k+\theta_l}{2}\Big)^2 \Big(2\sin\frac{\theta_k-\theta_l}{2}\Big)^2 . \end{aligned} \tag{3.28}$$

The normalization $K_N = \frac{2}{2^r r!}$ can be understood as follows. A factor $\frac{1}{r!}$ is due to the fact that with a $SO(N)$ constant gauge transformation one can permute the angles $\theta_k$ and there are $r$ angles in total. The remaining factor $\frac{2}{2^r}$ can be understood as follows. One could change any angle $\theta_k$ to $-\theta_k$ if parity would be allowed (i.e. reflections of a single coordinate) and this would give the factor $\frac{1}{2^r}$. Thus we introduce parity transformations, which is an invariance of (3.28), by enlarging the gauge group by a $Z_2$ factor and obtain the group $O(N)$. This justifies the identification of $\theta_k$ with $-\theta_k$ and explains the remaining factor 2; equivalently, within $SO(N)$ gauge transformations one can only change signs to pairs of angles simultaneously. It is perhaps more convenient to use some trigonometric identities and write the number of degrees of freedom as

$$\begin{aligned} Dof(D,N) &= \frac{2}{2^r r!}\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2} \\ &\times \prod_{k<l}\left[\left(2\cos\frac{\theta_k}{2}\right)^2-\left(2\cos\frac{\theta_l}{2}\right)^2\right]^2 . \end{aligned} \tag{3.29}$$

### 3.2.2 Odd case: $N = 2r + 1$

The case of odd $N$ describes a fermionic system in target space. In fact, the simplest example is for $N = 1$, which gives a spin 1/2 fermion. It has been treated in [20] on a general curved background, but there are no worldline gauge fields in this case. For odd $N > 1$ the rank of the gauge group is $r = \frac{N-1}{2}$ and the gauge field in the vector representation $a_{ij}$ can be gauge fixed to a constant matrix of the form

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 & . & 0 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & . & 0 & 0 & 0 \\ 0 & 0 & -\theta_2 & 0 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & \theta_r & 0 \\ 0 & 0 & 0 & 0 & . & -\theta_r & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \end{pmatrix} . \tag{3.30}$$

Then, in a way somewhat similar to the even case, one gets

$$\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec}) = \mathrm{Det}\,(\partial_\tau)\prod_{k=1}^{r}\mathrm{Det}\,(\partial_\tau + i\theta_k)\,\mathrm{Det}\,(\partial_\tau - i\theta_k) \tag{3.31}$$

and thus

$$\left(\mathrm{Det}\,(\partial_\tau - \hat{a}_{vec})_{ABC}\right)^{\frac{D}{2}-1} = 2^{\frac{D}{2}-1}\prod_{k=1}^{r}\left(2\cos\frac{\theta_k}{2}\right)^{D-2} . \tag{3.32}$$

Similarly for the determinant in the adjoint representation

$$\begin{aligned}\mathrm{Det}'\left(\partial_\tau-\hat{a}_{adj}\right)_{PBC} &= \prod_{k=1}^{r}\mathrm{Det}'\left(\partial_\tau\right)\mathrm{Det}\left(\partial_\tau+i\theta_k\right)\mathrm{Det}\left(\partial_\tau-i\theta_k\right)\\ &\times\prod_{k<l}\mathrm{Det}\left(\partial_\tau+i(\theta_k+\theta_l)\right)\mathrm{Det}\left(\partial_\tau-i(\theta_k+\theta_l)\right)\\ &\times\prod_{k<l}\mathrm{Det}\left(\partial_\tau+i(\theta_k-\theta_l)\right)\mathrm{Det}\left(\partial_\tau-i(\theta_k-\theta_l)\right)\end{aligned} \tag{3.33}$$

which gives

$$\begin{aligned}\mathrm{Det}'\left(\partial_\tau-\hat{a}_{adj}\right)_{PBC} &= \prod_{k=1}^{r}\left(2\sin\frac{\theta_k}{2}\right)^2\\ &\times\prod_{k<l}\left(2\sin\frac{\theta_k+\theta_l}{2}\right)^2\left(2\sin\frac{\theta_k-\theta_l}{2}\right)^2 .\end{aligned} \tag{3.34}$$

Thus, with a factor

$$K_N=\frac{1}{2^r r!} \tag{3.35}$$

one gets the formula

$$\begin{aligned}Dof(D,N) &= \frac{2^{\frac{D}{2}-1}}{2^r r!}\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2}\left(2\sin\frac{\theta_k}{2}\right)^2\\ &\times\prod_{k<l}\left(2\sin\frac{\theta_k+\theta_l}{2}\right)^2\left(2\sin\frac{\theta_k-\theta_l}{2}\right)^2 .\end{aligned} \tag{3.36}$$

In the expression for $K_N$ the factor 2 that appeared in the even case is now not included, since in the gauge (3.30) one can always reflect the last coordinate to obtain a $SO(N)$ transformation that changes $\theta_k$ into $-\theta_k$.

For explicit computations it is perhaps more convenient to write the number of degrees of freedom as

$$\begin{aligned}Dof(D,N) &= \frac{2^{\frac{D}{2}-1}}{2^r r!}\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2}\left(2\sin\frac{\theta_k}{2}\right)^2\\ &\times\prod_{k<l}\left[\left(2\cos\frac{\theta_k}{2}\right)^2-\left(2\cos\frac{\theta_l}{2}\right)^2\right]^2 .\end{aligned} \tag{3.37}$$

Let us resume, for commodity, the results we have obtained in this section

$$
\begin{aligned}
Dof(D,2r) &= \frac{2}{2^r r!}\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2} \\
&\times \prod_{1\le k<l\le r}\left[\left(2\cos\frac{\theta_l}{2}\right)^2-\left(2\cos\frac{\theta_k}{2}\right)^2\right]^2
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
Dof(D,2r+1) &= \frac{2^{\frac{D}{2}-1}}{2^r r!}\prod_{k=1}^{r}\int_0^{2\pi}\frac{d\theta_k}{2\pi}\left(2\cos\frac{\theta_k}{2}\right)^{D-2}\left(2\sin\frac{\theta_k}{2}\right)^2 \\
&\times \prod_{1\le k<l\le r}\left[\left(2\cos\frac{\theta_l}{2}\right)^2-\left(2\cos\frac{\theta_k}{2}\right)^2\right]^2
\end{aligned} \tag{3.39}
$$

with $N = 2r$ and $N = 2r+1$, respectively. It is obvious that $Dof(D,N)$ vanishes for an odd number of dimensions

$$
Dof(2d+1,N) = 0, \qquad \forall N > 1 \tag{3.40}
$$

as in such case the integrands are odd under the $Z_2$ symmetry $\frac{\theta}{2} \to \pi - \frac{\theta}{2}$. Only for $N = 0, 1$ these models have a non-vanishing number of degrees of freedom propagating in an odd-dimensional spacetime, as in such cases there are no constraints coming from the vector gauge fields. Also for $N = 2$ these models can have degrees of freedom propagating in odd-dimensional target spaces, provided a suitable Chern-Simons term is added to the worldline action [2]. However, Chern-Simons couplings are not possible for $N > 2$.

To compute (3.38) and (3.39) for an even-dimensional target space and for every $N$ we need to introduce a strong mathematical technique. In the next sections we will describe the orthogonal polynomials method and we will use it to solve explicitly the integral (3.38) and (3.39).

## 3.3 Orthogonal Polynomials method

Let us now briefly review some properties of the Van der Monde determinant and the orthogonal polynomials method. Further details and applications of the method can be found in Mehta's book on random matrices [58]. The Van der Monde determinant is defined by

$$
\Delta(x_i) = \prod_{1\le k<l\le r}(x_l - x_k) = \begin{vmatrix} {x_1}^0 & \cdots & {x_r}^0 \\ {x_1}^1 & \cdots & {x_r}^1 \\ \vdots & & \vdots \\ {x_1}^{r-1} & \cdots & {x_r}^{r-1} \end{vmatrix} \tag{3.41}
$$

where the second identity can be easily proved by induction; one can observe that:

1. the determinant on the right hand side vanishes if $x_r = x_i,\ i = 1, \ldots, r-1$
2. the coefficient of ${x_r}^{r-1}$ is the determinant of order $r-1$.

Furthermore, using basic theorems of linear algebra the Van der Monde determinant can be written as

$$\Delta(x_i) = \begin{vmatrix} p_0(x_1) & \cdots & p_0(x_r) \\ p_1(x_1) & \cdots & p_1(x_r) \\ \vdots & & \vdots \\ p_{r-1}(x_1) & \cdots & p_{r-1}(x_r) \end{vmatrix} \tag{3.42}$$

where $p_k(x)$ is an arbitrary, order$-k$ polynomial in the variable $x$, with the only constraint of being *monic*, that is $p_k(x) = x^k + a_{k-1}x^{k-1} + \cdots$.

Interesting properties are associated with the square of the Van der Monde determinant, which can be written as

$$\begin{aligned} \Delta^2(x_i) &= \det \begin{pmatrix} p_0(x_1) & \cdots & p_{r-1}(x_1) \\ p_0(x_2) & \cdots & p_{r-1}(x_2) \\ \vdots & & \vdots \\ p_0(x_r) & \cdots & p_{r-1}(x_r) \end{pmatrix} \begin{pmatrix} p_0(x_1) & \cdots & p_0(x_r) \\ p_1(x_1) & \cdots & p_1(x_r) \\ \vdots & & \vdots \\ p_{r-1}(x_1) & \cdots & p_{r-1}(x_r) \end{pmatrix} \\ &= \det K(x_i, x_j) \end{aligned} \tag{3.43}$$

where the kernel matrix $K$ reads as

$$K(x_i, x_j) = \sum_{k=0}^{r-1} p_k(x_i) p_k(x_j)\ . \tag{3.44}$$

The above polynomials can be chosen to be orthogonal with respect to a certain positive weight $w(x)$ in a domain $D$

$$\int_D dx\ w(x) p_n(x) p_m(x) = h_n \delta_{n,m}\ . \tag{3.45}$$

However, monic polynomials cannot in general be chosen to be ortho*normal*. Of course, one can relate them to a set of orthonormal polynomials $\tilde{p}_n(x)$

$$p_n(x) = \sqrt{h_n}\, \tilde{p}_n(x) \tag{3.46}$$

and the square of the Van der Monde determinant can be written in terms of a rescaled kernel

$$\Delta^2(x_i) = \prod_{k=0}^{r-1} h_k \det \tilde{K}(x_i, x_j) \tag{3.47}$$

with an obvious definition of the latter kernel in terms of the orthonormal polynomials. Thanks to the orthonormality condition, the rescaled kernel can be shown to satisfy the property

$$\int_D dz\ w(z) \tilde{K}(x, z) \tilde{K}(z, y) = \tilde{K}(x, y)\ , \tag{3.48}$$

that can be applied to prove (once again by induction) the following identity

$$\int_D dx_r\ w(x_r) \int_D dx_{r-1}\ w(x_{r-1}) \cdots \int_D dx_{h+1}\ w(x_{h+1})\ \det \tilde{K}(x_i, x_j)$$
$$= (r-h)! \det \tilde{K}^{(h)}(x_i, x_j)$$

where $\tilde{K}^{(h)}(x_i, x_j)$ is the order$-h$ minor obtained by removing from the kernel the last $r-h$ rows and columns. In particular

$$\int_D dx_r\ w(x_r) \cdots \int_D dx_1\ w(x_1)\ \det \tilde{K}(x_i, x_j)$$
$$= (r-1)! \int_D dx_1\ w(x_1) \tilde{K}(x_1, x_1) = r! \tag{3.49}$$

and

$$\frac{1}{r!} \int_D dx_r\ w(x_r) \cdots \int_D dx_1\ w(x_1)\, \Delta^2(x_i) = \prod_{k=0}^{r-1} h_k\ . \tag{3.50}$$

## 3.4 Counting degrees of freedom

We are ready now to solve explicitly (3.38) and (3.39). We first observe that the integrands are even under the aforementioned $Z_2$ symmetry, and thus we can restrict the range of integration

$$\begin{aligned} Dof(D, 2r) &= \frac{2}{r!} \prod_{k=1}^{r} \int_0^{\pi} \frac{d\theta_k}{2\pi} \left(2 \cos \frac{\theta_k}{2}\right)^{D-2} \\ &\times \prod_{1 \leq k < l \leq r} \left[ \left(2 \cos \frac{\theta_l}{2}\right)^2 - \left(2 \cos \frac{\theta_k}{2}\right)^2 \right]^2 , \end{aligned} \tag{3.51}$$

$$\begin{aligned} Dof(D, 2r+1) &= \frac{2^{\frac{D}{2}-1}}{r!} \prod_{k=1}^{r} \int_0^{\pi} \frac{d\theta_k}{2\pi} \left(2 \cos \frac{\theta_k}{2}\right)^{D-2} \left(2 \sin \frac{\theta_k}{2}\right)^2 \\ &\times \prod_{1 \leq k < l \leq r} \left[ \left(2 \cos \frac{\theta_l}{2}\right)^2 - \left(2 \cos \frac{\theta_k}{2}\right)^2 \right]^2 . \end{aligned} \tag{3.52}$$

Now, upon performing the transformations $x_k = \sin^2 \frac{\theta_k}{2}$, we get

$$\begin{aligned} Dof(2d, 2r) &= \frac{2^{2(d-1)r + (r-1)(2r-1)}}{\pi^r r!} \\ &\times \prod_{k=1}^{r} \int_0^1 dx_k\ x_k^{-1/2} (1-x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2 , \end{aligned} \tag{3.53}$$

$$\begin{aligned} Dof(2d, 2r+1) &= \frac{2^{(d-1) + r(2r+2d-3)}}{\pi^r r!} \\ &\times \prod_{k=1}^{r} \int_0^1 dx_k\ x_k^{1/2} (1-x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2 . \end{aligned} \tag{3.54}$$

We have made explicit in the integrands the square of the Van der Monde determinant: it is then possible to use the orthogonal polynomials method to perform the multiple integrals. Note in fact that in (3.53) and (3.54) the prefactors of the Van der Monde determinant have the correct form to be weights $w^{(p,q)}(x) = x^{q-1}(1-x)^{p-q}$ for the Jacobi polynomials $G_k^{(p,q)}$ with $(p,q) = (d-1, 1/2)$ and $(p,q) = (d, 3/2)$, respectively. The integration domain is also the correct one to set the orthogonality conditions

$$\int_0^1 dx\ w(x) G_k(x) G_l(x) = h_k(p,q)\,\delta_{kl} \tag{3.55}$$

with the normalizations given by

$$h_k(p,q) = \frac{k!\,\Gamma(k+q)\Gamma(k+p)\Gamma(k+p-q+1)}{(2k+p)\Gamma^2(2k+p)}\ , \tag{3.56}$$

see [59] for details about the known orthogonal polynomials. Since the Jacobi polynomials $G_k^{(p,q)}$ are all monic, the even$-N$ formula reduces to

$$\begin{aligned} Dof(2d,2r) &= \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r} \prod_{k=0}^{r-1} h_k(d-1,1/2) \\ &= 2^{(r-1)(2r+2d-3)} \frac{\Gamma(2d-1)}{\Gamma^2(d)} \frac{1}{\pi^{r-1}} \prod_{k=1}^{r-1} h_k(d-1,1/2) \end{aligned} \tag{3.57}$$

where in the second identity we have factored out the normalization of the zero-th order polynomial. It is straightforward algebra to get rid of all the irrational terms and reach the final expression

$$Dof(2d,2r) = 2^{r-1} \frac{(2d-2)!}{[(d-1)!]^2} \prod_{k=1}^{r-1} \frac{k\ (2k-1)!\ (2k+2d-3)!}{(2k+d-2)!\ (2k+d-1)!}\ . \tag{3.58}$$

For odd $N$ we have instead

$$\begin{aligned} Dof(2d,2r+1) &= \frac{2^{(d-1)+r(2r+2d-3)}}{\pi^r} \prod_{k=0}^{r-1} h_k(d,3/2) \\ &= \frac{2^{(2-d)+r(2r+2d-3)}}{d} \frac{\Gamma(2d-1)}{\Gamma^2(d)} \frac{1}{\pi^{r-1}} \prod_{k=1}^{r-1} h_k(d,3/2) \end{aligned} \tag{3.59}$$

which can be reduced to

$$Dof(2d,2r+1) = \frac{2^{d-2+r}}{d} \frac{(2d-2)!}{[(d-1)!]^2} \prod_{k=1}^{r-1} \frac{(k+d-1)\ (2k+1)!\ (2k+2d-3)!}{(2k+d-1)!\ (2k+d)!} \tag{3.60}$$

From these final expressions we can single out a few interesting special cases

$$(i) \quad Dof(2,N) = 1, \qquad \forall N \tag{3.61}$$

$$(ii) \quad Dof(4,N) = 2, \qquad \forall N \tag{3.62}$$

$$(iii) \quad Dof(2d,2) = \frac{(2d-2)!}{[(d-1)!]^2} \tag{3.63}$$

$$(iv) \quad Dof(2d,3) = \frac{2^{d-1}}{d}\frac{(2d-2)!}{[(d-1)!]^2} \tag{3.64}$$

$$(v) \quad Dof(2d,4) = \frac{1}{(2d-1)(2d+2)}\left(\frac{(2d)!}{[d!]^2}\right)^2 \tag{3.65}$$

$$(vi) \quad Dof(2d,5) = \frac{3\cdot 2^{d-2}}{(2d-1)(2d+4)(2d+1)^2}\left(\frac{(2d+2)!}{[(d+1)!]^2}\right)^2 \tag{3.66}$$

$$(vii) \quad Dof(2d,6) = \frac{12}{(2d-1)(2d+6)(2d+1)^2(2d+4)^2}\left(\frac{(2d+2)!}{[(d+1)!]^2}\right)^2 . \tag{3.67}$$

In particular, in $D=4$ one recognizes the two polarizations of massless particles of spin $N/2$. The cases of $N=3$ and $N=4$ correspond to free gravitino and graviton, respectively, but this is true only in $D=4$. In other dimensions one has a different field content compatible with conformal invariance.

Let us resume our results:

$$\boxed{Dof(2d,2r) = 2^{r-1}\frac{(2d-2)!}{[(d-1)!]^2}\prod_{k=1}^{r-1}\frac{k\ (2k-1)!\ (2k+2d-3)!}{(2k+d-2)!\ (2k+d-1)!}} \tag{3.68}$$

and

$$\boxed{Dof(2d,2r+1) = \frac{2^{d-2+r}}{d}\frac{(2d-2)!}{[(d-1)!]^2}\prod_{k=1}^{r-1}\frac{(k+d-1)\ (2k+1)!\ (2k+2d-3)!}{(2k+d-1)!\ (2k+d)!}} \tag{3.69}$$

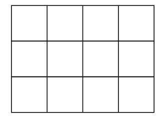
## 3.5 Rectangular $SO(D-2)$ Young tableaux

In the following we would like to prove that (3.68) coincide with the dimensions of the rectangular $SO(D-2)$ Young tableaux with $(D-2)/2$ rows and $N/2$ columns [1].

[1]Equivalently one can show that (3.69) coincide with the dimension of a spinorial rectangular $SO(D-2)$ Young tableaux with $(D-2)/2$ rows and $(N-1)/2$ columns; extension to the odd $N$ case is straightforward and will not be discussed in this section.

Let us start fixing the notation; our convention is to label a rectangular young tableaux with $n$ row and $\lambda$ column with

$$\underbrace{[\lambda, \lambda, ..., \lambda]}_{q \text{ times}} \equiv [\lambda^q] \equiv \tag{3.70}$$



We indicate with $d_{[\lambda^q]}(SO(D-2))$ the dimension of the representation $[\lambda^q]$ of the $SO(D-2)$ Lie group. In order to explicitly compute $d_{[r^{n-1}]}(SO(D-2))$ we use the hook rule algorithm (more details about this technique could be found in [60]). In particular after a straightforward and boring computation one can note some recursive relation that let us to write:

$$d_{[r^{n-1}]}(SO(D-2)) = \frac{Y_t}{Y_h} \tag{3.71}$$

where

$$Y_t = \prod_{i=1}^{n-1} \frac{(r+i-2)!(2r+2i-2)!}{(2i-2)!(2r+i-2)!} \tag{3.72}$$

and

$$Y_h = \prod_{i=1}^{n-1} \frac{(r+i-1)!}{(i-1)!} \tag{3.73}$$

We would like now to show that (3.71) reproduce (3.68) for every $r$ and $n$; to this aim we use a very simple trick. First of all it's easy to verify that (3.71)=(3.68) for the first few simple case.
Now we fix $r$ and if we calculate how $d_{[r^n]}(SO(2n-2))$ and $Dof(2n, 2r)$ scale when one increases the number of the dimensions; in particular we obtain:

$$\frac{d_{[r^{n-1}]}SO(2n-2)}{d_{[r^n]}SO(2n)} = \frac{Dof(2n, 2r)}{Dof(2n+2, 2r)} = \frac{2(n-1)(2r+2n-3)}{(2r+n-2)!(2n-2)!} \tag{3.74}$$

We reverse now the point of view. We fix $D$ and we calculate the scale factor when one $r$ increases:

$$\frac{d_{[(r+1)^{n-1}]}SO(2n)}{d_{[r^n]}SO(2n)} = \frac{Dof(2n, 2r+2)}{Dof(2n, 2r)} = \frac{2r(2r-1)!(2r+2n-3)}{(2r+n-2)!(2r+n-1)!} \tag{3.75}$$

This conclude our proof since $d_{[r^n]}(SO(2n-2)) = Dof(2n, 2r)$ fot $n = 1$ and $r = 1$ and they scale in the same way with respect $r$ and $n$.

## 3.6 The case of $N = 4$ and the Pashnev–Sorokin model

We would like now to analyze the so called $N = 4$ Pashnev and Sorokin model (PS) constructed in [22]; possible couplings to curved backgrounds have been studied in [61]. This model is constructed writing the $R$ symmetry group as $SO(4) = SU(2)_{\text{local}} \times SU(2)_{\text{global}}$. In the analysis of Pashnev and Sorokin the model corresponds to a conformal gravitational multiplet, and it was left undecided if the field content in $D = 4$ is that of a graviton plus three scalars (five degrees of freedom) or that of a graviton plus two scalars (four degrees of freedom). Thus, we apply the techniques discussed in the previous chapter, to compute the number of physical degrees of freedom to clarify the field content of the PS model.

In order to count the physical degrees of freedom let us consider the change of variables

$$\psi^i = \psi^{a\dot{a}} \left(\sigma^i\right)_{a\dot{a}} \tag{3.76}$$

where

$$\left(\bar{\sigma}^i\right)^{\dot{a}a} = (-i1, \sigma)^{\dot{a}a} \,, \qquad \left(\sigma^i\right)_{a\dot{a}} = (i1, \sigma)_{a\dot{a}} = -\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \left(\bar{\sigma}^i\right)^{\dot{b}b} \,. \tag{3.77}$$

The transformation (3.76) can be inverted as [2]

$$\psi^{a\dot{a}} = \frac{1}{2}\psi^i \left(\bar{\sigma}_i\right)^{\dot{a}a} \,. \tag{3.78}$$

The reality condition on $\psi^i$, along with the expressions (3.77), allows to write it also in the form

$$\psi^i = \bar{\psi}_{a\dot{a}} \left(\bar{\sigma}_i\right)^{\dot{a}a} \tag{3.79}$$

with

$$\bar{\psi}_{a\dot{a}} = -\epsilon_{ab}\, \epsilon_{\dot{a}\dot{b}} \psi^{b\dot{b}} \,. \tag{3.80}$$

Thus, the fermion part of the lagrangian can be written as

$$\frac{1}{2}\psi^i(\delta_{ij}\partial_\tau - a_{ij})\psi^j = \bar{\psi}_{a\dot{a}} \left(\delta^a{}_b \delta^{\dot{a}}{}_{\dot{b}} \partial_\tau - A^a{}_b{}^{\dot{a}}{}_{\dot{b}}\right) \psi^{b\dot{b}} \tag{3.81}$$

where

$$A^a{}_b{}^{\dot{a}}{}_{\dot{b}} = \frac{1}{2} a_{ij} \left(\bar{\sigma}^i\right)^{\dot{a}a} \left(\sigma^j\right)_{b\dot{b}} \tag{3.82}$$

and

$$a_{ij} = \frac{1}{2} \left(\sigma_i\right)_{a\dot{a}} \left(\bar{\sigma}_j\right)^{\dot{b}b} A^a{}_b{}^{\dot{a}}{}_{\dot{b}} \,. \tag{3.83}$$

The $SU(2) \times SU(2)$ gauge invariance of the action is now manifest. To gauge only a $SU(2)$ subgroup one may choose

$$A^a{}_b{}^{\dot{a}}{}_{\dot{b}} = \delta^a{}_b B^{\dot{a}}{}_{\dot{b}} \quad \Rightarrow \quad a_{ij} = \frac{1}{2}\text{tr}\left(\sigma_i B \bar{\sigma}_j\right) \tag{3.84}$$

[2] Here we make use of the well-known properties $\left(\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i\right)_a{}^b = 2\delta^{ij}\delta_a{}^b$, $\left(\bar{\sigma}^i\sigma^j + \bar{\sigma}^j\sigma^i\right)^{\dot{a}}{}_{\dot{b}} = 2\delta^{ij}\delta^{\dot{a}}{}_{\dot{b}}$, $\left(\sigma^i\right)_{a\dot{a}} \left(\bar{\sigma}_i\right)^{\dot{b}b} = 2\delta_a{}^b\delta_{\dot{a}}{}^{\dot{b}}$.

and gauge fix $B$ to

$$B^{\dot a}{}_{\dot b} = 2\theta\,(\frac{i}{2}\sigma^3)^{\dot a}{}_{\dot b} = i\theta\,(\sigma^3)^{\dot a}{}_{\dot b} \tag{3.85}$$

which gives

$$a_{ij} = \theta \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.86}$$

so that

$$\begin{aligned} \int D\psi\ \exp\left(-\frac{1}{2}\int \psi^i_\alpha(\partial_\tau\delta_{ij} - a_{ij})\psi^j_\alpha\right) &= \mathrm{Det}^D(\partial_\tau + i\theta)_{ABC}\ \mathrm{Det}^D(\partial_\tau - i\theta)_{ABC} \\ &= \left(2\cos\frac{\theta}{2}\right)^{2D} . \end{aligned} \tag{3.87}$$

The Faddeev-Popov determinant associated to the gauge-fixing of the $SU(2)$ gauge group reads

$$\mathrm{Det}\,(\partial_\tau 1_{adj} - B_{adj})_{PBC} = (2\sin\theta)^2 \tag{3.88}$$

since eq. (3.85) in the adjoint representation becomes

$$B_{adj} = 2\theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.89}$$

Finally, the Faddeev-Popov determinant associated to gauge-fixing the local supersymmetry reads

$$\mathrm{Det}^{-1}(\partial_\tau\delta_{ij} - a_{ij})_{ABC} = \left(2\cos\frac{\theta}{2}\right)^{-4} . \tag{3.90}$$

Assembling all determinants one gets (3.91), where the factor 1/2 is due to the parity transformation $\theta \to -\theta$.
We are ready now to write the counting $Dof$ formula that reads:

$$Dof(D,\mathrm{PS}) \ = \ \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}\left(2\cos\frac{\theta}{2}\right)^{2(D-2)}\left(2\sin\theta\right)^2 . \tag{3.91}$$

This can be cast in a form similar to those obtained in the previous section in order to extract the Van der Monde determinant

$$Dof(D,\mathrm{PS}) \ = \ \frac{2^{2D}}{2\pi}\int_0^1 dx\ (1-x)^{D-3/2}x^{1/2}. \tag{3.92}$$

Expilcitly computation gives

$$Dof(D,\mathrm{PS}) = 2^{D-1}\frac{(2D-3)!!}{D!} \tag{3.93}$$

producing $Dof(D,\mathrm{PS}) = (1,2,5,14,42,132,429,\dots)$ for $D = (2,3,4,5,6,7,8,\dots)$. Thus in $D = 4$ one gets 5 degrees of freedom, which must correspond to a graviton plus three scalars. Notice that the Pashnev–Sorokin model contains physical degrees of freedom also in spacetimes of odd dimensions.

# Chapter 4

# Higher spin generalized curvature in $(A)dS_{2n}$

In Chapter 1 we have studied spinning particle models with $SO(N)$ supergravity multiplet on the worldline, coupled with gravitational background; we have shown explicitly that only maximally symmetric spaces (i.e. $(A)dS$ space) preserve local worldline supersymmetry; we would like now to analyze the physical content of this model. The flat case was extensively analyzed in Chapter 2, where, in particular, we have shown that for every even dimension $D = 2n$ and for every integer spin (even $N$) a canonical analysis produces conformal HSC as physical sector of the Hilbert space; we have then used the *generalized Poincaré Lemma* to solve the differential Bianchi identity and introduce the HS gauge potential.

In the following we intend to quantize *á la Dirac* $SO(N)$ spinning particle model, with $N$ extended supergravity multiplet on the worldline, coupled with $(A)dS$ background and analyze carefully higher spin curvature in maximally symmetric background.

Let us recall, for commodity, the non linear sigma model we are going to study

$$S^{(1)} = \int d\tau [p_\mu \dot{x}^\mu + \frac{i}{2}\psi_i^\alpha \dot{\psi}_i^\beta \eta_{\alpha\beta} - eH - i\chi_i Q_i - \frac{i}{2} a_{ij} J_{ij}] \ . \tag{4.1}$$

In the previous action principle $H$, $Q_i$ and $J_{ij}$ are the constraints we have to impose as operatorial conditions on the Hilbert space states

$$\boxed{H \equiv \frac{1}{2}\pi^2 - \underbrace{\frac{1}{8}\psi_i^\alpha \psi_i^\beta \psi_j^\gamma \psi_j^\delta R_{\alpha\beta\gamma\delta}}_{\sim J^2} \approx 0} \tag{4.2}$$

$$\boxed{Q_i \equiv \psi_i^\mu p_\mu \approx 0} \tag{4.3}$$

$$\boxed{J_{ij} \equiv \psi_i^\alpha \psi_{j\alpha} \approx 0} \tag{4.4}$$

where $\mu, \nu, ...$ are curved indices, $\alpha, \beta, ...$ flat ones, $R_{\alpha\beta\gamma\delta} = b(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})$ is the $(A)dS$ curvature and $\pi_\mu$ is the covariant momentum. It's very important to note that in maximally symmetric space the term proportional to the curvature in the r.h.s. of (4.2) it's proportional to the product of two $SO(N)$ generators.

## 4.1 $(A)dS$ quantum Algebra

First of all we need to discuss the quantization of the supersymmetry algebra (1.39) described in Chapter 1. We will thus replace as usual, coordinate with operator and Poisson brackets with (anti)-commutators:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \qquad\qquad [\psi^\mu_i, \psi^\nu_j] = g^{\mu\nu}\delta_{ij} \ . \tag{4.5}$$

The Grassmann variables are respresented as Gamma matrices of a multiple Clifford algebra (more details could be found in Chapter 2); we recall to the reader that the operator $\sqrt{2}\psi^\alpha_i$ is the Gamma matrix $(\gamma^\alpha)_{a'_i a_i}$ in the basis $|a_1, \cdots, a_N\rangle$. The $SO(N)$ and susy generators become

$$J_{ij} = \frac{i}{2}(\psi_i \cdot \psi_j - \psi_j \cdot \psi_i) \tag{4.6}$$

$$Q_i = \psi^\alpha_i e^\mu_\alpha \pi_\mu \ . \tag{4.7}$$

The covariant momentum acts on the state of the Hilbert space as the covariant derivative

$$\pi_\mu = -i\partial_\mu - \frac{i}{2}\omega_{\mu\alpha\beta}\psi^\alpha \circ \psi^\beta = -i\nabla_\mu(\omega) \tag{4.8}$$

where the symbol $\circ$ means contraction over $SO(N)$ vector indices. The ordering in (4.7) is the one suggested by Einstein covariance, in other words the operators written in such a form will give rise to Einstein covariant (anti)-commutators. At this point one may start checking the algebra and identify a suitable quantum hamiltonian operator. We get the preliminary result

$$\begin{aligned}
[J_{ij}, J_{kl}] &= i\delta_{jk}J_{il} - i\delta_{ik}J_{jl} - i\delta_{jl}J_{ik} + i\delta_{il}J_{jk} \\
[J_{ij}, Q_k] &= i\delta_{jk}Q_i - i\delta_{ik}Q_j \\
[Q_i, Q_j] &= 2\delta_{ij}H_0 - \frac{1}{2}\psi^\alpha_i\psi^\beta_j R_{\alpha\beta\gamma\delta}M^{\gamma\delta}
\end{aligned} \tag{4.9}$$

where $M^{\gamma\delta}$ is the Lorenz generator defined as $M^{\gamma\delta} = \frac{1}{2}[\psi^\gamma_i, \psi^\delta_i]$ and $H_0$ is

$$H_0 = \frac{1}{2}\Big(\pi^\alpha\pi_\alpha - i\omega^\alpha{}_{\alpha\beta}\pi^\beta\Big); \tag{4.10}$$

note that this is the correct quantum ordering one must choose in order to reproduce the laplacian operator in curved space.
We restrict now ourselves to maximally symmetric spaces in which one can rewrite the term appearing in (4.9) as

$$\psi_i^\alpha \psi_j^\beta R_{\alpha\beta\gamma\delta} M^{\gamma\delta} = b\Big[\Big((2-N)\frac{D}{2} - \frac{D^2}{2}\Big)\delta_{ij} + J_{ik}J_{jk} + J_{jk}J_{ik}\Big]. \tag{4.11}$$

Let us write now the complete hamiltonian as $H = H_0 + \Delta H$; we have now to fix $\Delta H$ in order to close the algebra quadratically. We start computing

$$[H_0, Q_i] = i\frac{b}{2}(Q_k J_{ki} - J_{ik} Q_k) \ . \tag{4.12}$$

The clever choice

$$\Delta H = -\frac{b}{4} J_{ij} J_{ij} - \frac{bC}{4} \ , \qquad C = (2-N)\frac{D}{2} - \frac{D^2}{2} \tag{4.13}$$

produce the desired commutator

$$[H, Q_i] = 0 \ . \tag{4.14}$$

Thus the algebra is closed. The constant $C$ in (4.13) is needed to have a consistent first class algebra of constraints.

In summary, we have the quantum constraints

$$\begin{aligned} J_{ij} &= \frac{i}{2}[\psi_i^\alpha, \psi_{j\alpha}] \\ Q_i &= \psi_i^\alpha e_\alpha{}^\mu\Big(p_\mu - \frac{i}{2}\omega_\mu{}^{\beta\gamma}\psi_j^\beta\psi_j^\gamma\Big) \equiv \psi_i^\alpha \pi_\alpha \\ H &= \frac{1}{2}\Big(\pi^\alpha\pi_\alpha - i\omega^\alpha{}_{\alpha\beta}\pi^\beta\Big) - \frac{b}{4}J_{ij}J_{ij} - \frac{bC}{4} \end{aligned} \tag{4.15}$$

satisfying the following quadratic algebra

$$\begin{aligned} [J_{ij}, J_{kl}] &= i\delta_{jk}J_{il} - i\delta_{ik}J_{jl} - i\delta_{jl}J_{ik} + i\delta_{il}J_{jk} \\ [J_{ij}, Q_k] &= i\delta_{jk}Q_i - i\delta_{ik}Q_j \\ [Q_i, Q_j] &= 2\delta_{ij}H - \frac{b}{2}(J_{ik}J_{jk} + J_{jk}J_{ik} - \delta_{ij}J_{kl}J_{kl}) \ . \end{aligned} \tag{4.16}$$

One may check that this algebra coincides with the zero mode restriction in the Ramond sector of the nonlinear superconformal algebras discovered by Knizhnik and Bershadsky in two dimensions [62], [63].

### 4.1.1 The special even $N$ case

Reveals to be very useful for future analysis write the quantum algebra, discussed above, for even $N$, by using complex combinations of the $SO(N)$ indices.

$$\begin{array}{ccc} \psi_i & \rightarrow & (\lambda_I, \bar\lambda_I) \\ i = 1,2,...,2s & & I = 1,2,...,s \end{array}$$

so that[1]

$$[\lambda^\alpha_I, \bar\lambda^\beta_j] = \delta_{IJ}\eta^{\alpha\beta} \tag{4.17}$$

and the supercharges are

$$Q_I = -i\lambda^\gamma_I \; e^\mu_\gamma \left(\partial_\mu + \omega_{\mu\alpha\beta}\; \lambda^\alpha_J \frac{\partial}{\partial\lambda_{J\beta}}\right) \tag{4.18}$$

$$Q^I = -i\frac{\partial}{\partial\lambda_{I\gamma}} \; e^\mu_\gamma \left(\partial_\mu + \omega_{\mu\alpha\beta}\; \lambda^\alpha_J \frac{\partial}{\partial\lambda_{J\beta}}\right) \; . \tag{4.19}$$

The most interesting and useful (anti)commutators in this base are

$$\begin{aligned}
[J_I{}^K, J_J{}^P] &= i\delta_J{}^K J_I{}^P - i\delta_I{}^P J_J{}^K \\
[J_I{}^K J_{JP}] &= i\delta_J{}^K J_{IP} \\
[Q^I, Q^J] &= b(J^I_K J^{JK} + J^{IK} J^J_K) \\
[Q^I, Q^J] &= b(J_{KI} J_J{}^K + J_{KJ} J_I{}^K) \\
[Q_I, Q^J] &= b(J_{IK} J^{KJ} + J_I{}^K J_K{}^J) \qquad \text{with} \quad I \neq J \\
[Q_I, Q^I] &= 2H_0 + b(J_K{}^I J_I{}^K - J_{IK} J^{IK}) + ibJ_I{}^I - ib\sum_K J_K{}^K + \tilde A_s(D)
\end{aligned} \tag{4.20}$$

where $\tilde A_s(D) = b\frac{D}{2}(s + \frac{D}{2} - 1)$; note also that in the last anticommutator $I$ is a fixed index while $K$ runs from 1 to $s$ and just in some particular case one can write $\sum_K J_K{}^K = sJ_I{}^I$.

## 4.2 Canonical analysis and *the generalized Poincaré Lemma*

We are now ready to analyze the physical states of the Hilbert space. We focus our attention on the even $N$ (i.e. integer spin) case and even dimension $D = 2n$. We proceed in analogy with the flat case and we start imposing the condition $J_I{}^I|\mathcal{R}\rangle = 0$ and we obtain

$$J_I{}^I|\mathcal{R}\rangle = 0 \qquad \Rightarrow \qquad \mathcal{R}(x,\lambda) = \mathcal{R}_{\alpha_1..\alpha_n|...|\beta_1..\beta_n}(x)\, \lambda^{\alpha_1}_1..\lambda^{\alpha_n}_1...\lambda^{\beta_1}_s..\lambda^{\beta_n}_s \; . \tag{4.21}$$

From the other constraints constraints we learn:

[1] As in the flat case we realize the algebra by setting $\bar\lambda^\alpha_I = \frac{\partial}{\partial\lambda_{I\alpha}}$.

$$
\begin{array}{rcllcl}
\tilde{J}_I{}^J|\mathcal{R}\rangle & = & 0 & \Rightarrow & \mathcal{R}_{[n+1]_1[n-1]_2\ldots[n]_s} = 0 & \textbf{Algebraic Bianchi identity} \\
Q_I|\mathcal{R}\rangle & = & 0 & \Rightarrow & \nabla_{[\mu}\mathcal{R}_{\mu_1..\mu_n][n]_2\ldots[n]_s} = 0 & \mathcal{R}\ \nabla\textbf{-closed} \\
J^{IJ}|\mathcal{R}\rangle & = & 0 & \Rightarrow & \mathcal{R}^{\mu}{}_{[n-1]_1,\sigma[n-1]_2\ldots[n]_s} = 0 & \mathcal{R}\ \textbf{traceless}
\end{array}
$$

$$
\begin{array}{rcllcl}
J_{IJ}|\mathcal{R}\rangle & = & 0 & \Rightarrow & \sum c_k \boldsymbol{tr^k} R_{[n]_1\cdots[n]_s} = 0 & \textbf{Sum of traceless conditions} \\
Q^I|\mathcal{R}\rangle & = & 0 & \Rightarrow & \nabla^{\mu_1}\mathcal{R}_{\mu_1..\mu_n[n]_2\ldots[n]_s} = 0 & \mathcal{R}\ \nabla\textbf{-co-closed}
\end{array}
$$

Using the Young tableaux language we have:

$$
|\mathcal{R}\rangle \sim \mathcal{R}_{\alpha_1^1\cdots\alpha_n^1|\ldots|\alpha_1^s\cdots\alpha_n^s} \equiv \begin{array}{|c|c|c|c|}\hline \alpha_1^1 & \cdot & \cdot & \alpha_1^s \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \alpha_n^1 & \cdot & \cdot & \alpha_n^s \\ \hline \end{array} \tag{4.22}
$$

such that

$$
\nabla_{[\alpha}\mathcal{R}_{\alpha_1^1\cdots\alpha_n^1]|\ldots|\alpha_1^s\cdots\alpha_n^s} \equiv \begin{array}{|c|c|c|c|}\hline \alpha_1^1 & \cdot & \cdot & \alpha_1^s \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \alpha_n^1 & \cdot & \cdot & \alpha_n^s \\ \hline \nabla & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} = 0 \tag{4.23}
$$

Note that the relations we have obtained above are the minimal coupling of the flat one. In the following we will give a detailed analysis of the differential Bianchi identity in maximally symmetric space ($\nabla$-closed condition).
In the flat case we have solved the differential Bianchi identity by using the *the generalized Poincaré Lemma*. Unfortunately in $(A)dS$ we can't use this Lemma as a guide line. Let us emphasize that in literature it is well known how to make covariant and gauge invariant the Fronsdal-kinetic operator (see Appendix A fot more details and references), but it is not yet clear how to derive it starting from $(A)dS$ higher spin curvature.

In the following we will look for a solution of the differential Bianchi identity in curved space, but note that this could be just a particular solution and probably not the most generical one.

## 4.3 Warm up examples in D=2

We start considering some examples in $D = 2$.
Our strategy reads:

1. write the $(A)dS$ HSC as
$$\mathcal{R}^{(A)dS} = \mathcal{R}^{flat} + (A)dS \text{ corrections;} \tag{4.24}$$
note that in the r.h.s. of the previous relation we put all possible correction proportional to the $(A)dS$ curvature, preserving the symmetry of $\mathcal{R}$.

2. Impose closing condition (i.e. differentialBianchi identity) to fix the extra terms.

3. Use the traceless condition to find the HS gauge potential e.o.m..

**Spin 1 in two dimension:**

This is rather trivial. We start with $\mathcal{R}_\mu$. Then
$$\nabla_{[\mu}\mathcal{R}_{\nu]} = 0 \quad \rightarrow \quad \mathcal{R}_\mu = \partial_\mu \phi \tag{4.25}$$
so that
$$\nabla^\mu \mathcal{R}_\mu = 0 \quad \rightarrow \quad \nabla^2 \phi = 0 \tag{4.26}$$

**Spin 2 in two dimension:**

The HSC is now the symmetric tensor $\mathcal{R}_{\mu\nu}$. We relax the trace constraint and try to solve Bianchi by using the ansatz
$$\mathcal{R}_{\mu\nu} = \nabla_\mu \nabla_\nu \phi + \alpha g_{\mu\nu} \phi \ . \tag{4.27}$$
The differential Bianchi constraints fix $\alpha = 1$, i.e.
$$\nabla_{[\mu}\mathcal{R}_{\nu]\lambda} = 0 \quad \rightarrow \quad \alpha = 1 \ . \tag{4.28}$$
Imposing the trace constraint gives an e.o.m. on the potential
$$\mathcal{R}^\mu{}_\mu = 0 \quad \rightarrow \quad (\nabla^2 + 2)\phi = 0 \tag{4.29}$$
which corresponds to a nonminimally coupled scalar. Co-closing condition is consistently satisfied, as one can compute
$$\nabla^\mu \mathcal{R}_{\mu\nu} = 0 \quad \rightarrow \quad \nabla_\nu (\nabla^2 + 2)\phi = 0 \tag{4.30}$$

**Spin 3 in two dimension:**

In this case we use the following ansatz

$$\mathcal{R}_{\mu\nu\lambda} = (\nabla_{[\mu}\nabla_{\nu}\nabla_{\lambda]}\phi + \alpha g_{[\mu\nu}\nabla_{\lambda]}\phi) \tag{4.31}$$

Imposing Bianchi identities we find

$$\nabla_{[\alpha}\mathcal{R}_{\mu]\nu\lambda} = 0 \quad \rightarrow \quad \alpha = 4\ . \tag{4.32}$$

Finally we compute e.o.m. from the traceless condition

$$\mathcal{R}^{\mu}{}_{\mu\lambda} = 0 \quad \rightarrow \quad 3\nabla_{\lambda}(\nabla^2 + 6)\phi = 0 \tag{4.33}$$

that is also consistent with co-closing condition.

## 4.4 Fixing the strategy

In $(A)dS$ target space, in order to solve the differential Bianchi identity, we use as a guide line the results obtained in the flat limit case, resumed in Chapter 2. Let us emphasize that by using the algebra, the results we obtain do not depend on the target space dimension.
In particular our strategy for future analysis reads:

1. Write
$$|\mathcal{R}\rangle = (Q_{[1}....Q_{s]} + (A)dS \text{ corrections})|\phi\rangle\ ; \tag{4.34}$$
The HS potential $|\phi\rangle$ satisfies the condition
$$J_I{}^J|\phi\rangle = -i\delta_I^J|\phi\rangle \tag{4.35}$$
that
   - for $I \neq J$ implies that the potential satisfies the algebraic Bianchi identity
   - for $I = J$ one has that $|\phi\rangle = \phi_{[n-1]_1\cdots[n-1]_s}\ \lambda_1^{m_1}\cdots\lambda_1^{m_{n-1}}\cdots\lambda_s^{n_1}\cdots\lambda_s^{n_{n-1}}\ .$

   Note that the r.h.s. of (4.34) has to be $U(s)$ invariant and we have to take into account it in adding $(A)dS$ corrections, in order to preserve all the symmetry of the HSC.
2. Impose differential Bianchi identity (i.e. $Q_I|\mathcal{R}\rangle = 0 \quad \forall \quad I = 1, ....s$) to fix the extra terms

   and for the first few $N$ case
3. Analyze the gauge symmetry

4. Extract the higher derivative e.o.m.
5. Introduce the compensator field
6. Extract the linearized e.o.m.

In order to fix the notation and the idea we start considering the first easier example, $N = 4, 6, 8$ (i.e. spin $2, 3, 4$). Note that in $(A)dS$ supercharges do not anticommute; for this reason we prefer to start with an explicitly $U(s)$ invariant ansatz. The calculation become more complicated but also more controlable.

### 4.4.1 Spin 2 in $(A)dS_{2n}$

**Curvature:** The starting point is the manifest $U(2)$-invariant expression

$$|\mathcal{R}\rangle = \frac{1}{2!}\epsilon^{I_1 I_2}\left[Q_{I_1}Q_{I_2} + qJ_{I_1 I_2}\right]|\phi\rangle \ . \tag{4.36}$$

We impose now the differential Bianchi identity on the latter curvature

$$Q_I|\mathcal{R}\rangle = 0 \ . \tag{4.37}$$

This will suffice to fix the constant $q$. In particular, thanks to the symmetry property of (4.36) it will be enough to require $Q_1|\mathcal{R}\rangle = 0$. Let us rewrite (4.36) in a less elegant yet more convenient form. By making use of the commutators

$$[Q_I, Q_J] = -b\left(J_{IL}J_J{}^L + J_{JL}J_I{}^L\right) \tag{4.38}$$

$$[J_I{}^J, Q_K] = i\delta_K^J \ Q_I \tag{4.39}$$

one can easily get

$$|\mathcal{R}\rangle = \left[Q_1 Q_2 + q \ J_{12}\right]|\phi\rangle \tag{4.40}$$

and $Q_1|\mathcal{R}\rangle = 0$ uniquely fixes

$$q = ib \tag{4.41}$$

so that

$$|\mathcal{R}\rangle = \frac{1}{2!}\epsilon^{I_1 I_2}\left[Q_{I_1}Q_{I_2} + ib \ J_{I_1 I_2}\right]|\phi\rangle. \tag{4.42}$$

We conclude that the final form of the spin-2 curvature is

$$\boxed{|\mathcal{R}\rangle = \left[Q_1 Q_2 + ib \ J_{12}\right]|\phi\rangle} \tag{4.43}$$

**Gauge invariance:** Let us consider the transformation

$$\delta|\phi\rangle = Q_K \bar{V}^K |\xi\rangle \tag{4.44}$$

where $\bar{V}^K = V^m(x)\ \bar{\lambda}^K_m$; note that $|\xi\rangle$ and $|\phi\rangle$ have the same Young tableaux. In particular it's easy to verify that:

$$[J_I{}^J, \bar{V}^K] \ = \ -i\delta^K_I \bar{V}^J \tag{4.45}$$

and $Q_K\bar{V}^K$ is then a $U(2)$-scalar

$$[J_I{}^J, Q_K\bar{V}^K] \ = \ 0\ . \tag{4.46}$$

By using the previous relations it's thus easy to obtain:

$$\delta\Big(Q_1Q_2|\phi\rangle\Big) = -ib\ J_{12}\ Q_K\bar{V}^K \qquad \Longrightarrow \qquad \delta|\mathcal{R}\rangle = 0\ .$$

This prove that the spin 2 curvature is invariant with respect the gauge transformation (4.44).

**Equation of motion:** We derive now the equation of motion satisfied by the spin-2 potential. The curvature $|\mathcal{R}\rangle$ constructed previously satisfies all the required constraints except the trace $J^{IJ}|\mathcal{R}\rangle = 0$. Imposing the latter condition we will obtain the equation of motion satisfied by the potential; using the susy algebra we obtain

$$J^{12}|\mathcal{R}\rangle = 0$$

$$\Downarrow$$

$$i\Bigg[-2H_0 + Q_IQ^I - \tfrac{i}{2}Q_IQ_JJ^{IJ} + bJ_{IJ}J^{IJ} + bA_2(D)\Bigg]|\phi\rangle = 0 \tag{4.47}$$

where

$$A_2(D) = 4 - \tilde{A}_2(D)\ . \tag{4.48}$$

Into the bracket we recognize the spin 2 Fronsdal-Labastida kinetic operator

$$G_2^{(A)dS} = \underbrace{-2H_0 + Q_IQ^I - \frac{i}{2}Q_IQ_JJ^{IJ}}_{G^{flat}} + bJ_{IJ}J^{IJ} + bA_2(D) \tag{4.49}$$

**Graviton in $(A)dS_D$:** We would like now to show that this is the correct result by rederiving it starting from Einstein general relativity principles.
We consider a small perturbation to the background metric

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta} \tag{4.50}$$

Our convention for the connection and curvature reads:

$$\tilde{\Gamma}^{\delta}_{\alpha\beta}(g+h) = \Gamma^{\delta}_{\alpha\beta} + \delta\Gamma^{\delta}_{\alpha\beta}$$
$$\tilde{R}_{\alpha\beta\delta\sigma}(g+h) = R_{\alpha\beta\delta\sigma} + \delta R_{\alpha\beta\delta\sigma} + ...$$

The linearised contribution to the Christoffel connection can be written as

$$\delta\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma}\Big(-\nabla_\sigma h_{\alpha\beta} + \nabla_\beta h_{\sigma\alpha} + \nabla_\alpha h_{\beta\sigma}\Big) \tag{4.51}$$

and, since the Riemann tensor is given by

$$R_{\alpha\beta}{}^{\lambda}{}_{\sigma} = \nabla_\alpha \Gamma^{\lambda}_{\beta|\sigma|} - \nabla_\beta \Gamma^{\lambda}_{\alpha|\sigma|} \tag{4.52}$$

where the notation $|\cdots|$ means that the covariant derivative does not act on such index, we have

$$\delta R_{\alpha\beta}{}^{\lambda}{}_{\sigma} = \nabla_\alpha \delta\Gamma^{\lambda}_{\beta\sigma} - \nabla_\beta \delta\Gamma^{\lambda}_{\alpha\sigma} \tag{4.53}$$

that, using (4.51), reduces to

$$\begin{aligned} \delta R_{\alpha\beta}{}^{\lambda}{}_{\sigma} \;=\; & \frac{1}{2} g^{\lambda\rho}\Big(-\nabla_\alpha\nabla_\rho h_{\beta\sigma} + \nabla_\alpha\nabla_\sigma h_{\rho\beta} + \nabla_\alpha\nabla_\beta h_{\sigma\rho} \\ & + \nabla_\beta\nabla_\rho h_{\alpha\sigma} - \nabla_\beta\nabla_\sigma h_{\rho\alpha} - \nabla_\beta\nabla_\alpha h_{\sigma\rho}\Big) \end{aligned} \tag{4.54}$$

from which it's easy to compute the linearized Ricci tensor:

$$\delta R_{\alpha\beta} = -\frac{1}{2}\Big(\nabla^2 h_{\alpha\beta} + \nabla_\alpha\nabla_\beta h\Big) + \nabla^\lambda\nabla_{\{\alpha} h_{\beta\}\lambda} \tag{4.55}$$

$$= -\frac{1}{2}\Big(\nabla^2 h_{\alpha\beta} + \nabla_\alpha\nabla_\beta h\Big) + \nabla_{\{\alpha}\nabla^\lambda h_{\beta\}\lambda} + R^{\rho}{}_{\alpha\beta}{}^{\lambda} h_{\rho\lambda} + R^{\lambda}{}_{\{\alpha} h_{\beta\}\lambda} \tag{4.56}$$

In $(A)dS_D$ background the previous formula reduce to

$$\delta R_{\alpha\beta} = -\frac{1}{2}\Big(\nabla^2 h_{\alpha\beta} + \nabla_\alpha\nabla_\beta h\Big) + \nabla_{\{\alpha}\nabla^\lambda h_{\beta\}\lambda} + b\,\left(D h_{\alpha\beta} - g_{\alpha\beta} h\right)\,. \tag{4.57}$$

We are ready now to compute the the Einstein equation in $(A)dS$ that at the linearised level reads:

$$\delta R_{\alpha\beta} = (D-1) b h_{\alpha\beta} \tag{4.58}$$

We substitute now (4.50) and (4.57) in (4.58) and we find:

$$-\frac{1}{2}\Big(\nabla^2 h_{\alpha\beta} + \nabla_\alpha\nabla_\beta h\Big) + \nabla_{(\alpha}\nabla^\lambda h_{\beta)\lambda} + b\,\left(h_{\alpha\beta} - g_{\alpha\beta} h\right) = 0 \tag{4.59}$$

that is the equation of motion for a graviton in (A)dS$_D$ space. Note that in $D = 4$ (4.59) coincides with (4.47) and it is just written in a different language.

### 4.4.2 Spin 3 in $(A)dS_{2n}$

**Curvature:** Similarly to the previous section we start from the most generic $U(3)$-invariant expression

$$|\mathcal{R}\rangle = \frac{1}{3!}\epsilon^{I_1 I_2 I_3}\left[Q_{I_1}Q_{I_2}Q_{I_3} + pJ_{I_1 I_2}Q_{I_3}\right]|\phi\rangle\ . \tag{4.60}$$

We solve now the constraint $Q_1|R\rangle = 0$ that uniquely fixes

$$p = 4ib \tag{4.61}$$

so that one can write

$$\boxed{|\mathcal{R}\rangle = \frac{1}{3!}\epsilon^{I_1 I_2 I_3}\left[Q_{I_1}Q_{I_2}Q_{I_3} + i4b\ J_{I_1 I_2}Q_{I_3}\right]|\phi\rangle} \tag{4.62}$$

or equivalently

$$|\mathcal{R}\rangle \ = \ \left[Q_1 Q_2 Q_3 + ib\left(J_{12}Q_3 + J_{31}\ .Q_2\right) + i2b\ J_{23}Q_1\right]|\phi\rangle \tag{4.63}$$

We use the trace operator to extract the gauge potential e.o.m.. To this aim the following algebraic identities reveal to be very useful:

$$2iJ_{12}Q_3J^{23} \ \sim \ Q_1Q_1Q_KJ^{1K} + iQ_1J_{1K}J^{1K} + \text{"algebraic Bianchi identity"}$$

$$iJ_{1K}Q^K \ \sim \ Q_1Q_1Q^1 + \text{"algebraic Bianchi identity"}\ .$$

By using the previous relations we obtain a very elegant and explicitly $U(3)$ result:

$$J^{23}|\mathcal{R}\rangle = 0$$

$$\Downarrow$$

$$iQ_1\left[-2H_0 + Q_IQ^I - \tfrac{i}{2}Q_IQ_JJ^{IJ} + bJ_{IJ}J^{IJ} + bA_3(D)\right]|\phi\rangle = 0 \tag{4.64}$$

where

$$A_3(D) = 9 - \tilde{A}_3(D) \tag{4.65}$$

In the formula (4.64) we can read inside the square bracket the spin the spin 3 Fronsdal-Labastida kinetic operator

$$G_3^{(A)dS} = \underbrace{-2H_0 + Q_IQ^I - \frac{i}{2}Q_IQ_JJ^{IJ}}_{G^{flat}} + bJ_{IJ}J^{IJ} + bA_3(d) \tag{4.66}$$

**Gauge invariance:** Using the experience inherited from the flat case, we would like now to study the gauge invariance; we introduce also the compensator field

$$W^K W^J W^I |\rho\rangle \tag{4.67}$$

in order to linearize the e.o.m.. First of all, by using (4.63) and (4.64) we can write

$$G_3^{(A)dS} = (Q_I Q_J Q_K + i4bJ_{IJ}Q_K) W^K W^J W^I |\rho\rangle \tag{4.68}$$

The gauge transformation:

$$\delta|\phi\rangle = Q_K \bar{V}^K |\xi\rangle \tag{4.69}$$

leaves (4.63) invariant, while (4.66) trasforms as:

$$G_3^{(A)dS} \delta|\phi\rangle = \frac{i}{2}(Q_I Q_J Q_K + i4bJ_{IJ}Q_K) J^{\{[IJ}\bar{V}^{k]\}}|\xi\rangle \ . \tag{4.70}$$

From the previous expression is manifest that the Fronsdal-Labastida spin 3 kinetic operator isn't gauge invariant.

**Equation of motion:** Let us analyze how the compensator field transform under (4.69). After some straightforward computation it's easy to obtain:

$$\delta(W^K W^J W^I |\rho\rangle) = \frac{i}{2} J^{[IJ}\bar{V}^{k]}|\xi\rangle \ . \tag{4.71}$$

The previous expression plays a key role. We use it to gauge fix the compensator to zero; this gauge choice let us write

$$|\rho\rangle = 0 \qquad \Rightarrow \qquad G_3^{(A)dS}|\phi\rangle = 0 \tag{4.72}$$

that is the "linearized" spin 3 e.o.m. in $(A)dS$. Let us emphasize that this relation is gauge invariant only if we force the gauge parameter to be traceless.

### 4.4.3 Spin 4 in $(A)dS_{2n}$

**Curvature:** We start from the manifestly $U(4)$-invariant expression

$$|\mathcal{R}\rangle = \frac{1}{4!}\epsilon^{I_1 I_2 I_3 I_4}\left[Q_{I_1}Q_{I_2}Q_{I_3}Q_{I_4} + 3!\ p\ K_{I_1 I_2}Q_{I_3}Q_{I_4} + 3q\ K_{I_1 I_2}K_{I_3 I_4}\right]|\phi\rangle \tag{4.73}$$

and use the commutation rules to bring $Q_1$ in front of everything. After some straightforward yet tedious algebra one gets

$$\begin{aligned} |\mathcal{R}\rangle \ = \ & \Bigg[Q_1 Q_2 Q_3 Q_4 + \left(p - \frac{2i}{3}b\right)\Big(K_{12}Q_3Q_4 + K_{31}Q_4Q_2 + K_{14}Q_2Q_3\Big) + \\ & \left(p + \frac{4i}{3}b\right)K_{34}Q_1Q_2 + \left(p + \frac{i}{3}b\right)\Big(K_{42}Q_1Q_3 + K_{23}Q_1Q_4\Big) + \\ & q\Big(K_{12}K_{34} + K_{31}K_{24} + K_{14}K_{23}\Big)\Bigg]|\phi\rangle \end{aligned} \tag{4.74}$$

so that requiring $Q_1|\mathcal{R}\rangle = 0$, it yields $p = \frac{5i}{3}b$ and $q = -3b^2$. Hence one has

$$\begin{aligned} |\mathcal{R}\rangle \;=\; & \Big[Q_1Q_2Q_3Q_4 + ib\Big(K_{12}Q_3Q_4 + K_{31}Q_4Q_2 + K_{14}Q_2Q_3\Big) + i3b\;K_{34}Q_1Q_2 + \\ & i2b\;\Big(K_{42}Q_1Q_3 + K_{23}Q_1Q_4\Big) \\ & -3b^2\;\Big(K_{12}K_{34} + K_{31}K_{24} + K_{14}K_{23}\Big)\Big]|\phi\rangle \end{aligned} \tag{4.75}$$

or equivalently

$$\boxed{|\mathcal{R}\rangle = \frac{1}{4!}\epsilon^{I_1I_2I_3I_4}\Big[Q_{I_1}Q_{I_2}Q_{I_3}Q_{I_4} + i10b\;K_{I_1I_2}Q_{I_3}Q_{I_4} - 9b^2\;K_{I_1I_2}K_{I_3I_4}\Big]|\phi\rangle} \tag{4.76}$$

that is the final form of the spin-4 curvature.

**Equation of motion:** The traceless condition produces

$$(iQ_aQ_b - J_{ab})\Big[-2H_0 + Q_IQ^I - \frac{i}{2}Q_IQ_JJ^{IJ} + bJ_{IJ}J^{IJ} + bA_4(D)\Big]|\phi\rangle = 0 \tag{4.77}$$

where

$$A_4(D) = 16 - \tilde{A}_4(D) \tag{4.78}$$

In analogy with the other case we define

$$G_4^{(A)dS} = \underbrace{-2H_0 + Q_IQ^I - \frac{i}{2}Q_IQ_JJ^{IJ}}_{G^{flat}} + bJ_{IJ}J^{IJ} + bA_4(D) \tag{4.79}$$

### 4.4.4 Conjecture

The results we have obtained above suggest us that, for every spin and for every even dimension $D$, the Fronsdal-Labastida kinetic operator in $(A)dS$ becomes:

$$\boxed{G_s^{(A)dS} = \Big[-2H_0 + Q_IQ^I - \frac{i}{2}Q_IQ_JJ^{IJ} + bJ_{IJ}J^{IJ} + bA_s(D)\Big]} \tag{4.80}$$

where

$$A_s(D) = s^2 - \frac{D}{2}(s + \frac{D}{2} - 1). \tag{4.81}$$

Let us emphasize that in $D=4$ this operator reproduces the extension of the Fronsdal one in $(A)dS$ spaces: moreover if we force the HS field ($|\phi\rangle$) to be double traceless and the gauge parameter ($|\xi\rangle$) to be traceless it is invariant with respect the gauge tranformations:

$$\delta|\phi\rangle = Q_K \bar{V}^K |\xi\rangle \ .$$

## 4.5 HSC in $(A)dS_{2n}$

Before starting, for matter of convenience, we change our notation and we indicate with $K_{IJ}=J_{IJ}$ and $K^{IJ}=J^{IJ}$.
In this section we will look for for a generic $U(s)$-invariant expression ($N=2s, s\in N$) of the form:

$$|\mathcal{R}\rangle = \sum_{n=0}^{[s/2]} (ib)^n r_n(s)\mathcal{R}_n(s)\ |\phi\rangle\ , \quad \text{with} \quad r_0(s)\equiv 1 \tag{4.82}$$

where in this case $[s/2]$ means integer part and

$$\mathcal{R}_n(s) \equiv \frac{1}{s!}\epsilon^{I_1 I_2\cdots I_s} K_{I_1 I_2}\cdots K_{I_{2n-1}I_{2n}} Q_{I_{2n+1}}\cdots Q_{I_s}\ . \tag{4.83}$$

We fix the numerical coefficients $r_n(s)$ by requiring

$$Q_I|\mathcal{R}\rangle = 0\ . \tag{4.84}$$

In particular, thanks to the symmetry properties of (4.82) it will suffice to require $Q_1|\mathcal{R}\rangle=0$. In order to achieve such a task we shall need a few recursive relations that we now derive using the commutation relations

$$[Q_I, Q_J] \ = \ -b\left(K_{IL}J_J{}^L + K_{JL}J_I{}^L\right) \tag{4.85}$$

$$[J_I{}^J, Q_K] \ = \ i\delta^J_K\ Q_I \tag{4.86}$$

$$[K_{IJ}, K_{KL}] \ = \ [K_{IJ}, Q_K] = 0 \tag{4.87}$$

and the condition (4.35). Let us split now the $s$ indices into a "time-like" index 1 and $s-1$ "space-like" indices

$$I=(1,i)\ , \quad i=2,\ldots,s \tag{4.88}$$

and let us first define a shortcut notation that reveals to be extremely useful

$$\epsilon^{i_1\cdots i_{s-1}}\ Q_{i_1}\cdots Q_{i_n} Q_1 Q_{i_{n+1}}\cdots Q_{i_{s-1}}|\phi\rangle \ \equiv\ Q_{[n]}Q_1Q_{[s-1-n]} \tag{4.89}$$

$$\epsilon^{i_1\cdots i_{s-1}}\ K_{1i_1}\ Q_{i_3}\cdots Q_{i_n} Q_1 Q_{i_{n+1}}\cdots Q_{i_{s-1}}|\phi\rangle \ \equiv\ K_{1i_1}\ Q_{[n-2]}Q_1Q_{[s-1-n]} \tag{4.90}$$

and whenever we encounter a $K_{ij}$ tensor we use the commutation rules above and the antisymmetry provided by the $\epsilon$ tensor to bring them in front of everything and give them the first indices of the set $i_1, i_2, \ldots$.
After some algebraic manipulation, and by using the quantum algebra (4.20), it's easy to prove the following Lemma:

**Lemma 1.**

$$(-)^n Q_{[n]} Q_1 Q_{[s-1-n]} = Q_1 Q_{[s-1]} - ib\left[n(s-2) - \frac{n(n-1)}{2}\right] K_{1i_1} Q_{[s-2]} + ibK_{i_1i_2} \sum_{m=1}^{n} \sum_{k=m-1}^{s-3} (-)^k Q_{[k]} Q_1 Q_{[s-k-3]}. \qquad (4.91)$$

*Note now that the previous formula can be easily iterated by noting that the last term is just equal to the l.h.s. if one performs the substitution $s \to s-2$. The iteration process thus yields*

$$\sum_{n=0}^{s-1} (-)^n Q_{[n]} Q_1 Q_{[s-1-n]} = sQ_1 Q_{[s-1]}$$
$$-(ib)a_2(s)\Big(K_{1i_1} Q_{[s-2]} - K_{i_1i_2} Q_1 Q_{[s-3]}\Big)$$
$$-(ib)^2 a_4(s)\Big(K_{1i_1} K_{i_2i_3} Q_{[s-4]} - K_{i_1i_2} K_{i_3i_4} Q_1 Q_{[s-5]}\Big)$$
$$.$$
$$.$$
$$.$$
$$-(ib)^p a_{2p}(s) K_{1i_1} K_{i_2i_3} \cdots K_{i_{2(p-1)}i_{2p-1}} Q_{[s-2p]}$$
$$+(ib)^p \sum_{k_0=1}^{s-1} \sum_{m_1=1}^{k_0} \sum_{k_1=m_1-1}^{s-3} \cdots \sum_{m_p=1}^{k_{p-1}} \sum_{k_p=m_p-1}^{s-2p-1} (-)^{k_p} Q_{[k_p]} Q_1 Q_{[2-2p-1-k_p]} \qquad (4.92)$$

*with*

$$a_{2n}(s) \equiv \sum_{k_0=1}^{s-1} \sum_{m_1=1}^{k_0} \sum_{k_1=m_1-1}^{s-3} \cdots \sum_{m_u=1}^{k_{u-1}} \sum_{k_n=m_n-1}^{s-2n-1} 1 \ ; \qquad (4.93)$$

*the iterative expression (4.92) stops at the last-but-one entry if $s = 2p$ whereas it stops at the last if $s = 2p+1$.*

We will analyze carefully relation (4.93) In Appendix C, where, in particular, we will show that $a_{2n}(s)$ "factorize" as

$$a_{2n}(s) = f(n)\ (s-1)(s-2)\cdots(s-2n) \qquad (4.94)$$

for some function $f(n)$ defined as

$$f(n) = \sum_{\tilde{q}=0}^{n-2} \frac{(-)^{\tilde{q}}}{(2\tilde{q}+2)!} f(n-\tilde{q}-1) + \frac{(-)^{n-1}}{(2n-1)!}, \qquad \text{with} \qquad f(0) = 0\ . \qquad (4.95)$$

The $(A)dS$ quantum algebra could be used to derive other very useful relations. In particular it's not hard to prove that

**Lemma 2.**

$$Q_1^2 Q_{[s-1]} = -ibK_{1i_1}\sum_{n=0}^{s-2}(-)^n Q_{[n]}Q_1 Q_{[s-2-n]}; \tag{4.96}$$

*that by using **Lemma 1**, could be reduced to*

$$Q_1^2 Q_{[s-1]} = -ibK_{1i_1}\Big(a_0(s-1)\,Q_1 Q_{[s-2]} + ib\,a_2(s-1)\,K_{i_2i_3}Q_1 Q_{[s-4]} \\ + \cdots (ib)^{p-1}\,a_{2(p-1)}(s-1)\,K_{i_2i_3}\cdots K_{i_{2(p-1)}i_{2p-1}}Q_1\Big) \tag{4.97}$$

*noting that expressions containing $K_{1i_1}K_{1i_2}$ are vanishing thanks to the implied anti-symmetrization. In the latter we have defined $a_0(s) \equiv s$.*

It is easy now to convince ourselves that the zero-th order operator $\mathcal{R}_0(s)$ can be written as

$$s!\;\mathcal{R}_0(s)|\phi\rangle = \sum_{n=0}^{s-1}(-)^n Q_{[n]}Q_1 Q_{[s-1-n]} \tag{4.98}$$

so that making use of (4.92), and assuming for definiteness that $s = 2p$ one gets

$$\begin{aligned} s!\;Q_1\mathcal{R}_0(s)|\phi\rangle &= a_0(s)Q_1^2 Q_{[s-1]} \\ &\quad -(ib)a_2(s)\Big(K_{1i_1}Q_1Q_{[s-2]} - K_{i_1i_2}Q_1^2Q_{[s-3]}\Big) \\ &\quad -(ib)^2a_4(s)\Big(K_{1i_1}K_{i_2i_3}Q_1Q_{[s-4]} - K_{i_1i_2}K_{i_3i_4}Q_1^2Q_{[s-5]}\Big) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad -(ib)^p a_{2p}(s)K_{1i_1}K_{i_2i_3}\cdots K_{i_{2(p-1)}i_{2p-1}}Q_1\;. \end{aligned} \tag{4.99}$$

One can now easily use **Lemma 2** and get

$$s!\;Q_1\mathcal{R}_0(s)|\phi\rangle = -\sum_{n=1}^{s/2}(ib)^n\alpha_{2n}(s)Q_1 I_n(s) \tag{4.100}$$

where

$$I_n(s) \equiv K_{1i_1}K_{i_2i_3}\cdots K_{i_{2(n-1)}i_{2n-1}}Q_{[s-2n]} \tag{4.101}$$

and

$$\alpha_{2n}(s) \equiv \sum_{k=0}^{n} a_{2k}(s) a_{2(n-1-k)}(s-1-2k) \tag{4.102}$$

with $a_{-2}(u) \equiv 1$. This completes the first part of the analysis.

The next step will be to rewrite expression (4.100) in terms of $SU(s)$-covariant tensors. The covariantization of the tensors $I_n(s)$ is again an iterative process. Note in fact that one can write

$$I_n(s) = \frac{1}{2n}\left( J_n(s) - K_{i_1 i_2} \cdots K_{i_{2n-1} i_{2n}}(s-2n)!\; \mathcal{R}_0(s-2n) \right) \tag{4.103}$$

that using (4.100) and noting that

$$K_{i_1 i_2} \cdots K_{i_{2n-1} i_{2n}} I_m(s-2n) = I_{n+m}(s) \tag{4.104}$$

allows to write

$$I_n(s) = \frac{1}{2n}\left( J_n(s) + \sum_{m=n+1}^{s/2} (ib)^{m-n}\; \alpha_{2(m-n)}(s-2n)\; I_m(s) \right) . \tag{4.105}$$

that finally yields

$$Q_1 \sum_{n=0}^{[s/2]} (ib)^n r_n(s) \mathcal{R}_n(s) |\phi\rangle = 0 \tag{4.106}$$

with

$$r_n(s) = \frac{1}{2n} \sum_{k=1}^{n} r_{n-k}(s) \alpha_{2k}(s-2(n-k)) \; . \tag{4.107}$$

Note that in (4.106) we have replaced $s/2$ with its integer part: it is in fact not difficult to check that the latter holds for odd $s$ as well, with that precise replacement.

In order to save the iteration process reveals to be very important the following Lemma:

**Lemma 3.**

$$\alpha_{2n}(s) = a_{2n}(s+1) \tag{4.108}$$

*or equivalently*

$$\sum_{k=0}^{n-1} f(k+1) f(n-k) = (2n+1) f(n+1) \tag{4.109}$$

The proof of the previous Lemma can be found in Appendix C.

From the definition of $r_n(s)$, and by using (4.108) we find

$$r_n(s) = \frac{1}{2n}\sum_{k=1}^{n} r_{n-k}(s)\; a_{2k}(s-2(n-k)+1)\;. \tag{4.110}$$

We thus have

$$\begin{aligned}
r_0(s) &= 1\\
r_1(s) &= \frac{1}{2}a_2(s+1) = \frac{(s-1)s(s+1)}{6}\\
r_2(s) &= \frac{1}{4}\left(a_4(s+1)+\frac{1}{2}a_2(s+1)a_2(s-1)\right)\\
&= \frac{1}{5\cdot 2^3\cdot 3^2}(5s+7)(s+1)s(s-1)(s-2)(s-3)\\
r_3(s) &= \frac{1}{6}\Bigg(a_6(s+1)+\frac{1}{4}a_4(s+1)a_2(s-3)+\frac{1}{2}a_2(s+1)a_4(s-1)\\
&\qquad +\frac{1}{8}a_2(s+1)a_2(s-1)a_2(s-3)\Bigg)\\
&= \frac{1}{7\cdot 5\cdot 2^4\cdot 3^4}(35s^2+112s+93)(s+1)s(s-1)(s-2)(s-3)(s-4)(s-5)
\end{aligned} \tag{4.111}$$

Note now that substituting (4.110) and (4.111) into (4.82) we produce the correct result obtained in the previous sections (4.43), (4.63) and (4.76).
In addition we have the following predictions for $s=5,6$ and $s=7$

$$|R\rangle = \frac{1}{5!}\epsilon^{I_1\cdots I_5}\Big[Q_{I_1}\cdots Q_{I_5}+20ib\,K_{I_1I_2}Q_{I_3}\cdots Q_{I_5}-64b^2\,K_{I_1I_2}K_{I_3I_4}Q_{I_5}\Big]|\phi\rangle\;, \tag{4.112}$$

$$\begin{aligned}
|R\rangle = \frac{1}{6!}\epsilon^{I_1\cdots I_6}\Big[&Q_{I_1}\cdots Q_{I_6}+35ib\,K_{I_1I_2}Q_{I_3}\cdots Q_{I_6}-259b^2\,K_{I_1I_2}K_{I_3I_4}Q_{I_5}Q_{I_6}\\
&-225ib^3\,K_{I_1I_2}K_{I_3I_4}K_{I_5I_6}\Big]|\phi\rangle\;,
\end{aligned} \tag{4.113}$$

$$\begin{aligned}
|R\rangle = \frac{1}{7!}\epsilon^{I_1\cdots I_7}\Big[&Q_{I_1}\cdots Q_{I_7}+56ib\,K_{I_1I_2}Q_{I_3}\cdots Q_{I_7}-784b^2\,K_{I_1I_2}K_{I_3I_4}Q_{I_5}Q_{I_6}Q_{I_7}\\
&-2304ib^3\,K_{I_1I_2}K_{I_3I_4}K_{I_5I_6}Q_{I_7}\Big]|\phi\rangle\;.
\end{aligned} \tag{4.114}$$

More in general, starting from the relations (4.82), (4.94), (4.95) and (4.110) one can construct HSC for every integer spin in every even dimension $D=2n$.

# Chapter 5

# HK and QK $N = 4$ one dimensional SUGRA

Recently supersymmetric non linear sigma model on Quaternionic-Kähler manifold (QK) have attracted a great deal of attention in the context of studying radial quantization of BPS black-hole [50].
In this chapter chapter we intend to construct and analyze $N = 4$, one dimensional supergravity model on Hyper Kähler (HK) and QK background. In particular we will analyze the symmetries of these models and we will study the first class constraints algebra that in QK case reveals an interesting *"non Lie"* structure. In addition we construct the BRST charge and the gauge fixed action on the circle.

## 5.1 Special Geometry

Let us consider HyperKähler and Quaternionic Kähler geometries in dimension $4n$ and signature $(2n, 2n)$; they enjoy $sp(2n)$ and $sp(2) \otimes sp(2n)$ holonomy, respectively; more details on HK and QK geometry can be found in [64] [65] and references therein.
In this Chapter we will use the following notation

| | |
|---|---|
| Curved indices | $\mu, \nu = 1, 2, ..., 4n$ |
| Flat indices | $m, n = 1, 2, ..., 4n$ |
| Fund. representation of $sp(2n)$ | $A, B = 1, 2, ..., 2n$ |
| Fund. representation of $sp(2)$ | $\alpha, \beta = 1, 2$ |

The symplectic special holonomies allow us to decompose $SO(2n, 2n)$ tangent space indices with respect to the $sp(2) \otimes sp(2n)$ subgroup.

$$m = A\alpha. \tag{5.1}$$

The invariant $SO(2n, 2n)$ metric decomposes as

$$\eta_{mn} = \epsilon_{\alpha\beta}\epsilon_{AB} \tag{5.2}$$

where $\epsilon_{\alpha\beta}$ and $\epsilon_{AB}$ are the $sp(2)$ and $sp(2n)$ invariant, antysimmetric tensors, that we use to raise and lower $sp(2)$ and $sp(2n)$ indices by using the following rules

$$\begin{aligned} v_A &\equiv \epsilon_{AB} v^B & v^A &\equiv v_B \epsilon^{BA} \\ v_\alpha &\equiv \epsilon_{\alpha\beta} v^\beta & v^\alpha &\equiv v_\beta \epsilon^{\beta\alpha} \end{aligned} \tag{5.3}$$

and in our notation we take $\epsilon^\alpha{}_\beta = \delta^\beta_\alpha = -\epsilon_\beta{}^\alpha$. We denote the vielbein as $V^m_\mu$

$$V^m_\mu V_{\nu m} = g_{\mu\nu} \qquad V_{\mu m} V^\mu_n = \eta_{mn} \tag{5.4}$$

or equivalently

$$V^A_{\mu\alpha} V^\alpha_{\nu A} = -g_{\mu\nu}, \qquad V^A_{\mu\alpha} V^{\mu\beta}_B = -\delta^B_A \delta^\beta_\alpha. \tag{5.5}$$

Special holonomy implies also that

$$V^A_{\{\mu\alpha} V^\beta_{\nu\}A} = -\frac{1}{2} g_{\mu\nu} \delta^\beta_\alpha, \qquad V^A_{\{\mu\alpha} V^\alpha_{\nu\}} = -\frac{1}{2n} g_{\mu\nu} \delta^B_A \ . \tag{5.6}$$

The einbein covariantly constant condition reads

$$\nabla V^m \equiv dV^m + P_n{}^m \wedge V^n = 0 \tag{5.7}$$

where the spin connection $P_n{}^m$ decomposes as

$$P_n{}^m = \delta^A_B \omega_\beta{}^\alpha + \delta_\alpha{}^\beta \Omega^A_B \ . \tag{5.8}$$

The one-forms $\omega^\alpha_\beta$ and $\Omega^A_B$ are symmetric in their sympectic indices. On HK manifold just the $sp(2n)$ connection $\Omega^B_A$ is non vanishing while both are present for QK manifold. Let us also emphasize that the tangent bundle decompose into rank 2 and rank $2n$ vector bundles

$$TM = H \otimes E; \tag{5.9}$$

the connection acts on sections of $H$ and $E$ respecively as:

$$\nabla X^\alpha = dX^\alpha + \omega_\beta{}^\alpha X^\beta, \qquad \nabla X^A = dX^A + \omega_B{}^A X^B \ . \tag{5.10}$$

We need now another geometric ingredient, that is the Riemann tensor:

$$R^m{}_n \equiv dP^m{}_n + P^m{}_r P^r{}_n = \delta^\alpha_\beta R^A_B + \delta^A_B R^\alpha_\beta = \frac{1}{2} V^r \wedge V^s R^m{}_{nrs} \ . \tag{5.11}$$

It decompose as

$$R_{mnrs} = \lambda \epsilon_{(\alpha|\gamma|}\epsilon_{\beta)\delta}\epsilon_{AB}\epsilon_{CD} + \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}[\lambda \epsilon_{(A|C|}\epsilon_{B)D} + \Omega_{ABCD}] \ . \tag{5.12}$$

The $sp(2n)$ curvature $\Omega_{ABCD}$ is totally symmetric while the Bianchi identities force the $sp(2)$ curvature to vanish. The terms proportional to the constant $\lambda$ are present just in QK manifold and vanisch in HK case. It is important also to note that these are not proportional to $\eta_{r[m}\eta_{n]s}$, the constant curvature Riemann tensor, since constant curvature manifolds are not QK.

## 5.2 HK Sigma Model

Our starting point is the $N = 4$ susy algebra

$$\{Q^i_\alpha, Q^j_\beta\}_{pb} = -\epsilon^{ij}\epsilon_{\alpha\beta}; \tag{5.13}$$

the indices $i, j = 1, 2$ label the fundamental representation of an $sp(2)$ $R$-symmetry with invariant tensor $\epsilon^{ij}$. Let now $(\mathcal{M}, g_{\mu\nu})$ be the $4n$-dimensional HK target space. The field content of the model consists of bosonic worldline embedding coordinate $x^\mu(t)$, and fermionic spinning degrees of freedom $\psi^i_A(t)$. The model is governed by the simple action

$$S = \frac{1}{2}\int dt \Big[\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} + \psi^i_A \frac{\nabla\psi^A_i}{dt}\Big] \tag{5.14}$$

where we have defined the covariant derivative $\frac{\nabla\psi^A_i}{dt} \equiv \partial_t \psi^A_i + \dot{x}^\mu \Omega_{\mu C}{}^A \psi^C_i$. This model enjoys the rigid symmetry

**Worldline translation:**

$$\begin{aligned} \delta x^\mu &= \xi \dot{x}^\mu, \\ \delta\psi^i_A &= \xi \dot{\psi}^i_A \end{aligned} \tag{5.15}$$

$sp(2)$ **symmetry:**

$$\begin{aligned} \delta x^\mu &= 0, \\ \delta\psi^i_A &= \lambda^{ij}\psi_{Aj} \end{aligned} \tag{5.16}$$

$N = 4$ **supersymmetry:**

$$\begin{aligned} \delta x^\mu &= V^{\mu A}_\alpha \psi^i_A \epsilon^\alpha_i, \\ \delta\psi^i_A &= -\dot{x}^\mu V^\alpha_{\mu A}\epsilon^i_\alpha + V^{\mu B}_\beta \psi^j_B \epsilon^\beta_j \Omega_{\mu A}{}^C \psi^i_C \ . \end{aligned} \tag{5.17}$$

The model (5.14) in first order form reads

$$S^{(1)} = \int dt \Big[p_\mu \dot{x}^\mu + \frac{1}{2}\psi^i_A \dot{\psi}^A_i - \frac{1}{2}\pi_\mu \pi_\nu g^{\mu\nu}\Big] \tag{5.18}$$

with the canonical Poisson bracket

$$\{p_\mu, x^\nu\}_{pb} = -\delta^\nu_\mu, \qquad \{\psi^i_A, \psi^j_B\}_{pb} = -\epsilon^{ij}\epsilon_{AB} \ . \tag{5.19}$$

In the equations above, $\pi_\mu$ is the covariant momentum

$$\pi_\mu = p_\mu - \frac{i}{2} P_{\mu mn} M^{mn} \tag{5.20}$$

where $M^{mn}$ are quadratic in spinning degree of freedom and generate the Lorentz algebra

$$\{M^{mn}, M^{rs}\}_{pb} = -iM^{[m|s|}\eta^{n]r} + iM^{[m|r|}\eta^{n]s} \tag{5.21}$$

For special holonomy manifolds just a subgroup of the full orthogonal group need appear. In particular, in our case, for HK manifolds we have

$$P_{mn}M^{mn} = \Omega_{AB}T^{AB} \tag{5.22}$$

where $T^{AB}$ generates $sp(2n)$

$$\{T^{AB}, T^{CD}\} = -i\epsilon^{C\{A}T^{B\}D} - i\epsilon^{A\{C}T^{D\}B}. \tag{5.23}$$

Explicitly one has

$$T^{AB} \equiv i\psi^{i\{A}\psi_i^{B\}} \tag{5.24}$$

and

$$\pi_\mu = p_\mu - \frac{1}{2}\psi^i_A\Omega_{\mu B}{}^{A}\psi^B_i \ . \tag{5.25}$$

The diffeomorphism, $sp(2)$ and susy conserved charges associated to the symmetry (5.15), (5.16) and (5.17) are

$$H = \frac{1}{2}\pi^2 \qquad L_{ij} = -i\psi^{\{i}_A\psi^{j\}A} \qquad Q^i_\alpha = \psi^i_A V^{\mu A}_{\ \alpha}\pi_\mu \ . \tag{5.26}$$

It remains to compute the superalgebra. This computation is greatly simplified by using the central relations

$$\{\pi_{A\alpha}, \pi_{B\beta}\}_{pb} = \{\pi_{A\alpha}, \pi_{B\beta}\}_{pb} = \frac{i}{2}\epsilon_{\alpha\beta}\Omega_{ABCD}T^{CD} + \Omega_{B\beta|A}{}^{C}\pi_{C\alpha} - \Omega_{A\alpha|B}{}^{C}\pi_{C\beta} \tag{5.27}$$

$$\{\pi_{A\alpha}, \psi^C_j\}_{pb} = V^\mu_{A\alpha}\Omega_{\mu MN}\psi^M_j\epsilon^{NC} \tag{5.28}$$

Let us emphasize that in the r.h.s. of the previous relations, only the $sp(2n)$ component of the connection and curvature appear.
After some straightforward computation one finds

$$\begin{aligned} &\{Q^i_\alpha, Q^j_\beta\}_{pb} = -\epsilon^{ij}\epsilon_{\alpha\beta}H \qquad \{L^{ij}, Q^k_\alpha\}_{pb} = 2\epsilon^{k\{i}Q^{j\}}_\alpha \\ &\{L^{ij}, L^{kl}\}_{pb} = -i\epsilon^{ki}f^{jl} - i\epsilon^{kj}f^{il} - i\epsilon^{li}f^{jk} - i\epsilon^{lj}f^{ik} \\ &\{H, L^{ij}\}_{pb} = \{H, Q^i_\alpha\}_{pb} = 0 \ . \end{aligned} \tag{5.29}$$

## 5.3 QK $N = 4$ $d = 1$ SUGRA

We replace the HK target space with a QK one. First of all one has to observe that is no longer possible to maintain the $N = 4$ supersymmetry algebra (5.29); this is due to the fact that the central relations (5.27) changes when one turns on the $sp(2)$ connection:

$$\begin{aligned} \{\pi_{A\alpha}, \pi_{B\beta}\}_{pb} = \frac{i}{2}\epsilon_{\alpha\beta}\Omega_{ABCD}T^{CD} &+ \Omega_{B\beta|A}{}^{C}\pi_{C\alpha} - \Omega_{A\alpha|B}{}^{C}\pi_{C\beta} \\ &+ \omega_{B\beta|\alpha}{}^{\gamma}\pi_{A\gamma} - \omega_{A\alpha|\beta}{}^{\gamma}\pi_{B\gamma} \end{aligned} \tag{5.30}$$

The superalgebra (5.29) becomes:

$$\{Q_{i\alpha}, Q_{j\beta}\}_{pb} = (-\epsilon_{ij}\epsilon_{\alpha\beta})H + (-\psi_j^B \omega_{B\beta|\alpha}{}^{\gamma})Q_{i\gamma} + (-\psi_i^A \omega_{A\alpha|\beta}{}^{\gamma})Q_{j\gamma} \quad (5.31)$$

$$\{Q_{i\alpha}, H\}_{pb} = (\pi^{B\beta}\omega_{B\beta|\alpha}{}^{\gamma})Q_{i\gamma} + (\frac{1}{2}\psi_{iA}\psi^{jA})Q_{j\alpha} \quad (5.32)$$

where in the r.h.s. of (5.31) we have recognized the Hamiltonian

$$H = \tfrac{1}{2}\pi_{A\alpha}\pi^{A\alpha} + \tfrac{\lambda}{4}\psi^A \circ \psi^B \psi_A \circ \psi_B.$$

where $\psi^A \circ \psi^B = \psi^{iA}\psi_i^B$. The algebra, QK supersymmetry algebra (5.32), has some very important consequences:

- In QK manifold is no longer possible to maintain rigid supersymmetry, just a model with local susy is allowed.
- The algebra is no more a Lie algebra since structure functions appear in the r.h.s. of (5.31) (5.32).
- In QK manifold one has to introduce a curvature coupling with fermions.
- One can also observe that in (5.32) we have choosen to use just the supercharges but also the $sp(2)$ generator is present in the r.h.s.. The reason is just matter of convenience and will be clarified in 2 minutes.

In QK the bracket involving the operator $L_{ij}$ doesn't change with respect the one we have computed in HK.
Two possible interesting gauged model should be studied

| | Constraints | Gauge fields |
|---|---|---|
| 1. | $H = 0$ | one dimensional einbein $N$ |
| | $Q_i^\alpha = 0$ | one dimensional gravitini $\psi_i^\alpha$ |
| 2. | $H = 0$ | one dimensional einbein $N$ |
| | $L^{ij} = 0$ | Yang-Mills $f^{ij}$ |
| | $Q_i^\alpha = 0$ | one dimensional gravitini $\chi_i^\alpha$ |

In the following we concentrate our attention on the first one because intimately related to black hole supersymmetric quantum mechanics [50]. In particular we will study the first order action with $QK$ background

$$S^{(1)} = \int dt[p_\mu \dot{x}^\mu + \frac{1}{2}\psi_A^i \dot{\psi}_i^A - NH - \chi_i^\alpha Q_\alpha^i] \quad (5.33)$$

and after Legendre transformation one obtains

$$S=\int dt[\frac{1}{2N}(\dot{x}^{\mu}-V^{\mu A}_{\ \ \alpha}\psi^i_A\chi^{\alpha}_i)(\dot{x}^{\nu}-V^{\nu A}_{\ \ \alpha}\psi^i_A\chi^{\alpha}_i)g_{\mu\nu}+\frac{1}{2}\psi^i_A\frac{\nabla\psi^A_i}{dt}+\frac{\lambda N}{4}\psi^i_A\psi_{iB}\psi^{jA}\psi^B_j] \quad (5.34)$$

wich enjoys the symmeties

**Local worldline reparametrizations:**

$$\begin{aligned}\delta x^{\mu} &= \xi\dot{x}^{\mu},\\ \delta\psi^i_A &= \xi\dot{\psi}^i_A\\ \delta N &= \partial_t(\xi N)\\ \delta\chi^{\alpha}_i &= \partial_t(\xi\chi^{\alpha}_i)\end{aligned} \quad (5.35)$$

**Local $N=4$ supersymmetry:**

$$\begin{aligned}\delta x^{\mu} &= V^{\mu A}_{\ \ \alpha}\psi^i_A\epsilon^{\alpha}_i\\ \delta\psi^i_A &= -\frac{1}{N}(\dot{x}^{\mu}-V^{\mu B}_{\beta}\psi^i_B\chi^{\beta}_i)V_{\mu A}^{\ \ \alpha}\epsilon^i_{\alpha}+V^{\mu A}_{\ \ \gamma}\psi^j_A\epsilon^{\gamma}_j\Omega_{\mu A}^{\ \ B}\psi^i_B\\ \delta N &= \chi^i_{\alpha}\epsilon^{\alpha}_i\\ \delta\chi^i_{\alpha} &= \frac{\nabla\epsilon^i_{\alpha}}{dt}+\frac{\lambda N}{2}\psi^i_{\alpha}\psi^A_j\epsilon^j_{\alpha}+V^{\mu A}_{\ \ \gamma}\psi^j_A\epsilon^{\gamma}_j\omega_{\mu\alpha}^{\ \ \beta}\chi^i_{\beta}\end{aligned} \quad (5.36)$$

**Rigid $sp(2)$:**

$$\begin{aligned}\delta\psi^i_A &= \lambda^{ij}\psi_{Aj}\\ \delta\chi^i_{\alpha} &= \lambda^{ij}\chi_{\alpha j}\end{aligned} \quad (5.37)$$

## 5.4 BRST charge and gauge fixed action

We construct now the BRST charge associated to the algebra (5.31) and (5.32); note that this algebra is a *non Lie*. We define the fermi $(c,\rho)$ and bosonic $(\eta^{i\alpha},P_{k\alpha})$ fields corresponding to reparametrization and susy ghost, and ghost momenta respectively, equipped with the Poisson structure bracket

$$\{c,\rho\}=\{\rho,c\}=-1 \qquad \{\eta^{i\alpha},P_{k\beta}\}=-\{P_{k\beta},\eta^{i\alpha}\}=\delta^i_k\delta^{\alpha}_{\beta}\ . \quad (5.38)$$

The BRST charge reads:

$$\begin{aligned}Q^{brst} &= \eta^{i\alpha}Q_{i\alpha}+cH-\eta^{i\alpha}\eta^{j\beta}\psi^C_j\omega_{C\beta|\alpha}^{\ \ \ \gamma}P_{i\gamma}-\eta^{i\alpha}c\pi^{C\beta}\omega_{C\beta|\alpha}^{\ \ \ \gamma}P_{i\gamma}\\ &\quad -\frac{1}{2}\eta^{i\alpha}\eta^{j\beta}\epsilon_{ij}\epsilon_{\alpha\beta}\rho-\frac{\lambda}{2}\eta^{i\alpha}c\psi_{iC}\psi^{kC}P_{k\alpha}+\text{higher order terms};\end{aligned} \quad (5.39)$$

We have now to fix the higher order terms in ghost momenta. In order to do that we will use a very funny and elegant trick; in the previous BRST expansion, infact, we recognize a covariant structure underlining. We define the *ghost* generator

$$\begin{aligned} L^{gh}_{ij} &= \eta^{\alpha}_{\{i}P_{j\}\alpha} \\ \tau^{\alpha\beta} &= 2i\eta_i^{\{\alpha}P^{i\beta\}} \end{aligned} \tag{5.40}$$

The $sp(2)$ generators $L^{gh}_{ij}$ obey the $sp(2)$ algebra; the holonomy generators similarly are subject to

$$\{\tau^{\alpha\beta},\tau^{\gamma\delta}\} = \epsilon^{\gamma\alpha}\tau^{\beta\delta} + \epsilon^{\gamma\beta}\tau^{\alpha\delta} + \epsilon^{\delta\alpha}\tau^{\beta\gamma} + \epsilon^{\delta\beta}\tau^{\alpha\gamma}\ . \tag{5.41}$$

Therefore we can construct a covariant momentum in the *extended* (i.e. $sp(2)\otimes sp(2n)$) sense

$$\Pi_\mu = p_\mu - \frac{i}{2}\Omega_{\mu AB}T^{AB} - \frac{i}{2}\omega_{\mu\alpha\beta}\tau^{\alpha\beta}\ . \tag{5.42}$$

Observing carefully (5.39), one can observe that by adding the higher order terms proportional to $\tau^{\alpha\beta}\tau^{\delta\sigma}$, everything could be written in terms of the *extended* hamiltonian and supercharges

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\Pi^2 + \frac{\lambda}{4}\psi^A\circ\psi^B\psi_A\circ\psi_B \\ \mathcal{Q}^\alpha_i &= \psi_i^A\Pi_\mu V^{\mu\alpha}_{\ \ A} \end{aligned}$$

Therefore we can immediately construct the covariant, in the extended sense, BRST charge

$$\boxed{Q^{brst} = \eta^{i\alpha}\mathcal{Q}_{i\alpha} + c\mathcal{H} - \frac{1}{2}\eta^{i\alpha}\eta_{i\alpha}\rho - \frac{\lambda}{2}\eta^{i\alpha}c\psi_{iC}\psi^{kC}P_{k\alpha}} \tag{5.43}$$

and it's easy to verify that the higher order corrections we have added by using the argument proposed above, play a fundamental role in order to make (5.43) nilpotent.

We intend now to gauge fix the supergravity multiplet on the circle and construct the gauge fixed action.
The trasformation rule for gravitini and einbein are:

$$\begin{aligned} \delta e &= \dot{\xi} - \chi^{i\alpha}\epsilon^{j\beta}(\epsilon_{ij}\epsilon_{\alpha\beta}) \\ \delta\chi^{k\delta} &= \dot{\epsilon}^{k\delta} - \chi^{i\alpha}\epsilon^{j\beta}(\psi_j^C\omega_{C\beta|\alpha}{}^{\delta}\delta_i^k + \psi_i^C\omega_{C\alpha|\beta}{}^{\delta}\delta_j^k) \\ &\quad - e\epsilon^{i\alpha}(\pi^{B\beta}\omega_{B\beta|\alpha}{}^{\delta}\delta_i^k + \frac{\lambda}{2}\psi_{iC}\psi^{kC}\delta_\alpha^\delta) + \chi^{i\alpha}\xi(\pi^{B\beta}\omega_{B\beta|\alpha}{}^{\delta}\delta_i^k + \frac{\lambda}{2}\psi_{iC}\psi^{kC}\delta_\alpha^\delta)\ . \end{aligned} \tag{5.44}$$

We choose antiperiodic boundary condition for fermions and gravitini and this let us to gauge fix gravitini to zero. At this point the transformation rule for the einbein coincide with the one we have discussed in the $SO(N)$ spinning particle models. Thus the einbein could be

gauge fixed to the proper time $\beta$.
We introduce the gauge fixing fermion:

$$K = -\beta\rho \tag{5.45}$$

that let us to implement the gauge choice

$$(e, \chi_i^\alpha) = (\beta, 0) \tag{5.46}$$

The gauge fixed action can be computed by

$$H^{GF} = H + \{K, Q^{brst}\} \tag{5.47}$$

and explicitly we obtain the gauge fixed Hamiltonian

$$\begin{aligned} H^{GF} &= \frac{\beta}{2}(p_\mu - \frac{i}{2}\Omega_{\mu AB}T^{AB} - \frac{i}{2}\omega_{\mu\alpha\beta}\tau^{\alpha\beta})^2 \\ &\quad -\beta\frac{\lambda}{2}\psi_{iC}\psi^{kC}\eta^{i\alpha}P_{k\alpha} + \beta\frac{\lambda}{4}T_{AB}T^{AB} \ . \end{aligned} \tag{5.48}$$

# Appendix A

# Tour on higher spin gauge theory

In this appendix we intend to describe briefly some aspect of massless higher spin theory (HS gauge theory). In the first part we will focus our attention on the "linearized approach" to HS; we will derive, in $D = 4$, the Fronsdal (for bosons) and Fang-Fronsdal (for fermions) e.o.m., as the natural generalization of the KleinGordon/Einstein and Dirac/Rarita-Schwinger e.o.m., respectively.
We proceed analyzing an important subclass of HS field, *conformal HS gauge theory*, in various dimensions, because strictly related to $SO(N)$ spinning particle. We will focus our attention on the relation between geometrical approach to higher spin, that is higher derivative, and the linearized one; we will study the flat case and we will show how to obtain the e.o.m. and the Fronsdal-Labastida kinetic operator, which generalize those of Fronsdal for mixed symmetry fields.
Finally we will discuss how to modify the Fronsdal operator in $(A)dS_4$ background. Let us emphasize that in literature is well known how to make covariant and gauge invariant the Fronsdal kinetic operator, but is not clear how to derive it starting from geometrical objects: HSC or equivalently HS field strength. For a review on related topics and additional references see [33] [32] [29].

Let us start discussing the $D = 4$ case. In $D = 4$ symmetric tensor describe all possible representetion of the Poincaré group; to describe integer HS we use the completely symmetric tensor $\phi_{\alpha_1 \cdots \alpha_s}$ while physical states of half integer spin $s$ are indicated with $\psi^a_{\alpha_1 \cdots \alpha_{s-\frac{1}{2}}}$.
We resume, for commodity, quantum field theory for spin lower than 2, in the following

tableaux:

| Spin | Equation of motion | Gauge symmetry |
|---|---|---|
| Scalar field $\phi(x)$ $(s=0)$ | $\Box\phi = 0$ | No gauge symmetry |
| Spinor field $\psi_a$ $(s=\frac{1}{2})$ | $(\gamma^\alpha)^a{}_b\psi^b\partial_\alpha\psi^b = 0$ | No gauge symmetry |
| Maxwell field $A_\alpha$ $(s=1)$ | $\Box A_\alpha - \partial_\alpha\partial_\beta A^\beta = 0$ | $\delta A_\alpha = \partial_\alpha\xi$ |
| Rarita-Schwinger $\psi_a^\alpha$ $(s=\frac{3}{2})$ | $\gamma_{\alpha\beta\gamma}\partial^\beta\psi_a^\gamma = 0$ | $\delta\psi_a^\alpha = \partial^\alpha\xi_a$ |
| Graviton $h_{\alpha\beta}$ $(s=2)$ | $\partial_\alpha\partial_\beta h_{\alpha\delta} + \partial_\alpha\partial_\delta h_{\alpha\beta} - \Box h_{\delta\beta} - \partial_\beta\partial_\delta h = 0$ | $\delta h_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha$ |

Note now that except for the scalar and the spinor field, all other massless fields are gauge fields. For these reason seems to be reasonable assuming that all massless higher spin fields are also gauge fields.
HS gauge theory is constructed as the natural generalization of the well known massless quantum field theory; in the following, we will discuss gauge symmetry and equations of motion.

**Gauge transformations** are assumed to be

$$\delta\phi_{\alpha_1\cdots\alpha_s} = \partial_{\{\alpha_1}\xi_{a_2\cdots a_s\}} \tag{A.1}$$

$$\delta\psi^a_{\alpha_1\cdots\alpha_{s-\frac{1}{2}}} = \partial_{\{\alpha_1}\xi^a_{\alpha_2\cdots\alpha_{s-\frac{1}{2}}\}} \ . \tag{A.2}$$

Note that for $s = 1, \frac{3}{2}, 2$ the previous relations reproduce the well known transformations rule for Maxwell, Rarita-Schwinger and graviton field.

**Free equations of motion** of the HS are second order linear differential equations in the case of the integer spins and the first order differential equations in the case of the half integer spins. This is required by the unitary and ensures that the fields have a positive-definite norm. The massless higher spin equations have been derived from the massive higher spin equations [66] by Fronsdal for bosons [30] and by Fang and

Fronsdal for fermions [67], and studied in more detail in [31] .
In our convention the symbol $\sum$ denote symmetrized sum with respect free indices. In the bosonic case, the dynamics is governed by the so called Fronsdal e.o.m

$$\begin{aligned} G_{\alpha_1\cdots\alpha_s} &\equiv \Box\phi_{\alpha_1\cdots\alpha_s}(x) - \sum \partial_{\alpha_1}\partial_\beta \phi^\beta{}_{\alpha_2\cdots\alpha_s}(x) + \sum \partial_{\alpha_1}\partial_{\alpha_2}\phi^\beta{}_{\beta\alpha_3\cdots\alpha_s}(x) = 0 \\ &\equiv G\phi_{\alpha_1\cdots\alpha_s} \end{aligned} \tag{A.3}$$

where $G$ is the Fronsdal-Labastida kinetic operator that in an obvious notation we write as:

$$G \equiv \Box - \boldsymbol{grad\ div} + \boldsymbol{grad^2\ tr}\ . \tag{A.4}$$

In the fermionic sector the first order equations are a natural generalization of the Dirac and Rarita–Schwinger equation

$$G^a_{\alpha_1\cdots\alpha_{s-\frac{1}{2}}}(x) \equiv (\partial\!\!\!/\psi)^a_{\alpha_1\cdots\alpha_{s-\frac{1}{2}}} - \sum \partial_{\alpha_1}(\gamma^\beta\psi)^a_{\beta\alpha_2\cdots\alpha_{s-\frac{1}{2}}} = 0\,. \tag{A.5}$$

**Constraints on higher spin gauge parameters** is the next arguments we intend to analyze.
Let us now analyze how equations of motion (A.3) and (A.5) transform under gauge transformations (A.1) and (A.2). The direct computations gives

$$\delta G_{\alpha_1\cdots\alpha_s} = 3\sum \partial^3_{\alpha_1\alpha_2\alpha_3}\xi^\beta{}_{\beta\alpha_4\cdots\alpha_s}, \tag{A.6}$$

$$\delta G^\alpha_{\alpha_1\cdots\alpha_{s-\frac{1}{2}}} = -2\sum \partial^2_{\alpha_1\alpha_2}\gamma^{\beta a}{}_b\,\xi^b{}_{\beta\alpha_3\cdots\alpha_{s-\frac{1}{2}}}\,, \tag{A.7}$$

where $\partial^2_{\alpha_1\alpha_2} = \partial_{\alpha_1}\partial_{\alpha_2}$ and $\partial^3_{\alpha_1\alpha_2\alpha_3} = \partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}$.
Note now that the previous relations vanish only if the parameters of the transformations (A.6), for $s \geq 3$, are traceless, equivalently for (A.7) and $s \geq 5/2$ are $\gamma$-traceless.

$$\xi^\beta{}_{\beta\alpha_4\cdots\alpha_s} = 0 \tag{A.8}$$

$$(\gamma^\beta\xi)^\alpha_{\beta\alpha_3\cdots\alpha_{s-\frac{1}{2}}} = 0\,. \tag{A.9}$$

Another key ingredient we have to take into account is the Bianchi identities that imply that the traceless divergence on the left-hand-side of equations of motion must vanish; equivalently the currents of the matter fields if coupled to the gauge fields are conserved.

Let us consider Maxwell theory:

$$\partial_\alpha(\partial_\beta F^{\alpha\beta}) \equiv 0, \tag{A.10}$$

where $F_{\alpha\beta}$ is the field strength defined as $F_{\alpha\beta} = \partial_\alpha\phi_\beta - \partial_\beta\phi_\alpha$. Electric current into the Maxwell equations $\partial_\alpha F^{\alpha\beta} = J^\beta$ is conserved $\partial_\alpha J^\alpha = 0$.

In the spin 2 case (theory of gravity) coupled to matter fields and described by the Einstein equation

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta}$$

the energy-momentum conservation $\nabla_\alpha T^{\alpha\beta}=0$ is related to the Bianchi identity

$$\nabla_\alpha R^\alpha{}_\beta - \frac{1}{2}\nabla_\beta R^\alpha{}_\alpha \equiv 0\,. \tag{A.11}$$

Generalization of (A.11) to the case of the bosonic higher spin fields reads:

$$\partial_\beta G^\beta{}_{\alpha_2\cdots\alpha_s} - \frac{1}{2}\sum \partial_{\alpha_2} G^\beta{}_{\beta\alpha_3\cdots\alpha_s} = -\frac{3}{2}\sum \partial^3_{\alpha_2\alpha_3\alpha_4}\phi^{\beta\gamma}{}_{\beta\gamma\alpha_5\cdots\alpha_s}\,. \tag{A.12}$$

Note now that the previous relation vanishes only if the HS fields for $s\geq 4$ are double-traceless

$$\boldsymbol{tr^2}\phi_{\alpha_1\cdots\alpha_s} \equiv \phi^{\beta\gamma}{}_{\beta\gamma\alpha_5\cdots\alpha_s} = 0. \tag{A.13}$$

In analogy, in the fermionic sector we have

$$\begin{aligned}&\partial_\beta G^{a\beta}{}_{\alpha_2\cdots\alpha_{s-\frac{1}{2}}} - \frac{1}{2}\sum \partial_{\alpha_2} G^{a\beta}{}_{\beta\alpha_3\cdots\alpha_{s-\frac{1}{2}}} - \frac{1}{2}(\not\partial\gamma^n G)^a_{\beta\alpha_2\cdots\alpha_{s-\frac{1}{2}}}\\ &= \sum \partial^2_{\alpha_2\alpha_3}(\gamma^\beta\psi)^{a\gamma}{}_{\beta\gamma\alpha_4\cdots\alpha_{s-\frac{1}{2}}}\end{aligned} \tag{A.14}$$

that vanishes only if

$$(\gamma^\beta\gamma^\gamma\gamma^\sigma\psi)^a{}_{\beta\gamma\sigma\alpha_4\cdots\alpha_{s-\frac{1}{2}}} \equiv (\gamma^\beta\psi)^{a\gamma}{}_{\beta\gamma\alpha_4\cdots\alpha_{s-\frac{1}{2}}} = 0\,. \tag{A.15}$$

We can conclude that in order to obtain a consistent gauge theory we have to force the HS fields to be double traceless or gamma-triple traceless.
We would like now to show that the traceless condition on integer HS field is important in order to obtain the correct number of independent polarization a massless integer spin $s$ field field propagates. We start imposing the de Donder gauge condition

$$D_{\alpha_2\cdots\alpha_s} = \partial_\sigma\phi^\sigma{}_{\alpha_2\cdots\alpha_s} - \frac{1}{2}\sum\partial_{\alpha_2}\phi^\sigma{}_{\sigma\alpha_3\cdots\alpha_s} = 0\,. \tag{A.16}$$

Then the HS equation of motion reduces to the Klein Gordon one

$$\Box\phi_{\alpha_1\cdots\alpha_s} = 0. \tag{A.17}$$

Note now that under (A.1) de Donder gauge transforms as

$$\delta(\partial_\sigma\phi^\sigma{}_{\alpha_2\cdots\alpha_s} - \frac{1}{2}\sum\partial_{\alpha_2}\phi^\sigma{}_{\sigma\alpha_3\cdots\alpha_s}) = \Box\xi_{\alpha_1\cdots\alpha_{s-1}} \tag{A.18}$$

so we have further gauge freedom if $\Box\xi_{\alpha_1\cdots\alpha_{s-1}}=0$. We are ready now to count the number of degree of freedom the HS field propagates; a totally symmetric rank tensor in $D=4$ has $2(s^2+1)$ indipendent components. Note now that double traceless condition on HS implies that de Donder gauge is traceless:

$$D^\sigma{}_{\sigma\alpha_1\cdots\alpha_{s-4}} \sim \sum\partial_{\alpha_1}\phi^{\rho\sigma}{}_{\rho\sigma\alpha_2\cdots\alpha_{s-4}} = 0 \tag{A.19}$$

and contains $s^2$ independent constraints. Furthermore gauge fields components, not till fixed, satisfies wave function e.o.m. and we can use residual gauge freedom to eliminate further $s^2$ independent components. Finally we have $2s^2+2-2s^2=2$ degrees of freedom corresponding to the two polarization of a massless field.

**The HS action** are:

- Bosonic case

$$S_B = \int d^D x \left( \frac{1}{2} \phi^{\alpha_1 \cdots \alpha_s} G_{\alpha_1 \cdots \alpha_s} - \frac{1}{8} s(s-1)\, \phi_{\beta}{}^{\beta \alpha_3 \cdots \alpha_s} G^{\sigma}{}_{\sigma \alpha_3 \cdots \alpha_s} \right) \tag{A.20}$$

This action enjoys local symmetry (A.1) only if the gauge parameter is traceless and the gauge field double traceless.

- Fermionic case

$$\begin{aligned} S_F = \; & \int d^D x \Big( -\tfrac{1}{2} \bar{\psi}^{\alpha_1 \cdots \alpha_{s-\frac{1}{2}}} G_{\alpha_1 \cdots \alpha_{s-\frac{1}{2}}} + \tfrac{1}{4}\, s\, \bar{\psi}^{\alpha_2 \cdots \alpha_{s-\frac{1}{2}} n} \gamma_\beta \gamma^\sigma G_{\sigma \beta_2 \cdots \alpha_{s-\frac{1}{2}}} \\ & + \tfrac{1}{8} s(s-1) \bar{\psi}_\beta{}^{\beta \alpha_3 \cdots \alpha_{s-\frac{1}{2}}} G^{\sigma}{}_{\sigma \alpha_3 \cdots \alpha_{s-\frac{1}{2}}} \Big) , \end{aligned} \tag{A.21}$$

This action enjoys local symmetry (A.2) only if the gauge parameter is gamma-traceless the gauge field gamma-tripletraceless.

The presence of constraint on HS field and on gauge parameter suggest that the formulation discussed above is incomplete.
Different approach to analyze this problem have been suggested in literature. Let us analyze them briefly:

- In order to remove trecelessness constraints has been proposed to add an appropriate number of auxiliary fields, satisfying certain equation of motion. The main advantage of this approach is the fact that HS equation of motion remain lagrangian [69] [39].
- Another interesting way out was proposed by Francia and Sagnotti in [32] [34]. The key idea of this approach is the renounce of the locality of the theory. In particular they have shown that equations of motion for unconstrained HS and the HS actions can be implemented to an invariant form with respect unconstrained gauge parameter if enlarged with non local terms.
- Another approach to remove constraints on gauge fields and gauge parameters is based on the consideration that gauge theory can be easily constructed in terms of the field strength (HSC). Geometric formulation of free HS have been considered for the first time many years ago by Bargmann and Wigner [28] .The one advantage one has by using geometrical approach is the fact that the theory is manifestly gauge invariant in the unconstrained sense; however HS potential e.o.m. is higher derivative. Otherwise this does not spoil the unitarity of the theory and it's possible to show that, upon some manipulations, HS e.o.m. reduces to the Fronsdal-Labastida one.

In the next section we will focus our attention on the geometrical approach to HS. In particular we will discuss *conformal HS theory* that is an important and interesting subclass of HS field whose (at least linearized) equation of motion are conformally invariant.

## A.1 Conformal HS theory in various dimension

In $D = 4$ symmetric tensors describe all possible higher spin representation of the Poincaré group; in this case, in fact, all the irreducible massless representation of the Poicaré group are classified by using the group $SO(2)$ whose Young tableaux is a single rows; this implies that massless integer HS fields could be represented by totally symmetric tensor.

$$\phi_{\alpha_1\cdots\alpha_s} \equiv \boxed{\alpha_1}\boxed{\cdot}\boxed{\cdot}\boxed{\alpha_s} \tag{A.22}$$

In higher dimension, symmetric tensors do not describe all possible HS fields, and one has to take into account also tensor with mixed symmetry.
In this section we would like to construct conformal HSC for integer spin, and we will show how to obtain the linearized HS field e.o.m. (i.e. Fronsdal-Labastida e.o.m).

We will work in even $D = 2n$ dimension and we will use the following notation:

$$[n] \equiv \alpha_{[1} \cdots \alpha_{n]}, \tag{A.23}$$

stands for $n$ antisymmetrized indices. In particular, in this appendix, two cumulative indices which denote the same number of antisymmetric indices will be assumed to be symmetric

$$[n]_1[n]_2 = [n]_2[n]_1 \,. \tag{A.24}$$

We define the curvature (or the field strength) of a conformal integer spin $s$ as

$$R_{[n]_1\cdots[n]_s} = R_{\alpha_1^1\cdots\alpha_n^1,\cdots,\alpha_1^s\cdots\alpha_n^s}; \tag{A.25}$$

note that its Young diagram is a rectangle with $n$ rows and $s$ columns. The curvature tensor, by construction, is symmetric under exchange of any two blocks of antisymmetrized indices. In addition

- it satisfies the algebraic Bianchi identity

$$R_{[n+1]_1[n-1]_2[n]_3\cdots[n]_s} = 0.$$

- The curvature is closed (i.e. satisfies the differential Bianchi identity)

$$\partial_{[\beta}R_{\alpha_1\cdots\alpha_n][n]_2\cdots[n]_s} = 0\,, \tag{A.26}$$

- and is co-closed

$$\partial^\beta R_{\beta[n-1]_1[n]_2\cdots[n]_s} = 0\,. \tag{A.27}$$

Note now that that the Bianchi identity can be written as an exterior derivative acting on one of the groups of antisymmetric indices of the multiform $R_{[n]_1\cdots[n]_s}$

$$\partial_1 R_{[n]_1\cdots[n]_s} = 0\,, \tag{A.28}$$

where in our notation $\partial_i$ means derivative and the index of the derivative is antisymmetrized together with the $i^{th}$ group of antisymmetric indices (i.e. exterior derivative acting on the $i^{th}$ column):

$$\partial_i R_{[n]_1\cdots[n]_i\cdots[n]_s} \equiv \partial_{[\alpha^i_{n+1}} R_{[n]_1\cdots\alpha^i_1\cdots\alpha^i_n],[n]_{i+1}\cdots[n]_s} \; ; \tag{A.29}$$

note now that

$$\partial_i\partial_j = (1-\delta_{ij})\partial_j\partial_i \; . \tag{A.30}$$

We introduce now the differential operator

$$\partial \equiv \prod_{i=1}^{s} \partial_i \, . \tag{A.31}$$

It's not hard to convince ourselves, since $\partial_i^2 = 0$, that

$$\partial^{s+1} \equiv \partial_i\partial = 0 \qquad \forall\; i = 1, ..., s \; . \tag{A.32}$$

The *generalized Poincaré Lemma* [45], and Bianchi identity (A.26) implies that (at least locally) the curvature can be written as the $s$-th derivative of a potential:

$$R_{[n]_1\cdots[n]_s} = \partial_1\cdots\partial_s\phi_{[n-1]_1\cdots[n-1]_s} \; , \tag{A.33}$$

where the field $\phi \equiv \phi_{[n-1]_1\cdots[n-1]_s}$ is the conformal HS gauge potential of integer spin $s$ and is characterized by the rectangular Young diagram $s\otimes(n-1)$ and satisfies the algebraic Bianchi identity

$$\phi_{[n]_1\,[n-2]_2\,[n-1]_3\cdots[n-1]_s} = 0.$$

One can observe that (A.33) is the generalization to the higher spin $s$ case of the well known expression of the electromagnetic field strength in terms of the potential $F = \partial\, A$.
HSC, due to (A.30), are invariant under the following gauge transformations of the gauge potential [45]

$$\delta\phi_{[n-1]_1\cdots\,[n-1]_s} \;\; = \;\; \sum_{i=1}^{s} \partial_i\, \xi_{[n-1]_1\,\cdots\,[n-2]_i\cdots\,[n-1]_s} \; , \tag{A.34}$$

where $\xi(x)$ is an unconstrained gauge function characterized by the Young tableaux $(s,\cdots,s,s-1)\otimes[n-1]$.
Equation of motion are obtained by imposing tracelessness condition of the curvature tensor

$$\boldsymbol{tr}\, R_{[n]_1\cdots[n]_s} = 0\, . \tag{A.35}$$

Note that this is the field equation that generalizes the linearized Einstein equation $R_{\alpha\beta} = 0$.
In terms of the gauge field potential, equation (A.35) implies a generalization of the spin 3 Damour-Deser identity [68]

$$\boldsymbol{tr}\, R_{[n]_1\cdots[n]_s} = \partial_1\cdots\partial_{s-2}\, G_{[n-1]_1\cdots[n-1]_{s-2}\,[n-1]_{s-1}\,[n-1]_s} = 0 \tag{A.36}$$

where $G$ is the kinetic operator acting on the gauge field potential

$$G_{[n-1]_1\cdots[n-1]_s} = \Box\,\phi_{[n-1]_1\cdots[n-1]_s} - \sum_{i=1}^{s}\partial_i\,\partial^\beta\phi_{[n-1]_1\cdots,\,\beta[n-2]_i\,,\cdots[n-1]_s} + \sum_{j>i=1}^{s}\partial_i\,\partial_j\,\eta^{\alpha\beta}\,\phi_{[n-1]_1\cdots,\alpha[n-2]_i\,,\cdots,\beta[n-2]_j\,,\cdots[n-1]_s}\,, \qquad (A.37)$$

The left hand side of eq. (A.36) vanishes and this implies that $G$ is $\partial^{s-2}$-closed and the *generalized Poincaré lemma* [45] implies that $G$ is also $\partial^3$-exact

$$G_{[n-1]_1\cdots[n-1]_s} = \sum_{k>j>i=1}^{s}\partial_i\,\partial_j\,\partial_k\,\rho_{[n-1]_1\cdots,[n-2]_i\,,\cdots,[n-2]_j\,,\cdots[n-2]_k\,,\cdots[n-1]_s}\,, \qquad (A.38)$$

where we have introduced the so called 'compensator' field since its gauge transformation compensates the non-invariance of the kinetic operator $G(x)$ under the unconstrained local variations (A.34) of the gauge field potential $\phi(x)$.
The gauge variation of $G(x)$ is

$$\delta G_{[n-1]_1\cdots[n-1]_s} = \sum_{k>j>i=1}^{s}\partial_i\,\partial_j\,\partial_k\,\eta^{\alpha\beta}\,\xi_{[n-1]_1\cdots,[n-2]_i\,,\cdots,[n-2]_j\,\alpha,\cdots[n-2]_k\,\beta,\cdots[n-1]_s} \qquad (A.39)$$

and it is compensated by the gauge shift of the field $\rho(x)$ with the trace of the gauge parameter

$$\delta\rho_{[n-2]_1\,[n-2]_2\,[n-2]_3\,[n-1]_4\cdots[n-1]_s} = \eta^{\alpha\beta}\,\xi_{[n-2]_1\,\alpha\,,\,[n-2]_2\,\beta,\,[n-2]_3\,[n-1]_4\cdots[n-1]_s}\,. \qquad (A.40)$$

Now we can gauge fix the compensator to zero. Then the equations of motion of the gauge field $\phi(x)$ become the second order differential equations of Fronsdal-Labastida, which generalize those of Fronsdal for mixed symmetry fields

$$G_{[n-1]_1\cdots[n-1]_s} = G\;\phi_{[n-1]_1\cdots[n-1]_s} = 0\,. \qquad (A.41)$$

where $G$ is the Fronsdal-Labastida kinetic operator, that in $D=4$ coincide with the Fronsdal one (A.4). Note also that (A.41) is invariant under the gauge transformations (A.34) only if $\xi(x)$ is tracelss and the HS gauge field is double traceless.

## A.2 Integer HS in $(A)dS_4$

We intend now to analyze the dynamics of HS in $(A)dS_4$ background. In literature this problem has been extensively studied from the linearized point of view. In literature it is not yet clear how to derive the Fronsdal e.o.m. in $(A)dS_4$ from the $(A)dS$ HSC since in maximally symmetric space an extension of the *generalized Poincaré Lemma* is still lacking and one cannot use it as a guide line.

We will focus now our attention on integer spin in $(A)dS_4$. We assume also that the HS gauge field is double traceless and the gauge parameter is traceless.
In order to become familiar with our notation let us recall that covariant derivatives don't commute and in $(A)dS$ one finds that

$$R_{\mu\nu\rho\sigma} = b(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \qquad \rightarrow \qquad [\nabla_\mu, \nabla_\nu]V_\rho = R_{\mu\nu\rho}{}^\sigma V_\sigma = b(g_{\mu\rho}V_\nu - g_{\nu\rho}V_\mu) \quad \text{(A.42)}$$

for some vector $V^\mu$; we will replace now, as usual, ordinary derivative with the covariant one $\partial_\alpha \rightarrow \nabla_\mu$ in (A.3). Non commutativity implies that the covariant Fronsdal kinetic operator

$$G^{cov}_{\mu_1\ldots\mu_s} = \nabla^2\phi_{\mu_1\ldots\phi_s} - \sum \nabla_{\mu_1}\nabla_\rho\phi^\rho{}_{\mu_2\cdots\mu_s} + \sum\{\nabla_{\mu_1}, \nabla_{\mu_2}\}\phi^\rho{}_{\rho\mu_3\ldots\mu_s} \quad \text{(A.43)}$$

is not invariant with a suitable covariantization of (A.1):

$$\delta\phi_{\mu_1\cdots\mu_s}(x) \;\; = \;\; \nabla_{\{\mu_1}\xi_{\mu_2\cdots\mu_s\}} \quad \text{(A.44)}$$

In order to solve this problem one has to add a new covariant term that cancels the "extra contributions". Explicitly one obtains

$$G^{AdS}_{\mu_1\ldots\mu_s} \;\; = \;\; G^{cov}_{\mu_1\ldots\mu_s} + b[s^2 - 2(s+1)]\phi_{\mu_1\ldots\mu_s} + 2b\sum g_{\mu_1\mu_2}\phi^\rho{}_{\rho\mu_3\ldots\mu_s} \quad \text{(A.45)}$$

and the Fronsdal kinetic operator (A.4) in $(A)dS_4$ becomes

$$G^{(A)dS} \;\; \equiv \;\; (\nabla^2 - \boldsymbol{div}\ \boldsymbol{grad} + \boldsymbol{div^2}\ \boldsymbol{tr} + 2b\ \boldsymbol{g}\ \boldsymbol{tr} + bA_s) \quad \text{(A.46)}$$

where

$$A_s = s^2 - 2(s+1). \quad \text{(A.47)}$$

and the operator $\boldsymbol{g}$ acts on a completely symmetric tensor $V_{\mu_3\cdots\mu_s}$ as:

$$\boldsymbol{g}V_{\mu_3\cdots\mu_s} = \sum g_{\mu_1\mu_2}V_{\mu_3\cdots\mu_s}\ . \quad \text{(A.48)}$$

Note now that in (A.46) a mass term associated to the curvature appears. Further analysis about massless HS and partially massless HS in $(A)dS$ can be found in [41].

# Appendix B

# Koszul-Tate alghoritm and gauge fixed action

In this appendix we will describe a very useful technique to construct BRST charges, known as Koszul-Tate alghoritm. In particular we will use it to study an interesting class of non lie superalgebra, and to construct $SO(N)$ spinning particle in $(A)dS$ background gauge fixed action on the circle. More details about our analysis can be found in [53].

## B.1 Koszul-Tate differential

We start considering Hamiltonian system with first class constraints

$$G_A = 0 \ . \tag{B.1}$$

Let $z^F$ be a set of phase space variable (including bosons fermions and gauge fields); we extend now this space by introducing as many ghost $\eta^A$ and ghost momenta $\mathcal{P}_A$ as there are constraints $G_A$, and with opposite grading:

$$\begin{aligned} \epsilon(G_A) &= \epsilon_A \\ \epsilon(\mathcal{P}_A) &= \epsilon_A + 1 \\ \epsilon(\eta^A) &= \epsilon_A + 1 \ . \end{aligned}$$

In this extended phase space the Koszul-Tate differential $\delta$ acts in the following way:

$$\begin{aligned} \delta z^M &= 0 \\ \delta \eta^A &= 0 \\ \delta \mathcal{P}_A &= -G_A; \end{aligned} \tag{B.2}$$

on an arbitrary polymonial in the $\mathcal{P}_A$'s, $\delta$ acts as an odd-right derivative. Since $\delta^2$ vanishes on all the generators, $\delta$ is nilpotent. We use now this operator to construct the BRST generator $Q^{brst}$.
To this aim let us write

$$Q^{brst} \equiv \sum_{p\geq 0} \Omega^p \tag{B.3}$$

where $\Omega^0 = \eta^A G_A$ and

$$\begin{aligned} \Omega^p &= U^{a_1...a_p}\mathcal{P}_{a_p}.....\mathcal{P}_{a_1} \\ U^{a_1...a_p} &= \eta^{b_1}...\eta^{b_{p+1}}U^{a_1...a_p}_{b_1...b_{p+1}}. \end{aligned}$$

The coefficents "$U^{a_1...a_p}_{b_1...b_{p+1}}$" are only functions of the original phase space. Now one can rewrite the nilpotency condition $\{Q^{brst}, Q^{brst}\}_{pb} = 0$ as a set of equation for the unknown function $\Omega^p$:

$$\delta\Omega^{p+1} = -D^p \tag{B.4}$$

where

$$2D^p = \sum_{k=0}^{p}\{\Omega^k, \Omega^{p-k}\}_{pb} + \sum_{k=0}^{p-1}\{\Omega^{k+1}, \Omega^{p-k}\}_{ghost}. \tag{B.5}$$

In the previous formula $\{,\}_{pb}$ refers to the poisson bracket in the original phase space (without ghost), whereas $\{,\}_{ghost}$ refers to the poisson bracket acting only on the ghost and ghost momenta:

$$\{\mathcal{P}_A, \eta^B\}_{ghost} = -(-)^{(\epsilon_B+1)(\epsilon_A+1)}\{\eta^B, \mathcal{P}_A\}_{ghost} = -\delta^B_A \tag{B.6}$$

## B.2 Non linear supersymmetry algebra

Let $L_i$ and $T_\alpha$ be some bosonic and fermionic generators respectively and the superalgebra

$$\begin{aligned} &\{L_i, L_j\}_{pb} = C^k_{ij}L_k \\ &\{L_i, T_\alpha\}_{pb} = C^\beta_{i\alpha}T_\beta \\ &\{T_\alpha, T_\beta\}_{pb} = C^i_{\alpha\beta}L_i + D^{ij}_{\alpha\beta}L_iL_j \end{aligned} \tag{B.7}$$

with $C^k_{ij}$ $C^\beta_{i\alpha}$ $C^i_{\alpha\beta}$ $D^{ij}_{\alpha\beta}$ some structure constants.
We would like now to use the Koszul-Tate algorithm described in the previous section to construct the BRST generator associated to this algebra.
First of all we use the generalized Jacobi identity in order to extract the following very useful relations:

$$\begin{aligned} &C^m_{ij}C^p_{mk} + (-)^{\epsilon_k\epsilon_{ij}}C^m_{ki}C^p_{mj} + (-)^{\epsilon_i\epsilon_{kj}}C^m_{jk}C^p_{mi} = 0 \\ &C^\beta_{ij}C^\sigma_{\beta\alpha} + (-)^{\epsilon_\alpha\epsilon_{ij}}C^\beta_{\alpha i}C^\sigma_{\beta j} + (-)^{\epsilon_i\epsilon_{\alpha j}}C^\beta_{j\alpha}C^\sigma_{\beta i} = 0 \\ &C^\sigma_{i\alpha}C^k_{\sigma\beta} + (-)^{\epsilon_\beta\epsilon_{i\alpha}}C^\sigma_{\beta i}C^k_{\sigma\alpha} + (-)^{\epsilon_i\epsilon_{\alpha\beta}}C^j_{\alpha\beta}C^k_{ji} = 0 \\ &C^i_{\alpha\beta}C^\sigma_{i\delta} + (-)^{\epsilon_\delta\epsilon_{\alpha\beta}}C^i_{\delta\alpha}C^\sigma_{i\beta} + (-)^{\epsilon_\alpha\epsilon_{\delta\beta}}C^i_{\beta\delta}C^\sigma_{i\alpha} = 0 \\ &C^{\sigma_{i\alpha}}D^{kj}_{\sigma\beta} + (-)^{\epsilon_{\alpha\beta}+1}C^{\sigma_{i\beta}}D^{kj}_{\sigma\alpha} + (-)^{\epsilon_p(1+\epsilon_{\alpha\beta ij})\epsilon_j\epsilon_{\alpha\beta}}C^j_{pi}D^{kp}_{\alpha\beta} \\ &+(-)^{\epsilon_p(1+\epsilon_{\alpha\beta ik})+\epsilon_k\epsilon_{\alpha\beta}}C^k_{pi}D^{jp}_{\alpha\beta} = 0 \\ &D^{ij}_{\alpha\beta}C^\sigma_{j\delta} + (-)^{\epsilon_\delta\epsilon_{\alpha\beta}}D^{ij}_{\delta\alpha}C^\sigma_{j\beta} + (-)^{\epsilon_\alpha\epsilon_{\beta\delta}}D^{ij}_{\beta\delta}C^\sigma_{j\alpha} = 0 \end{aligned} \tag{B.8}$$

where $\epsilon_{i..j} = \epsilon_i + .. + \epsilon_j$.
Let us define $A = (i, \alpha)$ and $G_A = (L_i, T_\alpha)$; our starting point is the linear term

$$\Omega^0 = \eta^A G_A. \tag{B.9}$$

We use now (B.4) and relations (B.8), and we obtain:

$$\begin{aligned}
\Omega^1 &= -\frac{1}{2}(-)^{\epsilon_A}\eta^A\eta^B C^D_{BA}\mathcal{P}_D - \frac{1}{2}(-)^{\epsilon_\alpha}\eta^\alpha\eta^\beta D^{ij}_{\beta\alpha}L_i\mathcal{P}_j \\
\Omega^2 &= 0 \\
\Omega^3 &= -\frac{1}{24}(-)^{\epsilon_{\alpha\delta l}+\epsilon_i\epsilon_{kl}\epsilon_{\delta\sigma}\epsilon_{ij}}\eta^\delta\eta^\sigma\eta^\alpha\eta^\beta D^{ij}_{\beta\alpha}D^{kl}_{\sigma\delta}C^t_{ik}\mathcal{P}_t\mathcal{P}_l\mathcal{P}_j \\
\Omega^p &= 0 \qquad \forall p > 3
\end{aligned}$$

We conclude that the $Q^{brst}$ associated to the algebra (B.7) reads:

$$\begin{aligned}
Q^{brst} &= -\frac{1}{2}(-)^{\epsilon_A}\eta^A\eta^B C^D_{BA}\mathcal{P}_D - \frac{1}{2}(-)^{\epsilon_\alpha}\eta^\alpha\eta^\beta D^{ij}_{\beta\alpha}L_i\mathcal{P}_j \\
&\quad -\frac{1}{24}(-)^{\epsilon_{\alpha\delta l}+\epsilon_i\epsilon_{kl}\epsilon_{\delta\sigma}\epsilon_{ij}}\eta^\delta\eta^\sigma\eta^\alpha\eta^\beta D^{ij}_{\beta\alpha}D^{kl}_{\sigma\delta}C^t_{ik}\mathcal{P}_t\mathcal{P}_l\mathcal{P}_j
\end{aligned} \tag{B.10}$$

## B.3 Spinning particle in $(A)ds$: gauge fixing

We would like now to use the Koszul-Tate algorithm to analyze spinning particle model in $(A)dS$ space.[1] We rewrite the action principle (1.43) using euclidean convention[2]

$$\begin{aligned}
S &= \int d\tau [\frac{1}{2e}g_{\mu\nu}(\dot{x}^\mu - \chi_i\psi^\alpha_i e^\mu_\alpha)(\dot{x}^\nu - \chi_i\psi^\alpha_i e^\nu_\alpha) + \frac{1}{2}\psi^\alpha_i(\dot{\psi}_{i\alpha} - a_{ij}\psi_{j\alpha} + \dot{x}^\mu\omega_{\mu\alpha\beta}\psi^\beta_i) \\
&\quad -\frac{1}{8}e\psi^\alpha_i\psi^\beta_i\psi^\gamma_j\psi^\delta_j R_{\alpha\beta\gamma\delta}].
\end{aligned}$$

This action enjoys local supersymmetry (gauge parameter $\epsilon_i$) local $SO(N)$ (gauge parameter $\alpha_{ij}$) and worldline diffeomorphism (gauge parameter $\xi$). The corresponding constraints $Q_i, J_{ij}, H$ are first class and the algebra reads

$$\{Q_i, Q_j\} = -2iH\delta_{ij} + ibJ_{ik}J_{jk} - i\frac{b}{2}J_{lk}J_{lk}\delta_{ij} \tag{B.11}$$
$$\{Q_i, J_{jk}\} = 2\delta_{k[i}Q_{j]} \tag{B.12}$$
$$\{J_{ij}, J_{lm}\} = SO(N) \text{ algebra} . \tag{B.13}$$

The gauge field transforms as:

$$\begin{aligned}
\delta e &= \dot{\xi} + 2\chi_i\epsilon_i \\
\delta\chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j + \alpha_{ij}\chi_j \\
\delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im}a_{mj} - \alpha_{jm}a_{mi} \\
&\quad -\frac{b}{2}\psi^\alpha_j\psi_{k\alpha}(\chi_i\epsilon_k + \chi_k\epsilon_i) + \frac{b}{2}\psi^\alpha_i\psi_{k\alpha}(\chi_j\epsilon_k + \chi_k\epsilon_j) - b\psi^\alpha_i\psi_{j\alpha}\chi_k\epsilon_k
\end{aligned} \tag{B.14}$$

[1] We recall to the reader that in our convention $(A)dS$ curvature is $R_{\mu\nu\rho\sigma} = b(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$.
[2] We Wick rotate the time as $t \to -i\tau$ and $a_{ij} \to ia_{ij}$; note that we rotate also the gauge parameters $\epsilon_i \to -i\epsilon_i$ and $\xi \to -i\xi$.

One can note that, except for the susy $a_{ij}$ transformation rule (that we have emphasized writing it in red), relations (B.14) coincides with the one we have discussed in the flat case in Chapter 3 (3.2).

Let us restrict ourselves to periodic boundary conditions for bosons and antiperiodic boundary conditions for fermions and this let us gauge fix gravtini to zero, as was extensively discussed in Chapter 3. It is interesting to note that when gravitini vanish one leaves with the same flat case transformation rules for the $SO(N)$ gauge fields and for the einbein $e$. This simplify strongly the rest of the discussion; by using the same arguments we have used in Chapter 3 one can bring the gauge configuration $(e, \chi, a_{ij}) = (\beta, 0, \hat{a}_{ij})$ where

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 & . & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & . & 0 & 0 \\ 0 & 0 & -\theta_2 & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & \theta_r \\ 0 & 0 & 0 & 0 & . & -\theta_r & 0 \end{pmatrix} . \tag{B.15}$$

for even $N = 2r$ and

$$\hat{a}_{ij} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 & . & 0 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & . & 0 & 0 & 0 \\ 0 & 0 & -\theta_2 & 0 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & \theta_r & 0 \\ 0 & 0 & 0 & 0 & . & -\theta_r & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \end{pmatrix} . \tag{B.16}$$

for odd $N = 2r - 1$.

We would like now to construct the gauge fixed action. The extended phase space contains the original variables $(x^\mu, \psi_i^\alpha)$, the gauge field $\lambda^A = (e, \chi_i, a_{ij})$ and their momenta $\pi_A = (\pi_e, \pi_{\chi_i}, \pi_{a_{ij}})$ and the ghost $(\eta^1, \eta^{2i}, \eta^{3ij})$, and ghost momenta $(\mathcal{P}_1, \mathcal{P}_{2i}, \mathcal{P}_{3ij})$ associated with the first class constraints $(H, Q_i, J_{ij})$.

From equations (B.4) and (B.8)we find explicitly

$$\begin{aligned} Q^{brst} &= \eta^1 H + \eta^{2i} Q_i + \eta^{3ij} J_{ij} + \frac{i}{2r^2} \eta^{2i} \eta^{2j} (\frac{1}{2} \delta_{ij} \delta_{pk} \delta_{ql} - \delta_{ik} \delta_{pj} \delta_{ql}) J_{kl} \mathcal{P}_{3pq} \\ &\quad - i(\eta^{2i})^2 \mathcal{P}_1 - \eta^{2i} \eta^{3ij} \mathcal{P}_{2j} - \eta^{im} \eta^{mj} \mathcal{P}_{3ij} \\ &\quad + \frac{1}{24r^4} \eta^{2i} \eta^{2j} \eta^{2t} \eta^{2s} (\frac{1}{2} \delta_{ij} \delta_{pl} \delta_{qm} - \delta_{jp} \delta_{qm} \delta_{li}) (\frac{1}{2} \delta_{ts} \delta_{kr} \delta_{zw} - \delta_{sk} \delta_{tr} \delta_{zw}) (\delta_{v[z} \delta_{k][q} \delta_{p]h}) \mathcal{P}_{3hv} \mathcal{P}_{3rw} \mathcal{P}_{3lm} \\ &= \eta^1 H + \eta^{2i} Q_i + \eta^{3ij} J_{ij} + \frac{1}{2r^2} \eta^{2i} \eta^{2j} (\frac{1}{2} \delta_{ij} \delta_{pk} \delta_{ql} - \delta_{ik} \delta_{pj} \delta_{ql}) J_{kl} \mathcal{P}_{3pq} \\ &\quad - i(\eta^{2i})^2 \mathcal{P}_1 - \eta^{2i} \eta^{3ij} \mathcal{P}_{2j} - \eta^{im} \eta^{mj} \mathcal{P}_{3ij} \\ &\quad + \frac{1}{24r^4} \eta^{2i} \eta^{2j} \eta^{2t} \eta^{2s} (\delta_{ij} \delta_{ts} \mathcal{P}_{3hv} \mathcal{P}_{3mv} \mathcal{P}_{3lm} - 3 \delta_{ts} \mathcal{P}_{3jv} \mathcal{P}_{3mv} \mathcal{P}_{3im}) \end{aligned} \tag{B.17}$$

In order to implement the "temporal gauge fixing" $(e, \chi_i, a_{ij}) = (\beta, 0, \hat{a}_{ij})$, we choose the gauge fixing fermion

$$K = -\beta \mathcal{P}_1 - \hat{a}_{ij} \mathcal{P}_{3ij} \tag{B.18}$$

The gauge fixed Hamiltonian reads:

$$H^{GaugeFixed} \equiv H + [K, Q^{brst}] \tag{B.19}$$

$$\Downarrow$$

$$H^{GaugeFixed} = H + \beta H + \hat{a}_{ij} J_{ij} - \hat{a}^{ij} \eta_{2i} \mathcal{P}_{2j} \tag{B.20}$$

We are now ready to write the gauge fixed action and in particular we obtain

$$\begin{aligned} S^{GaugeFixed}[z^F, e, \chi_i, a_{ij}, \eta^A, \mathcal{P}_A] &= \int d\tau \left[ \dot{x}^\mu p_\mu + i \dot{\psi}_i^a \psi_i^b \eta_{ab} + \dot{\eta}^A \mathcal{P}_A - H^{GaugeFixed} \right] \\ &= S[z^F, e = \beta, \chi_i = 0, a_{ij} = \hat{a}_{ij}, \mathcal{P}_A] \\ &\quad + \int d\tau \mathcal{P}^{2i} \left( \frac{\partial}{\partial \tau} - \hat{a}_{ij} \right) \eta^{2j}. \end{aligned} \tag{B.21}$$

The last term is exactly the Faddeev-Popov determinant. This is not obvious at all because the algebra is non linear; otherwise one can notes that the higher order terms vanish when we gauge fix gravitini to zero, thus everything reduces to the usual Lie algebra case.

# Appendix C

# Summing with Mr. Pochhammer

In this appendix we will solve explicitly the series (4.93). The results we will obtain plays a fundamental role in constructing higher spin curvature, and reveals also a funny and interesting mathematical structure.

## C.1 Pochhammer function

We start defining *Pochhammer* function. Let $k, l, m, i \in \mathbb{N}$ and $x, y \in \mathbb{R}$. We define the function $P(x, k)$ as:

$$P(x,k) = \prod_{i=0}^{k}(x-i) \tag{C.1}$$

that is related to the well known *Pochhammer* function $\tilde{P}(x, k)$ by the relation

$$P(x,k) = (-)^{k+1}\tilde{P}(-x, k+1); \tag{C.2}$$

for matter of convenience we use $P$ instead of $\tilde{P}$ and, sometimes, we refer to it as ”modified *Pocchammer* function”. By definition one also has:

$$P(x,-1) = 1 \qquad \forall x \in \mathbb{R}; \tag{C.3}$$

by using (C.1) and (C.3) it's not hard to prove that $P(k, l)$ satisfies the following relations:

**Property 1:**

$$\boxed{\sum_{k=1}^{m} P(k,l) = \frac{1}{l+2}P(m+1, l+1)} \tag{C.4}$$

**Property 2:**

$$\boxed{P(S+2k, 2k) = P(S+2(k-l), 2(k-l))P(S+2k, 2l-1)} \tag{C.5}$$

## C.2 Solution by iteration

We consider now the function $a_{2n}(s)$, defined in (4.94) as

$$\begin{aligned} a_{2n}(s) &\equiv \sum_{k_0=1}^{s-1}\sum_{m_1=1}^{k_0}\sum_{k_1=m_1-1}^{s-3}\cdots\sum_{m_n-1=1}^{k_{n-2}}\sum_{k_{n-1}=m_{n-1}-1}^{s-2n-3}\sum_{m_n=1}^{k_{n-1}}\sum_{k_n=m_n-1}^{s-2n-1}1 \\ &= \sum_{k_0=1}^{s-1}\sum_{m_1=1}^{k_0}\sum_{k_1=m_1-1}^{s-3}\cdots\sum_{m_n-1=1}^{k_{n-2}}\sum_{k_{n-1}=m_{n-1}-1}^{s-2n-3}\sum_{m_n=1}^{k_{n-1}}(s-2n+1-m_n) \\ &= \sum_{k_0=1}^{s-1}\sum_{m_1=1}^{k_0}\sum_{k_1=m_1-1}^{s-3}\cdots\sum_{m_n-1=1}^{k_{n-2}}\sum_{k_{n-1}=m_{n-1}-1}^{s-2n-3}\underbrace{k_{n-1}}_{\natural}(s-2n)-\frac{1}{2}\underbrace{k_{n-1}(k_{n-1}-1)}_{\sharp}. \end{aligned} \tag{C.6}$$

Note that in the last line of the previous equation we recognize the modified *Pochhammer* functions $P(k_n,0)$ ($\natural$) and $P(k_n,1)$ ($\sharp$). This observation plays a key role in our analysis.
Let us start fixing the notation. We introduce the step parameter $z=0,1,...,n-1$; in our language doing a step means that we solve a couple of summatories :

$$\cdots\underbrace{\sum_{m_\mu=1}^{k_{\mu-1}}\sum_{k_\mu=m_\mu-1}^{s-2(\mu)-1}}_{\text{1 STEP}}\cdots \qquad \mu=n-z=0,1,...,n-1; \tag{C.7}$$

equivalently we define the $z^{th}$ step as the result we obtain after having solved $z$ couple of sommatories (i.e. after $z$ step)

$$\underbrace{\sum_{m_{n-z+1}=1}^{k_{n-z}}\sum_{k_{n-z+1}=m_{n-z+1}-1}^{s-2(n-z+1)-1}}_{z^{th}\text{couple}}\cdots\underbrace{\sum_{m_n-1=1}^{k_{n-2}}\sum_{k_{n-1}=m_{n-1}-1}^{s-2n-3}}_{2^{nd}\text{couple}}\underbrace{\sum_{m_n=1}^{k_{n-1}}\sum_{k_n=m_n-1}^{s-2n-1}}_{1^{st}\text{couple}}1. \tag{C.8}$$

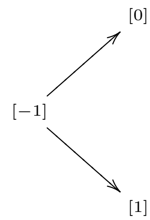
Sometimes, in the following, we prefer to use the shortcut notation

$$P(k_{n-z},q)=[q], \tag{C.9}$$

and the step parameter (i.e. $n-z$) will be clear from the context.
In order to become familiar with this language let us reanalyze, from this point of view, equation (4.82). In the first line of (4.82) we start summing "1" that could be thought as $P(k_n,-1)$. In the third line we have a linear combination of $P(k_{n-1},0)$ and $P(k_{n-1},1)$. Formally we say that $[-1]$ generates $[0]$ and $[1]$[1]; this could be resumed in the following

[1] In the first part of this analysis we don't care about the coefficients in front of the $P$ functions. we will fix them later.

diagram:

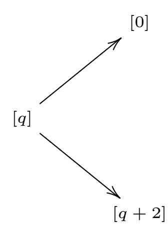



(C.10)

The above consideration could be generalized to every step $z$; by using the **Property 1**, infact, we have:

$$\sum_{m_{n-z}=1}^{k_{n-z-1}} \sum_{k_{n-z}=m_{n-z-1}}^{s-2(n-z)-1} P(k_{n-z}, q) = \frac{1}{q+2} P(s-2(n-z), q+1) P(k_{n-z-1}, 0) - \frac{(q+1)!}{(q+3)!} P(k_{n-z-1}, q+2). \quad (C.11)$$

The previous formula implies that, for every $[q]$ and $z$, the modified *Pochhammer* function $[q]$ generates in the $(z+1)^{th}$ step $[0]$ and $[q+2]$:




(C.12)

Start to become clear that after $z$ step we leave with a linear combination of $[q]$ function; this could be understood better looking at the diagramm (C.1):

At this point it's not hard to convince ourselves that, after $z$ steps, the parameter $q$ can take the values $2\tilde{q}$ with $\tilde{q} = 0, 1, ..., z-1$, or $2z-1$. In particular at the $z^{th}$ step we obtain:

$$\sum_{\tilde{q}=0}^{z-1} C(z, 2\tilde{q})[2\tilde{q}] + C(z, 2z-1)[2z-1] \quad (C.13)$$

for some function $C(\cdot,\cdot)$ that we have to evaluate step by step. From the diagram (C.1) we can observe that the "odd" coefficient $C(z, 2z-1)$ is obtained by itering $z$ times the second line of equation (C.11). In particular one finds:

$$C(z, 2z-1) = \frac{(-)^z}{(2z)!}. \quad (C.14)$$

Computing the other coefficients seems to be too hard because at every step we double the number of $[q]$ functions. Fortunately we can simplify the computation by using the following Lemma:

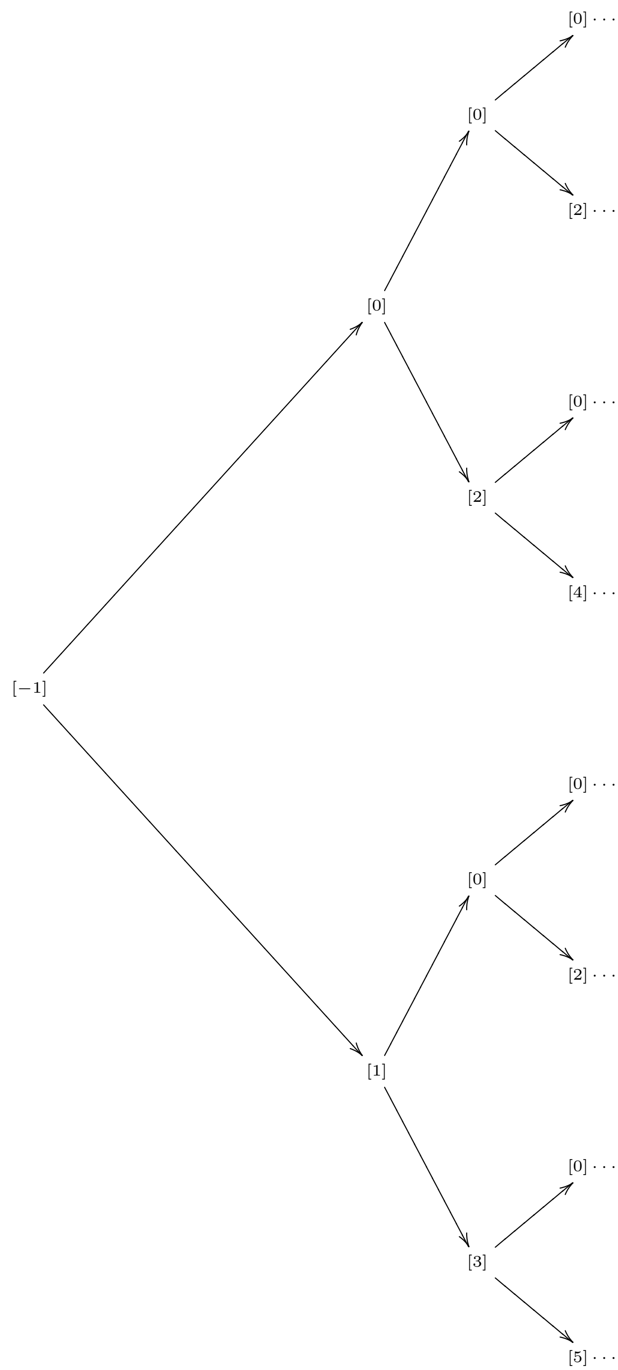



Figure C.1: Pochhammer family tree

**Lemma 4.**

*At every step $z$ one has*

$$C(z,0) = g(z)P(s-2(n-z+1), 2(z-1)) \qquad \forall z = 0, 1, ..., n-1 \tag{C.15}$$

*for some function $g(z)$.*

**Proof:**

We prove this Lemma by induction.[2]

Using equation (C.11) and the **Property 2** one can verify explicitly that for $z = 0, 1, 2, 3$ equation (C.15) is satisfied (*base of induction porcedure*).

Let us now suppose that (C.15) is satisfied till the $(z-1)^{th}$ step (*induction HP*). This assumption has an interesting consequence; it implies, infact, that the diagram (C.1) collaps into (C.2).

In particular, the modified *Pochhammer* function $P(k_{n-z}, 0)$ is generated by $P(k_{n-z-1}, 2\tilde{q})$ and $P(k_{n-z-1}, 2z-3)$. Observing carefully the diagram (C.2) we learn that the coefficients $P(k_{n-z-1}, 2\tilde{q})$ could be obtained just by applying $\tilde{q}$ times the second line of (C.11) to $C(z-1-\tilde{q}, 0)$. So we can conclude that

$$\begin{aligned} C(z-1, 2\tilde{q}) &= C(z-1-\tilde{q}, 0)\frac{(-)^{\tilde{q}}}{(2\tilde{q}+1)!} \\ &\equiv \frac{(-)^{\tilde{q}}}{(2\tilde{q}+1)!} g(z-1-\tilde{q})P(s-2(n-z+\tilde{q}+2), 2(z-\tilde{q}-2)) \end{aligned} \tag{C.16}$$

Using the first line of (C.11) and the **Property 2** it's easy to show that going from $z-1 \Rightarrow z$ one obtains (*induction step*)

$$\begin{aligned} C(z,0) &\backsim P(s-2(n-z+1), 2\tilde{q}+1)P(s-2(n-z+\tilde{q}+2), 2(z-\tilde{q}-2)) \\ &\backsim P(s-2(n-z+1), 2(z-1)) \end{aligned} \tag{C.17}$$

Let me emphasize that in the previous formula we don't care about the contribution arising from the odd modified *Pocchammer* function $[2z-3]$. Otherwise it is still proportional to $P(s-2(n-z+1), 2(z-1))$. This conclude the proof.

[2] In particular we are going to use the so called "extended" induction principle. Let $Prop(n)$ be some preposition depending on $n \in \mathbb{N}$.

1. $Prop(n_0)$ is true for some $n_0 \in \mathbb{N}$ (*base of induction porcedure*)
2. we suppose now that $\forall k \geqslant n_o$ $Prop(k)$ is true (*induction HP*)
3. if $Prop(k)$ true $\Rightarrow$ $Prop(k+1)$ true (*induction step*)

we can conclude that $Prop(n)$ is true for every $n \in \mathbb{N}$.

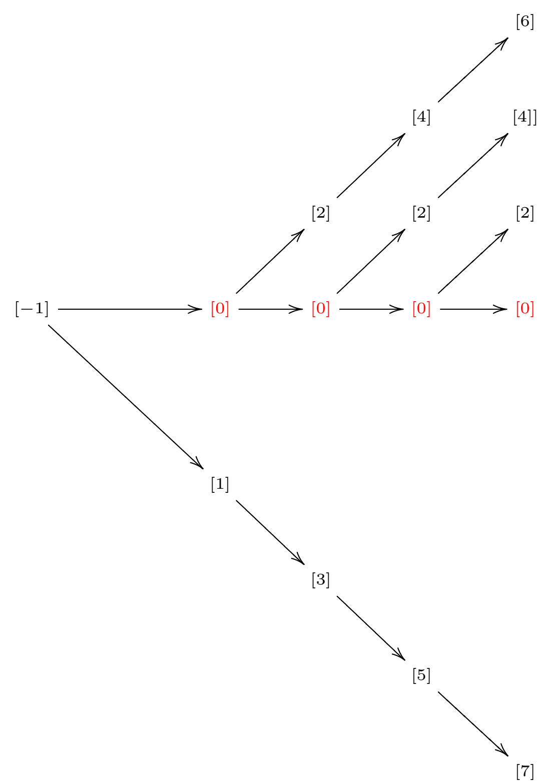



Figure C.2: Pochhammer family tree modified. Note that we have used the red square bracket [0] to emphasize that the coefficient that multiply the function [0], at every step, is just summed by using the hypothesis of the **Lemma 4**.

We resume now the results obtained from the **Lemma4**:

$$\begin{aligned} C(z, 2z-1) &= \frac{(-)^z}{(2z)!} && \text{(C.18)} \\ C(z, 2\tilde{q}) &= C(z-\tilde{q}, 0)\frac{(-)^{\tilde{q}}}{(2\tilde{q}+1)!} && \text{(C.19)} \\ C(z, 0) &= g(z)P(s-2(n-z+1), 2(z-1))\ ; && \text{(C.20)} \end{aligned}$$

in particular we also find that:

$$g(z) = \sum_{q=0}^{z-2} \frac{(-)^q}{(2q+2)!} g(z-1-q) + \frac{(-)^{z-1}}{(2z-1)!} \qquad \text{with} \qquad g(0) = 0\ . \tag{C.21}$$

We are ready now to calculate $a_{2n}(s)$. After $n$ step equation (C.11) reduces to:

$$\begin{aligned} a_{2n}(s) &= g(n)P(s-2n, 2(n-1)) \sum_{k_0}^{s-1} P(k_0, 0) \\ & \quad . \\ & \quad . \\ & \quad + C(n, 2\tilde{q}) \sum_{k_0=1}^{s-1} P(k_0, 2\tilde{q}) \\ & \quad . \\ & \quad . \\ & \quad \frac{(-)^n}{(2n)!} \sum_{k_0=1}^{s-1} P(k_0, 2n-1)\ . \end{aligned} \tag{C.22}$$

We use now **Property 1** and **Property 2**, and (C.18)-(C.20), in order to write the previous equation in a more compact and elegant form. In particular we can write the first $n$ lines of the previous equation as

$$\begin{aligned} & \frac{(-)^{\tilde{q}}}{(2\tilde{q}+1)!} g(n-\tilde{q}) P(s-2(\tilde{q}+1), 2(n-\tilde{q}-1)) \sum_{k_0}^{s-1} P(k_0, 2\tilde{q}) \\ = & \frac{(-)^{\tilde{q}}}{(2\tilde{q}+2)!} g(n-\tilde{q}) \underbrace{P(s-2(\tilde{q}+1), 2(n-\tilde{q}-1)) P(s, 2\tilde{q}+1)}_{P(s,2n)} \\ = & \frac{(-)^{\tilde{q}}}{(2\tilde{q}+2)!} g(n-\tilde{q}) P(s, 2n) \end{aligned} \tag{C.23}$$

We substitute now (C.23) into (C.22) and we find:

$$a_{2n}(s) = f(n)P(s, 2n) \tag{C.24}$$

where

$$f(n) = \sum_{\tilde{q}=0}^{n-1} \frac{(-)^{\tilde{q}}}{(2\tilde{q}+2)!} g(n-\tilde{q}) + \frac{(-)^n}{(2n+1)!} \ . \tag{C.25}$$

It's interesting to note that $f(n)$ is defined in terms of linear combination of function $g$. Everything become easier noting that $f(n) = g(n+1)$.
Finally we have:

$$a_{2n}(s) = f(n)s(s-1)(s-2)\cdots(s-2n+1)(s-2n) \tag{C.26}$$

where $f(n)$ is defined by recurrence as

$$f(n) = \sum_{\tilde{q}=0}^{n-2} \frac{(-)^{\tilde{q}}}{(2\tilde{q}+2)!} f(n-\tilde{q}-1) + \frac{(-)^{n-1}}{(2n-1)!}, \tag{C.27}$$

and this conclude the first part our analysis of equation (4.82).

In the second part of this appendix we would like to discuss an important property of the function $f(n)$. We introduce the shortcut notation:

$$f(n) = \sum_{q=0}^{n-2} A(q) f(n-q-1) + C(n) \qquad \text{with} \qquad \begin{array}{rcl} A(q) & = & \frac{(-)^q}{(2q+2)!} \\ C(n) & = & \frac{(-)^{n-1}}{(2n-1)!} \end{array} \tag{C.28}$$

One can observes that for the first few case the function $f(n)$ satisfies (*base of induction procedure*):

**Property 3:**

$$\boxed{\sum_{k=0}^{n-1} f(k+1) f(n-k) = (2n+1) f(n+1)} \tag{C.29}$$

We would like to prove that this relation is valid for every $n$ by using only (C.28) and the initial condition $f(0) = 0$ (i.e. $f(1) = 1$). In order to fix the notation and the ideas we start considering the case $f(6)$. Let us suppose that (C.29) is satisfied for every $n \leqslant 4$ (*induction HP*), or equivalentlly

$$\sum_{k=0}^{n-2} f(k+1) f(n-k-1) = (2n-1) f(n) \qquad \forall \leqslant 5 \tag{C.30}$$

We would like to recognize and isolate terms proportional to $(2q-1)f(q)$. In particular one has:

$$
\begin{aligned}
& 11f(6) \\
= & \quad \tfrac{11}{2}f(5) \quad -\tfrac{11}{4!}f(4) \quad +\tfrac{11}{6!}f(3) \quad -\tfrac{11}{8!}f(2) \quad +\tfrac{11}{10!}f(1) \quad -\tfrac{1}{10!} \\
= & \quad f(5)f(1) \quad +C(2)f(4) \quad +C(3)f(3) \quad +C(4)f(2) \quad +C(5)f(1) \\
& +A(0)9f(5) \quad +A(1)7f(4) \quad +A(2)5f(3) \quad +A(3)3f(2) \quad +A(4)3f(1)
\end{aligned}
$$

Note now that:

- Terms in red looks like $f(6-p)C(p)$
- Terms in blue looks like $f(p)(2p-1)$

We apply now the *induction HP* (C.30) on the blue terms and we can observe that everything collapses into

$$
\begin{aligned}
& f(1)\underbrace{(C(5)+A(0)f(4)+A(1)f(3)+A(2)f(2)+A(3)f(1))}_{f(5)} \\
+ \quad & f(2)\underbrace{(C(4)+A(0)f(3)+A(1)f(2)+A(2)f(1))}_{f(4)} \\
+ \quad & f(3)\underbrace{(C(3)+A(0)f(2)+A(1)f(1))}_{f(3)} \\
+ \quad & f(4)\underbrace{(C(2)+A(0)f(1))}_{f(2)} \\
+ \quad & f(5)\underbrace{(C(1)}_{f(1)}
\end{aligned} \tag{C.31}
$$

Explicitly one finds the desired result

$$
f(6) = (f(5)f(1) + f(4)f(2) + f(3)f(3) + f(2)f(4) + f(1)f(5)) \tag{C.32}
$$

We are ready now to attack the general $n$ case. In analogy with the example discussed above we suppose that

$$
\sum_{k=0}^{m-2} f(k+1)f(m-k-1) = (2m-1)f(m) \qquad \forall \;\; m \leqslant n-1. \tag{C.33}
$$

Another ingredient we need is the following relation:

$$
(2n-1)A(k) = (2(n-k-1)-1)A(k) + C(k+1). \tag{C.34}
$$

We write now $f(n)$ using the previous relation and its definition, in a clever way :

$$(2n-1)f(n) = \sum_{k=0}^{n-3} f(n-k-1)(2(n-k-1)-1)A(k) + \sum_{k=0}^{n-2} f(k+1)C(n-k-1). \quad \text{(C.35)}$$

We apply now the *induction HP* on the blue part of the previous relation

$$\sum_{k=0}^{n-3} \sum_{q=0}^{n-k-3} A(k)f(n-k-q-2)f(q+1) + \sum_{k=0}^{n-2} f(k+1)C(n-k-1). \quad \text{(C.36)}$$

It's very important to note that

$$\sum_{k=0}^{n-3} \sum_{q=0}^{n-k-3} A(k)f(n-k-q-2)f(q+1) = \sum_{k=0}^{n-3} f(k+1) \sum_{q=0}^{n-k-3} A(q)f(n-k-q-2) \quad \text{(C.37)}$$

Substituting (C.37) and (C.36) into (C.35) we obtain:

$$\begin{aligned}(2n-1)f(n) &= \sum_{k=0}^{n-3} f(k+1)\underbrace{\Big( C(n-k-1) + \sum_{q=0}^{n-k-3} A(q)f(n-k-q-2) \Big)}_{f(n-k-1)} + f(n-1)\underbrace{C(1)}_{f((1)} \\ &= \sum_{k=0}^{n-2} f(k+1)f(n-k-1) \quad \text{(C.38)}\end{aligned}$$

and this conclude our proof.

# Conclusions and outlook

In this thesis we have studied spinning particles with an $SO(N)$ extended supergravity multiplet on the worldline. In particular we have analyzed the symmetries of these models with flat and curved background and we have shown explicitly that in maximally symmetric space the first class constraints algebra, for every $N$, turns out to be closed, yet non linear; this approach has produced, as a byproduct, a more general coupling in $D = 2$ with certains spaces with non constant curvatures. An action principle with gauge symmetries (i.e. local symmetries), contains a $1^{st}$ class constraints algebra, but it need not be necessary a Lie algebra but could be more complicated. It would be interesting, by using the idea we have used in **Chapter 1**, to study $SO(N)$ spinning particles coupled to a more general background. In order to preserve local symmetries one can try to close $SO(N)$ extended supersymmetry algebra by admitting structure functions, instead of structure constant, or a more general non linear structure of the algebra.
This problem might be overcome also by adding one more constraint (constructed for example with four Grassmann variables), but note that at this point, in general, the physical meaning of the model could be modified and become less interesting.

In **Chapter 2** we have shown that, in every even space time dimensions $D = 2n$ and for every even $N$, a canonical analysis of $SO(N)$ spinning particles with flat background, produces as physical sector of the Hilbert space, tensors with mixed symmetry, and in particular with the symmetries of a rectangular Young tableaux with $s$ columns and $n$ rows (i.e. $s \otimes n$ Young tableaux). These tensors have to be closed and co-closed and we have recognized them as the conformal higher spin curvatures (HSC) describing the dynamic of conformal higher spin fields (HS) whose Young tableaux are $s \otimes (n-1)$ rectangles. We have also solved, by using the *generalized Poincaré Lemma*, the differential Bianchi identity (i.e. closing condition) and we have shown that the traceless condition on the HSC (that arise naturally from the quantization of spinning particles) produces an higher order derivative equation of motion for the HS fields. These equations, in virtue of the *generalized Poincaré Lemma* and after having introduced and gauge fixed to zero the compensator field, can be linearized; this analysis produces in particular, the Fronsdal-Labastida equation of motion, describing the dynamics of massless HS with mixed symmetry, and the correct traceless condition and double traceless conditions on the gauge parameters and gauge fields.

In **Chapter 3** we have studied the one loop quantization of $SO(N)$ spinning particles on the circle. We have considered propagation on a flat target space time and obtained the measure on the moduli space of the $SO(N)$ supergravity on the circle. We have used it to

compute the propagating physical degrees of freedom for every $N$ and every $D$, described by the spinning particles. In particular we have shown explicitly that spinning particles do not contain physical degrees of freedom in space times of odd dimensions. It would be very interesting, in a future project, to study the one loop quantization of a bigger class of models, orthosymplectic spinning particle models, describing tensors with mixed symmetry [70].
Let us also emphasize that the correct measure on the moduli space of the $SO(N)$ supergravity multiplet on the circle, we have obtained in this work, is necessary for computing more general quantum corrections arising when couplings to background fields are introduced. $SO(N)$ spinning particles, in fact, can consistently propagate in maximally symmetric space.

In **Chapter 4** we have studied the physical contents of $SO(N)$ spinning particles on $(A)dS$ backgrounds, via a canonical analysis. We have focused our attention on the integer spin case in every even dimension $D = 2n$ and we have shown explicitly that the quantization á la Dirac produces $(A)dS$ HSC as the physical states of the Hilbert space.
The dynamics of massless integer HS in $(A)dS$ background is described by the generalization of the Fronsdal-Labastida equation of motion in maximally symmetric space. However in the literature it is not yet clear how to derive these equations of motion starting from pure geometrical object (i.e. HSC); in curved space, in order to solve the differential Bianchi identity, one cannot use the *generalized Poincaré Lemma* as a guide line (covariant derivative do not commute). For this reasons we have proposed a different approach; we have written the $(A)dS$ HSC $|\mathcal{R}\rangle$ as

$$|\mathcal{R}\rangle = \sum_{m=0}^{[s/2]} (ib)^m r_m(s) \mathcal{R}_m(s) \ |\phi\rangle$$

where in this case $[s/2]$ means integer part, $\mathcal{R}_m(s)$ is an operator containing $s-2m$ covariant derivatives and $|\phi\rangle$ is the HS gauge potential. We have then imposed the Bianchi identity to fix the functions $r_m(s)$; these functions have been carefully evaluated by using the Pochhammer functions in **Appendix C** and we have found an elegant way to write them by using recursive relation with themselves.

More in general during this work, we have proposed a solution of the "$\nabla$-closed" condition of multiforms with the symmetries of rectangular $s \otimes n$ Young tableaux, living in maximally symmetric space; thus could be very interesting to study possible extensions to maximally symmetric space of the *generalized Poincaré Lemma*, by using our results as a guideline.

In the cases of $N = 4, 6, 8$ we have solved explicitly the differential Bianchi identity and we have shown that traceless condition on the HSC produces the Fronsdal-Labastida equation of motion in $(A)dS$ with the correct traceless and double traceless constraint on the gauge parameter and HS gauge potential.
It would be interesting, for further analysis, impose the traceless condition on the $(A)dS$ HSC for every integer spin. We expect that the higher derivative equations of motion for HS gauge fields reduce to a suitable generalization to curved space of the "$\partial^{s-2}$" closed condition

for the Fronsdal-Labastida operator $G_s^{(A)dS}$ that we write as

$$\sum_{m=0}^{[s/2-1]} (ib)^m r'_m(s) \mathcal{R}_m(s) G_s^{(A)dS} |\phi\rangle = 0$$

for some functions $r'_m(s)$; at this point one can try to introduce the compensator field and gauge fix it to zero and obtain the Fronsdal-Labastida equation of motion in $(A)dS$ background

$$G_s^{(A)dS} |\phi\rangle = 0 \; .$$

After having checked explicitly that $SO(N)$ spinning particle models describe, also in maximally symmetric background, the dynamics of conformal HS fields, it would be interesting to calculate the one loop partition function on the circle. To this aim we have constructed in **Appendix B**, by using the Koszul-Tate algorithm, the BRST charge associated to the $SO(N)$ extended supersymmetry algebra, that, in maximally symmetric space is a quadratic superalgebra; moreover we have gauge fixed the $SO(N)$ supergravity multiplet on the circle, and we have used the BRST charge to construct the gauge fixed action.

Note that in $D = 4$ flat target space, massless HS fields propagate 2 degrees of freedom associated to the two independent polarization, while massive fields with arbitrary spin $s$ propagate $2s+1$ independent polarizations. In $AdS_4$ partially massless states appear. These states propagate, for some particular values of the mass $m$ and the cosmological constant $\lambda$, $2s + 1 - 2c$ degrees of freedom, for some $c = 0, 1, 2, ..., s$ that is called step parameter [41].

For all the reasons discussed above another possible direction in which this work might be developed is the analysis of the massive $SO(N)$ spinning particles in $(A)dS$ target space. The computation of the partition function and the calculation of the degrees of freedom should be very interesting in order to study partially massless states in $D = 4$ and more in general in every dimension $D = 2n$.

In **Chapter 5** we have constructed $N = 4$ one dimensional supergravity models on Hyper Kähler (HK) and quaternionic Kähler (QK) manifolds. In particular we have analyzed the symmetries of these models and in the QK background case, we have constructed the BRST charge and the gauge fixed action on the circle. Thus, in the future, should be very interesting calculate the one loop partition function.

HK manifold endowed with homothetic killing vector have been extensively studied in literature (see for example [65] [71] and references therein); this is, in fact, an important subclass of HK manifolds admitting conformal symmetry and we refer to them as Hyper Kähler Cone (HKC). A map from a $4(n + 1)$ HKC to a $4n$ dimensional QK manifold for bosonic non linear sigma models, have been constructed and analyzed carefully in [72]. This procedure, known as superconformal reduction, is based on the gauging of the isometry of the HKC.

In a forthcoming paper [73] we will consider, as a toy model, supersymmetric (non linear) sigma model on flat and curved $D+1$ dimensional cones; in particular we propose a map for $SO(N)$ spinning particle and $U(1)$ spinning particle models, from the cone to the $D$ dimensional base manifold.
It would be interesting to extend our analysis to $sp(2)$ spinning particle models with $(4n+1)$ dimensional HKC background, and by gauging conformal and $sp(2)$ isometry, obtain $sp(2)$ spinning particle models propagating in $4n$ dimensional QK target space.

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