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Quantum gravity phenomenology: thermal dimension of quantum spacetime, causality and momentum conservation from Relative Locality

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Abstract

The original results presented in this thesis regard two very common topics of discussion in the quantum gravity debate: the dynamical dimensional reduction of spacetime and locality in quantum gravity regime. The dimensionality of the quantum spacetime is often understood in terms of the spectral dimension; here, a different notion of dimensionality, the thermal dimension, is proposed. I discuss its physical properties in relation to those of the spectral dimension through the study of specific models of quantum gravity, including preliminary results obtained in the case of models with relative locality. I show that, in those cases where the spectral dimension has puzzling properties, the thermal dimension gives a different and more meaningful picture. The statistical mechanics developed to define the thermal dimension is applied also to the study of the production of primordial cosmological perturbations assuming a running Newton constant and Rainbow Gravity. Concerning locality, I study in particular the theory of Relative Locality, a theoretical framework in which different observers may describe the same event as being local or non-local, depending whether it happens in the origin of their reference frame or far away from it, respectively. I show that requiring that locality is relative is enough to guarantee the objectivity of cause-effect relation in chains of events, the absence of causality-violating loops and processes violating the law of conservation of momentum.

Ai miei nonni,
e ai miei genitori.

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Introduction

The quantum gravity problem

The general relativistic description of gravitational phenomena and the quantum mechanics of the Standard Model of particles physics are the most fundamental physical theories known today. Each of them is spectacularly confirmed by experiments, but until now gravitational physics and quantum physics barely ”speak” to each other. In fact, GR has been confirmed by experiments on scales between 10^{-6} m and about 10^{20} m (at this scale one has to postulate the existence of dark matter in order to make general relativity agree with the experimental results), whereas the typical applications of QM and the SM concern physical phenomena at scales between 10^{-8} m and 10^{-20} m, the latter being the order of magnitude of the wavelength of the particles colliding at LHC. The gap between these two regimes covered by experiments comes from the fact that gravity is too weak at the energy scales at which quantum physics has been tested to detect its contribution in the measurements, whereas the other forces are either short range or their quantum properties averaged out at the scales at which gravitational interaction is relevant, as in the case of electromagnetic interaction. The goal of formulating a theory of Quantum Gravity originates not only from the discomfort that some might have in realizing that the two theories (GR and QM) are based on very different descriptions of the world, but is indeed justified by several genuine scientific arguments.

For example, as long as one ignores gravity, the SM gives definite predictions on the results of a scattering process between two particles each at energy of e.g. $\sim 10^{30}$ GeV. Such high energy processes are not presently within our technological reach, but contemplating them sheds light on the conceptual structure of our theories. It is known that the gravitational interaction for collisions between two particles of energy approximately (or greater than) the Planck energy $E_P = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{16}$ TeV cannot be neglected. Estimating the gravitational contribution to the scattering amplitudes (from some effective-field-theory formulation of gravitational interactions) one obtains unmanageable divergences.

Indeed, the attempts to formulate a local quantum field theory of gravity meet many problems, starting from the formalization of the microcausality postulate, *i.e.* that two local observables $A(x)$ and $B(y)$ must commute when x and y are separated by a spacelike interval. This postulate makes sense in the special-relativistic local quantum field theory since in that context the spacetime metric is fixed to be Minkowskian, whereas in GR the metric is a dynamical variable and therefore, in general, is not given at the beginning of the analysis. The standard approach is then to assume a background metric that fixes the spacetime intervals from the beginning and a perturbation of the metric that characterizes the gravitational interactions. The theory that one obtains from this procedure is non-renormalizable (at least in the standard sense; it will be considered in this thesis also the proposal first given by Steven Weinberg of Asymptotic Safety, which gives an alternative understanding of renormalizability in a broader sense).

It appears to be still possible developing QFT on a fixed background spacetime metric that is not Minkowskian. In this context, Hawking found the famous effect of black hole's radiation ([1]) studying this kind of theory on a Schwarzschild background metric. Hawking's result represents a serious theoretical challenge since it suggests that information is not conserved in the process of formation and evaporation of a black hole (see Ref. [2] for recent developments in the understanding of the problem).

An argument indicating rather clearly how QG requires a radical change in our description of Nature is Bronstein's argument on the measurability of the gravitational field. He applies to the gravitational field the measurement procedure considered by Landau and Peierls in their critique to the logical consistency of the newborn QED. In order to measure the electromagnetic field in a small region of spacetime (ideally a point), they studied the asymptotic states of a probe with electric charge e that interacts with the electromagnetic field in that region. What they found is that the uncertainty in the value of the field in that region is proportional to the ratio e/m_i where m_i is the inertial mass of the probe. So the ideal probe would have $e/m_i \sim 0$ and could be used to determine the electromagnetic field with arbitrary accuracy. As far as it is known today, there is not such ideal electromagnetic probe in Nature. Therefore, Landau and Peierls concluded that since it will never be possible to make a sharp measurement of the electromagnetic field, than QED, which admits also eigenstates of electromagnetic field as a basis of the Hilbert space, is logically inconsistent. It was then recognized by Bohr and Rosenfeld that QED is instead logically consistent, as the fact that there is no such ideal probe is to be taken as a technological limit, since the determination of the existence of such a particle is outside the scopes of QED[1]. Bronstein realized the importance of this argument for the case of quantum gravity: for the gravitational field the ratio e/m_i becomes m_g/m_i (m_g being the gravitational mass), but for the Equivalence Principle this is forced to equal 1. This means that the gravitational field is fundamentally not sharply measurable. QM formalism allows sharply measurable eigenvalues for all observable, it might only limits the accuracy of *simultaneous measurement of two observables.* Bronstein then argued that a new theoretical paradigm is needed to take this characteristic of gravity into account.

This new theoretical paradigm is likely to deal with effects that provide striking departures from our current theories. Unfortunately, today one can only speculate about such effects because experimental evidence of them is still missing. Actually, for a very long time it was a general conviction that QG effects were observable only for particles with Planck-scale energy, which is not accessible in laboratories neither at present nor in the foreseeable future. Even if it is not possible for present technology to accelerate particles to Planckian energies, it has been observed in the late 90's that it is possible to have indirect access to that scale by astrophysical and cosmological observations[2] (see Refs.[4],[5],[6],[7],[8],[9] and [10] for a recent review on quantum spacetime phenomenology). In particular, some effects due to the quantum structure of spacetime may sum up along the travel of a particle coming from a far away source. This includes possible modification to the energy-momentum relation

$$E^2 - p^2 = m^2$$

such as, for instance to leading order in Planck length $L_P = \frac{c\hbar}{E_P}$,

$$E^2 = p^2 + m^2 + \alpha L_P E p^2 + \mathcal{O}(L_P)^2, \tag{1}$$

where α is a dimensionless constant of order one. The typical effect that one expects from such modification is to observe an unexpected delay in the time of arrival of a very high energy particle and a low energy one coming from the same short-lived source at an astrophysical or cosmological distance. The quantitative prediction on the delay although depends on the details of the theory, in particular on how the Planck length is incorporated in the theoretical scheme in relation to Lorentz symmetry.

[1] The interest reader may find the complete report of this debate in Ref.[3].

[2] More recently it has been argued that quantum optics might be used to directly measure the canonical commutation relation (and the possible deformation due to the quantum structure of spacetime) of the center-of-mass mode of a mechanical oscillator with a mass close to the Planck mass (see Refs.[11], [12], [13] for a more complete discussion of this possibility).

In fact, the Planck length is often defined as $L_P = \sqrt{\frac{\hbar G}{c^3}}$, via the combination of three relativistic constants[3]. As long as this is the only operative definition of the Planck length, it is simply identifying a length scale and does not pose any problem to the relativistic picture of the theory. However, the moment it acquires a physical meaning as the length of *something* via an independent operative definition, for example via the deformed dispersion relation (1) and therefore independently measurable via the time-of-arrival delay of the kind mentioned above, one has to investigate if such operative definition is compatible or not with the other relativistic postulates in the proposed QG theory, as lengths are contracted by Lorentz transformations according to the relative motions of the observers. Then, a first possibility is that there is a preferred frame in which formulate our QG theory. Example of such theories are Hořava-Lifshitz gravity and Magueijo-Smolin formulation of Rainbow gravity. A second possibility is instead that the Lorentz transformations are just a low-energy approximation of a more complicated set of transformations that relates the measurements of two inertial observers and these transformations are such that Planck length is a relativistic invariant just as the speed of light is in Special Relativity. This is the general idea of Doubly Special Relativity (DSR). Some doubly-special-relativistic quantum gravity models are k-Minkowski non-commutative spacetime, $2+1$ gravity and Relative Locality. A third possibility considered in this thesis is that Lorentz transformation are still a valid symmetry of the physical laws and these are such that there is no contradiction between the existence of a different physical regime set by Planck scale and Lorentz symmetry. Such perspective is that of String Theory, some interpretations of Loop Quantum Gravity, Causal Sets and Asymptotic Safety, to mention the most popular ones. In this category, a model inspired by the Asymptotic Safety approach will be consider.

Two challenges for quantum spacetime research

Part of the work presented in this thesis wants to contribute to the development of theories formulated on a quantum spacetime. In fact, several argument suggest that our usual description of spacetime, which is strictly

[3] Although very different among each other: c is a relativistic invariant by postulate and Lorentz transformation respect this postulate in a non-trivial way, $\hbar$ invariance is related to the fact that it has dimension of an action and Newton constant is the outcome of a IR measurement ("infrared", *i.e.* for probes of wavelength much longer than the Planck length).

classical in GR as well as in QM and in QFT, needs to be deeply modified in QG, ultimately requiring the formulation of an appropriate notion of quantum geometry.

Consider for example the following argument. In QM an inertial observer can in principle operatively construct a coordinates system with labels on each spacetime point by setting up a dense array of pointlike synchronized clocks. Each clock marks the time coordinate of the event while space coordinates are given by the position of the clock and are all sharply measurable since position operators commutes with each other. For the Heisenberg principle, if each clock has finite mass, the observer should still worry about uncertainties in time evolution of the reference frame, since it is not possible to determine both position and velocity of each clock sharply, unless she uses clocks with infinite mass. By this it is really meant that it is possible to adopt a limiting procedure in which heavier and heavier clocks are used, so that, using a set of clocks with an appropriate mass, it is possible to construct a reference frame that is "classical enough" (*i.e.* the uncertainties in the time evolution of the position of each clock can be neglected) for any given sensibility of the experimental apparatus. Since QM ignores gravitational effects, this limiting procedure is legitimate and logically consistent within the theory. The same reasoning can be applied in the context of QFT, with the only difference that even if spacetime coordinates of events are sharply measurable, a particle with finite mass is just approximately localized in a region of radius equal to the particle's Compton wavelength, $\delta x \sim \hbar/cm$. If one tries to localize the position of the particle better than this by using probes with wavelength shorter than particle's Compton wavelength, other particles are produced in the measurement procedure and so this position measurement is actually meaningless.

Of course, when gravitational effects are taken into account the observer cannot use this construction of reference frames by infinitely massive clocks, since it can be shown that when a clock with mass $m \sim E_P/c^2$ is considered, then a probe cannot get closer to the clock than the Schwarzschild radius $R \sim L_P$. These arguments for an intrinsic limit in the localization of an event lead to a general conviction of the quantum gravity community that the description of spacetime as a Riemannian manifold must be replaced by a "quantum geometry" of "fuzzy" points.

This thesis deals with two different questions about quantum spacetime, very popular in the QG community: "what is the dimension of spacetime at scales of the order of the Planck length?" and "what happens to our usual notion of locality in the quantum gravity regime?"

The many alternative approaches to the study of the quantum-gravity problem are based on formalizations and physical pictures that are significantly different, in most cases offering very few opportunities to compare predictions between one approach and another. As a result, there is strong interest for the few features which are found to arise in several alternative models. In fact, the interest in the discussion about the number of dimension of spacetime at the Planck scale originates from the results obtained in the last decade by many groups, showing the common mechanism of "dynamical dimensional reduction": the familiar four-dimensional classical picture of spacetime in the IR is replaced by a quantum picture with an effective number of spacetime dimensions smaller than four in the UV ("ultraviolet", *i.e.* for probes of wavelength comparable to the Planck length). These exciting recent developments face the challenge that the standard concept of dimension of a spacetime, the "Hausdorff dimension", is inapplicable to a quantum spacetime [14, 68], and therefore one must rely on some suitable new concept. This challenge has been handled so far mostly[4] by resorting to the notion of "spectral dimension", whose key ingredient is the (modified) d'Alembertian of the theory[5] and for classical flat spacetimes reproduces the Hausdorff dimension. It was in terms of the spectral dimension that dynamical dimensional reduction was described for several approaches to the quantum-gravity problem, including the approach based on Causal Dynamical Triangulations [53], the Asymptotic-Safety approach [54], Hořava-Lifshitz gravity [55], the Causal-Sets approach [57], Loop Quantum Gravity [58, 59], Spacetime Noncommutativity [60] and theories with Planck-scale curvature of momentum space [61, 62].

The fact that so much of the intuition about the quantum-gravity realm is being attached to analyses based on the spectral dimension, which it is here argued not to be a physical characterization of a theory, should be reason of concern. For such precious cases where a feature is found in many approaches to the quantum-gravity problem, and therefore might be a "true feature" of the quantum-gravity realm, one should ask for no less than a fully physical characterization. *The first original result presented in this thesis work consists in the definition of such more physical characterization of quantum spacetime dimension, the "thermal dimension".*

Concerning the second question posed to the quantum spacetime, the fate of locality is another topic widely discussed in the community, a consistent part of which believes our usual notion of absolute locality will be lost. Here

[4]Other candidates for the characterization of the dimension of a quantum spacetime have been proposed in Refs. [68, 69, 70, 71, 72].

[5]There are cases, such as in Causal Dynamical Triangulations, where the d'Alembertian of the theory is not known, but it is possible to calculate the spectral dimension with other techniques. It has been established [73] that in these cases it is then possible to reconstruct the d'Alembertian.

I focus on the particular theory of Relative Locality, in which the Planck scale enters as the characteristic scale of the curvature of *momentum* space; the non-trivial geometry of momentum space has its *spacetime* counterpart in a weakening of locality. It will be shown, as a clarifying example of the origin of the basic idea of relative locality, how in the extensively studied noncommutative k-Minkowski spacetime two events may be coincident or not depending on the distance of the observer from the events. In this framework there is no notion of absolute locality, different observers see different spacetimes, and the spacetime they observe are energy and momentum-dependent. Locality, a coincidence of events, becomes relative: coincidences of events are still objective for all local observers, but they are not in general manifest in the spacetime coordinates constructed by distant observers.

There have been concerns [107],[109] that this notion of locality might have pathological implications for what concerns causality and momentum conservation. *Some original results of this thesis show that no such pathologies actually arise.*

Outline of the thesis

The first part of the thesis presents the different quantum gravity models that will be considered throughout the thesis, including Relative Locality. The focus then goes to the first question, regarding the characterization of the dimensional reduction of spacetime via the thermal dimension. Afterwards, the causality and momentum conservation topics in Relative Locality will be discussed.

Chapter 1 presents the theories in which Lorentz invariance is either preserved (as in Asymptotic Safety) or deformed that are of interest in the thesis work. It starts with some known results obtained in the study of scenarios for spacetime quantization, reviewed with the scope of highlighting the connection between noncommutative quantum spacetime and relativistic theories of interacting particles with nonlinear momentum space. The latter is the class of theories in which a considerable part of the original results presented in this thesis have been obtained. Section 1.1 presents an example of quantum spacetime, k-Minkowski. This noncommutative spacetime is used as a "storyteller" in the first part of the thesis and will lead to the concepts which are useful in the following. It will be recognized as a model of Doubly Special Relativity (DSR), where Planck length is a fundamental length scale consistent with the Principle of Relativity. Examples of DSR theories come from a notable source such as $2 + 1$ gravity coupled to matter, as quickly discussed in Section 1.3. Section 1.4 reviews the basic notions of Asymptotic Safety.

Chapter 2 introduces the concepts in Relative Locality which are relevant for this thesis. Section 2.1 shows quickly how k-Minkowski non-commutative spacetime is an example of spacetime with relative locality. The presentation of Relative Locality continues independently on any pre-existing model in Section 2.2, and in Section 2.3 the model of Relative Locality used in the rest of the thesis is introduced.

Chapter 3 introduces some already known proposal for some QG theories in which Planck scale breaks Lorentz invariance such as Hořava-Lifshitz gravity and Magueijo-Smolin Rainbow gravity, here reviewed in Section 3.1 and 3.2 respectively.

Chapter 4 introduces the first original contribution of this thesis; after reviewing the properties of the spectral dimension and its application in quantum gravity in Section 4.1, it is observed in Section 4.2 that some thermodynamical properties of radiation gas (such as the equation of state parameter and the scaling of temperature with energy density) could be used to assign a *thermal dimension* to the quantum spacetime. The good properties of this notion of dimension will be shown and discussed against those of the spectral dimension. Section 4.3 shows some preliminary results obtained so far in trying to extend the notion of thermal dimension of quantum spacetime with relative locality.

Then in Chapter 5 another original contribution is presented, consisting in the application of the modified statistical mechanics, introduced in the previous chapter, to the study of primordial cosmological perturbation in a rainbow universe with running Newton constant. It begins computing the Friedmann and scalar perturbations equations for a Rainbow metric associated to a dispersion relation of the Hořava-Lifshitz type in Sections 5.1 and 5.2. Then, Sections 5.3 and 5.4 compute the spectral index for both vacuum and hydrodynamical fluctuations respectively, noticing that the condition for obtaining the observed spectral index and solving the horizon problem is that Newton constant decreases in the UV. This is consistent with some precedent results where quantum gravity is responsible for solving the horizon problem without appealing to inflation.

Chapter 6 contains the original results obtained in the context of Relative Locality, beginning with the analysis of the causal behavior of the theory. Specifically, in Subsection 6.1.1 it is shown the objectivity of cause-and-effect relations and in Subsection 6.1.2 that the theory does not admit causally violating processes (causally violating loops). Section 6.2 discuss those processes in which the law of momentum conservation is violated, proving that they are not allowed in Relative Locality. Finally, Section 6.3 also shows that the theory does not admit even non-causally-violating loops (it must be stressed that the theory, as treated here, is classical, so these loops are not of the kind met in Feynman diagrams in perturbative Quantum Field Theory).

Chapter 7 briefly summarizes the original results presented in this work.

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Chapter 1

Theories preserving relativity of inertial frames

In the introduction few arguments suggesting that short-scale structure of spacetime might be characterized by a minimum length L_P, setting a limit on the localization of events, have been discussed. Other robust arguments indicate a second possible role of this length scale as that of wavelength at which new physical effects occur, while standard physics describes particles of larger wavelength. The latter proposal is often codified in deformed mass-shell relations such as, for example, $E^2 = c^2p^2 + c^4m^2 \pm cL_PEp^2$. Because of FitzGerald-Lorentz contractions, L_P cannot be a fundamental special-relativistic invariant scale in neither of the two possible roles (minimum length and characteristic wavelength), since two boosted observers will not agree on the fact that the minimum length/characteristic wavelength is equal to L_P. But the Relativity Principle demands that physical laws should be the same in all inertial frames, including the laws that attribute to L_P a fundamental role in the structure of spacetime. In the mid-1990s studies advocating a role for the Planck length in spacetime structure often ended up introducing (more or less explicitly) a preferred family of inertial observers (usually identified with the natural observers of the cosmic microwave background radiation), therefore breaking Lorentz symmetry (see, *e.g.* Ref.[18]).

The alternative possibility of introducing the Planck length in spacetime structure in a fully relativistic manner was proposed in 2000 ([19], [20]) and is the Doubly Special Relativity framework. A DSR theory requires the invariance of the minimum length/characteristic wavelength denoted by[1] L_{DSR} in addition to the request of invariance of the speed-of-light scale.

[1]Here the characteristic length scale is indicated as L_{DSR} rather than L_p to indicate a possible extra factor that multiplies L_p.

Section 1.1 introduces an example of quantum spacetime in which Lorentz symmetry is preserved, although the transformations are modified with respect to those of subPlanckian-energy physics. This provides guidance for getting some intuition for formulating a theory in which the speed of light scale and a length scale are both fundamental relativistic invariants (DSR). This general proposal is presented in Section 1.2. Section 1.3 discusses the case of $2+1$ gravity as a notable example of this kind of theory. Section 1.4 reviews the very different paradigm of asymptotic safety, where it is supposed that Lorentz symmetry is not modified and still a symmetry of physics.

1.1 k-Minkowski noncommutative spacetime

One of the most appealing realizations of the DSR concept is that of a Hopf-algebra scenario with k-Poincaré structure and the related k-Minkowski noncommutative spacetime. Noncommutative spacetimes are toy models where one tries to characterize the limitation in the localization of an event promoting spacetime coordinates to noncommuting operators. The physical regime considered might be that of a freely propagating particle whose energy is high enough to probe the quantum structure of spacetime, but its influence on the macroscopic scale structure of spacetime is still negligible. Therefore, the only contribution of gravity in determining the non-trivial structure of spacetime comes from this noncommutative character of the coordinates.

The characteristic spacetime-coordinate noncommutativity of k-Minkowski is given by

$$[\hat{x}_j, \hat{x}_0] = i\ell\hat{x}_j \tag{1.1}$$

$$[\hat{x}_j, \hat{x}_k] = 0 \tag{1.2}$$

where $\hat{x}_0$ is the time coordinate, $\hat{x}_j$ is the space coordinate ($j,\, k \in \{1,\, 2,\, 3\}$) and ℓ is a length scale. Functions of these noncommuting coordinates admit a "Fourier transform"

$$f(\hat{x}) = \int d^4k\, \tilde{f}(k) e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} \tag{1.3}$$

where the "Fourier parameters" k_0, k_i are ordinary commutative variables. It is therefore possible to characterize the action of transformations generators on the functions of noncommutative variables by studying their action directly on the basis exponentials $e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0}$.

A frequently used characterization of symmetry of k-Minkowski introduces the following definitions of generators of translations, space-rotations and boosts:

$$P_\mu \triangleright e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} = k_\mu e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0}, \tag{1.4}$$

$$M_j \triangleright e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} = \epsilon_{jkl} x_k k_l e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0}, \tag{1.5}$$

$$\begin{aligned} N_j \triangleright e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} = & - k_j e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} \hat{x}_0 + \\ & + \left[\hat{x}_j \left(\frac{1 - e^{2\ell k_0}}{2\ell} + \frac{\ell}{2} |\vec{k}|^2 \right) + \ell \hat{x}_l k_l k_j \right] e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0}. \end{aligned} \tag{1.6}$$

The fact that one here deals with a (k-Poincaré) Hopf algebra is essentially seen by acting with these generators on products of functions, observing, for example, that, from the k-Minkowski commutators (1.1),(1.2) and the Baker-Campbell-Hausdorff formula, one has

$$e^{-ik_j\hat{x}_j} e^{ik_0\hat{x}_0} e^{-iq_j\hat{x}_j} e^{iq_0\hat{x}_0} = e^{-ik_j\hat{x}_j} e^{-ie^{\ell k_0} q_j \hat{x}_j} e^{ik_0\hat{x}_0} e^{iq_0\hat{x}_0} = e^{-i(k_j + e^{\ell k_0} q_j)\hat{x}_j} e^{i(k_0+q_0)\hat{x}_0}. \tag{1.7}$$

Then the action of the translation generators is

$$P_\mu \triangleright e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} e^{-i\vec{q}\cdot\vec{\hat{x}}} e^{iq_0\hat{x}_0} = \left(k_\mu + e^{\ell k_0 \left(1 - \delta^0_\mu\right)} q_\mu \right) e^{-i\vec{k}\cdot\vec{\hat{x}}} e^{ik_0\hat{x}_0} e^{-i\vec{q}\cdot\vec{\hat{x}}} e^{iq_0\hat{x}_0}. \tag{1.8}$$

For a pair of functions $f(\hat{x})$ and $g(\hat{x})$ one finds

$$P_\mu \triangleright (f(\hat{x}) g(\hat{x})) = (P_\mu \triangleright f(\hat{x}))\, g(\hat{x}) + \left(e^{\ell P_0 \left(1 - \delta^0_\mu\right)} \triangleright f(\hat{x}) \right) (P_\mu \triangleright g(\hat{x})) \tag{1.9}$$

i.e. one finds a "non primitive coproduct[2]" $\Delta P_\mu = P_\mu \otimes 1 + e^{\ell P_0 \delta^1_\mu} \otimes P_\mu$, different from the "primitive coproduct" $\Delta P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu$ typical of ordinary differential operators. The coproduct has an important role in determining the form of generators reported above. Those generators in fact can be obtained assuming the standard action of translation and rotation generators (1.4), (1.5) and realizing then that using the undeformed boost does not allow getting the 10 generators closed Hopf algebra (the coproducts of undeformed boosts introduce an undesired generator of dilatation transformations) that would correspond to the Poincaré algebra of Minkowski spacetime symmetries. The deformed boosts action (1.6) is then obtained considering the most general deformation of boosts generators with the right classical limit admitted by the other symmetries, and requiring that together

[2] Given an algebra A, the coproduct is a linear map $\Delta : A \to A \otimes A$ that is "coassociative", that is $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.

with translation and rotation generators form a 10 generators closed Hopf algebra.

The commutators between the generators (1.4), (1.5), (1.6) are

$$[M_{\mu\nu},\, M_{\rho\tau}] = i\left(\eta_{\mu\tau}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\tau} + \eta_{\nu\rho}M_{\mu\tau} - \eta_{\nu\tau}M_{\mu\rho}\right),$$

$$[M_i,\, P_j] = i\epsilon_{ijk}P_k, \qquad [M_i,\, P_0] = 0,$$

$$[N_i,\, P_j] = i\delta_{ij}\left(\frac{1}{2\ell}\left(1 - e^{2\ell P_0}\right) + \frac{\ell}{2}P_iP_i\right) - i\ell P_iP_j,$$

$$[N_i,\, P_0] = iP_i,$$

$$[P_\mu,\, P_\nu] = 0,$$

where $P_\mu = (P_0,\, P_i)$ are the time and space components of the translations generators and $M_{\mu\nu}$ are modified Lorentz generators with rotations $M_k = \frac{1}{2}\epsilon_{ijk}M_{ij}$ and boosts $N_i = M_{0i}$.

With the coproducts (1.9) the commutators (1.1) and (1.2) are left invariant under the action of the generators in the sense that for translations, for example, one has[3]

$$\begin{aligned} &P_\mu \triangleright [\hat{x}_j,\, \hat{x}_0] = i\ell P_\mu \triangleright \hat{x}_j, \\ &P_\mu \triangleright [\hat{x}_j,\, \hat{x}_k] = 0. \end{aligned} \tag{1.10}$$

One also finds a deformed mass Casimir for this algebra, obtained from the generators given above

$$\mathcal{C}_\ell = \left(\frac{2}{\ell}\right)^2 \sinh^2\left(\frac{\ell}{2}P_0\right) - e^{-\ell P_0}P_iP_i. \tag{1.11}$$

The idea that this mathematics provides a possible basis for a DSR theory originates from the left-invariance of the k-Minkowski commutators under

[3]The interested reader can verified this commutator invariance straightforwardly by expressing $\hat{x}_\mu = \left(-i\frac{\partial}{\partial k^\mu}e^{ik^j\hat{x}_j}e^{ik^0\hat{x}_0}\right)_{|k=0}$, acting on the basis exponentials with the generators P_μ and then take $k^\nu = 0$. Indices are raised and lowered with Minkowski metric tensor $\eta_{\mu\nu} = (1, -1, -1, -1)$.

the action of the k-Poincaré generators as in Eq.(1.10) and the consequent identification of ℓ with L_{DSR}. Furthermore, the Casimir (1.11) can inspire a deformed on-shell relation for relativistic particles. For a low energy particle, at first order in ℓ, this takes the form

$$m^2 = P_0^2 - P_i P_i + \ell P_0 P_i P_i. \tag{1.12}$$

The generators are not the only nontrivial structure needed to implement symmetry transformations in k-Minkowski. Considering the case of translations, one of course wants that noncommuting variables $\hat{x}'_\mu$ used by a translated observer are obtainable from the old ones by a rule of the type $\hat{x}'_\mu = \hat{x}_\mu - \hat{a}_\mu$ and that these also satisfy the k-Minkowski commutators (1.1),(1.2). It is clear that the translation parameters $\hat{a}_\mu$ can not be commutative variables but must have noncommutative properties themselves, in particular one can adopt the following prescriptions

$$[\hat{a}_j, \hat{x}_0] = i\ell\hat{a}_j, \qquad [\hat{a}_\mu, \hat{x}_j] = 0, \qquad [\hat{a}_0, \hat{x}_0] = 0. \tag{1.13}$$

In this way the translation operator takes the familiar form

$$T = 1 + \mathbf{d}, \qquad \mathbf{d} = i\hat{a}_\mu P^\mu. \tag{1.14}$$

where $P^\mu = \eta^{\mu\nu} P_\nu$, $\eta^{\mu\nu}$ is the inverse of the Minkowski metric tensor.

The choice of the basis exponentials is arbitrary. For example, one could choose the basis $e^{ik^\mu \hat{x}_\mu}$ or $e^{ik^0\hat{x}_0} e^{ik^j \hat{x}_j}$. These different choices yield different form of the transformations generators, depending on the particular order one writes the basis exponentials. Consider for simplicity the translation generators. Denoting the translation generators used until now $P_{R\mu}$ (because the basis exponential with the time coordinate is to the right of that with spatial coordinates), one could define other translation generators by setting $P_{L\mu} \triangleright e^{ik_0\hat{x}_0} e^{-i\vec{k}\cdot\vec{\hat{x}}} = k_\mu e^{ik_0\hat{x}_0} e^{-i\vec{k}\cdot\vec{\hat{x}}}$. Then it is straightforward to verify that $P_{R\mu} e^{ik_0\hat{x}_0} e^{-i\vec{k}\cdot\vec{\hat{x}}} \neq P_{L\mu} e^{ik_0\hat{x}_0} e^{-i\vec{k}\cdot\vec{\hat{x}}}$, which implies $P_{R\mu} \neq P_{L\mu}$. However, this abundance of possible translation generators is not really a problem, since to each choice of ordering of the basis exponentials correspond also different translation parameters $\hat{a}_\mu$. Therefore, focusing on the two choices of time-to-the-right and time-to-the-left basis exponentials, one finds also that $\hat{a}_{R\mu} \neq \hat{a}_{L\mu}$, where $\hat{a}_{R\mu}$ denote the translation parameters related to the time-to-the-right basis whereas $\hat{a}_{L\mu}$ denote the translation parameters related to the time-to-the-left basis. It turns out that the translation operator T, defined in Eq.(1.14), is order-independent, *i.e.* its action on a function of noncommuting variables does not depend on the *arbitrary* choice of ordering of the basis exponentials when Fourier transforming (see Ref.[21] for more details).

It is also important highlighting that the possibility of removing all anomalies of the commutators by nonlinear redefinitions of the generators does not imply that one "recovers" Special Relativity. In fact, a proper description of Hopf algebra symmetries must take into account both commutators and coproducts of the generators; concurrently, a redefinition of the generators necessarily modifies also the coproducts in such a way that the physical differences between k-Minkowski and Special Relativity remain. Moreover, by using the whole machinery of commutators and coproducts it is possible ([21], [22]) to obtain conserved charges associated to the Hopf symmetries for a theory with classical fields in the noncommutative k-Minkowski spacetime, whereas other attempts to obtain conserved charged, ignoring the role of coproducts, had failed.

In k-Minkowski the description of translations necessarily requires some new structure, as it can be most elementarily seen by looking at the composition law of basis exponentials and the action of the translation generators on this product of functions, *i.e.* its coproduct (1.8). Cleary the spacetime noncommutativity is leading to a new composition of energy and momentum $(p, E) \oplus (q, \omega) = (p + e^{\ell E} q, E + \omega)$, which involves a clear non-linearity. This non-linear composition law of momenta might be seen as suggesting a non-linear geometry of momentum space. Indeed, it has been shown in Refs. [23], [24], [25], [26] that k-Poincaré Hopf algebra describes a curved momentum space with de Sitter metric, torsion and nonmetricity (the usual geometry of momentum space is recovered by letting $\ell \to 0$, so that ℓ (or L_{DSR}) might be seen as a deformation parameter). This geometry, in the appropriate regime in which relative locality is studied today, will be the basis for the explicit example of relative locality presented in Section 2.3.

1.2 The Doubly Special Relativity proposal

Besides k-Minkowski noncommutative spacetime there are many other DSR theories. It is therefore useful to describe here the general principles of the DSR proposal, independently on their specific formalization. A good starting point for introducing DSR is the analysis of the step from Galilean Relativity to Special Relativity as a solution to the problem of attributing to c the role of speed of light, a universal constant that is the same for every observer. From this perspective, one could regard Galilean Relativity as a theory based on the Relativity Principle and the assumption that there would be no fundamental scales of length or velocity.

The Relativity Principle introduced by Galilei can be stated as follows:

(R.P.) : *The laws of physics take the same form in all inertial frames (i.e. these laws are the same for all inertial observers).*

This principle has strong implications on geometry and kinematics when combined with the assumption of existence of fundamental scales. In fact, the hypothesis that there is some fundamental scale is to be regarded as a physical law itself. The Relativity Principle then implies that the relations between the measurements performed by different inertial observers must be such that every inertial observer agree with the value and the physical interpretation of this scale. Combining the Relativity Principle with the assumption that there are not absolute scales one can obtain the Galilean rules of transformation between observers. For example, if v is the velocity of a body with respect to an inertial observer, and a second observer moves with constant velocity v_0 with respect to the first observer, the velocity of the body with respect to the second observer, in absence of a fundamental velocity scale, can be only of the form $v' = f(v, v_0)$. Considering other reasonable assumptions ($f(v, 0) = v$, $f(0, v_0) = v_0$, $f(v, v_0) = f(v_0, v)$, $f(-v, -v_0) = -f(v, v_0)$), the well-known Galilean formula of composition of velocities $v' = v + v_0$ follows.

The step made by Einstein was introducing a fundamental velocity scale consistently with the Relativity Principle. To do so, it must be specified a unique experimental procedure that allows every inertial observer to measure the value of this fundamental scale. These two postulates might be summarized as follows:

(E.L.1) : *The laws of physics involve a fundamental scale of velocity c.*

(E.L.1b) : *The value of the fundamental velocity scale c can be measured by each inertial observer as the speed of light.*

One could have expected a more precise description of the measurement procedure to adopt in order to establish the value of c; for example, one could have expected the speed of light to depend on the velocity of the source or on the wavelength of the light. However, it is important to realize the role that the Relativity Principle and the postulate (E.L.1) have in determining the form of (E.L.1b): the specification of a wavelength dependence would have required a reference fundamental scale of length, whereas a dependence of the speed of light on the velocity of the source would be in conflict with the fundamental nature of c as a scale on which, according to the Relativity Principle, all inertial observers agree.

From the Relativity Principle, (E.L.1) and (E.L.1b) one can obtain the rules that relate the observations performed by different inertial observers, which are the Lorentz transformations. Famously, the transition from Galilean Relativity to Special Relativity requires the replacement of the simple formula of Galilean composition of velocities with a much richer special relativistic version

$$\vec{v}_1 \oplus \vec{v}_2 = \frac{1}{1+\frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}} \left(\vec{v}_1 + \frac{1}{\gamma_1}\vec{v}_2 + \frac{1}{c^2}\frac{\gamma_1}{1+\gamma_1}\left(\vec{v}_1 \cdot \vec{v}_2\right)\vec{v}_1 \right). \tag{1.15}$$

Furthermore, the introduction of c requires to abandon the concept of absolute simultaneity, which would contrast with the fact that the exchange of information between two clocks in relative motion is strongly constrained by (E.L.1) and (E.L.1b).

It is natural then, in order to introduce Planck length in a relativistic theory, to modify (E.L.1) and (E.L.1b) allowing for a fundamental length scale. (E.L.1) simply becomes:

(L.1) : *The laws of physics involve a fundamental length scale L_{DSR} and a fundamental velocity scale c.*

The new relativistic theory is defined once one gives the experimental procedures to measure c and L_{DSR} that substitute (E.L.1b). The introduction of L_{DSR} makes possible a wavelength dependence of the value of c; however, it is still possible that no such dependence occurs. Since experiments dealt only with wavelength much larger than L_{DSR}, one shall be cautious and modify (E.L.1b) as follows:

(L.1b) : *The value of the fundamental velocity scale c can be measured by each inertial observer as the speed of light with wavelength λ much larger than L_{DSR} (more rigorously, c is obtained as the $\lambda/L_{DSR} \to \infty$ limit of speed of light).*

The procedure (L.1c) by which every inertial observer can measure the value of L_{DSR} should be determined by experimental data. As already said, there are many theoretical arguments suggesting a role for the Planck length in the small-distance structure of spacetime. An example of a possible form for (L.1c) is

(L.1c*) : *Each inertial observer can establish the value of L_{DSR}, which is the same for all inertial observers, by determining the dispersion relation for photons. This takes the form $E^2 = c^2p^2 - f(E, p; L_{DSR})$, where the* *function f is the same for all inertial observers and in particular all inertial observers agree on the leading L_{DSR} dependence, which might be, for example, $f: f(E,p;L_{DSR}) \simeq L_{DSR}cp^2E$.*

The objective that motivates DSR research is that of coherently constructing a relativistic theory with *two fundamental scales, c and L_{DSR}, which are non-trivial relativistic invariants*. An example of what one refers to as trivial relativistic invariant is the rest mass of the electron. Another example of a trivial relativistic invariant is the Quantum Mechanics scale $\hbar$ that, as c does, establishes properties of the results of the measurements of certain observables; $\hbar$, for example, sets the minimum non zero value of angular momentum. But the discretization of angular momentum and the limitation in the measurement of its components does not affect spacetime symmetry under classical space-rotations, as shown in Ref.[27], since the measurements that QM allows are still subject to the same rules imposed by classical rotation symmetry. The reason is that $\hbar$ is not a scale pertaining to the spacetime structure of the rotation transformations, and in fact the introduction of $\hbar$ does not require any modification of the action of the rotation transformations. Galilei's boosts are necessarily deformed once c is introduced as a fundamental relativistic invariant and c itself has a role in the transformations that relate the measurements of two inertial observers in relative motion. In a DSR theory L_{DSR} must have a similar role to that of c in Special Relativity, *i.e.* it must participate in the transformations that relates the observations of two inertial observers.

Note that DSR is a very specific alternative to Special Relativity: only a certain class of deformations of Special Relativity is DSR compatible. For example, de Sitter Relativity is a deformation of Special Relativity by the scale of curvature. But de Sitter spacetime is a deformation of Minkowski spacetime by a long-distance scale (one can obtain Minkowski spacetime from de Sitter spacetime as the deformation length is sent to infinity), whereas one of the requirements for a DSR theory is that the deformation scale must be a short-distance scale (one should obtain Minkowski spacetime by sending to zero the deformation scale).

1.3 Aside on $2+1$ gravity

It is important to mention that it has been observed ([28],[29],[30],[31],[32]) that classical gravity for point particles in $2+1$ dimensions offers an example of DSR theory.

Of particular interest for the path followed in this thesis is the connection between the geometry of momentum space and spacetime noncommutativity.

In fact, in classical 2+1 gravity without cosmological constant the momentum space has anti-de Sitter geometry or, more precisely, it is the Lie group $SL(2,\mathbb{R})$, the group of linear transformations acting on $\mathbb{R}^2$ with determinant equal to one.

This follows from the fact that Einstein gravity in $2+1$ dimensions does not possess local degrees of freedom and a point particle is introduced as a topological defect surrounded by flat spacetime. For the case of a spinless particle of mass m one obtains the metric $ds^2 = -d\tau^2 + dr^2 + (1-4Gm)r^2 d\phi^2$, which describes a conical spacetime, the particle being located at the tip of the cone, $r = 0$. It is possible to show that vectors parallel transported along closed loops around the origin turns to be rotated by an angle $\alpha = 8\pi Gm$. This because the curvature vanishes everywhere except at the singularity $r = 0$. As in ordinary $2+1$ Minkowski spacetime one can characterize the physical momentum of the particle, once its mass is given, by specifying two additional parameters that describe the linear momentum and that are in one-to-one correspondence with boosts. Alternatively one can take three-momentum of the particle at rest (specified by its rest mass) and boost it to the appropriate value of the linear momentum. In this case three-momentum at rest is given by a vector in $2+1$ Minkowski space. This space is isomorphic to the Lorentz algebra $sl(2,\mathbb{R})$ as a vector space. In fact, when the particle is described by a conical defect, its mass (the three-momentum at rest) is determined by a rotation by the angle $\alpha = 8\pi Gm$, *i.e.* by $\exp(\alpha J_0) = g_0 \in SL(2,\mathbb{R})$, where J_0 is the generator of rotations. The physical momentum can be obtained by boosting the three-momentum at rest by conjugating g_0 by a Lorentz boost $L \in SL(2,\mathbb{R})$, that is $g = L^{-1} g_0 L$. Thus the kinematics of a massive particle is in this context determined by the set of rotation-like Lorentz transformations. The extended momentum space is given by the group manifold $SL(2,\mathbb{R})$.

In order to expose the anti-de Sitter geometry of momentum space, it is convenient to write the generic element $\mathbf{p}$ of $SL(2,\mathbb{R})$ as a combination of the identity matrix and of the elements of a basis of $sl(2,\mathbb{R})$, *i.e.* the Lie algebra of $SL(2,\mathbb{R})$[4]:

$$\mathbf{p} = u\mathbb{I} - 2\xi_\mu X^\mu. \tag{1.16}$$

Here $\mathbb{I}$ is the identity 2×2 matrix and the X^μ are

$$X^0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X^1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad X^2 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

[4]Throughout this section indices will be raised and lowered using the metric $\eta_{\mu\nu} = (-1, 1, 1)$.

which constitute a basis of $sl(2,\mathbb{R})$, and the requirement of having determinant equal to one ($\det \mathbf{p} = 1$) implies that the parameters u, ξ_μ must satisfy the constraint

$$u^2 - \xi^\mu \xi_\mu = 1. \tag{1.17}$$

This constraint provides, as announced, the definition of a 3 dimensional anti-de Sitter geometry.

Among the choices of coordinates for this momentum space geometry used in the 3D-gravity literature, particularly convenient for the purpose of this section is the choice of coordinates p^μ such that

$$\mathbf{p} = \sqrt{1 + \ell^2 p_\mu p^\mu}\mathbb{I} - 2\ell p_\mu X^\mu, \tag{1.18}$$

since it is then easy to obtain the (non-linear) composition law of momenta using the algebraic properties of X^μ matrices. Multiplying two elements

$$\mathbf{p} = \sqrt{1 + \ell^2 p_\mu p^\mu}\mathbb{I} - 2\ell p_\mu X^\mu,$$
$$\mathbf{q} = \sqrt{1 + \ell^2 q_\mu q^\mu}\mathbb{I} - 2\ell q_\mu X^\mu,$$

and using the identity

$$X^\mu X^\nu = \frac{1}{4}\eta^{\mu\nu}\mathbb{I} + \frac{1}{2}\epsilon^{\mu\nu}{}_\rho X^\rho, \tag{1.19}$$

where for the antisymmetric tensor $\epsilon_{\mu\nu\rho}$ the convection adopted is $\epsilon_{012} = -1$, one obtains a simple but non linear relation between the coordinates $(p \oplus q)_\mu$ of $\mathbf{pq}$ and the coordinates p_μ and q_μ of $\mathbf{p}$,$\mathbf{q}$ respectively:

$$(p \oplus q)_\mu = \sqrt{1 + \ell^2 q^\nu q_\nu} p_\mu + \sqrt{1 + \ell^2 p^\nu p_\nu} q_\mu - \ell \epsilon_\mu{}^{\nu\rho} p_\nu q_\rho. \tag{1.20}$$

Finally, the identity (1.19) implies that X^μ satisfy by construction (up to a dimensional constant) the commutation relations

$$[X^\mu, X^\nu] = \epsilon^{\mu\nu}{}_\rho X^\rho. \tag{1.21}$$

When one proceeds to the quantization of this theory (see for example Ref.[33]), the commutation rules (1.21) of the basis X^μ of $sl(2,\mathbb{R})$ contribute in the determination of the symplectic structure of the theory and one ends up with the same geometry for momentum space as in the classical theory and a noncommutative spacetime whose coordinates obey the commutation relations

$$[x^\mu, x^\nu] = i\hbar\ell\epsilon^{\mu\nu}{}_\rho x^\rho. \tag{1.22}$$

The DSR-relativistic symmetries of the emerging framework are already evident in the classical limit of the construction just described. In fact, the classical limit is characterized by spacetime coordinates with Poisson brackets given by

$$\{x^\mu, x^\nu\} = \ell\epsilon^{\mu\nu}{}_\rho x^\rho, \tag{1.23}$$

and by a momentum space with coordinates p_μ constrained on a mass shell governed by

$$\ell^{-2}\left(\arcsin\left(\sqrt{-\ell^2 p^\mu p_\mu}\right)\right)^2 = m^2, \tag{1.24}$$

and with law of composition

$$(p \oplus q)_\mu = \sqrt{1+\ell^2 q^\nu q_\nu}\, p_\mu + \sqrt{1+\ell^2 p^\nu p_\nu}\, q_\mu - \ell\epsilon_\mu{}^{\nu\rho} p_\nu q_\rho. \tag{1.25}$$

The relevant DSR-deformed relativistic symmetries are particularly simple since the action of Lorentz-sector generators on momenta remains undeformed. Indeed by posing

$$\{N_1, p_0\} = p_1, \qquad \{N_2, p_0\} = p_2, \qquad \{R, p_0\} = 0, \tag{1.26}$$

$$\{N_1, p_1\} = p_0, \qquad \{N_2, p_1\} = 0, \qquad \{R, p_1\} = -p_2, \tag{1.27}$$

$$\{N_1, p_2\} = 0, \qquad \{N_2, p_2\} = p_0, \qquad \{R, p_2\} = p_1, \tag{1.28}$$

one finds that the mass-shell (1.24) is invariant and the composition law (1.25) is covariant. So one here is dealing with a DSR-relativistic framework where the core aspect of the deformation is the action of translation transformation on multiparticles states. This was so far only left implicit by noticing that the momentum charges must be composed following the nonlinear law (1.25). Notice that this implies a deformed action of translation transformations on multiparticles states. Consider for example a system composed of only two particles, respectively with phase-space coordinates p_μ, x^μ and q_μ, y^μ: then a translation parametrized by b^ρ, and generated by the total-momentum charge $(p \oplus q)_\rho$, acts for example on the particle with phase-space coordinates p_μ, x^μ as follows

$$b^\rho\{(p \oplus q)_\rho, x^\nu\} \simeq b^\rho\{p_\rho, x^\nu\} - \ell b^\rho \epsilon_\rho{}^{\sigma\gamma} q_\gamma \{p_\sigma, x^\nu\} \tag{1.29}$$

where on the right-hand side it is shown only the leading-order Planck-scale modification.

Concerning translations acting on single-particle momenta one can notice that since the spacetime coordinates are such that $\{x^\mu, x^\nu\} = \ell\epsilon^{\mu\nu}{}_\rho x^\rho$, one could not possibly adopt the standard $\{p_\mu, x^\nu\} = -\delta^\nu_\mu$ since then the Jacobi identities would not be satisfied. Jacobi identities are satisfied if one adopts the description of translations acting on single-particle momenta given by

$$\{p_\mu, x^\nu\} = -\delta^\nu_\mu \sqrt{1 + \frac{\ell^2}{4} p^\rho p_\rho} + \frac{\ell}{2} \epsilon_\mu{}^{\nu\rho} p_\rho. \tag{1.30}$$

Another example is that treated in Ref.[28], where it was argued that quantum gravity in $2+1$ dimensions with vanishing cosmological constant must be invariant under some version of a k-Poincaré symmetry.

The argument there depends only on the assumption that quantum gravity in 2+1 dimensions with the cosmological constant $\Lambda = 0$ must be derivable from the $\Lambda \to 0$ limit of $2+1$ quantum gravity with non-zero cosmological constant; in fact, in many approaches it is necessary to include a bare cosmological constant in order to do perturbative calculations properly. Then, it is shown that the symmetry which characterizes transformations of excitations of the ground states of a quantum gravity theory in $2+1$ dimensions with $\Lambda > 0$ is actually quantum deformed de Sitter algebra $SO_q(3,1)$, with the quantum deformation parameter given by

$$z = \ln(q) \approx L_P \sqrt{\Lambda}.$$

The limit $\Lambda \to 0$ then involves the simultaneous limit $z \approx L_P\sqrt{\Lambda} \to 0$, and it is possible to see that this contraction of $SO_q(3,1)$ is not the classical Poincaré algebra, as would be the case if $q = 1$ throughout, but it is a modified Poincaré algebra with the dimensional parameter $k \approx L_P^{-1}$. Since some of these algebras provide a basis for DSR theory, it means that the theory is a DSR theory, and indeed all the features of DSR (relativity of inertial frames, non-linear action of boosts that preserve a preferred energy scale, non-linear modifications of energy-momentum relations...) has been seen in the literature of $2+1$ gravity.

The study of 2+1 gravity models, such as those with gravity coupled to N point particles, gives a class of non-trivial DSR theories that are completely explicit and solvable, both classically and quantum mechanically. The existence of these well-understood examples in the $2+1$ gravity context is a powerful tool for the conceptual analysis of DSR theories.

The debate on DSR often concerns whether these relativistic deformations should at all be considered in relation to the quantum gravity problem, and the fact that they necessarily arise in the $2+1$ quantum gravity context provides a strong element of support for the legitimacy of the study of DSR-deformed relativistic symmetries.

1.4 Asymptotic Safety

A QFT is said to be an "effective field theory" (EFT) if it breaks down at some energy scale, and "fundamental" or "UV complete" if it makes sense up to arbitrarily high energy scales. QCD is an example of the latter case. Before introducing the basic ideas of Asymptotic Safety, the reason for which Einstein theory of gravity is instead regarded as an EFT is here reviewed, in particular why it is not perturbatively renormalizable. Asymptotic safety, in fact, proposes a strategy to overcome this problem.

1.4.1 Non-renormalizability of General Relativity

The reason for which General Relativity is not perturbatively renormalizable, in the scheme of standard quantum field theory, can be understood by dimensional analyzing the degree of divergence of one-particle irreducible Feynman diagrams. The propagator of a field is the 4-dimensional Fourier transform of the vacuum expectation value of a time-ordered product of a pair of free fields, so a field ϕ with momentum dimensionality $\mathcal{D}_\phi$ has a propagator with dimensionality $d_{prop\,\phi} = -4 + 2\mathcal{D}_\phi$. An interaction term with $n_{\phi\,i}$ such fields and n_{der} derivatives has dimensionality $n_{der} + n_{\phi\,i}\mathcal{D}_\phi$. If different fields interact, this generalizes to $n_{der} + \sum_\phi n_{\phi\,i}\mathcal{D}_\phi$. Since the action must be dimensionless in our $\hbar = 1$ units, each term in the Lagrangian must be 4-dimensional to cancel the dimensionality -4 of the differential term d^4x. Hence the interaction must have a coupling constant g with dimension $d_g = 4 - n_{der} - \sum_\phi n_i\mathcal{D}_\phi$. If the Feynman diagram has $n_{ext\,\phi}$ external lines for a particular field ϕ, the amplitude in the momentum representation has dimension $\sum_\phi -4n_{ext\,\phi} + n_{ext\,\phi}\mathcal{D}_\phi$. Of this dimensionality -4 come from the momentum delta function and $n_{ext\,\phi}d_{prop\,\phi}$ come from the propagators of the external lines; the coupling constants for a given Feynman diagram with N_i vertices have total dimensionality $N_i d_g$, leaving the momentum space integral with dimensionality

$$\sum_\phi(-4n_{ext\,\phi}+n_{ext\,\phi}\mathcal{D}_\phi)-(-4)-\sum_\phi(n_{ext\,\phi}d_{prop\,\phi})-N_id_g = 4-\sum_\phi n_{ext\,\phi}\mathcal{D}_\phi-N_id_g.$$

In estimating the degree of divergence D of a diagram the interest goes mostly in the region of momentum space where all momenta go to infinity together. Then the degree of divergence coincides with the dimensionality of the diagram,

$$D = 4 - \sum_\phi n_{ext\,\phi}\mathcal{D}_\phi - N_i d_g. \tag{1.31}$$

If all interactions have $d_g > 0$, then Eq.(1.31) sets an upper limit on D that depends only on the number of external lines; that is, on the physical process in consideration,

$$D \leq 4 - \sum_{\phi} n_{ext\,\phi} \mathcal{D}_{\phi}. \tag{1.32}$$

This implies that only a finite number of external lines can yield superficially divergent integrals. In general one can show that a limited number of divergences appears in case $d_g \geq 0$ for all interactions and these are removed by redefinition of a finite number of physical constants and a renormalization of the fields.

On the other hand, if one has $d_g < 0$ the degree of divergence becomes larger and larger as more vertices are included. No matter how many external lines are added, eventually there will be enough vertices to make the integral divergent. This is the case of gravity, where Newton constant has dimension $[G_N] = -2$. The Feynman rules also involve the graviton propagator, which scales with the four momentum k_μ schematically as $k^{-2} = \frac{1}{E^2 - p^2}$. At increasing loop orders, the Feynman diagrams of the theory would require counterterms of ever-increasing degree in curvature. The resulting theory can still be treated as an effective quantum field theory, but it would still require a UV completion.

1.4.2 Asymptotically safe gravity

Asymptotic Safety gives an alternative notion of renormalizability ensuring UV completeness that may lead to a consistent theory of quantum gravity.

Let $g_i(\mu)$ denote the full set of all renormalized coupling parameters of a theory, defined at a renormalization point with momenta characterized by an energy scale μ. If $g_i(\mu)$ has momentum dimension of d_{g_i}, it can be replaced with a dimensionless coupling,

$$\tilde{g}_i(\mu) = \mu^{-d_{g_i}} g_i(\mu). \tag{1.33}$$

Any sort of partial or total reaction rate R may be written in the form

$$R = \mu^D f\left(\frac{E}{\mu}, X, \tilde{g}_i(\mu)\right) \tag{1.34}$$

where D is the ordinary dimensionality of R (e.g., for total cross section $D = -2$), E is some energy characterizing the process and X stands for all other dimensionless physical variables, including the ratios of energies. The central idea of the renormalization group methods is to recognize that the reaction rate cannot depend on the arbitrary choice of the renormalization point μ at which couplings are defined, so μ can be taken to be whatever is preferable, as in particular $\mu = E$, in which case one has,

$$R = E^D f\left(1, X, \tilde{g}_i(E)\right). \tag{1.35}$$

Thus, apart the factor E^D, the behavior of the reaction rates depends on the behavior of the couplings $\tilde{g}_i(\mu)$ as $\mu \to \infty$.

The emphasis here on reaction rates rather then off-shell Green's functions has a very important advantage. Mass-shell matrix elements and reaction rates do not depend on how the field are defined, so they are functions only of "essential" coupling parameters, i.e. those combinations of the coupling parameters in the Lagrangian that do not change when the field is subjected to a point transformation, such as $\phi \to \phi + \phi^2$ for a scalar field ϕ. In contrast, the off-shell Green's functions will of course reflect the definition of the fields involved and will therefore be functions of all the coupling parameters in the Lagrangian, including those inessential parameters that change under a redefinition of the fields. In the following, $\tilde{g}_i(\mu)$ are only the essential coupling parameters of the theory.

In order to clarify how to distinguish an essential parameter by an inessential parameter one can apply the following test. When one changes any unrenormalized coupling parameter γ by an infinitesimal amount ϵ the whole Lagrangian changes by

$$L \to L + \epsilon \frac{\partial L}{\partial \gamma}. \tag{1.36}$$

Suppose one tries to reproduce this change by a mere redefinition of the fields

$$\psi_n(x) \to \psi_n(x) + \epsilon F_n(\psi_n(x), \partial_\mu \psi_n(x), ...). \tag{1.37}$$

The change in L induced thereby is

$$\begin{aligned} \delta L &= \epsilon \sum_n \left(\frac{\partial L}{\partial \psi_n(x)} F_n + \left(\frac{\partial L}{\partial(\partial_\mu \psi_n(x))} \right) \partial_\mu F_n + ... \right) \\ &= \epsilon \sum_n \left(\frac{\partial L}{\partial \psi_n(x)} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi_n(x))} \right) + ... \right) F_n + \textit{total derivatives}. \end{aligned} \tag{1.38}$$

Thus a change in the Lagrangian due to a variation of the parameter γ can be reproduced by a redefinition of the fields by a function F_n such that

$$\frac{\partial L}{\partial \gamma} = \sum_n \left(\frac{\partial L}{\partial \psi_n(x)} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi_n(x))} \right) + ... \right) F_n + \textit{total derivatives}. \tag{1.39}$$

So the coupling parameter γ is an inessential coupling if and only if $\dfrac{\partial L}{\partial \gamma}$ vanishes or is a total derivative along the solutions of the equations of motion.

For example, in the renormalizable scalar field theory with Lagrangian

$$L = -\frac{1}{2} Z(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2) - \frac{1}{24} \lambda Z^2 \phi^4 \tag{1.40}$$

the field renormalization constant is an inessential coupling, because one can write

$$\frac{\partial L}{\partial Z} = -\frac{1}{2} \partial_\mu (\phi \partial^\mu \phi) \tag{1.41}$$

along the solution of the equations of motion. On the other hand, neither the mass m or the coupling λ are inessential. Working with essential coupling only allows one to formulate the condition for asymptotic safety in a very concise way.

Consider again the problem of determine the behavior of the essential couplings $\tilde{g}_i(\mu)$. The change in $\tilde{g}_i(\mu)$ under a given fractional change in μ is a dimensionless quantity, and can therefore depend on all the $\tilde{g}_i(\mu)$ but not on μ itself being the only dimensional parameter left after rescaling. Thus the rate of change of $\tilde{g}_i(\mu)$ with respect to rescaling of the renormalization point μ may be written as a generalized Gell-Mann-Low equation

$$\mu \frac{d}{d\mu} \tilde{g}_i(\mu) = \beta_i(\tilde{g}(\mu)). \tag{1.42}$$

Each specific theory is characterized by a trajectory in coupling constant space, generated by the solution of Eq.(1.42) with given initial conditions. If the coupling $\tilde{g}_i(\mu)$ approach a fixed point g^* as $\mu \to \infty$ then Eq.(1.35) gives a simple scaling behavior $R \to E^D$ for $E \to \infty$. In order for $\tilde{g}_i(\mu)$ to approach the fixed point it is necessary that the beta functions vanish at that point and also that the coupling lie on a trajectory $\tilde{g}_i(\mu)$ that actually hits the fixed point in the UV. The surface formed by such trajectories is called "ultraviolet critical surface", and theories lying on the UV critical surface have a sensible UV limit, since all the essential couplings hit the fixed point. In particular, if the UV critical surface is finite dimensional, the arbitrariness of the choice of the coupling constants is reduced to the choice of a finite number of them, which can be determined by a finite number of experiments. A theory will be called "asymptotically safe" if its essential coupling constants lie on the finite-dimensional ultraviolet critical surface of some fixed point, therefore being UV-complete and predictive. A perturbatively renormalizable, asymptotically free field theory such as QCD is a particular case of asymptotically safe theory. In that case the fixed point of the renormalization group is a Gaussian fixed point, where all couplings vanish, and the critical surface is spanned, near the fixed point, by the couplings which are perturbatively renormalizable.

Without entering in the detail of the discussion about the evidence for a fixed point, this subsection focuses on having an understanding of the running of Newton constant, following Ref. [35]. The coefficient of Einstein-Hilbert action is the square of Planck mass $M_{Pl}^2 = \frac{1}{16\pi G}$. In the quantum theory it is expected to diverge quadratically, leading to a beta function of the form

$$\mu \frac{d}{d\mu} M_{Pl}^2 = 2a\mu^2, \tag{1.43}$$

where a is a positive constant. This expectation comes from a number of different calculations that show that the beta function has this kind of behavior ([36]-[40]). Let $\tilde{G} = G\mu^2$ be the dimensionless Newton constant. Then, the beta function for $\tilde{G}$ is

$$\mu \frac{d\tilde{G}}{d\mu} = 2\tilde{G} - 32\pi a \tilde{G}^2. \tag{1.44}$$

This beta function has a IR attractive fixed point at $\tilde{G} = 0$ and, if $a > 0$, also a UV attractive nontrivial fixed point at $\tilde{G} = 1/16\pi a$. The solution of the RG equation (1.43) is

$$M_{Pl}^2(\mu) = M_{Pl}^2(0) + a\mu^2. \tag{1.45}$$

One can see then that for $\mu \ll M_{Pl}(0)$ the dimensionful G is constant while the dimensionless $\tilde{G}$ scales like μ^2. This is the regime experienced in everyday life. On the other hand, for $\mu \gg M_{Pl}(0)$ the dimensionful G scales as μ^{-2} and the dimensionless $\tilde{G}$ is constant. This is the UV fixed point regime.

Assuming that this is the true behavior of Newton constant and of all other couplings in the theory, it would seem that one can take the limit $\mu \to \infty$ and hence resolve arbitrarily small distance scales, in apparent conflict with all the arguments attributing a non classical, smooth geometry to spacetime at very small scales. Is this really the case? The point is that any dimensionful quantity such as μ does not have any intrinsic value, but one can attribute to it a value only when one measures it in some unity. So far μ has been used as a unity itself, but μ will always be equal to 1 in μ unity so, in order to give meaning to the limit $\mu \to \infty$, one has to use some other units. For example, one could use Planck units, where the value of μ is $\mu\sqrt{G}$, having set $c = \hbar = 1$. Since G is a running coupling, one should specify at what scale it is to be evaluated. If one wants to measure the size of objects at very small scales, then the value of G that is more relevant for this measurement is its value at the scale of the experiment. Therefore, one has that the more proper value of the cutoff in Planck units is $\mu\sqrt{G(\mu)} = \sqrt{\tilde{G}}$, which means that in the correct units μ is indeed limited. Since μ itself is the upper bound for the momenta one can talk about in the theory, one concludes that one cannot talk about momenta greater then Planck mass, or proper distances shorter than Planck length. Notice that using another coupling g_i of dimension d_i and g_i^{1/d_i} as a unit of mass gives the same result as using Planck units. In fact since the theory is asymptotically safe, $g_i\mu^d$ will still go to a constant value in the UV.

The very definition of asymptotically safe theory implies that if one restrict himself to "proper" measurements, one cannot probe distances shorter than the Planck length. The reason is that, since the theory is fundamental one cannot appeal to any external unit of mass or length. The unit has to be chosen within the theory, and in the fixed point regime all the possible candidates appear in constant, finite ratios between themselves and the cutoff. In this sense one can never have a "trans-Planckian" regime in Asymptotic Safety. After all, at the fixed point one has scale-invariance and in a fundamental, scale-invariant theory one cannot talk of distances. One can only speak about distances in the low energy, sub-Planckian regime, and in that regime the shortest length is the Planck distance.

Chapter 2

Preliminaries on Relative Locality

"Do we share the same time?". Probably, this question could never receive a different answer from "Of course we do!", if posed to someone that ignores Special Relativity. Independently on the fact that the answer turns to be the unexpected "no", Einstein taught us that such a question is not silly nor merely philosophical, but it is an experimental question. Then, once spacetime substitutes space and time, there is no reason for which one should not ask "how does an observer know that she lives in a spacetime? And if so how does she know that it is the same spacetime of any another observer?". These are the fundamental questions that Relative Locality poses as a starting point of reflection.

A local observer does not directly observe any event macroscopically distant from the measuring apparatus. The local observer could consider herself as a "calorimeter" with a clock. Her most fundamental measurements are the energies and angles of the quanta she emits and absorbs, and the time of these events. The idea that she lives in a spacetime is constructed by inferences from her measurements of energies and momenta. This was vividly illustrated by Einstein's procedure to give spacetime coordinates to distant events by exchanges of light signals. Adopting this procedure, the observer measures the time it takes the photon to travel forth and back but does not care about the energy of the photon, resulting in a projection into spacetime. When she does so, she presumes that the same spacetime is reconstructed by the exchange of light signals of different frequencies. One is also used to assume that different local observers, distant from each other, reconstruct the same spacetime by measurement of photons they send and receive.

But why should the information about the energy of the photon one uses to probe the spacetime be inessential? Might that be just a low energy approximation? And why should one presume that the same spacetime is reconstructed by two observers at a cosmological distance from each other?

One can see (following Refs.[102],[103]) that absolute locality, which postulates that all observers live in the same spacetime, is equivalent to the assumption that momentum space is a linear manifold. This corresponds to an idealization in which one throws away the information about the energy of the quanta one uses to probe spacetime and it can be transcended in a simple and powerful generalization of special relativistic physics which is motivated by considerations on unification of gravity and quantum physics such as those discussed previously. Locality will turn to be linked with the assumptions made about the geometry of momentum space. Thus, the concept of absolute locality is relaxed in a controlled manner by linking this to a new understanding of the geometry of momentum space. In this framework there is no notion of absolute locality, different observers see different spacetimes, and the spacetimes they observe are energy and momentum-dependent. Locality, a coincidence of events, becomes relative: coincidences of events are still objective for all local observers, but they are not in general manifest in the spacetime coordinates constructed by distant observers.

In the next section it will be shown how Relative Locality manifests in our "story teller" model, the k-Minkowski non-commutative spacetime. Then in Section 2.2, the basic principles and formulation of relative locality are given, independently on any pre-existing model. Then in Section 2.3 a specific realization of a theory with relative locality will be given. This will be the context in which the original results of this thesis are discussed in the following.

2.1 k-Minkowski fuzziness

For the original objective of spacetime noncommutativity, *i.e.* that of providing a characterization of spacetime fuzziness at the Planck length, the implication of the k-Minkowski commutators $[\hat{x}_j, \hat{x}_0] = i\ell\hat{x}_j$ remained unclear for relatively long time.

This section reports what might be significant steps forward in the comprehension of this problem made in Refs.[99], [100], [101]. The key in the strategy of analysis proposed is a new type of ”pregeometric representation” of k-Minkowski. The idea of pregeometric representation originates (see, *e.g.*, [98]) from the conjecture that k-Minkowski might be an effective description of particular physical regimes of a more fundamental theory of quantum gravity. From this perspective it might be natural to describe k-Minkowski noncommutativity in terms of standard Heisenberg quantum mechanics, introduced at some level of the description. Technically such a description allows reformulating the complexity of k-Minkowski commutation relations in terms of (a few copy of) the familiar Heisenberg algebra.

Before developing this pregeometric description, it is better to stop thinking on when and how one should make room for noncommutativity of spacetime coordinates, taking as starting point our current theories. Evidently the formalism of classical mechanics do not make room for noncommutative spacetime coordinates. There is no problem with this, since it is expected that classical mechanics would emerge as an approximate description in a regime for which one can consider $\hbar \to 0$, and this limiting procedure might be such that also the noncommutativity of spacetime coordinates is removed. The problem is that it is not straightforward to allow k-Minkowski spacetime noncommutativity also in ordinary quantum mechanics. This is due to the fact that in ordinary quantum mechanics time is not a self-adjoint operator but just an evolution parameter (therefore classical and commutative), whereas for k-Minkowski it should be an operator that does not commute with the space coordinates operators.

In Ref. [99], authors proposed to address this issue using the covariant formulation of quantum mechanics. In this formulation both the time coordinate and the spacial coordinates are well-defined operators on a "kinematical Hilbert space" and both play the same role of "partial observables". In the formulation of covariant quantum mechanics they commute with each other and do not commute with their respective conjugate momenta. The proposal is that this is the right point to introduce the k-Minkowski commutators (1.1),(1.2).

In this perspective, the kinematical Hilbert space plays a role within the covariant formulation of quantum mechanics that is analogous to the role that Minkowski spacetime plays in classical mechanics of special-relativistic particles. In fact, Minkowski spacetime is the arena where the dynamics of relativistic particles is determined by enforcing the Hamiltonian constraint. In the same way, the kinematical Hilbert space (that codifies the geometry of spacetime) is the arena where the dynamics of relativistic quantum particles is produced by enforcing the Hamiltonian quantum operator constraint.

After introducing the basic concepts of covariant quantum mechanics in the next subsection, the properties and in particular the relativistic symmetries of empty k-Minkowski spacetime will be analyzed in Subection 2.1.2. This analysis has its analogous in the study of the relativistic structure of Minkowski spacetime. Even if none of the properties of spacetime is directly observable (Minkowski spacetime properties are inferred from observation on the motion of classical relativistic particles in it), it is nevertheless an exercise that needs to be done since these formal properties affect the physical properties of the theories formulated on this spacetime. Similarly the properties of observables-operators on the kinematical Hilbert space are not themselves subjectable to measurement, but they usefully characterize the spacetime arena where then the quantum dynamics of particles on the physical Hilbert space takes place. Finally the description of a free particle propagating in this quantum spacetime will be discussed in Subection 2.1.3.

2.1.1 Covariant Quantum Mechanics

Here the basic concepts of covariant formulation of quantum mechanics that will be used in the following description of the fuzziness of k-Minkowski will be introduced. For more details the reader can refer to Refs. [93],[94],[95],[96], [97] and references therein.

Consider a free non-relativistic particle in one space dimension. Let $\psi(X,T)$ be its Schrödinger wave function, namely a solution of the free Schrödinger equation[1]

$$i\frac{\partial}{\partial T}\psi(X,T) = -\frac{1}{2m}\frac{\partial^2}{\partial X^2}\psi(X,T). \tag{2.1}$$

The Hilbert space $\mathcal{H}_0$ of the quantum theory is the space of normalizable solutions of the Schrödinger equation. It can be represented by the space $L^2[R]$ of square integrable functions on space alone[2]. The wavefunction $\psi(X,T)$ is represented by the square integrable function $\Psi(X) = \psi(X,0)$ at fixed time $T = 0$, and the state is denoted by $|\Psi\rangle$. In this representation the scalar product is

$$\langle\Psi|\Psi'\rangle = \int dX\overline{\Psi(X)}\Psi'(X). \tag{2.2}$$

The spacetime wavefunction ψ can be reconstructed from Ψ using the propagator. The generalized eigenstate of the position operator X is denoted by $|X\rangle$ and the generalized eigenstate of the unitarily evolving Heisenberg position operator $X(T)$ by $|X;T\rangle$ (so that $|X\rangle = |X;0\rangle$). Thus $\Psi(X) = \langle X|\Psi\rangle$

[1]Using units such that $\hbar = 1$.

[2]More precisely, the theory is defined on a rigged Hilbert space $\mathcal{S} \subset \mathcal{H}_0 \subset \mathcal{S}'$ formed by a Hilbert space $\mathcal{H}_0$, a proper subset $\mathcal{S}$ in $\mathcal{H}_0$ and its dual $\mathcal{S}'$, with their natural identifications. A manifold M and a measure $d\mu$ determines such a rigged Hilbert space $\mathcal{S}_M \subset \mathcal{H}_M \subset \mathcal{S}'_M$ where $\mathcal{S}_M$ is the space of smooth function on M with fast decrease (Schwarz space), $\mathcal{H}_M = L^2[M, d\mu]$, and $\mathcal{S}'_M$ is the space of tempered distributions on M.

and $\psi(X,T) = \langle X;T|\Psi\rangle$. The propagator of the Schrödinger equation is

$$\begin{aligned} W(X,T;X',T') &= \langle X;T|X';T'\rangle = \langle X|e^{-iH(T-T')}|X'\rangle \\ &= \int dp\,\langle X|e^{-iH(T-T')}|p\rangle\langle p|X'\rangle \\ &= \int dp\, e^{i[p(X-X')-\frac{p^2}{2m}(T-T')]} \\ &= \left(\frac{2\pi m}{i(T-T')}\right)^{\frac{1}{2}} \exp\left[i\frac{m(X-X')^2}{2(T-T')}\right], \end{aligned} \tag{2.3}$$

where H is the Hamiltonian and to solve the last integral one has to analytically continue time to the complex plane in order to render the integrand convergent, then to take limit for vanishing imaginary part of the complex time variable. When viewed as a function of X and T, with X' ant T' held fixed, this is a solution of the Schrödinger equation which at time $T = T'$ is a delta distribution centered at $X = X'$. Each function $\Psi(X)$ determines a solution of the Schrödinger equation by

$$\psi(X,\, T) = \int dX'\, W(X,T;X',0)\Psi(X'). \tag{2.4}$$

Thus the wavefunctions of the Schrödinger equation can be characterized by the functions $\Psi(X)$ of space only.

It is also convenient to consider the following states. Given any compact support complex function $f(X,T)$, the state

$$|f\rangle = \int dXdT\, f(X,T)|X;T\rangle \tag{2.5}$$

is in $\mathcal{H}_0$, for the Schrödinger wavefunction of $|f\rangle$ is

$$\begin{aligned} \psi_f(X,T) &= \langle X;T|f\rangle \\ &= \langle X;T|\int dX'dT'\, f(X',T')|X';T'\rangle \\ &= \int dX'dT'\, W(X,T;X',T')f(X',T') \end{aligned} \tag{2.6}$$

and it is a solution of the Schrödinger equation as well. $|f\rangle$ is called the "spacetime smeared state" of the function f. The scalar product of two spacetime smeared states is

$$\langle f|f'\rangle = \int dXdTdX'dT'\, \overline{f(X,T)}W(X,T;X',T')f'(X',T'). \tag{2.7}$$

These states generalize the usual wave packets for which $f(X,T) = f(X)\delta(T)$. Conventional wave packets can be thought as being associated with results of *instantaneous* position measurements with finite resolution in space. It can be shown that these spacetime smeared states can be associated with realistic measurements, where the measuring device has finite resolution both in space and in time.

A conventional Hamiltonian system, like the free particle is, is formulated in terms of a configuration space $\mathcal{C}_0$ and a Hamiltonian H_0 which is a function on the phase space $\Gamma_0 = T^*\mathcal{C}_0$, *i.e.* the cotangent bundle of the configuration space. The Hamiltonian generates the evolution of the system in an external (independent) variable T. The predictions of the theory are the values of the phase space variables as function of T as, for the example here considered, $X(T)$. Thus, more accurately, what the theory actually predicts are not the individual values of T and X, but rather the relations between these values. A basic example is the uniform motion $X(T) = vT$, which can be expressed by means of the two equations $X = s$, $vT = s$: although s is an arbitrary parameter, these two equations determine a relation between X and vT that is not arbitrary, and is the actual prediction of the theory. In the conventional dynamical system, the time variable can be naturally chosen as the evolution parameter, but in general this is not the case, as happens for example in General Relativity. One is then interested in a description of the system that establishes relations between values of T and X, and these relations are what an observer can compare with combined measurements of T and X. Thus, T and X are called "partial observables", whereas $X(T)$ is called a "complete observable". This suggests that, in order to reformulate this system in a covariant form, one should promote T to a configuration space variable: the extended configuration space (the space of partial observables) includes the conventional configuration space $\mathcal{C}_0$ and time T. So for the conventional Hamiltonian system one has $\mathcal{C} = \mathcal{C}_0 \times \mathbb{R}$, where the coordinate of $\mathbb{R}$ is identified with T. Also, one poses the general Hamiltonian to be $H = p_T + H_0$, where p_T is the conjugate momentum to T (that turns out to be minus the energy). Now, a relativistic system generally has an extended configuration space that is not reducible to the simple form $\mathcal{C} = \mathcal{C}_0 \times \mathbb{R}$ and the Hamiltonian would be a function on the extended phase space $\Gamma = T^*\mathcal{C}$ and $H \neq p_T + H_0$. This means that time is treated in the same way as the other configuration variables.

So, one is now interested in quantizing a system of the form $(\mathcal{C}, H)$. Since the kinematics of the classical system is defined by the extended configuration space, in order to proceed with its quantization it is natural to consider the "kinematical" rigged Hilbert space $\mathcal{S} \subset \mathcal{K} \subset \mathcal{S}'$ defined by $\mathcal{C}$ and the measure $dXdT$. That is, $\mathcal{S}$ is the space of smooth functions $f(X,T)$ on $\mathcal{C}$ with fast decrease, $\mathcal{K} = L^2[\mathcal{C}, dXdT]$, and $\mathcal{S}'$ is the space of tempered distributions on $\mathcal{C}$. $\mathcal{S}$ is the so-called "kinematical state space" and its elements $f(X,T)$ "kinematical states".

The quantum dynamics is determined by the "Wheeler-DeWitt" (WdW) equation

$$H\psi(X,T) = 0. \tag{2.8}$$

The Schrödinger equation can be written in this form, of course,

$$\left(i\frac{\partial}{\partial T} + \frac{1}{2m}\frac{\partial^2}{\partial X^2}\right)\psi(X,T) = 0, \tag{2.9}$$

but the WdW equation applies also for more general Hamiltonian functions for which $H \neq p_T + H_0$. The solutions of this equation form a linear space $\mathcal{H}$.

The key object for the relativistic quantum theory is the operator

$$P = \int d\tau\, e^{i\tau H} \tag{2.10}$$

from $\mathcal{S}$ to $\mathcal{S}'$. In what follows, it may also be denoted by $\delta(H)$. It can be shown that this operator maps arbitrary functions $f(X,T)$ of $\mathcal{S}$ into solutions of the WdW equation. For the case of the Schrödinger equation, for example, one has

$$\begin{aligned}
[Pf](X,T) &= \int d\tau\, e^{i\tau(i\partial/\partial T + \frac{1}{2m}\partial^2/\partial X^2)} f(X,T) \\
&= \int d\tau e^{i\tau(i\partial/\partial T + \frac{1}{2m}\partial^2/\partial X^2)} \int dpdE\, e^{i(pX-ET)}\tilde{f}(p,E) \\
&= \int d\tau \int dpdE\, e^{i\tau(E-\frac{p^2}{2m})} e^{i(pX-ET)}\tilde{f}(p,E) \\
&= \int dpdE\, \delta(E - \frac{p^2}{2m}) e^{i(pX-ET)}\tilde{f}(p,E) \\
&= \int dp\, e^{i(pX-\frac{p^2}{2m}T)}\tilde{f}(p,E(p))
\end{aligned} \tag{2.11}$$

which is a solution of the Schrödinger equation, indeed. One can also develop further the calculation and write

$$
\begin{aligned}
[Pf](X,T) &= \int dpdE\,\delta(E-\frac{p^2}{2m})e^{i(pX-ET)}\tilde{f}(p,E) \\
&= \int dpdE\,\delta(E-\frac{p^2}{2m})e^{i(pX-ET)}\int dX'dT'e^{-i(pX'-ET')}f(X',T') \\
&= \int dX'dT'\int dpdE\,\delta(E-\frac{p^2}{2m})e^{i[p(X-X')-E(T-T')]}f(X',T') \\
&= \int dX'dT'\,W(X,T;X',T')f(X',T').
\end{aligned}
\tag{2.12}
$$

The matrix elements of P,

$$
\langle f|P|f'\rangle_{\mathcal{K}} = \int dXdTdX'dT'\,\overline{f(X,T)}W(X,T;X',T')f'(X',T'), \tag{2.13}
$$

define a degenerate inner product in $\mathcal{S}$. Dividing $\mathcal{S}$ by the kernel of this inner product, that is, identifying f and f' if $Pf = Pf'$, and completing in norm, one obtains a Hilbert space that might be denoted $(\mathcal{S}, \langle\cdot|P|\cdot\rangle)$. But if $Pf = Pf'$, then f and f' define the same solution of the WdW equation. They define the solution that corresponds to the spacetime smeared state $|f\rangle$ defined previously (compare equations (2.12) and (2.6)). Therefore, an element of this Hilbert space $(\mathcal{S}, \langle\cdot|P|\cdot\rangle)$ corresponds to a solution of WdW equation: this Hilbert space $(\mathcal{S}, \langle\cdot|P|\cdot\rangle)$ can be identified with the space of the solutions of the WdW equation $\mathcal{H}$. So

$$
\begin{aligned}
P:\mathcal{S} &\to \mathcal{H} \\
f &\mapsto |f\rangle.
\end{aligned}
\tag{2.14}
$$

It follows that P equips the linear space $\mathcal{H}$ of the solutions of the WdW equation with a Hilbert space structure: if $\psi = Pf$ and $\psi' = Pf'$ are two solutions of the WdW equation, their scalar product is *defined* by

$$
\langle\psi|\psi'\rangle_{\mathcal{H}} \equiv \langle f|P|f'\rangle_{\mathcal{K}}. \tag{2.15}
$$

The partial observables T and X are described as self-adjoint operators on $\mathcal{K}$ which act simply by multiplication. Their common generalized eigenstates $|X,T\rangle$ are in $\mathcal{S}$. Notice that these states are different from the states $|X;T\rangle$, which are eigenstates of the complete observable $X(T)$ and determine solutions to the Schrödinger equation. The relation between the two is $|X;T\rangle = P|X,T\rangle$. These states $|X,T\rangle$ satisfy

$$
\langle X,T|P|X',T'\rangle_{\mathcal{K}} = W(X,T;X',T'). \tag{2.16}
$$

Notice that one also finds

$$W(X,T;X',T') = \langle X;T|X';T'\rangle_{\mathcal{H}} = \langle X,T|P^{\dagger}P|X',T'\rangle_{\mathcal{H}}, \qquad (2.17)$$

which is consistent with Eq.(2.16) because the definition of the scalar product in $\mathcal{H}$ is given by Eq.(2.15).

One can view these states $|X,T\rangle$ as "kinematical states" that do not know anything about dynamics. They correspond to a single "quantum event". Their ("kinematical") scalar product in $\mathcal{K}$, $\langle X,T|X',T'\rangle = \delta(X-X')\delta(T-T')$, expresses only their independence, while their "physical" scalar product (2.16) in $\mathcal{H}$ expresses the physical relation between the two events by mean of the presence of the particle propagator.

One can now propose the following axioms of a covariant quantum mechanics (only those axioms which are used in the following application to k-Minkowski are reported here):

- *Kinematical states*: Kinematical states form a space $\mathcal{S}$ in a rigged Hilbert space $S \subset \mathcal{K} \subset \mathcal{S}'$.

- *Partial observables*: A partial observable is represented by a self-adjoint operator in $\mathcal{K}$. Common eigenstates $|s\rangle$ of a complete set of commuting partial observables are denoted quantum events.

- *Dynamics*: The dynamics is determined by a self-adjoint operator H in $\mathcal{K}$, the (relativistic) Hamiltonian. The operator from $\mathcal{S}$ to $\mathcal{S}'$

  $$P = \int d\tau e^{i\tau H} \qquad (2.18)$$

  is (improperly) called "projector" and its matrix elements

  $$W(s,s') = \langle s|P|s'\rangle \qquad (2.19)$$

  are called transition amplitudes.

- *Physical states*: A physical state is a solution of the Wheeler-DeWitt equation

  $$H\psi = 0. \qquad (2.20)$$

  Equivalently, it is an element of the Hilbert space $\mathcal{H}$ defined by the quadratic form $\langle \cdot|P|\cdot\rangle$ on $\mathcal{S}$.

- *Complete observables*: A complete observable $\mathcal{A}$ is represented by a self-adjoint operator on $\mathcal{H}$. A self-adjoint operator A in $\mathcal{K}$ defines a complete observable if it commutes with the relativistic Hamiltonian H.

2.1.2 Pregeometry of k-Minkowski and fuzzy points

This section deals with the study of the properties of the noncommuting coordinates of a 1+1-dimensional k-Minkowski spacetime at the level of the kinematical Hilbert space of a covariant formulation of quantum mechanics. The units adopted are such that $c = \hbar = 1$ and the conventions for the Minkowski metric tensor $\eta_{\mu\nu} = \{1, -1\}$.

The pregeometric representation is given as follows. Given the phase space observables for the covariant formulation of 2D quantum mechanics,

$$[\hat{\pi}_0,\, \hat{q}_0] = i, \qquad [\hat{\pi}_0,\, \hat{q}_1] = 0,$$
$$[\hat{\pi}_1,\, \hat{q}_0] = 0, \qquad [\hat{\pi}_1,\, \hat{q}_1] = -i, \tag{2.21}$$

the k-Minkowski coordinates $\hat{x}_0$, $\hat{x}_1$ are described as

$$\hat{x}_0 = \hat{q}_0, \qquad \hat{x}_1 = \hat{q}_1 e^{\ell\hat{\pi}_0}, \tag{2.22}$$

that indeed satisfy (1.1) and (1.2). In fact, for example,

$$[\hat{x}_1, \hat{x}_0] = [\hat{q}_1 e^{\ell\hat{\pi}_0}, \hat{q}_0] = \hat{q}_1[e^{\ell\hat{\pi}_0}, \hat{q}_0] = i\ell\hat{q}_1 e^{\ell\hat{\pi}_0} = i\ell\hat{x}_1.$$

One finds in this pregeometric description also opportunities for describing the k-Minkowski differential calculus and the k-Poincaré transformations generators. For the translation generators, by posing

$$P_0 \triangleright f(\hat{x}_0,\, \hat{x}_1) \longleftrightarrow \left[\hat{\pi}_0,\, f(\hat{q}_0,\, \hat{q}_1 e^{\ell\hat{\pi}_0})\right],$$
$$P_1 \triangleright f(\hat{x}_0,\, \hat{x}_1) \longleftrightarrow e^{-\ell\hat{\pi}_0}\left[\hat{\pi}_1,\, f(\hat{q}_0,\, \hat{q}_1 e^{\ell\hat{\pi}_0})\right], \tag{2.23}$$

one does reproduce all the properties of k-Poincaré translation generators summarized earlier in Chapter 1. Notice that the properties of the elements $\hat{a}_\mu$ of the differential calculus given in (1.13) can be reproduced by combining ordinary parameters a_μ and the (partial) observable $\hat{\pi}_0$:

$$\hat{a}_0 = a_0, \qquad \hat{a}_1 = a_1 e^{\ell\hat{\pi}_0}. \tag{2.24}$$

In 2D k-Minkowski spacetime boost generator should satisfy the following properties of commutation with translation generators and of coproduct[3]:

$$-i\,[N,\,P_0]\triangleright f(\hat{x})\equiv P_1\triangleright f(\hat{x}),$$

$$-i\,[N,\,P_1]\triangleright f(\hat{x})\equiv\left(\frac{1-e^{-2\ell P_0}}{2\ell}-\frac{\ell}{2}P_1^2\right)\triangleright f(\hat{x}),$$

$$\Delta N=N\otimes 1+e^{-\ell P_0}\otimes N.$$

The boost operator takes the form

$$B=1+\mathbf{d}_N,\qquad \mathbf{d}_N=i\hat{\xi}N, \tag{2.25}$$

and the noncommutative boost-transformation parameter is

$$\left[\hat{\xi},\hat{x}_0\right]=i\ell\hat{\xi},\qquad \left[\hat{\xi},\,\hat{x}_1\right]=0. \tag{2.26}$$

The pregeometric description of boost parameter and generator is given by

$$\hat{\xi}=\xi e^{\ell\hat{\pi}_0},$$

$$N\triangleright f(\hat{x})\equiv e^{-\ell\hat{\pi}_0}\left[\hat{\eta},\,f(\hat{q}_0,\,\hat{q}_1e^{\ell\hat{\pi}_0})\right],$$

with

$$\hat{\eta}\equiv\left(\frac{e^{2\ell\hat{\pi}_0}-1}{2\ell}+\frac{\ell}{2}\hat{\pi}_1^2\right)\hat{q}_1-\hat{\pi}_1\hat{q}_0. \tag{2.27}$$

[3]Notice that in 2D k-Minkowski the coproduct of boost generator has the same form of the coproduct of translation generators. Then, since the noncommutativity properties of the transformation parameters are proven to be directly linked to the coproduct of the generators of the transformation, the properties of boost transformation parameters will immediately follow. In 4D this would no longer be the case, the coproducts of boosts generators being different from those of translation generators. This coincidence in the 2D case simplifies the analysis from a technical point of view, but conceptually there is no difference with the 4D k-Minkowski.

From these definitions one finds that under the action of boost

$$\begin{aligned}
\hat{\pi}'_0 =&\hat{\pi}_0 + i\hat{\xi}(N \triangleright \hat{\pi}_0)\\
=&\hat{\pi}_0 + \left(i\xi e^{\ell\hat{\pi}_0}\right) e^{-\ell\hat{\pi}_0} \left[\left(\frac{e^{2\ell\hat{\pi}_0}-1}{2\ell} + \frac{\ell}{2}\hat{\pi}_1^2\right)\hat{q}_1 - \hat{\pi}_1\hat{q}_0, \hat{\pi}_0\right]\\
=&\hat{\pi}_0 - \xi\hat{\pi}_1,\\
\\
\hat{\pi}'_1 =&\hat{\pi}_1 + i\hat{\xi}(N \triangleright \hat{\pi}_1)\\
=&\hat{\pi}_1 + \left(i\xi e^{\ell\hat{\pi}_0}\right) e^{-\ell\hat{\pi}_0} \left[\left(\frac{e^{2\ell\hat{\pi}_0}-1}{2\ell} + \frac{\ell}{2}\hat{\pi}_1^2\right)\hat{q}_1 - \hat{\pi}_1\hat{q}_0, \hat{\pi}_1\right]\\
=&\hat{\pi}_1 - \xi\left(\frac{e^{2\ell\hat{\pi}_0}-1}{2\ell} + \frac{\ell}{2}\hat{\pi}_1^2\right).
\end{aligned} \tag{2.28}$$

It has been already implicitly specified that the states of the kinematical Hilbert space for k-Minkowski will admit a representation (in the "pregeometric momentum space representation") as square-integrable functions of variables $\hat{\pi}_0$ and $\hat{\pi}_1$. In order to define properly the prescription of square-integrability one has to specify a measure on this kinematical Hilbert space. One shall characterize the scalar product in momentum space as

$$\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle = \int \mathcal{D}(\pi_\mu)\psi^\star(\pi_\mu)O(\pi_\mu)\psi(\pi_\mu), \tag{2.29}$$

where the measure (that must be invariant under the action of boost) is

$$\mathcal{D}(\pi_\mu) = d\pi_0 d\pi_1 e^{-\ell\pi_0}. \tag{2.30}$$

One sees that, with this measure, $\hat{\eta}$ is Hermitian, so the boost transformation operator (2.25) is unitary and preserves the scalar product:

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger(B)U(B)|\psi\rangle = \langle\psi|e^{i\xi\hat{\eta}}e^{-i\xi\hat{\eta}}|\psi\rangle = \langle\psi|\psi\rangle. \tag{2.31}$$

It is now time for describing fuzzy points of k-Minkowski and analyze this fuzziness from the perspective of distant observers in relative rest, observers connected by a pure translation. First one needs a description of these fuzzy points. Evidently within the pregeometric description a point of k-Minkowski will be described as a state in the pregeometric Hilbert space (the Hilbert space on which the pregeometric operators $\hat{q}_\mu$ and $\hat{\pi}_\mu$ are defined). It is indeed easy to see that no state in the pregeometric Hilbert space gives absolutely sharp values to $\hat{x}_0$ and $\hat{x}_1$ simultaneously: in light of $\hat{x}_0 = \hat{q}_0$, $\hat{x}_1 = \hat{q}_1 e^{\ell\hat{\pi}_0}$, in order to have a sharp value on $\hat{x}_0$ requires an eigenstate of $\hat{q}_0$ but, for such eigenstate, $\hat{\pi}_0$ is infinitely fuzzy ($\delta\pi_0 \approx \infty$), which in turn implies that $\hat{x}_1$ cannot be sharp. So all points in k-Minkowski must be fuzzy[4].

In order to study the properties of k-Minkowski fuzziness one can consider Gaussian states on the pregeometric Hilbert space. Adopting a pregeometric momentum-space representation this states take the form

$$\Psi_{\bar{q}_0,\bar{q}_1}(\pi_\mu;\,\bar{\pi}_\mu,\,\sigma_\mu) = N e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_1^2}}\, e^{i\pi_0\bar{q}_0 - i\pi_1\bar{q}_1} \tag{2.32}$$

with parameters $\bar{\pi}_0$, $\bar{\pi}_1$, σ_0, σ_1, and $\bar{q}_0$, $\bar{q}_1$, these being highlighted in the notation since the issue of localization of the particle is predominantly connected with those two parameters, which determine the expected values for the pregeometric position coordinates $\hat{q}_0$, $\hat{q}_1$. Essentially $\bar{\pi}_0$, $\bar{\pi}_1$ have the role of expected values for the pregeometric momenta $\hat{\pi}_0$, $\hat{\pi}_1$, whereas σ_0, σ_1 characterize the uncertainty for $\hat{\pi}_0$, $\hat{\pi}_1$. N is a normalization constant obtained by requiring $\langle\Psi|\Psi\rangle = 1$, from which

$$N^2 = \frac{e^{\ell\bar{\pi}_0} e^{-(\ell\sigma_0)^2/2}}{2\pi\sigma_0\sigma_1}. \tag{2.33}$$

The properties of points of k-Minkowski spacetime are characterized by evaluating in the Gaussian pregeometric state the mean values and the uncertainties of the operators $\hat{x}_0$, $\hat{x}_1$. Beginning with the time coordinate:

[4]This is true with the only exception of the origin $\hat{x}_0 = \hat{x}_1 = 0$ but this can be added as a limiting case for what is to be discussed in the following, where it is made evident that even if an observer describes the point in his origin as absolutely sharp, a distant observer describes that same point as fuzzy.

$$
\begin{aligned}
\langle\hat{x}_0\rangle =& N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}\Psi^*\left(-i\frac{\partial}{\partial\pi_0}\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{4\sigma_1^2}}e^{i\pi_0\bar{q}_0-i\pi_1\bar{q}_1}\\
=& N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}\Psi^*\left(\bar{q}_0+\frac{i}{2\sigma_0^2}(\pi_0-\bar{\pi}_0)\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{4\sigma_1^2}}e^{i\pi_0\bar{q}_0-i\pi_1\bar{q}_1}\\
=& \bar{q}_0+N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}\frac{i}{2\sigma_0^2}(\pi_0-\bar{\pi}_0)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}}\\
=& \bar{q}_0+\frac{i}{2\sigma_0^2}\frac{e^{\ell\bar{\pi}_0}e^{-(\ell\sigma_0)^2/2}}{2\pi\sigma_0\sigma_1}\int d\pi_0 d\pi_1 e^{-\ell\pi_0}(\pi_0-\bar{\pi}_0)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}}\\
=& \bar{q}_0+\frac{i}{2\sigma_0^2}\frac{e^{-(\ell\sigma_0)^2/2}}{\sqrt{2\pi}\sigma_0}\int d\pi_0 e^{-\ell(\pi_0-\bar{\pi}_0)}(\pi_0-\bar{\pi}_0)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}\\
=& \bar{q}_0+\frac{i}{2\sigma_0^2}\frac{e^{-(\ell\sigma_0)^2/2}}{\sqrt{2\pi}\sigma_0}\left(-\frac{\partial}{\partial\ell}\right)\int d\pi_0 e^{-\ell(\pi_0-\bar{\pi}_0)}e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}\\
=& \bar{q}_0+\frac{i}{2\sigma_0^2}e^{-(\ell\sigma_0)^2/2}\left(-\frac{\partial}{\partial\ell}\right)e^{(\ell\sigma_0)^2/2}\\
=& \bar{q}_0-i\frac{\ell}{2}.
\end{aligned}
$$

This constant contribution to $\bar{x}_0$ is expected on the basis of the fact that $\hat{q}_0$ is not Hermitian, and the Hermitian operator obtainable by $\hat{q}_0$ that can be used as k-Minkowski time coordinate is $\hat{x}_0^* = \hat{q}_0 - i\ell/2$. However, one can keep working with the previous choice of time coordinate for two main reasons: the first is that the physical properties of k-Minkowski will have to be formulated in terms of operators that commute with the Hamiltonian constraint, and k-Minkowski time coordinate is not one of these. The second is that, when one is interested in $\hat{x}_0$ as a partial observable on the physical Hilbert space, the most meaningful features are found to be inevitably formulated in terms of differences among values of this operator. Therefore this constant does not give any contribution.

Continuing the calculations one has

$$
\begin{aligned}
\langle \hat{x}_0^2\rangle =& N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}\Psi^*\left(-\frac{\partial^2}{\partial\pi_0^2}\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{4\sigma_1^2}}e^{i\pi_0\bar{q}_0-i\pi_1\bar{q}_1}\\
=& N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}\left(\bar{q}_0^2+\frac{1}{2\sigma_0^2}+\frac{i\bar{q}_0}{\sigma_0^2}(\pi_0-\bar{\pi}_0)-\frac{1}{4\sigma_0^4}(\pi_0-\bar{\pi}_0)^2\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}}\\
=& \bar{q}_0^2+\frac{1}{2\sigma_0^2}-i\ell\bar{q}_0-\frac{1}{4\sigma_0^4}N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}(\pi_0-\bar{\pi}_0)^2 e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}}\\
=& \bar{q}_0^2+\frac{1}{2\sigma_0^2}-i\ell\bar{q}_0-\frac{1}{4\sigma_0^4}\frac{e^{-(\ell\sigma_0)^2/2}}{\sqrt{2\pi}\sigma_0}\left(\frac{\partial^2}{\partial\ell^2}\right)\int d\pi_0 e^{-\ell(\pi_0-\bar{\pi}_0)}e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}\\
=& \bar{q}_0^2+\frac{1}{2\sigma_0^2}-i\ell\bar{q}_0-\frac{1}{4\sigma_0^4}e^{-(\ell\sigma_0)^2/2}\left(\sigma_0^2 e^{(\ell\sigma_0)^2/2}+\ell^2\sigma_0^4 e^{(\ell\sigma_0)^2/2}\right)\\
=& \bar{q}_0^2+\frac{1}{4\sigma_0^2}-i\ell\bar{q}_0-\frac{\ell^2}{4}.
\end{aligned}
$$

Then

$$
\delta\hat{x}_0=\sqrt{\langle\hat{x}_0^2\rangle-\langle\hat{x}_0\rangle^2}=\sqrt{\bar{q}_0^2+\frac{1}{4\sigma_0^2}-i\ell\bar{q}_0-\frac{\ell^2}{4}-(\bar{q}_0^2-i\ell\bar{q}_0-\frac{\ell^2}{4})}=\frac{1}{2\sigma_0}.
$$

Now for the spatial coordinate:

$$
\begin{aligned}
\langle \hat{x}_1\rangle =& \langle \hat{q}_1 e^{\ell\hat{\pi}_0}\rangle = N^2\int d\pi_0 d\pi_1 e^{-\ell\pi_0}\Psi^*\left(i\frac{\partial}{\partial\pi_1}e^{\ell\pi_0}\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{4\sigma_1^2}}e^{i\pi_0\bar{q}_0-i\pi_1\bar{q}_1}\\
=& N^2\int d\pi_0 d\pi_1\Psi^*\left(\bar{q}_1-\frac{i}{2\sigma_1^2}(\pi_1-\bar{\pi}_1)\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{4\sigma_1^2}}e^{i\pi_0\bar{q}_0-i\pi_1\bar{q}_1}\\
=& N^2\int d\pi_0 d\pi_1\left(\bar{q}_1-\frac{i}{2\sigma_1^2}(\pi_1-\bar{\pi}_1)\right)e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}}\\
=& \bar{q}_1\frac{e^{\ell\bar{\pi}_0}e^{-(\ell\sigma_0)^2/2}}{2\pi\sigma_0\sigma_1}\int d\pi_0 d\pi_1 e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}}e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}}\\
=& \bar{q}_1 e^{\ell\bar{\pi}_0}e^{-(\ell\sigma_0)^2/2};
\end{aligned}
$$

$$\begin{aligned}
\langle \hat{x}_1^2 \rangle =& \langle (\hat{q}_1 e^{\ell \hat{\pi}_0})^2 \rangle = N^2 \int d\pi_0 d\pi_1 e^{\ell \pi_0} \left(\bar{q}_1^2 + \frac{1}{2\sigma_1^2} - \frac{1}{4\sigma_1^4}(\pi_1 - \bar{\pi}_1)^2 \right) e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}} e^{-\frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}} \\
=& \left(\bar{q}_1^2 + \frac{1}{2\sigma_1^2} \left(1 + \frac{\partial}{\partial \beta} \right) \right)_{|\beta=1} N^2 \int d\pi_0 d\pi_1 e^{\ell \pi_0} e^{-\frac{(\pi_0-\bar{\pi}_0)^2}{2\sigma_0^2}} e^{-\beta \frac{(\pi_1-\bar{\pi}_1)^2}{2\sigma_1^2}} \\
=& \left(\bar{q}_1^2 + \frac{1}{2\sigma_1^2} \left(1 + \frac{\partial}{\partial \beta} \right) \right)_{|\beta=1} N^2 \sqrt{\frac{2\pi\sigma_1^2}{\beta}} \sqrt{2\pi\sigma_0^2}\, e^{(\ell\sigma_0)^2/2} e^{\ell\bar{\pi}_0} \\
=& \left(\bar{q}_1^2 + \frac{1}{2\sigma_1^2} \left(1 - \frac{1}{2} \right) \right) \frac{e^{\ell\bar{\pi}_0} e^{-(\ell\sigma_0)^2/2}}{2\pi\sigma_0\sigma_1} \sqrt{2\pi\sigma_1^2} \sqrt{2\pi\sigma_0^2}\, e^{(\ell\sigma_0)^2/2} e^{\ell\bar{\pi}_0} \\
=& \left(\bar{q}_1^2 + \frac{1}{4\sigma_1^2} \right) e^{2\ell\bar{\pi}_0};
\end{aligned}$$

Therefore one has

$$\begin{aligned}
\delta \hat{x}_1 =& \sqrt{\langle \hat{x}_1^2 \rangle - \langle \hat{x}_1 \rangle^2} = \sqrt{\left(\bar{q}_1^2 + \frac{1}{4\sigma_1^2} \right) e^{2\ell\bar{\pi}_0} - \bar{q}_1^2 e^{2\ell\bar{\pi}_0} e^{-(\ell\sigma_0)^2}} \\
=& e^{\ell\bar{\pi}_0} \left[\frac{1}{4\sigma_1^2} + \bar{q}_1^2 \left(1 - e^{-(\ell\sigma_0)^2} \right) \right]^{1/2}.
\end{aligned}$$

In summary, the following expression for mean values and uncertainties of the operators $\hat{x}_0$ and $\hat{x}_1$ have been found:

$$\bar{x}_0 = \langle \hat{q}_0 \rangle = \bar{q}_0 - i\frac{\ell}{2}, \tag{2.34a}$$

$$\delta \hat{x}_0 = \sqrt{\langle \hat{q}_0^2 \rangle - \bar{x}_0^2} = \frac{1}{2\sigma_0}, \tag{2.34b}$$

and

$$\bar{x}_1 = \langle \hat{q}_1 e^{\ell\hat{\pi}_0} \rangle = \bar{q}_1 e^{\ell\bar{\pi}_0} e^{-(\ell\sigma_0)^2/2}, \tag{2.35a}$$

$$\delta \hat{x}_1 = \sqrt{\langle (\hat{q}_1 e^{\ell\hat{\pi}_0})^2 \rangle - \bar{x}_1^2} = e^{\ell\bar{\pi}_0} \left[\frac{1}{4\sigma_1^2} + \bar{q}_1^2 \left(1 - e^{-(\ell\sigma_0)^2} \right) \right]^{1/2}. \tag{2.35b}$$

From these expressions one can already see that for fixed values of $\bar{q}_0$, $\bar{\pi}_0$, σ_0, σ_1 one finds larger fuzziness of $\hat{x}_1$ at large values of $\bar{q}_1$, because of the contribution to $\delta\hat{x}_1$ by the term with $\bar{q}_1^2$ in the last equation. However it is more interesting to study how distinct observers related by a pure translation characterize the fuzziness of the same point. To see this one has to implement a translation transformation on a fuzzy point of k-Minkowski. Within this pregeometric description the action of the operator $\mathbf{d}_P$ of Eq.(1.14) on a function $f(\hat{x})$ is easily found to be

$$\mathbf{d}_P \triangleright f(\hat{x}_0,\, \hat{x}_1) \longleftrightarrow ia^\mu[\hat{\pi}_\mu,\, f(\hat{q}_0,\, \hat{q}_1 e^{\ell\hat{\pi}})], \tag{2.36}$$

since

$$\mathbf{d}_P = i\hat{a}_\mu P^\mu = i\hat{a}_0 P_0 - i\hat{a}_1 P_1 = ia_0 P_0 - ia_1 e^{\ell\hat{\pi}_0} P_1$$

and then recalling the action (2.23) of translation generators. So this action involves only familiar commutative transformation parameters a^μ and standard translations (acting by commutators) at the pregeometric level. This allows implementing translation transformations straightforwardly:

$$\begin{aligned} T \triangleright \hat{x}_0 &= \hat{x}_0 + \mathbf{d}_P \triangleright \hat{x}_0 = \hat{x}_0 + ia^\mu[\hat{\pi}_\mu, \hat{q}_0] \\ &= \hat{x}_0 - a_0 = \hat{q}_0 - a_0, \end{aligned} \tag{2.37}$$

$$\begin{aligned} T \triangleright \hat{x}_1 &= \hat{x}_1 + ia^\mu[\hat{\pi}_\mu, \hat{q}_1 e^{\ell\hat{\pi}_0}] \\ &= \hat{x}_1 + ia^1[\hat{\pi}_1, \hat{q}_1]e^{\ell\hat{\pi}_0} \\ &= \hat{x}_1 - ia_1(-i)e^{\ell\hat{\pi}_0} \\ &= \hat{x}_1 - \hat{a}_1 = e^{\ell\hat{\pi}_0}(\hat{q}_1 - a_1). \end{aligned} \tag{2.38}$$

The mean values of uncertainties of $T \triangleright \hat{x}_\mu$ on the Gaussian state (2.32), are then immediately found:

$$\langle T \triangleright \hat{x}_0 \rangle = \bar{q}_0 - a_0 - i\frac{\ell}{2}, \tag{2.39a}$$

$$\delta(T \triangleright \hat{x}_0) = \frac{1}{2\sigma_0}, \tag{2.39b}$$

and

$$\langle T \triangleright \hat{x}_1 \rangle = (\bar{q}_1 - a_1)e^{\ell\bar{\pi}_0} e^{-\frac{\ell^2\sigma_0^2}{2}}, \tag{2.40a}$$

$$\delta(T \triangleright \hat{x}_1) = e^{\ell\bar{\pi}_0}\left[\frac{1}{4\sigma_1^2} + (\bar{q}_1 - a_1)^2\left(1 - e^{-\ell^2\sigma_0^2}\right)\right]^{1/2}. \tag{2.40b}$$

The interpretation here is of course that operators $\hat{x}_\mu$ are operators characterizing the distance of a given (fuzzy) point from the frame origin of some observer Alice, and $T \triangleright \hat{x}_\mu$ are the operators characterizing the distance of that point from the origin of another observer Bob, purely translated with respect to Alice. Comparing Eqs.(2.34),(2.35) with Eqs.(2.39), (2.40) one can recognize two main features:

- The same point appears to be more fuzzy to a distant observer than to a nearby observer.
- The point at Alice is not described as at Alice in the coordinatization of spacetime of observer Bob, and *vice versa* the point at Bob is not described as at Bob in the coordinatization of spacetime of observer Alice.

This second feature is characteristic of Relative Locality and will be discussed in detail in the following. As anticipated in the introduction, one can see that it is possible to formulate a consistent relativistic theory of interacting particles in which the concept of locality is weakened, from the absolute locality of the standard physics to a relative locality. In the first case all observers agree on characterizing all the interactions as local (there are no instantaneous-interaction-at-a-distance, the particles interact at one point of spacetime), independently on their distance from the interaction event or on their motion relative to the interacting particles; in the other case observers which are local ("near") to the interaction characterize it as local but distant observers might (erroneously) infer from their observations that the interaction is not local.

2.1.3 Fuzzy worldlines

The properties of boost strongly characterize the form of the on-shell condition, which in turn, as it has been seen in the section dedicated to the covariant formulation of quantum mechanics, through an appropriate Hamiltonian constraint governs the relationship between the kinematical Hilbert space and the physical Hilbert space. On the basis of the properties derived above one finds that the d'Alambertian operator that is invariant under the action of boosts is the ℓ-deformed

$$\Box_\ell = \left(\frac{2}{\ell}\right)^2 \sinh^2\left(\frac{\ell\hat{\pi}_0}{2}\right) - e^{-\ell\hat{\pi}_0}\hat{\pi}_1^2. \tag{2.41}$$

Then for massless particles the Hamiltonian operator that enforces the on-shellness condition and should vanish on physical states (WdW equation) is simply

$$H = \left(\frac{2}{\ell}\right)^2 \sinh^2\left(\frac{\ell\hat{\pi}_0}{2}\right) - e^{-\ell\hat{\pi}_0}\hat{\pi}_1^2. \tag{2.42}$$

One can proceed to study the physical scalar product $\langle\psi|\phi\rangle_{\mathcal{H}} = \langle\psi|\delta(H)\Theta(\pi_0)|\phi\rangle$, where $\Theta(\pi_0)$ specifies a restriction to positive-energy solutions only. In the momentum space representation this writes

$$\langle\psi|\phi\rangle_{\mathcal{H}} = \int d\pi_1 d\pi_0\, e^{-\ell\pi_0}\delta(H)\Theta(\pi_0)\psi^\star(\pi_\mu)\phi(\pi_\mu). \tag{2.43}$$

Here it will be now considered the case of a localized massless particle, describable in terms of the Gaussian state[5]

$$\Psi_{\bar{q}_0,\,\bar{q}_1}(\pi_\mu;\,\bar{\pi}_\mu,\,\sigma_\mu) = Ne^{-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_0^2}-\frac{(\pi_0-\bar{\pi}_0)^2}{4\sigma_1^2}}\,e^{i\pi_0\bar{q}_0-i\pi_1\bar{q}_1} \tag{2.44}$$

where N now is a new normalization constant that is computed by

$$N^{-2} = \int d\pi_1 d\pi_0\, e^{-\ell\pi_0}\delta(H)\Theta(\pi_0)|\Psi_{\hat{q}_0,\,\hat{q}_1}(\pi_\mu;\,\bar{\pi}_\mu,\,\sigma_\mu)|^2. \tag{2.45}$$

$\Psi_{\bar{q}_0,\,\bar{q}_1}$ is a state in the physical Hilbert space of relativistic free-particle quantum mechanics, so it identifies a worldline that is fuzzy, as will be clear shortly. The expectation in $\Psi_{\bar{q}_0,\,\bar{q}_1}$ of the measurable quantity described by the self-adjoint operator $\mathcal{O}$ is computed in terms of $\langle\Psi_{\bar{q}_0,\,\bar{q}_1}|\mathcal{O}|\Psi_{\bar{q}_0,\,\bar{q}_1}\rangle_{\mathcal{H}}$.

One now has to look for a well-defined complete observable suitable for the characterization of the fuzziness of the worldline. The apparently obvious choices $\hat{x}_0$, $\hat{x}_1$ are actually not suitable for this task because they are not self-adjoint operators on the physical Hilbert space (in particular they do not commute with H). One should expect this since these two operators are the k-Minkowski version of the partial observable time and position operators of covariant quantum mechanics. So what is really needed is a combination between these two quantities that gives a complete observable. Considering a free particle, classically speaking, one could imagine that it should go on a straight line. This line is determined completely once the intercept and the velocity are known. Authors in [101] found the following operator:

$$\mathcal{A} = e^{\ell\hat{\pi}_0}\left(\hat{q}_1 - \hat{\mathcal{V}}\hat{q}_0 - \frac{1}{2}[\hat{q}_0,\,\hat{\mathcal{V}}]\right), \tag{2.46}$$

where $\hat{\mathcal{V}}$ is defined as $\hat{\mathcal{V}} \equiv (\partial H/\partial\hat{\pi}^0)^{-1}\partial H/\partial\hat{\pi}^1$. $\mathcal{A}$ is self-adjoint and commutes with H, and so it is a good observable on the physical Hilbert space. Also, in the classical limit it evidently reduces to the intercept of the particle worldline with the x_1 axis. One may notice that $\mathcal{A}$ is describable as an ℓ-deformed Newton-Wigner operator, which is well known to being the best

[5] In the massless particle limit, one must proceed cautiously: $\Psi_{\hat{q}_0,\,\hat{q}_1}(\pi_\mu;\,\bar{\pi}_\mu,\,\sigma_\mu)$ must be replaced by $\Psi^\alpha_{\hat{q}_0,\,\hat{q}_1}(\pi_\mu;\,\bar{\pi}_\mu,\,\sigma_\mu) = \exp(-\alpha/\pi_0^2)\Psi_{\hat{q}_0,\,\hat{q}_1}(\pi_\mu;\,\bar{\pi}_\mu,\,\sigma_\mu)$ with α a small infrared regulator which never actually matters in the results here reported.

localization estimator within special-relativistic quantum mechanics (it can only be questioned for localization comparable to the Compton wavelength of the particle, but this conceptual limit is not very relevant for the level of localization achieved by particle production at, say, a quasar).

For conceptual clarity, the focus here is on the analysis of the properties of $\mathcal{A}$ for the case of $\Psi_{0,0}$, *i.e.* for $\bar{q}_0 = \bar{q}_1 = 0$. One finds that

$$\langle \Psi_{0,0} | \mathcal{A} | \Psi_{0,0} \rangle_{\mathcal{H}} = 0, \tag{2.47}$$

so this is a case where the particle intercepts the observer Alice in her origin. The fact that this intercept is fuzzy reflects the fuzziness of the worldline described by $\Psi_{0,0}$, and in particular the leading ℓ-dependent contribution to this fuzziness is characterized by

$$\delta\mathcal{A}^2_{[\ell]} = (\langle \Psi_{0,0} | \mathcal{A}^2 | \Psi_{0,0} \rangle_{\mathcal{H}})_{[\ell]} \approx \frac{\ell \langle \hat{\pi}_0 \rangle}{2\sigma^2}, \tag{2.48}$$

where for simplicity it has been assumed that σ_1 is small enough, in comparison with σ_0, $\bar{\pi}_1$ to allow a saddle point approximation in the π_1 integration; then σ (without indices) is the effective Gaussian width after the saddle point approximation in $\pi_1 : \sigma^{-2} \equiv \sigma_1^{-2} + \langle \hat{\mathcal{V}} \rangle^2 \sigma_0^{-2}$.

In the interpretation of the formalism proposed by the authors in Ref.[101] Eq.(2.48) gives the fuzziness of the worldline at the point where it crosses the origin of Alice's reference frame. It is of interest also considering the perspective given by observers reached by the particles at a cosmological distance from Alice. These observers are those connected to Alice by a pure translation, so that for them the state of the particle is Ψ_{a_0, a_1} and are such that $\langle \mathcal{A} \rangle = \langle \Psi_{a_0, a_1} | \mathcal{A} | \Psi_{a_0, a_1} \rangle_{\mathcal{H}} = 0$. Finding these observers consists in finding the translation parameters a_0, a_1 such that $\langle \Psi_{0,0} | T^{-1} \mathcal{A} T | \Psi_{0,0} \rangle_{\mathcal{H}} = 0$, where T is the translation operator previously defined. This leads to a one-parameter family of solutions (the family of observers on the worldline), which takes the form $a_1 = \langle \hat{\mathcal{V}} \rangle a_0$.

It is important to notice that these observers with vanishing expectation value for the intercept have values of the uncertainties of the intercept $\delta\mathcal{A}$ given by

$$\delta\mathcal{A}^2_{[\ell]} = (\langle \Psi_{a_0, \langle \hat{\mathcal{V}} \rangle a_0} | \mathcal{A}^2 | \Psi_{a_0, \langle \hat{\mathcal{V}} \rangle a_0} \rangle_{\mathcal{H}} \approx \left(\frac{\ell \langle \hat{\pi}_0 \rangle}{2\sigma^2} + \ell^2 \sigma^2 a_0^2 \right). \tag{2.49}$$

So a quantum spacetime picture is offered here: one can interpret our observer Alice, the observer on the worldline for whom the fuzziness of the intercept takes the minimum value, as the observer at the source (where the particle is produced); then the intercept of the particle worldline with the origin of the reference frame of a distant observer (which might detect the particle) has larger uncertainties. Notice that, since $\delta\mathcal{A}^2_{[\ell]}$ goes as $(\ell a_0)^2$, if the particle travels a long distance (a cosmological distance) its fuzziness ”benefits” of a sort of amplification. Therefore, from this formalization of k-Minkowski it is possible to extract (if one proceeds with the analysis) in principle observable phenomenological predictions as, for example, an anomalous blurring of images of distant quasars.

2.2 The principle of relative locality

The previous section showed how relativity of locality emerges in k-Minkowski non-commutative spacetime. Here the basic formulation of Relative Locality will be given, without relying on any specific model of quantum spacetime. In fact the main ingredient is the geometry of momentum space.

The approximation used in this study is that in which both $\hbar$ and G_{Newton} may be neglected while their ratio $M_{Pl} = \sqrt{\dfrac{\hbar}{G_{Newton}}}$ is held fixed[6]. In this approximation gravitational and quantum effects may both be neglected, but there may be new phenomena on scales of momentum or energy given by M_{Pl}. At the same time, because $L_P = \sqrt{\hbar G_{Newton}} \to 0$ no features of quantum spacetime geometry are expected to be relevant.

Since this approximation gives an energy scale, but not a length scale, one presumes that momentum space is more fundamental than spacetime, according to the operational point of view mentioned before. Thus, once the deformation of the geometry of momentum space by the scale M_{Pl} has been established, the properties of spacetime will be derived from the dynamics formulated in momentum space.

2.2.1 Defining the geometry of momentum space

The theoretical framework of Relative Locality takes an operational point of view in which one describes physics from the perspective of a local observer who is equipped with devices to measure energy and momenta of elementary particles in her vicinity. It is also supposed that the observer can measure a "local proper time" with a clock. She constructs the geometry of momentum space from measurements made of the dynamics of interacting particles. It is assumed that each choice of calorimeter is a preferred choice of local coordinates k_μ on momentum space. Notice that k_μ measure the energy and

[6]Units are such that $c = 1$.

momenta of excitations above the ground state, hence the origin of momentum space, $k_\mu = 0$, is physically well defined.

A local observer can make two kinds of measurements. One type of measurement can be done only with a single particle and it defines a metric on momentum space $\mathcal{P}$. In fact, it is assumed that the mass represents the geodesic distance from the origin of momentum space. This gives the dispersion relation

$$D^2(p) \equiv D^2(p,0) = m^2. \tag{2.50}$$

The observer can also measure the kinetic energy of a particle of mass m moving with respect to her but local to her. It is postulated that this measure defines the geodetic distance between a particle p at rest and a particle p' of identical mass and kinetic energy K, that is $D^2(p) = D^2(p') = m^2$ and

$$D^2(p,p') = -2mK. \tag{2.51}$$

The minus sign expresses the fact that the geometry of momentum space is Lorentzian.

The other type of measurement involves many particles and defines a connection. Consider a process in which n particles interact. Associated to each interaction there must be a combination rule for momenta, which will be in general non-linear. This rule for two particles is denoted by

$$(p,q) \to p'_\mu = (p \oplus q)_\mu. \tag{2.52}$$

Hence the momentum space has the structure of an algebra defined by the product rule "$\oplus$". It is assumed that more complicated processes are built up by iterations of this product (that in principle could be non-linear, non-commutative and non-associative). The inverse ("antipode") of "$\oplus$" is denoted by "$\ominus$" and satisfies $\ominus p \oplus p = p \oplus (\ominus p) = 0$. Then one has the conservation law for energy and momentum for any process, giving, for each type of interaction, four functions on $\mathcal{P}^n$, depending on momenta of interacting particles, which vanish

$$\mathcal{K}_\mu(k^I) = 0. \tag{2.53}$$

For example, for a process with three incoming particles with momenta p_μ, q_μ and k_μ one has

$$\mathcal{K}_\mu(p,q,k) = (p \oplus (q \oplus k))_\mu = 0. \tag{2.54}$$

These conservation laws will be discussed in the next section in greater detail.

From the algebra of combinations of momenta one can define an affine connection[7] on $\mathcal{P}$, in particular

$$\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q)_\rho|_{q,p=0} = -\Gamma^{\mu\nu}_\rho(0). \tag{2.55}$$

The torsion of the connection is a measure of the asymmetric part of the combination rule

$$-\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}((p \oplus q)_\rho - (q \oplus p)_\rho)|_{q,p=0} = T^{\mu\nu}_\rho(0). \tag{2.56}$$

Similarly the curvature of $\mathcal{P}$ is a measure of the lack of associativity of the combination rule

$$2\frac{\partial}{\partial p_{[\mu}}\frac{\partial}{\partial q_{\nu]}}\frac{\partial}{\partial k_\sigma}(((p \oplus q) \oplus k)_\rho - (p \oplus (q \oplus k))_\rho)|_{q,p=0} = R^{\mu\nu\sigma}_\rho(0), \tag{2.57}$$

where the brackets denote antisymmetrization.

Notice that there is no physical reason to expect a combination rule for momentum to be associative once it is non-linear. Indeed, the lack of associativity means that there is a physical distinction between the two processes of Fig.2.1, which is equivalent to saying that there is a definite microscopic causal structure. That is, *causal structure of the physics maps to nonassociativity of the combination rule for momentum which in turn maps to curvature of momentum space.* The curvature of momentum space makes microscopic causal orders distinguishable, hence meaningful.

To determine the connection, torsion and curvature away from the origin of momentum space one has to consider translations on momentum space, i.e. one can denote

$$p \oplus_k q = k \oplus ((\ominus k \oplus p) \oplus (\ominus k \oplus q)) \tag{2.58}$$

$$\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus_k q)_\rho|_{q,p=k} = -\Gamma^{\mu\nu}_\rho(k), \tag{2.59}$$

the identity for this product is at $0_k = k$.

Thus, the action of adding an infinitesimal momentum dq_μ from particle J to a finite momentum p_μ of particle I defines a parallel transport on $\mathcal{P}$

[7] One could also define other affine connection, for example, by defining an appropriate notion of parallel transport of the mass-geodesic of one particle along the mass-geodesic of a second particle and obtaining in this way the composite momentum (see Ref.[105]). These mathematical aspects are presently under investigation. In this thesis, however, these alternative definitions of affine connection are not considered.

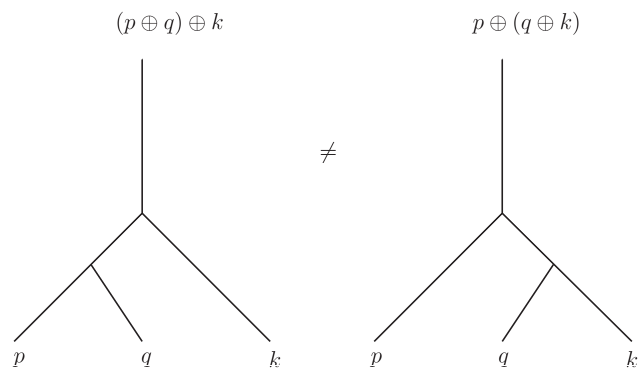

Figure 2.1: Curvature of the connection on momentum space produces nonassociativity of the composition rule.

$$p_\mu \oplus dq_\mu = p_\mu + dq_\nu \tau^\nu_\mu(p) \tag{2.60}$$

where $\tau^\nu_\mu(p)$ is the parallel transport operation from the identity to p. It can be expanded around $p = 0$

$$\tau^\nu_\mu(p) = \delta^\nu_\mu - \Gamma^{\nu\sigma}_\mu p_\sigma - \Gamma^{\nu\sigma\rho}_\mu p_\sigma p_\rho + ... \tag{2.61}$$

with

$$\Gamma^{\nu\sigma\rho}_\mu = \partial^\rho_p \Gamma^{\nu\sigma}_\mu - \Gamma^{\rho\nu}_\alpha \Gamma^{\alpha\sigma}_\mu - \Gamma^{\rho\sigma}_\alpha \Gamma^{\nu\alpha}_\mu . \tag{2.62}$$

The corresponding conservation law has the form to second order

$$\mathcal{K}_\mu(k) = \sum_I k^I_\mu + \sum_{J \in \mathcal{J}(I)} C_{I,J} \Gamma^{\nu\sigma}_\mu k^I_\nu k^J_\sigma + ... \tag{2.63}$$

where $\mathcal{J}(I)$ is the set of particles that interact with the I's one and $C_{I,J}$ are coefficients that depend on the form of the conservation law.

2.2.2 A variational principle

Here spacetime is viewed as an auxiliary concept that emerges when one seeks to define dynamics in momentum space. If the momenta of elementary particles are taken to be primary, then they themselves need momenta in order to develop a canonical dynamics. Momenta of momenta are quantities x^μ that live in the cotangent space of $\mathcal{P}^n$ at a point k_μ; these quantities are called *Hamiltonian spacetime coordinates*. The action proposed to define the dynamics of a free particle is

$$S_{free} = \int ds \left(x^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}(k) \right) \tag{2.64}$$

where s is an arbitrary evolution parameter and $\mathcal{N}_k$ is the Lagrange multiplier enforcing mass shell condition

$$\mathcal{C}(k) \equiv D^2(k) - m^2 = 0. \tag{2.65}$$

It shall be emphasized that the contraction $x^\mu k_\mu$ does not involve any metric, and the dynamics is given by constraints which are functions only of coordinates on $\mathcal{P}$ and depend only on the geometry of $\mathcal{P}$. This leads to the Poisson brackets

$$\{x^\mu_I,\, k^J_\nu\} = \delta^\mu_\nu \delta^J_I \tag{2.66}$$

where the indexes I, J identify the particle whose coordinates refer to.

One then has a phase space Γ of a single particle which is the cotangent bundle of $\mathcal{P}$. Note that there is neither an invariant projection to a spacetime $\mathcal{M}$, nor is defined any invariant spacetime metric. Still this structure is sufficient to describe the dynamics of free particles. Spacetime is also unnecessary to describe how particles interact.

Consider the following action:

$$S = \sum_J \int_{-\infty}^{0} ds\, \left(x^\mu_J \dot{k}^J_\mu + \mathcal{N}_J \mathcal{C}^J(k) \right) - \xi^\mu \mathcal{K}_\mu(k(s=0)). \tag{2.67}$$

It describes the simple (yet unrealistic) process in which n incoming particles interact at the interaction vertex (here the interaction is set to take place at the value $s = 0$ for each of the particles) and no outgoing particle is produced. One wants to impose conservation of momentum and this is done introducing the Lagrange multiplier ξ^μ enforcing this constraint.

To obtain the equations of motion one varies the action and, after integrating by parts in each of the free actions, one obtains

$$\delta S = \sum_J \int_{-\infty}^{0} ds \left(\delta x^\mu_J \dot{k}^J_\mu - \delta k^J_\mu \left[\dot{x}^\mu_J - \mathcal{N}_J \frac{\delta \mathcal{C}^J}{\delta k^J_\mu} \right] + \delta \mathcal{N}_J \mathcal{C}^J(k) \right) + \mathcal{R}. \tag{2.68}$$

Here $\mathcal{R}$ contains both the results of varying the interaction term and the boundary terms from the integration by parts. The equations of motion are the expected ones

$$\dot{k}^J_\mu = 0, \qquad \dot{x}^\mu_J = \mathcal{N}_J \frac{\delta \mathcal{C}^J}{\delta k^J_\mu}, \qquad \mathcal{C}^J(k) = 0. \tag{2.69}$$

One can fix $\delta k^J_\mu = 0$ at $s = -\infty$ and examine the remaining terms of the variation

$$\mathcal{R} = -\mathcal{K}_\mu(k)\delta\xi^\mu + \left(x^\mu_J(0) - \xi^\nu \frac{\delta \mathcal{K}_\nu}{\delta k^J_\mu} \right) \delta k^J_\mu. \tag{2.70}$$

Here x^μ_J and k^J_μ are taken for each particle at the value $s = 0$. $\mathcal{R}$ has to vanish as the variational principle must have a solution. From the vanishing of the coefficient of $\delta\xi^\mu$ one gets the four conservation laws of the interaction, $\mathcal{K}_\mu(k) = 0$. From the vanishing of the coefficient of δk^J_μ one finds $4n$ conditions that hold at the interaction

$$x^\mu_J(0) = \xi^\nu \frac{\delta\mathcal{K}_\nu}{\delta k^J_\mu}. \tag{2.71}$$

By using (2.63), this gives the conditions

$$x^\mu_J(0) = \xi^\mu - \xi^\nu \sum_{L\in\mathcal{J}(J)} C_{J,L}\Gamma^{\mu\sigma}_\nu k^L_\sigma + ... \tag{2.72}$$

This implies that to leading order, in which the nonlinearity of momentum space is ignored, all of the particles involved in the interaction meet at a single spacetime event, for they are all equal to ξ^μ (which in general should not be regarded as the event itself, but rather as an auxiliary variable that sets the observable relations between the $x^\mu_J(0)$). The choice of ξ^μ is not constrained and cannot be, for its variation gives the conservation laws $\mathcal{K}_\mu(k) = 0$. Thus, the usual notion that interaction of particles takes place at single spacetime event from the conservation of energy and momentum has been recovered.

However, considering the contributions due to the nonlinearity of momentum space, one finds that the interaction takes place at n distinct events, separated from ξ^μ by an interval

$$\Delta x^\mu_J(0) = -\xi^\nu \sum_{L\in\mathcal{J}(J)} C_{J,L}\Gamma^{\mu\sigma}_\nu k^L_\sigma + ... \tag{2.73}$$

These relations (2.72), (2.73) illustrate concisely the relativity of locality. For some fortunate observers the interaction takes place at the origin of their systems of coordinates, so that $\xi^\mu = x^\mu_J(0) = 0$ in which case the interaction is observed to be local. Any other observer, translated with respect to these, has a non-vanishing ξ^μ and hence sees the interaction to take place at a distant set of events. These are centered around ξ^μ but are not precisely at the same values of the coordinates.

Is it a real, physical non-locality or a new kind of coordinate artifact? It is easy to see that it is the latter, because the Δx^μ_J can be made to vanish by making a translation to the coordinates of another observer. In a canonical formulation, translations are generated by the laws of conservation of energy and momentum. Given any local observable in phase space $\mathcal{O}$ observed by a local observer, Alice, one can construct the observable as seen in coordinates constructed by another observer, Bob, distant from Alice, by a translation parameter b^μ

$$\delta_b \mathcal{O} = b^\nu \{\mathcal{K}_\nu, \mathcal{O}\}. \tag{2.74}$$

Since momentum space is curved, and $\mathcal{K}_\mu$ is non-linear, it follows that the "spacetime coordinates" x^μ_J of a particle translate in a way that is dependent on the energies and momenta of the particles it interacts with, $x^\mu_J \to x'^\mu_J(0) = x^\mu_J(0) + \delta_b x^\mu_J(0)$ where

$$\delta_b x^\mu_J(0) = b^\nu \{\mathcal{K}_\nu, x^\mu_J\} = -b^\mu + b^\nu \sum_{L \in \mathcal{J}(J)} C_{J,L} \Gamma^{\mu\sigma}_\nu k^L_\sigma + ... \tag{2.75}$$

This is a manifestation of the relativity of locality, *i.e.* local spacetime coordinates for one observer mix up energy and momenta on translation to the coordinates of a distant observer.

This mixing under translations effect also entirely accounts for the separation of an interaction into apparently distinct events, because with $b^\nu = -\xi^\nu$, one sees that Δx^μ_J of (2.73) is equal to $\delta_b x^\mu_J$ of (2.75). Thus, *the observer whose new coordinates one has translated to observes a single interaction taking place at* $x^\mu_J \to x'^\mu_J(0) = 0$.

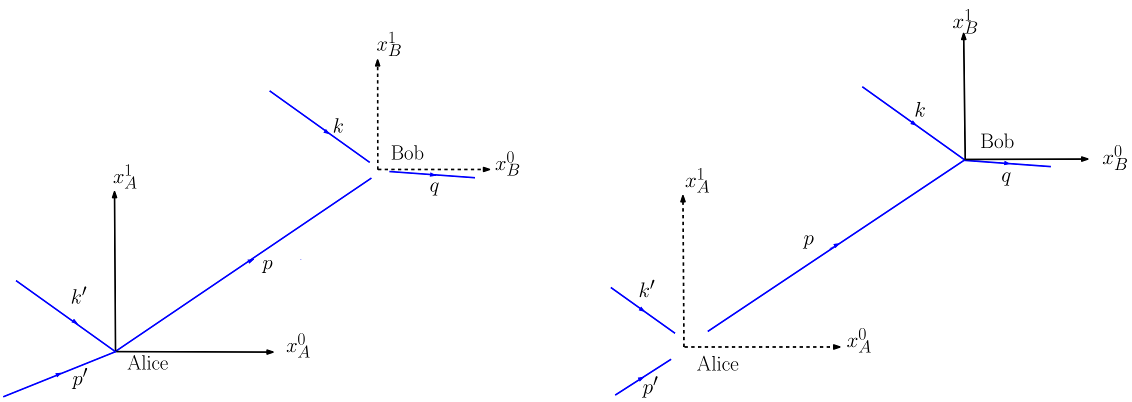

Figure 2.2: A process described in the relative locality framework by two observers: the figure on the left represents the description given by Alice, the one on the right represents Bob's description of the process.

Thus, if a local observer sees an interaction to take place via a collision at the origin of her coordinates system, a distant observer will generally see it in her own coordinates as spread out over a region of spacetime according to Eq.(2.73) and vice versa, as represented in Fig.2.2. There is not a physical non-locality since all momentum conserving interactions are seen as happening at a single spacetime event by some family of observers, who are local to the interaction. But it becomes impossible to localize distant interactions in an absolute manner: distant observers do not share the same spacetime. Furthermore, all observers related by a translation agree about the momenta of particles in the interaction, because under translations (2.74) $\delta_b k^J_\mu = 0$.

Note that if the curvature and torsion vanish there is no mixing of spacetime coordinates with momenta under translations, so there is an invariant definition of spacetime. Therefore, the flatness of momentum space is responsible for the notion of an absolute spacetime, just as the Galilean additivity of velocities allows Newtonian physics to have an absolute time.

2.3 k-de Sitter momentum space

In this section an explicit example of formalization of Relative Locality will be obtained. Again k-Minkowski is the source of inspiration in Subsection 2.3.1: in the relative locality regime the noncommutativity of spacetime coordinates is suppressed, but the non primitive coproduct of translation generators survives. From this one gets the affine connection of momentum space. The metric on momentum space is de Sitter, and the construction of the on-shell relation as the geodesic distance from the origin of momentum space is consistent with the relative locality limit of the mass Casimir of k-Minkowski. A particular effort is dedicated in Subsection 2.3.2 in discussing the role of the interaction terms in relation to the translational symmetry, highlighting that even though the same conservation laws of energy-momentum may be enforced by different interaction terms, different interaction terms lead to physically distinguished theories. The key concept is that *one can obtain a relativistic theory with curved momentum space (therefore, with relative locality) if the momentum space is maximally symmetric and the action is compatible with the symmetries of momentum space.* Finally in Subsection 2.3.3 it is introduced the strategy of analysis of the problem of determining the physical velocity of particles in Relative Locality, an exercise that is made conceptually less trivial than usual by the non trivial character of translation transformations and that will be largely used in the rest of the thesis.

2.3.1 Relative Locality limit of k-Minkowski

It has been shown in Refs. [23], [24], [25], [26] that k-Poincaré Hopf algebra describes a curved momentum space with de Sitter metric, torsion and nonmetricity. One can then study the properties of k-Minkowski momentum space in the Relative Locality regime. As it has been assumed in the previous section, the metric determines the distance of a point p_μ from the origin in momentum space $\mathcal{P}$. The composition law for momenta is determined by the composition law of basis exponentials $e^{-ip_j\hat{x}_j}e^{ip_0\hat{x}_0}$ of k-Minkowski. In fact, from the k-Minkowski commutators (1.1),(1.2) and the Baker-Campbell-Hausdorff formula one has, writing explicitly $\hbar$ and using the Planck length instead of ℓ in the definition of k-Minkowski commutators,

$$\begin{aligned} e^{-\frac{i}{\hbar}p_j\hat{x}_j}e^{\frac{i}{\hbar}p_0\hat{x}_0}e^{-\frac{i}{\hbar}q_j\hat{x}_j}e^{\frac{i}{\hbar}q_0\hat{x}_0} &= e^{-\frac{i}{\hbar}p_j\hat{x}_j}e^{-\frac{i}{\hbar}e^{\frac{L_P}{\hbar}p_0}q_j\hat{x}_j}e^{\frac{i}{\hbar}p_0\hat{x}_0}e^{\frac{i}{\hbar}q_0\hat{x}_0} \\ &= e^{-\frac{i}{\hbar}(p_j+e^{\frac{L_P}{\hbar}p_0}q_j)\hat{x}_j}e^{\frac{i}{\hbar}(p_0+q_0)\hat{x}_0}. \end{aligned} \tag{2.76}$$

Thus in the Relative Locality regime, where $\hbar \to 0$, $L_P \to 0$ while $\dfrac{\hbar}{L_P} = M_{Pl}$ is kept constant, the noncommutativity properties of spacetime coordinates disappear but the non primitive coproduct of translation generators remains. This expression can be used as the rule of composition of momenta:

$$(p \oplus q)_0 = p_0 + q_0, \qquad (p \oplus q)_i = p_i + e^{\ell p_0}q_i, \tag{2.77}$$

where it has been introduced the notation $M_{Pl}^{-1} = \ell = \lim_{\hbar, L_P \to 0} \dfrac{L_P}{\hbar}$, which is widely used in the relative locality literature and therefore will be used from now on. This deformed composition law is evidently noncommutative but it is found to be associative.

In what follows a particular attention will be dedicated in characterizing the non trivial geometry of momentum space only at leading order in ℓ, for it is unlikely that experiments would be sensible enough to determine corrections to standard physics phenomenology of greater orders. Therefore, one can use the composition law obtained developing the deformed sum of momenta in powers of ℓ:

$$(p \oplus q)_\mu \simeq p_\mu + q_\mu + \ell\delta^i_\mu p_0 q_i. \tag{2.78}$$

The exact antipode is

$$(\ominus p)_0 = -p_0, \qquad (\ominus p)_i = -e^{-\ell p_0}p_i, \tag{2.79}$$

while at leading order in ℓ it becomes

$$(\ominus p)_\mu \simeq -p_\mu + \ell\delta^i_\mu p_0 p_i. \tag{2.80}$$

In what follows it will be considered a 1+1-dimensional momentum space. The metric is

$$dk^2 = (dp_0)^2 - e^{-2\ell p_0}(dp_1)^2 \tag{2.81}$$

Solving the geodesic equation and computing the geodesic distance from the origin of momentum space for a generic momentum $p_\mu = (p_0,\, p_1)$ one has

$$D^2(p,\,0) = m^2 = p^2 + C^{\rho\mu\nu} p_\rho p_\mu p_\nu, \tag{2.82}$$

where $C^{\rho\mu\nu}$ are the Christoffel symbols for the metric. At leading order they are

$$\begin{aligned} C^{011} &= -\ell e^{-2\ell p_0} \simeq -\ell \\ C^{110} = C^{101} &= \ell \end{aligned} \tag{2.83}$$

and therefore,

$$D^2(p,\,0) = m^2 = p^2 + 2\ell p_0 p_1^2 - \ell e^{-2\ell p_0} p_0 p_1^2 \simeq p_0^2 - p_1^2 + \ell p_0 p_1^2. \tag{2.84}$$

Notice that this is also consistent with the expansion in powers of ℓ to first order of the k-Minkowski mass Casimir (1.11). Then, the action of the process considered in the previous section in the case $n = 2$ is

$$\mathcal{S} = \int_{-\infty}^{0} ds\ (x^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}(p)) + \int_{-\infty}^{0} ds\ (y^\mu \dot{q}_\mu + \mathcal{N}_q \mathcal{C}(q)) - \xi^\mu \mathcal{K}_\mu \tag{2.85}$$

with

$$\begin{aligned} \mathcal{K}_\mu &= p_\mu + q_\mu + \ell \delta_\mu^1 p_0 q_1, \\ \mathcal{C}(p) &= p_0^2 - p_1^2 + \ell p_0 p_1^2 - m_p^2, \\ \mathcal{C}(q) &= q_0^2 - q_1^2 + \ell q_0 q_1^2 - m_q^2. \end{aligned}$$

2.3.2 On the choice of the interaction terms $\mathcal{K}_\mu$

It is important now to focus on the sources of ambiguity in the choice of the laws of conservation of energy-momentum. One issue comes from the noncommutativity of the sum (2.78), which suggests that an ordering prescription for summing momenta should be given. However, it is easy to realize that the multiplicity of possible conservation laws is smaller than one may expect on the basis of the properties of the composition law. In fact, for arbitrary momenta p and q, from Eq.(2.78) one has $p \oplus q \neq q \oplus p$. Notice, however, that from $(p \oplus q)_\mu = 0$ one gets

$$0 = p_\mu + q_\mu + \ell \delta_\mu^1 p_0 q_1 = p_\mu + q_\mu + \ell \delta_\mu^1 (-q_0)(-p_1) = (q \oplus p)_\mu, \tag{2.86}$$

using leading order corrections only. Thus, when the composition rule (2.78) is used to write a conservation law, one actually does have

$$p \oplus q = 0 \Longleftrightarrow q \oplus p = 0. \tag{2.87}$$

Moreover, this is true for any choice of affine connection of momentum space, as one can see from the following chain properties:

$$p \oplus q = 0 \Longrightarrow p = \ominus q \Longrightarrow q \oplus p = q \oplus (\ominus q) = 0. \tag{2.88}$$

This observation also simplifies the description of a three-particles interaction:

$$p \oplus q \oplus k = 0 \Longleftrightarrow k \oplus p \oplus q = 0. \tag{2.89}$$

So, when the rule of composition of momenta is used for a conservation law it produces a conservation law with cyclicity, reducing then the possible independent choices for the law $\mathcal{K} = 0$.

A second issue regards interactions with incoming and outgoing particles. Until now in fact only incoming particles have been here considered. One could be tempted to write the conservation law of total momentum using antipodes to denote momenta of outgoing particles. Thus, for example, one could write

$$\begin{aligned} &\mathcal{K} = p \oplus q \oplus (\ominus p') \oplus (\ominus q'), \\ &\mathcal{K} = p \oplus q \oplus (\ominus(p' \oplus q')), \\ &\mathcal{K} = p \oplus q - (p' \oplus q'), \end{aligned} \tag{2.90}$$

where the prime denotes outgoing particles. The first two expressions differ from each other for it can be shown that $\ominus(p' \oplus q') = (\ominus q') \oplus (\ominus p')$. The last two expressions, when set equal to zero, give the same conservation laws, since

$$p \oplus q \oplus (\ominus(p' \oplus q')) = 0 \Longrightarrow p' \oplus q' = p \oplus q \Longrightarrow p \oplus q - p' \oplus q' = 0. \tag{2.91}$$

As will be clear shortly, these different forms of $\mathcal{K}$, even if they enforce the same conservation law, lead to physically different theories. It is of great importance to realize that a key concept of Relative Locality is that there must be a notion/prescription of translation transformations that makes the theory symmetric (as in the previous section) in order for the theory to be compatible with the relativity principle and to allow an interaction to be characterized as local for observers which are local to it, otherwise one would have a non-relativistic theory with physical non-locality, *i.e.* one that cannot be removed by a change of coordinates. Then, recalling the role that $\mathcal{K}$ has in determining the spacetime coordinates of the particles which participate in the interaction (2.71), the choice of $\mathcal{K}$ must ensure the symmetry of the action under a certain realization of translation transformations.

Consider the process shown in Fig.2.3. It might be described by the

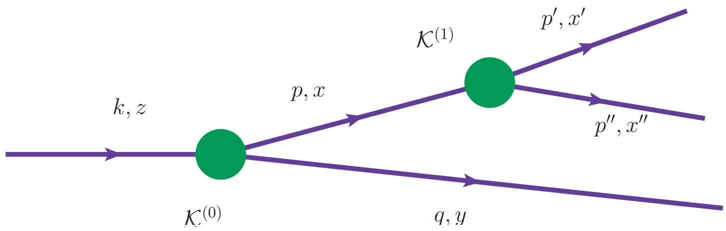

Figure 2.3: Example of process with both incoming and outgoing particles and a finite worldline.

following action, written by some observer Alice:

$$\begin{aligned}\mathcal{S}_A = &\int_{-\infty}^{s_0} ds\, (z_A^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}(k)) + \int_{s_0}^{s_1} ds\, (x_A^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}(p)) + \\ &+ \int_{s_0}^{\infty} ds\, (y_A^\mu \dot{q}_\mu + \mathcal{N}_q \mathcal{C}(q)) + \int_{s_1}^{\infty} ds\, (x_A'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} \mathcal{C}(p')) + \\ &+ \int_{s_1}^{\infty} ds\, (x_A''^\mu \dot{p}''_\mu + \mathcal{N}_{p''} \mathcal{C}(p'')) - \xi^\mu_{(0),A} \mathcal{K}^{(0)}_\mu - \xi^\mu_{(1),A} \mathcal{K}^{(1)}_\mu .\end{aligned} \tag{2.92}$$

The subscript A is omitted for momenta since they are invariant under translation transformations, which are generated by some combination of momenta. For what follows it is important to notice that Eq.(2.75) can be viewed as a prescription for translations generated by the "total momentum", which for that case corresponds to $\mathcal{K}_\mu$. In fact, one can write, for example

$$\delta x_b^\mu = b^\nu \{(p \oplus q)_\nu, x^\mu\} = b^\nu \{\mathcal{K}_\nu, x^\mu\} = -b^\nu \frac{\delta \mathcal{K}_\nu}{\delta p_\mu}. \tag{2.93}$$

Now it will be made evident the effect of different choices of the form of $\mathcal{K}$. Following Ref.[104], one might first start considering the expressions:

$$\begin{aligned}\mathcal{K}^{(0)}_\mu &= k_\mu - (p \oplus q)_\mu = k_\mu - p_\mu - q_\mu - \ell \delta^1_\mu p_0 q_1 \\ \mathcal{K}^{(1)}_\mu &= (p \oplus q)_\mu - (p' \oplus p'' \oplus q)_\mu \\ &= p_\mu - p'_\mu - p''_\mu + \ell \delta^1_\mu ((p_0 - p'_0 - p''_0) q_1 - p'_0 p''_1).\end{aligned} \tag{2.94}$$

Notice that it has been used the prescription of writing the deformed sum of the total momentum before and after the interaction. One could feel uncomfortable with the presence of momentum q in the vertex $\mathcal{K}^{(1)}_\mu$, which describes an interaction in which the particle with momentum q does not participate, but it is immediate to check that the conservation laws $\mathcal{K}^{(1)}_\mu = 0$

do not depend on q, for the only term of q that appears in these expressions is multiplied by $\mathcal{K}_0^{(1)}$. The equations of motion are

$$\dot{k}_\mu = 0, \qquad \dot{p}_\mu = 0, \qquad \dot{q}_\mu = 0, \qquad \dot{p}'_\mu = 0, \qquad \dot{p}''_\mu = 0,$$

$$\mathcal{C}(k) = 0, \qquad \mathcal{C}(p) = 0, \qquad \mathcal{C}(q) = 0, \qquad \mathcal{C}(p') = 0, \qquad \mathcal{C}(p'') = 0,$$

$$\mathcal{K}_\mu^{(0)} = 0, \qquad \mathcal{K}_\mu^{(1)} = 0$$

$$\begin{aligned}
\dot{z}_A^\mu &= \mathcal{N}_k \frac{\delta \mathcal{C}(k)}{\delta k_\mu} = \mathcal{N}_k(\delta_0^\mu(2k_0 + \ell k_1^2) + \delta_1^\mu(-2k_1 + 2\ell k_0 k_1)),\\
\dot{x}_A^\mu &= \mathcal{N}_p \frac{\delta \mathcal{C}(p)}{\delta p_\mu} = \mathcal{N}_p(\delta_0^\mu(2p_0 + \ell p_1^2) + \delta_1^\mu(-2p_1 + 2\ell p_0 p_1)),\\
\dot{y}_A^\mu &= \mathcal{N}_q \frac{\delta \mathcal{C}(q)}{\delta q_\mu} = \mathcal{N}_q(\delta_0^\mu(2q_0 + \ell q_1^2) + \delta_1^\mu(-2q_1 + 2\ell q_0 q_1)),\\
\dot{x}_A'^\mu &= \mathcal{N}_{p'} \frac{\delta \mathcal{C}(p')}{\delta p'_\mu} = \mathcal{N}_{p'}(\delta_0^\mu(2p'_0 + \ell p_1'^2) + \delta_1^\mu(-2p'_1 + 2\ell p'_0 p'_1)),\\
\dot{x}_A''^\mu &= \mathcal{N}_{p''} \frac{\delta \mathcal{C}(p'')}{\delta p''_\mu} = \mathcal{N}_{p''}(\delta_0^\mu(2p''_0 + \ell p_1''^2) + \delta_1^\mu(-2p''_1 + 2\ell p''_0 p''_1)),
\end{aligned}$$

while the boundary conditions are

$$\begin{aligned}
z_A^\mu(s_0) &= \xi_{(0),A}^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta k_\mu} = \xi_{(0),A}^\mu,\\
y_A^\mu(s_0) &= -\xi_{(0),A}^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta q_\mu} = \xi_{(0),A}^\mu + \ell \xi_{(0),A}^1 \delta_1^\mu p_0,\\
x_A^\mu(s_0) &= -\xi_{(0),A}^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta p_\mu} = \xi_{(0),A}^\mu + \ell \xi_{(0),A}^1 \delta_0^\mu q_1,\\
x_A^\mu(s_1) &= \xi_{(1),A}^\nu \frac{\delta \mathcal{K}_\nu^{(1)}}{\delta p_\mu} = \xi_{(1),A}^\mu + \ell \xi_{(1),A}^1 \delta_0^\mu q_1,\\
x_A'^\mu(s_1) &= -\xi_{(1),A}^\nu \frac{\delta \mathcal{K}_\nu^{(1)}}{\delta p'_\mu} = \xi_{(1),A}^\mu - \ell \xi_{(1),A}^1 \delta_0^\mu (q_1 + p''_1),\\
x_A''^\mu(s_1) &= -\xi_{(1),A}^\nu \frac{\delta \mathcal{K}_\nu^{(1)}}{\delta p''_\mu} = \xi_{(1),A}^\mu - \ell \xi_{(1),A}^1 (\delta_0^\mu q_1 + \delta_1^\mu p'_0).
\end{aligned}$$

Thanks to the form of the constraints $\mathcal{K}^{(i)}$ here considered, one can extend to an interaction in which participate both incoming and outgoing particles the rather standard prescription of translations generated by total momentum used previously in the case of incoming particles only. It is also immediate to see that these prescriptions on the form of $\mathcal{K}$ and translation transformations make the equations of motion and boundary terms symmetric under translations, and, furthermore, not only at first order in ℓ, but to all orders. In fact one has

$$\begin{aligned} z_B^\mu(s) &= z_A^\mu(s) + b^\nu\{k_\nu, z^\mu\} = z_A^\mu(s) + b^\nu\{k_\nu - (p\oplus q)_\nu, z^\mu\} \\ &= z_A^\mu(s) + b^\nu\{\mathcal{K}_\nu^{(0)}, z^\mu\} = z_A^\mu(s) - b^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta k_\mu}, \end{aligned}$$

where it has been exploited the property that the terms added in the second equality have null Poisson brackets with z. Using the same argument for the others particles one has[8]

$$\begin{aligned} z_B^\mu(s) &= z_A^\mu(s) - b^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta k_\mu}, \qquad & y_B^\mu(s) &= y_A^\mu(s) + b^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta q_\mu}, \\ x_B^\mu(s) &= x_A^\mu(s) + b^\nu \frac{\delta \mathcal{K}_\nu^{(0)}}{\delta p_\mu}, \qquad & x_B^\mu(s) &= x_A^\mu(s) - b^\nu \frac{\delta \mathcal{K}_\nu^{(1)}}{\delta p_\mu}, \\ x_B'^\mu(s) &= x_A'^\mu(s) + b^\nu \frac{\delta \mathcal{K}_\nu^{(1)}}{\delta p'_\mu}, \qquad & x_B''^\mu(s) &= x_A''^\mu(s) + b^\nu \frac{\delta \mathcal{K}_\nu^{(1)}}{\delta p''_\mu}. \end{aligned} \tag{2.95}$$

A direct calculation shows that, substituting these expression in action $\mathcal{S}_A$ one finds the same action for observer Bob[9] provided that one takes

$$\xi^\mu_{(i),B} = \xi^\mu_{(i),A} - b^\mu. \tag{2.96}$$

So this might be regarded as a prescription for "strong" translation transformations, that is the ξ's translate classically.

Furthermore, from Eqs.(2.95) for the finite worldline x^μ, one obtains a condition on the derivatives of $\mathcal{K}^{(i)}$ that must be satisfied for the theory to be

[8]For the worldline x^μ two different choices are possible, depending on what one adds to $\{(p\oplus q)_\nu, x^\mu\}$, either $0 = \{-k_\nu, x^\mu\}$ or $0 = \{-(p'\oplus p''\oplus q)_\nu, x^\mu\}$; thus, one can translate equivalently with $\mathcal{K}^{(0)}$ or with $\mathcal{K}^{(1)}$.

[9]Up to terms that do not add any other condition on the dynamical variables to those already obtained from the equations of motion and boundary conditions, so they can be safely neglected.

symmetric under this particular prescription for translation transformations. In fact, evaluating the two expressions at $s = s_0$ and $s = s_1$ one has

$$
\begin{aligned}
x^\mu_B(s_0) &= x^\mu_A(s_0) + b^\nu \frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} = -\xi^\nu_{(0),A} \frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} + b^\nu \frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} \\
&= x^\mu_A(s_0) - b^\nu \frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu} = -\xi^\nu_{(0),A} \frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} - b^\nu \frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu}, \\
x^\mu_B(s_1) &= x^\mu_A(s_1) + b^\nu \frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} = \xi^\nu_{(1),A} \frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu} + b^\nu \frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} \\
&= x^\mu_A(s_1) - b^\nu \frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu} = \xi^\nu_{(1),A} \frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu} - b^\nu \frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu},
\end{aligned}
$$

both requiring

$$
\frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu} = -\frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} \tag{2.97}
$$

in order for Alice and Bob to have boundary conditions of the same form $(x^\mu(s_i) = \pm \xi^\nu_{(i)} \frac{\delta \mathcal{K}^{(i)}_\nu}{\delta p_\mu})$ for the finite worldline x^μ. It is immediate seeing that this condition is indeed satisfied when the constraints $\mathcal{K}^{(i)}$ are written as in (2.94). In the special-relativistic limit $\ell \longrightarrow 0$ condition (2.97) is always trivial, for every non trivial term of the derivatives of $\mathcal{K}$ is proportional to ℓ; this aspect of these conditions will be further discussed during the analysis of the so-called Möbius diagram in section 6.2.

At this point it can be noticed that if one would have chosen to write the $\mathcal{K}$'s in the form

$$
\begin{aligned}
\mathcal{K}^{(0)}_\mu &= (k \oplus (\ominus p) \oplus (\ominus q))_\mu \\
&= k_\mu - p_\mu - q_\mu - \ell \delta^1_\mu ((k_0 - p_0) p_1 + q_1 (k_0 - p_0 - q_0))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}^{(1)}_\mu &= (p \oplus (\ominus p'') \oplus (\ominus p'))_\mu \\
&= p_\mu - p''_\mu - p'_\mu - \ell \delta^1_\mu ((p_0 - p''_0) p''_1 + p'_1 (p_0 - p'_0 - p''_0))
\end{aligned}
$$

condition (2.97) would not be satisfied, for

$$
\begin{aligned}
&\frac{\delta \mathcal{K}^{(1)}_\nu}{\delta p_\mu} = \delta^\mu_\nu - \ell \delta^1_\nu \delta^\mu_0 (p''_1 + p'_1) \neq \\
&\qquad \neq -\frac{\delta \mathcal{K}^{(0)}_\nu}{\delta p_\mu} = \delta^\mu_\nu - \ell \delta^1_\nu (\delta^\mu_0 (p_1 + q_1) - \delta^\mu_1 (k_0 - p_0)).
\end{aligned}
$$

The same must be said for the third possible choice of $\mathcal{K}$ previously considered, $\mathcal{K}^{(0)}_\mu = (k \oplus (\ominus(p \oplus q))_\mu$, $\mathcal{K}^{(1)}_\mu = (p \oplus (\ominus(p'' \oplus p'))_\mu$. Then, using the prescription for strong translation transformations (2.96), the only form of $\mathcal{K}^{(i)}$ suitable for a relativistic description of the finite worldline for Alice and Bob, purely translated with respect to each other, is that given in Eqs.(2.94). Otherwise, the theory obtained by adopting other expressions of $\mathcal{K}^{(i)}$ would describe non-local interactions. Thus, the role that $\mathcal{K}$ has in determining translation symmetry of the theory contributes to further reducing the possible sources of ambiguity in the choice of the appropriate form of $\mathcal{K}$, even among those which enforce equivalent conservation laws.

In Section 6 a weaker condition of the kind of (2.97) will be obtained from requiring that Alice and Bob, purely translated with respect to each other, describe finite worldlines in the same way, regardless of the specific form of constraints $\mathcal{K}^{(i)}$. These alternative translations are not explicitly constructed as it has done in this section, for it is an unnecessary exercise for the scope of this thesis, but it will be shown that, in principle, different prescriptions are admissible.

2.3.3 Physical velocity

The previous section presented some basic notions and key characterizing results of an explicit example of prescription for boundary terms, ensuring a relativistic description of distant observers within the Relative Locality framework, by a Lagrangian formulation of interacting particles. This section focuses on a first point of phenomenological relevance, concerning the observation of distant bursts of massless particles, which will be useful for subsequent discussions.

Consider the first part of the process studied in the previous section, that is the initial decay of the particle k, z in the particles p, x and q, y (vertex $\mathcal{K}^{(0)}$ in Fig. 2.3). For the scope of this section, the momenta q and p are assumed to be such that $|p| \gg |q|$, $\ell q \approx 0$ and $\ell p \neq 0$. Notice that this situation is also relevant for the description of observation of a gamma-ray burst, in which a high-energy pion (k, z) decays at the source into a high-energy ("hard") photon (p, x) and a low-energy ("soft") photon (q, y). It can be asked if and how the time of detection of the gamma ray depends on its momentum p, thereby obtaining a prediction for the large class of studies that are considering possible energy/time-of-arrival correlations for observations of gamma-ray bursts. The action describing the process is

$$\begin{aligned}\mathcal{S} = &\int_{-\infty}^{s_0} ds\,(z^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}(k)) + \int_{s_0}^{\infty} ds\,(x^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}(p)) + \\ &+ \int_{s_0}^{\infty} ds\,(y^\mu \dot{q}_\mu + \mathcal{N}_q \mathcal{C}(q)) - \xi^\mu_{(0)} \mathcal{K}^{(0)}_\mu,\end{aligned} \tag{2.98}$$

where again $\mathcal{K}^{(0)}_\mu = k_\mu - (p \oplus q)_\mu = k_\mu - p_\mu - q_\mu - \ell \delta^1_\mu p_0 q_1$. The equations of motion are exactly the same that were obtained in the previous section, of course. From the on-shell relation one finds the expression for p_0, at first order in ℓ,

$$p_0 = \sqrt{p_1^2 + m^2} - \frac{\ell}{2} p_1^2. \tag{2.99}$$

Then, for the massless case (or whenever $|\ell p_1| \gg m^2/p_1^2$) one finds the velocity

$$v^1 = \frac{\dot{x}^1}{\dot{x}^0} = \frac{-2p_1 + 2\ell p_0 p_1}{2p_0 + \ell p_1^2} \simeq -\frac{p_1}{|p_1|}(1 - \ell |p_1|). \tag{2.100}$$

For the choice of conventions here adopted, one needs $p_1 < 0$ in order to have $v^1 > 0$, and in such a case one has

$$v^1 = 1 + \ell p_1. \tag{2.101}$$

Then, Alice's description of the worldline of the particle (p, x) is $x^1_A(x^0_A) = \bar{x}^1_A + v^1(x^0_A - \bar{x}^0_A)$, with $\bar{x}^1, \bar{x}^0$ fixed. Assuming that both particles p, x and q, y are emitted at Alice's origin of spacetime coordinates, her description on the *inferred* propagation of the particles is simply

$$x^1_A(x^0_A) = (1 + \ell p_1) x^0_A, \qquad y^1_A(y^0_A) = y^0_A. \tag{2.102}$$

Since $-1 < \ell p_1 < 0$, from Alice's perspective the hard photon goes slower than the soft photon; therefore she infers that a distant observer Bob would measure a delay between the time-of-arrival of the two photons. But can this distant characterization of the relation between events be trusted? The two events that according to Alice are not coincident are the crossing of Bob's worldline with the worldline of the soft photon and the crossing of Bob's worldline with the worldline of the hard photon. To clarify the situation one should look at the two worldlines by Bob's perspective, since he is the one local to the detection.

For what concerns specifically the analysis of the problem so far reported in this section, the main challenge is related to the fact that one is used to read velocity from the formulae of worldlines, but this implicitly assumes that translation transformations are trivial. It is known that in classical spacetime with curvature the coordinate velocity may be affected by some coordinates artifact: for an observer in classical de Sitter spacetime, for example, the speed of a local photon is always 1, but this does not apply to the coordinate velocity that the observer attributes to distant photons. These features are not expected in a classical flat spacetime, where translations are trivial. In Relative Locality, however, the non triviality of translation transformations requires a more careful approach. Essentially, one is used to take the worldline written by Alice to describe both the emission of the photons "at Alice" (in Alice's origin) and their detection far away from Alice. The observer/detector Bob, who actually detects the photons, should be properly described by acting with a translation transformation on Alice's worldline. And the determination of the time-of-arrival at Bob should be determined on the basis of Bob's description of the worldline, just as much as the time-of-emission should be based on Alice's description of the worldline. When translations are trivial (translation generators conjugate to the spacetime coordinates) one can go by without worrying about this more careful level of discussion. This is because the naive argument based only on Alice's description of the worldline gives the same results as the more careful analysis using Alice's description of the worldline for the emission and Bob's description of the same worldline for detection. But when translations are nontrivial, as in Relative Locality, this luxury is lost. This will be shown for the case considered so far.

Bob's description of worldlines is, by dropping the contributions due to soft particles,

$$\begin{aligned} x^\mu_B(s) &= x^\mu_A(s) + b^\nu\{(p\oplus q)_\nu, x^\mu\} = x^\mu_A - b^\mu - \ell b^1\delta^\mu_0 q_1 \simeq x^\mu_A - b^\mu, \\ y^\mu_B(s) &= y^\mu_A(s) + b^\nu\{(p\oplus q)_\nu, y^\mu\} = y^\mu_A - b^\mu - \ell b^1\delta^\mu_1 p_0. \end{aligned} \tag{2.103}$$

Substituting these expressions in (2.102) one obtains

$$\begin{aligned} x^1_B(x^0_B) &= (1+\ell p_1)(x^0_B + b^0) - b^1, \\ y^1_B(y^0_B) &= (y^0_B + b^0) - b^1 - \ell b^1 p_0. \end{aligned} \tag{2.104}$$

One can then compute the delay between the two particles assuming that Bob detects the soft photon at its spacetime origin and the hard one at its spatial origin.

It is taken into account here that there are no relative-locality effects in the description given by Bob whenever the interactions occur "in the vicinity of Bob": the leading-order analysis assumes that the measuring apparatus has sensitivity sufficient to detect the manifestation of relativity of locality of order $\ell p_h L$ (where L is the distance from the interaction-event to the origin of the observer and p_h is a suitably high momentum), with L set in this case by the distance Alice-Bob; so even a hard-particle interaction which is at a distant d from Bob will be treated as absolutely local by Bob if $L \gg d$.

According to this, both "detection events" are absolutely local for Bob: of course this is true for the event of detection of the soft photon and it is also true for the interaction-event of "detection near Bob" of the hard photon. Ultimately this allows handling the time component of the coordinate four-vector as the actual delay that Bob measures between the detection times.

Thus, from the second of equations (2.104), setting $y_B^\mu = 0$ (detection at Bob's spacetime origin) one determines the translation parameter b^0 in terms of b^1: $b^0 = (1 + \ell p_0)b^1$. Substituting this in the first of equations (2.104) and setting $x_B^1 = 0$ (detection at Bob's spatial origin), one gets

$$(1 + \ell p_1)(x_B^0 + (1 + \ell p_0)b^1) - b^1 = 0 \tag{2.105}$$

from which, recalling the expression (2.99) and the sign convection on p_1,

$$x_B^0 = (1 - \ell p_1)b^1 - (1 + \ell p_0)b^1 = 0 + \mathcal{O}(\ell^2). \tag{2.106}$$

Therefore Bob does not measure any delay between the detections of the two photons, up to second order contributions. Only now one can conclude that the two particles have the same physical velocity, although they have different coordinate velocities.

The message that one should get from the discussion proposed in this section is not that massless particles have the same physical velocity under any conditions; the thesis author merely intended to discuss a representative example of the strategy of analysis of this kind of problems in Relative Locality that will be largely used in the following.

Chapter 3

Theories violating relativity of inertial frames

3.1 Hořava-Lifshitz gravity

In recent years, Hořava-Lifshitz gravity [41],[42] has attracted considerable interest in the quantum gravity community. Its basic idea is to break Lorentz symmetry through an anisotropic scaling between space and time in order to eliminate the divergences of the quantum field theory of gravity in the UV without ghosts. The next Subsection gives a very quick presentation of Hořava implementation of this idea in gravity, as in Subsection 3.1.2 it will be explained in greater clarity how anisotropic scaling can solve QFT divergences in a much simpler context such as an interacting scalar field theory.

3.1.1 Hořava proposal

It is known that an improved UV behavior of divergent quantum field theories, such as General Relativity, can be obtained if relativistic higher-derivatives corrections are added to the Lagrangian. Terms quadratic in spacetime curvature not only yield new interactions (with a dimensionless coupling), but they also modify the propagator. Schematically, denoting $p^2 = \omega^2 - k^2$, the propagator takes the form

$$\frac{1}{p^2} + \frac{1}{p^2} G_N p^4 \frac{1}{p^2} + \frac{1}{p^2} G_N p^4 \frac{1}{p^2} G_N p^4 \frac{1}{p^2} + ... = \frac{1}{p^2 - G_N p^4}. \tag{3.1}$$

At high energies it is dominated by the p^4 term. This cures the UV divergences, and in fact the calculations in Euclidean signature suggest that the theory exhibits asymptotic freedom. However, this cure simultaneously produces a new pathology, which prevents this modified theory from being a solution to the problem of quantum gravity. In fact, the propagator above exhibits two poles,

$$\frac{1}{p^2 - G_N p^4} = \frac{1}{p^2} - \frac{1}{p^2 - 1/G_N}. \tag{3.2}$$

One pole describes candidate massless gravitons, but the other corresponds to ghost exitations, which are states of negative norm. These are problematic because they can break unitarity, which is a key ingredient of quantum mechanics[1]. Violating unitarity in order to regularize the mathematical quantities may be regarded as quite a strong mutilation of the founding physical principles of the theory.

In contrast, breaking Lorentz symmetry to regularize the mathematical objects, while it is certainly a radical step, does not damage the logical foundations of the theory as it is more an experimental observation rather then a logical necessity. Hořava-Lifshitz gravity adopts this strategy to cancel the UV divergences of General Relativity, introducing an anisotropic scaling between space and time. This means that the theory will be symmetric under the transformation

$$\begin{aligned} \vec{x} &\to b\vec{x}, \\ t &\to b^z t. \end{aligned} \tag{3.3}$$

Such an anisotropic scaling is common in condensed matter systems, where the degree of anisotropy between space and time is characterized by the "dynamical critical exponent" z. Relativistic systems automatically satisfy $z = 1$ as a consequence of Lorentz invariance.

The techniques used in the construction of gravity models with anisotropic scaling in [42] follow methods developed in the theory of dynamical critical system [44],[45] and quantum criticality [46].

As a consequence of such anisotropy, the propagator of the graviton takes the form

$$\frac{1}{\omega^2 - c^2 k^2 - k^{2z} G} \tag{3.4}$$

where G is a coupling constant. In general there will be terms with powers of k^2 between 1 and z but one can simplify the discussion keeping the leading

[1]A way to include ghosts in the theory without breaking unitarity has been studied by Lee and Wick; in [43] they show that using a negative metric in quantum mechanics can lead to a unitary S-matrix, provided that all stable particle states are positive square length. In such a way, the negative-norm states are not asymptotic states and the unitarity of the S matrix is preserved.

term in the UV. In fact at high energy the propagator is clearly dominated by the anisotropic term $1/(\omega^2 - k^{2z}G)$. The high-energy behavior of the theory is controlled by a free-field fixed point with anisotropic scaling. For a suitably chosen z, this modification improves the short-distance behavior, shifting the dimension at which the theory is power counting renormalizable, so called "critical dimension". The ck^2 term in the propagator becomes important only at low energies. The massless dispersion relation $E^2 - p^2 - \ell^{2z-2}p^{2z} = 0$, suggested by this propagator, will be used in the later applications for the case of Hořava-Lifshitz gravity.

3.1.2 Lorentz symmetry breaking as a UV regulator

In order to obtain a basic understanding of how the anisotropic scaling between time and space can solve the divergences of the quantum theory of gravity, without getting lost in the huge algebra of the full theory, it is here briefly shown how how it works in a simple scalar field theory[2].

Consider the following action of a scalar field in flat $(d+1)$-dimensional spacetime

$$S_{free} = \int d^d x\, dt\, [\dot{\phi}^2 - \phi(-\Delta^z)\phi], \tag{3.5}$$

where $\Delta = \nabla^2$ is the spatial Laplacian. Notice that here the units are such that the coefficient in front of the kinetic term is the same as that of the spatial derivative term, which is not the common $c = 1$ set of units; Planck constant is set to be $\hbar = 1$. In these units one has that $[\partial_t] = [\nabla]^z$ and $[dt] = [dx]^z$. But since the action has to be dimensionless one has that $[\phi] = [dx]^{(z-d)/2}$. This suggests that the case $z = d$ will play a special role in the discussion, since the scalar field would then be dimensionless. It is convenient to define formal symbols κ and m having dimension of momentum and energy, $[\kappa] = [dx]^{-1}$ and $[m] = [dt]^{-1}$ respectively. It can also be noticed that $[\phi] = [\kappa]^{(d-z)/2} = [m]^{(d-z)/2z}$.

Consider now also the various sub-leading terms to this free Lagrangian

$$S_{free} = \int dt\, d^d x\, [\dot{\phi}^2 - \phi(m^2 - c^2\Delta + ... + (-\Delta)^z)\phi]. \tag{3.6}$$

Notice that $[c] = [dx/dt] = [dx]^{1-z} = [\kappa]^{z-1}$, which is the reason for which, with the choice of units explained earlier, one does not have the freedom to set $c = 1$, unless the trivial case $z = 1$ is under consideration.

[2]The interest reader can find a broader discussion of this topic in [47].

Consider now a polynomial interaction

$$S_{interaction} = \int dt\, d^d x\, P(\phi) = \int dt\, d^d x \sum_{n=1}^{N} g_n \phi^n. \qquad (3.7)$$

The couplings have dimension $[g_n] = [\kappa]^{d+z-n(d-z)/2}$. So the couplings have non negative dimension as long as

$$d + z - \frac{n(d-z)}{2} \geq 0. \qquad (3.8)$$

Since z, d and n are all positive integers by definition this is equivalent to either

$$n \leq \frac{2(d+z)}{d-z} \quad \text{if} \quad z < d,$$

or

$$z \leq \infty \quad \text{if} \quad z \geq d.$$

Consider now a generic Feynman diagram with L loops and I internal propagators. For each internal line one has a Lorentz violating propagator

$$G(\omega, \vec{k}) = \frac{1}{(\omega_L - \omega_e)^2 - (m^2 + c^2(\vec{k}_L - \vec{k}_e)^2 + ... + (\vec{k}_L - \vec{k}_e)^{2z})}, \qquad (3.9)$$

where ω_e and $\vec{k}_e$ are some linear combination of the external momenta, and ω_L and $\vec{k}_L$ are the loop energy and momentum. Each loop integral contributes to the total dimension as

$$\int d\omega\, d^d k \rightarrow [d\omega][dk]^d = [\kappa]^{d+z}$$

and for each propagator one has instead $[G(\omega, \vec{k})] = [\kappa]^{-2z}$. The total contribution for dimensionality coming from loop integrals for the entire Feynman diagram is

$$\delta = (d+z)L - 2zI = (d-z)L - 2(I-L)z, \qquad (3.10)$$

which reproduces the standard result in the case $z = 1$. Since the number of internal propagators I is always at least equal to the number of loops, one has

$$\delta \leq (d-z)L. \qquad (3.11)$$

It is a standard result that if the superficial degree of divergence is negative, and the superficial degree of divergence of every internal sub-graph is negative, then the Feynman diagram is convergent. Therefore, if one choose $d = z$ then one has $\delta \leq 0$ for any diagram, and the worse divergence one can meet is logarithmic, which can occur only when $L = I$ which are the so-called "rosette" Feynman diagram. This observation is enough to guarantee that the theory is power counting renormalizable.

3.2 Rainbow Gravity

As previously stated, the most promising opportunities for quantum gravity phenomenology come from the propagation of high-energy particles from a source at cosmological or astrophysical distance and it is therefore important to consider also the effects due to the geometry of spacetime on large scales. Indeed, the scope of Rainbow Gravity is to include Planck scale corrections to Einstein's theory of gravity. The next subsection is devoted to introducing the original proposal by Magueijo and Smolin. Subsection 3.2.2 briefly reviews the most recent approach to the original purpose of Rainbow gravity, using the technology of Finsler geometry.

3.2.1 Magueijo-Smolin Rainbow Gravity

Rainbow Gravity has first been proposed in [48] with the goal to extend the idea of DSR to General Relativity. The theory does not mean to be fundamental but rather a leading correction to a classical spacetime picture coming from a full quantum spacetime theory. Therefore, the main interest resides in computing effects at leading order in Planck scale on the propagation of quanta with energies smaller then the Planck scale E_P but with wavelengths much shorter than the local radius of curvature R. This latter assumption allows then not to take into account terms in $R\frac{\partial p}{p}$ which should be considered otherwise.

The starting point is the deformed dispersion relation

$$f^2(\ell E)E^2 - g^2(\ell E)p^2 = m^2, \tag{3.12}$$

where f and g are arbitrary functions and ℓ is a length scale which is assumed to be of the order of the Planck length. This can be obtained by the action of a non-linear map from momentum space to itself, denoted, $U : \mathcal{P} \to \mathcal{P}$, given by

$$U \cdot (E, p_i) = (U_0, U_i) = (f(\ell E)\, E, g(\ell E)\, p_i) \tag{3.13}$$

which implies that momentum space has a non-linear norm of the form

$$p^2 = \eta^{ab} U_a(p) U_b(p). \tag{3.14}$$

This norm is preserved by a non-linear realization of the Lorentz group, given by

$$\tilde{L}_a^b = U^{-1} \cdot L_a^b \cdot U \tag{3.15}$$

where L are the usual Lorentz generators.

Theories with deformed Lorentz transformations are usually formulated on momentum space. In order to develop the spacetime counter part, a suitable definition of the dual space has been looked for. This is a non trivial task due to the fact that the momentum transformation are non-linear (among the different answers proposed there are also non-commutative geometries, such as κ-Minkowksi non-commutative spacetime). Rainbow Gravity instead assumes that the research for a single dual space is not strictly necessary, since there is no single classical spacetime geometry when effects of order ℓE are taken into account. Instead, one has to consider a family of one-parameter spacetime metrics that describe the leading corrections to the classical spacetime, parametrized by ℓE. So, just as the properties of a material may depend on the energy of the phonon propagating through it, Rainbow Gravity adopts the view that the geometry of spacetime may depend on the energy of the particle moving in it. The Einstein equivalence principle can be maintained, with the specification that it is valid for regions of spacetime for which the radius of curvature is much larger then ℓ and that the particles moving in it have energies much below ℓ^{-1}. One further requires that in the limit $\ell E \to 0$ General Relativity is recovered.

It must be stressed that the parameter ℓE does not represent the energy of spacetime, but the energy scale at which it is probed according to a particular observer. Therefore, if an observer uses the motion of a particle or a system of particles to measure the geometry of the spacetime, E is the total energy of that particle or system of particles, as measured by that observer.

Another way to describe these properties is by saying that, in the absence of gravity, spacetime has an energy-dependent geometry, in the sense that particles of energy E move in a geometry given by an energy-dependent set of orthonormal frame fields,

$$e_0 = f^{-1}(\ell E)\tilde{e}_0, \quad e_i = g^{-1}(\ell E)\tilde{e}_i \tag{3.16}$$

where the tilde quantities represent energy-independent frame fields that specify the geometry probed by low energy particles. The metric given by

$$g(E) = \eta^{ab} e_a \otimes e_b \tag{3.17}$$

is flat for all E. The object $g(E)$ can be considered as a one-parameter family of flat rainbow metrics, parametrized by E. The metrics share the same set of inertial frames but, due to scalings, generally they do not share all their geodesics; instead, geodesics are generally energy-dependent. This is equivalent to saying that the energy-momentum relations are no longer quadratic.

The Rainbow Gravity picture is closely related to the work presented in [49] for constructing position space in DSR. In this approach one requires that free field theories in flat spacetime have plane waves solutions, even though the 4-momentum they carry satisfies deformed dispersion relations. For this to be possible the contraction between position and momentum providing the phase for such waves must remain linear, that is,

$$dx^a p_a = dx^0 p_0 + dx^i p_i. \tag{3.18}$$

If momentum transforms non-linearly then the dx^a transformations must be energy-dependent, as explained in [49]. Authors claim that, for a U of the form given above, spacetime dual has invariant

$$ds^2 = \frac{dt^2}{f^2(E)} - \frac{dx^2}{g^2(E)}. \tag{3.19}$$

Thus, the dual space dx^a is endowed with an energy-dependent quadratic invariant, that is an energy-dependent metric.

This example further elucidates the meaning of E in the metric. If a given observer sees a particle (or a plane wave, or a wave packet) with energy E, then he concludes that this particle is probing the metric $g(E)$. If the particle has energy $E' \neq E$ for a different observer, then the latter will assign to spacetime a different metric $g(E')$. Of course, as required by covariance, if the first observer probes the spacetime using two particles with different energies E and E' then it will attribute a different metric to each particle, even at the same spacetime coordinates.

Essentially, this construction justifies, in some sense, the naive guess that, if the dispersion relation is given in metric terms as $m^2 = g^{\mu\nu}(E)p_\mu p_\nu$ and is a (deformed) Lorentz scalar, then the spacetime metric is the tensor $g_{\mu\nu}(E)$ such that $g_{\mu\nu}(E)g^{\nu\sigma}(E) = \delta^\sigma_\mu$ and $ds^2 = g_{\mu\nu}(E)dx^\mu dx^\nu$ is also a scalar.

The reason for which this formulation of Rainbow Gravity breaks Lorentz symmetry is that the dispersion relation is indeed invariant under the deformed boosts, but the line element is not [50]. Consider for example the very commonly studied DSR dispersion relation

$$\mathcal{C} = a^{-2}(\eta)(\Omega^2 - \Pi^2) + \ell a^{-3}(\eta)(\gamma\Omega^3 + \beta\Omega\Pi^2) = m^2, \tag{3.20}$$

where (η, x) are the conformal coordinates on spacetime and (Ω, Π) are their conjugate momenta, $a(\eta)$ is the scale factor, β and γ two numerical parameters. Consider for simplicity of argument on the static case $a(\eta) = 1$ in two dimensions. Denoting the conjugate momenta in the flat case (p_0, p_1), one can write the dispersion relation as $\mathcal{C} = (1 + \ell\gamma p_0)p_0^2 - (1 - \ell\beta p_0)p_1^2$, and the line element associated with it is, at first order in ℓ,

$$ds^2 = (1 - \ell\gamma p_0)(dx^0)^2 - (1 + \ell\beta p_0)(dx^1)^2. \tag{3.21}$$

The dispersion relation (3.20) is invariant under a ℓ-deformed Lorentz boost

$$\mathcal{N} = x^0 p_1(1 - \ell\gamma p_0) + x^1 \left(p_0 + (\beta + \frac{\gamma}{2})\ell p_0^2 + \frac{\ell}{2}\beta p_1^2 \right), \tag{3.22}$$

as it can be shown that the Poisson bracket $\{\mathcal{N}, \mathcal{C}\} = 0$. This guarantees that the dispersion relation is in fact invariant also for a finite boost, since the action of a boost on an observable O can be expressed as

$$O' = O + \xi\{\mathcal{N}, O\} + \frac{\xi^2}{2!}\{\mathcal{N}, \{\mathcal{N}, O\}\} + ...$$

where ξ is the rapidity parameter. Under the same action of the boost, the line element (3.21) is not invariant, as it transform to

$$(ds^2)' = ds^2 - \ell\xi(\beta p_1(dx^1)^2 + \gamma p_1(dx^0)^2). \tag{3.23}$$

This non-invariance is evidently problematic from a relativistic point of view, as the norm of vectors would not be invariant under such transformation. This is the reason for which, even if the initial goal of Rainbow Gravity is to preserve the relativity of local inertial frames, it is in fact breaking Lorentz symmetry.

3.2.2 Connection with Finsler geometry

The original program of Rainbow Gravity has been further investigated and more rigorously understood in terms of a generalization of Riemannian geometry known as Finsler geometry in Ref.[51]. In Ref.[52],[50] the connection between Finsler geometries and DSR-relativistic theories has been clarified in greater details.

Finsler geometry fundamental ingredient is the norm $F(x, v)$, a real function of a spacetime point x and a tangent vector v, such that it satisfies the usual norm properties, that is

$$\begin{aligned} F(x, v) \neq 0 \;\text{ if }\; v \neq 0, \\ F(x, \lambda v) = |\lambda| F(x, v), \end{aligned} \tag{3.24}$$

where λ is a real number. From the norm squared $F^2(x, v)$ one can define the so called Finsler metric

$$g_{\mu\nu}(x, v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^\mu \partial v^\nu}, \tag{3.25}$$

which is required to be continuous and non-degenerate. Using Euler's theorem, stating that if $f(x)$ is a homogeneous function of degree r, then $x^i \frac{\partial f}{\partial x^i} = rf(x)$, it can be shown that (3.25) is equivalent to

$$F(x, v) = \sqrt{g_{\mu\nu}(x, v) v^\mu v^\nu}. \tag{3.26}$$

This shows that $g_{\mu\nu}(x, v)$ is a homogeneous function of degree zero of the vector v. Also, since by definition is non-degenerate, it admits an inverse $g^{\mu\nu}(x, v)$ such that $g_{\mu\nu}(x, v) g^{\nu\sigma}(x, v) = \delta_\mu^\sigma$. From the norm $F(x, v)$ one can also derive the norm for a form ω as

$$G(x, \omega) = F(x, v(\omega)), \tag{3.27}$$

and the metric on the dual space

$$h^{\mu\nu}(x, \omega) = \frac{1}{2} \frac{\partial G^2(x, \omega)}{\partial \omega_\mu \partial \omega_\nu} = g^{\mu\nu}(x, v(\omega)). \tag{3.28}$$

The action of a particle moving on a Finsler manifold is

$$\mathcal{S} = m \int F(x, \dot{x}) ds \tag{3.29}$$

which from (3.26) takes the form of a straightforward generalization of the standard relativistic particle action

$$\mathcal{S} = m \int \sqrt{g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu} ds.$$

Using Euler-Lagrange equations of motion one finds the momenta

$$p_\mu = m \frac{\partial F}{\partial \dot{x}^\mu} = m \frac{g_{\mu\nu}(x, \dot{x}) \dot{x}^\nu}{F}, \tag{3.30}$$

which satisfies the generalized on-shell relation

$$h^{\mu\nu}(x, p) p_\mu p_\nu = m^2 g^{\mu\nu} \frac{g_{\mu\rho} \dot{x}^\rho g_{\nu\sigma} \dot{x}^\sigma}{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} = m^2. \tag{3.31}$$

In order to deduce the Finsler spacetime metric corresponding to a particular dispersion relation, as in the spirit of Rainbow Gravity, one should start from the action

$$\mathcal{S} = \int ds [\dot{x}^\mu p_\mu - \lambda \left(\mathcal{C}_p - m^2 \right)] \tag{3.32}$$

where λ is a Lagrange multiplier enforcing on-shell relation $\mathcal{C}_p = m^2$. Using Hamilton equation

$$\dot{x}^\mu = \lambda \frac{\partial \mathcal{C}_p}{\partial p_\mu}, \tag{3.33}$$

one can express momenta p in terms of velocities $\dot{x}$ and find the action

$$\mathcal{S} = \int l(\dot{x}, \lambda). \tag{3.34}$$

Then by varying the action with respect to the Lagrange multiplier λ one can express it in terms of the velocities as well and obtaining the Lagrangian

$$\mathcal{S} = \int \mathcal{L}(\dot{x}, \lambda(\dot{x})), \tag{3.35}$$

from which one can identify Finsler norm

$$F(x, \dot{x}) = \frac{\mathcal{L}(x, \dot{x})}{m} \tag{3.36}$$

which satisfies the properties of a Finsler norm (3.24). Form this one can obtain the spacetime metric as already shown.

Chapter 4

Introducing the thermal dimension of quantum spacetime

This chapter is dedicated to one of the original results of this thesis [116], concerning the problem of the physical characterization of the dimension of spacetime at scales comparable to Planck length. The next session reviews what is a notion of dimensionality of spacetime which is broadly used in the QG community, the spectral dimension. The original proposal of thermal dimension of spacetime is then presented and its physical properties are compared with those of the spectral dimension, using examples of deformed dispersion relation inspired by the QG models reviewed in the previous chapters.

4.1 The spectral dimension

The spectral dimension has been proposed as a possible observable characterizing the geometry in discrete quantum gravity [53] and attracted a lot of interest in causal dynamical triangulations (CDTs) since finding meaningful observable in discrete geometry is a non trivial task. The hope of the communities working on discrete geometry is that such observable may provide the much needed connection between the discrete theory and its continuum limit. The spectral dimension can also be defined in continuum quantum gravity models and can be used to characterize and understand their short-distance behavior (see [55],[68],[66],[56]). Furthermore, it was shown in [55] that both CDTs and Hořava-Lifshitz gravity lead to a value of 2 for the spectral dimension in the UV, while it matches the value of the topological dimension in the IR. These results encouraged the use of the spectral dimension as a tool in the process of linking the discrete and continuum theories.

Here the basic definition of the spectral dimension, whose origin is Riemannian geometry, will be given; the following section will briefly show how it is linked to the dispersion relation of the theory in consideration. It has been shown in fact in [73] that, given a specified topological dimension n, it is possible to define a scale-dependent notion of spectral dimension for any arbitrary dispersion relation. Furthermore, also the other deductive way is possible: given a certain spectral dimension as a function of the diffusion time s, it is possible, in principle, to reconstruct the dispersion relation.

4.1.1 Basic definitions

The spectral dimension can be viewed as an effective notion of dimension defined through a fictitious diffusion process on a certain discrete geometry. In practice the diffusion process can be thought as a stochastic random walk, and the spectral dimension is defined in terms of the average return probability $P(s)$.

In the classical Brownian motion, the diffusion of the particle is described by the differential "heat" equation

$$\frac{\partial}{\partial t}K(x,y;t) - b\Delta_x K(x,y;t) = 0 \tag{4.1}$$

where b is a constant, t is the diffusion time, $K(x,y;t)$ is the probability density for the particle to diffuse from point x to point y in a time t and the initial condition $K(x,y;0) = \delta(x-y)$ indicates the point-like nature of the particle.

Similarly, the diffusion process on a n-dimensional Euclidean geometry with a fixed smooth metric $g_{\mu\nu}(x)$ is governed in fact by the equation the heat equation

$$\partial_s K(x,y;s) - \Delta_x K(x,y;s) = 0, \tag{4.2}$$

with the initial condition $K(x,y;0_+) = \delta(x-y)g^{-1/2}(x)$. Here $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplacian and ∇_ν is the covariant derivative. The parameter s plays the role of fictitious diffusion time and $K(x,y;s)$ is the probability density of diffusion from the event x to the event y in a "time" s. The return probability is then easily defined as

$$P(s) = \frac{\int d^n x\, g^{1/2} K(x,x;s)}{\int d^n x\, g^{1/2}} \approx \frac{1}{(4\pi s)^{n/2}} \sum_{i=0}^{\infty} a_i s^i, \tag{4.3}$$

where the coefficients are metric-dependent invariants which can be computed via recursion formulas, with $a_0 = 1$.

For an infinite flat space the solution to the heat equation is given by

$$K(x, y; s) = \frac{e^{-d_g^2(x,y)/4s}}{(4\pi s)^{n/2}} \tag{4.4}$$

where $d_g(x, y)$ is the geodesic distance between the two points. It follows that $\sqrt{s}$ is an effective measure of the spread of the Gaussian at diffusion time s. Because $P_g(s) = s^{-n/2}$ in the flat case, one can obtain the dimension n of the manifold by taking the logarithmic derivative of the return probability, defining the spectral dimension,

$$d_s \equiv -2\frac{\partial \log P_g(s)}{\partial \log s} = n, \tag{4.5}$$

where the last equality is true only in the flat case.

For curved spaces and/or finite spaces of volume V one can still use Eq. (4.5) to extract the dimension, but there will be correlations for sufficiently large s. For a curved space, probing a diffusion scale comparable or larger than the radius of curvature will affect the value of the spectral dimension via the details of the geometry of the space and the presence of gravitational sources. The spectral dimension then would deviate from the topological dimension as an effect of the curvature. At intermediate scales, smaller then the radius of curvature but larger then the Planck scale, the space is effectively flat and the spectral dimension has the same value of the topological dimension, as shown above. At scales comparable to Planck scale the deviation of the spectral dimension from the topological dimension is due to effects other then curvature.

4.1.2 Connection with the dispersion relation

Further interest in the spectral dimension comes from the work of Sotiriou, Visser and Weinfurtner [73], in which they demonstrated that the spectral dimension is not necessarily intrinsically geometric. At scales small enough for curvature effects to be negligible, its deviation from the topological dimension actually becomes an analytic property of the differential operator that one is using as input to define the fictitious diffusion process. In turn, this operator acts as the propagator of some dynamical degree of freedom in flat space. In this sense, the spectral dimension acts, at suitable scales, as a probe of the kinematics of the particular degree of freedom, allowing to deduce a dispersion relation; therefore the spectral dimension is an interesting observable even for those theories for which is difficult to find the return probability of a diffusion process on their quantum spacetimes, but which have a modified dispersion relation.

Consider in fact a $(n+1)$-dimensional spacetime and a dispersion relation $E = E(p)$. This can always be viewed as completely specified by the solution of the differential equation

$$D_L\Phi = (-\partial_t^2 - f(-\nabla^2))\Phi = 0, \tag{4.6}$$

where $f(p^2) = E(p)^2$. The reason for which the time derivative is only second order is that in this case the differential equation encoding the dispersion relation can typically be derived by a ghost-free Lagrangian,

$$L = \frac{1}{2}\Phi D_L \Phi. \tag{4.7}$$

In order to compute the spectral dimension one has first to Wick rotate the physical time t to consider Euclideanized differential operator D_E in $n+1$ topological dimension

$$D_E\Phi = (-\partial_t^2 + f(-\nabla^2))\Phi. \tag{4.8}$$

The diffusion process is governed by the equation

$$\frac{\partial}{\partial s}K(x, y; s) + D_E K(x, y; s) = 0, \tag{4.9}$$

with the initial condition $K(x, y; 0) = \delta^{n+1}(x - y)$. Again, x is the set $(t, \vec{x})$ and s is an auxiliary "fictitious diffusion time" or, more properly, a parameter characterizing the scale at which the particle is probing the spacetime. The general solution of the differential equation above is

$$K(x, y; s) = \int \frac{dE\, d^n p}{V(2\pi)^{n+1}} e^{i(\vec{p}\cdot(\vec{x}-\vec{y})+E(x^0-y^0))} e^{-s(E^2+f(p^2))}, \tag{4.10}$$

and the return probability is then

$$P(s) = \int d^n x\, K(x, x; s) = \int \frac{dE\, d^n p}{(2\pi)^{n+1}} e^{-s(E^2+f(p^2))}. \tag{4.11}$$

Factorizing it in the time-like and space-like contribution

$$P(s) = \int \frac{dE}{(2\pi)} e^{-sE^2} \int \frac{d^n p}{(2\pi)^n} e^{-sf(p^2)} = \frac{1}{\sqrt{4\pi s}} \int \frac{d^n p}{(2\pi)^n} e^{-sf(p^2)}, \tag{4.12}$$

one obtains

$$\ln P(s) = -\frac{1}{2}\ln s + \ln \int dp\, p^{n-1} e^{-sf(p^2)} + C \tag{4.13}$$

where C is a constant. Taking the derivatives with respect to $\ln s$ one gets the expression for the spectral dimension

$$d_S(s) = 1 + 2s\frac{\int dp\, p^{n-1} f(p^2) e^{-sf(p^2)}}{\int dp\, p^{n-1} e^{-sf(p^2)}}. \quad (4.14)$$

Recalling now that $E^2 = E(p)^2 = f(p^2)$, one can write

$$d_S(s) = 1 + 2s\frac{\int dp\, p^{n-1} E(p)^2 e^{-sE(p)^2}}{\int dp\, p^{n-1} e^{-sE(p)^2}}. \quad (4.15)$$

Note that the contribution 1 comes from the fact that the time derivatives appear only in the term ∂_t^2. If one has to consider more general operators such as $D\Phi = f(\partial_t^2, \nabla^2)\Phi$, the dispersion relation is expressed implicitly by $\mathcal{C}_p(E^2, p^2) = 0$. The return probability is then

$$P(s) = \int \frac{dE\, d^n p}{(2\pi)^{n+1}} e^{-s\mathcal{C}_p(E^2,p^2)}, \quad (4.16)$$

and therefore the spectral dimension is

$$d_S(s) = 2s\frac{\int dE\, d^n p\, \mathcal{C}_p(E^2, p^2) e^{-s\mathcal{C}_p(E^2,p^2)}}{\int dE\, d^n p\, e^{-s\mathcal{C}_p(E^2,p^2)}}. \quad (4.17)$$

This shows that from an arbitrary dispersion relation (but of the kind in which energy can be expressed in terms of the momentum) and specified topological dimension n a suitable differential operator can be construct that encode the dispersion relation and this can be used to define the corresponding spectral dimension. To show that the other way around is possible, one may notice that, defining the "partition function"

$$Z(s) = \int dp\, p^{n-1} e^{-sE(p)^2}, \quad (4.18)$$

one can write Eq.(4.15) as

$$d_S(s) = 1 - 2s\frac{dZ(s)}{ds}. \quad (4.19)$$

Note that the function $Z(s)$ encodes relatively simple information on the dispersion relation of the degree of freedom in consideration. If a theory gives us only the possibility to study the spectral dimension but it does not have a differential operator (as in the case of CDT) one can infer an effective dispersion relation by inverting formally Eq.(4.15) as a function of s,

$$\frac{Z(s)}{ds} = -\frac{d_S(s) - 1}{2s}, \quad (4.20)$$

from which

$$Z(s) = Z(s_0)e^{-\frac{1}{2}\int_{s_0}^{s} ds' \frac{d_S(s')-1}{s'}}. \tag{4.21}$$

The aim though is not to know just the function $Z(s)$ but to obtain the function $E(p)$. In order to get it, one can write the partition function as

$$Z(s) = \frac{1}{n}\int_0^\infty dE^2 \frac{dp^n(E)}{dE^2} e^{-sE^2}. \tag{4.22}$$

Integrating by parts one obtains

$$\int_0^\infty dE^2\, p^n(E) e^{-sE^2} = \frac{n}{s} Z(s), \tag{4.23}$$

which has the form of a Laplace transformation, in the variable E^2, of the function $p^n(E)$.

Implementing the inverse Laplace transformation via complex integration, one has

$$p^n(E) = \frac{1}{2\pi i}\int_C ds\, \frac{n}{s} Z(s) e^{sE^2}, \tag{4.24}$$

where C is an appropriate contour in the complex plane and $Z(s)$ is given by Eq.(4.21). Therefore, one can compute the effective dispersion relation for the degree of freedom in consideration when the spectral dimension is analytically known as a function of s on the complex plane.

4.2 Thermal dimension

As it has already been mentioned in the introduction, many different quantum gravity models share the common feature of "dynamical dimensional reduction": the familiar four-dimensional classical picture of spacetime in the IR is replaced by a quantum picture with an effective number of spacetime dimensions smaller than four in the UV.

This phenomenon has been studied mostly in terms of the spectral dimension, which provides a valuable characterization of properties of classical Riemannian geometries [60, 63], but its proposed applicability to the description of the dimension of a quantum spacetime involves some adaptations, as described in the previous section. In this section it will be shown that these adaptations are responsible for some of its inadequacies.

When the IR Hausdorff dimension of spacetime is $D+1$, and the Euclidean d'Alembertian of the theory is represented on momentum space as $\mathcal{C}_p^{Euc}(E,p)$, the return probability is given by[1]

$$P(s) \propto \int dE\, dp\, p^{D-1}\, e^{-s\mathcal{C}_p^{Euc}(E,p)}\,. \qquad (4.25)$$

The fact that the Euclidean version of the d'Alembertian intervenes should be cause of concern [2]. It is in fact well known that the Euclidean version of a quantum-gravity model can be profoundly different from the original model in Lorentzian spacetime (see, *e.g.*, Ref.[67]). Moreover, evidently in (4.25) an important role is played by off-shell modes, a role so important that, as it will be here shown, one can obtain wildly different values for the spectral dimension for different formulations of the same physical theory (cases where the formulations coincide on-shell but are different off-shell). It is also concerning the fact that evidently the $P(s)$ of (4.25) is invariant under active diffeomorphisms on momentum space (an active diffeomorphism on momentum space amounts to an irrelevant change of integration variable for $P(s)$). Since an active diffeomorphism can map a given physical theory into a very different one (also see here below), this degeneracy of the spectral dimension is worrisome.

While these concerns are very serious, it must be acknowledged that several analyses centered on the spectral dimension give rather meaningful results. Therefore, the guiding idea is that it is necessary to replace the spectral dimension with some other fully physical notion of dimensionality of a quantum spacetime, with the requirement that in most cases the new notion should agree with the spectral dimension. Only when the unphysical content of the spectral dimension plays a particularly significant role should the new notion differ significantly from the spectral dimension. The guidance adopted in searching for such a new notion is the observation reported in recent studies [76, 129, 78] (see also [79] for earlier related proposals) that in some instances the Stefan-Boltzmann law gives indications on the dimensionality of spacetime that are consistent with the spectral dimension. One can view the Stefan-Boltzmann law as an indicator of spacetime dimensionality since for a gas of radiation in a classical spacetime with $D+1$ dimensions the Stefan-Boltzmann law takes the form

$$U \propto T^{D+1}. \qquad (4.26)$$

[1] The thesis supported here is that even if (4.25) did describe the return probability (as usually assumed) still the spectral dimension would be unsatisfactory. It is interesting however that, as stressed in Ref. [64], the interpretation of (4.25) as return probability is not always applicable.

[2] Concerns for the Euclideanization involved were also raised in Ref.[65], within a study concerning the causal-set approach. Ref.[65] proposed a possible redefinition of the spectral dimension suitable for including Lorentzian signature and found that it gave different results with respect to the standard (Euclideanized) spectral dimension.

Actually several thermodynamical relations are sensitive to the dimensionality of spacetime, another example being the equation of state parameter $w \equiv P/\rho$, relating pressure P and energy density ρ, which for radiation in a classical spacetime with $D+1$ dimensions takes the form

$$w = \frac{1}{D}. \tag{4.27}$$

These observations inspire the proposal of assigning a "thermal dimension" to a quantum spacetime. The recipe presented in this thesis involves studying the thermodynamical properties of radiation with on-shellness characterized by the (deformed) d'Alembertian of the relevant quantum-spacetime theory (the same deformed d'Alembertian used when evaluating the spectral dimension, but in its Lorentzian form). By looking at the resulting Stefan-Boltzmann law and equation of state one can infer the effective dimensionality of the relevant quantum spacetime. This notion of dimensionality has the advantage of being directly observable, a genuine physical property of the quantum spacetime, and, as it will be here shown, fixes the shortcomings of the spectral dimension, while agreeing with it in some particularly noteworthy cases.

4.2.1 Application to generalized Hořava-Lifshitz scenarios

To start the quantitative part of the present study, consider a class of generalized Hořava-Lifshitz scenarios, which has been the most active area of research on dynamical dimensional reduction [55, 73, 61]. These are cases where the momentum-space representation of the deformed d'Alembertian takes the form

$$\mathcal{C}_{\gamma_t \gamma_x}(E,p) = E^2 - p^2 + \ell_t^{2\gamma_t} E^{2(1+\gamma_t)} - \ell_x^{2\gamma_x} p^{2(1+\gamma_x)} \,. \tag{4.28}$$

where E is the energy, p is the modulus of the spatial momentum, γ_t and γ_x are dimensionless parameters, and ℓ_t and ℓ_x are parameters with dimension of length (usually assumed to be of the order of the Planck length).

For this model it is known [61, 73] that the UV value of the spectral dimension, obtained from the Euclidean version of the above d'Alembertian $(E^2 + p^2 + \ell_t^{2\gamma_t} E^{2(1+\gamma_t)} + \ell_x^{2\gamma_x} p^{2(1+\gamma_x)})$, is

$$d_S(0) = \frac{1}{1+\gamma_t} + \frac{D}{1+\gamma_x} \,. \tag{4.29}$$

In deriving the thermal dimension for this case one can start from the logarithm of the thermodynamical partition function [80], written so that the integration is explicitly taken over the full energy-momentum space:

$$\log Q_{\gamma_t\gamma_x} = -\frac{2V}{(2\pi)^3}\int dE\, d^3p \left[\delta(\mathcal{C}_{\gamma_t\gamma_x})\,\Theta(E)\,\cdot \right.$$
$$\left. \cdot\, 2E\log\left(1-e^{-\beta E}\right)\right]. \qquad (4.30)$$

Here β is related to the Boltzmann constant k_B and temperature via $\beta = \frac{1}{k_B T}$, and the delta function $\delta(\mathcal{C}_{\gamma_t\gamma_x})$ enforces the on-shell relation $\mathcal{C}_{\gamma_t\gamma_x} = 0$.

From (4.30) one obtains the energy density and the pressure respectively as

$$\rho_{\gamma_t\gamma_x} \equiv -\frac{1}{V}\frac{\partial}{\partial\beta}\log Q_{\gamma_t\gamma_x}\ , \quad p_{\gamma_t\gamma_x} \equiv \frac{1}{\beta}\frac{\partial}{\partial V}\log Q_{\gamma_t\gamma_x}\,. \qquad (4.31)$$

Figure 4.1 shows (for a few choices of γ_x, γ_t) the resulting temperature dependence for the energy density and for the equation of state parameter. For the UV/high-temperature values of $\rho_{\gamma_t\gamma_x}$ and $w_{\gamma_t\gamma_x}$ one can easily establish the following behaviors at high temperature, in agreement with the content of Figure 4.1

$$\rho_{\gamma_t\gamma_x} \propto T^{1+3\frac{1+\gamma_t}{1+\gamma_x}}\ , \quad w_{\gamma_t\gamma_x} = \frac{1+\gamma_x}{3(1+\gamma_t)}\,. \qquad (4.32)$$

By comparison to (4.26) and (4.27) one sees that both of these results give a consistent prediction for the “thermal dimension" at high temperature, which is

$$d_T = 1 + 3\frac{1+\gamma_t}{1+\gamma_x}\,. \qquad (4.33)$$

Interestingly, in this case of generalized Hořava-Lifshitz scenarios the thermal dimension agrees with spectral dimension, Eq. (4.29), for $\gamma_t = 0$, but differs from the spectral dimension when $\gamma_t \neq 0$.

4.2.2 Implications of active diffeomorphisms on momentum space

Generalized Hořava-Lifshitz scenarios also give us an easy opportunity for comparing the properties of the thermal dimension and of the spectral dimension under active diffeomorphisms on momentum space. From this perspective the analysis is particularly simple for the case $\gamma_x = 0, \gamma_t = 1$, where one has

$$\mathcal{C}_{1,0}(E,p) = E^2 - p^2 + \ell_t^2 E^4\,. \qquad (4.34)$$

In light of the results reviewed and derived above it is known now that in this case the UV spectral dimension is $d_S = 3.5$, while the UV thermal dimension is $d_T = 7$.

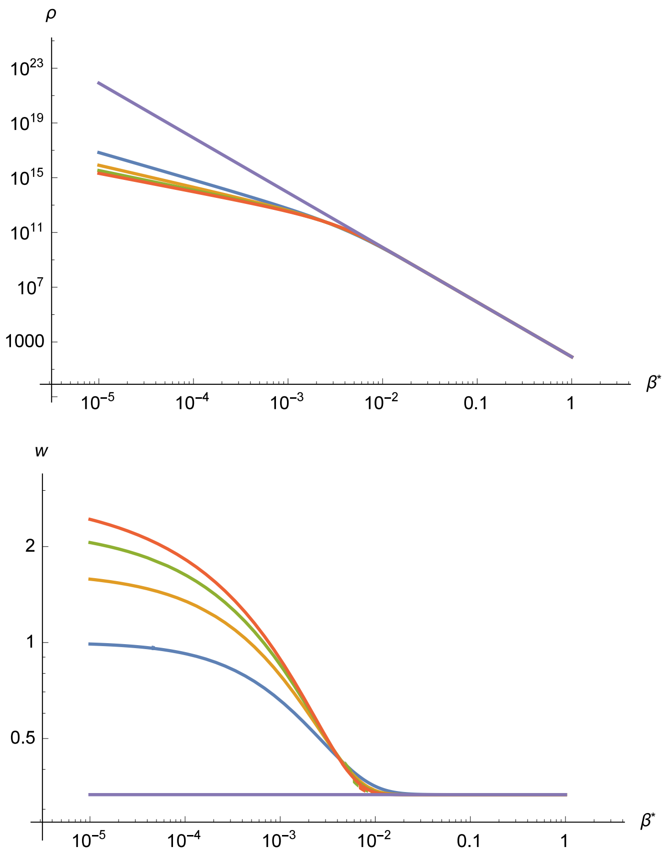

Figure 4.1: Behavior of the energy density ρ in arbitrary units (top panel) and of the equation of state parameter w (bottom panel) as a function of $\beta^* \equiv 10^{-3}\beta k_B T_P$, according to the partition function $Q_{\gamma_t\gamma_x}$, for $\gamma_t = 0$ and $\gamma_x = 2$ (blue), $\gamma_x = 4$ (orange), $\gamma_x = 6$ (green), $\gamma_x = 8$ (red). The purple line is the standard case, $\rho \propto T^4$ (top panel) and $w = 1/3$ (bottom panel).

Consider a simple diffeomorphism on momentum space, the following reparameterization of the energy variable: $E \to \tilde{E} = \sqrt{E^2 + \ell_t^2 E^4}$. In terms of $\tilde{E}$ the d'Alembertian takes the standard special-relativistic form, $\mathcal{C}_{1,0} = \tilde{E}^2 - p^2$, while the momentum space measure becomes non-trivial:

$$d\mu(\tilde{E}, p) = \frac{d\tilde{E}dp\,\sqrt{2}\ell_t p^2 \tilde{E}}{\sqrt{(1 + 4\ell_t^2\tilde{E}^2)(-1 + \sqrt{1 + 4\ell_t^2\tilde{E}^2})}} \tag{4.35}$$

When the above diffeomorphism on momentum space is an active one, the laws of physics are not invariant. This is indeed what is found when comparing the thermodynamical properties of the "$\tilde{E}, p$ theory" with d'Alembertian

$\tilde{E}^2 - p^2$ and momentum-space integration measure (4.35) and the "E, p theory" with (deformed) d'Alembertian $\mathcal{C}_{1,0}(E,p) = E^2 - p^2 + \ell_t^2 E^4$ and integration measure $dE\, d^3p$. In the "$\tilde{E}, p$ theory" the logarithm of the thermodynamical partition function is

$$\begin{aligned}\log \tilde{Q}_{act.} &= -\frac{2V}{(2\pi)^3}\int d\mu(\tilde{E},p)\Big[\delta(\tilde{E}^2 - p^2)\Theta(\tilde{E})\,\cdot \\ &\quad \cdot 2\tilde{E}\log\left(1 - e^{-\beta\tilde{E}}\right)\Big] \neq \log Q. \end{aligned} \tag{4.36}$$

Of course ultimately this leads to different values for the thermal dimension of these two theories. In fact, from the partition function (4.36) one can easily find that at high temperatures the energy density behaves as $\rho \sim T^{3.5}$, while the equation of state parameter is $w = 0.4$. These values point at a value of the UV thermal dimension of $d_T = 3.5$. Note that this result is different from the one that would follow from a passive diffeomorphism. In this case, the partition function in the $\tilde{E}, p$ variables would be straightforwardly obtained by a change of variables in Eq. (4.30):

$$\begin{aligned}\log \tilde{Q}_{pass.} &= -\frac{2V}{(2\pi)^3}\int d\mu(\tilde{E},p)\Big[\delta(\tilde{E}^2 - p^2)\,\cdot \\ &\quad \Theta(E(\tilde{E}))2E(\tilde{E})\log\left(1 - e^{-\beta E(\tilde{E})}\right)\Big] \\ &= \log Q\,. \end{aligned} \tag{4.37}$$

A passive diffeomorphism just relabels the same physical picture and of course the thermal dimension is not affected. On the other hand, it can be easily seen that the spectral dimension is not only invariant under passive diffeomorphisms but also under active diffeomorphisms on momentum space. In fact, active and passive diffeomorphisms have the same effect on the return probability $P(s)$, that of changing the integration variable (without changing the integral). Therefore the "$\tilde{E}, p$ theory" has the same UV spectral dimension ($d_s = 3.5$) as the "E, p theory".

In summary, one finds that the UV spectral dimension of both the "$\tilde{E}, p$ theory" and the "E, p theory" is 3.5, and 3.5 is also the value of the thermal dimension of the "$\tilde{E}, p$ theory", but the "E, p theory" has UV thermal dimension of 7. It should be evidently seen as advantageous for the thermal dimension[3] the fact that it assigns different UV dimension to the two very

[3]Previous works [61, 114, 81] contemplated the possibility of describing the dimension of a quantum spacetime in terms of the duality with momentum space, by resorting to the "Hausdorff dimension of momentum space". However, at least as formulated in [61, 114, 81], that notion is only applicable to theories of the type of the "$\tilde{E}, p$ theory", *i.e.* with undeformed d'Alembertian (but possibly deformed measure of integration on momentum space).

different "E, p theory" and "$\tilde{E}, p$ theory".

4.2.3 Application to $f(E^2 - p^2)$ scenarios

Another scenario of significant interest is the one where the d'Alembertian is deformed into a function of itself: $E^2 - p^2 \to f(E^2 - p^2)$. The structure of this scenario is very valuable for the purposes of the argument presented here, but it also has intrinsic interest since it has been proposed on the basis of studies of the Asymptotic-Safety approach [82] and of the approach based on Causal Sets [83]. This subsection considers a case which might deserve special interest from the quantum-gravity perspective, as stressed in Ref.[82], such that the deformed d'Alembertian takes the form

$$\mathcal{C}_\gamma(E, p) = E^2 - p^2 - \ell^{2\gamma}\left(E^2 - p^2\right)^{1+\gamma}, \tag{4.38}$$

where the parameter γ takes integer positive values and ℓ is a parameter with dimension of length.

For this case one easily finds that the UV spectral dimension is

$$d_S(0) = \frac{4}{1+\gamma}, \tag{4.39}$$

but the fact that this notion of the UV dimensionality of spacetime depends on γ is puzzling and points very clearly to the type of inadequacies of the spectral dimension that this study is concerned with. In fact, in the UV limit the parameter γ has no implications for the on-shell/physical properties of the (massless) theory. In general, massless particles governed by $\mathcal{C}_\gamma$ will be on-shell only either when

$$E^2 = p^2$$

or when

$$E^2 = p^2 + \frac{1}{\ell^2},$$

independently of the value of γ. At low energies only $E^2 = p^2$ is viable. For energies such that $E \geq 1/\ell$ also the second possibility, $E^2 = p^2 + \frac{1}{\ell^2}$, becomes viable. However, in the UV limit the two possibilities become indistinguishable, all particles are governed by $E \simeq p$ just like in any 4-dimensional spacetime, because as $E \to \infty$ one has that $p^2 + \frac{1}{\ell^2} \simeq p^2$. So without any need to resort to complicated analyses one knows that this theory in the UV limit must behave like a 4-dimensional theory, in contradiction with the mentioned result for the UV spectral dimension.

The UV value of the “thermal dimension" is correctly 4, independently of γ. This is easily seen by taking into account the deformation of d'Alembertian present in the $\mathcal{C}_\gamma$ of (4.38) for the analysis of the partition function:

$$\log Q_\gamma = -\frac{2V}{(2\pi)^3}\int dEd^3p\delta(\mathcal{C}_\gamma)\Theta(E)2E\log\left(1-e^{-\beta E}\right), \tag{4.40}$$

Using the fact that

$$\delta(\mathcal{C}_\gamma) = \frac{\delta(E-p)}{2p} + \frac{\delta(E-\sqrt{p^2+\frac{1}{\ell^2}})}{2\gamma\sqrt{p^2+\frac{1}{\ell^2}}}. \tag{4.41}$$

one easily finds that the UV behavior of thermodynamical quantities which is relevant to determine the thermal dimension is independent of γ, and in particular in the UV the Stefan-Boltzmann law and the equation-of-state parameter take the form known for a standard 4-dimensional spacetime:

$$\rho \propto T^4\ , \quad w = \frac{1}{3}. \tag{4.42}$$

So indeed in this scenario the UV value of the thermal dimension is 4. The theory does have “dynamical running of the dimensionality of spacetime" in a regime where the temperature is close to the Planckian temperature, as one should expect on the basis of the fact that the parameter γ does have a role in the theory for energies greater than $1/\ell$ but still small enough to distinguish between p^2 and $p^2+\frac{1}{\ell^2}$. This is shown in Figure 4.2, where the thermal dimension (inferred from the behaviour of the equation of state parameter and from the running of the energy density with temperature) is plotted as a function of β.

The disastrous failures of the spectral dimension in this case is to be attributed to a combination of its sensitivity to off-shell properties and its reliance on the Euclidean d'Alembertian. It is noteworthy that for the Euclidean d'Alembertian[4],

$$\mathcal{C}_\gamma^{[Euclidean]} = E^2+p^2+\ell^{2\gamma}(E^2+p^2)^{1+\gamma}\,, \tag{4.43}$$

in the UV limit one can neglect E^2+p^2 with respect to $\ell^{2\gamma}(E^2+p^2)^{1+\gamma}$. Instead for on-shell modes of the original Lorentzian $\mathcal{C}_\gamma$ one can never neglect E^2-p^2 with respect to $\ell^{2\gamma}(E^2-p^2)^{1+\gamma}$.

[4]Note that in order to have the Euclidean version of the d'Alembertian $\mathcal{C}_\gamma(E,p)$ one has to Wick-rotate also the parameter ℓ [84].

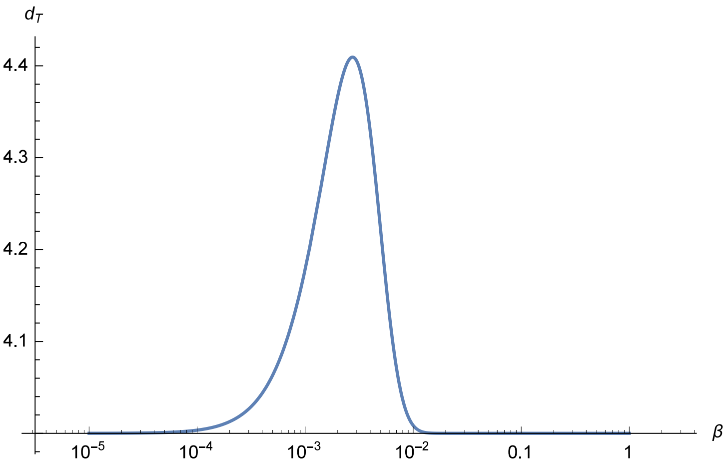

Figure 4.2: Behaviour of the thermal dimension d_T as a function of β. The thermal dimension is computed as $d_T = 1 + \frac{1}{w}$, where the equation of state parameter is the one associated with to the $\log Q_\gamma$, with $\gamma = 1$. β is in units of $10^3\beta_P$ (where $\beta_P = \frac{1}{k_B T_P}$ and T_P is the Planck temperature.

4.3 Thermal dimension in Relative Locality models

The present section shows the preliminary results obtained in trying to extend the notion of thermal dimension of quantum spacetime to models with relative locality. This would allow us to give further strength to the arguments developed in the previous section.

Two different sets of coordinates on the momentum space, the bicrossproduct and Judes Visser coordinates [110] of k-de Sitter momentum space, will be used to compute the relevant thermodynamical quantities and discuss the properties of thermal dimension. In particular the discussion about the different sensibility of spectral and thermal dimension under the action of active and passive diffeomorphism will be continued. Also, it is shown that the bicrossproduct and the Judes Visser coordinates describe in general different theories although they have equivalent on-shell relations; this allows a more detailed discussion of the problem of the off-shellness of the spectral dimension.

The starting point to study the thermodynamics is the definition of (logarithm of the) partition function, written in covariant form:

$$\log Q = -\frac{2V}{(2\pi)^3}\int d\mu(p_0,\vec{p})\delta(\mathcal{C}_p)\Theta(U^\mu p_\mu)2U^\mu p_\mu \times \\ \times \log\left(1 - e^{-\beta U^\mu p_\mu}\right). \tag{4.44}$$

Here, p_μ is the four-momentum of the photons in the radiation gas, U^μ is the four-velocity of the observer with respect to the system (so that the energy measured by the observer is $E = U^\mu p_\mu$), β is related to the Boltzmann constant k_B via $\beta = \frac{1}{k_B T}$, $\mathcal{C}_p$ is the on-shell relation and $d\mu(p_0, \vec{p})$ is the invariant measure on momentum space (these becomes $\mathcal{C}_p = p_0^2 - \vec{p}^2$ and $d\mu(p_0, \vec{p}) = d^4p$ in the undeformed case and (4.44) takes the usual form in the comoving reference frame $U^\mu = (1, \vec{0})$). Writing the partition function in covariant form allows to introduce non-trivial dispersion relations and curvature on momentum space consistently with the relativistic setup of the model. From this, following Section 4.2, all the thermodynamical quantities can be derived in the usual way. In particular, the main focus will be on the energy density

$$\rho \equiv -\frac{1}{V}\frac{\partial}{\partial\beta}\log Q \tag{4.45}$$

and the pressure

$$p \equiv \frac{1}{\beta}\frac{\partial}{\partial V}\log Q\,. \tag{4.46}$$

4.3.1 Thermal dimension of k-de Sitter in bicrossproduct coordinates

The metric on $(D+1)$-dimensional momentum space in bicrossproduct coordinates takes the form:

$$ds^2 = g^{\mu\nu}dp_\mu dp_\nu = dE^2 - e^{2\ell E}\sum_{j=1}^{D} dp_j^2\ , \tag{4.47}$$

so that the measure of integration of momentum space is (in 3+1 dimensions)

$$d\mu_{bp}(E,p) = \sqrt{-g}dEd^3p = p^2 e^{3\ell E}dEdp. \tag{4.48}$$

The momentum space representation of the mass Casimir operator gives the on-shell relation. This operator must of course be an invariant under the deformed symmetries of the model. The invariant that is mostly used in the literature is

$$\mathcal{C}_{bp} = \frac{4}{\ell^2}\sinh^2\left(\frac{\ell E}{2}\right) - e^{\ell E}|\vec{p}|^2\,, \tag{4.49}$$

and the on-shell relation is then given by

$$\frac{4}{\ell^2}\sinh^2\left(\frac{\ell E}{2}\right) - e^{\ell E}|\vec{p}|^2 = \frac{4}{\ell^2}\sinh^2\left(\frac{\ell m}{2}\right)\,, \tag{4.50}$$

where m is the rest energy. The Lorentz transformations are non linear transformations for these coordinates. Since the spacetime and momentum space here considered are isotropic one can work just with the modulus of spatial momentum, p.

The thermodynamical partition function for this model is

$$\log Q \propto \int dEdp\, p^2\, e^{3\ell E}\delta(\mathcal{C}_{bp})\Theta(E)2E\log\left(1-e^{-\beta E}\right). \tag{4.51}$$

The delta function can be rewritten as:

$$\delta(\mathcal{C}_{bp}) = \frac{\ell}{2(e^{\ell E}-1)}\delta\left(p - \frac{1-e^{-\ell E}}{\ell}\right) \tag{4.52}$$

from which one can see that the model has a maximum momentum, $p_{max} = \ell^{-1}$. The expression for the energy density, after integration over the p variable, reads:

$$\rho \propto \int dE\, e^{\ell E}\frac{e^{\ell E}-1}{\ell(e^{\beta E}-1)}E^2\,. \tag{4.53}$$

The integrand is divergent for $\beta < 2\ell$, from which one can deduce the existence of a maximal temperature, $T_{max} = 0.5\,T_P$, where T_P is Planck temperature. The same conclusion can be drawn from the examination of the expression for the pressure:

$$p \propto \frac{1}{\beta}\int dE\, e^{\ell E}\frac{e^{\ell E}-1}{\ell}E\log\left(1-e^{-\beta E}\right)\,. \tag{4.54}$$

So this is a case where the UV regime can not be defined by $T \to \infty$, but it will be then considered the $T \to T_{max}$ regime. When the temperature is close to its maximum the energy density behaves like:

$$\rho \sim (\beta - 2\lambda)^{-3} \tag{4.55}$$

and the equation of state parameter runs to the value

$$w = 0. \tag{4.56}$$

From the definition of thermal dimension given in Section 4.2, one can conclude that $d_T = \infty$. However it should be kept in mind that expressions (4.26) and (4.27) that link the exponent of the Stefan-Boltzmann law and the equation of state parameter to the number of dimension of spacetime do not contemplate a maximal temperature. One could be tempted, by looking at the expression (4.55), to claim that in presence of a non zero β_{min} what is the number of dimension of spacetime is not the exponential of β but that of $(\beta - \beta_{min})^{-1}$. So in this case, on the basis of (4.55) one would say that $D + 1 = 3$. One way to make this claim more reliable is to find an equation of state parameter coherent with this number of dimension. And since on the basis of our intuition is that the relavant combination is $\beta - \beta_{min}$, if one defines the pressure as

$$p \equiv \frac{1}{\beta - \beta_{min}} \frac{\partial}{\partial V} \log Q \,, \tag{4.57}$$

then one finds the value of the equation of state parameter which is coherent with the exponent of the Stefan-Boltzmann law,

$$w = 0.5 \,. \tag{4.58}$$

In this optics, both (4.55) and (4.58) suggest that the effective thermal dimension of the model, close to the maximal temperature, is the less problematic

$$d_T = 3 \,. \tag{4.59}$$

4.3.2 Thermal dimension of k-de Sitter in Judes Visser coordinates

The Judes Visser coordinates [110] $\epsilon(E,p)$ and $\pi(E,p)$ are defined in such a way that they transform as the usual 4-momentum under Lorentz transformation and the mass Casimir takes the standard form $\epsilon^2 - \pi^2 = \mu^2$. They are obtained as follows. Authors in Ref.[110] started from the expression of E and p as boosted rest energy m

$$\begin{aligned} e^{\ell E} &= e^{\ell m}(1 + \sinh(\ell m) e^{-\ell m}(\cosh\xi - 1)), \\ p &= \frac{1}{\ell} \frac{\sinh(\ell m) e^{-\ell m} \sinh\xi}{1 + \sinh(\ell m) e^{-\ell m}(\cosh\xi - 1)}, \end{aligned} \tag{4.60}$$

where ξ is the boost rapidity parameter. By inverting these relations to get the expression of the rapidity they get

$$\begin{aligned} \cosh\xi &= \frac{e^{\ell E} - \cosh(\ell m)}{\sinh(\ell m)}, \\ \sinh\xi &= \frac{\ell p e^{\ell E}}{\sinh(\ell m)}, \end{aligned} \tag{4.61}$$

and using the identity $\cosh\xi^2 - \sinh\xi^2 = 1$ rewrite the on-shell relation in the following way:

$$\cosh(\ell E) = \cosh(\ell m) + \frac{1}{2}\ell^2 p^2 e^{\ell E}. \tag{4.62}$$

Comparison with the standard dispersion relation fixes the relation between the rest energy m and the mass Casimir μ,

$$\cosh(\ell m) = 1 + \frac{1}{2}\ell^2\mu^2. \tag{4.63}$$

This fixes the ϵ and π coordinates as

$$\begin{aligned} \epsilon = \mu\cosh\xi &= \frac{e^{\ell E} - \cosh(\ell m)}{\ell\cosh(\ell m/2)}, \\ \pi = \mu\sinh\xi &= \frac{p e^{\ell E}}{\cosh(\ell m/2)}. \end{aligned} \tag{4.64}$$

It is here reported also the expression of the bicrossproduct coordinates in terms of the Judes Visser

$$\begin{aligned} E &= \frac{1}{\ell}\ln(1 + \ell\epsilon\sqrt{1 + \frac{\ell^2\mu^2}{4}} + \frac{\ell^2\mu^2}{2}), \\ p &= \frac{\pi\sqrt{1 + \frac{\ell^2\mu^2}{4}}}{1 + \ell\epsilon\sqrt{1 + \frac{\ell^2\mu^2}{4}} + \frac{\ell^2\mu^2}{2}}. \end{aligned} \tag{4.65}$$

The invariant measure (4.48) in these coordinates takes the form

$$dE\, dp\, p^2 e^{3\ell E} \to d\tilde{\mu}_{JV}(\epsilon, \pi, \mu) = d\epsilon\, d\pi \frac{\pi^2\left(1 + \frac{\ell^2\mu^2}{4}\right)}{1 + \ell\epsilon\sqrt{1 + \frac{\ell^2\mu^2}{4}} + \frac{\ell^2\mu^2}{2}}. \tag{4.66}$$

What it is important to notice for the following is that if one substitutes the expressions (4.65) into the bicrossproduct mass Casimir (4.49) one finds

$$\mathcal{C}_{JV}(\epsilon, \pi, \mu) = \frac{\epsilon^2 - \pi^2 + \ell\mu^2\epsilon\sqrt{1 + \frac{\ell^2\mu^2}{4}} + \frac{\ell^2\mu^2}{2}(\mu^2 + \epsilon^2 - \pi^2)}{1 + \ell\epsilon\sqrt{1 + \frac{\ell^2\mu^2}{4}} + \frac{\ell^2\mu^2}{2}}, \tag{4.67}$$

which reduces to the standard

$$C_{JV}(\epsilon, \pi) = \epsilon^2 - \pi^2 \tag{4.68}$$

only when one enforces the on-shell relation for the Judes Visser coordinates,

$$\epsilon^2 - \pi^2 = \mu^2. \tag{4.69}$$

So what one actually has is that the two set of coordinates give equivalent dispersion relations (4.50)-(4.69) but non equivalent d'Alambertians (4.49)-(4.67). The discussion will come back later on this important fact to discuss in more detail the different properties of spectral dimension and thermal dimension.

The "on-shell" expression of the bicrossproduct coordinates in terms of the Judes Visser coordinates are simply obtained by explicitly substituting $\mu^2 = \epsilon^2 - \pi^2$,

$$\begin{aligned} E &= \frac{1}{\ell}\ln(1+\ell\epsilon\sqrt{1+\frac{\ell^2(\epsilon^2-\pi^2)}{4}}+\frac{\ell^2(\epsilon^2-\pi^2)}{2}), \\ p &= \frac{\pi\sqrt{1+\frac{\ell^2(\epsilon^2-\pi^2)}{4}}}{1+\ell\epsilon\sqrt{1+\frac{\ell^2(\epsilon^2-\pi^2)}{4}}+\frac{\ell^2(\epsilon^2-\pi^2)}{2}}. \end{aligned} \tag{4.70}$$

Starting from these expression to compute the measure one obtains,

$$d\mu_{JV}(\epsilon,\pi) = d\epsilon\, d\pi\, \pi^2\left(1+\frac{\ell^2(\epsilon^2-\pi^2)}{4}\right). \tag{4.71}$$

For what concerns the thermal dimension, however, since the on-shellness is enforced by the Dirac delta function, this difference in the measures makes no difference in the final value of the integral since one can easily see that

$$\begin{aligned} &d\tilde{\mu}_{JV}(\epsilon,\pi,\mu)\delta(\mathcal{C}_{JV}(\epsilon,\pi,\mu)-\mu^2)\theta(\epsilon) = \\ &= d\mu_{JV}(\epsilon,\pi)\delta(\mathcal{C}_{JV}(\epsilon,\pi)-\mu^2)\theta(\epsilon) \\ &= d\epsilon\, d\pi\, \pi^2\left(1+\frac{\ell^2\mu^2}{4}\right)^2\delta(\epsilon^2-\pi^2-\mu^2)\theta(\epsilon). \end{aligned} \tag{4.72}$$

Notice than apart a constant factor the measure that ultimately enters in the relevant integral is the standard measure over minkowskian momentum space and it is exactly the standard one in the massless case $\mu = 0$, which is the case of interest to study the Stefan-Boltzmann law and equation of state parameter.

This result immediately tells us that the thermal dimension is sensible to the difference between active and passive diffeomorphisms on momentum space. In fact, if one switches coordinates from the bicrossproduct to the Judes Visser as a passive diffeomorphism, it actually is a mere change of coordinates in computing the relevant integrals, the resulting Stefan-Boltzmann law and w then being those computed in the previous section. However, if the Judes Visser coordinates are introduced as an active diffeomorphism, then the logarithm of the partition function is

$$\log Q \propto \int d\epsilon\, d\pi\, \pi^2 \delta(\epsilon^2 - \pi^2)\theta(\epsilon) 2\epsilon \log(1 - e^{\beta\epsilon}), \tag{4.73}$$

which therefore leads to the usual value $d_T = 4$. It is therefore evident that in this context physics is not invariant under active diffeomorphims on momentum space.

4.3.3 Spectral Dimension in Judes Visser coordinates

In the previous section it has been noticed that one has two set of transformations relating the bicrossproduct to the Judes Visser coordinates: the first, Eqs.(4.65), may called "off-shell Judes Visser", since the expression of the bicrossproduct mass Casimir (4.49) in terms of these coordinates takes the non standard form shown in Eq.(4.67); the second, Eqs.(4.70), may called "on-shell Judes Visser", since the expression of the bicrossproduct mass Casimir (4.49) in terms of these coordinates takes the standard form Eq.(4.68). The measure on momentum space in the two cases are respectively $d\tilde{\mu}_{JV}(\epsilon, \pi, \mu)$ and $d\mu_{JV}(\epsilon, \pi)$. It has been shown also that these different sets of coordinates still give the same value of thermal dimension because the integrals defining the thermodynamical quantities are computed on-shell. One could be interested in looking whether the value of thermal dimension computed for these models coincides with that of spectral dimension.

The UV spectral dimension for the Euclidean version [111] of this model can be computed using the return probability

$$P(s) \propto \int dE dp\, p^2\, e^{3\ell E}\, e^{-s\left(\frac{4}{\ell^2}\sinh^2\left(\frac{\ell E}{2}\right) + e^{\ell E}|\vec{p}|^2\right)}\,. \tag{4.74}$$

and turns out to be [60, 61]

$$d_S(0) = 6\,. \tag{4.75}$$

In the case of Judes Visser coordinates one has to deal with the fact that one has two alternatives, the off-shell and on-shell coordinates.

For the off-shell coordinates the return probability takes the form

$$P(s) \propto \int d\epsilon d\pi \frac{\pi^2}{1 + \ell\epsilon} e^{-s\frac{\epsilon^2 + \pi^2}{1 + \ell\epsilon}}, \tag{4.76}$$

giving the value $d_S(0) = 6$.

For the on-shell coordinates however this takes the form

$$P(s) \propto \int d\epsilon d\pi \pi^2 \left(1 + \frac{\ell^2(\epsilon^2 + \pi^2)}{4}\right) e^{-s(\epsilon^2 + \pi^2)}, \tag{4.77}$$

which gives again $d_S(0) = 6$.

This results should not surprise, as it has been already noticed that the spectral dimension is not sensible to the difference between active and passive diffeomorphisms. Therefore, no matter which coordinates one chooses for $P(s)$ in (4.74) it will give $d_S(0) = 6$.

4.4 Remarks on the thermal dimension

The exciting realization that the UV dimension of spacetime might be different from its IR dimension adds significance to the old challenge of describing the dimension of a quantum spacetime and it is argued that it is crucial to link this issue to observable properties. After all, what it is meant in physics by “dimension of spacetime" must inevitably be something one can measure. Moreover, only by relying on a truly physical/observable characterization one is assured to compare different theories in conclusive manner.

The inadequacy of the spectral dimension for these purposes has been fully exposed in the previous pages. The fact that this notion involves an unphysical Euclideanization could already lead to this conclusion. The observation about the undesirable invariance of the spectral dimension under active diffeomorphisms of momentum space should cast another shadow on the usefulness of the spectral dimension. The fact that one obtains different spectral dimensions for alternative formulations of the same physical theory as in Subsection 4.2.3 (formulations that differ only for what concerns unphysical off-shell modes) should leave no residual doubts.

The notion of thermal dimension presented here is free from the shortcomings of the spectral dimension, since it relies on the analysis of observable thermodynamical properties of radiation in the quantum spacetime. The next Chapter shows how the notion of thermal dimension of a quantum spacetime is not only physical but also particularly useful, at least for studies of the early universe, which is anyway the context where the UV dimension of spacetime should find its most significant applications [85, 86].

Chapter 5

Primordial perturbations in a rainbow universe with running Newton constant

The standard model of cosmology lacks of a causal explanation of the high degree of homogeneity seen at large scales in the universe, the sky being a mosaic of regions that have never been in causal contact but still are puzzling similar. Without a causal explanation for such homogeneity, it has to be given as extremely fine-tuned initial condition. This is the well known "horizon paradox". This weakness brought the development of different mechanism to solve the paradox, most notably inflation. There, a scalar field drives an exponential expansion of the universe, and the quantum vacuum fluctuations, in causal contact, are stretched and grown classical, becoming the seeds of the structures observed today.

Recent results suggest that the properties of the spectrum of primordial fluctuations might not need inflationary expansion to be explained, but could instead be a consequence of quantum-gravitational effects, which are relevant in the early universe [85, 112]. In particular in [85, 113, 114] it was shown that a scale invariant power spectrum can be obtained if the perturbations satisfy the Planck-scale-modified dispersion relation emerging in the high-energy regime of Horava-Lifshitz gravity [55]:

$$E^2 = p^2(1 + (\ell p)^4)\,. \tag{5.1}$$

As it has been shown earlier, this dispersion relation implies a running of spacetime dimensionality, so that the spacetime dimension in the deep Planckian regime is 2 [73, 115, 116]. The possibility of generalising this result to any theory with Planck-scale dimensional reduction to 2 was suggested in [86, 81]. These results rely on a number of assumptions, such as that the second order action for perturbations is the one of Einstein gravity and that the perturbations are produced in a quantum vacuum state. This rigidity in the assumptions makes it hard to find a mechanism that would produce the observed small departure from exact scale invariance.

This study [127] relaxes several of the assumptions previously made in the literature. Firstly, it is assumed the more general framework of rainbow gravity [48] previously introduced. The background cosmological evolution will then be described in terms of a metric which "runs" with the energy. For the dispersion relation:

$$f^2(E)E^2 - g^2(E)p^2 = m^2, \tag{5.2}$$

(where the continuous functions f and g approach the constant value 1 when the energy is well below the Planck energy), the associated rainbow line element is

$$ds^2 = \frac{dt^2}{f^2(E)} - \frac{1}{g^2(E)}\delta_{ij}dx^i dx^j \,. \tag{5.3}$$

Secondly, both perturbations of quantum origin for a vacuum state and perturbations that are originated in a thermal state [117, 118, 119, 120, 121] will be considered. In the latter case it will be assumed that the universe is filled with radiation and that both the background and the fluctuations are thermalized, so that they share the same (modified) thermodynamical properties [122]. Finally, it will be allowed for the Newton constant to also run with energy. This is motivated by results in Hořava-Lifshitz gravity and in Asymptotic Safety [123, 124, 125, 126], where the Newton constant tends to zero at super-Planckian energies.The Newton constant is allowed to both increase and decrease with energy. However, it will turn out that in order to solve the horizon problem and to produce perturbations with the required spectral index, the Newton constant must indeed be a decreasing function of energy at super-Planckian scales. This is true for both vacuum and thermal initial conditions for the perturbations.

Regarding the work on thermal fluctuations, the following motivating factors must be stressed. As it has been shown in the previous chapter, radiation obeying a deformed dispersion relation also has deformed thermodynamical properties [116, 128, 129]. This study of cosmological perturbations focuses on a generalization of the Hořava-Lifshitz dispersion relation (5.1):

$$E^2 = p^2(1 + (\ell p)^{2\gamma}) \,, \tag{5.4}$$

and it is here assumed to be in a regime where only the ultraviolet correction term is relevant, $E^2 \approx p^2(\ell p)^{2\gamma}$. According to the results obtained in the previous chapter, in this regime the associated Stefan-Boltzmann law and equation of state parameter $w \equiv P/\rho$ are:

$$\rho \propto T^{1+\frac{3}{1+\gamma}} \tag{5.5}$$
$$w = \frac{1+\gamma}{3}\,. \tag{5.6}$$

The present chapter is structured as follows. Section 5.1 starts by working out the evolution of the background, including modified thermodynamical relations. Section 5.2 obtains the equation for the evolution of primordial scalar perturbations, the constraints on the modified dispersion relation and on the running of the Newton constant which ensure an expanding universe and a solution to the horizon problem. Section 5.3 is devoted to the computation of the spectral index for perturbations generated in a quantum vacuum, while Section 5.4 shows the analogous results for perturbations with thermal initial conditions. Some conclusions are presented in Section 5.5.

5.1 Background evolution of a rainbow FLRW universe with deformed thermodynamics

The rainbow functions associated to the dispersion relation (5.4) are:

$$f^2 = 1 \quad g^2 = 1 + (\ell p)^{2\gamma}\,. \tag{5.7}$$

They enter in the rainbow line element for a FLRW spacetime in the following way [48, 129]:

$$ds^2 = \frac{dt^2}{f^2(E)} - \frac{a^2(t)}{g^2(E)}\delta_{ij}dx^i dx^j\,. \tag{5.8}$$

It is here assumed that the universe contains a perfect fluid, whose stress-energy tensor is $\mathcal{T}^\mu_\nu = (\rho + P)u^\mu u_\nu - P\delta^\mu_\nu$, where ρ is the energy density, P the pressure and u^μ the fluid four velocity [1]. Then the Friedmann equations read [48]:

$$\begin{aligned} H^2 &= \frac{8\pi G(E)}{3f^2}\rho \\ H^2 - \frac{\ddot{a}}{a} &= \frac{4\pi G(E)}{f^2}(\rho + P), \end{aligned} \tag{5.9}$$

[1] As mentioned in the introduction, a possible energy dependence of the Newton constant G is allowed.

where $H = \frac{da/dt}{a}$. From these the continuity equation follows

$$\dot{\rho} = -3H(\rho + P). \tag{5.10}$$

The solution of the continuity equation can be stated in terms of the equation of state parameter as usual, and if the universe is filled with radiation this translates into a dependence on the parameter γ appearing in the dispersion relation (5.4):

$$\rho = \bar{\rho} a^{-3(1+w)} = \bar{\rho} a^{-(4+\gamma)}. \tag{5.11}$$

Of course in the case of standard thermodynamics in four spacetime dimensions $d_T = 4$ and one recovers the usual scaling $\rho = \bar{\rho} a^{-4}$ in a radiation-filled universe.

Using the Stefan-Boltzmann law one finds that the deformed thermodynamics also affects the evolution of the temperature with the scale factor:

$$T \propto a^{-3w} = a^{-(1+\gamma)}\,. \tag{5.12}$$

5.2 Evolution of scalar perturbations in a rainbow universe and solution to the horizon problem

The perturbed rainbow FLRW metric in the longitudinal gauge[2] reads:

$$ds^2 = \frac{dt^2}{f^2(E)}(1 + 2\phi(t,x)) - \frac{a^2(t)}{g^2(E)}(1 - 2\psi(t,x))\delta_{ij} dx^i dx^j. \tag{5.13}$$

In order to work out the evolution equation for the perturbations one can introduce an energy-dependent time variable,

$$d\tilde{t} = \frac{dt}{f(E)}\,, \tag{5.14}$$

so that the time-dependent functions appearing in the metric read

$$\tilde{a}^2(E,\tilde{t}) = \frac{a^2(\tilde{t})}{g^2(E)},\;\; \tilde{\phi}(\tilde{t},x) = \phi(t,x),\;\; \tilde{\psi}(\tilde{t},x) = \psi(t,x)\,. \tag{5.15}$$

[2]By this it is here meant that in the limit where the energy dependence of the metric disappears, $f = g = 1$, one is left with the metric in longitudinal gauge.

The perturbed line element takes the standard form in terms of the new functions:

$$ds^2 = d\tilde{t}^2(1 + 2\tilde{\phi}(\tilde{t}, x)) - \tilde{a}^2(E, \tilde{t})(1 - 2\tilde{\psi}(\tilde{t}, x))\delta_{ij}dx^i dx^j \,. \tag{5.16}$$

Using these new variables one can just follow a standard procedure (see e.g. [122]) to obtain perturbation equations.

From the standard equations, with the prime denoting the derivative $\frac{d}{d\tilde{\eta}} \equiv \tilde{a}(E, \tilde{t})\frac{d}{d\tilde{t}}$,

$$\begin{aligned} \nabla^2\tilde{\phi} - 3\tilde{\mathcal{H}}(\tilde{\mathcal{H}}\tilde{\phi} + \tilde{\phi}') &= 4\pi G\tilde{a}^2\delta\tilde{\rho} \\ \left[\tilde{\mathcal{H}}\tilde{\phi} + \tilde{\phi}'\right]_{,i} &= 4\pi G\tilde{a}^2(\tilde{\rho} + \tilde{P})\delta\tilde{u}_i \\ \tilde{\phi}'' + 3\tilde{\mathcal{H}}\tilde{\phi}' + (2\tilde{\mathcal{H}}' + \tilde{\mathcal{H}}^2)\tilde{\phi} &= 4\pi G\tilde{a}^2\delta\tilde{P} \end{aligned} \tag{5.17}$$

one can combine the first and third equation of Eqs.(5.17) to get

$$\tilde{\phi}'' + 6\tilde{\phi}'\tilde{\mathcal{H}} + (2\tilde{\mathcal{H}}' + 4\tilde{\mathcal{H}}^2 - \nabla^2)\tilde{\phi} = 0, \tag{5.18}$$

where we set $\tilde{c}_s^2 = \frac{\delta\tilde{P}}{\delta\tilde{\rho}} = 1$. Defining the quantity

$$\tilde{\zeta} = \tilde{\phi}\frac{5 + 3w}{3(1 + w)} + \frac{\tilde{\phi}'}{\tilde{\mathcal{H}}}\frac{2}{3(1 + w)}, \tag{5.19}$$

Eq.(5.18) can be written as

$$\tilde{\zeta}' = \frac{2}{3}\frac{\nabla^2\tilde{\phi}}{\tilde{\mathcal{H}}(1 + w)} \tag{5.20}$$

and from this one can get the following

$$\tilde{\zeta}'' + 2\frac{\tilde{z}'}{\tilde{z}}\tilde{\zeta} - \nabla^2\tilde{\zeta} = 0, \tag{5.21}$$

with $\tilde{z} = \sqrt{\frac{3(1 + w)}{2}}\tilde{a}$. Finally, defining the quantity $\tilde{v} = \tilde{z}\tilde{\zeta}$, Eq.(5.18) takes the familiar form

$$\tilde{v}'' - \left(\nabla^2 + \frac{\tilde{z}''}{\tilde{z}}\right)\tilde{v} = 0. \tag{5.22}$$

Going back to the energy-independent time variable one finds that the curvature perturbation is left unchanged,

$$\tilde{\zeta} = \phi\frac{5 + 3w}{3(1 + w)} + \frac{ad\phi/dt}{da/dt}\frac{2}{3(1 + w)} = \zeta\,, \tag{5.23}$$

while

$$\tilde{z} = \sqrt{\frac{3(1+w)}{2}}\tilde{a} = \sqrt{\frac{3(1+w)}{2}}\frac{a}{g} = z/g. \tag{5.24}$$

Therefore, $v = \tilde{v}g$ satisfies the following evolution equation in Fourier space

$$v'' - \left(\frac{g^2}{f^2}k^2 + \frac{a''}{a}\right)v = 0\,. \tag{5.25}$$

From now on, the prime stands for the derivative with respect to the energy-independent conformal time, $\frac{d}{d\eta} \equiv a\frac{d}{dt}$. This equation is very similar to the standard one, with the factor $(f/g)^2$ which plays the role of an energy-dependent speed of sound.

Note that a possible energy dependence of the Newton constant does not appear explicitly in the evolution equations of the perturbations; however, it will be shown in the following that it affects the scale of the horizon and the conditions under which the horizon problem is solved within rainbow cosmology models.

A cosmological model that solves the horizon problem is such that modes start inside the horizon, where the first term in parentheses in the evolution equation (5.25) dominates, and subsequently exit the horizon, where the second term dominates [122, 130]. Here the conditions under which the horizon problem is solved are investigated specialising to the dispersion relation (5.4), with associated rainbow functions (5.7) and assuming to be in a regime where only the ultraviolet correction terms are relevant. It is important to bear in mind that the energy appearing in the rainbow functions is the physical one, related to the comoving k via $E = \left(\frac{\ell k}{a(\eta)}\right)^{2\gamma}$.

The behaviour of the two terms in parenthesis in Eq. (5.25) is governed by the evolution of the scale factor $a(\eta)$. This is found by integrating the first Friedmann equation (5.9), leading to

$$\eta^2 = \frac{a^{1+3w}}{(1+3w)^2}\frac{1}{\frac{2}{3}\pi\bar{\rho}G} = \frac{a^{2+\gamma}}{(2+\gamma)^2}\frac{1}{\frac{2}{3}\pi\bar{\rho}G}\,. \tag{5.26}$$

Here, $\bar{\rho}$ is the initial energy density and the relation between the equation of state parameter w and the deformation parameter γ is given by the modified thermodynamical relation (5.6). If the Newton constant is energy-independent, the scale factor evolves as:

$$a(\eta) = (C\eta^2)^{\frac{1}{2+\gamma}}\,, \tag{5.27}$$

where $C = G\frac{2}{3}\pi\bar{\rho}(2+\gamma)^2$ and η increases from 0 in order to have cosmological expansion with time. Then the two terms in parentheses in (5.25) take the form

$$k^2\left(\frac{\ell k}{a(\eta)}\right)^{2\gamma} = k^2(\ell k)^{2\gamma}C^{-\frac{2\gamma}{2+\gamma}}\eta^{-\frac{4\gamma}{2+\gamma}} \tag{5.28}$$

and

$$\frac{a''}{a} = \eta^{-2}\frac{2}{2+\gamma}\left(\frac{2}{2+\gamma}-1\right). \tag{5.29}$$

The horizon is then found at

$$\eta_H = \left(k^2(\ell k)^{2\gamma}C^{-\frac{2\gamma}{2+\gamma}}\frac{(2+\gamma)^2}{2\gamma}\right)^{\frac{2+\gamma}{2(\gamma-2)}}, \tag{5.30}$$

and in order to solve the horizon problem one needs

$$\gamma > 2\,. \tag{5.31}$$

If the Newton constant has a power-law dependence on energy in the ultraviolet regime,

$$G(E) = \ell^2(\ell E)^\alpha \sim \ell^2\left(\frac{\ell k}{a}\right)^{(1+\gamma)\alpha}, \tag{5.32}$$

then the evolution of the scale factor with time is

$$a(\eta) = (\bar{C}\eta^2(\ell k)^{(1+\gamma)\alpha})^{1/\nu}, \tag{5.33}$$

where $\nu = 2+\gamma+(1+\gamma)\alpha$ and $\bar{C} = \frac{2}{3}\pi\ell^2\bar{\rho}(2+\gamma)^2$. Note that depending on ν the conformal time can either be positive or negative. In fact, in order to have cosmological expansion with time if $\nu > 0$ then η must be positive and increasing from 0, while if $\nu < 0$ then η must be negative and approaching 0 from $-\infty$.

The terms in parenthesis in the perturbation equation (5.25) are now:

$$k^2\left(\frac{\ell k}{a(\eta)}\right)^{2\gamma} = \bar{C}^{-\frac{2\gamma}{\nu}}\eta^{-\frac{4\gamma}{\nu}}k^2(\ell k)^{\frac{2\gamma(2+\gamma)}{\nu}}, \tag{5.34}$$

and

$$\frac{a''}{a} = \frac{2}{\nu}\left(\frac{2}{\nu}-1\right)\eta^{-2}. \tag{5.35}$$

The horizon is then found at

$$\eta_H = \left(\frac{\nu\,\bar{C}^{-\frac{2\gamma}{\nu}}}{2\left(\frac{2}{\nu}-1\right)}k^2(\ell k)^{\frac{2\gamma(2+\gamma)}{\nu}}\right)^{\frac{\nu}{4\gamma-2\nu}} \tag{5.36}$$

and the horizon problem is solved for $\frac{4\gamma}{\nu} > 2$ if η is positive and for $\frac{4\gamma}{\nu} < 2$ otherwise. Then the overall conditions on α that ensure cosmological expansion and solution of the horizon problem are

$$-\frac{2+\gamma}{1+\gamma} < \alpha < \frac{\gamma-2}{1+\gamma} \tag{5.37}$$

for positive η and

$$\alpha < -\frac{2+\gamma}{1+\gamma}, \quad \alpha > \frac{\gamma-2}{1+\gamma} \tag{5.38}$$

for negative η. The latter possibility is obviously excluded. The first option correctly reduces to $\gamma > 2$ when $\alpha = 0$, while in general it constrains α to be in the range $-2 < \alpha < 1$.

5.3 Vacuum perturbations

One can study the power spectrum of vacuum fluctuations directly in the general case where the UV energy dependence of G is encoded in (5.32). The limit $\alpha = 0$ gives the results for energy-independent G.

The dynamics of modes inside the horizon is governed by the first term in parentheses in (5.25). Up to a phase, the vacuum fluctuations inside the horizon take the form [85, 113]:

$$v_V \sim \frac{a^{\gamma/2}}{\sqrt{\ell^\gamma k^{1+\gamma}}} \,. \tag{5.39}$$

The solution of (5.25) for modes outside the horizon can be cast in the ansatz:

$$v_V \sim F(k)a \,, \tag{5.40}$$

where the function F is found by asking that the two solutions match at the horizon:

$$F(k) = \frac{a^{\gamma/2-1}(\eta_H)}{\sqrt{\ell^\gamma k^{1+\gamma}}}. \tag{5.41}$$

The dimensionless power spectrum of curvature perturbations ζ is given by $k^3\mathcal{P}_\zeta \sim k^3 \left(\frac{v}{z}\right)^2 \equiv A^2 k^{n_s-1}$. Its spectral index n_s is found from (5.41) and (5.36):

$$n_s^V - 1 = \frac{(\gamma+4)(2-\gamma)}{2-\gamma+\alpha(1+\gamma)} \,. \tag{5.42}$$

Clearly $\gamma = 2$ gives a scale invariant power spectrum for any value of α allowed by the constraint (5.37), which for $\gamma = 2$ reads $-\frac{4}{3} < \alpha < 0$. The fact that scale invariance is achieved independently of how the Newton constant scales with energy is due to the time perturbations being already scale-invariant and proportional to the scale factor a inside the horizon. So the gluing procedure is trivial, bypassing whatever modified evolution of the background was introduced. Also a near-scale invariant power spectrum is allowed. In particular one can ask that $n_s^V = 0.968 \pm 0.006$, which is the present observational constraint from Planck [131], obtaining the allowed range of values shown in Fig. 5.1. Note that now the energy dependence of the Newton constant is relevant. In particular, the values of α that are selected by observational constraints are all negative, suggesting a vanishing Newton constant in the deep UV regime. On the other hand, from Eq. (5.35) one can see that observational constraints allow for both an accelerated or decelerated expansion. This is a crucial difference with respect to the constraints coming from thermal fluctuations, as shown in the following section.

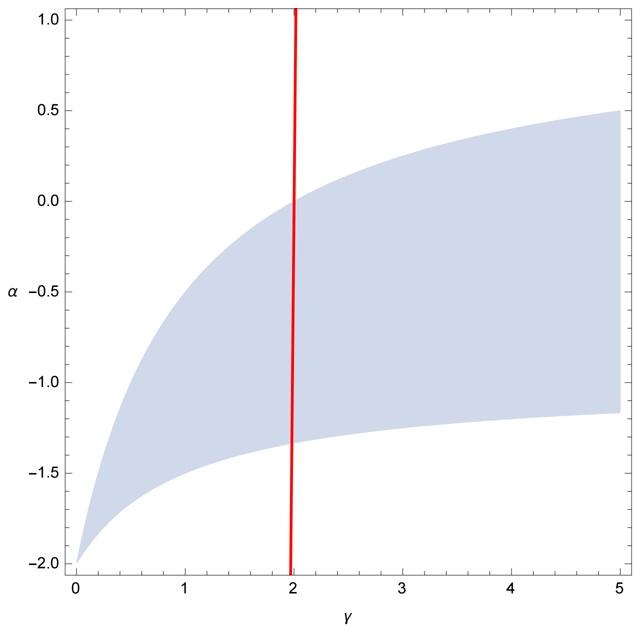

Figure 5.1: The constraint $n_s = 0.968 \pm 0.006$ is plotted in red , assuming vacuum fluctuations (the error bar is too small to be seen on the plot). The region satisfying the constraint ensuring solution of the horizon problem, Eq. (5.37) is plotted in blue .

In the limiting case $\alpha = 0$ (energy-independent Newton constant) the gluing condition at the horizon gives a spectral index which is far from scale invariance, $n_s^V - 1 = 4 + \gamma$. However, when $\gamma = 2$ both the terms governing the evolution of perturbations, (5.28) and (5.29), scale like η^{-2}. Therefore a mode is either inside or outside the horizon, unable to cross it. Whether a mode is inside or outside the horizon is set by the scale

$$k_H = \left(G \frac{8\pi}{3} \frac{\bar{\rho}}{\ell^4} \right)^{1/6} = H_0 \left(\frac{1}{(\ell H_0)^4} \frac{\bar{\rho}}{\rho_{cr}} \right)^{1/6} , \qquad (5.43)$$

where H_0 is the current value of the Hubble constant and ρ_{cr} is the critical energy density. If the modes are well inside the horizon, $k \gg k_H$, the perturbations behave like $v_V \sim \frac{a}{\sqrt{\ell^2 k^3}}$, and so they are scale-invariant, but never exit the horizon.

5.4 Thermal perturbations

Without an inflationary phase, there is no real reason to exclude the contribution to the perturbations power spectrum coming from thermalised perturbations, since this is not suppressed by a period of super-cooling [120, 117]. The thermal contribution to the power spectrum is here computed applying the method outlined in [118], but taking into account that in our model both background and perturbations are thermalised. This in particular means that the same thermodynamical constraints (5.6) hold for background and perturbations. The expectation value of a quantum operator is

$$\langle O \rangle = \frac{\sum_n \rho_{nn} \langle n | \hat{O} | n \rangle}{\sum_n \rho_{nn} \langle n | n \rangle} , \qquad (5.44)$$

where $|n\rangle$ is the n-particle state. It is here assumed that the density matrix follows the Boltzmann distribution $\rho_{nn} = e^{-\beta E_n}$, where $\beta = 1/k_B T$ and $E_n = p_n \sqrt{1 + (\ell p_n)^{2\gamma}}$ is the energy of a mode with occupation number n.

Then the correlation function of the quantised perturbation variable $\hat{v}$ is [120]

$$\langle \hat{v}(\vec{x}) \hat{v}(\vec{x} + \vec{r}) \rangle = \int \frac{d^3 k}{(2\pi)^{3/2}} |v_k(\eta)|^2 (2n(k,\eta) + 1) e^{i\vec{k}\cdot\vec{r}} , \qquad (5.45)$$

where the number density is given by the Bose-Einstein distribution:

$$n(k,\eta) = \frac{1}{e^{\beta E(k,\eta)} - 1} . \qquad (5.46)$$

The power spectrum of thermal perturbations imprinted at the horizon is therefore

$$\mathcal{P}_{Therm}(k) = \mathcal{P}_{Vac}(k)(2n(k,\eta_H) + 1). \qquad (5.47)$$

Since the regime of fluctuations being studied is in the Rayleigh-Jeans limit, one can set:

$$n(k, \eta_H) \approx (\beta E)^{-1} = \frac{k_B T_c \ell}{(\ell k)^{\gamma+1}}, \tag{5.48}$$

where the conformal temperature $T_c \equiv T a^{\gamma+1}$ is constant in time. As in [118, 132], the relation between the physical and conformal temperature is found by asking that the number density is independent of time. If c is k independent, this is just $T_c = Ta/c$. Here one should strip off the k dependence in c from the definition of T_c, so that it does not become k dependent.

Including the thermal contribution, the spectral index of perturbations becomes

$$n_s^T = n_s^V - 1 - \gamma. \tag{5.49}$$

Note that this result differs form the one in [133], because a mistake has been made there. In the Rayleigh-Jeans limit, $n \sim T/E$, not just T/k. The fact that c has an extra dependence in k is responsible for the last term in (5.49). This result is also independent of how the Newton constant runs with energy.

Using the value of the vacuum spectral index found in the previous section, Eq. (5.42), the thermal spectral index can be written as

$$n_s^T = \frac{4(2-\gamma) - \alpha\gamma\,(1+\gamma)}{2 - \gamma + \alpha(1+\gamma)}\,. \tag{5.50}$$

For energy-independent Newton constant, $\alpha = 0$, the thermal spectral index is

$$n_s^T = 4\,, \tag{5.51}$$

regardless of the value of γ. This result matches the one found in [120, 117] and of course it is ruled out by observational constraints.

For $\alpha \neq 0$, asking that the perturbations are scale invariant leads to a constraint linking α and γ. Asking in addition that the horizon problem is solved, Eq. (5.37), introduces an inferior bound $\gamma > 2$ on the allowed values of γ. Then the values of α that are compatible with scale invariance and which allow to solve the horizon problem fall in the range $-1/4 < \alpha < 0$.

It is also possible to match the spectral index to the Planck observed value $n_s = 0.968 \pm 0.006$ [131], giving the constraints shown in Fig. 5.2. According to Eq.(5.35), these observational constraint on α and γ only allow for a decelerating expansion of the universe.

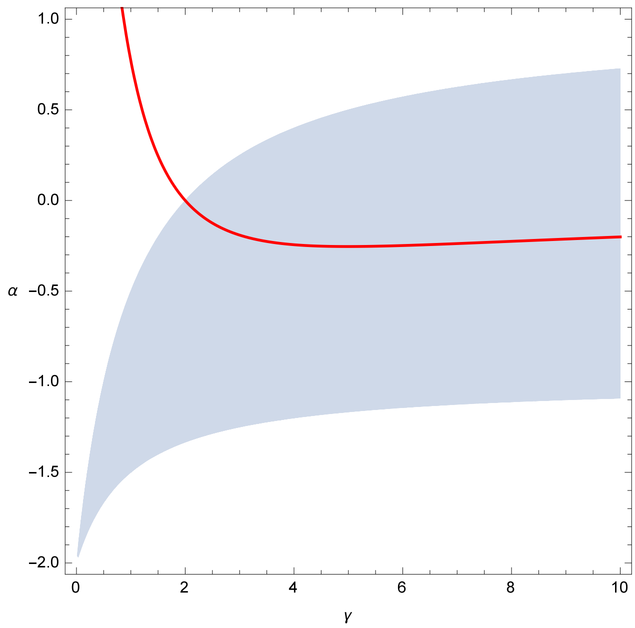

Figure 5.2: The constraint $n_s = 0.968 \pm 0.006$ is plotted in red, assuming thermal fluctuations (the error bar is too small to be seen on the plot). The region satisfying the constraint ensuring a solution of the horizon problem, Eq. (5.37) is plotted in blue.

5.5 Concluding remarks

It has been investigated in this chapter the possibility that a rainbow universe with running Newton constant can accommodate primordial perturbations whose spectral index matches current constraints, without relying on inflation to solve the horizon problem. Starting form a universe filled with radiation subject to deformed dispersion relations (of the Hořava-Lifshitz type), both vacuum and thermal initial conditions for the perturbations have been considered and a power-law dependence of the Newton constant on energy has been assumed. Crucially, it has been assumed that the background satisfies the thermodynamical relations peculiar to radiation subject to deformed dispersion relations.

For both kinds of initial conditions for the perturbations (vacuum and thermal) the running of the Newton constant is essential in achieving a viable picture. In particular, the Newton constant is constrained to be decreasing with energy in the ultraviolet regime. This is consistent with intuition from quantum gravity theories, such as Hořava-Lifshitz gravity and Asymptotic safety. It also resonates with the conjecture put forward in [86]. In scenarios considered, vacuum and thermal initial conditions can be distinguished because, while for the former the observational constraints are compatible with either an accelerating or decelerating expansion of the universe, for the latter only a decelerated expansion is allowed.

One may question the wisdom of enforcing thermodynamical constraints on the background as well as on the fluctuations. A counter-example is a scalar field, for which the background does not need to be thermalized even when the fluctuations are [118]. Nonetheless it is curious that when, for the sake of minimality, one imposes thermal conditions on both background and perturbations of a scalar field, one recovers the universal result $n_s^T = 4$ previously derived for a thermodynamical fluid [120]. Just as with [120] one needs to relax standard assumptions to evade this result. Here the running of Newton's constant was the crucial ingredient.

Chapter 6

Analysis of causality and momentum conservation with Relative Locality

6.1 Causality from Relative locality

The present section offers a discussion on causality in Relative Locality. In Subsection 6.1.1 it is shown that the relativity of locality does not imply a relativity of causal relations: the causal connection between events is objective even in the relative locality framework. The only difference with respect to the standard case is that now the observer should not trust the inferences about distant events obtained from her coordinatization, but rather use translation transformations in combination with her description of worldlines. This is done analyzing a case of two causally disconnected chains of processes which are, nevertheless, tangled in such a way that a single observer would obtain a completely misleading picture of the process if she adopts only her own coordinatization to describe the process. A careful analysis shows that with the help of a proper use of translation transformations she can completely disentangle the two chains.

After this, in Subsection 6.1.2, opposite to what has been claimed in a recent paper ([107]), it is shown that causal loops, which in general are not excluded by the equations of motion in curved-momentum-space theories, are indeed excluded as soon as the extra requirement of relativity of locality is enforced in this class of theories. In fact, for a generic theory with curved momentum space, it is possible to obtain general conditions on the derivatives of the $\mathcal{K}$'s that must be satisfied in order for that theory to be symmetric under an appropriate notion of translation transformation. These conditions are translated into conditions on energies and momenta of the interacting particles. If these conditions are not satisfied, the causal loop is allowed, whereas when these are satisfied the only solution of the equations of motion is that the whole loop collapses to a single event.

6.1.1 Cause and effect, with relative locality

Consider a situation where two pairs of causally-linked events are present, arranged in such a way that the coordinatization by an observer may not render manifest the causal link (then finding that awareness of the form of translation transformations allows decoding the causal link). Specifically this situation consists in two atoms, that are excited by two photons, propagate and finally de-excite, each re-emitting a photon. Since it will be important in the subsequent analysis, it must be remarked that each pair of causally-linked events are causally independent from the other. It is also assumed that there is an observer Alice which is local to the excitation of the atoms, for which the two excitation events coincide, and an observer Bob, which is local to the de-excitation of the two atoms. Alice and Bob are taken in relative rest and the relation between their coordinazation of the worldlines of the particles is given by a translation transformation. Fig. 6.1 shows the two pairs of causally-linked events, together with the observers local to them.

For purposes of this section, two conditions on the energies of the particles must be satisfied. The first one is that the energies of the incoming photons are such that both atoms in the excited states can be considered as ultra-relativistic *i.e.* $p'_0 \gg m_{p'}$, $q'_0 \gg m_{q'}$. The other one is that some particles have their energy negligible with respect to the energy scale of the theory ℓ^{-1} while the energy of the other particles cannot be neglected. The first kind of particles is called "soft" and the second "hard". In Fig. 6.1 solid lines stand for hard particles while dashed lines stand for soft ones. In particular both atoms before excitations are soft particles, then the one labeled as (p', x') becomes hard when it absorbs the hard photon (p, x) and after propagating it re-emits the hard photon (p'', x'').

Now the relative locality framework inspired by the κ-momentum space with "time-to-the-right" coordinates is introduced (see [104]). This implies that the on-shell relation for a particle of momentum p and mass m is

$$\mathcal{C}_p = p_0^2 - p_1^2 + \ell p_0 p_1^2 - m^2 = 0 \;, \tag{6.1}$$

while the composition of two momenta p, q is

$$\begin{aligned}(p \oplus q)_0 &= p_0 + q_0 \;, \\ (p \oplus q)_1 &= p_1 + q_1 + \ell p_0 q_1 \;.\end{aligned} \tag{6.2}$$

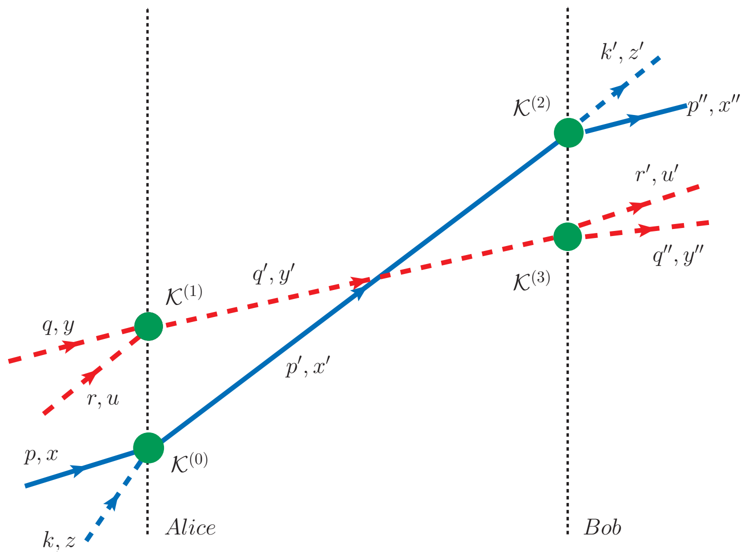

Figure 6.1: A process involving two causally-linked pairs of events. Different pairs are distinguished by different colors, while solid lines stand for "hard" particles and dashed lines for "soft" particles.

Then the process of Fig. 6.1 is described by the following action

$$\begin{aligned}\mathcal{S} = &\int_{-\infty}^{s_0} ds \left(z^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}_k\right) + \int_{-\infty}^{s_0} ds \left(x^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}_p\right) + \int_{-\infty}^{s_1} ds \left(y^\mu \dot{q}_\mu + \mathcal{N}_q \mathcal{C}_q\right) + \\ &+ \int_{-\infty}^{s_1} ds \left(u^\mu \dot{r}_\mu + \mathcal{N}_r \mathcal{C}_r\right) + \int_{s_0}^{s_3} ds \left(x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} \mathcal{C}_{p'}\right) + \int_{s_1}^{s_2} ds \left(y'^\mu \dot{q}'_\mu + \mathcal{N}_{q'} \mathcal{C}_{q'}\right) \\ &+ \int_{s_2}^{+\infty} ds \left(y''^\mu \dot{q}''_\mu + \mathcal{N}_{q''} \mathcal{C}_{q''}\right) + \int_{s_2}^{+\infty} ds \left(u'^\mu \dot{r}'_\mu + \mathcal{N}_{r'} \mathcal{C}_{r'}\right) + \int_{s_3}^{+\infty} ds \left(x''^\mu \dot{p}''_\mu + \mathcal{N}_{p''} \mathcal{C}_{p''}\right) \\ &+ \int_{s_3}^{+\infty} ds \left(z'^\mu \dot{k}'_\mu + \mathcal{N}_{k'} \mathcal{C}_{k'}\right) - \xi^\mu_{(0)} \mathcal{K}^{(0)}_\mu - \xi^\mu_{(1)} \mathcal{K}^{(1)}_\mu - \xi^\mu_{(2)} \mathcal{K}^{(2)}_\mu - \xi^\mu_{(3)} \mathcal{K}^{(3)}_\mu \ ,\end{aligned} \tag{6.3}$$

where the $\mathcal{K}^{(i)}_\mu$ appearing in the boundary terms are defined as

$$\begin{aligned}\mathcal{K}^{(0)}_\mu &= (k \oplus p)_\mu - p'_\mu \ , \\ \mathcal{K}^{(1)}_\mu &= (r \oplus q)_\mu - q'_\mu \ , \\ \mathcal{K}^{(2)}_\mu &= p'_\mu - (k' \oplus p'')_\mu \ , \\ \mathcal{K}^{(3)}_\mu &= q'_\mu - (r' \oplus q'')_\mu \ .\end{aligned} \tag{6.4}$$

Before going on it can be noticed that the action can be split into the sum of two parts, each describing one pair of causally-linked events.

By varying the action (6.3), one obtains the following equations of motion

$$\dot{p}_\mu = 0 \ , \quad \dot{q}_\mu = 0 \ , \quad \dot{k}_\mu = 0 \ , \quad \dot{r}_\mu = 0 \ , \quad \dot{p}'_\mu = 0 \ ,$$
$$\dot{q}'_\mu = 0 \ , \quad \dot{p}''_\mu = 0 \ , \quad \dot{q}''_\mu = 0 \ , \quad \dot{k}'_\mu = 0 \ , \quad \dot{r}'_\mu = 0 \ ,$$

$$\mathcal{C}_p = 0 \ , \quad \mathcal{C}_q = 0 \ , \quad \mathcal{C}_k = 0 \ , \quad \mathcal{C}_r = 0 \ , \quad \mathcal{C}_{p'} = 0 \ ,$$
$$\mathcal{C}_{q'} = 0 \ , \quad \mathcal{C}_{p''} = 0 \ , \quad \mathcal{C}_{q''} = 0 \ , \quad \mathcal{C}_{k'} = 0 \ , \quad \mathcal{C}_{r'} = 0 \ ,$$

$$\mathcal{K}^{(0)}_\mu = 0 \ , \quad \mathcal{K}^{(1)}_\mu = 0 \ , \quad \mathcal{K}^{(2)}_\mu = 0 \ , \quad \mathcal{K}^{(3)}_\mu = 0 \ ,$$

$$\dot{x}^\mu = \mathcal{N}_p \frac{\partial \mathcal{C}_p}{\partial p_\mu} \ , \quad \dot{y}^\mu = \mathcal{N}_q \frac{\partial \mathcal{C}_q}{\partial q_\mu} \ , \quad \dot{z}^\mu = \mathcal{N}_k \frac{\partial \mathcal{C}_k}{\partial k_\mu} \ , \quad \dot{u}^\mu = \mathcal{N}_r \frac{\partial \mathcal{C}_r}{\partial r_\mu} \ , \quad \dot{x}'^\mu = \mathcal{N}_{p'} \frac{\partial \mathcal{C}_{p'}}{\partial p'_\mu} \ ,$$
$$\dot{y}'^\mu = \mathcal{N}_{q'} \frac{\partial \mathcal{C}_{q'}}{\partial q'_\mu} \ , \quad \dot{x}''^\mu = \mathcal{N}_{p''} \frac{\partial \mathcal{C}_{p''}}{\partial p''_\mu} \ , \quad \dot{y}''^\mu = \mathcal{N}_{q''} \frac{\partial \mathcal{C}_{q''}}{\partial q''_\mu} \ , \quad \dot{z}'^\mu = \mathcal{N}_{k'} \frac{\partial \mathcal{C}_{k'}}{\partial k'_\mu} \ , \quad \dot{u}'^\mu = \mathcal{N}_{r'} \frac{\partial \mathcal{C}_{r'}}{\partial r'_\mu} \ ,$$

and the following boundary conditions for the endpoints of the worldlines

$$x^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\mu} \ , \quad y^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial q_\mu} \ , \quad z^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial k_\mu} \ ,$$
$$u^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial r_\mu} \ , \quad x'^\mu(s_0) = -\xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\mu} \ , \quad x'^\mu(s_3) = \xi^\nu_{(3)} \frac{\partial \mathcal{K}^{(3)}_\nu}{\partial p'_\mu} \ ,$$
$$y'^\mu(s_1) = -\xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial q'_\mu} \ , \quad y'^\mu(s_2) = \xi^\nu_{(2)} \frac{\partial \mathcal{K}^{(2)}_\nu}{\partial q'_\mu} \ , \quad x''^\mu(s_3) = -\xi^\nu_{(3)} \frac{\partial \mathcal{K}^{(3)}_\nu}{\partial p''_\mu} \ ,$$
$$y''^\mu(s_2) = -\xi^\nu_{(2)} \frac{\partial \mathcal{K}^{(2)}_\nu}{\partial q''_\mu} \ , \quad z'^\mu(s_3) = -\xi^\nu_{(3)} \frac{\partial \mathcal{K}^{(3)}_\nu}{\partial k'_\mu} \ , \quad u'^\mu(s_2) = -\xi^\nu_{(2)} \frac{\partial \mathcal{K}^{(2)}_\nu}{\partial r'_\mu} \ .$$

It is easy to check that the above equations of motion and boundary conditions are invariant under the following translation transformation:

$$\begin{aligned}
x^\mu_B &= x^\mu_A + b^\nu\{(k\oplus p)_\nu, x^\mu\}\ ,\\
y^\mu_B &= y^\mu_A + b^\nu\{(r\oplus q)_\nu, y^\mu\}\ ,\\
z^\mu_B &= z^\mu_A + b^\nu\{(k\oplus p)_\nu, z^\mu\}\ ,\\
u^\mu_B &= u^\mu + b^\nu\{(r\oplus q)_\nu, u^\mu\}\ ,\\
x'^\mu_B &= x'^\mu_A + b^\nu\{p'_\nu, x'^\mu\}\ ,\\
y'^\mu_B &= y'^\mu_A + b^\nu\{q'_\nu, y'^\mu\}\ ,\\
x''^\mu_B &= x''^\mu_A + b^\nu\{(k'\oplus p'')_\mu, x''^\mu\}\ ,\\
y''^\mu_B &= y''^\mu_A + b^\nu\{(r'\oplus q'')_\nu, y''^\mu\}\ ,\\
z'^\mu_B &= z'^\mu_A + b^\nu\{k'\oplus p'')_\mu, z'^\mu\}\ ,\\
u'^\mu_B &= u'^\mu_A + b^\nu\{(r'\oplus q'')_\nu, u'^\mu\}\ ,
\end{aligned} \tag{6.5}$$

where b^μ are the translation parameters.

Now it is supposed that the two atoms are excited at Alice's spacetime origin *i.e.* $x'^\mu_A = y'^\mu_A = 0$ and the soft atom de-excites at Bob's spacetime origin *i.e* $y'^\mu_B = 0$. It is supposed instead that the hard atom de-excites just in the space origin of Bob *i.e.* $x'^1_B = 0$. At first order in ℓ, the equations of motion yield

$$\frac{\dot{x}'^1}{\dot{x}'^0} = 1 + \ell p'_1\ , \qquad \frac{\dot{y}'^1}{\dot{y}'^0} = 1\ , \tag{6.6}$$

where it has been considered that $p'_0 \gg m_{p'}$, $q'_0 \gg m_{q'}$ (being $p'_1, q'_1 < 0$, with the conventions adopted). So Alice describes the worldlines of the two excited atoms as

$$\begin{aligned}
x'^1_A &= (1+\ell p'_1)x'^0_A\ ,\\
y'^1_A &= y'^0_A\ .
\end{aligned} \tag{6.7}$$

To compute at which times Bob sees these events to happen, one should use the worldlines in Bob's coordinazation, as it has been explained in Section 2.3.3. These worldlines can be obtained by introducing in (6.7) the translation trasformation which relates the coordinatization of Alice and Bob. For the coordinates of the two excited atoms the translation transformation is undeformed:

$$\begin{aligned}
x'^\mu_B(s) &= x'^\mu_A(s) + b^\nu\{p'_\nu, x'^\mu\} = x'^\mu_A(s) - b^\mu\ ,\\
y'^\mu_B(s) &= y'^\mu_A(s) + b^\nu\{q'_\nu, y'^\mu\} = y'^\mu_A(s) - b^\mu\ .
\end{aligned} \tag{6.8}$$

So the worldlines in Bob's coordinatization are

$$\begin{aligned}
x'^1_B &= (1+\ell p'_1)x'^0_B - b^1 + b^0 + b^0\ell p'_1\ ,\\
y'^1_B &= y'^0_B - b^1 + b^0\ .
\end{aligned} \tag{6.9}$$

Imposing $y'^{\mu}_B = 0$, it is found that $b^0 = b^1$; then, using $x^1_B = 0$, one gets

$$x'^0_B = -b^1 \ell p'_1 \ . \tag{6.10}$$

So the result is that Bob sees the hard atom arriving after the soft one in his space origin, with a time delay between them given by $\Delta t = -b^1 \ell p'_1$.

The attention can now be focused on what Alice infers about the two processes of de-excitation happening locally at Bob. It will be found that there are some puzzling features in her inferences. First of all one notices that the translations (6.8) are undeformed, so that Alice infers the same time delay measured by Bob as the time delay between the arrival of the soft and hard atoms in Bob's space origin. Then it is necessary to look at the boundary conditions in Alice's coordinatization for the particles involved in those processes:

$$\begin{aligned} y''^{\mu}_A(s_3) &= u'^{\mu}_A(s_3) = \xi^{\mu}_{(2)A} = (b^1, b^1) \ , \\ x''^0_A(s_2) &= b^1 - b^1 \ell p'_1 \ , \\ x''^1_A(s_2) &= b^1 + b^1 \ell k'_1 \ \approx b^1, \\ z'^0_A(s_2) &= b^1 - b^1 \ell p'_1 + b^1 \ell p''_1 \ , \\ z'^1_A(s_2) &= b^1 \ . \end{aligned} \tag{6.11}$$

Figg. 6.2 and 6.3 give a pictorial representation of the processes as seen and inferred by the two observers, Alice and Bob. Notice that, according to Alice's description, a hard photon is emitted by the hard atom, which actually after the de-excitation appears to be far from the place where the emission of the photon took place. More precisely it appears to emerge from the process of de-excitation of the soft atom ($p'_\mu \approx p''_\mu$).

Through this analysis it has been shown that two pairs of causally-connected events can provide a puzzling picture to observer Alice if she trusts her inferences about distant events: one could arrange the two events at Bob to be simultaneous, according to Bob and , since the two events appear to be delocalized in Alice's coordinates, then Alice might get misleading input in her analysis of causal links. However, if Alice uses in her analysis the translation transformations, so that she can establish how the two events distant from her actually appear to the nearby observer Bob, then Alice can cleanly disentangle the causal links.

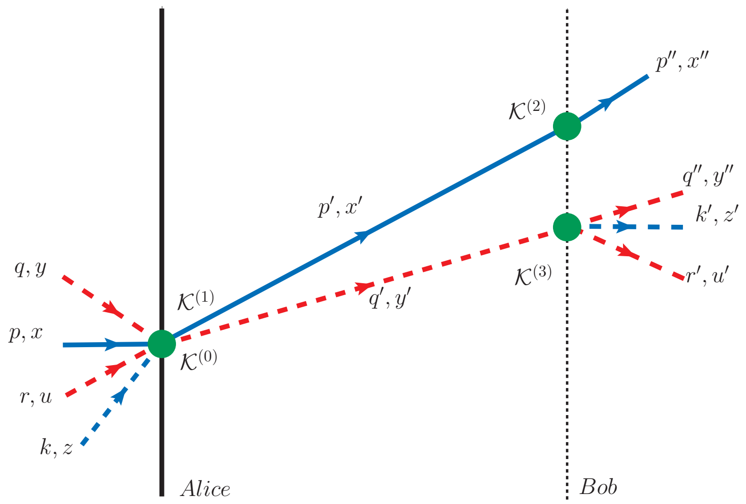

Figure 6.2: The two pairs of causally-linked events as seen (if local) or inferred (if distant) by Alice.

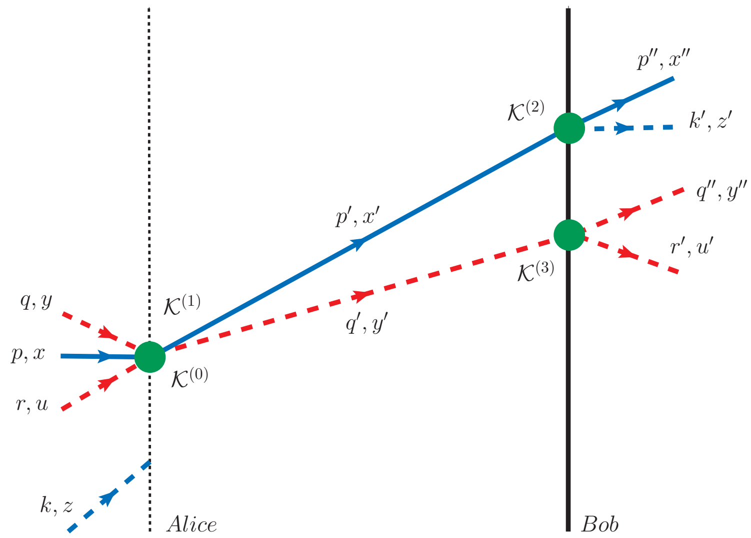

Figure 6.3: The two pairs of causally-linked events as seen (if local) or inferred (if distant) by Bob.

6.1.2 Causal Loop

The next task is to test causality beyond simple causal chains, *i.e.* considering the possibility of causality-violating loops (which for short shall often be labeled as "causal loops"). This is a possibility which was already considered in Ref. [107], yet by a perspective somewhat different from that discussed in Sections 2.2 and 2.3.

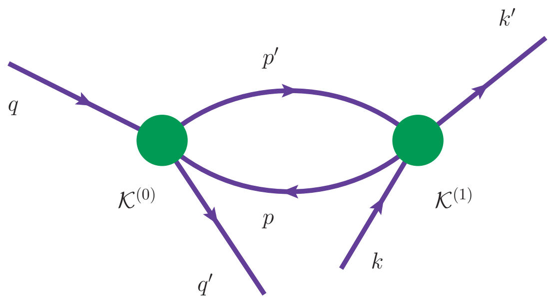

Figure 6.4: A causal chain which describe a causal loop as proposed in [107].

An action that reproduces the equations of motion and the boundary conditions that lead to the emergence of causal loops as described in [107] must be found. The results obtained shall be that causal loops are indeed in general allowed in theories with curved momentum spaces but they cannot be present when the theory with curved momentum space enjoys relative locality. Referring to Fig. 6.4, the action reads

$$S = \int_{-\infty}^{s_0} ds\, y^\mu \dot{q}_\mu + \mathcal{N}_q \mathcal{C}(q) + \int_{s_0}^{\infty} ds\, y'^\mu \dot{q}'_\mu + \mathcal{N}_{q'} \mathcal{C}(q') + \int_{-\infty}^{s_1} ds\, z^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}(k) +$$

$$+ \int_{s_1}^{\infty} ds\, z'^\mu \dot{k'}_\mu + \mathcal{N}_{k'} \mathcal{C}(k') + \int_{s_0}^{s_1} ds\, x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} \mathcal{C}(p') + \int_{s_1}^{s_0} ds\, x^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}(p) -$$

$$- \xi^\mu_{(0)} \mathcal{K}_\mu{}^{(0)} - \xi^\mu_{(1)} \mathcal{K}_\mu{}^{(1)},$$

where $\mathcal{K}^{(0)} = q \oplus p \oplus (\ominus(p' \oplus q'))$ and $\mathcal{K}^{(1)} = p' \oplus k \oplus (\ominus(k' \oplus p))$. Notice that the last integral, which stands for the free propagation of the particle that is traveling back in time, has inverted integration extrema. By varying this action one obtains the following equations of motion

$$\dot{p}_\mu = 0, \qquad \dot{p}'_\mu = 0, \qquad \dot{q}_\mu = 0, \qquad \dot{q}'_\mu = 0, \qquad \dot{k}_\mu = 0, \qquad \dot{k}'_\mu = 0,$$

$$\mathcal{C}_p = 0, \qquad \mathcal{C}_{p'} = 0, \qquad \mathcal{C}_q = 0, \qquad \mathcal{C}_{q'} = 0, \qquad \mathcal{C}_{k'} = 0, \qquad \mathcal{C}_k = 0,$$

$$\dot{x}^\mu(s) = \mathcal{N}_p \frac{\partial \mathcal{C}_p}{\partial p_\mu}, \qquad \dot{x}'^\mu(s) = \mathcal{N}_{p'} \frac{\partial \mathcal{C}_{p'}}{\partial p'_\mu}, \qquad \dot{y}^\mu(s) = \mathcal{N}_q \frac{\partial \mathcal{C}_q}{\partial q_\mu},$$
$$\dot{y}'^\mu(s) = \mathcal{N}_{q'} \frac{\partial \mathcal{C}_{q'}}{\partial q'_\mu}, \qquad \dot{z}^\mu(s) = \mathcal{N}_k \frac{\partial \mathcal{C}_k}{\partial k_\mu}, \qquad \dot{z}'^\mu(s) = \mathcal{N}_{k'} \frac{\partial \mathcal{C}_{k'}}{\partial k'_\mu},$$

and boundary terms

$$\mathcal{K}^{(0)}_\mu = 0, \qquad \mathcal{K}^{(1)}_\mu = 0,$$

$$y^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial q_\mu}, \qquad y'^\mu(s_0) = -\xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial q'_\mu}, \qquad z^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial k_\mu},$$
$$z'^\mu(s_1) = -\xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial k'_\mu}, \qquad x'^\mu(s_0) = -\xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\mu}, \qquad x'^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu},$$
$$x^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\mu}, \qquad x^\mu(s_1) = -\xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p_\mu}.$$

In this way the first goal has been reached: proposing an action that seems to reproduce the causal loop process anylized in [107]. In order to understand the properties of this action a step by step analysis is undertaken, first studying its Special Relativistic limit, then taking into account the deformations induced by the curvature over momentum space.

Notice that with this choice of the constraints $\mathcal{K}$, this action does not satisfy the prescriptions that guarantee translational invariance used in Section 2.3. Translation symmetry has a key role in distinguishing non-local theories from relative locality theories. Therefore, the calculations will continue in what follows taking care of finding an alternative prescription that makes this action symmetric under translations.

Causal loop in Special Relativity

In this subsection a $1+1$ spacetime with metric $\eta_{00} = 1$, $\eta_{11} = -1$ is considered. It is first worth remarking the equations of motion that will be needed for the subsequent analysis. Consider, as an example, the worldline of the particle of momentum p (for the other particles the same remark holds). Since, in the special relativistic limit, the dispersion relation reduces to $\mathcal{C}_p = p_0^2 - p_1^2 - m_p^2 = 0$, the equation of motion for the particle with momentum p becomes

$$\dot{x}^\mu(s) = 2\mathcal{N}_p p^\mu. \tag{6.12}$$

One can notice that

$$\dot{x}^\mu \dot{x}_\mu = 4\mathcal{N}_p^2 p^\mu p_\mu = 4\mathcal{N}_p^2 m_p^2, \tag{6.13}$$

so that

$$\mathcal{N}_p = \frac{(\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}}}{2m_p} \tag{6.14}$$

and one can then rewrite the equation (6.12) in the following way

$$\dot{x}^\mu(s) = (\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}} \frac{p^\mu}{m_p}. \tag{6.15}$$

Now the system is asked to satisfy two requirements:

1. All particles involved in the process travel along timelike worldlines; the velocity $\dot{x}^\mu$ (defined with respect to the arbitrary parameter s) and the momentum p_μ must satisfy that $\dot{x}^2 > 0,\ \dot{x}^0 > 0;\ p^2 = m_p^2 > 0,\ p^0 \geq m_p > 0$. This states simply that exotic particles are not considered in this discussion.

2. The class of physical reference frames considered here is that of all those that can be mutually obtained by means of a proper orthochronous Lorentz transformation ($\det\Lambda = 1,\ {\Lambda^0}_0 \geq 1$), *i.e.* the class of transformations that do not change the direction of time in going from a reference frame to another one; this means that two observers, each traveling in relative rest with respect to one of the two particles that form the loop, have clocks that go in the same direction. Furthermore, observers connected by an antichronous transformation (${\Lambda^0}_0 \leq -1$), would also disagree on the sign of the particles' energies.

These may be seen as too limiting assumptions to admit the possibility of causal loops. Nevertheless, these come from the particular kind of causal loop that has been studied in Ref.[107], that is one in which two observers, each local to a particular vertex of interaction of the loop, do not detect any anomaly; the anomaly of the process as a whole is reconstructed *a posteriori*.

Proper time, as usual, is defined by

$$d\tau = ds\,(\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}} = ds\dot{x}^0 \sqrt{1-\left(\frac{\dot{x}^1}{\dot{x}^0}\right)^2} = ds\dot{x}^0 \sqrt{1-\left(\frac{p^1}{p^0}\right)^2} = ds\dot{x}^0 \gamma_p^{-1},$$

where γ_p is the usual Lorentz factor and in the third equality the equation (6.15) was used.

For the (p', x') worldline which travels from $x'^\mu(s_0)$ to $x'^\mu(s_1)$ the following chain of equalities holds

$$\begin{aligned} x'^\mu(s_1) - x'^\mu(s_0) &= \int_{s_0}^{s_1} ds\,\frac{dx'^\mu}{ds} = \int_{s_0}^{s_1} ds\,\left(\dot{x}'^\mu \dot{x}'_\mu\right)^{\frac{1}{2}} \frac{p'^\mu}{m_{p'}} = \\ &= \int_{\tau'(s_0)}^{\tau'(s_1)} d\tau'\,\frac{p'^\mu}{m_{p'}} = \Delta\tau' u'^\mu, \end{aligned} \tag{6.16}$$

with $u'^\mu = \dfrac{p'^\mu}{m_{p'}}$. Similarly, for the (p, x) worldline, which travels from $x^\mu(s_0)$ to $x^\mu(s_1)$, holds

$$\begin{aligned} x^\mu(s_0) - x^\mu(s_1) &= \int_{s_1}^{s_0} ds\,\frac{dx^\mu}{ds} = \int_{s_1}^{s_0} ds\,(\dot{x}^\nu \dot{x}_\nu)^{\frac{1}{2}} \frac{p^\mu}{m_p} = \\ &= \int_{\tau(s_1)}^{\tau(s_0)} d\tau \frac{p^\mu}{m_p} = \Delta\tau\ u^\mu, \end{aligned} \tag{6.17}$$

In the Special Relativistic limit the terms enforcing the conservation laws take the simple form $\mathcal{K}_\mu^{(0)} = q_\mu + p_\mu - p'_\mu - q'_\mu$ and $\mathcal{K}_\mu^{(1)} = p'_\mu + k_\mu - k'_\mu - p_\mu$, giving for the particles inside the loop the boundary terms

$$\xi^\mu_{(0)} = x'^\mu(s_0), \tag{6.18}$$

$$\xi^\mu_{(0)} = x^\mu(s_0), \tag{6.19}$$

$$\xi^\mu_{(1)} = x'^\mu(s_1), \tag{6.20}$$

$$\xi^\mu_{(1)} = x^\mu(s_1). \tag{6.21}$$

Subtracting (6.20) from (6.18) and (6.19) from (6.21) and using the equations (6.16) and (6.17) the following relations are obtained

$$\xi^\mu_{(1)} - \xi^\mu_{(0)} = x'^\mu(s_1) - x'^\mu(s_0) = \Delta\tau' u'^\mu, \tag{6.22}$$

$$\xi^\mu_{(0)} - \xi^\mu_{(1)} = x^\mu(s_0) - x^\mu(s_1) = \Delta\tau\ u^\mu, \tag{6.23}$$

which imply

$$\Delta\tau\ u^\mu + \Delta\tau' u'^\mu = 0. \tag{6.24}$$

After the definition of causal loop stated before, the only solution to (6.24) is $\Delta\tau = \Delta\tau' = 0$ and $\xi^\mu_{(0)} = \xi^\mu_{(1)} = 0$.

It is also observed that computing directly the proper time interval of the particles inside the loop, one obtains

$$\Delta\tau = \int_{s_1}^{s_0} ds\, \dot{x}^0 \gamma_p^{-1} = \gamma_p^{-1}\left(x^0(s_0) - x^0(s_1)\right) = \gamma_p^{-1}\left(\xi^0_{(0)} - \xi^0_{(1)}\right), \tag{6.25}$$

$$\Delta\tau' = \int_{s_0}^{s_1} ds\, \dot{x}'^0 \gamma_{p'}^{-1} = \gamma_{p'}^{-1}\left(x'^0(s_1) - x'^0(s_0)\right) = \gamma_{p'}^{-1}\left(\xi^0_{(1)} - \xi^0_{(0)}\right). \tag{6.26}$$

and, imposing (from the second requirement) $\Delta\tau \geq 0$, $\Delta\tau' \geq 0$, gets $\xi^0_{(0)} = \xi^0_{(1)}$. Equations of motion imply that particles connect only events whose coordinates satisfy $\left(\xi_{(1)} - \xi_{(0)}\right)^2 \geq 0$ thus the loop collapses to a single event $\xi^\mu_{(0)} = \xi^\mu_{(1)}$.

Causal loop with curved momentum space

The next step is to take into account the deformations induced by the curvature of the momentum space. The second requirement above must be slightly modified in order to allow DSR-deformed relativistic transformations.

In order to perform quantitative computations the well-known κ-momentum space and its DSR-relativistic symmetries is chosen. Thus spacetime is Minkowskian with metric $\eta_{\mu\nu} = diag(1, -1)$, but the dispersion relation at leading order reads as

$$\mathcal{C}_p = p_0^2 - p_1^2 + \ell p_0 p_1^2 - m_p^2 = 0, \tag{6.27}$$

while conservation laws at first order become

$$\mathcal{K}^{(0)}_0 = q_0 + p_0 - q'_0 - p'_0, \tag{6.28a}$$

$$\mathcal{K}^{(0)}_1 = q_1 + p_1 - q'_1 - p'_1 + \ell\left(q_0 p_1 - \mathcal{K}^{(0)}_0 p'_1 - (q_0 + p_0 - q'_0)q'_1\right), \tag{6.28b}$$

$$\mathcal{K}^{(1)}_0 = p'_0 + k_0 - p_0 - k'_0, \tag{6.28c}$$

$$\mathcal{K}^{(1)}_1 = p'_1 + k_1 - p_1 - k'_1 + \ell\left(p'_0 k_1 - \mathcal{K}^{(1)}_0 k'_1 - (p'_0 + k_0 - p_0)p_1\right). \tag{6.28d}$$

Taking as before, for example, the first of (6.1.2)[1], one obtains[2]

$$\dot{x}^\mu(s) = \mathcal{N}_p\left[2p^\mu + \ell\left(\delta^\mu_0 p_1^2 + \delta^\mu_1 2 p_0 p_1\right)\right]. \tag{6.29}$$

[1] The computations for the other worldlines are still the same.
[2] All computations are made at first order in ℓ.

where the notation $p^\mu \equiv \eta^{\mu\nu} p_\nu$ has been introduced. Similarly, introducing $x_\mu \equiv \eta_{\mu\nu} x^\nu$, the norm of both sides can be computed

$$\dot{x}^\mu \dot{x}_\mu = 4\mathcal{N}_p^2 \left(m_p^2 + 2\ell p_0 p_1^2 \right), \tag{6.30}$$

so

$$\mathcal{N}_p = \frac{(\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}}}{2m_p} \left(1 - \ell \frac{p_0 p_1^2}{m_p^2} \right) \tag{6.31}$$

and finally

$$\dot{x}^\mu(s) = (\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}} \frac{p^\mu}{m_p} - \ell \frac{(\dot{x}^\mu \dot{x}_\mu)^{\frac{1}{2}}}{2m_p} \left(2\frac{p_0 p_1^2}{m_p^2} p^\mu - \delta_0^\mu p_1^2 - \delta_1^\mu 2 p_0 p_1 \right) = (\dot{x}^\nu \dot{x}_\nu)^{\frac{1}{2}} u^\mu, \tag{6.32}$$

with $u^\mu = \dfrac{p^\mu}{m_p} - \dfrac{\ell}{2m_p} \left(2\dfrac{p_0 p_1^2}{m_p^2} p^\mu - \delta_0^\mu p_1^2 - \delta_1^\mu 2 p_0 p_1 \right)$.

Following the same pattern used in (6.16) and (6.17) one obtains that

$$x'^\mu(s_1) - x'^\mu(s_0) = \Delta\tau' u'^\mu, \tag{6.33}$$

$$x^\mu(s_0) - x^\mu(s_1) = \Delta\tau \; u^\mu. \tag{6.34}$$

Manipulating the boundary terms related to the particles (p, x) and (p', x'), it follows that[3]

$$\xi^\nu_{(0)} = -x'^\mu(s_0) \left(\frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\mu} \right)^{-1} = x^\mu(s_0) \left(\frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\mu} \right)^{-1}, \tag{6.35}$$

$$\xi^\nu_{(1)} = x'^\mu(s_1) \left(\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu} \right)^{-1} = -x^\mu(s_1) \left(\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p_\mu} \right)^{-1}. \tag{6.36}$$

Equation (6.35) combined with (6.34) implies

$$-x'^\mu(s_0) \left(\frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\mu} \right)^{-1} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\rho} = x^\rho(s_0) = x^\rho(s_1) + \Delta\tau u^\rho_p, \tag{6.37}$$

---

[3] Here and in the following $\left(\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu} \right)^{-1}$ denotes the (ν, μ) element of the matrix made of the derivatives of the different components of $\mathcal{K}^{(1)}$ with respect to the different components of p', $\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu}$. That is, $\left(\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\rho} \right) \left(\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu} \right)^{-1} = \delta^\rho_\nu$. Another possible notation in substitution of $\left(\frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu} \right)^{-1}$ could have been $\left(\frac{\partial \mathcal{K}^{(1)\,-1}}{\partial p'} \right)^\nu_\mu$.

while equation (6.36) combined with (6.33) implies

$$\begin{aligned} x^\rho(s_1) &= -x'^\mu(s_1)\left(\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p'_\mu}\right)^{-1}\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_\rho} = \\ &= -(x'^\mu(s_0)+\Delta\tau' u'^\mu)\left(\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p'_\mu}\right)^{-1}\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_\rho}. \end{aligned} \tag{6.38}$$

Finally, replacing the value of $x^\rho(s_1)$, given by equation (6.38), in equation (6.37), one obtains the same condition given in [107]:

$$\begin{aligned} &\left[\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_\rho}\left(\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p'_\mu}\right)^{-1} - \frac{\partial\mathcal{K}^{(0)}_\nu}{\partial p_\rho}\left(\frac{\partial\mathcal{K}^{(0)}_\nu}{\partial p'_\mu}\right)^{-1}\right] x'^\mu(s_0) = \\ &= -\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_\rho}\left(\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p'_\mu}\right)^{-1}\Delta\tau' u'^\mu + \Delta\tau u^\rho. \end{aligned} \tag{6.39}$$

Keeping only terms up to the first order in ℓ, it becomes

$$\begin{aligned} &\ell\left[\delta^\rho_0\left(k_1-q'_1\right)+\delta^\rho_1\left(q_0-k'_0\right)\right]x'^1(s_0) = \\ &= \Delta\tau u^\rho + \Delta\tau'\left[u'^\rho + u'^1\ell\left(\delta^\rho_1 k'_0 - \delta^\rho_0 k_1\right)\right]. \end{aligned} \tag{6.40}$$

This (6.40) is what replaces (6.24) when the causal loop is analyzed on a curved momentum space without enforcing relative locality. Notice that this (6.40), when its left-hand side does not vanish, can have solutions with positive $\Delta\tau$ and $\Delta\tau'$ and positive zero components of the four-velocities, which was not possible with (6.24). This means that contrary to the special-relativistic case (Minkowski momentum space) causal loops are possible on a curved momentum space, at least if one does not enforce relative locality.

Some interesting equalities follow from (6.40) and therefore must hold for the causal loop to be allowed

$$\Delta\tau = -\Delta\tau'\frac{u'^0}{u^0} + \ell x'^1(s_0)\left(\frac{q'_1-k_1}{u^0}\right) - \ell\Delta\tau'\left(\frac{u'^1 k_1}{u^0}\right), \tag{6.41}$$

$$\ell x'^1(s_0) = \Delta\tau'\frac{u^1u'^0 - u^0u'^1 + \ell u'^1(k_1u^1 + k'_0u^0)}{u^0(q_0-k'_0)+u^1(q'_1-k_1)} \tag{6.42}$$

and imply that in order for (6.41) to have acceptable solutions one must have that

$$x'^1(s_0) > \frac{\Delta\tau'(u'^0+\ell u'^1k_1)}{\ell|q'_1-k_1|}. \tag{6.43}$$

This is in good agreement with the results of Ref. [107], but it is useful to add some observations to those reported in Ref. [107]. A first point to notice is that Eq. (6.43) appears to suggest that x'^1 should take peculiarly large values, as in some of the estimates given in Ref. [107], since x'^1 has magnitude set by a formula with the small scale ℓ in the denominator. If one could conclude that only cases with ultralarge x'^1 allowed such a causal loop, then the violations of causality would be to some extent less concerning (if confined to a range of values of x'^1 large enough to fall outside our observational window). However, it is easy to see that (6.43) does not really impose any restriction on the size of x'^1: one will have that typically x'^1 is much larger than $\Delta\tau'$ but there are causal loops for any value of x'^1 (under the condition of taking suitable values of $\Delta\tau'$ and $\Delta\tau$). So when momentum space is curved and one does not enforce the relativity of spacetime locality the violations of causality are rather pervasive.

There is also a technical point that deserves some comments and is related to this pervasiveness of the violations of causality: it might appear to be surprising that within a perturbative expansion, assuming small ℓ, one arrives at a formula like (6.43), with ℓ in the denominator. This is however not so surprising considering the role of x'^1 in this sort of analysis. The main clarification comes from observing that in the unperturbed theory (the $\ell = 0$ theory, *i.e.* special relativity) x'^1 is completely undetermined: as shown in the previous subsection the only causal loops allowed in special relativity are those that collapse (no violation of causality) and such collapsed causal loops are allowed for any however large or however small value of x'^1. As stressed above this fact that x'^1 can take any value is preserved by the ℓ corrections. The apparently surprising factor of $1/\ell$ only appears in a relationship between x'^1 and $\Delta\tau'$. If x'^1 and $\Delta\tau'$ both had some fixed finite value in the $\ell = 0$ theory than at finite small ℓ their values should change by very little. But since in the $\ell = 0$ theory x'^1 is unconstrained (in particular it could take any however large value) and its value is not linked in any way to the value $\Delta\tau'$, then it is not surprising that the ℓ corrections take the form shown for example in (6.43).

Causal loop analysis in 3+1 dimensions

So far the 1+1-dimensional case has been examined, but it is rather evident that the features discussed in the previous subsection are not an artifact of that dimensional reduction. Nonetheless it is worth pausing briefly in this subsection for verifying that indeed those features are still present in $3 + 1$ dimensions. In this case the on-shellness is governed by $\mathcal{C}_p = p_0^2 - \vec{p}^2 - \ell p_0 \vec{p}^2$

while conservation laws at first order take the form

$$\mathcal{K}^{(0)}_{\ 0} = q_0 + p_0 - q'_0 - p'_0 \ , \tag{6.44a}$$

$$\mathcal{K}^{(0)}_{\ i} = q_i + p_i - q'_i - p'_i - \ell\delta^j_i \left[q_0 p_j - (q_0 + p_0 - q'_0 - p'_0)p'_j - (q_0 + p_0 - q'_0)q'_j\right] \ , \tag{6.44b}$$

$$\mathcal{K}^{(1)}_{\ 0} = p'_0 + k_0 - p_0 - k'_0 \ , \tag{6.44c}$$

$$\mathcal{K}^{(1)}_{\ i} = p'_i + k_i - p_i - k'_i - \ell\delta^j_i \left[p'_0 k_j - (p'_0 + k_0 - p_0 - k'_0)k'_j - (p'_0 + k_0 - p_0)p_j\right] \ , \tag{6.44d}$$

where $i, j = 1, 2, 3$.

Adopting these expressions, eq.(6.39), keeping only terms up to first order in ℓ in the matrices like $\frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\rho}$ and their products, takes the form

$$\ell\left[\delta^\rho_i\left(k'_0 - q_0\right) + \delta^\rho_0\left(q'_i - k_i\right)\right] x'^i(s_0) = \left[u'^\rho + u'^i \ell\left(\delta^\rho_0 k_i - \delta^\rho_i k'_0\right)\right]\Delta\tau' + u^\rho \Delta\tau \ , \tag{6.45}$$

or, more clearly, using the energy conservation laws,

$$\begin{aligned}
\ell(q'_1 - k_1)x'^1(s_0) + \ell(q'_2 - k_2)x'^2(s_0) + \ell(q'_3 - k_3)x'^3(s_0) &= (u'^0 + \ell k_1 u'^1 + \ell k_2 u'^2 + \ell k_3 u'^3)\Delta\tau' + \\
&\quad + u^0 \Delta\tau, \\
\ell(k_0 - q'_0)x'^1(s_0) &= (1 - \ell k'_0)u'^1 \Delta\tau' + u^1 \Delta\tau, \\
\ell(k_0 - q'_0)x'^2(s_0) &= (1 - \ell k'_0)u'^2 \Delta\tau' + u^2 \Delta\tau, \\
\ell(k_0 - q'_0)x'^3(s_0) &= (1 - \ell k'_0)u'^3 \Delta\tau' + u^3 \Delta\tau.
\end{aligned} \tag{6.46}$$

Without really loosing any generality one can analyze the implications of this for an observer orienting her axis of the reference frame so that $p_i = 0$ and $p'_i = 0$ for $i = 2, 3$. As a result one also has that $u^i = 0$ and $u'^i = 0$ for $i = 2, 3$. For what concerns the other momenta involved in the analysis, q, q', k, k'. this choice of orientation of axis only affects rather mildly the conservation laws:

$$\begin{aligned}
&q_2 = q'_2 - \ell p'_0 q'_2, \qquad q_3 = q'_3 - \ell p'_0 q'_3, \qquad q'_2 = q_2 + \ell p'_0 q_2, \qquad q'_3 = q_3 + \ell p'_0 q_3, \\
&k_2 = k'_2 + \ell p'_0 k'_2, \qquad k_3 = k'_3 + \ell p'_0 k'_3, \qquad k'_2 = k_2 - \ell p'_0 k_2, \qquad k'_3 = k_3 - \ell p'_0 k_3.
\end{aligned}$$

Since $u^i = 0$ and $u'^i = 0$ for $i = 2, 3$ the last two equations of eq.(6.46) imply $x'^2 = 0$ and $x'^3 = 0$, which in turn (looking then at the first two equations of eq.(6.46)) takes the computation back to (6.41)-(6.42)

$$\Delta\tau = -\Delta\tau' \frac{u'^0}{u^0} + \ell x'^1(s_0)\left(\frac{q'_1 - k_1}{u^0}\right) - \ell\Delta\tau'\left(\frac{u'^1 k_1}{u^0}\right),$$

$$\ell x'^1(s_0) = \Delta\tau' \frac{u^1 u'^0 - u^0 u'^1 + \ell u'^1(k_1 u^1 + k'_0 u^0)}{u^0(q_0 - k'_0) + u^1(q'_1 - k_1)} \ .$$

Evidently then all the features discussed for the 1+1-dimensional in the previous subsection are also present in the 3+1-dimensional case.

Enforcing Relative Locality

It will be now shown that there are no causal loops in theories with curved momentum spaces if these theories have relative locality. Relative locality is evidently a weaker notion than absolute locality but is still strong enough as to enforce causality.

By definition [102] Relative Locality is such that the locality of events may not be manifest in coordinatizations by distant observers, but for the coordinatizations of observers near an event (ideally at the event) it enforces locality in a way that is not weaker than ordinary locality.

It shall also be noticed that the definition of Relative Locality imposes that translation transformations be formalized in the theory: since one must verify that events are local according to nearby observers (while they may be described as nonlocal by distant observers), these need to use translation transformations in order to ensure that the Principle of Relative Locality is enforced.

In Ref. [104] it has been introduced a prescription for having a very powerful implementation of translational invariance in relative-locality theories. One can easily see that the causal loop described in the previous subsections is not compatible with that strong implementation of translational invariance. Evidently then causality is preserved in theories with curved momentum spaces if the strong notion of translational invariance of Ref. [104] is enforced by postulate.

What is here intended to be shown is that, however, causal loops are forbidden even without enforcing such a strong notion of translational invariance. Causal loops are forbidden even by a minimal notion of translational invariance, *i.e.* the bare minimum needed in order to contemplate relative locality (as stressed just above, one cannot even speak of relative locality in lack of a notion of translational invariance).

Consistently with this objective, it is only required the availability of some translation generator (possibly momentum-dependent) that can enforce the covariance of the equations of motion and the boundary conditions. Consider a first observer, Alice, and a second one, Bob, purely translated by a parameter b^μ with respect to Alice. For the particle involved inside the loop one has

$$x_B^\mu(s) = x_A^\mu(s) - b^\nu \mathcal{T}_\nu^\mu, \tag{6.47}$$

$$x_B'^\mu(s) = x_A'^\mu(s) - b^\nu \mathcal{T}_\nu'^\mu. \tag{6.48}$$

Combining the boundaries (6.1.2) with the transformation (6.47) one obtains

$$-\xi^\nu_{B(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p_\mu} = x_B^\mu(s_1) = x_A^\mu(s_1) - b^\nu \mathcal{T}_\nu^\mu = -\xi^\nu_{A(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p_\mu} - b^\nu \mathcal{T}_\nu^\mu \tag{6.49}$$

$$\xi^\nu_{B(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p_\mu} = x_B^\mu(s_0) = x_A^\mu(s_0) - b^\nu \mathcal{T}_\nu^\mu = \xi^\nu_{A(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p_\mu} - b^\nu \mathcal{T}_\nu^\mu. \tag{6.50}$$

Defining $\delta\xi^\nu_{(i)} = \xi^\nu_{B(i)} - \xi^\nu_{A(i)}$, equations (6.49) and (6.50) read as

$$b^\nu \mathcal{T}_\nu^\mu = \delta\xi^\nu_{(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p_\mu}, \tag{6.51}$$

$$b^\nu \mathcal{T}_\nu^\mu = -\delta\xi^\nu_{(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p_\mu}. \tag{6.52}$$

So

$$\delta\xi^\nu_{(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p_\mu} = -\delta\xi^\nu_{(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p_\mu} \tag{6.53}$$

Similarly, combining the last two boundaries of (6.1.2) with the transformation (6.48) one obtains

$$-\xi^\nu_{B(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p'_\mu} = x_B'^\mu(s_0) = x_A'^\mu(s_0) - b^\nu \mathcal{T}_\nu'^\mu = -\xi^\nu_{A(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p'_\mu} - b^\nu \mathcal{T}_\nu'^\mu, \tag{6.54}$$

$$\xi^\nu_{B(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p'_\mu} = x_B'^\mu(s_1) = x_A'^\mu(s_1) - b^\nu \mathcal{T}_\nu'^\mu = \xi^\nu_{A(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p'_\mu} - b^\nu \mathcal{T}_\nu'^\mu, \tag{6.55}$$

from which it follows that

$$-\delta\xi^\nu_{(1)} \frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p'_\mu} = \delta\xi^\nu_{(0)} \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p'_\mu}. \tag{6.56}$$

Before going on with the analysis it can be noticed that the equations (6.53) and (6.56) lead to conditions already analyzed in literature. Writing the conservation laws as $\bigoplus_{i=1}^{i=n} P^i_{in} - \bigoplus_{i=1}^{i=m} P^i_{out}$, where P^i_{in} are the ingoing momenta in a vertex and P^i_{out} are the outgoing momenta, one obtains the same conditions found in [108], while assuming that $\delta\xi^{\nu}_{(1)} = \delta\xi^{\nu}_{(0)} = -b^{\nu}$ the same conditions found in [104] are derived.

Going back to our analysis of the causal loop, from Eq.(6.56) one gets

$$\delta\xi^{\nu}_{(0)} = -\delta\xi^{\sigma}_{(1)} \frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p'_{\mu}} \left(\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p'_{\mu}}\right)^{-1}, \tag{6.57}$$

replacing it in equation (6.53) gives

$$\delta\xi^{\sigma}_{(1)} \left[\frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p_{\rho}} - \frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p'_{\mu}} \left(\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p'_{\mu}}\right)^{-1} \frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p_{\rho}}\right] = 0, \tag{6.58}$$

and finally, imposing $\delta\xi^{\sigma}_{(1)} \neq 0$,

$$\frac{\partial\mathcal{K}^{(1)}_{\nu}}{\partial p_{\rho}} \left(\frac{\partial\mathcal{K}^{(1)}_{\nu}}{\partial p'_{\mu}}\right)^{-1} - \frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p_{\rho}} \left(\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p'_{\mu}}\right)^{-1} = 0. \tag{6.59}$$

Equation (6.59) is then a condition on the boundary terms which comes from insisting that the theory be compatible with the enforcement of relative locality and, therefore, be compatible with a least the weakest possible notion of translational invariance. Using it into equation (6.39) it is observed that indeed the dependence on the position disappears. With the choice of the conservation laws made in [107], equation (6.59) becomes a condition on the momenta involved in the process. Explicitly, keeping only terms up to first order equations (6.59) becomes

$$\ell\delta^{1}_{\mu} \left[\delta^{\rho}_{0} \left(q'_{1} - k_{1}\right) - \delta^{\rho}_{1} \left(k'_{0} - q_{0}\right)\right] = 0, \tag{6.60}$$

which implies that $k'_0 = q_0 + \mathcal{O}(\ell)$ and $q'_1 = k_1 + \mathcal{O}(\ell)$.

The fact that then the causal loop is forbidden can then be seen easily for example by looking back at equation (6.40), now enforcing (6.60): one obtains

$$\Delta\tau u^{\rho} + \Delta\tau' \left[u'^{\rho} + u'^{1}\ell \left(\delta^{\rho}_{1} k'_{0} - \delta^{\rho}_{0} k_{1}\right)\right] = 0. \tag{6.61}$$

Analyzing it for $\rho = 0$, it is evident that in order to have solutions, either one between $\Delta\tau$ and $\Delta\tau'$ must be negative, or the zeroth component of one of the two 4-velocity must be negative as it is found in the Special Relativistic case. This because the terms proportional to ℓ is only a small correction which cannot cause a change of sign of the coefficient of $\Delta\tau'$. The only acceptable solution is then $\Delta\tau = \Delta\tau' = 0$.

The values of $\Delta\tau$ and $\Delta\tau'$ can also be computed directly. Following equations (6.25) and (6.26) the interval of proper times[4] between the two events for the two particles inside the loop are

$$\Delta\tau = \int_{s_1}^{s_0} ds\ \dot{x}^0 \gamma_p^{-1} = \gamma_p^{-1}\left(x^0(s_0) - x^0(s_1)\right), \tag{6.62}$$

$$\Delta\tau' = \int_{s_0}^{s_1} ds\ \dot{x}'^0 \gamma_{p'}^{-1} = \gamma_{p'}^{-1}\left(x'^0(s_1) - x'^0(s_0)\right). \tag{6.63}$$

The two Lorentz factor can be computed as in the Special Relativistic case: $\gamma_p = \frac{1}{\sqrt{1-\beta_p^2}}$. The only difference is that now $\beta_p = \frac{\dot{x}^1}{\dot{x}^0}$, where one has to use for the $\dot{x}^\mu$s the expression (6.32). Using the boundary conditions (6.1.2) and (6.1.2) the expressions for the interval of proper times at leading order becomes

$$\Delta\tau = \left[\sqrt{1-\frac{p_1^2}{p_0^2}}\left(1+\frac{\ell}{2}\frac{2p_0^2p_1^2+p_1^4}{p_0^3-p_1^2p_0}\right)\right]\left(\xi^\nu_{(1)}\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_0}+\xi^\nu_{(0)}\frac{\partial\mathcal{K}^{(0)}_\nu}{\partial p_0}\right), \tag{6.64}$$

$$\Delta\tau' = \left[\sqrt{1-\frac{p_1'^2}{p_0'^2}}\left(1+\frac{\ell}{2}\frac{2p_0'^2p_1'^2+p_1'^4}{p_0'^3-p_1'^2p_0'}\right)\right]\left(\xi^\nu_{(1)}\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_0'}+\xi^\nu_{(0)}\frac{\partial\mathcal{K}^{(0)}_\nu}{\partial p_0'}\right). \tag{6.65}$$

They are positive provided that

$$\begin{cases} \xi^\nu_{(1)}\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p_0}+\xi^\nu_{(0)}\frac{\partial\mathcal{K}^{(0)}_\nu}{\partial p_0} \geq 0 \\ \xi^\nu_{(1)}\frac{\partial\mathcal{K}^{(1)}_\nu}{\partial p'_0}+\xi^\nu_{(0)}\frac{\partial\mathcal{K}^{(0)}_\nu}{\partial p'_0} \geq 0. \end{cases} \tag{6.66}$$

At leading-order in ℓ, this system becomes

$$\begin{cases} \xi^0_{(1)} - \xi^0_{(0)} - \ell p_1'\left(\xi^1_{(1)} - \xi^1_{(0)}\right) \geq 0 \\ \xi^0_{(0)} - \xi^0_{(1)} - \ell p_1'\left(\xi^1_{(0)} - \xi^1_{(1)}\right) - \ell\left(\xi^1_{(0)}q_1' - \xi^1_{(1)}k_1\right) \geq 0. \end{cases} \tag{6.67}$$

[4] The physical meaning of this affine parameter called here "proper time" is related to the geometry of momentum space: for geometries that do not deform the composition law for energy (as in Special Relativity and κ-Minkowski) there are not effects of relative locality for pure time translations, *i.e.* those translations in which the only non null parameter is b^0. In such cases, one can attribute to the interval $\Delta\tau$ the usual meaning of time interval measured by a clock at rest relative to that reference frame. If there is relative locality also for pure time translations, the measurement of $\Delta\tau$ involves a local measurement and an inference. Then τ would not be an observable any more.

Then the $\xi^\mu_{(i)}$s are expanded into powers of ℓ, i.e. $\xi^\mu_{(i)} = \xi^{\mu[0]}_{(i)} + \ell \xi^{\mu[1]}_{(i)}$. In this way it is known that the zeroth order of the expansion assumes the Special Relativistic value of the $\xi^\mu_{(i)}$. Substituting this expansion in the system (6.67), and using the Special Relativistic result $\xi^{\mu[0]}_{(1)} = \xi^{\mu[0]}_{(0)}$, one obtains

$$\begin{cases} \ell\left(\xi^{0[1]}_{(1)} - \xi^{0[1]}_{(0)}\right) \geq 0 \\ \ell\left(\xi^{0[1]}_{(0)} - \xi^{0[1]}_{(1)}\right) - \ell\xi^{1[0]}_{(0)}\left(q_1' - k_1\right) \geq 0. \end{cases} \tag{6.68}$$

It is recalled now that the translational covariance is recovered by imposing the condition $q_1' = k_1 + \mathcal{O}(\ell)$, so the system (6.68) becomes

$$\begin{cases} \ell\left(\xi^{0[1]}_{(1)} - \xi^{0[1]}_{(0)}\right) \geq 0 \\ \ell\left(\xi^{0[1]}_{(0)} - \xi^{0[1]}_{(1)}\right) \geq 0, \end{cases} \tag{6.69}$$

which implies that $\xi^{0[1]}_{(1)} = \xi^{0[1]}_{(0)} + \mathcal{O}(\ell)$ and then $\xi^0_{(1)} = \xi^0_{(0)} + \mathcal{O}(\ell^2)$. From this condition it follows that $\Delta\tau = \Delta\tau' = 0 + \mathcal{O}(\ell^2)$. Now it can be shown that from the equations of motion one gets also $\xi^{1[1]}_{(1)} = \xi^{1[1]}_{(0)} + \mathcal{O}(\ell)$. In fact one has

$$\begin{aligned} 0 + \mathcal{O}(\ell) = \Delta\tau' u'^1 &= x'^1(s_1) - x'^1(s_0) = \\ &= \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p_1'} + \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_1'} = \\ &= \xi^{1[1]}_{(0)} - \xi^{1[1]}_{(1)} - \ell\xi^{1[0]}_{(0)}(q_0 + p_0 - p_0' - q_0') = \\ &= \xi^{1[1]}_{(0)} - \xi^{1[1]}_{(1)} \end{aligned} \tag{6.70}$$

where in the second equality has been exploited that the zeroth order terms of the ξs coincide and in the last that the term in parenthesis is exactly $\mathcal{K}^{(0)}_0$.

The same thing can be verified considering the other worldline, for which one finds that

$$\begin{aligned} 0 + \mathcal{O}(\ell) = \Delta\tau u^1 &= x^1(s_1) - x^1(s_0) = \\ &= \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p_1} + \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_1} = \\ &= \xi^1_{(0)} - \xi^1_{(1)} + \ell\xi^1_{(0)} q_0 - \ell\xi^1_{(1)}(p_0 + k_0 - p_0') = \\ &= \ell(\xi^{1[1]}_{(0)} - \xi^{1[1]}_{(1)}) - \ell\xi^{1[0]}_{(0)}(p_0 + k_0 - p_0' - q_0). \end{aligned} \tag{6.71}$$

Since for the covariance under translations $q_0 = k_0'$, $\ell\xi^{1[0]}_{(0)}$ is multiplied again by $\mathcal{K}^{(1)}_0$, from which the result follows.

Summarizing, it has been demonstrated that $\xi^\mu_{(1)} = \xi^\mu_{(0)} + \mathcal{O}(\ell^2)$, so the request of translational covariance of the system leads to the collapse of the causal loop into a single event (up to the second order in ℓ) in the Relative Locality framework as well as in Special Relativity. This causal loop is indeed forbidden once Relative Locality is enforced.

6.2 Momentum conservation from Relative Locality

Having shown that causal loop of Ref. [107] is indeed allowed in generic theories on curved momentum spaces, but is forbidden when relative spacetime locality is enforced, it is time to move on to the next announced task which concerns two other species of loops: those that violates conservation of momentum and those that are non-causally violating.

This section focuses on a translational-invariance-violating diagram studied in Ref. [109]. There, the author considered theories on a curved momentum space, without enforcing relative spacetime locality, and showed that in general the diagram shown in Fig. 6.5 can produce violations of global momentum conservation. These violations take the shape [109] of $k' \neq k$, *i.e.* the momentum incoming into the diagram is not equal to the momentum outgoing from the diagram. Similarly to what has been shown in the previous section for a causal loop, it will be found that these violations of global momentum conservation from the diagram in Fig. 6.5 do not occur if one enforces relative spacetime locality.

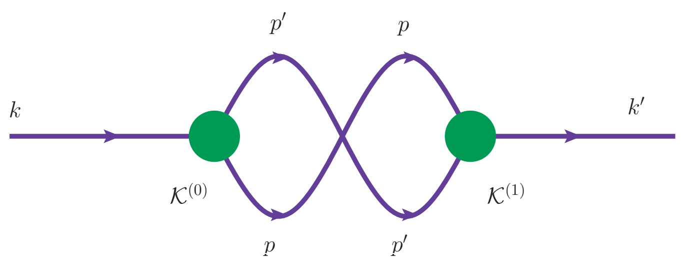

Figure 6.5: A Möbius diagram loop process.

6.2.1 Möbius diagram and translational invariance

The relative-locality-framework description of the diagram in Fig. 6.5 is obtained through the action

$$\begin{aligned}
\mathcal{S} &= \int_{-\infty}^{s_0} ds \left(z^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}_k\right) + \int_{s_1}^{+\infty} ds \left(z'^\mu \dot{k}'_\mu + \mathcal{N}_{k'} \mathcal{C}_{k'}\right) + \\
&+ \int_{s_0}^{s_1} ds \left(x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} \mathcal{C}_{p'}\right) + \int_{s_0}^{s_1} ds \left(x^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}_p\right) + \\
&- \xi^\mu_{(0)} \mathcal{K}^{(0)}_\mu - \xi^\mu_{(1)} \mathcal{K}^{(1)}_\mu ,
\end{aligned} \tag{6.72}$$

where the conservation law is given by the same functions considered in Ref. [109]

$$\begin{aligned}
\mathcal{K}^{(0)}_\mu &= \left(k \oplus (\ominus (p \oplus p'))\right)_\mu \\
&\simeq k_\mu - p_\mu - p'_\mu - \delta^1_\mu \ell \left[p_1 (k_0 - p_0 - p'_0) + p'_1 (k_0 - p'_0)\right], \\
\mathcal{K}^{(1)}_\mu &= \left((p' \oplus p) \oplus (\ominus k')\right)_\mu \\
&\simeq p'_\mu + p_\mu - k'_\mu - \delta^1_\mu \ell \left[k'_1 (p'_0 + p_0 - k'_0) - p'_0 p_1\right].
\end{aligned} \tag{6.73}$$

From the structure of (6.73) it is clear why the diagram in Fig. 6.5 has been labelled "Möbius diagram": the laws of conservation at the two vertices are setup in such a way to use the noncommutativity of the composition law in such a way that the particle outgoing from the first vertex with momentum appearing on the right-hand side of the composition law enters the second vertex with momentum appearing on the left-hand side of the composition law (Of course, the opposite applies to the other particle exchanged between the vertices). If one then draws the diagram with the convention that the orientation of pairs of legs entering/exiting a vertex consistently reflects the order in which the momenta are composed then the only way to draw the diagram makes it resemble a Möbius strip.

Evidently there is no room for such a structure when the momentum space has composition law which is commutative. In particular there is no way to contemplate such a Möbius diagram in Special Relativity. But on k-momentum space this structure is possible and its implication surely need to be studied.

Consistently with what has been reported in the previous section, the interest of this section is into understanding how the properties of the Möbius diagram are affected if one enforces relative spacetime locality in theories on the k-momentum space. In particular, it will be here shown that $k' = k$ (no violation of global momentum conservation) is required by relative spacetime locality.

And, as also already stressed above, relative spacetime locality in a relativistic theory on curved momentum space necessarily requires at least a weak form of translational invariance. This insistence on at least the weakest possible notion of translational invariance yield equations (6.53) and (6.56) for the causal loop, and, as one can easily verify, for the case of the Möbius diagram it leads to the equations

$$\delta\xi^{\nu}_{(0)}\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p_{\mu}} = -\delta\xi^{\nu}_{(1)}\frac{\partial\mathcal{K}^{(1)}_{\nu}}{\partial p_{\mu}}, \tag{6.74a}$$

$$\delta\xi^{\nu}_{(0)}\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p'_{\mu}} = -\delta\xi^{\nu}_{(1)}\frac{\partial\mathcal{K}^{(1)}_{\nu}}{\partial p'_{\mu}}. \tag{6.74b}$$

Explicating, for example, $\delta\xi^{\nu}_{(0)}$ in the second condition and substituting it back in the first, one can obtain the equation

$$\delta\xi^{\sigma}_{(1)}\left[\frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p_{\mu}} - \frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p'_{\rho}}\left(\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p'_{\rho}}\right)^{-1}\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p_{\mu}}\right] = 0. \tag{6.75}$$

Since translated observers must coordinatize the same event in different ways, one can impose $\delta\xi^{\sigma}_{(i)} \neq 0$. So the term in parenthesis of equation (6.75) have to be zero. This is clearly a condition over the momenta that are now analyzed at first order in ℓ. Writing first the expression of the matrices involved in the equation (6.75)

$$\frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p_{\mu}} = \delta^{\mu}_{\sigma} - \ell\delta^{1}_{\sigma}\left(\delta^{\mu}_{0}k'_{1} - \delta^{\mu}_{1}p'_{0}\right), \tag{6.76a}$$

$$\frac{\partial\mathcal{K}^{(1)}_{\sigma}}{\partial p'_{\rho}} = \delta^{\rho}_{\sigma} - \ell\delta^{1}_{\sigma}\delta^{\rho}_{0}\left(k'_{1} - p_{1}\right), \tag{6.76b}$$

$$\left(\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p'_{\rho}}\right)^{-1} = -\delta^{\nu}_{\rho} + \ell\delta^{1}_{\rho}\left[\delta^{\nu}_{1}\left(k_{0} - p'_{0}\right) - \delta^{\nu}_{0}\left(p_{1} + p'_{1}\right)\right], \tag{6.76c}$$

$$\frac{\partial\mathcal{K}^{(0)}_{\nu}}{\partial p_{\mu}} = -\delta^{\mu}_{\nu} + \ell\delta^{\mu}_{0}\delta^{1}_{\nu}p_{1}. \tag{6.76d}$$

So from (6.75) one finds the condition

$$\ell\left[\delta^{\mu}_{1}k_{0} - \delta^{\mu}_{0}\left(p_{1} + p'_{1}\right)\right] = 0 \tag{6.77}$$

Using this result in combination with the conservation laws $\mathcal{K}^{(0)}_{\mu} = 0$ and $\mathcal{K}^{(1)}_{\mu} = 0$ one can easily establish that

$$p_{\mu} + p'_{\mu} = 0 + \mathcal{O}(\ell)\ , \tag{6.78}$$

and one can also rewrite those conservation laws as follows

$$0 = k_\mu - p_\mu - p'_\mu - \delta^1_\mu \ell p'_1 p'_0, \tag{6.79}$$
$$0 = p'_\mu + p_\mu - k'_\mu - \delta^1_\mu \ell p'_0 p_1 \ . \tag{6.80}$$

Summing these (6.79) and (6.80), also using (6.78), we get to the sought result

$$k_\mu = k'_\mu + \mathcal{O}(\ell^2) \ , \tag{6.81}$$

showing that indeed by insisting on a having a translational invariant picture with associated relativity of spacetime locality, one finds no global violation of momentum conservation (at least at order in ℓ, which is the level of accuracy pursued in this work). Were it not a limitation on a leading-order-in-ℓ analysis, one could perhaps characterize this result on the Möbius diagram even more strongly: at leading order translational invariance essentially forbids Möbius diagrams. This can be seen in particular from Eq.(6.77) which also imposes[5] $\ell k_0 = 0$. So, up to possible corrections of order ℓ^2, Möbius diagrams are anly allowed if the energies of the incoming and outgoing particles vanish. We interpret this as implying that, at least to leading order, translational invariance essentially forbids Möbius diagrams.

The same results hold when the Möbius diagram is obtained using the prescriptions for constructing the constraints $\mathcal{K}$ given in [104]:

$$\begin{aligned} \mathcal{K}^{(0)}_\mu &= k_\mu - (p \oplus p')_\mu \simeq k_\mu - p_\mu - p'_\mu - \ell \delta^1_\mu p_0 p'_1, \\ \mathcal{K}^{(1)}_\mu &= (p' \oplus p)_\mu - k'_\mu \simeq p'_\mu + p_\mu - k_\mu + \ell \delta^1_\mu p'_0 p_1. \end{aligned} \tag{6.82}$$

In this case, in fact, one replaces Eq. (6.76) with

[5]It should be underlined that this condition $\ell k_0 = 0$ is a striking manifestation of how Möbius diagrams are foregn to translationally invariant implementations of the relative locality framework. The implied requirement $k_0 = 0$ is not a smooth correction to $\ell = 0$ theory, where k_0 is free (that is, can take any value). This is a similar mechanism to the one described after Eq.(6.43): a quantity which was completely free in the original theory (Special Relativity, with $\ell = 0$) ends up being governed by an equation in the deformed theory, or else the diagram must be discarded.

$$\frac{\partial \mathcal{K}^{(1)}_\sigma}{\partial p_\mu} = \delta^\mu_\sigma + \ell \delta^1_\sigma \delta^\mu_1 p'_0, \tag{6.83a}$$

$$\frac{\partial \mathcal{K}^{(1)}_\sigma}{\partial p'_\rho} = \delta^\rho_\sigma + \ell \delta^1_\sigma \delta^\rho_0 p_1, \tag{6.83b}$$

$$\left(\frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\rho}\right)^{-1} = -\delta^\nu_\rho + \ell \delta^1_\rho \delta^\nu_1 p_0, \tag{6.83c}$$

$$\frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\mu} = -\delta^\mu_\nu - \ell \delta^\mu_0 \delta^1_\nu p'_1. \tag{6.83d}$$

So from (6.75) one finds the condition

$$\ell \left(\delta^\mu_1 \left(p_0 + p'_0\right) - \delta^\mu_0 \left(p_1 + p'_1\right)\right) = 0 \tag{6.84}$$

From $\mu = 1$ and from $\mu = 0$ one finds that $p_\mu + p'_\mu = 0 + O(\ell)$. Summing the conservation laws enforced by the constraints (6.88) one has

$$0 = k_\mu - k'_\mu - \ell p_0 p'_1 + \ell p'_0 p_1.$$

The condition $p_\mu + p'_\mu = 0 + O(\ell)$ then again implies conservation of the spatial momentum $k_\mu = k'_\mu + \mathcal{O}(\ell^2)$.

6.2.2 Possible implications for the quantum theory: Fuzzy Momentum conservation

The results presented in the previous sections suggest that causality and global momentum conservation are protected by relative locality in theories with curved momentum spaces. It should be noticed that the objective of enforcing relative spacetime locality led to the introduction of some restrictions on the choice of boundary terms, particularly for causally connected interactions. The relevant class of theories has been studied so far only in the context of classical mechanics and therefore such prescriptions concerning boundary terms are meaningful and unproblematic, as they can be enforced by principle, as a postulate. The quantum version of Relative Locality is still not known, but if one tries to imagine which shape it might take, it seems that enforcing the principle of relative locality in a quantum theory might be very challenging: think in particular of quantum field theories formulated in terms of a generating functional. There is no specific result addressing this point to report here, but it is still worthy to provide evidence for the fact that combinations of diagrams on curved momentum space might have fewer anomalous properties, even without enforcing relative locality, than single diagrams.

Essentially it is here observed that the violations of causality and global translational symmetry that arise on curved momentum spaces (if one does not enforce relative locality) are not systematic, in the sense that for each diagram contributing an effect of a certain magnitude and sign there is always another equally acceptable diagram that gives effects of the same magnitude and opposite sign. This may be indeed relevant for quantum field theory since there one cannot choose which diagrams connect a given "in" state to a given "out" state: the formalism automatically takes into account all the diagrams that possibly connect the "in" state to the "out" state.

In an appropriate sense it is here attempted to provide first elements in support of a picture that might ultimately be somewhat analogous to what happens, for example, in the analysis of the gauge invariance of the first contribution to the matrix element of the Compton scattering $e^- + \gamma \to e^- + \gamma$ in standard QED. In fact in that case there are only two Feynman diagrams and the invariant matrix element is given by

$$\mathcal{M}_{fi} = (-ie)^2 \left(\bar{u}_{p'} \not{\epsilon}(q') \frac{i}{\not{p} + \not{q} - m} \not{\epsilon}(q) u_p + \bar{u}_{p'} \not{\epsilon}(q) \frac{i}{\not{p} - \not{q}' - m} \not{\epsilon}(q') u_p \right) \tag{6.85}$$

where p and q are the momenta of the electron and the photon respectively, in the initial state, p' and q' are the momenta of the electron and the photon respectively, in the final state, u_p and $\bar{u}_p$ are Dirac spinors, ϵ_μ the photon polarization 4-vector. For a free photon described in the Lorentz gauge by a plane wave $A_\mu(x) \propto \epsilon_\mu(k) e^{\pm i k_\nu x^\nu}$, the gauge transformation $A^\Lambda_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$ with $\Lambda(x) = \tilde{\Lambda}(k) e^{\pm i k_\nu x^\nu}$ corresponds to a transformation of the polarization 4-vector $\epsilon^\Lambda_\mu(k) = \epsilon_\mu(k) - i k_\mu \tilde{\Lambda}(k)$. Then the contribution to the matrix element due to this transformation of, for example, 4-vector $\epsilon_\mu(q)$ is (apart from a common factor) for the first term

$$\bar{u}_{p'} \not{\epsilon}(q') \frac{i}{\not{p} + \not{q} - m} \not{q} u_p = \bar{u}_{p'} \not{\epsilon}(q') \frac{i}{\not{p} + \not{q} - m} (\not{p} + \not{q} - m) u_p = i \bar{u}_{p'} \not{\epsilon}(q') u_p, \tag{6.86}$$

where the relation $(\not{p} - m) u_p = 0$ has been used. The second term gives the contribution

$$\bar{u}_{p'}\not{q}\frac{i}{\not{p}-\not{q}'-m}\not{\epsilon}(q')u_p = \bar{u}_{p'}(\not{q}-\not{p}'+m)\frac{i}{\not{p}-\not{q}'-m}\not{\epsilon}(q')u_p$$
$$= \bar{u}_{p'}(\not{q}-\not{p}'+m)\frac{i}{\not{p}'-\not{q}-m}\not{\epsilon}(q')u_p = -i\bar{u}_{p'}\not{\epsilon}(q')u_p, \tag{6.87}$$

where in the first equality $\bar{u}_{p'}(\not{p}'-m)=0$ has been used and in the second the equality $p-q'=q-p'$ has been used, which comes from global momentum conservation. Thus the matrix element is indeed gauge invariant even though the Feynman diagrams are not gauge invariant by themselves.

A conclusive evidence that a similar mechanism is at work for causality and global momentum conservation is of course still to be found (it would be impossible without knowing how to formulate such a quantum field theory), but it may nonetheless be interesting to note that one can find points of connection, at least at intuition level, with the story such as gauge invariance for Compton scattering.

For definiteness and simplicity, the explicit analysis in this section is for global translational symmetries, and therefore, the Möbius diagrams. In the previous subsection this case has been analyzed using the the choice of boundary terms adopted in Ref.[109] since the appreciation of the presence of a challenge due to Möbius diagrams originated from the study reported there. Here however the argument evolves beyond the scopes of Ref.[109] and it is therefore adopted the convention on boundary terms preferred by the author, which allows also to streamline the derivation of the results, the one given in [104]. Consider the Möbius diagram obtained using the prescriptions for constructing the constraints $\mathcal{K}$ given in [104]:

$$\begin{aligned}\mathcal{K}^{(0)}_\mu &= k_\mu - (p\oplus p')_\mu \simeq k_\mu - p_\mu - p'_\mu - \ell\delta^1_\mu p_0 p'_1,\\ \mathcal{K}^{(1)}_\mu &= (p'\oplus p)_\mu - k'_\mu \simeq p'_\mu + p_\mu - k_\mu + \ell\delta^1_\mu p'_0 p_1.\end{aligned} \tag{6.88}$$

From the conservation of four-momentum at each vertex $\mathcal{K}^{(0)}_\mu=0$, $\mathcal{K}^{(1)}_\mu=0$ one gets

$$k_\mu - k'_\mu = -\ell\delta^1_\mu(p'_0p_1 - p_0p'_1) = -\ell\delta^1_\mu\left(\frac{m_p^2 p'_1}{2p_1} - \frac{m_{p'}^2 p_1}{2p'_1}\right) \equiv -\ell\delta^1_\mu\Delta \tag{6.89}$$

where, since the energy-momentum of the particles here considered are such that $\ell^{-1}\gg p_\mu \gg m$, from the on-shell condition (6.1) the energy of the particles has been expressed in terms of the spatial momentum[6] $p_0=\sqrt{p_1^2+m^2}-\frac{\ell p_1^2}{2}\approx -p_1-\frac{m^2}{2p_1}-\frac{\ell p_1^2}{2}$ and only the leading correction terms have been kept.

[6]The readers should remind that the conventions adopted here are such that $p_1<0$.

Evidently, the only alternative possible Möbius diagram is obtained from the other form of the constraints $\mathcal{K}$ compatible with our prescription, that is by changing the order of p and p':

$$\begin{aligned}\tilde{\mathcal{K}}^{(0)}_\mu &= k_\mu - (p' \oplus p)_\mu \simeq k_\mu - p'_\mu - p_\mu - \ell\delta^1_\mu p'_0 p_1,\\ \tilde{\mathcal{K}}^{(1)}_\mu &= (p \oplus p')_\mu - k'_\mu \simeq p'_\mu + p_\mu - k'_\mu + \ell\delta^1_\mu p_0 p'_1.\end{aligned} \tag{6.90}$$

Proceeding as for the previous one, one gets

$$k_\mu - k'_\mu = \ell\delta^1_\mu\Delta. \tag{6.91}$$

Of course, in light of what it has been established in the previous subsection, both Möbius diagrams must be excluded if one enforces the principle of relative spacetime locality. But is it interesting to notice that if we were to allow these Möbius diagrams, the violation of global momentum conservation produced by one of them, (6.89), is exactly the opposite of the one produced by the other one, (6.91). In a quantum field theory version of the classical theories analyzed here, one might have to include these opposite contributions together, in which case it is here conjectured that the net result would not be some systematic prediction of violation of global momentum conservation, but rather something of the sort rendering global momentum still conserved but fuzzy.

Of course, the main challenge for the development of this novel research program is the construction of a quantum field theory. A general framework for introducing such quantum field theories was recently proposed in Ref. [134]. While presently this proposal appears to be still at too early and too formal a stage of development for addressing the challenges that were here of interest, it is legitimate to hope that, as its understanding deepens, a consistent quantum picture of causality and momentum conservation with curved momentum spaces will arise.

Going back to the classical mechanics version of these theories, it is amusing to notice that a chain composed of two Möbius diagrams considered in this subsection would have as a net result no violation of global momentum.

6.3 Non-causality-violating loops

A second species of loop, the so-called non-causality-violating loops represented in Fig.6.6, is analyzed in the present section. In Special Relativity, with its absolute locality, loops of this kind are trivial: they describe in some sense a composite of two parts at rest, with the two parts "splitting" for a while and then "recombining". This is a case of "history without a history":

all that one has is a single composite at rest throughout the history of the system, having allowed, for mere language, a split/recombination storyline. It is shown that relative locality is a strong-enough notion of locality to preserve this aspect of triviality of the non-causality-violating loops of the species shown in Fig.6.6.

Consider the following action describing the process of Fig. 6.6

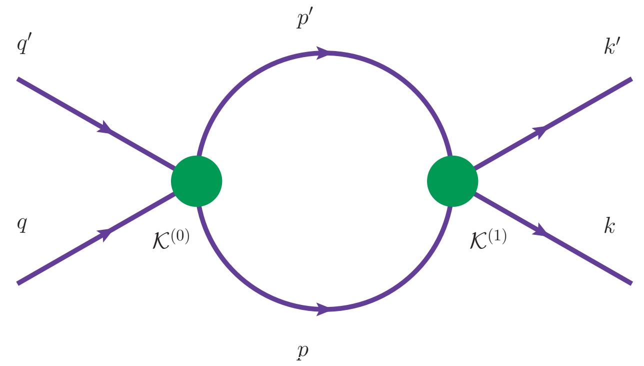

Figure 6.6: An example of non-causality-violating loop.

$$
\begin{aligned}
\mathcal{S} = & \int_{-\infty}^{s_0} ds \left(y^\mu \dot{q}_\mu + \mathcal{N}_q \mathcal{C}_q\right) + \int_{-\infty}^{s_0} ds \left(y'^\mu \dot{q}'_\mu + \mathcal{N}_{q'} \mathcal{C}_{q'}\right) + \\
& + \int_{s_1}^{+\infty} ds \left(z^\mu \dot{k}_\mu + \mathcal{N}_k \mathcal{C}_k\right) + \int_{s_1}^{+\infty} ds \left(z'^\mu \dot{k}'_\mu + \mathcal{N}_{k'} \mathcal{C}_{k'}\right) + \\
& + \int_{s_0}^{s_1} ds \left(x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} \mathcal{C}_{p'}\right) + \int_{s_0}^{s_1} ds \left(x^\mu \dot{p}_\mu + \mathcal{N}_p \mathcal{C}_p\right) + \\
& - \xi^\mu_{(0)} \mathcal{K}_\mu{}^{(0)} - \xi^\mu_{(1)} \mathcal{K}_\mu{}^{(1)},
\end{aligned}
\tag{6.92}
$$

with

$$
\mathcal{K}^{(0)}_\mu = \left(q' \oplus q\right)_\mu - \left(p' \oplus p\right)_\mu \simeq q'_\mu + q_\mu - p'_\mu - p_\mu + \ell \delta^1_\mu \left(q'_0 q_1 - p'_0 p_1\right), \tag{6.93}
$$

$$
\mathcal{K}^{(1)}_\mu = \left(p' \oplus p\right)_\mu - \left(k' \oplus k\right)_\mu \simeq p'_\mu + p_\mu - k'_\mu - k_\mu + \ell \delta^1_\mu \left(p'_0 p_1 - k'_0 k_1\right). \tag{6.94}
$$

The equations of motion are then

$$\dot{p}_\mu = 0, \qquad \dot{p}'_\mu = 0, \qquad \dot{q}_\mu = 0, \qquad \dot{q}'_\mu = 0, \qquad \dot{k}_\mu = 0, \qquad \dot{k}'_\mu = 0,$$

$$\mathcal{C}_p = 0, \qquad \mathcal{C}_{p'} = 0, \qquad \mathcal{C}_q = 0, \qquad \mathcal{C}_{q'} = 0, \qquad \mathcal{C}_{k'} = 0, \qquad \mathcal{C}_k = 0,$$

$$\dot{x}^\mu(s) = \mathcal{N}_p \frac{\partial \mathcal{C}_p}{\partial p_\mu}, \qquad \dot{x}'^\mu(s) = \mathcal{N}_{p'} \frac{\partial \mathcal{C}_{p'}}{\partial p'_\mu}, \qquad \dot{y}^\mu(s) = \mathcal{N}_q \frac{\partial \mathcal{C}_q}{\partial q_\mu}, \tag{6.95}$$

$$\dot{y}'^\mu(s) = \mathcal{N}_{q'} \frac{\partial \mathcal{C}_{q'}}{\partial q'_\mu}, \qquad \dot{z}^\mu(s) = \mathcal{N}_k \frac{\partial \mathcal{C}_k}{\partial k_\mu}, \qquad \dot{z}'^\mu(s) = \mathcal{N}_{k'} \frac{\partial \mathcal{C}_{k'}}{\partial k'_\mu}, \tag{6.96}$$

and the boundary terms are

$$\mathcal{K}^{(0)}_\mu = 0, \qquad \mathcal{K}^{(1)}_\mu = 0,$$

$$y^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial q_\mu}, \qquad y'^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial q'_\mu}, \qquad z^\mu(s_1) = -\xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial k_\mu},$$

$$z'^\mu(s_1) = -\xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial k'_\mu}, \qquad x'^\mu(s_0) = -\xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\mu}, \qquad x'^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu},$$

$$x^\mu(s_0) = -\xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\mu}, \qquad x^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p_\mu}.$$

As it has been done in the previous section, the process is first analyzed in Special Relativity, then in Relative Locality. It is shown now that in Relative Locality, as well as in Special Relativity, only trivial loops are allowed by the kinematics.

An example of trivial loop is the following: consider a molecule of hydrogen. Its motion may be described as that of a single particle. The loop starts when the motion of the molecule is described in terms of the motions of its two atoms and ends once one goes back to the description of the motion of the molecule as that of a single particle.

6.3.1 Non-causality violating loop in Special Relativity

In Special Relativity the analysis of the problem is simple. As one could expect, the loop might happen provided that x and x' travel in the same direction with the same velocity. Indeed, in the Special Relativistic limit the conservation laws (6.93) and (6.94) take the simple form

$$\mathcal{K}^{(0)}_{\mu} = q'_{\mu} + q_{\mu} - p'_{\mu} - p_{\mu}, \tag{6.97}$$
$$\mathcal{K}^{(1)}_{\mu} = p'_{\mu} + p_{\mu} - k'_{\mu} - k_{\mu}. \tag{6.98}$$

From the boundary terms related to the particles forming the loop, it follows that

$$x^{\mu}(s_0) = \xi^{\mu}_{(0)}, \qquad x'^{\mu}(s_0) = \xi^{\mu}_{(0)}, \tag{6.99}$$
$$x^{\mu}(s_1) = \xi^{\mu}_{(1)}, \qquad x'^{\mu}(s_1) = \xi^{\mu}_{(1)}. \tag{6.100}$$

Using the conditions (6.99), the equations of motion for the (p, x) and (p', x') particles can be written as

$$\begin{cases} x^1 = \dfrac{p_1}{p_0}\left(x^0 - \xi^0_{(0)}\right) + \xi^1_{(0)} \\ x'^1 = \dfrac{p'_1}{p'_0}\left(x'^0 - \xi^0_{(0)}\right) + \xi^1_{(0)}. \end{cases} \tag{6.101}$$

Enforcing then the conditions (6.100), one obtains that the equations of motion (6.101) imply that

$$\frac{p_1}{p_0} = \frac{p'_1}{p'_0}, \tag{6.102}$$

which means that the two particles must obviously travel with the same speed. Computing the invariant mass of the system composed by these two particles, from the dispersion relations one has

$$\frac{m_{p'}}{m_p} = \frac{p'_0\sqrt{1 - \left(\frac{p'_1}{p'_0}\right)^2}}{p_0\sqrt{1 - \left(\frac{p_1}{p_0}\right)^2}} = \frac{p'_0}{p_0}, \tag{6.103}$$

then

$$\begin{aligned} M^2 =& \left(p^{\mu} + p'^{\mu}\right)\left(p_{\mu} + p'_{\mu}\right) = m_p^2 + m_{p'}^2 + 2p^{\mu}p'_{\mu} \\ =& m_p^2 + m_{p'}^2 + 2\left(p_0 p'_0 - p_1 p'_1\right) \\ =& m_p^2 + m_{p'}^2 + 2\left(p_0 p'_0 - \frac{p'_0}{p_0}p_1^2\right) \\ =& m_p^2 + m_{p'}^2 + 2\frac{p'_0}{p_0}m_p^2 = \left(m_p + m_{p'}\right)^2. \end{aligned} \tag{6.104}$$

Equation (6.104), combined with equation (6.102), reveals what are the kinematical properties of a loop in Special Relativity. From equation (6.102) it is known that the two particles must have the same speed; moreover, from equation (6.104), it is understood that they must be in relative rest since the invariant mass of the system is given only by the sum of their masses. So, in Special Relativity, if the laboratory is at rest with respect to the two particles, the non-causality-violating loop reduces to the description of two particles standing at the same point, which before and after the loop are considered as a whole.

6.3.2 Non-causality-violating loop in Relative Locality

Relative Locality requires a more careful analysis. However, one still looks for a condition of equal *physical* velocities (which would not come from a condition of equal *coordinates* velocities, as an effect of the non trivial translations [106]) and it is expected that this will imply again $M^2 = (m + m')^2$. Since in Relative Locality only local observations are meaningful, two observers are needed to reconstruct that the loop effectively took place: one local with the emission of the two particles, and a second observer local with the absorption of them. One could deduce that the loop occurred if for the first observer, Alice, holds $x^\mu_A(s_0) = {x'_A}^\mu(s_0) = 0$ and for the second observer, Bob, purely translated with respect to Alice by a vector b^μ, holds $x^\mu_B(s_1) = {x'_B}^\mu(s_1) = 0$. This is, evidently, the condition of equal physical velocities. The relation between the two observers, using the prescription for translations used [104], is then

$$\begin{cases} x^\mu_B = x^\mu_A + b^\nu \left\{(p' \oplus p)_\nu \, , x^\mu\right\} \simeq x^\mu_A - b^\mu - \delta^\mu_1 b^1 \ell p'_0 \\ {x'_B}^\mu = {x'_A}^\mu + b^\nu \left\{(p' \oplus p)_\nu \, , x'^\mu\right\} \simeq {x'_A}^\mu - b^\mu - \delta^\mu_0 b^1 \ell p_1 . \end{cases} \tag{6.105}$$

Using the dispersion relation (6.27), the first of the (6.95) becomes

$$\dot{x}^0 = \mathcal{N}_p \left(-2p_1 + 2\ell p_1 p_0\right), \qquad \dot{x}^1 = \mathcal{N}_p \left(2p_0 + \ell p_1^2\right), \tag{6.106}$$

so the coordinate velocity for the (p, x) worldline is

$$\begin{aligned} v =& \frac{\dot{x}^1}{\dot{x}^0} = \frac{-2p_1 + 2\ell p_1 p_0}{2p_0 + \ell p_1^2} \simeq \frac{-2p_1 (1 - \ell p_0)}{2p_0} \left(1 - \ell \frac{p_1^2}{2p_0}\right) \\ =& -\frac{p_1}{p_0} \left(1 - \ell p_0 - \ell \frac{p_1^2}{2p_0}\right). \end{aligned} \tag{6.107}$$

In what follows it is more useful to make the substitution $p_1^2 = p_0^2 - m_p^2 + \ell p_0 \left(p_0^2 - m_p^2\right)$, which comes from the dispersion relation, thus the relation

(6.107) becomes

$$v = -\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2} \right). \tag{6.108}$$

With exactly the same computations the coordinate velocity for the (p', x') worldline reads as

$$v' = -\frac{p_1'}{p_0'} - \ell p_1' \left(\frac{m_{p'}^2}{2{p_0'}^2} - \frac{3}{2} \right). \tag{6.109}$$

Now one can write the coordinate description performed by Alice of the two particles

$$x_A^1 = v x_A^0, \tag{6.110}$$

$${x'_A}^1 = v' {x'_A}^0. \tag{6.111}$$

Using the transformations (6.105), one finds the description made by Bob

$$x_B^1 = v\left(x_B^0 + b^0\right) - b^1 - b^1 \ell p_0', \tag{6.112}$$

$${x'_B}^1 = v'\left({x'_B}^0 + b^0 + b^1 \ell p_1\right) - b^1. \tag{6.113}$$

Enforcing the condition $x_B^\mu(s_1) = {x'_B}^\mu(s_1) = 0$, one finds at leading order the two conditions

$$\begin{cases} b^1 = b^0 v \left(1 - \ell p_0'\right) \\ v' = v\left[1 - \ell(p_0' + v p_1)\right]. \end{cases} \tag{6.114}$$

Focusing on the second one of these equations, after explicating the velocities, it becomes

$$-\frac{p_1'}{p_0'} - \ell p_1' \left(\frac{m_{p'}^2}{2p_0'^2} - \frac{3}{2} \right) = \frac{-\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2} \right) + \ell \frac{p_1}{p_0} p_0'}{\left(1 - \ell \frac{p_1^2}{p_0}\right)}. \tag{6.115}$$

The second member of (6.115) is manipulated as follows:

$$\begin{aligned}
\frac{-\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2}\right) + \ell \frac{p_1}{p_0} p_0'}{\left(1 - \ell \frac{p_1^2}{p_0}\right)} &= \left(-\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2}\right) + \ell \frac{p_1}{p_0} p_0'\right)\left(1 + \ell \frac{p_1^2}{p_0}\right) \\
&= -\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2}\right) + \ell \frac{p_1}{p_0} p_0' - \ell \frac{p_1}{p_0^2} p_1^2 \\
&= -\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2} - \frac{p_0'}{p_0} + \frac{p_1^2}{p_0^2}\right) \\
&= -\frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} - \frac{3}{2} - \frac{p_0'}{p_0} + 1 - \frac{m_p^2}{p_0^2}\right) \\
&= -\frac{p_1}{p_0} + \ell p_1 \left(\frac{m_p^2}{2p_0^2} + \frac{p_0'}{p_0} + \frac{1}{2}\right).
\end{aligned} \tag{6.116}$$

A convenient way to express the first member is

$$-\frac{p_1'}{p_0'} - \ell p_1' \left(\frac{m_{p'}^2}{2p_0'^2} - \frac{3}{2}\right) = -\frac{p_1'}{p_0'}\left(1 + \ell \left(\frac{m_{p'}^2 - 3p_0'^2}{2p_0'}\right)\right).$$

Eq.(6.115) then becomes

$$\frac{p_1'}{p_0'}\left(1 + \ell \left(\frac{m_{p'}^2 - 3p_0'^2}{2p_0'}\right)\right) = \frac{p_1}{p_0} - \ell p_1 \left(\frac{m_p^2}{2p_0^2} + \frac{p_0'}{p_0} + \frac{1}{2}\right) \tag{6.117}$$

from which one can explicit p_1', after some manipulations:

$$\begin{aligned}
p_1' =& \frac{p_1}{p_0} p_0' - \ell p_1 p_0' \left(\frac{m_p^2}{2p_0^2} + \frac{p_0'}{p_0} + \frac{1}{2}\right) - \ell \left(\frac{m_{p'}^2 - 3p_0'^2}{2p_0'}\right) \frac{p_1}{p_0} p_0' \\
=& \frac{p_1}{p_0} p_0' - \ell \frac{1}{2} \frac{p_1}{p_0} \left(m_p^2 \frac{p_0'}{p_0} + 2p_0'^2 + p_0 p_0'\right) - \ell \frac{1}{2} \frac{p_1}{p_0} \left(m_{p'}^2 - 3p_0'^2\right) \\
=& \frac{p_1}{p_0} p_0' - \ell \frac{1}{2} \frac{p_1}{p_0} \left(m_p^2 \frac{p_0'}{p_0} + p_0 p_0' + m_{p'}^2 - {p_0'}^2\right),
\end{aligned} \tag{6.118}$$

which is clearly a deformation at the leading-order of the Special Relativistic expression (6.102) as expected.

Now it is possible to compute the invariant mass of the system, similarly to what has been done in the previous subsection

$$\begin{aligned}M^2 =& (p' \oplus p)_0^2 - (p' \oplus p)_1^2 + \ell\, (p' \oplus p)_0\, (p' \oplus p)_1^2 = \\ &= {p'_0}^2 + p_0^2 + 2p'_0 p_0 - {p'_1}^2 - p_1^2 - 2p'_1 p_1 - \\ &\quad - 2\ell p'_0 p'_1 p_1 - 2\ell p'_0 p_1^2 + \ell\, (p'_0 + p_0) \left({p'_1}^2 + p_1^2 + 2p'_1 p_1\right) = \\ &= m_p^2 + m_{p'}^2 + 2p'_0 p_0 - 2p'_1 p_1 + \ell \left(p_0 {p'_1}^2 + 2p'_1 p_1 p_0 - p'_0 p_1^2\right) =\end{aligned} \tag{6.119}$$

now using the equation (6.118), yields

$$\begin{aligned}M^2 =& m_p^2 + m_{p'}^2 + 2p'_0 p_0 - 2\frac{p_1^2}{p_0} p'_0 + \\ &+ \ell \frac{p_1^2}{p_0} \left(m_p^2 \frac{p'_0}{p_0} + p_0 p'_0 + m_{p'}^2 - {p'_0}^2 \right) + \ell \left(\frac{p_1^2}{p_0} {p'_0}^2 + 2p_1^2 p'_0 - p'_0 p_1^2 \right) \\ =& m_p^2 + m_{p'}^2 + 2m_p \left(\frac{m_p p'_0}{p_0} + \ell \frac{m_p p_1^2 p'_0}{2p_0^2} + \ell \frac{m_{p'}^2 p_1^2}{2m_p p_0} \right) = m_p^2 + m_{p'}^2 + 2m_p m_{p'}.\end{aligned} \tag{6.120}$$

The last equality comes from the following chain of equalities

$$\begin{aligned}m_{p'}^2 &= {p'_0}^2 - {p'_1}^2 + \ell p'_0 {p'_1}^2 \\ &= {p'_0}^2 - \left[\frac{p_1^2}{p_0^2} {p'_0}^2 - \ell \frac{p_1^2}{p_0^2} p'_0 \left(m_p^2 \frac{p'_0}{p_0} + p_0 p'_0 + m_{p'}^2 - {p'_0}^2 \right) \right] + \ell \frac{p_1^2}{p_0^2} {p'_0}^3 \\ &= {p'_0}^2 - \frac{p_1^2}{p_0^2} {p'_0}^2 + \ell \frac{p_1^2}{p_0} {p'_0}^2 + \ell \frac{p_1^2 {p'_0}^2}{p_0^3} m_p^2 + \ell \frac{p_1^2}{p_0^2} p'_0 m_{p'}^2 \\ &= \frac{{p'_0}^2}{p_0^2} m_p^2 + \ell \frac{p_1^2 {p'_0}^2}{p_0^3} m_p^2 + \ell \frac{p_1^2}{p_0^2} p'_0 m_{p'}^2 \\ &= \frac{{p'_0}^2}{p_0^2} m_p^2 \left[1 + \ell \frac{p_1^2}{p_0} + \ell \frac{p_1^2}{p'_0} \left(\frac{m_{p'}}{m_p} \right)^2 \right],\end{aligned} \tag{6.121}$$

so

$$\begin{aligned}m_{p'} =& \frac{p'_0}{p_0} m_p \left(1 + \ell \frac{p_1^2}{2p_0} + \ell \frac{p_1^2}{2p'_0} \left(\frac{m_{p'}}{m_p} \right)^2 \right) \\ =& \frac{m_p p'_0}{p_0} + \ell \frac{m_p p_1^2 p'_0}{2p_0^2} + \ell \frac{m_{p'}^2 p_1^2}{2m_p p_0}.\end{aligned} \tag{6.122}$$

From equation (6.120) has been found that the two particles must be in relative rest in the Relative Locality framework too in order to produce a loop. So this loop is trivial for the same argument that applies to the Special Relativistic case.

Chapter 7

Conclusions

This thesis tackled two main topics of research in quantum gravity: quantum spacetime dimensionality and the departures from absolute locality of events due to the structure of spacetime at the Planck scale.

The observation that the dimension of spactime at very short scales may be different from 4 (typically less), which has been found in many different approaches to quantum gravity, is of extreme interest, as it may point towards a "true feature" of quantum spacetime that our current models try to grasp. The analysis of this phenomenon relied mostly on the spectral dimension of quantum spacetime, which is a notion of dimension adapted for the scope from its original definition in Riemannian geometry. It is here argued that the spectral dimension is not a reliable physical observable, as the modifications to its definition employed for its use in describing a quantum spacetime are such that its physical meaning is severely weakened. For such an interesting *common* feature as running spacetime dimension one should look for a robust physical characterization of the phenomenon. For this scope, it has been here proposed a notion of spacetime dimension, the thermal dimension, which is based on thermodynamical observables related to the behavior of a gas of radiation at very high temperature. It has been shown, by detailed study of a variety of quantum gravity models, how its properties are physically more appealing with respect to those of the spectral dimension. It is therefore argued that the thermal dimension could be a valuable physical observable to test the behavior of running spacetime dimension, in particular for those theories whose dispersion relation is such that the physical meaning of the spectral dimension is particularly unclear.

A further application of the deformed thermodynamics of high-energy radiation is the investigation of the production of primordial perturbations in a universe described by Rainbow Gravity with a running Newton constant. Both vacuum and thermal initial conditions for the perturbations have been considered and a power-law dependence of the Newton constant on energy has been assumed, together with the fact that the background satisfies the thermodynamical relations peculiar to radiation subject to deformed dispersion relations. This model is then able to produce primordial scalar perturbations whose spectral index respects the constraint set recently by the Planck satellite. For both kinds of initial conditions for the perturbations (vacuum and thermal) the running of the Newton constant is essential in achieving a viable picture. In particular, the Newton constant is constrained to be decreasing with energy in the ultraviolet regime. This is consistent with intuition from quantum gravity theories, such as Hořava-Lifshitz gravity and Asymptotic safety. It also resonates with the conjecture put forward in [86] and deserves further investigations.

Concerning the possible departure from absolute locality of standard physics, some aspects of the theory of Relative Locality has been analyzed. This theory is studied in its classical-mechanics formulation, where Planck mass plays the role of *relativistic invariant* (in the sense of DSR) scale of curvature of momentum space. Relativity of spacetime locality is then a reflection of the introduction of this new relativistic invariant: as the introduction of a relativistic invariant speed of light implied the relativity of simultaneity (relativity of *time* coincidence of events), the introduction of a relativistic invariant curvature of momentum space implies the relativity of locality (relativity of *spacetime* coincidence of events). As original results, it has first been shown that the relativity of spacetime locality does not spoil the objectivity of cause-effect relation in a chain of events. This has been shown considering a couple of disconnected chains of events, set up in such a way that an observer may infer a very misleading pictures if she relies on a description of the events based only on her coordinates. A proper use of translation transformations gives her back the correct, objective picture. Secondly, it has been shown that those phenomena that may be pathological for what concerns causality (causal loops) or violation of momentum conservation ("Möbius loops"), while may occur in generic theories with curved momentum space, are excluded when the theory is formulated in such a way that the (deformed) relativistic symmetries are satisfied, as is Relative Locality. In fact, for a generic theory with curved momentum space, it is possible to obtain general conditions on the generators of translation transformations that must be satisfied in order for that theory to be symmetric under an appropriate notion of translation transformation. These conditions are translated into conditions on energies and momenta of the interacting particles. If these conditions are not satisfied, the causal loop is allowed, whereas when these are satisfied the only solution of the equations of motion is that the whole loop collapses to a single event. The same applies to the

Möbius diagrams.

It has then been proposed a point of reflection on the possible mechanism that may guarantee the relativity of spacetime locality even in the quantum version of the theory, which is still unknown. In a similar way to what happens on standard QED, where gauge-symmetry-violating Feynman diagrams add up to give a gauge symmetric matrix element (see, for example, the Compton scattering), symmetry-violating diagrams such as Möbius diagram may add up to give a symmetric matrix element.

Finally, it has been shown how non-causality-violating loops are trivial in Relative Locality, as well as they are in Special Relativity.

Acronyms and symbols

QM : Quantum Mechanics.

GR : General Relativity.

SM : Standard Model.

QFT : Quantum Field Theory.

QG : Quantum Gravity.

DSR : Doubly Special Relativity.

ℓ : Deformation parameter.

L_P : Planck length.

L_{DSR} : Fundamental relativistic invariant length scale.

$\triangleright$: Right action.

$\hat{x}$: Noncommutative coordinate.

$\oplus$: Deformed sum.

$\ominus$: Inverse of the deformed sum.

Greek indices take the value $\{0, ..., D\}$ where D is the number of spatial dimensions of spacetime. Latin indices take the value $\{1, ..., D\}$.

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