Alma Mater Studiorum · Università di Bologna

DOTTORATO DI RICERCA IN MATEMATICA
CICLO XXIX

Settore Concorsuale di afferenza: 01/A3

Settore Scientifico disciplinare: MAT/05

LOCAL SOLVABILITY OF A CLASS OF DEGENERATE SECOND ORDER OPERATORS

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Esame finale 2017

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Introduction

This thesis deals with the local solvability problem related to some degenerate second order partial differential operators with smooth and non-smooth coefficients. The main class analyzed, which is the one with smooth coefficients, is given by operators of the form

$$P = \sum_{j=1}^{N} X_j^* f^p X_j + iX_0 + a_0, \tag{0.0.1}$$

where $p \geq 1$ is an integer, $X_j(x, D)$, $0 \leq j \leq N$ $(D = -i\partial)$, are *homogeneous first order differential operators* (i.e. with no lower order terms) with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$ and with a *real* principal symbol (in other words, the $iX_j(x, D)$ are *real* vector fields), a_0 is a smooth possibly complex-valued function, and $f \colon \Omega \longrightarrow \mathbb{R}$ a smooth function with $f^{-1}(0) \neq \emptyset$ and $df\big|_{f^{-1}(0)} \neq 0$. In addition we suppose that $iX_0 f > 0$ on $S = f^{-1}(0)$, therefore the vector field $iX_0(x, D)$ is nonvanishing on S and can only be zero at some point of $\Omega \setminus S$. Instead the vector fields iX_j, $1 \leq j \leq N$, are allowed to be zero at some point of S as well.

Due to the vanishing of the function f, and also to the degeneracy due to the characteristic set of the system of first order operators, we have that (0.0.1) represents a class of degenerate second order partial differential operators with multiple characteristics whose degree of degeneracy depends on the parity of the exponent p. The choice of the exponent affects in several ways the properties of the operators in (0.0.1). In fact we need to distinguish between the case determined by odd degeneracy (p odd) and that one characterized by even degeneracy (p even).

The most interesting situation occurs when one considers p an odd integer. In this case we deal with operators having a changing sign principal symbol in the neighborhood of the set S where the function f vanishes (more precisely the principal symbol changes sign in the neighborhood of the fiber of S that we denote by $\pi^{-1}(S)$). This changing sign property can negatively affect the local solvability of the operator near the points of S around which the principal symbol changes sign. This is indeed the case for the celebrated Kannai operator $P = x_1 \sum_{j=2}^n D_j^2 - iD_1$, which is unsolvable at the set $S = \{x \in \mathbb{R}^n; x_1 = 0\}$ near which its principal symbol changes sign (see [13]).

This example is also remarkable in order to explore the relation between hypoellipticity and local solvability. In fact the Kannai operator is hypoelliptic on the whole $\mathbb{R}^n$ but is not locally solvable at the set S, where the principal symbol vanishes to change sign. This means that the hypoellipticity property does not imply the local solvability of the operator itself. On the other hand, the hypoellipticity property of an operator has as a consequence the local solvability of its formal adjoint. Because of that the adjoint of the Kannai operator is locally solvable over the whole $\mathbb{R}^n$. Thus, also near S where its principal symbol changes sign as well.

These observations motivate the study of the local solvability of classes of operators having a changing sign principal symbol in the neighborhood of the set where the changing of sign occurs.

For the adjoint of the Kannai operator we get the local solvability property as a consequence of the hypoellipticity, and we can also use this regularity property (as we shall see in Chapter 4) to prove that the adjoint is locally solvable with a loss of one derivative. However, we can not relate the local solvability to the hypoellipticity, since locally solvable ψDO (pseudodifferential operators) are not even adjoints of hypoelliptic ones.

A first class of solvable operators having a changing sign principal symbol being a generalization of the Kannai operator (or, more precisely, of the adjoint of the Kannai operator) was studied by Colombini, Cordaro and Pernazza in [3], in which they obtain some local solvability results by requiring a kind of condition (ψ).

In [6] with A.Parmeggiani we generalized the class introduced in [3]. Thanks to the local solvability results contained in [6] we have a criterion to distinguish whenever an operator of the form (0.0.1) is locally solvable at S where the principal symbol vanishes to change sign. Unfortunately, there is not yet a way to recover when this kind of operators are unsolvable, since, having multiple characteristics, the well-known results available for principal type ψDO are not suitable in this context.

As we mentioned before the local solvability problem is linked to the hypoellipticity problem, problem which is not yet completely solved. In fact there is not a general rule to establish when an operator has this property, and, also, it is very hard and not always possible to define the sharp hypoelliptic loss of derivatives of a hypoelliptic pseudodifferential operator (see [19]).

In the multiple characteristics setting we have results by Mendoza [15], Mendoza and Uhlmann [16], Müller and Ricci [18], Peloso and Ricci [22], Tréves [24], and J.J. Kohn [12] (complex vector fields). We also have a recent work by Parenti and

Parmeggiani [20] concerning the semi-global solvability in presence of multiple transverse symplectic characteristics. There are also results by Beals and Fefferman [2] (see also Zuily [26] and Akamatsu [1]) in which they give some conditions for the hypoellipticity of operators of the form P^* where P is contained in the class (0.0.1). Their results for P^* agrees with our local solvability results for P in the class (0.0.1). However, our results are not optimal, giving sufficient and not necessary conditions for the local solvability around S. Moreover, the characterization of the loss of derivatives is still open, the difficulties being given by the high degeneracy and the control of commutators. For instance, Parenti and Rodino in [21] proved the *anisotropic* hypoellipticity with loss of one derivative of a class of operators, which gives, for operators of the form $P = t^{2k+1}D_x^2 + iD_t$ having adjoints P^* in the class considered in [21], a more precise local solvability result.

Concerning the even degeneracy case associated with (0.0.1), note that we do not have the changing sign property of the principal symbol anymore, but we still deal with highly degenerate operators with multiple characteristics, for which the local solvability is not guaranteed. In this case also the technique we use to prove the result is different than that used in the odd degeneracy case, and, in particular, we use Carleman estimates to get the result, which as we will see is on the one hand less precise than the one we get in the odd setting, but, on the other, less demanding as regards the requests on the operator.

Let me mention that also the local solvability of (0.0.1) far from S is analyzed, since it is, again, not granted a priori.

Inspired by [6] I studied in [5] the local solvability of two models of PDO with non-smooth coefficients which are a variation of the main class presented above. Once again, the problem is studied around a set S where the principal symbol changes sign or where the operator is degenerate (or both). The class considered is invariant under affine changes of variables.

We conclude this introduction by giving the plan of the thesis.

In Chapter 1 we introduce the main class mentioned above and we show the invariance of some properties required to the latter.

Chapter 2 is devoted to the proof of the local solvability results for the main class. First we treat the problem around S in presence of both odd and even degeneracy. Then also the problem far from S in presence of any parity in analyzed. Finally we give a generalization of the result in the odd setting in a complex case, that is when all the vector fields in the second order part are supposed to be complex, a step in the generalization of [12] in the case of a changing sign principal symbol.

In Chapter 3 we study the two non-smooth coefficients classes inspired by (0.0.1) mentioned above. We will obtain some local solvability results when the coefficients in the higher order part of the operator are real or complex.

Chapter 4 is devoted to the proof of the local solvability with loss of one derivative of the adjoint of the Kannai operator (this also is interesting to see).

Chapter 1

The main class

1.1 Setting and hypotheses

This thesis starts with the introduction of the main class

$$P = \sum_{j=1}^{N} X_j^* f^p X_j + iX_0 + a_0, \tag{1.1.1}$$

where $p \geq 1$ is an integer, $X_j(x, D)$, $0 \leq j \leq N$ $(D = -i\partial)$, are *homogeneous first order differential operators* (i.e. with no lower order terms) with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$ and with a *real* principal symbol (in other words, the $iX_j(x, D)$ are *real* vector fields, so in the sequel we will often call the X_j vector fields; however, we shall also consider the case of $X_1, \ldots, X_N$ being *complex*), a_0 is a smooth possibily complex-valued function, and $f \colon \Omega \longrightarrow \mathbb{R}$ a smooth function with $f^{-1}(0) \neq \emptyset$ and $df\big|_{f^{-1}(0)} \neq 0$. The other models we are going to treat later will be a variation of that introduced here.

The local solvability of P is studied first in the neighborhood of the zeros of the function f where the principal symbol changes sign, and next also far from the zeros of f.

The changing sign property of the principal symbol of an operator is a very important property to look at when we deal with the local solvability problem, since it can produce the non solvability of the operator itself, like in the important Kannai example. It is therefore interesting to find some models with the changing sign property which are still locally solvable in the neighborhood of the set where the changing of sign of the principal symbol happens. This is one of the main reasons why we are interested in the study of the class of operators represented by P. Note also that in the neighborhood of the set $S = f^{-1}(0)$ the operator is degenerate, due to the presence of f in the second order part. Moreover P has multiple characteristics, that is the characteristic set of P contains multiple zeros of the principal symbol (in this case, since the operator has order two, we have double zeros of the principal symbol in the characteristic set, thus P has double characteristics).

In order to prove a solvability result for P we need to impose some conditions on the symbols of the operators X_j and on the first order part represented by iX_0 (which is the subprincipal part of P). We state here the hypotheses in the real case, that is when all the vector fields iX_j are supposed to be real, since in the complex case, as we will see, we consider a suitable generalization of the real case.

We shall denote by $X_j(x, \xi)$ the principal symbol of X_j. We shall also denote by $\Sigma_j = \{(x, \xi) \in T^*\Omega \setminus 0;\ X_j(x, \xi) = 0\}$ the characteristic set of each X_j and put

$$\Sigma := \bigcap_{j=0}^{N} \Sigma_j \tag{1.1.2}$$

for the *characteristic set* of the *system of first order operators* $(X_0, \ldots, X_N)$. Finally, writing $H_{X_j} = H_j$ for the Hamilton vector fields associated with the symbols X_j, for $\rho \in \Sigma$ we write $V(\rho) := \operatorname{Span}\{H_0(\rho), \ldots, H_N(\rho)\}$. Let also $\pi\colon T^*\Omega \longrightarrow \Omega$ be the canonical projection.

The hypotheses on the class (in the real case), that we call (H1), (H2) and (H3), are stated in the following way:

(H1) $iX_0 f(x) > 0$ *for all* $x \in S := f^{-1}(0) \neq \emptyset$*;*

(H2) *For all* $j = 1, \ldots, N$ *and for all compact* $K \subset \Omega$ *there exists* $C_{K,j} > 0$ *such that*

$$\{X_j, X_0\}(x, \xi)^2 \leq C_{K,j} \sum_{j'=0}^{N} X_{j'}(x, \xi)^2, \quad \forall (x, \xi) \in K \times (\mathbb{R}^n \setminus \{0\}).$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket with respect to the standard symplectic from σ on $T^*\Omega$.

Now, for $\rho \in \Sigma$ let $J = J(\rho) \subset \{0, \ldots, N\}$ be the set of indices for which the vectors $H_j(\rho)$, $j \in J$, form a basis of $V(\rho)$. Say that $\#J = r$. Let $M(\rho)$ be the $r \times r$ real matrix defined as

$$M(\rho) = \big[\{X_j, X_{j'}\}(\rho)\big]_{j,j' \in J}. \tag{1.1.3}$$

Definition 1.1.1. *We shall say that hypothesis (H3) is satisfied at a point x_0 if $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and*

$$\text{(H3)} \qquad \operatorname{rank} M(\rho) \geq 2, \quad \forall \rho \in \pi^{-1}(x_0) \cap \Sigma.$$

Hence hypothesis (H3) yields the existence of a (connected) neighborhood $W \subset \Omega$ of x_0 such that

$$\operatorname{rank} M(\rho) \geq 2, \quad \forall \rho \in \pi^{-1}(W) \cap \Sigma.$$

It is important to remark, as we shall prove in Lemma 2.4.3 below, that condition $(H3)$ is independent of the choice of the basis of $V(\rho)$.

Except for condition $(H1)$, the other requirements are imposed on the Poisson brackets of the principal symbols of the vector fields, that is on the principal symbols of the commutators $i[X_j, X_{j'}](x, D)$ with $j,\, j' \in \{0, ..., N\}$. This suggests that geometric relations among the vector fields determine the kind of solvability we can expect for P, that is, as shown later, they affect the loss of derivatives of the operator. Observe moreover that nondegeneracy conditions are not imposed on the vector fields iX_j for $j \neq 0$, therefore our model presents a degeneracy due both to the degeneracy of the function f (which is required to be such that $f^{-1}(0) \neq \emptyset$) and to the degeneracy of the system of vector fields $\{iX_j\}_{1 \leq j \leq N}$, if present. Instead, for $j = 0$, we require the nondegeneracy condition $iX_0(x, D)f(x) > 0$ around the set $S = f^{-1}(0)$ given by (H1), which does not affect the main degeneracy of the operator, since this vector field is not contained in the leading term of P.

It is interesting to make a few observations concerning hypothesis (H2), which is a fundamental assumption.

(i) The first one is that (H2) is *equivalent* to requiring that *there exist functions $\beta_{jk} \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R}^n)$ such that for each $j, k = 0 \ldots, N$*

$$\{X_j, X_0\} = \sum_{k=0}^{N} \beta_{jk} X_k. \tag{1.1.4}$$

In fact, it suffices to take

$$\beta_{jk} = \{X_j, X_0\} X_k \Big/ \Big(\sum_{\ell=0}^{N} X_\ell^2\Big) \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R}^n) \tag{1.1.5}$$

(with, say, $\beta_{jk} = 0$ on $\Sigma \cup (\Omega \times \{0\})$, which has zero measure on $\Omega \times \mathbb{R}^n$). Notice that such β_{jk} are then *smooth* outside the characteristic set Σ and are *homogeneous of degree* 0 in the fibers.

(ii) Notice that in general condition (1.1.4) *does not* imply a similar relation among the $[X_0, X_j]$ and the iX_j when they are thought of as *vector fields*. However, as a consequence of Lemma 1.1.2 below, we have that, for instance when $N = 1$ and X_1 is nonzero near a point x_0, in a neighborhood of x_0 the function β_{11} is indeed smooth, whence we get that necessarily the vector fields iX_1 and iX_0 must be tangent to the integral submanifold of $\mathbb{R}^n$ defined, near x_0, by the involutive distribution $\operatorname{Span}\{iX_1, iX_0\}$, when the dimension of the latter is constant (which is the case when $df(iX_1) = 0$ on S, i.e. iX_1 is tangent to S, for by hypothesis we require $df(iX_0) \neq 0$ there).

In connection to point (ii) above, we have the following lemma. Recall that $X_j(x, \xi) = \langle \alpha_j(x), \xi \rangle$, $0 \leq j \leq N$.

Lemma 1.1.2. *For each $x \in \Omega$ let $V(x) = \operatorname{Span}\{\alpha_0(x), \dots, \alpha_N(x)\}$. Suppose there exists $x_0 \in \Omega$ and an open neighborhood $\tilde{U} \subset \Omega$ of x_0 such that $\dim V(x) = 1$ for all $x \in \tilde{U}$. Then there is an open neighborhood $U \subset \tilde{U}$ of x_0 such that hypothesis (H2) is satisfied iff there are $\mu_{jk} \in C^\infty(U)$ such that*

$$\{X_0, X_j\}(x, \xi) = \sum_{k=0}^{N} \mu_{jk}(x) X_k(x, \xi),\ \forall (x, \xi) \in \pi^{-1}(U) = U \times \mathbb{R}^n,\ 1 \leq j \leq N.$$

Proof. The sufficiency is clear. We need only prove the necessity. As already seen in point (i) above, we have that $\{X_0, X_j\}(x, \xi) = \sum_{k=0}^{N} \beta_{jk}(x, \xi) X_k(x, \xi)$ with $\beta_{jk} \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$, which are then *smooth* outside Σ, and *homogeneous of degree* 0 *in the fibers* ξ. Put, for short, $Z_j(x, \xi) := \{X_0, X_j\}(x, \xi) = \langle z_j(x), \xi \rangle$. Then, by assumption (possibly by shrinking $\tilde{U}$ around x_0), we have a vector bundle $V \to \tilde{U}$ of rank 1 with fibers $V(x)$ and may find an open neighborhood U of x_0 and a smooth normalized section $\zeta\colon U \ni x \longmapsto \zeta(x) \in V(x)$ of $V \to U$ which generates $V(x)$ for every $x \in U$. We thus have that on U

$$X_j(x, \xi) = \langle \xi, \zeta(x) \rangle \langle \alpha_j(x), \zeta(x) \rangle, \quad j = 0, \dots, N,$$

and the same holds for Z_j. Let $\Pi_0(x)\colon \mathbb{R}^n \longrightarrow V(x) \subset \mathbb{R}^n$ be the orthogonal projection onto $V(x)$. Hence the map $U \times \mathbb{R}^n \ni (x, \xi) \longmapsto \Pi_0(x)\xi \in \mathbb{R}^n$ is smooth. Now, by virtue of (H2) we have that $Z_j(x, \xi) = 0$ wherever all the $X_k(x, \xi)$ vanish, the latter being the case for all $\xi \in V(x)^\perp$ with $x \in U$. Therefore, for all $(x, \xi) \in \pi^{-1}(U)$, $\xi \neq 0$,

$$Z_j(x, \xi) = Z_j(x, \Pi_0(x)\xi) = \sum_{k=0}^{N} \beta_{jk}(x, \Pi_0(x)\xi) X_k(x, \Pi_0(x)\xi)$$

$$= \sum_{k=0}^{N} \beta_{jk}(x, \Pi_0(x)\xi) X_k(x,\xi) = \sum_{k=0}^{N} \beta_{jk}(x, \zeta(x)) X_k(x,\xi),$$

where we have used the homogeneity of degree 0 of the β_{jk} in ξ. Finally, we specialize the above equality when $\xi = \zeta(x)$, $x \in U$, so obtaining

$$Z_j(x, \zeta(x)) = \langle z_j(x), \zeta(x) \rangle = \sum_{k=0}^{N} \beta_{jk}(x, \zeta(x)) \langle \alpha_k(x), \zeta(x) \rangle.$$

Since $\sum_{j=0}^{N} X_j(x,\zeta(x))^2 = \sum_{j=0}^{N} \langle \alpha_j(x), \zeta(x) \rangle^2 > 0$ for all $x \in U$, we have from (1.1.5) that for all $1 \le j \le N$ and all $0 \le k \le N$,

$$\mu_{jk}(x) := \beta_{jk}(x, \zeta(x)) = \frac{Z_j(x,\zeta(x)) X_k(x,\zeta(x))}{\sum_{\ell=0}^{N} X_\ell(x,\zeta(x))^2},$$

thus proving that the μ_{jk} are smooth. □

In the sequel we will focus our attention on the invariance properties of the class described by P. In particular we will first show that the expression of the operator P and hypotheses (H1), (H2) and (H3) are invariant through changes of variables in P. Next we will prove another property of P, namely

$$\mathrm{sub}(P)(x,\xi) = iX_0(x,\xi),$$

where $\mathrm{sub}(P)(x,\xi)$ is the subprincipal symbol of P.

1.2 Invariance through changes of variables

In the last section we listed the hypotheses on the class (1.1.1). However, in order to show that these objects effectively form a class of operators, we need to show that the required properties are invariant under changes of variables.

Since conditions (H2) and (H3) are imposed on the principal symbols of the vector fields $i[X_j, X_{j'}](x, D)$, j, $j' \in \{0, ..., N\}$, they are trivially satisfied even if we perform a change of variables in the operator P, being the principal symbol an invariant of partial differential operators. Therefore, to characterize these operators as a class, we prove here the invariance through changes of variables of the expression of the operator and of condition (H1). Moreover we can consider here the general case in which all the vector fields iX_j, with $j \neq 0$, are complex, since, also in this case, all the other requirements are imposed on symbols, whence they are trivially invariant under changes of variables.

Let $F\colon \Omega \longrightarrow \Omega'$ be a C^∞ diffeomorphism, with $y = F(x)$. We denote by $F'(x)$ the tangent map of F at x and by $y \longmapsto G(y) = x$ the inverse F^{-1}. We have that, denoting by $\tilde{u} := u \circ G$,

$$\Bigl(X_j(x, D_x)u(x)\Bigr)\Big|_{x=G(y)} = \tilde{X}_j(y, D_y)\tilde{u}(y), \quad 1 \le j \le N,$$

so that

$$\tilde{X}_j(y, D_y)\tilde{u}(y) = \sum_{k'=1}^{n}\Bigl(\sum_{k=1}^{n} \alpha_{k,j}(G(y))\frac{\partial F_{k'}}{\partial x_k}(G(y))\Bigr)D_{y_{k'}}\tilde{u}(y) = \sum_{k'=1}^{n} \tilde{\alpha}_{k'j}(y)D_{y_{k'}}\tilde{u}(y),$$

and

$$\mathsf{d}_{\tilde{X}_j}(y) := \sum_{k'=1}^{n} \frac{\partial \tilde{\alpha}_{k'j}}{\partial y_{k'}}(y)$$

$$= \sum_{k,k',\ell=1}^{n} \frac{\alpha_{kj}}{\partial x_\ell}(G(y))\Bigl[\frac{\partial G_\ell}{\partial y_{k'}}(y)\frac{\partial F_{k'}}{\partial x_k}(G(y)) + \alpha_{kj}(G(y))\frac{\partial^2 F_{k'}}{\partial x_k \partial x_\ell}(G(y))\frac{\partial G_\ell}{\partial y_{k'}}(y)\Bigr]$$

$$= \sum_{k=1}^{n} \frac{\alpha_{kj}}{\partial x_k}(G(y)) + \sum_{k=1}^{n} \alpha_{kj}(G(y)) \underbrace{\sum_{k',\ell=1}^{n} \frac{\partial^2 F_{k'}}{\partial x_k \partial x_\ell}(G(y))\frac{\partial G_\ell}{\partial y_{k'}}(y)}_{=\frac{\partial}{\partial x_k}\ln|\det(F'(x))|\big|_{x=G(y)}}$$

$$= \mathsf{d}_{X_j}(G(y)) + \sum_{k=1}^{n} \alpha_{kj}(G(y))\frac{\partial}{\partial x_k}\ln|\det(F'(x))|\Big|_{x=G(y)}.$$

Next, for any given $u, v \in C_0^\infty(\Omega)$ we have

$$(Pu, v) = \sum_{j=1}^{N} \int f(x)^p X_j(x, D_x)u(x)\overline{X_j(x, D_x)v(x)}dx$$

$$+ \int iX_0(x, D_x)u(x)\overline{v(x)}dx + \int a_0(x)u(x)\overline{v(x)}dx$$

$(x = G(y),\ \tilde{u}(y) = u(G(y)))$

$$= \sum_{j=1}^{N} \int f(G(y))^p \tilde{X}_j(y, D_y)\tilde{u}(y)\overline{\tilde{X}_j(y, D_y)\tilde{v}(y)}|\det G'(y)|dy$$

$$+\int i\tilde{X}_0(x,D_y)\tilde{u}(y)\overline{\tilde{v}(y)}|\det G'(y)|dy+\int a_0(G(y))\tilde{u}(y)\overline{\tilde{v}(y)}|\det G'(y)|dy,$$

whence, with $\tilde{f}(y)=f(G(y))$, $\tilde{a}_0(y)=a_0(G(y))$ and $g(y)=|\det G'(y)|$, we have that P goes over, in the new coordinates and by taking adjoints with respect to the Lebesgue measure dy, to the operator

$$\tilde{P}=\sum_{j=1}^{N}\tilde{X}_j(y,D_y)^*g(y)\tilde{f}(y)^p\tilde{X}_j(y,D_y)+ig(y)\tilde{X}_0(y,D_y)+g(y)\tilde{a}_0(y)$$

$$=\sum_{j=1}^{N}\tilde{X}_j(y,D_y)^*\tilde{f}_{\mathrm{new}}(y)^p\tilde{X}_j(y,D_y)+i\tilde{X}_{0_{\mathrm{new}}}(y,D_y)+g(y)\tilde{a}_0(y).$$

The operator $\tilde{P}$ maintains the same structure as P and satisfies **exactly** the same hypotheses as P, in particular it still satisfies condition (H1) (on $\tilde{f}_{\mathrm{new}}^{-1}(0)$). In fact, since $i\tilde{X}_{0_{\mathrm{new}}}=ig(y)\tilde{X}_0(y,D_y)$ and $g(y)>0$, then we have

$$i\tilde{X}_{0_{\mathrm{new}}}(y,D_y)\tilde{f}_{\mathrm{new}}(y)=g(y)^2i\tilde{X}_0(y,D_y)\tilde{f}+g(y)\tilde{f}(y)i\tilde{X}_0(y,D_y)g(y),$$

which is still strictly greater than zero on $\tilde{f}_{\mathrm{new}}^{-1}(0)=\tilde{f}^{-1}(0)$.

If one considers the transformed operator with respect to the pull-back Lebesgue measure $g(y)dy$ one gets the operator (a more invariant setting) as follows. We have to compute the adjoint $\tilde{X}_j(y,D_y)^\star$ of $\tilde{X}_j(y,D_y)$ with respect to the measure $g(y)dy$. Let $\tilde{X}(y,D_y)=\langle\tilde{\alpha}(y)D_y\rangle$ and let $\varphi,\psi\in C_0^\infty$ in the variables $y\in\tilde{\Omega}$. Then

$$(\tilde{X}(y,D_y)\varphi,\psi)_{L^2(gdy)}=\sum_{j=1}^{n}\int_{\tilde{\Omega}}(\tilde{\alpha}_j(y)D_{y_j}\varphi(y))\overline{\psi(y)}g(y)dy$$

$$=\sum_{j=1}^{n}\int_{\tilde{\Omega}}\varphi(y)\overline{\left(\overline{\tilde{\alpha}_j(y)}D_{y_j}(g(y)\psi(y))+D_{y_j}\overline{\tilde{\alpha}_j(y)}\ \psi(y)g(y)\right)}dy$$

$$=\sum_{j=1}^{n}\int_{\tilde{\Omega}}\varphi(y)\overline{\left(\overline{\tilde{\alpha}_j(y)}D_{y_j}\varphi(y)+\frac{\overline{\tilde{\alpha}_j(y)}}{g(y)}(D_{y_j}g(y))\psi(y)+(D_{y_j}\overline{\tilde{\alpha}_j(y)})\psi(y)\right)}g(y)dy$$

$$=\int_{\tilde{\Omega}}\varphi(y)\overline{\tilde{X}(y,D_y)^\star\varphi(y)}g(y)dy=(\varphi,\tilde{X}(y,D_y)^\star\psi)_{L^2(gdy)},$$

that is, in general, with $\tilde{X}(y,D_y)=\langle\tilde{\alpha}(y),D_y\rangle$

$$\tilde{X}(y,D_y)^\star=\langle\overline{\tilde{\alpha}(y)},D_y\rangle+\frac{1}{g(y)}\overline{\tilde{X}(y,D_y)g(y)}-i\,\mathrm{div}(\overline{\tilde{\alpha}(y)}).$$

Hence, in this case, if $\tilde{X}(y, D_y)^*$ is the adjoint with respect to dy we have

$$\tilde{X}(y, D_y)^\star = \tilde{X}(y, D_y)^* + \frac{\tilde{X}(y, D_y)g(y)}{g(y)}.$$

Therefore the transformed operator $\tilde{P}_g$ with respect to the measure $g(y)dy$ is the operator

$$\tilde{P}_g = \sum_{j=1}^{N} \tilde{X}_j(y, D_y)^* \tilde{f}(y)^p \tilde{X}_j(y, D_y) + i\tilde{X}_{0_{\rm new}}(y, D_y) + \tilde{a}_0(y),$$

where

$$i\tilde{X}_{0_{\rm new}}(y, D_y) = i\tilde{X}_0(y, D_y) - i\sum_{j=1}^{N} \overline{\frac{\tilde{X}_j(y, D_y)g(y)}{g(y)}} \tilde{f}(y)^p \tilde{X}_j(y, D_y).$$

Once again condition (H1) is satisfied by $\tilde{P}_g$ (on $\tilde{f}^{-1}(0)$), due to the presence of $\tilde{f}$ in the expression of $i\tilde{X}_{0_{\rm new}}$.

1.3 The subprincipal symbol

The other property we are going to show about P regards its subprincipal symbol. We have seen that condition (H1) is imposed on the first order part represented by $iX_0(x, D)$ (which is not the total first order part of the operator P in (1.1.1)), and that it is invariant through changes of variables. Even if (H1) is invariant in general, it is remarkable that it is actually a condition imposed on the subprincipal part of P. In fact we will show here that the subprincipal symbol of P is exactly given by $iX_0(x, \xi)$. Furthermore, it is useful to see how the invariants of the operator look like, especially for further generalizations (e.g. the pseudodifferential case) or improvements of the results obtained for P. In particular it is important to give conditions which are invariant (e.g. invariant under changes of variables), and in general this reduces to imposing conditions on the invariants of the operator (principal symbol, subprincipal symbol, etc.).

Given a partial differential operator P of order m with poly-homogeneous symbol $p(x, \xi) = \sum_{j=0}^{m} p_{m-j}(x, \xi)$, we denote by $p^s_{m-1}(x, \xi)$ the subprincipal symbol of P obtained from $p(x, \xi)$ in the following way

$$p^s_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_j \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, \xi),$$

where $p_{m-1}(x,\xi)$ is the homogeneous part of order $m-1$ in the variable ξ in $p(x,\xi)$, while $p_m(x,\xi)$ is the principal symbol (that is the homogeneous part of order m in ξ). This object is invariant on the set of double zeros of the principal symbol, that is where both $p_m = 0$ and $dp_m = 0$.

We denote by $p^w(x,\xi)$ the Weyl symbol of P given by

$$p^w(x,\xi) = e^{\langle D_x, D_\xi\rangle/2i} p(x,\xi) \sim \sum_{j\geq 0} \Big(\sum_{l+r=j} \frac{1}{r!} \Big(\frac{1}{2i} \langle D_x, D_\xi \rangle \Big)^r p_{m-l}(x,\xi) \Big),$$

and since we have

$$p^s_{m-1}(x,\xi) = p^w_{m-1}(x,\xi),$$

then we can use the Weyl calculus on the Weyl symbol to immediately recover the subprincipal part of P. The choice of the Weyl symbol instead of the classical symbol to obtain the subprincipal part allows us to make the computations easier by using the rules of the Weyl calculus. In particular we can easily recover the Weyl symbol of the adjoint of an operator just by complex conjugation of the Weyl symbol of the starting operator.

This section is therefore devoted to the proof of the following property of the subprincipal symbol of P:

$$\mathrm{sub}(P)(x,\xi) = iX_0(x,\xi),$$

where, recall, $X_0(x,\xi)$ is the symbol of $X_0(x,D)$.

Note also that, even if we perform a change of variables in P, we get a new term of the kind $i\tilde{X}_0(x,D)$, and the subprincipal part is still given by this term. Whence $\mathrm{sub}(P)(x,\xi)$ is invariant (it is not just invariant on double zeros of the principal symbol of P), since, in our case, it coincides with the symbol of an operator.

Since $X_j(x,D) = \langle \alpha_j(x), D\rangle = \sum_{k=1}^n \alpha_{j,k}(x)D_k$, we have that $X_j(x,\xi) = \sum_{k=1}^n \alpha_{j,k}(x)\xi_k$ and $X_j^w(x,\xi) = X_j(x,\xi) + i\mathrm{div}(iX_j)(x)$, where $\mathrm{div}(iX_j)(x) = \sum_{k=1}^n \partial_{x_k}\alpha_{j,k}(x)$. Therefore $(X_j^*)^w(x,\xi) = \overline{X_j^w(x,\xi)} = X_j(x,\xi) - i\mathrm{div}(iX_j)(x)$, where both $X_j(x,\xi)$ and $\mathrm{div}(iX_j)(x)$ are real. Hence the differential operator P is the Weyl quantization of $p^w(x,\xi)$, that is $P = (p^w)^W(x,D)$, where

$$p^w(x,\xi) = \sum_{j=1}^N \overline{X_j^w} \sharp f^p \sharp X_j^w(x,\xi) + iX_0^w(x,\xi),$$

with $\sharp$ denoting the composition of symbols in the Weyl calculus.

Using the computation rules for the composition, and calling $r_0(x)$ a general symbol of order 0 (that, in our case, is a smooth function in x), we have for all j

$$\overline{X_j^w}\sharp f^p \sharp X_j^w(x,\xi) = \overline{X_j^w}\sharp(f^p\sharp X_j^w)(x,\xi) = \overline{X_j^w}\sharp(f^p X_j^w + \frac{i}{2}\{f, X_j^w\})$$

$$= f^p|X_j^w|^2 + \frac{i}{2}\{\overline{X_j^w}, f^p X_j^w\} + \frac{i}{2}\overline{X_j^w}\{f^p, X_j^w\} + \frac{i}{2}\{\overline{X_j^w}, \frac{i}{2}\{f^p, X_j^w\}\} + r_0$$

$$= f^p|X_j^w|^2 + \frac{i}{2}\{\overline{X_j^w}, f^p\}X_j^w + \frac{i}{2}f^p\{\overline{X_j^w}, X_j^w\} + \frac{i}{2}\{f^p, X_j^w\}\overline{X_j^w} + r_0,$$

and since $\overline{X_j^w} = X_j(x,\xi) + i\mathrm{div}(X_j)(x)$ is a sum of a symbol of order one and one of order zero, we have, keeping all the zeroth order terms in r_0,

$$\overline{X_j^w}\sharp f^p \sharp X_j^w(x,\xi) = f^p X_j^2 + \frac{i}{2}\{X_j, f^p\}X_j + \frac{i}{2}\{f^p, X_j\}X_j + r_0 = f^p X_j^2 + r_0.$$

Plugging that in $p^w(x,\xi)$, and recalling that $X_0^w(x,\xi) = X_0(x,\xi) + i\mathrm{div}(X_0)$, where the zeroth order part is still left in r_0, we find

$$p^w(x,\xi) = f(x)^p \sum_{j=1}^N X_j(x,\xi)^2 + iX_0(x,\xi) + r_0,$$

and finally have

$$\mathrm{sub}(P)(x,\xi) = p_1^w(x,\xi) = iX_0(x,\xi).$$

Chapter 2

Local solvability results for the main class

In this chapter we are going to present some local solvability results for the operator P of the form (1.1.1) previously introduced.

Before starting with the argument, we need to clarify what we exactly mean with *local solvability* of an operator and the technique used to prove it for P, that is, the *technique of a priori estimates*. Therefore, first we will give some basic definitions about the local solvability of a general partial differential operator, and we will explain how to recover the local solvability property starting from an a priori estimate true for P. Then we will give the statement of the first result concerning the local solvability of P around S when p is odd in the real case, that is when all the operators X_j, $0 \leq j \leq N$, in the expression of P are supposed to have real principal symbols. There will follow some examples of operators contained in the class under consideration, also remarking the difference with other similar models discussed in [1] and [26]. We will complete the analysis of the real case by giving a local solvability result far from the set $S = f^{-1}(0)$. Finally, we will also give a generalization of the local solvability result around the set S in the case p odd in a complex setting.

Summarizing, in the following chapter, we are going to prove the results listed below:

- Solvability near $S = f^{-1}(0)$ in the *real* case (that is when all the vector fields iX_j are real) when:
  - $p = 2k + 1$, $k \in \mathbb{Z}_+$;
  - $p = 2k$, $k \in \mathbb{Z}_+$;
- Solvability off $S = f^{-1}(0)$ when $p \in \mathbb{Z}_+$, in the *real* case.

- Solvability near $S = f^{-1}(0)$ when $p = 2k+1$, $k \in \mathbb{Z}_+$ in the general *complex* case, that is when all the vector fields iX_j, with $j \neq 0$, are complex (generalization of the *real* case)

2.1 Local solvability

In this section we give some basic definitions about local solvability of a partial differential operator and we explain the connection between local solvability and some a priori estimates.

Throughout we shall denote by $\mathring{K}$ the interior of the compact set K, and by $\langle\cdot,\cdot\rangle$, $(\cdot,\cdot)$, $\|\cdot\|_s$ the dual inner product, the L^2-inner product, and the H^s-Sobolev norm respectively. Moreover we will call P a general partial differential operator of order m on an open set $\Omega \subseteq \mathbb{R}^n$.

Definition 2.1.1 (Local solvability)**.** *We say that P is locally solvable at $x_0 \in \Omega$ if there exists an open set V containing x_0, $V \subseteq \Omega$, such that for all $f \in C^\infty(\Omega)$ there is $u \in \mathscr{D}'(\Omega)$ for which $Pu = f$ in V.*

Definition 2.1.2 (Local solvability in the sense H^s to $H^{s'}$)**.** *We say that P is H^s to $H^{s'}$ locally solvable at $x_0 \in \Omega$ if there exists $K \subset \Omega$, $x_0 \in \mathring{K} = U$, such that for all $f \in H^s_{\mathrm{loc}}(\Omega)$ there is $u \in H^{s'}_{\mathrm{loc}}(\Omega)$ which solves $Pu = f$ in U.*

Clearly the equality $Pu = f$ is to be considered in the sense of distributions.
It is important to remark that the previous definitions still hold in general for pseudodifferential operators. Moreover, attached to the general H^s to $H^{s'}$ local solvability, we have the *loss of derivatives* property of a locally solvable operator. This property reflects how far an operator is from being elliptic, that is, describes the kind of regularity we have for the solution of the problem $Pu = f$ depending on the regularity of the given source term f.

Definition 2.1.3 (Loss of derivatives)**.** *Let P, Ω be as above, and let $x_0 \in \Omega$ and $\mu \geq 0$. We shall say that P is locally solvable with loss of μ derivatives at x_0 if for every $s \in \mathbb{R}$ there exists a compact set $K \subset \Omega$, $x_0 \in \mathring{K} = U$, such that*

$$\forall f \in H^s_{\mathrm{loc}}(\Omega), \quad \exists u \in H^{s+m-\mu}_{\mathrm{loc}}(\Omega), \quad \text{with } Pu = f \text{ in } U.$$

The two lemmas below show that the local solvability property of an operator is equivalent to the validity of an a priori estimate.

Lemma 2.1.4. *Let Ω be an open set of $\mathbb{R}^n$, $x_0 \in \Omega$, and let $P \in \Psi^m_{ps}(\Omega)$ be a pseudodifferential operator solvable at x_0 in the sense H^s to $H^{s'}$. Then there exists an open neighborhood V of x_0 and a positive constant C such that, for all $u \in C_0^\infty(V)$,*

$$\|P^* u\|_{-s'} \geq C\|u\|_{-s},$$

where P^* denotes the formal adjoint of P.

Proof. Since, by hypothesis, P is H^s to $H^{s'}$ locally solvable at x_0, we can take an open set U as in Definition 2.1.2.

We consider, for any compact $K \subset U$, the space $C_K^\infty(U) = \{v \in C^\infty(U);\, \mathrm{supp}(v) \subset K\}$ with the norm $\|P^* v\|_{-s'}$. Note that, due to the local solvability property of P, this is a normed space. In fact, if we consider $v_0 \in C_0^\infty(U)$ such that $P^* v_0 = 0$, then, by Definition 2.1.2, we have for all $\varphi \in C_0^\infty(U)$ the existence of $u \in \mathscr{D}'(\Omega)$ (in particular, in this case, we have $u \in H^{s'}_{\mathrm{loc}}(\Omega)$) such that $Pu = \varphi$ in U. Whence, for all $\varphi \in C_0^\infty(U)$,

$$\langle Pu, v_0\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle u, P^* v_0\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = 0,$$

which gives $v_0 = 0$ so that the norm is well defined.

We then consider a compact set $\tilde{K}$ of U which is a neighborhood of a given compact $K \subset U$, and take the Hilbert space $H^s_{\tilde{K}}(U) = \{\varphi \in H^s(U);\, \mathrm{supp}(\varphi) \subset \tilde{K}\}$. We show now that the bilinear form

$$H^s_{\tilde{K}}(U) \times C_K^\infty(U) \ni (\varphi, v) \mapsto \langle \varphi, v\rangle = \int \varphi(x)\bar{v}(x)dx$$

is continuous, and then, as a consequence, that the estimate in the statement holds.

Observe first that, for a fixed $v \in C_K^\infty(U)$, the mapping $H^s_{\tilde{K}}(U) \ni \varphi \mapsto \langle \varphi, v\rangle$ is continuous, since $|\langle \varphi, v\rangle| \leq \|\varphi\|_s \|v\|_{-s}$. In addition, for a fixed $\varphi \in H^s_{\tilde{K}}(U)$, also the mapping $C_K^\infty(U) \ni v \mapsto \langle \varphi, v\rangle$ is continuous for the topology on $C_K^\infty(U)$ given by the norm $\|P^* v\|_{-s'}$. In fact, since P is H^s to $H^{s'}$ locally solvable at x_0, and since we considered U as in Definition 2.1.2, we have for a fixed $\varphi \in H^s_{\tilde{K}}(U)$ the existence of $u \in H^{s'}_{\mathrm{loc}}(\Omega)$ such that $Pu = \varphi$ in U. Thus, taking a function $\chi \in C_0^\infty(\Omega)$, $\chi = 1$ near the support of $P^*(C_K^\infty(U))$, we have

$$|\langle \varphi, v\rangle| = |\langle Pu, v\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}| = |\langle u, \chi P^* v\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}| \leq \|\chi u\|_{s'} \|P^* v\|_{-s'},$$

which proves the continuity of the linear form on $C_K^\infty(U)$ with respect to the given topology. In particular this gives that the bilinear form $H^s_{\tilde{K}}(U) \times C_K^\infty(U) \ni (\varphi, v) \mapsto \langle \varphi, v\rangle$ is separately continuous on the product of a Fréchet space with a metrizable space, whence it is continuous on $H^s_{\tilde{K}}(U) \times C^\infty_K(U)$. As a consequence, there exists a positive constant C such that, for all $\varphi \in H^s_{\tilde{K}}(U)$ and for all $v \in C^\infty_K(U)$,

$$|\langle \varphi, v\rangle| \leq C\|P^* v\|_{-s'}\|\varphi\|_s.$$

Now we finally use the inclusion $K \Subset \tilde{K}$, together with the continuity of the bilinear form $H^s_{\tilde{K}}(U) \times C^\infty_K(U) \ni (\varphi, v) \mapsto \langle \varphi, v\rangle$, to prove that, for all $v \in C^\infty_K(U)$, we have $v \in H^{-s}(\mathbb{R}^n)$ and $\|v\|_{-s} \leq C\|P^* v\|_{-s'}$, where C is a new suitable constant.

We define, for all $v \in C^\infty_K(U)$, the linear form on $H^s(\mathbb{R}^n)$ given by $\ell_v(\varphi) := \langle \varphi, v\rangle$, $\varphi \in H^s(\mathbb{R}^n)$. Since $K \Subset \tilde{K}$, we consider $\chi \in C^\infty_0(\tilde{K})$ such that $\chi = 1$ on $\mathrm{supp}(v)$. Hence, by the continuity of the bilinear form $H^s_{\tilde{K}}(U) \times C^\infty_K(U) \ni (\varphi, v) \mapsto \langle \varphi, v\rangle$, we get, for all $v \in C^\infty_K(U)$,

$$|\ell_v(\varphi)| = |\langle \varphi, (\chi + 1 - \chi)v\rangle| = |\langle \chi\varphi, v\rangle| \leq C\|P^* v\|_{-s'}\|\chi\varphi\|_s \leq C\|P^* v\|_{-s'}\|\varphi\|_s,$$

where C is a new positive constant. Then, for all $v \in C^\infty_K(U)$, we have that ℓ_v, which is independent on the choice of χ, depends continuously on $\varphi \in H^s(\mathbb{R}^n)$, and, consequently, for all $v \in C^\infty_K(U)$ $\ell_v \in (H^s(\mathbb{R}^n))^*$. Finally, since $(H^s(\mathbb{R}^n))^* = H^{-s}(\mathbb{R}^n)$, by the Riesz representation Theorem we get, for all $v \in C^\infty_K(U)$,

$$\|v\|_{-s} = \|\ell_v\|_{(H^s(\mathbb{R}^n))^*} = \sup_{0 \neq \varphi \in H^s(\mathbb{R}^n)} \frac{|\langle \varphi, v\rangle|}{\|\varphi\|_s} = \sup_{0 \neq \varphi \in H^s(\mathbb{R}^n)} \frac{|\langle \chi\varphi, v\rangle|}{\|\varphi\|_s} \leq C\|P^* v\|_{-s'}.$$

To conclude the proof it suffices now to choose K containing x_0 in its interior $\mathring{K}$ and consider $V = \mathring{K}$.

□

Lemma 2.1.5 (Sufficient conditions for H^s to $H^{s'}$ local solvability)**.** *Let Ω be an open set of $\mathbb{R}^n$, $x_0 \in \Omega$, and let $P \in \Psi^m_{ps}(\Omega)$. Assume that there exists an open neighborhood $U \subset \Omega$ of x_0 and a positive constant C such that, for all $u \in C^\infty_0(U)$,*

$$C\|P^* u\|_{-s'} \geq \|u\|_{-s}.$$

Then, for all $f \in H^s_{\mathrm{loc}}(\Omega)$, there exists $u \in H^{s'}_{\mathrm{loc}}(\Omega)$ which solves $Pu = f$ in U.

Proof. Since $\|P^* u\|_{-s'} \geq C\|u\|_{-s}$ for all $u \in C^\infty_0(U)$, P^* is injective on $C^\infty_0(U)$. Then, assuming $U \Subset \Omega$ (we can always make this assumption by shrinking U around x_0 to an open set that we keep denoting by U and which is strictly contained in Ω), we get that the space $E := P^*(C^\infty_0(U)) = \{P^* v; v \in C^\infty_0(U)\}$ is a subspace of $C^\infty_K(\Omega)$, where K is a suitable compact set of Ω. Now, fixing $f_0 \in H^s_{\mathrm{loc}}(\Omega)$, and considering

E as a subspace of $H^{-s'}_{\mathrm{comp}}(\Omega)$, we define on E the linear form $\ell(P^*v) := \langle v, f_0\rangle$. Therefore, taking a function $\chi \in C_0^\infty(\Omega)$, with $\chi = 1$ on $\overline{U}$, we have

$$|\ell(P^*v)| = |\langle v, f_0\rangle| \leq \|v\|_{-s}\|\chi f_0\|_s \leq C\|P^*v\|_{-s'}\|\chi f_0\|_s,$$

which gives the continuity of ℓ on the subspace E. Now, by the Hahn-Banach Theorem, we can extend ℓ to a linear form ℓ' over the whole $H^{-s'}_{\mathrm{comp}}(\Omega)$ such that there exists a distribution $u_0 \in \mathscr{D}'(\Omega)$ satisfying $\ell'(\psi) = \langle \psi, u_0\rangle_{\mathscr{D}(\Omega),\mathscr{D}'(\Omega)}$ for all $\psi \in C_0^\infty(\Omega)$. Note also that, if $\varphi \in C_0^\infty(\Omega)$, then the map $\psi \mapsto \langle \psi, \varphi u_0\rangle_{\mathscr{D}(\Omega),\mathscr{D}'(\Omega)} = \ell'(\varphi\psi)$ depends continuously on $\psi \in C_0^\infty(\Omega)$ for the topology induced by $H^{-s'}(\mathbb{R}^n)$, whence $\varphi u_0 \in H^{s'}(\mathbb{R}^n)$ and $u_0 \in H^{s'}_{\mathrm{loc}}(\Omega) = (H^{-s'}_{\mathrm{comp}}(\Omega))^*$.

In conclusion, by the previous arguments, for all $v \in C_0^\infty(U)$

$$\langle v, f_0\rangle = \ell'(P^*v) = \langle P^*v, u_0\rangle = \langle v, Pu_0\rangle,$$

which finally gives $Pu_0 = f_0$ on U.

□

We can thus rephrase the definition of H^s to $H^{s'}$ local solvability in terms of a priori estimate in the following way.

Corollary 2.1.6 (Definition of local solvability via a priori estimates)**.** *We have that P is H^s to $H^{s'}$ locally solvable at $x_0 \in \Omega$ if there exists a compact set $K \subset \Omega$, $x_0 \in \mathring{K} = U$, and a positive constant C such that, for all $u \in C_0^\infty(U)$,*

$$C\|P^*u\|_{-s'} \geq \|u\|_{-s}. \tag{2.1.1}$$

We will often refer to (2.1.1) as the *solvability estimate*.
By using functional analysis arguments we proved that the previous statement, that we take for simplicity as a definition, is equivalent to the formal Definition 2.1.2 given before. We choose to take it as a definition especially to remark that the local solvability property of an operator is equivalent to an a priori estimate true for it. In fact we will obtain our solvability results for P of the form (1.1.1), and also for the other classes treated in this thesis, by proving some estimates of the form (2.1.1). That is why the technique we use to prove the local solvability property is called *technique of a priori estimates*.

It is important to remark that this is an extremely powerful tool. In fact, even when a fundamental solution is available (that is, even if we now that an operator is solvable), it is difficult to control its properties, since its expression is not always known. Therefore, once again, one can use this technique, because it gives information both on the solvability and on the regularity properties of the operator.

Remark 2.1.7. *Observe that, given a hypoelliptic partial differential operator P, we have that its formal adjoint P^* is always locally solvable, due to the fact that the solvability estimate (with suitable value of s and s') is always true for P^*. This is exactly the case when we consider the operator P equal to the Kannai operator, that is, its formal adjoint P^* is locally solvable due to the hypoellipticity property of P. In particular P^*, which is in the class* (1.1.1)*, is also locally solvable in the neighborhood of the set where the principal symbol of P changes sign, and this property, in this case, comes from the hypoellipticity of P (which is hypoelliptic over the whole $\mathbb{R}^n$), but can also be proved by using our solvability result given in Section 2.2.*

However, the hypoellipticity property of an operator implies the local solvability of the adjoint, but does not imply the local solvability of the operator itself. Therefore, it is restrictive to connect the solvability problem to the hypoellipticity problem, since, in general, a locally solvable operator is not even the adjoint of a hypoelliptic operator. This remarks the difference between the problem we study here and that analyzed in [1] and [26], where the hypoellipticity property of operators of the form P^*, with P of the kind (1.1.1)*, is considered. Hence, in the context of the local solvability problem, the solvability results we obtain for P in the class* (1.1.1) *are more general than the solvability property that follows from the results in [1] and [26]. We shall see this difference through an example in Subsection 2.2.1.*

2.2 Solvability near S with odd degeneracy in the real case: the statement

In this section we give the statement of the first result concerning the local solvability of P of the form (1.1.1) around the set $S = f^{-1}(0)$ when p is odd in the real case. In Section 2.9 we will also give the result in a more general setting, that is, when the operators X_j $(1 \leq j \leq N)$ in the second order part of P are allowed to be complex. The proof of the result in the real case is given in a slightly different way than in the complex case. That is why we present the two statements separately (even if the complex one generalizes the real one), since the tools we use in the real case, as we shall remark later, could also work in a more general case, that is in presence of pseudodifferential operators in the expression of P.

We recall here the setting of the problem using the notations introduced in Section 1.2.

Let $\Omega \subset \mathbb{R}^n$ be open. Let $\alpha_0, \ldots, \alpha_N \in C^\infty(\Omega; \mathbb{R}^n)$ and consider the *homogeneous first order partial differential operators with real smooth coefficients*, and hence no zeroth order term,

$$X_j = X_j(x, D) = \langle \alpha_j(x), D\rangle, \quad 0 \le j \le N,$$

where $D = (D_1, D_2, \ldots, D_n)$, $D_j = -i\partial_{x_j}$, and $\langle v, w\rangle = \sum_\ell v_\ell w_\ell$. (Hence, the $iX_j(x, D)$ are smooth *real* vector fields.) Let also $f \in C^\infty(\Omega; \mathbb{R})$ and $a_0 \in C^\infty(\Omega; \mathbb{C})$. On Ω we consider the second order operator

$$P = \sum_{j=1}^{N} X_j^* f^{2k+1} X_j + iX_0 + a_0, \tag{2.2.2}$$

where $k \ge 0$ is an integer.

Our goal is to prove that operators of this form are locally solvable around the set $S := f^{-1}(0)$. Note also that in this case we deal with an operator with odd degeneracy, since we assumed S to be different from the empty set.

Recall that we suppose (see Section 1.2 for details and notation):

(H1) *$iX_0 f(x) > 0$ for all $x \in S := f^{-1}(0) \neq \emptyset$;*

(H2) *For all $j = 1, \ldots, N$ and for all compact $K \subset \Omega$ there exists $C_{K,j} > 0$ such that*

$$\{X_j, X_0\}(x, \xi)^2 \le C_{K,j} \sum_{j'=0}^{N} X_{j'}(x, \xi)^2, \quad \forall (x, \xi) \in K \times (\mathbb{R}^n);$$

(H3) *Given $x_0 \in S$, with $\pi^{-1}(x_0) \bigcap \Sigma \neq \emptyset$, we have*

$$\operatorname{rank} M(\rho) \ge 2, \quad \forall \rho \in \pi^{-1}(x_0) \cap \Sigma.$$

After this summary about the hypotheses on the class described by P we are ready to give the statement of the first result.

Theorem 2.2.1. *Let the operator P in (2.2.2) satisfy hypotheses (H1) and (H2).*

(i) Let $k = 0$. Then for all $x_0 \in S$ with $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and at which hypothesis (H3) is fulfilled, there exists a compact $K \subset W$ with $x_0 \in U = \mathring{K}$ such that for all $v \in H^{-1/2}_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have $H^{-1/2}$ to L^2 local solvability).

(ii) Let $k = 0$. Then, for all $x_0 \in S$ for which $\pi^{-1}(x_0) \cap \Sigma = \emptyset$ there exists a compact $K \subset \Omega$ with $x_0 \in U = \mathring{K}$ such that for all $v \in H^{-1}_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have H^{-1} to L^2 local solvability).

(iii) *If $k \geq 1$ and x_0 is any given point of S, or $k = 0$ and $x_0 \in S$ (with $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$) is such that (H3) is not satisfied at x_0, then there exists a compact $K \subset \Omega$ with $x_0 \in U = \overset{\circ}{K}$ such that for all $v \in L^2_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have L^2 to L^2 local solvability).*

As we said before, depending on the geometric relations among the vector fields iX_j, we clearly have a better kind of solvability when stronger conditions are satisfied. By Definition 2.1.6, we will have results in (i), (ii) and (iii), when the a priori estimate (2.1.1) is true with $s' = 0$ and s respectively equal to $s = -1/2$, $s = -1$ and $s = 0$. These estimates are obtained starting from an estimate true for P^*, that we will call *main estimate* and which is proved in Section 2.3, involving an operator that we will call $\widehat{P}_0$. Depending on the properties of $\widehat{P}_0$, related to hypotheses given in (i), (ii) and (iii), we will be able to perform the Melin inequality, the Gårding inequality, and the Fefferman-Phong inequality respectively, gaining, this way, the estimate (2.1.1) with different Sobolev norms.

Remark 2.2.2.

- *Of course, hypothesis (H3) in Theorem 2.2.1 (i) is non-empty only when $\Sigma \neq \emptyset$, that is, when the operator $\sum_{j=0}^{N} X_j^* X_j$ is not elliptic. The elliptic case $\Sigma = \emptyset$ is covered by point (ii) of the theorem.*

- *Observe that we do not suppose that $X_1(x, D), \ldots, X_N(x, D)$ be non-singular everywhere (that is, we do not assume that the vectors $\alpha_j(x) \neq 0$, $1 \leq j \leq N$, for all $x \in \Omega$; this is not the case for X_0). Of course, the validity of hypothesis (H3) permits only certain kinds of degeneracies of the vector fields.*

- *Note that, when $k = 0$, we can have different kind of solvability depending on the point $x_0 \in S$ we are looking at. For instance, even if we always have L^2 to L^2 local solvability at each point of S, we can also have that, for some $x_0 \in S$, we are in case (i) or (ii) of Theorem 2.2.1, and thus we have more than L^2 to L^2 local solvability around that point, that is, we have $H^{-1/2}$ to L^2 or H^{-1} to L^2 local solvability.*

- *In general, our class of operators and assumptions are invariant under symplectomorphisms induced by diffeomorphisms of the base manifold.*

2.2.1 Examples

We give here some example of operators in the class described by (1.1.1), showing how Theorem 2.2.1 applies on different operators with odd degeneracy. We shall observe that when the odd exponent is $p = 1$ different possibilities do arise. A case when $p = 1$ is also treated in order to show the generality of our solvability result with respect to the solvability as a consequence of the hypoellipticity property obtained in [26].

Example 2.2.3. Consider $\mathbb{R}^2$ with coordinates $x = (x_1, x_2)$ (it will be then immediately clear to the reader how to generalize the example to the case $x = (x_1, x_2, x') \in \mathbb{R}^{2+n'}$ with $n' \geq 1$). Let $g = g(x_2) = 1 + x_2^2$, and $f(x) = x_1 - (x_2 + x_2^3/3)$. Consider the 2×2 symmetric matrix $A(x) = [a_{jj'}(x)]_{1 \leq j, j' \leq 2} = \begin{bmatrix} g & 1 \\ 1 & 1/g \end{bmatrix}$. Then $A(x) \geq 0$ and $\dim \operatorname{Ker} A(x) = 1$ for all $x \in \mathbb{R}^2$. Consider

$$P = \sum_{j,j'=1}^{2} D_j \Big(f(x)^{2k+1} a_{jj'}(x) D_{j'} \Big) + iX_0(x, D) + a_0(x),$$

where $X_0(x, \xi) = \alpha \xi_1 + \xi_2 / g(x_2)$, with $\alpha > 1$ a constant. Consider the orthogonal projection $\Pi_1(x) \colon \mathbb{R}^2 \longrightarrow \operatorname{Ker} A(x)^{\perp}$. Then we may consider the smooth nonnegative square root $A_{1/2}(x)$ of $A(x)$ (it is easily seen that $A_{1/2} = \sqrt{g/(1+g^2)}\, A$) and define the symbols $X_j(x, \xi)$, $j = 1, 2$, by

$$A_{1/2}(x) \Pi_1(x) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} X_1(x, \xi) \\ X_2(x, \xi) \end{bmatrix},$$

so that

$$X_1(x, \xi) = \sqrt{g(x_2)} \frac{g(x_2)\xi_1 + \xi_2}{\sqrt{1 + g(x_2)^2}}, \quad X_2(x, \xi) \frac{1}{\sqrt{g(x_2)}} \frac{g(x_2)\xi_1 + \xi_2}{\sqrt{1 + g(x_2)^2}}.$$

We shall also put $X(x, \xi) = g(x_2)\xi_1 + \xi_2$, and write $X_j =: g_j X$, $j = 1, 2$. Hence

$$\langle A(x)\xi, \xi \rangle = X_1(x, \xi)^2 + X_2(x, \xi)^2, \quad \forall (x, \xi) \in T^*\mathbb{R}^2,$$

and

$$P = \sum_{j=1}^{2} X_j^* f^{2k+1} X_j + iX_0 + a_0.$$

Now, since $\{X_0, X_j\} = \{X_0, g_j\} X + g_j \{X_0, X\}$, to check whether (H2) holds or not, it suffices to study $\{X_0, X\}$. We have (using the fact that $g = g(x_2)$)

$$\{X_0, X\}(x, \xi) = \{\alpha \xi_1 + \frac{\xi_2}{g}, g\xi_1 + \xi_2\} = \frac{\{\xi_2, g\}}{g^2} X(x, \xi). \tag{2.2.3}$$

Hence (H2) holds. We finally deal with (H1): we have

$$iX_0 f(x) = \alpha - 1 > 0$$

by assumption. Since hypothesis (H3) cannot hold at any point (notice that the Hamilton vector fields H_{X_1} and H_{X_2} are always linearly dependent, that the Hamilton vector fields H_X and H_{X_0} are linearly independent, but that $\{X_0, X\} = 0$ on the set $\{X = 0\}$), Theorem 2.2.1 yields local solvability in L^2 near $f^{-1}(0)$ for all $k \geq 0$. In this case we have that the vector fields iX_1 and iX_2 are tangent to $f^{-1}(0)$.

Observe that the condition $\alpha > 1$ makes the vector field iX_0 transverse to $f^{-1}(0)$.

Observe also that in this example hypothesis (H2) holds because of the stronger property $\{X_0, X\} = \mu(x)X$, with μ smooth. This is not by chance. As a consequence of Lemma 1.1.2 one has that, for instance when $N = 1$ as in this example, near a non-singular point x_0 of X (that is a point at which $X \neq 0$ as a vector field), hypothesis (H2) is equivalent to the existence of a smooth μ such that $\{X_0, X\} = \mu(x)X$.

Example 2.2.4. The second example is an elaboration of the first one. We take the operator

$$P = \sum_{j=1}^{2} X_j^* f^{2k+1} X_j + iX_0 + a_0,$$

where X_1 and X_2 are as in Example 2.2.3, and where this time $f(x) = x_1 + x_2 + x_2^3/3$ and $X_0(x, \xi) = \alpha(x)X(x, \xi)$, α smooth with $\alpha(x) > 0$ on $f^{-1}(0)$. Hypothesis (H1) is readily seen to be satisfied, and the same holds for hypothesis (H2), because X_0, X_1 and X_2 are all multiples of X. Since hypothesis (H3) cannot hold at any point for all $k \geq 0$ and since $\Sigma \neq \emptyset$ over $f^{-1}(0)$, we then have local solvability of P in L^2 near $f^{-1}(0)$. In this case the vector fields iX_1 and iX_2 are *not* tangent to $f^{-1}(0)$.

Example 2.2.5. As a third example, consider in $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_2)$ the vector fields iX_1, iX_2 and iX_3 where (recall that $D = -i\partial$)

$$X_1 = D_{x_1} - \frac{x_2}{2} D_{x_3},\ X_2 = D_{x_2} + \frac{x_1}{2} D_{x_3},\ X_3 = D_{x_3}.$$

The vector fields iX_1, iX_2, iX_3 realize a Heisenberg group structure on $\mathbb{R}^3$. Let $X_0(x, D) = \alpha X_3(x, D)$, where $\alpha > 0$ is a constant. Let $f(x) = x_3 - (x_1^2 + x_2^2)$. Consider the operator $P = \sum_{j=1}^{2} X_j^* f X_j + iX_0 + a_0$ (which is a sort of degenerate sub-Laplacian). One immediately sees that $iX_0 f(x) = \alpha > 0$, for all x, and also that hypothesis (H2) is trivially satisfied. Since $\Sigma = \Sigma_0 \cap \Sigma_1 \cap \Sigma_2 = \emptyset$, we have that P is *H^{-1} to L^2 locally solvable* near $f^{-1}(0)$, that is (recall), for all $x_0 \in f^{-1}(0)$ there exists a compact K containing x_0 in its interior U such that for all $v \in H^{-1}_{\mathrm{loc}}$ we may find $u \in L^2_{\mathrm{loc}}(\Omega)$ solving $Pu = v$ in U.

Remark 2.2.6. *Note that in Example 2.2.5 we have $iX_1 f = -2x_1 - x_2/2$, so that it is not always zero on $f^{-1}(0)$. Hence, Akamatsu's hypoellipticity condition (see [1]) on the system of vector fields $X_0, fX_1, \dots, fX_N$ does not always hold in this case.*

Example 2.2.7. In the last example we show an instance in which hypothesis (H3) is not identically satisfied, changing therefore the solvability according to where $\pi^{-1}(x_0)$, with $x_0 \in f^{-1}(0)$, is located with respect to Σ. Consider in $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ an open set Ω which contains the plain $x_1 = -1$ and the first order operators $X_j(x, D)$, $1 \leq j \leq 3$, with symbols

$$X_1(x,\xi) = \xi_1 - x_3\xi_3, \quad X_2(x,\xi) = (1 + x_1)\xi_3, \quad X_3(x,\xi) = \xi_2 - x_1\xi_3.$$

Let $\Omega_+ := \{x \in \Omega;\ x_1 > -1\}$ and $\Omega_- = \{x \in \Omega;\ x_1 < -1\}$. We then have

$$\{X_1, X_3\} = -X_2, \quad \{X_1, X_2\} = (2 + x_1)\xi_3, \quad \{X_2, X_3\} = 0. \tag{2.2.4}$$

Let then $f(x) = x_2 + x_2^3/3 - x_1x_3$ and $X_0(x, D) = X_3(x, D)$, say. Consider the operator $P = \sum_{j=1}^{2} X_j^* f X_j + iX_0 + a_0$. Notice that, because of the choice of Ω, the characteristic set Σ (see (1.1.2)) is such that we always have $\xi_3 \neq 0$ for points $(x, \xi) \in \Sigma$, and

(i) $\pi^{-1}(\Omega_\pm) \cap \Sigma = \emptyset$,

(ii) while if $x_0 = (-1, x_2^0, x_3^0) \in \Omega$, then

$$\pi^{-1}(x_0) \cap \Sigma = \{(x_0, \xi) \in T^*\Omega \setminus 0;\ \xi_1 = x_3^0\xi_3,\ \xi_2 = -\xi_3,\ \xi_3 \neq 0\}.$$

Hence, in case (ii), in the fiber over $f^{-1}(0) \cap \Omega$ we always find characteristic points. It is readily seen that $iX_0 f(x) = 1 + x_1^2 + x_2^2 > 0$ for all x, so that hypothesis (H1) is fulfilled. Hypothesis (H2) is satisfied everywhere in view of the first and third of relations (2.2.4). As for (H3), in view of case (i) we have that, if $W \subset \Omega$ is a (connected) neighborhood of $x_0 = (x_1^0 = -1, x_2^0, x_3^0)$, on $\pi^{-1}(W) \cap \Sigma$ the Hamilton vector fields H_{X_0}, H_{X_1} and H_{X_2} are linearly independent and the relations (2.2.4) grant the validity of (H3) at x_0. Therefore:

- if $x_0 = (x_1^0, x_2^0, x_3^0) \in f^{-1}(0) \cap \Omega_\pm$ (case (i)) we have H^{-1} *to* L^2 *local solvability* near x_0,
- whereas if $x_0 \in \{x \in \Omega \cap f^{-1}(0);\ x_1 = -1\}$ (case (ii)) we have $H^{-1/2}$ *to* L^2 *local solvability* in Ω near x_0.

Example 2.2.8. This example shows that the class considered contains also operators whose adjoint is not hypoelliptic. Let $g \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with $g \not\equiv 0$ and $g(0) \neq 0$. Let $1 \leq k \in \mathbb{N}$ and let

$$L = iD_{x_2} - (x_2 - g(x_1))^k D_{x_1}^2, \quad (x_1, x_2) = x \in \mathbb{R} \times \mathbb{R}.$$

By Zuily [26], L is (C^∞) hypoelliptic iff $k \neq 1$, whence for such values of k the operator L^* is locally solvable. Put

$$X_0 = D_{x_2} + kg'(x_1)(x_2 - g(x_1))^{k-1} D_{x_1}, \quad X_1 = D_{x_1},$$

$$f(x) = x_2 - g(x_1), \quad a_0(x) = k\frac{\partial}{\partial x_1}\big(g'(x_1)(x_2 - g(x_1))^{k-1}\big),$$

and let

$$P = X_1 f^k X_1 + iX_0 + a_0.$$

Then $P^* = -L$, whence for $k \geq 1$ the operator P is locally solvable because of the hypoellipticity of L. However, when $k = 1$ and supposing that $|g'(x_1)| \leq c > 1$ for all x_1, one has from Theorem 2.2.1 that P is H^{-1} to L^2 *locally solvable* near $f^{-1}(0)$, since in this case we have that $\Sigma = \emptyset$, the vector fields iX_0 and iX_1 being linearly independent.
It is important to remark that this example shows that Theorem 2.2.1 covers cases which are not covered by Zuily's hypoellipticity results in [26]: the operator P when $k = 1$ is H^{-1} to L^2 locally solvable near $f^{-1}(0)$ even if P^* is *not* hypoelliptic.

2.3 The main estimate

As mentioned earlier, the technique of a priori estimates represents the method we use to treat the solvability problem for P of the form (1.1.1). The starting point to obtain the solvability result when the exponent p is odd, both in the real and in the complex case, is given by an estimate that we shall call *main estimate*. This section is therefore devoted to the proof of the *main estimate* stated in the Proposition 2.3.1 given below.

Recall that throughout we denote by $(\cdot, \cdot)$ and $\|\cdot\|_0$, respectively, the L^2 inner product and norm, and by $\mathring{K}$ the interior of K.

Proposition 2.3.1. *[Main Estimate] Let P be as in (2.2.2) and satisfy (H1). For all $x_0 \in S$ there exists a compact $K_0 \subset \Omega$, with $x_0 \in \mathring{K}_0$, constants $c = c(K_0), C = C(K_0) > 0$ and $\varepsilon_0 = \varepsilon_0(K_0)$ with $\varepsilon_0(R) \to 0$ as the compact $R \searrow \{x_0\}$, such that for all compact $K \subset K_0$*

$$\|P^* u\|_0^2 \geq \frac{1}{8}\|X_0 u\|_0^2 + c\Big(\widehat{P}_0(x, D)u, u\Big) - C\|u\|_0^2, \tag{2.3.5}$$

for all $u \in C_0^\infty(K)$, where

$$\widehat{P}_0 = X_0^* X_0 + \sum_{j=1}^{N} (X_j^* f^{2k} X_j - \varepsilon_0^2 [X_j, X_0]^* f^{2k} [X_j, X_0]). \tag{2.3.6}$$

Proof. In the first place we have $X_0^* = X_0 + d_{X_0}$, where the function $d_{X_0} := \sum_{k=1}^n D_k(\alpha_{0,k}) = -i\,\mathrm{div}(iX_0)$ is smooth and taking values in $i\mathbb{R}$, and for all compact $K \subset \Omega$

$$\|P^* u\|_0^2 \geq \frac{1}{2}\|(\sum_{j=1}^{N} X_j^* f^{2k+1} X_j - iX_0^*)u\|_0^2 - \|a_0\|_{L^\infty(K)}^2 \|u\|_0^2, \ \forall u \in C_0^\infty(K). \tag{2.3.7}$$

For the first term in the right-hand side we have

$$\|(\sum_{j=1}^{N} X_j^* f^{2k+1} X_j - iX_0^*)u\|_0^2 = \|\sum_{j=1}^{N} X_j^* f^{2k+1} X_j u\|_0^2$$

$$+\|X_0^* u\|_0^2 - 2\sum_{j=1}^{N} \mathsf{Re}(X_j^* f^{2k+1} X_j u, iX_0^* u).$$

Next, we have $(1 \leq j \leq N)$

$$-2\mathsf{Re}(X_j^* f^{2k+1} X_j u, iX_0^* u) = -2\mathsf{Im}(X_j^* f^{2k+1} X_j u, X_0^* u),$$

and consider

$$(X_j^* f^{2k+1} X_j u, X_0 u) = (f^{2k+1} X_j u, X_0 X_j u) + (f^{2k+1} X_j u, [X_j, X_0]u).$$

One has

$$\begin{aligned} 2i\mathsf{Im}(f^{2k+1} X_j u, X_0 X_j u) &= (f^{2k+1} X_j u, X_0 X_j u) - (X_0 X_j u, f^{2k+1} X_j u) \\ &= (X_0^* f^{2k+1} X_j u, X_j u) - (X_0 X_j u, f^{2k+1} X_j u) \end{aligned}$$

$$
\begin{aligned}
&= (d_{X_0} f^{2k+1} X_j u, X_j u) + ([X_0, f^{2k+1}] X_j u, X_j u) \\
&\quad + (f^{2k+1} X_0 X_j u, X_j u) - (X_0 X_j u, f^{2k+1} X_j u) \\
&= (d_{X_0} f^{2k+1} X_j u, X_j u) - i(2k+1)((iX_0 f) f^{2k} X_j u, X_j u),
\end{aligned}
$$

where it is important to observe that the latter term is purely imaginary. Therefore

$$
\begin{aligned}
\mathsf{Im}(f^{2k+1} X_j u, X_0 X_j u) &= \frac{1}{2i}(d_{X_0} f^{2k+1} X_j u, X_j u) - \frac{2k+1}{2}((iX_0 f) f^{2k} X_j u, X_j u) \\
&= \frac{1}{2}\mathsf{Im}(d_{X_0} f^{2k+1} X_j u, X_j u) - \frac{2k+1}{2}((iX_0 f) f^{2k} X_j u, X_j u),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathsf{Im}(X_j^* f^{2k+1} X_j u, X_0^* u) \\
&= \mathsf{Im}(f^{2k+1} X_j u, X_j(d_{X_0} u)) + \mathsf{Im}(f^{2k+1} X_j u, X_0 X_j u) + \mathsf{Im}(f^{2k+1} X_j u, [X_j, X_0] u) \\
&= \mathsf{Im}(f^{2k+1} X_j u, (X_j d_{X_0}) u) + \mathsf{Im}(f^{2k+1} X_j u, d_{X_0} X_j u) + \frac{1}{2}\mathsf{Im}(d_{X_0} f^{2k+1} X_j u, X_j u) \\
&\qquad - \frac{2k+1}{2}(f^{2k}(iX_0 f) X_j u, X_j u) + \mathsf{Im}(f^{2k+1} X_j u, [X_j, X_0] u) \\
&= \mathsf{Im}(f^{2k+1} X_j u, (X_j d_{X_0}) u) - \frac{2k+1}{2}(f^{2k}(iX_0 f) X_j u, X_j u) \\
&\qquad - \frac{1}{2}\mathsf{Im}(d_{X_0} f^{2k+1} X_j u, X_j u) + \mathsf{Im}(f^{2k+1} X_j u, [X_j, X_0] u).
\end{aligned}
$$

We may thus choose a compact $K_0 \subset \Omega$ containing $x_0 \in S$ in its interior such that $iX_0 f\big|_{K_0} \geq c_0 > 0$. We therefore have that for any given compact $K \subset K_0$ with $x_0 \in \overset{\circ}{K}$ and for all $u \in C_0^\infty(K)$

$$
\begin{aligned}
&-2\mathsf{Im}(X_j^* f^{2k+1} X_j u, X_0^* u) \geq (2k+1)((iX_0 f) f^k X_j u, f^k X_j u) \\
&-2\|f\|_{L^\infty(K)} \|f^k X_j d_{X_0}\|_{L^\infty(K)} \|f^k X_j u\|_0 \, \|u\|_0 - 2\|f\|_{L^\infty(K)} \|f^k X_j u\|_0 \|f^k [X_j, X_0] u\|_0 \\
&\qquad\qquad - \|d_{X_0}\|_{L^\infty(K)} \|f\|_{L^\infty(K)} \|f^k X_j u\|_0^2 \\
&\geq (2k+1)\Big[c_0 - \|f\|_{L^\infty(K_0)}\Big(\|f^k X_j d_{X_0}\|_{L^\infty(K_0)} + \|d_{X_0}\|_{L^\infty(K_0)} + 1\Big)\Big]\Big(X_j^* f^{2k} X_j u, u\Big) \\
&\quad - \|f\|_{L^\infty(K_0)}\Big([X_j, X_0]^* f^{2k} [X_j, X_0] u, u\Big) - \|f\|_{L^\infty(K_0)} \|f^k X_j d_{X_0}\|_{L^\infty(K_0)} \|u\|_0^2.
\end{aligned}
$$

Since $\|f\|_{L^\infty(K)} \to 0$ as $K \searrow \{x_0\}$, we shrink K_0 around x_0, keeping x_0 in its interior, so as to have

$$c_0 - \|f\|_{L^\infty(K_0)}\Big(\|f^k X_j d_{X_0}\|_{L^\infty(K_0)} + \|d_{X_0}\|_{L^\infty(K_0)} + 1\Big) \geq \frac{c_0}{2(2k+1)},$$

and may choose

$$\varepsilon_0 = \varepsilon_0(K_0) := \|f\|^{1/2}_{L^\infty(K_0)}/\sqrt{c}, \quad c := \min(\frac{c_0}{4}, \frac{1}{8}). \tag{2.3.8}$$

Observe that in general $\varepsilon_0(R) \to 0$ as the compact $R \searrow \{x_0\}$. Therefore, once we fix K_0 with the above properties, then for all compact $K \subset K_0$

$$\|P^*u\|_0^2 \geq \frac{1}{2}\|X_0^*u\|_0^2 + c\Big(\sum_{j=1}^{N}(X_j^* f^{2k} X_j - \varepsilon_0^2[X_j, X_0]^* f^{2k}[X_j, X_0])u, u\Big)$$

$$-c\varepsilon_0^2\|f^k X_j d_{X_0}\|_{L^\infty(K_0)}\|u\|_0^2 - \|a_0\|^2_{L^\infty(K_0)}\|u\|_0^2, \ \forall u \in C_0^\infty(K).$$

Since $\|X_0^*u\|_0^2 \geq \|X_0u\|_0^2/2 - \|d_{X_0}\|^2_{L^\infty(K_0)}\|u\|_0^2$ for all $u \in C_0^\infty(K)$, and $c \leq 1/8$, we finally get the desired inequality. □

We have the following corollary of hypothesis (H2) and Proposition 2.3.1 which will be fundamental.

Corollary 2.3.2. *Let K_0 be the compact of Proposition 2.3.1. We may **shrink** K_0 around $x_0 \in S$ to a compact containing x_0 in its interior (that we keep denoting by K_0), in such a way that the principal symbol*

$$\widehat{p}_0(x,\xi) := X_0(x,\xi)^2 + f(x)^{2k}\sum_{j=1}^{N}\Big(X_j(x,\xi)^2 - \varepsilon_0^2\{X_j, X_0\}(x,\xi)^2\Big)$$

*of $\widehat{P}_0$ is **nonnegative** for all $(x,\xi) \in \pi^{-1}(K_0)$, where (recall) $\varepsilon_0 = \varepsilon_0(K_0)$ is given in (2.3.8). We actually have that for a suitable constant $C > 0$ (depending only on K_0)*

$$C^{-1}\Big(X_0^2 + f^{2k}\sum_{j=1}^{N}X_j^2\Big) \leq \widehat{p}_0 \leq C\Big(X_0^2 + f^{2k}\sum_{j=1}^{N}X_j^2\Big), \ \forall (x,\xi) \in \pi^{-1}(K_0).$$

Proof. The proof is an immediate consequence of the equality

$$X_0(x,\xi)^2 + f(x)^{2k}\sum_{j=1}^{N}\Big(X_j(x,\xi)^2 - \varepsilon_0^2\{X_j,X_0\}(x,\xi)^2\Big)$$

$$= (1-f(x)^{2k})X_0(x,\xi)^2 + f(x)^{2k}\sum_{j=0}^{N}\Big(X_j(x,\xi)^2 - \varepsilon_0^2\{X_j,X_0\}(x,\xi)^2\Big).$$

□

Remark 2.3.3. *From the corollary it follows that if we consider the parameter-dependent family*

$$\widehat{P}_{0,\varepsilon} = X_0^*X_0 + \sum_{j=1}^{N}(X_j^* f^{2k} X_j - \varepsilon^2[X_j,X_0]^* f^{2k}[X_j,X_0]),$$

where we now allow ε *to vary in* $\mathbb{R}$*, then for all compact* $K\subset K_0$ *and all* $|\varepsilon|\le\varepsilon_0(K_0)$*, we still have that*

$$\widehat{p}_{0,\varepsilon}(x,\xi) := X_0^2 + f^{2k}\sum_{j=1}^{N}(X_j^2-\varepsilon^2\{X_j,X_0\}^2)\ge 0,\ \forall (x,\xi)\in\pi^{-1}(K),$$

where $\widehat{p}_{0,\varepsilon}$ *is the principal symbol of* $\widehat{P}_{0,\varepsilon}$*. Notice that* $\widehat{P}_0=\widehat{P}_{0,\varepsilon_0}$ *(see* (2.3.6)*).*

It is clear that in order to obtain the solvability estimate (2.1.1) starting from the main estimate (2.3.5) we need to estimate from below $(\widehat{P}_0u,u)$ and $\|X_0u\|_0$, and then cancel the L^2-error given by $-C\|u\|_0^2$. Since by condition (H1) the vector field iX_0 is nondegenerate around $S=f^{-1}(0)$, we can use a Poincaré inequality for non-degenerate vector fields to estimate this term from below. A version of the Poincaré inequality for nondegenerate vector fields is given in Appendix A.

Recalling that point (i) of Theorem 2.2.1 gives $H^{-1/2}$ to L^2 local solvability, then, in this case, we need to gain a $H^{1/2}$ Sobolev norm from the term $(\widehat{P}_0u,u)$, since the Poincaré inequality for iX_0 just gives an estimate between L^2 norms. Likewise, point (ii) of Theorem 2.2.1 requires the gain of a H^1 Sobolev norm, while in the case (iii) we just need to estimate $(\widehat{P}_0u,u)$ from below with a L^2 norm to recover the solvability estimate.

Before giving the proof of Theorem 2.2.1, we focus our attention on condition (H3) and on its connection with the estimate true for $\widehat{P}_0$ in this particular case, that is the

Melin inequality. We also show the equivalence of (H3) and the Hörmander condition at step 2, which gives the same result as the Melin inequality (but requiring a stronger condition). Afterwards, we will prove that the Fefferman-Phong inequality always holds for $\widehat{P}_0$ (by suitably shrinking the compact K_0 around $x_0 \in S$), and finally we give the proof of Theorem 2.2.1 by combining the main estimate with the inequalities true for $\widehat{P}_0$.

2.4 A focus on condition (H3)

In the last section we presented (as we shall see) the key point in the proof of Theorem 2.2.1, namely, the *main estimate* true for P^* and involving a new operator that we denoted by $\widehat{P}_0$. The second step to prove Theorem 2.2.1 consists in the analysis of $\widehat{P}_0$, that is, in particular, in finding some estimates satisfied by the latter. As said earlier, different estimates hold for $\widehat{P}_0$ if we are under the hypotheses contained in the case (i), (ii) or (iii) of Theorem 2.2.1.

In this section we focus our attention on condition (H3) required in (i), and we will see that it has as a consequence the validity of the Melin inequality for $\widehat{P}_0$.

Thus, we suppose now to be under hypotheses of point (i) of Theorem 2.2.1, that is, we analyze the solvability of P when $k = 0$ around a point $x_0 \in S$ which satisfies condition (H3), namely, x_0 is such that $\pi^{-1}(x_0) \bigcap \Sigma \neq \emptyset$ and rank $M(\rho) \geq 2$ for all $\rho \in \pi^{-1}(x_0) \cap \Sigma$. Note that, if $\pi^{-1}(x_0) \bigcap \Sigma \neq \emptyset$, then there exists a sufficiently small compact set $K \subset \Omega$, containing x_0 in its interior U, such that rank $M(\rho) \geq 2$ for all $\rho \in \pi^{-1}(U) \cap \Sigma$. Therefore throughout we consider $U = \mathring{K}_0$ (containing the point x_0 in which we have (H3)), where K_0 is the compact of Corollary 2.3.2, namely, it is such that $\widehat{p}_0(x, \xi)$ in nonnegative on $\pi^{-1}(K_0)$.

As we said before, the result in (i) strictly depends on the term $(\widehat{P}_0 u, u)$ in (2.3.5), and in particular depends on the validity of the *Melin inequality* for $\widehat{P}_0$, since it allows the gain of the $H^{1/2}$ Sobolev norm. Hence, provided the condition for the *strong Melin inequality* (see Hörmander [9], Thm. 22.3.2) holds, that is, *if on* $\pi^{-1}(U) \cap \Sigma$ (which is then the characteristic set of $\widehat{p}_0$ in $U \times (\mathbb{R}^n \setminus \{0\})$, $\widehat{p}_0$ denoting the principal symbol of $\widehat{P}_0$), where $\widehat{p}_0 \geq 0$, one has *the (strong) Melin trace-+ condition*

$$\mathrm{sub}(\widehat{P}_0)(\rho) + \mathrm{Tr}^+ F(\rho) > 0, \quad \forall \rho \in \pi^{-1}(U) \cap \Sigma, \tag{2.4.9}$$

then for all compact $K' \subset U$ *there exist constants* $c, C > 0$ *such that*

$$(\widehat{P}_0 u, u) \geq c\|u\|_{1/2}^2 - C\|u\|_0^2, \quad \forall u \in C_0^\infty(K'). \tag{2.4.10}$$

Here $\mathrm{sub}(\widehat{P}_0)$ is the *subprincipal* symbol of $\widehat{P}_0$ and Tr^+F the *positive trace* of the *Hamilton map* $F = F_{\widehat{P}_0}$ of $\widehat{P}_0$ (the linearization of the Hamilton flow of $\widehat{p}_0$ at the characteristic points, also called *fundamental matrix*). All of them are *symplectic invariants* of $\widehat{P}_0$ (see Hörmander [11]). Notice that since $\widehat{P}_0$ is a *differential* operator, symmetry yields that (2.4.9) is equivalent to

$$|\mathrm{sub}(\widehat{P}_0)(\rho)| < \mathrm{Tr}^+F(\rho), \quad \rho \in \pi^{-1}(U) \cap \Sigma.$$

Hypothesis (H3) will yield $\mathrm{Tr}^+F\big|_{\pi^{-1}(U)\cap\Sigma} > 0$, when U containing x_0 is sufficiently small, and since (as we shall see) $\mathrm{sub}(\widehat{P}_0)\big|_{\pi^{-1}(U)\cap\Sigma} = 0$, Melin's condition (2.4.9) will be fulfilled if (H3) holds. Therefore, in order to prove that (H3) implies the Melin inequality for $\widehat{P}_0$, we first prove that $\mathrm{sub}(\widehat{P}_0)\big|_{\pi^{-1}(U)\cap\Sigma} = 0$ in the general case $k \geq 0$, and not just when $k = 0$ (that is the case covered by (i) of Theorem 2.2.1).

Thus recall that $\widehat{P}_0$ is given by

$$\begin{aligned}
\widehat{P}_0 &= X_0^* X_0 + \sum_{j=1}^{N} (X_j^* f^{2k} X_j - \varepsilon_0^2 [X_j, X_0]^* f^{2k} [X_j, X_0]) \\
&= X_0^* X_0 + \sum_{j=1}^{N} \Big((f^k X_j)^* (f^k X_j) - \varepsilon_0^2 (f^k [X_j, X_0])^* (f^k [X_j, X_0]) \Big),
\end{aligned}$$

and that the subprincipal part coincides with the first order part of its Weyl symbol, that we denote by $\widehat{p}_0^w$. Note that in our notation $\widehat{p}_0^w$ stands for the *total* Weyl symbol, and $\widehat{p}_0$ for the principal symbol of $\widehat{P}_0$, which coincides with the principal symbol of $\widehat{p}_0^w$. For short, we call $p_0(x,\xi)$ the Weyl symbol of $X_0(x,D)$, and $p_j(x,\xi)$, $q_j(x,\xi)$, $1 \leq j \leq N$, the Weyl symbols of $X_j(x,D)$ and $i[X_j, X_0](x,D)$ respectively, that is,

$$\begin{aligned}
p_0(x,\xi) &= e^{-i\langle D_x, D_\xi\rangle/2} X_0(x,\xi) = X_0(x,\xi) + i\ell_0, \\
p_j(x,\xi) &= e^{-i\langle D_x, D_\xi\rangle/2} f^k X_j(x,\xi) = X_j(x,\xi) + i\ell_j, \\
q_j(x,\xi) &= e^{-i\langle D_x, D_\xi\rangle/2} f^k \{X_j, X_0\}(x,\xi) = \{X_j, X_0\}(x,\xi) + ib_j,
\end{aligned}$$

where the ℓ_0, ℓ_j, and b_j are *real*, smooth functions, while $X_j(x,\xi)$, $0 \leq j \leq N$, $\{X_j, X_0\}(x,\xi)$, $1 \leq j \leq N$, are real symbols of order one contained in the symbol class $S^1(\Omega \times \mathbb{R}^n)$. Hence, once again as in the computation in Section 1.3, we have that $\widehat{p}_0^w = \sum_{j=0}^{N} (\bar{p}_j \# p_j - \varepsilon_0^2 \bar{q}_j \# q_j)$, where $q_0 = 0$.

By virtue of the form of $\widehat{p}_0^w$, it suffices to compute in general $(\alpha_1 - i\alpha_0)\#(\alpha_1 + i\alpha_0)$ where $\alpha_1 \in S^1(\Omega \times \mathbb{R}^n)$ is a first order differential symbol with real coefficients and

$\alpha_0 \in S^0(\Omega \times \mathbb{R}^n)$ is a smooth real valued function of x. We have

$$(\alpha_1 - i\alpha_0)\#(\alpha_1 + i\alpha_0) = \alpha_1\#\alpha_1 + i\alpha_1\#\alpha_0 - i\alpha_0\#\alpha_1 + \alpha_0\#\alpha_0$$

$$= \alpha_1^2 + i(\alpha_1\alpha_0 - \alpha_0\alpha_1) + (\text{smooth function in } x),$$

whence

$$\widehat{p}_0^w = X_0^2 + f^{2k}\sum_{j=1}^{N}\big(X_j^2 - \varepsilon_0^2\{X_j, X_0\}^2\big) + r_0,$$

where $r_0 = r_0(x)$ is a smooth real valued function over Ω. Finally, since no first order term is contained in the expression of $\widehat{p}_0^w$, we conclude that $\mathrm{sub}(\widehat{P}_0) = 0$ on $\pi^{-1}(\Omega)$, and thus also on $\pi^{-1}(U) \cap \Sigma$.

Going back to the *the (strong) Melin trace-+ condition* on $\pi^{-1}(U) \cap \Sigma$, we have now that it is equivalent to

$$\mathrm{Tr}^+ F(\rho) > 0, \quad \forall \rho \in \pi^{-1}(U) \cap \Sigma,$$

therefore we have to prove that condition (H3) implies the previous inequality. So, we take $x_0 \in S$ and work inside the compact K_0 determined in Corollary 2.3.2. *Recall that in this case we are supposing* $k = 0$. Hence the operator $\widehat{P}_0$, acting on $C_0^\infty(K_0)$, is given by $\widehat{P}_0 = \sum_{j=0}^N (X_j^* X_j - \varepsilon_0^2 [X_j, X_0]^*[X_j, X_0])$, where $\varepsilon_0 = \varepsilon_0(K_0) > 0$ (see (2.3.8); of course, $[X_0, X_0] = 0$) and the principal symbol $\widehat{p}_0$ of $\widehat{P}_0$ is *nonnegative* on $\pi^{-1}(K_0)$. It makes hence sense to consider the Hamilton map $F(\rho)$ of $\widehat{p}_0$ at points $\rho \in \pi^{-1}(\mathring{K}_0) \cap \Sigma$. For simplicity, we will denote by F_1 the Hamilton map of the operator $\sum_j X_j^* X_j$ and by F_2 that of the operator $\sum_j [X_j, X_0]^*[X_j, X_0]$. Therefore the Hamilton map of $\widehat{P}_0$, and the corresponding *positive trace*, are given respectively by

$$F(\rho) = F_1(\rho) - \varepsilon_0^2 F_2(\rho), \quad \rho \in \pi^{-1}(\mathring{K}_0) \cap \Sigma,$$

and

$$\mathrm{Tr}^+ F(\rho) = \mathrm{Tr}^+\Big(F_1(\rho) - \varepsilon_0^2 F_2(\rho)\Big).$$

Recall that the positive trace of a second order symbol is positively homogeneous of degree 1 in the fibers.

We will show first the connection of (H3) to the condition $\mathrm{Tr}^+ F_1 > 0$, and then to the condition $\mathrm{Tr}^+ F > 0$ (by suitably shrinking K_0 around $x_0 \in S$). Finally we will prove the invariance of $(H3)$ under changes of the basis of the vector space $V(\rho) = \mathrm{Span}\big\{H_0(\rho), \dots, H_N(\rho)\big\}$, for $\rho \in \Sigma$. So, we start by showing that (H3) yields $\mathrm{Tr}^+ F_1(\rho) > 0$ for all $\rho \in \pi^{-1}(W) \cap \Sigma$.

Since the principal symbol of $\sum_j X_j^* X_j$ is given by $\sum_j X_j(x,\xi)^2$, we have (denoting by σ the standard symplectic form on $T^*\Omega$)

$$F_1(\rho)v = \sum_{j=0}^{N} \sigma(v, H_j)H_j, \quad \forall v \in T_\rho T^*\Omega.$$

Let $r = r(\rho) \geqslant 1$ be the dimension of $V(\rho)$, and let $J = J(\rho)$, be a set of indices such that $\{H_j(\rho)\}_{j\in J}$ is a basis of $V(\rho)$, with $\#J = r$. We then have

$$H_j(\rho) = \sum_{k\in J} \gamma_{kj}(\rho)H_k(\rho), \quad j \notin J,$$

so that we may write (dropping for a moment the dependence on ρ in the computations)

$$\begin{aligned}
F_1 v &= \sum_{j=0}^{N} \sigma(v, H_j)H_j = \sum_{j\in J} \sigma(v, H_j)H_j + \sum_{j\notin J} \sigma(v, H_j)H_j \\
&= \sum_{j\in J} \sigma(v, H_j)H_j + \sum_{j\notin J} \Big(\sum_{k\in J} \gamma_{kj}\sigma(v, H_k) \sum_{\ell\in J} \gamma_{\ell j} H_\ell \Big) \\
&= \sum_{j\in J} \sigma(v, H_j)H_j + \sum_{k,\ell\in J} \underbrace{\Big(\sum_{j\notin J} \gamma_{kj}\gamma_{\ell j} \Big)}_{=:\Gamma_{\ell k}=\Gamma_{k\ell}} \sigma(v, H_k)H_\ell \\
&= \sum_{j\in J} \sigma(v, H_j)H_j + \sum_{k,\ell\in J} \Gamma_{\ell k}\sigma(v, H_k)H_\ell \\
&= \sum_{j\in J} \Big(\sigma(v, H_j) + \sum_{k\in J} \Gamma_{jk}\sigma(v, H_k) \Big) H_j \\
&= \sum_{j\in J} \Big(\sigma(v, H_j) + \big(\Gamma\sigma(v, \vec{H})\big)_j \Big) H_j,
\end{aligned}$$

where $\vec{H}$ is the column-vector with entries H_j, $j \in J$, and Γ is the $r \times r$ matrix $[\Gamma_{jk}]_{j,k\in J}$. Obviously, Γ is symmetric and *nonnegative*, for, if we denote by γ the $(N+1-r)\times r$ matrix $[\gamma_{k\ell}]_{k\in J,\ell\notin J}$, then $\Gamma = {}^t\gamma\gamma$. It follows that $I + \Gamma > 0$ because, denoting by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the inner product and norm in $\mathbb{R}^r$,

$$\langle (I+\Gamma)w, w\rangle = |w|^2 + \langle \Gamma w, w\rangle \geq |w|^2, \quad \forall w \in \mathbb{R}^r.$$

Consider now the linear map

$$L = L(\rho)\colon T_\rho T^*\Omega \ni v \longmapsto \begin{bmatrix} \sigma(v, H_{j_1}) \\ \vdots \\ \sigma(v, H_{j_r}) \end{bmatrix} = [\sigma(v, H_j)]_{j\in J} \in \mathbb{R}^r,$$

which is surjective with $\operatorname{Ker} L = \operatorname{Ker} F_1$, and the linear map

$$T = T(\rho)\colon \mathbb{R}^r \ni \zeta = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_r \end{bmatrix} \longmapsto \sum_{j\in J} \Big(\zeta_j + (\Gamma\zeta)_j\Big) H_j \in T_\rho T^*\Omega.$$

Since $T\zeta = \sum_{j\in J}((I+\Gamma)\zeta)_j H_j$ and $I+\Gamma > 0$ we get that T is injective and *surjective* onto $\operatorname{Im} F_1$, that is, $T\colon \mathbb{R}^r \longrightarrow \operatorname{Im} F_1$ is an isomorphism (for each $\rho \in \Sigma$). It is now easy to see, recalling from Section 1.2 the skew-symmetric $r \times r$ matrix M (see (1.1.3)), that

$$(L \circ T)\zeta = -M(I+\Gamma)\zeta, \quad (T \circ L)v = F_1 v.$$

Therefore, considering the *complexified* linear transformations,

$$F_1 v = \lambda v \Longleftrightarrow (T \circ L)v = \lambda v, \tag{2.4.11}$$

for $\lambda \in \mathbb{C}$ and $v \neq 0$ in the complexified $\mathbb{C}T_\rho T^*\Omega = \mathbb{C} \otimes T_\rho T^*\Omega$ of $T_\rho T^*\Omega$. Hence, to study the purely imaginary eigenvalues in the spectrum of F_1 we may use the map $T \circ L$. Since the eigenvectors of F_1 must belong to the complex vector space $\mathbb{C}\operatorname{Im} F_1$, we may take v of the form $v = T\zeta$, $\zeta \in \mathbb{C}^r$, so that the eigenvalue problem (2.4.11) becomes

$$T(L \circ T)\zeta = \lambda T\zeta \Longleftrightarrow -M(I+\Gamma)\zeta = \lambda\zeta,$$

which, by the positivity of $I+\Gamma$ and setting $w := (I+\Gamma)^{1/2}\zeta$, is in turn equivalent to

$$-(I+\Gamma)^{1/2} M (I+\Gamma)^{1/2} w = \lambda w, \quad w \neq 0.$$

Hence $\operatorname{Tr}^+ F_1 = \operatorname{Tr}^+(-(I+\Gamma)^{1/2} M (I+\Gamma)^{1/2})$, and if $\operatorname{rank} M \geq 2$ then $\operatorname{Tr}^+(-(I+\Gamma)^{1/2} M (I+\Gamma)^{1/2}) > 0$ and the same holds for F_1. This shows the connection of (H3) to $\operatorname{Tr}^+ F_1 > 0$.

We next show the connection of (H3) to $\operatorname{Tr}^+ F > 0$ (recall that $k = 0$).

Lemma 2.4.1. *Let $x_0 \in S$ be such that (H3) holds. We may then shrink K_0 to a compact containing x_0 in its interior, that we keep denoting by K_0, in such a way that* $\operatorname{Tr}^+ F(\rho) > 0$ *for all $\rho \in \pi^{-1}(\mathring{K}_0) \cap \Sigma$.*

Proof. Since our argument is perturbative, it will be convenient to let ε vary and consider the family $\widehat{P}_{0,\varepsilon}$, $|\varepsilon| \leq \varepsilon_0(K_0)$, introduced in Remark 2.3.3. We may therefore consider the Hamilton map $F(\rho, \varepsilon)$ of $p_{0,\varepsilon}$ at points $\rho \in \pi^{-1}(\mathring{K}_0) \cap \Sigma$, for $|\varepsilon| \leq \varepsilon_0(K_0)$. Hence

$$F(\rho, \varepsilon) = F_1(\rho) - \varepsilon^2 F_2(\rho), \ \rho \in \Sigma, \ |\varepsilon| < \varepsilon(K_0),$$
$$\mathrm{Tr}^+ F(\rho, \varepsilon) = \mathrm{Tr}^+ \Big(F_1(\rho) - \varepsilon^2 F_2(\rho) \Big),$$

and, for $\varepsilon = 0$,

$$\mathrm{Tr}^+ F(\rho, 0) = \mathrm{Tr}^+ F_1(\rho).$$

We now observe the following. With $U = \mathring{K}_0$, let $\rho_0 \in \pi^{-1}(U) \cap \Sigma =: \Sigma_U$ (the piece of Σ over U) with $x_0 = \pi(\rho_0) \in S$, and suppose that ρ_0 is a point such that $\mathrm{Tr}^+ F_1(\rho_0) > 0$. By the continuity of $(\rho, \varepsilon) \longmapsto \mathrm{Tr}^+ F(\rho, \varepsilon)$, there exist $0 < \delta_0 \leq \varepsilon_0(K_0)$ and a conic (in the fibers) neighborhood $\mathscr{W}(\rho_0) \subset \Sigma_U$ of ρ_0 with a relatively compact base (containing x_0 in its interior), such that

$$\mathrm{Tr}^+ F(\rho, \varepsilon) \geq \mathrm{Tr}^+ F_1(\rho_0)/2, \ \forall (\rho, \varepsilon) \in \mathscr{W}(\rho_0) \times (-\delta_0, \delta_0).$$

Let next $W \subset U$ be a neighborhood of $x_0 \in S$ on which (H3) holds. For all $\rho \in \pi^{-1}(W) \cap \Sigma$ we may find a conic neighborhood $\mathscr{W}(\rho) \subset \Sigma_U$ and $0 < \delta \leq \varepsilon_0(K_0)$, such that $\mathrm{Tr}^+ F(\rho', \varepsilon') \geq \mathrm{Tr}^+ F_1(\rho)/2$ for all $(\rho', \varepsilon') \in \mathscr{W}(\rho) \times (-\delta, \delta)$. Take then a compact $K \subset W$ containing x_0 in its interior, and consider $\pi^{-1}(K) \cap \mathbb{S}^*\Sigma$, where $\mathbb{S}^*\Sigma = \{(x, \xi) \in \Sigma; \ |\xi| = 1\}$ is the cosphere of Σ. Since $\pi^{-1}(K) \cap \mathbb{S}^*\Sigma = \{(x, \xi) \in K \times \mathbb{S}^{n-1}; \ X_j(x, \xi) = 0, \ 0 \leq j \leq N\}$ is compact, we may find an integer $N_0 \geq 1$, a family $\{(\rho_\nu, \delta_\nu)\}_{1 \leq \nu \leq N_0}$ with $\rho_\nu \in \pi^{-1}(K) \cap \mathbb{S}^*\Sigma$ and $\delta_\nu > 0$, and conic neighborhoods $\mathscr{W}(\rho_\nu) \subset \Sigma_U$ as above, $1 \leq \nu \leq N_0$, that form an open covering of $\pi^{-1}(K) \cap \Sigma$ and for which $\mathrm{Tr}^+ F(\rho, \varepsilon) > 0$ for all $(\rho, \varepsilon) \in \mathscr{W}(\rho_\nu) \times (-\delta_\nu, \delta_\nu)$, for all $\nu = 1, \ldots, N_0$. Therefore for all $\rho \in \pi^{-1}(K) \cap \Sigma$ and $|\varepsilon| < \delta_{\min} = \min\{\delta_1, \ldots, \delta_{N_0}\} \leq \varepsilon_0(K_0)$ we have that $\mathrm{Tr}^+ F(\rho, \varepsilon) > 0$. To conclude, we must shrink K, if necessary, in such a way that $\varepsilon_0(K) < \delta_{\min}$. This is possible by Proposition 2.3.1. Thus for the positive trace of the Hamilton map of the operator $\widehat{P}_0$ acting on $C_0^\infty(K)$, we have $\mathrm{Tr}^+ F(\rho) = \mathrm{Tr}^+ F(\rho, \varepsilon_0) > 0$ for all $\rho \in \pi^{-1}(K) \cap \Sigma$. This shows that we may shrink K_0, with $x_0 \in \mathring{K}_0$, in such a way that (H3) implies $\mathrm{Tr}^+ F > 0$ on $\pi^{-1}(K_0) \cap \Sigma$. $\square$

Due to the vanishing of the subprincipal symbol, we have that Lemma 2.4.1 gives the validity of the *(strong) Melin trace-+condition* for $\widehat{P}_0$, therefore, together with the positivity of $\widehat{p}_0(x, \xi)$, we get the following corollary.

Corollary 2.4.2 (Melin Inequality for $\widehat{P}_0$). *Let $x_0 \in S$ be such that (H3) holds at x_0, and let $\widehat{P}_0$ be as in* (2.3.6). *Then there exists a compact set $K_0 \subset \Omega$ containing*

x_0 in its interior such that, for all $K \subset K_0$, there exist two positive constants c and C such that

$$(\widehat{P}_0 u, u) \geq c\|u\|_{1/2}^2 - C\|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.4.12}$$

We finally show the invariance of hypothesis (H3).

Lemma 2.4.3. *For all $\rho \in \Sigma$, hypothesis (H3) does not depend on the choice of the basis of $V(\rho) = \operatorname{Span}\{H_0(\rho), \dots, H_N(\rho)\}$.*

Proof. We keep $\rho \in \Sigma$ fixed, and therefore drop it in the computations. Denote henceforth by J and J', with $\#J = \#J' = r$, two sets of indices such that $\mathsf{H} = \{H_j\}_{j\in J}$ and $\mathsf{H}' = \{H_k\}_{k\in J'}$ are two distinct bases of V. We shall prove that, denoting by M' the skew-symmetric $r \times r$ matrix $[\{X_j, X_{j'}\}]_{j,j'\in J'}$,

$$\operatorname{rank} M = \operatorname{rank} M',$$

and

$$\operatorname{Spec}((I+\Gamma)^{1/2} M (I+\Gamma)^{1/2}) = \operatorname{Spec}((I+\Gamma')^{1/2} M' (I+\Gamma')^{1/2}),$$

where Γ' is the symmetric $r \times r$ matrix constructed using the basis H'. Denote also by T' and L' the linear maps corresponding to the basis H'. Since H' is another basis of V, there esists an invertible $r \times r$ matrix $S = [s_{kj}]_{k\in J', j\in J}$ such that

$$H_j = \sum_{k\in J'} s_{kj} H_k, \quad \forall j \in J.$$

But then we have

$$M_{j\ell} = \sigma(H_j, H_\ell) = \sum_{k,k'\in J'} s_{kj} s_{k'\ell} \sigma(H_k, H_{k'}) = ({}^t S M' S)_{j\ell},$$

that is

$$M = {}^t S M' S, \tag{2.4.13}$$

which proves, S being an isomorphism, that M and M' have the same rank at each fixed ρ.

We finally show the main property concerning the two matrices M and M', that is, that either one can be equivalently used to compute the positive trace of F_1. In the above framework, one has

$$F_1 v = \sum_{j=0}^N \sigma(v, H_j) H_j = \sum_{j\in J} \Big(\sigma(v, H_j) + \sum_{\ell\in J} \Gamma_{j\ell} \sigma(v, H_\ell)\Big) H_j.$$

Therefore,

$$
\begin{aligned}
F_1 v &= \sum_{j\in J}\Big(\sum_{k\in J'} s_{kj}\sigma(v,H_k) + \sum_{\ell\in J}\sum_{k'\in J'}\Gamma_{j\ell}s_{k'\ell}\sigma(v,H_{k'})\Big)\sum_{h\in J'} s_{hj}H_h \\
&= \sum_{h\in J'}\sum_{j\in J}\sum_{k\in J'} s_{kj}s_{hj}\sigma(v,H_k)H_h + \sum_{h,k'\in J'}\sum_{j,\ell\in J} s_{hj}\Gamma_{j\ell}s_{k'\ell}\sigma(v,H_{k'})H_h \\
&= \sum_{h\in J'}\big(S\,{}^t\!S\sigma(v,\vec{H}')\big)_h H_h + \sum_{h\in J'}\big(S\Gamma\,{}^t\!S\sigma(v,\vec{H}')\big)_h H_h \\
&= \sum_{h\in J'}\big(S(I+\Gamma)\,{}^t\!S\sigma(v,\vec{H}')\big)_h H_h,
\end{aligned}
$$

where $\vec{H}'$ is the column vector with entries H_j, $j\in J'$. Since we also have that

$$
F_1 v = \sum_{h\in J'}\big((I+\Gamma')\sigma(v,\vec{H}')\big)_h H_h,
$$

we therefore get the relation

$$
S(I+\Gamma)\,{}^t\!S = I+\Gamma'. \tag{2.4.14}
$$

Hence, by (2.4.13) and (2.4.14), the eigenvalue equation $F_1v=\lambda v$ is equivalent to $-M(I+\Gamma)\zeta=\lambda\zeta$ and to $-M'(I+\Gamma')\zeta'=\lambda\zeta'$, where $\zeta=Tv$ and $\zeta'=T'v$, and this concludes the proof of the lemma. □

2.4.1 Equivalence of (H3) and the Hörmander condition at step 2

We prove here the equivalence of hypothesis (H3) and the Hörmander condition at step 2. The general Hörmander condition at step r on a system of vector fields $\{iX_j\}_{0\le j\le N}$ yields an a priori estimate that we can use to derive the solvability estimate (2.1.1) with $s'=0$ and $s=-1/2$. However, in order to use it to gain a $H^{1/2}$ Sobolev norm, we also need to use the Fefferman-Phong inequality on $\widehat{P}_0$, which is not required if we directly use the Melin inequality to get the result. This is just one reason why the use of the Melin inequality is more efficient. The other motivation is due to the fact that, as we shall see, the validity of the Fefferman-Phong inequality in our case strictly depends on the vanishing of the subprincipal symbol of $\widehat{P}_0$, while in the Melin inequality there is hope to control that part, in case it is nonvanishing, with the positive trace of the fundamental matrix if the latter is nonzero. In general the sub($\widehat{P}_0$) is nonvanishing in the complex case (i.e. when all the vector fields iX_j, for all $j \neq 0$, are supposed to be complex) if no other conditions are imposed. Moreover, the use of the Melin inequality is natural in the more general context of pseudodifferential operators, so the previous approach is more suitable for further generalizations of the result in the pseudodifferential setting. Anyway it is interesting to observe the connection of (H3) with other geometric conditions on the vector fields.

Given a system of vector fields $\{iX_j\}_{0\leq j\leq N}$ on an open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathfrak{g}(\{iX_j\})_{0\leq j\leq N}$ the Lie algebra (over $\mathbb{R}$) generated by the vector fields iX_j (with respect to the usual commutation bracket $[X,Y] = XY - YX$). In other words $\mathfrak{g}(\{iX_j\})_{0\leq j\leq N}$ is the real vector space spanned by all the successive brackets of the vector fields iX_j. We then consider their symbols, so that at any given $x \in \Omega$ we can identify the vector fields (i.e. operators of the first order with no zeroth order term) with *linear forms* on $\mathbb{R}^n$. This way, by freezing the coefficients of the vector fields at a point $x_0 \in \Omega$, we identify $\mathfrak{g}(\{iX_j\})_{0\leq j\leq N}$ (at x_0) with a linear subspace $\mathscr{L}(x_0)$ of the dual of $\mathbb{R}^n$ (which is $\mathbb{R}^n$ itself), and we call *rank* of $\mathfrak{g}$ at the point x_0 the dimension of $\mathscr{L}(x_0)$.

Throughout we shall consider a general system of smooth real vector fields that we denote by $\{iX_j\}_{0\leq j\leq N}$, and we shall call bracket of length r a bracket of the form

$$[iX_{j_1},[iX_{j_2}[...[iX_{j_{r-1}},iX_{j_r}]...]]], \quad 0 \leq j_1,...,j_r \leq N.$$

Definition 2.4.4. *We say that the system $\{iX_j\}_{0\leq j\leq N}$ satisfies the Hörmander condition at step r at $x_0 \in \Omega$ if at x_0 we have rank $\mathfrak{g}(\{iX_j\})_{0\leq j\leq N} = n$ (i.e. $\dim \mathscr{L}(x_0) = n$), and $\mathfrak{g}(\{iX_j\})_{0\leq j\leq N}$ is generated by all the successive brackets of the vector fields iX_j up to length r.*

Remark 2.4.5. *Note that, if the system $\{iX_j\}_{0\leq j\leq N}$ satisfies the Hörmander condition at step r at $x_0 \in \Omega$, then we can find a neighborhood U of x_0 such that the condition still holds in U.*

Recall that $X_j(x,\xi) = \langle \alpha_j(x), \xi\rangle$ denotes the symbol of the first order operator X_j, $0 \leq j \leq N$, and, of course,

$$\{X_j, X_k\}(x,\xi) = \langle [\alpha_j, \alpha_k](x), \xi\rangle$$

is the symbol of the first order operator given by the commutator $i[X_j, X_k]$.

Observe that, if the Hörmander condition at step 2 is true for $\{iX_j\}_{0\leq j\leq N}$ at a point x_0, then $\mathrm{Span}\{iX_0,...,iX_N,[iX_j,iX_k],\ 0 \leq j,k \leq N\}(x_0) = T_{x_0}\Omega$ or, equivalently, $\mathscr{L}(x_0) = \mathrm{Span}\{\alpha_0(x_0),...,\alpha_N(x_0),[\alpha_j,\alpha_k](x_0),\ 0 \leq j,k \leq N\} = \mathbb{R}^n$.

Now, for any given $x \in \Omega$, let

$$\mathscr{L}_2(x) := \mathrm{Span}_{\mathbb{R}}\{\alpha_j(x), [\alpha_j, \alpha_k](x); \quad 0 \leq j, k \leq N\},$$

where $\mathscr{L}_2(x) \subseteq \mathscr{L}(x)$ in general, since we are considering just commutators of length 2 in the generating set, and let

$$\{\alpha_{i_1}(x), \ldots, \alpha_{i_d}(x),\ [\alpha_{j_\ell}, \alpha_{k_\ell}](x);\ 0 \leq \ell \leq m\}$$

be a basis for $\mathscr{L}_2(x)$. In particular we choose d maximal, that is $d = d(x)$ is the largest number of linearly independent vectors in $\{\alpha_j(x)\}_{j=0,\ldots,N}$ at x, and $m = m(x)$ is the number of linearly independent vectors in $\{[\alpha_j, \alpha_k](x)\}_{j,k=0,\ldots,N}$ at x being also linearly independent of $\alpha_{i_1}, ..., \alpha_{i_d}$.Then we consider

$$W(x) := \mathrm{Span}\{\alpha_{i_1}(x), \ldots, \alpha_{i_d}(x)\},$$

$$W_1(x) := \mathrm{Span}\{[\alpha_{j_\ell}, \alpha_{k_\ell}](x);\ 1 \leq \ell \leq m,\ \ 0 \leq j_\ell, k_\ell \leq N\},$$

such that

$$\mathscr{L}_2(x) = W(x) \oplus W_1(x).$$

Note that

$$(x, \xi) \in \Sigma \subset T^*\Omega \setminus 0 \Longleftrightarrow \xi \in W(x)^\perp,\ \xi \neq 0,$$

and that

$$\left\{ \begin{array}{l} \dim W(x_0) < n \\ \dim \mathscr{L}_2(x_0) = n \end{array} \right. \Longleftrightarrow \begin{array}{c} \{iX_j\}_{0 \leq j \leq N} \textit{ satisfies the} \\ \textit{Hörmander condition at step 2 at } x_0. \end{array}$$

Remark 2.4.6 (Equivalence of (H3) at x_0 and $m(x_0) \geq 1$)**.** *Recall that condition (H3) is satisfied at a point x_0 when $\pi^{-1}\{x_0\} \cap \Sigma \neq \emptyset$ and $\operatorname{rank} M(\rho) \geq 2$ for all $\rho \in \pi^{-1}\{x_0\} \cap \Sigma$. Moreover, due to the form of the matrix M, at any point $\rho \in T^*\Omega$ we have $\operatorname{rank} M(\rho) \geq 2 \Longleftrightarrow m(\pi(\rho)) \geq 1$. In fact, to get that $m(\pi(\rho)) \geq 1$ is a sufficient condition to the rank of $M(\rho)$ to be greater than 2, we can observe that, if for some ρ_0 there is at least a pair j, k such that $\{X_j, X_k\}(\rho_0) \neq 0$, that is, if $m(\pi(\rho_0)) \geq 1$, then, by the skew-symmetry of M, we have $\operatorname{rank} M(\rho_0) \geq 2$. Conversely, if $\operatorname{rank} M(\rho_0) \geq 2$ for some ρ_0, then, trivially, one gets $m(\pi(\rho_0)) \geq 1$. Whence, by the previous argument, we get (H3) at $x_0 \Longleftrightarrow m(x_0) \geq 1$.*

Remark 2.4.7. *If $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$, which is always the case when we are under hypothesis (H3), then we have $\dim W(x_0) < n$ (the case $\dim W(x_0) = n$ is covered by point (ii) of Theorem 2.2.1). In fact $\pi^{-1}(x_0) \cap \Sigma = \{x_0\} \times (W(x_0)^\perp \setminus \{0\})$, therefore,* *if* $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$, *then* $\dim W(x_0)^\perp \geq 1$ *and* $\dim W(x_0) < n$. *Conversely, if* $\dim W(x_0) < n$, *then* $\{x_0\} \times (W(x_0)^\perp \setminus \{0\}) = \pi^{-1}(x_0) \cap \Sigma \neq \emptyset$, *thus*

$$\pi^{-1}(x_0) \cap \Sigma \neq \emptyset \iff \dim W(x_0) < n.$$

Therefore Hörmander's condition at step 2 at x_0 *(but not step 1, otherwise* $\dim W(x_0) = n$*) for the system* $\{iX_j\}_{0\leq j\leq N}$ *gives* $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$.

Lemma 2.4.8. *If for some* $x_0 \in \Omega$ *we have* $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ *(i.e.* $\dim W(x_0) < n$*) and* $\dim \mathscr{L}_2(x_0) = n$ *(i.e. if the Hörmander condition at step 2 holds at* x_0 *for* $\{iX_j\}_{0\leq j\leq N}$*), then for every* $\xi \in W(x_0)^\perp$, $\xi \neq 0$, *there exists* $j, k \in \{0, \dots, N\}$ *such that*

$$\{X_j, X_k\}(x_0, \xi) = \langle [\alpha_j, \alpha_k](x_0), \xi\rangle \neq 0.$$

In other words,

$$\begin{cases} \dim W(x_0) < n \\ \dim \mathscr{L}_2(x_0) = n \end{cases} \implies \text{(H3) at } x_0.$$

Proof. Otherwise we would have that there exists $\xi_0 \in W(x_0)^\perp \setminus \{0\}$ such that

$$\xi_0 \in \operatorname{Span}\{[\alpha_j, \alpha_k](x_0);\ 0 \leq j, k \leq N\}^\perp \subset W_1(x_0)^\perp.$$

But then

$$0 \neq \xi_0 \in W(x_0)^\perp \cap W_1(x_0)^\perp = \mathscr{L}_2(x_0)^\perp = \{0\},$$

which is a contradiction. Therefore, since there is at least a pair j, k such that $\{X_j, X_k\}(x_0, \xi) \neq 0$, then $m(x_0) \geq 1$ and (H3) follows by Remark 2.4.6. □

Lemma 2.4.9. *Given* $x_0 \in \Omega$ *we have that*

$$\text{(H3) at } x_0 \implies \begin{cases} \dim W(x_0) < n \\ \dim \mathscr{L}_2(x_0) = n \end{cases}.$$

Proof. Since (H3) holds at x_0 when $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ (see Definition 1.1.1), then, by Remark 2.4.7, we have $\dim W(x_0) < n$.

Suppose now by contradiction that $\dim \mathscr{L}_2(x_0) = k < n$. If $d(x_0) = k$, that is $m(x_0) = 0 \iff W_1(x_0) = \{0\}$, then $\{X_j, X_k\}(x_0, \xi) = 0$ for all $\xi \in W(x_0)^\perp$, which contradics (H3) and we are done.

Suppose then that $d(x_0) < k$ and $m(x_0) = k - d(x_0) \geq 1$. In this case we may find $0 \neq \xi_0 \in W(x_0)$ such that

$$\langle [\alpha_{j_\ell}, \alpha_{k_\ell}](x_0), \xi_0\rangle = 0,\ \ \forall \ell = 1, \dots, m,$$

where $\{[\alpha_{j_\ell}, \alpha_{k_\ell}](x_0)\}_{\ell=1,\dots,m}$ is a basis of $W_1(x_0)$. This follows by writing

$$[\alpha_{j_\ell}, \alpha_{k_\ell}](x_0) = w_\ell + w'_\ell, \quad w_\ell \in W(x_0)^\perp, \; w'_\ell \in W(x_0),$$

where $w_\ell \neq 0$ (for every $\ell = 1, ..., m$) by construction.

Since we supposed $\dim \mathscr{L}_2(x_0) < n$, then $\mathrm{Span}\{w_1, \dots, w_m\} \subsetneq W(x_0)^\perp$ (recall that $\dim \mathrm{Span}\{w_1, \dots, w_m\} = m < n - d = \dim W(x_0)^\perp$), and $W(x_0) \subsetneq \mathrm{Span}\{w_1, \dots, w_m\}^\perp$. It therefore suffices to take any given

$$0 \neq \xi_0 \in W(x_0)^\perp \cap \mathrm{Span}\{w_1, \dots, w_m\}^\perp$$

to get that $\langle [\alpha_{j_\ell}, \alpha_{k_\ell}](x_0), \xi_0 \rangle = 0, \; \forall \ell = 1, \dots, m$. But then, once more, it follows that

$$\{X_j, X_k\}(x_0, \xi_0) = 0, \quad \forall j, k = 1, \dots, N,$$

again contradicting (H3). $\square$

Lemma 2.4.8 and 2.4.9 therefore give the following proposition.

Proposition 2.4.10. *We have that*

$$\begin{array}{c} \{iX_j\}_{0\le j\le N} \text{ satisfies the} \\ \text{Hörmander condition at step 2 at } x_0 \\ \text{(but not step 1)} \end{array} \iff \begin{cases} \dim W(x_0) < n \\ \dim \mathscr{L}_2(x_0) = n \end{cases} \iff \text{(H3) at } x_0.$$

The equivalence given in the previous proposition allows us to prove the solvability estimate (2.1.1) with $s' = 0$ and $s = -1/2$ by using the following Hörmander inequality.

Proposition 2.4.11. *Let $\Omega \subseteq \mathbb{R}^n$ and $\{iX_j\}_{0\le j\le N}$ be a system of vector fields over Ω satisfying the Hörmander condition at step r at a point $x_0 \in \Omega$. Then there exists a neighborhood $U \subset \Omega$ of x_0 and a positive constant $C(r, U)$, such that, for all $u \in C_0^\infty(U)$,*

$$\|u\|_{1/r}^2 \le C\Big(\sum_{j=0}^{N} \|X_j u\|_0^2 + \|u\|_0^2\Big). \tag{2.4.15}$$

The use of (2.4.15) permits to derive for $\widehat{P}_0$ the same estimate given by the Melin inequality, since, in our case, a condition at step 2 is satisfied. However, in order to do that, we also need to apply the Fefferman-Phong inequality on $\widehat{P}_0$, which is always true in virtue of the form of $\widehat{P}_0$, that is, in particular, due to the vanishing of its subprincipal symbol.

Therefore, before giving a complete description of the procedure consisting in the use of the Hörmander inequality, we first prove in the next section the Fefferman-Phong inequality (see [7] and [9]) for $\widehat{P}_0$, which is also needed in the proof of the general case (iii) of Theorem 2.2.1.

2.5 The Fefferman-Phong inequality for $\widehat{P}_0$

This section is devoted to the proof of the Fefferman-Phong inequality for the operator $\widehat{P}_0$ involved in the *main estimate* (when $k \geq 0$), where, recall,

$$\widehat{P}_0 = X_0^* X_0 + \sum_{j=1}^{N} (X_j^* f^k X_j - \varepsilon_0^2 [X_j, X_0]^* f^k [X_j, X_0]).$$

The estimate we are going to prove represents a powerful instrument in the proof of Theorem 2.2.1, since it holds just requiring conditions (H1) and (H2) on the system of vector fields $\{iX_j\}_{0\leq j\leq N}$ contained in the expression of P and $\widehat{P}_0$, and in general in presence of an exponent $k \geq 0$.

Note that we have stronger results for P, that is, we have more than L^2 to L^2 local solvability around S, only when $k = 0$, and also by requiring additional conditions on $\{iX_j\}_{0\leq j\leq N}$.

Recall also that we are studying the local solvability of P around S, thus all our statements are local around each $x_0 \in S$. In particular, we are now working in the compact set K_0 of Corollary 2.3.2, which is chosen in such a way that $\widehat{p}_0(x, \xi)$ in nonnegative on $\pi^{-1}(K_0)$, and which is also such that our main estimate holds for all $u \in C_0^\infty(K_0)$.

Lemma 2.5.1. *We may shrink K_0 to a compact containing x_0 in its interior, that we keep denoting by K_0, so as to have that for any given $k \geq 0$ there exists $C > 0$ such that*

$$(\widehat{P}_0 u, u) \geq -C\|u\|_0^2, \quad \forall u \in C_0^\infty(K_0).$$

Proof. Recall that in $\widehat{P}_0$ we fixed $\varepsilon_0 = \varepsilon_0(K_0)$ so that $\widehat{p}_0$ is nonnegative on $\pi^{-1}(K_0)$. By shrinking K_0 around x_0, keeping x_0 in its interior, and then considering a compact neighborhood K_0' of K_0, we may suppose that $\widehat{p}_0$ is nonnegative on the bigger set $\pi^{-1}(K_0')$. Observe that $[X_j(x, D), X_0(x, D)] = (1/i)\{X_j, X_0\}(x, D)$, and that of course $[X_0(x, D), X_0(x, D)] \equiv 0$. The plan is to be in a position to apply the Fefferman-Phong inequality. To this purpose we extend $\widehat{P}_0$ to an operator with symbol in $S^2_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. We shall then be able to use the Weyl calculus and finish the proof. Let henceforth $\chi \in C_0^\infty(K_0')$ with $\chi \equiv 1$ near K_0 and $0 \le \chi \le 1$, and let

$$L_j(x,D) = f^k \chi X_j(x,D), \quad B_j(x,D) = f^k \chi [X_j(x,D), X_0(x,D)],\ 1 \le j \le N,$$

$$L_0(x,D) = \chi X_0(x,D), \quad B_0(x,D) = 0.$$

Then, with $\mathrm{OPS}^m(\mathbb{R}^n)$ denoting the space of pseudodifferential operators (ψdos, for short) whose symbols are in $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we clearly have that $L_j(x,D), B_j(x,D)$, $0 \le j \le N$, all belong to $\mathrm{OPS}^1(\mathbb{R}^n)$. Furthermore, for all $\varphi \in C_0^\infty(K_0)$ we have $L_0(x,D)\varphi = X_0(x,D)\varphi$ and

$$L_j(x,D)\varphi = f^k X_j(x,D)\varphi,\ B_j(x,D)\varphi = f^k [X_j(x,D), X_0(x,D)]\varphi,$$

for all j. The same holds for their (formal) adjoints. Set

$$A = \sum_{j=0}^{N} \big(L_j^* L_j - \varepsilon_0^2 B_j^* B_j \big) \in \mathrm{OPS}^2(\mathbb{R}^n)$$

(where $B_0 = 0$). Since the L_j and B_j are local operators (that is, they decrease supports), as well as the X_j and X_0, we get that

$$\widehat{P}_0 \varphi = A\varphi, \quad \forall \varphi \in C_0^\infty(K).$$

With $L_0(x,\xi) = \chi(x) X_0(x,\xi)$, resp. $B_0(x,\xi) = 0$, denoting the symbol of $L_0(x,D)$, resp. $B_0(x,D)$, and for $1 \le j \le N$

$$L_j(x,\xi) = f(x)^k \chi(x) X_j(x,\xi),\ \text{resp. } B_j(x,\xi) = -i f(x)^k \chi(x) \{X_j, X_0\}(x,\xi)$$

denoting the symbol of $L_j(x,D)$, resp. $B_j(x,D)$, let us now consider

$$p_0(x,\xi) = e^{-i\langle D_x, D_\xi\rangle/2} L_0(x,\xi) = \chi(x) X_0(x,\xi) + i\ell_0(x),$$

and for $1 \le j \le N$

$$p_j(x,\xi) = e^{-i\langle D_x, D_\xi\rangle/2} L_j(x,\xi) = f(x)^k \chi(x) X_j(x,\xi) + i\ell_j(x),$$

$$q_j(x,\xi) = i e^{-i\langle D_x, D_\xi\rangle/2} B_j(x,\xi) = f(x)^k \chi(x) \{X_j, X_0\}(x,\xi) + i b_j(x),$$

where the ℓ_0, ℓ_j, and b_j are *real*, smooth and supported in K_0'. Then (see Hörmander [11])

$$L_j(x,D) = p_j^{\mathrm{w}}(x,D) =: P_j, \quad B_j(x,D) = -i q_j^{\mathrm{w}}(x,D) =: -iQ_j,\ Q_0 = 0,$$

and

$$A = \sum_{j=0}^{N} \big(P_j^* P_j - \varepsilon_0^2 Q_j^* Q_j\big) = \sum_{j=0}^{N} \big((\bar{p}_j \# p_j)^{\mathrm{w}}(x, D) - \varepsilon_0^2 (\bar{q}_j \# q_j)^{\mathrm{w}}(x, D)\big),$$

with # denoting the symbol composition in the Weyl calculus. Hence the *differential operator* $A = a^{\mathrm{w}}(x, D)$, where $a = \sum_j (\bar{p}_j \# p_j - \varepsilon_0^2 \bar{q}_j \# q_j) \in S^2_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, the principal part of $a\big|_{\pi^{-1}(K_0)}$ is $\widehat{p}_0\big|_{\pi^{-1}(K_0)}$. We next write out the symbol a. By virtue of the form of A, as we have already seen in Section 2.4, it suffices to compute in general $(\alpha_1 - i\alpha_0)\#(\alpha_1 + i\alpha_0)$ where $\alpha_1 \in S^1_{1,0}$ is a first order differential symbol with real coefficients and $\alpha_0 \in S^0_{1,0}$ is a smooth compactly supported real valued function of x. Recall that we have

$$(\alpha_1 - i\alpha_0)\#(\alpha_1 + i\alpha_0) = \alpha_1 \# \alpha_1 + i\alpha_1 \# \alpha_0 - i\alpha_0 \# \alpha_1 + \alpha_0 \# \alpha_0$$

$$= \alpha_1^2 + i(\alpha_1 \alpha_0 - \alpha_0 \alpha_1) + (\text{smooth function in } x),$$

whence

$$a = \chi^2 X_0^2 + f^{2k} \sum_{j=1}^{N} \big(\chi^2 X_j^2 - \varepsilon_0^2 \chi^2 \{X_j, X_0\}^2\big) + r_0,$$

where $r_0 = r_0(x)$ is a smooth real valued function compactly supported in K_0'. Therefore $a(x, \xi) \geq -c$ for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and we may apply the Fefferman-Phong inequality. This concludes the proof. $\square$

Remark 2.5.2. *The validity of the Fefferman-Phong inequality for $\widehat{P}_0$ is allowed by the vanishing of* $\mathrm{sub}(\widehat{P}_0)$.

Recall that the Fefferman-Phong inequality holds for an operator P if its (global) Weyl symbol $p^w(x, \xi)$ is such that there exists a positive constant C for which $p^w(x, \xi) \geq -C$ for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

In our case, since $\mathrm{sub}(\widehat{P}_0) = 0$, *the Weyl symbol p_0^w of $\widehat{P}_0$ does not have part of order one, thus, to get the required estimate for p_0^w, we control the principal part by using Corollary 2.3.2. Conversely, when the subprincipal part is nonzero, like, for example, in the general complex case if no other conditions for the vanishing of the subprincipal part are imposed, also the lower order part of the Weyl symbol counts (the first order part is not zero anymore), thus one could not have the required control of the symbol, and, consequently, the Fefferman-Phong inequality could be untrue.*

2.5.1 The use of the Hörmander inequality on $\widehat{P}_0$

In Section 2.4.1 we proved that, given a system of smooth real vector fields $\{iX_j\}_{0\leq j\leq N}$, we have that condition (H3) at a point x_0 is equivalent to the Hörmander condition at step 2 at x_0 for $\{iX_j\}_{0\leq j\leq N}$. In particular, by Proposition 2.4.11, this means that the Hörmander inequality (2.4.15) holds for the system of vector fields. Moreover, still in Section 2.4.1, we proved that when the odd exponent $p = 2k + 1$ in (1.1.1) is equal to 1, that is when $k = 0$, then if (H3) is satisfied at a point $x_0 \in S$ the Melin inequality holds for $\widehat{P}_0$ in a suitable neighborhood of x_0.

In this subsection we finally show how to use the equivalence of (H3) and the Hörmander condition at step 2 and the Feffermann-Phong inequality to get for $\widehat{P}_0$ the same result as the Melin inequality (recall, when $k = 0$).

Let us consider a point $x_0 \in S$ such that (H3) is satisfied at x_0, and let K_0 be the compact set given in Corollary 2.3.2 and containing x_0 in its interior. Since we are now in the case $k = 0$, we can rewrite the expression of $\widehat{P}_0$ in (2.3.6) in the following equivalent way:

$$\widehat{P}_0 = \frac{1}{2}\sum_{j=0}^{N} X_j^* X_j + \widehat{P}_0'(x, D),$$

where

$$\widehat{P}_0'(x, D) = \sum_{j=0}^{N} \Big(\frac{1}{2} X_j^* X_j - \varepsilon_0^2 [X_0, X_j]^* [X_0, X_j]\Big).$$

Note that $\widehat{P}_0'$ has exactly the same form of $\widehat{P}_0$, thus, possibly by shrinking the compact set K_0 around x_0 in such a way that Corollary 2.3.2 is true for $\widehat{P}_0'$, that is, in such a way that now ε_0^2 is sufficiently small to get the control of the commutator part, we can apply the Fefferman-Phong inequality on $\widehat{P}_0'$ so that, for all $u \in C_0^\infty(K_0)$,

$$(\widehat{P}_0 u, u) = \frac{1}{2}\sum_{j=0}^{N}(X_j u, X_j u) + (\widehat{P}_0'(x, D)u, u) \geq \frac{1}{2}\sum_{j=0}^{N} \|X_j u\|_0^2 - C\|u\|_0^2.$$

Since condition (H3) is true at x_0, which is equivalent to say that the system $\{iX_j\}_{0\leq j\leq N}$ satisfies the Hörmander condition at step 2 at x_0, then, possibly by shrinking (again) K_0 around x_0 to a compact set that we keep denoting by K_0, we can use the inequality (2.4.15) (with $r = 2$) to get, for all $u \in C_0^\infty(K_0)$,

$$(\widehat{P}_0 u, u) \geq C'\|u\|_{1/2}^2 - C''\|u\|_0^2, \quad C', C'' > 0,$$

which is the same estimate given by the Melin Inequality.

Remark 2.5.3. *The same procedure (still when $k = 0$) can be applied when the system of vector fields satisfies the Hörmander condition at step $r \neq 2$, which is not related to condition (H3). In that case we can get a local solvability result for P in* (1.1.1) *in the sense $H^{1/r}$ to L^2, which is clearly less than the one given by using the Melin Inequality or, equivalently, by using a combination of the Fefferman-Phong inequality and the Hörmander condition at step 2. This generalization, given by the combination of the Fefferman-Phong inequality and the Hörmander condition at step r, will be given in the statement of the general complex version of Theorem 2.2.1.*

2.6 Solvability near S with odd degeneracy in the real case: the proof

This section is devoted to the proof of Theorem 2.2.1 concerning the local solvability of P of the form (1.1.1) around the set S when p is odd. As we shall see, the case p even is solved in a different way than the odd case.

Recall also that around the set S the principal symbol of the operator changes sign, therefore it is more difficult and interesting to study the local solvability here.

The proof of the theorem consists in the application of the results given in the previous sections and subsections, therefore, by keeping all together these results in a consistent way we get the proof given below.

Theorem 2.6.1. *Let the operator P in (2.2.2) satisfy hypotheses (H1) and (H2).*

(i) Let $k = 0$. Then for all $x_0 \in S$ with $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ and at which hypothesis (H3) is fulfilled, there exists a compact $K \subset W$ with $x_0 \in U = \overset{\circ}{K}$ such that for all $v \in H^{-1/2}_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have $H^{-1/2}$ to L^2 local solvability).

(ii) Let $k = 0$. Then, for all $x_0 \in S$ for which $\pi^{-1}(x_0) \cap \Sigma = \emptyset$ there exists a compact $K \subset \Omega$ with $x_0 \in U = \overset{\circ}{K}$ such that for all $v \in H^{-1}_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have H^{-1} to L^2 local solvability).

(iii) If $k \geq 1$ and x_0 is any given point of S, or $k = 0$ and $x_0 \in S$ (with $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$) is such that (H3) is not satisfied at x_0, then there exists a compact $K \subset \Omega$ with $x_0 \in U = \overset{\circ}{K}$ such that for all $v \in L^2_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have L^2 to L^2 local solvability).

Proof. The starting point of the proof is given by the use of the *main estimate*. Thus, by Proposition 2.3.1, for all $x_0 \in S$ there exists a compact $K_0 \subset \Omega$, with

$x_0 \in \mathring{K}_0$, constants $c = c(K_0), C = C(K_0) > 0$ and $\varepsilon_0 = \varepsilon_0(K_0)$ with $\varepsilon_0(R) \to 0$ as the compact $R \searrow \{x_0\}$, such that for all compact $K \subset K_0$

$$\|P^* u\|_0^2 \geq \frac{1}{8}\|X_0 u\|_0^2 + c\Big(\widehat{P}_0(x, D)u, u\Big) - C\|u\|_0^2, \tag{2.6.16}$$

for all $u \in C_0^\infty(K_0)$.

By the definition given in Corollary 2.1.6 we have that P of the form (1.1.1) is H^s to $H^{s'}$ locally solvable at x_0 when there exists a compact set $K \subset \Omega$, containing x_0 in its interior $\mathring{K} = U$, such that, for all $u \in C_0^\infty(U)$, the *solvability estimate* (2.1.1) is satisfied. Whence, in order to prove the result, the point is to pass from the *main estimate* to the *solvability estimate* (2.1.1) with suitable values of s and s' according to the case (i), (ii) or (iii).

By using the inequalities proved for $\widehat{P}_0$, we have the following intermediate estimate:

- *For all compact $K \subset K_0$ there are constants $c_1 \geq 0$ and $C_1 > 0$ such that*

  $$\|P^* u\|_0^2 \geq \frac{1}{8}\|X_0 u\|_0^2 + c_1\|u\|_s^2 - C_1\|u\|_0^2, \quad \forall u \in C_0^\infty(K), \tag{2.6.17}$$

  where

  - (a) $c_1 > 0$ *and* $s = 1/2$, *in case the hypotheses of point (i) of Theorem 2.2.1 are fulfilled so that we can apply in* (2.6.16) *the Melin inequality for* $\widehat{P}_0$ *given in Corollary 2.4.2;*
  - (b) $c_1 > 0$ *and* $s = 1$, *in case the hypotheses of point (ii) of Theorem 2.2.1 are fulfilled so that we can apply in* (2.6.16) *the Gårding inequality (see [9], [14]) for* $\widehat{P}_0$ *(due to the ellipticity of* $\widehat{P}_0$*);*
  - (c) $c_1 = 0$ *in case the hypotheses of point (iii) of Theorem 2.2.1 are fulfilled so that we can apply in* (2.6.16) *the Fefferman-Phong inequality for* $\widehat{P}_0$ *given in Lemma 2.5.1.*

The final step to obtain the *solvability estimate* from (2.6.16) is to use the fact that $X_0 \neq 0$ near S and control all the L^2-error terms by means of the following Poincaré inquality (see Appendix A).

Lemma 2.6.2. *We may shrink K_0 around x_0 so that there exists $C_2 = C_2(K_0) > 0$ such that for all compact $K \subset K_0$*

$$\|u\|_0 \leq C_2 \operatorname{diam}(K)\|X_0 u\|_0, \quad \forall u \in C_0^\infty(K). \tag{2.6.18}$$

Therefore, finally, we shrink K_0 to that of Lemma 2.6.2 and it is important to observe at this point that $\varepsilon_0 = \varepsilon_0(K_0)$ is *finally fixed* in relation to K_0. There will not be in fact any additional need to choose it again even if we will have to shrink K_0 further around x_0.

We hence get that for all $K \subset K_0$ (with c_1, C_1 and C_2 the constants in the above lemmas)

$$\|P^*u\|_0^2 \geq \big(\frac{1}{16} - C_1C_2^2 \operatorname{diam}(K)^2\big)\|X_0u\|_0^2 + \frac{1}{16}\|X_0u\|_0^2 + c_1\|u\|_s^2, \quad \forall u \in C_0^\infty(K).$$

At last, we *choose* a compact $K \subset K_0$ so as to have $x_0 \in \mathring{K}$ and

$$0 < \operatorname{diam}(K)^2 \leq \frac{1}{16C_1C_2^2},$$

and get the estimate

$$\|P^*u\|_0^2 \geq c_1\|u\|_s^2 + \frac{1}{16C_2^2 \operatorname{diam}(K)^2}\|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.6.19}$$

This completes the proof of the theorem. $\square$

Remark 2.6.3. *Observe that when we take the Sobolev exponent $-s$, $s > 0$, from (2.6.18) we trivially get $\|u\|_{-s} \leq C_2 \operatorname{diam}(K)\|X_0u\|_0$, for all $u \in C_0^\infty(K)$, and hence from (2.6.19) the estimate $\|P^*u\|_0 \geq C\|u\|_{-s}$, which proves the local solvability near S of $Pu = v$, with $v \in H^s_{\mathrm{loc}}(\Omega)$ and $u \in L^2_{\mathrm{loc}}(\Omega)$. Hence, we do have solvability in Sobolev spaces with positive exponent s, but our approach loses a large number of derivatives.*

Remark 2.6.4. *Observe that our method of proof yields also solvability near S but outside of it.*

2.7 Solvability near S with even degeneracy in the real case

We show in this section that when the degeneracy carried by f is of even type, that is, f^{2k}, we have L^2 *to* L^2 *local solvability near* S. We thus deal in this section with the second order degenerate differential operator (for some integer $k \geq 1$)

$$P = \sum_{j=1}^N X_j^* f^{2k} X_j + iX_0 + a_0, \tag{2.7.2}$$

where the X_j, X_0, a_0 and f are as before. As for the main assumptions, we just assume hypothesis (H4):

(H4) $iX_0 f(x) \neq 0$ *for all* $x \in S$.

Recall that $S = f^{-1}(0) \neq \emptyset$, that $D = -i\partial$, and that $\mathring{K}$ denotes the interior part of the set K.

Remark 2.7.1. *Note that, in this case, the principal symbol of the operator does not change sign in the neighborhood of the set S (more precisely, in the neighborhood of the set $\pi^{-1}(S)$ that is the fiber of S). However, in order to give a complete characterization of the class of degenerate operators we are considering, it is also important to describe the local solvability property when the changing of sign of the principal symbol does not occur, which corresponds to the case $p = 2k$.*

In addition operators in (2.7.2) *could exibhit a degeneracy due to the interplay of the degeneracy of f and that of the system of vector fields $\{iX_j\}_{1\leq j\leq N}$ ($j \neq 0$), and, consequently, the local solvability is not guaranted.*

Theorem 2.7.2. *Let the operator P in (2.7.2) satisfy hypothesis (H4). Then for all $x_0 \in S$ there exists a compact $K \subset \Omega$ with $x_0 \in U = \mathring{K}$ such that for all $v \in L^2_{\rm loc}(\Omega)$ there exists $u \in L^2(\Omega)$ solving $Pu = v$ in U (hence we have L^2 to L^2 local solvability).*

Proof. Since the reduction of the problem to the nonnegativity of an operator $\widehat{P}_0$ is not possible in this case, we shall prove the result through a Carleman estimate, that is, for $\lambda \in \mathbb{R}$ and $\varphi \in C_0^\infty$ we shall estimate $|\mathsf{Re}(P^*\varphi, e^{-2\lambda f}\varphi)|$ from below.

It will be useful to have the following lemma. Recall that our first order operators X_j, $0 \leq j \leq N$, have the form

$$Z(x,D) = \sum_{j=1}^{n} \zeta_j(x) D_{x_j}, \quad \zeta_j \in C^\infty(\Omega;\mathbb{R}).$$

Hence

$$Z(x,D)^* = Z(x,D) + \frac{1}{i}\sum_{j=1}^{n} \frac{\partial \zeta_j}{\partial x_j}(x) =: Z(x,D) - i\mathsf{d}_Z(x),$$

where $\mathsf{d}_Z = \mathrm{div}(iZ) \in C^\infty(\Omega;\mathbb{R})$ is the divergence of the real vector field iZ, which is therefore real-valued. We need the following lemma.

Lemma 2.7.3. *If $g \in C^\infty(\Omega;\mathbb{R})$ and $Z(x,D)$ is as above, we have the formula*

$$2\mathsf{Re}\,(Z\varphi, ig\varphi,) = (\mathsf{d}_Z g\varphi, \varphi) + ((iZg)\varphi, \varphi), \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.7.3}$$

(Observe that d_Z and iZg are both real-valued.)

Proof (of the lemma). In fact,

$$\begin{aligned}(Z\varphi, ig\varphi) &= (\varphi, Z^*(ig\varphi)) = (\varphi, -i\mathsf{d}_Z ig\varphi) + (\varphi, Z(ig\varphi)) \\ &= (\varphi, (iZg)\varphi) - \overline{(Z\varphi, ig\varphi)} + (\varphi, \mathsf{d}_Z g\varphi) \\ &= ((iZg)\varphi, \varphi) - \overline{(Z\varphi, ig\varphi)} + (\mathsf{d}_Z g\varphi, \varphi),\end{aligned}$$

which proves formula (2.7.3). □

Let $\kappa_0 := \mathsf{sgn}(iX_0 f(x_0))$. We next fix a compact $K_0 \subset \Omega$ with $x_0 \in \mathring{K}_0$ such that, by virtue of (H4), we have

$$\mathsf{sgn}(iX_0 f(x)) = \kappa_0, \text{ and } |iX_0 f(x)| \geq |iX_0 f(x_0)|/2 =: c_0 > 0, \quad \forall x \in K_0.$$

Next, since $P^* = \sum_{j=1}^N X_j^* f^{2k} X_j - iX_0^* + \bar{a}_0$, we have for all $\varphi \in C_0^\infty(K_0)$

$$\begin{aligned}\mathsf{Re}\,(P^*\varphi, e^{-2\lambda f}\varphi) &= \sum_{j=1}^N \mathsf{Re}\,(f^{2k} X_j \varphi, X_j(e^{-2\lambda f}\varphi)) \\ &\quad -\mathsf{Re}\,(iX_0^*\varphi, e^{-2\lambda f}\varphi) + \mathsf{Re}\,(\bar{a}_0\varphi, e^{-2\lambda f}\varphi) \\ &= \sum_{j=1}^N \|f^k e^{-\lambda f} X_j \varphi\|_0^2 - 2\lambda \sum_{j=1}^N \mathsf{Re}\,(f^{2k} X_j\varphi, (X_j f) e^{-2\lambda f}\varphi) \\ &\quad +\mathsf{Re}\,(X_0\varphi, ie^{-2\lambda f}\varphi) - \mathsf{Re}\,(\mathsf{d}_{X_0}\varphi, e^{-2\lambda f}\varphi) + \mathsf{Re}\,(\bar{a}_0\varphi, e^{-2\lambda f}\varphi)\end{aligned}$$

(by (2.7.3))

$$\begin{aligned}&= \sum_{j=1}^N \|f^k e^{-\lambda f} X_j \varphi\|_0^2 - 2\lambda \sum_{j=1}^N \mathsf{Re}\,(f^{2k} X_j\varphi, (X_j f) e^{-2\lambda f}\varphi) \\ &+\frac{1}{2}(\mathsf{d}_{X_0} e^{-2\lambda f}\varphi, \varphi) - \lambda((iX_0 f) e^{-2\lambda f}\varphi, \varphi) - \mathsf{Re}\,(\mathsf{d}_{X_0}\varphi, e^{-2\lambda f}\varphi) + \mathsf{Re}\,(\bar{a}_0\varphi, e^{-2\lambda f}\varphi) \\ &= \sum_{j=1}^N \|f^k e^{-\lambda f} X_j \varphi\|_0^2 - 2\lambda \sum_{j=1}^N \mathsf{Re}\,(f^{2k} X_j\varphi, (X_j f) e^{-2\lambda f}\varphi) \\ &\quad -\frac{1}{2}(\mathsf{d}_{X_0} e^{-2\lambda f}\varphi, \varphi) - \lambda((iX_0 f) e^{-2\lambda f}\varphi, \varphi) + \mathsf{Re}\,(\bar{a}_0\varphi, e^{-2\lambda f}\varphi),\end{aligned}$$

whence it follows

$$|\mathsf{Re}\,(P^*\varphi, e^{-2\lambda f}\varphi)| \geq \mathsf{Re}\,(P^*\varphi, e^{-2\lambda f}\varphi) \tag{2.7.4}$$

$$= \sum_{j=1}^{N} \|f^k e^{-\lambda f} X_j \varphi\|_0^2 - 2\lambda \sum_{j=1}^{N} \mathsf{Re}\,(f^{2k} X_j\varphi, (X_j f) e^{-2\lambda f}\varphi)$$
$$-\frac{1}{2}\,(\mathsf{d}_{X_0} e^{-2\lambda f}\varphi, \varphi) - \lambda\kappa_0\,(|iX_0 f| e^{-2\lambda f}\varphi, \varphi) + \kappa_0 \mathsf{Re}\,(\bar{a}_0\varphi, e^{-2\lambda f}\varphi).$$

Case $\kappa_0 = 1$**.** In this case we take $\lambda = -|\lambda| < 0$ and get from (2.7.4), for any given $\delta > 0$ and for all $\varphi \in C_0^\infty(K_0)$,

$$|\mathsf{Re}\,(P^*\varphi, e^{2|\lambda| f}\varphi)| \geq \big(1 - \delta|\lambda|\big) \sum_{j=1}^{N} \|f^k e^{|\lambda| f} X_j\varphi\|_0^2$$
$$-\frac{|\lambda|}{\delta} \sum_{j=1}^{N} \|f^k (X_j f) e^{|\lambda| f}\varphi\|_0^2 + c_0|\lambda| \Big(1 - \frac{\|\mathsf{d}_{X_0}\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{c_0|\lambda|}\Big) \|e^{|\lambda| f}\varphi\|_0^2.$$

We next choose $|\lambda| = \lambda_0 \geq 1$ so large that $1 - \frac{\|\mathsf{d}_{X_0}\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{c_0\lambda_0} \geq 1/2$ and $\delta = \delta_0 > 0$ so small that $1 - \delta_0\lambda_0 \geq 1/2$. Hence *for every compact $K \subset K_0$ and all $\varphi \in C_0^\infty(K)$ we have*

$$|\mathsf{Re}\,(P^*\varphi, e^{2\lambda_0 f}\varphi)| \geq \Big(\frac{c_0\lambda_0}{2} - \frac{\lambda_0}{\delta_0}\big(\sum_{j=1}^{N} \|X_j f\|_{L^\infty(K_0)}^2\big)\|f\|_{L^\infty(K)}^{2k}\Big) \|e^{\lambda_0 f}\varphi\|_0^2.$$

We now choose a compact $K \subset K_0$ with $x_0 \in \mathring{K}$ such that

$$\frac{c_0}{2} - \frac{1}{\delta_0}\big(\sum_{j=1}^{N} \|X_j f\|_{L^\infty(K_0)}^2\big)\|f\|_{L^\infty(K)}^{2k} \geq \frac{c_0}{4},$$

and get

$$e^{2\lambda_0\|f\|_{L^\infty(K_0)}} \|P^*\varphi\|_0 \|\varphi\|_0 \geq \frac{c_0\lambda_0}{4} e^{-2\lambda_0\|f\|_{L^\infty(K_0)}} \|\varphi\|_0^2, \quad \forall \varphi \in C_0^\infty(K),$$

whence the existence of a constant $C = c_0 e^{-4\lambda_0\|f\|_{L^\infty(K_0)}}/4 > 0$ such that

$$\|P^*\varphi\|_0 \geq C\|\varphi\|_0, \quad \forall \varphi \in C_0^\infty(K). \tag{2.7.5}$$

Case $\kappa_0 = -1$**.** In this case we take $\lambda = |\lambda| > 0$ and, by exactly the same considerations we have just seen, get from (2.7.4) that for the same compact K determined in the previous case and the same constant $C > 0$ the local L^2-solvability estimate (2.7.5). This concludes the proof of the theorem. □

Remark 2.7.4. *It is interesting to see what one obtains by using this method in the case the degeneracy p is of odd type, i.e. of the kind f^{2k+1}, under assumption (H1). Consider therefore $P = \sum_{j=1}^{N} X_j^* f^p X_j + iX_0 + a_0$, p odd, and write*

$$\begin{aligned}
\mathsf{Re}(P^*\varphi, e^{-2\lambda f}\varphi) &= \sum_{j=1}^{N}(f^p X_j\varphi, X_j(e^{-2\lambda f}\varphi)) \\
&\quad -\mathsf{Re}(iX_0^*\varphi, e^{-2\lambda f}\varphi) + \mathsf{Re}(\bar{a}_0 e^{-\lambda f}\varphi, e^{-\lambda f}\varphi) \\
&= \sum_{j=1}^{N}\Big[\underbrace{(f^p X_j\varphi, e^{-2\lambda f}X_j\varphi)}_{=:I_1} - 2\lambda\underbrace{(f^p X_j\varphi, e^{-2\lambda f}(X_j f)\varphi)}_{=:I_2}\Big] \\
&\quad +\mathsf{Re}(X_0\varphi, ie^{-2\lambda f}\varphi) - \mathsf{Re}(\mathsf{d}_{X_0}\varphi, e^{-2\lambda f}\varphi) + \mathsf{Re}(\bar{a}_0 e^{-\lambda f}\varphi, e^{-\lambda f}\varphi).
\end{aligned}$$

The problem is now to use the Cauchy-Schwarz inequality to control I_2 by I_1, which must then necessarily be positive. But a term $I_1^{(j)}$ in I_1 is of the form

$$I_1^{(j)} = \int_K f^p |X_j\varphi|^2 e^{-2\lambda f} dx = \int_K \mathsf{sgn}(f)\Big(|f|^{p/2}|X_j\varphi|\Big)^2 e^{-2\lambda f} dx,$$

and we have $I_1^{(j)} > 0$ for any given $\varphi \in C_0^\infty(K)$ (recall that $x_0 \in \mathring{K}$, so K has a non-empty interior part) for sure in case

$$\mathsf{sgn}(f) = 1 \quad \textit{almost everywhere in } \mathring{K}.$$

Hence, the solvability estimate may still hold for large λ in case p is odd and (H1) is fulfilled only provided we choose very particular compact sets K, namely those compact K that, roughly speaking, are concentrated (since f is smooth and has a regular zero-set) in the set $f^{-1}(0, +\infty)$.

2.8 Solvability off S in the real case

In this section we prove the L^2 to L^2 local solvability of P in the complement of S in Ω (regardless the parity of p), so we consider, with $p \geq 1$ an integer,

$$P = \sum_{j=1}^{N} X_j^* f^p X_j + iX_0 + a_0. \tag{2.8.2}$$

The only assumption in this case is:

(H5) *For every $x_0 \in \Omega \setminus S$ there exists j_0, $1 \leq j_0 \leq N$, such that $\alpha_{j_0}(x_0) \neq 0$ (i.e., the vector field iX_{j_0} is nonsingular at x_0).*

Remark 2.8.1. *The principal symbol of an operator of the form* (2.8.2) *does not change sign in a sufficiently small neighborhood of $\Omega \setminus S$ (to be precise, in a sufficiently small neighborhood of $\pi^{-1}(\Omega \setminus S)$). However, even if the degeneracy due to the function f is not present here, the class described by* (2.8.2) *also contains degenerate operators, since we do not impose nondegeneracy conditions on the system vector fields $\{iX_j\}_{1 \leq j \leq N}$ ($j \neq 0$).*

Theorem 2.8.2. *Let the operator P in (2.8.2) satisfy hypothesis (H5). Then P is L^2 to L^2 locally solvable in $\Omega \setminus S$.*

Proof. Let $x_0 \in \Omega \setminus S$. Let $U \subset \Omega \setminus S$ be an open connected neighborhood of x_0 on which iX_{j_0} is nonsingular for all $x \in U$ and take $K_0 \subset U$ to be a connected compact, with $x_0 \in \mathring{K}_0$, on which we have the Poincaré inequality (A.0.2) for iX_{j_0}, for all compact sets $K \subset K_0$.

Since for all $x \in K_0$ we may write $f(x) = \mathsf{sgn}(f(x_0))|f(x)|$, we have

$$\forall x \in K_0,\ f(x)^p = (\mathsf{sgn}(f(x_0)))^p |f(x)|^p = \kappa_0 |f(x)|^p,$$

where

$$\kappa_0 = \begin{cases} 1, \text{ if } p \text{ is even,} \\ \mathsf{sgn}(f(x_0)), \text{ if } p \text{ is odd.} \end{cases}$$

Hence, on K_0, writing $\tilde{X}_j := |f(x)|^{p/2} X_j$, $1 \leq j \leq N$ (whence $i\tilde{X}_{j_0}$ is nonsingular as well as iX_{j_0} on K_0), we have

$$P = \kappa_0 \sum_{j=1}^{N} X_j^* (|f(x)|^{p/2})^2 X_j + iX_0 + a_0 = \kappa_0 \sum_{j=1}^{N} \tilde{X}_j^* \tilde{X}_j + iX_0 + a_0.$$

We next choose $g \in C^\infty(U; \mathbb{R})$, and put $\gamma_0 := \max_{x \in K_0} |g(x)|$. By the same method we followed in the preceding section, we have, for all $\varphi \in C_0^\infty(K_0)$,

$$|\mathsf{Re}(\kappa_0 P^* \varphi, e^{2g} \varphi)| \geq \sum_{j=1}^{N} \|e^g \tilde{X}_j \varphi\|_0^2 + 2 \sum_{j=1}^{N} \mathsf{Re}(\tilde{X}_j \varphi, (\tilde{X}_j g) e^{2g} \varphi)$$

$$- \frac{\kappa_0}{2} (\mathsf{d}_{X_0} e^{2g} \varphi, \varphi) + \kappa_0 ((iX_0 g) e^{2g} \varphi, \varphi) + \kappa_0 \mathsf{Re}(\bar{a}_0 \varphi, e^{2g} \varphi)$$

$$\geq \frac{1}{2}\sum_{j=1}^{N} \|e^g \tilde{X}_j\varphi\|_0^2 - 2\sum_{j=1}^{N} \|e^g(\tilde{X}_j g)\varphi\|_0^2$$

$$- \underbrace{\Big(\frac{1}{2}\|\mathsf{d}_{X_0}\|_{L^\infty(K_0)} + \|iX_0 g\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}\Big)}_{=:C_1} \|e^g\varphi\|_0^2$$

$$\geq \frac{1}{2}e^{-2\gamma_0}\|\tilde{X}_{j_0}\varphi\|_0^2 - \Big(\underbrace{2N \max_{1\leq j\leq N} \|\tilde{X}_j g\|^2_{L^\infty(K_0)}}_{=:C_2} + C_1\Big)\|e^g\varphi\|_0^2$$

Now, using the Poincaré inequality (A.0.2) for $i\tilde{X}_{j_0}$ we have that for a constant $C_0 = C_0(K_0) > 0$ and all compact $K \subset K_0$

$$|\mathsf{Re}(\kappa_0 P^*\varphi, e^{2g}\varphi)| \geq \frac{1}{2}e^{-2\gamma_0}C_0 \mathrm{diam}(K)^{-2}\|\varphi\|_0^2 - (C_1 + C_2)e^{2\gamma_0}\|\varphi\|_0^2$$

$$= \mathrm{diam}(K)^{-2}\Big(\frac{C_0 e^{-2\gamma_0}}{2} - \mathrm{diam}(K)^2 e^{2\gamma_0}(C_1 + C_2)\Big)\|\varphi\|_0^2, \quad \forall\varphi \in C_0^\infty(K).$$

Hence, we may choose $K \subset K_0$ with $x_0 \in \mathring{K}$ such that

$$\frac{C_0 e^{-2\gamma_0}}{2} - \mathrm{diam}(K)^2 e^{2\gamma_0}(C_1 + C_2) \geq \frac{C_0}{4e^{2\gamma_0}},$$

whence

$$\|P^*\varphi\|_0 e^{2\gamma_0}\|\varphi\|_0 \geq \frac{C_0}{4e^{2\gamma_0}\mathrm{diam}(K)^2}\|\varphi\|_0^2, \quad \forall\varphi \in C_0^\infty(K),$$

that is,

$$\|P^*\varphi\|_0 \geq \frac{C_0}{4e^{4\gamma_0}\mathrm{diam}(K)^2}\|\varphi\|_0, \quad \forall\varphi \in C_0^\infty(K),$$

which concludes the proof. □

2.9 Odd degeneracy: the general complex case

In this section we wish to extend Theorem 2.2.1 to a case in which the vector fields are complex. More precisely, we consider operators $X_0(x, D)$, $X_1(x, D), \ldots, X_N(x, D)$ as before where $iX_0(x, D)$ is *real* and satisfies (H1) but where the $iX_1(x, D), \ldots, iX_N(x, D)$ are now allowed to be *complex*. So we write $X_0(x, \xi) = \langle\alpha_0(x), \xi\rangle$ and

$$X_j(x, \xi) = \langle\alpha_{2j-1}(x) + i\alpha_{2j}(x), \xi\rangle, \quad \alpha_0, \alpha_1, \ldots, \alpha_{2N} \in C^\infty(\Omega, \mathbb{R}^n).$$

Besides (H1) we shall assume:

(H2') *For all* $j = 1, \dots, N$ *and for all compact* $K \subset \Omega$ *there exists* $C_{K,j} > 0$ *such that*

$$|\{X_j, X_0\}(x,\xi)|^2 \leq C_{K,j} \sum_{j'=0}^{N} |X_{j'}(x,\xi)|^2, \quad \forall (x,\xi) \in \pi^{-1}(K) \setminus 0;$$

(H3') *For all compact* $K \subset \Omega$ *there exists* $C_K > 0$ *such that*

$$|\sum_{j=1}^{N} \{\bar{X}_j, X_j\}(x,\xi)|^2 \leq C_K \sum_{j=0}^{N} |X_j(x,\xi)|^2, \quad \forall (x,\xi) \in \pi^{-1}(K) \setminus 0;$$

(H4') *For all compact* $K \subset \Omega$ *there exists* $C_K > 0$ *such that*

$$|\sum_{j=1}^{N} \{\overline{\{X_j, X_0\}}, \{X_j, X_0\}\}(x,\xi)|^2 \leq C_K \sum_{j=0}^{N} |X_j(x,\xi)|^2, \quad \forall (x,\xi) \in \pi^{-1}(K) \setminus 0.$$

Remark 2.9.1. *Note that condition (H4') is, by the Jacobi identity, a condition on commutators of length* 4*, that is of the kind* $[a,[b,[c,d]]]$*. In fact we have* $[[a,b],[c,d]] = [a,[b,[c,d]]] - [b,[a,[c,d]]]$.

Given $r \geq 1$ an integer and $x \in \Omega$, we denote by

$$\mathscr{L}_r(x) = \mathrm{Span}_{\mathbb{R}}\{\alpha_0(x), \dots, \alpha_{2N}(x),$$
$$[\alpha_{j_1}, [\alpha_{j_2}, \dots, [\alpha_{j_{h-1}}, \alpha_{j_h}] \dots]](x);\ 0 \leq j_1, \dots, j_h \leq 2N,\ 1 \leq h \leq r\}$$

the Lie algebra generated by the vector fields $\alpha_0, \dots, \alpha_{2N}$ and their commutators of length at most r at x. Hence, if $\dim \mathscr{L}_r(x) = n$ at every point of Ω, the sum of squares of vector fields $\sum_{j=0}^{N} X_j^* X_j$ satisfies the Hörmander condition at step r and hence we have the subelliptic estimate of Proposition 2.4.11, that we recall here this way: *For all compact* $K \subset \Omega$ *there exists* $C_K > 0$ *such that*

$$\|u\|_{1/r}^2 \leq C_K \Big((\sum_{j=0}^{N} X_j^* X_j u, u) + \|u\|_0^2 \Big), \quad \forall u \in C_0^\infty(K). \tag{2.9.3}$$

Notice that, once we have a compact K on which (2.9.3) holds, then the same constant may be used for all compact $K' \subset K$.

We shall prove in this section the following theorem (recall that $S = f^{-1}(0)$).

Theorem 2.9.2. *Let* P *be given as in (2.2.2) satisfying hypotheses (H1), (H2'), (H3') and (H4') above.*

- *If $k \geq 0$, then for all $x_0 \in S$ there exists a compact $K \subset \Omega$ with $x_0 \in \mathring{K} = U$ such that L^2 to L^2 local solvability holds on U.*
- *If $k = 0$, suppose in addition that there exists $r \geq 1$ such that for all $x_0 \in S$ for which $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$ there exists a neighborhood $W_{x_0} \subset \Omega$ of x_0 such that*

  $$\dim \mathscr{L}_r(x) = n, \quad \forall x \in W_{x_0}. \tag{2.9.4}$$

  Then for each $x_0 \in S$ there exists a compact $K \subset \Omega$ with $x_0 \in \mathring{K} = U$ such that we have $H^{-1/r}$ to L^2 solvability on U.

Of course, as we have already seen, at the points x_0 such that $\pi^{-1}(x_0) \cap \Sigma = \emptyset$ we have H^{-1} to L^2 local solvability. Therefore in proving the second part of Theorem 2.9.2 we may restrict ourselves to considering only points with fiber intersecting the characteristic set Σ of the system of first order operators $(X_0, \ldots, X_N)$. The proof will use the main estimate (2.3.5), which holds *also* in this case of complex vector fields, the main point being that the symbol $X_0(x, \xi)$ is still *real*, and a blend of the subelliptic estimate and the Fefferman-Phong inequality to control the operator $\widehat{P}_0$ given in (2.3.6).

It will be convenient to have the following two lemmas, which extend the Fefferman-Phong inequality in case of addition of first order operators, that will be stated directly in the Weyl-Hörmander calculus.

Lemma 2.9.3. *Let g be a Hörmander metric on $\mathbb{R}^n \times \mathbb{R}^n$ and let h be the associated Planck's function. Let $p_2 \in S(h^{-2}, g)$ and $p_1 \in S(h^{-1}, g)$ be real symbols. Suppose that there are constants $c_0, c_1, c_2 > 0$ such that*

$$p_2(x, \xi) \geq -c_0, \quad p_1(x, \xi)^2 \leq c_1(p_2(x, \xi) + c_2), \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Then there exists $C > 0$ such that $p_2^{\mathrm{w}}(x, D) + p_1^{\mathrm{w}}(x, D) \geq -C$.

Proof. We simply write

$$p_2(x, \xi) + p_1(x, \xi) = p_2(x, \xi) - \frac{p_1(x, \xi)^2}{c_1} + \Big(\frac{p_1(x, \xi)}{\sqrt{c_1}} + \frac{\sqrt{c_1}}{2}\Big)^2 - \frac{1}{4}.$$

Thus, by using the hypothesis giving the control of $p_1(x, \xi)^2$, we get

$$p_2(x, \xi) + p_1(x, \xi) \geq -c_2 - \frac{c_1}{4} := -C,$$

therefore the Fefferman-Phong inequality holds for $p_2^{\mathrm{w}}(x, D) + p_1^{\mathrm{w}}(x, D)$ and the result follows. ∎

Lemma 2.9.4. *Let $p_j \in S(h^{-1}, g)$, $0 \le j \le N$. Suppose that p_0 is real whereas $p_1, \dots, p_N$ are complex. Suppose there are constants $C_1, C_2 > 0$ such that*

$$\text{(i)} \qquad |\{p_j, p_0\}|^2 \le C_1 \sum_{j'=0}^{N} |p_{j'}|^2 + C_2, \quad 1 \le j \le N;$$

$$\text{(ii)} \qquad |\sum_{j=1}^{N} \{\bar{p}_j, p_j\}|^2 \le C_1 \sum_{j'=0}^{N} |p_{j'}|^2 + C_2;$$

$$\text{(iii)} \qquad |\sum_{j=1}^{N} \{\overline{\{p_j, p_0\}}, \{p_j, p_0\}\}|^2 \le C_1 \sum_{j'=0}^{N} |p_{j'}|^2 + C_2.$$

Put $P_j = p_j^{\mathrm{w}}(x, D)$, $Q_j = \{p_j, p_0\}^{\mathrm{w}}(x, D)$, $0 \le j \le N$ (note that $Q_0 = 0$). Then there exists $\varepsilon_* \in (0, 1]$ and $C > 0$ such that for all $\varepsilon_0 \in (0, \varepsilon_*)$ we have

$$A := a^{\mathrm{w}}(x, D) = \sum_{j=0}^{N} (P_j^* P_j - \varepsilon_0^2 Q_j^* Q_j) \ge -C.$$

Proof. By the calculus, we have $P_j^* P_j = (\bar{p}_j \sharp p_j)^{\mathrm{w}}(x, D)$, and likewise for $Q_j^* Q_j$. Since

$$\bar{p}_j \sharp p_j = |p_j|^2 - \frac{i}{2} \{\bar{p}_j, p_j\} + r_{0,j},$$

$$\overline{\{p_j, p_0\}} \sharp \{p_j, p_0\} = |\{p_j, p_0\}|^2 - \frac{i}{2} \{\overline{\{p_j, p_0\}}, \{p_j, p_0\}\} + r'_{0,j},$$

where the errors $r_{0,j}, r'_{0,j} \in S(1, g)$ are *real-valued*, we have that $a = a_2 + a_1$ where

$$a_2 = \sum_{j=0}^{N} \Big(|p_j|^2 - \varepsilon_0^2 |\{p_j, p_0\}|^2 \Big) \in S(h^{-2}, g),$$

$$a_1 = -\frac{i}{2} \sum_{j=0}^{N} \Big(\{\bar{p}_j, p_j\} - \varepsilon_0^2 \{\overline{\{p_j, p_0\}}, \{p_j, p_0\}\} \Big) + S(1, g) \in S(h^{-1}, g).$$

Note that a_1 is *real-valued.* For ε_0 sufficiently small we thus have, for some constants $c_1, c_2, c_3, c_4 > 0$,

$$a_2 \ge c_1 \sum_{j=0}^{N} |p_j|^2 - c_2, \quad a_1^2 \le c_3 \sum_{j=0}^{N} |p_j|^2 + c_4$$

and hence

$$a_1^2 \leq \frac{c_3}{c_1}(a_2 + c_2) + c_4.$$

Therefore Lemma 2.9.3 yields the result. □

Remark 2.9.5. *Note that, since $a_1(x,\xi)$ is not identically zero, the subprincipal symbol of $\widehat{P}_0$ is in general different from zero in the complex setting. In fact, in this case, unlike the real one where the subprincipal symbol of $\widehat{P}_0$ is always zero, we need additional hypotheses on the system of vector fields $\{iX_j\}_{0\leq j\leq N}$ in order to apply Lemma 2.9.3 and get the Fefferman-Phong inequality for $\widehat{P_0}$, which is fundamental to prove the local solvability result.*

Proof of Theorem 2.9.2. The proof follows exactly the same lines of the proof of Theorem 2.2.1. The main estimate (2.3.5) holds also in this complex case (by virtue of the fact that $X_0(x,\xi)$ is still supposed to be *real*), so that for all $x_0 \in S$ there exists a compact $K_0 \subset \Omega$ with $x_0 \in \overset{\circ}{K}_0$, constants $c, C > 0$ and $\varepsilon_0 = \varepsilon_0(K_0) > 0$ as in Proposition 2.3.1 such that for all compact $K \subset K_0$

$$\|P^*u\|_0^2 \geq \frac{1}{8}\|X_0u\|_0^2 + c(\widehat{P}_0u, u) - C\|u\|_0^2, \quad \forall u \in C_0^\infty(K), \tag{2.9.5}$$

where

$$\widehat{P}_0 = X_0^*X_0 + \sum_{j=1}^N \Big(X_j^* f^{2k} X_j - \varepsilon_0^2 [X_j, X_0]^* f^{2k} [X_j, X_0]\Big).$$

When $k = 0$ (hence no f is present in the expression of $\widehat{P}_0$) we write

$$\widehat{P}_0 = \frac{1}{2}\sum_{j=0}^N X_j^*X_j + \sum_{j=0}^N \Big(\frac{1}{2}X_j^*X_j - \varepsilon_0^2[X_j, X_0]^*[X_j, X_0]\Big) =: \widehat{P}_0' + \widehat{P}_0''. \tag{2.9.6}$$

Now, proceeding exactly as in Lemma 2.5.1, using hypoetheses (H2'), (H3') and (H4') and Lemmas 2.9.3 and 2.9.4 we have that, suitably shrinking K_0 (and hence ε_0) and for a suitable constant $C_0 > 0$,

$$(\widehat{P}_0''u, u) \geq -C_0\|u\|_0^2, \quad \forall u \in C_0^\infty(K_0).$$

Therefore, whatever $k \in \mathbb{Z}_+$, we have from (2.9.5) that for all compact $K \subset K_0$

$$\|P^*u\|_0^2 \geq \frac{1}{8}\|X_0u\|_0^2 + \frac{c}{2}\sum_{j=0}^N \|f^kX_ju\|_0^2 - (cC_0 + C)\|u\|_0^2, \quad \forall u \in C_0^\infty(K). \tag{2.9.7}$$

One therefore ends the proof as in Theorem 2.2.1 by using a Poincaré inequality for X_0 to select a compact $K \subset K_0$ on which the L^2 error-term can be controlled, so as to obtain ($C_K > 0$)

$$\|P^*u\|_0^2 \geq C_K\|u\|_0^2, \quad \forall u \in C_0^\infty(K),$$

which gives L^2 to L^2 local solvability in the interior of K.

It remains to see what happens when $k = 0$ and condition (2.9.4) holds. In this case if $\pi^{-1}(x_0) \cap \Sigma = \emptyset$, the system of vector fields $(iX_0, \ldots, iX_N)$ is elliptic and the Gårding inequality gives, using the Poincaré inequality for X_0 to control L^2 errors as before, the existence of a compact $K \subset K_0$ on which the estimate for the H^{-1} to L^2 local solvability holds as in the proof of Theorem 2.2.1, which in particular says that we also have $H^{-1/r}$ to L^2 local solvability in the interior of K with $r = 1$, since $\dim \mathscr{L}_r(x) = n$ near x_0 then, and we are done. Otherwise, if $\pi^{-1}(x_0) \cap \Sigma \neq \emptyset$, we exploit hypothesis (2.9.4) to obtain the subelliptic estimate (2.9.3) and hence get from (2.9.7) (on a suitable compact $K_0 \subset W_{x_0}$, containing x_0 in its interior, on which also the Fefferman-Phong inequality can be used) the estimate

$$\|P^*u\|_0^2 \geq \frac{1}{8}\|X_0u\|_0^2 + C_{K_0}\|u\|_{1/r}^2 - (cC_0 + C)\|u\|_0^2, \quad \forall u \in C_0^\infty(K),$$

for all compact sets $K \subset K_0$ which contain x_0 in their interior. Therefore, using once more the Poincaré estimate for X_0 to control L^2 errors, the existence of a suitable compact $K \subset K_0 \subset W_{x_0}$ and of a constant $C_K > 0$ such that

$$\|P^*u\|_0^2 \geq C_K\|u\|_{1/r}^2, \quad \forall u \in C_0^\infty(K),$$

thus obtaining the $H^{-1/r}$ to L^2 local solvability in the interior of K. This ends the proof of the theorem. □

Remark 2.9.6. *It is important to remark that Theorem 2.2.1 may be viewed as a corollary of Theorem 2.9.2. This is indeed the case. However, hypothesis (H3) in Theorem 2.2.1 when $k = 0$ is expressed in terms of the symplectic geometry of the vector fields $H_{X_0}, \ldots, H_{X_N}$ and are invariant under symplectic transformations. This point of view will be important when generalizing the operator P to cases in which the first order operators $X_j(x, D)$ are replaced by first order pseudodifferential operators $P_j(x, D)$.*

Chapter 3

The non smooth coefficients case

This chapter is devoted to the analysis of the local solvability property of two classes of degenerate second order partial differential operators with non smooth coefficients similar to (1.1.1).

First we will consider the second order partial differential operator on $\mathbb{R}^n$ of the form

$$P = \sum_{j=1}^{N} X_j^* g|g| X_j + iX_0 + a_0, \tag{3.0.2}$$

where $X_j = X_j(D)$, $1 \leq j \leq N$, are homogeneous first order differential operators (in other words iX_j, $0 \leq j \leq N$ are vector fields) with real or complex constant coefficients (the two cases will be analyzed separately), $iX_0 = iX_0(x, D)$ is a real vector field with *affine* coefficients, $g \not\equiv 0$ is an affine function, and a_0 is a continuous function on $\mathbb{R}^n$ with complex values.

The purpose here is to study the L^2-local solvability of P in a neighborhood of the zeros of the function g, where the principal symbol of the operator can possibly change sign (that is, the principal symbol can change sign in the neighborhood of the fiber of $g^{-1}(0)$).We will see that P is a class of locally solvable second order operator across S having respectively $C^{1,1}$ coefficients if the vector fields iX_j are tangent to $S = g^{-1}(0) \neq \emptyset$ for all indices j, $1 \leq j \leq N$, or $C^{0,1}$ coefficients if there is at least an index $k \neq 0$ such that iX_k is transverse to S.

The proof follows the approach used in Chapter 2, in which a solvability result for degenerate second order operators with smooth coefficients analogous to P is proved. In fact, the class considered here is an elaboration of that introduced in Chapter 1 and it differs (here is the novelty and interest) from that class in the regularity assumption on the coefficients which are assumed to be less regular.

The second class we are going to treat is described by the operator

$$P=\sum_{j=1}^{N} X_j^*|f|X_j+iX_0+a_0, \tag{3.0.3}$$

where $X_j=X_j(x,D)$, $0\leq j\leq N$, are homogeneous first order differential operators with smooth coefficients (in other words iX_j, $0\leq j\leq N$ are smooth vector fields) defined on an open set $\Omega\subseteq\mathbb{R}^n$ and with a real principal symbol, $f:\Omega\longrightarrow\mathbb{R}^n$ is a C^1 function with $f^{-1}(0)\neq\emptyset$ and $df|_S\neq 0$, and a_0 is a continuous possibly complex valued function.

Note that, unlike the operator in (3.0.2), we do not have for (3.0.3) the changing sign property of the principal symbol. However, to give a complete characterization of operators of this form, it is interesting to analyze the local solvability property of operators like (3.0.3), having non smooth coefficients and being degenerate. Moreover, once more like in (1.1.1), we can have an interplay between the degeneracy due to the function f and that due to the system of first order operators appearing in the second order part of the operator.

Again, we are interested in the L^2-local solvability property of (3.0.3) around the set $S=f^{-1}(0)$ where the operator is degenerate and the coefficients are non smooth, since, far from S, operators like (3.0.2) and (3.0.3) are still in the class considered before, that is in (1.1.1), thus we can extend, at least in the real case, Theorem 2.8.2 which is available for operators in (1.1.1).

Since we deal with operators with non smooth coefficients, and since we are interested in the L^2-local solvability property, we give below the definition of L^2-locally solvable PDO which is suitable in this context (for more information about solvability see [10] and [14]).

Definition 3.0.1. *Given a partial differential operator P, defined on an open set $\Omega\subseteq\mathbb{R}^n$, such that both P and its adjoint P^* have at least L^∞ coefficients, we say that P is L^2-locally solvable in Ω if for any given $x_0\in\Omega$ there is a compact set $K\subset\Omega$ with $x_0\in U=\overset{\circ}{K}$ (where $\overset{\circ}{K}$ denotes the interior of K) such that for all $f\in L^2_{\mathrm{loc}}(\Omega)$ there exists $u\in L^2(U)$ such that for every compact $K\subset U$*

$$(u,P^*\varphi)=(f,\varphi),\quad \forall\varphi\in C_0^\infty(K),$$

where $(\cdot,\cdot)$ is the L^2 inner product.

Remark 3.0.2. *Note that in Definition 3.0.1 we require at least L^∞ coefficients not just for the operator itself but also for its adjoint P^*. This is indeed the case for our class of operators, but it is worth to remark that without the L^∞ regularity assumption* *for P^* the previous definition is in general ill posed. In fact, given a general partial differential operator P of order m defined on an open set $\Omega \subseteq \mathbb{R}^n$ of the form*

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where a_α is at least $L^\infty(\Omega)$ for every α, we have that P^* may have coefficients which are merely distributions of order m, and consequently the L^2 inner product $(u, P^*\varphi)$ used in Definition 3.0.1 may be not defined. However, as said earlier, this problem does not occur in our class of operators (which always have adjoints with at least L^∞ coefficients), therefore throughout we shall refer to the previous definition when talking about L^2-local solvability and solution of the problem mentioned before.

Let us end this introduction by giving the plan of the chapter.

In Section 3.1 we make the setting precise, introduce the main hypotheses, and give the proof of their invariance under affine changes of variables of the latter.

In Section 3.2 we prove a fundamental estimate, corresponding to the main estimate in Section 2.3, that will be the crucial step in the proof of the *solvability estimate* (3.2.3) below.

In Section 3.3 we prove a solvability result for the operator (3.0.2) in the real coefficients case. Here we shall use the estimate of Section 3.2 to derive the solvability estimate from which the result follows.

In Section 3.4 we study (3.0.2) in the complex coefficients case. Again, by using the estimate of Section 3.2, we obtain a solvability result.

Finally in Section 3.5 we look at the model operator (3.0.3) that differs from the operator in (3.0.2) in that the function responsible for the extra degeneracy of the symbol *does not* change sign across its zero set but the coefficients are less regular. Here, unlike the other cases listed, the solvability result is not based on the fundamental estimate proved in Section 3.2, but it follows by using a Carleman estimate.

3.1 Hypotheses

Let P be a linear second order partial differential operator as in the introduction, then the first order partial differential operators $X_j, 1 \le j \le N$, and X_0 in the expression of P are of the form

$$X_j(x, D) = X_j(D) = \langle \alpha_j, D \rangle, \quad X_0(x, D) = \langle \beta(x), D \rangle$$

where $D = (D_1, D_2, \ldots, D_n)$, $D_j = -i\partial_{x_j}$, $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n}) \in \mathbb{C}^n$, and $\beta(x) = (\beta_1(x), \ldots, \beta_n(x))$, where $\beta_j(x)$, $j = 1, \ldots, n$, are affine real functions of the form $\beta_j(x) = \sum_{k=1}^n \beta_{j,k} x_k + \beta_{j,0}$, and $\beta_{j,k}, \beta_{j,0} \in \mathbb{R}$ for all $j, k = 1, \ldots, n$. Moreover g is an affine real function over $\mathbb{R}^n$, thus we have $g(x) = \sum_{j=1}^n g_j x_j + g_0$, with $g_j, g_0 \in \mathbb{R}$ for all $j = 1, \ldots, n$, and g is such that $S = g^{-1}(0) \neq \emptyset$, $g \not\equiv 0$. Note also that the commutator $[X_j, X_0]$, for all $1 \leq j \leq N$, is a first order homogenous partial differential operator with complex *constant* coefficients. In addition we suppose:

(H1) $iX_0 g(x) > 0$ for all $x \in S := g^{-1}(0)$;

(H2) for all $1 \leq j \leq N$ there exists a constant $C > 0$ such that

$$|\{X_j, X_0\}(\xi)|^2 \leq C \sum_{j=1}^N |X_j(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n,$$

where the $\{X_j, X_0\}(\xi)$ and the $X_j(\xi)$ are the total (principal because of homegeneity) symbols of $i[X_j, X_0]$ and X_j respectively, $\{\cdot, \cdot\}$ denoting the Poisson bracket.

First of all we show that the analysis of the local solvability of

$$P = \sum_{j=1}^N X_j^* g|g| X_j + iX_0 + a_0$$

can always be reduced, after an affine change of variables, to that of

$$\tilde{P} = \sum_{j=1}^N \tilde{X}_j^* y_1 |y_1| \tilde{X}_j + i\tilde{X}_0 + \tilde{a}_0,$$

where $\tilde{P}$ (in the new variables) is of the same kind of P, and the new quantities still satisfy hypotheses (H1) and (H2). After that we will focus our attention on the local solvability of P in a neighborhood of the points of $S = g^{-1}(0)$, where, by the previous argument, we can assume $g(x) = x_1$. In this way we deal with the operator in a form that is simpler to study.

Observe that hypothesis (H1) is explicitly stated as

$$iX_0 g(x) = \langle \beta(x), \bigtriangledown g \rangle = \textstyle\sum_{j=1}^n \beta_j(x) g_j > 0 \text{ on } S.$$

We may suppose that $\frac{\partial g}{\partial x_1} = g_1 \neq 0$ (otherwise we would have $\frac{\partial g}{\partial x_j} = g_j \neq 0$ for some index j, and we could repeat the argument below with respect to the variable x_j). Under this assumption the function $\chi : (x_1, \ldots, x_n) \mapsto (g(x), x_2, \ldots, x_n)$ is an affine diffeomorphism of $\mathbb{R}^n$, and we can choose $(y_1, \ldots, y_n) = \chi(x_1, \ldots, x_n)$ as new coordinates. Changing variables we have

$$\tilde{P} = \sum_{j=1}^{N} \tilde{X}_j^* y_1 |y_1| \tilde{X}_j + i\tilde{X}_0 + \tilde{a}_0,$$

where

$$\begin{aligned}
\tilde{X}_j(D_y) &= \sum_{k=1}^{n} \alpha_{j,k} g_k D_{y_1} + \sum_{k=2}^{n} \alpha_{j,k} D_{y_k}, \\
\tilde{X}_0(y, D_y) &= \sum_{j=1}^{n} (\beta_j \circ \chi^{-1})(y) g_j D_{y_1} + \sum_{j=2}^{n} (\beta_j \circ \chi^{-1})(y) D_{y_j}, \\
\tilde{a}_0(y) &= (a_0 \circ \chi^{-1})(y), \\
\tilde{g}(y) &= (g \circ \chi^{-1})(y) = y_1.
\end{aligned}$$

It is important to note that $\tilde{X}_j$, $1 \leqslant j \leqslant N$, and $\tilde{X}_0$ are still first order homogeneous partial differential operators, and they still have, respectively, constant and affine coefficients.

Now we look at conditions (H1),(H2), and we see that if they are satisfied by the X_j, $1 \leqslant j \leqslant N$, and X_0, then the same holds for the $\tilde{X}_j$, $1 \leqslant j \leqslant N$, and $\tilde{X}_0$. In fact, since

$$\tilde{X}_j \tilde{g} = \tilde{X}_j y_1 = X_j g, \quad \tilde{X}_0 \tilde{g} = \tilde{X}_0 y_1 = X_0 g, \tag{3.1.4}$$

then our hypothesis (H1) is trivially invariant with respect to affine changes of variables. As for condition (H2), there is nothing to prove, since the Poisson bracket is an invariant of partial differential operators. Observe moreover that the first identity in (3.1.4) means that if iX_j, $1 \leq j \leq N$, is tangent or transverse to S, then the same holds for $i\tilde{X}_j$. All this proves that after performing an affine change of variables in P, what we get is an operator with a simpler expression and of the same type of P.

3.2 The fundamental estimate

By the argument of Section 3.1, we can therefore reduce our problem to the analysis of the local solvability of operators of the form

$$P = \sum_{j=1}^{N} X_j^* x_1 |x_1| X_j + iX_0 + a_0, \tag{3.2.2}$$

where the X_j, $1 \leqslant j \leqslant N$, X_0 and a_0 are assumed to be as before.

By the definition given in Corollary 2.1.6 we have that to obtain a local L^2-solvability result for an operator P on $\mathbb{R}^n$ the main point is to get the following a priori estimate: there exist a compact set K and a positive constant C such that

$$\|P^* u\| \geqslant C\|u\|, \quad \forall u \in C_0^\infty(K), \tag{3.2.3}$$

where P^* is the formal adjoint of P, and $\|\cdot\|$ is the L^2-norm. If this inequality holds for P^* then, using standard arguments, we have for all $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ the existence of $u \in L^2(\mathring{K})$ solving $Pu = v$ in $U = \mathring{K}$, where $\mathring{K}$ denotes the interior of K.

Consequently, our goal is to obtain the solvability estimates (3.2.3) for our operator P of the form (3.2.2) in a neighborhood of S. To this aim, we need some further preliminary estimates. In particular we will derive in this section a fundamental estimate that will be useful both in the real and in the complex coefficients case.

Proposition 3.2.1. *Let $S = \{x \in \mathbb{R}^n; x_1 = 0\}$. Then for all $x_0 \in S$ there exist a compact set K_0 containing x_0 in its interior and three positive constants $C = C(K_0), c = c(K_0)$ and $\varepsilon_0 = \varepsilon_0(K_0)$, with $\varepsilon_0 \to 0$ as $K_0 \searrow \{x_0\}$, such that for all compact sets $K \subset K_0$*

$$\|P^* u\|^2 \geqslant \frac{1}{4}\|X_0 u\|^2 + c(P_0 u, u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K), \tag{3.2.4}$$

where

$$P_0 = \sum_{j=1}^{N} (X_j^* |x_1| X_j - \varepsilon_0^2 [X_j, X_0]^* |x_1| [X_j, X_0]). \tag{3.2.5}$$

Proof. First of all we observe that $X_0^* = X_0 + d_{X_0}$, where $d_{X_0} = \sum_{k=1}^n D_k(\beta_k) \in i\mathbb{R}$, $D_k = -i\partial_{x_k}$, and $X_j^* = \langle \overline{\alpha}_j, D\rangle$, $1 \leqslant j \leqslant N$ (we are considering the general case in which $\alpha_j \in \mathbb{C}^n$).

Moreover, since $P^* = \sum_{j=1}^N X_j^* x_1 |x_1| X_j - iX_0^* + \overline{a}_0$, for all compact $K \subset \mathbb{R}^n$ we have

$$\|P^* u\|^2 \geq \frac{1}{2}\|\sum_{j=1}^{N} (X_j^* x_1 |x_1| X_j - iX_0^*) u\|^2 - \|a_0\|^2_{L^\infty(K)} \|u\|^2$$

for all $u \in C_0^\infty(K)$, where

$$\begin{aligned}
\frac{1}{2}\|\sum_{j=1}^N (X_j^* x_1|x_1|X_j - iX_0^*)u\|^2 &= \frac{1}{2}\big(\|X_0^* u\|^2 + \|\sum_{j=1}^N X_j^* x_1|x_1|X_j u\|^2 \\
&- 2\sum_{j=1}^N \mathsf{Re}(X_j^* x_1|x_1|X_j u, iX_0^* u)\big) \\
&\geq \frac{1}{2}\|X_0^* u\|^2 - \sum_{j=1}^N \mathsf{Re}(X_j^* x_1|x_1|X_j u, iX_0^* u).
\end{aligned}$$

Since

$$\sum_{j=1}^N \mathsf{Re}(X_j^* x_1|x_1|X_j u, iX_0^* u) = \sum_{j=1}^N \mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0^* u),$$

we then estimate the imaginary part. Thus, for each index j, we have

$$\begin{aligned}
\mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0^* u) &= \mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0 u) + \mathsf{Im}(X_j^* x_1|x_1|X_j u, d_{X_0} u) \\
&= \mathsf{Im}(x_1|x_1|X_j u, X_j X_0 u) + \mathsf{Im}(x_1|x_1|X_j u, (X_j d_{X_0})u) \\
&+ \mathsf{Im}(x_1|x_1|X_j u, d_{X_0} X_j u),
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{Im}(x_1|x_1|X_j u, X_j X_0 u) &= \mathsf{Im}(x_1|x_1|X_j u, [X_j, X_0]u) + \mathsf{Im}(x_1|x_1|X_j u, X_0 X_j u) \\
&= \mathsf{Im}(x_1|x_1|X_j u, [X_j, X_0]u) + \frac{1}{2i}\big[(x_1|x_1|X_j u, X_0 X_j u) \\
&- (X_0 X_j u, x_1|x_1|X_j u)\big] \\
&= \mathsf{Im}(x_1|x_1|X_j u, [X_j, X_0]u) + \frac{1}{2i}\big[(|x_1|(X_0 x_1)X_j u, X_j u) \\
&+ (x_1(X_0|x_1|)X_j u, X_j u) + (x_1|x_1|X_0 X_j u, X_j u) \\
&+ (d_{X_0} x_1|x_1|X_j u, X_j u) - (X_0 X_j u, x_1|x_1|X_j u)\big] \\
&= \mathsf{Im}(x_1|x_1|X_j u, [X_j, X_0]u) - (|x_1|(iX_0 x_1)X_j u, X_j u) \\
&+ \frac{1}{2}\mathsf{Im}(d_{X_0} x_1|x_1|X_j u, X_j u).
\end{aligned}$$

Putting the last expression inside the term $\mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0^* u)$ gives

$$-\mathsf{Im}(X_j^* x_1|x_1|X_j u, X_0^* u) = -\mathsf{Im}(x_1|x_1|X_j u, [X_j, X_0]u) + (|x_1|(iX_0 x_1)X_j u, X_j u)$$

$$+\frac{1}{2}\mathsf{Im}(d_{X_0}x_1|x_1|X_ju, X_ju) - \mathsf{Im}(x_1|x_1|X_ju, (X_jd_{X_0})u).$$

By hypothesis (H1), for each $x_0 \in S$ we can find a compact set K_1 such that $x_0 \in \mathring{K}_1$ and $iX_0g(x) > c_0$ in K_1, with $c_0 > 0$ and $g(x) = x_1$. We then work in a fixed compact set K_1 containing the point $x_0 \in S$ in its interior, and we get

$$-\mathsf{Im}(X_j^* x_1|x_1|X_ju, X_0^*u) \geq c_0(|x_1|^{1/2}X_ju, |x_1|^{1/2}X_ju)$$

$$-\mathsf{Im}(x_1|x_1|^{1/2}X_ju, |x_1|^{1/2}[X_j, X_0]u) + \frac{1}{2}\mathsf{Im}(d_{X_0}x_1|x_1|^{1/2}X_ju, |x_1|^{1/2}X_ju)$$

$$-\mathsf{Im}(x_1|x_1|^{1/2}X_ju, |x_1|^{1/2}(X_jd_{X_0})u),$$

for all $u \in C_0^\infty(K_1)$. Therefore, for each $K \subset K_1$ with $x_0 \in \mathring{K}$, we have

$$-\mathsf{Im}(X_j^* x_1|x_1|X_ju, X_0^*u) \geq c_0\||x_1|^{1/2}X_ju\|^2$$

$$-2\|x_1\|_{L^\infty(K)}\||x_1|^{1/2}X_ju\|\||x_1|^{1/2}[X_j, X_0]u\|$$

$$-\frac{1}{2}\|x_1\|_{L^\infty(K)}\|d_{X_0}\|_{L^\infty(K_1)}\||x_1|^{1/2}X_ju\|^2$$

$$-2\|x_1\|_{L^\infty(K)}\||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_1)}\|u\|\||x_1|^{1/2}X_ju\|$$

$$\geq \||x_1|^{1/2}X_ju\|^2\Big(c_0 - \|x_1\|_{L^\infty(K)}\big(1 + \frac{1}{2}\|d_{X_0}\|_{L^\infty(K_1)} + \||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_1)}\big)\Big)$$

$$-\|x_1\|_{L^\infty(K)}\||x_1|^{1/2}[X_j, X_0]u\|^2 - \|x_1\|_{L^\infty(K)}\||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_1)}\|u\|^2.$$

Since $\|x_1\|_{L^\infty(K)} \longrightarrow 0$ when $K \searrow \{x_0\}$, $x_0 \in S$, we can find a compact set $K_0 \subset K_1$ containing x_0 in its interior such that, for all $K \subset K_0$ with $x_0 \in \mathring{K}$, we have

$$\begin{aligned} c_0 - \|x_1\|_{L^\infty(K)}\big(1 + \frac{1}{2}\|d_{X_0}\|_{L^\infty(K)} + \||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K)}\big) &\geqslant \\ c_0 - \|x_1\|_{L^\infty(K_0)}\big(1 + \frac{1}{2}\|d_{X_0}\|_{L^\infty(K_0)} + \||x_1|^{1/2}(X_jd_{X_0})\|_{L^\infty(K_0)}\big) &\geqslant \frac{c_0}{2} \end{aligned}$$

since $\|x_1\|_{L^\infty(K)} \leq \|x_1\|_{L^\infty(K_0)}$ for every compact set $K \subset K_0$. Then, calling $c = c(K_0) := c_0/2$, we may choose

$$\varepsilon_0 = \varepsilon_0(K_0) := (\|x_1\|_{L^\infty(K_0)}/c)^{1/2}, \quad \text{with} \quad \varepsilon_0 \to 0 \quad \text{as} \quad K_0 \searrow \{x_0\},$$

which depends only on K_0. By the previous arguments we get that, for all $K \subset K_0$, and all $u \in C_0^\infty(K)$

$$\|P^*u\|^2 \geq \frac{1}{2}\|X_0^*u\|^2 + c(P_0u, u) - c\varepsilon_0^2\||x_1|^{1/2}(X_j d_{X_0})\|^2_{L^\infty(K_0)}\|u\|^2 - \|a_0\|^2_{L^\infty(K_0)}\|u\|^2.$$

Finally, since $\|X_0^*u\|^2 \geq (1/2)\|X_0u\|^2 - \|d_{X_0}\|^2_{L^\infty(K_0)}\|u\|^2$ for all $u \in C_0^\infty(K)$, for all $K \subset K_0$ (containing x_0 in its interior), we obtain inequality (3.2.4). □

Looking at (3.2.4), it is obvious that we need to estimate (P_0u, u) to have the solvability estimate (3.2.3). For this reason we have to distinguish between the real and the complex coefficients case, since we need different hypotheses on the vector fields iX_j in order to obtain the appropriate estimate for the term (P_0u, u).

3.3 Local solvability result in the real coefficients case

Let us start with the real case, that is, we assume that $iX_j = iX_j(D)$, for each $j \neq 0$, is a vector field with real constant coefficients, and $iX_0 = iX_0(x, D)$ is a vector field with real affine coefficients. The plan is to use (3.2.4) to derive (3.2.3) by estimating P_0 from below in L^2 and then by using a Poincaré inequality for iX_0.

A key point in order to pass from (3.2.4) to (3.2.3) is represented by (H2). Condition (H2), which is an estimate between symbols, does not imply the same relation between the relative operators (see Section 1.1). In this case, however, (H2) gives strong properties.

Therefore, before proving an estimate for P_0, we give the following consequence of hypothesis (H2).

Lemma 3.3.1. *If condition (H2) holds, then, for each index $j \in \{1, \cdots, N\}$, we have*

$$i\,[X_j, X_0] = \sum_{k=1}^{N} c_k X_k, \quad c_k \in \mathbb{R}. \tag{3.3.2}$$

Proof. Recall that the iX_j and $[X_j, X_0]$, $1 \leq j \leq N$, have real constant coefficients, and that $\alpha_j \in \mathbb{R}^n$ is the vector associated to X_j, $1 \leq j \leq N$. Now we consider two cases. The first one is when there exist n linearly independent elements $\alpha_{j_1}, \cdots, \alpha_{j_n}$

of $\mathbb{R}^n$ (associated with $X_{j_1}, \cdots, X_{j_n}$), with $j_1, \cdots, j_n \in \{1, \cdots, N\}$. This means essentially that $\mathbb{R}^n = \mathrm{Span}\{\alpha_{j_1}, \cdots, \alpha_{j_n}\}$, and thus, for each index j, we have

$$i\,[X_j, X_0] = \sum_{k=1}^{n} c_k X_{j_k} = \sum_{k=1}^{N} c_k X_k, \quad c_k \in \mathbb{R},\, c_k = 0,\ \forall k \notin \{j_1, \cdots, j_n\}.$$

The second case is when there are $m < n$ linearly independent elements $\alpha_{j_1}, \cdots, \alpha_{j_m}$ of $\mathbb{R}^n$ (associated with $X_{j_1}, \cdots, X_{j_m}$), with $j_1, \cdots, j_m \in \{1, \cdots, N\}$.
Since $X_j(x, D) = X_j(D) = \langle \alpha_j, D\rangle$ and $i[X_j, X_0](x, D) = i[X_j, X_0](D) = \langle \gamma_j, D\rangle$, we shall denote by V_k and W_k the sets

$$V_k = \{\xi \in \mathbb{R}^n;\ X_k(\xi) = 0\} = \mathrm{Span}_{\mathbb{R}}\{\alpha_k\}^{\perp},$$

$$W_k = \{\xi \in \mathbb{R}^n;\ \{X_k, X_0\}(\xi) = 0\} = \mathrm{Span}_{\mathbb{R}}\{\gamma_k\}^{\perp},$$

and also by Σ_{X_k}, $\Sigma_{[X_k,X_0]}$ the characteristic sets of X_k, $i[X_k, X_0]$ respectively, so that

$$\Sigma_{X_k} = V_k \setminus \{0\}, \quad \Sigma_{i[X_k,X_0]} = W_k \setminus \{0\}.$$

In this situation condition (H2) states that, for all $1 \le j \le N$, there exists a constant $C > 0$ such that

$$|\langle \gamma_j, \xi\rangle|^2 \le C \sum_{k=1}^{m} |\langle \alpha_{j_k}, \xi\rangle|^2, \quad \forall \xi \in \mathbb{R}^n,$$

which implies

$$\bigcap_{k=1}^{m} V_{j_k} \subseteq W_j, \quad 1 \le j \le N.$$

The latter inclusion shows that, passing to the orthogonal complements, we have

$$\Big(\bigcap_{k=1}^{m} V_{j_k}\Big)^{\perp} \supseteq W_j^{\perp}. \tag{3.3.3}$$

Now, applying in (3.3.3) the well-known relations

$$\Big(\bigcap_{i=1}^{m} V_i\Big)^{\perp} \ = \ V_1^{\perp} + \cdots + V_m^{\perp} \tag{3.3.4}$$

we have

$$V_{j_1}^{\perp} + \cdots + V_{j_m}^{\perp} \supseteq W_j^{\perp}, \quad \forall j = 1, \cdots, N,$$

which is equivalent to

$$\mathrm{Span}_{\mathbb{R}}\{\alpha_{j_1},\cdots,\alpha_{j_m}\}\supseteq \mathrm{Span}_{\mathbb{R}}\{\gamma_j\},\quad \forall j=1,\cdots,N.$$

Finally, by the latter inclusion, we have

$$i[X_j,X_0]=\langle\gamma,D\rangle=\langle\sum_{k=1}^{m}c_k\alpha_{j_k},D\rangle=\sum_{k=1}^{m}c_kX_{j_k}=\sum_{k=1}^{N}c_kX_k,$$

where $c_k\in\mathbb{R}$, and $c_k=0,\ \forall k\notin\{j_1,\cdots,j_m\}$. □

Next we prove the following lemma (the analogue of the Fefferman-Phong inequality in this case with non-smooth coefficients).

Lemma 3.3.2. *Consider $x_0\in S$ and K_0 as in Proposition 3.2.1. Then, suitably shrinking K_0 to a compact set containing x_0 in its interior, and that we still denote by K_0, we have that for all $K\subset K_0$, with $x_0\in\mathring{K}$, we have*

$$(P_0u,u)\geqslant 0,\quad \forall u\in C_0^\infty(K).$$

Proof. From the definition of P_0 (see (3.2.5)) we have

$$(P_0u,u)=\sum_{j=1}^{N}\||x_1|^{1/2}X_ju\|^2-\varepsilon_0^2\sum_{j=1}^{N}\||x_1|^{1/2}[X_j,X_0]u\|^2,$$

where $K\subset K_0$ and $u\in C_0^\infty(K)$.

Observe now, in view of Lemma 3.3.1, that

$$\||x_1|^{1/2}[X_j,X_0]u\|^2=\int|x_1||[X_j,X_0]u|^2dx=\int|x_1||\sum_{k=1}^{N}c_kX_ku|^2dx\leq$$

$$\leq N\sum_{k=1}^{N}\int|x_1||c_kX_ku|^2dx\leq N(\max_k c_k^2)\sum_{k=1}^{N}\int|x_1||X_ku|^2dx$$

$$=C(j)\sum_{k=1}^{N}\||x_1|^{1/2}X_ku\|^2,$$

where $C(j) = N \max_k c_k^2 > 0$.The latter inequality yields

$$\begin{aligned}
\sum_{j=1}^N \||x_1|^{1/2}[X_j, X_0]u\|^2 &\leq \sum_{j=1}^N C(j) \sum_{k=1}^N \||x_1|^{1/2} X_k u\|^2 \\
&\leq N(\max_j C(j)) \sum_{k=1}^N \||x_1|^{1/2} X_k u\|^2 \\
&= C \sum_{k=1}^N \||x_1|^{1/2} X_k u\|^2,
\end{aligned}$$

where $C = N \max_j C(j) > 0$. Therefore

$$\begin{aligned}
(P_0 u, u) &= \sum_{j=1}^N \||x_1|^{1/2} X_j u\|^2 - \varepsilon_0^2 \sum_{j=1}^N \||x_1|^{1/2}[X_j, X_0]u\|^2 \\
&\geqslant (1 - \varepsilon_0^2 C) \sum_{j=1}^N \||x_1|^{1/2} X_j u\|^2, \qquad \forall K \subset K_0,\ \forall u \in C_0^\infty(K).
\end{aligned}$$

and since $\varepsilon_0^2 = \varepsilon_0(K_0)^2$, we can shrink K_0 to a compact set K_0' containing x_0 in its interior in such a way that $C\varepsilon_0(K_0')^2 \leqslant 1/2$. Finally, denoting K_0' by K_0 again, the result follows. □

Remark 3.3.3. *Lemma 3.3.2 still holds for all compact $K \subset K_0$ not necessarily containing x_0 in its interior but sufficiently close to S.*

Remark 3.3.4. *Summarizing, by Proposition 3.2.1 and Lemma 3.3.2, for every compact $K \subset K_0$ (containing x_0 in its interior) we have that there exists a positive constant $C = C(K_0)$ such that*

$$\|P^* u\|^2 \geqslant \frac{1}{4}\|X_0 u\|^2 - C\|u\|^2, \quad \forall u \in C_0^\infty(K). \tag{3.3.5}$$

To conclude the solvability estimate (3.2.3) we need to apply the Poincaré inequality (2.6.18) on X_0, which is true being $iX_0 \neq 0$ near S (see Appendix A).

Then, in view of (2.6.18) and (3.3.5), for all $K \subset K_0$ (K_0 suitably shrunk so that Lemma 2.6.2 holds)

$$\|P^* u\|^2 \geqslant \Big(\frac{1}{4} - CC_2^2 \mathrm{diam}(K)^2\Big)\|X_0 u\|^2, \quad \forall u \in C_0^\infty(K).$$

We finally choose a compact set $K \subset K_0$ (which is a shrinking of K_0 containing x_0 in its interior) such that

$$\mathrm{diam}(K) \leqslant \Big(\frac{1}{8CC_2^2}\Big)^{1/2},$$

and we obtain the solvability estimate

$$\|P^*u\|^2 \geq \frac{1}{8}\|X_0u\|^2 \geq \frac{1}{C_2^2 \mathrm{diam}(K)^2}\|u\|^2, \quad \forall u \in C_0^\infty(K). \tag{3.3.6}$$

We have essentially proved the following result.

Theorem 3.3.5. *Let P be of the form (3.0.2) such that all the vector fields iX_j have real coefficients and hypotheses (H1),(H2) are satisfied, and let S be the zero set of g. Then for all $x_0 \in S$ there exists a compact set $K \subset \mathbb{R}^n$ with $U = \mathring{K}$ and $x_0 \in U$, such that for all $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U.*

Proof. After reducing P of the form (3.0.2) to the form (3.2.2) (see Section 3.1) the proof follows directly by the solvability estimate (3.3.6) using classical arguments. □

Remark 3.3.6. *Theorem 3.3.5 means that, for all $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, there exists a solution $u \in L^2(U)$ of the equation $Pu = v$ in U in the sense of Definition 3.0.1, that is, for all compact $K \subset U$,*

$$(u, P^*\varphi) = (v, \varphi), \quad \forall \varphi \in C_0^\infty(K).$$

3.4 The complex coefficients case

Also in the complex coefficients case it is possible to prove a solvability result when P is given in the general form

$$P = \sum_{j=1}^{N} X_j^* g|g| X_j + iX_0 + a_0, \tag{3.4.2}$$

when $iX_j = iX_j(D)$ (or simply X_j in this case), for all $j \neq 0$, are vector fields with complex constants coefficients, and $iX_0 = iX_0(x, D)$ is a real vector field with *real* affine coefficients, that is,

$$X_0 = \langle \beta(x), D \rangle, \quad \beta(x) \in \mathbb{R}^n,$$

in which

$$\beta_j(x) = \beta_{j,0} + \sum_{i=1}^{n} \beta_{j,i} x_i, \quad \beta_{j,i} \in \mathbb{R} \quad \forall i, j = 1, ..., n.$$

Once more we may reduce matters to the case $g(x) = x_1$.

We assume now the following hypotheses, which we state for $g(x) = x_1$, since they are invariant:

(H1) $iX_0 x_1 > 0$;

(H2) for all $1 \leq j \leq N$ there exists a constant $C > 0$ such that

$$|\{X_j, X_0\}(\xi)|^2 \leq C \sum_{j=1}^{N} |X_j(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n,$$

(H3) $X_j g = X_j x_1 = 0, \quad \forall j = 1, ..., N,$

where (H3) means that each vector field X_j, with $j \neq 0$, is tangent to $S = \{x \in \mathbb{R}^n;\, x_1 = 0\} = g^{-1}(0)$, while (H1) states that iX_0 is transverse to S.

Our goal now is to prove the analogue of Theorem 3.3.5 in this case.

The solvability result still follows by the a priori estimate (3.2.3) that we recall here: for all $x_0 \in S$ there exist a compact set K which contains x_0 in its interior and a positive constant C such that

$$\|P^* u\| \geqslant C\|u\|, \quad \forall u \in C_0^\infty(K).$$

First of all note that the main estimate (3.2.4) still holds for P^* even if P^* has complex coefficients in the second order part, thus we have that for all $x_0 \in S$ there exist a compact set K_0 containing x_0 in its interior and three positive constants $C = C(K_0), c = c(K_0)$ and $\varepsilon_0 = \varepsilon_0(K_0)$, with $\varepsilon_0 \to 0$ as $K_0 \searrow \{x_0\}$, such that for all compact $K \subset K_0$

$$\|P^* u\|^2 \geqslant \frac{1}{4}\|X_0 u\|^2 + c(P_0 u, u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K),$$

where

$$P_0 = \sum_{j=1}^{N} (X_j^* |x_1| X_j - \varepsilon_0^2 [X_j, X_0]^* |x_1| [X_j, X_0]).$$

Since we need to control the term $(P_0 u, u)$ from below we will use hypotheses (H2),(H3) to obtain some useful results to conclude the desired estimate.

Corollary 3.4.1. *Consider $x_0 \in S$ and K_0 ($x_0 \in \mathring{K}_0$) as in Proposition 3.2.1. We then can shrink K_0 to a compact set that we keep denoting by K_0, with $x_0 \in \mathring{K}_0$, so that*

$$\varepsilon_0^2 \sum_{j=1}^{N} |\{X_j, X_0\}(\xi)|^2 \leq \sum_{j=1}^{N} |X_j(\xi)|^2, \quad \forall \xi \in \pi^{-1}(K_0), \tag{3.4.3}$$

where $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection.

Proof. By condition (H2) we have

$$\sum_{j=1}^{N} |X_j(\xi)|^2 - \varepsilon_0^2 \sum_{j=1}^{N} |\{X_j, X_0\}(\xi)|^2 \geq (1 - CN\varepsilon_0^2) \sum_{j=1}^{N} |X_j(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n.$$

By Proposition 3.2.1 we can shrink K_0 to a compact set, that we keep denoting by K_0, with $x_0 \in \mathring{K}_0$, so that $CN\varepsilon_0(K_0)^2 \leqslant 1/2$ and (3.4.3) holds. □

We shall work throughout in the compact set K_0 of Corollary 4.0.3, and we shall consequentely fix $\varepsilon_0 = \varepsilon_0(K_0)$.

Remark 3.4.2. *Recall that by (H3) we have $X_j g(x) = 0$ for each index $1 \leqslant j \leqslant N$, where $g(x) = x_1$. Therefore, if we write $\xi = (\xi_1, \xi')$, $\xi' \in \mathbb{R}^{n-1}$, we have $X_j(\xi) = X_j(\xi_1, 0) + X_j(0, \xi') = X_j(0, \xi')$. Moreover, since condition (H2) holds, we also have $\{X_j, X_0\}(\xi) = \{X_j, X_0\}(0, \xi')$. Then, by Corollary 4.0.3,*

$$\varepsilon_0^2 \sum_{j=1}^{N} |\{X_j, X_0\}(0, \xi')|^2 \leq \sum_{j=1}^{N} |X_j(0, \xi')|^2, \quad \forall \xi' \in \pi_{\xi'}(\pi^{-1}(K_0')),$$

where $\pi_{\xi'}$ is the projection on the component ξ'.

Now we prove the following lemma.

Lemma 3.4.3. *Consider $x_0 \in S$ and K_0 as in Corollary 4.0.3. Then for all $K \subset K_0$ with $x_0 \in \mathring{K}$ we have*

$$(P_0 u, u) \geqslant 0, \quad \forall u \in C_0^\infty(K).$$

Proof. Observe that

$$(P_0 u, u) \geqslant 0, \quad \forall u \in C_0^\infty(K)$$

is equivalent to

$$\sum_{j=1}^{N} \||x_1|^{1/2} X_j u\|^2 - \varepsilon_0^2 \sum_{j=1}^{N} \||x_1|^{1/2} [X_0, X_j] u\|^2 \geq 0, \quad \forall u \in C_0^\infty(K), \tag{3.4.4}$$

therefore we prove the latter.

We write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Since for all $K \subset K_0$ we have

$$\begin{aligned} \sum_{j=1}^{N} \||x_1|^{1/2} X_j u\|^2 &= \sum_{j=1}^{N} \int |x_1| \big|X_j u(x_1, x')\big|^2 dx \\ &= \sum_{j=1}^{N} \int |x_1| \|X_j u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})} dx_1, \quad \forall u \in C_0^\infty(K), \end{aligned}$$

to have (3.4.4) it suffices to prove the pointwise estimate

$$\sum_{j=1}^{N} \|X_j u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})} \geq \varepsilon_0^2 \sum_{j=1}^{N} \|[X_j, X_0] u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})}, \tag{3.4.5}$$

where x_1 is thought of as a parameter.

Denoting by $\widehat{f}(x_1, \xi')$ the partial Fourier transform in the x' variable of a function $f(x) = f(x_1, x')$ then, by the Plancherel theorem and (3.4.3), we get

$$\begin{aligned} \sum_{j=1}^{N} \|X_j u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})} &= \frac{1}{(2\pi)^{n-1}} \sum_{j=1}^{N} \|\widehat{X_j u}(x_1, \xi')\|^2_{L^2(\mathbb{R}^{n-1}_{\xi'})} \\ &= \frac{1}{(2\pi)^{n-1}} \sum_{j=1}^{N} \int |X_j(0, \xi')|^2 |\widehat{u}(x_1, \xi')|^2 d\xi' \\ &\underset{(3.4.3)}{\geq} \frac{\varepsilon_0^2}{(2\pi)^{n-1}} \sum_{j=1}^{N} \int |\{X_j, X_0\}(0, \xi')|^2 |\widehat{u}(x_1, \xi')|^2 d\xi' \\ &= \varepsilon_0^2 \sum_{j=1}^{N} \|[X_j, X_0] u(x_1, \cdot)\|^2_{L^2(\mathbb{R}^{n-1}_{x'})}, \end{aligned}$$

which is exactly (3.4.5), whence (3.4.4) holds. □

Remark 3.4.4. *Summarizing, since Proposition 3.2.1 holds in K_0, $x_0 \in \mathring{K}_0$, $x_0 \in S$, and since we have shrunk K_0 in such a way that Lemma 3.4.3 holds, then for all $K \subset K_0$ (containing x_0 in its interior) we have that there exists a positive constant $C = C(K_0)$ such that*

$$\|P^* u\|^2 \geqslant \frac{1}{4} \|X_0 u\|^2 - C\|u\|^2, \quad \forall u \in C_0^\infty(K). \tag{3.4.6}$$

Now, exactly as in the real case, by applying the Poincaré inequality (2.6.18) on X_0 we get the solvability estimate.

We have therefore proved the following theorem.

Theorem 3.4.5. *Let P be of the form (3.4.2) such that hypotheses (H1) to (H3) are satisfied, and let S be the zero set of g. Then for all $x_0 \in S$ there exist a compact set $K \subset \mathbb{R}^n$ with $U = \mathring{K}$ and $x_0 \in U$, such that for all $v \in L^2_{\text{loc}}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U in the sense of Definition 3.0.1.*

3.4.1 A remark on the difference between the complex and the real case

The complex coefficients case is in general more difficult to solve. In fact, in order to prove Theorem 3.4.5 for P of the form (3.4.2), we require that the vector fields X_j, for all $j \neq 0$, be tangent to S (i.e $X_j g = 0$ for all $j \neq 0$). This assumption is needed in order to prove that the term $\sum_{j=1}^N \||x_1|^{1/2}[X_j, X_0]\|$ is controlled by $\sum_{j=1}^N \||x_1|^{1/2} X_j\|$. This control is necessary to obtain the solvability estimate that yields the result.

In the real coefficients case we do not suppose this tangency condition, since, by Lemma 3.3.1, condition (H2) gives that, for all j, $i[X_j, X_0] \in \operatorname{Span}_{\mathbb{R}}\{X_1, ..., X_N\}$, which, in particular, allows the desired control on the norms.

To have the analogue of Lemma 3.3.1 in the complex coefficients case, that is, to have the solvability result without asking to the complex vector fields to be tangent to S, we should replace condition (H2) with the following condition (H2$'$):

(H2$'$) for all $1 \leq j \leq N$ there exists a constant $C > 0$ such that

$$|\{X_j, X_0\}(\zeta)|^2 \leq C \sum_{j=1}^N |X_j(\zeta)|^2, \quad \forall \zeta \in \mathbb{C}^n,$$

the point being that we need this inequality for ζ *complex*.
With this assumption we would have that, for all j, $[X_j, X_0] \in \operatorname{Span}_{\mathbb{C}}\{X_1, ..., X_N\}$, and, once again, we would get the control on the commutators and finally the solvability estimate. However condition (H2$'$) is stronger than (H2). In fact (H2) does not imply (H2$'$) as shown by this counterexample.

Example 3.4.6. We consider the operator $P = X_1^* x_1 |x_1| X_1 + i X_0$ of the form (3.4.2) with $N = 1$ and $n = 2$, where

$$X_1(D) = (1+i)D_1 + (2+i)D_2 \quad \text{and} \quad X_0(x, D) = (3x_1 + 1)D_1 + (6x_1 - x_2)D_2.$$

First we will show that condition (H1) and (H2) are satisfied by P, next we will prove that condition (H2$'$) is not fulfilled even if (H2) holds for P.

Note that (H1) is trivially satisfied, since $iX_0(x,D)x_1 = 1 > 0$, and also that $X_1(D)$ is not tangent to $S = \{x \in \mathbb{R}^2; x_1 = 0\}$, since $X_1(D)x_1 \neq 0$. This means in particular that (H3) is not true in this case, so we are not allowed to use Theorem 3.4.5 here.

As regards (H2), since $X_1(\xi) = (1+i)\xi_1 + (2+i)\xi_2$ and $X_0(x,\xi) = (3x_1+1)\xi_1 + (6x_1 - x_2)\xi_2$, we have $\{X_1, X_0\}(\xi) = (3+3i)\xi_1 + (4+5i)\xi_2$, and, denoting by $\langle \cdot, \cdot \rangle$ the euclidean scalar product in $\mathbb{R}^2$ (note that $\xi \in \mathbb{R}^2$ now),

$$|\{X_1, X_0\}(\xi)|^2 = |\langle (3,4), (\xi_1, \xi_2)\rangle + i\langle (3,5), (\xi_1,\xi_2)\rangle|^2 =$$

$$|\langle (3,4), (\xi_1, \xi_2)\rangle|^2 + |\langle (3,5), (\xi_1,\xi_2)\rangle|^2 =$$

$$|\langle (1,2), (\xi_1, \xi_2)\rangle + 2\langle (1,1), (\xi_1,\xi_2)\rangle|^2 + |2\langle (1,2), (\xi_1, \xi_2)\rangle + \langle (1,1), (\xi_1,\xi_2)\rangle|^2 \leq$$

$$10(|\langle (1,2), (\xi_1, \xi_2)\rangle|^2 + |\langle (1,1), (\xi_1,\xi_2)\rangle|^2) = 10|X_1(\xi)|^2, \quad \forall \xi \in \mathbb{R}^2.$$

By the latter inequality we conclude that (H2) is satisfied by P.

We focus now our attention on condition (H2$'$). If (H2$'$) holds for P, then, for all $\zeta \in \mathbb{C}^2$ such that $X_1(\zeta) = \langle \alpha, \zeta \rangle = 0$, where $\alpha = (1+i, 2+i)$, also $\{X_1, X_0\}(\zeta) = \langle \gamma, \zeta \rangle = 0$, where $\gamma = (3+3i, 4+5i)$. We then consider $\zeta_0 = (-13+i, 8+2i)$ and note that

$$X_1(\zeta_0) = \langle (1+i, 2+i), (-13+i, 8+2i)\rangle = 0,$$

but

$$\{X_1, X_0\}(\zeta_0) = \langle 3+3i, 4+5i), (-13+i, 8+2i)\rangle = -20 + 12i \neq 0.$$

Whence (H2$'$) is not verified by P satisfying (H2), therefore (H2) $\not\Rightarrow$ (H2$'$).

Remark 3.4.7. *Recall that in Theorem 3.4.5 we require that the complex system of vector fields* $\{X_j\}_{1 \leq j \leq N}$ *be tangent to* S *and satisfies (H2). This is not the same as requiring* (H2$'$)*, that is, the tangency condition together with (H2) does not imply* (H2$'$)*. Summarizing we have that* (H2) $\not\Rightarrow$ (H2$'$) *in general, but also* (H2) + *the tangency condition* $\not\Rightarrow$ (H2$'$) *as Example 3.4.8 shows. Thus, in the proof of Theorem 3.4.5, we do not use* (H2$'$) *to prove the result.*

Example 3.4.8. We consider now the operator $P = X_1^* x_1|x_1|X_1 + iX_0$ of the form (3.4.2) with $N = 1$ and $n = 3$, where

$$X_1(D) = (2+i)D_2 + D_3 \quad \text{and} \quad X_0(x,D) = D_1 + x_2 D_2.$$

Note that (H1) is satisfied by X_0, since $iX_0(x,D)x_1 = 1 > 0$, and that $X_1(D)$ is now tangent to S. Also condition (H2) is trivially satisfied, since $X_1(\xi) = (2+i)\xi_2 + \xi_3$, $X_0(x,\xi) = \xi_1 + x_2\xi_2$, $\{X_1, X_0\}(\xi) = (2+i)\xi_2$, and

$$|\{X_1, X_0\}(\xi)|^2 = 5|\xi_2|^2 \leq 5(|2\xi_2 + \xi_3|^2 + |\xi_2|^2) = 5|X_1(\xi)|^2, \quad \forall \xi \in \mathbb{R}^3.$$

Since (H1) and (H2) are true for P and the complex vector field X_1 is tangent to S, we have the local solvability of P around S by Theorem 3.4.5.

Again, if (H2$'$) were true for P, then, for all $\zeta \in \mathbb{C}^3$ such that $X_1(\zeta) = 0$, we would also have $\{X_1, X_0\}(\zeta) = 0$.

We call $\alpha = (0, 2+i, 1)$ the vector in $\mathbb{C}^3$ such that $X_1(\zeta) = \langle \alpha, \zeta \rangle$, and $\gamma = (0, 2+i, 0)$ that one such that $\{X_1, X_0\}(\zeta) = \langle \gamma, \zeta \rangle$ (we are considering $\zeta \in \mathbb{C}^3$ and we are still denoting by $\langle \cdot, \cdot \rangle$ the euclidean scalar product). Next we consider $\zeta_0 = (0, 1+i, -1-3i)$, and observe that we have

$$X_1(\zeta_0) = \langle (0, 2+i, 1), (0, 1+i, -1-3i) \rangle = 0$$

and

$$\{X_1, X_0\}(\zeta_0) = \langle (0, 2+i, 0), (0, 1+i, -1-3i) \rangle = 1 + 3i \neq 0.$$

This shows that, even if both the tangency condition and (H2) are satisfied by the system of complex vector fields (here we have an easier situation since we are considering just one complex vector field), we do not have as a consequence the validity of (H2$'$).

Remark 3.4.9. *However, still under hypotheses* $(H1)$ *and* $(H2)$*, it is possible to obtain the same solvability result in a specific case in which the tangency property is not required. The case under consideration is when* $N = 1$*, that is, when just a vector field is involved in the second order part of* P *(precisely* $P = X_1^* g|g|X_1 + iX_0 + a_0$ *where* X_1 *has complex constant coefficients and* iX_0 *has affine real coefficients), and the coefficients of* $X_1(D) = \langle \alpha, D \rangle$*,* $\alpha \in \mathbb{C}^n$*, are such that* $\mathsf{Re}(\alpha)$ *and* $\mathsf{Im}(\alpha)$ *are two linearly dependent vectors in* $\mathbb{R}^n$*. Under this assumption (and still under hypotheses* $(H1)$*-*$(H2)$*) we have that there exists* $z_0 \in \mathbb{C}$ *such that* $[X_1, X_0](D) = z_0 X_1(D)$*. In fact, since* $\{X_1, X_0\}(\xi) = \langle \gamma, \xi \rangle = \langle \gamma_1 + i\gamma_2, \xi \rangle$*,* γ_1*,* $\gamma_2 \in \mathbb{R}^n$*, and* $X_1(\xi) = \langle \alpha, \xi \rangle = \langle (1+i\lambda)\alpha_1, \xi \rangle$*,* $\alpha_1 \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$ *(we used the assumption that* $\mathsf{Re}(\alpha)$ *and* $\mathsf{Im}(\alpha)$ *are linearly dependent), then by condition* $(H2)$ *we have*

$$|\langle \gamma_1 + i\gamma_2, \xi \rangle|^2 \leq C|\langle (1+i\lambda)\alpha_1, \xi \rangle|^2, \quad \forall \xi \in \mathbb{R}^n,$$

which in particular means that $\mathrm{Span}_{\mathbb{R}}\{\gamma_1, \gamma_2\}^\perp \supseteq \mathrm{Span}_{\mathbb{R}}\{\alpha_1\}^\perp$*. Finally, passing to the orthogonal complements we get* $\mathrm{Span}_{\mathbb{R}}\{\gamma_1, \gamma_2\} \subseteq \mathrm{Span}_{\mathbb{R}}\{\alpha_1\}$*, and therefore can find a unique suitable* $z_0 \in \mathbb{C}$ *such that* $\gamma = z_0\alpha$ *and* $[X_1, X_0](D) = z_0 X_1(D)$*.*

This last property is sufficient to prove that $P = X_1^* g|g|X_1 + iX_0 + a_0$ *(where* X_1 *is not tangent to* S *anymore but has coefficients with a suitable structure) is* L^2*-locally solvable in a neighborhood of* S*. In fact, since we may still assume* $g(x) = x_1$*, and since Proposition 3.2.1 still holds in this case, again we have that for all* $x_0 \in S$ *there exist a compact set* K_0*, containing* x_0 *in its interior, and three positive constants* $C(K_0), c(K_0), \varepsilon_0(K_0)$ *such that, for every compact set* $K \subset K_0$

$$\|P^*u\|^2 \geqslant \frac{1}{4}\|X_0u\|^2 + c(P_0u,u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K),$$

where now P_0 *is given by*

$$P_0 = X_1^*|x_1|X_1 - \varepsilon_0^2[X_1,X_0]^*|x_1|[X_1,X_0].$$

Once more to prove the L^2*-local solvability it is sufficient to prove the solvability estimate (3.2.3). Thus, by showing that* $(P_0u,u) \geqslant 0$ *and by using the Poincaré inequality for* iX_0*, the estimate (3.2.3) will follow. So we start by looking at* (P_0u,u) *which is, due to the property* $[X_1,X_0](D) = z_0X_1(D)$*, given by*

$$\begin{aligned}(P_0u,u) &= \||x_1|^{1/2}X_1u\|^2 - \varepsilon_0^2\||x_1|^{1/2}[X_1,X_0]u\|^2 \\ &= (1-\varepsilon_0^2|z_0|^2)\||x_1|^{1/2}X_1u\|^2\end{aligned}$$

for all $u \in C_0^\infty(K)$*. Since* $\varepsilon_0 = \varepsilon_0(K_0)$*, and in particular* ε_0 *shrinks when* K_0 *is shrunk (see Proposition 3.2.1), we can then suitably shrink* K_0 *to a compact set that we still denote by* K_0 *and which contains* x_0 *in its interior, so that* $\varepsilon_0 \leqslant 1/(2|z_0|^2)$*. Choosing* K_0 *in this way we have, for all compact* $K \subset K_0$*, that* $(P_0u,u) \geqslant 0$ *for all* $u \in C_0^\infty(K)$*, and moreover*

$$\|P^*u\|^2 \geqslant \frac{1}{4}\|X_0u\|^2 - C\|u\|^2, \quad \forall u \in C_0^\infty(K).$$

Then one ends the proof using Lemma 2.6.2 as before.

Remark 3.4.10. *It is important to observe that in the special case shown in Remark 3.4.9 we have that* (H2′) *is always satisfied, since* $[X_0,X_1](D) = z_0X_1(D)$ *for a suitable* $z_0 \in \mathbb{C}$*. Therefore, when* $N = 1$*,* X_1 *has linearly dependent real and imaginary part and (H2) is true then we trivially get that also* (H2′) *is true. This suggests that when we consider a general system of complex vector fields (not necessarily tangent) in the second order part of the operator* (3.4.2) *we need to require the stronger condition* (H2′) *to get the* L^2 *to* L^2 *local solvability result. This shows the difference between the real and the complex coefficients case and the difficulties carried by the complex case.*

On the other hand, by Remark 3.4.9, we conclude that there is an explicit subclass of operators in (3.4.2) *always satisfying* (H2′)*, which is in general a too strong condition to require.*

3.5 A further class

In this final section we study the solvability of a class of operators similar to the previous one, that is

$$P = \sum_{j=1}^{N} X_j^* |f| X_j + iX_0 + a_0, \tag{3.5.2}$$

where $X_j = X_j(x, D)$, $0 \leq j \leq N$, are homogeneous first order differential operators with smooth coefficients defined on an open set $\Omega \subseteq \mathbb{R}^n$ and with a real principal symbol, $f : \Omega \longrightarrow \mathbb{R}$ is a C^1 function with $f^{-1}(0) \neq \emptyset$ and $df|_S \neq 0$, and a_0 is a continuous possibly complex valued function.

This class is more "general" in the sense that the vector fields iX_j (or simply X_j in this case, since they can also be complex), $1 \leq j \leq N$, are not necessarily with *constant* coefficients but they are given in general with variable coefficients, and iX_0 is not required to have affine real coefficients but smooth variable coefficients. Moreover note that, in this case, the coefficients of our operator P could have $C^{0,1}$ or L^∞ regularity depending on the tangency or transversality, respectively, to the zero set of f of the vector fields X_j, $1 \leq j \leq N$, which is less demanding as far as the regularity of the coefficients in the preceding examples is concerned.

Our purpose is to prove also in this case an L^2 to L^2 local solvability result in a neighborhood of the zero set of the function f, that we keep denoting by S and which is non-empty by hypothesis. The method used here is that of Carleman estimates.

We assume now only the following assumption

(H1) $X_0 f \neq 0$ for all $x \in S := f^{-1}(0) \neq \emptyset$.

Theorem 3.5.1. *Let the operator P in (3.5.2) satisfy hypothesis* (H1)*. Then for all $x_0 \in S$ there exists a compact set $K \subset \Omega$ with $U = \overset{\circ}{K}$ and $x_0 \in U$, such that for all $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ there exists $u \in L^2(U)$ solving $Pu = v$ in U in the sense of Definition 3.0.1.*

Proof. We take $e^{2\lambda f}$, where f is the function appearing in P and λ is a real number that we will choose later. Observe that, for all $u \in C_0^\infty(\Omega)$, we have

$$2\mathrm{Re}(P^* u, e^{2\lambda f} u) = \sum_{j=1}^{N} \mathrm{Re}(X_j^* |f| X_j u, e^{2\lambda f} u)$$
$$-\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) + \mathrm{Re}(\bar{a}_0 u, e^{2\lambda f} u)$$

$$= 2\lambda \sum_{j=1}^{N} \mathrm{Re}(|f|X_j u, (X_j f)e^{2\lambda f} u) + \sum_{j=1}^{N} \mathrm{Re}(|f|X_j u, e^{2\lambda f} X_j u)$$

$$-\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) + \mathrm{Re}(\bar{a}_0 u, e^{2\lambda f} u)$$

$$= \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2 + 2\lambda \sum_{j=1}^{N} \mathrm{Re}(e^{\lambda f}|f|^{1/2} X_j u, e^{\lambda f}|f|^{1/2}(X_j f)u)$$

$$-\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) + \mathrm{Re}(\bar{a}_0 u, e^{2\lambda f} u).$$

Furthermore, since $X_0^* = X_0 + d_{X_0}$, where $d_{X_0} = -i\,\mathrm{div}(iX_0)$,

$$\begin{array}{rcl}
\mathrm{Re}(iX_0^* u, e^{2\lambda f} u) & = & \frac{1}{2}\big[(iX_0^* u, e^{2\lambda f} u) + (e^{2\lambda f} u, iX_0^* u)\big] \\
& = & \frac{1}{2}\big[2\lambda(u, (-iX_0 f)e^{2\lambda f} u) + (e^{2\lambda f} u, -iX_0 u) \\
& + & (e^{2\lambda f} u, iX_0 u) + (e^{2\lambda f} u, id_{X_0} u)\big] \\
& = & \lambda\big((iX_0 f)e^{\lambda f} u, e^{\lambda f} u\big) + \frac{1}{2}(e^{\lambda f} u, ie^{\lambda f} d_{X_0} u),
\end{array}$$

and therefore

$$2\mathrm{Re}(P^* u, e^{2\lambda f} u) \geq \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2$$

$$-2|\lambda| \sum_{j=1}^{N} \Big|\mathrm{Re}(e^{\lambda f}|f|^{1/2} X_j u, |f|^{1/2} e^{\lambda f}(X_j f)u)\Big| - \lambda\big((iX_0 f)e^{\lambda f} u, e^{\lambda f} u\big)$$

$$-\frac{1}{2}(e^{\lambda f} u, ie^{\lambda f} d_{X_0} u) - \|a_0\|\|e^{\lambda f} u\|^2, \quad \forall u \in C_0^\infty(\Omega).$$

Recall that (H1) states that $iX_0 f \neq 0$ on S, which yields that for all $x_0 \in S$ there exists a compact $K_0 \subset \Omega$ containing x_0 in its interior such that $iX_0 f \neq 0$ on K_0, and in particular it has a constant positive or negative sign in K_0. Hence, if $iX_0 f > 0$ in K_0, then we choose λ negative so that $-\lambda(iX_0 f) = |\lambda||iX_0 f| > c_0$ in K_0 for some positive constant c_0, otherwise if $iX_0 f < 0$ in K_0 we choose λ positive so that $-\lambda(iX_0 f) = |\lambda||iX_0 f| > c_0$ in K_0. Thus, by choosing λ having the appropriate sign, we have

$$2\mathrm{Re}(P^* u, e^{2\lambda f} u) \geq \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2 - \delta|\lambda| \sum_{j=1}^{N} \||f|^{1/2} e^{\lambda f} X_j u\|^2$$

$$
\begin{aligned}
&-\frac{|\lambda|}{\delta}\||f|^{1/2}\|^2_{L^\infty(K_0)}\sum_{j=1}^N \|X_j f\|^2_{L^\infty(K_0)}\|e^{\lambda f}u\|^2 + c_0|\lambda|\|e^{\lambda f}u\|^2\\
&-\|d_{X_0}\|_{L^\infty(K_0)}\|e^{\lambda f}u\|^2 - \|a_0\|_{L^\infty(K_0)}\|e^{\lambda f}u\|^2\\
&= (1-\delta|\lambda|)\sum_{j=1}^N \||f|^{1/2}e^{\lambda f}X_j u\|^2 + |\lambda|\Big(c_0\\
&-\frac{1}{\delta}\||f|^{1/2}\|^2_{L^\infty(K_0)}\sum_{j=1}^N \|X_j f\|^2_{L^\infty(K_0)} - \frac{\|d_{X_0}\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{|\lambda|}\Big)\|e^{\lambda f}u\|^2,
\end{aligned}
$$

for all $u \in C_0^\infty(K_0)$. Now we fix $\lambda := \lambda_0$ (with the sign previously chosen) such that $|\lambda_0|$ is so big that

$$
c_0 - \frac{\|d_{X_0}\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{|\lambda|} \geq \frac{c_0}{2}.
$$

In addition we choose $\delta := 1/(2|\lambda_0|)$ so that

$$
\begin{aligned}
\operatorname{Re}(P^*u, e^{2\lambda_0 f}u) &\geq \frac{1}{2}\sum_{j=1}^N \||f|^{1/2}e^{\lambda_0 f}X_j u\|^2 + |\lambda_0|\Big(\frac{c_0}{2}\\
&-2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(K_0)}\sum_{j=1}^N \|X_j f\|^2_{L^\infty(K_0)}\Big)\|e^{\lambda_0 f}u\|^2\\
&\geq |\lambda_0|\Big(\frac{c_0}{2} - 2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(K_0)}\sum_{j=1}^N \|X_j f\|^2_{L^\infty(K_0)}\Big)\|e^{\lambda_0 f}u\|^2,
\end{aligned}
$$

for all $u \in C_0^\infty(K_0)$, and in particular for all $u \in C_0^\infty(K)$ for every compact $K \subset K_0$. Since $x_0 \in K_0$ and $f(x_0) = 0$, we can find a compact set $K \subset K_0$ sufficiently small and containing x_0 in its interior such that

$$
\begin{aligned}
&\frac{c_0}{2} - 2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(K_0)}\sum_{j=1}^N \|X_j f\|^2_{L^\infty(K_0)}\\
&\geq \frac{c_0}{2} - 2|\lambda_0|\||f|^{1/2}\|^2_{L^\infty(\tilde{K})}\sum_{j=1}^N \|X_j f\|^2_{L^\infty(K_0)} \geq \frac{c_0}{4},
\end{aligned}
$$

whence

$$
|\operatorname{Re}(P^*u, e^{2\lambda_0 f}u)| \geq \operatorname{Re}(P^*u, e^{2\lambda_0 f}u) \geq |\lambda_0|\frac{c_0}{4}\|e^{\lambda_0 f}u\|^2, \quad \forall u \in C_0^\infty(K),
$$

and finally

$$e^{2\lambda_0\|f\|_{L^\infty(\tilde{K})}}\|P^*u\|\|u\| \geq |\lambda_0|\frac{c_0}{4}e^{-2\lambda_0\|f\|_{L^\infty(\tilde{K})}}\|u\|^2, \quad \forall u \in C_0^\infty(K).$$

In conclusion, we have shown that for all $x_0 \in S$ there exist a compact set K containing x_0 in its interior and a positive constant (depending on the compact K) $C = |\lambda_0|\frac{c_0}{4}e^{-4\lambda_0\|f\|_{L^\infty(K)}}$ such that

$$\|P^*u\| \geq C\|u\|, \quad \forall u \in C_0^\infty(K).$$

The latter inequality is exactly the solvability estimate (3.2.3) that we were searching for, thus the proof follows directly by once more using standard functional analysis arguments. $\square$

Chapter 4

Loss of one derivative for the adjoint of the Kannai operator

Let us consider the Kannai operator on $\mathbb{R}^n$

$$P = x_1|D_{x'}|^2 - iD_1, \tag{4.0.1}$$

where $x' = (x_2, ..., x_n)$, $|D_{x'}|^2 = \sum_{j=2}^n D_j^2$, and $D_j = -i\partial_{x_j}$.

As we said before, this operator is a fundamental example to look at when we deal with the local solvability of an operator with a changing sign principal symbol. In particular P is a hypoelliptic operator on the whole $\mathbb{R}^n$ having a changing sign principal symbol in the neighborhood of the set $S = \{x \in \mathbb{R}^n; x_1 = 0\}$ at which the operator is not locally solvable.

Note also that, even if P has an expression of the form (1.1.1), it is not in the class of operators considered before, that is, in the class consisting of operators of the form (1.1.1) satisfying conditions (H1) and (H2) (or (H1), (H2) and (H3)), since condition (H1) is not satisfied by the latter, and, consequently, Theorem 2.2.1 does not apply on P.

It is also worth to remark that Theorem 2.2.1 gives sufficient conditions for the local solvability of the class (1.1.1), thus we are not allowed to say that P is unsolvable at S because the hypotheses of Theorem 2.2.1 are violated. This shows that the nonsolvability of P comes from the changing sign property of its principal symbol and has no relations with the fact that P is not in the class (1.1.1).

Conversely, due to the hypoellipticity of P, the adjoint P^* is a locally solvable partial differential operator over the whole $\mathbb{R}^n$ which has a changing sign principal symbol in the neighborhood of the set $S = \{x \in \mathbb{R}^n; x_1 = 0\}$. Of course, since P^* is in the class (1.1.1) (containing operators which are not necessarily adjoints of hypoelliptic ones), we can get the local solvability of P^* around S also by using Theorem 2.2.1, and the local solvability of P^* off the set S by using Theorem 2.8.2.

Moreover, by virtue of the hypoellipticity property of P in (4.0.1), we can say more about the regularity property of its adjoint P^*. In fact, once more by using the technique of a priori estimates, we shall prove in this chapter that P^* is locally solvable in $\mathbb{R}^n$ with loss of one derivative (that is, H^s to H^{s+1} locally solvable $\forall s \in \mathbb{R}$).

Our starting point will be the estimate given by the following Proposition.

Proposition 4.0.1. *For every $x_0 \in \mathbb{R}^n$ there exists a compact set K, which contains x_0 in its interior that we denote by $\mathring{K} = U$, and a positive constant C such that*

$$\|Pu\|_0 \geq C\|u\|_1, \quad \forall u \in C_0^\infty(U), \tag{4.0.2}$$

where $\|\cdot\|_s$ denotes the norm in the Sobolev space H^s.

The previous inequality yields (see Definition given by Corollary 2.1.6) that for all $f \in H^{-1}_{\text{loc}}(\mathbb{R}^n)$ there exists a function $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that $P^*u = f$ in $U = \mathring{K}$, where $\mathring{K}$ denotes the interior of K.

Proof. Let K be an arbitrary compact set such that $x_0 \in \mathring{K}$, then, for every $u \in C_0^\infty(K)$, we have

$$\|Pu\|_0^2 = \|D_1u\|_0^2 + \sum_{j=2}^{n} \|x_1 D_j^2 u\|_0^2 + \sum_{j=2}^{n} 2\mathsf{Re}(-iD_1u, x_1D_j^2u). \tag{4.0.3}$$

Since

$$2\mathsf{Re}(-iD_1u, x_1D_j^2u) = (-iD_1u, x_1D_j^2u) + (x_1D_j^2u, -iD_1u) = \|D_ju\|^2, \quad \forall u \in C_0^\infty(K),$$

this means that

$$\|Pu\|_0^2 \geq \sum_{j=1}^{n} \|D_ju\|_0^2, \quad \forall u \in C_0^\infty(K),$$

which is equivalent to

$$\|Pu\|_0^2 \geq \sum_{j=1}^{n} \|D_ju\|_0^2 + \|u\|_0^2 - \|u\|_0^2 = \|u\|_1^2 - \|u\|_0^2, \quad \forall u \in C_0^\infty(K).$$

By the Poincaré inequality, for all $u \in C_0^\infty(K)$, there exists a positive constant C such that $\|u\|_0^2 \leq C\text{diam}(K)^2\|u\|_1^2$. Therefore

$$\|Pu\|_0^2 \geq (1 - C\text{diam(K)}^2)\|u\|_1^2, \quad \forall u \in C_0^\infty(K).$$

Finally, by suitable shrinking K around x_0 to a compact set, that we keep denoting by K, in such a way that $1 - C\mathrm{diam}(\mathrm{K})^2 \geq 1/2$, we get the result. $\square$

To prove that P^* is locally solvable (on $\mathbb{R}^n$) with a loss of one derivative we need to show that: *for all $x_0 \in \mathbb{R}^n$ there exists a compact set K, which contains x_0 in its interior U, such that for all $s \in \mathbb{R}$ there exists a positive constant C for which we have*

$$\|Pu\|_s \geq C\|u\|_{s+1}, \quad \forall u \in C_0^\infty(U). \tag{4.0.4}$$

Let us consider an arbitrary $x_0 \in \mathbb{R}^n$ and a suitable compact set K, with $x_0 \in \mathring{K} = U$, such that (4.0.2) holds for all $u \in C_0^\infty(U)$ (we will work throughout in this fixed compact set K).

Let Λ_s and $\Lambda_{s,\varepsilon}$, with $s, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be the pseudodifferential operators of order s with total symbols $\lambda_s(\xi) = (1 + \xi_1^2 + |\xi'|^2)^{s/2}$ and $\lambda_{s,\varepsilon}(\xi) = (1 + \varepsilon^2\xi_1^2 + |\xi'|^2)^{s/2}$ respectively, and note that, for every fixed $\varepsilon > 0$, we have the following equivalent norms in the Sobolev space H^s

$$\|\cdot\|_s = \|\Lambda_s(\cdot)\|_0 \sim \|\cdot\|_{s,\varepsilon} = \|\Lambda_{s,\varepsilon}(\cdot)\|_0.$$

We take now three compact sets K', K'', K''', with $x_0 \in \mathring{K}'$, and three open neighborhoods V', V'', V''' of K', K'' and K''' respectively, such that $K' \Subset V' \Subset K'' \Subset V'' \Subset K''' \Subset V''' \Subset U = \mathring{K}$. We call $U' = \mathring{K}'$, $U'' = \mathring{K}''$ and $U''' = \mathring{K}'''$, and we take three smooth functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ such that

$$\begin{aligned} &\alpha \in C_0^\infty(U''), \quad && \alpha \equiv 1 \text{ on } V',\ 0 \leq \alpha \leq 1, \\ &\beta \in C_0^\infty(U'''), && \beta \equiv 1 \text{ on } V'',\ 0 \leq \beta \leq 1, \\ &\gamma \in C_0^\infty(U), && \gamma \equiv 1 \text{ on } V''',\ 0 \leq \gamma \leq 1. \end{aligned} \tag{4.0.5}$$

Finally we define the properly supported pseudodifferential operators $E'_{s,\varepsilon}$, $E''_{s,\varepsilon}$, $E''_{s,\varepsilon}$ of order s with total symbols

$$\begin{aligned} e'_{s,\varepsilon}(x,\xi) &= \alpha(x)\lambda_{s,\varepsilon}(\xi) \\ e''_{s,\varepsilon}(x,\xi) &= \beta(x)\lambda_{s,\varepsilon}(\xi) \\ e'''_{s,\varepsilon}(x,\xi) &= \gamma(x)\lambda_{s,\varepsilon}(\xi) \end{aligned} \tag{4.0.6}$$

respectively. We recall that $E'_{s,\varepsilon}$ maps $C_0^\infty(U'')$ in itself, $E''_{s,\varepsilon}$ maps $C_0^\infty(U''')$ in itself, and $E'''_{s,\varepsilon}$ maps $C_0^\infty(U)$ in itself.

Note also that

$$
\begin{aligned}
E'_{s,\varepsilon} &= \Lambda_{s,\varepsilon} + R'_{s,\varepsilon} \quad \text{on } \mathcal{E}'(K'),\\
E''_{s,\varepsilon} &= \Lambda_{s,\varepsilon} + R''_{s,\varepsilon} \quad \text{on } \mathcal{E}'(K''),\\
E'''_{s,\varepsilon} &= \Lambda_{s,\varepsilon} + R'''_{s,\varepsilon} \quad \text{on } \mathcal{E}'(K'''),
\end{aligned}
\tag{4.0.7}
$$

where $R'_{s,\varepsilon}$, $R''_{s,\varepsilon}$, and $R'''_{s,\varepsilon}$ (which depend on s and ε) are regularizing when acting on compactly supported distributions on the compact sets K', K'' and K''', respectively. In fact we have that $R'_{s,\varepsilon} = (\alpha(x) - 1)\Lambda_{s,\varepsilon}$, and its kernel is given by the oscillatory integral

$$
\mathsf{K}_{R'_{s,\varepsilon}}(x,y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle}(1-\alpha(x))\lambda_{s,\varepsilon}(\xi)\chi(y)d\xi,
$$

where $\chi \in C_0^\infty(V')$ is such that $\chi \equiv 1$ on $\operatorname{supp}(u)$ for all $u \in C_0^\infty(K')$. Since $1-\alpha(x) \equiv 0$ on V', we have that $\mathsf{K}_{R'_{s,\varepsilon}}$ is supported outside the diagonal of $K' \times K'$, whence it is regularizing on $\mathcal{E}'(K')$. By using the same argument, we find the analogous property for $R''_{s,\varepsilon}$ and $R'''_{s,\varepsilon}$.

In the sequel we will use the following identities

$$
\begin{aligned}
E'''_{s,\varepsilon}E''_{-s,\varepsilon} &= I + \tilde{R}_{s,\varepsilon} \quad \text{on } \mathcal{E}'(K''),\\
E''_{-s,\varepsilon}E'_{s,\varepsilon} &= I + \tilde{R}'_{s,\varepsilon} \quad \text{on } \mathcal{E}'(K'),
\end{aligned}
\tag{4.0.8}
$$

where $\tilde{R}_{s,\varepsilon}$, $\tilde{R}'_{s,\varepsilon}$ are regularizing on $\mathcal{E}'(K'')$ and $\mathcal{E}'(K')$ respectively.

To prove the first equality in (4.0.8) we compute the symbol of the pseudodifferential operator of order zero $E'''_{s,\varepsilon}E''_{-s,\varepsilon}$, which is $e'''_{s,\varepsilon}\sharp e''_{-s,\varepsilon}(x,\xi)$ (modulo $S^{-\infty}$), where $\sharp$ denotes the composition in the symbolic calculus. We have

$$
\begin{aligned}
e'''_{s,\varepsilon}\sharp e''_{-s,\varepsilon}(x,\xi) &= \sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial_\xi^\alpha e'''_{s,\varepsilon}(x,\xi)D_x^\alpha e''_{-s,\varepsilon}(x,\xi)\\
&= \gamma(x)\lambda_{s,\varepsilon}(\xi)\beta(x)\lambda_{-s,\varepsilon}(\xi) + \sum_{\alpha\geq 1}\frac{1}{\alpha!}\partial_\xi^\alpha\big(\gamma(x)\lambda_{s,\varepsilon}(\xi)\big)D_x^\alpha\big(\beta(x)\lambda_{-s,\varepsilon}(\xi)\big).
\end{aligned}
\tag{4.0.9}
$$

Observe now that $E'''_{s,\varepsilon}E''_{-s,\varepsilon} = \gamma(x)\beta(x)I + \tilde{R}_{s,\varepsilon}$, where $\gamma(x)\beta(x) = 1$ on V'', and $\tilde{R}_{s,\varepsilon}$ is smoothing (by the same argument used above in the proof of (4.0.7)), thus the first relation in (4.0.8) follows, and also the second one by the same arguments.

We will call throughout $C_{s,\varepsilon}$ every constant depending on s and ε possibly different at each appearance. Since $E'_{s,\varepsilon} = \Lambda_{s,\varepsilon} + R'_{s,\varepsilon}$ on $C_0^\infty(U')$, for all $s \in \mathbb{R}$ and for every fixed $\varepsilon > 0$ there exists a positive constant $C_{s,\varepsilon}$ such that, for all $u \in C_0^\infty(U')$,

$$
\|u\|_{s+1,\varepsilon} \leq \|E'_{s+1,\varepsilon}u\|_0 + C_{s,\varepsilon}\|u\|_{s,\varepsilon}. \tag{4.0.10}
$$

Note also that when $0 < \varepsilon \leq 1$ then by (4.0.7) there is a positive constant (which depends on s and ε and that we keep denoting by $C_{s,\varepsilon}$) such that for all $u \in C_0^\infty(U')$

$$\|E'_{s+1,\varepsilon}u\|_0 = \|\Lambda_{s+1,\varepsilon}u + R'_{s+1,\varepsilon}u\|_0 \underset{(0<\varepsilon\leq 1)}{\leq} \|\Lambda_{s,\varepsilon}u\|_1 + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \underset{(4.0.7)}{\leq} \|E'_{s,\varepsilon}u\|_1 + C_{s,\varepsilon}\|u\|_{s,\varepsilon}.$$

Therefore, taking $\varepsilon \in (0,1]$ and using the previous inequality in (4.0.10), we have that for every $s \in \mathbb{R}$ there exists $C > 0$ such that for all $u \in C_0^\infty(U')$

$$\begin{aligned}
\|u\|_{s+1,\varepsilon} &\leq \|E'_{s,\varepsilon}u\|_1 + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \\
&\underset{(4.0.2)}{\leq} C\|PE'_{s,\varepsilon}u\|_0 + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \\
&\underset{(4.0.8)}{=} C\|(I + \tilde{R}_{s,\varepsilon} - \tilde{R}_{s,\varepsilon})PE'_{s,\varepsilon}u\|_0 + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \\
&\leq C\|E'''_{s,\varepsilon}E''_{-s,\varepsilon}PE'_{s,\varepsilon}u\|_{s,\varepsilon} + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \\
&\underset{(4.0.7)}{\leq} C\|E''_{-s,\varepsilon}PE'_{s,\varepsilon}u\|_{s,\varepsilon} + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \\
&= C\|E''_{-s,\varepsilon}[P, E'_{s,\varepsilon}]u + E''_{-s,\varepsilon}E'_{s,\varepsilon}Pu\|_{s,\varepsilon} + C_{s,\varepsilon}\|u\|_{s,\varepsilon} \\
&\underset{(4.0.8)}{\leq} C(\|E''_{-s,\varepsilon}[P, E'_{s,\varepsilon}]u\|_{s,\varepsilon} + \|Pu\|_{s,\varepsilon}) + C_{s,\varepsilon}\|u\|_{s,\varepsilon}.
\end{aligned} \tag{4.0.11}$$

We focus now our attention on the term $\|E''_{-s,\varepsilon}[P, E'_{s,\varepsilon}]u\|_{s,\varepsilon}$.

We call for short L the pseudodifferential operator of order one given by $L := E''_{-s,\varepsilon}[P, E'_{s,\varepsilon}]$. Observe that, denoting by $p(x,\xi) = x_1|\xi'|^2 - i\xi_1$ the symbol of P, we have that the part $l_1(x,\xi)$ of the symbol $l(x,\xi)$ of L which contains the principal symbol of L is given by

(4.0.12)

$$\begin{aligned}
l_1(x,\xi) &= e''_{-s,\varepsilon}\sharp\{p, e'_{s,\varepsilon}\}(x,\xi) \\
&= e''_{-s,\varepsilon}(x,\xi)\{p, e'_{s,\varepsilon}\}(x,\xi) + \textstyle\sum_{|\alpha|\geq 1} \frac{1}{\alpha!}\partial_\xi^\alpha e''_{-s,\varepsilon}(x,\xi) D_x^\alpha\{p, e'_{s,\varepsilon}\}(x,\xi) \\
&= e''_{-s,\varepsilon}(x,\xi)\{x_1|\xi'|^2, e'_{s,\varepsilon}\}(x,\xi) - e''_{-s,\varepsilon}(x,\xi)\{i\xi_1, e'_{s,\varepsilon}\}(x,\xi) \\
&+ \textstyle\sum_{|\alpha|\geq 1} \partial_\xi^\alpha e''_{-s,\varepsilon}(x,\xi) D_x^\alpha\{p, e'_{s,\varepsilon}\}(x,\xi),
\end{aligned}$$

where

(4.0.13)
$$\begin{aligned}
e''_{-s,\varepsilon}(x,\xi)\{x_1|\xi'|^2, e'_{s,\varepsilon}\}(x,\xi) &= \beta(x)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{-s/2}\{x_1|\xi'|^2, \alpha(x)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{s/2}\} \\
&= \beta(x)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{-s/2}\Big(2x_1(\sum_{k=2}^n \xi_k\partial_{x_k}\alpha(x))(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{s/2} \\
&\qquad - s|\xi'|^2\alpha(x)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{s/2-1}\varepsilon^2\xi_1\Big) \\
&= 2x_1\beta(x)\Big(\sum_{k=2}^n \partial_{x_k}\alpha(x)\xi_k\Big) - \varepsilon\beta(x)\alpha(x)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{-1}(\varepsilon\xi_1)|\xi'|^2.
\end{aligned}$$

By using (4.0.12) and (4.0.13) we can write $l_1(x,\xi) = l_1'(x,\xi) + l_1''(x,\xi) + l_0(x,\xi)$, where

$$\begin{aligned}
& l_1'(x,\xi) = 2x_1\beta(x)\Big(\sum_{k=2}^n (\partial_{x_k}\alpha(x))\xi_k\Big), \\
\text{(4.0.14)}\quad & l_1''(x,\xi) = -\varepsilon\beta(x)\alpha(x)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^{-1}(\varepsilon\xi_1)|\xi'|^2 \\
& l_0(x,\xi) = -e''_{-s,\varepsilon}(x,\xi)\{i\xi_1, e'_{s,\varepsilon}\}(x,\xi) + \sum_{\alpha\geq 1}\partial_\xi^\alpha e''_{-s,\varepsilon}(x,\xi)D_x^\alpha\{p, e'_{s,\varepsilon}\}(x,\xi),
\end{aligned}$$

and this allows us to write $L = L_1' + L_1'' + L_0$, where L_1', L_1'', L_0 are the pseudodifferential operators with symbols $l_1'(x,\xi)$, $l_1''(x,\xi)$ and $l(x,\xi) - l_1'(x,\xi) - l_1''(x,\xi)$, respectively. Note that L_1' is a differential operator of order one, thus, by (4.0.5), $L_1'u = 0$ for every $u \in C_0^\infty(U')$, L_1'' is a pseudodifferential operator of order one, and L_0 is a pseudodifferential operator of order zero. This means that, for a suitable positive constant $C_{s,\varepsilon}$, we have

$$\|Lu\|_{s,\varepsilon} = \|L_1''u + L_0u\|_{s,\varepsilon} \leq \|L_1''u\|_{s,\varepsilon} + \|L_0u\|_{s,\varepsilon} \leq \|L_1''u\|_{s,\varepsilon} + C_{s,\varepsilon}\|u\|_{s,\varepsilon}, \tag{4.0.15}$$

for all $u \in C_0^\infty(U')$.
Since $\|L_1''u\|_{s,\varepsilon} \leq \|L_1''\Lambda_{s,\varepsilon}u\|_0 + \|[\Lambda_{s,\varepsilon}, L_1'']u\|_0 \leq \|L_1''\Lambda_{s,\varepsilon}u\|_0 + C_{s,\varepsilon}\|u\|_{s,\varepsilon}$, then, for a suitable (new) constant $C_{s,\varepsilon} > 0$,

$$\|Lu\|_{s,\varepsilon} \leq \|L_1''\Lambda_{s,\varepsilon}u\|_0 + C_{s,\varepsilon}\|u\|_{s,\varepsilon}, \quad \forall u \in C_0^\infty(U'). \tag{4.0.16}$$

We now estimate the term $\|L_1''\Lambda_{s,\varepsilon}u\|_0$.

Observe that, since the symbol $l_1''(x,\xi)\lambda_{s,\varepsilon}(\xi)$ can be written as

$$l_1''(x,\xi)\lambda_{s,\varepsilon}(\xi) = \varepsilon(-2\beta(x)\alpha(x))\frac{(\varepsilon\xi_1)|\xi'|^2}{\lambda_{3,\varepsilon}(\xi)}\lambda_{s+1,\varepsilon}(\xi) := \varepsilon a(x)b_\varepsilon(\xi)\lambda_{s+1,\varepsilon}(\xi),$$

then, denoting by B_ε the pseudodifferential operator of order zero with symbol $b_\varepsilon(\xi)$, we have

$$\|L_1''\Lambda_{s,\varepsilon}u\|_0 = \varepsilon\|aB_\varepsilon\Lambda_{s+1,\varepsilon}u\|_0, \quad \forall u \in C_0^\infty(U').$$

Moreover $a \in C_0^\infty(U'')$ and $w := B_\varepsilon\Lambda_{s+1,\varepsilon}u \in \mathscr{S}(\mathbb{R}^n)$ for all $u \in C_0^\infty(U')$. Thus $\varepsilon\|aw\|_0 \le \varepsilon\|a\|_\infty\|w\|_0 = \varepsilon C_a\|B_\varepsilon\Lambda_{s+1,\varepsilon}u\|_0$ for all $u \in C_0^\infty(U')$, where $C_a = \|a\|_\infty$.
We now write $B_\varepsilon\Lambda_{s+1,\varepsilon}u$ as $B_\varepsilon v$, where $v := \Lambda_{s+1,\varepsilon}u \in \mathscr{S}(\mathbb{R}^n)$ for all $u \in C_0^\infty(U')$. Since $b_\varepsilon(\xi)\widehat{v}(\xi) = \widehat{B_\varepsilon v}(\xi)$, where $\widehat{B_\varepsilon v}$ denotes the Fourier transform of $B_\varepsilon v$, we have by Parseval's identity

$$\varepsilon C_a\|B_\varepsilon v\|_0 = \varepsilon C_a(2\pi)^{-n/2}\|\widehat{B_\varepsilon v}\|_0 = \varepsilon C_a(2\pi)^{-n/2}\|b_\varepsilon\widehat{v}\|_0, \quad \forall v \in \mathscr{S}(\mathbb{R}^n).$$

In order to prove (4.0.10) we need only show that $\|b_\varepsilon\widehat{v}\|_0 \le C\|v\|_0$, where C is a positive constant independent of ε. In fact

$$\begin{aligned}
\|b_\varepsilon\widehat{v}\|_0^2 &= \int \frac{\varepsilon^2\xi_1^2|\xi'|^4}{(1+\varepsilon^2\xi_1^2+|\xi'|^2)^3}|\widehat{v}(\xi)|^2 d\xi \\
&\le \int \frac{(1+\varepsilon^2\xi_1^2+|\xi'|^2)(1+\varepsilon^2\xi_1^2+|\xi'|^2)^2}{(1+\varepsilon^2\xi_1^2+|\xi'|^2)^3}|\widehat{v}(\xi)|^2 d\xi \qquad (4.0.17)\\
&= \int |\widehat{v}(\xi)|^2 d\xi = \|\widehat{v}\|_0^2 = (2\pi)^n\|v\|_0^2, \quad \forall v \in \mathscr{S}(\mathbb{R}^n).
\end{aligned}$$

Therefore $\varepsilon C_a\|B_\varepsilon\Lambda_{s+1,\varepsilon}u\|_0 \le \varepsilon C_a\|\Lambda_{s+1,\varepsilon}u\|_0 = \varepsilon C_a\|u\|_{s+1,\varepsilon}$ for all $u \in C_0^\infty(U')$, and

$$\|L_1''\Lambda_{s,\varepsilon}u\|_0 = \varepsilon\|aB_\varepsilon\Lambda_{s+1,\varepsilon}u\|_0 \le \varepsilon C_a\|\Lambda_{s+1,\varepsilon}u\|_0 = \varepsilon C_a\|u\|_{s+1,\varepsilon}, \quad \forall u \in C_0^\infty(U').$$

Summarizing, by the previous inequality and (4.0.16), we have

$$\|Lu\|_{s,\varepsilon} \le \varepsilon C_a\|u\|_{s+1,\varepsilon} + C_{s,\varepsilon}\|u\|_{s,\varepsilon}, \quad \forall u \in C_0^\infty(U').$$

Using the last inequality in (4.0.11) we find

$$\|u\|_{s+1,\varepsilon} \le C\|Pu\|_{s,\varepsilon} + \varepsilon C_a\|u\|_{s+1,\varepsilon} + C_{s,\varepsilon}\|u\|_{s,\varepsilon}, \quad \forall u \in C_0^\infty(U').$$

We now fix $\varepsilon := \varepsilon_0 \in (0,1]$ such that $(1-\varepsilon_0 C_a) \ge 1/2$, and finally, with suitable new constants (in which we drop the dependence on ε_0 that is fixed now),

$$C\|Pu\|_{s,\varepsilon_0} + C_s\|u\|_{s,\varepsilon_0} \ge \|u\|_{s+1,\varepsilon_0}, \quad \forall u \in C_0^\infty(U').$$

Since for every fixed $\varepsilon > 0$ we have $\|\cdot\|_{s,\varepsilon} \sim \|\cdot\|_s$, the latter inequality also holds with the standard Sobolev norm $\|\cdot\|_s$ instead of $\|\cdot\|_{s,\varepsilon_0}$ (with new positive constants C and C_s).

In conclusion we have proved the following proposition.

Proposition 4.0.2. *For every $x_0 \in \mathbb{R}^n$ there exists a compact set K, containing x_0 in its interior $\mathring{K} = U$, such that for every $s \in \mathbb{R}$ there exist two positive constants C_1 and C_2 such that*

$$C_1 \|Pu\|_s + C_2 \|u\|_s \geq \|u\|_{s+1}, \quad \forall u \in C_0^\infty(U). \tag{4.0.18}$$

From the latter Proposition we can easily derive the following Corollary.

Corollary 4.0.3. *For every $s > -n/2$ and for every $x_0 \in \mathbb{R}^n$, there exists a compact set K, containing x_0 in its interior $\mathring{K} = U$, such that for all $f \in H^{-s}_{\mathrm{loc}}(\mathbb{R}^n)$ there exists a function $u \in H^{-(s-1)}_{\mathrm{loc}}(\mathbb{R}^n)$ for which $Pu = f$ in $U = \mathring{K}$.*

Proof. The proof follows directly from (4.0.18) and a Poincaré type inequality (see [14] Lemma 4.3.5) by suitable shrinking the compact set K of Proposition 4.0.2 around x_0. □

Remark 4.0.4. *Note that the result in Corollary 4.0.3 does not depend on the hypoellipticity of P, and it still holds for every operator which satisfies the hypotheses of Proposition 4.0.2. In general, given an arbitrary operator L for which* (4.0.18) *is true, we can prove more about the regularity properties of its solutions (depending on the regularity of the source term f) if the space $N(K) = \{u \in \mathscr{E}'(K); Lu = 0\}$ is both a subspace of $C_0^\infty(K)$ and finite dimensional. In order to prove these properties for $N(K)$, and thus to have a more general solvability result (even under hypotheses of Proposition 4.0.2), we need to use, if possible, either the hypoelliticity of the operator or a propagation of singularity argument.*

We can finally use the hypoellipticity of P and the result in Proposition 4.0.2 to prove the solvability estimate (4.0.4) we are looking for.

Theorem 4.0.5. *For every $x_0 \in \mathbb{R}^n$ there exists a compact set K, containing x_0 in its interior $\mathring{K} = U$, such that for every $s \in \mathbb{R}$ the estimate* (4.0.4) *holds for all $u \in C_0^\infty(K)$. In particular for every $s \in \mathbb{R}$ and for all $f \in H^{-s}_{\mathrm{loc}}(\mathbb{R}^n) \cap N(K)^\perp$, where $N(K) = \{u \in \mathscr{E}'(K); Pu = 0\}$ is such that $N(K) \subset C_0^\infty(K)$ and* $\dim N(K) < +\infty$, *there exists a function $u \in H^{-s+1}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $P^* u = f$ in U.*

Proof. Since P is hypoelliptic we have that

$$N(K) = \{u \in \mathscr{E}'(K); Pu = 0\}$$

is a closed subspace of $C_0^\infty(K)$. Moreover, by considering the identity map $\mathsf{id} : (N(K), C^\infty) \longrightarrow (N(K), L^2)$, from $N(K)$ with the C^∞ topology to $N(K)$ with the

L^2 topology, then we have by the closed graph theorem that the two topologies are equivalent on $N(K)$ (since also the inverse map id^{-1} is continuous). Moreover a closed ball in $(N(K), C^\infty)$ is a bounded and closed set in $C^\infty(K)$, therefore it is compact by the Heine-Borel property of $C^\infty(K)$ (see [23]). Hence the closed unit ball in $(N(K), L^2)$ is compact (by the equivalence of the L^2 and C^∞ topology over there) and therefore $\dim N(K) < +\infty$.

We next consider a supplementary space of $N(K)$ in $H^{s+1} \cap \mathscr{E}'(K)$ that we denote by V. If estimate (4.0.4) were not true we could find a sequence $\{v_j\} \subset V$ such that

$$\|v_j\|_{s+1} = 1, \quad \|Pv_j\|_s \to 0.$$

Since $H^{s+1} \cap \mathscr{E}'(K)$ is compactly embedded in $H^s \cap \mathscr{E}'(K)$, then there exists a subsequence $\{v_{j_k}\}$ of $\{v_j\}$ which converges in H^s to an element $v \in V$ so that $v \in H^{s+1} \cap \mathscr{E}'$ (since $v_{j_k} \xrightarrow{w} v \in V$ and $v_{j_k} \longrightarrow v$ in H^s) and $Pv = 0$. Finally this yields a contradiction, since by (4.0.18) we have $C_2\|v\|_s \geq 1$, and hence $0 \neq v \in V \cap N(K) = \{0\}$. The solvability statement of the theorem now follows by standard functional analysis arguments.

$\square$

Appendix A

The Poicaré inequality for nondegenerate vector fields

In this chapter we give the proof of the Poincaré inequality for nondegenerate vector fields stated in Lemma 2.6.2. In order to prove the result we need the following preliminary lemma.

Lemma A.0.1. *Let* $K \subset \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$ *be a compact set. Let* $\delta_K := \operatorname{diam} \pi_t(K)$*, where* $\pi_t : \mathbb{R}^{1+n}_{t,x} \longrightarrow \mathbb{R}_t$ *is the* t*-projection, so that* $K \subset I_K \times \mathbb{R}^n$*, with* $\operatorname{diam}(I_K) = \delta_K$*. We have*

$$\|u\|_0 \leq 2\delta_K \|\partial_t u\|_0, \quad \forall u \in C_0^\infty(K). \tag{A.0.1}$$

Proof. Let us assume that u is a real valued function, since, in the complex case, we simply apply the argument below to the real and to the imaginary part separately. By using the previous assumption we have that, for all $u \in C_0^\infty(K)$,

$$u(t,x)^2 = 2\int_{-\infty}^{t} \partial_s u(s,x) u(s,x) ds$$

$$\leq 2\Big(\int_{-\infty}^{+\infty} \partial_s u(s,x)^2 ds\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} u(s,x)^2 ds\Big)^{1/2}$$

$$= 2\|\partial_t u(\cdot,x)\|_{L^2(dt)} \|u(\cdot,x)\|_{L^2(dt)},$$

whence, since $K \subset I_K \times \mathbb{R}^n$,

$$\int u(t,x)^2 dt = \|u(\cdot,x)\|^2_{L^2(dt)} = \int_{I_K} u(t,x)^2 dt \leq 2\delta_K \|\partial_t u(\cdot,x)\|_{L^2(dt)} \|u(\cdot,x)\|_{L^2(dt)}.$$

Therefore, denoting by $\pi_x : \mathbb{R}^{1+n}_{t,x} \longrightarrow \mathbb{R}^n_x$ the x-projection, we get

$$\|u\|_0^2 = \int_{\pi_x(K)} \|u(\cdot,x)\|^2_{L^2(dt)} dx = \int_{I_K \times \pi_x(K)} u(t,x)^2 dt dx$$

$$\leq 2\delta_K \int_{\pi_x(K)} \|\partial_t u(\cdot,x)\|_{L^2(dt)} \|u(\cdot,x)\|_{L^2(dt)} dx \leq 2\delta_K \|\partial_t u\|_0 \|u\|_0,$$

by the Cauchy-Schwartz inequality. Thus, finally,

$$\|u\|_0 \leq 2\delta_K \|\partial_t u\|_0 \leq 2\delta_K \| \nabla_{t,x} u\|_0.$$

□

We recall below some basic definitions which will be useful in the sequel.

Definition A.0.2. *Let Ω and Ω' be two open sets in $\mathbb{R}^n$. Let $\phi : \Omega \mapsto \Omega'$ be a smooth diffeomorphism, and f a smooth function over Ω'. We call pull-back of f under ϕ the smooth function $\phi^* f$ over Ω' given by $\phi^* f := f \circ \phi$.*

Definition A.0.3. *Let Ω and Ω' be two open sets in $\mathbb{R}^n$. Let $\phi : \Omega \mapsto \Omega'$ be a smooth function, and $X \in C^\infty(\Omega, T\Omega)$ a smooth vector field. The push-forward of X is a smooth vector field $\phi_* X \in C^\infty(\Omega', T\Omega')$ defined by*

$$\phi_* X f \Big|_{\phi(x)} = X(f \circ \phi) \Big|_x, \quad \forall f \in C^\infty(\Omega'), \quad \forall x \in \Omega.$$

We can also write $\phi_* X(f) = X(\phi^* f)$.

By means of Lemma A.0.1 we now prove the Poincaré inequality for nondegenerate vector fields used in Lemma 2.6.2.

Lemma A.0.4 (Poincaré inequality for nondegenerate vector fields)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let X be a smooth real vector field on Ω. Let $x_0 \in \Omega$ be such that $X(x_0) \neq 0$. Then there exists a compact set $K_0 \subset \Omega$, with $x_0 \in K_0$, and a constant $C = C(K_0) > 0$ such that for all compact $K \subset K_0$,*

$$\|u\|_0 \leq C \operatorname{diam}(K) \|Xu\|_0, \quad \forall u \in C_0^\infty(K). \tag{A.0.2}$$

Proof. By the *flow box theorem* there exists $U \subset \Omega$ open and a smooth diffeomorphism $\Phi : U \longrightarrow (-T,T) \times U'$, $\Phi(x) = (y_1, y')$, $U' \subset \mathbb{R}^{n-1}$ open, such that

$$\Phi_* X \Big|_{\phi(x)} = \frac{\partial}{\partial y_1} \Big|_{\phi(x)}, \quad \forall x \in U.$$

Now we take two compact sets K and K_0 such that $x_0 \in K \subset K_0 \subset U$, and consider the compact $\Phi(K)$. Then, given any $u \in C_0^\infty(K)$, we have that the function

$$\tilde{u}(y) := u(\Phi^{-1}(y)) = (\Phi^{-1})^* u(y),$$

is such that $\tilde{u} \in C_0^\infty(\Phi(K))$. Whence, by Lemma A.0.1, we get

$$\|\tilde{u}\|_0 \leq 2\mathrm{diam}(\Phi(K))\|\Phi_* X\tilde{u}\|_0, \quad \forall \tilde{u} \in C_0^\infty(\Phi(K)).$$

On the other hand we may find constants c_1, C_1, $C_2 > 0$, depending on K_0, such that

$$c_1^2 \leq |\det J_\Phi(x)| \leq C_1^2, \quad \forall x \in K_0,$$

and

$$|\Phi(x') - \Phi(x'')| \leq C_2|x' - x''|, \quad \forall x', x'' \in K_0.$$

The latter inequality yields that

$$\mathrm{diam}(\Phi(K)) = \sup_{x', x'' \in K} |\Phi(x') - \Phi(x'')| \leq C_2 \mathrm{diam}(K).$$

Therefore

$$c_1^2\|u\|_0^2 \leq \int |u(x)|^2 |\det J_\Phi(x)| dx = \int |\tilde{u}(y)|^2 dy \leq C_1^2 \|\tilde{u}\|_0^2,$$

and likewise for $\|Xu\|_0$. Hence

$$c_1^2\|u\|_0^2 \leq C_1^2\|\tilde{u}\|_0^2 \leq 2C_1^2 \mathrm{diam}(\Phi(K))^2 \|\Phi_* X\tilde{u}\|_0^2 \leq 2C_1^4 C_2^2 \mathrm{diam}(K)^2 \|Xu\|_0^2,$$

and thus

$$\|u\|_0 \leq 2\frac{C_1^2 C_2}{c_1} \mathrm{diam}(K)\|Xu\|_0,$$

which concludes the proof. $\square$

Acknowledgements

At the end of this experience I wish to thank some people who represented an important part of this path.

Thanks to my family for the constant support, especially to my mother and my sister, who shared with me an unlimited number of positive and negative moments.

Thanks to my boyfriend Giuseppe, who has always been by my side from the first time we met each other.

A special thanks goes to my supervisor Alberto Parmeggiani, who has been an irreplaceable guide. With his talent he constantly raised my passion for math, and I will always remember all our discussions. Thank you for introducing me to the problem treated in this thesis that I enjoyed so much. Without you all this work would not have been possible.

Thanks to Gerardo Mendoza, that I estimate both as mathematician and as person, for interesting discussions and suggestions.

Thanks to Gigliola Staffilani, who has been my guide for a short period of my studies. It was a great pleasure to work with such a great mathematician and person.

I wish to thank Ferruccio Colombini, Paulo Cordaro and Daniele Del Santo for their interest on my work and useful suggestions.

Of course, I can not forget my colleagues, who shared with me all the feelings characterizing this period. In particular I want to mention three of them, Erika Battaglia, Benedetta Franceschiello and Lorenzo Ruffoni, since we have been immediately so close each other.

Thanks to all the Math Department of the University of Bologna for having been part of this trip.

Thanks to all friends and others who in one way or another were part of this wonderful experience.

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