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Potential Analysis for Hypoelliptic Second Order PDEs with Nonnegative Characteristic Form

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A Ilprof,
per la paternità con la quale
mi è stato Maestro.

Contents

Introduction

The results presented in this Thesis give some contributions to the theory of second order Partial Differential Equations with non-negative characteristic form. This class of PDEs was introduced by M.Picone, who named them elliptic-parabolic equations. Some decades after Picone's works, G.Fichera carried out a first systematic study of boundary value problems for wide classes of elliptic parabolic operators, and introduced his celebrated classification of the boundary points. Subsequently, O.A. Oleinik&E.V. Radkevic, and J.J. Kohn&L. Nierenberg, proved several existence and regularity results, in terms of the properties of the Fichera's boundary. In 1967 L. Hörmander proved a celebrated Theorem giving a sufficient condition of hypoellipticity for operators sums of squares of vector fields. Soon after, O.A. Oleinik&E.V. Radkevic extended Hörmander's Theorem to elliptic-parabolic operators in general form. These theorems, giving hypoellipticity conditions in terms of suitable Lie algebra related to the involved operators, opened a research field: the analysis of elliptic-parabolic equations with underlying algebraic-geometric structures of *sub-Riemannian type*. In the recent papers [14] and [15], A.Bonfiglioli, E.Lanconelli athnd A.Tommasoli, started a Potential Analysis of elliptic-parabolic operators with smooth coefficients, only assuming, instead, *hypoellipticity and existence of a well behaved global fundamental solution* for the relevant operators. In this Thesis we follow this new kind of axiomatic approach, and we give, in this setting, several original contributions of Potential Analysis-type.

We want to stress that the hypothesis of the global existence on $\mathbb{R}^N \times \mathbb{R}^N$ of

a fundamental solution Γ can be removed by replacing $\mathbb{R}^N$ with a bounded open set Ω and replacing Γ with the Green function $G_\Omega(x,y)$ for Ω.
Finally, we point out some conditions from which it follows the existence of a global fundamental solution and examples of classes of operators to which our results can be applied.

(a) Sub-Laplacians on Carnot groups (see Section 5.1) are a particular case of operators which we deal with in our work. Among the results of the Thesis for our general operators, the Lebesgue-type Theorem of Chapter 4 is a new result also for sub-Laplacians on Carnot groups.

(b) Given an open set O of $\mathbb{R}^N$, we consider

$$\mathcal{L} = div(A(x)\nabla), \quad x \in O$$

Let $X_1, ..., X_N$ be the vector fields which are the columns of the matrix $A(x)$. If $rank(Lie\{X_1, ..., X_N\})(x) = N$ for every $x \in O$, by [53] it follows that the operator $\mathcal{L}$ is hypoelliptic. Then, by [19, Part I, Section 2], there exists a matrix $B(x)$, $x \in \mathbb{R}^N$, non-negative and with smooth entries, such that

$$B(x) = A(x) \quad \text{in } O_1 \subset\subset O \quad \text{and} \quad B(x) = I \quad \text{in a neighborhood of } \infty.$$

Then

$$\hat{\mathcal{L}} = div(B(x)\nabla), \quad x \in \mathbb{R}^N$$

satisfies all conditions of operators considered in this thesis. Its fundamental solution has locally the following behavior

$$\Gamma(x,y) \approx \frac{d^2(x,y)}{|B(x, d(x,y)|}$$

where d is the control distance associated with the matrix.

(c) The operators which we deal with in our work are in *divergence form,* so that they are formally self-adjoint. Then, the hypoellipticity assumption, together with a condition of not total degeneracy, implies

the existence of a *local* fundamental solution (see [68], se also [17]). An argument, as the one used by Folland in [28], shows that the local fundamental solution can be extended to a global one if the involved operator is homogeneous with respect to a *group of dilations*.
An example is the operator of Grushin type

$$\Delta_x + |x|^{2m}\Delta_y, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$$

(d) Other sufficient conditions for the existence of a global fundamental solutions have been given by Bonfiglioli and Lanconelli in [13].

We close this Introduction by giving a general outline of the thesis and by briefly describing our main results. A more detailed summary of the contents will be presented at the beginning of each Chapter.

Chapter 1 contains the complete list of assumptions on the operator $\mathcal{L}$, its properties deriving from those assumptions, main definitions, and recalls from [14] on characterizations of subharmonic functions related to $\mathcal{L}$, as solutions of the inequality $\mathcal{L}u \geq 0$ in the weak sense of distributions and as sub-mean functions w.r.t. appropriate mean-value integral operators $M_r(u)$, generalizing the mean-integral in the classical case of the Laplace operator .

In *Chapter 2* we establish some *representation Theorems of Riesz-type for $\mathcal{L}$-subharmonic functions*, both in general open sets and in $\mathbb{R}^N$. Our Riesz representation theorems are expressed in terms of $\mathcal{L}$-Green potentials of Radon measures, and require the analysis of $\mathcal{L}$-Green function for arbitrary open set $\Omega \subseteq \mathbb{R}^N$, introduced and investigated in this chapter. Moreover, starting from Riesz theorems, we investigate the Poisson-Jensen formula for $\mathcal{L}$-regular domains, from which we obtain Mean value formulas for $\mathcal{L}$-subharmonic functions. The results contained in this chapter are new and extend analogous results for the Sub-Laplacians on Stratified Lie groups. They will be gathered up in a work that will be submitted to a journal for its publication.

Chapter 3 contains results published in the joint paper with A. Bonfiglioli [1]. By the well-known *Mean Value Theorem*, classical harmonic functions are characterized by means of Euclidean balls: indeed, a function is harmonic if and only if it verifies the Mean Value Formula. But it holds also an *Inverse Mean Value Theorem*, whereby Euclidean balls are characterized by means of harmonic functions: they are, indeed, the unique sets for which it holds a Mean Value Formula for every harmonic function. In this chapter it is proved an Inverse Mean Value Theorem for our operators $\mathcal{L}$, where the rôle that Euclidean balls have in the classical Riemannian case is played, in our sub-Riemannian setting, by the superlevel sets of the fundamental solution.

Chapter 4 is dedicated to a *Lebesgue-type result for the Perron-Wiener generalized solution*, as well published in the reference [1]. It is an extension to our operators $\mathcal{L}$ of a result proved, for sub-Laplacians on Carnot groups, in the Thesis of Master Degree [2]. Given a bounded open set $\Omega \subset \mathbb{R}^N$, we suppose that the boundary datum is the restriction to $\partial\Omega$ of a continuous function defined in Ω and we take its M_r-mean, and then iteratively the M_r-mean of the function $x \mapsto M_r(u)(x)$ and so on. In this way, we construct a sequence of functions converging to the Perron-Wiener solution of the Dirichlet problem.

In *Chapter 5* we *compare Perron-Wiener and weak variational solutions* of the Dirichlet problem, under specific hypothesis on the boundary datum, extending to a more general framework a result by Arendt and Daners [7], related to the classical Laplacian in $\mathbb{R}^N$. We generalize it first of all to the Sub-Laplacians on Carnot groups, and then to our operators $\mathcal{L}$ for which, however, we have to require stronger hypothesis. The achieved results are contained in the note [3] submitted to a journal for the publication, and then to our operators $\mathcal{L}$ for which, however, we have to require stronger hypothesis.

In the *Appendix A* we give some topics from the theory of Abstract Harmonic spaces.

In the *Appendix B* we state a remarkable *Phillips and Sarason result* ([58]) on the square root of a symmetric, non-negative and C^2 matrix, which we will use to prove that our operators $\mathcal{L}$ are *uniformly X-elliptic operators*, in the sense of Lanconelli and Kogoj [44] (see also Gutierrez and Lanconelli, [34]) and can be written as a sort of *sum of squares of vector fields.*

Chapter 1

Main assumptions and Recalls

In this first chapter, we present the operators which with we deal with in our work. They are divergence-form PDO, with a matrix symmetric, nonnegative definite, and with smooth entries in $\mathbb{R}^N$. Besides a non-total degeneracy assumption on A (we assume that one of the $a_{i,i}$ is everywhere positive), we require that $\mathcal{L}$ is a C^∞-hypoelliptic operator. We also ask for $\mathcal{L}$ to possess a global fundamental solution $\Gamma(x, y)$ with well-behaved properties inspired by those holding true in the case of sub-Laplacians on stratified Lie groups (in the sense of Folland and Stein [28, 29]): for instance we require that Γ is positive out of the diagonal, $\Gamma(x, \cdot)$ blows up at x and $-\mathcal{L}\Gamma(x, \cdot)$ is the Dirac mass at x in the distributional sense. By the results in [49, 65], it is known that these properties are satisfied by a very large class of PDOs, the Hörmander sums of squares of vector fields $\mathcal{L} = \sum_{j=1}^{m} X_j^2$ (with $\mathrm{div}(X_j) = 0$ so that $\mathcal{L}$ is in divergence form). We here require the extra assumption that $\Gamma(x, \cdot)$ is globally defined and it vanishes at infinity (which is true, e.g., in the stratified group case).

Moreover, in this chapter, several recalls of general notions and of results taken from the paper [14] are shown. Besides a list of the properties of the operators and of their fundamental solution, we fix the main notations and the basic definitions: $\mathcal{L}$-regular open sets, $\mathcal{L}$-subharmonic and $\mathcal{L}$-superharmonic functions, the mean-integral operator related to $\mathcal{L}$, the generalized solution

in the sense of Perron-Wiener. We stress that the generalized solvability is a consequence of the axiomatic Potential Theory for $\mathcal{L}$, the relevant harmonic-space axioms being satisfied thanks to our assumptions on $\mathcal{L}$ and Γ (in particular, hypoellipticity of $\mathcal{L}$ plays a key rôle).
Finally, we recall the Mean-Value formula for $\mathcal{L}$ and some characterizations of subharmonic functions related to $\mathcal{L}$, which we will use several times over the thesis.

1.1 The operator

Let

$$\mathcal{L} := \sum_{i,j=1}^{N} \partial_{x_i} \left(a_{i,j}(x) \partial_{x_j} \right) = div(A(x)\nabla) \tag{1.1}$$

be a linear second order PDO in $\mathbb{R}^N$, in divergence form, with C^∞ coefficients and such that the matrix $A(x) = (a_{i,j})_{i,j\leq N}$ is *symmetric* and *non-negative definite* at any point $x = (x_1, ..., x_N) \in \mathbb{R}^N$. In (1.1), ∇ denotes the Euclidean gradient operator $\nabla = (\partial_{x_1}, ..., \partial_{x_N})^T$. The operator $\mathcal{L}$ is formally self-adjoint and it is (possibly) degenerate elliptic. However, we always assume without further comments that $\mathcal{L}$ is everywhere *not totally degenerate.* Precisely, we assume that the following condition holds:

$$\text{there exists } i \leq N \text{ such that } a_{i,i} > 0 \text{ for all } x \in \mathbb{R}^N \tag{1.2}$$

Our main assumptions is that $\mathcal{L}$ is a C^∞*-hypoelliptic differential operator*, that is, for every open set $\Omega \subseteq \mathbb{R}^N$ and for every $f \in C^\infty(\Omega, \mathbb{R})$, if u is a distributional solution of $\mathcal{L}u = f$, then u coincides almost everywhere with a C^∞ function on Ω.
Furthermore, we assume that $\mathcal{L}$ is equipped with a *global fundamental solution* Γ, that is, there exists a function

$$\Gamma : D = \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y \right\} \longrightarrow \mathbb{R}$$

with the following properties:

1. $\Gamma \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R}^N) \cap C^2(D, \mathbb{R}), \Gamma(x, y) > 0$ for every $(x, y) \in D$;

2. for every fixed $x \in \mathbb{R}^N$, we have $\lim_{y \to x} \Gamma(x, y) = \infty$ and $\lim_{y \to \infty} \Gamma(x, y) = 0$

3. for every $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$,

$$\int_{\mathbb{R}^N} \Gamma(x, y) \mathcal{L}\varphi(y) dy = -\varphi(x), \text{ for any } x \in \mathbb{R}^N \tag{1.3}$$

1.2 Definitions, properties of the operator and recalls

If $\Omega \subseteq \mathbb{R}^N$ is open, then we say that u is *$\mathcal{L}$-harmonic* on Ω if $u \in C^2(\Omega, \mathbb{R})$ and $\mathcal{L}u = 0$ in Ω. We denote by $\mathcal{H}(\Omega)$ the family of $\mathcal{L}$-harmonic functions in Ω.

A bounded open set $V \subset \mathbb{R}^N$ is said to be *$\mathcal{L}$-regular* if the following property is satisfied:
for every $f \in C(\partial V, \mathbb{R})$, there exists a (unique) $\mathcal{L}$-harmonic function in V, denoted by H_f^V, satisfying $\lim_{y \to x} H_f^V(y) = f(x)$ for every $x \in \partial V$. The function H_f^V is called the *classical solution* of the Dirichlet problem

$$D(f, V) \quad \begin{cases} \mathcal{L}u = 0 & \text{in } V \\ u = f & \text{on } \partial V \end{cases}$$

An upper semicontinuous function (u.s.c function, for short) $u : \Omega \to [-\infty, \infty)$ [1] will called *$\mathcal{L}$-hypoharmonic* in Ω if it satisfies the following property:
for every $\mathcal{L}$-regular open set $V \subset \overline{V} \subset \Omega$ and for every $f \in C(\partial V, \mathbb{R})$ such

[1] $u : \Omega \to [-\infty, \infty)$ is called *upper semicontinuous* at $x \in \Omega$ if

$$u(x) = \limsup_{y \to x} u(y) := \inf_{V \in \mathcal{U}_x} \left(\sup_{V \cap \Omega} u \right)$$

where $\mathcal{U}_x$ denotes the family of the neighborhoods of x.

that $u \leq f$ on ∂V, it holds that $u \leq H_f^V$ in V. We shall denote by $\underline{\mathcal{H}}(\Omega)$ the family of $\mathcal{L}-$ hypoharmonic functions in Ω. Any function in $-\underline{\mathcal{H}}(\Omega) := \overline{\mathcal{H}}(\Omega)$ will be called $\mathcal{L}$-*hyperharmonic* in Ω.

A u.s.c function $u : \Omega \to [-\infty, \infty)$ will be called $\mathcal{L}$-*subharmonic* in Ω if it is $\mathcal{L}$-hypoharmonic on Ω and if the set $\{x \in \Omega : u(x) > -\infty\}$ cointains at least one point of every connected component of Ω. We shall denote by $\underline{\mathcal{S}}(\Omega)$ the family of $\mathcal{L}$-subharmonic functions in Ω. Any functions in $-\underline{\mathcal{S}}(\Omega) =: -\overline{\mathcal{S}}(\Omega)$ will be said to be $\mathcal{L}$-*superharmonic* in Ω.

Following the theory of classical harmonic functions, we give the next definition.

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Given $f : \partial\Omega \to [-\infty, \infty]$, we set*

$$\overline{\mathcal{U}}_f^\Omega := \left\{ u \in \overline{\mathcal{H}}(\Omega) : \inf_\Omega u > -\infty, \liminf_{x \to y} u(x) \geq f(y) \;\; \forall\, y \in \partial\Omega \right\}$$

$$\underline{\mathcal{U}}_f^\Omega := \left\{ u \in \underline{\mathcal{H}}(\Omega) : \sup_\Omega u < \infty, \limsup_{x \to y} u(x) \leq f(y) \;\; \forall\, y \in \partial\Omega \right\}$$

We will, respectively, refer to the real extended functions

$$\overline{\mathcal{H}}_f^\Omega := \inf \left\{ u : u \in \overline{\mathcal{U}}_f^\Omega \right\}, \;\; \underline{\mathcal{H}}_f^\Omega := \sup \left\{ u : u \in \underline{\mathcal{U}}_f^\Omega \right\}$$

as the upper solution *and the* lower solution *of the Dirichlet Problem*

$$D(f, \Omega) \qquad \begin{cases} \mathcal{L}u = 0 & in\ \Omega \\ u = f & on\ \partial\Omega \end{cases}$$

Moreover, f is called resolutive *if $\overline{\mathcal{H}}_f^\Omega = \underline{\mathcal{H}}_f^\Omega \in \mathcal{H}(\Omega)$. In this case, these coinciding functions are denoted by H_f^Ω and this referred to as* generalized solution, *in the sense of Perron-Wiener-Brelot (from now of PWB), of $D(f, \Omega)$, or simply as the PWB solution of $D(f, \Omega)$.*
If Ω id Dirichlet regular, then H_f^Ω coincides with the classical solution of $D(f, \Omega)$.

The following is a list of consequences of our assumptions on $\mathcal{L}$.

1. The condition (1.2), together with $A(x) \geq 0$ implies *Picone's Weak Maximum Principle* for $\mathcal{L}$:
   If $V \subset \mathbb{R}^N$ is open and bounded and $u \in C^2(V, \mathbb{R})$ satisfies
   $$\mathcal{L}u \geq 0 \text{ in } V \text{ and } \limsup_{x \to y} u(x) \leq 0 \text{ for every } y \in \partial V,$$
   then $u \leq 0$ in V. (See [42, Corollary 1.3]).
   We will show in Theorem 3.2.1 that $\mathcal{L}$ also satisfies the *Strong Maximum Principle.*

2. The function $y \to \Gamma(x, y)$ is smooth. Besides, the property (1.3), since $\mathcal{L}^* = \mathcal{L}$, can be restated ad follows: $-\mathcal{L}\Gamma(x, \cdot)$ equals the Dirac measure at $\{x\}$, in the sense of distributions. This in particular implies that the function $y \to \Gamma(x, y)$ is $\mathcal{L}$-harmonic in $\mathbb{R}^N \setminus \{x\}$. As a consequence, since $-\Gamma(x, y) \to \infty$ as $y \to x$, an easy application of the Weak Maximum Principle shows that $-\Gamma(x, \cdot)$ is $\mathcal{L}-$subharmonic in $\mathbb{R}^N$.

3. Since $\mathcal{L}$ is self-adjoint, the hypoellipticity of $\mathcal{L}$ ensures that the fundamental solution for $\mathcal{L}$ is *symmetric*, i.e., $\Gamma(x, y) = \Gamma(y, x)$ for every $x \neq y$ (see Bony [17, Section 6]).

4. The fundamental solution for $\mathcal{L}$ is *unique*. Indeed, if Γ, Γ' are two fundamental solutions, then for every fixed $x \in \mathbb{R}^N$, the function $h := \Gamma(x, \cdot) - \Gamma'(x, \cdot)$ solves $\mathcal{L}h = 0$ on $\mathbb{R}^N$ in the sense of distributions. Hence h coincides with a smooth $\mathcal{L}$-harmonic function $\tilde{h}$ on $\mathbb{R}^N$, which vanishes at infinity. The weak maximum principle easily implies that $\tilde{h} \equiv 0$ on $\mathbb{R}^N$, that is $\Gamma(x, y) = \Gamma'(x, y)$ for every $y \in \mathbb{R}^N \setminus \{x\}$. By the symmetry result above, we infer that $\Gamma \equiv \Gamma'$.

5. The *Doob convergence property*: If $\{u_n\}_n$ is a monotone increasing sequence of $\mathcal{L}$-harmonic functions on an open set $\Omega \subseteq \mathbb{R}^N$, then $u := \sup_n u_n$ is $\mathcal{L}$-harmonic in Ω, provided that u is finite in a dense subset of Ω. By using the hypoellipticity of $\mathcal{L}$ and the positivity of Γ on D, one

can prove a weak form of the Harnack inequality (as in [17, Théoréme 7,1, page 298]) which, in its turn, easily implies the Doob property. The weak form of the Harnack inequality will be used in Chapter 3 to prove the *Brelot convergence property* (Theorem 2.2.8).

6. The *regularity axiom*: The $\mathcal{L}$-regular open sets form a basis of the Euclidean topology (see Bony [18]; see also [16, Section 7.1]).

7. The map $\Omega \mapsto \mathcal{H}(\Omega)$ is a harmonic sheaf and $(\mathbb{R}^N, \mathcal{H})$ is a σ^**-harmonic space* (see Appendix A), which we call the $\mathcal{L}$*-harmonic space.* Indeed, the functions of the type $\max\{-\Gamma(x,\cdot), -k\}$ (with $k \in \mathbb{N}$) provide non-positive continuous $\mathcal{L}$-subharmonic functions separating points of $\mathbb{R}^N$. Moreover, the function $s_{x_0} := (\Gamma(\cdot, x_0))^{-1}$ belongs to $C(\mathbb{R}^N) \cap \underline{S}(\mathbb{R}^N)$, it vanishes at x_0 only and is positive elsewhere.

8. The following *Wiener resolutivity theorem* holds true: given any bounded open set $\Omega \subset \mathbb{R}^N$, every continuous functions $f : \partial\Omega \to \mathbb{R}$ is resolutive, in the sense of Definition 1.2.1. (see e.g., [16, Theorem 6.8.4])

If V is any $\mathcal{L}$-regular open set and $x \in V$, the map $C(\partial V, \mathbb{R}) \ni f \mapsto H_f^V(x) \in \mathbb{R}$ is linear and it is nonnegative on nonnegative f's. Hence, there exists a unique Radon measure μ_x^V on ∂V such that

$$H_f^V = \int_{\partial V} f(y) d\mu_x^V(y) \qquad \text{for every } f \in C(\partial V, \mathbb{R}) \tag{1.4}$$

One says that μ_x^V is the $\mathcal{L}-$*harmonic measure* related to V and x.

For any given $x \in \mathbb{R}^N$ and $r > 0$, we set

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^N \,:\, \Gamma(x,y) > 1/r \right\}, \tag{1.5}$$

with the convention that $\Gamma := \infty$. We also assume that, for every $x \in \mathbb{R}^N$ and $r > 0$,

$$\nabla(\Gamma(x,\cdot)) \neq 0 \text{ on } \partial\Omega_r(x), \tag{1.6}$$

whence $\partial\Omega_r(x)$ is a manifold of class C^∞ of dimension $N-1$.

Note that any $\Omega_r(x)$ is a bounded open neighborhood of x and

$$\bigcap_{r>0} \Omega_r(x) = \{x\}, \qquad \bigcup_{r>0} \Omega_r(x) = \mathbb{R}^N. \tag{1.7}$$

Here and in the sequel, if E is any (Lebesgue-)measurable subset of $\mathbb{R}^N$, we denote by $|E|$ its Lebesgue measure. Moreover, dy and $d\sigma(y)$ will respectively denote, without possibility of ambiguity, the Lebesgue measure and the surface measure in $\mathbb{R}^N$, the latter being the Hausdorff $(N-1)$-dimensional measure. By the Bouligand regularity theorem, which holds true in any $\mathfrak{S}^*$-harmonic space (see Appendix A), the set $\Omega_r(x)$ is $\mathcal{L}$-regular, for every $r > 0$ and every $x \in \mathbb{R}^N$. Indeed, the function $y \mapsto \Gamma(x_0, y) - 1/r$ is an $\mathcal{H}$-barrier function (see Appendix A) at any point x_0 of $\partial\Omega_r(x)$.

***Remark* 1.2.2.** Our assumptions on the fundamental solution of $\mathcal{L}$ imply that *every $\mathcal{L}$-subharmonic function is finite in a dense subset of its domain.* Indeed, let $u \in \underline{S}(\Omega)$ and assume, by contradiction, that $u \equiv -\infty$ in an open set $O \subseteq \Omega$. Then there exists a super-level set of Γ, $\Omega_r(x) := \{y : \Gamma(x, y) > 1/r\} \cup \{y\}$, whose closure is contained in O. As we shall prove in a moment, it follows that $\Omega_s(x) \subseteq O$, whenever $s > r$ and $\overline{\Omega_s(x)} \subset \Omega$. A connection argument will prove that $u \equiv -\infty$ on the connected component of Ω containing x, in contradiction with the definition of $\mathcal{L}$-subharmonic function. To complete the proof, let $s > r$ be such that $\overline{\Omega_s(x)} \subset \Omega$. Letting $V := \Omega_s(x) \setminus \overline{\Omega_r(x)}$, and, for any $n \in \mathbb{N}$,

$$h_n := \sup_{\overline{\Omega_s(x)}} u - n\left(\Gamma(x, \cdot) - 1/s\right),$$

we have the following facts:

(i) $M := \sup_{\overline{\Omega_s(x)}} u \in \mathbb{R}$ since u is u.s.c., $\overline{\Omega_s(x)}$ is compact and u is not identically $-\infty$ in $\overline{\Omega_s(x)}$ (otherwise the proof will be trivial);

(ii) $u \le M \le h_n$ on $\partial\Omega_s(x)$;

(iii) $u = -\infty < h_n$ on $\partial\Omega_r(x)$;

(iv) h_n is $\mathcal{L}$-harmonic in V.

Then, since u is $\mathcal{L}$-subharmonic, $u \le h_n$ in V. Letting $n \to \infty$, we get $u \equiv -\infty$ in V. Hence $u \equiv -\infty$ in $\Omega_s(x) = V \cup \overline{\Omega_r(x)}$.

We next introduce an important integral operator, which we shall use in the next chapters of this thesis.

Definition 1.2.3 (Mean-integral operator related to $\mathcal{L}$)**.** *Le $\mathcal{L}$ be the differential operator in (1.1), let A be the associated matrix, and let $x \in \mathbb{R}^N$. We consider the functions, defined for $y \in \mathbb{R}^N \setminus \{x\}$,*

$$\Gamma_x(y) := \Gamma(x,y), \quad \mathcal{K}_x(y) := \frac{\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle}{|\nabla\Gamma_x(y)|} \tag{1.8}$$

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and suppose $u : \Omega \to [-\infty, \infty)$ is u.s.c. For every fixed $\alpha > 0$, and every $x \in \mathbb{R}^N$ and $r > 0$ such that $\overline{\Omega_r(x)} \subset \Omega$, we introduce the following integrals:

$$m_r(u)(x) = \int_{\partial\Omega_r(x)} u(y)\mathcal{K}_x(y)\, d\sigma(y), \quad M_r(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha m_\rho(u)(x)\, d\rho.$$

An alternative representation for $M_r(u)(x)$ is given by the following formula

$$M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y) K_\alpha(x,y) dy \tag{1.9}$$

where we have set

$$K_\alpha(x,y) := \frac{\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle}{\Gamma_x^{2+\alpha}(y)}. \tag{1.10}$$

We say that m_r is the Surface mean-integral operator related to $\mathcal{L}$ *and M_r^α is the* Solid mean-integral operator related to $\mathcal{L}$*. Throughout the following, $\alpha > 0$ will be fixed and we use the simpler notation M_r and K instead of M_r^α and K.*

***Remark* 1.2.4.** *The above definitions are well-posed.* Indeed, note that $m_r(u)(x)$ is well-posed because $\partial\Omega_r(x)$ is a compact subset of Ω (see also hypothesis (2) on the fundamental solution), and u is bounded from above on the compact sets (since it is upper semicontinuous).
Moreover, in the hypothesis of the above definition, we claim that the map

$r \mapsto m_r(u)(x)$ is upper semicontinuous, so that $M_r(u)(x)$ is well posed too. The claim follows from the following argument: being u u.s.c. and being $\partial\Omega_r(x)$ compact, there exists a decreasing sequence of continuous functions $\{u_j\}_j$ on $\partial\Omega_r(x)$ converging pointwise to u; it easily seen that $r \mapsto m_r(u_j)(x)$ is continuous (for every $j \in \mathbb{N}$) and that $m_r(u)(x) = \lim_{j\to\infty} m_r(u_j)(x)$. Hence $r \mapsto m_r(u)(x)$ is upper semicontinuous.

We recall, for later use, remarkable results proved by A. Bonfiglioli and E.Lanconelli in [14]. We use the notation $R(x) = \sup\{r > 0 \, : \, \Omega_r(x) \subseteq \Omega\}$. The first result show the peculiar form of the Mean-Integral operators in the case of the fundamental solution Γ. .

Theorem 1.2.5. *Let $r > 0$ and let $x, z \in \mathbb{R}^N$. Then we have*

$$m_r(\Gamma(\cdot, z))(x) = \min\{\Gamma(x,z), 1/r\}$$

and, for every $\alpha > 0$,

$$M_r(\Gamma(\cdot, z))(x) = \begin{cases} \frac{\alpha+1}{\alpha r} & \text{if } x = z \\ \frac{1}{\alpha r}\left(\alpha + 1 - \frac{1}{(r\Gamma(x,z))^\alpha}\right) & \text{if } x \in \Omega_r(z),\ x \neq z, \\ \Gamma(x,z) & \text{if } x \notin \Omega_r(z). \end{cases}$$

The following theorems contains, respectively, mean-value formulas generalizing the classical Gauss-Green formulas for Laplace's operator and characterizations of $\mathcal{L}$-subharmonicity.

Theorem 1.2.6 (Mean-Value Formulas for $\mathcal{L}$)**.** *Let m_r, M_r be the average operators in Definition 1.2.3. Let also $x \in \mathbb{R}^N$ and $r > 0$.*
Then, for every function u of class C^2 on an open set containing $\overline{\Omega_r(x)}$, we have the following $\mathcal{L}$-representation formulas:

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x,y) - \frac{1}{r}\right) \mathcal{L}u(y)\, dy \tag{1.11}$$

$$u(x) = M_r(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x,y) - \frac{1}{\rho}\right) \mathcal{L}u(y)\, dy \right) d\rho. \tag{1.12}$$

We shall refer to (1.11) as the Surface Mean-value Formula for $\mathcal{L}$, *whereas (1.12) will be called the* Solid Mean-Value Formula for $\mathcal{L}$.

Taking $u \equiv 1$ in (1.12) it follows that

$$M_r(1)(x) = 1, \quad \text{for every } r > 0 \text{ and } x \in \mathbb{R}^N, \tag{1.13}$$

which shows the local integrability of $K(x, \cdot)$, for every $x \in \mathbb{R}^N$.

Theorem 1.2.7. *Let Ω be an open subset of $\mathbb{R}^N$ and let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function, finite in at least one point of every connected component of Ω. The following conditions are equivalent:*

1. *$u \in \underline{S}(\Omega)$ with respect to $\mathcal{L}$.*
2. *$u(x) \leq M_r(u)(x)$, for every $x \in \Omega$ and $r \in (0, R(x))$.*
3. *$u(x) \leq m_r(u)(x)$, for every $x \in \Omega$ and $r \in (0, R(x))$.*
4. *$u \in L^1_{\text{loc}}(\Omega)$, $\mathcal{L}u \geq 0$ in the sense of distributions, and $\lim_{r \to 0} M_r(u)(x) = u(x)$ for all $x \in \Omega$.*
5. *$u \in L^1_{\text{loc}}(\Omega)$, $\mathcal{L}u \geq 0$ in the sense of distributions, and $\lim_{r \to 0} m_r(u)(x) = u(x)$ for all $x \in \Omega$.*

If in condition (iii) we remove the hypothesis $\lim_{r \to 0} M_r(u)(x) = u(x)$, we can still conclude that u is equal almost everywhere to an $\mathcal{L}$-subharmonic function, which is precisely given by the map $x \mapsto \lim_{r \to 0} M_r(u)(x)$.

Le u be an $\mathcal{L}$-subharmonic function in an open set $\Omega \subseteq \mathbb{R}^N$. By Theorem 1.2.7 $u \in L^1_{loc}(\Omega)$ and $\mathcal{L}u \geq 0$ in the weak sense of distributions. Then[2] there exists a Radon measure μ in Ω such that

$$\mathcal{L}u = \mu$$

[2] Here we used the following result. *Given a linear map $L : C_0^\infty \to \mathbb{R}$ such that $L(\varphi) \geq 0$ whenever $\varphi \geq 0$, there exists a Radon measure μ on Ω such that $L(\varphi) = \int \varphi \, d\mu$ for every $\varphi \in C_0^\infty(\Omega)$.* For a proof of this result, it suffices to rerun the proof of the classical Riesz representation theorem of positive functionals on C_0 as presented, e.g. in [63]

in the weak sense of distributions. The measure μ will be called the $\mathcal{L}$-*Riesz measure of* u.
If u is $\mathcal{L}$-superharmonic in Ω, the $\mathcal{L}$-Riesz measure related to $-u$ will be referred to as the $\mathcal{L}$-Riesz measure of u. In this case, it holds $\mathcal{L}u = -\mu$, in the weak sense of distributions.
With reference to the above definition, we shall sometimes also write $\mu[u]$ or μ_u instead of μ.

***Remark* 1.2.8.** If $u = \Gamma$, then $\mu[\Gamma]$ =Dirac$_0$, the Dirac mass supported at $\{0\}$. Indeed, by the property 3. of the fundamental solution,

$$\int_{\mathbb{R}^N} (-\Gamma)\mathcal{L}\varphi = \varphi(0) = \int_{\mathbb{R}^N} Dirac_0\varphi \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

Chapter 2

Representation Theorems

The aim of this Chapter is to make a deep analysis of the $\mathcal{L}$-Green function for arbitrary open sets, and of its applications to the Representation Theorems of Riesz-type for $\mathcal{L}$-subharmonic and $\mathcal{L}$-superharmonic functions. This analysis is similar to the one presented in the monograph [16, Chapter 9].

We start by introducing the $\mathcal{L}$-Green function G_Ω, first for an $\mathcal{L}$-regular domain Ω, then for general open sets. We stress that, in order to prove the symmetry of G_Ω in the latter case, we prove a very remarkable result: the possibility of approximating from the inside every open set by $\mathcal{L}$-regular set, result which we will use also in the Chapter 5.

We then come to the core of the Chapter, by introducing the $\mathcal{L}$-Green potentials and proving several *Riesz-type representation theorems for $\mathcal{L}$-subharmonic and $\mathcal{L}$-superharmonic functions, in general open sets and in the space.*

Finally we give an application of the above results: we prove the Poisson-Jensen formula for $\mathcal{L}$-regular domains.

The proofs mostly rely on the use of appropriate techniques relevant to the Potential Theory for $\mathcal{L}$ and are inspired by the methods presented in the monograph [16].

2.1 $\mathcal{L}$-Green functions for $\mathcal{L}$- regular Domains

Let Ω be a bounded $\mathcal{L}$-regular open subset of $\mathbb{R}^N$. We call $\mathcal{L}$-Green function of Ω with pole at $x \in \Omega$, the function $G_\Omega(x, \cdot) : \Omega \to] - \infty, \infty]$ defined as follows

$$G_\Omega(x, y) := \Gamma(x, y) - h_x(y),$$

where Γ is the fundamental solution for $\mathcal{L}$, and h_x denotes the solution to the boundary value problem

$$\begin{cases} \mathcal{L}h = 0 & \text{in } \Omega \\ h(z) = \Gamma(x, z) & \text{for every } z \in \partial\Omega, \end{cases}$$

With the above definition, we have (we recall that $\Gamma(x, y)$ is $\mathcal{L}$-harmonic in $\mathbb{R}^N \setminus \{x\}$)

$$G_\Omega(x, \cdot) \text{ is } \mathcal{L}\text{-harmonic in } \Omega \setminus \{x\} \tag{2.1}$$

$$G_\Omega(x, y) \longrightarrow 0 \text{ as } y \to z, \text{ for every } z \in \partial\Omega \tag{2.2}$$

and, by (1.4),

$$G_\Omega(x, y) = \Gamma(x, y) - \int_{\partial\Omega} \Gamma(x, z)\, d\mu_y^\Omega(z), \quad x, y \in \Omega \tag{2.3}$$

We recall that μ_y^Ω denotes the $\mathcal{L}$-harmonic measure related to (the $\mathcal{L}$-regular open set) Ω and the point y.

The following theorem states some other important properties of the $\mathcal{L}$-Green function.

Theorem 2.1.1. *For every $x, y \in \Omega$, $x \neq y$, we have:*

1. $G_\Omega(x, y) \geq 0$,
2. $G_\Omega(x, y) > 0$ *iff x and y belong to the same connected component of* Ω,
3. $G_\Omega(x, y) = G_\Omega(y, x)$.

Proof. 1. Since $\Gamma(x, y) \to \infty$ as $z \to x$ and $h_x \in C^2(\Omega, \mathbb{R})$, then

$$\lim_{z \to x} G_\Omega(x, z) = \infty$$

and then there exists $r > 0$ such that $G_\Omega(x, y) > 0$ for every $z \in \overline{\Omega_r(x)}$. Moreover,

$$\begin{cases} \mathcal{L} G_\Omega(x, \cdot) = 0 & \text{in } \Omega \setminus \overline{\Omega_r(x)} \\ \lim_{z \to \zeta} G_\Omega(x, y) \geq 0 & \text{for every } \zeta \in \partial\left(\Omega \setminus \overline{\Omega_r(x)}\right), \end{cases}$$

so that, by the Picone's Maximum Principle, $G_\Omega(x, z) \geq 0$ in $\Omega \setminus \overline{\Omega_r(x)}$. Thus, $G_\Omega(x, z) \geq 0$ for any $z \in \Omega$. In particular, $G_\Omega(x, y) \geq 0$.

2. Suppose $x, y \in \Omega_0$, with $\Omega_0 \subseteq \Omega$ open and connected. Assume by contradiction $G_\Omega(x, y) = 0$. Then, since $G_\Omega(x, \cdot)$ is non-negative and $\mathcal{L}$-harmonic in $\Omega_0 \setminus \{x\}$, by the Strong Maximum Principle (Theorem 3.2.1)
$$G_\Omega(x, z) = 0 \qquad \text{for every } z \in \Omega_0 \setminus \{x\}.$$
This is impossible, because $G_\Omega(x, z) \to \infty$ as $z \to x$. Let us now suppose $y \in \Omega_1$, being Ω_1 a connected component of Ω not containing x. Then $z \mapsto \Gamma(x, z)$ is $\mathcal{L}$-harmonic in an open set containing $\overline{\Omega_1}$, so that $h_x(z) = \Gamma(x, z)$ for every $z \in \Omega_1$. It follows that $G_\Omega(x, \cdot) = 0$ in Ω_1. In particular, $G_\Omega(x, y) = 0$.

3. Let $y \in \Omega$ be fixed. Denote by g_y the Γ-potential of μ_y^Ω, i.e.
$$g_y(z) := \int_{\partial\Omega} \Gamma(\zeta, z)\, d\mu_y^\Omega(\zeta) = \int_{\partial\Omega} \Gamma(z, \zeta)\, d\mu_y^\Omega(\zeta), \quad z \in \mathbb{R}^N$$
The function g_y is $\mathcal{L}$-harmonic in Ω because, for every $\varphi \in C_0^\infty(\Omega)$
$$\begin{aligned} \int_\Omega \left(\int_{\partial\Omega} \Gamma(x, \zeta)\, d\mu_y^\Omega(\zeta) \right) \mathcal{L}\varphi(x)\, dx &= \int_{\partial\Omega} \left(\int_\Omega \Gamma(x, \zeta) \mathcal{L}\varphi(x)\, dx \right) d\mu_y^\Omega(\zeta) \\ &= - \int_{\partial\Omega} \varphi(\zeta)\, d\mu_y^\Omega(\zeta) \end{aligned}$$
where the last equality derives from the definition of fundamental solution (property 3.). Now, since $\varphi \in C_0^\infty(\Omega)$, we have $\varphi(\zeta) = 0$ for every

$\zeta \in \partial\Omega$, and then $\mathcal{L}(g_y) = 0$ in Ω. On the other hand (2.3), together with the positivity of G_Ω, gives

$$g_y(z) \le \Gamma(z, y) \quad \forall\, z \in \Omega$$

It follows that $\limsup_{z\to\zeta} g_y(z) \le \Gamma(\zeta, y)$ for every $\zeta \in \partial\Omega$. The definition of h_y and the Picone's Maximum Principle imply $g_y(z) \le h_y(z)$ for every $z \in \Omega$. In particular, $g_y(x) \le h_y(x)$. Then

$$\Gamma(x, y) - G_\Omega(x, y) = g_y(x) \le h_y(x) = \Gamma(y, x) - G_\Omega(y, x),$$

so that, since $\Gamma(x, y) = \Gamma(y, x)$,

$$G_\Omega(x, y) \ge G_\Omega(y, x).$$

By interchanging the rôles of x and y, we also get $G_\Omega(y, x) \ge G_\Omega(x, y)$. Hence, $G_\Omega(x, y) = G_\Omega(y, x)$.

□

***Remark* 2.1.2.** We know that $\Omega_r(x)$ is an $\mathcal{L}$-regular domain. Since $\Gamma(x, y) = 1/r$ if $y \in \partial\Omega_r(x)$, we have $h_x \equiv 1/r$. Then

$$G_{\Omega_r(x)}(x, y) = \Gamma(x, y) - \frac{1}{r} \tag{2.4}$$

2.2 $\mathcal{L}$-Green Function for General Domains

We first recall a general result from classical and abstract Potential Theory (see [16, Section 6.9]).
If $u \in \overline{S}(\Omega)$ has a $\mathcal{L}$-subharmonic minorant u_0 in Ω, then the family $\{v \in \underline{S}(\Omega) : u_0 \le v \le u\}$ has a maximum $h \in \mathcal{H}(\Omega)$. It is called *the greatest $\mathcal{L}$-harmonic minorant of u in Ω*. We have

Proposition 2.2.1. *Let $u_1, u_2 \in \overline{S}(\Omega)$ and assume u_1, u_2 have a $\mathcal{L}$-subharmonic minorant. Then $u_1 + u_2$ has a greatest $\mathcal{L}$-harmonic minorant given by $h_1 + h_2$, where h_i is the greatest $\mathcal{L}-$harmonic minorant of u_i.*

Now we extend the notion of $\mathcal{L}$-Green function to general open sets.
Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $x \in \Omega$. The function $y \mapsto \Gamma(x,y)$ is $\mathcal{L}$-superharmonic and non-negative in Ω. Then it has a greatest $\mathcal{L}$-harmonic minorant in Ω: let us denote it by h_x.
The function

$$\Omega \times \Omega \ni (x,y) \mapsto G_\Omega(x,y) := \Gamma(x,y) - h_x(y) \in [0,\infty]$$

is the $\mathcal{L}$-*Green function for* Ω.

We explicity remark that h_x and $G_\Omega(x,\cdot)$ are $\mathcal{L}$-harmonic, respectively, in Ω and in $\Omega \setminus \{x\}$. Moreover, $G_\Omega(x,\cdot)$ is $\mathcal{L}$-superharmonic in Ω and

$$h_x = \sup\left\{v \in \underline{S}(\Omega) | v \leq \Gamma(x,\cdot)\right\}.$$

As a consequence, $0 \leq h_x \leq \Gamma(x,\cdot)$ and $G_\Omega \geq 0$.

For future references it is worth starting the following proposition.

Proposition 2.2.2. *Let $x \in \Omega$, and let $v \in \underline{S}(\Omega)$ be such that*

$$v \leq G_\Omega(x,\cdot) \qquad \text{in } \Omega$$

Then $v \leq 0$. Hence, the null function is the greatest $\mathcal{L}$-harmonic minorant of the function $\Omega \ni x \mapsto G_\Omega(x,\cdot)$.

Proof. The hypothesis implies $v + h_x \leq \Gamma(x,\cdot)$. Then, since $v + h_x \in \underline{S}(\Omega)$, we infer $v + h_x \leq h_x$, that is $v \leq 0$. The second part of the proposition trivially follows from the first one. $\square$

***Remark* 2.2.3.** The $\mathcal{L}$-Green function for $\mathbb{R}^N$ is

$$G_{\mathbb{R}^N}(x,y) = \Gamma(x,y), \qquad x,y \in \mathbb{R}^N.$$

Indeed, since $0 \leq h_x \leq \Gamma(x,\cdot)$ and $\Gamma(x,y) \to 0$ as $y \to \infty$, we have $h_x \equiv 0$ and $G_{\mathbb{R}^N}(x,\cdot) = \Gamma(x,\cdot)$.

When Ω is bounded, G_Ω can be expressed in terms of the Perron-Wiener-Brelot operator. Indeed, the following theorem holds.

Theorem 2.2.4. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. Then, for every $x \in \Omega$, the greatest $\mathcal{L}$-harmonic minorant in Ω of the map $x \mapsto \Gamma(x, y)$ is the Perron-Wiener-Brelot solution of the Dirichlet problem*

$$\begin{cases} \mathcal{L}h = 0 & \text{in } \Omega \\ h|_{\partial\Omega} = \Gamma(x, \cdot). \end{cases}$$

Proof. Let $x \in \Omega$ be fixed, and let $\varphi := \Gamma(x, \cdot)|_{\partial\Omega}$. Let $v \in \underline{S}(\Omega)$. From the Picone's Maximum Principle we obtain

$$\limsup_{\partial\Omega} v \le \varphi \quad \text{iff} \quad v \le \Gamma(x, \cdot) \ \text{ in } \Omega.$$

Then, since φ is resolutive,

$$h_x = \sup\{v \in \underline{S}(\Omega) \,:\, v \le \Gamma(x, \cdot)\} = \sup \underline{\mathcal{U}}_\varphi^\Omega = \underline{H}_\varphi^\Omega = H_\varphi^\Omega$$

as we aimed to prove. $\square$

***Remark* 2.2.5.** From this theorem it follows that

$$G_\Omega(x, y) = \Gamma(x, y) - \int_{\partial\Omega} \Gamma(x, z)\, d\mu_y^\Omega(z), \qquad x, y \in \Omega \tag{2.5}$$

where, as usual, μ_y^Ω denotes the $\mathcal{L}$-harmonic measure related to Ω and y. When Ω is $\mathcal{L}$-regular, this formula gives back (2.3).

The $\mathcal{L}$-Green function G_Ω is non-negative in $\Omega \times \Omega$ and such that, for every fixed $x \in \Omega$, $G_\Omega(x, y)$ is the sum of $\Gamma(x, \cdot)$ plus an $\mathcal{L}$-harmonic function on Ω. We now show that $\Gamma(x, \cdot)$ does not exceed any other function sharing the same property.

Proposition 2.2.6. *Let $x \in \Omega$, and let $u \in \overline{S}(\Omega)$, $u \ge 0$, be such that*

$$u = \Gamma(x, \cdot) + v$$

with $v \in \overline{S}(\Omega)$. Then

$$u \ge G_\Omega(x, \cdot).$$

Proof. The condition $u \geq 0$ implies $-v \leq \Gamma(x, \cdot)$, so that (since $-v \in \underline{S}(\Omega)$) $-v \leq h_x$. Hence $\Gamma(x, \cdot) - u \leq h_x$, and the assertion follows. $\square$

Corollary 2.2.7. *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N$. Then*

$$G_{\Omega_1} \leq G_{\Omega_2}$$

Proof. Let $x \in \Omega_1$. The function $G_{\Omega_2}(x, \cdot)|_{\Omega_1}$ is $\mathcal{L}$-superharmonic and non-negative in Ω_1 and is the sum of $\Gamma(x, \cdot)$ plus an $\mathcal{L}$-harmonic function. Proposition 2.2.6 implies $G_{\Omega_1}(x, \cdot) \leq G_{\Omega_2}(x, \cdot)$. $\square$

In order to prove an approximation theorem, we need the following Thereom, the *Brelot convergence property*

Theorem 2.2.8. *Let $\Omega \subseteq \mathbb{R}^N$ be open and connected. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{L}$-harmonic functions in Ω. Assume that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is monotone increasing and*

$$\sup \{u_n(x_0)\} < \infty \tag{2.6}$$

at some point $x_0 \in \Omega$. Then there exists an $\mathcal{L}$-harmonic function u in Ω such that $\{u_n\}_{n \in \mathbb{N}}$ is uniformly convergent on every compact subset of Ω to u.

Proof. Let K be a compact set of Ω. Since Ω is connected, by a weak form of the Harnack inequality (as in [17, Théorème 7.1]), there exists a constant c independent of n and m and there exists $x_0 \in \Omega$ such that

$$\sup_K (u_n - u_m) \leq c \cdot (u_n(x_0) - u_m(x_0)) \quad \text{for every } n \geq m \geq 1$$

Then, by condition (2.6), $\{u_n\}_n$ is uniformly convergent on K. Since K is an arbitrary compact subset of Ω, $\{u_n\}_n$ is locally uniformly convergent to a continuous function $u : \Omega \to \mathbb{R}$. On the other hand, by the solid mean value Theorem (Theorem 1.2.6), for every $x \in \Omega$ and $r > 0$ such that $\overline{\Omega_r(x)} \subset \Omega$, we have

$$u_n(x) = M_r(u_n)(x) \quad \forall\, n \in \mathbb{N}$$

Letting n tend to infinity (by the uniform convergence $u_n \to u$), we get

$$u(x) = M_r(u)(x) \qquad \forall\, x \in \Omega,\, r > 0 \,:\, \overline{\Omega_r(x)} \subset \Omega$$

and now the Koebe-type Theorem (see [11]) implies that u is $\mathcal{L}$-harmonic in Ω. □

The following approximation theorem holds.

Theorem 2.2.9. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence of open sets, and let*

$$\Omega := \bigcup_{n\in\mathbb{N}} \Omega_n.$$

Then

$$\lim_{n\to\infty} G_{\Omega_n} = G_\Omega \tag{2.7}$$

Proof. Since $\Omega_n \subseteq \Omega_{n+1} \subseteq \Omega$, Corollary 2.2.7 gives

$$G_{\Omega_n} \leq G_{\Omega_{n+1}} \leq G_\Omega$$

Hence the limit in (2.7) exists and is $\leq G_\Omega$. To prove the opposite inequality, we fix $x \in \Omega$ and consider $n \in \mathbb{N}$ such that $\Omega_n \ni x$. Then

$$G_{\Omega_n}(x,\cdot) = \Gamma(x,\cdot) - h_n, \qquad h_n := h_{\Omega_n,x}$$

and then

$$h_n \geq h_{n+1} \geq 0 \quad \text{in } \Omega_n$$

By the Brelot convergence property (Theorem 2.2.8), there exists a $\mathcal{L}$-harmonic function $h \in \mathcal{H}(\Omega)$ such that $h_n \downarrow h$. It follows that

$$\lim_{n\to+\infty} G_{\Omega_n} = U := \Gamma(x,\cdot) - h$$

is non-negative in Ω since $G_{\Omega_n} \geq 0$ in Ω_n. Moreover, $G_{\Omega_n} \in \overline{S}(\Omega)$ and $h \in \mathcal{H}(\Omega)$, then, by Proposition 2.2.6,

$$G_\Omega(x,\cdot) \leq U = \lim_{n\to\infty} G_{\Omega_n}(x,\cdot).$$

This completes the proof. □

In order to prove the symmetry of the $\mathcal{L}$-Green function for general domains, we need a result which is of independent interest, concerning the *approximation from the inside by $\mathcal{L}$-regular open sets.*

Lemma 2.2.10. *Given an open set $\Omega \subset \mathbb{R}^N$, there exists a monotone increasing sequence of bounded $\mathcal{L}$-regular open sets $(\Omega_n)_{n\in\mathbb{N}}$ such that*

$$\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$$

Proof. We first assume that Ω is bounded. For every $n \in \mathbb{N}$, let us cover $\partial\Omega$ by a finite family of level sets

$$\left\{\Omega_{r_j^n}(x_j^n)\right\}_{j=1,\dots,p_n} \qquad \text{with } 0 < r_j^n < 1/n$$

We choose the level sets in a such way that

$$\overline{\Omega_{r_j^{n+1}}(x_j^{n+1})} \subseteq \bigcup_{i=1}^{p_n} \Omega_{r_i^n}(x_i^n)$$

The, defining

$$\Omega_n := \Omega \setminus \bigcup_{i=1}^{p_n} \overline{\Omega_{r_i^n}(x_i^n)}$$

we have $\Omega_n \subseteq \Omega_{n+1}$ and $\bigcup_n \Omega_n = \Omega$. The open sets Ω_n are $\mathcal{L}$-regular. Indeed, if $z_0 \in \partial\Omega_n$, there exists $j \in \{1, ..., p_n\}$ such that $z_0 \in \partial\Omega_{r_j^n}(x_j^n)$. The function

$$w = \frac{1}{r_j^n} - \Gamma((x_j^n), \cdot)$$

is $\mathcal{L}$-harmonic and strictly positive in the complement of $\Omega_{r_j^n}(x_j^n)$, hence in Ω_n. Moreover, $\lim_{z\to z_0} w(z) = 0$. Thus w is an $\mathcal{L}$-barrier for Ω_n at z_0, i.e. z_0 is $\mathcal{L}-$regular for Ω_n.

Let us now suppose that Ω is unbounded and put

$$O_n := \Omega \cap \Omega_n(0), \qquad n \in \mathbb{N}.$$

Since O_n is bounded, we can find an increasing sequence of bounded $\mathcal{L}$-regular open sets $\left(O_n^k\right)_{k\in\mathbb{N}}$ such that $\bigcup_{k\in\mathbb{N}} O_n^k = O_n$. Using the first part of the proof, we can choose O_n^k such that $O_n^k \subseteq O_{n+1}^k$ for every n and for every k. It follows that $O_n^k \subseteq O_m^m$ if $k, n \leq m$. Let us now put

$$\Omega_n = O_n^n$$

Then Ω_n is bounded and $\mathcal{L}$-regular. Moreover,

$$\Omega = \bigcup_n O_n = \bigcup_n \left(\bigcup_k O_n^k \right) \subseteq \bigcup_n O_n^n \subseteq \Omega.$$

Hence $\Omega = \bigcup_n \Omega_n$, and the proof is complete. □

This lemma, together with Theorem 2.1.1, immediately proves the following proposition on *symmetry of the $\mathcal{L}$-Green function.*

Proposition 2.2.11. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then*

$$G_\Omega(x, y) = G_\Omega(y, x) \qquad \textit{for every } x, y \in \Omega.$$

In particular, for every fixed $y \in \Omega$, the function

$$x \mapsto G_\Omega(x, y)$$

is $\mathcal{L}$-harmonic in $\Omega \setminus \{y\}$.

***Remark* 2.2.12.** From (2.5) and the above Proposition 2.2.11 we immediately get the following formula

$$G_\Omega(x, y) = \Gamma(y, x) - \int_{\partial\Omega} \Gamma(y, \eta)\, d\mu_x^\Omega(\eta) \qquad \forall\, x, y \in \Omega.$$

By collecting together some of the results of this section, we have the following characterizations of the $\mathcal{L}$-Green function, which we state for future reference.

Proposition 2.2.13. *Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $x \in \Omega$ be fixed. Let us denote by h_x the greatest $\mathcal{L}$-harmonic minorant of $\Gamma(x, \cdot)$ in Ω. Then the following facts hold:*

1. *The $\mathcal{L}$-Green function $G_\Omega(x,y) = \Gamma(x,y) - h_x(y)$ is a symmetric function, i.e. $G_\Omega(x,y) = G_\Omega(y,x)$ for all $x, y \in \Omega$. Moreover, G_Ω is continuous (in the extended sense) on $\Omega \times \Omega$, and the greatest $\mathcal{L}$-harmonic minorant of $G_\Omega(x,\cdot)$ in Ω is the null function.*

2. *It holds*
$$h_x = \sup\{u \in \underline{S}(\Omega) \, : \, u \le \Gamma(x,\cdot) \quad \textit{on} \quad \Omega\}.$$

3. *If Ω is a bounded domain, then $h_x = H^\Omega_{\Gamma(x,\cdot)}$, in the sense of Perron-Wiener-Brelot, or equivalently*
$$h_x = \sup\left\{u \in \underline{S}(\Omega) \, : \, \limsup_{z \to \zeta} u(z) \le \Gamma(x,\zeta) \quad \textit{for all} \quad \zeta \in \partial\Omega\right\}.$$

4. *If Ω is a $\mathcal{L}$-regular domain, then h_x is the solution (in the classical sense) to*
$$\mathcal{L}u = 0 \;\; \textit{in } \Omega \quad \textit{and} \quad u = \Gamma(x,\cdot) \;\; \textit{on } \partial\Omega.$$

5. *An equivalent definition of the $\mathcal{L}$-Green function is the following one: G_Ω is a non-negative function on $\Omega \times \Omega$ such that (for every $x \in \Omega$) the function $G_\Omega(x,\cdot)$ is the sum of $\Gamma(x,\cdot)$ plus a $\mathcal{L}$-harmonic function on Ω and, moreover, $G_\Omega(x,\cdot)$ does not exceed any other non-negative $\mathcal{L}$-superharmonic function on Ω which is the sum of $\Gamma(x,\cdot)$ plus a $\mathcal{L}$-superharmonic function on Ω.*

2.3 Potentials of Radon Measures

Let $\Omega \subseteq \mathbb{R}^N$ be open and let G_Ω be its $\mathcal{L}$-Green function. Let μ be a Radon measure in Ω. The function
$$G_\Omega * \mu : \Omega \to [0,\infty], \quad (G_\Omega * \mu)(x) := \int_\Omega G_\Omega(x,y)\, d\mu(y)$$
is well defined and l.s.c. It is called the *G_Ω-potential of μ*.

The following theorem holds.

Theorem 2.3.1. *Suppose Ω is connected. Then $G_\Omega * \mu \in \overline{S}(\Omega)$ if and only if there exists $x_0 \in \Omega$ such that $(G_\Omega * \mu)(x_0) < \infty$.*

Proof. The "only if" part is trivial: actually, if $G_\Omega * \mu \in \overline{S}(\Omega)$, then, by Remark 1.2.2,

$$G_\Omega * \mu < \infty \qquad \text{in a dense subset of } \Omega.$$

To prove the "if" part, by Theorem 1.2.7, it is enough to check that $G_\Omega * \mu$ is super-mean. Given $x \in \Omega$ and $r > 0$ such that $\Omega_r(x) \subseteq \Omega$, we have

$$\begin{aligned}
M_r(G_\Omega * \mu)(x) &= \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} K_\alpha(x,y)(G_\Omega * \mu)(y)\, dy \\
&= \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} K_\alpha(x,y) \left(\int_\Omega G_\Omega(y,z)\, d\mu(z) \right) dy \\
&= \int_\Omega M_r(G_\Omega(y,z))(x)\, d\mu(z) \\
&\quad \text{since } y \mapsto G_\Omega(y,z) \text{ is } \mathcal{L}\text{-superharmonic,} \\
&\quad \text{hence super-mean (see Theorem 1.2.7)} \\
&\le \int_\Omega G_\Omega(x,z)\, d\mu(z) = (G_\Omega * \mu)(x).
\end{aligned}$$

Then $G_\Omega * \mu$ is super-mean, and the proof is complete. □

Corollary 2.3.2. *Let μ a Radon measure in Ω such that*

$$\mu(\Omega) < \infty.$$

Then $G_\Omega * \mu \in \overline{S}(\Omega)$.

Proof. Let $\overline{\Omega_r(x)} \subseteq \Omega$. Since

$$\begin{aligned}
\int_{\Omega_r(x_0)} (G_\Omega * \mu)(x)\, dx &= \int_{\Omega_r(x_0)} \left(\int_\Omega G_\Omega(x,y)\, d\mu(y) \right) dx \\
&= \int_\Omega \left(\int_{\Omega_r(x_0)} G_\Omega(x,y)\, dx \right) d\mu(y) \\
&\le \mu(\Omega) \sup_{y \in \Omega} \int_{\Omega_r(x_0)} G_\Omega(x,y)\, dx < \infty
\end{aligned}$$

then $G_\Omega * \mu < \infty$ almost everywhere in $\Omega_r(x_0)$. Then the assertion follows from the previous theorem. □

By using Harnack inequality, Corollary 2.3.2 can be improved as follows.

Corollary 2.3.3. *Let $\Omega \subseteq \mathbb{R}^N$ be open and connected, and let μ be a Radon measure in Ω such that $K := supp(\mu)$ is a compact subset of Ω. For every fixed $y_0 \in K$ and every bounded and connected open set U such that*

$$K \subseteq U, \quad \overline{U} \subseteq \Omega,$$

there exists a positive constant $C = C(U, y_0, \mu(K))$ such that

$$(G_\Omega * \mu)(x) \le C\, G_\Omega(x, y_0) \qquad \forall\, x \in \Omega \setminus U.$$

Proof. For every $x \in \Omega \setminus U$, the function $y \mapsto G_\Omega(x, y)$ is $\mathcal{L}$-harmonic and non-negative in U. Then, by a weak form of the Harnack inequality (as in [17, Théorème 7.1]), there exists a positive constant C independent of x and there exists $y_0 \in \Omega$ such that $G_\Omega(x, y) \le C\, G_\Omega(x, y_0)$ for every $y \in K$. As a consequence,

$$(G_\Omega * \mu)(x) = \int_K G_\Omega(x, y)\, d\mu(y) \le C\, \mu(K)\, G_\Omega(x, y_0)$$

for every $x \in \Omega \setminus U$. □

Theorem 2.3.4. *Under the hypothesis of Theorem 2.3.1, we have*

$$\mathcal{L}(G_\Omega * \mu) = -\mu \qquad \text{in the weak sense of distributions}$$

In particular, $G_\Omega * \mu$ is $\mathcal{L}$-harmonic in $\Omega \setminus supp(\mu)$.

Proof. Since $G_\Omega * \mu \in \overline{S}(\Omega)$, according to Theorem 1.2.7, $G_\Omega * \mu \in L^1_{loc}(\Omega)$. Morevoer, for every $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\int_\Omega (G_\Omega * \mu)(x) \mathcal{L}\varphi(x) dx &= \int_\Omega \left(\int_\Omega (\Gamma(y, x) - h_x(y)) \mathcal{L}\varphi(x) dx \right) d\mu(y) \\
(\text{since } \mathcal{L} h_y = 0) &= \int_\Omega \left(\int_\Omega \Gamma(y, x) \mathcal{L}\varphi(x) dx \right) d\mu(y) \\
&= -\int_\Omega \varphi(y)\, d\mu(y).
\end{aligned}$$

This proves the first part of the theorem. The second part follows from the hypoellipticity of $\mathcal{L}$. □

From this theorem and Corollary 2.3.3 we obtain a corollary that will be used very soon.

Corollary 2.3.5. *Let μ be a compactly supported Radon measure in Ω. Then:*

1. $G_\Omega * \mu \in \overline{S}(\Omega)$,

2. $G_\Omega * \mu$ *is* $\mathcal{L}$*-harmonic in* $\Omega\setminus$ *supp*(μ),

3. *if* $v \in \underline{S}(\Omega)$ *and* $v \leq G_\Omega * \mu$, *then* $v \leq 0$.

Proof. 1. and 2. directly follow from Corollary 2.3.2 and Theorem 2.3.4. To prove 3., we assume that Ω is connected (this is not restrictive) and use Corollary 2.3.3. First of all, if $v \in \underline{S}(\Omega)$ and $v \leq G_\Omega * \mu$, there exists $h \in \mathcal{H}(\Omega)$ such that $v \leq h \leq G_\Omega * \mu$ (h is the greatest $\mathcal{L}$-harmonic minorant of $G_\Omega * \mu$). For a fixed $y_0 \in K := \text{supp}(\mu)$ and a connected bounded open set $U \supseteq K$, $\overline{U} \subseteq \Omega$, we have

$$h(x) \leq (G_\Omega * \mu)(x) \leq C\, G_\Omega(x, y_0) \qquad \forall\, x \in \Omega \setminus U$$

where C is a positive constant $C = C(U, y_0)$. Since $G_\Omega(\cdot, y_0)$ is $\mathcal{L}$-harmonic, hence continuous, in $\Omega \setminus \{y_0\}$, we have

$$\lim_{U \ni x \to \xi} (C\, G_\Omega(x, y_0) - h(x)) = C\, G_\Omega(\xi, y_0) - h(\xi) \geq 0$$

for every $\xi \in \partial U$. Moreover:

$$\lim_{U\setminus\{y_0\} \ni x \to y_0} (C\, G_\Omega(x, y_0) - h(x)) = \infty$$

Then, by Picone's Maximum Principle, $C\, G_\Omega(\cdot, y_0) \geq h$ in U. Summing up:

$$h \leq C\, G_\Omega(\cdot, y_0) \qquad \text{in } \Omega$$

so that, by Proposition 2.2.2, we have $h \leq 0$. Hence $v \leq 0$, and the proof is complete. □

If a $G_\Omega-$potential is $\mathcal{L}$-superharmonic, then its $\mathcal{L}$-harmonic minorants are non-positive constant functions. Indeed, the following theorem holds.

Theorem 2.3.6. *Let μ be a Radon measure in an open and connected set Ω such that $(G_\Omega * \mu)(x_0) < \infty$ for some $x_0 \in \Omega$. Let h be a $\mathcal{L}$-harmonic function in Ω such that*

$$h(x) \le (G_\Omega * \mu)(x) \qquad \forall\, x \in \Omega. \tag{2.8}$$

Then $h \le 0$.

Proof. Let $\{K_n\}$ be a sequence of compact subsets of Ω such that

$$K_n \subseteq K_{n+1}, \quad \bigcup_n K_n = \Omega$$

For every $n \in \mathbb{N}$, we have

$$h \le G_\Omega * \mu = G_\Omega * (\mu|_{K_n}) + G_\Omega * \left(\mu|_{\Omega \setminus K_n}\right) =: v_n + w_n.$$

The functions v_n and w_n are non-negative and $\mathcal{L}$-superharmonic in Ω (see Theorem 2.3.1). Moreover, the greatest $\mathcal{L}$-harmonic minorant of v_n is the zero function (see Corollary 2.3.5). Then, by Proposition 2.2.1, h is less than the greatest $\mathcal{L}$-harmonic minorant of w_n. In particular,

$$h \le w_n \quad \text{in } \Omega \;\; \forall\, n \in \mathbb{N}. \tag{2.9}$$

On the other hand, by the monotone convergence Theorem, we infer

$$v_n \uparrow G_\Omega * \mu,$$

so that

$$w_n = G_\Omega * \mu - v_n \downarrow 0, \quad \text{as } n \to \infty.$$

This, together with (2.9), implies $h \le 0$ and completes the proof. □

2.4 Riesz Representation Theorems for $\mathcal{L}$-subharmonic Functions

Let u be an $\mathcal{L}$-subharmonic function in an open set $\Omega \subseteq \mathbb{R}^N$. We recall that the $\mathcal{L}$-Riesz measure of u is a Radon measure μ in Ω such that $\mathcal{L}u = \mu$. When μ is compactly supported, a representation theorem easily follows.

Theorem 2.4.1 (Riesz representation. I.)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in \underline{S}(\Omega)$. Let μ be the $\mathcal{L}$-Riesz measure of u. Assume that supp(μ) is a compact subset of Ω. Then there exists an $\mathcal{L}$-harmonic function h in Ω satisfying the identity*

$$u = h - G_\Omega * \mu$$

Proof. By Corollary 2.3.2 and Theorem 2.3.4, $v := G_\Omega * \mu$ is $\mathcal{L}$-superharmonic in Ω, and $\mathcal{L}v = -\mu$ in the weak sense of distributions. It follows that $\mathcal{L}(u+v) = 0$ in Ω in the weak sense of distributions. Since $\mathcal{L}$ is hypoelliptic, there exists a function h, $\mathcal{L}$-harmonic in Ω, such that $h(x) = u(x) + v(x)$ almost everywhere in Ω. As a consequence, for every $x \in \Omega$ and for every $r > 0$ such that $\overline{\Omega_r(x)} \subseteq \Omega$

$$M_r(u)(x) = -M_r(v)(x) + h(x)$$

where M_r is the solid average operator. Here we have used the $\mathcal{L}$-harmonicity of h to write h in place of $M_r(h)$ (by Theorem 1.2.7). Letting r tend to zero in the last identity and using Theorem 1.2.7, we get

$$u(x) = -v(x) + h(x) \quad \forall\, x \in \Omega.$$

This completes the proof. □

For the future reference, we explicitly show the following theorem, which can be proved as Theorem 2.4.1.

Theorem 2.4.2. *Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in \underline{S}(\Omega)$. Let μ be the $\mathcal{L}$-Riesz measure of u. Then, for every bounded open set Ω_1 such that $\overline{\Omega_1} \subset \Omega$, there exists a function h, $\mathcal{L}$-harmonic in Ω_1, satisfying the identity*

$$u(x) = -\int_{\overline{\Omega_1}} \Gamma(y,x)\, d\mu(y) + h(x) \quad \forall\, x \in \Omega_1.$$

Proof. The function

$$v(x) := -\int_{\overline{\Omega_1}} \Gamma(y,x)\, d\mu(y), \quad x \in \mathbb{R}^N$$

is $\mathcal{L}$-subharmonic in $\mathbb{R}^N$ and satisfies $\mathcal{L}v = \mu|_{\overline{\Omega_1}}$ in the weak sense of distributions. Indeed:

$$\begin{aligned}\int_{\mathbb{R}^N} \left(-\int_{\overline{\Omega_1}} \Gamma(y,x)\, d\mu(y) \right) \mathcal{L}\varphi(x)\, dx &= \int_{\overline{\Omega_1}} \left(-\int_{\mathbb{R}^N} \Gamma(y,x) \mathcal{L}\varphi(x)\, dx \right) d\mu(y) \\ &= \int_{\overline{\Omega_1}} \varphi(y) d\mu(y) = \int_{\mathbb{R}^N} \varphi(y)\, d\left(\mu|_{\overline{\Omega_1}}\right)(y)\end{aligned}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$.
Therefore, $\mathcal{L}(u - v) = 0$ in $\mathcal{D}'(\Omega_1)$. Then, just proceeding as in the proof of the previous theorem, we show the existence of an $\mathcal{L}$-harmonic function in Ω_1 such that $u = v + h$ in Ω_1. □

Lemma 2.4.3. *Let $u \in \overline{S}(\Omega)$, let μ be its $\mathcal{L}$-Riesz measure and let $K \subseteq \Omega$ be compact. There exists $w \in \overline{S}(\Omega)$*

$$u = G_\Omega * (\mu|_K) + w \quad \textit{in } \Omega.$$

Proof. Let $H \subseteq \Omega$ be a compact and such that $K \subseteq \mathrm{Int}(H)$. By Theorem 2.4.1 there exists $h \in \mathcal{H}(\mathrm{Int}(H))$ such that

$$u = h + G_\Omega * (\mu|_H) \quad \text{in } \mathrm{Int}(H)$$

Define

$$w : \Omega \to [-\infty, \infty],\ w := \begin{cases} h + G_\Omega * (\mu|_{H\setminus K}) & \text{in } \mathrm{Int}(H) \\ u - G_\Omega * (\mu|_K) & \text{in } \Omega \setminus K. \end{cases}$$

This definition is well-posed since the function $G_\Omega * (\mu|_K)$ is $\mathcal{L}$-harmonic in $\mathrm{Int}(H) \setminus K$, hence real-valued, and

$$h + G_\Omega * (\mu|_{H\setminus K}) = h + G_\Omega * (\mu|_H) - G_\Omega * (\mu|_K) = u - G_\Omega * (\mu|_K).$$

Moreover, w is $\mathcal{L}$-superharmonic in Ω since it is $\mathcal{L}$-superharmonic in $\mathrm{Int}(H)$ and in $\Omega \setminus K$, and $\Omega = \mathrm{Int}(H) \cup (\Omega \setminus K)$. □

In order to enunciate and prove the main Riesz representation theorem, we prove the following theorem on the superharmonicity of $G_\Omega * \mathcal{L}u$.

Theorem 2.4.4. *Let $u \in \overline{S}(\Omega)$, and let μ be its $\mathcal{L}$-Riesz measure. Then:*

$$G_\Omega * \mu \in \overline{S}(\Omega) \tag{2.10}$$

if and only if there exists $v \in \underline{S}(\Omega)$ such that

$$v \leq u$$

.

Proof. We first prove the "if" part and assume $u \geq v$ in Ω with $v \in \underline{S}(\Omega)$. Let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact sets such that $\bigcup_n K_n = \Omega$. Let us put $\mu_n = \mu|_{K_n}$. By Lemma 2.4.3, there exists $w_n \in \overline{S}(\Omega)$ such that $u = w_n + G_\Omega * \mu_n$. The hypothesis implies $0 \leq u - v = (w_n - v) + G_\Omega * \mu_n$ in Ω, so that

$$G_\Omega * \mu_n \geq v - w_n.$$

By Corollary 2.3.5, we have $v - w_n \leq 0$ in Ω. Hence

$$u - v \geq G_\Omega * \mu_n \quad \text{in } \Omega \;\; \forall n \in \mathbb{N}.$$

Letting n tend to infinity, we get $u - v \geq G_\Omega * \mu$. Since $u - v \in \overline{S}(\Omega)$, this implies (2.10): Vice versa, assume (2.10) is true. Then by Theorem 2.3.4,

$$\mathcal{L}(G_\Omega * \mu - u) = 0 \quad \text{in } \Omega, \text{ in the weak sense of distributions.}$$

Since $\mathcal{L}$ is hypoelliptic, there exists a function $h \in \mathcal{H}(\Omega)$ such that $G_\Omega * \mu = u + h$ a.e. in Ω. Then for every $x \in \Omega$ and for every $r > 0$ such that $\overline{\Omega_r(x)} \subseteq \Omega$ we have $M_r(G_\Omega * \mu)(x) = M_r(u + h)(x)$, where M_r is the solid average operator. Letting r tend to zero in the last identity and using Theorem 1.2.7, we get $G_\Omega * \mu = u + h$ everywhere in Ω. Since $G_\Omega * \mu \geq 0$, we have $u \geq -h$, and the proof is complete. □

Theorem 2.4.5 (The Riesz representation)**.** *Let $u \in \overline{S}(\Omega)$, and let μ be the $\mathcal{L}$-Riesz measure of u. The following statements are equivalent:*

(i) there exists $h \in \mathcal{H}(\Omega)$ such that

$$u = G_\Omega * \mu + h \quad \text{in } \Omega, \tag{2.11}$$

(ii) there exists $v \in \underline{S}(\Omega)$ such that $v \leq u$ in Ω,

(iii) every connected component of Ω contains a point x_0 such that

$$(G_\Omega * \mu)(x_0) < \infty.$$

Moreover, (2.11) holds with $h \in \mathcal{H}(\Omega)$ if and only if h is the greatest $\mathcal{L}$-harmonic minorant of u in Ω.

Proof. $(i) \Rightarrow (ii)$. If (2.11) holds, then h is an $\mathcal{L}$-harmonic minorant of u (hence an $\mathcal{L}$-subharmonic function), since $G_\Omega * \mu \geq 0$.
$(ii) \Leftrightarrow (iii)$. This follows from Theorems 2.4.4 and 2.3.1.
$(iii) \Rightarrow (i)$. It is not restrictive to assume that Ω is connected. Let $x_0 \in \Omega$ be such that $(G_\Omega * \mu)(x_0) < \infty$, and let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of bounded open sets such that

$$\overline{\Omega_n} \subset \Omega_{n+1}, \;\; \overline{\Omega_{n+1}} \subset \Omega, \;\; \bigcup_{n\in\mathbb{N}} \Omega_n = \Omega.$$

Define

$$\mu_n := \mu|_{\overline{\Omega_n}}, \quad n \in \mathbb{N}.$$

Then, by the representation Theorem 2.4.1, there exists an $\mathcal{L}$-harmonic function h_n such that

$$u(x) = (G_\Omega * \mu_n)(x) + h_n(x) \quad \forall\, x \in \Omega_n, \; \forall\, n \in \mathbb{N}. \tag{2.12}$$

Since $G_\Omega * \mu_n \nearrow G_\Omega * \mu$, we have

$$h_n(x) \geq h_{n+k}(x) \geq u(x) - (G_\Omega * \mu)(x) \quad \forall\, x \in \Omega_n, \; \forall\, n, k \in \mathbb{N}$$

On the other hand, by Theorem 2.3.1, $G_\Omega * \mu \in \overline{S}(\Omega)$. Then, keeping in mind that $u(x) > -\infty$ for every $x \in \Omega$ (since $u \in \overline{S}(\Omega)$) and that $-G_\Omega * \mu > -\infty$ in a dense subset of Ω (since $-G_\Omega * \mu \in \underline{S}(\Omega)$), for every $n \in \mathbb{N}$ we have

$$\inf_K h_{n+k} \geq u(x) - (G_\Omega * \mu)(x) > -\infty \qquad \text{in a dense subset of } \Omega_n$$

By using Theorem 2.2.8, we infer the existence of a function $h : \Omega \to \mathbb{R}$, $\mathcal{L}$-harmonic in Ω, such that

$$h(x) = \lim_{k\to\infty} h_{n+k}(x) \qquad \forall\, x \in \Omega_n,\ \forall\, n \in \mathbb{N}.$$

Then (2.11) follows from (2.12).

We are left with the proof of the second part of the theorem. Assume (2.11) holds with $h \in \mathcal{H}(\Omega)$. Then, if $k \in \mathcal{H}(\Omega)$ and $k \le u$, we have $k - h \le G_\Omega * \mu$, so that, by Theorem 2.3.6, we have $k - h \le 0$, i.e. $k \le h$. Vice versa, assume h is the greatest $\mathcal{L}$-harmonic minorant of u. Then, by (ii), there exists $k \in \mathcal{H}(\Omega)$ such that $u = G_\Omega * \mu + k$. This implies that k is the greatest $\mathcal{L}$-harmonic minorant of u, i.e. $k = h$. Thus $u = G_\Omega * \mu + h$. The proof is complete. □

As an application of this theorem, we get the following result

Corollary 2.4.6 (Riesz representation in space). *Let $u \in \overline{S}(\mathbb{R}^N)$ be such that $U := \inf_{\mathbb{R}^N} u > -\infty$. Then, there exists $h \in \mathcal{H}(\mathbb{R}^N)$, $h \ge 0$; satisfying*

$$u = U + \Gamma * \mu + h \tag{2.13}$$

where μ is the $\mathcal{L}$-Riesz measure of u.

Proof. Since U is a $\mathcal{L}$-harmonic minorant of u, from the previous Theorem 2.4.5 and from the Remark 2.2.3 it follows that $u = \Gamma * \mu + k$, where k is the greatest $\mathcal{L}$-harmonic minorant of u in $\mathbb{R}^N$. We have $U \le k$, i.e. $h := k - U \ge 0$, $h \in \mathcal{H}(\mathbb{R}^N)$. Thus Equation (2.13) holds. □

We remark that a stronger version of the previous corollary will be proved in Theorem 2.6.1.

Corollary 2.4.7 (Riesz representation. III.). *Let $u \in \underline{S}(\Omega)$ be such that $\mathcal{L}u = 0$ outside a compact set $K \subset \Omega$. Then there exists an $\mathcal{L}$-harmonic function h in Ω such that*

$$u = -G_\Omega * \mu + h \quad \text{in } \Omega.$$

Proof. Since supp(μ)$\subseteq K$, we have, by Corollary 2.3.5, $G_\Omega * \mu \in \overline{S}(\Omega)$. Then $G_\Omega * \mu < \infty$ in a dense subset of Ω, and the assertion follows from Theorem 2.4.5. $\square$

For the future references, we explicitly write the following consequence of Theorems 2.3.1 and 2.3.4.

Corollary 2.4.8 (Riesz representation. IV)**.** *Let $u \in \underline{S}(\Omega)$, and let μ be the $\mathcal{L}$- Riesz measure of u. Assume $(\Gamma * \mu)(x_0) < \infty$ at some point $x_0 \in \mathbb{R}^N$. Then there exists an $\mathcal{L}$-harmonic function h in Ω such that*

$$u(x) = -(\Gamma * \mu)(x) + h(x) \quad \forall\, x \in \Omega.$$

Proof. We can consider μ as a measure in the whole space, by extending it with 0 outside Ω. By Theorem 2.3.4, the hypothesis on $\Gamma * \mu$ implies that $\mathcal{L}(\Gamma * \mu) = \mu$ in Ω in the weak sense of distributions, so that $\mathcal{L}(u + \Gamma * \mu) = 0$ in Ω in the weak sense of distributions. Then, proceeding as in the proof of Theorem 2.4.1, we show the existence of $h \in \mathcal{H}(\Omega)$ such that $u = -\Gamma * \mu + h$ in Ω. $\square$

2.5 The Poisson-Jensen Formula

The next theorem, when $\mathcal{L} = \Delta$ is the classical Laplace operator, will give back the classical Poisson-Jensen formula (see, e.g. [36, Theorem 3.14]).

Theorem 2.5.1. *Let U, Ω be open subsets of $\mathbb{R}^N$, $\overline{\Omega} \subset U$ and Ω be $\mathcal{L}$-regular. Let $u \in \underline{S}(\Omega)$ and $\mu = \mathcal{L}u$ be its $\mathcal{L}$-Riesz measure. Then*

$$u(x) = \int_{\partial\Omega} u(y)\, d\mu_x^\Omega(y) - \int_\Omega G_\Omega(y, x)\, d\mu(y), \quad x \in \Omega. \tag{2.14}$$

Here G_Ω is the $\mathcal{L}$-Green function of Ω and μ_x^Ω is the $\mathcal{L}$-harmonic measure related to Ω.

Proof. Let O be a bounded open set such that $\overline{\Omega} \subset O \subset \overline{O} \subset U$. By the Riesz representation Theorem 2.4.2 there exists an $\mathcal{L}$-harmonic function h in O such that

$$u(x) = -\int_{\overline{O}} \Gamma(y,x)\, d\mu(y) + h(x) =: v(x) + h(x) \quad \forall\, x \in O.$$

We have $v \in \underline{S}(\mathbb{R}^N)$ and $\mathcal{L}v = \mu|_{\overline{O}}$. Then, since

$$h(x) = \int_{\partial\Omega} h(y)\, d\mu_x^\Omega(y) \quad \forall\, x \in \Omega,$$

it suffices to prove (2.14) with u replaced by v. We can also suppose that $v(x) > -\infty$. Indeed, if $v(x) = -\infty$, then $u(x) = -\infty$ and

$$\int_\Omega G_\Omega(y,x)\, d\mu(y) \geq \int_\Omega \Gamma(y,x)\, d\mu(y) = -u(x) - h(x) = \infty$$

Moreover, since $u \in \underline{S}(\Omega)$, the function $x \mapsto \int_{\partial\Omega} u(y)\, d\mu_x^\Omega(y)$ is $\mathcal{L}$-harmonic, hence real-valued. Thus, in this case, (2.14) trivially holds.
Let us fix $x \in \Omega$. We have

$$\int_{\partial\Omega} v(y)\, d\mu_x^\Omega(y) = -\int_{\overline{O}} \left(\int_{\partial\Omega} \Gamma(z,y)\, d\mu_x^\Omega(y) \right) d\mu(z).$$

The crucial part of the proof is to show that

$$\int_{\partial\Omega} \Gamma(z,y)\, d\mu_x^\Omega(y) = \begin{cases} \Gamma(z,x), & z \in \overline{O} \setminus \Omega, \\ h(z) = \Gamma(x,z), & z \in \Omega. \end{cases} \tag{2.15}$$

With (2.15) at hand, and keeping in mind the assumption $v(x) > -\infty$ which implies that $z \mapsto \Gamma(z,x)$ is μ-summable, we get the assertion. Indeed,

$$\begin{aligned} \int_{\partial\Omega} v(y)\, d\mu_x^\Omega(y) &= \int_{\overline{O}} \Gamma(z,x)\, d\mu(z) + \int_\Omega G_\Omega(z,x)\, d\mu(z) \\ &= v(x) + \int_\Omega G_\Omega(z,x)\, d\mu(z). \end{aligned}$$

Then, it remains to prove (2.15). If $z \in \overline{O} \setminus \overline{\Omega}$, the function $\Gamma(z,\cdot)$ is harmonic in $O \setminus \{z\}$, so that

$$\Gamma(z,x) = \int_{\partial\Omega} \Gamma(z,y)\, d\mu_x^\Omega(y).$$

If $z \in \Omega$, the function $\Gamma(z,\cdot)$ is continuous in $\partial\Omega$, hence the solution h_x to the Dirichlet problem

$$\begin{cases} \mathcal{L}h(x) = 0, & x \in \Omega, \\ h(y) = \Gamma(z,y), & y \in \partial\Omega, \end{cases}$$

is given by

$$h_z(y) := \int_{\partial\Omega} \Gamma(z,y)\, d\mu_x^\Omega(y).$$

Then, by the definition of the $\mathcal{L}$-Green function,

$$\int_{\partial\Omega} \Gamma(z,y)\, d\mu_x^\Omega(y) = h_z(x) = \Gamma(z,x) - G_\Omega(z,x).$$

Finally, we fix $z_0 \in \partial\Omega$. Let us prove that

$$\int_{\partial\Omega} \Gamma(y,z_0)\, d\mu_x^\Omega(y) = \Gamma(x,z_0). \tag{2.16}$$

Since $\Gamma(x,z_0) = \Gamma(z_0,x)$, this will give (2.15) in the case $z_0 \in \partial\Omega$. Let us define

$$w(z) := \Gamma(x,z) - \int_{\partial\Omega} \Gamma(y,z)\, d\mu_x^\Omega(y), \quad z \in \mathbb{R}^N \setminus \{x\}$$

The function w is $\mathcal{L}$-subharmonic in $\mathbb{R}^N \setminus \{x\}$ and

$$\limsup_{z \notin \partial\Omega,\, z \to \to \zeta} w(z) = 0 \quad \forall\, \zeta \in \partial\Omega. \tag{2.17}$$

Indeed, (2.15) holds in $O \setminus \partial\Omega$ and $G_\Omega(z,x) \to 0$ as $z \to \zeta$ from inside of Ω, since Ω is $\mathcal{L}$-regular and G_Ω is symmetric. In order to prove (2.16), we have to show that $w \equiv 0$ on $\partial\Omega$. First of all, we observe that

$$w(\zeta) \geq \limsup_{z \to \zeta} w(z) \geq \limsup_{z \notin \partial\Omega,\, z \to \zeta} w(z) = 0 \quad \forall\, \zeta \in \partial\Omega.$$

Suppose, by contradiction, that $w > 0$ somewhere in $\partial\Omega$. Then we have $\max_{\partial\Omega} w > 0$. From (2.17) it follows that there exists an open set $V \subseteq O$ such that $\partial\Omega \subset V$, $x \notin V$ and $\max_V w = \max_{\partial\Omega} w$. Let $\zeta_0 \in \partial\Omega$ be such that $w(\zeta_0) = \max_V w$. Since w is $\mathcal{L}$-subharmonic in V, by the Strong Maximum Principle for $\mathcal{L}$-subharmonic functions, Theorem 3.2.1, $w \equiv w(\zeta_0)$ in the connected component of V containing ζ_0. Thus,

$$\lim_{z \notin \partial\Omega,\, z \to \zeta_0} w(z) = w(\zeta_0) = \max_V w = \max_{\partial\Omega} w > 0.$$

in contradiction with (2.17). This completes the proof of (2.15). $\square$

If, in the previous theorem, we take $\Omega = \Omega_r(x)$, we obtain an extension of the mean value formulas (Theorem 1.2.6) to the $\mathcal{L}$-subharmonic functions. The same Theorem will be prove in Chapter 3 (Theorem 3.3.4), throughout another proof, more laborious, which will note use the $\mathcal{L}$-Green function.

Theorem 2.5.2 (Mean value formulas for $\mathcal{L}$-subharmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $u \in \underline{S}(\Omega)$. Then, for every $x \in \Omega$, and $r > 0$ such that $\overline{\Omega_r(x)} \subset \Omega$, we have*

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x,y) - \frac{1}{r}\right) d\mu(y) \tag{2.18}$$

and

$$u(x) = M_r(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x,y) - \frac{1}{\rho}\right) d\mu(y) \right) d\rho, \tag{2.19}$$

where $\mu := \mathcal{L}u$ is the $\mathcal{L}$-Riesz measure of u.

Proof. Take $\Omega = \Omega_r(x)$ in Poisson-Jensen's Theorem 2.5.1. By Lemma 5.8 in [14] for every $x \in \mathbb{R}^N$ and every $r > 0$, we have

$$d\mu_x^{\Omega_r(x)} = \mathcal{K}(x,y)\, d\sigma(y)$$

where $d\sigma$ denotes, the $N-1$-dimensional Hausdorff measure in $\mathbb{R}^N$. Then

$$\int_{\partial\Omega_r(x)} u(y)\, d\mu_x^{\Omega_r(x)} = m_r(u)(x).$$

Moreover, by Remark 2.1.2, we have

$$G_{\Omega_r(x)} = \Gamma(x,y) - \frac{1}{r}$$

Then (2.18) follows from (2.14).
Identity (2.19) follows from (2.18) keeping in mind that

$$M_r(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha m_\rho(u)(x)\, d\rho$$

This completes the proof. $\square$

2.6 Bounded-above $\mathcal{L}$-subharmonic Functions in $\mathbb{R}^N$

Theorem 2.6.1 (The $\mathcal{L}$-Riesz measure of a bounded-above $u \in \underline{S}(\mathbb{R}^N))$). *Let μ be a Radon measure in $\mathbb{R}^N$, and let $x_0 \in \mathbb{R}^N$. If the following condition holds:*

$$\int_0^\infty \frac{\mu(\Omega_\rho(x_0)}{\rho^2}\, d\rho < \infty \tag{2.20}$$

then there is a $\mathcal{L}$-subharmonic function u in $\mathbb{R}^N$ having the $\mathcal{L}$-Riesz measure μ, the least upper bound $U < \infty$ and such that $u(x_0) > -\infty$, given by

$$u(x) = U - \int_{\mathbb{R}^N} \Gamma(y, x)\, d\mu(y) \tag{2.21}$$

Besides, if v is another function satisfying (2.21), then, by Corollary 2.4.6,

$$u = v + h$$

where $h \in \mathcal{H}(\Omega)$, $h \leq 0$.

Proof. It is not restrictive to assume $x_0 = 0$. Consider the function

$$u(x) := U - \int_{\mathbb{R}^N} \Gamma(y, x)\, d\mu(y), \quad x \in \mathbb{R}^N.$$

We shall show that

(i) $u(0) > -\infty$,

(ii) $\sup_{\mathbb{R}^N} u = U$

Since (i) implies $u \in \underline{S}(\mathbb{R}^N)$ and $\mathcal{L}u = \mu$ (see Theorems 2.3.1 and 2.3.4), this will prove the theorem.
To prove (i) we observe first of all that, by (2.20),

$$\mu(\{0\}) = \lim_{t \to 0} \mu(\Omega_t(0)) = 0$$

Then

$$\begin{aligned}
u(0) - U &= -\int_{\mathbb{R}^N \setminus \{0\}} \Gamma(y,0)\, d\mu(y) \\
&= -\lim_{\lambda \downarrow 0} \int_{\left\{0 < \Gamma(y,0) \le \frac{1}{\lambda}\right\}} \Gamma(y,0)\, d\mu(y) \\
&= -\lim_{\lambda \downarrow 0} \int_{\left\{0 < \Gamma \le \frac{1}{\lambda}\right\}} \left(\int_0^{\Gamma(y,0)} ds \right) d\mu(y) \\
&= -\lim_{\lambda \downarrow 0} \int_0^{\frac{1}{\lambda}} \left(\int_{\Omega_{\frac{1}{s}}(0) \setminus \Omega_\lambda(0)} d\mu(y) \right) ds \\
&= -\lim_{\lambda \downarrow 0} \int_0^{\frac{1}{\lambda}} \mu\left(\Omega_{\frac{1}{s}}(0) - \Omega_\lambda(0) \right) ds \\
&= -\int_0^{+\infty} \mu\left(\Omega_{\frac{1}{s}}(0) - \{0\} \right) ds \\
&= -\int_0^{+\infty} \mu\left(\Omega_{\frac{1}{s}}(0) \right) ds \\
&= \int_0^{+\infty} \frac{\mu\left(\Omega_\rho(0)\right)}{\rho^2} d\rho > -\infty
\end{aligned}$$

This implies that $u(0) > -\infty$.
To prove (ii), for a fixed $R > 0$, let us split $u - U$ as follows

$$u(x) - U = u_R(x) + u_R^\infty(x) \tag{2.22}$$

where

$$u_R(x) := -\int_{\Omega_R(0)} \Gamma(y,x)\, d\mu(y)$$

$$u_R^\infty(x) := -\int_{\mathbb{R}^N \setminus \Omega_R(0)} \Gamma(y,x)\, d\mu(y)$$

We have

$$\lim_{x \to +\infty} u_R(x) = 0 \tag{2.23}$$

On the other hand, since u_R^∞ is the Γ-potential of the measure $\mu|_{\mathbb{R}^N \setminus \Omega_R(0)}$ and

$$u_R^\infty(0) \ge u(0) - U \ge -\infty$$

by Theorem 2.3.1, u_R^∞ is $\mathcal{L}$-subharmonic in $\mathbb{R}^N$. Hence, by Theorem 1.2.7 u_R^∞ is sub-mean. Thus, for every $r > 0$, we have

$$\begin{aligned}
m_r(u_R^\infty(0)) \geq u_R^\infty(0) &= -\int_{\mathbb{R}^N \setminus \Omega_R(0)} \Gamma(y,0)\, d\mu(y) \\
&= \int_{\left\{\Gamma(y,0)<\frac{1}{R}\right\}} \Gamma(y,0)\, d\mu(y) \\
&= -\int_{\left\{\Gamma<\frac{1}{R}\right\}} \left(\int_0^{\Gamma(y,0)} ds\right) d\mu(y) \\
&= -\int_0^{\frac{1}{R}} \left(\int_{\left\{s<\Gamma<\frac{1}{R}\right\}} d\mu(y)\right) ds \\
&= -\int_0^{\frac{1}{R}} \mu\left(\Omega_{\frac{1}{s}}(0) - \Omega_R(0)\right) ds \\
&\geq -\int_0^{\frac{1}{R}} \mu\left(\Omega_{\frac{1}{s}}(0)\right) ds \\
&= -\int_R^{+\infty} \frac{\mu\left(\Omega_\rho(0)\right)}{\rho^2} d\rho > -\infty
\end{aligned}$$

This implies the existence of at least one point $y(r,R) \in \partial\Omega_r(0)$ such that

$$u_r^\infty(y(r,R)) \geq -\int_R^{+\infty} \frac{\mu\left(\Omega_\rho(0)\right)}{\rho^2} d\rho > -\infty \tag{2.24}$$

Since $d(y(r,R)) = r$ for every $R > 0$, the fact that $\sup u_{\mathbb{R}^N} = U$ follows from (2.22)-(2.24) and condition (2.20). $\square$

***Remark* 2.6.2.** If we consider the case of sub-Laplacians on Carnot groups, condition (2.20) becomes condition (9.29) of the Theorem 9.6.1 in [16]. Indeed, since

$$\begin{aligned}
\Omega_\rho(x_0) &= \left\{\Gamma > \frac{1}{\rho}\right\} = \left\{\frac{1}{d^{Q-2}} > \frac{1}{\rho}\right\} \\
&= \left\{d^{Q-2} < \rho\right\} = \left\{d < \rho^{\frac{1}{Q-2}}\right\} = B_d\left(x_0, \rho^{\frac{1}{Q-2}}\right),
\end{aligned}$$

then

$$\begin{aligned}
&\int_0^\infty \frac{\mu\left(\Omega_\rho(x_0)\right)}{\rho^2}\, d\rho = \int_0^\infty \frac{\mu\left(B_d\left(x_0, \rho^{\frac{1}{Q-2}}\right)\right)}{\rho^2}\, d\rho \\
&= \int_0^\infty \frac{\mu\left(B_d\left(x_0,t\right)\right) t^{Q-1}}{t^{2Q-2}}\, dt = \int_0^\infty \frac{\mu\left(B_d\left(x_0,t\right)\right)}{t^{Q-1}}\, dt.
\end{aligned}$$

Chapter 3

The inverse mean value Theorem

The main result of this Chapter is an inverse mean value theorem characterizing the sub-Riemannian "balls" $\Omega_r(x)$ by means of the $\mathcal{L}$-harmonic functions.
We first briefly describe our investigation of the *positivity set of the kernel K of the mean integral operator*, because our main theorem makes crucial use of it. Whereas in the classical harmonic case the kernel K is identically 1 (this is the only case where K can be constant, see [12]), the simplest case of a non-elliptic operator $\mathcal{L}$, namely the Kohn-Laplacian on the Heisenberg group in $\mathbb{R}^3$, shows that K may be non-trivial and may admit infinite zeroes. Indeed, in the Heisenberg group case the associated function $x \mapsto K(0, x)$ vanishes precisely on the x_3-axis. However (as the latter example confirms) we shall prove that for any of our operators $\mathcal{L}$ and for any given $x \in \mathbb{R}^N$, the set $\{y : K(x, y) > 0\}$ is an open *dense* set in $\mathbb{R}^N$ (see Theorem 3.1.1).
The proof of this result exploits in a crucial way the hypoellipticity of $\mathcal{L}$ and some geometrical sub-Riemannian properties of $\mathcal{L}$, so it deserves to be mentioned in this introduction. We first prove that the zeroes of $K(x, \cdot)$ are given by the solutions of the PDE system

$$X_1\Gamma_x = \cdots = X_N\Gamma_x = 0,$$

where $\Gamma_x = \Gamma(x, \cdot)$ is the fundamental solution of $\mathcal{L}$ with pole at x, and $X_1, \ldots, X_N$ are the vector fields associated to the rows of A. Now, thanks to some results by Amano [5], the hypoellipticity of $\mathcal{L}$ ensures that the frame $X_1, \ldots, X_N$ satisfies Hörmander's rank condition on an open dense set of $\mathbb{R}^N$. This fact will allow us to prove that if the zero-set of $K(x, \cdot)$ had nonempty interior, the function Γ_x would be constant on some open set, contrarily to our assumptions on the fundamental solution Γ.

We remark that by means of our result on the zeroes of K, we are able to provide a *very simple proof of the Strong Maximum Principle for $\mathcal{L}$-subharmonic functions* (see the proof of Theorem 3.2.1). The study of the 'smallness' of this set of zeroes is furthermore enhanced by Proposition 3.1.2 which proves that, if $X = \{X_1, \ldots, X_N\}$, the X-characteristic set of the manifold $\partial\Omega_r(x)$ is equal to the subset of $\partial\Omega_r(x)$ where $K(x, \cdot)$ vanishes. By a result of Derridj [21], this proves that the latter set has vanishing $(N-1)$-dimensional Hausdorff measure, relative to the dense open set where X is a Hörmander system (see also [10, 31, 48]).

Let us pass to discuss our version of the *inverse mean value theorem* on sub-Riemannian "balls" $\Omega_r(x)$. By the well-known Mean Value Theorem for classical harmonic functions, if $B = B(0, r)$ denotes the Euclidean ball centred at $0 \in \mathbb{R}^N$ with radius $r > 0$ and if H_N denotes Lebesgue N-dimensional measure, then

$$u(0) = \frac{1}{H_N(B)} \int_B u(y) \, \mathrm{d}H_N(y), \tag{3.1}$$

for every integrable harmonic function u on B. Conversely, if B is a bounded open neighborhood of 0 and if the above identity holds for every integrable harmonic function u on B, then B is necessarily a Euclidean ball centred at the origin. This notable *inverse mean value theorem* is a 1972 result by Kuran [41] (actually, Kuran does not suppose that B is bounded, but he only assumes B has finite Lebesgue measure). A comprehensive bibliography on similar spherical-symmetry results, under different assumptions, may be found in the survey paper by Netuka and Veselý [51]. The fact that Euclidean balls are superlevel sets of the fundamental solution of the Laplace operator

obviously plays a fundamental rôle.
More recently, an inverse mean value theorem has been proved by Suzuki and Watson [67] in the case of the Heat equation: Superlevel sets B of the fundamental solution of the Heat operator in $\mathbb{R}^{N+1}$ are characterized by means of identities of the form

$$u(0) = \frac{1}{\mu(B)} \int_B u(y)\,\mathrm{d}\mu(y),$$

it sufficing that u belongs to a suitable subclass of caloric functions on B. Here the measure μ is of the form $\mathrm{d}\mu = \psi\,\mathrm{d}H_{N+1}$, for a certain nonnegative density ψ deeply related to the Pini-Watson's mean value theorem for temperatures [59, 69]. In the Heat operator case, suitable further assumptions on B are needed (for the precise statement, see [69, Theorem at p.2710]).
Very recently, an inverse mean value theorem has been established for sub-Laplacians on Carnot groups by Lanconelli [43]. Lanconelli extended to the framework of sub-Laplacians the following result by Aharonov, Schiffer, Zalcman [4], which is in its turn a generalization of Kuran's theorem: Let B be a bounded open neighborhood of 0 in $\mathbb{R}^N$, $N \geq 3$; assume that, for some real constants a, b

$$\int_B |x-y|^{2-N}\,\mathrm{d}H_N(y) = a\,|x|^{2-N} + b, \qquad \forall\, x \notin B. \tag{3.2}$$

Then $a = H_N(B)$, $b = 0$ and B is a Euclidean ball centred at the origin. We observe that $|x|^{2-N}$ is the fundamental solution (with pole at the origin) of the Laplace operator in $\mathbb{R}^N$, $N \geq 3$; moreover, the family $\{y \mapsto |x-y|^{2-N} : x \notin B\}$ is a subclass of harmonic functions on B. Lanconelli's result [43, Theorem 1.1] proves the following fact: Let $\mathbb{G} = (\mathbb{R}^N, *)$ be a Carnot group (with homogeneous dimension $Q \geq 3$) and let $\mathcal{L} = \sum_{j=1}^m X_j^2$ be a sub-Laplacian on $\mathbb{G}$ with fundamental solution (with pole at the origin) $\Gamma(x) = d^{2-Q}(x)$; let $B \subset \mathbb{R}^N$ be a bounded open neighborhood of 0 and assume that, for some real constants a, b

$$\int_B d^{2-Q}(y^{-1} * x)\,\mathrm{d}\mu(y) = a\,d^{2-Q}(x) + b, \qquad \forall\, x \notin B. \tag{3.3}$$

Then $a = \mu(B)$, $b = 0$ and B is a superlevel set of Γ. Here $\mathrm{d}\mu = \psi\,\mathrm{d}H_N$ is the measure on $\mathbb{G}$ with density $\psi = (\sum_{j=1}^m (X_j d)^2)^{1/2}$. Notice that, since $\Gamma(x,y) = d^{2-Q}(y^{-1} * x)$ is the fundamental solution of $\mathcal{L}$ with pole at x, (3.3) is a generalization of (3.2). We remark that the density function ψ is the kernel appearing in a suitable version, for sub-Laplacians, of the classical Mean Value Theorem (see e.g. [16, Chapter 5]). At the same time, ψ is a particular case of our previous kernels K_α introduced in Chapter 1: precisely, $\psi = K_\alpha(0,\cdot)$ with $\alpha = 2/(Q-2)$.

In this chapter we shall be dealing with a generalization of the original Kuran's version of the inverse mean value theorem. We shall consider our operators $\mathcal{L}$, hence our setting covers the case of sub-Laplacians treated in [43], but not the Heat operator case in [67], for in this latter case the pole of Γ is on the boundary of the relevant superlevel set, which is excluded by our set of assumptions on Γ. More precisely, if $\mathcal{L}$ and Γ satisfy the requirements cited at the beginning of the introduction, together with a further integrability assumption on Γ (see hypothesis (H) in Section 3.3, a natural assumption since it is satisfied by all Hörmander sums of squares of vector fields), we prove the following result:

Let B be a bounded open neighborhood of 0. Suppose that

$$u(0) = \frac{1}{\mu(B)} \int_B u(y)\,\mathrm{d}\mu(y),$$

for every u which is $\mathcal{L}$-harmonic and μ-integrable on B. Then B is a superlevel set of $\Gamma(0,\cdot)$, that is $B = \Omega_r(0)$ for some $r > 0$.

Here, as in [43], $\mathrm{d}\mu = \psi\,\mathrm{d}H_N$, with $\psi(y) = K(0,y)$. Actually, our proof only assumes that

$$\Gamma(x,0) = \frac{1}{\mu(B)} \int_B \Gamma(x,y)\,\mathrm{d}\mu(y), \qquad \forall\ x \notin B,$$

which is a weaker condition if compared to Kuran's assumption (3.1) (indeed a smaller family of $\mathcal{L}$-harmonic functions is involved), and, at the same time, a generalization (when $b = 0$) of Lanconelli's hypothesis (3.3).

We follow the potential-theoretic approach used in [43] (which in its turn uses the techniques in [41] and [67]). Although that approach is perfectly suited to prove the above result, the proof of our inverse mean value theorem is unexpectedly more delicate. Indeed, with respect to the Carnot group case considered in [43], we do not benefit of some features which appear to play a crucial rôle in that setting (hence in the Euclidean case too) for the proof of continuity of the Γ-potential functions. For instance, in Carnot groups the following properties are satisfied: the 'left-invariance' of Γ (namely, the identity $\Gamma(x,y) = \Gamma(y^{-1} * x, 0)$ for $x \neq y$); the presence of suitable dilations δ_λ; ψ is δ_λ-homogeneous of degree zero, hence *bounded*. These properties easily ensure the continuity of the Γ-potential function

$$x \mapsto \int_B \Gamma(x,y)\, \psi(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^N.$$

Instead, in the more general context of the present paper, additional assumptions and new proofs are required to obtain this continuity property.
Furthermore, we need to prove that the superlevel and sublevel sets of $\Gamma(x,\cdot)$ are connected sets (see Proposition 3.3.3): whereas this is obvious in the Euclidean case, whilst it is a consequence of the existence of dilations in the Carnot group case, in our more general framework it seems nontrivial. We shall prove it as a consequence of the Weak Maximum Principle for $\mathcal{L}$.

3.1 Kernel of the mean value operator

The aim of this section is to prove the following result.

Theorem 3.1.1 (Positivity of the kernel of M_r)**.** *Following the notation in Definition 1.2.3, for every $x \in \mathbb{R}^N$ the set*

$$U_x := \big\{ y \in \mathbb{R}^N \setminus \{x\} :\, K(x,y) = 0 \big\} \tag{3.4}$$

is relatively closed in $\mathbb{R}^N \setminus \{x\}$ and it has empty interior, whence $K(x,\cdot)$ is positive on a dense open subset of $\mathbb{R}^N \setminus \{x\}$.

Proof. We begin by recalling that, given a symmetric nonnegative-definite real matrix A, the set $I := \{\xi \in \mathbb{R}^N : \langle A\xi, \xi\rangle = 0\}$ coincides with the kernel of A.

Let $x \in \mathbb{R}^N$ be fixed and let U_x be as in the assertion. By the continuity of $K(x, \cdot)$ on $\mathbb{R}^N \setminus \{x\}$, it follows that U_x is relatively closed in $\mathbb{R}^N \setminus \{x\}$. Moreover we obviously have

$$U_x = \{y \in \mathbb{R}^N \setminus \{x\} : \langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle = 0\}.$$

Hence, by the linear-algebraic remark at the beginning of the proof, we have

$$U_x = \{y \in \mathbb{R}^N \setminus \{x\} : A(y)\nabla\Gamma_x(y) = 0\}. \tag{3.5}$$

Let us consider the smooth vector fields (i.e., linear first-order PDOs) associated to the N rows of the matrix $A = (a_{i,j})$ in (1.1), that is we set $X_i := \sum_{k=1}^N a_{i,k}\,\partial_k$, for $i = 1, \dots, N$. By means of the vector fields X_i, we can rewrite $A(y)\nabla\Gamma_x(y) = \big(X_1\Gamma_x(y) \;\cdots\; X_N\Gamma_x(y)\big)^T$. Consequently, by (3.5) we infer that

$$U_x = \{y \in \mathbb{R}^N \setminus \{x\} \,\big|\, X_1\Gamma_x(y) = \cdots = X_N\Gamma_x(y) = 0\}. \tag{3.6}$$

Our aim is to show that the set where the vector fields $X_1\Gamma_x, \dots, X_N\Gamma_x$ are all identically vanishing has empty interior. We shall do this by a connectivity theorem (along the integral curves of the X_i's) and by exploiting the hypoellipticity of $\mathcal{L}$.

We denote by $\mathrm{Lie}\{X_1, \dots, X_N\}(z)$ the subspace of $\mathbb{R}^N$ obtained by evaluating at $z \in \mathbb{R}^N$ the vector fields of the Lie algebra generated by $\{X_1, \dots, X_N\}$. By gathering together Theorems 1 and 2 in Amano's paper [5, pag.113], we deduce that the hypoellipticity of $\mathcal{L} = \mathrm{div}(A\nabla)$ ensures that the set $\{z \in \mathbb{R}^N : \dim(\mathrm{Lie}\{X_1, ..., X_N\}(z)) < N\}$ is a closed set with empty interior in $\mathbb{R}^N$. As a consequence, the set

$$\Omega := \{z \in \mathbb{R}^N : \dim(\mathrm{Lie}\{X_1, ..., X_N\}(z)) = N\} \tag{3.7}$$

is an open dense set in $\mathbb{R}^N$. We claim that

$$U_x \cap \Omega \text{ has empty interior.} \tag{3.8}$$

This easily shows that U_x has empty interior, since Ω is dense in $\mathbb{R}^N$.
Hence we are left to prove (3.8). To this end, we argue by contradiction assuming the existence of a non-empty open Euclidean ball $B \subseteq U_x \cap \Omega$.
On B we have $X_i\Gamma_x = 0$ for $i = 1, ..., N$, hence every commutator applied on Γ_x vanishes on B. Then, since, by (3.7), $\{X_1, ..., X_N\}$ is a Hörmander system of vector fields on B, it follows that every derivative of Γ_x vanishes on B and then Γ_x is constant on B. This fact is in contradiction with (1.6). □

We next prove another special feature of the set where the kernel $K(x, \cdot)$ vanishes:

Proposition 3.1.2. *Let $x \in \mathbb{R}^N$ be fixed and let U_x be the set defined in (3.4). Let $X = \{X_1, \dots, X_N\}$ be the vector fields associated to the rows of $A(x)$ in (1.1).*
Then, for every $r > 0$, the X-characteristic set of the manifold $\partial\Omega_r(x)$ is equal to the subset of $\partial\Omega_r(x)$ contained in U_x.

We recall that, given a set of vector fields $X = \{X_1, \dots, X_N\}$, and an $(N-1)$-dimensional smooth submanifold M of $\mathbb{R}^N$, we say that the set

$$\mathrm{Char}_X(M) := \big\{p \in M \,\big|\, X_i(p) \in T_p(M),\ \forall\ i = 1, ..., N\big\}$$

is the X-characteristic set of M. Here and in the proof below, we denote by $T_p(M)$ the tangent space to the manifold M at the point $p \in M$.

Proof. As $\partial\Omega_r(x)$ is a level set of Γ_x, the hypothesis (1.6) yields that $T_y\big(\partial\Omega_r(x)\big) = \big\{\xi \in \mathbb{R}^N \,:\, \langle\nabla\Gamma_x(y), \xi\rangle = 0\big\}$, for every $y \in \partial\Omega_r(x)$. As a consequence we have

$$\begin{aligned}\mathrm{Char}_X(\partial\Omega_r(x)) &= \big\{y \in \partial\Omega_r(x) \,\big|\, \langle\nabla\Gamma_x(y), X_i(y)\rangle = 0,\ \forall\ i = 1, ..., N\big\}\\ &= \big\{y \in \partial\Omega_r(x) \,\big|\, X_i\Gamma_x(y) = 0,\ \forall\ i = 1, ..., N\big\} = U_x \cap \partial\Omega_r(x).\end{aligned}$$

In the last equality we used (3.6). □

As a motivation of the next result, we recall that, in the case of the sub-Laplacian operators $\mathcal{L} = \sum_{j=1}^m Z_j^2$ on Carnot groups, we have the following special properties of the kernel K (proved e.g., in [16, Section 5.5]):

1. the kernel K related to the operator M_r can be written as $\|\nabla_{\mathcal{L}} d\|^2 := \sum_{j=1}^m |Z_j d|^2$, where d is the gauge function obtained from a suitable negative power of Γ_x;

2. M_r is a mean-integral operator over the d-ball $\{y : d(x,y) < r\}$.

The next proposition shows that analogues of the above properties also holds true in the more general context of our subelliptic operators $\mathcal{L}$ in (1.1). As usual, $\alpha > 0$ is fixed and M_r is the mean-value operator in (1.9).

Proposition 3.1.3. *If $\mathcal{L}$ and A are as in (1.1), we set $\|\nabla_{\mathcal{L}} f(y)\|^2 := \big\langle A(y)\nabla f(y), \nabla f(y)\big\rangle$, for every C^1 function f. Let us also set*

$$d_\alpha(x,y) := \Gamma^{-\alpha/2}(x,y), \quad x \neq y. \tag{3.9}$$

Then, for every u.s.c. function $u : \Omega \to [-\infty, \infty)$ and whenever $\Omega_r(x) \Subset \Omega$, we have

$$M_{r^{2/\alpha}}(u)(x) = \frac{4(\alpha+1)}{\alpha^2\, r^{2+\frac{2}{\alpha}}} \int_{d(x,y)<r} u(y)\, \|\nabla_{\mathcal{L}} d_\alpha(x,\cdot)\|^2(y)\, \mathrm{d}y. \tag{3.10}$$

Proof. Let $\beta = 2/\alpha$ and let $d_x(y) := \Gamma^{-1/\beta}(x,y)$. Since $\nabla\Gamma_x = \nabla(d_x^{-\beta}) = -\beta\, d_x^{-\beta-1}\, \nabla d_x$, we have the following computation:

$$\begin{aligned}
\frac{\alpha+1}{r^{\beta\,(\alpha+1)}} \cdot \frac{\big\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\big\rangle}{\Gamma_x^{2+\alpha}(y)} &= \frac{(\alpha+1)\beta^2}{r^{\beta(\alpha+1)}} \cdot \frac{d_x^{-2\beta-2}(y)}{d_x^{-\beta(2+\alpha)}(y)} \cdot \big\langle A(y)\nabla d_x(y), \nabla d_x(y)\big\rangle \\
&= \frac{4(\alpha+1)}{\alpha^2\, r^{2+\frac{2}{\alpha}}} \cdot \|\nabla_{\mathcal{L}} d_x(y)\|^2.
\end{aligned}$$

Formula (3.10) now follows from the equality $\{\Gamma_x > 1/r^\beta\} = \{d_x < r\}$. □

3.2 Strong Maximum Principle

We provide a very simple proof of the Strong Maximum Principle for $\mathcal{L}$-subharmonic functions, by means of the sub-mean characterization in Theorem 1.2.7 and by the aid of Theorem 3.1.1.

Theorem 3.2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set and let $u \in \underline{\mathcal{S}}(\Omega)$. If there exists $x \in \Omega$ such that $u(x) = \max_\Omega u$, then $u \equiv u(x)$ in Ω.*

Proof. Let $x_0 \in \Omega$ be such that $u(x_0)$ is the maximum of u in Ω. We may suppose $u(x_0) \neq -\infty$. Since u is $\mathcal{L}$-sub-mean, by Theorem 1.2.7 we have (see also (1.13))

$$0 \leq M_r(u)(x_0) - u(x_0) = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x_0)} \big(u(y) - u(x_0)\big)\, K(x_0, y)\, \mathrm{d}y,$$

for some $r > 0$. Thus, since $K \geq 0$ and $u(y) \leq u(x_0)$,

$$\big(u(y) - u(x_0)\big)\, K(x_0, y) = 0 \quad \text{for almost every } y \text{ in } \Omega_r(x_0).$$

On the other hand, by Theorem 3.1.1, $K(x_0, \cdot) > 0$ in a dense open subset of $\Omega_r(x_0)$. Since the intersection of an *open* dense set with a set whose complement has vanishing Lebesgue measure is a dense set, we infer that $u(y) = u(x_0)$ for y in a dense subset of $\Omega_r(x_0)$. From the upper semicontinuity of u we deduce that $u \geq u(x_0)$ on the whole of $\Omega_r(x_0)$. Since $u(x_0)$ is the maximum of u, we get $u \equiv u(x_0)$ on $\Omega_r(x_0)$. A connectivity argument finally proves that $\Omega = \{x_0 \in \Omega : \ u(x_0) = \max_\Omega u\}$, this set being non-empty by hypothesis. $\square$

3.3 Main result

In order to establish the main result of this section, we need a further hypothesis on the fundamental solution Γ of $\mathcal{L}$:

(H) There exists $\alpha > 0$ such that, for every compact set $F \subset \mathbb{R}^N$, the sequence of functions

$$f_n(x) := \int_{\Omega_{1/n}(x)} \Gamma(x, y)\, K_\alpha(0, y)\, \mathrm{d}y$$

vanishes as $n \to \infty$, uniformly in $x \in F$.

In order to confirm that hypothesis (H) is not too restrictive, we show that *it is satisfied for all the Hörmander sums of squares of vector fields* $\mathcal{L} = \sum_{j=1}^{m} X_j^2$ which fall into the class of operators considered in Chapter 1 (see also Remarks 4.0.11 and 3.3.8 for other examples):

***Remark* 3.3.1.** Let $\mathcal{L} = \sum_{j=1}^{m} X_j^2$ be a Hörmander sum of squares of vector fields on $\mathbb{R}^N$, $N \geq 3$. We suppose the vector fields $X_1, \ldots, X_m$ have vanishing divergence, so that $\mathcal{L}$ is a divergence form operator (1.1). Let $d(x,y)$ denote the Carnot-Carathéodory metric associated to the system of vector fields $X_1, \ldots, X_m$, and let $B(x,r)$ denote the associated d-ball centred at x with radius r. We suppose $\mathcal{L}$ has a global fundamental solution Γ such that $\Gamma(x,\cdot)$ vanishes at infinity; all the other assumptions on Γ made in Chapter 1 are satisfied thanks to the results in [49] and [65]. By some fundamental results in the latter papers, we know that, on compact sets, one has the estimates (for some constant $C > 0$)

$$\begin{aligned} &C^{-1} \frac{d^2(x,y)}{|B(x,d(x,y))|} \leq \Gamma(x,y) \leq C \frac{d^2(x,y)}{|B(x,d(x,y))|}, \\ &|X_j \Gamma(x,y)| \leq C \frac{d(x,y)}{|B(x,d(x,y))|} \qquad (j = 1, \ldots, m). \end{aligned} \tag{3.11}$$

With the notation in (1.10), it turns out that our kernel K_α is given by $K_\alpha(x,y) = \sum_{j=1}^{m} (X_j \Gamma_x(y))^2 / \Gamma_x^{2+\alpha}(y)$. Hence, by the above estimates one gets (for y in a compact set)

$$|K_\alpha(0,y)| \leq C \, \frac{|B(0,d(y))|^\alpha}{d(y)^{2+2\alpha}}, \tag{3.12}$$

where $d(y) = d(0,y)$. Since in general $d(x,y) \geq C\,\|x-y\|$ (for x,y in a compact set, $\|\cdot\|$ denoting the Euclidean norm) one deduces that $|B(0,d(y))| \leq C'\,d(y)^N$, whence (3.12) gives $|K_\alpha(0,y)| \leq C\,d(y)^{N\alpha-2-2\alpha}$. If α is large enough, namely $\alpha \geq 2/(N-2)$, the kernel $K_\alpha(0,y)$ is locally bounded. For such α, in order to prove (H) it suffices to show that $\int_{B(x,\varepsilon)} \Gamma(x,y)\,\mathrm{d}y$

vanishes as $\varepsilon \to 0$, uniformly for x in a compact set. To this aim we have

$$\begin{aligned}
\int_{B(x,\varepsilon)} \Gamma(x,y)\,\mathrm{d}y &= \sum_{k=0}^{\infty} \int_{\varepsilon/2^{k+1} \le d(x,y) < \varepsilon/2^k} \Gamma(x,y)\,\mathrm{d}y \\
&\le C \sum_{k=0}^{\infty} \int_{\varepsilon/2^{k+1} \le d(x,y) < \varepsilon/2^k} \frac{d^2(x,y)}{|B(x,d(x,y))|}\,\mathrm{d}y \\
&\le C \sum_{k=0}^{\infty} \int_{\varepsilon/2^{k+1} \le d(x,y) < \varepsilon/2^k} \frac{(\varepsilon/2^k)^2}{|B(x,\varepsilon/2^{k+1})|}\,\mathrm{d}y \\
&\le C \sum_{k=0}^{\infty} \frac{\varepsilon^2}{2^{2k}} \frac{|B(x,\varepsilon/2^k)|}{|B(x,\varepsilon/2^{k+1})|} \\
&\le C'\,\varepsilon^2 \sum_{k=0}^{\infty} \frac{1}{4^k} \xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

In the last inequality we used the fact that x lies in a compact set, together with the doubling property of d (see e.g., [49], recalling that $X_1, \dots, X_m$ is a Hörmander system).

***Remark* 3.3.2.** Thanks to Remark 3.3.1, we observe that all sub-Laplacians on Carnot groups do satisfy hypothesis (H) for large α's. Actually this holds true for $\alpha \ge \frac{2}{Q-2}$. Indeed, in this case K_α is locally bounded from above, so that (for x in the compact set F and for a constant c_F)

$$|f_n(x)| \le c_F \int_{\Omega_{1/n}(x)} \Gamma(x,y)\,\mathrm{d}y.$$

By left-invariance of Γ and of Lebesgue measure, the above integral equals $\int_{\Omega_{1/n}(0)} \Gamma(0,y)\,\mathrm{d}y$, which vanishes as $n \to \infty$ since Γ is locally integrable.

We observe that the choice $\alpha = \frac{2}{Q-2}$ is the one made by Lanconelli in [43].

The aim of this chapter is to prove, under the above hypothesis (H), a generalization to our subelliptic operators $\mathcal{L}$ of a result due to Kuran [41] for the classical Laplace operator. Before stating our result, whose proof in our generality is delicate, we need preliminary results. The following one, deriving from the Weak Maximum Principle, has an interest in its own right.

Proposition 3.3.3. *Let $x \in \mathbb{R}^N$ and $r > 0$ be fixed. Then both $\Omega_r(x)$ and $\mathbb{R}^N \setminus \overline{\Omega_r(x)}$ are connected open sets.*

Proof. I. Let us prove that $O := \Omega_r(x)$ is connected. By contradiction, suppose that $O = O_1 \cup O_2$, where O_1, O_2 are disjoint non-empty open sets. Only one of these sets, let O_2 say, contains x. Thus the function $u := \Gamma(x, \cdot)$ is $\mathcal{L}$-harmonic on an open set containing $\overline{O_1}$ and $u \equiv 1/r$ on ∂O_1. By the Weak Maximum Principle (observe that O_1 is bounded as it is a subset of O), we infer $u \leq 1/r$ on O_1. This is absurd since $u = \Gamma(x, \cdot) > 1/r$ on $O \supset O_1$.

II. Let us prove that $\mathbb{R}^N \setminus \overline{\Omega_r(x)}$ is connected. We need a preliminary simple result:

Let $O_1, O_2 \subset \mathbb{R}^N$ be disjoint non-empty open sets, and suppose that they are both unbounded. Then ∂O_1 and ∂O_2 are unbounded.

We provide the proof for completeness. To make a choice, we prove that ∂O_1 is unbounded. By hypothesis, we can select points $a_n \in O_1$ and $b_n \in O_2$ such that $\|a_n\|, \|b_n\| \to \infty$ as $n \to \infty$. For every $n \in \mathbb{N}$, there exists a continuous curve $\gamma_n : [0, 1] \to \mathbb{R}^N$ such that $\gamma_n(0) = a_n$, $\gamma_n(1) = b_n$ and $\|\gamma_n(t)\| \geq \min\{\|a_n\|, \|b_n\|\}$ for every $t \in [0, 1]$. Since $\gamma_n([0, 1])$ is a connected set containing $a_n \in O_1$ and $b_n \in O_2$ (which is contained in the exterior of O_1), there necessarily exists $t_n \in [0, 1]$ such that $\gamma_n(t_n) \in \partial O_1$. Since we have $\|\gamma_n(t_n)\| \geq \min\{\|a_n\|, \|b_n\|\} \xrightarrow{n\to\infty} \infty$, this proves that ∂O_1 is unbounded.

We can now prove that $O := \mathbb{R}^N \setminus \overline{\Omega_r(x)} = \{y : \Gamma(x, y) < 1/r\}$ is connected. Let us suppose, by contradiction, that $O = O_1 \cup O_2$, where O_1, O_2 are disjoint non-empty open sets. We have two cases: O_1 and O_2 are both unbounded, O_1 or O_2 is bounded.

We show that the first case cannot occur. Indeed, if O_1 and O_2 are unbounded, by what we proved above we infer that ∂O_1 and ∂O_2 are unbounded. As a consequence (since O_1, O_2 are disjoint open sets) $\partial O = \partial(O_1 \cup O_2) = \partial O_1 \cup \partial O_2$ is unbounded, but this is absurd since $\partial O = \partial \Omega_r(x)$. We can therefore suppose that, say, O_1 is bounded. Thus the function $u := -\Gamma(x, \cdot)$ is $\mathcal{L}$-harmonic on an open set containing $\overline{O_1}$ and $u \equiv -1/r$

on $\partial O_1 \subseteq \partial O$. Since O_1 is bounded, we can apply again the the Weak Maximum Principle and deduce that $u \leq -1/r$ on O_1. This is equivalent to $\Gamma(x, \cdot) \geq 1/r$ on O_1, which is absurd since $\Gamma(x, \cdot) < 1/r$ on $O \supset O_1$. $\square$

Let $u \in \underline{\mathcal{S}}(\Omega)$, and let μ_u is the $\mathcal{L}$-Riesz measure of u. By Theorem 2.4.2, for every bounded open set O such that $O \Subset \Omega$, there exists $h \in \mathcal{H}(O)$ such that

$$u(x) = h(x) - \int_{\overline{O}} \Gamma(x, y)\, \mathrm{d}\mu_u(y), \tag{3.13}$$

for every $x \in O$.

We need Theorem 2.5.2, that now we prove in another way.

Proposition 3.3.4 (of Poisson-Jensen type for $\mathcal{L}$)**.** *Let $u \in \underline{\mathcal{S}}(\Omega)$ and suppose that $\Omega_r(x) \Subset \Omega$. If μ_u is the $\mathcal{L}$-Riesz measure of u on Ω, we have the following representation formula:*

$$u(x) = M_r(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \Big(\Gamma(x, y) - \frac{1}{\rho}\Big)\, \mathrm{d}\mu_u(y) \right) \mathrm{d}\rho. \tag{3.14}$$

Proof. Suppose $\Omega_r(x) \Subset \Omega$ and let also $s > r$ be such that $\Omega_s(x) \Subset \Omega$. Applying (3.13) with $O := \Omega_s(x)$, we get

$$u(z) = h(z) - \int_{\overline{O}} \Gamma(z, y)\, \mathrm{d}\mu_u(y), \qquad \forall\, z \in O, \tag{3.15}$$

for a suitable $\mathcal{L}$-harmonic function h on O. We set for brevity $v(z) := -\int_{\overline{O}} \Gamma(z, y)\, \mathrm{d}\mu_u(y)$, for $z \in \mathbb{R}^N$. By definition of fundamental solution, it is very simple to show that

$$\int_{\mathbb{R}^N} v\, \mathcal{L}\varphi = \int_{\mathbb{R}^N} \varphi\, \mathrm{d}(\mu_u|_{\overline{O}}), \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

This proves simultaneously that $v \in \underline{\mathcal{S}}(\mathbb{R}^N)$ and that its $\mathcal{L}$-Riesz measure is $\mu_v = \mu_u|_{\overline{O}}$ (i.e., the restriction of μ_u to $\overline{O}$, prolonged to be 0 outside $\overline{O}$).

Let us suppose that (3.14) had been proved in the particular $u = v$. Then we would have

$$
\begin{aligned}
u(x) &\overset{(3.15)}{=} h(x) + v(x) = M_r(h)(x) + v(x) \\
&= M_r(h)(x) + M_r(v)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \Big(\int_{\Omega_\rho(x)} \Big(\Gamma(x,y) - \frac{1}{\rho} \Big) \,\mathrm{d}\mu_v(y) \Big) \mathrm{d}\rho \\
&\overset{(3.15)}{=} M_r(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \Big(\int_{\Omega_\rho(x)} \Big(\Gamma(x,y) - \frac{1}{\rho} \Big) \,\mathrm{d}\mu_u(y) \Big) \mathrm{d}\rho.
\end{aligned}
$$

In the second equality we have used the $\mathcal{L}$-harmonicity of h; in the fourth equality we have exploited the fact that $\mu_v = \mu_u|_{\overline{O}}$, together with $\Omega_\rho \subset O$, since $\rho < r < s$. The above chain of equalities proves (3.14). We are then left to show that (3.14) is fulfilled when $u = v$.

We know that

$$
M_r(v)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \, m_\rho(v)(x) \,\mathrm{d}\rho. \tag{3.16}
$$

Clearly, (3.16) shows that (3.14) holds true if we are able to show that

$$
v(x) = m_r(v)(x) - \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r} \Big) \,\mathrm{d}\mu_v(y). \tag{3.17}
$$

Identity (3.17) is equivalent to

$$
\int_{\overline{O}} \Gamma(x,y) \,\mathrm{d}\mu_u(y) = m_r\Big(\int_{\overline{O}} \Gamma(\cdot,y) \,\mathrm{d}\mu_u(y) \Big)(x) + \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r} \Big) \,\mathrm{d}\mu_v(y). \tag{3.18}
$$

Now, the first summand after the equality sign is given by

$$
\begin{aligned}
& m_r\Big(\int_{\overline{O}} \Gamma(\cdot,y) \,\mathrm{d}\mu_u(y) \Big)(x) = \int_{\overline{O}} m_r\big(\Gamma(\cdot,y)\big)(x) \,\mathrm{d}\mu_u(y) \\
& \Big(\text{by Theorem 1.2.5, } m_r\big(\Gamma(\cdot,y)\big)(x) = \min\{\Gamma(x,y), 1/r\}\Big) \\
& = \int_{\overline{O}} \min\{\Gamma(x,y), 1/r\} \,\mathrm{d}\mu_u(y) = \int_{\Omega_r(x)} \frac{1}{r} \,\mathrm{d}\mu_u(y) + \int_{\overline{\Omega_s(x)} \setminus \Omega_r(x)} \Gamma(x,y) \,\mathrm{d}\mu_u(y).
\end{aligned}
$$

If we insert the above identity in the right-hand side of (3.18) we get

$$
\begin{aligned}
& \int_{\Omega_r(x)} \frac{1}{r} \,\mathrm{d}\mu_u(y) + \int_{\overline{\Omega_s(x)} \setminus \Omega_r(x)} \Gamma(x,y) \,\mathrm{d}\mu_u(y) + \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r} \Big) \,\mathrm{d}\mu_v(y) \\
& = \int_{\overline{\Omega_s(x)} \setminus \Omega_r(x)} \Gamma(x,y) \,\mathrm{d}\mu_u(y) + \int_{\Omega_r(x)} \Gamma(x,y) \,\mathrm{d}\mu_v(y) = \int_{\overline{\Omega_s(x)}} \Gamma(x,y) \,\mathrm{d}\mu_u(y).
\end{aligned}
$$

This demonstrates (3.18) and the proof is complete. □

We are ready to state, under hypothesis (H) of the beginning of the section, the generalization to our operators $\mathcal{L}$ of a symmetry result contained in [41]. To this end, we observe that formula (1.12) in Theorem 1.2.6 gives, in the particular case when u is $\mathcal{L}$-harmonic on an open set containing the closure of $\Omega_r(0)$, the identity

$$u(0) = M_r(u)(0) = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(0)} u(y)\, K(0,y)\, \mathrm{d}y. \tag{3.19}$$

Henceforth we fix any $\alpha > 0$ such that hypothesis (H) of the beginning of this section is fulfilled. For brevity, we introduce the density and the measure

$$\psi(y) := K_\alpha(0,y), \qquad \mathrm{d}\mu(y) := \psi(y)\, \mathrm{d}y. \tag{3.20}$$

The notation ψ_α will sometimes occur to recall the dependence of ψ on α. From (3.22), we can rewrite (3.19) as follows

$$u(0) = \frac{1}{\mu(D)} \int_D u(y)\, \psi(y)\, \mathrm{d}y,$$

with $D = \Omega_r(0)$, and this formula obviously holds for every $\mathcal{L}$-harmonic and μ-integrable function on D. As a reverse fact, we are here interested in the following symmetry result, characterizing the "balls" $\Omega_r(0)$:

Theorem 3.3.5 (Inverse mean value theorem for $\mathcal{L}$)**.** *Suppose there exists $\alpha > 0$ such that hypothesis* (H) *is satisfied, and let $\psi = \psi_\alpha$ and μ be as in* (3.20)*.*

Let D be a bounded open neighborhood of 0*. Suppose that*

$$u(0) = \frac{1}{\mu(D)} \int_D u(y)\, \psi(y)\, \mathrm{d}y, \tag{3.21}$$

for every u which is $\mathcal{L}$-harmonic and μ-integrable on D. Then $D = \Omega_r(0)$ for some $r > 0$.

More precisely, it suffices to suppose that (3.21) *holds for the family of $\mathcal{L}$-harmonic functions on D of the form $D \ni y \mapsto \Gamma(y,x)$, for $x \notin D$.*

This theorem extends a previous result by Kuran [41] proved for the Laplace operator. Furthermore, Theorem 3.3.5 also provides a generalization of a recent result by Lanconelli (see [43, Theorem 1.1] for $b = 0$), proved for sub-Laplacians on Carnot groups, a subclass of our operators $\mathcal{L}$ satisfying hypothesis (H), see Remark 3.3.1. Indeed, in [43] the choice $\alpha = \frac{2}{Q-2}$ is made, and this ensures that hypothesis (H) is fulfilled, as observed in Remark 3.3.2.

***Remark* 3.3.6.** We remark that the density function ψ is nonnegative (see (1.10) and recall that $A \geq 0$), and that, thanks to our Theorem 3.1.1, it vanishes on a set with empty interior. Moreover, μ is a Borel measure and (by the mentioned non-vanishing property of ψ) μ is positive on every set with non-empty interior. Furthermore, μ is a Radon measure, for it is finite on bounded sets; indeed, any bounded set is contained in some $\Omega_r(0)$ and (by (1.13))

$$\mu(\Omega_r(0)) = \frac{r^{\alpha+1}}{\alpha+1}. \tag{3.22}$$

We provide another preliminary result: it is here (precisely in the proof of the continuity property) that we exploit our hypothesis (H).

Lemma 3.3.7. *Suppose there exists $\alpha > 0$ such that hypothesis* (H) *is satisfied, and let $\psi = \psi_\alpha$ and μ be as in* (3.20)*. Then for every bounded set D, the function*

$$v(x) := -\int_D \Gamma(y,x)\,\psi(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^N$$

is continuous and $\mathcal{L}$-subharmonic in $\mathbb{R}^N$.

Proof. With the notation in (3.20), we denote by ν the Radon measure obtained by setting $\nu(E) := \mu(E \cap D)$, for every measurable set E. Then we obviously have $v(x) = -\int_{\mathbb{R}^N} \Gamma(y,x)\,\mathrm{d}\nu(y)$, for any $x \in \mathbb{R}^N$. Firstly, v is u.s.c. on $\mathbb{R}^N$ thanks to Fatou's Lemma. Secondly, by using the definition of fundamental solution we easily get

$$\int v(x)\,\mathcal{L}\varphi(x)\,\mathrm{d}x = \int \varphi(x)\,\mathrm{d}\nu(x), \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

This proves that $\mathcal{L}v \geq 0$ in the sense of distributions. Actually it proves more: the $\mathcal{L}$-Riesz measure of v is precisely ν. If we show that v is continuous on $\mathbb{R}^N$, Theorem 1.2.7 will yield $v \in \underline{\mathcal{S}}(\mathbb{R}^N)$. To this end, for every $n \in \mathbb{N}$ we fix $\varphi_n \in C_0^\infty([0,\infty),\mathbb{R})$ such that $0 \le \varphi_n \le 1$ and $\varphi_n(t) = 1$ if $t \le n/2$, $\varphi_n(t) = 0$ if $t \ge n$. Then we set $\eta_n(x,y) := \varphi_n(\Gamma(x,y))$. This gives $\eta_n(x,y) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ and $0 \le \eta_n \le 1$; moreover it holds that

$$\eta_n(x,y) = 0 \quad \text{for } y \in \Omega_{1/n}(x), \quad \text{and} \quad \eta_n(x,y) = 1 \quad \text{for } y \notin \Omega_{2/n}(x).$$

We will show that, setting

$$v_n(x) := -\int_D \Gamma(y,x)\,\psi(y)\,\eta_n(x,y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^N, \tag{3.23}$$

(1) for every fixed $n \in \mathbb{N}$, v_n is continuous on $\mathbb{R}^N$,

(2) $v_n \longrightarrow v$ as $n \to \infty$, uniformly on compact sets of $\mathbb{R}^N$.

For every fixed $n \in \mathbb{N}$ and every $x_0 \in \mathbb{R}^N$, we show that $v_n(x) \to v_n(x_0)$ as $x \to x_0$. This is a simple consequence of Lebesgue's dominated convergence theorem. Indeed, thanks to our choice of $\eta_n(x,y)$, the integrand function in (3.23) is continuous w.r.t. (x,y) (hence it is bounded by some constant $C_n(D,K)$, if y belongs to the bounded set D and x belongs to some compact neighborhood K of x_0). This proves property (1).

As for property (2), we argue as follows. Suppose K is a compact subset of $\mathbb{R}^N$ and let $x \in K$. We have (recall that $\psi = K(0,\cdot)$ is nonnegative)

$$|v_n(x) - v(x)| \le \int_D \Gamma(y,x)\,\psi(y)\,|\eta_n(x,y) - 1|\,\mathrm{d}y \le \int_{\Omega_{2/n}(x)} \Gamma(y,x)\,\psi(y)\,\mathrm{d}y.$$

Note that the right-hand side vanishes as $n \to \infty$ (uniformly in $x \in K$), when x lies in a compact set K, precisely when hypothesis (H) is satisfied (recall that $\Gamma(x,y) = \Gamma(y,x)$). □

We are finally ready to give the:

Proof. (of Theorem 3.3.5). Suppose D is a bounded open neighborhood of 0. In particular, (3.21) is valid for u of the form $u := \Gamma(\cdot, x)$ with $x \notin D$. Indeed, in this case the μ-integrability of u follows from the following argument.

Chosen $R \gg 1$ so large that the bounded set D is contained in $\Omega_R(0)$ (recall (1.7)), we have

$$\int_D \Gamma(y,x)\,\psi(y)\,\mathrm{d}y \le \int_{\Omega_R(0)} \Gamma(y,x)\,K(0,y)\,\mathrm{d}y = \frac{R^{\alpha+1}}{\alpha+1}\cdot M_R^{\alpha}(\Gamma(\cdot,x))(0)$$

(by Theorem 1.2.5)

$$= \frac{R^{\alpha+1}}{\alpha+1}\cdot \begin{cases} \frac{1}{\alpha R}\big(\alpha+1-R^{-\alpha}\Gamma^{-\alpha}(0,x)\big), & \text{if } x\in\Omega_R(0), \\ \Gamma(0,x), & \text{if } x\notin\Omega_R(0). \end{cases}$$

Since $x \neq 0$ (for D is a neighborhood of 0 and $x \notin D$), the above far right-hand is finite.

The assumptions of our theorem hence imply the validity of the identity

$$\Gamma(0,x) = \frac{1}{\mu(D)} \int_D \Gamma(y,x)\,\psi(y)\,\mathrm{d}y, \qquad x\notin D. \tag{3.24}$$

This identity will suffice to prove the whole theorem. Since $\mu(\Omega_r(0)) = \frac{r^{\alpha+1}}{\alpha+1}$ (by (3.22)) and $\mu(D) \in (0,\infty)$ (by Remark 3.3.6), there exists a unique $r>0$ such that

$$\mu(\Omega_r(0)) = \mu(D). \tag{3.25}$$

Throughout the sequel, r will be as above and B will denote $\Omega_r(0)$. We set

$$v(x) := -\int_D \Gamma(y,x)\,\psi(y)\,\mathrm{d}y, \qquad u(x) := \mu(D)\,\Gamma(0,x) + v(x); \tag{3.26a}$$

$$v_0(x) := -\int_B \Gamma(y,x)\,\psi(y)\,\mathrm{d}y, \qquad u_0(x) := \mu(B)\,\Gamma(0,x) + v_0(x). \tag{3.26b}$$

By Lemma 3.3.7, we have $v, v_0 \in C(\mathbb{R}^N)\cap\underline{\mathcal{S}}(\mathbb{R}^N)$. Moreover, as we showed in the proof of the cited Lemma, the $\mathcal{L}$-Riesz measures of v and v_0 are given by the restrictions of μ to D and to B, respectively. Since $\Gamma(0,\cdot)$ is $\mathcal{L}$-harmonic in $\mathbb{R}^N\setminus\{0\}$, we also have $u, u_0 \in C(\mathbb{R}^N\setminus\{0\})\cap\underline{\mathcal{S}}(\mathbb{R}^N\setminus\{0\})$; more precisely we have the following identities of $\mathcal{L}$-Riesz measures:

$$\mu_u = \mu_v \text{ and } \mu_{u_0} = \mu_{v_0} \text{ relatively to the open set } \mathbb{R}^N\setminus\{0\}.$$

Finally, by (3.24) we get

$$u(x) = 0 \quad \text{for every } x\notin D. \tag{3.27}$$

As for v_0, u_0 we have more information: by (1.9), (3.20), we infer $v_0(x) = -\frac{r^{\alpha+1}}{\alpha+1} \cdot M_r(\Gamma(\cdot, x))(0)$. This identity, again by the aid of Theorem 1.2.5, yields the explicit formula

$$u_0(x) = \begin{cases} \frac{r^{\alpha+1}}{\alpha+1}\,\Gamma(0,x) - \frac{r^\alpha}{\alpha} + \frac{\Gamma^{-\alpha}(0,x)}{(\alpha+1)\alpha}, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases} \tag{3.28}$$

A direct estimate (based on the inequality $\alpha\, R + R^{-\alpha} > \alpha + 1$ for $R > 1$) gives

$$u_0 \geq 0 \quad \text{on } \mathbb{R}^N \text{ and } \quad u_0 > 0 \quad \text{on } B. \tag{3.29}$$

We now claim that

$$D \subseteq \overline{B}. \tag{3.30}$$

This fact will complete the proof of $D = B$. Indeed, if $D \subseteq \overline{B}$ then $D = \mathrm{Int}(D) \subseteq \mathrm{Int}(\overline{B}) = B$. (Here we have used (1.6), which ensures that $\mathrm{Int}(\overline{B}) = B$.) Thus $D \subseteq B$. Vice versa, we show that $B \subseteq D$. Indeed, since $\mu(B) = \mu(D)$ (see (3.25)), by the definition (3.20) of μ we have $\chi_B\, \psi = \chi_D\, \psi$ almost everywhere on $\mathbb{R}^N$ (here χ_B, χ_D are the characteristic functions of B, D). We thus infer that $v \equiv v_0$ so that, again by (3.25), $u \equiv u_0$. Since $u = 0$ outside D (see (3.27)) and $u_0 > 0$ on B (see (3.29)), the last identity gives $B \subseteq D$.

We are left with the proof of the claimed (3.30). We prove this by contradiction assuming that $D \setminus \overline{B} \neq \emptyset$. Let us consider the decomposition $\mathbb{R}^N \setminus \overline{B} = [D \setminus \overline{B}] \cup [\partial D \setminus \overline{B}] \cup [\mathbb{R}^N \setminus (\overline{D} \cup \overline{B})]$. The first set in the right-hand side is open and nonempty by our contradiction assumption, whilst the third one is open and nonempty since $\overline{D}$ and $\overline{B}$ are bounded sets. As a consequence, *since* $\mathbb{R}^N \setminus \overline{B}$ *is connected* as we proved in Proposition 3.3.3, we deduce that $\partial D \setminus \overline{B} \neq \emptyset$. Hence there exists $x_0 \in \partial D$ and $R > 0$ such that $\Omega_R(x_0) \cap \overline{B} = \emptyset$.

We know that $u \in \underline{\mathcal{S}}(\mathbb{R}^N \setminus \{0\})$, and since B is a neighborhood of 0 and $\Omega_R(x_0) \cap \overline{B} = \emptyset$, we deduce that u is $\mathcal{L}$-subharmonic on a neighborhood of $\overline{\Omega_R(x_0)}$. We can therefore apply the Poisson-Jensen formula (3.14) in Proposition 3.3.4 and obtain that

$$u(x_0) = M_R(u)(x_0) - \frac{\alpha+1}{R^{\alpha+1}} \int_0^R \rho^\alpha \bigg(\int_{\Omega_\rho(x_0)} \Big(\Gamma(x_0, y) - \frac{1}{\rho} \Big)\, \mathrm{d}\mu_u(y) \bigg) \mathrm{d}\rho.$$

Since $\mu_u = \mu_v$ is equal to the measure $\chi_D(y)\,\psi(y)\,\mathrm{d}y$, we see that the double integral in the above right-hand side is positive (here we have also used the fact that $x_0 \in \partial D$ and the positivity property of $\psi = K(0,\cdot)$ in Theorem 3.1.1). Thus $u(x_0) \lneqq M_R(u)(x_0)$. Since $u = 0$ outside D (see (3.27)) we derive $M_R(u)(x_0) > 0$, whence u is positive on some set $P \subseteq D \cap \Omega_R(x_0)$. Moreover, by (3.28), $u_0 = 0$ outside B, and in particular this is true on $D \cap \Omega_R(x_0) \subset \Omega_R(x_0) \subset \mathbb{R}^N \setminus \overline{B}$. This shows that

$$h := u - u_0 = u > 0 \quad \text{on } P. \tag{3.31}$$

By collecting together (3.25), (3.26a) and (3.26b), we get $h = v - v_0$, so that h is continuous on $\mathbb{R}^N$, since this is true of v and v_0. Furthermore, in the weak sense of distributions we have

$$\mathcal{L}h = \mathcal{L}(v - v_0) = \mathcal{L}v - \mathcal{L}v_0 = \chi_D\,\psi\,\mathrm{d}y - \chi_B\,\psi\,\mathrm{d}y.$$

In particular, in the sense of distributions on the open set D we have $\mathcal{L}h = (1 - \chi_{B\cap D})\,\psi\,\mathrm{d}y$. Thus $\mathcal{L}h \geq 0$ in the weak sense of distributions on D. Since h is continuous, by Theorem 1.2.7 we deduce that h is $\mathcal{L}$-subharmonic on D.

Let us set $m := \sup_{\overline{D}} h$. Clearly, $m \in \mathbb{R}$ since h is continuous and D is bounded. Moreover, $m > 0$ thanks to (3.31). We claim that h attains m at an interior point of D. Indeed,

$$x \in \partial D \quad \Longrightarrow \quad h(x) = u(x) - u_0(x) \overset{(3.27)}{=} -u_0(x) \overset{(3.29)}{\leq} 0, \tag{3.32}$$

whereas, for $x \in P$, we have

$$h(x) = u(x) - u_0(x) = u(x) \overset{(3.31)}{>} 0,$$

since $P \subset \Omega_R(x_0) \subseteq \mathbb{R}^N \setminus B$ and $u_0 = 0$ outside B, by (3.28). This proves that h attains m on D. Since $h \in \underline{\mathcal{S}}(D)$, the Strong Maximum Principle for $\mathcal{L}$ in Theorem 3.2.1 ensures that $h \equiv m$ on a connected component of D, whence there exists $\xi \in \partial D$ such that $h(\xi) = m$. By the argument in (3.32) this gives $0 < m = h(\xi) \leq 0$, and this contradiction completes the proof. $\square$

***Remark* 3.3.8.** The inverse mean-value theorem can be generalized to operators possessing a fundamental solution with estimates analogous (in the sense explained below) to (3.11), since these estimates imply the validity of our assumption (H), as the argument in Remark 3.3.1 shows. More precisely, some remarks are in order:

(a) If $\mathcal{L}$ is as in (1.1), let $d(x,y)$ denote the distance associated with $\mathcal{L}$ in the sense of Fefferman and Phong [23]. Let R be a symmetric matrix (with Lipschitz-continuous entries) such that $R^2 = (a_{i,j})$ where $(a_{i,j})$ is as in (1.1), and let us denote by $X_1, \dots, X_N$ the vector fields associated with the rows of R (see Appendix B for the existence of R). As in [23], assume that the following geometric condition is satisfied

$$\exists\, \varepsilon, C > 0 : \qquad B_{\mathrm{Eu}}(x,r) \subseteq B_d(x, C\, r^{\epsilon}), \quad \text{for small } r, \tag{3.33}$$

(here $B_{\mathrm{Eu}}(x,r), B_d(x,r)$ denote, respectively, the Euclidean ball and the d-ball of centre x and radius r). Then, by the results on sub-ellipticity in [23] and by the estimates in [24] by Fefferman and Sánchez-Calle, $\mathcal{L}$ possesses a (local) fundamental solution Γ such that Γ and $X_j\Gamma$ satisfy the estimates (3.11), where $d(x,y)$ is as above. This is a consequence of the fact that the vector field X_j is subunit w.r.t. $\mathcal{L}$ (in the sense of [23]) so that we can apply [24, Theorem 2, page 263] to estimate $|X_j\Gamma|$. Arguing precisely as in our Remark 3.3.1, hypothesis (H) is fulfilled; indeed, by the choice of R, our kernel $K_\alpha(x,y)$ can be written as follows

$$K_\alpha(x,y) = \frac{\big\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\big\rangle}{\Gamma_x^{2+\alpha}(y)} = \frac{\sum_{j=1}^N |X_j\Gamma_x(y)|^2}{\Gamma_x^{2+\alpha}(y)}.$$

This shows that our technical hypothesis (H) is satisfied in all the meaningful subelliptic settings where the geometric condition (3.33) holds true. We also remark that operators $\mathcal{L}$ as in (1.1) satisfying (3.33) need not be sum of squares of vector fields.

(b) Motivated by function theory in several complex variables, Jerison and Sánchez-Calle [39] consider operators as in (4.17) satisfying some subelliptic estimates, which are indeed equivalent to (3.33) (see conditions (1.1) and (1.2) in [39, page 46]). These estimates guarantee in particular hypoellipticity, the strong maximum principle, Harnack's inequality and estimates on the size of the fundamental solution Γ and of its derivatives along subunit vector fields. Taking into account the results of Fefferman and Sánchez-Calle mentioned in (a) above, one has all the ingredients to extend the results of our paper to these operators.

(c) Without entering the details, our results in this Chapter can be used to characterize the geodesic balls on some subclasses of the so-called harmonic Riemannian manifolds, since in this setting the Laplace-Beltrami operator Δ possesses a radial-symmetric (in a suitable sense) fundamental solution Γ (see e.g., [6] and [20]). For instance, as it is shown in [64] by Saloff-Coste, if one assumes the joint validity of the Poincaré inequality and the doubling condition, this ensures bounds on the fundamental solution of the heat operator $\Delta - \partial_t$, from which one can expect to obtain estimates as in (3.11). We plan to develop this topic in a future work.

Chapter 4

A Lebesgue-type result for the PWB generalized solution

Given a bounded open set Ω and a function $f : \partial\Omega \to \mathbb{R}$, let us consider the homogeneous Dirichlet problem

$$D(f,\Omega): \qquad \mathcal{L}u = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega.$$

As is well-known, $D(f,\Omega)$ is not always solvable in the classical sense without further assumptions on Ω and f. Nonetheless, this is always the case (at least for continuous f's) if we seek for a *generalized solution* of $D(f,\Omega)$, denoted by H_f^Ω, *in the sense of Perron-Wiener-Brelot* (PWB solution, for short).
Our main result on PWB solutions (tracing back to an idea of Lebesgue [47]) makes use of a "regularizing" property of the mean value operator M_r. Given a bounded open set $\Omega \subset \mathbb{R}^N$ and any $x \in \Omega$, we suppose that the boundary datum f is the restriction to $\partial\Omega$ of a continuous function $F : \overline{\Omega} \to \mathbb{R}$, and we take its M_r-mean, and then iteratively the M_r-mean of the function $x \mapsto M_r(u)(x)$ and so on. In this way, we build a *sequence of functions converging to the Perron-Wiener solution of the Dirichlet problem.* The proof of this result, which shows how the mean value operator M_r may be profitably used in the investigation of Dirichlet problems, exploits potential-theoretic techniques and makes essential use of the characterizations of $\mathcal{L}$-subharmonicity given in [14], together with some abstract Potential Theory

(see e.g., [16, Chapter 6]). The rôle of mean-integral operators similar to M_r in smoothing problems is classical (see [9, Section 3.3] for the Laplace operator case), but it has also recently appeared in sub-elliptic frameworks (see [14, Theorem 7.1]; see also [32, Theorem 6.1] for heat-type operators). Lebesgue-type theorems on *unbounded* domains of the Heisenberg group first appeared in [45] and [46], where they are used to solve the Dirichlet problem of the sub-Laplacian on half-spaces.

The *Lebesgue-type result for the Perron-Wiener generalized solution* is the following one.

Theorem 4.0.9. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. For $x \in \Omega$ we define $r(x)$ by requiring that $(r(x))^{-1} = \sup_{y \notin \Omega} \Gamma(x, y)$.*

For every $u : \Omega \to [-\infty, \infty]$ locally bounded from above (or from below), we finally set

$$T(u)(x) := M_{r(x)}(u)(x), \qquad x \in \Omega.$$

Given $F \in C(\overline{\Omega}, \mathbb{R})$, we consider the iterative sequence of functions $\{T^k(F)\}_k$ on Ω defined by

$$T^1(F) = T(F), \qquad T^{k+1}(F) = T(T^k F), \quad k \geq 1.$$

Then, setting $f := F|_{\partial\Omega}$, we have

$$\lim_{k\to\infty} T^k(F)(x) = H_f^\Omega(x), \qquad x \in \Omega,$$

where H_f^Ω is the generalized solution in the Perron-Wiener-Brelot sense of the Dirichlet problem

$$\mathcal{L}u = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega.$$

The proof of this theorem is laborious and is split into several steps.

STEP I. We begin by observing that *the function $\Omega \ni x \mapsto r(x)$ is continuous and $\Omega_{r(x)}(x) \Subset \Omega$ for every $x \in \Omega$.*

The only non-trivial fact is the continuity of $r(x)$. This will follow if we show that $x \mapsto \rho(x) := \sup_{y \notin \Omega} \Gamma(x,y)$ is continuous. First we claim that

$$\rho(x) = \max_{y \in \partial\Omega} \Gamma(x,y). \tag{4.1}$$

The proof of this claim is quite delicate. From the decomposition $\mathbb{R}^N \setminus \{x\} = \bigcup_{r>0} \left\{y \in \mathbb{R}^N : \Gamma(x,y) = \frac{1}{r}\right\} = \bigcup_{r>0} \partial\Omega_r(x)$, we obtain

$$\sup_{y \notin \Omega} \Gamma(x,y) = \sup_{r>0} \left\{\tfrac{1}{r} \,\middle|\, \partial\Omega_r(x) \setminus \Omega \neq \emptyset\right\} = \inf_{r>0} \left\{r \,\middle|\, \partial\Omega_r(x) \setminus \Omega \neq \emptyset\right\}.$$

By continuity arguments, it is simple to show that the latter infimum is attained at some radius $r_0 > 0$. Moreover it clearly holds that

$$\partial\Omega_s(x) \subset \Omega \quad \forall\, s < r_0. \tag{4.2}$$

Let $y \in \partial\Omega_{r_0}(x) \setminus \Omega$. In order to prove (4.1), it suffices to show that $y \in \partial\Omega$. Since $y \notin \Omega$, we need to prove that every neighborhood of y contains points of Ω. By contradiction, let us assume the existence of a Euclidean ball U centered at y not containing points of Ω. Due to (4.2), U does not intersect $\partial\Omega_s(x)$, for every $s < r_0$. Thus, for every $z \in U$ one has $\Gamma(x,z) \neq \frac{1}{s}$ for any $s < r_0$. This gives

$$\Gamma(x,z) \leq \frac{1}{r_0} \quad \text{for every } z \in U, \quad \text{and} \quad \Gamma(x,y) = \frac{1}{r_0}.$$

This proves that $U \ni y \mapsto \Gamma(x,y)$ attains its maximum at an interior point of U. Thanks to the Strong Maximum Principle for $\mathcal{L}$ in Theorem 3.2.1, $\Gamma(x,\cdot) \equiv 1/r_0$ on U (recall that $\Gamma(x,\cdot)$ is $\mathcal{L}$-harmonic in $\mathbb{R}^N \setminus \{x\}$). Thus $\partial\Omega_{r_0}(x)$ contains a ball, in contradiction with (1.6).

The above arguments prove (4.1). Next, the continuity of ρ in (4.1) is a consequence of the continuity of $\Gamma(\cdot,\cdot)$ away from the diagonal, which easily gives

$$\max_{y \in \partial\Omega} \Gamma(x_k,\cdot) \longrightarrow \max_{y \in \partial\Omega} \Gamma(x_0,\cdot) \qquad \text{as } k \to \infty,$$

whenever x_k is a sequence in Ω converging to $x_0 \in \Omega$ as $k \to \infty$.

Step II. We prove the well posedness of the sequence $\{T^k(F)\}_k$ in the assertion of Theorem 4.0.9 by showing the validity of the following result:

If $u : \Omega \to (-\infty, \infty]$ *is locally bounded from below, then* $T(u) : \Omega \to (-\infty, \infty]$ *is also locally bounded from below. Moreover, if* $u \in C(\Omega, \mathbb{R})$ *then* $T(u) \in C(\Omega, \mathbb{R})$.

We first prove that, for every compact $K \subset \Omega$, there exists a compact set $\widetilde{K} \subset \Omega$ such that

$$\bigcup_{x \in K} \Omega_{r(x)}(x) \subseteq \widetilde{K}. \tag{4.3}$$

By contradiction, let us suppose that $A := \bigcup_{x \in K} \Omega_{r(x)}(x)$ contains a sequence $\{a_j\}_j$ which escapes every compact subset of Ω. Since Ω is bounded, the same is true of A (which is contained in Ω; see Step I). We can therefore extract a subsequence from $\{a_j\}_j$ (which we still denote by $\{a_j\}_j$) converging to a point $a \in \partial\Omega$. From $a_j \in A$, there exists $x_j \in K$ such that $a_j \in \Omega_{r(x_j)}(x_j)$. From the compactness of K we can extract a subsequence $\{x_{j_n}\}_n$ from $\{x_j\}_j$ which converges to the point $\xi \in K$. By the continuity of $x \mapsto r(x)$ we derive $r(x_{j_n}) \longrightarrow r(\xi)$, as $n \to \infty$. Since $a_{j_n} \in \Omega_{r(x_{j_n})}(x_{j_n})$, we have $\Gamma(x_{j_n}, a_{j_n}) > 1/r(x_{j_n})$, for every $n \in \mathbb{N}$. Letting $n \to \infty$, and using the continuity of $\Gamma(\cdot, \cdot)$ out of the diagonal (note that $\xi \neq a$ for $\xi \in K$ and $a \in \partial\Omega$) we deduce that $\Gamma(\xi, a) \geq 1/r(\xi)$. Hence $a \in \overline{\Omega_{r(\xi)}(\xi)} \subseteq \Omega$, in contradiction with the assumption $a \in \partial\Omega$, and (4.3) is proved.

Let now $u : \Omega \to (-\infty, \infty]$ be locally bounded from below. From (4.3) we get, for all $x \in K$,

$$\begin{aligned} T(u)(x) \; &\geq \frac{\alpha + 1}{(r(x))^{\alpha+1}} \inf_{\Omega_{r(x)}(x)} u \cdot \int_{\Omega_{r(x)}(x)} K(x, y) \mathrm{d}y = \inf_{\Omega_{r(x)}(x)} u \cdot M_{r(x)}(1)(x) \\ &\overset{(1.13)}{=} \inf_{\Omega_{r(x)}(x)} u \geq \inf_{A} u \overset{(4.3)}{\geq} \inf_{\widetilde{K}} u > -\infty. \end{aligned}$$

We finally prove that $u \in C(\Omega, \mathbb{R})$ implies $T(u) \in C(\Omega, \mathbb{R})$. Let us set $V(x, r) := M_r(u)(x)$, for (x, r) such that $\Omega_r(x) \Subset \Omega$. We prove the continuity of $V(x, r)$ in the couple (x, r). Once this is proved, by using the continuity of $r(x)$ is Step I we will obviously deduce the continuity of $T(u)(x) = V(x, r(x))$. Fixed (ξ, ρ) such that $\Omega_\rho(\xi) \Subset \Omega$, it suffices to prove the continuity of V in a small neighborhood of (ξ, ρ). To this end, we consider a cut-off function

$\eta_n(x,y) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ such that $0 \leq \eta_n \leq 1$ and such that

$$\eta_n(x,y) = 0 \quad \text{for } y \in \Omega_{1/n}(x), \quad \text{and} \quad \eta_n(x,y) = 1 \quad \text{for } y \notin \Omega_{2/n}(x). \tag{4.4}$$

We will show that, setting

$$V_n(x,r) := \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y)\, K(x,y)\, \eta_n(x,y)\, \mathrm{d}y, \tag{4.5}$$

1. V_n is continuous near (ξ, ρ), for every fixed n sufficiently large;
2. $V_n(x,r) \to V(x,r)$ as $n \to \infty$, uniformly for (x,r) belonging to a neighborhood of (ξ,ρ).

This will clearly provide the needed continuity of V. It is non-restrictive to prove these properties for $\widetilde{V}_n(x,r) = \frac{r^{\alpha+1}}{\alpha+1} V_n(x,r)$ and $\widetilde{V}(x,r) = \frac{r^{\alpha+1}}{\alpha+1} V(x,r)$ in place of $V_n(x,r)$ and $V(x,r)$ respectively.

Let us prove (1). Given any (x_0, r_0) near (ξ,ρ), and given any sequence (x_j, r_j) converging to (x_0, r_0) as $j \to \infty$, we have

$$|\widetilde{V}_n(x_j,r_j) - \widetilde{V}_n(x_0,r_0)| \leq \int_{\Omega_{r_j}(x_j)} |u(y)| \Big| K(x_j,y)\, \eta_n(x_j,y) - K(x_0,y)\, \eta_n(x_0,y) \Big| +$$

$$+ \int_{\Delta_j} |u(y)|\, K(x_0,y)\, \eta_n(x_0,y)\, \mathrm{d}y = \mathrm{I}_j + \mathrm{II}_j,$$

where $\Delta_j = (\Omega_{r_j}(x_j) \cup \Omega_{r_0}(x_0)) \setminus (\Omega_{r_j}(x_j) \cap \Omega_{r_0}(x_0))$. Now, $\lim_{j\to\infty} \mathrm{I}_j = 0$ by a simple dominated convergence argument (indeed one can replace $\Omega_{r_j}(x_j)$ with a compact set independent of j; note also that $K\,\eta_n$ is continuous for every fixed n). Moreover, $\lim_{j\to\infty} \mathrm{II}_j = 0$ follows immediately from the fact that the Lebesgue measure of Δ_j vanishes as $j \to \infty$, together with the integrability of the integrand function in II_j.

Let us finally prove (2). By (4.4) we have

$$|\widetilde{V}_n(x,r) - \widetilde{V}(x,r)| \leq \int_{\Omega_{2/n}(x)} |u(y)|\, K(x,y)\, |\eta_n(x,y) - 1|\, \mathrm{d}y$$

$$\leq C_u \int_{\Omega_{2/n}(x)} K(x,y)\, \mathrm{d}y = C_u\, \frac{(\frac{2}{n})^{\alpha+1}}{\alpha+1}\, M_{2/n}(1)(x) \overset{(1.13)}{=} C_u\, \frac{(\frac{2}{n})^{\alpha+1}}{\alpha+1} \xrightarrow{n\to\infty} 0.$$

Here C_u is an upper bound for $|u|$ on a suitable neighborhood of ξ. $\square$

STEP III. We observe that, given two functions u and v, locally bounded from below and such that $u \le v$, it follows that $T(u) \le T(v)$ (as $K \ge 0$). As a consequence, if $u \in \overline{\mathcal{S}}(\Omega)$, from Theorem 1.2.7, we have $u(x) \ge M_{r(x)}(u)(x) = T(u)(x)$, for every $x \in \Omega$. This proves that

$$u \ge T(u) \quad \text{for every } u \in \overline{\mathcal{S}}(\Omega). \tag{4.6}$$

By these results we infer that, for every $u \in \overline{\mathcal{S}}(\Omega)$,

$$T^k(u) \ge T^{k+1}(u) \ge u, \quad \text{for every } k \in \mathbb{N}. \tag{4.7}$$

If $u \in \overline{\mathcal{S}}(\Omega)$ and $v \in \underline{\mathcal{S}}(\Omega)$ are such that $v \le u$, then (by the monotonicity of T and by the analogue of (4.7) for $\mathcal{L}$-subharmonic functions) it is easily seen that

$$v \le T^k(v) \le T^{k+1}(v) \le T^{k+1}(u) \le T^k(u) \le u, \quad \text{for every } k \in \mathbb{N}. \tag{4.8}$$

STEP IV. We claim that, given $u \in \overline{\mathcal{S}}(\Omega)$ and $v \in \underline{\mathcal{S}}(\Omega)$ with $v \le u$, there exist $u^*, v^* \in L^1_{\text{loc}}(\Omega)$ such that $T^{(k)}(v) \uparrow v^*$ and $T^k(u) \downarrow u^*$, and moreover

$$v \le T(v^*) = v^* \le u^* = T(u^*) \le u. \tag{4.9}$$

Thanks to (4.8), the sequence $\{T^k(u)\}_k$ is monotone non-increasing, so that there exists $u^* : \Omega \to [-\infty, \infty[$ such that, pointwise in Ω, $u^* := \lim_{k\to\infty} T^k(u) = \inf_k T^k(u)$. Moreover, (4.8) also gives $v \le u^* \le u$, whence $u^* \in L^1_{\text{loc}}$ (see also Theorem 1.2.7). Analogous assertions hold true for $v^* = \lim_{k\to\infty} T^k(v)$. Clearly, (4.8) also gives $v^* \le u^*$. In order to prove that v^* and u^* are fixed points of T, we argue as follows: From $T^k(u) \downarrow u^* \in L^1_{\text{loc}}(\Omega)$, we are allowed to use the monotone convergence theorem, so that

$$T(u^*)(x) = \lim_{k\to\infty} \frac{\alpha+1}{(r(x))^{\alpha+1}} \int_{\Omega_{r(x)}(x)} T^k(u)(y)\, K(x,y)\, \mathrm{d}y = \lim_{k\to\infty} T^{k+1}(u)(x) = u^*(x).$$

One analogously shows that $T(v^*) = v^*$.

STEP V. Our next step consists in showing the following version of the Strong Minimum Principle for the solutions of the inequality $T(u) \le u$. The previous Theorem 3.1.1 on the positivity of K is here required.

Lemma 4.0.10. *Let $\Omega \subseteq \mathbb{R}^N$ be open and connected. Let $u : \Omega \to (-\infty, \infty]$ be a l.s.c. function such that $T(u) \leq u$ on Ω. If u attains its minimum at a point of Ω, then u is constant.*

Proof. Let u be as in the assertion and suppose that $x_0 \in \Omega$ is such that $u(x_0) = \min_\Omega u$. Let us set $\Omega_0 = \big\{x \in \Omega \,:\, u(x) = u(x_0)\big\}$. We shall prove that $\Omega = \Omega_0$ by showing that Ω_0 is nonempty, open and closed relatively to Ω. Clearly Ω_0 contains at least x_0. Moreover, let $\{x_j\}_j$ be a sequence in Ω_0 converging to $y_0 \in \Omega$. We will show that $y_0 \in \Omega_0$. Exploiting the lower semicontinuity of u, we have $\min_\Omega u = \liminf_{j\to\infty} u(x_j) \geq \liminf_{y \to y_0} u(y) \geq u(y_0) \geq \min_\Omega u$. This gives at once $u(y_0) = \min_\Omega u$.

We next show that Ω_0 is open. To this aim, let $z \in \Omega_0$; from the hypothesis $T(u)(z) \leq u(z)$ and $T(1) \equiv 1$ (see (1.13)) we derive

$$\frac{\alpha+1}{(r(z))^{\alpha+1}} \int_{\Omega_{r(z)}(z)} K(z,y)\,(u(y) - u(z))\,\mathrm{d}y \leq 0. \tag{4.10}$$

On the other hand, the above integrand function is nonnegative, since $u(z) = \min_\Omega u$ and $K \geq 0$. As a consequence of (4.10), the above integral is actually 0 and, again from the nonnegativity of the integrand, we get

$$F(y) := K(z,y)\,(u(y) - u(z)) = 0 \qquad \text{for almost every } y \text{ in } \Omega_{r(z)}(z).$$

This shows that $F = 0$ in a dense subset of $\Omega_{r(z)}(z)$. Since $K(z,\cdot) > 0$ on an *open* dense subset of $\mathbb{R}^N \setminus \{z\}$ (see Theorem 3.1.1), we can conclude that $u(y) - u(z) = 0$ for y in a dense subset of $\Omega_{r(z)}(z)$. We next show that $u \equiv u(z)$ in a neighborhood of z. By contradiction, let us suppose this is false: as $u \geq u(z)$, this is equivalent to suppose the existence of a sequence z_j such that $z_j \longrightarrow z$ and $u(z_j) > u(z)$ for all $j \in \mathbb{N}$. We can suppose that $z_j \in \Omega_{r(z)}(z)$ for every j. By the lower semicontinuity of u, every z_j is hence equipped with an open neighborhood where $u > u(z)$, thus contradicting the fact that $u \equiv u(z)$ in a dense set of $\Omega_{r(z)}(z)$. $\square$

Step VI. Together with the Strong Minimum Principle in Lemma 4.0.10, in a standard way one can prove the following Weak Minimum Principle for the solutions of $T(u) \leq u$.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Let $u : \Omega \to (-\infty, \infty]$ be a l.s.c. function such that $T(u) \leq u$ on Ω and such that $\liminf_{x \to y} u(x) \geq 0$ for every $y \in \partial\Omega$. Then $u \geq 0$ on Ω.

STEP VII. We prove the following result: *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, and let $u \in \overline{\mathcal{S}}(\Omega) \cap C(\overline{\Omega})$. If $f := u|_{\partial\Omega}$ then $\lim_{k\to\infty} T^k(u) = H_f^\Omega$, where H_f^Ω is the generalized solution, in the sense of Perron-Wiener-Brelot, of the Dirichlet problem $D(f, \Omega)$ in Definition 1.2.1.*

By Step IV, $u^* := \lim_{k\to\infty} T^k(u)$ is well defined. We begin to prove that, following the notations in Definition 1.2.1,

$$v \leq u^* \leq w, \qquad \text{for every } v \in \underline{\mathcal{U}}_f^\Omega \text{ and every } w \in \overline{\mathcal{U}}_f^\Omega. \tag{4.11}$$

This will immediately give $\underline{H}_f^\Omega \leq u^* \leq \overline{H}_f^\Omega$. Thus, by Wiener's resolutivity theorem (note that f is continuous), we derive $\underline{H}_f^\Omega = u^* = \overline{H}_f^\Omega$, so that $H_f^\Omega = u^* = \lim_{k\to\infty} T^k(u)$, which is what we aimed to prove. We next turn to prove (4.11). To this end, it is clearly non-restrictive to suppose that Ω is connected. Let $v \in \underline{\mathcal{U}}_f^\Omega$. We can suppose that v is not identically $-\infty$, otherwise (4.11) is trivial. Therefore $v \in \underline{\mathcal{S}}(\Omega)$. Since $u \in \overline{\mathcal{S}}(\Omega)$, we get $u - v \in \overline{\mathcal{S}}(\Omega)$, whence, by (4.6), $T(u - v) \leq u - v$. Moreover, for every $y \in \partial\Omega$ we have (from $u \in C(\overline{\Omega})$ and $f = u|_{\partial\Omega}$)

$$\liminf_{x\to y} (u - v)(x) \geq \liminf_{x\to y} u(x) + \liminf_{x\to y} (-v(x)) = f(y) - \limsup_{x\to y} v(x) \geq 0.$$

We are in a position to apply the Weak Minimum Principle in Step VI, which gives $v \leq u$ in Ω. This fact, together with $v \in \underline{\mathcal{S}}(\Omega)$ and $u \in \overline{\mathcal{S}}(\Omega)$, allows us to use (4.9) in Part IV, giving

$$v \leq v^* \leq u^* \leq u. \tag{4.12}$$

In particular this yields $v \leq u^*$, for every $\in \underline{\mathcal{U}}_f^\Omega$, which is the first inequality in (4.11). To prove the second inequality we need a slightly different argument. Let $w \in \overline{\mathcal{U}}_f^\Omega$. We can clearly suppose that $w \in \overline{\mathcal{S}}(\Omega)$. By (4.6) in Step III we have $T(w) \leq w$, whilst, by Step IV, we have $T(u^*) = u^*$. We hence derive that

$$T(w - u^*) \leq w - u^*. \tag{4.13}$$

Furthermore, from the last inequality in (4.12) and by the continuity of u up to $\partial\Omega$, for every $y \in \partial\Omega$ one has $\liminf_{x\to y}(w(x)-u^*(x)) \geq \liminf_{x\to y}(w(x)-u(x)) \geq \liminf_{x\to y} w(x) - f(y) \geq 0$. We claim that

$$w - u^* \text{ is lower semicontinuous in } \Omega. \tag{4.14}$$

Once this claim is proved, we are fully entitled to apply Step VI and conclude that $w - u^* \geq 0$ in Ω, which is the second inequality in (4.11).

We are therefore left to prove the claimed (4.14). Since $w \in \overline{\mathcal{S}}(\Omega)$, we derive that w is l.s.c., so it suffices to show that $-u^*$ is l.s.c., or equivalently, u^* is u.s.c. As u is continuous, by Step II we inductively infer that $T^k(u)$ is continuous for every $k \in \mathbb{N}$. Recalling that $u^* = \lim_{k\to\infty} T^k(u)$ and $\{T^k(u)\}_k$ is *monotone non-increasing* (see Step IV), well-known results on semicontinuity imply that u^* is u.s.c.

Step VIII. Our final step towards the proof of Theorem 4.0.9 consists in removing the hypothesis $u \in \overline{\mathcal{S}}(\Omega)$ in the previous Step VII. More precisely, we are ready to prove that, *if $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, if $F \in C(\overline{\Omega}, \mathbb{R})$ and $f := F|_{\partial\Omega}$, then $\lim_{k\to\infty} T^k(F) = H_f^\Omega$.*

To this aim, we let $\mathcal{A} := \left\{u - v \;\middle|\; u,\, v \in \overline{\mathcal{S}}(\mathbb{R}^N) \cap C(\mathbb{R}^N),\ u, v \geq 0\right\}$.

By means of the general potential-theoretic result contained in Proposition 6.8.3 of [16, page 364] (holding true in the $\mathfrak{S}^*$-harmonic space $(\mathbb{R}^N, \mathcal{H})$), for every $\varepsilon > 0$ there exists $p \in \mathcal{A}$ such that $\sup_{\overline{\Omega}} |F-p| < \varepsilon$. In particular, for every $n \in \mathbb{N}$ there exists $p_n \in \mathcal{A}$, $p_n = u_n - v_n$ (with u_n, $v_n \in \overline{\mathcal{S}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, u_n, $v_n \geq 0$) such that on the whole Ω one has

$$u_n - v_n - \frac{1}{n} < F < u_n - v_n + \frac{1}{n} \qquad \forall\, n \in \mathbb{N}. \tag{4.15}$$

Thus, by linearity and monotonicity of T (see Step III) and since $T(1) = 1$ (see (1.13)) we derive

$$T^k(u_n) - T^k(v_n) - \frac{1}{n} \leq T^k(F) \leq T^k(u_n) - T^k(v_n) + \frac{1}{n}.$$

By means of the previous Step VII and by setting $g_n := (u_n - w_n)|_{\partial\Omega}$, we get

$$H_{g_n}^\Omega - \frac{1}{n} \leq \liminf_{k\to\infty} T^k(F) \leq \limsup_{k\to\infty} T^k(F) \leq H_{g_n}^\Omega + \frac{1}{n}, \qquad \forall\, n \in \mathbb{N}. \tag{4.16}$$

On the other hand, from (4.15) we also obtain $H^{\Omega}_{g_n} - \frac{1}{n} \leq H^{\Omega}_f \leq H^{\Omega}_{g_n} + \frac{1}{n}$, for every $n \in \mathbb{N}$, and this immediately implies that $H^{\Omega}_f = \lim_{n\to\infty} H^{\Omega}_{g_n}$. This last fact, together with (4.16), yields $H^{\Omega}_f = \lim_{k\to\infty} T^k(F)$. The proof is complete. □

***Remark* 4.0.11.** We close the chapter by pointing out classes of operators (arising from relevant geometrical contexts) to which our results can be applied or easily extended. The results can be generalized to the following cases:

(i) The assumption (1.1) on the form of L can be easily extended to include operators like:

$$L = \frac{1}{V} \sum_{i,j} \partial_{x_i}(V\, a_{i,j}\, \partial_{x_j}), \tag{4.17}$$

where V is a smooth positive function. The only modification is that Lebesgue measure in our mean-value operator (1.9) has to be replaced by the measure $V(x)\,\mathrm{d}x$ (the absolutely continuous measure w.r.t. Lebesgue measure with density V); equivalently, without any change on the measure, the kernel (1.10) may be replaced by $V(y)\, K_\alpha(x,y)$.

(ii) We remember that the hypothesis of the global existence on $\mathbb{R}^N \times \mathbb{R}^N$ of a fundamental solution Γ can be removed by replacing $\mathbb{R}^N$ with a bounded open set Ω and replacing Γ with the Green function $G_\Omega(x,y)$ for Ω. This does not affect the results of this thesis: it will indeed suffice to embed the bounded set Ω in the Lebesgue-type Theorem of this chapter or the bounded set D in the inverse mean-value Theorem of chapter 3 in some overlying bounded open set O.

(iii) Operators of the form (4.17) arise in many contexts of geometrical relevance, like in Riemannian Geometry (e.g., Laplace-Beltrami operator has the form $\Delta = \sqrt{|g|^{-1}}\partial_i(\sqrt{|g|}g^{ij}\partial_j))$, or in the sub-Riemannian setting of self-adjoint left-invariant operators on Lie groups. For example, in the latter case, if $\mathbb{R}^N$ is endowed with a Lie group structure $\mathbb{G} = (\mathbb{R}^N, \cdot)$, any sub-Laplacian $\mathcal{L} = -\sum_{j=1}^m X_j^* X_j$ on $\mathbb{G}$ (here X_j^* denotes the adjoint of $X_j \in \mathrm{Lie}(\mathbb{G})$ with respect to the Haar measure μ of $\mathbb{G}$) has precisely the form (4.17) where V is the density of μ w.r.t. Lebesgue measure and $(a_{i,j})$ is a suitable nonnegative symmetric matrix.

We explicitly remark that, in the two mentioned examples, the hypotheses on Γ made in this Chapter are satisfied (if one also takes into account remark (ii) above), since Δ and $\mathcal{L}$ are hypoelliptic (this is true of Δ since it is an elliptic operator with smooth coefficients, whilst $\mathcal{L}$ is hypoelliptic since, by definition of sub-Laplacian, $X_1, \ldots, X_m$ generate $\mathrm{Lie}(\mathbb{G})$ whence Hörmander's condition is satisfied).

(iv) Other classes of operators to which our theory can be adapted are the so-called X-elliptic operators on Lie groups $\mathbb{G}$, i.e., with the notation in (iii) above, operators of the form $-\sum_{i,j=1}^{m} X_i^*(\alpha_{i,j}\, X_j)$ where $(\alpha_{i,j})$ is a uniformly elliptic positive-definite matrix with smooth coefficients. All these operators are of the form (4.17) and they are hypoelliptic.

Chapter 5

Comparison between Variational and Perron-Wiener solutions

In this chapter we will compare Perron-Wiener and weak variational solutions of the Dirichlet problem, extending to a more general framework the following result by Arendt and Daners [7], related to the classical Laplacian in $\mathbb{R}^N$.

Let Ω be a bounded open set in $\mathbb{R}^N$. Let $\varphi \in C(\partial\Omega)$ and assume that φ has an extension $\phi \in C(\overline{\Omega})$ such that $\Delta\phi \in H^{-1}(\Omega)$. Let $u \in H_0^1(\Omega)$ be the unique solution of Poisson's equation

$$-\Delta u = \Delta\phi \quad in \;\; \mathcal{D}(\Omega)'$$

Then $u + \phi = H_\varphi^\Omega$ is the Perron solution of the Dirichlet problem

$$\Delta h = 0 \quad in \;\; \Omega, \quad h|_{\partial\Omega} = \varphi$$

We specify that $H^1(\Omega)$ is the first Sobolev space, $H_0^1(\Omega)$ is the clousure of the test functions $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $\mathcal{D}(\Omega)'$ is the space of all distributions on Ω and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

Arendt and Daners's theorem in its turn extends results by Hildebrandt and Simader. Hildebrandt [38, Theorem 1] shows that for every $\phi \in H^1(\Omega) \cap C(\overline{\Omega})$ the minimiser of

$$\min \left\{ \int_\Omega |\nabla w|^2 \, dx \, : \, w \in H^1(\Omega), \, w - \phi \in H_0^1(\Omega) \right\} \tag{5.1}$$

assumes the boundary values φ for all regular points $z \in \partial\Omega$. Thus, if $\phi \in H^1 \cap C(\overline{\Omega})$ and if Ω is *Dirichlet regular*, it follows that the minimiser of (5.1) coincides with the solution of Perron Wiener, result which is also proved by Simader [66, Theorem 1.6]. We remark that if $\phi \in H^1(\Omega)$ then $\Delta\phi \in H^{-1}(\Omega)$.
The main result of this Chapter is the generalization of Arendt and Daners's theorem to

1. the Sub-Laplacians on Carnot groups, a framework in which it is natural to consider both Perron solution and variational solution (see Theorem 5.1.4);

2. general operators (1.1) (with properties introduced in Chapter 1), for which, however, we have to require more strong hypothesis: we assume that $\varphi \in C(\partial\Omega)$ has an extension $\phi \in C(\overline{\Omega})$ such that $\phi \in H^1(\Omega)$ (see Theorem 5.2.2).

We want to stress that a crucial rôle in the proofs of Theorem 5.1.4 and Theorem 5.2.2 is played by the possibility of approximating every open set $\Omega \subseteq \mathbb{R}^N$ through a monotone increasing sequence of bounded Dirichler regular open sets. Whereas this approximation is quite obvious in the Euclidean case, in our more general frameworks it is not trivial at all.

5.1 Stratified Lie groups

We begin by recalling some basic facts concerning sub-Laplacians on Carnot groups. All the details can be found in [16].
We say that a Lie group on $\mathbb{R}^N$, $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$, is a homogeneous Carnot group if the following properties hold:

(C1) $\mathbb{R}^N$ can be split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ and, for every $\lambda > 0$, the dilatation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, ..., \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbb{R}^{N_i}$$

is an automorphism of the group $\mathbb{G}$, for every $\lambda > 0$.

(C2) If N_1 is as above, let $Z_1, ..., Z_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $Z_j(0) = \partial/\partial_{x_j}|_0$, for $j = 1, ..., N_1$. Then

$$rank(Lie\,\{Z_1, ..., Z_{N_1}\})(x) = N \text{ for every } x \in \mathbb{R}^N$$

where $\mathrm{rank}(Lie\,\{U\}\,(x)) = \dim_{\mathbb{R}}\{XI(x)\,|\,X \in Lie\,\{U\}\}$.
If the previous properties are satisfied, we shall say that the triple $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ is a *homogeneus Carnot group.*

Any operators

$$\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$$

where $X_1, ..., X_{N_1}$ is a basis of $span\,\{Z_1, ..., Z_{N_1}\}$ is called a *sub-Laplacian* in $\mathbb{G}$. The vector valued operator

$$\nabla_{\mathcal{L}} = (X_1, ..., X_{N_1})$$

is called the $\mathcal{L}$-gradient.

We would like to list some basic properties of the sub-Laplacians:

1. $\mathcal{L}$ is *hypoelliptic*, i.e., every distributional solution of $\mathcal{L}u = f$ is of class C^∞ whenever f is of class C^∞. This follows from the rank-condition (C2) and the *Hörmander hypoellipticity theorem*.

2. $\mathcal{L}$ is *invariant with respect to the left translations* on $\mathbb{G}$, i.e. for every fixed $\alpha \in \mathbb{G}$

   $$\mathcal{L}(u(\alpha \circ x)) = (\mathcal{L}u)(\alpha \circ x) \qquad \text{for every}\, x \in \mathbb{G}\, \text{and every}\, u \in C^\infty(\mathbb{R}^N).$$

   This holds since the X_j's are left-translation invariant on $\mathbb{G}$.

3. $\mathcal{L}$ is a formally *self-adjoint* operator. This holds since $X_j^* = -X_j$.

We esplicity remark that $\circ$ has a particular nice form: indeed, for every $j \in \{1, ..., N\}$, the j-th coordinate of $x \circ y$ is given by

$$(x \circ y)_j = x_j + y_j + Q_j(x_1, ..., x_{j-1}, y_1, ..., y_{j-1}) \tag{5.2}$$

for a suitable polynomial function Q_j (here $Q_j = 0$ for $j = 1, ..., N_1$). Thus, the Jacobian matrices of the right and the left translations on $\mathbb{G}$ are triangular matrices with 1's on the main diagonal. This ensures that the *Lebesgue measure on* $\mathbb{R}^N$ *is a bi-invariant Haar measure for* $\mathbb{G}$.
Besides, it is not difficult to prove that (5.2) implies that *any sub-Laplacian is a divergence form operator.* Indeed, if we make the position

$$X_j = \sum_{i=1}^{N} \sigma_{i,j}(x) \frac{\partial}{\partial x_i}, \quad j = 1, ..., m$$

and we consider the $N \times m$ matrix $\sigma(x) = (\sigma_{i,j}(x))_{i \le N, j \le m}$, we have

$$\mathcal{L} = \operatorname{div}\left(A(x)\nabla\right), \quad \text{where} A(x) = \sigma(x) \cdot \sigma^T(x).$$

It can be easily proved that $a_{1,1}$ is a positive constant, so that (1.2) in Chapter 1 holds. Note also that $A(x)$ is positive semidefinite and it is definite if only if $m = N$, in which case we obtain the Euclidean group $\mathbb{G} = \left(\mathbb{R}^N, +\right)$ and $\mathcal{L}$ is a (strictly) elliptic operator with constant coefficients (for example, the classical Laplace operator $\Delta = \sum_{j=1}^{N} (\partial_j)^2$).

The results in [16, Chapter 5] ensure that any sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ admits a unique fundamental solution Γ with the properties enumerated in Chapter 1. Furthemore, the results in [16, Chapter 7] prove that the set of the $\mathcal{L}$-harmonic functions endows $\mathbb{R}^N$ with the structure of a σ^*-harmonic space (see Appendix A). Hence, *the sub-Laplacians on Carnot groups fall into the class of operators considered in Chapter 1 and all the definitions and results of this thesis can be applied to them.*
In particular, it holds the definition (1.2.1) of *Perron Wiener solution* of the Dirichlet problem.

5.1.1 Notations

Throughout this Chapter, we will denote with $C_C(\Omega)$ the space of all continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$.
Besides, we will denote by $H^1(\Omega)$ the first Sobolev space, i.e. the set of the functions $u \in L^2(\Omega)$ such that $X_j u \in L^2(\Omega)$, provided with the norm $\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u|^2 dx\right)^{1/2} + \left(\int_\Omega |\nabla_{\mathcal{L}} u|^2 dx\right)^{1/2}$. We will denote by $H_0^1(\Omega)$ the closure of the test functions space $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. By virtue of Poincarè's inequality, $\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla_{\mathcal{L}} u|^2 dx\right)^{1/2}$ defines an equivalent norm on $H_0^1(\Omega)$, generated by the inner product $\langle u, v\rangle_{H_0^1} = \left(\int_\Omega \nabla u \nabla v \, dx\right)^{1/2}$ (see [16, Section 5.9]).
Finally, we denote by $\mathcal{D}(\Omega)'$ the space of all distributions on Ω and with $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

5.1.2 Variational Solution

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\phi \in C(\bar{\Omega})$. We say that $\mathcal{L}\phi \in H^{-1}(\Omega)$ if there exists a positive constant c such that

$$\left|\int_\Omega \phi \mathcal{L} v\right| \leq c \, \|v\|_{H_0^1(\Omega)} \qquad \forall \, v \in \mathcal{D}(\Omega)$$

Then, if $\mathcal{L}\phi \in H^{-1}(\Omega)$, by Hahn-Banach Theorem there exists $F^* \in H^{-1}(\Omega)$ such that

$$\int_\Omega \phi \mathcal{L} v \, dx = F^*(v) \qquad \forall\, v \in \mathcal{D}(\Omega)$$

On the other hand by Fréchet-Riesz representation Theorem, there exists $u \in H_0^1(\Omega)$ satisfying

$$F^*(v) = \langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla_{\mathcal{L}} u \nabla_{\mathcal{L}} v \qquad \forall\, v \in H_0^1(\Omega)$$

In particular, for all $v \in \mathcal{D}(\Omega)$

$$\int_\Omega \nabla_{\mathcal{L}} u \nabla_{\mathcal{L}} v = \int_\Omega \phi \mathcal{L} v$$

An integration by parts at the left side gives

$$-\int_\Omega u \mathcal{L} v = \int_\Omega \phi \mathcal{L} v \qquad \forall\, v \in \mathcal{D}(\Omega) \tag{5.3}$$

Here we have used the fact that $X_j^* = -X_j$.
The integral identity (5.3) tells us that

$$\mathcal{L} u = -\mathcal{L}\phi \qquad \text{in the weak sense of distribution.}$$

Thus, we have found one solution $u \in H_0^1(\Omega)$ of

$$\begin{cases} \mathcal{L} u = -\mathcal{L}\phi & \text{in } \mathcal{D}(\Omega)' \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This solution is unique. Indeed, if we consider another function $w \in H_0^1(\Omega)$ such that

$$\begin{cases} \mathcal{L} w = -\mathcal{L}\phi & \text{in } \mathcal{D}(\Omega)' \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

then we have $u - w \in H_0^1(\Omega)$ and $\mathcal{L} u = \mathcal{L} w$, i.e.,

$$\int_\Omega u \mathcal{L}^* \varphi = \int_\Omega w \mathcal{L}^* \varphi \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

from which

$$\langle u - w, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla_{\mathcal{L}} (u - w) \nabla_{\mathcal{L}} \varphi = 0 \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ then we have $u - w = 0$.

***Remark* 5.1.1.** If $\mathcal{L}\phi \in L^2(\Omega)$ then $\mathcal{L}\phi \in H^{-1}(\Omega)$.
Indeed, for all $v \in \mathcal{D}(\Omega)$, we have

$$\left|\int \phi\, \mathcal{L}v\right| = \left|\int \mathcal{L}\phi\, v\right| \leq \|\mathcal{L}\phi\|_{L^2} \cdot \|v\|_{L^2} \leq \underbrace{c \cdot \|\mathcal{L}\phi\|_{L^2}}_{\text{cost}} \cdot \|v\|_{H_0^1}$$

where the last inequality derives from Poincarè inequality.

***Remark* 5.1.2.** If $\phi \in H^1(\Omega)$ then $\mathcal{L}\phi \in H^{-1}(\Omega)$.
Indeed, for all $v \in \mathcal{D}(\Omega)$, we have

$$\left|\int \phi\, \mathcal{L}v\right| = \left|\int \nabla_{\mathcal{L}}\phi \cdot \nabla_{\mathcal{L}}v\right| \leq \|\phi\|_{H_0^1} \cdot \|v\|_{H_0^1}$$

***Remark* 5.1.3.** Obviously, if the open set Ω is Dirichlet regular, then $\mathcal{L}H_\varphi^\Omega \in H^{-1}(\Omega)$, because $H_\varphi^\Omega \in C(\overline{\Omega}) \cap \mathcal{H}(\Omega)$.

5.1.3 Main result

Our main result is the following.

Theorem 5.1.4. *Let Ω be a bounded open set in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $\varphi \in C(\partial\Omega)$ and assume that φ has an extension $\phi \in C(\bar{\Omega})$ such that $\mathcal{L}\phi \in H^{-1}(\Omega)$. Let $u \in \mathrm{H}_0^1(\Omega)$ be the unique solution of Poisson's equation*

$$-\mathcal{L}u = \mathcal{L}\phi \text{ in } \mathcal{D}(\Omega)'$$

Then u can be modified in a set of measure zero such that $u + \phi = h_\varphi$ is the Perron solution of the Dirichlet problem $D(\varphi, \Omega)$.

The proof of this theorem requires some prerequisites. First of all, we need the Lemma 2.2.10, which allows us to prove this proposition, which is of independent interest.

Proposition 5.1.5. *Let Ω be a bounded open set in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $(\Omega_n)_{n\in\mathbb{N}}$ the sequence of Dirichlet regular open sets of the Lemma 2.2.10. Let $\phi \in C(\bar{\Omega})$ and denote $\varphi := \phi|_{\partial\Omega}$. Let h_n the solution of $D(\Omega_n, \phi|_{\partial\Omega_n})$,* *i.e.,* $h_n \in C(\overline{\Omega}_n) \cap \mathcal{H}(\Omega_n)$ *such that* $h_n = \phi$ *on* $\partial\Omega_n$*. Suppose that* $h_n = \phi$ *in* $\mathbb{R}^N \setminus \Omega_n$*. Then* h_n *is pointwise convergent in* Ω *and*

$$\lim_{n\to+\infty} h_n(x) = H_\varphi^\Omega(x) \tag{5.4}$$

for all $x \in \Omega$*.*

Proof. First of all, we observe that the approximating Dirichlet regular open sets can be taken such that

$$\forall\, \delta > 0 \;\; \exists\, \bar{n} \in \mathbb{N} \;\text{ such that }\; \partial\Omega_n \subset \{x \in \Omega : dist(x, \partial\Omega) < \delta\} \;\text{ for all } n > \overline{n} \tag{5.5}$$

Let $u \in \overline{U}_\varphi^\Omega$. Then

$$\liminf_{x\to y} u(x) \geq \phi(y) \text{ for all } y \in \partial\Omega,$$

i.e.,

$$\lim_{\rho\to 0}\left(\inf_{B(y,\rho)\cap\Omega} u\right) \geq \phi(y) \text{ for all } y \in \partial\Omega.$$

We deduce from this that

$$\forall y \in \partial\Omega \;\; \forall\, \varepsilon > 0 \;\; \exists\, \rho = \rho(y,\varepsilon) > 0 \quad \text{such that} \quad \inf_{B(y,\rho)\cap\Omega} u > \phi(y) - \varepsilon.$$

Then there exists r, $0 < r < \rho$, such that

$$\inf_{B(y,\rho)\cap\Omega} u > \phi(z) - \varepsilon \quad \forall\, z \in B(y,r) \cap \Omega$$

from which

$$\inf_{B(y,r)\cap\Omega} u > \phi(z) - \varepsilon \quad \forall\, z \in B(y,r) \cap \Omega$$

that is

$$u(x) + \varepsilon > \phi(z) \quad \forall\, x, z \in B(y,r) \cap \Omega \tag{5.6}$$

Then there exist $y_1, ..., y_p \in \partial\Omega$ such that

$$\partial\Omega \subseteq \bigcup_{j=1}^{p} B(y_j, r_j).$$

Define

$$O := \bigcup_{j=1}^{p} \left(B\left(y_j, r_j\right) \cap \Omega\right).$$

From Lemma 2.2.10 and from (5.5) it follows that

$$\exists \ \overline{n}_\varepsilon \quad \text{such that} \quad \partial\Omega_n \subset O \quad \forall\, n > \overline{n}_\varepsilon.$$

Fix $n > \overline{n}_\varepsilon$ and let $z \in \partial\Omega_n$.
Then there exists j such that $z \in \Omega \cap B(y_j, r_j)$ and, from (5.6)

$$u(x) + \varepsilon > \phi(z) \quad \forall\, x \in \Omega \cap B(y_j, r_j)$$

from which we obtain

$$\liminf_{x \to z} \left(u(x) + \varepsilon\right) \geq \phi(z).$$

Thus

$$u + \varepsilon \in \overline{U}^{\Omega_n}_{\phi/\partial\Omega_n} \qquad \forall\, n \geq \overline{n}_\varepsilon$$

and so

$$u + \varepsilon \geq h_n \ \text{ in } \Omega_n. \tag{5.7}$$

Proceeding similarly, we prove that, given $v \in \underline{U}^{\Omega}_{\varphi}$,

$$\forall\, \varepsilon > 0 \quad \exists\, \underline{n}_\varepsilon \in \mathbb{N} \quad \text{such that} \quad v - \varepsilon \in \underline{U}^{\Omega_n}_{\phi/\partial\Omega_n} \quad \forall \ \ n > \underline{n}_\varepsilon$$

and so

$$v - \varepsilon \leq h_n \ \text{ in } \Omega_n.$$

Let $K \subset \Omega$ a compact subset of Ω. From Lemma 2.2.10 and from (5.5) it follows that

$$\exists\, n'' \in \mathbb{N} \quad \text{such that} \quad K \subset \Omega_n \quad \forall\, n > n''.$$

Setting $\overline{n''}_\varepsilon = \max\left\{\overline{n}_\varepsilon, n''\right\}$, we have from (5.7)

$$u(x) + \varepsilon \geq h_n(x) \quad \forall\, x \in K \subset \Omega_n,\ \forall\, n > \overline{n''}_\varepsilon,$$

from which it follows that

$$u(x) + \varepsilon \geq \sup_{j \geq n}(h_j(x)) \quad \forall\, x \in K, \forall\, n > \overline{n''}_\varepsilon.$$

Letting $n \to +\infty$ we obtain

$$u(x) + \varepsilon \geq \limsup_{n\to+\infty} h_n(x) \text{ for all } x \in K, \text{ for all } u \in \overline{U}^{\Omega_n}_{\phi/\partial\Omega_n}$$

and taking the infimum with respect to u we have

$$H^{\Omega}_{\varphi}(x) + \varepsilon \geq \limsup_{n\to+\infty} h_n(x) \text{ for all } x \in K.$$

Since K is arbitrary

$$H^{\Omega}_{\varphi} + \varepsilon \geq \limsup_{n\to+\infty} h_n \text{ in } \Omega$$

and since ε is arbitrary

$$H^{\Omega}_{\varphi} \geq \limsup_{n\to+\infty} h_n \text{ in } \Omega.$$

Proceeding analogously, we prove that

$$H^{\Omega}_{\varphi} \leq \liminf_{n\to+\infty} h_n \text{ in } \Omega.$$

Hence h_n is pointwise convergent in Ω and

$$H^{\Omega}_{\varphi}(x) = \lim_{n\to+\infty} h_n(x) \text{ for all } x \in \Omega.$$

□

In order to prove Theorem 5.1.4 we need other propositions.

Proposition 5.1.6. *Let $u \in H^1(\Omega)$ and let $\alpha \in C_0^\infty(\Omega)$. Then $\alpha \cdot u \in H_0^1(\Omega)$.*

Proof. Since $u \in H^1(\Omega)$, from [30] and [33], it follows that there exists a sequence $u_n \in C^\infty(\Omega) \cap H^1(\Omega)$ such that

$$u_n \to u \quad \text{in } L^2(\Omega) \quad \text{and} \quad X_j u_n \to X_j u \quad \text{in } L^2(\Omega) \tag{5.8}$$

We show that

$$\alpha u_n \to \alpha u \quad \text{in } H^1(\Omega) \tag{5.9}$$

Indeed, from (5.8) and from the boundedness of α it follows that

$$\alpha u_n - \alpha u = \alpha(u_n - u) \to 0 \quad \text{in } L^2(\Omega).$$

and

$$X_j(\alpha u_n) = (X_j\alpha)u_n + \alpha(X_j u_n) \to (X_j\alpha)u + \alpha(X_j u) = X_j(\alpha u) \quad \text{in} \quad L^2(\Omega).$$

Since $\alpha u_n \in C_0^\infty(\Omega)$, from (5.9) it follows that $\alpha u \in H_0^1(\Omega)$. □

Proposition 5.1.7. *Let $u \in \mathrm{H}^1_{loc}(\Omega)$, and assume that $supp(u)$ is a compact subset of Ω. Then $u \in H_0^1(\Omega)$.*

Proof. Fix an open set ω such that $supp(u) \subset \omega \subset\subset \Omega$ and choose $\alpha \in C_0^\infty(\Omega)$ such that $\alpha = 1$ su ω; thus $\alpha u = u$ in Ω and, by Proposition 5.1.6 $\alpha u \in H_0^1(\Omega)$. So $u \in H_0^1(\Omega)$. □

Proposition 5.1.8. *Let Ω be a bounded open set and $u \in C_C(\Omega)$. Suppose that $\mathcal{L}u \in L^2(\Omega)$. Then $u \in \mathrm{H}_0^1(\Omega)$.*

Proof. Let $v \in \mathrm{H}_0^1(\Omega)$ such that $-\mathcal{L}v = \mathcal{L}u$. Then $h = u + v \in \mathcal{H}(\Omega)$ and, consequently, h belongs to $C^\infty(\Omega)$. Consequently, $u = h - v \in \mathrm{H}^1_{loc}(\Omega)$. Let $\varepsilon > 0$. Since $u \in C_C(\Omega)$, it follows that $(u-\varepsilon)^+ \in C_0(\Omega)$. On the other hand $(u-\varepsilon)^+ \in H^1_{loc}(\Omega)$ ([34]); then, from Proposition 5.1.7, it follows that $(u-\varepsilon)^+ \in \mathrm{H}_0^1(\Omega)$. Let $\omega \subset\subset \Omega$ such that $supp(u-\varepsilon)^+ \subset \omega$. If we let $f := -\mathcal{L}u$, we have by hypothesis $f \in L^2(\Omega)$ and then, for all $w \in \mathcal{D}(\omega)$,

$$\int_\Omega \nabla_{\mathcal{L}} u \nabla_{\mathcal{L}} w dx = -\int_\Omega u \mathcal{L} w dx = \int_\Omega f w dx$$

hence

$$\int_\Omega \nabla_{\mathcal{L}} u \nabla_{\mathcal{L}} w = \int_\Omega f w dx.$$

This identity remains true for $w \in \mathrm{H}_0^1(\omega)$. Take $w = (u-\varepsilon)^+$. Then

$$\int \left|\nabla_{\mathcal{L}}(u-\varepsilon)^+\right|^2 dx = \int \nabla_{\mathcal{L}} u \nabla_{\mathcal{L}}(u-\varepsilon)^+ dx = \int f(u-\varepsilon)^+ dx$$

$$\leq \|f\|_2 \left\|(u-\varepsilon)^+\right\|_2 \leq \|f\|_2 |\Omega|^{1/2} \|u\|_\infty$$

where the first equality follows from ([34, Corollary 2.2])

$$\begin{cases} \nabla_{\mathcal{L}}(u-\varepsilon)^+ = 0 & \text{when } u < \varepsilon \\ \nabla_{\mathcal{L}}(u-\varepsilon)^+ = \nabla_{\mathcal{L}} u & \text{when } u \geq \varepsilon. \end{cases}$$

Thus $\{(u-\varepsilon)^+ : \varepsilon \in (0,1]\}$ is bounded in $\mathrm{H}_0^1(\Omega)$. Hence there exists sequence $\varepsilon_j \searrow 0$ and $v \in H_0^1(\Omega)$ such that $(u-\varepsilon_j)^+ \to v$ weakly in $H_0^1(\Omega)$. By the weakly convergence, we can suppose the convergence in L^2, which in turn implies the convergence almost everywhere. Then we have $(u-\varepsilon_j)^+ \to u^+$ in $L^2(\Omega)$ and it follows that $u^+ = v \in \mathrm{H}_0^1(\Omega)$.
Applying this to $-u$ instead of u, we obtain that also $u^- \in H_0^1(\Omega)$. Then $u \in H_0^1(\Omega)$. □

The following Proposition follows easily from Proposition 5.1.8.

Proposition 5.1.9. *Let Ω be Dirichlet regular and $\phi \in C^2(\overline{\Omega})$, $\varphi = \phi|_{\partial\Omega}$. Let h be the solution of $D(\varphi, \Omega)$. Then $u = h - \phi \in \mathrm{H}_0^1(\Omega)$.*

We are now ready to prove Theorem 5.1.4.

Proof. (of Theorem 5.1.4)
Let σ_n be a mollifier, that is, $0 \le \sigma_n \in C_0^\infty(\mathbb{R}^N)$, $\mathrm{supp}(\sigma_n) \subset B_\rho\left(0, \frac{1}{n}\right)$ and $\int_{\mathbb{R}^\mathbb{N}} \sigma_n(x)dx = 1$. Extend ϕ to a uniformly continuous function on $\mathbb{R}^N$, which we still denote by ϕ. We define

$$\phi_n = \sigma_n *_{\mathbb{G}} \phi = \int_{\mathbb{R}^N} \sigma_n(x \circ z^{-1}) f(z) dz$$

Then

$$\phi_n \longrightarrow \phi \quad \text{uniformly on } \mathbb{R}^N. \tag{5.10}$$

Let $(\Omega_n)_{n\in\mathbb{N}}$ the sequence of Dirichlet regular open sets of the Lemma 2.2.10. Let k_n the solution of $D(\Omega_n, \phi_n|_{\partial\Omega_n})$, i.e., $k_n \in \mathcal{H}(\Omega_n) \cap C(\overline{\Omega}_n)$ such that $k_n = \phi_n$ on $\partial\Omega_n$. We show that

$$k_n \longrightarrow h_\varphi \tag{5.11}$$

pointwise on Ω. Indeed, let $K \subset \Omega$ be compact and denote with h_n the solution of $D(\Omega_n, \phi_{\partial\Omega_n})$. By Lemma 2.2.10 there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n \ge n_0$. Then, by the Maximum Principle and by (5.10), we have

$$\sup_K |k_n - h_n| \le \sup_{\overline{\Omega}_n} |k_n - h_n| \le \sup_{\partial\Omega_n} |k_n - h_n| = \sup_{\partial\Omega} |\phi_n - \phi| \to 0$$

as $n \to \infty$. Hence $(k_n - h_n) \to 0$ as $n \to \infty$, uniformly on compact subsets of Ω and, consequently, pointwise on Ω. From this and from Proposition 5.1.5 it follows that $k_n \to h_\varphi$ pointwise in Ω. Consider the function

$$u_n := k_n - \phi_n$$

Then $u_n \in C_C(\Omega_n)$ and $-\mathcal{L}u_n = \mathcal{L}\phi_n$ in Ω_n. It follows from Proposition 5.1.9 that $u_n \in H^1_0(\Omega_n)$.

Now, from the hypothesis $\mathcal{L}\phi \in \mathrm{H}^{-1}(\Omega)$, we deduce that there exists a constant $c > 0$ such that

$$\left| \int_\Omega \phi \, \mathcal{L}v \, dx \right| \le c \left(\int_\Omega |\nabla_{\mathcal{L}} v|^2 \right)^{1/2} \tag{5.12}$$

for all $v \in \mathcal{D}(\Omega)$. This will allow us to prove that

$$\left(\int_\Omega |\nabla_{\mathcal{L}} u_n|^2 \, dx \right)^{1/2} \le c \tag{5.13}$$

for all $n \in \mathbb{N}$.

In order to prove (5.13) fix $n \in \mathbb{N}$. Let $v \in \mathcal{D}(\Omega_n)$. Then:

$$\int_{\Omega_n} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v dx = -\int_{\Omega_n} u_n \, \mathcal{L}v \, dx = \int_{\Omega_n} (\phi_n - k_n) \, \mathcal{L}v \, dx = \int_{\Omega_n} \phi_n \mathcal{L}v dx$$

where the last equality follows from the fact that $\mathcal{L} = \mathcal{L}^*$ and k_n is $\mathcal{L}$-harmonic.

Now,

$$\begin{aligned}
\int_{\Omega_n} \phi_n(x) \mathcal{L}v(x) dx &= \int_{\Omega_n} \left(\int_{\mathbb{R}^N} \sigma_n(x \circ z^{-1}) \phi(z) dz \right) \mathcal{L}v(x) dx \\
&= \int_{\mathbb{R}^N} \phi(z) \left(\int_{\mathbb{R}^N} \sigma_n(x \circ z^{-1}) \mathcal{L}v(x) dx \right) dz \\
&= \int_{\mathbb{R}^N} \phi(z) \left(\int_{\mathbb{R}^N} \sigma_n(y) \left(\mathcal{L}v\right)(y \circ z) dy \right) dz
\end{aligned}$$

where in the third equality we have used the change of variables $y = x \circ z^{-1}$, with $dx = dy$ thanks to the invariance of the Lebesgue measure on $\mathbb{R}^N$ with respect to the right translations on $\mathbb{G}$. Now, from the invariance of the operator with respect to the left translations on $\mathbb{G}$, it follows that the last integral is equal to

$$\int_{\mathbb{R}^N} \phi(z) \left(\int_{\mathbb{R}^N} \sigma_n(y) \mathcal{L}_z \left(v(y \circ z) \right) dy \right) dz$$
$$= \int_{\mathbb{R}^N} \phi(z) \mathcal{L}_z \left(\int_{\mathbb{R}^N} \sigma_n(y) v(y \circ z) dy \right) dz$$

Hence

$$\int_{\Omega_n} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v dx = \int_{\mathbb{R}^N} \phi(x) \mathcal{L} v_n(x) dx$$

where $v_n(z) = \int_{\mathbb{R}^N} \sigma_n(y) v(y \circ z) dy$. Thus $v_n \in \mathcal{D}(\Omega)$ for n large enough and it follows from (5.12) that

$$\left| \int_{\mathbb{R}^N} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v \, dx \right| = \left| \int_{\mathbb{R}^N} \phi \mathcal{L} v_n dx \right| \leq c \left(\int_{\mathbb{R}^N} |\nabla_{\mathcal{L}} v_n|^2 dx \right)^{1/2}. \qquad (5.14)$$

On the other hand

$$\nabla_{\mathcal{L}} v_n(z) = \nabla_{\mathcal{L}} \left(\int_{\mathbb{R}^N} \sigma_n(y) v(y \circ z) \, dy \right) = \int_{\mathbb{R}^N} \sigma_n(y) \left(\nabla_{\mathcal{L}} v \right) (y \circ z) dy,$$

hence

$$\left(\int_{\mathbb{R}^N} |\nabla_{\mathcal{L}} v_n|^2 dx \right)^{1/2} \leq \left(\int_{\mathbb{R}^N} |\nabla_{\mathcal{L}} v|^2 dx \right)^{1/2}. \qquad (5.15)$$

From (5.14) and (5.15) it follows that

$$\left| \int_{\mathbb{R}^N} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v \, dx \right| \leq c \left(\int_{\mathbb{R}^N} |\nabla_{\mathcal{L}} v|^2 dx \right)^{1/2} \qquad (5.16)$$

for all $v \in \mathcal{D}(\Omega)$ and for n large enough. From this inequality and from the fact that $u_n \in H_0^1(\Omega_n)$ the claim (5.13) follows. Indeed, since $\mathcal{D}(\Omega_n)$ is dense in $H_0^1(\Omega_n)$, for all $v \in H_0^1(\Omega_n)$ there exists $v_j \in \mathcal{D}(\Omega_n)$ such that $v_j \to v$ in $H_0^1(\Omega_n)$. Therefore:

$$\left| \int_{\mathbb{R}^N} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v \, dx \right| = \left| \langle u_n, v \rangle_{H_0^1} \right| \leq |\langle u_n, v - v_j \rangle| + |\langle u_n, v_j \rangle| \leq$$

$$\leq \|u_n\|_{H_0^1} \|v - v_j\|_{H_0^1} + c \|v_j\|_{H_0^1}$$

where the last inequality follows from the Cauchy-Schwartz inequality and from (5.14). Letting $j \to \infty$ we obtain

$$\left| \langle u_n, v \rangle_{H_0^1} \right| \le c \, \|v\|_{H_0^1}$$

and, thus, taking $v = u_n$, the claim (5.13).
Now we define $\tilde{u}_n = u_n(x)$ if $x \in \Omega_n$ and $\tilde{u}_n(x) = 0$ if $x \notin \Omega_n$. Since $u_n \in H_0^1(\Omega_n)$, then $\tilde{u}_n \in H_0^1(\Omega)$ and $\nabla_{\mathcal{L}} \tilde{u}_n = \tilde{\nabla}_{\mathcal{L}} u_n$. We identify u_n and $\tilde{u}_n$ to simplify the notation. By (5.13) the sequence u_n is bounded in $H_0^1(\Omega)$. Hence there exists a subsequence u_{n_m} converging weakly to a function $u \in H_0^1(\Omega)$ as $m \to +\infty$. Since

$$u_{n_m} + \phi_{n_m} = k_{n_m} \tag{5.17}$$

we have

$$-\int_{\Omega_{n_m}} \nabla_{\mathcal{L}} u_{n_m} \, \nabla_{\mathcal{L}} v + \int_{\Omega_{n_m}} \phi_{n_m} \, \mathcal{L} v = \int_{\Omega_{n_m}} k_{n_m} \, \mathcal{L} v = 0$$

for all $v \in \mathcal{D}(\Omega_{n_m})$. Letting $m \to \infty$ we conclude that

$$-\int_{\Omega} \nabla_{\mathcal{L}} u \, \nabla_{\mathcal{L}} v + \int_{\Omega} \phi \, \mathcal{L} v = 0$$

for all $v \in \mathcal{D}(\Omega) = \bigcup_{m \in \mathbb{N}} \mathcal{D}(\Omega_{n_m})$. Thus u is the solution of

$$u \in H_0^1(\Omega), \quad -\mathcal{L} u = \mathcal{L} \phi \quad \text{in } \mathcal{D}(\Omega)'$$

On the other hand, u_{n_m} converges almost everywhere to u (as a consequence of the convergence in L^2, which in turn derives from the weakly convergence in H_0^1), and ϕ_{n_m} converges pointwise to ϕ (as a consequence of the uniformly convergence), therefore

$$u_{n_m} + \phi_{n_m} \to u + \phi \qquad \text{almost everywhere on } \Omega$$

Besides by (5.11) we have that k_{n_m} converges pointwise to H_{φ}^{Ω}, thus it follows from (5.17) that

$$u + \phi = H_{\varphi}^{\Omega}$$

almost everywhere on Ω. This completes the proof of the Theorem 5.1.4. $\square$

5.2 General operators

Let $\mathcal{L}$ an operator of the form (1.1), introduced in Chapter 1.

5.2.1 Assumptions

Let us denote by $X_1, ..., X_N$ the vector fields associated with the columns of the square root of the matrix A. We have (see Appendix B)

$$\langle A(x)\xi, \xi\rangle = \sum_{j=1}^{N} \langle X_j I(x), \xi\rangle^2 \tag{5.18}$$

$$\mathcal{L} = -\sum_{j=1}^{N} X_j^* X_j \tag{5.19}$$

Definition 5.2.1. *We say that an absolutely continuous curve $\gamma : [0, T] \to \Omega$ is a sub-unit curve with respect to $X = (X_1, ..., X_N)$ if for any $\xi \in \mathbb{R}^N$*

$$\langle \dot{\gamma}(t), \xi\rangle^2 \leq \sum_{j=1}^{N} \langle X_j(\gamma(t)), \xi\rangle^2 = \langle A(\gamma(t))\xi, \xi\rangle$$

for a.e. $t \in [0, T]$ (where the last equality derives from (5.18)). If $x_1, x_2 \in \Omega$, we define

$$d(x_1, x_2) = \inf\{T > 0 \,:\, \exists\, \gamma : [0; T] \to \Omega, \text{ sub-unit curve } \gamma(0) = x_1,\ \gamma(T) = x_2\}$$

If the above set of curves is empty, we put $d(x_1, x_2) = \infty$.

We will assume the following hypothesis hold:

1. $d(x, y) < \infty$ for any $x, y \in \Omega$, so that d is a distance in Ω. Moreover, the distance d is continuous with respect to the usual topology of $\mathbb{R}^N$.

   If $x \in \Omega$ and $r > 0$ we will denote by $B_r(x) = \{y \in \Omega \,:\, d(x, y) < r\}$ the metric balls with respect to d.

2. For any compact $K \subset \Omega$ there exists $r_K > 0$ such that for any $r < r_K$ there exists a positive constant C_K such that

$$|B_{2r}(x)| \leq C_K |B_r(x)|$$

for any $x \in K$ and $r < r_K$.
This property is known as *doubling property* of d.

3. Balls of distance d satisify Poincaré inequality, i.e.,

$$\int_{B_r} |u - u_r|^2 \, dx \leq C\, r^2 \int_{B_r} |Xu|^2 \, dx, \quad \forall\, u \in C^1(\overline{B_r})$$

where for every ball B_r with radius r in the distance d, u_r denotes the mean integral of u on B_r, Xu denotes the intrinsic gradient in $(\mathbb{R}^N,\, d_X)$ and C is a constant independent of u.

5.2.2 Main result

With the previous hypothesis on the distance, it holds the following theorem. Notations are the same of Subsection 5.1.1.

Theorem 5.2.2. *Let $\varphi \in C(\partial\Omega)$ and assume that φ has an extension $\phi \in C(\bar{\Omega}) \cap H^1(\Omega)$. Let $u \in \mathrm{H}^1_0(\Omega)$ be the unique solution of Poisson's equation*

$$-\mathcal{L}u = \mathcal{L}\phi \text{ in } D(\Omega)'$$

Then $u + \phi = H^\Omega_\varphi$ is the Perron solution of the Dirichlet problem $D(\varphi, \Omega)$.

It holds the definition (1.2.1) of *Perron-Wiener solution* of the Dirichlet problem. The definition of *Variational Solution of the Dirichlet problem* and the proof of its existence and uniqueness are the same of Subsection 5.1.2 (we remark that if $\phi \in H^1(\Omega)$ then $\mathcal{L}\phi \in H^{-1}(\Omega)$), with the only difference that (5.3) doesn't follow from $X_j = -X_j^*$, but it follows from (5.19).
Proposition 5.1.7 and Proposition 5.1.5 hold, and the proofs are the same. Proposition 5.1.6 holds, because thanks to assumptions on the distance we can apply also in its proof results of [30] and of [33].

In this framework we don't have a structure of group, then we can't define here the regularization of the function ϕ: for this reason we have been compelled to change the hypothesis of the theorem (hypotesis $\phi \in H^1(\Omega)$ in this theorem is stronger that the hypotesis $\mathcal{L}\phi \in H^{-1}(\Omega)$ in Theorem 5.1.4) and its proof is quite different from the proof of Theorem 5.1.4.

Proof. (of Theorem 5.2.2) Let $(\Omega_n)_{n\in\mathbb{N}}$ the sequence of Dirichlet regular open sets of the Lemma 2.2.10. Let h_n the solution of $D(\Omega_n, \phi|_{\partial\Omega_n})$. Consider the function

$$u_n := h_n - \phi.$$

Let $\varepsilon > 0$. Since $u_n \in C_C(\Omega_n)$, it follows that $(u_n - \varepsilon)^+ \in C_C(\Omega_n)$. On the other hand $(u_n - \varepsilon)^+ \in H^1_{loc}(\Omega_n)$ (see [34], remembering that $\mathcal{L}$ is a X-elliptic operator, as proved in Appendix B); then, from Proposition 5.1.7, $(u_n - \varepsilon)^+ \in H^1_0(\Omega_n)$.
We have, for all $v \in D(\Omega_n)$,

$$\begin{aligned}\int_{\Omega_n} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v \, dx &= \int_{\Omega_n} \nabla_{\mathcal{L}}(k_n - \phi) \nabla_{\mathcal{L}} v \, dx \\ &= \int_{\Omega_n} \nabla_{\mathcal{L}}\phi \nabla_{\mathcal{L}} v \, dx \\ &\leq \|\nabla_{\mathcal{L}}\phi\|_{L^2(\Omega_n)} \|\nabla_{\mathcal{L}} v\|_{L^2(\Omega_n)}\end{aligned}$$

where the second equality derives from (5.19).
The inequality proved remains true for $v \in H^1_0(\Omega_n)$. Take $v = (u_n - \varepsilon)^+$. Then

$$\begin{aligned}\int_{\Omega_n} \left|\nabla_{\mathcal{L}}(u_n - \varepsilon)^+\right|^2 &= \int_{\Omega_n} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}}(u_n - \varepsilon)^+ \, dx \\ &\leq \|\nabla_{\mathcal{L}}\phi\|_{L^2(\Omega_n)} \left\|\nabla_{\mathcal{L}}(u_n - \varepsilon)^+\right\|_{L^2(\Omega_n)}\end{aligned}$$

where the first equality follows from ([34, Corollary 2.2])

$$\begin{cases} \nabla_{\mathcal{L}}(u_n - \varepsilon)^+ = 0 & \text{when } u_n < \varepsilon \\ \nabla_{\mathcal{L}}(u_n - \varepsilon)^+ = \nabla_{\mathcal{L}} u_n & \text{when } u_n \geq \varepsilon. \end{cases}$$

Thus $\{(u_n-\varepsilon)^+ : \varepsilon \in (0,1]\}$ is bounded in $H_0^1(\Omega_n)$. Hence there exists a sequence $\varepsilon_j \searrow 0$ and $v_n \in H_0^1(\Omega_n)$ such that $(u_n-\varepsilon_j)^+ \to v_n$ weakly in $H_0^1(\Omega_n)$. By the weakly convergence we can suppose the convergence in L^2, which in turn implies the convergence almost everywhere. Then we have $(u_n-\varepsilon_j)^+ \to u_n^+$ in $L^2(\Omega_n)$. It follows that $u_n^+ = v_n \in H_0^1(\Omega_n)$.
Applying this to $-u_n$ instead of u_n, we obtain that also $u_n^- \in H_0^1(\Omega_n)$. Then $u_n \in H_0^1(\Omega_n)$.
Now, from the hypothesis $\phi \in H^1(\Omega)$, we deduce that there exists a constant $c > 0$ such that

$$\left| \int_\Omega \phi\, \mathcal{L}v\, dx \right| \leq c \left(\int_\Omega |\nabla_{\mathcal{L}} v|^2 \right)^{1/2} \tag{5.20}$$

for all $v \in D(\Omega)$. This will allow us to prove that

$$\left(\int_\Omega |\nabla_{\mathcal{L}} u_n|^2\, dx \right)^{1/2} \leq c \tag{5.21}$$

for all $n \in \mathbb{N}$.
In order to prove (5.21) fix $n \in \mathbb{N}$. Let $v \in D(\Omega_n)$. Then:

$$\int_{\Omega_n} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v dx = -\int_{\Omega_n} u_n\, \mathcal{L}v\, dx = \int_{\Omega_n} (\phi - h_n)\, \mathcal{L}v\, dx = \int_{\Omega_n} \phi \mathcal{L} v dx$$

where the last equality follows from the fact that $\mathcal{L} = \mathcal{L}^*$ and h_n is $\mathcal{L}$-harmonic.
Thus it follows from (5.20) that

$$\left| \int_{\mathbb{R}^N} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v\, dx \right| = \left| \int_{\mathbb{R}^N} \phi \mathcal{L} v\, dx \right| \leq c \left(\int_{\mathbb{R}^N} |\nabla_{\mathcal{L}} v|^2\, dx \right)^{1/2} \tag{5.22}$$

for all $v \in D(\Omega_n)$.
From this inequality and from the fact that $u_n \in H_0^1(\Omega_n)$ the claim (5.21) follows. Indeed, since $D(\Omega_n)$ is dense in $H_0^1(\Omega_n)$, for all $v \in H_0^1(\Omega_n)$ there exists $v_j \in D(\Omega_n)$ such that $v_j \to v$ in $H_0^1(\Omega_n)$. Therefore:

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \nabla_{\mathcal{L}} u_n \nabla_{\mathcal{L}} v\, dx \right| &= \left| \langle u_n, v \rangle_{H_0^1} \right| \leq |\langle u_n, v - v_j \rangle| + |\langle u_n, v_j \rangle| \\ &\leq \|u_n\|_{H_0^1} \|v - v_j\|_{H_0^1} + c\, \|v_j\|_{H_0^1} \end{aligned}$$

where the last inequality follows from the Cauchy-Schwartz inequality and from (5.22). Letting $j \to \infty$ we obtain

$$\left| \langle u_n, v \rangle_{H_0^1} \right| \leq c \, \|v\|_{H_0^1}$$

and, thus, taking $v = u_n$, the claim (5.21).
Now we define $\tilde{u}_n(x) = u_n(x)$ if $x \in \Omega_n$ and $\tilde{u}_n(x) = 0$ if $x \notin \Omega_n$. Since $u_n \in H_0^1(\Omega_n)$, then $\tilde{u}_n \in H_0^1(\Omega)$ and $\nabla_L \tilde{u}_n = \tilde{\nabla}_L u_n$. We identify u_n and $\tilde{u}_n$ to simplify the notation. By (5.21) the sequence u_n is bounded in $H_0^1(\Omega)$; hence there exists a subsequence u_{n_m} converging weakly to a function $u \in H_0^1(\Omega)$ as $m \to +\infty$. Since

$$u_{n_m} + \phi = h_{n_m} \tag{5.23}$$

we have

$$-\int_{\Omega_{n_m}} \nabla_L u_{n_m} \, \nabla_L v + \int_{\Omega_{n_m}} \phi \, Lv = \int_{\Omega_{n_m}} h_{n_m} \, Lv = 0$$

for all $v \in D(\Omega_{n_m})$. Letting $m \to \infty$ we conclude that

$$-\int_{\Omega} \nabla_L u \, \nabla_L v + \int_{\Omega} \phi \, Lv = 0$$

for all $v \in D(\Omega) = \bigcup_{m \in \mathbb{N}} D(\Omega_{n_m})$. Thus u is the solution of

$$u \in H_0^1(\Omega), \quad -Lu = L\phi \quad \text{in } D(\Omega)'$$

On the other hand, u_{n_m} converges almost everywhere to u (as a consequence of the convergence in L^2, which in turn derives from the weakly convergence in H_0^1), therefore

$$u_{n_m} + \phi \to u + \phi \qquad \text{almost everywhere on } \Omega$$

Besides by (5.4) we have that h_{n_m} converges pointwise to H_φ^Ω, thus it follows from (5.23) that

$$u + \phi = H_\varphi^\Omega$$

almost everywhere on Ω This completes the proof of the Theorem 5.2.2. $\square$

Appendix A

Abstract Harmonic Spaces

In this Appendix we present some topics from the *theory of Abstract Harmonic Space.* For details and proofs see [16, Chapter 6].
Throughout the Appendix (E, τ) will denote a topological Hausdorff space, locally connected and locally compact, and we assume that the topology τ has a countable base.

Definition A.0.3 (Harmonic sheaf). *A sheaf of function $\mathcal{H}$ in E is called* harmonic sheaf *if for every open set $\Omega \subset E$ the set $\mathcal{H}(\Omega)$ is a linear subspace of $C(\Omega, \mathbb{R})$, the vector space of the real continuous functions defined on Ω. When $\mathcal{H}$ is a harmonic sheaf on E and V is an open set in τ, a function $u \in \mathcal{H}(V)$ will be called* $\mathcal{H}$-harmonic.

Definition A.0.4 ($\mathcal{H}$-regular set). *Let $\mathcal{H}$ be a harmonic sheaf in E. We say that an open set $V \in$ is* $\mathcal{H}$-regular *if the following conditions are satisfied:*

(R1) $\overline{V}$ is compact and $\partial V \neq \emptyset$;

(R2) for every continuous function $f : \partial V \to \mathbb{R}$, there exists a unique $\mathcal{H}$-harmonic function in V, denoted by H_f^V, such that

$$\lim_{x \to y} H_f^V(x) = f(y) \quad \textit{for every } y \in \partial V;$$

(R3) if $f \geq 0$*, then* $H_f^V \geq 0$.

When V is $\mathcal{H}$-regular, from the linearity of $\mathcal{H}(V)$ and the uniqueness assumption in condition $(R2)$ it follows that

$$H_{f+g}^V = H_f^V + H_g^V, \quad H_{\lambda f}^V = \lambda H_f^V$$

for every $f, g \in C(\partial V, \mathbb{R})$, and for every $\lambda \in \mathbb{R}$. Then, also keeping in mind (R3), for every $\mathcal{H}$-regular open set V and for every $x \in V$, the map

$$C(\partial V, \mathbb{R}) \ni f \mapsto H_f^V(x) \in \mathbb{R}$$

is linear and positive. Hence, the following definition is well posed.

Definition A.0.5 ($\mathcal{H}$-harmonic measure)**.** *Let* $\mathcal{H}$ *be a harmonic sheaf on* E*. Let* $V \in$ *be an* $\mathcal{H}$*-regular set. Then there exists a Radon measure* μ_x^V *on* $C(V, \mathbb{R})$ *such that*

$$H_f^V(x) = \int_{\partial V} f(y)\, d\mu_x^V(y) \quad \forall\, f \in C(\partial V, \mathbb{R})$$

The measure μ_x^V *is called the* $\mathcal{H}$-harmonic measure *related to* V *and* x*.*

We provide a definition which we will be used throughout the sequel.

Definition A.0.6 ($\mathcal{H}$-hyperharmonic function)**.** *Let* $\mathcal{H}$ *be a harmonic sheaf on* (E, τ)*. Let* $\Omega \in \tau$*. A function* $u : \Omega \to (-\infty, \infty]$ *is called* $\mathcal{H}$-hyperharmonic *in* Ω *if:*

(i) u *is lower semi-continuous;*

(ii) for every $\mathcal{H}$*-regular open set* $V \subset \overline{V} \subseteq \Omega$*, one has*

$$u(x) \geq \int_{\partial V} u(y)\, d\mu_x^V(y) \quad \forall\, x \in V$$

We shall denote by $\mathcal{H}^*(\Omega)$ *the set of the* $\mathcal{H}$*-hyperharmonic functions in* Ω*.*

Since

$$\int_{\partial V} u \, d\mu_x^V = \sup \left\{ \int_{\partial V} \varphi \, d\mu_x^V \mid \varphi \in C(\partial V, \mathbb{R}), \ \varphi \leq u \right\}$$

condition (ii) can be rewritten as follows

(ii)' for every $\mathcal{H}$-regular open set $V \subseteq \overline{V} \subseteq \Omega$ and for every $\varphi \in C(\partial V, \mathbb{R})$ such that $\varphi \leq u|_{\partial V}$, one has

$$H_\varphi^V \leq u|_V.$$

A function $v : \Omega \to [-\infty, \infty)$ will be called $\mathcal{H}$-hypoharmonic *if $-v \in \mathcal{H}^*(\Omega)$. We denote by*

$$\mathcal{H}_*(\Omega) := -\mathcal{H}^*(\Omega)$$

the family of the $\mathcal{H}$-hypoharmonic functions in Ω.

Definition A.0.7 (Harmonic space). *Let $\mathcal{H}$ be a harmonic sheaf on E. We say that $(E, \mathcal{H})$ is a* harmonic space *if the following axioms are satisfied*

(A1) (Positivity). *For every relatively compact open set $K \subseteq E$ there exists $h_0 \in \mathcal{H}^*(K)$ and $k_0 \in \mathcal{H}_*(K)$ satisfying:*

$$\inf_K h_0, \ \inf_K h_0 > 0 \quad \textit{and} \ \ h_0 < \infty \ \ \forall x \in K.$$

(A2) (Positivity). *If $\{u_n\}_{n \in \mathbb{N}}$ is a monotone increasing sequence of $\mathcal{H}$-harmonic functions on an open set $\Omega \in$ such that*

$$\left\{ x \in \Omega \ : \ \sup_{n \in \mathbb{N}} u_n(x) < \infty \right\}$$

is dense in Ω, then

$$u := \lim_{n \to \infty} u_n$$

is $\mathcal{H}$-harmonic in Ω.

(A3) (Regularity). *The family τ_r of the $\mathcal{H}$-regular open sets is a basis for the topology τ.*

(A4) (Separation). *For every $x, y \in E$ with $x \neq y$, there exist $u, v \in \mathcal{H}^*(E)$ such that*

$$u(x)v(y) \neq u(y)v(x).$$

Definition A.0.8 ($\mathcal{H}$-super- and $\mathcal{H}$-sub-harmonic function). *Let $(E, \mathcal{H})$ be a harmonic space, and let $\Omega \subseteq E$ be open. A function $u \in \mathcal{H}^*(\Omega)$ will be said* $\mathcal{H}$-superharmonic *if, for every $\mathcal{H}$-regular open set $V \subseteq \overline{V} \subseteq \Omega$, the function*

$$V \ni x \mapsto \int_{\partial V} u \, d\mu_x^V$$

is $\mathcal{H}$-harmonic in V. The set of the $\mathcal{H}$-superharmonic functions in Ω will be denoted by

$$\overline{S}(\Omega).$$

A function $v : \Omega \to [-\infty, \infty)$ will be $\mathcal{H}$-subharmonic *in Ω if $-v \in \overline{S}(\Omega)$. We shall denote by*

$$\underline{S}(\Omega) := -\overline{S}(\Omega)$$

the set of the $\mathcal{H}$-subharmonic functions in Ω.

Definition A.0.9 (Upper and lower functions and solutions). *Let $(E, \mathcal{H})$ be a harmonic space, and let $\Omega \subseteq E$ be open and such that $\overline{\Omega}$ is compact and $\partial\Omega \neq \emptyset$. Given a function $f : \partial\Omega \to [-\infty, \infty]$, we set*

$$\overline{\mathcal{U}}_f^\Omega := \left\{ u \in \mathcal{H}^*(\Omega) \ : \ \liminf_{\partial\Omega} u \geq f, \ \inf u > -\infty \right\}$$

and

$$\underline{\mathcal{U}}_f^\Omega := \left\{ v \in -\mathcal{H}^*(\Omega) \ : \ \limsup_{\partial\Omega} u \leq f, \ \sup v < \infty \right\}.$$

The families $\overline{\mathcal{U}}_f^\Omega$ and $\underline{\mathcal{U}}_f^\Omega$ will be called, respectively, the family of the upper functions *and of the* lower functions *related to f and Ω.*

The real extended functions

$$\overline{H}_f^\Omega := \inf \overline{\mathcal{U}}_f^\Omega, \quad \underline{H}_f^\Omega := \sup \underline{\mathcal{U}}_f^\Omega$$

will be called the upper solution *and the* lower solution, *respectively, to the Dirichlet problem*

$$(\mathcal{H} - D) \quad \begin{cases} u \in \mathcal{H}(\Omega) \\ \lim_{x \to y} u(x) = f(y) \quad \textit{per ogni } y \in \partial\Omega \end{cases}$$

Definition A.0.10 (Resolutive function and generalized solution)**.** *Let* $(E, \mathcal{H})$ *be a harmonic space, and let* $\Omega \subseteq E$ *be an open set with compact closure and non-empty boundary. A real extended function* $f : \partial\Omega \to [-\infty, \infty]$ *will be said* resolutive *if:*

(i) $\overline{H}_f^\Omega = \underline{H}_f^\Omega$

(ii) $\overline{H}_f^\Omega \in \mathcal{H}(\Omega)$.

In this case, we set

$$H_f^\Omega := \overline{H}_f^\Omega \, (= \underline{H}_f^\Omega)$$

and we say that H_f^Ω *is the generalized solution, in the sense of Perron-Wiener-Brelot, to the Dirichlet problem (*$\mathcal{H}$*-D). We also call* H_f^Ω *the PWB function related to* Ω *and* f*. The set of the resolutive functions* $f : \partial\Omega \to [-\infty, \infty]$ *will be denoted by* $\mathcal{R}(\partial\Omega)$*,*

$$\mathcal{R}(\partial\Omega) := \{f : \partial\Omega \to [-\infty, \infty] \,|\, f \text{ is resolutive}\}$$

The connection between H_f^Ω and the Dirichlet problem ($\mathcal{H}$-D) is showed by the following proposition

Proposition A.0.11. *Let the hypothesis of Definition A.0.10 hold. Let* $f : \partial\Omega \to [-\infty, \infty]$ *be a bounded function. Then the following statements are equivalent:*

(i) f *is resolutive and* $\lim_{x \to y} H_f^\Omega(x) = f(y)$ *for every* $y \in \partial\Omega$*.*

(ii) There exists $u \in \mathcal{H}(\Omega)$ *such that* $\lim_{x \to y} u(x) = f(y)$ *for every* $y \in \partial\Omega$*.*

In this latter case, $u = H_f^\Omega$*.*

Definition A.0.12 (σ-harmonic Space)**.** *A harmonic space* $(E, \mathcal{H})$ *will be said* $\sigma-$harmonic *if the family*

$$\overline{S}_C^+(E) := \left\{u \in \overline{S}(E) \cap C(E, \mathbb{R}) \,:\, u \geq 0\right\}$$

separates the points of E*, that is,*

for every $x, y \in E$, $x \neq y$, there exists $u, v \in \overline{S}_C^+(E)$ such that
$u(x)v(y) \neq u(y)v(x)$

In σ-harmonic spaces it holds the following Wiener resolutivity theorem.

Theorem A.0.13 (Wiener resolutivity theorem)**.** *Let $(E, \mathcal{H})$ be a σ-harmonic space, and let $\Omega \subseteq E$ be an open set with compact closure and non-empty boundary. Every continuous function $f : \partial\Omega \to \mathbb{R}$ is resolutive.*

Definition A.0.14 (σ^*-harmonic space)**.** *A $\sigma-$harmonic space $(E, \mathcal{H})$ will be said σ^*-harmonic if the following property holds:*

For every $x_0 \in E$ there exists $s_{x_0} \in \underline{S}_C^+(E)$ such that $s_{x_0} = 0$ and
$\inf_{E \setminus V} s_{x_0} > 0$ *for every neighborhood V of x_0*

We know from the Wiener resolutivity theorem that every continuous function $f : \partial\Omega \to \mathbb{R}$ is resolutive, so that the Perron-Wiener-Brelot function H_f^Ω is $\mathcal{H}$-harmonic in Ω. However, in general, we cannot expext a good behavior of H_f^Ω at the boundary points of Ω.

Definition A.0.15 ($\mathcal{H}$-regular point)**.** *A point $y \in \partial\Omega$ will be called $\mathcal{H}$-*regular *if*

$$\lim_{\Omega \ni x \to y} H_f^\Omega(x) = f(y) \quad \forall\, f \in C(\partial\Omega, \mathbb{R}).$$

Obviously, the function H_f^Ω is the solution of the Dirichlet problem ($\mathcal{H}$-D) for every $f \in C(\partial\Omega, \mathbb{R})$ if and only if all the boundary points of Ω are $\mathcal{H}$-regular points.
Unfortunately, we have to expect that, in general, $\partial\Omega$ contains boundary points which are not $\mathcal{H}$-regular.
The notion of $\mathcal{H}$-barrier function will allow us to give a necessary and sufficient condition for the $\mathcal{H}$-regularity.

Definition A.0.16 ($\mathcal{H}$-barrier function)**.** *Let $y \in \partial\Omega$. A $\mathcal{H}$-barrier function for Ω at y is a function ω defined in $\Omega \cap V$, being V a suitable open neighborhood of y, such that:*

(i) $\omega \in \overline{S}(\Omega \cap V)$,

(ii) $\omega(x) > 0$ *for every* $x \in \Omega \cap V$,

(iii) $\lim_{x \to y} \omega(x) = 0$.

The link between $\mathcal{H}$-regularity and $\mathcal{H}$-barrier functions is given by the following theorem.

Theorem A.0.17 (Bouligand's theorem)**.** *Let* $(E, \mathcal{H})$ *be a* σ^**-harmonic space, and let* $\Omega \subseteq E$ *be a relatively compact open set with non-empty boundary. A point* $x_0 \in \partial\Omega$ *is* $\mathcal{H}$*-regular for* Ω *if and only if there exists an* $\mathcal{H}$*-barrier function for* Ω *at* x_0*.*

Appendix B

X-elliptic operators

In this Appendix we first state a remarkable Philipps and Sarason result ([58]): every symmetric, non-negative and C^2 matrix has a square root locally uniformly Lipschitz continuous. By this result we will prove that our operator

$$\mathcal{L} = \sum_{i,j=1}^{N} \partial_{x_i} \left(a_{i,j} \partial_{x_j} \right)$$

introduced in Chapter 1, is a uniformly X-elliptic operator, in the sense of Gutierrez and Lanconelli ([34]), where $X = \{X_1, ..., X_N\}$ are the vector fields associated with the columns of the square root B of the matrix A. Besides, we will prove that it can be written as a sort of sum of squares of vector fields

$$\mathcal{L} = -\sum X_j X_j^*$$

Theorem B.0.18 (Philips and Sarason's result)**.** *Let A be a symmetric, non-negative and C^2 matrix in an open set $\Omega \subseteq \mathbb{R}^N$. Let Ω_1 be relatively compact in Ω. Then $B = \sqrt{A}$ is uniformly Lispschitz continuous in Ω_1.*

The proof of this theorem requires several Lemmas.

Lemma B.0.19. *Let f be a function such that*

$$f : (0, \delta) \to \mathbb{R}, \quad f(t) = a - bt + ct^2,$$

with $0 < a \leq c$, $b > 0$ *and* $f(t) \geq 0$ *for all* $t \in (0, \delta)$. *Then there exists a constant* $C = C(\delta)$ *such that*

$$b \leq C \cdot \sqrt{ac}$$

Proof. From the hypothesis $a - bt + ct^2 \geq 0$ for all $t \in (0, \delta)$, it follows that

$$b \leq \frac{a}{t} + ct \quad \text{for all } t \in (0, \delta) \tag{B.1}$$

Observing that the function $g(t) = \frac{a}{t} + ct$ has the positive minimum at $t_0 = \sqrt{\frac{a}{c}}$, let we pose in (B.1) $t = \delta \cdot \sqrt{\frac{a}{c}}$. It follows that

$$b \leq \frac{a}{\delta}\sqrt{\frac{c}{a}} + c\delta\sqrt{\frac{a}{c}} = \left(\frac{1}{\delta} + \delta\right)\sqrt{ac}.$$

$\square$

Lemma B.0.20. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set. We consider the function*

$$F : \Omega \to \mathbb{R}, \quad F \in C^2(\Omega), \quad F \geq 0.$$

Let $\Omega_1,\ \Omega_2 \subseteq \mathbb{R}^N$ *be open sets such that*

$$\overline{\Omega_1} \subseteq \Omega_2 \subseteq \overline{\Omega_2} \subseteq \Omega.$$

and, for every $x_0 \in \Omega_1$, *let*

$$c := |F(x_0)| + \sum_{i,j} \sup_{\Omega_2} |D_{ij}F|$$

Then there exists a constant $C = C(\delta)$ *such that*

$$|\nabla F(x_0)| \leq C \cdot \sqrt{F(x_0)c}. \tag{B.2}$$

Proof. Let $\delta := \mathrm{dist}(\Omega_1, \mathbb{R}^N \setminus \Omega_2)$ and, for every $x_0 \in \Omega_1$, let $x = x_0 + ty$, with $|y| \leq 1$, $t \leq \delta$. Then $x \in \Omega_2$ and, by Taylor formula,

$$0 \leq F(x) = F(x_0) + \langle \nabla F(x_0), x - x_0 \rangle + \frac{1}{2} \langle \mathcal{H}_F(z)(x - x_0), x - x_0 \rangle$$

where $\mathcal{H}_F$ is the Hessian matrix related to F.
If $|\nabla F(x_0)| \neq 0$ we take $y = -\frac{\nabla F(x_0)}{|\nabla F(x_0)|}$ and then we have

$$0 \leq F(x) = F(x_0) - t\,|\nabla F(x_0)| + ct^2 \quad \text{for all } t \in (0, \delta).$$

From Lemma B.0.19 it follows (B.2) □

Lemma B.0.21. *Let A be a symmetric and non-negative matrix, with entries*

$$\alpha^{ij} = \langle Ae_i, e_j \rangle$$

Then

$$\left|\alpha^{ij}\right| \leq \frac{1}{2} \left(\left| \left\langle A(e^i + e^j), e^i + e^j \right\rangle \right| + \left| \left\langle A(e^i - e^j), e^i - e^j \right\rangle \right| \right). \tag{B.3}$$

Proof. We have

$$\begin{aligned}
\left|\alpha^{ij}\right| &= \frac{1}{2} \left|\alpha^{ij} + \alpha^{ji}\right| \\
&= \frac{1}{2} \left| \left\langle Ae^j, e^i \right\rangle + \left\langle Ae^i, e^j \right\rangle \right| \\
&= \frac{1}{2} \left| \left\langle A(e^i + e^j), e^i + e^j \right\rangle - \left\langle A(e^i - e^j), e^i - e^j \right\rangle \right| \\
&\leq \frac{1}{2} \left(\left| \left\langle A(e^i + e^j), e^i + e^j \right\rangle \right| + \left| \left\langle A(e^i - e^j), e^i - e^j \right\rangle \right| \right).
\end{aligned}$$

□

Lemma B.0.22. *Let A be a symmetric and non-negative matrix. Let $B = \sqrt{A}$. Then*

$$\nabla b^{ij} = \frac{\nabla \alpha^{ij}}{\sqrt{\alpha^{jj}} + \sqrt{\alpha^{ii}}} \tag{B.4}$$

at each point x_0 where A is diagonal.

Proof. The matrix $B = \sqrt{A}$ is given by

$$B = \frac{1}{2\pi i} \int_\Gamma \sqrt{\lambda}(\lambda - A)^{-1}\, d\lambda,$$

where Γ is a closed curve surrounding the spectrum of A and contained in $\{Re\, \lambda > 0\}$. If $A = A(x)$, with $x \in \mathbb{R}$, then

$$B(x) = \frac{1}{2\pi i} \int_\Gamma \sqrt{\lambda}(\lambda - A(x))^{-1}\, d\lambda.$$

Deriving $B(x)$ with respect to x, we have

$$\frac{B(x+h) - B(x)}{h} = \frac{1}{2\pi i} \int_\Gamma \sqrt{\lambda} \left((\lambda - A(x+h))^{-1} - (\lambda - A(x))^{-1}\right)\, d\lambda.$$

Since

$$\begin{aligned}
(\lambda - &A(x+h))^{-1} - (\lambda - A(x))^{-1} \\
&= (\lambda - A(x+h))^{-1} \left\{I - (\lambda - A(x+h))\,(\lambda - A(x))^{-1}\right\} \\
&= (\lambda - A(x+h))^{-1} \left\{(\lambda - A(x)) - (\lambda - A(x+h))\right\} (\lambda - A(x))^{-1},
\end{aligned}$$

then

$$\begin{aligned}
&\frac{(\lambda - A(x+h))^{-1} - (\lambda - A(x))^{-1}}{h} \\
&= (\lambda - A(x+h))^{-1} \frac{A(x+h) - A(x)}{h} (\lambda - A(x))^{-1}
\end{aligned}$$

which tends, for $h \to 0$, to

$$(\lambda - A(x+h))^{-1} A'(x) (\lambda - A(x))^{-1}.$$

Then, if $x \in \mathbb{R}^N$ we have

$$\nabla B(x) = \frac{1}{2\pi i} \int_\Gamma \sqrt{\lambda}\, (\lambda - A(x))^{-1}\, \nabla A(x)\, (\lambda - A(x))^{-1}\, d\lambda. \tag{B.5}$$

At each point x_0 where the matrix A is diagonal we set

$$A(x_0) = \operatorname{diag}\left(\alpha^{11}(x_0), ..., \alpha^{nn}(x_0)\right) =: \operatorname{diag}\Delta$$

Then we have

$$\nabla B(x_0) = \frac{1}{2\pi i} \int_\Gamma \sqrt{\lambda}\, (\lambda - \operatorname{diag}\Delta)^{-1}\, \nabla A(x_0)\, (\lambda - \operatorname{diag}\Delta)^{-1}\, d\lambda.$$

Since

$$\begin{aligned}(\lambda - \text{diag}\Delta)^{-1} &= \left(\text{diag}(\lambda - \alpha^{11}(x_0), ..., \lambda - \alpha^{nn}(x_0))\right)^{-1}\\ &= \text{diag}\left((\lambda - \alpha^{11}(x_0))^{-1}, ..., (\lambda - \alpha^{nn}(x_0))^{-1}\right)\end{aligned}$$

and since for every matrix $C = (c^{ij})$ it holds

$$\text{diag}(\mu_1, ..., \mu_n) \cdot C \cdot \text{diag}(\mu_1, ..., \mu_n) = \left(\mu_i \mu_j c^{ij}\right)$$

it follows that, at each point x_0 where the matrix A is diagonal

$$\begin{aligned}\nabla b^{ij} &= \frac{1}{2\pi i} \int_\Gamma \sqrt{\lambda} \frac{\nabla \alpha^{ij}}{(\lambda - \alpha^{ii})(\lambda - \alpha^{jj})}\, d\lambda\\ &= \frac{1}{2\pi i} \nabla \alpha^{ij} \int_\Gamma \frac{\sqrt{\lambda}}{(\lambda - \alpha^{ii})(\lambda - \alpha^{jj})}\, d\lambda\\ &= \nabla \alpha^{ij} \left(\text{Res}(\alpha^{ii}) + \text{Res}(\alpha^{jj})\right)\\ &= \nabla \alpha^{ij} \cdot \left(\frac{\sqrt{\alpha^{ii}}}{\alpha^{ii} - \alpha^{jj}} + \frac{\sqrt{\alpha^{jj}}}{\alpha^{jj} - \alpha^{ii}}\right).\end{aligned}$$

Then it holds (B.4). $\square$

Now we are ready to prove the *Theorem B.0.18.*

Proof. We shall derive an estimate for the C^1 norm of B in terms of the C^2 norm of A under the assumption that A is not singular. For singular A this gives a uniform estimate for $(A + \varepsilon I)^{1/2}$, $0 < \varepsilon \leq 1$. In the limit as $\varepsilon \searrow 0$, we get the Lipschitz continuity of B.
Without loss, then, assume that A is non-singular.
Suppose also that A is diagonal at $x_0 \in \Omega_1$.
By Lemma B.0.22, at x_0 it holds

$$\nabla b^{ij} = \frac{\nabla \alpha^{ij}}{\sqrt{\alpha^{jj}} + \sqrt{\alpha^{ii}}}. \tag{B.6}$$

In order to apply (B.6), choose a subdomain Ω_2 relatively compact in Ω and containing $\overline{\Omega_1}$ and suppose that $\delta > 0$ is less than the distance from Ω_1 to the compliment of Ω_2. Let $\tau(x)$ be a non-negative C^2 function in $\overline{\Omega_2}$ and let

$$c := \|\tau(x)\|_{C^2}$$

Then, by Lemma B.0.20 there exists a constant $C = C(\delta)$ such that

$$|\nabla\tau| \leq C \cdot \sqrt{c\tau} \quad \text{in } \Omega_1 \tag{B.7}$$

Next, let $\{u^i\}$ be the set of diagonalizing vectors for A at x_0. Applying (B.7) at (B.6) with

$$\tau = \left\langle Au^i, u^i \right\rangle = A^{ii}$$

gives

$$\left|\nabla b^{ii}\right| = \frac{|\nabla\alpha^{ii}|}{2\left(\alpha^{ii}\right)^{1/2}} \leq C \cdot \frac{(c\alpha^{ii})^{1/2}}{2(\alpha^{ii})^{1/2}} \leq \frac{C}{2} \cdot c^{1/2} \tag{B.8}$$

Using Lemma B.0.21 and applying (B.7) to $\tau = \langle A(u^i \pm u^j), u^i \pm u^j\rangle$ we obtain

$$\begin{aligned}
4\left|\nabla\alpha^{ij}\right| &\leq 2\left(\left|\left\langle\nabla A(u^i + u^j), u^i + u^j\right\rangle\right| + \left|\left\langle\nabla A(u^i - u^j), u^i - u^j\right\rangle\right|\right)\\
&\leq Cc^{1/2}\left[\left\langle\nabla A(u^i + u^j), u^i + u^j\right\rangle^{1/2} + \left\langle\nabla A(u^i - u^j), u^i - u^j\right\rangle^{1/2}\right]\\
&\leq C(2c)^{1/2}\left[\left\langle\nabla A(u^i + u^j), u^i + u^j\right\rangle + \left\langle\nabla A(u^i - u^j), u^i - u^j\right\rangle\right]^{1/2}\\
&= 2Cc^{1/2}\left[\left\langle Au^i, u^j\right\rangle + \left\langle Au^j, u^j\right\rangle\right]^{1/2}\\
&= 2Cc^{1/2}\left[\alpha^{ii} + \alpha^{jj}\right]^{1/2} \leq 2Cc^{1/2}\left[(\alpha^{ii})^{1/2} + (\alpha^{jj})^{1/2}\right].
\end{aligned}$$

Inserting this in (B.6) we obtain

$$\left|\nabla b^{ij}\right| \leq 2Cc^{1/2} \tag{B.9}$$

The estimates (B.8) and (B.9) prove the theorem, under the hypothesis of A diagonal at x_0.

If A is not diagonal at x_0, then there exists an ortogonal matrix U such that

$$U^{-1}AU \;\text{ is diagonal.}$$

Then, we can deduce an estimate for $|\nabla b^{ij}(x_0)|$ appying to

$$\sqrt{U^{-1}AU} = U^{-1}BU$$

the estimate obtained in the previuous part of the proof and observing that

$$|\nabla B(x_0)| \leq \left|\nabla(U^{-1}BU)(x_0)\right|.$$

□

Now, we give the definition of X-elliptic operators.

Definition B.0.23 (X-elliptic operators)**.** *Let $\{X_1, ..., X_m\}$ be a family of vector fields in $\mathbb{R}^N$, $X_j = (b_{j1}, ..., b_{jN})$, $j = 1, ..., n$, where $b_{jk}(x)$ are locally Lipschitz continuous functions in $\mathbb{R}^N$. As usual, we identify the vector field X_j with the first order differential operator*

$$\sum_{k=1}^{N} b_{jk}\partial_k$$

We consider

$$\mathcal{L}u = \sum_{i,j=1}^{N} \partial_i \left(c_{ij}\partial_j u + e_i u\right) + \sum_{i=1}^{N} c_i \partial_i u + du, \tag{B.10}$$

where $c_{ij}(x) = c_{ji}(x)$, e_i, c_i and d are measurable functions. Set $C = (c_{ij})$, $e = (e_1, ..., e_N)$ and $c = (c_1, ..., c_N)$.

We say that the operator $\mathcal{L}$ is X-elliptic in an open subset Ω of $\mathbb{R}^N$ *if it satisfies the following conditions:*

(1) There exists a constant $\lambda > 0$ such that

$$\lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 \leq \langle C(x)\xi, \xi\rangle, \quad \textit{for all } \xi \in \mathbb{R}^N, x \in \Omega, \tag{B.11}$$

where $\langle C(x)\xi, \xi\rangle$ is the characteristic form of $\mathcal{L}$ given by

$$\langle C(x)\xi, \xi\rangle = \sum_{i,j=1}^{N} c_{ij}(x)\xi_i\xi_j; \tag{B.12}$$

(2) There exists a function $\gamma(x) \geq 0$ such that

$$\langle e(x), \xi \rangle^2 + \langle c(x), \xi \rangle^2 \leq \gamma(x)^2 \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 , \quad \textit{for all } \xi \in \mathbb{R}^N,\ x \in \Omega. \tag{B.13}$$

We say that $\mathcal{L}$ is uniformly X-elliptic in Ω *if $\mathcal{L}$ is X-elliptic in Ω and in addition there exists a positive constant Λ such that*

$$\langle C(x)\xi, \xi \rangle \leq \Lambda \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 , \quad \textit{for all } \xi \in \mathbb{R}^N,\ x \in \Omega. \tag{B.14}$$

Consider now our operator

$$\mathcal{L} := \sum_{i,j=1}^{N} \partial_{x_i} \left(a_{i,j}(x) \partial_{x_j} \partial_{x_j} \right) = \operatorname{div}(A(x)\nabla)$$

with properties introduced in Chapter 1.
Let us denote by

$$\begin{aligned} &X_1, ..., X_N \textit{ the vector fields associated with the columns} \\ &\qquad\qquad \textit{of the square root } B \textit{ of the matrix } A = (a_{i,j}). \end{aligned} \tag{B.15}$$

Thanks to the Phillips and Sarason result (B.0.18) B is a symmetric matrix with *locally Lipschitz-continuous entries.*
Since

$$\langle A(x)\xi, \xi \rangle = \left\langle B(x)^2 \xi, \xi \right\rangle = \left\langle B(x)^T B(x) \xi, \xi \right\rangle = \langle B(x)\xi, B(x)\xi \rangle = \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2$$

it follows that $\mathcal{L}$ satisfies (B.12) (with the equality and the constant $\lambda = 1$), it satisfies (B.13) (because $\mathcal{L}$ is in *principal form*, that is, $\mathcal{L}$ has the form (B.11) and the coefficient c_i, e_i and d are identically zero), and finally it satisfies (B.14) (with the equality and the constant $\Lambda = 1$).

Then, our operator $\mathcal{L}$ *is unifomly X-elliptic in* Ω.

Moreover, since

$$X_j = \sum_i b_{ij}\partial_i \quad \text{and} \quad X_j^* = -\sum_i \partial_i \left(b_{ij}\cdot\right)$$

we have

$$\begin{aligned}
\sum_j X_j^* X_j &= \sum_j \left(-\sum_i \partial_i \left(b_{ji} \sum_k b_{jk}\partial_k \right)\right) \\
&= -\sum_{i,j} \partial_i b_{ji} \cdot \sum_k b_{jk}\partial_k - \sum_{i,j} b_{ji} \sum_k \partial_i \left(b_{jk}\partial_k\right) \\
&= \sum_{i,j,k} \partial_i b_{ji} \cdot b_{jk}\partial_k - \sum_{i,j,k} b_{ji}\partial_i \left(b_{jk}\partial_k\right) \\
&= -\sum_{i,j,k} \partial_i b_{ji} \cdot b_{jk}\partial_k - \sum_{i,j,k} b_{ji}b_{jk}\partial_i\partial_k - \sum_{i,j,k} b_{ji}\partial_i b_{jk} \cdot \partial_k
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\operatorname{div}\left(A(x)\nabla^T\right) &= \sum_i \partial_i \left(\sum_k A_{ik}\partial_k \right) \\
&= \sum_{i,k} A_{i,k}\partial^2_{i,k} + \sum_{i,j,k} \partial_i \left(b_{ji}b_{jk}\right)\partial_k \\
&= \sum_{i,j,k} b_{ji}b_{jk}\partial_i\partial_k + \sum_{i,j,k} \partial_i b_{ji} \cdot b_{jk}\partial_k + \sum_{i,j,k} b_{ji}\partial_i b_{jk}\partial_k
\end{aligned}$$

Then it follows that

$$\mathcal{L} = \operatorname{div}\left(A(x)\nabla^T\right) = -\sum X_j X_j^* \tag{B.16}$$

where X_j are the vector fields introduced in (B.15).

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