HIGHER CHERN-SIMONS GAUGE THEORY

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Introduction

Higher gauge theory is a branch of mathematical physics which studies the generalization of ordinary gauge theory to higher algebraic structures. The latter are higher extensions of the concepts of Lie groups, Lie algebras and principal bundles, such as n-groupoids, L_∞-algebras and gerbes. In particular, higher gauge theory deals with the development of parallel transport along higher dimensional surfaces instead of just paths. This is usually called higher parallel transport.

This subject firstly appeared in string theory. String theory introduced fundamental objects which are no longer points, and therefore have nonzero dimension, such as strings and branes. Instead of a one dimensional worldline, these objects trace hypersurfaces (worldsheets for strings and worldvolumes for higher dimensional branes) during their time evolution. Therefore if we want to define a suitable notion of "Wilson surfaces" for strings and branes, to generalize the Wilson loops associated with charged particles, a theory of higher parallel transport is needed.

The first evidence of this was in 1986, when Gawedtzki [1] showed that the Kalb-Ramond (also known as Neveu-Schwarz) field B, which is a 2-form that generalizes the electromagnetic potential to strings, can be seen as a connection on a bundle gerbe, although the precise mathematical introduction of the concept of a *gerbe* came into this field only later with the works of Carey, Johnson and Murray [2] and of Gawedtzki and Reis [3]. Freed and Witten [4] exploited this viewpoint to understand the anomaly cancellation in the worldsheet path integral in superstring theory.

Another important occurrence of a higher gauge structure in string theory comes with the String structure. This is an higher analog of the Spin structure, a lift of the group $SO(n)$, which arises because in order to define spinors in Dirac theory a $Spin(n)$ principal bundle is needed. In superstring theory the worldsheet anomaly cancellation implies that the $Spin(n)$ bundle must be improved further to a lift of $Spin(n)$ which is called $String(n)$ [5]. The resulting $String(n)$ bundle is called a String structure. The group $String(n)$ is topological and infinite dimensional, but it was realized that it can be seen as the nerve of a smooth 2-group. This 2-group remains somewhat mysterious and very hard to describe finite dimensionally [6], but it can be treated with differential geometry being smooth and its infinitesimal version, the 𝔰𝔱𝔯𝔦𝔫𝔤 2-algebra is very simple. Recently another step was added in this ladder by Sati, Schreiber, and Stasheff, [7],[8], who found that the anomaly freedom of the spinors on the fivebrane's worldvolume in M-theory requires the target manifold to carry an higher analog of a String structure, which they call Fivebrane structure, obtained by lifting $String(n)$ to $Fivebrane(n)$. The latter is a smooth 6-group they introduced.

Recently, Fiorenza, Sati and Schreiber [9] proposed a higher seven dimensional Chern-Simons model, whose field is a connection 2-form valued in the string group,

as the AdS_7/CFT_6 counterpart of the $\mathcal{N} = (2, 0)$ six dimensional $M5$-brane theory in M theory. Other links between M-theory and higher gauge theory and in particular higher twisted cohomology are exposed in [10]. Other application to supergravity are found in [11].

Alongside with these connections with superstring theory, higher gauge theory has found several other applications in theoretical physics, for example in loop quantum gravity and in spin foam models. Representations of the Poincaré 2-group can be used to build spin foam models that describe quantum field theory in 4d Minkowski space-time [12],[13], although this can't be generally be seen as resulting from the quantization of some classical higher gauge theory with gauge structure the Poincaré 2-group. 2-Connections for a closely related 2-group, the tangent 2-group of the Lorentz group, are the solutions to topological gravity.

The Kalb-Ramond field coupled with a bosonic string was also seen to be connected with multisymplectic geometry, as was shown by Baez, Hoffnung and Rogers [14]. Multisymplectic geometry is the higher analog of symplectic geometry, where symplectic forms of degree greater than two are introduced.

BF theory can be reinterpreted as a higher gauge theory with structure 2-group the tangent 2-group of a Lie group G. In particular, the solution 4-dimensional BF theory can be seen as a 2-connection on a 2-bundle. When G is the Lorentz group it provides a 4-dimensional model for topological gravity. Observables are 1-dimensional holonomies around worldsheets as well as 2-dimensional holonomies around worldlines of the Aharonov-Bohm kind, whose particle interpretation is still problematic. Palatini formulations of gravity also admits a higher gauge interpretation in these terms.

The BF theory with cosmological constant can also be viewed as a theory of connections of a particular 2-group, namely the Inner automorphism 2-group of a group G. The quantization of this theory is conjectured to give a spin foam model called the Crane-Yetter model [15],[16]. These and other interesting applications of higher gauge theory to physics can be found discussed in much more detail in [17].

Mathematically higher gauge theory finds its natural environment in the theory of higher categories. A category is roughly a collection of objects and of arrows between objects. These arrows, called morphisms, can be composed in an associative manner and they admit an identity for every object. The categorical interpretation of various aspects of gauge field theories is very well acknowledged, the most trivial example of this being a group seen as a category with just one object and only invertible morphisms. Going from ordinary categories to higher categories means to add another level of morphisms, called 2-morphisms, which are arrows between arrows, and which have to obey several axioms. One gets so a 2-category. We can go further and define 3-morphisms, 4-morphisms and so on, leading to the general concepts of n-categories and ∞-categories. This ladder provides a straightforward and simple way to generalize physical concepts which have a categorical interpretation: we can define higher groups, higher bundles, higher connections and so on. The latter are connections with form degree greater than one, and these are exactly the objects that are found in higher gauge theories.

Many features of higher gauge theory remain unknown. In this thesis we explore the possible definition of a higher gauge field theory and we try to study it. We define a higher version of the Chern-Simons theory and investigate its canonical quantiza-

tion. Chern-Simons theory is one of the most renowned quantum field theory. It is a topological gauge theory, it can be solved exactly and it has been shown to have topological invariants as observables. Furthermore it has remarkable links with conformal field theory in two dimensions. The only dynamical field is a connection on a principal bundle, thus it is particularly suitable for the generalization to higher gauge theory. The field content of our model is a connection with values in a 2-term L_∞ algebra, which is the lowest nontrivial higher generalization of a Lie algebra, on a four dimensional smooth manifold, and the action is built in such a way that the classical equations of motion enforce the flatness of this connection, mimicking what happens in ordinary Chern-Simons. Understanding the gauge structure and invariance, the possible quantization schemes and the class of observables of this model should provide several important clues about other higher generalizations and higher gauge theory in general.

Unluckily, the proneness of category theory to the generalization of concepts, stemming from its intrinsic abstractness, corresponds on the other hand to a theoretically heavy and hard-to-handle machinery which makes it cumbersome to use in concrete computations. Indeed, despite its great achievements in some fields of mathematics which are of physical interest, category theory remains unused and unknown among physicists. In our work, my advisor and I tried to face a higher gauge theory model of our construction, the 2-term L_∞ algebra Chern-Simons theory defined in 4 dimensions, with methods and techniques which belong to usual quantum field theory. In this way we were able to touch concrete aspects of this higher gauge theory. On the other hand, we had to face difficulties arising from the incomplete understanding of many mathematical features of higher gauge theory at the state of the art and the partial inadequacy of our methods to solve some of the problems we met.

The biggest obstacle to overcome is the unclear relation between 2-groups and 2-term L_∞ algebras, which should play the role of infinitesimal counterpart to 2-groups, see sect. 2.5. Up to now, there is no way to relate 2-groups to 2-term L_∞ algebras which is viable for our purposes. This poses serious problems to a direct approach to the generalizations of gauge theories to 2-groups, because ordinary gauge theories rely heavily on both the finite and the infinitesimal version of the gauge group. Namely, one needs a Lie algebra to define a local connection 1-form and the Lie group integrating it to define gauge transformations. To circumvent this difficulty, we exploit an idea by Zucchini firstly introduced in [21], whose point of view we adopt throughout this thesis. The main point is to reformulate local ordinary gauge theory in such a way that only the Lie algebra is essential. Given a Lie algebra $\mathfrak{g}$ and a smooth manifold M, which can be taken to be contractible since we are only interested in local aspects, a $\mathfrak{g}$-connection on M is a 1-form

$$\omega \in \Omega^1(M, \mathfrak{g}). \tag{0.0.1}$$

Usually gauge transformations of connections are governed by smooth maps $\gamma : M \to G$ where G is the Lie group integrating $\mathfrak{g}$, and they act on connections as

$$\omega \to {}^\gamma\omega = \gamma\omega\gamma^{-1} - d\gamma\gamma^{-1}. \tag{0.0.2}$$

This can be rephrased by saying that a gauge transformation consists of a map $g :$

$M \to \mathrm{Aut}(\mathfrak{g})$ and a 1-form $\sigma_g \in \Omega^1(M, \mathfrak{g})$ which satisfy the relations

$$g^{-1}dg(\cdot) - [\sigma_g, \cdot] = 0, \tag{0.0.3a}$$

$$d\sigma_g + \frac{1}{2}[\sigma_g, \sigma_g] = 0, \tag{0.0.3b}$$

and that they act on a connection ω as

$$\omega \to {}^g\omega = g(\omega - \sigma_g), \tag{0.0.4}$$

see sect 3.1.1. This form of gauge transformations is prone to the generalization to the higher setting, because it suffices to substitute $\mathfrak{g}$ with the desired 2-term L_∞ algebra and the job is done. Every mention to the gauge group has disappeared.

Notice that this formulation, while being useful towards higher gauge theory, spoils some aspects of ordinary gauge theory. This new formulation of gauge transformations includes the old one as a particular case, but it is more general. That's why we call them extended gauge transformations. First of all, the automorphism g is not required to be an inner automorphism, as happens ordinarily. This is because we have no way to generalize the concept of inner automorphism to the higher setting. Secondly, the flat connection σ_g is not required to be the pull-back of a Maurer-Cartan form, for the same reason. This means that the gauge transformations we define for 2-term L_∞ gauge could be too general too, but this is the best we can do in order to have something to work with in a gauge field theory.

Another more subtle point is important regarding gauge transformations. The data (g, σ_g) encoding our extended gauge transformations hide the data γ which constitutes the usual gauge transformations, but we can't extract γ from (g, σ_g). Generalizing the extended gauge transformations to the higher setting as they stand, we are probably ending up with some data which hide some more fundamental objects too. This makes our higher gauge transformations not fully useful to completely comprehend some aspects of the higher Chern-Simons theory, such as the possible quantization of the gauge anomaly, which has been determined only for particular cases (see sect. 6.4).

Nevertheless, this approach proved useful under many points of view. We were able to build a well defined action, to study its gauge invariance and to perform canonical quantization on it. We found that our higher Chern-Simons displays remarkable similarities with the ordinary one. Concerning the classical level, the equations of motion imply the flatness of the connection fields and the gauge anomaly shows features resembling the anomaly of ordinary Chern-Simons theory. At the quantum level canonical quantization is carried out exactly as in ordinary Chern-Simons theory, except for the polarization scheme which remains a bit mysterious in some aspects. We also speculate that Wilson surfaces can be used as observables for the theory in order to compute higher knot invariants.

This thesis is divided into three parts. Part I is a self-contained introduction to the mathematical tools that are needed in subsequent parts. In chapter 1 we discuss the basis of category theory, and we give the main definitions that we will use thereafter, with particular emphasis on the theory of 2-categories and double categories. In chapter 2 L_∞ algebras are defined and we study in some detail 2-term L_∞ algebras, which are the main algebraic ingredient of higher gauge theory.

Part II is devoted to the development of the 2-term L_∞ gauge theory. In chapter 3 we generalize connection and curvature forms from the ordinary setting to the case of a gauge structure governed by a 2-term L_∞ algebra, introducing gauge transformations for such forms and all key elements for the definition of a 2-term L_∞ gauge field theory. In chapter 4 we focus on higher parallel transport, and we display a technical framework for surface holonomies which makes use of double categories and double groupoids.

Part III faces Chern-Simons theory. Chapter 5 is a short review of some aspects of the ordinary Chern-Simons theory, in preparation of its higher partner. Chapter 6 targets the main subject of the thesis, dealing with the definition of a 2-term L_∞ algebra Chern-Simons theory and with the study of its gauge structure and its possible quantization schemes. Finally in the last chapter we discuss what remains unclear and what requires further study in this model.

The original results are concentrated in some sections of chapter 4 and, which is taken from [18], and in chapter 6, which is taken from [19].

Part I

Mathematical preliminaries

Chapter 1

Category Theory

Category theory was first introduced in the 1945 by Eilenberg and Mac Lane and it has since then immensely developed, growing fast to an enormous extent. It was firstly used in homological algebra and in algebraic geometry, but it is now influential in almost every field of mathematics. Its versatility comes from the fact that it captures the abstract essence of mathematical concepts in order to work with few simple and basilar axioms. This makes it easy to obtain results in full generality, or to further generalize ideas and definitions already known.

Uses of category theory in mathematical physics have also been made, for example in the study of axiomatic Topological Quantum Field Theories (TQFT) introduced by Atiyah in [20], where TQFT's are viewed as maps between categories, or in the construction of Higher Gauge Theories ([17], [21]), where categories are employed to define a generalization of gauge transformations, which is the case of study in this thesis.

In this chapter we will briefly expose the notions we need about category theory. Due to the largeness of this field and to the great variety of tasks that can be pursued studying it, we will not attempt a full discussion of the subject, which is out of our reach for lack of space and knowledge, nor will we achieve a complete introduction to the basics of categories. There is plenty of good references for the interest reader, for example [22], [23] and [24].

1.1 Basics of category theory

Categories abstractly generalize the concepts of sets and functions, and they are defined as collection of objects (sets) and arrows between them (functions). In order to make things work similarly as in set theory, they must be endowed with an associative composition of arrows.

Definition 1. *a **Category** $\mathcal{C}$ consists of a set of objects $Obj_{\mathcal{C}}$ and a set of morphisms $Hom_{\mathcal{C}}$, such that for every morphism f there is a unique source object X and a unique target object Y in the category, and we will write $f : X \rightarrow Y$. The set of all morphisms going from X to Y is denoted $Hom_{\mathcal{C}}(X, Y)$. Moreover for every couple of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ such that the target of the first is equal to the source of the second there exists a **composite morphism** $g \circ f : X \rightarrow Z$, and for every object X*

*there is an **identity morphisms** $i_X : X \to X$ such that:*

$$(h \circ g) \circ f = h \circ (g \circ f), \tag{1.1.1}$$

$$i_Y \circ f = f \circ i_X = f \; for \; f : X \to Y. \tag{1.1.2}$$

What we have defined is more often called a *small* category. In mathematical literature the collections of all objects and of all morphisms in a category are classes instead of sets. A category C is then called *locally small* if for every couple of objects X and Y the morphisms $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ form a set, and *small* if it is locally small and if the class of all objects does be a set.

Generally, there is no requirement about the invertibility of the morphisms, which may or may not be invertible. Given a morphism $f : X \to Y$ a left (right) inverse to f is a morphism $g : Y \to X$ such that $g \circ f = 1_X$ ($f \circ g = 1_Y$). A morphism which has both a left and a right inverse, which must therefore coincide, is said to be an isomorphism.

It is very useful to introduce diagrams while working with categories. Diagrams in category theory are pictorial representations of objects and morphisms, the former drawn as points and the latter drawn as arrows connecting points, for example

$$X \xrightarrow{\;f\;} Y \tag{1.1.3}$$

is a morphism $f : X \to Y$. A concatenation of several morphisms means composition:

$$X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z \xrightarrow{\;h\;} W. \tag{1.1.4}$$

with the arc labeled $h \circ g \circ f$ over it.

The great help of diagrams is that through them one is able to write relations between morphisms in a quite concise and understandable way. This is done thanks to commuting diagrams: these are diagrams where two or more morphisms or sequences of morphisms go from a chosen object to another. The fact that the diagram commutes means that these sequences of morphisms are equal. For instance, the associativity axiom for the composition can be restated by requiring that, given any $f : X \to Y$, $g : X \to Z$ and $h : Z \to W$ morphisms in the category, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle g \circ f} \downarrow & {\scriptstyle g}\swarrow\;\nearrow & \downarrow {\scriptstyle h \circ g} \\ Z & \xrightarrow[h]{} & W \end{array} \tag{1.1.5}$$

commutes. Similar diagrams that encode the neutrality of the identity can be easily drawn: given any $f : X \to Y$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle 1_X}\downarrow & \searrow {\scriptstyle f} & \downarrow {\scriptstyle 1_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{1.1.6}$$

commutes. These examples may not be very illuminating on the utility of commuting diagrams, but we will meet later on diagrams whose transposition in usual written equalities would be terribly cumbersome and rather incomprehensible (see for example the pentagon identity (1.2.2)).

We turn now to some basic operations that can be done on categories:

Definition 2. *Let C be a category. Then the **opposite category** C^{op} is defined as the category which has the same objects and morphisms of C, where the target and the source of a morphism as well as the arguments of composition are exchanged.*

Definition 3. *Let C and D be two categories. The **product category** of C and D denoted $C \times D$ is the category whose objects are pairs of one object in C and one in D and whose morphisms are pairs of one morphism in C and one in D, where composition acts component-wise:*

$$(f, g) \circ (h, k) = (f \circ h, g \circ k), \tag{1.1.7}$$

$$i_{(X,Y)} = (i_X, i_Y). \tag{1.1.8}$$

We are now going to introduce other very important objects, i.e. the functors. These are just homomorphisms of categories, because they are map from one category to another preserving the structure and the properties of the composition of morphisms.

Definition 4. *Given two categories C and D a **Fuctor** F from C to D, also denoted $F : C \to D$, is a map that associates to every object X in C an object $F(X)$ in D and to every morphism $f : X \to Y$ in C a morphism $F(f) : F(X) \to F(Y)$ in D such that*

$$F(i_X) = i_{F(X)}, \tag{1.1.9}$$

$$F(g \circ f) = F(g) \circ F(f). \tag{1.1.10}$$

As maps in set theory can be divided into several kinds depending on how their image is close to their domain or their codomain, that is they can be injective, surjective or bijective, similar distinctions are defined for functors. Since functors really are couples of maps, namely a map between sets of objects and a map between sets of morphisms, injectivity and surjectivity can be defined in two different context. Actually, only injectivity or surjectivity at the level of morphisms is usually used, because functors find their interestingness on their action on morphisms:

Definition 5. *A functor $F : C \to D$ between locally small categories is said to be **faithful** if for every X, Y objects in C the map $F : Hom_C(X, Y) \to Hom_D(F(X), F(Y))$ is injective, is said **full** if the same map is surjective and it is **fully faithful** if it is both full and faithful.*

Analogous definitions can be made at the level of objects, leading to the definitions of an *injective-on-objects functor*, a *surjective-on-objects functor* and a *bijective-on-objects functor*, but these definition are not widely employed.

Another interesting definition concerning functors introduces contravariant functors:

Definition 6. *A functor $F : C \to D$ is also called a **covariant functor** from C to D. A functor $F : C^{op} \to D$ is called a **contravariant functor** from C to D.*

Strictly speaking, a contravariant functor $F : C \to D$ is *not* a functor from C to D, because it reverses the direction of morphisms and it doesn't meet the criteria of definition (4). Nevertheless, there are so many interesting examples of contravariant functors that they are usually just regarded as functors with a little abuse of language.

Functors can be obviously composed. Given two functors $F : C \to D$ and $G : D \to E$ their composition on objects is defined by $(G \circ F)(X) = G(F(X))$ and on morphisms by $(G \circ F)(f) = G(F(f))$. There is also an identity functor from a category to itself which sends every object and every morphism to itself. These two elements allow us to make the first example of a category through the following definition:

Definition 7. `Cat` *is the category whose objects are small categories and whose morphisms are functors between them.*

There are many other examples of categories:

- `Sets` is the category whose objects are sets and whose morphisms are functions. This is the inspiring example for all category theory.

- Every kind of algebraic structure on a set makes it possible to define a category with those algebraic sets as objects and homomorphisms between them as morphisms. Hence `Grp` is the category of all groups and group homomorphisms, `Vect`$_k$ is the category of vector spaces on the field k and linear maps, $R - $`Mod` is the category of modules over a ring R and module homomorphisms and so on.

- `Top` is the category of topological spaces and continuous maps, and every topological space X can be regarded as a category `Top`(X) whose objects are the open subsets and the morphisms are the inclusions.

- Geometric objects such as manifolds can also be gathered in a category. `SmoothMfld` is the category of smooth manifolds and smooth maps, `HolMfld` is the category of holomorphic manifolds and holomorphic maps, `Bund` is the category of fiber bundles and bundle maps, `LieGrp` is the category of Lie groups and smooth maps preserving the group structure.

- `Cmplx` is the category of differential complexes and chain maps.

Notably, these are not a small categories, because the collections of all small categories, all sets, all smooth manifolds and so on do not form sets but rather classes. This avoids logical problems of the kind of the Russell's paradox, since `Cat` can't be an object of itself.

Concerning functors, there are some examples:

- A category `C` is a **subcategory** of another category `D` if all objects and morphisms of `C` are also objects and morphisms of `D`. The *inclusion functor* $i : C \to D$ is the identity functor on `D` restricted to `C`.

- A presheaf of R-modules on X can be seen as a functor `Top`$(X) \to R - $`Mod`.

- The de Rham complex of differential forms is a contravariant functor from `SmoothMfld` to `Cmplx`.

- Given a small category `C` and X an object in `C`, taking the morphisms in `C` from X provides a functor from `C` to `Sets`. This is called the Hom-Functor. Namely, an object Y in `C` is mapped to $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ in `Sets`, and a morphism $f : Y \to Z$ in `C` is mapped to the morphisms $\mathrm{Hom}_{\mathsf{C}}(X, f) : \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{C}}(X, Z)$ that sends $g : X \to Y$ to $f \circ g : X \to Z$. A similar construction which takes morphisms "to X" instead of morphisms "from X" generates the contravariant Hom-Functor.

We come now to another central concept in category theory, that are natural transformations. As functors are morphisms between categories, natural transformations are morphisms between functors, in that they transform a functor between two fixed categories into another functor with the same target and source category.

Definition 8. *Given two categories C, D and two functors $F, G : \mathsf{C} \to \mathsf{D}$, a **natural transformation** from F to G, denoted $\eta : F \Rightarrow G$, is a map that associates to every object X of C a morphism in D $\eta(X) : F(X) \to G(X)$ such that, for every morphism $f : X \to Y$ in C we have that the following diagram*

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
{\scriptstyle \eta(X)}\downarrow & & \downarrow{\scriptstyle \eta(Y)} \\
G(X) & \xrightarrow[G(f)]{} & G(Y)
\end{array}
\tag{1.1.11}
$$

*commutes. The last property is also usually called naturality. A natural transformation η such that $\eta(X)$ is an isomorphism for every object X is said a **natural isomorphism**.*

Natural transformations can be composed in two different ways, horizontally and vertically. Let us define both of these operations:

Definition 9. *Let C, D be categories, $F, G, H : \mathsf{C} \to \mathsf{D}$ functors, $\eta : F \Rightarrow G$ and $\theta : G \Rightarrow H$ natural transformations. The **vertical composition** $\theta \cdot \eta : F \Rightarrow H$ of θ and η is the natural transformation defined by $(\theta \cdot \eta)(X) = \theta(X) \circ \eta(X) : F(X) \to H(X)$ for X object in C.*

Definition 10. *Let $\mathsf{C}, \mathsf{D}, \mathsf{E}$ be categories, $F, G : \mathsf{C} \to \mathsf{D}$ and $H, K : \mathsf{D} \to \mathsf{E}$ functors, $\eta : F \Rightarrow G$ and $\theta : H \Rightarrow K$ natural transformations. The **horizontal composition** $\theta \circ \eta : H \circ F \Rightarrow K \circ G$ of θ and η is the natural transformation defined by $(\theta \circ \eta)(X) = K(\eta(X)) \circ \theta(F(X)) = \theta(G(X)) \circ H(\eta(X)) : H(F(X)) \to K(G(X))$ for X and object in C*

Natural transformations provide a way to define a notion of equivalence in category theory, a generalization of the concept of isomorphism. One is tempted to extend the definition of isomorphism from set theory to category theory, saying that two categories `C` and `D` are isomorphic if there are functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F = 1_{\mathsf{C}}$ and $F \circ G = 1_{\mathsf{D}}$. Unluckily, this is a very restrictive definition, and in

category theory it is very difficult to find such two functors between similar categories. Instead, there is a weaker notion of isomorphisms that suits well for categories, the notion of equivalence, which rises when we relax the requirements $G \circ F = 1_C$ and $F \circ G = 1_D$.

Definition 11. *Let C and D be categories. C and D are said to be equivalent if there are functors $F : C \to D$ and $G : D \to C$ and natural isomorphisms $\epsilon : F \circ G \Rightarrow 1_D$ and $\eta : G \circ F \Rightarrow 1_C$.*

In this way, two categories are equivalent if they are connected by two functors whose compositions send objects to other objects which are isomorphic to the starting ones. This concept is very useful and widely employed in category theory. For instance, it is easy to see that it is an equivalence relation on categories: if C is equivalent to D and D to E, then C is equivalent to E, and every category is obviously equivalent to itself.

1.2   Monoidal categories

Tensor product over a field k between vector spaces provides $\texttt{Vect}_k$ with a product that composes two objects to obtain a new object. This finds a generalization in the monoidal categories, which are categories equipped with some notion of tensor product.

Definition 12. *A **monoidal category** C is a category together with a functor $\otimes : C \times C \to C$ called product, an object 1 called unit object, natural isomorphisms $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, $l_x : 1 \otimes x \to x$ and $r_x : x \otimes 1 \to x$ called the associator, the left and the right unitors, such that the following diagrams commute:*

$$
\begin{array}{ccc}
(x \otimes 1) \otimes y & \xrightarrow{\ a_{x,1,y}\ } & x \otimes (1 \otimes y) \\
& & \\
{\scriptstyle r_x \otimes 1_y} \searrow & & \swarrow {\scriptstyle 1_x \otimes l_y} \\
& x \otimes y &
\end{array}
\tag{1.2.1}
$$

$$
\begin{array}{ccccc}
& & (x \otimes y) \otimes (z \otimes w) & & \\
& {\scriptstyle a_{x \otimes y, z, w}} \nearrow & & \searrow {\scriptstyle a_{x, y, z \otimes w}} & \\
((x \otimes y) \otimes z) \otimes w & & & & x \otimes (y \otimes (z \otimes w)) \\
{\scriptstyle a_{x,y,z} \circ 1_w} \searrow & & & & \nearrow {\scriptstyle 1_x \circ a_{y,z,w}} \\
& (x \otimes (y \otimes z)) \otimes w & \xrightarrow{\ a_{x, y \otimes z, w}\ } & x \otimes ((y \otimes z) \otimes w) &
\end{array}
\tag{1.2.2}
$$

A monoidal category can also be denoted as $(C, \otimes, 1, a, l, r)$ to clarify the notation of all the monoidal structure.

In this section we will use small letters to denote objects for convenience.

The aim of the natural isomorphisms $a_{x,y,z}$, l_x and r_x is to relax the usual properties of a tensor product. Here the neutrality of the identity object 1 and the associativity of the product $\otimes$ are asked to hold on objects only up to invertible morphisms.

The two axioms are coherence requirements for the unitors and the associators. Relation (1.2.2) is called the pentagon equation. It is possible to prove a coherence theorem that states that if the axioms of a monoidal category are satisfied then every diagram made up with associators, unitors and identity morphisms commutes [25].

There is also a notion of monoidal category in which the tensor product is associative and the unit object is a true unit with respect to the tensor product.

Definition 13. *A **strict monoidal category** $(C, \otimes, 1)$ is a monoidal category where the associator and the unitors are identity morphisms. A monoidal category which is not strict is called **weak**.*

This distinction between weak and strict categories is very common in category theory, and we shall meet it again later.

Functors that respect the monoidal structure of two monoidal categories are called monoidal functors. In the following definition we assume that the target and source categories are strict. Everything applies to weak monoidal categories with few more requirements. We refer the interested reader to [26].

Definition 14. *Let $(C, \otimes_C, 1_C)$ and $(D, \otimes_D, 1_D)$ be strict monoidal categories. A **lax monoidal functor** from C to D is a functor $F : C \to D$ equipped with a natural transformation $\eta_{x,y} : F(x) \otimes_D F(y) \to F(x \otimes_C y)$ and a morphism in D $\phi : 1_D \to F(1_C)$ such that the following diagrams:*

$$
\begin{array}{ccc}
F(x) \otimes_D F(y) \otimes_D F(z) & \xrightarrow{\;1_{F(x)} \otimes_D \eta_{y,z}\;} & F(x) \otimes_D F(y \otimes_C z) \\
{\scriptstyle \eta_{x,y} \otimes_D 1_{F(z)}} \downarrow & & \downarrow {\scriptstyle \eta_{x,y \otimes_C z}} \\
F(x \otimes_C y) \otimes_D F(z) & \xrightarrow[\eta_{x \otimes_C y,z}]{} & F(x \otimes_C y \otimes_C z)
\end{array}
\qquad (1.2.3)
$$

$$
\begin{array}{ccc}
1_D \otimes_D F(x) & \xrightarrow{\;l_{F(x)}\;} & F(x) \\
{\scriptstyle \phi \otimes_D 1_{F(x)}} \downarrow & & \uparrow {\scriptstyle F(l_x)} \\
F(1_C) \otimes_D F(x) & \xrightarrow[\eta_{1_C,x}]{} & F(1_C \otimes_D x)
\end{array}
\qquad (1.2.4)
$$

*and an analogous diagram for r_x commute. If $\eta_{x,y}$ and ϕ are a natural isomorphism and an isomorphism then F is called a **strong monoidal functor**. If they are identities then F is a **strict monoidal functor**.*

There is also a notion of monoidal natural transformations between monoidal functors:

Definition 15. *Let $(C, \otimes_C, 1_C)$ and $(D, \otimes_D, 1_D)$ be strict monoidal categories and $(F, \eta, \phi), (G, \xi, \psi)$ lax monoidal functors from C to D. A **monoidal natural transformation** $\alpha : F \Rightarrow$*

G is a natural transformation from F to G such that the diagrams

$$\begin{array}{ccc} F(x) \otimes_D F(y) & \xrightarrow{\alpha(x) \otimes_D \alpha(y)} & G(x) \otimes_D G(y) \\ {\scriptstyle \eta_{x,y}} \downarrow & & \downarrow {\scriptstyle \xi_{x,y}} \\ F(x \otimes_C y) & \xrightarrow[\alpha(x \otimes_C y)]{} & G(x \otimes_C y) \end{array} \qquad (1.2.5)$$

$$\begin{array}{ccc} & 1_D & \\ {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \psi} \\ F(1_C) & \xrightarrow[\alpha(1_C)]{} & G(1_C) \end{array} \qquad (1.2.6)$$

commute.

1.3   Higher Categories

As categories encode the general idea of having objects and arrows connecting them, like sets and functions, higher categories generalize this concept by adding the notion of higher morphisms which connect morphisms that are one step below. So we have objects, morphisms between objects, 2-morphisms between morphisms, 3-morphisms between 2-morphisms and so on. There is number of good references for higher category theory, among which we mention [27] and the more advanced [28]. Other introductions to higher category theory which use a notation very similar to the one adopted here can be found in the appendixes of [39] and [40].

In this chapter we will be only interested in 2-categories. As said, 2-categories add to usual categories the notion of morphisms between morphisms, or 2-morphisms. In diagrams these are usually drawn as doubled arrows, for example the diagram

$$X \underset{g}{\overset{f}{\Rrightarrow\alpha}} Y \qquad (1.3.1)$$

represents a 2-morphisms α going from $f : X \to Y$ to $g : X \to Y$.

Analogously to what happens for monoidal categories, 2-Categories admit two definition, depending on how we want to generalize the axioms for ordinary categories. We may want to extend them straightforwardly as they stand, just adding rules for 2-morphism, and we would get what is called a strict 2-category. Otherwise we may think of a 2-morphism as a homotopy between morphisms, and we can ask that objects and ordinary morphisms respect the axioms of a category only up to these higher homotopies. In this case we get a weak or non-strict 2-category.

Definition 16. *A **weak 2-category** C consists of a set of objects; for every two objects X, Y a set of 1-morphisms $1 - Hom_C(X, Y)$, where given a 1-morphism f we denote $f : X \to Y$; for every two 1-morphisms f, g with the same source and target a set of 2-morphisms $2 - Hom_C(f, g)$, where given a 2-morphism α we denote $\alpha : f \Rightarrow g$.*

For every couple of 1-morphisms $f : X \to Y$, $g : Y \to Z$ there is a 1-morphism $g \circ f : X \to Z$ called the composition of f and g. For every couple of 2-morphisms

$\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, for $f, g, h : X \to Y$, there is a 2-morphism $\beta \bullet \alpha : f \Rightarrow h$ called the vertical composition of α and β. For every couple of 2-morphisms $\alpha : f \Rightarrow g$, $\beta : h \Rightarrow k$, such that $f, g : X \to Y$, $h, k : Y \to Z$, there is a 2-morphism $\beta \circ \alpha : h \circ f \Rightarrow k \circ g$ called the horizontal composition of α and β.

Furthermore, for every object X there is an identity 1-morphism $1_X : X \to X$. For every triple of 1-morphisms $f : X \to Y, g : Y \to Z, h : Z \to W$ there is an invertible 2-morphism $a_{h,g,f} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$. For every 1-morphism $f : X \to Y$, there are three invertible 2-morphisms $l_f : f \circ i_X \Rightarrow f$, $r_f : 1_Y \circ f \Rightarrow f$ called left and right unifiers and $1_f : f \Rightarrow f$.

These data must satisfy the following axioms:

1. $\gamma \bullet (\beta \bullet \alpha) = (\gamma \bullet \beta) \bullet \alpha$ whenever possible.

2. $\alpha \bullet 1_f = 1_g \bullet \alpha = \alpha$ and $1_f \circ 1_g = 1_{f \circ g}$ whenever possible.

3. $(1_g \circ r_f) \bullet a_{g,1_Y,f} = l_g \circ 1_f$ whenever possible, or equivalently the following diagram of 2-morphisms commutes:

$$
\begin{array}{ccc}
(g \circ 1_Y) \circ f & \xrightarrow{\ a_{g,1_Y,f}\ } & g \circ (1_Y \circ f) \\
& \searrow_{l_g \circ 1_f} \qquad \swarrow_{1_g \circ r_f} & \\
& g \circ f &
\end{array}
\tag{1.3.2}
$$

4. $(\delta \circ \gamma) \bullet (\beta \circ \alpha) = (\delta \bullet \beta) \circ (\gamma \bullet \alpha)$ whenever possible.

5. The associator 2-morphisms satisfy the pentagon identity, that is the following diagram of 2-morphisms

$$
\begin{array}{ccccc}
& & (k \circ h) \circ (g \circ f) & & \\
& \overset{a_{k \circ h, g, f}}{\nearrow} & & \overset{a_{k,h,g \circ f}}{\searrow} & \\
((k \circ h) \circ g) \circ f & & & & k \circ (h \circ (g \circ f)) \\
\big\downarrow_{a_{k,h,g} \circ 1_f} & & & \nearrow_{1_k \circ a_{h,g,f}} & \\
(k \circ (h \circ g)) \circ f & \xrightarrow{\ a_{k,h \circ g, f}\ } & k \circ ((h \circ g) \circ f) & &
\end{array}
\tag{1.3.3}
$$

commutes.

Sometimes the 1-morphisms and the 2-morphisms are called 1-cells and 2-cells respectively, and the objects are called 0-cells.

Let us clarify the meaning of these axioms. Axioms 1 and 2 just say that vertical composition of 2-morphisms is associative and that the 2-morphism 1_f is an identity under the composition of 2-morphisms, as happens in ordinary categories for morphisms and 1_X. Axiom 3 is a coherence relation between unifiers and associators, and it says that there is only one 2-morphism that plays the role of the unifier for a sequence of several 1-morphisms containing identity 1-morphisms, independent of the

order according to which these identities are absorbed if the order of the remaining compositions is preserved. Axiom 4 is known as the *exchange law*, and it states that the two different ways of composing the following four 2-morphisms:

$$X \xrightarrow{\ \ g\ \ } Y \xrightarrow{\ \ l\ \ } Z \tag{1.3.4}$$

namely, first composing horizontally $\beta \circ \alpha$ and $\delta \circ \gamma$ and then composing vertically the two resulting 2-morphisms, or first composing vertically $\gamma \bullet \alpha$ and $\delta \bullet \beta$ and then composing horizontally the two resulting 2-morphisms, obtaining a 2-morphisms $k \circ f \Rightarrow m \circ h$, lead to the same result. Thanks to this, it is possible to paste together diagrams, where the surfaces enclosed by some arrows are understood as the 2-morphisms connecting the 1-morphisms that edge it. Axiom 5 is a coherence relation similar to (1.2.2), and it assures that any diagram of 2-morphisms made up with associators and unitors commutes.

Notice that in diagrams (1.3.2) and (1.3.3) the vertexes are morphisms instead of objects and the arrows are 2-morphisms instead of ordinary morphisms. Nevertheless, these diagrams are treated exactly as the diagrams that we saw earlier with objects and morphisms, i.e. they simply picture equalities of 2-morphisms. This similitude hints the following definition:

Definition 17. *Let C be a 2-category. Then, for every couple of objects X, Y in C there is a small category $C(X, Y)$ called the **Hom-category** of X and Y, whose set of objects is $1 - Hom_C(X, Y)$ and whose morphisms are the 2-morphisms of C with target and source in $1 - Hom_C(X, Y)$. Composition of morphisms in $C(X, Y)$ is the vertical composition of 2-morphisms in C.*

It is simple to see that the axiom of a category are satisfied by $C(X, Y)$. Making a comparison with ordinary categories, we may say that if a locally small category is *enriched over sets*, i.e. the collection of arrows between two chosen objects is a set, any 2-category is a *category enriched over categories*, where all kinds of morphisms between any two objects define a category instead of a set.

In weak 2-categories it is useful to introduce the notion of weak inverse for 1-morphisms. Since everything at the level of 1-morphisms is expected to hold only up to 2-morphisms, it makes sense to define a weaker notion of invertibility than the one usually employed.

Definition 18. *Let $f : X \to Y$ be a 1-morphism in a weak 2-category C. A **weak inverse** for f is a 1-morphism $\bar{f} : Y \to X$ in C such that there are 2-isomorphisms $f \circ \bar{f} \Rightarrow 1_Y$ and $\bar{f} \circ f \Rightarrow 1_X$. An 1-morphism which has a weak inverse is also called an **equivalence***

As mentioned earlier, there is also a notion of strict 2-category, where the axioms of an ordinary category are extended to 2-morphisms without being relaxed for 1-morphisms.

Definition 19. *a* **strict 2-category** $\mathcal{C}$ *is a weak 2-category whose associator and unifier 2-morphisms are the identity 2-morphisms.*

This distinction between *strict* and *weak* definitions is very common in category theory, and indeed it is the same difference that we encountered in the last section between weak monoidal categories and strict monoidal categories. The main difference between weak and strict objects (monoidal categories, 2-categories etc.) is that the latter are much easier to handle and study, while the former enclose a much wider and more useful list of examples and cases of interest.

The first example of a strict 2-category comes from the previous sections: we call `2-Cat` the 2-category having small categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms. This is clearly strict, since composition of functors is associative and the identity functor is a true unit under this composition. This category is the extension of `Cat` obtained by adding natural transformations as 2-morphisms. This 2-category also allows us to show another example of category: given two categories `C` and `D`, `Fun(C,D)` is the category having functors from `C` to `D` as objects and natural transformations as morphisms. This is the Hom-category of `2-Cat`.

Definition (17) makes it possible to show another example of 2-category which illustrates a construction that will be used again later on. Let `C` be a monoidal category. Then $B\texttt{C}$ is the 2-category which has only one object, the objects of `C` as 1-morphisms and the morphisms of `C` as 2-morphisms. Horizontal composition of both 1- and 2-morphisms is given by the product $\otimes$ in `C`, while vertical composition of 2-morphisms is the usual composition of morphisms in `C`. Remarkably, $B\texttt{C}$ is strict as a 2-category if and only if `C` is strict as a monoidal category.

Conversely, it can be shown that the Hom-category of a 2-category inherits a monoidal structure from the horizontal composition on 1-morphisms of the original 2-category. The monoidal structure is strict if and only if the 2-category is strict.

Since in what follows we will be interested only in strict categories, by now we will outline other concepts related to 2-categories, such as 2-functors, only in the strict case. Nevertheless, this is not the most general framework, and many of the definitions and results that we will explain have a non trivial generalization for weak 2-categories.

As functors are maps between categories, 2-functors are maps between 2-categories. Even in this case there is a distinction between weak and strict 2-functors.

Definition 20. *Let $\mathcal{C}$ and $\mathcal{D}$ be two (strict) 2-categories. A* **weak 2-functor** *F from $\mathcal{C}$ to $\mathcal{D}$, also denoted $F : \mathcal{C} \to \mathcal{D}$, is a map that associates to every object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, to every 1-morphism f in $\mathcal{C}$ a 1-morphisms $F(f)$ in $\mathcal{D}$ and to every 2-morphism α in $\mathcal{C}$ a 2-morphism $F(\alpha)$ in $\mathcal{D}$, together with a 2-isomorphism $u_X : F(1_X) \Rightarrow 1_{F(X)}$ in $\mathcal{D}$ for every object X in $\mathcal{C}$, and a 2-isomorphism $m_{f,g} : F(f) \circ F(g) \Rightarrow F(f \circ g)$ in $\mathcal{D}$ for every couple of composable morphisms in $\mathcal{C}$, such that the following axioms are satisfied:*

   1. $F(1_f) = 1_{F(f)}$ and $F(\alpha \bullet \beta) = F(\alpha) \bullet F(\beta)$.

2. For any $\alpha : f \Rightarrow h$ and $\beta : g \Rightarrow k$ in C, the following diagram commutes:

$$
\begin{array}{ccc}
F(f) \circ F(g) & \xrightarrow{\ \ F(\alpha) \circ F(\beta)\ \ } & F(h) \circ F(k) \\
\Big\Vert{\scriptstyle m_{f,g}} & & \Big\Vert{\scriptstyle m_{h,k}} \\
F(f \circ g) & \xrightarrow[\ \ F(\alpha \circ \beta)\ \ ]{} & F(h \circ k)
\end{array}
\qquad (1.3.5)
$$

3. The following diagram commutes:

$$
\begin{array}{ccc}
F(f) \circ F(g) \circ F(h) & \xrightarrow{\ \ 1_{F(f)} \circ m_{g,h}\ \ } & F(f) \circ F(g \circ h) \\
\Big\Vert{\scriptstyle m_{f,g} \circ 1_{F(h)}} & & \Big\Vert{\scriptstyle m_{f,g\circ h}} \\
F(f \circ g) \circ F(h) & \xrightarrow[\ \ m_{f\circ g,h}\ \ ]{} & F(f \circ g \circ h)
\end{array}
\qquad (1.3.6)
$$

4. the following relations are satisfied:

$$
m_{f,1_X} = 1_{F(f)} \circ u_X \ \text{ and } \ m_{1_Y,f} = u_Y \circ 1_{F(f)}.
\qquad (1.3.7)
$$

Definition 21. *A 2-functor $F : C \to D$ is called **strict** if u_X and $m_{f,g}$ are identity 2-morphisms for every X object in C and for every f, g composable 1-morphisms in C.*

Notice that, as mentioned earlier, both these definitions employ strict 2-categories. Only weak 2-functors can be defined for weak 2-categories, while there is no sensible definition of a strict 2-functor between weak 2-categories. There is also a more general notion of 2 functors that is a *lax 2-functor*. This is a weak 2-functor whose u_X and $m_{f,g}$ need not to be natural isomorphisms, but we will not deal with these objects.

Now we define the generalization of natural transformations for 2-categories.

Definition 22. *Let C and D be (strict) 2-categories, and $F, G : C \to D$ 2-functors. A **pseudonatural transformation** η from F to G, also denoted $\eta : F \Rightarrow G$, associates to every object X in C a 1-morphism $\eta(X) : F(X) \to G(X)$ in D and to every 1-morphism $f : X \to Y$ in C a 2-isomorphism $\eta(f) : \eta(Y) \circ F(f) \Rightarrow G(f) \circ \eta(X)$, such that the following axioms are satisfied:*

1. For every $f : X \to Y$ and $g : Y \to Z$ in C, the following diagram commutes:

$$
\begin{array}{ccc}
\eta(Z) \circ F(g) \circ F(f) & \xrightarrow{\ \ 1_{\eta(Z)} \circ m^{(F)}_{g,f}\ \ } & \eta(Z) \circ F(g \circ f) \\
\Big\Vert{\scriptstyle \eta(g) \circ 1_{F(f)}} & & \\
G(g) \circ \eta(Y) \circ F(f) & & \Big\Vert{\scriptstyle \eta(g\circ f)} \\
\Big\Vert{\scriptstyle 1_{G(g)} \circ \eta(f)} & & \\
G(g) \circ G(f) \circ \eta(X) & \xrightarrow[\ \ m^{(G)}_{g,f} \circ 1_{\eta(X)}\ \ ]{} & G(g \circ f) \circ \eta(X)
\end{array}
\qquad (1.3.8)
$$

where $m^{(F)}$ and $m^{(G)}$ are respectively the 2-morphisms for the composition of F and G.

2. *For every 2-morphism $\alpha : f \to g$ in C the following diagram commutes:*

$$\eta(Y) \circ F(f) \overset{\eta(f)}{\Longrightarrow} G(f) \circ \eta(X) \tag{1.3.9}$$
$$1_{\eta(Y)} \circ F(\alpha) \Big\Downarrow \qquad\qquad \Big\Downarrow G(\alpha) \circ 1_{\eta(X)}$$
$$\eta(Y) \circ F(g) \underset{\eta(g)}{\Longrightarrow} G(g) \circ \eta(X)$$

This definition is closely related to the definition of natural transformations. The assignment $f \to \eta(f)$ is best understood in the following diagram:

$$F(X) \overset{F(f)}{\longrightarrow} F(Y) \tag{1.3.10}$$
$$\eta(X) \Big\downarrow \quad \overset{\eta(f)}{\nearrow} \quad \Big\downarrow \eta(Y)$$
$$G(X) \underset{G(f)}{\longrightarrow} G(Y)$$

The commutativity of this diagram at the level of 1-morphisms is the naturality property of natural transformation. Here it is relaxed so that it holds only up to an isomorphism. Furthermore, naturality strictly holds at the level of 2-morphisms, thanks to axiom (2).

There is another definition which is needed and which relates two pseudonatural transformations:

Definition 23. *Let C and D be (strict) 2-categories, $F, G : C \to D$ 2-functors and $\eta, \xi : F \Rightarrow G$ pseudonatural transformations. A **modification** $\mathcal{M}$ from η to ξ, also denoted $\mathcal{M} : \eta \Rightarrow \xi$, is a map that assigns to every object X in C a 2-morphism $\mathcal{M}(X) : \eta(X) \Rightarrow \xi(X)$ in D such that the following diagram*

$$\eta(Y) \circ F(f) \overset{\eta(f)}{\Longrightarrow} G(f) \circ \eta(X) \tag{1.3.11}$$
$$\mathcal{M}(Y) \circ 1_{F(f)} \Big\Downarrow \qquad\qquad \Big\Downarrow 1_{G(f)} \circ \mathcal{M}(X)$$
$$\xi(Y) \circ F(f) \underset{\xi(f)}{\Longrightarrow} G(f) \circ \xi(X)$$

commutes.

All these objects admit several composition rules.

Definition 24. *Let C, D and E be strict 2-categories and $F : C \to D$ and $G : D \to E$ be weak 2-functors. The **horizontal composition** of G and F is the weak 2-functor $G \circ F : C \to E$ that sends every object X in C to $G(F(X))$ in E, every 1-morphism f in $catC$ to $G(F(f))$ in $catE$ and every 2-morphism α in C to $G(F(\alpha))$ in E, and whose natural isomorphisms for composition and identities are given by*

$$u_X^{G \circ F} = u_{F(X)}^G \bullet G(u_X^F) \tag{1.3.12}$$

$$m_{f,g}^{G \circ F} = G(m_{f,g}^F) \bullet m_{F(f),F(g)}^G \tag{1.3.13}$$

The restriction of this composition to strict 2-functor is obvious. This composition is also associative thanks to the associativity of vertical composition of 2-morphisms in a 2-category.

Regarding pseudonatural transformations, they can be composed both horizontally and vertically, as happens for ordinary natural transformations.

Definition 25. *Let C and D be strict 2-categories, $F, G, H : C \to D$ be 2-functors and $\eta : F \Rightarrow G$ and $\xi : G \Rightarrow H$ pseudonatural transformations. The **vertical composition** of ξ and η is a pseudonatural transformation $\xi \bullet \eta : F \to H$ which sends every object X in C to the 1-morphism $(\xi \bullet \eta)(X) = \xi(X) \circ \eta(X) : F(X) \to H(X)$ in D and every 1-morphism $f : X \to Y$ in C to the 2-isomorphism*

$$(\xi \bullet \eta)(f) = (\xi(f) \circ 1_{\eta(X)}) \bullet (1_{\xi(Y)} \circ \eta(f)) : (\xi \bullet \eta)(Y) \circ F(f) \Rightarrow H(f) \circ (\xi \bullet \eta)(X) \quad (1.3.14)$$

in D.

Horizontal composition of pseudonatural transformations is more delicate to define. Recall that while defining horizontal composition of natural transformations (see definition (10)) there are two ways to construct the morphism in the target category that links the two functors. With ordinary categories these two ways are equivalent, but in higher categories this is not the case, due to the fact that naturality does not hold strictly at the level of 1-morphism. Namely, suppose that we have $\eta : F \Rightarrow G$ and $\xi : H \Rightarrow K$ pseudonatural transformations for $F, G : C \to D$ and $H, K : D \to E$ 2-functors, and we want to define a pseudonatural transformation $(\xi \circ \eta) : H \circ F \Rightarrow K \circ G$. We must look for a map that sends an object X in C to a 1-morphism $(\xi \circ \eta)(X) : H(F(X)) \to K(G(X))$ in E, but this can be naturally done in two different ways: $\xi(G(X)) \circ H(\eta(X))$ and $K(\eta(X)) \circ \xi(F(X))$. The result is that we have *two* possible horizontal compositions for pseudnatural transformations. We thus get to the following definition:

Definition 26. *Let C, D and E be strict 2-categories, $F, G : C \to D$ and $H, K : D \to E$ 2-functors and $\eta : F \Rightarrow H$ and $\xi : H \Rightarrow K$ pseudonatural transformations. There are two **horizontal compositions** of ξ and η, pseudonatural transformations $(\xi \circ \eta)_i : H \circ F \Rightarrow K \circ G$ for $i = 0, 1$, given by*

$$(\xi \circ \eta)_0(X) = \xi(G(X)) \circ H(\eta(X)) \quad (1.3.15)$$

$$(\xi \circ \eta)_0(f) = (\xi(G(f)) \circ 1_{H(\eta(X))}) \bullet \left[1_{\xi(G(Y))} \circ \left(m^H_{G(f),\eta(X)}{}^{-1} \bullet H(\eta(f)) \bullet m^H_{\eta(f),F(f)} \right) \right] \quad (1.3.16)$$

$$(\xi \circ \eta)_1(X) = K(\eta(X)) \circ \xi(F(X)) \quad (1.3.17)$$

$$(\xi \circ \eta)_1(f) = \left[\left(m^K_{G(f),\eta(X)}{}^{-1} \bullet K(\eta(f)) \bullet m^K_{\eta(Y),F(f)} \right) \circ 1_{\xi(F(X))} \right] \bullet (1_{K(\eta(Y))} \circ \xi(F(f))) \quad (1.3.18)$$

These two pseudonatural transformations are not completely unrelated, as they differ only up to a 2-isomorphisms in the target category. Namely we have the following result:

Proposition 1. *There is an invertible modification* $\tau : (\xi \circ \eta)_0 \Rightarrow (\xi \circ \eta)_1$ *given by* $\tau(X) = \xi(\eta(X))$.

Modifications can also be composed in various manners, but we will not see them.

1.4 Double categories

This section is taken from the appendix of [18]. Here we present the basic notions and results of double category theory, which is required by our cocycle based formulation of parallel transport theory (see chapter 4). Most of the material is not original and is included to help the reader. (See for instance [34].) However, to the best of our knowledge, the notions of double natural transformation and modification we present and use in the main body of the paper are original. We also define the plane rectangle double groupoid playing an essential role in our construction and recall the definition of the edge symmetric double groupoid of a crossed module for its relevance.

1.4.1 Double categories

Double categories are categories internal to the category of categories [35]. They are however more conveniently defined as follows.

Definition 27. *A **double category** D consists of the following elements*

1. A set of objects a, b, c, $\ldots$.

2. For each pair of objects a, b a set of horizontal and vertical arrows,

$$b \overset{x}{\longleftarrow} a \qquad \qquad \begin{array}{c} b \\ \uparrow x \\ a \end{array} \tag{1.4.1}$$

3. For each quadruple of objects a, b, c, d, pair of horizontal arrows $b \overset{y}{\longleftarrow} a$, $d \overset{u}{\longleftarrow} c$ and pair of vertical arrows $c \overset{x}{\longleftarrow} a$, $d \overset{v}{\longleftarrow} b$ (here written horizontally for convenience), a set of arrow squares

$$\begin{array}{ccc} d & \overset{u}{\longleftarrow} & c \\ v \uparrow & X \swarrow & \uparrow x \\ b & \overset{}{\longleftarrow} & a \\ & y & \end{array} \tag{1.4.2}$$

Objects and horizontal arrows form an ordinary category with composition $\circ_h$ and identity assigning map id_h. Similarly, objects and vertical arrows form a category with composition $\circ_v$ and identity assigning map id_v. Furthermore, arrow squares can be

composed both horizontally and vertically compatibly with the composition of horizontal and vertical arrows,

$$
\begin{array}{ccc}
f \xleftarrow{\ v\ } e \xleftarrow{\ u\ } d & & f \xleftarrow{v\circ_h u} d \\
t\uparrow\ Y\!\!\diagup\ \ s\uparrow\ X\!\!\diagup\ \ \uparrow r & = & t\uparrow\ \ Y\circ_h X\!\!\diagup\ \ \uparrow r \\
c \xleftarrow{\ y\ } b \xleftarrow{\ x\ } a & & c \xleftarrow{y\circ_h x} a
\end{array}
\tag{1.4.3}
$$

$$
\begin{array}{ccc}
f \xleftarrow{\ z\ } e & & f \xleftarrow{\ z\ } e \\
t\uparrow\ Y\!\!\diagup\ \uparrow r & = & t\circ_v s\uparrow\ \ Y\circ_v X\!\!\diagup\ \ \uparrow r\circ_v q \\
d \xleftarrow{\ y\ } c & & b \xleftarrow{\ x\ } a \\
s\uparrow\ X\!\!\diagup\ \uparrow q & & \\
b \xleftarrow{\ x\ } a & &
\end{array}
$$

Compatible horizontal and vertical identity arrow squares are also defined,

$$
\begin{array}{cc}
b \xleftarrow{\mathrm{id}_{hb}} b & b \xleftarrow{\ x\ } a \\
x\uparrow\ \mathrm{Id}_{hx}\!\!\diagup\ \uparrow x & \mathrm{id}_{vb}\uparrow\ \mathrm{Id}_{vx}\!\!\diagup\ \uparrow \mathrm{id}_{va} \\
a \xleftarrow{\mathrm{id}_{ha}} a & b \xleftarrow{\ x\ } a
\end{array}
\tag{1.4.4}
$$

Vertical arrows and arrow squares connecting them form an ordinary category with composition $\circ_h$ and identity assigning map Id_h. Similarly, horizontal arrows and arrow squares form a category with composition $\circ_v$ and identity assigning map Id_v. Finally the exchange law holds, which means that the result of the composition of the four arrow squares of the form

$$
\begin{array}{ccc}
i \xleftarrow{\ v\ } h \xleftarrow{\ u\ } g \\
s\uparrow\ U\!\!\diagup\ q\uparrow\ Z\!\!\diagup\ \uparrow n \\
f \xleftarrow{\ w\ } e \xleftarrow{\ z\ } d \\
r\uparrow\ Y\!\!\diagup\ p\uparrow\ X\!\!\diagup\ \uparrow m \\
c \xleftarrow{\ y\ } b \xleftarrow{\ x\ } a
\end{array}
\tag{1.4.5}
$$

does not depend on whether the horizontal composition of the bottom and top pairs of squares or the vertical composition of the right and left pairs of squares is performed first.

The transpose of a double category D, which switches the vertical and horizontal arrows, is again a double category TD.

Definition 28. *A **double groupoid** D is a double category in which the horizontal and vertical arrow categories are groupoid with inverse operations $^{-1_h}$, $^{-1_v}$, respectively, and each arrow square has an horizontal and vertical inverse compatible with the arrow inversions*

$$
\begin{array}{cc}
c \xleftarrow{u^{-1_h}} d & b \xleftarrow{\ y\ } a \\
x\uparrow\ X^{-1_h}\!\!\diagup\ \uparrow v & v^{-1_v}\uparrow\ X^{-1_v}\!\!\diagup\ \uparrow x^{-1_v} \\
a \xleftarrow{y^{-1_h}} b & d \xleftarrow{\ u\ } c
\end{array}
\tag{1.4.6}
$$

1.4.2 Double functors

Double functors are structure preserving maps of double categories.
Let D, E be double categories.

Definition 29. *A **double functor** $F : D \to E$ consists of the following elements*

1. *A mapping $a \longmapsto F(a)$ of the set of objects of D into that of E.*

2. *Mappings*

$$
b \xleftarrow{\;x\;} a \qquad \longmapsto \qquad F(b) \xleftarrow{\;F(x)\;} F(a) \qquad\qquad
\begin{array}{c} b \\ \uparrow x \\ a \end{array}
\qquad \longmapsto \qquad
\begin{array}{c} F(b) \\ \uparrow F(x) \\ F(a) \end{array}
\qquad (1.4.7)
$$

of the sets of horizontal and vertical arrows of D into those of E, respectively, compatible with the mapping of objects.

3. *A mapping*

$$
\begin{array}{ccc}
d & \xleftarrow{\;u\;} & c \\
{\scriptstyle v}\uparrow & {\scriptstyle X}\!\!\diagup\!\!\diagup & \uparrow{\scriptstyle x} \\
b & \xleftarrow[y]{} & a
\end{array}
\qquad \longmapsto \qquad
\begin{array}{ccc}
F(d) & \xleftarrow{\;F(u)\;} & F(c) \\
{\scriptstyle F(v)}\uparrow & {\scriptstyle F(X)}\!\!\diagup\!\!\diagup & \uparrow{\scriptstyle F(x)} \\
F(b) & \xleftarrow[F(y)]{} & F(a)
\end{array}
\qquad (1.4.8)
$$

of the set of arrow squares of D into that of E compatible with the mappings of objects and arrows.

These mappings are required to preserve all types of compositions and units.

Let D, E be double groupoids.

Definition 30. *A **double groupoid functor** $F : D \to E$ is a double category functor that preserves all types of inverses.*

Proposition 2. *Small double categories and double functors with the obvious composition and identity assigning map constitute a category. Small double groupoids and double functors form a full subcategory of it.*

1.4.3 Edge 2–categories of double categories

Edge categories are 2–categories canonically associated with double categories playing an important role in many double categorical constructions.

Proposition 3. *With a double category D there are associated two strict 2–categories HD and VD, called edge 2–categories of D.*
The 2–category HD is defined as follows.

1. *The 0–cells of HD are the objects of D.*

2. *The 1–cells of HD are the horizontal arrows of D.*

3. *The 2–cells of HD are the arrow squares of D of the form*

$$\begin{array}{ccc} b\;\overset{y}{\overset{\curvearrowleft}{\underset{x}{X\Downarrow}}}\;a & \equiv & \begin{array}{ccc} b & \overset{y}{\longleftarrow} & a \\ \mathrm{id}_{vb}\uparrow & X\!\!\swarrow & \uparrow\mathrm{id}_{va} \\ b & \underset{x}{\longleftarrow} & a \end{array} \end{array} \qquad (1.4.9)$$

The composition of two 1–cells of HD is the composition of the corresponding horizontal arrows of D. The identity 1–cells of HD are the horizontal identity arrows of D. The horizontal composition of two 2–cells of HD is the horizontal composition of the corresponding arrow squares of D. The vertical composition of two 2–cells of HD is the vertical composition of the corresponding arrow squares of D. The unit 2–cells of HD are the vertical unit squares of D.

The 2–category VD is defined as follows.

1. The 0–cells of VD are the objects of D.

2. The 1–cells of VD are the vertical arrows of D.

3. The 2–cells of VD are the arrow squares of D of the form

$$\begin{array}{ccc} b\;\overset{x}{\overset{\curvearrowleft}{\underset{y}{X\Downarrow}}}\;a & \equiv & \begin{array}{ccc} b & \overset{\mathrm{id}_{hb}}{\longleftarrow} & b \\ y\uparrow & X\!\!\swarrow & \uparrow x \\ a & \underset{\mathrm{id}_{ha}}{\longleftarrow} & a \end{array} \end{array} \qquad (1.4.10)$$

The composition of two 1–cells of VD is the composition of the corresponding vertical arrows of D. The identity 1–cells of VD are the vertical identity arrows of D. The horizontal composition of two 2–cells of VD is the vertical composition of the corresponding arrow squares of D. The vertical composition of two 2–cells of VD is the horizontal composition of the corresponding arrow squares of D. The unit 2–cells of VD are the horizontal unit squares of D.

We denote by HD_0 and VD_0 the ordinary categories underlying HD and VD. HD_0 is the category whose 0– and 1– cells are the objects and horizontal arrows of D with the composition $\circ_h$ and identity assigning map id_h inherited from D. Similarly, VD_0 is the category whose 0– and 1 cells are the objects and vertical arrows of D with the composition $\circ_h$ and identity assigning map id_h inherited from D.

Proposition 4. *If D is a double groupoid, then HD and VD are 2–groupoids.*
The inverse of a 1–cell of HD is the inverse of the corresponding horizontal arrow of D. The horizontal inverse of a 2–cells of HD is the horizontal inverse of the corresponding arrow square of D. The vertical inverse of a 2–cells of HD is the vertical inverse of the corresponding arrow square of D.
The inverse of a 1–cell of VD is the inverse of the corresponding vertical arrow of D. The horizontal inverse of a 2–cells of VD is the vertical inverse of the corresponding arrow square of D. The vertical inverse of a 2–cells of VD is the horizontal inverse of the corresponding arrow square of D.

In such a case, HD_0 and VD_0 are ordinary groupoids.

Definition 31. *A double category D is said **edge symmetric** if there is an invertible 2–functor $S : VD \to HD$. Similarly, for a double groupoid D.*

S induces an invertible functor $S_0 : VD_0 \to HD_0$.

Proposition 5. *A double functor $F : D \to E$ of two double categories or groupoids D, E induces strict 2–functors $HF : HD \to HE$, $VF : VD \to VE$ of the associated horizontal and vertical 2–categories or 2–groupoids HD, HE and VD, VE, respectively.*

The edge 2–categories of double categories enter in an essential way in the definition of the notion of folding.

1.4.4   Folding of edge symmetric double categories

Let D be an edge symmetric double category or a double groupoid. Then, as we explained in subapp. 1.4.3, we have an invertible functor of VD_0 into HD_0,

$$(1.4.11)$$

Definition 32. *A **horizontal folding** of D consists of a single datum.*

1. A mapping of the set arrow squares of D into that of 2–cells of HD

$$(1.4.12)$$

The following axioms

$$(1.4.13)$$

$$(1.4.14)$$

$$
\begin{array}{ccc}
\begin{array}{ccc}
b & \xleftarrow{\ \mathrm{id}_{hb}\ } & b \\
\big\uparrow x & \overset{\mathrm{Id}_{hx}}{\diagup\!\!\diagup} & \big\uparrow x \\
a & \xleftarrow{\ \mathrm{id}_{ha}\ } & a
\end{array}
& \longmapsto &
\begin{array}{ccc}
b & \xleftarrow{\ \tilde{x}\ } & a \\
\big\uparrow \mathrm{id}_{vb} & \overset{\mathrm{Id}_{v\tilde{x}}}{\diagup\!\!\diagup} & \big\uparrow \mathrm{id}_{va} \\
b & \xleftarrow{\ \tilde{x}\ } & a
\end{array}
\end{array}
\tag{1.4.15}
$$

must be fulfilled. For a double groupoid D we have further

$$
\begin{array}{ccc}
\begin{array}{ccc}
c & \xleftarrow{\ u^{-1_h}\ } & d \\
\big\uparrow x & \overset{X^{-1_h}}{\diagup\!\!\diagup} & \big\uparrow v \\
a & \xleftarrow{\ y^{-1_h}\ } & b
\end{array}
& \longmapsto &
\begin{array}{ccc}
c & \xleftarrow{\ u^{-1_h \circ_h \tilde{v}}\ } & b \\
\big\uparrow \mathrm{id}_{vc} & \overset{\mathrm{Id}_{v\tilde{x}}\,\circ_h\,\tilde{X}^{-1_h}\,\circ_h\,\mathrm{Id}_{v\tilde{v}}}{\diagup\!\!\diagup} & \big\uparrow \mathrm{id}_{vb} \\
c & \xleftarrow{\ \tilde{x}\circ_h y^{-1_h}\ } & b
\end{array}
\end{array}
\tag{1.4.16}
$$

$$
\begin{array}{ccc}
\begin{array}{ccc}
b & \xleftarrow{\ y\ } & a \\
\big\uparrow v^{-1_v} & \overset{X^{-1_v}}{\diagup\!\!\diagup} & \big\uparrow x^{-1_v} \\
d & \xleftarrow{\ u\ } & c
\end{array}
& \longmapsto &
\begin{array}{ccc}
b & \xleftarrow{\ y\circ_h \tilde{x}^{-1_h}\ } & c \\
\big\uparrow \mathrm{id}_{vb} & \overset{\mathrm{Id}_{v\tilde{v}^{-1_h}}\,\circ_h\,\tilde{X}^{-1_v}\,\circ_h\,\mathrm{Id}_{v\tilde{x}^{-1_h}}}{\diagup\!\!\diagup} & \big\uparrow \mathrm{id}_{vc} \\
b & \xleftarrow{\ \tilde{v}^{-1_h \circ_h u}\ } & c
\end{array}
\end{array}
\tag{1.4.17}
$$

A vertical folding is defined similarly.

We shall consider only horizontal foldings aiming to define double natural transformations.

1.4.5 Double natural transformations

In double category theory, there is a standard notion of double natural transformation, which has two variants. This notion however does not fit our purposes. Here, we present a new one, which is original to the best of our knowledge.

Let D, E be double categories or groupoids. Further, let E be edge symmetric and equipped with a folding (cf. subsections 1.4.3, 1.4.4). Let $F, G : D \to E$ be two double functors (cf. subsection 1.4.2).

Definition 33. *A **double natural transformation** $\rho : F \Rightarrow G$ consists of the following data.*

1. *A mapping of the set of object of D into the set of vertical arrows of E,*

$$
a \ \longmapsto \
\begin{array}{c}
G(a) \\
\big\uparrow \rho(a) \\
F(a)
\end{array}
\tag{1.4.18}
$$

2. *Two compatible functors from the horizontal and vertical arrow categories of D into the horizontal truncation category E_h of E.*

$$
\Big(\, b \xleftarrow{\ x\ } a \,\Big) \ \longmapsto \
\begin{array}{ccc}
G(b) & \xleftarrow{\ G(x)\ } & G(a) \\
\big\uparrow \rho(b) & \overset{\rho(x)}{\diagup\!\!\diagup} & \big\uparrow \rho(a) \\
F(b) & \xleftarrow{\ F(x)\ } & F(a)
\end{array}
\tag{1.4.19}
$$

$$
\begin{array}{ccc}
b & & G(b) \xleftarrow{\widetilde{G(x)}} G(a) \\
\big\uparrow{\scriptstyle x} & \longmapsto & {\scriptstyle\rho(b)}\big\uparrow \ {\scriptstyle\bar\rho(x)}\!\!\diagup\!\!\diagup \ \big\uparrow{\scriptstyle\rho(a)} \\
a & & F(b) \xleftarrow[\widetilde{F(x)}]{} F(a)
\end{array}
$$

Above E_h is the category whose objects are the vertical arrows of E and whose morphisms are the arrow squares of E connecting them with the composition $\circ_h$ and identity assigning map Id_h inherited form E. The data must fulfill a special naturality condition. For any arrow square

$$
\begin{array}{ccc}
d & \xleftarrow{\;y\;} & b \\
{\scriptstyle u}\big\uparrow & {\scriptstyle X}\!\!\diagup\!\!\diagup & \big\uparrow{\scriptstyle x} \\
c & \xleftarrow[\;z\;]{} & a
\end{array}
\tag{1.4.20}
$$

one has

$$
\begin{array}{c}
\begin{array}{ccccc}
G(d) & \xleftarrow{G(y)} & G(b) & \xleftarrow{\widetilde{G(x)}} & G(a) \\
{\scriptstyle\rho(d)}\big\uparrow & {\scriptstyle\rho(y)}\!\diagup\!\diagup & {\scriptstyle\rho(b)}\big\uparrow & {\scriptstyle\bar\rho(x)}\!\diagup\!\diagup & \big\uparrow{\scriptstyle\rho(a)} \\
F(d) & \xleftarrow{F(y)} F(b) & \xleftarrow{\widetilde{F(x)}} & F(a) \\
& \widetilde{F(X)} & & \\
{\scriptstyle\mathrm{id}_{vF(d)}}\big\uparrow & & & & \big\uparrow{\scriptstyle\mathrm{id}_{vF(a)}} \\
F(d) & \xleftarrow[\widetilde{F(u)}\circ_h F(z)]{} & & & F(a)
\end{array}
\end{array}
\;=\;
\begin{array}{c}
\begin{array}{ccccc}
G(d) & \xleftarrow{G(y)\circ_h \widetilde{G(x)}} & & & G(a) \\
{\scriptstyle\mathrm{id}_{vG(d)}}\big\uparrow & & {\scriptstyle\widetilde{G(X)}} & & \big\uparrow{\scriptstyle\mathrm{id}_{vG(a)}} \\
G(d) & \xleftarrow{\widetilde{G(u)}} G(c) & \xleftarrow{G(z)} & G(a) \\
{\scriptstyle\rho(d)}\big\uparrow & {\scriptstyle\bar\rho(u)}\!\diagup\!\diagup & {\scriptstyle\rho(c)}\big\uparrow & {\scriptstyle\rho(z)}\!\diagup\!\diagup & \big\uparrow{\scriptstyle\rho(a)} \\
F(d) & \xleftarrow[\widetilde{F(u)}]{} & F(c) & \xleftarrow[F(z)]{} & F(a)
\end{array}
\end{array}
\tag{1.4.21}
$$

The conventionally defined double natural transformations do not require a prior assignment of a folding. Further, they can be either horizontal or vertical. The naturality condition they satisfy mimics that of the ordinary natural transformations with arrows replaced by arrow squares of the form (1.4.19) and arrow composition replaced by the horizontal and vertical square compositions, respectively.

If we forget the distinction between horizontal and vertical arrows of E exploiting the edge symmetry of the latter, the naturality condition can be viewed as the requirement of commutativity of the cube diagram

$$
\tag{1.4.22}
$$

Here, we have dropped all double arrows in order not to clog the diagram.

1.4.6   Double modifications

The non standard definition of double modification given below is dictated by the non standard notion of double natural transformation of the last subsection.

Let D, E be double categories or groupoids with E edge symmetric and folded (cf. subsections 1.4.3, 1.4.4). Let $F, G : D \to E$ be double functors and $\sigma, \sigma : F \Rightarrow G$ be double natural transformations (cf. subsections 1.4.2, 1.4.5).

Definition 34. *A **double modification** $\rho \Rightarrow \sigma$ consists of a single datum.*

1. A mapping of the set of objects of D into the set of 2–cells of VD,

$$
a \quad \longmapsto \quad
\begin{array}{c}
G(a) \xleftarrow{\mathrm{id}_{hG(a)}} G(a) \\
\sigma(a) \uparrow \quad \overset{T(a)}{\Longrightarrow} \quad \uparrow \rho(a) \\
F(a) \xleftarrow[\mathrm{id}_{hF(a)}]{} F(a)
\end{array}
\tag{1.4.23}
$$

This must satisfy the modification axioms. For any horizontal arrow of D

$$
b \xleftarrow{\ x\ } a
\tag{1.4.24}
$$

one has

$$
\begin{array}{c}
G(b) \xleftarrow{\mathrm{id}_{hG(b)}} G(b) \xleftarrow{G(x)} G(a) \\
\sigma(b) \uparrow \overset{T(b)}{\Longrightarrow} \rho(b) \uparrow \overset{\rho(x)}{\Longrightarrow} \uparrow \rho(a) \\
F(b) \xleftarrow[\mathrm{id}_{hF(b)}]{} F(b) \xleftarrow[F(x)]{} F(a)
\end{array}
=
\begin{array}{c}
G(b) \xleftarrow{G(x)} G(a) \xleftarrow{\mathrm{id}_{hG(a)}} G(a) \\
\sigma(b) \uparrow \overset{\sigma(x)}{\Longrightarrow} \sigma(a) \uparrow \overset{T(a)}{\Longrightarrow} \uparrow \rho(a) \\
F(b) \xleftarrow[F(x)]{} F(a) \xleftarrow[\mathrm{id}_{hF(a)}]{} F(a)
\end{array}
\tag{1.4.25}
$$

For any vertical arrow of D

$$
\begin{array}{c}
b \\
\uparrow x \\
a
\end{array}
\tag{1.4.26}
$$

one has

$$
\begin{array}{c}
G(b) \xleftarrow{\mathrm{id}_{hG(b)}} G(b) \xleftarrow{\widetilde{G(x)}} G(a) \\
\sigma(b) \uparrow \overset{T(b)}{\Longrightarrow} \rho(b) \uparrow \overset{\bar{\rho}(x)}{\Longrightarrow} \uparrow \rho(a) \\
F(b) \xleftarrow[\mathrm{id}_{hF(b)}]{} F(b) \xleftarrow[\widetilde{F(x)}]{} F(a)
\end{array}
=
\begin{array}{c}
G(b) \xleftarrow{\widetilde{G(x)}} G(a) \xleftarrow{\mathrm{id}_{hG(a)}} G(a) \\
\sigma(b) \uparrow \overset{\bar{\sigma}(x)}{\Longrightarrow} \sigma(a) \uparrow \overset{T(a)}{\Longrightarrow} \uparrow \rho(a) \\
F(b) \xleftarrow[\widetilde{F(x)}]{} F(a) \xleftarrow[\mathrm{id}_{hF(a)}]{} F(a)
\end{array}
\tag{1.4.27}
$$

The axioms can be interpreted as the commutativity condition of the following cylinder diagrams

$$
\tag{1.4.28a}
$$

$$(1.4.28b)$$

Above all double arrows have been dropped. Further the identity morphisms of the modification arrow squares have been collapsed.

1.4.7   The double groupoid of plane rectangles

Rectangles in $\mathbb{R}^2$ can be organized in a double groupoid.

Proposition 6. *There is a double groupoid* $\mathbb{GR}^2$ *defined as follows.*

1. *For each* $x, y \in \mathbb{R}$, *there is an object* (x, y) *of* $\mathbb{GR}^2$.

2. *For each* $x, x', y \in \mathbb{R}$ *there is a unique horizontal arrow*

$$(x', y) \longleftarrow (x, y) \tag{1.4.29}$$

   For each $x, y, y' \in \mathbb{R}$ *there is a unique vertical arrow*

$$(x, y') \tag{1.4.30}$$
$$\uparrow$$
$$(x, y)$$

3. *For each quadruple* $x, x', y, y' \in \mathbb{R}$ *there is a unique arrow square*

$$\tag{1.4.31}$$

The horizontal and vertical composition of arrows and arrow squares are codified in the diagrams

$$\tag{1.4.32}$$

respectively. The horizontal and vertical composition identity arrows and arrow squares are similarly encoded in the diagrams

$$
\begin{array}{ccc}
(x,y') \longleftarrow (x,y') & \qquad & (x',y) \longleftarrow (x,y) \\
\uparrow \quad \diagup \quad \uparrow & & \uparrow \quad \diagup \quad \uparrow \\
(x,y) \longleftarrow (x,y) & & (x',y) \longleftarrow (x,y)
\end{array}
\tag{1.4.33}
$$

respectively. Finally, the horizontal and vertical inverses of arrows and arrow squares in (1.4.31) are

$$
\begin{array}{ccc}
(x,y') \longleftarrow (x',y') & \qquad & (x',y) \longleftarrow (x,y) \\
\uparrow \quad \diagup \quad \uparrow & & \uparrow \quad \diagup \quad \uparrow \\
(x,y) \longleftarrow (x',y) & & (x',y') \longleftarrow (x,y')
\end{array}
\tag{1.4.34}
$$

1.4.8 The double groupoid of a crossed module

Let (G, H) be a crossed module with target map $t : H \to G$ and G action $m : G \times H \to H$ (see definition 51).

Proposition 7. *There is a double groupoid $B(G, H)$ defined as follows.*

1. *There is a unique object $*$.*

2. *For each element $x \in G$, there is one horizontal and one vertical arrow,*

$$
* \xleftarrow{\;x\;} * \qquad \qquad
\begin{array}{c}
* \\ \uparrow x \\ *
\end{array}
\tag{1.4.35}
$$

3. *For each quadruple $x, y, u, v \in G$ and each $X \in H$ satisfying the target matching condition*

$$
vy = uxt(X)
\tag{1.4.36}
$$

there is one arrow square

$$
\begin{array}{c}
* \xleftarrow{\;u\;} * \\
v \uparrow \;\; X \diagup \;\; \uparrow x \\
* \xleftarrow{\;y\;} *
\end{array}
\tag{1.4.37}
$$

The horizontal and vertical composition of arrows and arrow squares are codified in the diagrams

$$
\begin{array}{c}
* \xleftarrow{\;v\;} * \xleftarrow{\;u\;} * \\
t \uparrow \; Y \diagup \; \uparrow s \; X \diagup \; \uparrow r \\
* \xleftarrow{\;y\;} * \xleftarrow{\;x\;} *
\end{array}
\quad = \quad
\begin{array}{c}
* \xleftarrow{\quad\quad vu \quad\quad} * \\
t \uparrow \;\; Xm(x^{-1})(Y) \diagup \;\; \uparrow r \\
* \xleftarrow{\quad\quad yx \quad\quad} *
\end{array}
\tag{1.4.38}
$$

$$
\begin{array}{ccc}
\ast \xleftarrow{z} \ast & & \ast \xleftarrow{z} \ast \\
t \uparrow \;\; \overset{Y}{\nearrow\!\!\!\nearrow} \;\; \uparrow r & = & ts \uparrow \;\; m(q^{-1})(Y)X \;\; \uparrow rq \\
\ast \xleftarrow{y} \ast & & \ast \xleftarrow{x} \ast \\
s \uparrow \;\; \overset{X}{\nearrow\!\!\!\nearrow} \;\; \uparrow q & & \\
\ast \xleftarrow{x} \ast & &
\end{array}
$$

respectively. The horizontal and vertical identity arrows and arrow squares are similarly encoded in the diagrams

$$
\begin{array}{cc}
\ast \xleftarrow{1_G} \ast & \ast \xleftarrow{x} \ast \\
x \uparrow \;\; \overset{1_H}{\nearrow\!\!\!\nearrow} \;\; \uparrow x & 1_G \uparrow \;\; \overset{1_H}{\nearrow\!\!\!\nearrow} \;\; \uparrow 1_G \\
\ast \xleftarrow{1_G} \ast & \ast \xleftarrow{x} \ast
\end{array}
\tag{1.4.39}
$$

respectively. Finally the horizontal and vertical inverses of arrows and arrow squares in (1.4.37) are

$$
\begin{array}{cc}
\ast \xleftarrow{u^{-1}} \ast & \ast \xleftarrow{y} \ast \\
x \uparrow \;\; m(y)(X^{-1}) \;\; \uparrow v & v^{-1} \uparrow \;\; m(x)(X^{-1}) \;\; \uparrow x^{-1} \\
\ast \xleftarrow{y^{-1}} \ast & \ast \xleftarrow{u} \ast
\end{array}
\tag{1.4.40}
$$

We remark that the target matching condition (1.4.36) is essential for the exchange law (1.4.5) to be satisfied.

Proposition 8. *The double groupoid $B(G,H)$ is edge symmetric.*

The invertible functor $VB(G,H) \rightarrow HB(G,H)$ implementing edge symmetry is defined as

$$
\begin{array}{ccc}
\ast \xleftarrow{1_G} \ast & & \ast \xleftarrow{y} \ast \\
y \uparrow \;\; \overset{X}{\nearrow\!\!\!\nearrow} \;\; \uparrow x & \longmapsto & 1_G \uparrow \;\; \overset{X^{-1}}{\nearrow\!\!\!\nearrow} \;\; \uparrow 1_G \\
\ast \xleftarrow{1_G} \ast & & \ast \xleftarrow{x} \ast
\end{array}
\tag{1.4.41}
$$

Proposition 9. *The mapping*

$$
\begin{array}{ccc}
\ast \xleftarrow{u} \ast & & \ast \xleftarrow{ux} \ast \\
v \uparrow \;\; \overset{X}{\nearrow\!\!\!\nearrow} \;\; \uparrow x & \longmapsto & 1_G \uparrow \;\; \overset{X}{\nearrow\!\!\!\nearrow} \;\; \uparrow 1_G \\
\ast \xleftarrow{y} \ast & & \ast \xleftarrow{vy} \ast
\end{array}
\tag{1.4.42}
$$

defines a folding of $B(G,H)$.

Chapter 2

L_∞ algebras

In this chapter we will introduce L_∞-algebras and higher groups, central concepts in higher gauge theory. They are higher generalizations of Lie algebras and groups, and can thus be used to define gauge theories with higher gauge structure. We will focus mainly on 2-term L_∞ algebras, which are the basic algebraic ingredient of the higher Chern-Simons model. In the last section we will briefly discuss some important issues in the generalization of the Lie theory for L_∞ algebras.

2.1 Review of Lie groups theory

In this section we will briefly summarize the theory of Lie groups and Lie algebras, so that everything that we will do in the following will be a clear generalization of the classical Lie theory. We start with the two central definitions:

Definition 35. *A **group** is a set G with a multiplication $m : G \times G \to G$, also denoted $m(g, h) = gh$, such that there is an identity $e \in G$, m is associative and every element of G is invertible. A **Lie group** is a group which is also a smooth manifold, with the multiplication m and the inversion $\cdot^{-1} : G \to G$ smooth maps.*

Definition 36. *A **Lie algebra** is a vector space $\mathfrak{g}$ endowed with an antisymmetric bilinear bracket $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity:*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{2.1.1}$$

If $\{e_a\}$ is a basis for $\mathfrak{g}$, the Lie brackets can be expanded in the following way: $[e_a, e_b] = f_{ab}^c e_c$, with an understood sum over repeated indexes. The real coefficients f_{ab}^c are called the structure constants of $\mathfrak{g}$, and they uniquely define the Lie algebra $\mathfrak{g}$. The Jacobi identity in this basis expansion reads

$$f_{ad}^e f_{bc}^d + f_{bd}^e f_{ca}^d + f_{cd}^e f_{ab}^d = 0. \tag{2.1.2}$$

Groups are a cornerstone in physics and mathematics, due to the fact that they abstractly describe symmetries. Lie algebras encode the infinitesimal structure of Lie groups, as the following well known result shows:

Proposition 10. *Given a Lie group G, its tangent space at the identity $\mathfrak{g} := T_e G$ is a Lie algebra. It is called the Lie algebra associated with G, and is denoted $Lie(G)$.*

Let us give some quick details about the construction of the Lie algebra structure on $\mathfrak{g}$. Denote L_g for $g \in G$ the left multiplication map: $L_g(h) = gh$. It is a smooth map $L_g : G \to G$. A *left invariant vector field* on G is a vector field V satisfying $L_{g_*} V|_h = V|_{gh}$. Given two vectors $x, y \in \mathfrak{g}$, we can build two unique left-invariant vector fields V_x and V_y on G such that $V_x|_e = x$ and $V_y|_e = y$: $V_x|_g := L_{g_*} x$ and similarly for V_y. Their commutator $[V_x, V_y]$ is again a left invariant vector field. The Lie bracket on $\mathfrak{g}$ is then defined as $[x, y] := [V_x, V_y]|_e$.

Moreover Lie's third theorem or Cartan-Lie theorem states that every real finite dimensional Lie algebra can be integrated:

Theorem 1. *Given a Lie algebra $\mathfrak{g}$, there is a simply connected Lie group G such that $Lie(G) = \mathfrak{g}$.*

The exponential map $\exp : \mathfrak{g} \to G$ integrates the Lie algebra to the Lie group, sending every vector in $\mathfrak{g}$ to a finite group element. Given $x \in \mathfrak{g}$, the left invariant vector field V_x generates a one-parameter group of transformations of G: $\sigma_x(t, g) : \mathbb{R} \times G \to G$, such that $\sigma_x(t, \sigma_x(t', g)) = \sigma_x(t + t', g)$, $\sigma_x(0, g) = g$. The exponential map is defined as $\exp(x) := \sigma_x(1, e)$. In the case of a matrix group, this map is the exponential map of matrices:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2.1.3}$$

We can associate with every Lie algebra $\mathfrak{g}$ a differential complex called the Chevalley-Eilenberg complex of $\mathfrak{g}$. This is a very important construction that we will use later on in the definition of a L_∞ algebra (see section (2.4)). Given a Lie algebra $\mathfrak{g}$, its algebraic dual $\mathfrak{g}^*$ is naturally equipped with a bilinear product $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \otimes \mathfrak{g} \to \mathbb{R}$. This pairing can be canonically extended to the exterior algebras $\wedge^\bullet \mathfrak{g}^* \cong (\wedge^\bullet \mathfrak{g})^*$ and $\wedge^\bullet \mathfrak{g}$: given $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \wedge^k \mathfrak{g}^*$ and $x = x_1 \wedge \cdots \wedge x_h \in \wedge^h \mathfrak{g}$ it is defined as

$$\langle \xi, x \rangle = \begin{cases} \sum_{\sigma \in S_h} (-1)^{k(k-1)/2} (-1)^\sigma \langle \xi_1, x_{\sigma(1)} \rangle \cdots \langle \xi_k, x_{\sigma(k)} \rangle & \text{if } k = h \\ 0 \text{ if } k \neq h \end{cases}. \tag{2.1.4}$$

Here $(-1)^\sigma$ denotes the signature of the permutation σ. This pairing together with the Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$ defines an operator $Q_{CE} : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$:

$$\langle Q_{CE} \xi, x \wedge y \rangle := \langle \xi, [x, y] \rangle. \tag{2.1.5}$$

We can now extend the operator Q_{CE} to all the exterior algebra $\wedge^\bullet \mathfrak{g}^*$, so that $Q_{CE} : \wedge^k \mathfrak{g}^* \to \wedge^{k+1} \mathfrak{g}^*$, through the graded Leibniz identity:

$$Q_{CE}(\xi_1 \wedge \xi_2) = (Q_{CE} \xi_1) \wedge \xi_2 + (-1)^{\deg \xi_1} \xi_1 \wedge (Q_{CE} \xi_2), \tag{2.1.6}$$

where deg is the natural grading of the exterior algebra: $\deg \xi = k$ if $\xi \in \wedge^k \mathfrak{g}^*$. Also the Lie bracket of $\mathfrak{g}$ can be extended from $\wedge^2 \mathfrak{g}$ to $\wedge^\bullet \mathfrak{g}$:

$$[x_1 \wedge \cdots \wedge x_k] := \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k, \tag{2.1.7}$$

where the hat means a missing entry. In this way $[\cdot] : \wedge^\bullet \mathfrak{g} \to \wedge^{\bullet-1}\mathfrak{g}$, with $[x] = 0$ for $x \in \mathfrak{g}$. It is possible to show that these extended brackets are dual to the operator Q_{CE}, in fact for any $\xi \in \wedge^k \mathfrak{g}^*$ and $x \in \wedge^{k+1}\mathfrak{g}$ we have

$$\langle Q_{CE}\xi, x \rangle = \langle \xi, [x] \rangle. \tag{2.1.8}$$

The operator Q_{CE} enjoys the following property:

Proposition 11.

$$Q_{CE}^2 = 0. \tag{2.1.9}$$

Proof. Given $\xi = \xi_1 \wedge \cdots \wedge \xi_k$ an element of $\wedge^k \mathfrak{g}^*$ with $\xi_i \in \mathfrak{g}$, we have that

$$Q_{CE}^2 \xi = Q_{CE}(\sum_{i=1}^{k}(-1)^{i+1}\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge \xi_k) =$$

$$= \sum_{i=1}^{k}(-1)^{i+1} \left(\sum_{j=1}^{i-1}(-1)^{j+1}\xi_1 \wedge \cdots \wedge d\xi_j \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge \xi_k + \right.$$

$$\left. +(-1)^{i+1}\xi_1 \wedge \ldots d^2\xi_i \wedge \cdots \wedge \xi_k + \sum_{j=i+1}^{k}(-1)^{j}\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_j \wedge \ldots \xi_k \right) =$$

$$= \sum_{i=1}^{k}\xi_1 \wedge \cdots \wedge d^2\xi_i \wedge \ldots \xi_k. \tag{2.1.10}$$

Using the coordinate expression

$$d\xi^a = -\frac{1}{2}f^a_{bc}\xi^b \wedge \xi^c, \tag{2.1.11}$$

where $\{\xi^a\}$ is a basis of $\mathfrak{g}^*$, it is easy to compute

$$\langle d^2\xi, x \wedge y \wedge z \rangle = \langle \xi, [[x,y]z] + [[y,z],x] + [[z,x],y]\rangle = 0 \tag{2.1.12}$$

and (2.1.9) follows. $\square$

This makes Q_{CE} a differential and $\wedge^\bullet\mathfrak{g}^*$ a differential complex. We can now give this definition:

Definition 37. *Given a Lie algebra $\mathfrak{g}$, the cochain complex $CE^\bullet(\mathfrak{g}) := \wedge^\bullet\mathfrak{g}^*$ is called the **Chevalley-Eilenberg complex** associated with $\mathfrak{g}$, and Q_{CE} is the **Chevalley-Eilenberg differential**.*

What is more, given any differential of degree 1 on the wedge power of some vector space, namely $Q : C \to C$ with $C = \oplus_n \wedge^n V$, that satisfies the graded Leibniz rule, we can find a Lie algebra structure on the algebraic dual of V. Expanding the differential on a basis $\{\xi^a\}$ of V we obtain the usual expression

$$Q(\xi^a) = -\frac{1}{2}f^a_{bc}\xi^b \wedge \xi^c, \tag{2.1.13}$$

and the fact that Q squares to zero implies that the constants f_{bc}^a satisfy (2.1.2). Since they are intrinsically antisymmetric, they define Lie bracket $[e_a, e_b] = f_{ab}^c e_c$ on V^*. This shows that Lie algebras and Chevalley-Eilenberg complexes are in one-to-one correspondence.

Through the definition $\pi := \xi^a \otimes e_a$, with $\{e_a\}$ basis of $\mathfrak{g}$ and $\{\xi^a\}$ dual basis of $\mathfrak{g}^*$, we can write the Chevalley-Eilenberg differential in the concise form

$$Q_{CE}\pi = -\frac{1}{2}[\pi, \pi]. \tag{2.1.14}$$

We will give a generalization of this formula for a 2-term L_∞ algebra.

Remarkably, the correspondence between Lie algebras and Chevalley-Eilenberg complexes holds at the level of morphisms. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, a homomorphism of Lie algebras is a linear map $f : \mathfrak{g} \to \mathfrak{h}$ such that $[f(x), f(y)]_\mathfrak{h} = f([x, y]_\mathfrak{g})$ for any $x, y \in \mathfrak{g}$. Homomorphisms of a Lie algebra generate chain maps of the Chevalley-Eilenberg complex:

Proposition 12. *Homomorphisms of a Lie algebra $\mathfrak{g}$ and chain maps of the Chevalley-Eilenberg complex $CE^\bullet(\mathfrak{g})$ are in 1-to-1 correspondence*

The correspondence is based on the fact that every linear map $f : \mathfrak{g} \to \mathfrak{g}$ generates a dual map $f^* : \mathfrak{g}^* \to \mathfrak{g}^*$ which can be naturally extended to the complex $CE^\bullet(\mathfrak{g})$. The fact that f preserves the Lie bracket of $\mathfrak{g}$ implies that the dual map f^* respects the Chevalley-Eilenberg differential, $f^*Q_{CE} = Q_{CE}f^*$, and vice versa.

It is worth to notice that the Chevalley-Eilenberg complex of a Lie algebra $\mathfrak{g}$ can be generalized to take values in an arbitrary module carrying a representation of $\mathfrak{g}$.

Definition 38. *Given a real Lie algebra $\mathfrak{g}$ and a real module m, a **representation of $\mathfrak{g}$ on** m is a bilinear map $\varphi : \mathfrak{g} \times m \to m$, also denoted $\varphi(x, a) = x \cdot a$, such that*

$$x \cdot (y \cdot a) - y \cdot (x \cdot a) = [x, y] \cdot a \ , \ \forall x, y \in \mathfrak{g}, \forall a \in m \tag{2.1.15}$$

In this case the cochain complex is defined to be $CE_\varphi^\bullet(\mathfrak{g}, m) := \wedge^\bullet \mathfrak{g}^* \otimes m$. Through the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ this complex is the set of linear maps from $\wedge^\bullet \mathfrak{g}$ to m: Given $\xi \in \wedge^\bullet \mathfrak{g}^*$, $a \in m$ and $x \in \wedge^\bullet \mathfrak{g}$ the pairing is

$$\langle \xi \otimes a, x \rangle' = \langle \xi, x \rangle a. \tag{2.1.16}$$

The Chevalley-Eilenberg differential is also slightly modified according to this expression:

$$\langle Q_{CE,\varphi}(\xi \otimes a), x_1 \wedge \cdots \wedge x_{k+1} \rangle' := \langle Q_{CE}\xi, x_1 \wedge \cdots \wedge x_{k+1} \rangle a -$$

$$- \sum_{j=1}^{k+1} (-1)^{k+j} \langle \xi, x_1 \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_{k+1} \rangle x_j \cdot a, \tag{2.1.17}$$

where $Q_{CE,\varphi}$ is the Chevalley-Eilenberg differential of $CE_\varphi^\bullet(\mathfrak{g}, m)$ and Q_{CE} is the Chevalley-Eilenberg differential that we have previously defined. The complex $CE^\bullet(\mathfrak{g})$ is the particular case of $CE_\varphi^\bullet(\mathfrak{g}, m)$ for $m = \mathbb{R}$ and φ the trivial zero representation.

2.2 Groups and Groupoids

In this section we discuss some extensions of the concept of group, namely groupoids, higher groupoids and higher groups, with a special focus on 2-groups.

Groupoids are a categorical generalization of groups. The inspiring example for groupoids, which is also of interest for our purposes (see sect. 4.3.1), is the path groupoid: given a smooth manifold M and $x, y \in M$, a path in M from x to y is a smooth map $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The set of all paths in M is called the path space. There is a composition defined on this space, which is concatenation. Without going into details, which will be discussed later, it is possible to show that if we mod out a certain homotopy equivalence on the path space, then composition acquires nice properties: first of all it becomes associative; at any point $x \in M$, the constant path $\gamma(t) = x$ is a unit for the composition; every path has an inverse. These properties resemble the ones of a group, but with a substantial difference: in the path space not every two elements can be composed, it depends on their source and target points. There are two maps s, t going from the path space to M, namely the starting point and the target point of a path, and two paths γ and γ' admit a composition $\gamma \circ \gamma'$ if and only if $s(\gamma) = t(\gamma')$. Thus, the path space isn't a group, and the algebraic structure that it reveals is instead a groupoid. This is very similar to what happens in category theory, where only consecutive morphisms con be composed. Capturing the essence of these properties, we arrive at the following definition:

Definition 39. *A **groupoid** is a category where every morphism is invertible.*

This definition is very simple, but it sums up all the properties we want. In our previous example, the set of objects is M and the set of morphisms is the path space. The axioms of a category provide all the algebraic structure which is needed, except for the invertibility. The relation between groups and groupoids is clarified by the next definition and proposition:

Definition 40. *Given G a group, the **delooping** of G is the category BG which has only one object and G as set of morphisms, with 1_G as identity morphism and the group multiplication as composition.*

Proposition 13. *Given $\mathcal{G}$ a groupoid with a single object, there is a group G such that $\mathcal{G} = BG$:*

This simple result illustrates the difference between a group and a groupoid: the former is a groupoid whose morphisms are concentrated on a single object.

In the same way as higher categories extend the concept of categories, higher groupoids extend the concept of groupoids, and the same happens for higher groups, which are particular higher groupoids in exactly the same way as groups are particular groupoids. There are n-groupoids for arbitrary n, where the meaning of the natural number n is the same as in higher category theory, and n-groups are n-groupoids with just one object. We will limit our discussion to the case $n = 2$.

2-groupoids and 2-groups can be either weak or strict, depending on the underlying categorical structure. Thus we distinguish two different notions of 2-groupoids:

Definition 41. *A **weak 2-groupoid** is a 2-category in which every 1-morphism is an equivalence and every 2-morphism is invertible under vertical composition.*

Definition 42. *A **strict 2-groupoid** is a weak 2-groupoid whose underlying 2-category is strict.*

We can now come to the definition of weak or strict 2-groups:

Definition 43. *A **weak (strict) 2-group** is a weak (strict) 2-groupoid with only one object.*

The relation between 2-groups and 2-groupoids (actually between n-groups and n-groupoids) according to this definition is the same as the relation between groups and groupoids, with a slight difference: mimicking the ordinary case, we would have stated that a groupoid with a single object is $B\mathcal{G}$ for some 2-group $\mathcal{G}$. A 2-group would then have been defined as the Hom category of a 2-groupoid with a single object, which we know is a monoidal category. Indeed, another more self contained definition of 2-groups which doesn't require 2-groupoids says that 2-groups are monoidal categories whose objects are all weakly invertible and whose morphisms are all invertible. This is the definition employed for example in [36].

In [36] Baez and Lauda define another kind of 2-groups, which they call *coherent 2-groups*. To proceed similarly to before, we modify slightly their definition and we adopt the point of view of 2-categories instead of the one of monoidal categories:

Definition 44. *A **coherent 2-group** is a weak 2-category with one object in which every 2-morphism is invertible and every 1-morphism f is equipped with a triple $(\bar{f}, i_f, e_f)$ where $\bar{f}$ is a 1-morphism and $i_f : 1 \Rightarrow f \circ \bar{f}$ and $e_f : \bar{f} \circ f \Rightarrow 1$ are 2-isomorphisms.*

The difference between a weak 2-group and a coherent 2-group is that in the latter we assign a precise weak inverse together with all its structure to every 1-morphism instead of just saying that it exists. Nevertheless every weak 2-group can be enhanced to become a coherent one, and every coherent 2-group becomes a weak 2-group just by forgetting the extra structure. Indeed, theorem 17 in [36] states that there is an equivalence between the 2-category of weak 2-groups and the 2-category of coherent 2-groups.

There are other definitions of strict 2-groups which are used in literature. We will not go through all the details of these definitions, nor we will prove that they are all equivalent, but we'll mention them for completeness. More details on these definitions and for the proofs of the equivalences between most of them, see [49].

2-groups can be introduced as group objects in Cat, the category of categories. Given a category C with products and a terminal object 1, a group object in C is defined as an object G in C together with morphisms $m : G \times G \to G$, $i : G \to G$ and $e : 1 \to G$ in C that fulfill some relations resembling the axiom of associativity, invertibility and identity of usual groups. Such an object in the category of categories is a monoidal category, with the tensor product provided by m, where every object and every morphism is invertible, with i giving the inverse function, and the image of e is the identity object and morphism (the terminal object in Cat is the trivial category with one object and the identity morphism on it only).

Another definition adopts the opposite point of view, as it says that 2-groups are internal categories in `Grp`, the category of groups and group homomorphisms. An internal category in a category `C` is defined as a pair of objects O and M in `C`, the former called object of objects and the second object of morphisms, together with morphisms $s, t : M \to O$, $id : O \to M$, $\circ : M \times_O M \to M$ in `C` that satisfy axioms similar to those of a category. An internal category in the category of groups consists of a two groups O and M with source and target homomorphisms $s, t : M \to O$. Horizontal composition is given by the group law on O and by $\circ : M \times_O M \to M$ on M and vertical composition is given by the group law on M.

The definitions we gave up to now are very concise but a bit implicit. Let us unpack them and make more precise definitions with which it will be easier to work. Since we will deal only with the strict version of higher groups and groupoids, we will restrict ourself to them:

Definition 45. *A **groupoid** $\mathcal{G}$ consists of the following set of data:*

1. *a set of objects $\mathcal{G}_0$;*

2. *for each pair of objects x, y, a set of 1-morphisms $\mathcal{G}_1(x, y)$;*

3. *for each triple of objects x, y, z, a composition law of 1–morphisms $\circ : \mathcal{G}_1(x, y) \times \mathcal{G}_1(y, z) \to \mathcal{G}_1(x, z)$;*

4. *for each pair of objects x, y, a inversion law of 1–morphisms $^{-1\circ} : \mathcal{G}_1(x, y) \to \mathcal{G}_1(y, z)$;*

5. *for each object x, a distinguished unit 1–morphisms $1_x \in \mathcal{G}_1(x, x)$;*

These are required to satisfy the following axioms.

$$(c \circ b) \circ a = c \circ (b \circ a), \tag{2.2.1a}$$
$$a^{-1\circ} \circ a = 1_x, \qquad a \circ a^{-1\circ} = 1_y, \tag{2.2.1b}$$
$$a \circ 1_x = 1_y \circ a = a, \tag{2.2.1c}$$

Here and in the following, $x, y, z, \cdots \in \mathcal{G}_0$, $a, b, c, \cdots \in \mathcal{G}_1$, where $\mathcal{G}_1$ denotes the set of all 1–morphisms. All identities hold whenever defined.

Definition 46. *A **strict 2-group** $\mathcal{G}$ consists of the following set of data:*

1. *a set of 1-morphisms $\mathcal{G}_1$;*

2. *a composition law of 1–morphisms $\circ : \mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_1$;*

3. *a inversion law of 1–morphisms $^{-1\circ} : \mathcal{G}_1 \to \mathcal{G}_1$;*

4. *a distinguished unit 1–morphism $1 \in \mathcal{G}_1$;*

5. *for each pair of 1–morphisms $a, b \in \mathcal{G}_1$, a set of 2–morphisms $\mathcal{G}_2(a, b)$;*

6. *for each quadruple of 1–morphisms $a, b, c, d \in \mathcal{G}_1$, a horizontal composition law of 2–morphisms $\circ : \mathcal{G}_2(a, c) \times \mathcal{G}_2(b, d) \to \mathcal{G}_2(b \circ a, d \circ c)$;*

7. *for each pair of 1–morphisms $a, b \in \mathcal{G}_1$, a horizontal inversion law of 2–morphisms $^{-1\circ} : \mathcal{G}_2(a, b) \rightarrow \mathcal{G}_2(a^{-1\circ}, b^{-1\circ})$;*

8. *for each triple of 1–morphisms $a, b, c \in \mathcal{G}_1$, a vertical composition law of 2– morphisms $\bullet : \mathcal{G}_2(a, b) \times \mathcal{G}_2(b, c) \rightarrow \mathcal{G}_2(a, c)$;*

9. *for each pair of 1–morphisms $a, b \in \mathcal{G}_1$, a vertical inversion law of 2–v $^{-1\bullet} : \mathcal{G}_2(a, b) \rightarrow \mathcal{G}_2(b, a)$;*

10. *for each 1–morphism a, a distinguished unit 2–morphism $1_a \in \mathcal{G}_2(a, a)$.*

These are required to satisfy the following axioms.

$$(c \circ b) \circ a = c \circ (b \circ a), \tag{2.2.2a}$$
$$a^{-1\circ} \circ a = a \circ a^{-1\circ} = 1, \tag{2.2.2b}$$
$$a \circ 1 = 1 \circ a = a, \tag{2.2.2c}$$
$$(C \circ B) \circ A = C \circ (B \circ A), \tag{2.2.2d}$$
$$A^{-1\circ} \circ A = A \circ A^{-1\circ} = 1_1, \tag{2.2.2e}$$
$$A \circ 1_1 = 1_1 \circ A = A, \tag{2.2.2f}$$
$$(C \bullet B) \bullet A = C \bullet (B \bullet A), \tag{2.2.2g}$$
$$A^{-1\bullet} \bullet A = 1_a, \qquad A \bullet A^{-1\bullet} = 1_b, \tag{2.2.2h}$$
$$A \bullet 1_a = 1_b \bullet A = A, \tag{2.2.2i}$$
$$(D \bullet C) \circ (B \bullet A) = (D \circ B) \bullet (C \circ A). \tag{2.2.2j}$$

All identities involving the vertical composition and inversion hold whenever defined. Relation (2.2.2j) is the exchange law.

Definition 47. *A **strict 2-groupoid** $\mathcal{G}$ consists of the following set of data:*

1. *a set of objects $\mathcal{G}_0$;*

2. *for each pair of objects x, y, a set of 1-morphisms $\mathcal{G}_1(x, y)$;*

3. *for each triple of objects x, y, z, a composition law of 1–morphisms $\circ : \mathcal{G}_1(x, y) \times \mathcal{G}_1(y, z) \rightarrow \mathcal{G}_1(x, z)$;*

4. *for each pair of objects x, y, a inversion law of 1–morphisms $^{-1\circ} : \mathcal{G}_1(x, y) \rightarrow \mathcal{G}_1(y, z)$;*

5. *for each object x, a distinguished unit 1–morphisms $1_x \in \mathcal{G}_1(x, x)$;*

6. *for each pair of objects x, y and for each pair of 1–morphisms $a, b \in \mathcal{G}_1(x, y)$, a set of 2–morphisms $\mathcal{G}_2(a, b)$;*

7. *for each triple of objects x, y, z and for each pair of 1–morphisms $a, c \in \mathcal{G}_1(x, y)$ and for each pair of 1–morphisms $b, d \in \mathcal{G}_1(y, z)$, a horizontal composition law of 2–morphisms $\circ : \mathcal{G}_2(a, c) \times \mathcal{G}_2(b, d) \rightarrow \mathcal{G}_2(b \circ a, d \circ c)$;*

8. *for each pair of objects x, y and for each pair of 1–morphisms $a, b \in \mathcal{G}_1(x, y)$, a horizontal inversion law of 2–morphisms $^{-1\circ} : \mathcal{G}_2(a, b) \rightarrow \mathcal{G}_2(a^{-1\circ}, b^{-1\circ})$;*

9. *for each pair of objects x, y and for each triple of 1–morphisms $a, b, c \in \mathcal{G}_1(x, y)$, a vertical composition law of 2–morphisms $\bullet : \mathcal{G}_2(a, b) \times \mathcal{G}_2(b, c) \to \mathcal{G}_2(a, c)$;*

10. *for each pair of objects x, y and for each pair of 1–morphisms $a, b \in \mathcal{G}_1(x, y)$, a vertical inversion law of 2–morphisms $^{-1\bullet} : \mathcal{G}_2(a, b) \to \mathcal{G}_2(b, a)$;*

11. *for each pair of objects x, y and for each 1–morphism $a \in \mathcal{G}_1(x, y)$, a distinguished unit 2–cell $1_a \in \mathcal{G}_2(a, a)$.*

These are required to satisfy the following axioms.

$$(c \circ b) \circ a = c \circ (b \circ a), \tag{2.2.3a}$$

$$a^{-1\circ} \circ a = 1_x, \qquad a \circ a^{-1\circ} = 1_y, \tag{2.2.3b}$$

$$a \circ 1_x = 1_y \circ a = a, \tag{2.2.3c}$$

$$(C \circ B) \circ A = C \circ (B \circ A), \tag{2.2.3d}$$

$$A^{-1\circ} \circ A = 1_{1_x} \qquad A \circ A^{-1\circ} = 1_{1_y}, \tag{2.2.3e}$$

$$A \circ 1_{1_x} = 1_{1_y} \circ A = A, \tag{2.2.3f}$$

$$(C \bullet B) \bullet A = C \bullet (B \bullet A), \tag{2.2.3g}$$

$$A^{-1\bullet} \bullet A = 1_a, \qquad A \bullet A^{-1\bullet} = 1_b, \tag{2.2.3h}$$

$$A \bullet 1_a = 1_b \bullet A = A, \tag{2.2.3i}$$

$$(D \bullet C) \circ (B \bullet A) = (D \circ B) \bullet (C \circ A). \tag{2.2.3j}$$

Here and in the following, $x, y, z, \cdots \in \mathcal{G}_0$, $a, b, c, \cdots \in \mathcal{G}_1$, $A, B, C, \cdots \in \mathcal{G}_2$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the set of all 1– and 2–morphisms, respectively. All identities involving the horizontal and vertical composition and inversion hold whenever defined. Relation (2.2.3j) is again the exchange law.

Morphisms of higher groups and higher groupoids are just higher functors between them. This leads to the following definitions:

Definition 48. *Given two groupids $\mathcal{G}$ and $\mathcal{H}$, a **groupid morphism** $F : \mathcal{G} \to \mathcal{H}$ consists of the following set of data:*

- *a map $F_0 : \mathcal{G}_0 \to \mathcal{H}_0$;*

- *for every couple of objects $x, y \in \mathcal{G}_0$, a map $F_1(x, y) : \mathcal{G}_1(x, y) \to \mathcal{H}_1(F_0(x), F_0(y))$.*

These are required to satisfy the following axioms.

$$F_1(x, x)(1_x) = 1_{F_0(x)}, \tag{2.2.4a}$$

$$F_1(x, z)(a \circ_{\mathcal{G}} b) = F_1(x, y)(a) \circ_{\mathcal{H}} F_1(y, z)(b). \tag{2.2.4b}$$

In the following, $F_1(x, y)$ will be denoted simply as F_1 for notational convenience.

Definition 49. *Given two strict 2-groups $\mathcal{G}$ and $\mathcal{H}$, a strict **2-group morphism** $F : \mathcal{G} \to \mathcal{H}$ consists of the following set of data:*

- *a map $F_1 : \mathcal{G}_1 \to \mathcal{H}_1$;*

- *for every couple of 1-morphisms $a, b \in \mathcal{G}_1$ a map $F_2(a, b) : \mathcal{G}_2(a, b) \to \mathcal{H}_2(a, b)$.*

These are required to satisfy the following axioms.

$$F_1(1_\mathcal{G}) = 1_\mathcal{H}, \tag{2.2.5a}$$

$$F_1(a \circ_\mathcal{G} b) = F_1(a) \circ_\mathcal{H} F_1(b), \tag{2.2.5b}$$

$$F_2(a, a)(1_a) = 1_{F_1(a)}, \tag{2.2.5c}$$

$$F_2(a \circ_\mathcal{G} c, b \circ_\mathcal{G} d)(A \circ_\mathcal{G} B) = F_2(a, b)(A) \circ_\mathcal{H} F_2(c, d)(B), \tag{2.2.5d}$$

$$F_2(a, c)(A \bullet_\mathcal{G} B) = F_2(a, b)(A) \bullet_\mathcal{H} F_2(b, c)(B). \tag{2.2.5e}$$

In the following $F_2(a, b)$ will be denoted simply as F_2 for notational convenience.

Definition 50. *Given two strict 2-groupoids $\mathcal{G}$ and $\mathcal{H}$, a strict **2-groupoid morphism** $F : \mathcal{G} \to \mathcal{H}$ consists of the following set of data:*

- *a map $F_0 : \mathcal{G}_0 \to \mathcal{H}_0$;*

- *for every couple of objects $x, y \in \mathcal{G}_0$ a map $F_1(x, y) : \mathcal{G}_1(x, y) \to \mathcal{H}_1(x, y)$;*

- *for every couple of 1-morphisms $a, b \in \mathcal{G}_1(x, y)$ a map $F_2(a, b) : \mathcal{G}_2(a, b) \to \mathcal{H}_2(a, b)$.*

These are required to satisfy the following axioms

$$F_1(x, x)(1_x) = 1_{F_0(x)}, \tag{2.2.6a}$$

$$F_1(x, z)(a \circ_\mathcal{G} b) = F_1(x, y)(a) \circ_\mathcal{H} F_1(y, z)(b), \tag{2.2.6b}$$

$$F_2(a, a)(1_a) = 1_{F_1(a)}, \tag{2.2.6c}$$

$$F_2(a \circ_\mathcal{G} c, b \circ_\mathcal{G} d)(A \circ_\mathcal{G} B) = F_2(a, b)(A) \circ_\mathcal{H} F_2(c, d)(B), \tag{2.2.6d}$$

$$F_2(a, c)(A \bullet_\mathcal{G} B) = F_2(a, b)(A) \bullet_\mathcal{H} F_2(b, c)(B). \tag{2.2.6e}$$

In the following $F_1(x, y)$ and $F_2(a, b)$ will be denoted simply as F_1 and F_2 respectively for notational convenience.

2.3   Crossed modules

We are turning now to a very interesting definition which is closely related to 2-groups:

Definition 51. *A **crossed module** is a quadruple (G, H, m, t) where G and H are groups, $m : G \to Hom(H, H)$ and $t : H \to G$ are group homomorphisms such that the following relations are satisfied:*

$$t(m(g)(h)) = g t(h) g^{-1} \ , \ \forall g \in G, h \in H, \tag{2.3.1}$$

$$m(t(h))(h') = h h' h^{-1} \ , \ \forall h, h' \in H. \tag{2.3.2}$$

Relation (2.3.2) is also called the Peiffer idendity.

The link with 2-groups is shown in the next proposition:

Proposition 14. *There is one-to-one correspondence between crossed modules and strict 2-groups*

Proof. Let $\mathcal{G}$ be a strict 2-group with $\mathcal{G}_1$ set of 1-morphisms and $\mathcal{G}_2$ set of 2-morphisms. We need to find a quadruple (G, H, m, t) starting from $\mathcal{G}$. Since $\mathcal{G}_1$ is itself a group with the multiplication given by the horizontal composition, we set $G = \mathcal{G}_1$. Denoting 1 the identity 1-morphism in $\mathcal{G}_1$ and $s : \mathcal{G}_2 \to \mathcal{G}_1$ the source map of the 2-group on 2-morphisms, we will prove that $s^{-1}(1)$ is a group too. It is the set of all 2-morphisms starting at 1. Two such 2-morphisms can be composed horizontally to give another 2-morphism in $s^{-1}(1)$, and this composition provides the group law on $s^{-1}(1)$. This composition is associative since horizontal composition of 2-morphisms is associative, and it has 1_1, the identity 2-morphism on the identity 1-morphism in $\mathcal{G}$, as identity. We can thus set $H = s^{-1}(1)$.

The map m is constructed in this way: given $g \in G = \mathcal{G}_1$, the action of $m(g)$ on $h \in H = s^{-1}(1)$ is given by

$$m(g)(h) = 1_g \circ h \circ 1_{g^{-1}}. \tag{2.3.3}$$

This map is well defined since $s(1_g \circ h \circ 1_{g^{-1}}) = g \circ 1 \circ g^{-1} = 1$. This is a homomorphism on H because $m(g)(h) \circ m(g)(h') = 1_g \circ h \circ 1_{g^{-1}} 1_g \circ h \circ 1_{g^{-1}} = 1_g \circ h \circ h' \circ 1_{g^{-1}} = m(g)(h \circ h')$ and it is a homomorphism from G to $\mathrm{Hom}(H, H)$ because $m(g)(m(g')(h)) = 1_g \circ 1_{g'} \circ h \circ 1_{g'^{-1}} \circ 1_{g^{-1}} = 1_{g \circ g'} \circ h \circ 1_{(g \circ g')^{-1}} = m(g \circ g')(h)$.

The map $t : H \to G$ is simply the restriction to H of the target map $t : \mathcal{G}_2 \to \mathcal{G}_1$. It is a group homomorphism because the target map of the horizontal composition of two 2-morphisms is the horizontal composition of the targets of the composed 2-morphisms.

We must now check the axioms of a crossed module. Relation (2.3.1) is very simple and follows straightforwardly from our definitions: $t(m(g)(h)) = t(1_g \circ h \circ 1_{g^{-1}}) = g \circ t(h) \circ g^{-1}$. Relation (2.3.2) is a bit trickier:

$$m(t(h))(h') = 1_{t(h)} \circ h' \circ 1_{t(h)^{-1}} = \left(1_{t(h)} \circ h' \circ 1_{t(h)^{-1}}\right) \bullet \left(h \circ 1_1 \circ h^{-1}\right) =$$

$$= \left(1_{t(h)} \bullet h\right) \circ \left(h' \bullet 1_1\right) \circ \left(1_{t(h)^{-1}} \bullet h^{-1}\right) = h \circ h' \circ h^{-1}. \tag{2.3.4}$$

Here we used the exchange law (4) for the 2-group $\mathcal{G}$, as shown in this diagram:

$$\tag{2.3.5}$$

This proves that $(\mathcal{G}_1, s^{-1}(1), m, t)$ is a crossed module.

Conversely, given (G, H, m, t) a crossed module we can define a strict 2-group $\mathcal{G}$ as $\mathcal{G}_1 = G$ and $\mathcal{G}_2 = H \rtimes G$. Horizontal composition on 1-morphisms is the group multiplication on G with the corresponding identity. Given a 2-morphisms (h, g), we set $s(h, g) = g$ and $\tau(h, g) = t(h)g$ (here we denoted $\tau : \mathcal{G}_2 \to \mathcal{G}_1$ the target map of $\mathcal{G}$ not to cause confusion with the homomorphism t of the crossed module). Horizontal and vertical compositions of 2-morphisms are defined as follows: $(h', t(h)g) \bullet (h, g) = (h'h, g)$ and $(h', g') \circ (h, g) = (h'm(g', h), g'g)$. The identity 2-morphism on g is $(1_H, g)$. it is easy to show that this is a strict 2-category: horizontal composition is well defined since $s((h', g') \circ (h, g)) = s(h'm(g', h), g'g) = g'g$ and $\tau((h', g') \circ (h, g)) = t(h'm(g', h))g'g = t(h')g't(h)g'^{-1}g'g = t(h')g't(h)g$. Both vertical and horizontal compositions are associative and have identities as 1_G and $i_g = (1_H, g)$. The exchange law is readily checked:

$$((k', t(h')g') \circ (k, t(h)g)) \bullet ((h', g') \circ (h, g)) =$$
$$= (k'h'm(g', k)h'^{-1}, t(h')g't(h)g) \bullet (h'm(g', h), g'g) =$$
$$= (k'h'm(g', kh), g'g) = (k'h', g') \circ (kh, g) =$$
$$= ((k', t(h')g') \bullet (h', g')) \circ ((k, t(h)g) \bullet (h, g)). \tag{2.3.6}$$

Furthermore every morphism is invertible: $g^{-1\circ} = g^{-1}$, $(h, g)^{-1\bullet} = (h^{-1}, t(h)g)$ and $(h, g)^{-1\circ} = (m(g^{-1}, h^{-1}), g^{-1})$. Thus $\mathcal{G}$ is a strict 2-group. $\square$

The last proposition is very important because it means that at the core strict 2-groups theory reduces to usual group theory. All the techniques which are known in group theory can be used to study in full depth strict 2-groups. This is not true for non strict 2-groups, which appear to be extremely different from ordinary groups: unluckily, in the non strict case there isn't any result analogous to proposition (14), and usual group theory is of little or no use in this case.

What we have said up to now generalizes the algebraic structure of a group. In order to implement the continuity and the differential structure of Lie groups we have to combine higher groupoids with smooth manifolds and maps. This is done by defining obvious generalizations of Lie groups, which are called Lie groupoids:

Definition 52. *A **Lie** n-**groupoid** is a n-groupoid whose sets of objects and of i-morphisms, $i = 1, \ldots, n$, are smooth manifolds, with all the target and source maps, the identity and inversion maps and all the composition maps being smooth maps.*

2.4 L_∞ algebras

2.4.1 L_∞ algebra cohomology

L_∞ algebras are a generalization of Lie algebras. They appeared with the name of *strong homotopy Lie algebras* in [50], see also [51], and since then they have been largely explored, see [52], [37] and references therein. They play a pivotal role in the present work, therefore we will try to discuss them in some depth. In this subsection we will explain how they are related to Lie algebras and where their definition comes from.

To begin we will outline some basic concepts of graded algebra and fix some notations that will be of use later on.

Definition 53. A **graded vector space** is a direct sum $V = \oplus_{n \in \mathbb{Z}} V_n$. An element $x \in V_n$ is said to be in degree n, and we write $|x| = n$. A graded vector space is said to be positively (negatively) graded if $V_n = \{0\}$ for $n < 0$ $(n > 0)$.

Definition 54. A **graded linear map of degree** k is a linear map $f : V \to W$ between graded vector spaces V and W such that $|f(x)| = |x| + k$.

Given a graded vector space V, we can define the k-shifted graded vector space $V[k]$ which is defined as $V[k]_i := V_{i+k}$. Basically $V[k]$ has the same elements of V with the degree increased by k.

It is possible to make the usual operations between vector space on graded vector spaces. If we have two graded vector space $V = \oplus_{n \in \mathbb{Z}} V_n$ and $W = \oplus_{n \in \mathbb{Z}} W_n$ we can define their direct sum and their tensor product, which also are graded vector spaces, according to the following relations:

$$(V \oplus W)_n := V_n \oplus W_n, \tag{2.4.1}$$

$$(V \otimes W)_n := \bigoplus_{p+q=n} (V_p \otimes W_q). \tag{2.4.2}$$

The symmetric group S_n acts on $\otimes^n V$ for V a vector space in two manners according to the two 1-dimensional representations of S_n. The trivial representation gives the symmetric action:

$$\sigma : v_1 \otimes v_2 \otimes \cdots \otimes v_n \to v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}, \tag{2.4.3}$$

while the signature gives the antisymmetric action:

$$\sigma : v_1 \otimes v_2 \otimes \cdots \otimes v_n \to (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{2.4.4}$$

Taking the orbits of these two actions of the permutation group leads respectively to the symmetric algebra $S^n(V)$ and the exterior algebra $\wedge^n V$. It is possible to generalize this action to graded vector spaces.

Definition 55. Given the tensor algebra $\otimes^\bullet V$ of a graded vector space V, we define the **symmetric algebra of** V $S^\bullet(V)$ as the quotient of $\otimes^\bullet V$ with respect to the ideal of elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x, \tag{2.4.5}$$

and we define the **exterior algebra of** V $\wedge^\bullet V$ as the quotient of the tensor algebra with respect to the ideal of elements of the form

$$x \otimes y + (-1)^{|x||y|} y \otimes x = 0. \tag{2.4.6}$$

These two spaces come naturally equipped with a graded symmetric (respectively graded skew-symmetric) bilinear product given by the tensor product. The grading on $S^\bullet(V)$ is defined in the following way:

$$|x_1 \otimes x_2 \otimes \cdots \otimes x_n| = |x_1| + |x_2| + \cdots + |x_n|. \tag{2.4.7}$$

Instead we have two kinds of degrees on $\wedge^\bullet V$. One derives from the original grading of V, it is denoted $|\cdot|$ and is computed as in (2.4.7), the other is the *exterior degree*, it is denoted $\deg(x)$ and is computed according to $\deg(x) = k \Leftrightarrow x \in \wedge^k V$. The wedge product on $\wedge^\bullet V$ is graded commutative in the following way:

$$x \wedge y = (-1)^{|x||y| + \deg(x)\deg(y)} y \wedge x. \tag{2.4.8}$$

A permutation σ acts on these spaces sending a string of vectors $x_1 \otimes \cdots \otimes x_n$ to $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ times an appropriate sign. The following definition is very useful to keep track of the signs arising from permutations of graded elements:

Definition 56. *Let V be a graded vector space and $\sigma \in S_n$ a permutation. The **Koszul sign** of σ relative to the set $x_1, \ldots, x_n$ of elements of V, denoted $K(\sigma; x_1, \ldots, x_n)$ or $K(\sigma)$ for brevity, is defined as*

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n = K(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)} \tag{2.4.9}$$

where $x_i \in V$ and $\otimes$ is the symmetric product in $S^\bullet(V)$.

A permutation $\sigma \in S_n$ acts on $\wedge^n V$ in the following manner:

$$\sigma : x_1 \wedge x_2 \wedge \cdots \wedge x_n = (-1)^\sigma K(\sigma) x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(n)}. \tag{2.4.10}$$

Before facing a precise but obscure definition on L_∞ algebras, let us explain how they are linked to Lie algebras. Instead of adopting the category theory point of view, we will show how they rise from a generalization of the Chevalley-Eilenberg complex. We know from section (2.1) that, given a vector space V, the presence of a differential on the complex $C = \wedge^\bullet V^*$ furnishes a Lie algebra structure to V. What is more, Lie algebras and Chevalley-Eilenberg complexes are in one-to-one correspondence. We can now assume that the real vector space V is substituted by a negatively graded real vector space $V = \oplus_{n \leq 0} V_n$. The dual V^* is then positively graded $V^* = \oplus_{n \geq 0} V_n^*$, so that the sum of the degrees of two dual elements is zero. In this way the natural pairing $V \otimes V^* \to \mathbb{R}$ preserves the grading, since the real numbers have degree 0 by definition. To construct an analog of the Chevalley-Eilenberg complex for V we define:

$$C = S^\bullet(V^*[1]) = V_0^*[1] \oplus (V_0^*[1] \otimes V_0^*[1] \oplus V_1^*[1]) \oplus (V_0^*[1] \otimes V_1^*[1] \oplus V_2^*[1]) \oplus \cdots \tag{2.4.11}$$

If we take V concentrated in degree 0, i.e. $V_n = 0$ for $n \neq 0$, the complex defined in (2.4.11) is just $\wedge^\bullet V^*$. We can now give a differential Q of degree 1 to C as happens for Chevalley-Eilenberg complexes. Its action on the basis $\{e_{(n)}^a\}$ of $V_n^*[1]$ can be written in full generality as

$$Qe_{(0)}^a = -\frac{1}{2} f_{bc}^a e_{(0)}^b e_{(0)}^c + \phi_b^a e_{(1)}^b \tag{2.4.12}$$

$$Qe_{(1)}^a = -\rho_{bc}^a e_{(0)}^b e_{(1)}^c + \frac{1}{6} R_{bcd}^a e_{(0)}^b e_{(0)}^c e_{(0)}^d - \psi_b^a e_{(2)}^b$$

$$Qe_{(2)}^a = -\epsilon_{bc}^a e_{(0)}^b e_{(2)}^c + \frac{1}{2} T_{bc}^a e_{(1)}^b e_{(1)}^c - \frac{1}{2} G_{bcd}^a e_{(0)}^b e_{(0)}^c e_{(1)}^d + \frac{1}{24} H_{bcde}^a e_{(0)}^b e_{(0)}^c e_{(0)}^d e_{(0)}^e - S_b^a e_{(3)}^b \cdots$$

$$\vdots$$

(Here the signs are put by convention, and we denoted the indexes of the different set of basis $\{e_{(n)}^a\}$ with the same small latin letters $a, b, c, \ldots$ just for notational simplicity). The coefficients $f_{bc}^a, \phi_b^a, \rho_{bc}^a, R_{bcd}^a, \ldots$ are called the structure constants and can be used to define several multilinear brackets between the V_n's. For example, f_{bc}^a defines a bracket $[\cdot, \cdot] : V_0 \wedge V_0 \to V_0$, as happens for an usual Lie algebra, and the constant G_{bcd}^a defines a trilinear bracket $[\cdot, \cdot, \cdot] : V_0 \wedge V_0 \otimes V_1 \to V_2$. The constraint $Q^2 = 0$ implies then several algebraic identities that must be satisfied by the structure constants, and in turn this implies that there are the same amount of algebraic relations that must be satisfied by the brackets. These relations can then be viewed as generalizations of the Jacobi identity for a Lie algebra, because the Jacobi identity is the relation we would get if V is concentrated in degree 0. The graded vector space V with this infinite tower of multibrackets is called L_∞ algebra.

We turn now to a more concise definition. Define the $i(n-i)$-**unshuffle** $S_{i(n-i)}$ as the subset of S_n made of all permutations σ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots \sigma(n)$. They are all permutations of n elements that do not change the ordering of the first i elements nor of the last $n-i$ elements. We can now define an L_∞ algebra:

Definition 57. *A L_∞ **algebra** is a negatively graded vector space $\mathfrak{v}$ with multilinear skewsymmetric brackets $l_n : \wedge^n \mathfrak{v} \to \mathfrak{v}$ for $n \in \mathbb{N}$ of degree $\deg(l_n) = 2 - n$ such that they satisfy the following relations:*

$$\sum_{i+j=n+1} \sum_{\sigma \in S_{j,n-j}} (-1)^\sigma K(\sigma)(-1)^{i(j+1)} l_i(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) = 0$$

(2.4.13)

*for $n \geq 1$. If $\mathfrak{v}_k = 0$ for $k \leq n$ then the L_∞ algebra is called a n-**term** L_∞ **algebra**.*

We remark that there are different conventions for the signs in formula (2.4.13) and other choices can be found in the literature. The differential chain complex which is naturally associated to a L_∞ algebra $\mathfrak{v}$ as described above will be called the **Chevalley-Eilenberg complex** of $\mathfrak{v}$ and denoted $CE^\bullet(\mathfrak{v})$.

Let us examine in detail some examples of equation (2.4.13). For $n = 1$ we need just one element $x \in \mathfrak{v}$ and the equation becomes

$$l_1(l_1(x)) = 0, \tag{2.4.14}$$

stating that the linear operator $l_1 : \mathfrak{v}^\bullet \to \mathfrak{v}^{\bullet+1}$ squares to zero. This makes $(\mathfrak{v}, l_1)$ a chain complex with l_1 a differential of degree 1. As a convention, l_1 is denoted ∂. For $n = 2$ the relation (2.4.13) becomes

$$\partial l_2(x_1, x_2) - l_2(\partial x_1, x_2) - (-1)^{|x_1|} l_2(x_1, \partial x_2) = 0, \tag{2.4.15}$$

which is exactly a graded Leibniz identity, so that ∂ acts as a derivation of degree 1 with respect to the operator l_2. Another interesting case is $n = 3$, which reads

$$[[x_1, x_2], x_3] + (-1)^{|x_1|(|x_2|+|x_3|)}[[x_2, x_3], x_1] + (-1)^{|x_3|(|x_1|+|x_2|)}[[x_3, x_1], x_2] +$$

$$+ \partial l_3(x_1, x_2, x_3) + l_3(\partial x_1, x_2, x_3) + (-1)^{|x_1|} l_3(x_1, \partial x_2, x_3) + (-1)^{|x_1|+|x_2|} l_3(x_1, x_2, \partial x_3) = 0,$$

(2.4.16)

where we used square brackets $[\cdot, \cdot]$ to denote the operator $l_2 : \mathfrak{v}^p \wedge \mathfrak{v}^q \to \mathfrak{v}^{p+q}$. The first line of this relation is the Jacobi identity for the square brackets, and the second line can be viewed as an homotopy equivalence between the Jacobi identity and zero. This means that in a L_∞ algebra the Jacobi identity for l_2 holds only up to a higher homotopy governed by a higher operator (l_3) and the differential ∂. Remarkably, the failure of the Jacobi identity to hold is equal to the failure of the differential ∂ to fulfill a graded Leibniz rule with respect to l_3. This is why in the works of Stasheff et al. [50] and [51] L_∞ algebra were called *strong homotopy Lie algebras*. Notice that if $\mathfrak{v} = \mathfrak{v}_0$ relation (2.4.16) would actually be the Jacobi identity for $[\cdot, \cdot]$, because if $\mathfrak{v}$ is concentrated in degree 0 there is no room for ∂ and l_3 which would vanish identically. This again confirms that a 1-term L_∞ algebra is just a Lie algebra.

So far we gave two different definitions of L_∞ algebras. We will now prove that these two definitions coincide [51]. First of all we define a homomorphisms $\eta : \wedge^\bullet V \to S^\bullet(V[-1])$ for V a negatively graded vector space. To avoid confusion between these two spaces, we will denote with x_i elements of V and with v_i the corresponding elements in $V[-1]$, so that $|v_i| = |x_i| - 1$. The map η can then be defined as:

$$\eta(x_1 \wedge \cdots \wedge x_n) = (-1)^{\sum_{i=1}^{n-1}(n-i)|x_i|} v_1 \otimes \cdots \otimes v_n. \qquad (2.4.17)$$

This is an algebra homomorphism which preserves the action of S_n. It is enough to check this on a transposition of two elements:

$$\eta(t_{12}(x_1 \wedge x_2)) = (-1)^{|x_1||x_2|+1}\eta(x_2 \wedge x_1) = (-1)^{|x_2|(|x_1|+1)+1} v_2 \otimes v_1, \qquad (2.4.18)$$

$$t_{12}(\eta(x_1 \wedge x_2)) = (-1)^{|x_1|}t_{12}(v_1 \otimes v_2) = (-1)^{|x_1|+(|x_1|-1)(|x_2|-1)} v_2 \otimes v_1. \qquad (2.4.19)$$

The map η doesn't have a fixed degree. If we call η_k the restriction of η to $\wedge^k V$, then $|\eta_k| = -k$.

Now take $\mathfrak{v}$ an L_∞ algebra as in definition (57). We can use the map η to define multilinear operators on $\mathfrak{v}[-1]$:

$$\hat{l}_n : S^n(\mathfrak{v}[-1]) \to \mathfrak{v}[-1]; \; , \; \hat{l}_n := \eta \circ l_n \circ \eta^{-1}. \qquad (2.4.20)$$

Notice that $|\hat{l}_n| = |\eta_1| + |l_n| + |\eta_n^{-1}| = 1$. We can extend the operators $\hat{l}_n$ to the whole $S^\bullet(\mathfrak{v}[-1])$:

$$\tilde{l}_k(v_1 \otimes \cdots \otimes v_n) := \begin{cases} \sum_{\sigma \in S_{k(n-k)}} K(\sigma, v) \hat{l}_k(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) \otimes v_{\sigma(k+1)} \otimes \cdots \otimes v_{\sigma(n)} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} . \qquad (2.4.21)$$

These linear maps define by duality an operator Q on $S^\bullet(\mathfrak{v}^*[1])$ in the following way:

$$\langle Q\xi, v \rangle := \langle \xi, \sum_{n \geq 1} \tilde{l}_n(v) \rangle, \qquad (2.4.22)$$

where $\xi \in S^\bullet(\mathfrak{v}^*[1])$ and $v \in S^\bullet(\mathfrak{v}[-1])$. The operator Q has degree 1. $S^\bullet(\mathfrak{v}^*[1])$ is the Chevalley-Eilenberg complex of the L_∞ algebra $\mathfrak{v}$. The following result shows that Q defines a differential on it:

Theorem 2.

$$Q^2 = 0. \qquad (2.4.23)$$

Proof. We have that

$$\langle Q^2\xi, v\rangle = \langle \xi, \sum_{i,j} \tilde{l}_i(\tilde{l}_j(v))\rangle, \tag{2.4.24}$$

thus we have to show that $\sum_{i,j} \tilde{l}_i(\tilde{l}_j(v))$ vanishes. Suppose $v = v_1 \otimes \cdots \otimes v_n$ with $v_i \in \mathfrak{v}[-1]$ for $i = 1, 2, \ldots, n$. The sum can be rearranged in the following way:

$$\sum_{i,j\geq 1} \tilde{l}_i(\tilde{l}_j(v_1 \otimes \cdots \otimes v_n)) = \sum_{k\geq 1} \sum_{i+j=k+1} \tilde{l}_i(\tilde{l}_j(v_1 \otimes \cdots \otimes v_n)). \tag{2.4.25}$$

In the last sum there are two kinds of terms, namely those appearing like

$$\hat{l}_i(\hat{l}_j(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(j+i-1)}) \otimes \cdots \otimes v_{\sigma(n)} \tag{2.4.26}$$

and those like

$$\hat{l}_i(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \otimes \hat{l}_j(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(i+j)}) \otimes \cdots \otimes v_{\sigma(n)}, \tag{2.4.27}$$

times an appropriate sign, where σ is the composition of the two unshuffles arising from $\tilde{l}_i$ and $\tilde{l}_j$, see (2.4.21). Let us analyze the terms like (2.4.26):

$$K(\sigma, v)\hat{l}_i(\hat{l}_j(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(j+i-1)}) \otimes \cdots \otimes v_{\sigma(n)} =$$

$$= K(\sigma, v)(-1)^{\sum_{p=1}^{j}(j-p)|x_{\sigma(p)}|}\hat{l}_i(\eta_1 l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(j+i-1)}) \otimes \cdots \otimes v_{\sigma(n)} =$$

$$= K(\sigma, v)(-1)^{\sum_{p=1}^{j}(j-p)|x_{\sigma(p)}|+(i-1)|l_j(x_{\sigma(1)},\ldots,x_{\sigma(j)})|+\sum_{q=2}^{i}(i-q)|x_{\sigma(j+q-1)}|} \times$$

$$\times \eta_1 l_i(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(j+i-1)}) \otimes \cdots \otimes v_{\sigma(n)}. \tag{2.4.28}$$

Using the fact that $|l_j(x_1, \ldots, x_j)| = 2 - j + \sum_{p=1}^{j} |x_p|$, we find that the sign in the last term is equal to

$$K(\sigma, v)(-1)^{(i-1)j+\sum_{p=1}^{i+j-1}(i+j-1-p)|x_{\sigma(p)}|}. \tag{2.4.29}$$

Now notice that for a string of n elements of $\mathfrak{v}$ $x_1, \ldots, x_n$ and a permutation $\sigma \in S_n$, the following technical result holds:

$$(-1)^{\sum_{p=1}^{n}(n-p)|x_{\sigma(p)}|}K(\sigma, v) = (-1)^{\sum_{p=1}^{n}(n-p)|x_p|}(-1)^\sigma K(\sigma, x). \tag{2.4.30}$$

As usual, it is enough to check it on a transposition $t_{i,i+1}$, for which a simple computation shows the result. Finally terms like (2.4.26) can be written as

$$(-1)^{\sum_{p=1}^{n}(n-p)|x_p|}(-1)^\sigma K(\sigma, x)(-1)^{(i+1)j} \times$$

$$\times \eta_1 l_i(l_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(j+i-1)}) \otimes \cdots \otimes v_{\sigma(n)}. \tag{2.4.31}$$

Since the map η and the tensor product are linear and the sign in front doesn't depend on σ, i or j, they can be factored out of the sum over i and j, which vanishes due to (2.4.13).

We turn now on terms as in (2.4.27). They simply cancel in pairs because each of them appears two times with opposite signs. First of all notice that since $\hat{l}_j$ has odd degree, (2.4.21) can be cast in the form

$$\tilde{l}_j(v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_{j(n-j)}} K(\sigma, v)(-1)^{\sum_{p=1}^{i} |v_{\sigma(p)}|} \times$$

$$\times v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)} \otimes \hat{l}_j(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(i+j)}) \otimes \cdots \otimes v_{\sigma(n)}. \tag{2.4.32}$$

If we use this to move $\hat{l}_j$ i times with $\hat{l}_i$ left in the first place in $\tilde{l}_i(\tilde{l}_j(v))$, we see that we get a term like (2.4.27) with the sign

$$K(\sigma, v)(-1)^{\sum_{p=1}^{i} |v_{\sigma(p)}|}. \tag{2.4.33}$$

Consider now the summand $\tilde{l}_j(\tilde{l}_i(v))$, take the same permutation σ and leave at first position $\hat{l}_i$ while moving $\hat{l}_j$ of 1 step so that it goes at the right of $\hat{l}_i(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)})$. The result is again the term in (2.4.27), but the sign is

$$K(\sigma, v)(-1)^{|\hat{l}_i(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)})|}, \tag{2.4.34}$$

which is exactly the opposite of (2.4.33).   □

This theorem proves the equivalence between the two definitions that we gave for L_∞ algebra: the one that defines it as a graded vector space with an infinite number of brackets satisfying (2.4.13) and the one that defines it as the dual of a graded Chevalley-Eilenberg differential complex.

2.4.2   2-term L_∞ algebras

We will now focus on 2-term L_∞ algebras, which are the simplest example of L_∞ algebras which are not Lie algebras [37]. Above we defined general L_∞ algebras including also the case of 2-term L_∞ algebra. Nevertheless, let us give another more pedantic definition for 2-term L_∞ algebra, which is a main character in this thesis. The equivalence between the two definitions is a trivial check.

Definition 58. *A **2-term L_∞ algebra** $\mathfrak{v}$ consists of two real vector spaces $\mathfrak{v}_0$ and $\mathfrak{v}_1$ together with the following linear maps:*

- $\partial : \mathfrak{v}_1 \to \mathfrak{v}_0$

- $[\cdot, \cdot] : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_0$

- $[\cdot, \cdot] : \mathfrak{v}_0 \otimes \mathfrak{v}_1 \to \mathfrak{v}_1$

- $[\cdot, \cdot, \cdot] : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_1$

which are required to satisfy the following relations:

$$\partial[x, X] - [x, \partial X] = 0 \tag{2.4.35}$$

$$[\partial X, Y] + [\partial Y, X] = 0 \tag{2.4.36}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] - \partial[x, y, z] = 0 \tag{2.4.37}$$

$$[x, [y, X]] - [y, [x, X]] - [[x, y], X] - [x, y, \partial X] = 0 \tag{2.4.38}$$

$$[x, y, [z, t]] + [x, z, [t, y]] + [x, t, [y, z]] - [y, z, [t, x]] - [z, t, [y, x]] - [t, y, [z, x]] -$$
$$- [x, [y, z, t]] + [y, [z, t, x]] - [z, [t, x, y,]] + [t, [x, y, z]] = 0 \tag{2.4.39}$$

for all $x, y, z, t \in \mathfrak{v}_0$ and $X, Y \in \mathfrak{v}_1$.

The algebraic structure is very complicated already at this level, since going from a Lie algebra to a 2-term L_∞ algebra the number of brackets goes from 1 to 4, and the number of constraints that these brackets must satisfy grows even more. It appears that it is very uncomfortable to handle L_∞ algebras with more than 2 terms in such an explicit way.

Notice that in this definition both $\mathfrak{v}_0$ and $\mathfrak{v}_1$ have degree 0, and everything reduces to standard linear algebra. We also denoted $\mathfrak{v}_1$ instead of $\mathfrak{v}_{-1}$ for notational convenience. The original grading of definition (57) here is hidden in the relative signs that appear in the axioms and in the symmetry properties of the brackets. To avoid confusion later on while dealing with 2-term L_∞ algebras, we introduce the following notation: we will write $\mathfrak{v}$, $\mathfrak{v}_0$ and $\mathfrak{v}_1$ for the linear vector spaces without any grading, or equivalently with zero grading, as in definition 58; we will write $\tilde{\mathfrak{v}}$, $\tilde{\mathfrak{v}}_0$ and $\tilde{\mathfrak{v}}_1$ for the graded version of the 2-term L_∞ algebra. Practically the only difference is that $\deg \mathfrak{v}_1 = 0$ while $\deg \tilde{\mathfrak{v}}_1 = -1$.

The Chevalley-Eilenberg cohomology of a 2-term L_∞ algebra $\mathfrak{v}$ can be written concisely in the following way. The complex is generated by $\mathfrak{v}_0^*[1] \oplus \mathfrak{v}_1^*[2]$, where the second grading takes into account the fact that here $\mathfrak{v}_1$ has degree 0 instead of -1. Taking $\{e_a\}$ and $\{E_A\}$ basis for $\mathfrak{v}_0$ and $\mathfrak{v}_1$ respectively, and dual basis $\{\xi^a\}$ and $\{\Xi^A\}$ of $\mathfrak{v}_0^*[1]$ and of $\mathfrak{v}_1^*[2]$, we define $\pi := \xi^a \otimes e_a$ and $\Pi = \Xi^A \otimes E_A$, where a sum over repeated indexes is understood. The Chevalley-Eilenberg differential can then be written as

$$Q_{CE}\pi = -\frac{1}{2}[\pi, \pi] + \partial \Pi, \tag{2.4.40a}$$

$$Q_{CE}\Pi = -[\pi, \Pi] + \frac{1}{6}[\pi, \pi, \pi]. \tag{2.4.40b}$$

This formalism can be used to write shorter forms of the identities involving multi-brackets. For example relation (2.4.39) can be rewritten as

$$6[\pi, \pi, [\pi, \pi]] - 4[\pi, [\pi, \pi, \pi]] = 0. \tag{2.4.41}$$

The expressions for Q_{CE} can be readily cast into coordinates:

$$Q_{CE}\xi^a = -\frac{1}{2}f_{bc}^a \xi^b \xi^c + \partial_B^a \Xi^B, \tag{2.4.42}$$

$$Q_{CE}\Xi^A = -f_{aB}^A \xi^a \Xi^B + \frac{1}{6}R_{abc}^A \xi^a \xi^b \xi^c, \tag{2.4.43}$$

where the constants ∂_A^a, f_{bc}^a, f_{aB}^A and R_{abc}^A define the multilinear brackets of $\mathfrak{v}$.

It is worth to remark that 2-term L_∞ algebra can be viewed in a different fashion, through which the analogy with Lie algebras and especially the categorical setting become more apparent. Indeed, 2-term L_∞ algebras are equivalent to *Lie 2-algebras*, which are a categorical generalization of Lie algebras. They consist of the following data:

- a category `L` internal to `Vect`, that is a category whose sets of objects and morphisms are vector spaces and whose composition, source, target and identity maps are linear maps;

- a bilinear antisymmetric functor $\{\cdot,\cdot\} : \mathsf{L} \times \mathsf{L} \to \mathsf{L}$;

- a natural isomorphism $J_{x,y,z} : \{\{x,y\},z\} \Rightarrow \{x,\{y,z\}\} + \{\{x,z\},y\}$ called the *Jacobiator*;

all these structures are required to satisfy some axioms, see [37] for details. Lie 2-algebras are immediately seen as generalized Lie algebras: the underlying vector space is replaced by a category internal to vector spaces, the bracket becomes then a functor and the Jacobi identity is asked to hold up to a natural isomorphism. In [37] it was shown that the category of all Lie 2-algebras and the category of all 2-term L_∞ algebras are equivalent. Even more, these two objects are in one to one correspondence. Given a 2-term L_∞ algebra $\mathfrak{m}$ we can construct a Lie 2-algebra in the following way:

- $\mathsf{L}_0 := \mathfrak{v}_0$,

- $\mathsf{L}_1 := \mathfrak{v}_0 \oplus \mathfrak{v}_1$,

- $s(x,X) := x$,

- $t(x,X) := x + \partial X$,

- $i(x) := (x,0)$,

- $\{x,y\} := [x,y]$,

- $\{(x,X),(y,Y)\} := ([x,y],[x,Y]-[y,X]+[\partial X,Y])$,

- $J_{x,y,z} = ([[x,y],z],[x,y,z])$,

where L_0 and L_1 are respectively the space of objects and of morphisms of L. On the other hand, given a Lie 2-algebra L we can define a 2-term L_∞ algebra in this way:

- $\mathfrak{v}_0 := \mathsf{L}_0$,

- $\mathfrak{v}_1 := \ker s \subseteq \mathsf{L}_1$,

- $\partial X := t(X)$,

- $[x,y] := \{x,y\}$,

- $[x,X] := \{1_x,X\}$,

- $[x,y,z] := J_{x,y,z} - 1_{[[x,y],z]}$.

These definitions fulfill all axioms of both 2-term L_∞ algebra and of Lie 2-algebra, again we refer the interested reader to [37] for the details. Furthermore these two maps between the set of Lie 2-algebras and te set of 2-term L_∞ algebras are one the inverse of the other.

2.4.3 Differential Lie crossed module

The L_∞ algebra we have defined is also called *semistrict L_∞ algebra*. Analogously to what happens in higher category theory, there is also a notion of *strict L_∞* algebra, which we introduce now.

Definition 59. *An L_∞ algebra $\mathfrak{v}$ is called **strict** if $l_n = 0$ for $n \geq 2$.*

A strict L_∞ algebra turns then out to be simply a differential chain complex $\partial : \mathfrak{v}_i \to \mathfrak{v}_{i-1}$ together with a bilinear graded skewsymmetric bracket $[\cdot, \cdot]$ of degree zero, such that the differential ∂ enjoys the graded Leibniz identity (2.4.15) and the graded Jacobi identity holds (2.4.16).

The most interesting example is that of a 2-term strict L_∞ algebra. In this case we recover a differential Lie crossed module:

Definition 60. *A **differential Lie crossed module** is a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ together with two Lie algebra homomorphisms $\tau : \mathfrak{h} \to \mathfrak{g}$ and $\mu : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{h})$, where $\mathfrak{Der}(\mathfrak{h})$ is the Lie algebra of derivations of $\mathfrak{h}$, such that the following relations hold:*

$$\tau(\mu(g)(h)) = [g, \tau(h)], \tag{2.4.44}$$

$$\mu(\tau(h))(h') = [h, h']. \tag{2.4.45}$$

Proposition 15. *strict 2-term L_∞ algebras and differential Lie crossed modules are in one to one correspondence.*

Proof. The correspondence is given by the following expressions:

- $\mathfrak{g} = \mathfrak{v}_0$,

- $\mathfrak{h} = \mathfrak{v}_1$,

- $[x, y]_\mathfrak{g} = [x, y]_\mathfrak{v}$,

- $[X, Y]_\mathfrak{h} = [\partial X, Y]_\mathfrak{v}$,

- $\tau = \partial$,

- $\mu(x)(X) = [x, X]_\mathfrak{v}$.

These relations provide a strict 2-term L_∞ algebra if a differential Lie crossed module is given, and a differential Lie crossed module if a strict 2-term L_∞ algebra is given. It's easy to check that all axioms are satisfied. $\square$

As their name suggests, differential Lie crossed modules are the infinitesimal version of Lie crossed modules: givenn a Lie crossed module (G, H, t, m), its differential Lie crossed module is $(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$ where $\mathfrak{g}$, $\mathfrak{h}$ are the Lie algebras of G, H, respectively, and τ and μ are the differential of the maps t and m repsectively:

$$\tau(X) = \left.\frac{dt(C(v))}{dv}\right|_{v=0}, \tag{2.4.46}$$

$$\mu(x)(X) = \frac{\partial}{\partial u}\left(\left.\frac{\partial m(c(u))(C(v))}{\partial v}\right|_{v=0}\right)\Big|_{u=0}, \tag{2.4.47}$$

where $x \in \mathfrak{g}$, $X \in \mathfrak{h}$, $c(u)$ is any curve in G such that $c(u)\big|_{u=0} = 1_G$ and $dc(u)/du\big|_{u=0} = x$ and $C(v)$ is any curve in H such that $C(v)\big|_{v=0} = 1_H$ and $dC(v)/dv\big|_{v=0} = X$.

Much as for their finite counterparts, differential Lie crossed modules are very simple to handle because well known Lie algebra techniques are all is needed to manipulate them, but they are only a particular case of semistrict L_∞ algebras, and do not display all their interesting features.

Together with the usual adjoint representation of G on $\mathfrak{g}$ and of H on $\mathfrak{h}$, due to the extra structure of crossed modules there are other ways in which the groups G and H act on the differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$. We have an action of G on $\mathfrak{h}$, denoted $\dot{m} : G \times \mathfrak{h} \to \mathfrak{h}$, defined as

$$\dot{m}(g)(X) := \frac{d}{ds}\left(m(g)(h(s))\right)|_{s=0}, \tag{2.4.48}$$

where $g \in G$ and $h(s) : \mathbb{R} \to H$ is a smooth curve such that $h(0) = 1_H$ and $(dh(s)/ds)|_{s=0} = X \in \mathfrak{h}$. Differentiating the same map m on the first argument leads instead to an operator $Q : H \times \mathfrak{g} \to \mathfrak{h}$:

$$Q(h)(x) := \frac{d}{ds}\left(m(g(s))(h)\right)|_{s=0}, \tag{2.4.49}$$

where $h \in H$ and $g(s) : \mathbb{R} \to G$ is a smooth curve such that $g(0) = 1_G$ and $(dg(s)/ds)|_{s=0} = x \in \mathfrak{g}$.

The relation between Lie crossed modules and differential Lie crossed modules is the only case in the theory of higher groups in which Lie theory has found a full generalization. Nevertheless this is of little use and interest, since the differentiation of a Lie crossed module or the integration of a differential Lie crossed module simply exploit usual Lie differentiation or Lie integration of usual Lie groups and Lie algebras. The semistrict case is extremely more complicated and obscure. Despite some notable efforts, at the state of the art we lack a satisfactory theory for the integration of L_∞ algebras or for the differentiation of higher groups, and this poses severe obstacles to the theoretical development of higher gauge theories. We will come back to this in later subsections.

2.4.4 L_∞ algebra morphisms

Through the Chevalley-Eilenberg complex it is possible to define what a L_∞ algebra morphism is. Recall that the chain maps of $CE(\mathfrak{g})$ for $\mathfrak{g}$ a Lie algebra are dual to the homomorphisms of $\mathfrak{g}$. We can generalize this equivalence:

Definition 61. *Given $\tilde{\mathfrak{v}}$ and $\tilde{\mathfrak{w}}$ two L_∞ algebras, a L_∞ **algebra homomorphism** from $\tilde{\mathfrak{v}}$ to $\tilde{\mathfrak{w}}$ is a linear map $\phi : \wedge^\bullet \tilde{\mathfrak{v}} \to \wedge^\bullet \tilde{\mathfrak{w}}$ such that the dual map $\phi^* : S^\bullet(\tilde{\mathfrak{w}}^*[1]) \to S^\bullet(\tilde{\mathfrak{v}}^*[1])$ is a graded algebra homomorphism of degree 0 which induces a chain map from $CE(\mathfrak{w})$ to $CE(\mathfrak{v})$, i.e.:*

$$\phi^* Q_{CE(\mathfrak{w})} = Q_{CE(\mathfrak{v})} \phi^*. \tag{2.4.50}$$

For 2-term L_∞ algebras morphisms take this form:

Proposition 16. *Given two 2-term L_∞ algebras $\mathfrak{v}$ and $\mathfrak{w}$, a morphisms $\phi : \mathfrak{v} \to \mathfrak{w}$ consists of a triple of linear maps (ϕ_0, ϕ_1, ϕ_2):*

- *$\phi_0 : \mathfrak{v}_0 \to \mathfrak{w}_0$,*

- *$\phi_1 : \mathfrak{v}_1 \to \mathfrak{w}_1$,*

- *$\phi_2 : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{w}_1$,*

that satisfy the following relations:

$$\partial_\mathfrak{w} \phi_1(\Pi) = \phi_0 \partial_\mathfrak{v}(\Pi), \tag{2.4.51}$$

$$\phi_0([\pi,\pi]_\mathfrak{v}) - [\phi_0(\pi), \phi_0(\pi)]_\mathfrak{w} - \partial_\mathfrak{w} \phi_2(\pi,\pi) = 0, \tag{2.4.52}$$

$$\phi_1([\pi,\Pi]_\mathfrak{v}) - [\phi_0(\pi), \phi_1(\Pi)]_\mathfrak{w} - \phi_2(\pi, \partial_\mathfrak{v}\Pi) = 0, \tag{2.4.53}$$

$$\phi_1([\pi,\pi,\pi]_\mathfrak{v}) - [\phi_0(\pi), \phi_0(\pi), \phi_0(\pi)]_\mathfrak{w} - 3[\phi_0(\pi), \phi_2(\pi,\pi)]_\mathfrak{w} - 3\phi_2(\pi, [\pi,\pi]_\mathfrak{v}) = 0. \tag{2.4.54}$$

Proof. Despite being very easy, let us carry out all the proof of this proposition. It is best done in coordinates. Assume ϕ^* to be a chain map from $CE(\mathfrak{w})$ to $CE(\mathfrak{v})$. Since it has to be a graded algebra homomorphism also, it is determined by its action on a basis of $S^1(\mathfrak{w}^*[1])$: taking $\{e_a\}$, $\{E_A\}$, $\{h_i\}$, $\{H_I\}$ basis for $\mathfrak{v}_0$, $\mathfrak{v}_1$, $\mathfrak{w}_0$ and $\mathfrak{w}_1$ considered in degree 0, and dual basis $\{\xi^a\}$, $\{\Xi^A\}$, $\{\chi^i\}$ and $\{X^I\}$ for $\mathfrak{v}_0^*[1]$, $\mathfrak{v}_1^*[2]$, $\mathfrak{w}_0^*[1]$ and $\mathfrak{w}_1^*[2]$ respectively, the action of ϕ^* can be written as

$$\phi^*(\chi^i) = (\phi_0)_a^i \xi^a \quad , \quad \phi^*(X^I) = (\phi_1)_A^I \Xi^A - \frac{1}{2}(\phi_2)_{ab}^I \xi^a \xi^b, \tag{2.4.55}$$

where ϕ_0, ϕ_1 and ϕ_2 are constants. It is a straightforward computation to check that the condition $Q_{CE(\mathfrak{v})} \phi^* = \phi^* Q_{CE(\mathfrak{w})}$ on χ^i and X^J implies the following relations:

$$-\frac{1}{2}(\phi_0)_a^i (f_\mathfrak{v})_{bc}^a \xi^a \xi^b + (\phi_0)_a^i (\partial_\mathfrak{v})_A^a \Xi^A = -\frac{1}{2}(f_\mathfrak{w})_{jk}^i (\phi_0)_a^j (\phi_0)_b^k \xi^a \xi^b +$$

$$+ (\partial_\mathfrak{w})_I^i (\phi_1)_A^I \Xi^A - \frac{1}{2}(\partial_\mathfrak{w})_I^i (\phi_2)_{ab}^I \xi^a \xi^b, \tag{2.4.56}$$

$$-(\phi_1)_A^I (f_\mathfrak{v})_{aB}^A \xi^a \Xi^B + \frac{1}{6}(\phi_1)_A^I (R_\mathfrak{v})_{abc}^A \xi^a \xi^b \xi^c - \frac{1}{2}(\phi_2)_{ab}^I (f_\mathfrak{v})_{cd}^b \xi^a \xi^c \xi^d - (\phi_2)_{ab}^I (\partial_\mathfrak{v})_A^a \Xi^A \xi^b =$$

$$= -(f_\mathfrak{w})_{iJ}^I (\phi_0)_a^i (\phi_1)_A^J \xi^a \Xi^A + \frac{1}{2}(f_\mathfrak{w})_{jk}^i (\phi_0)_a^j (\phi_2)_{bc}^k \xi^a \xi^b \xi^c + \frac{1}{6}(R_\mathfrak{w})_{ijk}^I (\phi_0)_a^i (\phi_0)_b^j (\phi_0)_c^k \xi^a \xi^b \xi^c, \tag{2.4.57}$$

which are equivalent to relations (2.4.51)-(2.4.54). The converse is also true in virtue of the same computation: given a triple (ϕ_0, ϕ_1, ϕ_2) as in the hypothesis of this proposition, through formula (2.4.55) we can find a chain map between the Chevalley-Eilenberg complexes of $\mathfrak{v}$ and $\mathfrak{w}$. Therefore the two things are completely equivalent. $\square$

Looking only at the algebraic structure of L_∞ algebras, homomorphisms do not look like true *homomorphisms* according to the usual naive meaning of this word, in that they do not preserve any bracket, and they can be interpreted in this way only considering the Chevalley-Eilenberg complex. Nevertheless relations (2.4.51)-(2.4.54)

can be given a meaningful understanding. Relation (2.4.51) states that, viewing 2-term L_∞ algebras as differential complexes, homomorphisms are chain maps between them. The remaining relations say that homomorphisms come equipped with a homotopy that measures how much the preservation of the other brackets fails. Notice that this is true even for a differential Lie crossed module: relations (2.4.52) and (2.4.53) imply that if $\phi : (\mathfrak{g}, \mathfrak{h}) \to (\mathfrak{g}', \mathfrak{h}')$ is a homomorphisms between differential Lie crossed modules, then neither $\phi_0 : \mathfrak{g} \to \mathfrak{g}'$ or $\phi_1 : \mathfrak{h} \to \mathfrak{h}'$ need to be Lie algebra homomorphisms.

L_∞ algebras homomorphisms can be composed. Given 2-term L_∞ algebras $\mathfrak{v}$, $\mathfrak{w}$ and $\mathfrak{z}$, and homomorphisms $\phi : \mathfrak{v} \to \mathfrak{w}$ and $\psi : \mathfrak{w} \to \mathfrak{z}$, their composition is the homomorphism $\psi \circ \phi : \mathfrak{v} \to \mathfrak{z}$ described by the triple

$$(\psi \circ \phi)_0(x) = \psi_0(\phi_0(x)), \tag{2.4.58}$$

$$(\psi \circ \phi)_1(X) = \psi_1(\phi_1(X)), \tag{2.4.59}$$

$$(\psi \circ \phi)_2(x, y) = \psi_1(\phi_2(x, y)) + \psi_2(\phi_0(x), \phi_0(y)). \tag{2.4.60}$$

The form of the composed homomorphism is obtained by looking at the composition of the dual chain maps ϕ^* and ψ^*. This triple satisfies conditions (2.4.51)-(2.4.54), as can be readily checked by a simple computation.

A 2-term L_∞ algebra homomorphism $\phi : \mathfrak{v} \to \mathfrak{w}$ is invertible if there is a second homomorphism $\phi^{-1} : \mathfrak{w} \to \mathfrak{v}$ such that $\phi \circ \phi^{-1} = (1_{\mathfrak{w}_0}, 1_{\mathfrak{w}_1}, 0)$ and $\phi^{-1} \circ \phi = (1_{\mathfrak{v}_0}, 1_{\mathfrak{v}_1}, 0)$. It is simple to see that a homomorphism ϕ is invertible if and only if ϕ_0 and ϕ_1 are invertible as linear maps. In such a case, the inverse homomorphism is the triple

$$(\phi^{-1})_0 = (\phi_0)^{-1}, \tag{2.4.61}$$

$$(\phi^{-1})_1 = (\phi_1)^{-1}, \tag{2.4.62}$$

$$(\phi^{-1})_2(x, y) = -\phi_1^{-1}\phi_2(\phi_0^{-1}(x), \phi_0^{-1}(y)). \tag{2.4.63}$$

This triple fulfills requirements (2.4.51)-(2.4.54) and thus defines a honest 2-term L_∞ algebra homomorphism. Given a 2-term L_∞ algebra $\mathfrak{v}$, all invertible homomorphisms from $\mathfrak{v}$ to itself are called **automorphism of $\mathfrak{v}$**. Their set, denoted $\mathrm{Aut}_1(\mathfrak{v})$, is a Lie group, with the group law, unit and inversion described above.

We can also define 2-morphisms between 2-term L_∞ algebras homomorphisms. Homomorphisms have been defined as chain maps between the Chevalley-Eilenberg complexes of the L_∞ algebras, therefore it is natural to define 2-morphisms as homotopies between these chain maps.

Definition 62. *Given $\tilde{\mathfrak{v}}$ and $\tilde{\mathfrak{w}}$ L_∞ algebras and $\phi, \psi : \tilde{\mathfrak{v}} \to \tilde{\mathfrak{w}}$ homomorphisms between them, a **2-morphisms F from ϕ to ψ**, also denoted $F : \phi \Rightarrow \psi$, is a linear map $F : \wedge^\bullet \tilde{\mathfrak{v}} \to \wedge^\bullet \tilde{\mathfrak{w}}$ such that the dual map $F^* : \wedge^\bullet \tilde{\mathfrak{w}}^*[1] \to \wedge^\bullet \tilde{\mathfrak{v}}^*[1]$ is a degree -1 homotopy between ϕ^* and ψ^*:*

$$\psi^* - \phi^* = Q_{CE(\mathfrak{v})}F^* + F^*Q_{CE(\mathfrak{w})}. \tag{2.4.64}$$

In the case of a 2-term L_∞ algebra we have the following result concerning 2-morphisms:

Proposition 17. *Given $\mathfrak{v}$ and $\mathfrak{w}$ 2-term L_∞ algebras and $\phi, \psi : \mathfrak{v} \to \mathfrak{w}$ homomorphisms between them, a 2-morphism $F : \phi \Rightarrow \psi$ is determined by a linear map*

$F : \mathfrak{v}_0 \to \mathfrak{w}_1$ such that:

$$\phi_0 - \psi_0 = \partial_{\mathfrak{w}} F, \tag{2.4.65}$$

$$\phi_1 - \psi_1 = F \partial_{\mathfrak{v}}, \tag{2.4.66}$$

$$\phi_2(x, y) - \psi_2(x, y) = F([x, y]) - [\phi_0(x), F(y)] + [\psi_0(y), F(x)]. \tag{2.4.67}$$

Proof. We follow the same notations as in proposition (16). Take F^* to be a degree 1 homotopy between ϕ^* and ψ^* as is definition (62). F^* can't be a homomorphism of algebras because this is not compatible with the non vanishing degree of F^*. Notice that if ϕ^* and ψ^* are two degree 0 algebra homomorphisms, then their difference $K := \phi^* - \psi^*$ enjoys the following property:

$$K(\xi \otimes \zeta) = K(\xi) \otimes \frac{1}{2}(\phi^* + \psi^*)(\zeta) + \frac{1}{2}(\phi^* + \psi^*)(\xi) \otimes K(\zeta). \tag{2.4.68}$$

In our case $K = Q_{CE(\mathfrak{v})} F^* + F^* Q_{CE(\mathfrak{w})}$. This forces F^* to fulfill the relation

$$F^*(\xi \otimes \zeta) = F^*(\xi) \otimes \frac{1}{2}(\phi^* + \psi^*)(\zeta) + (-1)^{|\xi|} \frac{1}{2}(\phi^* + \psi^*)(\xi) \otimes F^*(\zeta). \tag{2.4.69}$$

In this way F^* is still determined by its action on a basis of $S^1(\tilde{\mathfrak{v}}^*[1])$. Since it has degree -1, the only possibility is

$$F^*(\chi^i) = 0, \quad F^*(X^I) = F_a^I \xi^a. \tag{2.4.70}$$

With these formulas we see that (2.4.64) is equivalent to the relations

$$(\phi_0)_a^i \xi^a - (\psi_0)_a^i \xi^a = (\partial_{\mathfrak{w}})_I^i F_a^I \xi^a, \tag{2.4.71}$$

$$(\phi_1)_A^I \Xi^A - \frac{1}{2}(\phi_2)_{ab}^I \xi^a \xi^b - (\psi_1)_A^I \Xi^A + \frac{1}{2}(\psi_2)_{ab}^I \xi^a \xi^b =$$

$$= -\frac{1}{2} F_c^I (f_{\mathfrak{v}})_{ab}^c \xi^a \xi^b + F_a^I (\partial_{\mathfrak{v}})_A^a \Xi^A + \frac{1}{2}(f_{\mathfrak{w}})_{iJ}^I \left((\phi_0)_a^i + (\psi_0)_a^i\right) F_b^J \xi^a \xi^b. \tag{2.4.72}$$

It's easy to check that these relations are equivalent to (2.4.65)-(2.4.67) for F defined by $F(e_a) = F_a^I H_I$. $\qquad\square$

Notice that the right hand side of (2.4.67) has to be antisymmetric in x and y because the left hand side is, although it is not apparent. Manifest antisymmetry can be restored combining (2.4.67) with (2.4.65). The right hand side can then be put in the following form:

$$F([x, y]) - [\phi_0(x), F(y)] + [\phi_0(y), F(x)] + [\partial F(x), F(y)], \tag{2.4.73}$$

where antisymmetry emerges more evidently.

2-term L_∞ algebras 2-morphisms can be composed in two ways, horizontally and vertically. Given two 2-morphisms $F : \phi \Rightarrow \psi$ and $G : \psi \Rightarrow \gamma$ for $\phi, \psi \gamma : \mathfrak{v} \to \mathfrak{w}$, their vertical composition is the 2-morphism $G \bullet F : \phi \Rightarrow \gamma$ defined by the map

$$F + G : \mathfrak{v}_0 \to \mathfrak{w}_1. \tag{2.4.74}$$

Given homomorphisms $\phi, \phi' : \mathfrak{v} \to \mathfrak{w}$ and $\psi, \psi' : \mathfrak{w} \to \mathfrak{z}$ and 2-morphisms $F : \phi \Rightarrow \phi'$ and $G : \psi \Rightarrow \psi'$, their horizontal composition is the 2-morphism $G \circ F : \psi \circ \phi \Rightarrow \psi' \circ \phi'$ defined by the map

$$G\phi_0 + \psi_1' F = G\phi_0' + \psi_1 F : \mathfrak{v}_0 \to \mathfrak{z}_1. \tag{2.4.75}$$

Defining the identity 2-morphism 1_ϕ for ϕ a homomorphism as the zero map, we can define invertibility for 2-morphisms. Notice that every 2-morphism is vertically invertible, with the inverse defined by

$$F^{-1\bullet} = -F, \tag{2.4.76}$$

while a 2-morphism $F : \phi \Rightarrow \psi$ is horizontally invertible if and only if both ϕ and ψ are invertible homomorphisms. In such a case the horizontal inverse $F^{-1\circ} : \phi^{-1} \Rightarrow \psi^{-1}$ is given by

$$F^{-1\circ} = -\psi_1^{-1} F \phi_0^{-1} = -\phi_1^{-1} F \psi_0^{-1}. \tag{2.4.77}$$

Altogether these compositions make up the structure of a strict 2-groupoid, whose objects are 2-term L_∞ algebras, whose morphisms are invertible homomorphisms of 2-term L_∞ algebras and whose 2-morphisms are 2-morphisms between these. In particular, for every 2-term L_∞ algebra we call $\mathrm{Aut}(\mathfrak{v})$ the 2-group of the automorphisms of $\mathfrak{v}$. The set of 1-morphisms $\mathrm{Aut}_1(\mathfrak{v})$ is the set of all automorphisms of $\mathfrak{v}$ and the set of 2-morphisms $\mathrm{Aut}_2(\mathfrak{v})$ collects the 2-morphisms between these automorphisms. In the spirit of 2-category theory, from now on we will call a L_∞ algebra homomorphism simply a 1-morphism. Notice that $\mathrm{Aut}_2(\mathfrak{v})$ is not simply the set of all maps $F : \mathfrak{v}_0 \to \mathfrak{v}_1$ such that there exist two 1-morphisms satisfying (2.4.65)-(2.4.67). In principle one such map F could link several different pairs of 1-morphisms, if there are several couples of 1-morphisms satisfying relations (2.4.65)-(2.4.67) for F. Such a map will appear in $\mathrm{Aut}_2(\mathfrak{v})$ a number of times equal to the number of pairs of 1-morphisms it connects, labeled by its source and target. In the following we will denote by F both a 2-morphism and the linear map $F : \mathfrak{v}_0 \to \mathfrak{v}_1$ that is associated with it. It will be hopefully clear whether we mean one or the other.

The 2-group $\mathrm{Aut}(\mathfrak{v})$ is strict for every 2-term L_∞ algebra $\mathfrak{v}$, be $\mathfrak{v}$ strict or not. Thus it can be viewed as a crossed module (G, H, m, t). The group of 1-morphisms G is $\mathrm{Aut}_1(\mathfrak{v})$, the group of all 1-morphisms of $\mathfrak{v}$, with the associated identity, composition and inversion. The group H is a subset of $\mathrm{Aut}_2(\mathfrak{v})$ and we will denote it $\mathrm{Aut}_2^*(\mathfrak{v}) \subset \mathrm{Aut}_2(\mathfrak{v})$. It is the group of all 2-morphisms whose source is the identity 1-morphism. Once the source of a 2-morphism $F \in \mathrm{Aut}_2(\mathfrak{v})$ is specified together with the linear map defining the 2-morphism, its target is also determined: if $F : 1_\mathfrak{v} \Rightarrow t(F)$, then

$$t(F)_0 = 1_{\mathfrak{v}_0} - \partial F, \tag{2.4.78a}$$
$$t(F)_1 = 1_{\mathfrak{v}_1} - F\partial, \tag{2.4.78b}$$
$$t(F)_2(x, y) = [x, F(y)] - [y, F(x)] - F([x, y]). \tag{2.4.78c}$$

A map $F : \mathfrak{v}_0 \to \mathfrak{v}_1$ then belongs to $\mathrm{Aut}_2^*(\mathfrak{v})$ if it represents a 2-morphism in $\mathrm{Aut}_2(\mathfrak{v})$, namely if its target is an invertible 1-morphism. Notice that if $t(F)_0$ is invertible, then so is $t(F)_1$: using $Ft(F)_0 = t(F)_1 F$ we find that the inverse maps are related in this way:

$$t(F)_1^{-1} = 1_{\mathfrak{v}_1} + Ft(F)_0^{-1}\partial. \tag{2.4.79}$$

We can then characterize $\mathrm{Aut}_2^*(\mathfrak{v})$ as follows:

$$\mathrm{Aut}_2^*(\mathfrak{v}) = \{F : \mathfrak{v}_0 \to \mathfrak{v}_1, \text{ such that } (1_{\mathfrak{v}_0} - \partial F) : \mathfrak{v}_0 \to \mathfrak{v}_0 \text{ is an invertible map}\}. \tag{2.4.80}$$

The composition in this group is just the horizontal composition of 2-morphisms in $\mathrm{Aut}(\mathfrak{v})$, which is defined by relation (2.4.75). Given two maps $F, G \in \mathrm{Aut}_2^*(\mathfrak{v})$ this expression takes the simpler form

$$G \circ F = F + G - G\partial F. \tag{2.4.81}$$

The identity is $1_{1_\mathfrak{v}}$ which is the trivial map from $\mathfrak{v}_0$ to $\mathfrak{v}_1$. The inversion is defined by

$$F^{-1_\circ} = -\sum_{n=0}^{\infty} F(\partial F)^n = -(1 - F\partial)^{-1}F = -F(1 - \partial F)^{-1}. \tag{2.4.82}$$

The map t of the crossed module is the restriction of the target map of the associated 2-category (see section (2.2)), and is thus described in formulas (2.4.78a)-(2.4.78c). It's readily checked that it is a group homomorphism.

The map m is defined in (2.3.3), and in this case it becomes

$$m(\phi)(F) = \phi_1 F \phi_0^{-1}. \tag{2.4.83}$$

2.4.5   2-term L_∞ algebra derivation

A derivation of a 2-term L_∞ algebra $\mathfrak{v}$ is the infinitesimal version of an automorphism of $\mathfrak{v}$. They must therefore obey linearized versions of relations (2.4.51)-(2.4.54):

Definition 63. *Given a 2-term L_∞ algebra $\mathfrak{v}$, a **1-derivation** α **of** $\mathfrak{v}$ is a triple of linear maps $(\alpha_0, \alpha_1, \alpha_2)$:*

- $\alpha_0 : \mathfrak{v}_0 \to \mathfrak{v}_0$,

- $\alpha_1 : \mathfrak{v}_1 \to \mathfrak{v}_1$,

- $\alpha_2 : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_1$,

such that the following relations are satisfied:

$$\alpha_0(\partial X) - \partial \alpha_1(X) = 0, \tag{2.4.84}$$
$$\alpha_0([\pi, \pi]) - 2[\alpha_0(\pi), \pi] - \partial \alpha_2(\pi, \pi) = 0, \tag{2.4.85}$$
$$\alpha_1([\pi, \Pi]) - [\alpha_0(\pi), \Pi] - [\pi, \alpha_1(\Pi)] - \alpha_2(\pi, \partial \Pi) = 0, \tag{2.4.86}$$
$$3[\pi, \alpha_2(\pi, \pi)] + 3\alpha_2(\pi, [\pi, \pi]) + 3[\pi, \pi, \alpha_0(\pi)] - \alpha_1([\pi, \pi, \pi]) = 0. \tag{2.4.87}$$

the set of all derivations of $\mathfrak{v}$ is denoted $\mathfrak{aut}_0(\mathfrak{v})$.

It is possible to define also 2-derivations, which are infinitesimal 2-morphisms. Their definition is quite simple:

Definition 64. *Given a 2-term L_∞ algebra, a **2-derivation of** $\mathfrak{v}$ is a linear map $\Gamma : \mathfrak{v}_0 \to \mathfrak{v}_1$. The set of all 2-derivations of $\mathfrak{v}$ is denoted $\mathfrak{aut}_1(\mathfrak{v})$.*

As 1-morphisms and 2-morphisms of a 2-term L_∞ algebra are a strict 2-group or Lie crossed module, 1-derivations and 2-derivations of a 2-term L_∞ algebra are a strict 2-term L_∞ algebra or differential Lie crossed module. This algebra is denoted $\mathfrak{aut}(\mathfrak{v})$ and is called the derivation 2-term L_∞ algebra of $\mathfrak{v}$. The brackets are defined as

$$(\partial_{\mathfrak{aut}}\Gamma)_0 = -\partial\Gamma, \tag{2.4.88}$$

$$(\partial_{\mathfrak{aut}}\Gamma)_1 = -\Gamma\partial, \tag{2.4.89}$$

$$(\partial_{\mathfrak{aut}}\Gamma)_2(\pi,\pi) = 2[\pi,\Gamma(\pi)] - \Gamma([\pi,\pi]), \tag{2.4.90}$$

$$([\alpha,\beta]_{\mathfrak{aut}})_0 = \alpha_0\beta_0 - \beta_0\alpha_0, \tag{2.4.91}$$

$$([\alpha,\beta]_{\mathfrak{aut}})_1 = \alpha_1\beta_1 - \beta_1\alpha_1, \tag{2.4.92}$$

$$([\alpha,\beta]_{\mathfrak{aut}})_2(\pi,\pi) = \alpha_1(\beta_2(\pi,\pi)) + 2\alpha_2(\beta_0(\pi),\pi) - \beta_1(\alpha_2(\pi,\pi)) - 2\beta_2(\alpha_0(\pi),\pi), \tag{2.4.93}$$

$$[\alpha,\Gamma]_{\mathfrak{aut}} = \alpha_1\Gamma - \Gamma\alpha_0. \tag{2.4.94}$$

The three-bracket is zero because this 2-term L_∞ algebra is strict. The differential Lie crossed module is $(\mathfrak{aut}_0(\mathfrak{v}), \mathfrak{aut}_1(\mathfrak{v}), \tau, \mu)$ where the Lie bracket of the Lie algebra $\mathfrak{aut}_0(\mathfrak{v})$ are written in (2.4.91)-(2.4.93), while the Lie bracket associated with $\mathfrak{aut}_1(\mathfrak{v})$ are defined as

$$[\Gamma,\Xi]_{\mathfrak{aut}_1(\mathfrak{v})} := [\partial_{\mathfrak{aut}}\Gamma,\Xi]_{\mathfrak{aut}} = -\Gamma\partial\Xi + \Xi\partial\Gamma, \tag{2.4.95}$$

and the maps $\tau : \mathfrak{aut}_1(\mathfrak{v}) \to \mathfrak{aut}_0(\mathfrak{v})$ and $\mu : \mathfrak{aut}_0(\mathfrak{v}) \times \mathfrak{aut}_1(\mathfrak{v}) \to \mathfrak{aut}_1(\mathfrak{v})$ are

$$\tau(\Gamma) := \partial_{\mathfrak{aut}}\Gamma, \tag{2.4.96}$$

$$\mu(\alpha)(\Gamma) := [\alpha,\Gamma]_{\mathfrak{aut}}. \tag{2.4.97}$$

$\mathfrak{aut}_0(\mathfrak{v})$ and $\mathfrak{aut}_1(\mathfrak{v})$ are the Lie algebras of $\mathrm{Aut}_1(\mathfrak{v})$ and $\mathrm{Aut}_2^*(\mathfrak{v})$ respectively. Thus there is naturally defined an adjoint action of $\mathrm{Aut}_1(\mathfrak{v})$ on $\mathfrak{aut}_0(\mathfrak{v})$: given a 1-automorphism g and a 1-derivation α the 1-derivation $g\alpha g^{-1}$ is defined by the triple

$$(g\alpha g^{-1})_0 = g_0\alpha_0 g_0^{-1}, \tag{2.4.98}$$

$$(g\alpha g^{-1})_1 = g_1\alpha_1 g_1^{-1}, \tag{2.4.99}$$

$$(g\alpha g^{-1})_2(\pi,\pi) = -g_1\alpha_1 g_1^{-1}g_2(g_0^{-1}(\pi),g_0^{-1}(\pi)) + \\ + g_1\alpha_2(g_0^{-1}(\pi),g_0^{-1}(\pi)) + 2g_2(\alpha_0 g_0^{-1}(\pi),g_0^{-1}\pi). \tag{2.4.100}$$

There is also an action of $\mathrm{Aut}_1(\mathfrak{v})$ on $\mathfrak{aut}_1(\mathfrak{v})$ and on operator $Q : \mathrm{Aut}_2^*(\mathfrak{v}) \times \mathfrak{aut}_0(\mathfrak{v}) \to \mathfrak{aut}_1(\mathfrak{v})$ according to definitions (2.4.48) and (2.4.49):

$$\dot{m}(\phi)(\Gamma) = \phi_1\Gamma\phi_0^{-1}, \tag{2.4.101}$$

$$Q(F)(\alpha) = \alpha_1 F - F\alpha_0. \tag{2.4.102}$$

It is evident that relations (2.4.91)-(2.4.94) are the linearization of these actions.

It is possible to explicitly integrate a derivation of $\mathfrak{v}$ to an automorphism of $\mathfrak{v}$. Namely, a 1-derivation can be integrated to a 1-automorphism and a 2-derivation to a 2-automorphism. This is done through the exponential map.

Definition 65. *Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be a 1-derivation of a 2-term L_∞ algebra $\mathfrak{v}$. We define e^α as the triple $((e^\alpha)_0, (e^\alpha)_1, (e^\alpha)_2)$, where*

- $(e^\alpha)_0 := e^{\alpha_0} : \mathfrak{v}_0 \to \mathfrak{v}_0,$

- $(e^\alpha)_1 := e^{\alpha_1} : \mathfrak{v}_1 \to \mathfrak{v}_1,$

- $(e^\alpha)_2(\pi, \pi) := \int_0^1 dt \, e^{-t\alpha_1} \alpha_2 \left(e^{(1-t)\alpha_0}(\pi), e^{(1-t)\alpha_0}(\pi) \right).$

Proposition 18. *Given α a 1-derivation of $\mathfrak{v}$, the triple e^α defines a 1-automorphism of $\mathfrak{v}$.*

Proof. The maps $(e^\alpha)_0$ and $(e^\alpha)_1$ are invertible because every exponential of a linear map is invertible, with inverses $e^{-\alpha_0}$ and $e^{-\alpha_1}$. To be actually a 2-term L_∞ algebra morphism from $\mathfrak{v}$ to itself e^α has to enjoy relations (2.4.51)-(2.4.54). These proofs are easy but tedious computations implying power series. Let us just show the computation for (2.4.52). For α a 1-derivation and for $n \geq 1$, the following formula can be shown by induction:

$$\alpha_0^n([x,y]) = \sum_{m=0}^{n} \binom{n}{m} [\alpha_0^m(x), \alpha_0^{n-m}(y)] +$$

$$+ \partial \sum_{k=0}^{n-1} \sum_{m=0}^{n-1-k} \binom{n-1-k}{m} \alpha_1^k(\alpha_2(\alpha_0^m(x), \alpha_0^{n-1-k-m}(y))). \tag{2.4.103}$$

After few manipulations involving this expression, we have that

$$e^{\alpha_0}([x,y]) - [e^{\alpha_0}(x), e^{\alpha_0}(y)] = \sum_{n=0}^{\infty} \frac{\alpha_0^n}{n!}([x,y]) - \sum_{q,p=0}^{\infty} \frac{1}{q!p!}[\alpha_0^q(x), \alpha_0^p(y)] =$$

$$= \partial \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \sum_{m=0}^{n-1-k} \frac{(n-1-k)!}{n!m!(n-1-k-m)!} \alpha_1^k(\alpha_2(\alpha_0^m(x), \alpha_0^{n-1-k-m}(y))) =$$

$$= \partial \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(s+t)!}{s!t!(r+s+t+1)!} \alpha_1^r(\alpha_2(\alpha_0^s(x), \alpha_0^t(y))). \tag{2.4.104}$$

Applying the identity

$$\int_0^1 dt(1-t)^n t^m = \frac{n!m!}{(n+m+1)!}, \quad m,n \in \mathbb{N} \tag{2.4.105}$$

to the power expansion of the definition of $(e^\alpha)_2$, we see that it is equal to (2.4.104). This proves axiom (2.4.52) for e^α. The other needed relations are demonstrated analogously. $\qquad\square$

2-derivations can also be exponentiated to 2-morphisms:

Definition 66. *Let Γ be a 2-derivation of the 2-term L_∞ algebra $\mathfrak{v}$. Then we define e^Γ as the map*

$$e^\Gamma := \sum_{n=0}^{\infty} (-1)^n \Gamma(\partial\Gamma)^n = \Gamma \frac{1_{\mathfrak{v}_0} - e^{-\partial\Gamma}}{\partial\Gamma} = \int_0^1 dt \, \Gamma \, e^{-t\partial\Gamma} : \mathfrak{v}_0 \to \mathfrak{v}_1. \tag{2.4.106}$$

Proposition 19. *For Γ a 2-derivation of a 2-term L_∞ algebra $\mathfrak{v}$, e^Γ is a 2-automorphism of $\mathfrak{v}$, i.e. $e^\Gamma \in Aut_2^*(\mathfrak{v})$.*

Proof. All we have to prove is that $(1_{\mathfrak{v}_0} - \partial e^\Gamma)$ is invertible, but this follows immediately:

$$(1_{\mathfrak{v}_0} - \partial e^\Gamma)^{-1} = (e^{-\partial\Gamma})^{-1} = e^{\partial\Gamma}. \qquad (2.4.107)$$

$\square$

As happens for ordinary Lie algebras, there is a notion of adjoint representation of a 2-term L_∞ algebra of itself.

Proposition 20. *Let $\mathfrak{v}$ be a 2-term L_∞ algebra. Then there is 2-term L_∞ algebra morphism $\mathrm{ad} : \mathfrak{v} \to \mathfrak{aut}(\mathfrak{v})$ consisting of the triple $(\mathrm{ad}_0, \mathrm{ad}_1, \mathrm{ad}_2)$:*

- $\mathrm{ad}_0 : \mathfrak{v}_0 \to \mathfrak{aut}_0(\mathfrak{v})$,

- $\mathrm{ad}_1 : \mathfrak{v}_1 \to \mathfrak{aut}_1(\mathfrak{v})$,

- $\mathrm{ad}_2 : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{aut}_1(\mathfrak{v})$,

defined as:

$$(\mathrm{ad}_0(x))_0(y) := [x, y], \qquad (2.4.108)$$
$$(\mathrm{ad}_0(x))_1(X) := [x, X], \qquad (2.4.109)$$
$$(\mathrm{ad}_0(x))_2(y, z) := [x, y, z], \qquad (2.4.110)$$
$$(\mathrm{ad}_1(X))(x) := [x, X], \qquad (2.4.111)$$
$$(\mathrm{ad}_2(x, y))(z) := [x, y, z]. \qquad (2.4.112)$$

*This is called the **adjoint representation** of $\mathfrak{v}$ on itself.*

Proof. We have to show that the map ad is well defined.

First of all we have to show that $\mathrm{ad}_0(x)$ does belong to $\mathrm{aut}_0(\mathfrak{v})$ for every $x \in \mathfrak{v}_0$, namely we have to show that the triple $((\mathrm{ad}_0(x))_0, (\mathrm{ad}_0(x))_1, (\mathrm{ad}_0(x))_2)$ satisfies axioms (2.4.84)-(2.4.87). This follows by the properties of the L_∞ brackets of $\mathfrak{v}$. Let us prove axiom (2.4.87) as an example:

$$-3[\pi, (\mathrm{ad}_0(\pi))_2(\pi, \pi)] + 3(\mathrm{ad}_0(\pi))_2(\pi, [\pi, \pi]) +$$

$$+3[\pi, \pi(\mathrm{ad}_0(\pi))_0(\pi)] - (\mathrm{ad}_0(\pi))_1([\pi, \pi, \pi]) =$$

$$= -3[\pi, [\pi, \pi, \pi]] + 3[\pi, \pi, [\pi, \pi]] + 3[\pi, \pi, [\pi, \pi]] - [\pi, [\pi, \pi, \pi]] = 0, \qquad (2.4.113)$$

in virtue of axiom (2.4.39). The minus sign in the first term of this equation is due to the grading of π, which renders $\mathrm{ad}_0(\pi)$ an odd derivation which has to anticommute with other odd elements. The other relations which define a 1-derivation are shown in a similar manner.

Next we need to prove that the map ad fulfills relations (2.4.51)-(2.4.54) that define a 2-term L_∞ algebra morphism. Again, this follows from the algebra of the 2-term L_∞ algebra brackets. Let us show just axiom (2.4.54):

$$(\mathrm{ad}_1([\pi, \pi, \pi]))(\pi) - [(\mathrm{ad}_0(\pi)), (\mathrm{ad}_0(\pi)), (\mathrm{ad}_0(\pi))]_{\mathfrak{aut}}(\pi) -$$

$$-3[(\mathrm{ad}_0(\pi)),(\mathrm{ad}_2(\pi,\pi))]_{\mathfrak{aut}}(\pi) - 3(\mathrm{ad}_2(\pi,[\pi,\pi]))(\pi) =$$
$$= (\mathrm{ad}_1([\pi,\pi,\pi]))(\pi) - 3(\mathrm{ad}_0(\pi))_1(\mathrm{ad}_2(\pi,\pi))(\pi) +$$
$$+ 3(\mathrm{ad}_2(\pi,\pi))(\mathrm{ad}_0(\pi))_0(\pi) - 3(\mathrm{ad}_2(\pi,[\pi,\pi]))(\pi) =$$
$$= -[\pi,[\pi,\pi,\pi]] - 3[\pi,[\pi,\pi,\pi]] + 3[\pi,\pi,[\pi,\pi]] - 3[\pi,[\pi,\pi],\pi] = 0, \qquad (2.4.114)$$

again due to (2.4.39). Here we used the fact that $[\cdot,\cdot,\cdot]_{\mathfrak{aut}} = 0$. The other relations are proved with similar computations. □

We can give more explicit formulas for the exponential of an adjoint derivation in the case of a differential Lie crossed module $(\mathfrak{g},\mathfrak{h})$. In this case we have exponential maps relative to the Lie algebras that are in the crossed module: $\exp : \mathfrak{g} \to G$, $\exp : \mathfrak{h} \to H$, where (G,H) is the Lie crossed module integrating $(\mathfrak{g},\mathfrak{h})$. Given $x \in \mathfrak{g}$, the associated adjoint 1-derivation is the triple

$$[x,\cdot]_{\mathfrak{g}} \ , \ \mu(x)(\cdot) \ , \ 0. \qquad (2.4.115)$$

The exponentiated 1-morphism can be expressed as the triple

$$\mathrm{ad}\gamma \ , \ \dot{m}(\gamma)(\cdot) \ , \ 0, \qquad (2.4.116)$$

where γ is the element of G that integrates x. For $X \in \mathfrak{h}$ the adjoint 2-derivation reads

$$\mu(\cdot)(X), \qquad (2.4.117)$$

and the integrated 2-morphism is the operator

$$Q(h)(\cdot), \qquad (2.4.118)$$

recall definition (2.4.49), where again $h \in H$ is the exponential of X.

2.4.6   Invariant form

We wish now to generalize the concept of an invariant form on a Lie algebra to an L_∞ algebra.

Definition 67. *An L_∞ algebra V is **cyclic** if it is endowed with a non degenerate graded symmetric bilinear form $C : V \otimes V \to \mathbb{R}$ such that for every $k \geq 1$ the map*

$$C(l_k(\cdot,\ldots,\cdot),\cdot) : V^{\otimes(k+1)} \to \mathbb{R} \qquad (2.4.119)$$

is graded antisymmetric.

The axiom (2.4.119) is an invariance requirement for the bilinear form. It states that

$$C\left(l_k(x_1,x_2,\ldots,x_{k-1},x_k),x_{k+1}\right) = (-1)^{|x_k||x_{k+1}|+1}C\left(l_k(x_1,x_2,\ldots,x_{k-1},x_{k+1}),x_k\right). \qquad (2.4.120)$$

It is immediately seen that if the cyclic L_∞ algebra $V = V_0$ is a Lie algebra then the cyclicity condition for C reduces to the usual invariance condition for a bilinear form on a Lie algebra:

$$C([x,y],z) = -C([x,z],y). \qquad (2.4.121)$$

We turn now to 2-term L_∞ algebra. We define a more restrictive notion of cyclic 2-term L_∞ algebra than in the general case:

Definition 68. *A 2-term L_∞ algebra $\mathfrak{v}$ is **balanced** if $\dim \mathfrak{v}_0 = \dim \mathfrak{v}_1$. A **balanced bilinear form on** $\mathfrak{v}$ is a non degenerate pairing $(\cdot, \cdot) : \mathfrak{v}_0 \otimes \mathfrak{v}_1 \to \mathbb{R}$ such that*

$$(\partial X, Y)(\partial Y, X) = 0, \tag{2.4.122}$$

$$([x, y], X) + (y, [x, X]) = 0, \tag{2.4.123}$$

$$([x, y, z], t) + (z, [x, y, t]) = 0. \tag{2.4.124}$$

A balanced 2-term L_∞ algebra with invariant form is a cyclic 2-term L_∞ algebra with the bilinear form restricted to have off-diagonal non degenerate blocks. The reason why we need this particular notion emerges in studying the higher gauge Chern Simons model we will define later: in this field theory the non degeneracy of the pairing between $\mathfrak{v}_0$ and $\mathfrak{v}_1$ is a necessary requirement in order to have sensible equations of motion.

The constraint that $\mathfrak{v}_0$ and $\mathfrak{v}_1$ must have the same dimension may seem very restrictive, but there is the possibility to extend every 2-term L_∞ algebra $\mathfrak{v}$ to a balanced one $\mathfrak{v}^\sim$ perturbing it minimally. By this, we mean:

- $\mathfrak{v}$ is contained in $\mathfrak{v}^\sim$;

- $\dim\mathfrak{v}^\sim$ is minimal;

- $\mathfrak{v}^\sim$ is as trivial as possible outside $\mathfrak{v}$.

Let us sketch the construction of such a $\mathfrak{v}^\sim$. Suppose we start with a 2-term L_∞ algebra such that $\dim\mathfrak{v}_0 < \dim\mathfrak{v}_1$. We then define $\mathfrak{v}_0^\sim := \mathfrak{v}_0 \oplus \mathfrak{w}$ with $\mathfrak{w}$ a vector space such that $\dim\mathfrak{w} = \dim\mathfrak{v}_1 - \dim\mathfrak{v}_0$ and $\mathfrak{v}_1^\sim := \mathfrak{v}_1$. The brackets on $\mathfrak{v}^\sim$ are defined in the following way: for $x, y, z \in \mathfrak{v}_0$, $a, b, c \in \mathfrak{w}$ and $X \in \mathfrak{v}_1$

$$\partial^\sim X := \partial X \oplus 0, \tag{2.4.125}$$

$$[x \oplus a, y \oplus b]^\sim := [x, y] \oplus 0, \tag{2.4.126}$$

$$[x \oplus a, X]^\sim := [x, X], \tag{2.4.127}$$

$$[x \oplus a, y \oplus b, z \oplus c]^\sim := [x, y, z]. \tag{2.4.128}$$

On the other hand, if $\dim\mathfrak{v}_0 > \dim\mathfrak{v}_1$ we take $\mathfrak{w}$ such that $\dim\mathfrak{w} = \dim\mathfrak{v}_0 - \dim\mathfrak{v}_1$ and we define $\mathfrak{v}_0^\sim := \mathfrak{v}_0$ and $\mathfrak{v}_1^\sim := \mathfrak{v}_1 \oplus \mathfrak{w}$. For $x, y, z \in \mathfrak{v}_0$, $X \in \mathfrak{v}_1$ and $A \in \mathfrak{w}$ the brackets are

$$\partial^\sim (X \oplus A) := \partial X, \tag{2.4.129}$$

$$[x, y]^\sim := [x, y], \tag{2.4.130}$$

$$[x, X \oplus A]^\sim := [x, X] \oplus 0, \tag{2.4.131}$$

$$[x, y, z]^\sim := [x, y, z] \oplus 0. \tag{2.4.132}$$

Such an extended $\mathfrak{v}^\sim$ is unique up to non canonical isomorphism.

Next we study morphisms that preserve the bilinear form of a balanced 2-term L_∞ algebra. These are said orthogonal, and play a central role in 2-term L_∞ algebra gauge theory.

Definition 69. *Let $\mathfrak{v}$ be a balance 2-term L_∞ algebra with invariant bilinear form $(\cdot, \cdot)$. A 1-automorphism ϕ of $\mathfrak{v}$ is* **orthogonal** *if*

$$(\phi_0(x), \phi_1(X)) = (x, X), \tag{2.4.133}$$

$$(\phi_0(x), \phi_2(y, z)) + (\phi_0(z), \phi_2(y, x)) = 0. \tag{2.4.134}$$

The set of orthogonal 1-automorphisms of $\mathfrak{v}$ is denoted $\mathrm{OAut}_1(\mathfrak{v})$. Since it is closed under composition, it is a subgroup of $\mathrm{Aut}_1(\mathfrak{v})$.

We can extend this definition to 2-automorphism:

Definition 70. *Let $\mathfrak{v}$ be a balance 2-term L_∞ algebra with invariant bilinear form $(\cdot, \cdot)$. A 2-automorphism is* **orthogonal** *if both its source and its target are orthogonal 1-automorphisms.*

The set of orthogonal 2-automorphisms is called $\mathrm{OAut}_2(\mathfrak{v})$. This is a subset of $\mathrm{Aut}_2(\mathfrak{v})$ and since it is closed under horizontal and vertical composition, we have that $\mathrm{OAut}(\mathfrak{v}) = (\mathrm{OAut}_1(\mathfrak{v}), \mathrm{OAut}_2(\mathfrak{v}))$ is a 2-subgroup of $\mathrm{Aut}(\mathfrak{v})$. This 2-group can be described as a Lie crossed module. The two underlying groups are $\mathrm{OAut}_1(\mathfrak{v})$ and $\mathrm{OAut}_2^*(\mathfrak{v})$. The latter is the subgroup of $\mathrm{Aut}_2^*(\mathfrak{v})$ formed by all elements whose target is an orthogonal 1-automorphism. It can be characterized as the group of all maps $F : \mathfrak{v}_0 \to \mathfrak{v}_1$ belonging to $\mathrm{Aut}_2^*(\mathfrak{v})$ such that

$$(\partial F(x), X) + (x, F(\partial X)) - (\partial F(x), F(\partial X)) = 0, \tag{2.4.135}$$

$$(y + \partial F(y), [x, F(z)] + [z, F(x)]) + (x - \partial F(x), F([y, z])) + (z - \partial F(z), F([y, x])) = 0. \tag{2.4.136}$$

The strict 2-term L_∞ algebra associated with $\mathrm{OAut}(\mathfrak{v})$ (or analogously the differential Lie crossed module associated with $(\mathrm{OAut}_1(\mathfrak{v}), \mathrm{OAut}_2^*(\mathfrak{v}))$) is denoted $\mathfrak{oaut}(\mathfrak{v})$ (or $(\mathfrak{oaut}_0(\mathfrak{v}), \mathfrak{oaut}_1(\mathfrak{v}))$), and it's a subalgebra of $\mathfrak{aut}(\mathfrak{v})$. Its elements are orthogonal 1- and 2-derivations, and they are defined as follows:

Definition 71. *Let $\mathfrak{v}$ be a balance 2-term L_∞ algebra with invariant bilinear form $(\cdot, \cdot)$. A 1-derivation α is said* **orthogonal** *if*

$$(\alpha_0(x), X) + (x, \alpha_1(X)) = 0, \tag{2.4.137}$$

$$(x, \alpha_2(y, z)) + (z, \alpha_2(y, x)) = 0. \tag{2.4.138}$$

Definition 72. *Let $\mathfrak{v}$ be a balance 2-term L_∞ algebra with invariant bilinear form $(\cdot, \cdot)$. A 2-derivation Γ is said* **orthogonal** *if*

$$(\partial \Gamma(x), X) + (x, \Gamma(\partial X)) = 0, \tag{2.4.139}$$

$$(y, [x\Gamma(z)] + [z, \Gamma(x)]) + (x, \Gamma([y, z])) + (z, \Gamma([y, x])) = 0. \tag{2.4.140}$$

It is evident that the derivations just defined are infinitesimal version of the orthogonal morphisms defined in definition (69) and (70), since the axioms they obey are the linearization of the axioms of orthogonal 1- and 2-morphisms. The exponential map $\exp : \mathfrak{oaut}(\mathfrak{v}) \to \mathrm{OAut}(\mathfrak{v})$ is just the restriction of the map $\exp : \mathfrak{aut}(\mathfrak{v}) \to \mathrm{Aut}(\mathfrak{v})$ to orthogonal derivations.

The next proposition shows that an adjoint morphism in a balanced 2-term L_∞ algebra with invariant form is always orthogonal:

Proposition 21. *Let $\mathfrak{v}$ be a balanced 2-term L_∞ algebra with invariant form $(\cdot,\cdot)$. Then the image of the adjoint representation lies in $\mathfrak{oaut}(\mathfrak{v})$, i.e.* $\mathrm{ad} : \mathfrak{v} \to \mathfrak{oaut}(\mathfrak{v})$.

Proof. First, we need to prove that a 1-derivation of the form $([x,\cdot],[x,\cdot],[x,\cdot,\cdot])$ for some $x \in \mathfrak{v}_0$ satisfies relations (2.4.137) and (2.4.138). This follows straightforwardly from the invariance of the bilinear form, (2.4.123)-(2.4.124).

Then we have to show that the 2-derivations $[\cdot,X]$ for $X \in \mathfrak{v}_1$ and $[x,y,\cdot]$ for $x,y \in \mathfrak{v}_0$ fulfill (2.4.139) and (2.4.140). Again, for each 2-derivation both the relations follow from (2.4.122)-(2.4.124). $\square$

It follows from this proposition that the exponential of an adjoint derivation is an orthogonal morphism.

2.4.7   Examples

There are several interesting examples of 2-term L_∞ algebras.

- Any pre-Lie algebra $\mathfrak{h}$ can be cast in the form of a non-strict 2-term L_∞ algebra. Recall that a pre-Lie algebra is a vector space endowed with a bilinear operation $[\cdot,\cdot]_{\mathfrak{h}} : \mathfrak{h} \wedge \mathfrak{h} \to \mathfrak{h}$ which is not required to satisfy the Jacobi identity. We can define a 2-term L_∞ algebra $\mathfrak{v}$ with $\mathfrak{v}_0 = \mathfrak{v}_1 = \mathfrak{h}$, $\partial = 1_{\mathfrak{h}}$, both the 2-brackets are equal to $[\cdot,\cdot]_{\mathfrak{h}}$ and the 3-bracket is the *jacobiator*:

$$[x,y,z] := [x,[y,z]_{\mathfrak{h}}]_{\mathfrak{h}} + [y,[z,x]_{\mathfrak{h}}]_{\mathfrak{h}} + [z,[x,y]_{\mathfrak{h}}]_{\mathfrak{h}}. \qquad (2.4.141)$$

  As a subexample, every non-associative algebra can be used to define such a 2-term L_∞ algebra, because every non-associative algebra $\mathfrak{a}$ can be given the structure of a pre-Lie algebra by endowing it with the antisymmetric bracket

$$[x,y] = xy - yx, \qquad (2.4.142)$$

  which in general do not fulfill the Jacobi identity due to the non-associativity. An interesting example of this construction is given by the octonions' algebra.

- A very important example is the so called *string 2-algebra*. Given a Lie algebra $\mathfrak{g}$ with an invariant bilinear form $\langle\cdot,\cdot\rangle$ and a real number $k \in \mathbb{R}$, the string 2-algebra $\mathfrak{string}_k(\mathfrak{g})$ is defined as the 2-term L_∞ algebra having $\mathfrak{string}_k(\mathfrak{g})_0 = \mathfrak{g}$ and $\mathfrak{string}_k(\mathfrak{g})_1 = \mathbb{R}$. The bracket $[\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ are the Lie bracket of the Lie algebra $\mathfrak{g}$, the linear map ∂ as well as the bracket $[\cdot,\cdot] : \mathfrak{g} \otimes \mathbb{R} \to \mathbb{R}$ vanish, while the 3-bracket $[\cdot,\cdot,\cdot] : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R}$ are defined as

$$[x,y,z] = k\langle x,[y,z]\rangle. \qquad (2.4.143)$$

- Rather than an example, we will now illustrate a particular kind of balanced 2term L_∞ algebras. Let $\mathfrak{v}$ be a balanced 2-term L_∞ algebra with a non-degenerate invariant bilinear form $(\cdot,\cdot)$. Suppose that $\partial = 0$. This means that the 2-bracket $[\cdot,\cdot]$ on $\mathfrak{v}_0$ satisfy the Jacobi identity, therefore $\mathfrak{v}_0$ is a Lie algebra. The non-degeneracy of the pairing $(\cdot,\cdot)$ between $\mathfrak{v}_0$ and $\mathfrak{v}_1$ canonically identifies $\mathfrak{v}_1$ with the algebraic dual of $\mathfrak{v}_0$. Furthermore the invariance of the bilinear form implies that the 2-bracket $[\cdot,\cdot] : \mathfrak{v}_0 \otimes \mathfrak{v}_1 \to \mathfrak{v}_1$ realizes the coadjoint action of $\mathfrak{v}_0$ on its dual

$\mathfrak{v}_1$. Due to property (2.4.39), the 3-bracket $[\cdot, \cdot, \cdot]$ defines a Chevalley-Eilenberg cocycle $\phi \in \mathrm{CE}^3(\mathfrak{v}_0, \mathfrak{v}_0^*)$. Since the bilinear form $(\cdot, \cdot)$ is invariant, ϕ belongs to the subcomplex of the *cyclic Chevalley-Eilenberg cochains*, i.e. those elements $\psi \in \mathrm{CE}^p(\mathfrak{v}_0, \mathfrak{v}_0^*)$ satisfying

$$\Big(x, \psi(y, \pi, \ldots, \pi)\Big) + \Big(y, \psi(x, \pi, \ldots, \pi)\Big) = 0. \qquad (2.4.144)$$

Since if ψ is a cyclic cochain then $Q_{CE}\psi$ is also cyclic, cyclic cochains form a subcomplex of the Chevalley-Eilenberg complex, denoted $cCE^\bullet(\mathfrak{v}_0, \mathfrak{v}_0^*)$. It is a fact that there is an isomorphism

$$cCE^\bullet(\mathfrak{v}_0, \mathfrak{v}_0^*) \cong CE^\bullet[-1](\mathfrak{v}_0) \qquad (2.4.145)$$

for $\mathfrak{v}_0$ a Lie algebra [57]-[58]. The correspondence is given by

$$\psi(\pi, \ldots, \pi) \to \Big(\pi, \psi(\pi, \ldots, \pi)\Big). \qquad (2.4.146)$$

Since it commutes with the Chevalley-Eilenberg differential, this isomorphism descends to cohomology.

We have thus shown that every balanced 2-term L_∞ algebra with invariant form and vanishing ∂ is determined by a Lie algebra $\mathfrak{v}_0$ together with a closed element of $CE^4(\mathfrak{v}_0)$. On the converse given a Lie algebra $\mathfrak{g}$ we can construct a balanced 2-term L_∞ algebra $\mathfrak{v}$ with $\mathfrak{v}_0 = \mathfrak{g}$ and $\mathfrak{v}_1 = \mathfrak{g}^*$ where the 3-bracket is given by a cocycle in $CE^4(\mathfrak{g})$. It can be shown that this 2-term L_∞ algebra is non-strict provided that this 4-cocycle defines a non-vanishing cohomology class in the Chevalley-Eilenberg cohomology of $\mathfrak{g}$.

2.5 Integrating L_∞ algebras

In this section we present the state of the art of Lie theory applied to L_∞ algebras. This is a very technical and complicated subject which we can't face in full depth, and since it is not of primary interest to our purpose we will just sketch the main ideas.

As every Lie algebra can be integrated to a Lie group, every n-term L_∞ algebra can be integrated to a smooth n-group. The general integration procedure (see [29],[30]) takes a L_∞ algebra V to build a simplicial manifold $\int V$. A simplicial manifold M is an infinite tower of manifolds M_k for $k \in \mathbb{N}$ together with smooth maps $d_i : M_k \to M_{k-1}$ for $i = 0, \ldots, k$ called *face maps* and morphisms $s_i : M_k \to M_{k+1}$ for $i = 0, \ldots, k$ called *degeneracy maps* which must obey several axioms. In this case, $\int V$ is defined as

$$\left(\textstyle\int V\right)_k = \mathrm{Hom}_{\mathsf{DGA}}(CE^\bullet(V), \Omega^\bullet(\Delta^k)). \qquad (2.5.1)$$

This means that $(\int V)_k$ is the space of all differential graded algebra homomorphisms from the Chevalley-Eilenberg complex of V and the de Rham complex on the standard k-dimensional simplex Δ^k. Elements in $(\int V)_k$ are roughly V-connections on Δ^k. The face and degeneracy maps for $\int V$ are defined by the restriction of such a connection on a face Δ^{k-1} of the k-simplex or by its trivial extension to the $k+1$-simplex. It can be shown that $\int V$ is a *Kan complex*. This means that it can be obtained as the

nerve of a smooth higher group. Given a n-groupoid $\mathcal{G}$, its nerve is a simplicial set (simplicial manifold if the groupoid is smooth) which has k-morphisms at the level $k < n$ and sequences of composable morphisms for $k > n$, and face and degeneracy maps are the target and source maps of $\mathcal{G}$ and the composition. For more details on simplicial sets and Kan complexes see [28] and [31]. Furthermore, it can be proved that all the information carried by $\int V$ lies in the first $n+1$ manifolds for V a n-term L_∞ algebra, so that a finite number of manifolds and maps is enough to extract a smooth n-group from $\int V$. This n-group is the one integrating V.

This integration procedure doesn't work for L_∞ algebras as nicely as standard Lie integration does for Lie algebras. The main difference is that the n-group arising from a n-term L_∞ algebras is generally infinite dimensional, as can be deduced from (2.5.1). This is not fully satisfactory, not only because infinite dimensional objects are much harder to control than finite ones, but mostly because it is reasonable to expect that this is not the most precise result. It is likely that the finite counterpart of a n-term L_∞ algebra is a finite dimensional n-group which is somehow contained in the infinite dimensional one we are able to construct. Evidence for this arises from some better results that have been achieved for particular cases, most notably for the string 2-group [6] and for nilpotent L_∞ algebras [30],[32]. Another good hint for this lies in the strict case: strict n-term L_∞ algebras can be integrated to finite dimensional strict n-groups. For example, differential Lie crossed module are integrated to Lie crossed modules. It is very unnatural that passing from strict to non strict algebras forces us to consider infinite dimensional higher groups: as shown in [33], the two integration procedures of strict and non-strict 2-term L_∞ algebras are morita equivalent. For instance, even if V is a Lie algebra the algorithm we readily described produces an infinite dimensional space where the Lie group integrating V is hidden. Furthermore, this picture leaves no room for an infinitesimal counterpart to finite dimensional non strict higher groups.

There is another approach to the problem which is promising and deserves to be mentioned. A way of defining global semistrict higher gauge theory has been recently investigated by Jurčo, Saemann and Wolf in [74], inspired by a previous work of Ševera [75]. In this paper the authors focus on the differentiation of semistrict 2-groups using descent data associated with 2-bundles with values in these 2-groups. Infinitesimally these descent data are seen to encode a semistrict 2-term L_∞ algebra. This is used in [74] to define gauge transformations for 2-groups connections in terms of maps and forms with values in the gauge 2-group, achieving a description which is more elementary than ours (see chapter 3). The drawback of this approach could be the heavier presence of the categorical framework at every stage, making it harder to make concrete computations.

Part II

2-gauge theory

Chapter 3

2-term L_∞ algebra gauge theory

In this chapter we will develop the basic ingredients of 2-term L_∞ algebra gauge theory. We will first of all recall the main features of ordinary gauge theories, then we will adapt them to define a suitable generalization which takes into account the possibility of a gauge structure that is encoded in a 2-term L_∞ algebra instead of an usual Lie algebra. The subject covered here is taken from [21] and [19]. We will be mainly focused on the local theory, neglecting global issues, which will be briefly targeted at the end of the chapter.

3.1 Ordinary gauge theory

In this section we review ordinary gauge theory. The basic ingredients in ordinary gauge theory are connection forms on principal bundles. Since, as mentioned, we are not interested in discussing global topology, we will only deal with a trivial principal bundle $M \times G$ with M a smooth orientable manifold diffeomorphic to $\mathbb{R}^n$ for some n and G a Lie group. M is the *base manifold* and G is the *structure group* or *gauge group*. The fields are usually differential forms with values in $\mathfrak{g} := \text{Lie}(G)$. We can make the following definition:

Definition 73. *A **field of bidegree** (m, n) is an element of $\Omega^m(M, \mathfrak{g}[n])$.*

Of central importance is the connection 1-form, which is a bidegree $(1, 0)$ field denoted $\omega \in \Omega^1(M, \mathfrak{g})$.

ω is characterized by its *curvature* f, which is the bidegree $(2, 0)$ field given by

$$f = d\omega + \frac{1}{2}[\omega, \omega]. \tag{3.1.1}$$

From its definition, f satisfies the standard *Bianchi identity*

$$df + [\omega, f] = 0. \tag{3.1.2}$$

The connection ω is said to be flat if the curvature 2-form vanishes, $f = 0$. The relation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \tag{3.1.3}$$

that realizes this condition is also called *Maurer-Cartan equation.*

Of central importance is the concept of gauge transformations. These are symmetries of the theory encoded in the action of the gauge group G on the principal bundle. Given a smooth map $\gamma \in \mathrm{Map}(M, G)$ this induces a gauge transformation whose action on a generic bidegree (n, m) field ϕ is

$$\phi \to \phi' = \gamma \phi \gamma^{-1} = \mathrm{Ad}\, \gamma \phi. \tag{3.1.4}$$

The connection 1-form behaves differently under a gauge transformation, and it is shifted in the following way:

$$\omega \to \omega' = \gamma \omega \gamma^{-1} - d\gamma\gamma^{-1} = \mathrm{Ad}\, \gamma \omega - d\gamma\gamma^{-1}. \tag{3.1.5}$$

$\mathrm{Map}(M, G)$ is the set which governs gauge transformations, and it is a group with the multiplication in G as group law and the trivial map $\gamma \equiv 1_G$ as identity. Gauge transformation action is a left action of $\mathrm{Map}(M, G)$ on the space of $\mathfrak{g}$-connections $\Omega^1(M, \mathfrak{g})$.

The term $-d\gamma\gamma^{-1}$ is a bidegree $(1, 0)$ field. If we make the change $\gamma \to \xi\gamma$ for $\xi \in \mathrm{Map}(M, G)$ the 1-form $-d\gamma\gamma^{-1}$ becomes $\mathrm{Ad}\, \xi(-d\gamma\gamma^{-1}) - d\xi\xi^{-1}$. Moreover it satisfies identically the Maurer-Cartan equation, therefore it can be regarded itself as a flat connection.

Notice that the the curvature 2-form behaves covariantly under a gauge transformation:

$$f \to f' = \mathrm{Ad}\, \gamma f. \tag{3.1.6}$$

Instead the de Rham differential of a field ϕ doesn't, as the adjoint action of the gauge transformation doesn't commute with d: $d\phi' \neq \mathrm{Ad}\, \gamma d\phi$. Covariant expressions which include the de Rham differential, essential in defining sensible gauge theories, are obtained through the *covariant derivative* of a field ϕ, which is given by the well–known expression

$$D\phi = d\phi + [\omega, \phi], \tag{3.1.7}$$

for ϕ any bidegree (n, m) field. The covariant derivative satisfies the standard Ricci identity:

$$DD\phi = [f, \phi]. \tag{3.1.8}$$

What is more, as its name suggests it transforms covariantly under gauge transformations:

$$D\phi \to D'\phi' = \mathrm{Ad}\, \gamma D\phi \tag{3.1.9}$$

The Bianchi identity (3.1.2) obeyed by f can be written compactly through the covariant derivative as

$$Df = 0. \tag{3.1.10}$$

In standard gauge theory, gauge symmetry is most efficiently analyzed concentrating on infinitesimal gauge transformation of the adjoint type. Infinitesimal gauge transformations are contained in $\mathrm{Map}(M, \mathfrak{g})$, the Lie algebra of $\mathrm{Map}(M, G)$. The action of an element $\xi \in \mathrm{Map}(M, \mathfrak{g})$ on a connection is the linearization of a finite gauge transformation:

$$\omega \to \omega' = \omega - [\omega, \xi] - d\xi = \omega - D\xi, \tag{3.1.11}$$

or

$$\delta_\xi \omega := \omega' - \omega = -D\xi. \tag{3.1.12}$$

Since infinitesimal gauge transformations are a Lie algebra, a commutator of two infinitesimal gauge transformations is again an infinitesimal gauge transformation:

$$[\delta_\xi, \delta_\eta]\omega = (\delta_\xi \delta_\eta - \delta_\eta \delta_\xi)\omega = -D[\xi, \eta] = \delta_{[\xi,\eta]}\omega. \tag{3.1.13}$$

Infinitesimal gauge transformations can then be elevated to an odd differential, called the BRST operator, whose cohomology classifies the observables of the theory. This is done by introducing a bidegree $(0, 1)$ ghost field $c \in \Omega^0(M, \mathfrak{g}[1])$ which parametrizes the ghost degree 1 infinitesimal gauge transformation. This shifted infinitesimal gauge transformation gives the odd BRST operator s. Its action on the connection ω is

$$s\omega = -Dc \tag{3.1.14}$$

To make s nilpotent we have to suitably define the variation sc of c. Since by (3.1.14) we have

$$s^2\omega = D\Big(sc + \frac{1}{2}[c, c]\Big), \tag{3.1.15}$$

we can enforce $s^2\omega = 0$ by setting

$$sc = -\frac{1}{2}[c, c]. \tag{3.1.16}$$

$s^2 c = 0$, as is readily verified, and so s is nilpotent as required,

$$s^2 = 0. \tag{3.1.17}$$

For completeness, we report also the BRST variation of the curvature f of ω which, by (3.1.39), reads

$$sf = -[c, f]. \tag{3.1.18}$$

3.1.1 The Weil algebra and the extended gauge transformations

In order to extend these concepts to higher gauge structures such as a 2-term L_∞ algebra, we need to reformulate them in a different fashion. Connection and curvature differential forms have a useful interpretation in term of differential graded commutative algebras which we will now illustrate ([21]). This approach makes use of the Weil algebra. The theory of the Weil algebra is classical and well-established [53],[54],[55],[56]. The way we are going to present it here is by no means general, since it is a particular case that applies to trivial fiber bundles and it is of use for our scopes.

Definition 74. *Given a Lie algebra* $\mathfrak{g}$, *the **Weil algebra of** $\mathfrak{g}$ $W(\mathfrak{g})$ is the graded commutative algebra defined as*

$$W^\bullet(\mathfrak{g}) := S^\bullet(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2]). \tag{3.1.19}$$

It is possible to turn this algebra into a differential complex by defining a differential on it. Notice that the Chevalley-Eilenberg complex of $\mathfrak{g}$ sits into $W^\bullet(\mathfrak{g})$ as a subalgebra. The Weil differential is defined on this subalgebra as the sum of the Chevalley-Eilenberg differential plus a shift operator:

$$Q_W = Q_{CE} + \sigma. \tag{3.1.20}$$

Here $\sigma : \mathfrak{g}^*[1] \to \mathfrak{g}^*[2]$ is an operator that acts as the identity on the vector space $\mathfrak{g}^*$ but increases the degree by 1. Denoting $\{e_a\}$ a basis for $\mathfrak{g}$, $\{\pi^a\}$ a basis for $\mathfrak{g}^*[1]$ and $\{\gamma^a\}$ a basis for $\mathfrak{g}^*[2]$, and setting $\pi := \pi^a \otimes e_a$ and $\gamma := \gamma^a \otimes e_a$ the action of Q_W on $\mathfrak{g}^*[1]$ is summarized in the formula

$$Q_W \pi = -\frac{1}{2}[\pi, \pi] + \gamma. \tag{3.1.21}$$

Knowing that $Q_{CE}^2 \pi = \sigma^2 \pi = 0$, to achieve $Q_W^2 \pi = 0$ we need $(Q_{CE}\sigma + \sigma Q_{CE})\pi = 0$. This determines the action of Q_W on γ:

$$Q_W \gamma = Q_{CE} \gamma = -\sigma Q_{CE} \pi = -[\pi, \gamma]. \tag{3.1.22}$$

With a straightforward computation we see that $Q_W^2 \gamma = 0$ due to the Jacobi identity and the fact that $[\gamma, \gamma] = 0$ by antisymmetry. This turns $W^\bullet(\mathfrak{g})$ into a differential graded commutative algebra (dgca).

The usefulness of the Weil algebra lies in that it captures the algebraic properties of connection and curvature differential forms. The link between theses objects is cleared in the next proposition:

Proposition 22. *Given a Lie algebra $\mathfrak{g}$ and a manifold $M \cong \mathbb{R}^n$ for some n, a dgca homomorphisms $\mathcal{A} : W^\bullet(\mathfrak{g}) \to \Omega^\bullet(M)$ uniquely defines a $\mathfrak{g}$-connection on M and vice-versa.*

Proof. Any homomorphism $\mathcal{A} : W^\bullet(\mathfrak{g}) \to \Omega^\bullet(M)$ defines by its action on the generators $\{\pi^a\}$ a 1-form $\omega \in \Omega^1(M, \mathfrak{g})$:

$$\omega^a := \mathcal{A}(\pi^a) \in \Omega^1(M), \tag{3.1.23}$$

which is a $\mathfrak{g}$-connection on M.

On the other hand, since the Weil algebra is free, to determine the dgca homomorphism $\mathcal{A}$ we need both a $\mathfrak{g}$-connection ω and a 2-form $f \in \Omega^2(M, \mathfrak{g})$ which gives the action of $\mathcal{A}$ on the generators $\{\gamma^a\}$:

$$\mathcal{A}(\gamma^a) := f^a \in \Omega^2(M). \tag{3.1.24}$$

But since $\mathcal{A}$ is a dgca homomorphism, we need to impose $d\mathcal{A} = \mathcal{A}Q_W$. Applying this to (3.1.21) and (3.1.22) we obtain the constraints

$$d\omega = -\frac{1}{2}[\omega, \omega] + f, \tag{3.1.25}$$

$$df = -[\omega, f]. \tag{3.1.26}$$

These relations identify the 2-form $f \in \Omega^2(M, \mathfrak{g})$ as the curvature of the connection ω, which is therefore enough to completely define $\mathcal{A}$. $\qquad\square$

This proposition furnishes a powerful tool to deal with connections and curvatures in a merely algebraic way. Actually, this result can be used to define a connection as a dgca homomorphism from a Weil algebra to a de Rham complex, and to define the curvature 2-form and the covariant derivative looking at the relations induced by the Weil differential.

We shall also give a different definition of gauge transformations:

Definition 75. *Given a smooth manifold $M \cong \mathbb{R}^n$ and a Lie group G, an **extended gauge transformation** (g, σ_g) consists of*

1. *a map $g \in \mathrm{Map}(M, \mathrm{Aut}(\mathfrak{g}))$,*

2. *a flat connection σ_g,*

$$d\sigma_g + \frac{1}{2}[\sigma_g, \sigma_g] = 0, \qquad (3.1.27)$$

such that

$$g^{-1}dg(\pi) - [\sigma_g, \pi] = 0, \qquad (3.1.28)$$

where as usual $\pi = \pi^a \otimes e_a$. We shall denote by $\mathrm{Gau}(M, \mathfrak{g})$ the set of all extended gauge transformations.

We shall denote the gauge transformation by (g, σ_g) or simply by g, having in mind that now σ_g is not determined by g but participates with g in the transformation.

The definition of gauge transformation given here is more general than the one we gave previously. If G is a Lie group exponentiating $\mathfrak{g}$ and $\gamma \in \mathrm{Map}(M, G)$, then the pair $(\mathrm{Ad}\,\gamma, \gamma^{-1}d\gamma)$ is a gauge transformation in the sense just defined. However, not every extended gauge transformation (g, σ_g) is of this form, since there is no requirement that the automorphism g is the adjoint action, nor that the connection σ_g is the pullback of the left invariant Maurer-Cartan form on G. Nevertheless, extended gauge transformations can be defined disregarding the gauge group G and using as fundamental algebraic datum only the Lie algebra $\mathfrak{g}$. This makes it much easier to extend them to higher algebraic structures bypassing the challenging problem of integrating an L_∞ algebra to a higher group.

$\mathrm{Gau}(M, \mathfrak{g})$ substitutes $\mathrm{Map}(M, G)$, which is the group that contains all gauge transformations defined by (3.1.5). As already mentioned, the latter is included in the former according to the correspondence $\gamma \to (\mathrm{Ad}\,\gamma, \gamma^{-1}d\gamma)$. The action of an extended gauge transformation $g \in \mathrm{Gau}(M, \mathfrak{g})$ on a connection $\omega \in \Omega^1(M, \mathfrak{g})$ must then be coherent with (3.1.5). Then, the gauge transform ${}^g\omega$ of ω reads

$$ {}^g\omega = g(\omega - \sigma_g). \qquad (3.1.29) $$

The gauge transform ${}^g f$ of the curvature curvature f of ω is

$$ {}^g f = g(f), \qquad (3.1.30) $$

due to (3.1.27)-(3.1.28). In the case of a gauge transformation of the form $(\mathrm{Ad}\,\gamma, -\gamma^{-1}d\gamma)$ this is compatible with (3.1.6).

Gau$(M, \mathfrak{g})$ is an infinite dimensional Lie group, which contains Map(M, G) as a proper subgroup. The composition, denoted $\diamond$, can be determined by performing two gauge transformations:

$$^{h \diamond g}\omega = (h \diamond g)(\omega - \sigma_{h \diamond g}), \tag{3.1.31}$$

$$^{h \diamond g}\omega = {}^h({}^g\omega) = h(g(\omega - \sigma_g) - \sigma_h). \tag{3.1.32}$$

Comparing the two terms one obtains

$$h \diamond g = hg, \tag{3.1.33a}$$

$$\sigma_{h \diamond g} = \sigma_g + g^{-1}(\sigma_h). \tag{3.1.33b}$$

It is readily checked that these relations define an extended gauge transformation which respects (3.1.27)-(3.1.28).

The inversion and the unit of Gau$(M, \mathfrak{g})$ can be obtained in a similar manner and are defined by the relations

$$g^{-1 \diamond} = g^{-1}, \tag{3.1.33c}$$

$$\sigma_{g^{-1 \diamond}} = -g(\sigma_g), \tag{3.1.33d}$$

$$i = \mathrm{id}_{\mathfrak{g}}, \tag{3.1.33e}$$

$$\sigma_i = 0, \tag{3.1.33f}$$

where $g, h \in$ Gau$(M, \mathfrak{g})$ and, in (3.1.33a), (3.1.33c), (3.1.33e), the composition, inversion and unit in the right hand side are those of Aut$(\mathfrak{g})$ thought of as holding pointwise on M.

The form of (3.1.29) ensures that covariant differentiation is gauge covariant (cf. eq. (3.1.7)):

$$^gD^g\phi = d(g(\phi)) + [g(\omega - \sigma_g), g(\phi)] = g(D\phi), \tag{3.1.34}$$

and that gauge transformation action is a left action of the group Gau$_1(M, \mathfrak{g})$ on the space of $\mathfrak{g}$–connections as required.

Ordinary gauge transformation can be studied in infinitesimal form, and so do extended gauge transformations. An *infinitesimal extended gauge transformation* is an extended gauge transformation in linearized form. It consists of:

1. a map $u \in$ Map$(M, \mathfrak{aut}(\mathfrak{g}))$,

2. a linearized flat connection $\dot{\sigma}_u$,

$$d\dot{\sigma}_u = 0, \tag{3.1.35}$$

obeying the relation

$$du(\pi) - [\dot{\sigma}_u, \pi] = 0, \tag{3.1.36}$$

as follows from expanding (3.1.27), (3.1.28) to first order around the unit transformation i. We shall denote the transformation as $(u, \dot{\sigma}_u)$, understanding as usual that $\dot{\sigma}_u$ is the partner of u in the gauge transformation, or simply as u. We shall denote the set of all infinitesimal gauge transformations by $\mathfrak{gau}(M, \mathfrak{g})$.

$\mathfrak{gau}(M, \mathfrak{g})$ is an infinite dimensional Lie algebra, in fact that of the gauge transformation Lie group $\mathrm{Gau}(M, \mathfrak{g})$. The brackets of $\mathfrak{gau}(M, \mathfrak{g})$ are defined by

$$[u, v]_\diamond = [u, v], \tag{3.1.37a}$$

$$\dot{\sigma}_{[u,v]_\diamond} = u(\dot{\sigma}_v) - v(\dot{\sigma}_u), \tag{3.1.37b}$$

where $u, v \in \mathfrak{gau}(M, \mathfrak{g})$. In (3.1.37a), the brackets in the right hand side are those of $\mathfrak{aut}(\mathfrak{g})$ thought of as holding pointwise on M.

Given an infinitesimal extended gauge transformation $(u, \dot{\sigma}_u) \in \mathfrak{gau}(M, \mathfrak{g})$, the gauge variation $\delta_u \omega$ of ω is

$$\delta_u \omega = u(\omega) - \dot{\sigma}_u. \tag{3.1.38}$$

The gauge variation $\delta_u f$ of f reads similarly as

$$\delta_u f = u(f). \tag{3.1.39}$$

Since $\mathrm{Gau}(M, \mathfrak{g})$ contains $\mathrm{Map}(M, G)$ as a subgroup, $\mathfrak{gau}(M, \mathfrak{g})$ contains $\mathrm{Map}(M, \mathfrak{g})$ as a subalgebra. The infinitesimal extended gauge transformations corresponding to elements of $\mathrm{Map}(M, \mathfrak{g})$ are of the form

$$u = \mathrm{ad}\, s, \tag{3.1.40a}$$

$$\dot{\sigma}_u = ds, \tag{3.1.40b}$$

where $s \in \mathrm{Map}(M, \mathfrak{g})$. In (3.1.40a), the adjoint operator in the right hand side is that of $\mathfrak{g}$ holding pointwise on M. Formulas (3.1.40) are the infinitesimal version of $(\mathrm{Ad}\,\gamma, \gamma^{-1}d\gamma)$ for $\gamma \in \mathrm{Map}(M, G)$.

Infinitesimal extended gauge transformation can be exponentiated to finite ones. The exponential map $\exp_\diamond : \mathfrak{gau}(M, \mathfrak{g}) \to \mathrm{Gau}(M, \mathfrak{g})$ is given by

$$\exp_\diamond(u) = \exp(u), \tag{3.1.41a}$$

$$\sigma_{\exp_\diamond(u)} = \frac{1_\mathfrak{g} - \exp(-u)}{u}(\dot{\sigma}_u), \tag{3.1.41b}$$

where $u \in \mathfrak{gau}(M, \mathfrak{g})$. In (3.1.41a), the exponentiation in the right hand side is that of $\mathfrak{aut}(\mathfrak{g})$ thought of as holding pointwise on M. As one expects, the exponentiation of an infinitesimal extended gauge transformation of the adjoint kind, as in (3.1.40), gives an ordinary gauge transformation in $\mathrm{Map}(M, G)$. For $s \in \mathrm{Map}(M, \mathfrak{g})$, we have

$$\exp(\mathrm{ad}\, s) = \mathrm{Ad}\exp(s) = \mathrm{Ad}\,\gamma, \tag{3.1.42}$$

where $\gamma \in \mathrm{Map}(M, G)$ is pointwise an element of the gauge group exponentiating s. Furthermore

$$\frac{1_\mathfrak{g} - \exp(-\mathrm{ad}\, s)}{\mathrm{ad}\, s}(ds) = \int_0^1 dt \exp(-t\,\mathrm{ad}\, s)(ds) = \exp(-s) \int_0^1 dt \exp((1-t)s) ds \exp(ts) =$$

$$= \exp(-s) d \exp(s) = \gamma^{-1} d\gamma. \tag{3.1.43}$$

3.1.2   Orthogonal gauge transformation

The notion of extended gauge transformation as it stands so far is not precise enough to be of real use in gauge theory. Usually to build an action functional an invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, most often a trace over some representation, is taken. To have an action invariant under gauge transformations it is crucial to require invariance of the bilinear form:

$$(\operatorname{Ad}\gamma(x), \operatorname{Ad}\gamma(y)) = (x, y) \tag{3.1.44}$$

for any $x, y \in \mathfrak{g}$. If we are dealing with extended gauge transformation we admit arbitrary automorphisms of $\mathfrak{g}$, without restricting ourself to adjoint automorphisms. General automorphisms are not guarantee to respect invariance of the bilinear form. Thus we have to impose this by hand.

Given a Lie algebra $\mathfrak{g}$ equipped with an invariant bilinear form $(\cdot, \cdot)$, an *orthogonal automorphism* g of $\mathfrak{g}$ is an automorphism of $\mathfrak{g}$ such that

$$(g(x), g(y)) = (x, y) \tag{3.1.45}$$

for any $x, y \in \mathfrak{g}$. The set of orthogonal automorphisms of $\mathfrak{g}$ is denoted $\operatorname{OAut}(\mathfrak{g})$, and it is in fact a Lie subgroup of $\operatorname{Aut}(\mathfrak{g})$. An extended gauge transformation (g, σ_g) of $\operatorname{Gau}(M, \mathfrak{g})$ is said *orthogonal* if g is pointwise orthogonal,

1. $g \in \operatorname{Map}(M, \operatorname{OAut}(\mathfrak{g}))$.

We shall denote by $\operatorname{OGau}(M, \mathfrak{g})$ the set of all orthogonal elements $g \in \operatorname{Gau}(M, \mathfrak{g})$. $\operatorname{OGau}(M, \mathfrak{g})$ is an infinite dimensional Lie proper subgroup of the gauge Lie group $\operatorname{Gau}(M, \mathfrak{g})$, nevertheless it still contains $\operatorname{Map}(M, G)$ as a subgroup.

An infinitesimal extended gauge transformation $(u, \dot{\sigma}_u)$ of $\mathfrak{gau}(M, \mathfrak{g})$ is accordingly orthogonal if u is pointwise orthogonal,

1. $u \in \operatorname{Map}(M, \mathfrak{oaut}(\mathfrak{g}))$.

Here $\mathfrak{oaut}(\mathfrak{g})$ is the Lie algebra of $\operatorname{OAut}(\mathfrak{g})$, and it is defined by all derivations u of $\mathfrak{g}$ such that

$$(u(x), y) + (x, u(y)) = 0 \tag{3.1.46}$$

for any $x, y \in \mathfrak{g}$. We let $\mathfrak{ogau}(M, \mathfrak{g})$ be the set of all orthogonal elements $u \in \mathfrak{gau}(M, \mathfrak{g})$. $\mathfrak{ogau}(M, \mathfrak{g})$ is an infinite dimensional Lie subalgebra of the gauge Lie algebra $\mathfrak{gau}(M, \mathfrak{g})$. $\mathfrak{ogau}(M, \mathfrak{g})$ is also the Lie algebra of the orthogonal gauge Lie group $\operatorname{OGau}(M, \mathfrak{g})$.

For $s \in \Omega^0(M, \mathfrak{g})$, the adjoint type infinitesimal gauge transformation $\operatorname{ad}_M s \in \mathfrak{gau}(M, \mathfrak{g})$ is clearly orthogonal, $\operatorname{ad}_M s \in \mathfrak{ogau}(M, \mathfrak{g})$, since it belongs to $\operatorname{Map}(M, \mathfrak{g})$, the prdinary infinitesimal gauge trasfnormations which are orthogonal by construction.

The exponential map $\exp_\diamond : \mathfrak{ogau}(M, \mathfrak{g}) \to \operatorname{OGau}(M, \mathfrak{g})$ of $\mathfrak{ogau}(M, \mathfrak{g})$ is simply the restriction of the exponential map $\exp_\diamond : \mathfrak{gau}(M, \mathfrak{g}) \to \operatorname{Gau}(M, \mathfrak{g})$ of $\mathfrak{gau}(M, \mathfrak{g})$ to $\mathfrak{ogau}(M, \mathfrak{g})$. In particular, the orthogonal exponential is still computed by the expressions (3.1.40).

3.2 Semistrict higher gauge theory

The expression *semistrict higher gauge theory* stands for a gauge theory whose algebraic structure is encoded in a (2-term) L_∞ algebra which is generally taken to be non- strict. In this section we define suitable generalizations of the central concepts of gauge theory to 2-term L_∞ algebras. Namely, we will define fields and connection forms with value in a 2-term L_∞ algebra $\mathfrak{v}$, gauge transformations for such fields and, in the spirit of the higher categorical setting of higher gauge theory, 2-gauge transformations, which have no analog in ordinary gauge theory.

Again, we will limit ourself to the local theory. We work on a smooth manifold M diffeomorphic to $\mathbb{R}^n$. Here we intentionally avoid any mention to a flat higher bundle which should replace $G \times M$, or to a structure or gauge 2-group which should take the place of G. Our formulation of higher gauge theory works consistently without needing these delicate concepts.

3.2.1 Field and connection doublets

First of all we have to define the field content. In semistrict higher gauge theory with structure Lie 2–algebra $\mathfrak{v}$, fields are organized in *field doublets*, due to the fact that $\mathfrak{v}$ is a direct sum of two vector spaces, $\mathfrak{v}_0$ and $\mathfrak{v}_1$.

Definition 76. *A **bidegree** (m, n) **field doublet** is a couple of differential forms*

$$(\phi, \Phi_\phi) \in \Omega^m(M, \mathfrak{v}_0[n]) \times \Omega^{m+1}(M, \mathfrak{v}_1[n]), \tag{3.2.1}$$

where $-1 \leq m \leq d$. If $m = -1$, the first component of the doublet vanishes. If $m = d$, the second component does.

Above, we attached a suffix ϕ to Φ_ϕ to indicate that Φ_ϕ is the partner of ϕ in the doublet, not to mean that Φ_ϕ depends on ϕ in any way. This allows us to concisely denote the doublet (ϕ, Φ_ϕ) simply as ϕ in many instances.

The forms entering in a doublet have not the same form degree, the second component of the doublet being of form degree greater than the degree of the first component by 1. The reason lies in the grading of the 2-term L_∞ algebra $\mathfrak{v}$. A field doublet of bidegree (m, n) is just an homogeneous degree $m+n$ element of $\Omega^\bullet(M) \otimes \tilde{\mathfrak{v}}[n]$, see subsection 2.4.2. The way it is defined in (3.2.1) makes it explicit the difference between the component that takes value in $\mathfrak{v}_0$ and the one with value in $\mathfrak{v}_1$.

To define connection and curvature forms for semistrict higher gauge theory, we employ the Weil algebra, in parallel to subsection 3.1.1. Given a 2-term L_∞ algebra, its Weil algebra is given by

$$W^\bullet(\mathfrak{v}) = S^\bullet(\mathfrak{v}_0^*[1] \oplus \mathfrak{v}_1^*[2] \oplus \mathfrak{v}_0^*[2] \oplus \mathfrak{v}_1^*[3]). \tag{3.2.2}$$

We pick basis $\{e_a\}$, $\{E_A\}$, $\{\pi^a\}$, $\{\Pi^A\}$, $\{\gamma^a\}$ and $\{\Gamma^A\}$ for $\mathfrak{v}_0$, $\mathfrak{v}_1$, $\mathfrak{v}_0^*[1]$, $\mathfrak{v}_1^*[2]$, $\mathfrak{v}_0^*[2]$ and $\mathfrak{v}_1^*[3]$ respectively, and as usual we define $\pi := \pi^a \otimes e_a$, $\Pi := \Pi^A \otimes E_A$, $\gamma := \gamma^a \otimes e_a$ and $\Gamma := \Gamma^A \otimes E_A$. The Weil differential Q_W is defined as the sum of the Chevalley-Eilenberg differential Q_{CE} plus a shift operator σ. The action of Q_W on elements of

$\mathfrak{v}_0^*[1]$ and of $\mathfrak{v}_1^*[2]$ is thus obtained looking at (2.4.40) and is defined as

$$Q_W \pi = -\frac{1}{2}[\pi, \pi] + \partial\Pi + \gamma, \tag{3.2.3}$$

$$Q_W \Pi = -[\pi, \Pi] + \frac{1}{6}[\pi, \pi, \pi] + \Gamma. \tag{3.2.4}$$

The action of Q_W on elements of $\mathfrak{v}_0^*[2]$ and $\mathfrak{v}_1^*[3]$ is instead computed by enforcing $Q_W^2 = 0$ on π and Π:

$$Q_W \gamma = -[\pi, \gamma] - \partial\Gamma, \tag{3.2.5}$$

$$Q_W \Gamma = -[\pi, \Gamma] + [\gamma, \Pi] - \frac{1}{2}[\pi, \pi, \gamma]. \tag{3.2.6}$$

Now we extend the correspondence between connection forms and dgca algebra homomorphism between the Weil algebra and the de Rham complex on M from ordinary gauge theory to the present setting. Thus a $\mathfrak{v}$-connection on M is determined by a homomorphism $\mathcal{A}: W^\bullet(\mathfrak{v}) \to \Omega^\bullet(M)$. Such a homomorphism is defined by its action on the generators π and Π. Since $\omega := \mathcal{A}(\pi)$ and $\Omega := \mathcal{A}(\Pi)$ belong to $\Omega^1(M, \mathfrak{v}_0)$ and $\Omega^2(M, \mathfrak{v}_1)$ respectively, we make the following definition:

Definition 77. *Given a 2-term L_∞ algebra $\mathfrak{v}$, a $\mathfrak{v}$-**connection on** M is a bidegree (1,0) doublet (ω, Ω_ω).*

Analogously to what happens in ordinary gauge theory, the image of γ and Γ under the action of the homomorphism $\mathcal{A}$ defines a 2-form $f \in \Omega^2(M, \mathfrak{v}_0)$ and a 3-form $F \in \Omega^3(M, \mathfrak{v}_1)$ which are the curvatures. The forms f and F are completely determined by ω and Ω_ω, due to the fact that $\mathcal{A}$ preserves the differential, and this tells us how curvature and connection forms are related:

Definition 78. *Given a connection doublet (ω, Ω_ω), its **curvature** is the bidegree (2,0) doublet (f, F_f) defined by*

$$f = d\omega + \frac{1}{2}[\omega, \omega] - \partial\Omega_\omega, \tag{3.2.7}$$

$$F_f = d\Omega_\omega + [\omega, \Omega_\omega] - \frac{1}{6}[\omega, \omega, \omega]. \tag{3.2.8}$$

*A connection doublet is said **flat** if $f = F_f = 0$.*

The 2-form f is also known as the *fake curvature* of the connection doublet. The reason is that some properties of the curvature in ordinary gauge theory apply to F_f in higher gauge theory provided that $f = 0$, as we shall see later.

The last information we can obtain from the Weil algebra and from the dgca algebra homomorphism $\mathcal{A}$ is the shape of the covariant derivative. We saw in the last section that the covariant derivative in ordinary gauge theory can be guessed by looking at the preservation of the differential acting on γ (or f), which is translated to the Bianchi identity in the language of differential forms. In higher gauge theory we use the same argument. From (3.2.5) and (3.2.6) we find that

$$df + [\omega, f] + \partial F_f = 0, \tag{3.2.9}$$

$$dF_f + [\omega, F_f] - [f, \Omega_\omega] + \frac{1}{2}[\omega, \omega, f] = 0. \tag{3.2.10}$$

These are the Bianchi identities for the curvature doublet (f, F_f). It is easy to check that they hold from the definition of the curvature (3.2.7)-(3.2.8). We wish to write them as

$$Df = 0, \tag{3.2.11}$$

$$DF_f = 0, \tag{3.2.12}$$

as is the case for the ordinary Bianchi identity, (3.1.10). Therefore we make the following definition:

Definition 79. *Let* (ϕ, Φ_ϕ) *be a field doublet of bidegree* (p, q). *The* **covariant derivative doublet** *of* (ϕ, Φ_ϕ) *is the field doublet* $(D\phi, D\Phi_\phi)$ *of bidegree* $(p+1, q)$ *given by*

$$D\phi = d\phi + [\omega, \phi] + (-1)^{p+q} \partial \Phi_\phi, \tag{3.2.13a}$$

$$D\Phi_\phi = d\Phi_\phi + [\omega, \Phi_\phi] - (-1)^{p+q}[\phi, \Omega_\omega] + \frac{(-1)^{p+q}}{2}[\omega, \omega, \phi]. \tag{3.2.13b}$$

The sign $(-1)^{p+q}$ is conventional, since the relative sign of ϕ, Φ_ϕ cannot be fixed in any natural manner. The covariant derivative doublet of (ϕ, Φ_ϕ) should be properly written as $(D\phi, D\Phi_{D\phi})$. We shall write it as $(D\phi, D\Phi_\phi)$ for simplicity.

From (3.2.13), we deduce easily the appropriate version of the Ricci identities,

$$DD\phi = [f, \phi], \tag{3.2.14a}$$

$$DD\Phi_\phi = [f, \Phi_\phi] - [\phi, F] - [\phi, \omega, f]. \tag{3.2.14b}$$

The explicit appearance of the connection component ω in the right hand side of (3.2.14b) is a consequence of the presence of a term quadratic in ω in (3.2.13b).

3.2.2   2-term L_∞ algebra higher gauge transformations

The definition of gauge transformations for field and connection doublets is very delicate and complicated. Unlike connection and curvature forms, for which the Weil algebra provided a safe and easy path to the generalizations done in the last subsection, there is no straightforward way to extend the concept of gauge transformation from the ordinary setting to the higher one. Nevertheless, it is possible to build a sensible and useful notion of higher gauge transformation which we shall now show [21]. While proceeding we will try to explain which are the main reasons that lead to this definition.

We start from the extended gauge transformations defined for ordinary gauge theory. Since they are the only kind of gauge transformations that we will be able to treat in the higher setting, from now on we will call them simple gauge transformations, forgetting about the distinction between them and the original kind of gauge transformations typical of usual gauge theory.

There are two basic ingredients in the ordinary gauge transformations: an automorphism of the Lie algebra $\mathfrak{g}$ (local on M) g and a flat connection σ_g. These data can be generalized as they stand with no difficulties. We pick

1. a smooth map $g = (g_0, g_1, g_2) \in \mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$,

2. a flat connection doublet $(\sigma_g, \Sigma_g) \in \Omega^1(M, \mathfrak{v}_0) \times \Omega^2(M, \mathfrak{v}_1)$,

$$d\sigma_g + \frac{1}{2}[\sigma_g, \sigma_g] - \partial\Sigma_g = 0, \qquad (3.2.15a)$$

$$d\Sigma_g + [\sigma_g, \Sigma_g] - \frac{1}{6}[\sigma_g, \sigma_g, \sigma_g] = 0. \qquad (3.2.15b)$$

g and σ_g are not unrelated. The pullback by g of the left invariant Maurer-Cartan form on $\mathfrak{aut}(\mathfrak{g})$ must be equal to the adjoint of σ_g, (3.1.28). First of all we have to find the left invariant Maurer-Cartan form for $\mathrm{Aut}_1(\mathfrak{v})$, which takes values in $\mathfrak{aut}_0(\mathfrak{v})$. Given $g \in \mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$, the triplet

$$g_0^{-1}dg_0, \qquad (3.2.16)$$

$$g_1^{-1}dg_1, \qquad (3.2.17)$$

$$g_1^{-1}dg_2(x, y) - g_1^{-1}g_2(g_0^{-1}dg_0(x), y) - g_1^{-1}g_2(x, g_0^{-1}dg_0(y)). \qquad (3.2.18)$$

defines a 1-form on M with values in $\mathfrak{aut}_0(\mathfrak{v})$. It is easy to check that it satisfies the Maurer-Cartan equation and that it is invariant under the left composition in $\mathrm{Aut}_1(\mathfrak{v})$. This triplet can then be regarded as the pullback by g of the left invariant Maurer-Cartan form, and we will denote it simply as $g^{-1}dg$. It is now tempting to require that

$$g^{-1}dg - \mathrm{ad}_0(\sigma_g) = 0, \qquad (3.2.19)$$

in the spirit of (3.1.28). Unfortunately, this is not a good choice. In the ordinary case, the flatness of σ_g as a connection was equivalent to the Maurer-Cartan equation for $g^{-1}dg$. In the higher setting this is no longer true. Since $\mathrm{Aut}_1(\mathfrak{v})$ is an ordinary Lie group the Maurer-Cartan equation for $g^{-1}dg$ isn't different from the ordinary one, but the flatness condition for σ_g is changed due to the fact that $\mathfrak{v}$ isn't a Lie algebra. (3.2.19) means

$$d\mathrm{ad}_0(\sigma_g) + \frac{1}{2}[\mathrm{ad}_0(\sigma_g), \mathrm{ad}_0(\sigma_g)]_{\mathfrak{aut}} = 0, \qquad (3.2.20)$$

which in turn implies the following three constraints:

$$\partial(\frac{1}{2}[\sigma_g, \sigma_g, x] - [x, \Sigma_g]) = 0, \qquad (3.2.21)$$

$$\frac{1}{2}[\sigma_g, \sigma_g, \partial X] - [\partial X, \Sigma_g] = 0, \qquad (3.2.22)$$

$$[x, \frac{1}{2}[\sigma_g, \sigma_g, y] - [y, \Sigma_g]] - [y, \frac{1}{2}[\sigma_g, \sigma_g, x] - [x, \Sigma_g]] - \frac{1}{2}[\sigma_g, \sigma_g, [x, y]] + [[x, y], \Sigma_g] = 0, \qquad (3.2.23)$$

for $x, y \in \mathfrak{v}_0$, $X \in \mathfrak{v}_1$. These can be satisfied by putting

$$\frac{1}{2}[\sigma_g, \sigma_g, \cdot] - [\cdot, \Sigma_g] = \frac{1}{2}\mathrm{ad}_2(\sigma_g, \sigma_g) - \mathrm{ad}_1(\Sigma_g) = 0, \qquad (3.2.24)$$

but this purely algebraic constraint on the flat connection doublet (σ_g, Σ_g) is very unnatural and doesn't fit into any interpretation.

To circumvent this obstacle, we choose to relax equation (3.2.19). As is usual in higher category theory, we ask that (3.2.19) holds only up to higher homotopy. Since

the equation takes value in $\Omega^1(M, \mathfrak{aut}_0(\mathfrak{v}))$, we pick an element $\tau_g \in \Omega^1(M, \mathfrak{aut}_1(\mathfrak{v}))$ and we set

$$g^{-1}dg - \mathrm{ad}_0(\sigma_g) - \partial_{\mathfrak{aut}}\tau_g = 0. \tag{3.2.25}$$

Now the Maurer-Cartan equation for $g^{-1}dg$ translates into the following differential constraint on τ_g:

$$d\tau_g + \frac{1}{2}[\partial_{\mathfrak{aut}}\tau_g, \tau_g]_{\mathfrak{aut}} + [\mathrm{ad}_0(\sigma_g), \tau_g]_{\mathfrak{aut}} - \mathrm{ad}_1(\Sigma_g) + \frac{1}{2}\mathrm{ad}_2(\sigma_g, \sigma_g) = 0. \tag{3.2.26}$$

To sum up, we have the following definition:

Definition 80. *A **higher 1–gauge transformation** consists of the following data:*

1. *a map $g \in \mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$,*

2. *a flat connection doublet (σ_g, Σ_g),*

3. *an element τ_g of $\Omega^1(M, \mathfrak{aut}_1(\mathfrak{v}))$ satisfying*

$$d\tau_g(\pi) + [\sigma_g, \tau_g(\pi)] - [\pi, \Sigma_g] + \frac{1}{2}[\sigma_g, \sigma_g, \pi] + \tau_g([\sigma_g, \pi] + \partial\tau_g(\pi)) = 0. \tag{3.2.27}$$

g, σ_g, Σ_g, τ_g are required to satisfy a number of relations. If $g = (g_0, g_1, g_2)$, these relations read:

$$g_0{}^{-1}dg_0(\pi) - [\sigma_g, \pi] - \partial\tau_g(\pi) = 0, \tag{3.2.28a}$$
$$g_1{}^{-1}dg_1(\Pi) - [\sigma_g, \Pi] - \tau_g(\partial\Pi) = 0, \tag{3.2.28b}$$
$$g_1{}^{-1}(dg_2(\pi, \pi) - 2g_2(g_0{}^{-1}dg_0(\pi), \pi)) - [\sigma_g, \pi, \pi] - \tau_g([\pi, \pi]) - 2[\pi, \tau_g(\pi)] = 0. \tag{3.2.28c}$$

We shall denote the set of all higher 1–gauge transformations by $\mathrm{Gau}_1(M, \mathfrak{v})$.

In the following, we are going to denote a 1–gauge transformation such as the above as $(g, \sigma_g, \Sigma_g, \tau_g)$ or simply as g. Again, in so doing, we are not implying that σ_g, Σ_g, τ_g are determined by g, but only that they are the partners of g in the gauge transformation.

This definition of higher gauge transformation gives a generalization of $\mathrm{Gau}(M, \mathfrak{g})$, but still we have to define its action on field and connection doublets in order to see its validity and to employ it in a higher gauge field theory.

To argue the action of $\mathrm{Gau}_1(M, \mathfrak{v})$ on doublets we start from some requirements. First of all, we want that higher gauge transformations contain ordinary gauge transformations as a special case. In the particular case of a 2-term L_∞ algebra which actually is a Lie algebra, i.e. $\mathfrak{v}_1 \equiv 0$, a field doublet (ϕ, Φ_ϕ) reduces simply to ϕ and a 1–gauge transformation $(g, \sigma_g, \Sigma_g, \tau_g)$ boils down to (g_0, σ_g). Thus, we require that the first component of a doublet transforms as

$$^g\phi = g_0(\phi), \tag{3.2.29}$$

because this is the most general linear transformation rule that reduces to the ordinary case if $\mathfrak{v}_1 = 0$. A similar argument fixes the transformation law for the connection 1-form ω:

$$^g\omega = g_0(\omega - \sigma_g). \tag{3.2.30}$$

The transformation of the second component Φ_ϕ of a doublet as well as of the connection 2-form Ω_ω cannot be guessed in this way. Asking the gauge action to be linear on fields and completely general on connections, the most general form it can take on Φ_ϕ and Ω_ω reads

$$^g\Phi_\phi = g_1(\Phi_\phi) + \kappa(\phi), \tag{3.2.31}$$

$$^g\Omega_\omega = g_1(\Omega_\omega) + \eta(\omega) + \Xi + \frac{1}{2}\xi(\omega,\omega), \tag{3.2.32}$$

where $\kappa, \eta \in \Omega^1(M, \mathrm{Map}(\mathfrak{v}_0, \mathfrak{v}_1))$, $\Xi \in \Omega^2(M, \mathfrak{v}_1)$ and $\xi \in \Omega^0(M, \mathrm{Map}(\mathfrak{v}_0 \wedge \mathfrak{v}_0, \mathfrak{v}_1))$ are undetermined parameters, which we now wish to fix as functions of the gauge elements g, (σ_g, Σ_g) and τ_g. In order to do so we adopt another requirement: we impose to our gauge transformations to render the covariant derivative D we defined in (3.2.13) really *covariant*, that is to commute with the gauge action. Applied to the first component ϕ of a (p, q) bidegree doublet (ϕ, Φ_ϕ) this translates into

$$^gD^g\phi = g_0(D\phi), \tag{3.2.33}$$

or, more explicitly,

$$dg_0(\phi) + [g_0(\omega - \sigma_g), g_0(\phi)] + (-1)^{p+q}\partial(g_1(\Phi_\phi + \kappa(\phi))) = g_0(d\phi) + g_0([\omega, \phi]) + (-1)^{p+q}g_0(\partial\Phi_\phi). \tag{3.2.34}$$

This relation is fulfilled if

$$\kappa = -(-1)^{p+q}g_1(\tau_g(\cdot)) + (-1)^{p+q}g_2(\omega - \sigma_g, \cdot). \tag{3.2.35}$$

The transformation law for the second component of the doublet is thus totally determined:

$$^g\Phi_\phi = g_1(\Phi_\phi - (-1)^{p+q}\tau_g(\phi)) + (-1)^{p+q}g_2(\omega - \sigma_g, \phi). \tag{3.2.36}$$

This formula shows some new features if compared to the ordinary gauge transformations of fields. Most evidently it shuffles the components of the doublet, but this is of no surprise if we think of the two fields entering the doublet as just two component of the same vector in $\tilde{\mathfrak{v}}$. What is more, here we have an explicit appearance of the connection 1-form ω. Therefore in semistrict higher gauge theory, the gauge action on the fields is not independent of the choice of the connection doublet, as was the case in ordinary gauge theory.

If we try to impose covariance on the second component of the covariant derivative of a doublet, we run into problems. The formula

$$^gD^g\Phi_\phi = g_1(D\Phi_\phi - (-1)^{p+q+1}\tau_g(D\phi)) + (-1)^{p+q+1}g_2(\omega - \sigma_g, D\phi) \tag{3.2.37}$$

cannot hold as it stands, even if we adjust the maps η, Ξ and ξ ad hoc. The point is that on the left-hand side the term $g_2(f, \phi)$ appears, with f the fake curvature, and it can't be canceled by any term on the right-hand side. Therefore we have to relax the form of covariance obeyed by the second component of a covariant derivative to

$$^gD^g\Phi_\phi = g_1(D\Phi_\phi - (-1)^{p+q+1}\tau_g(D\phi)) + (-1)^{p+q+1}g_2(\omega - \sigma_g, D\phi) + (-1)^{p+q}g_2(f, \phi). \tag{3.2.38}$$

This can be achieved, and it fixes the free parameters entering in the gauge transformation of Ω_ω:

$$\eta(\cdot) = g_1(\tau_g(\cdot)) + g_2(\sigma_g, \cdot), \tag{3.2.39}$$

$$\Xi = -g_1(\tau_g(\sigma_g) + \Sigma_g) - \frac{1}{2}g_2(\sigma_g, \sigma_g), \tag{3.2.40}$$

$$\xi(\cdot, \cdot) = -g_2(\cdot, \cdot). \tag{3.2.41}$$

The gauge transformation action of the connection doublet can then be completely written:

$$^g\omega = g_0(\omega - \sigma_g), \tag{3.2.42a}$$

$$^g\Omega_\omega = g_1(\Omega_\omega - \Sigma_g + \tau_g(\omega - \sigma_g)) - \frac{1}{2}g_2(\omega - \sigma_g, \omega - \sigma_g). \tag{3.2.42b}$$

As a last consistency requirement, we have to check that these gauge transformations are coherent with the definition of curvature doublet. Namely, we want that the curvature 2-from and 3-form transform as a bidegree $(2, 0)$ doublet according to (3.2.29) and (3.2.36). This is indeed the case: the gauge transform of the curvature doublet $f = (f, F_f)$ of ω is computed as

$$^g f = g_0(f), \tag{3.2.43a}$$

$$^g F_f = g_1(F_f - \tau_g(f)) + g_2(\omega - \sigma_g, f). \tag{3.2.43b}$$

Here we can make an interesting remark on why f is also called fake curvature. The point is that if $f = 0$, two important features of ordinary gauge theory straightforwardly extend to the semistrict case: the covariance of the covariant derivative and of the curvature (3-form in this case). Indeed, if $f = 0$ then (3.2.37) holds, and so

$$^g D^g \Phi_\phi = {}^g(D\Phi_\phi). \tag{3.2.44}$$

Moreover if $f = 0$ the transformation law for F becomes

$$^g F = g_1(F). \tag{3.2.45}$$

In analogy to higher category theory, in higher gauge theory it is possible to define a notion of 2-gauge transformation. These can be interpreted as *gauge for gauge symmetry* in the language of ordinary field theory. We will present now how they can be constructed within our framework.

We will build 2-gauge transformations as all deformation that transform a gauge transformation $(g, \sigma_g, \Sigma_g, \tau_g)$ into another gauge transformation $(h, \sigma_h, \Sigma_h, \tau_h)$. The former will be called the source and the latter the target of the 2-gauge transformation.

It is legit to take as first datum of a 2-gauge transformation a map $F \in \Omega^0(M, \mathrm{Aut}_2(\mathfrak{v}))$, such that point-wise on M F is a 2-morphism in $\mathrm{Aut}_2(\mathfrak{v})$ going from g to h, $F : g \Rightarrow h$. We can think of h as the 2-gauge transformed of g. We have:

$$h_0 = g_0 - \partial F, \tag{3.2.46}$$

$$h_1 = g_1 - F\partial, \tag{3.2.47}$$

$$h_2(\pi, \pi) = g_2(\pi, \pi) - F([\pi, \pi]) + 2[g_0(\pi), F(\pi)] - [\partial F(\pi), F(\pi)]. \tag{3.2.48}$$

Next, we perturb the flat connection doublet (σ_g, Σ_g) by shifting it. It is an easy computation to show that, in order to have a transformed doublet which is again flat as a connection, the shifted doublet must be

$$\sigma_h = \sigma_g - \partial A_F, \tag{3.2.49}$$

$$\Sigma_h = \Sigma_g - dA_F - [\sigma_g, A_F] + \frac{1}{2}[\partial A_F, A_F], \tag{3.2.50}$$

with $A_F \in \Omega^1(M, \mathfrak{v})$. The transformed of τ_g, denoted τ_h, is determined by forcing equations (3.2.27)-(3.2.28) to hold for the transformed gauge transformation h. Putting all together, we come to the following definition:

Definition 81. *Let g and h be gauge transformations. A **2-gauge tranformations from g to** h, also denoted $F : g \Rightarrow h$, consists of the following data.*

1. *a map $F \in \mathrm{Map}(M, \mathrm{Aut}_2(\mathfrak{v}))(g, h)$, where $\mathrm{Map}(M, \mathrm{Aut}_2(\mathfrak{v}))(g, h)$ is the space of sections of the fiber bundle $\bigcup_{m \in M} \mathrm{Aut}_2(\mathfrak{v})(g(m), h(m)) \to M$*

2. *an element $A_F \in \Omega^1(M, \mathfrak{v}_1)$.*

F, A_F are required to satisfy the relations,

$$\sigma_g - \sigma_h = \partial A_F, \tag{3.2.51a}$$

$$\Sigma_g - \Sigma_h = dA_F + [\sigma_g, A_F] - \frac{1}{2}[\partial A_F, A_F], \tag{3.2.51b}$$

$$\tau_g(\pi) - \tau_h(\pi) = -[\pi, A_F] + g_1^{-1}\big(dF(\pi) - F([\sigma_h, \pi] + \partial\tau_h(\pi))\big). \tag{3.2.51c}$$

The set of all 2-gauge transformations is denoted $\mathrm{Gau}_2(M, \mathfrak{v})$, and the set of all 2-gauge transformations from g to h is denoted $\mathrm{Gau}_2(M, \mathfrak{v})(g, h)$.

In the following, we are going to denote a 2–gauge transformation like the above as (F, A_F), meaning that A_F is the partner of F in the transformation, or simply as F.

In the following, we will call gauge transformations *1-gauge transformations*, to make the relashionship between them and 2-gauge transformations explicit.

We just defined 2-gauge transformations as having a source and a target. In the spirit of gauge field theory, we could think of them as being intrinsically independent of 1-gauge transformation, on which they act, much as a 1-gauge transformation acting on a connection doublet is independent on the connection itself (recall that this is not true if one considers also field doublets). In this case, the set of 2-gauge transformations is the subset of $\mathrm{Gau}_2(M, \mathfrak{v})$ having the identity 1-gauge transformation as source, denoted $\mathrm{Gau}_2^*(M, \mathfrak{v})$. This set can be characterized as the set of pairs (F, A_F) with:

1. $F \in \mathrm{Map}(M, \mathrm{Aut}_2{}^*(\mathfrak{v}))$;

2. $A_F \in \Omega^*(M, \mathfrak{v}_1)$.

A 2-gauge transformation $G \in \mathrm{Gau}_2^*(M, \mathfrak{v})$ has a target 1-gauge transformation $t(G)$ defined by the quadruple

$$t(G), \tag{3.2.52}$$

$$\sigma_{t(G)} = -\partial A_G, \tag{3.2.53}$$

$$\Sigma_{t(G)} = -dA_G + \frac{1}{2}[\partial A_G, A_G], \tag{3.2.54}$$

$$\tau_{t(G)}(\pi) = [\pi, A_G] + \tag{3.2.55}$$

where the t map in the first line is the crossed module homomorphism of $\mathrm{Aut}(\mathfrak{v})$, see (2.4.78). Such a 2-gauge transformation G acts on a 1-gauge transformation g as:

$$^G g = t(G)g, \tag{3.2.56a}$$

$$\sigma_{^G g} = \sigma_g - \partial g_1^{-1}(A_G), \tag{3.2.56b}$$

$$\Sigma_{^G g} = \Sigma_g - d(g_1^{-1}(A_G)) - [\sigma_g, g_1^{-1}(A_G)] + \frac{1}{2}[\partial g_1^{-1}(A_G), g_1^{-1}(A_G)], \tag{3.2.56c}$$

$$\tau_{^G g}(\pi) = \tau_g(\pi) + [\pi, g_1^{-1}(A_G)] - g_1^{-1}(1_{\mathfrak{v}_1} - G\partial)^{-1} dG g_0(\pi). \tag{3.2.56d}$$

3.2.3 The categorical structure of higher gauge transformations

In ordinary gauge theory, gauge transformations can be viewed as the action of a group on the set of all connections. Semistrict higher gauge theory is richer under this perspective, and it admits several interpretations.

As our notation could hint, $\mathrm{Gau}_1(M, \mathfrak{v})$ and $\mathrm{Gau}_2(M, \mathfrak{v})$ combine to form an infinite dimensional strict Lie 2–group, the gauge transformation 2–group of the theory, denoted $\mathrm{Gau}(M, \mathfrak{v})$. 1-gauge transformations in $\mathrm{Gau}_1(M, \mathfrak{v})$ are the 1-morphisms and 2-gauge transformations in $\mathrm{Gau}_2(M, \mathfrak{v})$ are the 2-morphisms. The composition (denoted $\diamond$) and inversion laws and the unit 1-gauge transformation and the horizontal ($\diamond$) and vertical ($\bullet$) composition and inversion laws and the unit 2–gauge transformations of $\mathrm{Gau}(M, \mathfrak{v})$ are defined by

$$h \diamond g = h \circ g, \tag{3.2.57a}$$

$$\sigma_{h \diamond g} = \sigma_g + g_0^{-1}(\sigma_h), \tag{3.2.57b}$$

$$\Sigma_{h \diamond g} = \Sigma_g + g_1^{-1}\left(\Sigma_h + \frac{1}{2} g_2(g_0^{-1}(\sigma_h), g_0^{-1}(\sigma_h))\right) - \tau_g(g_0^{-1}(\sigma_h)), \tag{3.2.57c}$$

$$\tau_{h \diamond g}(\pi) = \tau_g(\pi) + g_1^{-1}\left(\tau_h(g_0(\pi)) - g_2(g_0^{-1}(\sigma_h), \pi)\right), \tag{3.2.57d}$$

$$g^{-1\diamond} = g^{-1\circ}, \tag{3.2.57e}$$

$$\sigma_{g^{-1\diamond}} = -g_0(\sigma_g), \tag{3.2.57f}$$

$$\Sigma_{g^{-1\diamond}} = -g_1(\Sigma_g + \tau_g(\sigma_g)) - \frac{1}{2} g_2(\sigma_g, \sigma_g), \tag{3.2.57g}$$

$$\tau_{g^{-1\diamond}}(\pi) = -g_1(\tau_g(g_0^{-1}(\pi))) - g_2(\sigma_g, g_0^{-1}(\pi)), \tag{3.2.57h}$$

$$i = \mathrm{id}, \tag{3.2.57i}$$

$$\sigma_i = 0, \tag{3.2.57j}$$

$$\Sigma_i = 0, \tag{3.2.57k}$$

$$\tau_i(\pi) = 0, \tag{3.2.57l}$$

$$G \diamond F = G \circ F, \tag{3.2.57m}$$

$$A_{G \diamond F} = A_F + h^{-1}{}_1(A_G) - g_1{}^{-1}Fh_0{}^{-1}(\sigma_k), \tag{3.2.57n}$$

$$F^{-1 \diamond} = F^{-1 \circ}, \tag{3.2.57o}$$

$$A_{F^{-1 \diamond}} = -g_1(A_F) - F(\sigma_h), \tag{3.2.57p}$$

$$K \bullet H = K \cdot H, \tag{3.2.57q}$$

$$A_{K \bullet H} = A_H + A_K, \tag{3.2.57r}$$

$$H^{-1 \bullet} = H^{-1 \cdot}, \tag{3.2.57s}$$

$$A_{H^{-1 \bullet}} = -A_H, \tag{3.2.57t}$$

$$I_g = \mathrm{Id}_g, \tag{3.2.57u}$$

$$A_{I_g} = 0, \tag{3.2.57v}$$

where $g, h, k, l \in \mathrm{Gau}_1(M, \mathfrak{v})$ and $F, G, H, K \in \mathrm{Gau}_2(M, \mathfrak{v})$, with $F : g \Rightarrow h$, $G : k \Rightarrow l$ and H, K composable. In (3.2.57a), (3.2.57e), (3.2.57i), the composition, inversion and unit in the right hand side are those of $\mathrm{Aut}_1(\mathfrak{v})$ thought of as holding pointwise on M. In (3.2.57m), (3.2.57o), (3.2.57q), (3.2.57s), (3.2.57u), the horizontal and vertical composition and inversion and the units in the right hand side are those of $\mathrm{Aut}_2(\mathfrak{v})$ thought of as holding pointwise on M.

All these 2-group laws can be determined by comparing the consecutive action of two transformations, as we did for the ordinary case. For example, equations (3.2.57a)-(3.2.57d) can be obtained requiring

$$^h({}^g\omega) = {}^{h \diamond g}\omega \ , \ ^h({}^g\Omega_\omega) = {}^{h \diamond g}\Omega_\omega. \tag{3.2.58}$$

Therefore the gauge action is the action of the group $\mathrm{Gau}_1(M, \mathfrak{v})$ on the set of all connection doublets.

The strict 2–group $\mathrm{Gau}(M, \mathfrak{v})$ can be described also as a crossed module. The two groups underlying it are $\mathrm{Gau}_1(M, \mathfrak{v})$ and $\mathrm{Gau}_2{}^*(M, \mathfrak{v})$. The crossed module multiplications, inversions, units, target map and action are linked to the 2-group laws according to proposition 14, and they are given by the expressions

$$h \diamond g = h \circ g, \tag{3.2.59a}$$

$$\sigma_{h \diamond g} = \sigma_g + g_0{}^{-1}(\sigma_h), \tag{3.2.59b}$$

$$\Sigma_{h \diamond g} = \Sigma_g + g_1{}^{-1}\left(\Sigma_h + \frac{1}{2}g_2(g_0{}^{-1}(\sigma_h), g_0{}^{-1}(\sigma_h))\right) - \tau_g(g_0{}^{-1}(\sigma_h)), \tag{3.2.59c}$$

$$\tau_{h \diamond g}(\pi) = \tau_g(\pi) + g_1{}^{-1}\left(\tau_h(g_0(\pi)) - g_2(g_0{}^{-1}(\sigma_h), \pi)\right), \tag{3.2.59d}$$

$$g^{-1 \diamond} = g^{-1 \circ}, \tag{3.2.59e}$$

$$\sigma_{g^{-1 \diamond}} = -g_0(\sigma_g), \tag{3.2.59f}$$

$$\Sigma_{g^{-1 \diamond}} = -g_1(\Sigma_g + \tau_g(\sigma_g)) - \frac{1}{2}g_2(\sigma_g, \sigma_g), \tag{3.2.59g}$$

$$\tau_{g^{-1 \diamond}}(\pi) = -g_1(\tau_g(g_0{}^{-1}(\pi))) - g_2(\sigma_g, g_0{}^{-1}(\pi)), \tag{3.2.59h}$$

$$i = \mathrm{id}, \tag{3.2.59i}$$

$$\sigma_i = 0, \tag{3.2.59j}$$

$$\Sigma_i = 0, \tag{3.2.59k}$$

$$\tau_i(\pi) = 0, \tag{3.2.59l}$$

$$G \diamond F = G \circ F, \tag{3.2.59m}$$

$$A_{G \diamond F} = A_F + (1_{\mathfrak{v}_1} - F\partial)^{-1}(A_G), \tag{3.2.59n}$$

$$F^{-1\diamond} = F^{-1\circ}, \tag{3.2.59o}$$

$$A_{F^{-1\diamond}} = -(1_{\mathfrak{v}_1} - F\partial)(A_F), \tag{3.2.59p}$$

$$I = \mathrm{Id}_i, \tag{3.2.59q}$$

$$\mathrm{t}(F) = t(F) \tag{3.2.59r}$$

$$\sigma_{\mathrm{t}(F)} = -\partial A_F, \tag{3.2.59s}$$

$$\Sigma_{\mathrm{t}(F)} = -dA_F + \frac{1}{2}[\partial A_F, A_F], \tag{3.2.59t}$$

$$\tau_{\mathrm{t}(F)}(\pi) = [\pi, A_F] - (1_{\mathfrak{v}_1} - F\partial)^{-1}dF(\pi) \tag{3.2.59u}$$

$$A_{I_g} = 0, \tag{3.2.59v}$$

$$\mathrm{m}(g)(F) = m(g)(F), \tag{3.2.59w}$$

$$A_{\mathrm{m}(g)(F)} = g_1(A_F - F(1_{\mathfrak{v}_0} - \partial F)^{-1}(\sigma_g)), \tag{3.2.59x}$$

where $g, h \in \mathrm{Gau}_1(M, \mathfrak{v})$ and $F, G \in \mathrm{Gau}_2{}^*(M, \mathfrak{v})$. In (3.2.59a) (3.2.59e), (3.2.59i), the composition, inversion and unit in the right hand side are those of $\mathrm{Aut}_1(\mathfrak{v})$ thought of as holding pointwise on M. In (3.2.59m), (3.2.59o), (3.2.59q), the composition, inversion and unit in the right hand side are those of $\mathrm{Aut}_2{}^*(\mathfrak{v})$ thought of as holding pointwise on M. In (3.2.59r), the target map in the right hand side is that of $\mathrm{Aut}_2{}^*(\mathfrak{v})$ thought of as holding pointwise on M. Finally, in (3.2.59w), the crossed module action in the right hand side is that of $\mathrm{Aut}_1(\mathfrak{v})$ on $\mathrm{Aut}_2{}^*(\mathfrak{v})$ thought of as holding pointwise on M.

The action of $\mathrm{Gau}_2^*(M, \mathfrak{v})$ on $\mathrm{Gau}_1(M, \mathfrak{v})$ (3.2.56) is also compatible with the composition $\diamond$ in $\mathrm{Gau}_2^*(M, \mathfrak{v})$, because

$$^G(^Fg) = {}^{G \diamond F}g. \tag{3.2.60}$$

Under this point of view, 2-gauge transformations behave on 1-gauge transformations exactly as 1-gauge transformations do on connections.

This is not the only interpretation we can give to the higher gauge structure. Another interesting point of view, which we didn't discuss in the ordinary case, employs 2-groupoids instead of 2-groups, absorbing connection doublets as the objects. This is particularly useful when dealing with field doublets, because gauge transformations cannot act on the second component of a doublet without knowing which is the chosen connection 1-form. We can therefore define the 2-groupoid $\mathrm{Conn}(M, \mathfrak{v})$ in the following way:

- objects are connection doublets (ω, Ω_ω) on M;

- given two objects ω and ω', the 1-morphisms going from ω to ω' are all those 1-gauge transformations $g \in \mathrm{Gau}_1(M, \mathfrak{v})$ such that $\omega' = {}^g\omega$;

- given two 1-morphisms g and h, the set of 2-morphisms going from g to h is $\mathrm{Gau}_2(M, \mathfrak{g})(g, h)$.

All the compositions, units and inversions are those of $\mathrm{Gau}(M, \mathfrak{v})$. Notice that the gauge structure encoded in $\mathrm{Conn}(M, \mathfrak{v})$ is slightly different to that encoded in $\mathrm{Gau}(M, \mathfrak{v})$. First of all, any 1-gauge transformation in $\mathrm{Gau}_1(M, \mathfrak{v})$ appears several times in $\mathrm{Conn}(M, \mathfrak{v})$,

with different source and target connections. Furthermore, not every 2-gauge transformation belonging to $\mathrm{Gau}_2(M, \mathfrak{v})$ has room in $\mathrm{Conn}(M, \mathfrak{v})$, due to the target matching condition of the 2-groupoid. In $\mathrm{Conn}(M, \mathfrak{v})$, given two 2-gauge transformations g and h with source the connection doublet ω, they can be linked by a 2-gauge transformation $F : g \Rightarrow h$ only if they share also the same target connection doublet, that is if

$$^{g}\omega = {}^{h}\omega. \tag{3.2.61}$$

In $\mathrm{Gau}(M, \mathfrak{v})$ this restriction lacks, and we can meet 2-gauge transformations between totally unrelated 1-gauge transformations. These are not present in $\mathrm{Conn}(M, \mathfrak{v})$. 2-gauge transformations between 1-gauge transformations with the same source and target can be nicely characterized provided that the fake curvature is zero. Let us define the 2-groupoid $\mathrm{Conn}_f(M, \mathfrak{v})$, which is identical to $\mathrm{Conn}(M, \mathfrak{v})$ but restricted to object whose fake curvature vanishes. This is well defined because $f = 0$ is a gauge invariant condition. It can be shown that if $f = 0$, given a 2-gauge transformation $F : g \to h$ between two 1-gauge transformations with the same source ω and the same target connection doublet, the condition $^{g}\omega = {}^{h}\omega$ is equivalent to the condition

$$A_F = h_1^{-1} F(\omega - \sigma_g). \tag{3.2.62}$$

3.2.4   The 2-term L_∞ algebra of infinitesimal gauge transformations

In higher gauge theory, as in ordinary gauge theory, many aspects of gauge symmetry are often conveniently studied by switching to the infinitesimal form of gauge transformations.

Consider a higher gauge theory with symmetry 2-term L_∞ algebra $\mathfrak{v}$. A *infinitesimal higher 1–gauge transformation* is a 1–gauge transformation in linearized form as in the ordinary case. Expanding (3.2.15), (3.2.27) around the unit transformation i to first order reveals that it consists of a set of data of the following form:

1. a map $u \in \mathrm{Map}(M, \mathfrak{aut}_0(\mathfrak{v}))$;

2. a linearized flat connection doublet $(\dot{\sigma}_u, \dot{\Sigma}_u)$,

$$d\dot{\sigma}_u - \partial \dot{\Sigma}_u = 0, \tag{3.2.63a}$$

$$d\dot{\Sigma}_u = 0; \tag{3.2.63b}$$

3. an element $\dot{\tau}_u$ of $\Omega^1(M, \mathfrak{aut}_1(\mathfrak{v}))$ such that

$$d\dot{\tau}_u(\pi) - [\pi, \dot{\Sigma}_u] = 0. \tag{3.2.64}$$

u, $\dot{\sigma}_u$, $\dot{\Sigma}_u$, $\dot{\tau}_u$ are required to satisfy the relations stemming from (3.2.28) by linearization. If $u = (u_0, u_1, u_2)$ then these read

$$du_0(\pi) - [\dot{\sigma}_u, \pi] - \partial \dot{\tau}_u(\pi) = 0, \tag{3.2.65a}$$

$$du_1(\Pi) - [\dot{\sigma}_u, \Pi] - \dot{\tau}_u(\partial \Pi) = 0, \tag{3.2.65b}$$

$$du_2(\pi, \pi) - [\dot{\sigma}_u, \pi, \pi] - \dot{\tau}_u([\pi, \pi]) - 2[\pi, \dot{\tau}_u(\pi)] = 0. \tag{3.2.65c}$$

In the following, we shall denote the infinitesimal 1–gauge transformation as $(u, \dot{\sigma}_u, \dot{\Sigma}_u, \dot{\tau}_u)$, indicating as usual $\dot{\sigma}_u$, $\dot{\Sigma}_u$, $\dot{\tau}_u$ as the partners of u in the gauge transformation data, or simply as u. We shall denote the set of all infinitesimal Lie 2–algebra 1–gauge transformations by $\mathfrak{gau}_0(M, \mathfrak{v})$.

Elements of $\mathfrak{gau}_0(M, \mathfrak{v})$ act on connections and fields, generating infinitesimal gauge variations. For an infinitesimal 1–gauge transformation $u \in \mathfrak{gau}_0(M, \mathfrak{v})$, the gauge variation $(\delta_u \omega, \delta_u \Omega_\omega)$ of (ω, Ω_ω) obeys (3.2.42) in infinitesimal form, which reads

$$\delta_u \omega = u_0(\omega) - \dot{\sigma}_u, \tag{3.2.66a}$$

$$\delta_u \Omega_\omega = u_1(\Omega_\omega) - \dot{\Sigma}_u + \dot{\tau}_u(\omega) - \frac{1}{2} u_2(\omega, \omega). \tag{3.2.66b}$$

The gauge variation $(\delta_u \phi, \delta_u \Phi_\phi)$ of a field doublet (ϕ, Φ) of bidegree (p, q) is (see (3.2.29)-(3.2.36))

$$\delta_u \phi = u_0(\phi), \tag{3.2.67}$$

$$\delta_u \Phi_\phi = u_1(\Phi_\phi) - (-1)^{p+q} \dot{\tau}_u(\phi) + (-1)^{p+q} u_2(\omega, \phi). \tag{3.2.68}$$

The gauge variation $(\delta_u f, \delta_u F_f)$ of (f, F_f) reads similarly as

$$\delta_u f = u_0(f), \tag{3.2.69a}$$

$$\delta_u F_f = u_1(F_f) - \dot{\tau}_u(f) + u_2(\omega, f). \tag{3.2.69b}$$

2-gauge transformations can be put in infinitesimal form too. Expansion around the unit transformation I_i to first order shows that an *infinitesimal higher 2–gauge transformation* consists of the data

1. a map $P \in \mathrm{Map}(M, \mathfrak{aut}_1(\mathfrak{v}))$;

2. an element $\dot{A}_P \in \Omega^1(M, \mathfrak{v}_1)$.

There are no further relations these objects must obey. We shall denote the infinitesimal 2–gauge transformation as $(P, \dot{A}_P)$, indicating $\dot{A}_P$ as the partner of P in the gauge transformation, or simply as P. We shall denote the set of all infinitesimal higher 2–gauge transformations by $\mathfrak{gau}_1(M, \mathfrak{v})$.

The action of an infinitesimal 2–gauge transformation $P \in \mathfrak{gau}_1(M, \mathfrak{v})$ on a 1–gauge transformation $g \in \mathrm{Gau}_1(M, \mathfrak{v})$ correspondingly is

$$g^{-1} \delta_P g = \tau_\circ P, \tag{3.2.70a}$$

$$\delta_P \sigma_g = -\partial g_1^{-1}(\dot{A}_P), \tag{3.2.70b}$$

$$\delta_P \Sigma_g = -d(g_1^{-1}(\dot{A}_P)) - [\sigma_g, g_1^{-1}(\dot{A}_P)], \tag{3.2.70c}$$

$$\delta_P \tau_g(\pi) = [\pi, g_1^{-1}(\dot{A}_P)] - g_1^{-1} dP g_0(\pi). \tag{3.2.70d}$$

This in turn induces an action of P on an infinitesimal 1–gauge transformation $u \in \mathfrak{gau}_0(M, \mathfrak{v})$ given by

$$\delta_P u = \tau_\circ P, \tag{3.2.71a}$$

$$\delta_P \dot{\sigma}_u = -\partial \dot{A}_P, \tag{3.2.71b}$$

$$\delta_P \dot{\Sigma}_u = -d\dot{A}_P, \tag{3.2.71c}$$

$$\delta_P \dot{\tau}_u(\pi) = [\pi, \dot{A}_P] - dP(\pi). \tag{3.2.71d}$$

2–gauge symmetry represents gauge for gauge symmetry, that is gauge symmetry of 1–gauge transformation.

As one might expect, $\mathfrak{gau}_0(M, \mathfrak{v})$ and $\mathfrak{gau}_1(M, \mathfrak{v})$ combine together to form an infinite dimensional 2-term L_∞ algebra, denoted $\mathfrak{gau}(M, \mathfrak{v})$, in fact that of the gauge transformation Lie 2–group $\mathrm{Gau}(M, \mathfrak{v})$. Since $\mathrm{Gau}(M, \mathfrak{v})$ is strict as a 2-group, $\mathfrak{gau}(M, \mathfrak{v})$ is strict as a 2-term L_∞ algebra. The boundary map and the brackets of $\mathfrak{gau}(M, \mathfrak{v})$ are given by the expressions

$$\partial_\diamond P = \partial_\circ P, \tag{3.2.72a}$$

$$\dot\sigma_{\partial_\diamond P} = -\partial \dot A_P, \tag{3.2.72b}$$

$$\dot\Sigma_{\partial_\diamond P} = -d\dot A_P, \tag{3.2.72c}$$

$$\dot\tau_{\partial_\diamond P}(\pi) = [\pi, \dot A_P] - dP(\pi), \tag{3.2.72d}$$

$$[u, v]_\diamond = [u, v]_\circ, \tag{3.2.72e}$$

$$\dot\sigma_{[u,v]_\diamond} = u_0(\dot\sigma_v) - v_0(\dot\sigma_u), \tag{3.2.72f}$$

$$\dot\Sigma_{[u,v]_\diamond} = u_1(\dot\Sigma_v) - v_1(\dot\Sigma_u) + \dot\tau_u(\dot\sigma_v) - \dot\tau_v(\dot\sigma_u), \tag{3.2.72g}$$

$$\dot\tau_{[u,v]_\diamond}(\pi) = u_1\dot\tau_v(\pi) - v_1\dot\tau_u(\pi) + \dot\tau_u v_0(\pi) - \dot\tau_v u_0(\pi) + u_2(\dot\sigma_v, \pi) - v_2(\dot\sigma_u, \pi), \tag{3.2.72h}$$

$$[u, P]_\diamond = [u, P]_\circ, \tag{3.2.72i}$$

$$\dot A_{[u,P]_\diamond} = u_1(\dot A_P) - P(\dot\sigma_u), \tag{3.2.72j}$$

$$[u, v, w]_\diamond = [u, v, w]_\circ = 0, \tag{3.2.72k}$$

where $u, v, w \in \mathfrak{gau}_0(M, \mathfrak{v})$ and $P \in \mathfrak{gau}_1(M, \mathfrak{v})$. In (3.2.72a), (3.2.72e), (3.2.72i), (3.2.72k), the boundary and the brackets in the right hand side are those of $\mathfrak{aut}(\mathfrak{v})$ thought of as holding pointwise on M.

The strict Lie 2–algebra $\mathfrak{gau}(M, \mathfrak{v})$ can also be described as a differential Lie crossed module. The two underlying Lie algebras are $\mathfrak{gau}_0(M, \mathfrak{v})$ and $\mathfrak{gau}_1(M, \mathfrak{v})$. The differential Lie crossed module Lie brackets, target map and action are given by the expressions

$$[u, v]_\diamond = [u, v]_\circ, \tag{3.2.73a}$$

$$\dot\sigma_{[u,v]_\diamond} = u_0(\dot\sigma_v) - v_0(\dot\sigma_u), \tag{3.2.73b}$$

$$\dot\Sigma_{[u,v]_\diamond} = u_1(\dot\Sigma_v) - v_1(\dot\Sigma_u) + \dot\tau_u(\dot\sigma_v) - \dot\tau_v(\dot\sigma_u), \tag{3.2.73c}$$

$$\dot\tau_{[u,v]_\diamond}(\pi) = u_1\dot\tau_v(\pi) - v_1\dot\tau_u(\pi) + \dot\tau_u v_0(\pi) \tag{3.2.73d}$$
$$- \dot\tau_v u_0(\pi) + u_2(\dot\sigma_v, \pi) - v_2(\dot\sigma_u, \pi),$$

$$[P, Q]_\diamond = [P, Q]_\circ \tag{3.2.73e}$$

$$\dot A_{[P,Q]_\diamond} = -P(\partial \dot A_Q) + Q(\partial \dot A_P) \tag{3.2.73f}$$

$$\tau_\diamond P = \tau_\circ P, \tag{3.2.73g}$$

$$\dot\sigma_{\tau_\diamond P} = -\partial \dot A_P, \tag{3.2.73h}$$

$$\dot\Sigma_{\tau_\diamond P} = -d\dot A_P, \tag{3.2.73i}$$

$$\dot\tau_{\tau_\diamond P}(\pi) = [\pi, \dot A_P] - dP(\pi), \tag{3.2.73j}$$

$$\mu_\diamond(u)(P) = \mu_\circ(u)(P), \tag{3.2.73k}$$

$$\dot A_{\mu_\diamond(u)(P)} = u_1(\dot A_P) - P(\dot\sigma_u), \tag{3.2.73l}$$

where $u, v \in \mathfrak{gau}_0(M, \mathfrak{v})$ and $P, Q \in \mathfrak{gau}_1(M, \mathfrak{v})$. In (3.2.73a), (3.2.73e), (3.2.73g), (3.2.73k), the brackets, the target map and the Lie algebra morphism in the right hand side are those of $\mathfrak{aut}(\mathfrak{v})$ thought of as holding pointwise on M. Note that eqs. (3.2.71) can be concisely written as $\delta_P u = \tau_\diamond u$ by (3.2.73g)–(3.2.73j).

For any $s \in \Omega^0(M, \mathfrak{v}_0)$, an element $\mathrm{ad}_M s \in \mathfrak{gau}_0(M, \mathfrak{v})$,

$$\mathrm{ad}_M s = \mathrm{ad}_0 s, \tag{3.2.74a}$$

$$\dot{\sigma}_{\mathrm{ad}_M s} = ds, \tag{3.2.74b}$$

$$\dot{\Sigma}_{\mathrm{ad}_M s} = 0, \tag{3.2.74c}$$

$$\dot{\tau}_{\mathrm{ad}_M s}(\pi) = 0 \tag{3.2.74d}$$

is defined, the adjoint of s. In (3.2.74a), the adjoint operator in the right hand side is that of $\mathfrak{v}_0$ holding pointwise on M. Similarly, with any $s, t \in \Omega^0(M, \mathfrak{v}_0)$ and any $S \in \Omega^0(M, \mathfrak{v}_1)$, there are associated elements $\mathrm{ad}_M s \wedge t, \mathrm{ad}_M S \in \mathfrak{gau}_1(M, \mathfrak{v})$ by

$$\mathrm{ad}_M s \wedge t = \mathrm{ad}_2(s, t), \tag{3.2.75a}$$

$$\dot{A}_{\mathrm{ad}_M s \wedge t} = 0, \tag{3.2.75b}$$

$$\mathrm{ad}_M S = \mathrm{ad}_1 S, \tag{3.2.75c}$$

$$\dot{A}_{\mathrm{ad}_M S} = 0, \tag{3.2.75d}$$

the adjoints of s, t and S, respectively. In (3.2.75a), (3.2.75c), the adjoint operators in the right hand side are those of $\mathfrak{v}_1$ holding pointwise on M (cf. proposition 20).

Infinitesimal Lie 2–algebra gauge transformation can be exponentiated to finite ones. The exponential map $\exp_\diamond : \mathfrak{gau}(M, \mathfrak{v}) \to \mathrm{Gau}(M, \mathfrak{v})$ can be described explicitly. We have

$$\exp_\diamond(u) = \exp_\circ(u), \tag{3.2.76a}$$

$$\sigma_{\exp_\diamond(u)} = \frac{1_{\mathfrak{v}_0} - \exp(-u_0)}{u_0}(\dot{\sigma}_u), \tag{3.2.76b}$$

$$\Sigma_{\exp_\diamond(u)} = \frac{1_{\mathfrak{v}_1} - \exp(-u_1)}{u_1}(\dot{\Sigma}_u), \tag{3.2.76c}$$

$$- \int_0^1 ds \, \exp(-su_1)\dot{\tau}_u \frac{1_{\mathfrak{v}_0} - \exp(-(1-s)u_0)}{u_0}(\dot{\sigma}_u)$$

$$+ \int_0^1 ds \int_0^s dt \, \exp(-(s-t)u_1)$$

$$\times u_2\left(\exp(-tu_0)(\dot{\sigma}_u), \exp(-tu_0)\frac{1_{\mathfrak{v}_0} - \exp(-(1-s)u_0)}{u_0}(\dot{\sigma}_u)\right),$$

$$\tau_{\exp_\diamond(u)}(\pi) = \int_0^1 ds \, \exp(-su_1)\dot{\tau}_u \exp(su_0)(\pi) \tag{3.2.76d}$$

$$+ \int_0^1 ds \, \exp(-su_1)u_2\left(\exp(su_0)(\pi), \frac{1_{\mathfrak{v}_0} - \exp(-(1-s)u_0)}{u_0}(\dot{\sigma}_u)\right),$$

$$\exp_\diamond(P) = \exp_\circ(P), \tag{3.2.76e}$$

$$\dot{A}_{\exp_\diamond(P)} = \frac{\exp(P\partial) - 1_{\mathfrak{v}_1}}{P\partial}(\dot{A}_P) \tag{3.2.76f}$$

where $u \in \mathfrak{gau}_0(M, \mathfrak{v})$, $P \in \mathfrak{gau}_1(M, \mathfrak{v})$. In (3.2.76a), the exponentiation in the right hand side is that of $\mathfrak{aut}_0(\mathfrak{v})$ thought of as holding pointwise on M. Similarly, in (3.2.76e), the exponentiation in the right hand side is that of $\mathfrak{aut}_1(\mathfrak{v})$ pointwise on M.

3.2.5   Orthogonal gauge transformations

At the end of the previous section, we introduced orthogonal extended gauge transformations for ordinary gauge theories. The necessity of this concept rises from the necessity of having an invariant bilinear form on the Lie algebra $\mathfrak{g}$ in order to define a sensible theory. This is true in semistrict higher gauge theory: to build a field theory we need to extract from our algebraic datum a number, so that it is possible to define an action functional. Thus we need an invariant form.

We consider now a semistrict higher gauge theory having as symmetry algebra a balanced 2-term L_∞ algebra $\mathfrak{v}$ equipped with an invariant bilinear form $(\cdot, \cdot)$.

Definition 82. *A 1–gauge transformation $(g, \sigma_g, \Sigma_g, \tau_g)$ of $\mathrm{Gau}_1(M, \mathfrak{v})$ is said **orthogonal** if:*

    1. $g \in \mathrm{Map}(M, \mathrm{OAut}_1(\mathfrak{v}))$;

    2. $\tau_g \in \Omega^1(M, \mathfrak{oaut}_1(\mathfrak{v}))$, *i.e.*

$$(x, \tau_g(y)) + (y, \tau_g(x)) = 0. \tag{3.2.77}$$

We shall denote by $\mathrm{OGau}_1(M, \mathfrak{v})$ the set of all orthogonal elements $g \in \mathrm{Gau}_1(M, \mathfrak{v})$.

Condition (3.2.77) is at first glance a bit mysterious, but it emerges naturally in many contexts and is a necessary condition for orthogonal symmetry invariance in higher Chern–Simons theory.

A 2–gauge transformation (F, A_F) of $\mathrm{Gau}_2(M, \mathfrak{v})(g, h)$, $g, h \in \mathrm{Gau}_1(M, \mathfrak{v})$ being two 1–gauge transformations, is said orthogonal if both g, h are orthogonal. For $g, h \in \mathrm{OGau}_1(M, \mathfrak{v})$, we shall set $\mathrm{OGau}_2(M, \mathfrak{v})(g, h) = \mathrm{Gau}_2(M, \mathfrak{v})(g, h)$. We further set $\mathrm{OGau}_2(M, \mathfrak{v}) = \bigcup_{g, h \in \mathrm{OGau}_1(M, \mathfrak{v})} \mathrm{Gau}_2(M, \mathfrak{v})(g, h)$.

Remarkably, $\mathrm{OGau}(M, \mathfrak{v}) = (\mathrm{OGau}_1(M, \mathfrak{v}), \mathrm{OGau}_2(M, \mathfrak{v}))$ is a Lie 2–subgroup of the strict Lie 2–group $\mathrm{Gau}(M, \mathfrak{v}) = (\mathrm{Gau}_1(M, \mathfrak{v}), \mathrm{Gau}_2(M, \mathfrak{v}))$, meaning that $\mathrm{OGau}(M, \mathfrak{v})$ is closed under all 2–group operations of $\mathrm{Gau}(M, \mathfrak{v})$ (cf. subsect. 3.2.2).

$\mathrm{OGau}(M, \mathfrak{v})$ can be described as a crossed module. The two groups underlying it are $\mathrm{OGau}_1(M, \mathfrak{v})$ and $\mathrm{OGau}_2{}^*(M, \mathfrak{v}) = \bigcup_{g \in \mathrm{OGau}_1(M, \mathfrak{v})} \mathrm{Gau}_2(M, \mathfrak{v})(i, g)$. $\mathrm{OGau}_2{}^*(M, \mathfrak{v})$ can be characterized as the set of pairs (F, A_F) with:

    1. $F \in \mathrm{Map}(M, \mathrm{OAut}_2{}^*(\mathfrak{v}))$ and

$$(x, dF(y)) + (y, dF(x)) - d(\partial F(x), F(y)) = 0, \tag{3.2.78}$$

    for $x, y \in \mathfrak{v}_0$.

    2. $A_F \in \Omega^1(M, \mathfrak{v}_1)$.

Condition (3.2.78) is required by compatibility with (3.2.77). In this description, as expected, $\mathrm{OGau}(M,\mathfrak{v})$ is a Lie crossed submodule of the Lie crossed module $\mathrm{Gau}(M,\mathfrak{v})$ (cf. subsect. 3.2.2).

An infinitesimal higher 1–gauge transformation $(u,\dot{\sigma}_u,\dot{\Sigma}_u,\dot{\tau}_u)$ of $\mathfrak{gau}_0(M,\mathfrak{v})$ is orthogonal if:

1. $u \in \mathrm{Map}(M,\mathfrak{oaut}_0(\mathfrak{v}))$;

2. For $x,y \in \mathfrak{v}_0$, one has
$$(x,\dot{\tau}_u(y)) + (y,\dot{\tau}_u(x)) = 0. \tag{3.2.79}$$

(3.2.79) arises from (3.2.77) by linearization around i. We shall denote by $\mathfrak{ogau}_0\,(M,\mathfrak{v})$ the set of all orthogonal elements $u \in \mathfrak{gau}_0(M,\mathfrak{v})$.

An infinitesimal 2–gauge transformation $(P,\dot{A}_P)$ of $\mathfrak{gau}_1(M,\mathfrak{v})$ is said orthogonal if:

1. $P \in \mathrm{Map}(M,\mathfrak{oaut}_1(\mathfrak{v}))$ and
$$(x,dP(y)) + (y,dP(x)) = 0, \tag{3.2.80}$$
  for $x,y \in \mathfrak{v}_0$.

(3.2.80) stems from (3.2.78) through linearization around I_i. We shall denote by $\mathfrak{ogau}_1(M,\mathfrak{v})$ the set of all orthogonal elements $P \in \mathfrak{gau}_1(M,\mathfrak{v})$.

$\mathfrak{ogau}(M,\mathfrak{v}) = (\mathfrak{ogau}_0(M,\mathfrak{v}),\mathfrak{ogau}_1(M,\mathfrak{v}))$ is an infinite dimensional strict Lie 2–subalgebra of the gauge algebra $\mathfrak{gau}(M,\mathfrak{v}) = (\mathfrak{gau}_0(M,\mathfrak{v}),\mathfrak{gau}_1(M,\mathfrak{v}))$, meaning that $\mathfrak{ogau}(M,\mathfrak{v})$ is closed under all 2–algebra operations of $\mathfrak{gau}(M,\mathfrak{v})$. Furthermore, $\mathfrak{ogau}(M,\mathfrak{v})$ is the strict Lie 2–algebra of the orthogonal gauge Lie 2–group $\mathrm{OGau}(M,\mathfrak{v})$.

For $s \in \Omega^0(M,\mathfrak{v}_0)$, the infinitesimal 1–gauge transformation $\mathrm{ad}_M\,s \in \mathfrak{gau}_0(M,\mathfrak{v})$ is orthogonal, $\mathrm{ad}_M\,s \in \mathfrak{ogau}_0(M,\mathfrak{v})$ (cf. eqs. (3.2.74)). Likewise, for and $s,t \in \Omega^0(M,\mathfrak{v}_0)$ and any $S \in \Omega^0(M,\mathfrak{v}_1)$, the infinitesimal 2–gauge transformations $\mathrm{ad}_M\,s \wedge t, \mathrm{ad}_M\,S \in \mathfrak{gau}_1(M,\mathfrak{v})$ are orthogonal, $\mathrm{ad}_M\,s \wedge t, \mathrm{ad}_M\,S \in \mathfrak{oaut}_1(M,\mathfrak{v})$ (cf. eqs. (3.2.75)).

The exponential map $\exp_{\diamond} : \mathfrak{ogau}(M,\mathfrak{v}) \to \mathrm{OGau}(M,\mathfrak{v})$ of $\mathfrak{ogau}(M,\mathfrak{v})$ is simply the restriction of the exponential map $\exp_{\diamond} : \mathfrak{gau}(M,\mathfrak{v}) \to \mathrm{Gau}(M,\mathfrak{v})$ of $\mathfrak{gau}(M,\mathfrak{v})$ to $\mathfrak{ogau}(M,\mathfrak{v})$. In particular, the orthogonal exponential is still computed by the expressions (3.2.76).

3.2.6 BRST cohomology in semistrict higher gauge theory

In semistrict higher gauge theory, analogously to ordinary gauge theory, higher gauge symmetry is most efficiently analyzed concentrating on higher infinitesimal gauge transformation of the adjoint type. Infinitesimal higher 1–gauge transformation is codified by a bidegree $(0,1)$ ghost field doublet (c,C_c) through the ghost degree 1 infinitesimal 1–gauge transformation $w \in \mathfrak{gau}_0(M,\mathfrak{v})[1]$ given by $w = -\mathrm{ad}_M\,c$ and $\dot{\sigma}_w = dc - \partial C_c, \dot{\Sigma}_w = dC_c$ and $\dot{\tau}_w(\pi) = -[\pi,C_c]$ (cf. eqs. (3.2.74) for a special case) and is implemented by the odd BRST operator $s_1 = \delta_w$. Infinitesimal 2–gauge transformations turn out to be field dependent necessitating the specification of a connection doublet (ω,Ω_ω) by the requirement of BRST nilpotence. It is codified by a bidegree $(-1,2)$ ghost field doublet $(0,\Gamma)$ through the ghost degree 2 infinitesimal 2–gauge

transformation $W \in \mathfrak{gau}_1(M, \mathfrak{v})[2]$ given by $W = -\operatorname{ad}_M \Gamma$ and $\dot{A}_W = -[\omega, \Gamma]$ (cf. eqs. (3.2.75a), (3.2.75b) for a special case) and is implemented by the odd BRST operator $s_2 = \delta_W$. The total BRST operator is therefore given by

$$s = s_1 + s_2. \tag{3.2.81}$$

By (3.2.66a), (3.2.66b), then,

$$s_1 \omega = -Dc, \tag{3.2.82a}$$

$$s_1 \Omega_\omega = -DC_c \tag{3.2.82b}$$

(cf. eqs. (3.2.13a), (3.2.13b)). As 2–gauge transformations are inert on ω, Ω_ω,

$$s_2 \omega = 0, \tag{3.2.83a}$$

$$s_2 \Omega_\omega = 0, \tag{3.2.83b}$$

trivially. In conclusion, we have

$$s\omega = -Dc, \tag{3.2.84a}$$

$$s\Omega_\omega = -DC_c. \tag{3.2.84b}$$

We can try to make s nilpotent by suitably defining the variations sc, sC_c of c, C_c. From (3.2.82a), (3.2.82b), we find the relation

$$s_1{}^2 \omega = D\left(s_1 c + \frac{1}{2}[c, c]\right), \tag{3.2.85a}$$

$$s_1{}^2 \Omega = D\left(s_1 C_c + [c, C_c] - \frac{1}{2}[\omega, c, c]\right) + \frac{1}{2}[f, c, c], \tag{3.2.85b}$$

Here the covariant derivatives are applied to the pair $(s_1 c + \frac{1}{2}[c, c], s_1 C_c + [c, C_c] - \frac{1}{2}[\omega, c, c])$ considered as a field doublet. This suggests to set

$$s_1 c = -\frac{1}{2}[c, c], \tag{3.2.86a}$$

$$s_1 C_c = -[c, C_c] + \frac{1}{2}[\omega, c, c]. \tag{3.2.86b}$$

Of course, this is not enough to eventually make $s^2\Omega$ vanish unless $f = 0$, but it is the best we can do. From (3.2.71a)–(3.2.71d), we find the relations

$$[s_2 c - \partial \Gamma, \pi] = 0, \tag{3.2.87a}$$

$$d(s_2 c - \partial \Gamma) + \partial(s_2 C + D\Gamma) = 0, \tag{3.2.87b}$$

$$d(s_2 C + D\Gamma) = 0, \tag{3.2.87c}$$

$$[\pi, s_2 C + D\Gamma] = 0 \tag{3.2.87d}$$

which reveal that

$$s_2 c = \partial \Gamma \tag{3.2.88a}$$

$$s_2 C_c = -D\Gamma. \tag{3.2.88b}$$

From (3.2.86), (3.2.88), we conclude that

$$sc = -\frac{1}{2}[c, c] + \partial \Gamma \tag{3.2.89a}$$

$$sC_c = -[c, C_c] + \frac{1}{2}[\omega, c, c] - D\Gamma. \tag{3.2.89b}$$

We can now check that, with above definition of sc, sC_c, one has $s^2\omega = 0$ and $s^2\Omega = 0$ for connection doublets (ω, Ω_ω) satisfying the condition $f = 0$, called vanishing fake curvature condition in the literature. To make s nilpotent, we have to suitably define also the variation $s\Gamma$ of Γ. To this end, we note that

$$s^2 c = \partial\Big(s\Gamma + [c, \Gamma] - \frac{1}{6}[c, c, c]\Big), \tag{3.2.90a}$$

$$s^2 C_c = D\Big(s\Gamma + [c, \Gamma] - \frac{1}{6}[c, c, c]\Big). \tag{3.2.90b}$$

Thus, we succeed to enforce $s^2 c = 0$ and $s^2 C_c = 0$ by requiring that

$$s\Gamma = -[c, \Gamma] + \frac{1}{6}[c, c, c]. \tag{3.2.91}$$

$s^2\Gamma = 0$ as wished.

In conclusion s is nilpotent as desired

$$s^2 = 0, \tag{3.2.92}$$

provided we restrict to connection doublets (ω, Ω_ω) such that $f = 0$. We note here that the ghost sector here is not pure, as the BRST variation sC_c explicitly depends on the connection component ω.

For completeness, we report the BRST variation of curvature doublet (f, F_f) of (ω, Ω_ω), which by (3.2.69), (3.2.69b) read

$$sf = -[c, f], \tag{3.2.93a}$$

$$sF_f = -[c, F_f] + [f, C_c] - [c, \omega, f]. \tag{3.2.93b}$$

We expect BRST cohomology to play the same basic role in semistrict higher gauge theory, which it does in ordinary gauge theory.

The results of above analysis keep holding with no modifications in the case where the Lie 2–algebra $\mathfrak{v}$ is balanced and equipped with an invariant bilinear form, the gauge 2–group $\mathrm{Gau}(M, \mathfrak{v})$ and the gauge Lie 2–algebra $\mathfrak{gau}(M, \mathfrak{v})$ being replaced by their orthogonal counterparts $\mathrm{OGau}(M, \mathfrak{v})$ and $\mathfrak{ogau}(M, \mathfrak{v})$, respectively (cf. subsect. 3.2.5). In particular, no additional restriction on the ghost fields c, C_c and Γ is required.

3.2.7   Crossed module gauge transformations

In [65] and [66] formulas for the gauge transformations of connections on a principal 2-bundle with gauge structure encoded in a strict 2-group (or crossed module) were found. Without going into details, let us say that this was made by viewing a connection on a principal bundle as a functor from the path groupoid of the base manifold to the delooped version of the structure group, so that a principal 2-bundle (see section 3.3) with connection can be defined as a 2-functor from the path 2-groupoid of the base manifold to the delooped version of the structure 2-group. Gauge transformations for the higher connection can then be extracted from the theory of higher bundles.

Here we are only interested in the local theory. Given a smooth manifold M and a crossed module (G, H), a (G, H)-connection is given by a couple (A, B) with

$$A \in \Omega^1(M, \mathfrak{g}) \ , \ B \in \Omega^2(M, \mathfrak{h}). \tag{3.2.94}$$

The curvature is a couple of differential forms $f \in \Omega^2(M, \mathfrak{g})$, $F \in \Omega^3(M, \mathfrak{h})$ given by

$$f = dA + \frac{1}{2}[A, A] - \dot{t}(B). \tag{3.2.95}$$

$$F = dB + [A, B]. \tag{3.2.96}$$

A 1-gauge transformation is parametrized by a smooth function $\gamma : M \to G$ and a 1-form $\chi_\gamma \in \Omega^1(M, \mathfrak{h})$. It acts on the connection doublet (A, B) as

$$A' = \gamma A \gamma^{-1} - d\gamma \gamma^{-1} - \dot{t}(\chi_\gamma), \tag{3.2.97}$$

$$B' = \dot{m}(\gamma)(B) - \hat{m}(A')(\chi_\gamma) - d\chi_\gamma - \frac{1}{2}[\chi_\gamma, \chi_\gamma]. \tag{3.2.98}$$

Under this transformation the curvature forms change as

$$f' = \gamma f \gamma^{-1}, \tag{3.2.99}$$

$$F' = \dot{m}(\gamma)(F) - \hat{m}(f')(\chi_\gamma). \tag{3.2.100}$$

Crossed module 2-gauge transformations are determined by a smooth function $\theta : M \to H$ and a 1-form $\Xi_\theta \in \Omega^1(M, \mathfrak{h})$. The action on a 1-gauge transformation γ is

$$\gamma' = t(\theta)\gamma, \tag{3.2.101}$$

$$\chi_{\gamma'} = \chi_\gamma - \Xi_\theta. \tag{3.2.102}$$

1-gauge transformations and 2-gauge transformations make up a 2-group called $\mathrm{Gau}(M, G, H)$. 1-morphisms are 1-gauge transformations (γ, χ_γ) and 2-morphisms are 2-gauge transformations $(\theta, \Xi_\theta) : \gamma \to \gamma'$ (here we write γ for the whole doublet (γ, χ_γ)). The identity and composition for 1-morphisms is as follows:

$$1 = (1_G, 0), \tag{3.2.103}$$

$$\gamma' \diamond \gamma = \gamma'\gamma, \tag{3.2.104}$$

$$\chi_{\gamma' \diamond \gamma} = \chi_{\gamma'} + \dot{m}(\gamma')(\chi_\gamma), \tag{3.2.105}$$

$$\gamma^{-1\diamond} = \gamma^{-1}, \tag{3.2.106}$$

$$\chi_{\gamma^{-1\diamond}} = -\dot{m}(\gamma^{-1})(\chi_\gamma). \tag{3.2.107}$$

Instead vertical and horizontal composition of 2-morphisms and the respective identities and inverses are as follows:

$$1_\gamma = (1_H, 0), \tag{3.2.108}$$

$$\eta \diamond \theta = \eta m(\zeta)(\theta), \tag{3.2.109}$$

$$\Xi_{\eta \diamond \theta} = \Xi_\eta + \dot{m}(\zeta)(\Xi_\theta) + Q(\zeta \dot{t}(\chi_{\gamma'})\zeta^{-1}, \eta), \tag{3.2.110}$$

$$\theta^{-1\diamond} = m(\gamma^{-1})(\theta^{-1}), \tag{3.2.111}$$

$$\Xi_{\theta^{-1\diamond}} = -\dot{m}(\gamma^{-1})(\Xi_\theta) + \dot{m}(\gamma^{-1})(Q(\dot{t}(\theta^{-1}\chi_{\gamma'}\theta), \theta)), \tag{3.2.112}$$

$$\theta' \bullet \theta = \theta'\theta, \tag{3.2.113}$$

$$\Xi_{\theta' \bullet \theta} = \Xi_{\theta'} + \Xi_\theta, \tag{3.2.114}$$

$$\theta^{-1\bullet} = \theta^{-1}, \tag{3.2.115}$$

$$\Xi_{\theta^{-1\bullet}} = -\Xi_\theta. \tag{3.2.116}$$

Here $\theta : \gamma \to \gamma'$, $\theta' : \gamma' \to \gamma''$ and $\eta : \zeta \to \zeta'$.

Since a Lie crossed module has as infinitesimal version a differential Lie crossed module, that we know to be equivalent to a strict 2-term L_∞ algebra, one expects that these result are compatible with the theory developed in section 3.2, otherwise what we have done lacks in consistency. Indeed, the formulas above are a particular case of the relations found previously for the general case of a gauge theory for a semistrict 2-term L_∞ algebra $\mathfrak{v}$. Since the strict 2-term L_∞ algebra corresponding to a differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$ consists as a couple of vector spaces of the two Lie algebras $(\mathfrak{g}, \mathfrak{h})$, the connection doublet for the Lie crossed module and the connection doublet for the strict 2-term L_∞ algebra $(\mathfrak{g}, \mathfrak{h})$ agree:

$$\omega = A, \tag{3.2.117}$$

$$\Omega = B. \tag{3.2.118}$$

Concerning gauge transformation, there is a 2-group morphism Φ from $\mathrm{Gau}(M, G, H)$ to $\mathrm{Gau}(M, \mathfrak{v})$, where $\mathfrak{v} = (\mathfrak{g}, \mathfrak{h})$ is the strict 2-term L_∞ algebra associated with the Lie crossed module (G, H). This 2-group morphism is defined by the relations:

$$\Phi(\gamma)_0 = \mathrm{ad}_\gamma, \tag{3.2.119}$$

$$\Phi(\gamma)_1 = \dot{m}(\gamma)(\cdot), \tag{3.2.120}$$

$$\Phi(\gamma)_2 = 0, \tag{3.2.121}$$

$$\sigma_{\Phi(\gamma)} = \gamma^{-1}d\gamma + \gamma^{-1}\dot{t}(\chi_\gamma)\gamma, \tag{3.2.122}$$

$$\Sigma_{\Phi(\gamma)} = \dot{m}(\gamma^{-1})\left(d\chi_\gamma + \frac{1}{2}[\chi_\gamma, \chi_\gamma]\right), \tag{3.2.123}$$

$$\tau_{\Phi(\gamma)}(x) = \hat{m}(x)(\dot{m}(\gamma^{-1})(\chi_\gamma)), \tag{3.2.124}$$

$$\Phi(\theta)(x) = Q(\gamma x \gamma^{-1}, \theta), \tag{3.2.125}$$

$$A_{\Phi(\theta)} = \dot{m}(\gamma^{-1})(-\theta^{-1}d\theta + \chi_\gamma - \theta^{-1}\chi_{\gamma'}\theta^{-1}). \tag{3.2.126}$$

It's easy to check that this is indeed a 2-group morphism, i.e. all the compositions are preserved.

3.3   Global higher gauge theory

In this section we will target the global aspects of higher gauge theory, explaining how the 2-term L_∞ algebra gauge theory developed in the previous sections can be generalized to non-trivial base manifolds, and why our version of semistrict higher gauge theory can't fill into any higher bundle theoretic interpretation, which instead work well for the strict case. Our presentation will follow the analysis made in sections (3.7)-(3.8) of [21].

3.3.1   Higher bundles

Going from the local point of view to the global one is a matter of gluing local data on different patches. Let us show how it works in ordinary gauge theories. Given a non trivial smooth manifold M, we pick a good open cover $U = \{U_i\}$, with $M = \cup_i U_i$. On each open subset U_i we can define fields and connections in local form, as elements of $\Omega^\bullet(U_i)$ with values in some linear space, usually the Lie algebra $\mathfrak{g}$ of the gauge group G. Fields and connections defined in this way define global fields and connections if they transform in a suitable way on the overlaps $U_{ij} := U_i \cap U_j$. These transformations are governed by the gauge transformations: to every overlap U_{ij} we associate a gauge transformation $g_{ij} : U_{ij} \to G$, and given a collection of local connections $\omega_i \in \Omega^1(U_i, \mathfrak{g})$, one on every open subset in the cover, for a global connection to be consistently defined we have to require that

$$\omega_j = g_{ij}\omega_i g_{ij}^{-1} - dg_{ij}g_{ij}^{-1} \tag{3.3.1}$$

on every U_{ij}. Simlarly, given a collection of fields $\phi_i \in \Omega^p(U_i, \mathfrak{g})$, this defines a global field if we have that

$$\phi_j = g_{ij}\phi_i g_{ij}^{-1} \tag{3.3.2}$$

on every U_{ij}. The gauge transformations $\{g_{ij}\}$ must satisfy a coherence relation on triple overlaps, to avoid non-uniqueness paradoxes. This is called the *cocycle condition*, and it reads

$$g_{ij}g_{jk} = g_{ik} \tag{3.3.3}$$

on every $U_{ijk} := U_i \cap U_j \cap U_k$. Collection of functions $\{g_{ij}\}$ satisfying (3.3.3) are called *transition functions*.

Transition functions are the codifying data of principal bundles, which come out to be the basic geometric ingredient for ordinary gauge theory. The easiest way to define them is by saying that a principal bundle on a base manifold M is given by a smooth surjection $p : P \to M$, such that P, the total space, is locally on M isomorphic to $M \times G$, for G a Lie group called the structure group. More precisely, for any good open cover $\{U_i\}$ of M there exist diffeomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times G$ such that the following diagram

$$p^{-1}(U_i) \xrightarrow{\phi_i} U_i \times G \tag{3.3.4}$$
$$\downarrow{\scriptstyle p} \swarrow$$
$$U_i$$

is commutative, where the map on the right is the obvious projection on the first factor. This property of the maps ϕ_i makes it possible to define *transition functions*

$g_{ij} : U_{ij} := U_i \cap U_j \to G$ by requiring that

$$\phi_j \circ \phi_i^{-1} = \mathrm{id}_{U_{ij}} \times L_{g_{ij}} : U_{ij} \times G \to U_{ij} \times G. \tag{3.3.5}$$

Here L_g means the left multiplication by g in G.

A connection can then be defined as a separation of the tangent bundle of the principal bundle into a vertical and a horizontal component, and fields can be defined as sections of the associated vector bundle.

It is a well-known classical fact that given a set of transition functions on a cover $U = \{U_i\}$ of a manifold M satisfying the cocycle condition (3.3.3) then a unique principal bundle on M is determined by these data. This can be made precise by defining a particular groupoid, which we call $P(U, G)$, which is associated to the open cover U:

- objects in $P(U, G)$ are collections of transition functions $\{g_{ij}\}$ satisfying the cocycle condition. The set of objects is denoted by $P_0(U, G)$;

- given objects $\{g_{ij}\}$ and $\{g'_{ij}\}$ in $P(U, G)$, a morphism $\{g_{ij}\} \to \{g'_{ij}\}$ is a collection $\{h_i\}$ of maps $h_i : U_i \to G$, one for every open subset in the covering, such that

$$h_i g_{ij} = g'_{ij} h_j. \tag{3.3.6}$$

  Composition is given by multiplication in G, and the identity morphism is given by the collection $\{1_G\}$ which associated to every U_i the constant map to the identity in G. The set of morphisms is denoted $P_1(U, G)$.

It's easy to see that every such morphism is invertible, so that this is a well defined groupoid. From this groupoid we can go to Čech cohomology. It can be shown that the set of isomorphism classes of objects of $P(U, G)$, which is denoted $H^1(U, G)$, is the first Čech cohomology of the covering U with values in the Lie group G. Two transition functions which are related by a morphism in $P(U, G)$ define, according to (3.3.1)-(3.3.2), global fields which differ by a global gauge transformation. Data $\{h_i\}$ thus define an isomorphism between the principal bundles defined by its source and target transition functions. This means that $H^1(U, G)$ is the set of isomorphisms classes of principal bundles associated with the covering U. To disregard the dependence on the particular cover we define

$$H^1(M, G) := \lim_{\overrightarrow{U}} H^1(U, G), \tag{3.3.7}$$

where the limit is an inductive limit on cover refinement. Namely, $H^1(M, G)$ is the set of equivalence classes induced by the following equivalence relation on $\cup_U H^1(U, G)$: on representatives, two transition functions $\{g_{ij}\}$ and $\{g_{ij}\}$ on coverings $U = \{U_i\}$ and $U' = \{U'_i\}$ are equivalent if there is a covering $V = \{V_a\}$ which is a refinement for both U and U' with a set of transition functions $\{\gamma_{ab}\}$ such that the restriction of every g_{ij} or g'_{kl} to some $V_{ab} \subset U_{ij}, U'_{kl}$ agrees with γ_{ab}. The group $H^1(M, G)$ is the set of diffeomorphism classes of principal G-bundles on M.

Furthermore, given two transition functions $g = \{g_{ij}\}$ and $g' = \{g'_{ij}\}$, the set of morphism $h = \{h_i\} : g \to g'$, denoted $H^2(U, G)(g, g')$, depends only on the common

equivalence class of g and g' in $H^1(U, G)$. Again, taking the limit under covering refinement gives the group

$$H^2(P) := \varinjlim_{\vec{U}} H^2(U, G)(g, g'), \qquad (3.3.8)$$

where P is the principal G-bundle associated with the isomorphism class of g and g'. The group $H^2(P)$ is the set of the automorphisms of P.

All this can be reformulated in terms of groupoid morphisms. This view will be the ideal path towards the generalization to the higher setting. Given an open cover $\{U_i\}$ of a manifold M, we can define its **Čech groupoid** $\check{C}(U)$. It is the smooth groupoid defined as follows:

- the set of objects is $\check{C}(U)_0 := \coprod_i U_i = \{(x, i), x \in U_i\}$;

- the set of morphisms is $\check{C}(U)_1 := \coprod_{i,j} U_{ij} = \{(x, i, j), x \in U_{ij}\}$.

A morphism (x, i, j) goes from (x, j) to (x, i), with the identity morphism on (x, i) given by (x, i, i) and the composition by

$$(x, i, j) \circ (x, j, k) = (x, i, k). \qquad (3.3.9)$$

The usefulness of this groupoid is made explicit in the following proposition:

Proposition 23. *The groupoids $P(U, G)$ and $\mathbf{Fun}(\check{C}(U), BG)$ are isomorphic. Explicitly, transition functions $\{g_{ij}\}$ on U and functors $F : \check{C}(U) \to BG$ are in one-to-one correspondence, and gauge transformations $\{h_i\}$ from $\{g_{ij}\}$ to $\{g'_{ij}\}$ are in one-to-one correspondence with natural transformations between the functors corresponding to $\{g_{ij}\}$ and $\{g'_{ij}\}$*

Proof. The proof is very easy: BG is trivial at the level of objects, so the functor F is determined by its action on morphisms, which is equivalent to a collection of functions from the U_{ij} to G:

$$g_{ij}(x) := F(x, i, j). \qquad (3.3.10)$$

The cocycle condition is the conservation of the composition of morphisms by F.

A natural transformation $\eta : F \Rightarrow F'$ is specified by a map from $\check{C}(U)_0$ to G, namely

$$h_i(x) := \eta(x, i), \qquad (3.3.11)$$

and relation (3.3.6) is the naturality condition. $\square$

This proposition underlines how all the informations about principal bundles on M are contained in the functor category $\mathbf{Fun}(\check{C}(U), BG)$.

This notion can be extended straightforwardly to the strict higher setting, and this leads to the so called *higher bundles*, or more precisely *2-bundles* if we are dealing with 2-groups. These were first introduced by Bartels in [67] and then further developed in [65]-[66]. Here, we employ the equivalent functorial definition which can be found for example in [63].

Let us mention that in literature there are many other objects that are related to higher bundles and that aim to a generalization of (some aspects of) ordinary principal bundles, such as *gerbes*,[70],[72], *bundle gerbes* [70] and *nonabelian bundle gerbes* [68],

which we will not be interested in. For details on these topics and deep comparisons between different approaches see [63],[69],[65],[73] and references therein. Let us now go back into the discussion of 2-bundles.

For $\mathcal{G}$ a strict 2-group we define $\mathcal{G}$-2-bundles on M as (equivalence classes of) 2-functors from a higher version of the Čech groupoid to $B\mathcal{G}$. The Čech 2-groupoid, denoted $\check{C}_2(U)$ is a trivial extension of its lower version $\check{C}(U)$ obtained by adding trivial 2-morphisms, which are only identities on 1-morphisms.

Now we define as higher generalization of $P(U, G)$ the 2-groupoid $\mathtt{2\text{-}Fun}(\check{C}_2(U), B\mathcal{G})$ ([21]). In this way we also go one steep ahead than in ordinary gauge theory, taking into account 2-gauge transformations, which are encoded in the 2-morphisms of $\mathtt{2\text{-}Fun}(\check{C}_2(U), B\mathcal{G})$, or modifications.

We define a transition function on a covering $\{U_i\}$ with values in a 2-group $\mathcal{G}$ as a 2-functor from $\check{C}_2(U)$ to $B\mathcal{G}$. The key point is that we assume this 2-functor to be (possibly) non-strict. Recalling that a strict 2-group is equivalent to a crossed module (G, H), and definition 20, we see that such a functor F is determined by set of functions $\{g_{ij}, \xi_i, W_{ijk}\}$ with $g_{ij} : U_{ij} \to G$, $\xi_i : U_i \to H$ and $W_{ijk} : U_{ijk} \to H$. Functions g_{ij} are the images $F(x, i, j)$ of 1-morphisms, the W_{ijk} and the ξ_i are the H part of the isomorphisms m and u of definition 20 respectively. Relations (1.3.7) are equivalent to

$$\xi_i = m(g_{ij}^{-1})(W_{ijj}), \qquad (3.3.12)$$

so that the functions $\{\xi_i\}$ are determined by the $\{g_{ij}\}$ and the $\{W_{ijk}\}$ and are thus inessential. Target matching condition for the isomorphism m implies

$$t(W_{ijk})g_{ij}g_{jk} = g_{ik} \text{ on } U_{ijk}, \qquad (3.3.13)$$

while axiom (1.3.6) translates into the relation

$$W_{ijl}m(g_{ij})(W_{jkl}) = W_{ikl}W_{ijk} \text{ on } U_{ijkl}. \qquad (3.3.14)$$

These relation are now taken as the defining cocycle conditions for transition functions $\{g_{ij}, W_{ijk}\}$ with values in a strict 2-group $\mathcal{G}$. The first relation is similar to the cocycle relation (3.3.6) for ordinary transition functions, but it is required to hold only up to higher morphisms. The second is a coherence relation that makes the triple product $g_{ij}g_{jk}g_{kl}$ uniquely defined.

Continuing the analogy with ordinary principal bundles, gauge transformations are represented by pseudonatural transformations of 2-functors from $\check{C}_2(U)$ to $B\mathcal{G}$. Given two such 2-functors F and F' defined by the transition functions $\{g_{ij}, W_{ijk}\}$ and $\{g'_{ij}, W'_{ijk}\}$, a pseudonatural transformation $\eta : F \Rightarrow F'$ is defined by the functions $\{h_i\}$ and $\{J_{ij}\}$ with $h_i : U_i \to G$ and $J_{ij} : U_{ij} \to H$ which have to fulfill the following relations:

$$t(J_{ij})h_ig_{ij} = g'_{ij}h_j \text{ on } U_{ij}, \qquad (3.3.15)$$

$$J_{ij}m(g_{ij})(J_{jk})W_{ijk} = m(h_i)(W'_{ijk})J_{ik}. \qquad (3.3.16)$$

which are the target matching condition for $\eta(x, i, j)$ and axiom (1.3.8) in definition 22. The first relation again is similar to (3.3.6), but it is relaxed and it is valid only up to higher morphisms. The second relation states that the two 2-morphisms $g_{ik} \circ h_k \Rightarrow h_i \circ g'_{ij} \circ g'_{jk}$ which can be built using J and W have to be equal.

We can go beyond and define 2-gauge transformations, which are described by modifications between pseudonatural transformations. If η and η' are pseudonatural transformations defined by the functions $\{h_i\}$, $\{J_{ij}\}$ and $\{h'_i\}$, $\{J'_{ij}\}$, between the 2-functors F and F', defined by the transition functions $\{g_{ij}\}$, $\{W_{ijk}\}$ and $\{g'_{ij}\}$, $\{W'_{ijk}\}$, a modification from η to η' is determined by functions $\{K_i\}$, $K_i : U_i \to H$ satisfying

$$h'_i = t(K_i)h_i, \tag{3.3.17}$$

$$J'_{ij}K_i = m(g'_{ij})(K_j)J_{ij} \text{ on } U_{ij}, \tag{3.3.18}$$

The first relation is just the target matching condition intrinsic in the definition of a modification, the second relation is axiom (1.3.11).

If we define $H^1(U, \mathcal{G})$ as the set of 1-isomorphisms classes of objects in $\mathtt{Fun}(\check{C}_2(U), B\mathcal{G})$, then the set of isomorphisms classes of principal $\mathcal{G}$-2-bundles on M is given by the inductive limit under covering refinement

$$\varinjlim_{U} H^1(U, \mathcal{G}) =: H^1(M, \mathcal{G}). \tag{3.3.19}$$

Analogously, if we define $H^2(U, \mathcal{G}, F)$ to be set of 2-isomorphisms classes of 1-morphisms in $\mathtt{Fun}(\check{C}_2(U), B\mathcal{G})$ going from F to F, then the inductive limit

$$\varinjlim_{U} H^2(U, \mathcal{G}, F) =: H^2(M, \mathcal{G}, P). \tag{3.3.20}$$

is the group of isomorphisms classes of automorphisms of P, where P is the principal $\mathcal{G}$-2-bundle defined by the equivalence class of F. If we define $H^3(U, \mathcal{G}, \eta)$ to be set of 2-morphisms in $\mathtt{Fun}(\check{C}_2(U), B\mathcal{G})$ going from η to η, then the inductive limit

$$\varinjlim_{U} H^3(U, \mathcal{G}, \eta) =: H^3(M, \mathcal{G}, f). \tag{3.3.21}$$

is the group of 2-automorphisms of f, which is the automorphism of principal $\mathcal{G}$-2-bundles defined by the equivalence class of η.

3.3.2 Global semistrict higher gauge theory

So far we have achieved a definition of principal 2-bundle for strict higher gauge theory. This leads to the gauge theory with gauge structure encoded in a strict 2-group or crossed module, see subsection 3.2.7. We can try to repeat this analysis for the semistrict case.

Let M be a smooth manifold endowed with an open cover $U = \{U_i\}$, and let $\mathfrak{v} = (\mathfrak{v}_0, \mathfrak{v}_1)$ be a 2-term L_∞ algebra. On every open subset U_i we can define local field doublets (ϕ_i, Φ_{ϕ_i}) and local connection doublets $(\omega_i, \Omega_{\omega_i})$ as elements of $\Omega^\bullet(U_i, \mathfrak{v}_0) \oplus \Omega^{\bullet+1}(U_i, \mathfrak{v}_1)$. To glue together these local doublets, we assign to every open intersection U_{ij} a 1-gauge transformation $g_{ij} \in \mathrm{Gau}_1(U_{ij}, \mathfrak{v})$, consisting of data $(g_{ij}, \sigma_{g_{ij}}, \Sigma_{g_{ij}}, \tau_{g_{ij}})$ that satisfy the axioms of definition 80, and to every triple intersection U_{ijk} a 2-gauge transformation $W_{ijk} \in \mathrm{Gau}_2(U_{jik}, \mathfrak{v})$, such that $W_{ijk} : g_{ij} \diamond g_{jk} \Rightarrow g_{ik}$, consisting of data $(W_{ijk}, A_{W_{ijk}})$ satisfying the axioms of definition 81 and fulfilling the relation

$$W_{ijl} \bullet \left(1_{g_{ij}} \diamond W_{jkl}\right) = W_{ikl} \bullet \left(W_{ijk} \diamond 1_{g_{kl}}\right). \tag{3.3.22}$$

Mimicking what is done in the strict case, we use this as starting point for the definition of a 2-groupoid, which we denote $P(U, \mathfrak{v})$, which should contain the information governing the gluing structure, the 1-isomorphism and the 2-isomorphisms of the geometric framework of $\mathfrak{v}$ gauge theory.

The objects of this groupoid are collections of 1- and 2-gauge transformations $\{g_{ij}, W_{ijk}\}$ as formerly mentioned. We define 1-morphisms of $P(U, \mathfrak{v})$ going from $\{g_{ij}, W_{ijk}\}$ to $\{g'_{ij}, W'_{ijk}\}$ as collections $\{h_i, J_{ij}\}$, with $h_i \in \text{Gau}_1(U_i, \mathfrak{v})$ and $J_{ij} \in \text{Gau}_2(U_{ij}, \mathfrak{v})$, explicitly made up by the elements $\{h_i, \sigma_{h_i}, \Sigma_{h_i}, \tau_{h_i}, J_{ij}, A_{J_{ij}}\}$, such that

$$J_{ij} : h_i \diamond g_{ij} \Rightarrow g'_{ij} \diamond h_j \text{ on } U_{ij}, \tag{3.3.23}$$

$$J_{ik} \bullet (1_{h_i} \diamond W_{ijk}) = \left(W'_{ijk} \diamond 1_{h_k}\right) \bullet \left(1_{g'_{ij}} \diamond J_{jk}\right) \bullet \left(J_{ij} \diamond 1_{g_{jk}}\right). \tag{3.3.24}$$

Given two 1-morphisms $\{h_i, J_{ij}\}$ and $\{h'_i, J'_{ij}\}$, a 2-morphism in $P(U_i, \mathfrak{v})$ between them is a collection $\{K_i\}$ of 2-gauge transformations $K_i \in \text{Gau}_2(U_i, \mathfrak{v})$, with $K_i : h_i \Rightarrow h'_i$ consisting of (K_i, A_{k_i}), such that

$$\left(1_{g'_{ij}} \diamond K_j\right) \bullet J_{ij} = J'_{ij} \bullet \left(K_i \diamond 1_{g_{ij}}\right). \tag{3.3.25}$$

All compositions are inherited by the compositions in $\text{Gau}(U, \mathfrak{v})$, and the identities are the obvious ones.

Unfortunately, this 2-groupoid can't be cast into the form $\text{2-Fun}(\check{C}(U), B\mathcal{G})$ for some 2-group $\mathcal{G}$. The reason lies in the inner structure of the relations obeyed by the objects and the morphisms in $P(U, \mathfrak{v})$: part of the data are differential forms on some open subsets of M obeying differential constraints, for example

$$d\sigma_{g_{ij}} + \frac{1}{2}[\sigma_{g_{ij}}, \sigma_{g_{ij}}] - \partial \Sigma_{g_{ij}} = 0 \text{ on } U_{ij}. \tag{3.3.26}$$

We wouldn't meet this obstruction if we were doing strict higher gauge theory. If the 2-group $\text{Gau}(U, \mathfrak{v})$ is substituted by its strict version $\text{Gau}(U, G, H)$, containing the crossed module gauge transformations explained in subsection 3.2.7 for some Lie crossed module (G, H), then no differential equations appear into the constraints. The basic data are functions γ and a to G and H respectively and a 1-form χ_γ with values in $\mathfrak{h}$. Just forgetting about χ_γ provides a way to go from $\text{Gau}(U, G, H)$ to $\text{2-Fun}(\check{C}(U), B(G, H))$.

The best we can do in the semistrict case is to forget about the differential forms, but in this case we don't obtain maps into a semistrict 2-group representing the structure 2-group, but instead we get maps into $\text{Aut}(\mathfrak{v})$, which is not the structure 2-group.

The point is that the approach we have used previously in defining higher gauge transformations is not convenient for dealing with the global theory. This can be seen more clearly by adopting our point of view in the ordinary case: if we use extended gauge transformations, the gluing data would be represented by collections $\{g_{ij}, \sigma_{g_{ij}}\}$, with values in $\text{Gau}(U, \mathfrak{g})$. A complete integration of these transition functions is impeded by the differential constraints imposed by the flatness of $\sigma_{g_{ij}}$. If we forget about $\sigma_{g_{ij}}$, we are left with $\{g_{ij}\}$ which are not maps into the gauge group, but into $\text{Aut}(\mathfrak{g})$, which lacks information about the center of the gauge group.

Nevertheless, global semistrict higher gauge theory can be defined just by requiring gauge covariance. It will lack a bundle theoretic interpretation, but it will be usable to the definition of field theories.

Chapter 4

Higher parallel transport

A connection on a principal bundle separates the tangent space of the bundle into a vertical and a horizontal space. Therefore it makes it possible to define parallel transport, which is a way to 'move' in the principal bundle along paths on the base manifold keeping our tangent vector fully horizontal. Parallel transport along closed loops gives the holonomy of the loop, which is a gauge invariant and is an observable in gauge theories.

In this chapter, which is taken from [18], we extend the notion of parallel transport to strict higher gauge theory. Many papers have been written about the precise and rigorous definition of parallel transport. We have in mind in particular for the influence they had on our work the papers by Schreiber and Waldorf [38, 39, 40] and Martins and Picken [41, 42, 43]. Recent contributions include [44] and [45, 46]. Here we propose a new formulation of parallel transport in strict higher gauge theory. We do not claim any new results but we only offer a new perspective from which to view old ones, which hopefully may provide new insight. Our interest in this subject has been prompted by our formulation of semistrict higher gauge theory aimed to higher Chern–Simons theory, in which we circumvent the difficulties related to the integration of the underlying semistrict Lie 2–algebra to a semistrict 2–group, when possible, by relying on the automorphism 2–group of the Lie 2–algebra, which is always strict [21, 19]. (See also [74] for an alternative approach.)

Our formulation is based on an original notion of Lie crossed module cocycle and cocycle 1– and 2–gauge transformation with a non standard double category theoretic interpretation. (See [41, 42] and [44] for related approaches.)

4.1 Main idea

In this introductory subsection, we want to convey an intuitive idea of our formulation of higher parallel transport theory by reviewing first the cocycle approach to the ordinary theory and then outlining the higher generalization of it we propose. Here, we have no pretension of full mathematical rigor. Everything we say below holds in the smooth category.

Let G be a Lie group. A G–cocycle is a map $f : \mathbb{R}^2 \to G$ obeying

$$f(x'', x')f(x', x) = f(x'', x). \tag{4.1.1}$$

A G–connection a is just a $\mathfrak{g}$–valued 1–form on $\mathbb{R}$. G–cocycles are in one–to–one correspondence with G–connections. The G–connection a_f corresponding to a G–cocycle f is defined by

$$a_{fx}(x) = -d_{x'}f(x', x)f(x', x)^{-1}\big|_{x'=x}. \tag{4.1.2}$$

The G–cocycle f_a corresponding to a G–connection a is given by $f_a(x, x_0) = u_{x_0}(x)$, where u_{x_0} is the unique solution of the differential problem

$$d_x u_{x_0}(x)u_{x_0}(x)^{-1} = -a_x(x), \qquad u_{x_0}(x_0) = 1_G. \tag{4.1.3}$$

A G–gauge transformation is simply a mapping $\varkappa : \mathbb{R} \to G$. G–gauge transformations act on G–cocycles and G–connections. The gauge transform of a cocycle f by a gauge transformation $\varkappa$ is

$$^{\varkappa}f(x', x) = \varkappa(x')f(x', x)\varkappa(x)^{-1}. \tag{4.1.4}$$

The gauge transform of a connection a by a gauge transformation $\varkappa$ is given by the familiar relation

$$^{\varkappa}a_x(x) = \mathrm{Ad}\,\varkappa(x)(a_x(x)) - d_x\varkappa(x)\varkappa(x)^{-1}. \tag{4.1.5}$$

These actions are furthermore compatible with the cocycle to connection correspondence. The above has a categorical formulation. Let $\mathbb{GR}$ be the oriented segment groupoid of $\mathbb{R}$, the familiar groupoid of pairs of elements $\mathbb{R}$, and BG be the delooping of G, the one object groupoid whose morphisms set is G. Then, a G–cocycle f can be viewed as a functor $f : \mathbb{GR} \to BG$. Further, any G–gauge transformation $\varkappa$ encodes a natural transformation $\varkappa : f \Rightarrow {}^{\varkappa}f$.

Parallel transport in a gauge theory with gauge group G on a manifold M can now be defined as follows. For simplicity we assume that the background principal G–bundle is trivial. A G–connection θ is then simply a $\mathfrak{g}$–valued 1–form on M. Given two points p_0, p_1 of M a curve $\gamma : p_0 \to p_1$ in M with sitting instants joining them, the pull–back $\gamma^*\theta$ is a G–connection in the sense defined in the previous paragraph. With this, there is associated a G–cocycle $f_{\gamma^*\theta}$. The parallel transport induced by θ along γ is then given by

$$F_{\theta}(\gamma) = f_{\gamma^*\theta}(1, 0). \tag{4.1.6}$$

A G–gauge transformation is just a G–valued map g on M. It acts on a G–connection θ in the well–known way,

$$^{g}\theta = \mathrm{Ad}\,g(\theta) - dgg^{-1}. \tag{4.1.7}$$

The associated parallel transport transforms correspondingly as

$$F_{g\theta}(\gamma) = g(p_1)F_{\theta}(\gamma)g(p_0)^{-1}. \tag{4.1.8}$$

since g yields a G–gauge transformation γ^*g on $\gamma^*\theta$ in the sense defined in the preceding paragraph. From a categorical point of view, it is found that F_{θ} defines a functor $F_{\theta} : (M, P_1M) \to BG$ from the path groupoid (M, P_1M) of M to BG and that g defines a natural transformation $g : F_{\theta} \Rightarrow F_{g\theta}$.

In the next sections, we show that the cocycle based formulation of parallel transport of ordinary gauge theory outlined above admits a non trivial extension to strict higher gauge theory. Let (G, H) be a Lie crossed module. In sect. 4.2, we introduce

the notion of (G, H)–cocycle, a triple of three maps $f : \mathbb{R}^3 \to G$, $g : \mathbb{R}^3 \to G$ and $W : \mathbb{R}^4 \to H$ obeying relations extending (4.1.1) and a target matching condition relating f g and W, and recall that of (G, H)–connection doublet, a pair of a $\mathfrak{g}$–valued 1–form a and a $\mathfrak{h}$–valued 2–form B on $\mathbb{R}^2$ satisfying the so–called zero fake curvature condition familiar in higher gauge theory. We show then that there is a one–to–one correspondence between (G, H)–cocycles and (G, H)–connection doublets analogous to (4.1.2), (4.1.3). We introduce next the notion of integral (G, H)–1–gauge transformation, a triple constituted by three maps $\kappa : \mathbb{R}^2 \to G$ and $\Psi : \mathbb{R}^3 \to H$, $\Phi : \mathbb{R}^3 \to H$ obeying certain cocycle relations, and of differential (G, H)–1–gauge transformation, a pair of a G–valued map $\varkappa$ and a $\mathfrak{h}$–valued 1–form Γ on $\mathbb{R}^2$. We prove then the existence of a one–to–one correspondence between integral and differential (G, H)–1– gauge transformations. Integral (G, H)–1–gauge transformations are next shown to act on (G, H) cocycles by an extension of (4.1.5) and, similarly, differential (G, H)–1– gauge transformations on (G, H)–connection doublets through the usual higher gauge theoretic prescription generalizing (4.1.6) and these actions are found to be compatible with the correspondences between cocycles and connection doublets and integral and differential gauge transformations. Finally, we introduce the notion of (G, H)–2–gauge transformation, a single mapping $A : \mathbb{R}^2 \to H$, and show that (G, H)–2–gauge transformations act both on integral and differential (G, H)–1–gauge transformations in a way that is compatible with the correspondence between the two.

The above construction has a remarkable double categorical interpretation. The basic ingredients of this are the double groupoid $\mathbb{G}\mathbb{R}^2$ of oriented rectangles of $\mathbb{R}^2$ and the edge symmetric double groupoid $B(G, H)$ canonically associated to the Lie crossed module (G, H). A (G, H)–cocycle amounts to a double functor $\mathbb{R}^2 \to B(G, H)$. An integral (G, H)–1–gauge transformation encodes a form of double natural transformation between a (G, H)–cocycle and its 1–gauge transform. Finally, a (G, H)–2–gauge transformation yields a double modifications between an integral (G, H)–1–gauge transformation and its 2–gauge transform. The notion of double natural transformation and modification we use are not standard and are precisely defined in section 1.4. This may be of some interest in category theory.

In sect. 4.3, we rederive higher parallel transport theory originally obtained in the references recalled above using higher cocycle theory. We consider a strict higher gauge theory with gauge crossed module (G, H) on a manifold M for a trivial (G, H) 2–bundle. A (G, H) connection doublet is a pair of a $\mathfrak{g}$–valued 1–form θ and a $\mathfrak{h}$– valued 2–form Υ on M satisfying the zero fake curvature condition. If γ_0, γ_1 are curves with the same endpoints and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a surface connecting them, all with sitting instants, then $\Sigma^*\theta$, $\Sigma^*\Upsilon$ constitute a (G, H) connection doublet in the sense defined two paragraphs above with which there is associated a (G, H)–cocycle $f_{\Sigma^*\theta\Sigma^*\Upsilon}$, $g_{\Sigma^*\theta\Sigma^*\Upsilon}$, $W_{\Sigma^*\theta\Sigma^*\Upsilon}$. The 1–parallel transport along the γ_i and the 2–parallel transport along Σ are $F_{\theta,\Upsilon}(\gamma_i) = f_{\Sigma^*\theta\Sigma^*\Upsilon|i}(1, 0)$ and $F_{\theta,\Upsilon}(\Sigma) = W_{\Sigma^*\theta\Sigma^*\Upsilon}(0, 1; 1, 0)$, extending the prescription (4.1.6). Next, a (G, H)–1–gauge transformation is a pair of a G– valued map g and a $\mathfrak{h}$–valued 1–form J on M. (G, H)–1–gauge transformations act on (G, H)–connection doublets θ, Υ according the higher gauge theoretic prescription generalizing (4.1.7) and thus on parallel transport. This action comes through the action of the integral (G, H)–1–transformation $\kappa_{\Sigma^*g,\Sigma^*J}$, $\Psi_{\Sigma^*g,\Sigma^*J}$, $\Phi_{\Sigma^*g,\Sigma^*J}$ associated to the differential (G, H)–1–gauge transformation Σ^*g, Σ^*J on the (G, H)–cocycle

$f_{\Sigma^*\theta\Sigma^*\Upsilon}$, $g_{\Sigma^*\theta\Sigma^*\Upsilon}$, $W_{\Sigma^*\theta\Sigma^*\Upsilon}$ and leads to the appropriate extension of (4.1.8). Similar considerations hold for 2–gauge transformations.

We find that the higher parallel transport operation constructed in this way agrees with that developed in earlier literature [47, 48, 38, 39, 40, 41, 42, 43]. In particular, we recover the remarkable interpretation of the higher transport $F_{\theta,\Upsilon}$ as a 2–functor $F_{\theta,\Upsilon} : (M, P_1M, P_2M) \to B_0(G, H)$ from the path 2–groupoid (M, P_1M, P_2M) of M to the strict 2–group $B_0(G, H)$ corresponding to (G, H) and of (G, H)–1– and 2–gauge transformation as pseudonatural transformations and modifications, respectively.

4.2 Lie crossed module cocycle theory

In this section, we expound our theory of Lie crossed module cocycles. Hints of this approach were already present in refs. [38, 39, 40], to which we are indebted for inspiration. We illustrate our construction stressing its being an extension of the ordinary Lie group cocycle theory. The theory of Lie crossed module 1– and 2–gauge transformations is presented on the same lines.

4.2.1 Lie crossed module cocycles

Cocycle theory plays a basic tole in higher holonomy theory and gauge theory. We begin by recalling the definition and main properties of Lie group cocycles and then move to state the definition and study the properties of Lie crossed module cocycles.

Let G be a Lie group.

Definition 83. *A G–cocycle is a map $f \in \mathrm{Map}(\mathbb{R}^2, G)$ such that*

$$f(x'', x) = f(x'', x')f(x', x), \tag{4.2.1}$$

for $x, x', x'' \in \mathbb{R}$. We denote the set of G–cocycles as $\mathrm{Cyc}(G)$.

A few basic properties of cocycles follow immediately from the definition.

Proposition 24. *If f is a G–cocycle, then*

$$f(x, x) = 1_G, \tag{4.2.2a}$$

$$f(x, x') = f(x', x)^{-1}, \tag{4.2.2b}$$

for $x, x' \in \mathbb{R}$.

Lie group cocycles have a categorical interpretation. Though this is well known, we review it here since it points to and justifies the less known generalization to Lie crossed module cocycle presented below.

The segment groupoid $\mathbb{GR}$ has one object for each real number $x \in \mathbb{R}$ and one arrow for each pair of real numbers $x, x' \in \mathbb{R}$

$$x' \longleftarrow x. \tag{4.2.3}$$

Composition of arrows is carried out by concatenation at their common end. The identity arrows are those with equal ends. Inversion of an arrow is performed by exchange of its ends. $\mathbb{G}\mathbb{R}$ is evidently isomorphic to the pair groupoid of $\mathbb{R}$.

A Lie group G can be viewed as a one object groupoid BG, the delooping of G, with one arrow for each element of $g \in G$

$$* \xleftarrow{\ g\ } *. \tag{4.2.4}$$

Composition is given by group multiplication. The identity arrow is that corresponding to the neutral element 1_G. Inversion is the same as group inversion.

Proposition 25. *A G–cocycle f is equivalent to a smooth functor $\mathbb{G}\mathbb{R} \to BG$*

$$x' \longleftarrow x \qquad \longmapsto \qquad * \xleftarrow{\ f(x',x)\ } *. \tag{4.2.5}$$

Proof. The cocycle relation (4.2.1) is a necessary and sufficient condition for the functoriality of the above mapping. □

Every Lie group cocycle yields and can be reconstructed from a Lie valued differential form datum.

Definition 84. *A G–connection is a form $a \in \Omega^1(\mathbb{R}, \mathfrak{g})$. We denote the set of G–connections by $\mathrm{Conn}(G)$.*

The following theorem holds [39].

Theorem 3. *There is a canonical one–to–one correspondence between the set $\mathrm{Cyc}(G)$ of G–cocycles and that $\mathrm{Conn}(G)$ of G–connections. The G–connection a_f corresponding to a G–cocycle f is*

$$a_{fx}(x) = -d_{x'} f(x', x) f(x', x)^{-1}\big|_{x'=x}. \tag{4.2.6}$$

The G–cocycle f_a corresponding to a G–connection a is

$$f_a(x, x_0) = u_{x_0}(x), \tag{4.2.7}$$

where u_{x_0} is the unique solution

$$d_x u_{x_0}(x) u_{x_0}(x)^{-1} = -a_x(x) \tag{4.2.8}$$

with $u_{x_0} : \mathbb{R} \to G$ smooth and satisfying the initial condition

$$u_{x_0}(x_0) = 1_G. \tag{4.2.9}$$

Proof. If f is a G–cocycle, then (4.2.6) clearly defines a G–connection a_f. If a is a G–connection, then the solution u_{x_0} of the differential problem (4.2.8), (4.2.9) exists, is unique and smooth in x_0. The G–valued maps

$$u_1(x) = f_a(x, x_1) f_a(x_1, x_0), \tag{4.2.10a}$$

$$u_2(x) = f_a(x, x_0) \tag{4.2.10b}$$

solve the differential equation $d_x u(x) u(x)^{-1} = -a_x(x)$ with initial condition $u(x_1) = f_a(x_1, x_0)$, by (4.2.7)–(4.2.9). As this differential problem has only one solution, we have $u_1 = u_2$. From (4.2.10), it follows then that f_a obeys the cocycle condition (4.2.1). (4.2.8) implies immediately that $a_{f_a} = a$. By (4.2.6) and (4.2.2a), $f = f_{a_f}$. The mappings $f \to a_f$ and $a \to f_a$ are thus reciprocally inverse. $\square$

We now present the definition of Lie crossed module cocycle. Let (G, H, t, m) be a Lie crossed module.

Definition 85. *A (G, H)–cocycle consists of three mappings $f \in \mathrm{Map}(\mathbb{R}^2 \times \mathbb{R}, G)$, $g \in \mathrm{Map}(\mathbb{R} \times \mathbb{R}^2, G)$ and $W \in \mathrm{Map}(\mathbb{R}^2 \times \mathbb{R}^2, H)$ satisfying the target matching condition*

$$t(W(x', x; y', y)) = g(x; y', y)^{-1} f(x', x; y')^{-1} g(x'; y', y) f(x', x; y) \qquad (4.2.11)$$

and the relations

$$f_{|y}(x'', x) = f_{|y}(x'', x') f_{|y}(x', x), \qquad (4.2.12a)$$

$$g_{|x}(y'', y) = g_{|x}(y'', y') g_{|x}(y', y), \qquad (4.2.12b)$$

$$W_{|y', y}(x'', x) = W_{|y', y}(x', x) m(f_{|y}(x', x)^{-1})(W_{|y', y}(x'', x')), \qquad (4.2.12c)$$

$$W_{|x', x}(y'', y) = m(g_{|x}(y', y)^{-1})(W_{|x', x}(y'', y')) W_{|x', x}(y', y) \qquad (4.2.12d)$$

for $x, x', x'', y, y', y'' \in \mathbb{R}$. We denote the set of (G, H)–cocycles as $\mathrm{Cyc}(G, H)$.

Above, we have set $f_{|y}(x', x) = f(x', x; y)$, $g_{|x}(y', y) = g(x; y', y)$ and $W_{|y', y}(x', x) = W_{|x', x}(y', y) = W(x', x; y', y)$ for convenience.

Lie crossed module cocycles enjoy a number of properties generalizing (4.2.2).

Proposition 26. *If (f, g, W) is a (G, H)–cocycle, then*

$$f_{|y}(x, x) = 1_G, \qquad (4.2.13a)$$

$$f_{|y}(x, x') = f_{|y}(x', x)^{-1}, \qquad (4.2.13b)$$

$$g_{|x}(y, y) = 1_G, \qquad (4.2.13c)$$

$$g_{|x}(y, y') = g_{|x}(y', y)^{-1}, \qquad (4.2.13d)$$

$$W_{|y', y}(x, x) = W_{|x', x}(y, y) = 1_H, \qquad (4.2.13e)$$

$$W_{|y', y}(x, x') = m(f_{|y}(x', x))(W_{|y', y}(x', x)^{-1}), \qquad (4.2.13f)$$

$$W_{|x', x}(y, y') = m(g_{|x}(y', y))(W_{|x', x}(y', y)^{-1}) \qquad (4.2.13g)$$

for $x, x', x'', y, y', y'' \in \mathbb{R}$.

As we announced above, Lie crossed module cocycles enjoy a categorical interpretation analogous to and extending that of ordinary Lie group cocycles. Its statement requires basic notions of double category theory that are reviewed in sect. 1.4.

The rectangle double groupoid $\mathbb{GR}^2$ has one object (x, y) for each $x, y \in \mathbb{R}$, one horizontal arrow

$$(x', y) \longleftarrow (x, y), \qquad (4.2.14)$$

for each $x, x', y \in \mathbb{R}$, one vertical arrow

$$
\begin{array}{c}
(x, y'), \\
\uparrow \\
(x, y)
\end{array}
\tag{4.2.15}
$$

for each $x, y, y' \in \mathbb{R}$ and one arrow square

$$
\begin{array}{ccc}
(x', y') & \longleftarrow & (x, y'), \\
\uparrow & \nearrow & \uparrow \\
(x', y) & \longleftarrow & (x, y)
\end{array}
\tag{4.2.16}
$$

for each quadruple $x, x', y, y' \in \mathbb{R}$. The various operations of composition, identity assignment and inversion of arrows and arrow squares are defined in subsect. 1.4.7. Arrow operations are essentially the same as those of the segment groupoid. Intuitively, arrow square operations go by concatenation through a common arrow, identification of opposite arrows and exchange of opposite arrows in either the horizontal or the vertical direction.

With a Lie crossed module (G, H) there is canonically associated a double groupoid $B(G, H)$ in many ways analogous to the delooping of a Lie group. $B(G, H)$ has a single object $*$, one horizontal arrow and one vertical arrow

$$
\begin{array}{ccc}
& & * \\
& & \uparrow x \\
* \xleftarrow{\ x\ } * & & *
\end{array}
\tag{4.2.17}
$$

for each element $x \in G$ and one arrow square

$$
\begin{array}{ccc}
* & \xleftarrow{\ u\ } & * \\
v \uparrow & X \nearrow & \uparrow x \\
* & \xleftarrow{\ y\ } & *
\end{array}
\tag{4.2.18}
$$

for each $x, y, u, v \in G$ and $X \in H$ satisfying the target matching condition

$$
vy = uxt(X). \tag{4.2.19}
$$

The various operations of composition, identity assignment and inversion of arrows and arrow squares are defined in subsect. 1.4.8. Arrow operations are essentially the same as those of the delooping BG of G. Arrow square operations involve the full crossed module structure of (G, H). The target matching condition is required for the exchange law to hold.

Proposition 27. *A (G, H)–cocycle (f, g, W) is equivalent to a smooth double functor $\mathbb{GR}^2 \to B(G, H)$*

$$
\begin{array}{ccc}
(x', y') & \longleftarrow & (x, y') \\
\uparrow & \nearrow & \uparrow \\
(x', y) & \longleftarrow & (x, y)
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
* & \xleftarrow{\ f(x', x; y')\ } & * \\
g(x'; y', y) \uparrow & W(x', x; y', y) \nearrow & \uparrow g(x; y', y) \\
* & \xleftarrow{\ f(x', x; y)\ } & *
\end{array}
\tag{4.2.20}
$$

Proof. Inspection of the double groupoid operations of $\mathbb{G}\mathbb{R}^2$, $B(G, H)$ (cf. subsects. 1.4.7,1.4.8) reveals that the cocycle relations (4.2.12) are an equivalent to the double functoriality of the above mapping (cf. subsect. 1.4.2). □

Analogously to ordinary Lie group cocycles, any Lie crossed module cocycle yields and can be reconstructed from differential Lie crossed module valued differential form data.

Definition 86. *A (G, H)–connection doublet is a pair of forms $(a, B) \in \Omega^1(\mathbb{R}^2, \mathfrak{g}) \times \Omega^2(\mathbb{R}^2, \mathfrak{h})$ satisfying the zero fake curvature condition*

$$da + \frac{1}{2}[a, a] - \dot{t}(B) = 0. \tag{4.2.21}$$

We denote the set of (G, H)–connection doublets by $\mathrm{Conn}(G, H)$.

The following theorem holds.

Theorem 4. *There is a canonical one–to–one correspondence between the set $\mathrm{Cyc}(G, H)$ of (G, H)–cocycles and the set $\mathrm{Conn}(G, H)$ of (G, H)–connection doublets. The connection doublet $(a_{f,g,W}, B_{f,g,W})$ corresponding to a (G, H)–cocycle (f, g, W) is given by*

$$a_{f,g,Wx}(x, y) = -\partial_{x'} f(x', x; y) f(x', x; y)^{-1}\big|_{x'=x}, \tag{4.2.22a}$$

$$a_{f,g,Wy}(x, y) = -\partial_{y'} g(x; y', y) g(x; y', y)^{-1}\big|_{y'=y},$$

$$B_{f,g,Wxy}(x, y) = -\partial_{x'}(\partial_{y'} W(x', x; y', y) W(x', x; y', y)^{-1})\big|_{x'=x,y'=y} \tag{4.2.22b}$$

$$= -\partial_{y'}(W(x', x; y', y)^{-1}\partial_{x'} W(x', x; y', y))\big|_{x'=x,y'=y}.$$

The (G, H)–cocycle $(f_{a,B}, g_{a,B}, W_{a,B})$ corresponding to a (G, H)–connection doublet (a, B) is given by

$$f_{a,B}(x, x_0; y) = u_{|y,x_0}(x), \tag{4.2.23a}$$

$$g_{a,B}(x; y, y_0) = v_{|x,y_0}(y), \tag{4.2.23b}$$

$$W_{a,B}(x, x_0; y, y_0) = E_{|x_0,y_0}(x, y), \tag{4.2.23c}$$

where $u_{|y,x_0}$, $v_{|x,y_0}$, $E_{|x_0,y_0}$ are the unique solution of the differential problem

$$\partial_x u_{|y,x_0}(x) u_{|y,x_0}(x)^{-1} = -a_x(x, y), \tag{4.2.24a}$$

$$\partial_y v_{|x,y_0}(y) v_{|x,y_0}(y)^{-1} = -a_y(x, y), \tag{4.2.24b}$$

$$\partial_x(\partial_y E_{|x_0,y_0}(x, y) E_{|x_0,y_0}(x, y)^{-1}) \tag{4.2.24c}$$

$$= -\dot{m}(v_{|x_0,y_0}(y)^{-1} u_{|y,x_0}(x)^{-1})(B_{xy}(x, y)) \quad or$$

$$\partial_y(E_{|x_0,y_0}(x, y)^{-1} \partial_x E_{|x_0,y_0}(x, y))$$

$$= -\dot{m}(u_{|y_0,x_0}(x)^{-1} v_{|x,y_0}(y)^{-1})(B_{xy}(x, y))$$

with $u_{|y,x_0}$, $v_{|x,y_0}(y)$ and $E_{|x_0,y_0}$ smooth and satisfying the initial conditions

$$u_{|y,x_0}(x_0) = 1_G, \tag{4.2.25a}$$

$$v_{|x,y_0}(y_0) = 1_G, \tag{4.2.25b}$$

$$E_{|x_0,y_0}(x_0, y) = E_{|x_0,y_0}(x, y_0) = 1_H \tag{4.2.25c}$$

(cf. eq. (2.4.48)). The two forms of the differential problem (4.2.24c) with the initial condition (4.2.25c) are equivalent: any solution of one is automatically solution of the other.

Proof. If (f, g, W) is a (G, H)–cocycle, then (4.2.22a), (4.2.22b) clearly define a $\mathfrak{g}$–valued 1–form $a_{f,g,W}$ and a $\mathfrak{h}$–valued 2–form $B_{f,g,W}$ on $\mathbb{R}^2$. The identity of the two expressions of $B_{f,g,W}$ follows from the relation

$$\partial_{x'}(\partial_{y'}W(x', x; y', y)W(x', x; y', y)^{-1}) \tag{4.2.26}$$

$$= \operatorname{Ad}W(x', x; y', y)(\partial_{y'}(W(x', x; y', y)^{-1}\partial_{x'}W(x', x; y', y)))$$

and (4.2.13e). Using relations (4.2.22a), (4.2.22b) and the target matching condition (4.2.11), we find,

$$\dot{t}(B_{f,g,W\,xy}(x, y)) \tag{4.2.27}$$

$$= -\partial_{x'}(\partial_{y'}t(W(x', x; y', y))t(W(x', x; y', y))^{-1})\big|_{x'=x,y'=y}$$

$$= -\partial_{x'}\big(\partial_{y'}(g(x; y', y)^{-1}f(x', x; y')^{-1}g(x'; y', y)f(x', x; y))$$

$$\times f(x', x; y)^{-1}g(x'; y', y)^{-1}f(x', x; y')g(x; y', y))\big|_{x'=x,y'=y}$$

$$= -\partial_x(\partial_{y'}g(x; y', y)g(x; y', y)^{-1}\big|_{y'=y}) + \partial_y(\partial_{x'}f(x', x; y)f(x', x; y)^{-1}\big|_{x'=x})$$

$$+ [\partial_{x'}f(x', x; y)f(x', x; y)^{-1}\big|_{x'=x}, \partial_{y'}g(x; y', y)g(x; y', y)^{-1}\big|_{y'=y}]$$

$$= \partial_x a_{f,g,W\,y}(x, y) - \partial_y a_{f,g,W\,x}(x, y) + [a_{f,g,W\,x}(x, y), a_{f,g,W\,y}(x, y)]$$

verifying the zero fake curvature condition (4.2.21). Thus, the pair $(a_{f,g,W}, a_{f,g,W})$ is a (G, H)–connection doublet. This shows the first part of the theorem.

Proving the second part of the theorem requires some preparatory work. We assume that r, l are G–valued maps and D is an $\mathfrak{h}$–valued 2–form on $\mathbb{R}^2$ satisfying the differential relations

$$\partial_x(r(x, y)^{-1}\partial_y r(x, y) - l(x, y)^{-1}\partial_y l(x, y)) \tag{4.2.28a}$$

$$+ [r(x, y)^{-1}\partial_x r(x, y), r(x, y)^{-1}\partial_y r(x, y) - l(x, y)^{-1}\partial_y l(x, y)] = \dot{t}(D_{xy}(x, y)),$$

$$\partial_y(r(x, y)^{-1}\partial_x r(x, y) - l(x, y)^{-1}\partial_x l(x, y)) \tag{4.2.28b}$$

$$+ [l(x, y)^{-1}\partial_y l(x, y), r(x, y)^{-1}\partial_x r(x, y) - l(x, y)^{-1}\partial_x l(x, y)] = \dot{t}(D_{xy}(x, y))$$

and the initial conditions

$$r(x_0, y)l(x_0, y)^{-1} = r(x, y_0)l(x, y_0)^{-1} = 1_G. \tag{4.2.29}$$

The differential problem

$$\partial_x(\partial_y R(x, y)R(x, y)^{-1}) = \dot{m}(r(x, y))(D_{xy}(x, y)), \tag{4.2.30}$$

$$R(x_0, y) = R(x, y_0) = 1_H \tag{4.2.31}$$

with R a smooth H–valued map on $\mathbb{R}^2$ has a unique solution, since it is equivalent to the differential problem

$$\partial_y R(x, y)R(x, y)^{-1} = \int_{x_0}^x d\xi \, \dot{m}(r(\xi, y))(D_{xy}(\xi, y)), \tag{4.2.32}$$

$$R(x, y_0) = 1_H, \tag{4.2.33}$$

which does. Similarly, the differential problem

$$\partial_y(L(x, y)^{-1}\partial_x L(x, y)) = \dot{m}(l(x, y))(D_{xy}(x, y)), \tag{4.2.34}$$

$$L(x_0, y) = L(x, y_0) = 1_H \tag{4.2.35}$$

with L a smooth H–valued map on $\mathbb{R}^2$ has a unique solution by being equivalent to the problem

$$L(x, y)^{-1}\partial_x L(x, y) = \int_{y_0}^y d\eta \, \dot{m}(l(x, \eta))(D_{xy}(x, \eta)), \tag{4.2.36}$$

$$L(x_0, y) = 1_H. \tag{4.2.37}$$

Suppose that Q is an H–valued map on $\mathbb{R}^2$ such that

$$t(Q(x, y)) = r(x, y)l(x, y)^{-1}. \tag{4.2.38}$$

Then, $R(x, y) = Q(x, y)$ solves the differential problem (4.2.30), (4.2.31) if and only if $L(x, y) = Q(x, y)$ does that (4.2.34), (4.2.35), by the relation

$$\partial_x(\partial_y Q(x, y)Q(x, y)^{-1}) = \operatorname{Ad} Q(x, y)(\partial_y(Q(x, y)^{-1}\partial_x Q(x, y))) \tag{4.2.39}$$

and the Peiffer identity.

The auxiliary differential problem

$$\partial_x(\partial_y \rho(x, y)\rho(x, y)^{-1}) = \operatorname{Ad} r(x, y))(\dot{t}(D_{xy}(x, y))), \tag{4.2.40}$$

$$\rho(x_0, y) = \rho(x, y_0) = 1_G \tag{4.2.41}$$

with ρ a smooth G–valued map on $\mathbb{R}^2$ has has a unique solution, by a reasoning completely analogous to that indicated two paragraphs above. Similarly, the auxiliary differential problem

$$\partial_y(\lambda(x,y)^{-1}\partial_x\lambda(x,y)) = \operatorname{Ad} l(x,y))(\dot{t}(D_{xy}(x,y))), \tag{4.2.42}$$

$$\lambda(x_0,y) = \lambda(x,y_0) = 1_G \tag{4.2.43}$$

with λ a smooth G–valued map on $\mathbb{R}^2$ has has a unique solution.

Suppose that Q is an H–valued map on $\mathbb{R}^2$ such that $R(x,y) = Q(x,y)$ solves the differential problem (4.2.30), (4.2.31). Then, $\rho(x,y) = t(Q(x,y))$ solves (4.2.40), (4.2.41). Using (4.2.28a) and (4.2.29), it is straightforward to verify that $\rho(x,y) = r(x,y)l(x,y)^{-1}$ also solves (4.2.40), (4.2.41). By uniqueness, it then follows that (4.2.38) holds. Similarly, by using (4.2.28b) and (4.2.29) and making reference to the problem (4.2.42), (4.2.43) instead, one finds that when Q is an H–valued map on $\mathbb{R}^2$ such that $L(x,y) = Q(x,y)$ solves the differential problem (4.2.34), (4.2.35) and that (4.2.38) holds. We conclude that, under the assumptions (4.2.28) and (4.2.29), the differential problems (4.2.30), (4.2.31) and (4.2.34), (4.2.35) have a unique solution and that this solution is the same for both and obeys (4.2.38).

We can now complete the proof of the second part of the theorem. Let (a,B) be a (G,H)–connection doublet. The solution $u_{|y,x_0}$ of the differential problem (4.2.24a), (4.2.25a) exists, is unique and is smooth in y and x_0. Similarly, the solution $v_{|x,y_0}$ of the differential problem (4.2.24b), (4.2.25b) exists, is unique and is smooth in x and y_0. Using (4.2.24a), (4.2.24b) and (4.2.25a), (4.2.25b) and the zero fake curvature condition (4.2.21), it is straightforward to check that the G–valued maps r, l and the $\mathfrak{h}$–valued 2–form D on $\mathbb{R}^2$ defined by

$$r(x,y) = v_{|x_0,y_0}(y)^{-1}u_{|y,x_0}(x)^{-1}, \tag{4.2.44a}$$

$$l(x,y) = u_{|y_0,x_0}(x)^{-1}v_{|x,y_0}(y)^{-1}, \tag{4.2.44b}$$

$$D_{xy}(x,y) = -B_{xy}(x,y) \tag{4.2.44c}$$

obey relations (4.2.28a), (4.2.28b) and (4.2.29). Therefore, by what was shown above, the solution $E_{|x_0,y_0}$ of the twin differential problems (4.2.24c), (4.2.25c) exists, is unique and is smooth in x_0, y_0 and furthermore it is the same for both and satisfies

$$t(E_{|x_0,y_0}(x,y)) = v_{|x_0,y_0}(y)^{-1}u_{|y,x_0}(x)^{-1}v_{|x,y_0}(y)u_{|y_0,x_0}(x). \tag{4.2.45}$$

Relations (4.2.23a)–(4.2.23c) define in this way a G–valued map $f_{a,B}$ on $\mathbb{R}^2 \times \mathbb{R}$, a G–valued map $g_{a,B}$ on $\mathbb{R} \times \mathbb{R}^2$ and an H–valued map W on $\mathbb{R}^2 \times \mathbb{R}^2$ fulfilling the target matching condition (4.2.11). We have now to show that these objects satisfy

the cocycle relations (4.2.12). Consider the G– and H–valued maps

$$u_1(x) = f_{a,B|y}(x, x_1) f_{a,B|y}(x_1, x_0), \tag{4.2.46a}$$

$$u_2(x) = f_{a,B|y}(x, x_0), \tag{4.2.46b}$$

$$v_1(y) = g_{a,B|x}(y, y_1) g_{a,B|x}(y_1, y_0), \tag{4.2.46c}$$

$$v_2(y) = g_{a,B|x}(y, y_0), \tag{4.2.46d}$$

$$E_1(x, y) = W_{a,B|y,y_0}(x_1, x_0) m(f_{a,B|y_0}(x_1, x_0)^{-1})(W_{a,B|y,y_0}(x, x_1)), \tag{4.2.46e}$$

$$E_2(x, y) = W_{a,B|y,y_0}(x, x_0), \tag{4.2.46f}$$

$$E_3(x, y) = m(g_{a,B|x_0}(y_1, y_0)^{-1})(W_{a,B|x,x_0}(y, y_1)) W_{a,B|x,x_0}(y_1, y_0), \tag{4.2.46g}$$

$$E_4(x, y) = W_{a,B|x,x_0}(y, y_0). \tag{4.2.46h}$$

By (4.2.23a), (4.2.24a), (4.2.25a), u_1, u_2 both solve the differential equation $d_x u(x) u(x)^{-1} = -a_x(x, y)$ with initial condition $u(x_1) = f_{a,B|y}(x_1, x_0)$. By the uniqueness of the solution of this differential problem, $u_1 = u_2$. By (4.2.46a), (4.2.46b), then, $f_{a,B|y}$ fulfills the cocycle condition (4.2.12a) as required. Similarly, by (4.2.23b), (4.2.24b), (4.2.25b), v_1, v_2 both solve the differential equation $d_y v(y) v(y)^{-1} = -a_y(x, y)$ with initial condition $v(y_1) = g_{a,B|x}(y_1, y_0)$, so that $v_1 = v_2$. By (4.2.46c), (4.2.46d), then, $g_{a,B|x}$ fulfills the cocycle condition (4.2.12b). By (4.2.23c), (4.2.24c), (4.2.25c), E_1, E_2 both solve the differential equation

$$E(x, y)^{-1} \partial_x E(x, y) = -\int_{y_0}^{y} d\eta \, \dot{m}(f_{a,B|y_0}(x, x_0)^{-1} g_{a,B|x}(\eta, y_0)^{-1})(B_{xy}(x, \eta))$$

with initial condition $E(x_1, y) = W_{a,B|y,y_0}(x_1, x_0)$. Again by the uniqueness of the solution of this differential problem, we have $E_1 = E_2$, from which through (4.2.46e), (4.2.46f) it follows that $W_{a,B}$ obeys the cocycle condition (4.2.12c). By considering instead the equation

$$\partial_y E(x, y) E(x, y)^{-1} = -\int_{x_0}^{x} d\xi \, \dot{m}(g_{a,B|x_0}(y, y_0)^{-1} f_{a,B|y}(\xi, x_0)^{-1})(B_{xy}(\xi, y))$$

one finds that $E_3 = E_4$. from which through (4.2.46g), (4.2.46h) it follows that $W_{a,B}$ also obeys the condition (4.2.12d).

To conclude the proof of the theorem, we have to show that the mappings $(f, g, W) \to (a_{f,g,W}, B_{f,g,W})$ and $(a, B) \to (f_{a,B}, g_{a,B}, W_{a,B})$ are reciprocally inverse. For a given doublet (a, B), inserting the (4.2.23) into the (4.2.22) and using (4.2.24), (4.2.25), it is immediately verified that $a_{f_{a,B},g_{a,B},W_{a,B}} = a$, $B_{f_{a,B},g_{a,B},W_{a,B}} = B$. For a given cocycle (f, g, W), from the (4.2.22), using the cocycle relations (4.2.12), it is relatively straightforward to check that $u_{|y,x_0}(x) = f(x, x_0; y)$, $v_{|x,y_0}(y) = g(x; y, y_0)$ and $E_{|x_0,y_0}(x, y) = W(x, x_0; y, y_0)$ solve the differential problem (4.2.24), (4.2.25) with $a = a_{f,g,W}$, $B = B_{f,g,W}$, so that $f_{a_{f,g,W},B_{f,g,W}} = f$, $g_{a_{f,g,W},B_{f,g,W}} = g$, $W_{a_{f,g,W},B_{f,g,W}} = W$. The claim is so shown. $\square$

We have so achieved our first goal, the formulation of a Lie crossed module cocycle theory naturally relating to higher gauge theory.

4.2.2 Lie crossed module 1–gauge transformations

In ordinary as in higher gauge theory, parallel transport must be gauge covariant. It is important therefore to have the appropriate notion of gauge transformations of cocycles. We review first gauge transformation of ordinary group cocycles and then we define gauge transformation of crossed module cocycles.

Let G be a Lie group.

Definition 87. *A G–gauge transformation is a map $\varkappa \in \mathrm{Map}(\mathbb{R}, G)$. The G– gauge transformations form a set $\mathrm{Gau}(G)$.*

The following proposition is basic.

Proposition 28. *For any G–cocycle f and any G–gauge transformation $\varkappa$, the mapping $^{\varkappa}f \in \mathrm{Map}(\mathbb{R}, G)$ defined by the expression*

$$^{\varkappa}f(x', x) = \varkappa(x') f(x', x) \varkappa(x)^{-1}. \tag{4.2.47}$$

is also a G–cocycle, the gauge transform of f by $\varkappa$.

Proof. It is readily checked that $^{\varkappa}f$ obeys the cocycle relation (4.2.1). □

As we showed in subsect. 4.2.1, every Lie group cocycle represents secretly a smooth functor form the segment groupoid to the delooping groupoid of the Lie group. In the same spirit, every gauge transformation defines a natural transformation between a Lie group cocycle and its gauge transform.

Proposition 29. *If f is G–cocycle and $\varkappa$ is a G–gauge transformation, then $\varkappa$ yields a natural transformation $\varkappa : f \Rightarrow {}^{\varkappa}f$ of the functors $f, {}^{\varkappa}f : \mathbb{GR} \to BG$.*

Proof. By (4.2.47), a gauge transformation $\varkappa$ amounts to a mapping

$$x \;\longmapsto\; \begin{matrix} * \\ \uparrow \varkappa(x) \\ * \end{matrix} \tag{4.2.48}$$

of the objects of $\mathbb{GR}$ to the arrows of BG such that for each arrow

$$y \longleftarrow x \tag{4.2.49}$$

of $\mathbb{GR}$, the diagram of BG

$$\begin{matrix} * & \xleftarrow{\;{}^{\varkappa}f(x',x)\;} & * \\ {\scriptstyle\varkappa(y)}\big\uparrow & & \big\uparrow{\scriptstyle\varkappa(x)} \\ * & \xleftarrow[\;f(x',x)\;]{} & * \end{matrix} \tag{4.2.50}$$

commutes. This is precisely the statement that $\varkappa$ is a natural transformation $f \Rightarrow {}^{\varkappa}f$ of the functors $f, {}^{\varkappa}f : \mathbb{GR} \to BG$. □

By theor. 3, there is one–to–one correspondence between G–cocycles f and G–connections a. Hence, the action of a G–gauge transformation $\varkappa$ on f must translate into one on the form a_f.

Theorem 5. *Let f be a G cocycle and $\varkappa$ be a gauge transformation. Then, the form $a_{\varkappa f}$ associated with the gauge transformed cocycle $^\varkappa f$ is*

$$a_{\varkappa f} = \operatorname{Ad} \varkappa(a_f) - d\varkappa\varkappa^{-1}. \tag{4.2.51}$$

Proof. This follows readily from inserting (4.2.47) into (4.2.6). See also ref. [39]. □
    An action of G–gauge transformations on G–connections is so yielded.

Definition 88. *Let a be a G–connection. For a G–gauge transformation $\varkappa$,*

$$^\varkappa a = \operatorname{Ad} \varkappa(a) - d\varkappa\varkappa^{-1}. \tag{4.2.52}$$

We now extend the above to a Lie crossed module (G, H, t, m).

Definition 89. *Let (f, g, W) be a (G, H)–cocycle. An (f, g, W)–1–gauge transformation, or an integral (G, H)–1–gauge transformation when (f, g, W) is understood, consists of three maps $\kappa \in \operatorname{Map}(\mathbb{R} \times \mathbb{R}, G)$, $\Psi \in \operatorname{Map}(\mathbb{R}^2 \times \mathbb{R}, H)$, $\Phi \in \operatorname{Map}(\mathbb{R} \times \mathbb{R}^2, H)$ satisfying the relations*

$$\Psi_{|y}(x'', x) = \Psi_{|y}(x', x)m(f_{|y}(x', x)^{-1})(\Psi_{|y}(x'', x')), \tag{4.2.53a}$$

$$\Phi_{|x}(y'', y) = \Phi_{|x}(y', y)m(g_{|x}(y', y)^{-1})(\Phi_{|x}(y'', y')), \tag{4.2.53b}$$

where we have set $\Psi_{|y}(x', x) = \Psi(x', x; y)$ and $\Phi_{|x}(y', y) = \Phi(x; y', y)$ for clarity. The (f, g, W)–1–gauge transformations form a set $\operatorname{Gau}_{1 f,g,W}(G, H)$.

The following properties of crossed module cocycles are immediately proven.

Proposition 30. *If (f, g, W) is a (G, H)–cocycle and (κ, Ψ, Φ) is an (f, g, W) –1–gauge transformation, then*

$$\Psi_{|y}(x, x) = 1_H, \tag{4.2.54a}$$

$$\Psi_{|y}(x, x') = m(f_{|y}(x', x))(\Psi_{|y}(x', x)^{-1}), \tag{4.2.54b}$$

$$\Phi_{|x}(y, y) = 1_H, \tag{4.2.54c}$$

$$\Phi_{|x}(y, y') = m(g_{|x}(y', y))(\Phi_{|x}(y', y)^{-1}) \tag{4.2.54d}$$

for $x, x', x'', y, y', y'' \in \mathbb{R}$.

Just as ordinary gauge transformation act on group cocycles 1–gauge transformations act on crossed module cocycles.

Proposition 31. *Let (f, g, W) be a (G, H)–cocycle and (κ, Ψ, Φ) be an (f, g, W) –gauge transformation. Then, the mappings $^{\kappa,\Psi,\Phi}f \in \operatorname{Map}(\mathbb{R}^2 \times \mathbb{R}, G)$, $^{\kappa,\Psi,\Phi}g \in \operatorname{Map}(\mathbb{R} \times \mathbb{R}^2, G)$ and $^{\kappa,\Psi,\Phi}W \in \operatorname{Map}(\mathbb{R}^2 \times \mathbb{R}^2, H)$ defined by the expressions*

$$^{\kappa,\Psi,\Phi}f_{|y}(x', x) = \kappa_{|y}(x')f_{|y}(x', x)t(\Psi_{|y}(x', x))^{-1}\kappa_{|y}(x)^{-1}, \tag{4.2.55a}$$

$$^{\kappa,\Psi,\Phi}g_{|x}(y', y) = \kappa_{|x}(y')g_{|x}(y', y)t(\Phi_{|x}(y', y))^{-1}\kappa_{|x}(y)^{-1}, \tag{4.2.55b}$$

$$^{\kappa,\Psi,\Phi}W(x', x; y', y) = m(\kappa(x; y))\big(\Phi_{|x}(y', y)m(g_{|x}(y', y)^{-1})(\Psi_{|y'}(x', x)) \tag{4.2.55c}$$

$$\times\, W(x', x; y', y)m(f_{|y}(x', x)^{-1})(\Phi_{|x'}(y', y))^{-1}\Psi_{|y}(x', x)^{-1}\big),$$

where we have set $\kappa_{|x}(y) = \kappa_{|y}(x) = \kappa(x;y)$ for clarity, constitute a (G,H)–cocycle $({}^{\kappa,\Psi,\Phi}f,\,{}^{\kappa,\Psi,\Phi}g,\,{}^{\kappa,\Psi,\Phi}W)$, the gauge transform of (f,g,W) by (κ,Ψ,Φ).

Proof. Exploiting the (4.2.53), one checks that $({}^{\kappa,\Psi,\Phi}f,{}^{\kappa,\Psi,\Phi}g,{}^{\kappa,\Psi,\Phi}W)$ satisfies the target matching condition (4.2.11) and the cocycle relations (4.2.12) whenever (f,g,W) does. $\square$

As we showed in subsect. 4.2.1, every Lie crossed module cocycle represents secretly a smooth functor from the rectangle double groupoid to the delooping double groupoid of the Lie crossed module. Analogously to the ordinary case, every 1–gauge transformation defines a double natural transformation between a Lie crossed module cocycle and its gauge transform. The notion of double natural transformation we use, however, is not the customary one and presupposes that the target category is edge symmetric and folded (cf. subsects. 1.4.3, 1.4.4, 1.4.8).

Proposition 32. *If (f,g,W) is (G,H)–cocycle and (κ,Ψ,Φ) is a (f,g,W)–1–gauge transformation, then (κ,Ψ,Φ) is equivalent to a double natural transformation $(f,g,W) \Rightarrow ({}^{\kappa,\Psi,\Phi}f,{}^{\kappa,\Psi,\Phi}g,{}^{\kappa,\Psi,\Phi}W)$ of the double functors $(f,g,W),({}^{\kappa,\Psi,\Phi}f,\,{}^{\kappa,\Psi,\Phi}g,\,{}^{\kappa,\Psi,\Phi}W):\mathbb{GR}^2 \to B(G,H)$.*

Proof. The data of a (f,g,W)–1–gauge transformation (κ,Ψ,Φ) are equivalent to a mapping of the set of object of $\mathbb{GR}^2$ into the set of vertical arrows of $B(G,H)$,

$$(x,y) \quad \longmapsto \quad \begin{matrix}*\\ \uparrow{\scriptstyle\kappa(x;y)}\\ *\end{matrix} \qquad\qquad (4.2.56)$$

and two compatible functors from the horizontal and vertical arrow groupoids of $\mathbb{GR}^2$ into the horizontal truncation groupoid $B(G,H)_h$ of $B(G,H)$

$$(x',y) \longleftarrow (x,y) \quad \longmapsto \quad \begin{matrix} * & \xleftarrow{\ {}^{\kappa,\Psi,\Phi}f(x',x;y)\ } & * \\ {\scriptstyle\kappa(x';y)}\uparrow & {\scriptstyle\Psi(x',x;y)}\!\!\swArrow & \uparrow{\scriptstyle\kappa(x;y)} \\ * & \xleftarrow{\ f(x',x;y)\ } & * \end{matrix} \qquad (4.2.57)$$

$$\begin{matrix}(x,y')\\ \uparrow\\ (x,y)\end{matrix} \quad \longmapsto \quad \begin{matrix} * & \xleftarrow{\ {}^{\kappa,\Psi,\Phi}g(x;y',y)\ } & * \\ {\scriptstyle\kappa(x;y')}\uparrow & {\scriptstyle\Phi(x;y',y)}\!\!\swArrow & \uparrow{\scriptstyle\kappa(x;y)} \\ * & \xleftarrow{\ g(x;y',y)\ } & * \end{matrix}.$$

(cf. eqs. (1.4.18), (1.4.19)). The fulfillment of the target matching condition (1.4.36) is guaranteed by relations (4.2.55a), (4.2.55b). The functoriality of the mappings (4.2.57) is equivalent to relations (4.2.53a), (4.2.53b) and the ensuing relations (4.2.54a)-(4.2.54d). (4.2.56), (4.2.57) are precisely the data required for a double natural transformation from the first to the second of the double functors $(f,g,W),({}^{\kappa,\Psi,\Phi}f,\,{}^{\kappa,\Psi,\Phi}g,{}^{\kappa,\Psi,\Phi}W):\mathbb{GR}^2 \to B(G,H)$. The only thing left to check is the double naturality condition (1.4.21). Using the expressions of the operations of the double groupoid

$B(G, H)$ of subsect. 1.4.8, it is easily checked that this is equivalent to relation (4.2.55c) written in the form

$$\Phi(x; y', y)m(g(x; y', y)^{-1})(\Psi(x', x; y'))W(x', x; y', y) \tag{4.2.58}$$

$$= m(\kappa(x; y)^{-1})(^{\kappa, \Psi, \Phi}W(x', x; y', y))\Psi(x', x; y)m(f(x', x; y)^{-1})(\Phi(x'; y', y)).$$

Intuitively, the double naturality condition can be interpreted as the requirement that the cube diagram of $B(G, H)$

$$\tag{4.2.59}$$

commutes for any arrow square of $\mathbb{G}\mathbb{R}^2$,

$$\tag{4.2.60}$$

where we have dropped all double arrows in order not to clog the diagram (cf. eq. (1.4.22)). The precise meaning of this statement is given by the diagrammatic identity (1.4.21) adapted to the edge symmetric folded groupoid $B(G, H)$.   □

In contrast to ordinary gauge transformations, a crossed module 1–gauge transformation yields and can be reconstructed from differential Lie crossed module valued differential form data.

Definition 90. *A differential (G, H)–1–gauge transformation is a pair $(\varkappa, \Gamma) \in \mathrm{Map}(\mathbb{R}^2, G) \times \Omega^1(\mathbb{R}^2, \mathfrak{h})$. We denote the set of differential (G, H)–1–gauge transformation by $\mathrm{Gau}_1(G, H)$.*

The following theorem holds.

Theorem 6. *For a fixed (G, H)–cocycle (f, g, W), there is a canonical one–to–one correspondence between the set $\mathrm{Gau}_{1\,f,g,W}(G, H)$ of (f, g, W)–1–gauge transformations and the set $\mathrm{Gau}_1(G, H)$ differential (G, H)–1–gauge transformations. The differential (G, H)–1–gauge transformation $(\varkappa_{\kappa, \Psi, \Phi}, \Gamma_{\kappa, \Psi, \Phi})$ corresponding to a (f, g, W)–1–gauge*

transformation (κ, Ψ, Φ) *is given by*

$$\varkappa_{\kappa,\Psi,\Phi}(x,y) = \kappa(x;y), \tag{4.2.61a}$$

$$\Gamma_{\kappa,\Psi,\Phi\,x}(x,y) = -\dot{m}(\kappa(x;y))(\Psi(x',x;y)^{-1}\partial_{x'}\Psi(x',x;y)\big|_{x'=x}), \tag{4.2.61b}$$

$$\Gamma_{\kappa,\Psi,\Phi\,y}(x,y) = -\dot{m}(\kappa(x;y))(\Phi(x;y',y)^{-1}\partial_{y'}\Phi(x;y',y)\big|_{y'=y})$$

(cf. eq. (2.4.48)). Conversely, the (f,g,W)*–1–gauge transformation* $(\kappa_{\varkappa,\Gamma}, \Psi_{\varkappa,\Gamma}, \Phi_{\varkappa,\Gamma})$ *corresponding to a differential* (G,H)*–1–gauge transformation* $(\varkappa, \Gamma)$ *is*

$$\kappa_{\varkappa,\Gamma}(x;y) = \varkappa(x,y), \tag{4.2.62a}$$

$$\Psi_{\varkappa,\Gamma}(x,x_0;y) = \Lambda_{|y,x_0}(x), \tag{4.2.62b}$$

$$\Phi_{\varkappa,\Gamma}(x;y,y_0) = \Xi_{|x,y_0}(y), \tag{4.2.62c}$$

where $\Lambda_{|y,x_0}$, $\Xi_{|x,y_0}$ *are the unique solutions of the differential problem*

$$\Lambda_{|y,x_0}(x)^{-1}\partial_x\Lambda_{|y,x_0}(x) = -\dot{m}(f(x,x_0;y)^{-1}\varkappa(x,y)^{-1})(\Gamma_x(x,y)), \tag{4.2.63a}$$

$$\Xi_{|x,y_0}(y)^{-1}\partial_{|y}\Xi_{|x,y_0}(y) = -\dot{m}(g(x;y,y_0)^{-1}\varkappa(x,y)^{-1})(\Gamma_y(x,y)) \tag{4.2.63b}$$

with the initial conditions

$$\Lambda_{|y,x_0}(x_0) = 1_H, \tag{4.2.64a}$$

$$\Xi_{|x,y_0}(y_0) = 1_H. \tag{4.2.64b}$$

Proof. If (κ, Ψ, Φ) is an (f,g,W)–1–gauge transformation, then (4.2.61a), (4.2.61b) clearly define a G–valued map $\varkappa_{\kappa,\Psi,\Phi}$ and an $\mathfrak{h}$–valued 1–form $\Gamma_{\kappa,\Psi,\Phi}$ on $\mathbb{R}^2$, so a differential 1–gauge transformation. This shows the first part of the theorem.

Let $(\varkappa, \Gamma)$ be a differential 1–gauge transformation. The solution $\Lambda_{|y,x_0}$ of the differential problem (4.2.63a), (4.2.64a) exists, is unique and is smooth in y and x_0. Similarly, the solution $\Xi_{|x,y_0}$ of the differential problem (4.2.63b), (4.2.64b) exists, is unique and is smooth in x and y_0. Relations (4.2.62a), (4.2.62b) define in this way a G–valued map $\kappa_{\varkappa,\Gamma}$ on $\mathbb{R} \times \mathbb{R}$ and two H–valued maps $\Psi_{\varkappa,\Gamma}$ and $\Phi_{\varkappa,\Gamma}$ on $\mathbb{R}^2 \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R}^2$, respectively. We have now to show that the cocycle relations (4.2.53) are identically obeyed. Consider the H–valued maps

$$\Lambda_1(x) = \Psi_{\varkappa,\Gamma|y}(x_1,x_0)m(f_{|y}(x_1,x_0)^{-1})(\Psi_{\varkappa,\Gamma|y}(x,x_1)), \tag{4.2.65a}$$

$$\Lambda_2(x) = \Psi_{\varkappa,\Gamma|y}(x,x_0), \tag{4.2.65b}$$

$$\Xi_1(y) = \Phi_{\varkappa,\Gamma|x}(y_1,y_0)m(g_{|x}(y_1,y_0)^{-1})(\Phi_{\varkappa,\Gamma|x}(y,y_1)), \tag{4.2.65c}$$

$$\Xi_2(y) = \Phi_{\varkappa,\Gamma|x}(y,y_0). \tag{4.2.65d}$$

In virtue of (4.2.62b), (4.2.63a), (4.2.64a), Λ_1, Λ_2 are both solution of the differential equation

$$\Lambda(x)^{-1}d_x\Lambda(x) = -\dot{m}(f_{|y}(x,x_0)^{-1}\varkappa(x,y)^{-1})(\Gamma_x(x,y))$$

with initial condition $\Lambda(x_1) = \Psi_{\varkappa,\Gamma|y}(x_1,x_0)$. By the uniqueness of the solution of this differential problem, $\Lambda_1 = \Lambda_2$. By (4.2.65a), (4.2.65b), then, $\Psi_{\varkappa,\Gamma|y}$ fulfills the cocycle condition (4.2.53a) as required. Similarly, by (4.2.62c), (4.2.63b), (4.2.64b), Ξ_1, Ξ_2 are both solution of the differential equation

$$\Xi(y)^{-1}d_y\Xi(y) = -\dot{m}(g_{|x}(y,y_0)^{-1}\varkappa(x,y)^{-1})(\Gamma_y(x,y))$$

with initial condition $\Xi(y_1) = \Phi_{\varkappa,\Gamma|x}(y_1,y_0)$, so that $\Xi_1 = \Xi_2$. By (4.2.65c), (4.2.65d), then, $\Phi_{\varkappa,\Gamma|x}$ fulfills the cocycle condition (4.2.53b).

To conclude the proof of the theorem, we have to show that the mappings $(\kappa,\Psi,\Phi) \to (\varkappa_{\kappa,\Psi,\Phi}, \Gamma_{\kappa,\Psi,\Phi})$ and $(\varkappa,\Gamma) \to (\kappa_{\varkappa,\Gamma}, \Psi_{\varkappa,\Gamma}, \Phi_{\varkappa,\Gamma})$ are reciprocally inverse. For a given differential 1–gauge transformation $(\varkappa,\Gamma)$, inserting the (4.2.62) into the (4.2.61) and using (4.2.63), (4.2.64), it is immediately verified that

$$\varkappa_{\kappa_{\varkappa,\Gamma},\Psi_{\varkappa,\Gamma},\Phi_{\varkappa,\Gamma}} = \varkappa,$$

$$\Gamma_{\kappa_{\varkappa,\Gamma},\Psi_{\varkappa,\Gamma},\Phi_{\varkappa,\Gamma}} = \Gamma.$$

For a given integral 1–gauge transformation (κ,Ψ,Φ), from the (4.2.61), using the cocycle relations (4.2.53), it is straightforwardly checked that $\Lambda_{|y,x_0}(x) = \Psi(x,x_0;y)$, $\Xi_{|x,y_0}(y) = \Phi(x;y,y_0)$ solve the differential problem (4.2.63), (4.2.64) with $\varkappa = \varkappa_{\kappa,\Psi,\Phi}$, $\Gamma = \Gamma_{\kappa,\Psi,\Phi}$, so that

$$\kappa_{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}} = \kappa,$$

$$\Psi_{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}} = \Psi,$$

$$\Phi_{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}} = \Phi.$$

The claim is so shown.   □

Remark 4.2.2.1. *Since $\varkappa$, Γ do not obey any conditions, the sets $\mathrm{Gau}_{1\,f,g,W}(G,H)$ with varying cocycle (f,g,W) are all in canonical one–to–one correspondence.*

By theor. 4, there exists one–to–one correspondence between (G,H)–cocycles (f,g,W) and connection doublets (a,B). Hence, the action of a (f,g,W) –1–gauge transformation (κ,Ψ,Φ) must translate into one on the associated doublet $(a_{f,g,W}, B_{f,g,W})$.

Theorem 7. *Let (f,g,W) be a (G,H)–cocycle and (κ,Ψ,Φ) be an (f,g,W)– 1–gauge transformation. Then, the (G,H)–connection doublet $(a_{\kappa,\Psi,\Phi f, \kappa,\Psi,\Phi g, \kappa,\Psi,\Phi W}, B_{\kappa,\Psi,\Phi f, \kappa,\Psi,\Phi g, \kappa,\Psi,\Phi W})$ associated with the gauge transformed cocycle $(^{\kappa,\Psi,\Phi}f, ^{\kappa,\Psi,\Phi}g, ^{\kappa,\Psi,\Phi}W)$ is given by the expressions*

$$a_{\kappa,\Psi,\Phi f, \kappa,\Psi,\Phi g, \kappa,\Psi,\Phi W} = \mathrm{Ad}\,\varkappa_{\kappa,\Psi,\Phi}(a_{f,g,W}) - d\varkappa_{\kappa,\Psi,\Phi}\varkappa_{\kappa,\Psi,\Phi}^{-1} - \dot{t}(\Gamma_{\kappa,\Psi,\Phi}), \quad\quad (4.2.66a)$$

$$B_{\kappa,\Psi,\Phi f, \kappa,\Psi,\Phi g, \kappa,\Psi,\Phi W} = \dot{m}(\varkappa_{\kappa,\Psi,\Phi})(B_{f,g,W}) - d\Gamma_{\kappa,\Psi,\Phi} - \frac{1}{2}[\Gamma_{\kappa,\Psi,\Phi}, \Gamma_{\kappa,\Psi,\Phi}] \quad\quad (4.2.66b)$$

$$- \widehat{m}(\mathrm{Ad}\,\varkappa_{\kappa,\Psi,\Phi}(a_{f,g,W}) - d\varkappa_{\kappa,\Psi,\Phi}\varkappa_{\kappa,\Psi,\Phi}^{-1} - \dot{t}(\Gamma_{\kappa,\Psi,\Phi}), \Gamma_{\kappa,\Psi,\Phi})$$

Proof. These relations follow from substituting the (4.2.55) into the (4.2.22) through a relatively straightforward calculation. See also ref. [39]. □

If we take the (G, H)–connection doublets and the differential (G, H)–1–gauge transformations as basic cocycle and gauge transformation data relying on theors. 4, 6, then the (4.2.66) define an action of differential 1–gauge transformations on connection doublets.

Definition 91. *Let (a, B) be a (G, H)–connection doublet. For a differential (G, H)–1–gauge transformations $(\varkappa, \Gamma)$ let*

$$^{\varkappa, \Gamma} a = \mathrm{Ad}\, \varkappa(a) - d\varkappa \varkappa^{-1} - \dot{t}(\Gamma), \tag{4.2.67a}$$

$$^{\varkappa, \Gamma} B = \dot{m}(\varkappa)(B) - d\Gamma - \frac{1}{2}[\Gamma, \Gamma] - \widehat{m}(\mathrm{Ad}\, \varkappa(a) - d\varkappa \varkappa^{-1} - \dot{t}(\Gamma), \Gamma). \tag{4.2.67b}$$

It can be checked that this gauge transformation is compatible with the zero fake curvature condition (4.2.21).

We have in this way achieved our second goal, the incorporation of gauge transformation into Lie crossed module cocycle theory in a manner that naturally relates to gauge invariance in higher gauge theory.

4.2.3   Lie crossed module 2–gauge transformations

We consider now 2–gauge transformations, which have no nontrivial counterpart in ordinary gauge theory.

Definition 92. *A (G, H)–2–gauge transformation is a mapping $A \in \mathrm{Map}(\mathbb{R} \times \mathbb{R}, H)$. We denote by $\mathrm{Gau}_2(G, H)$ the set of all (G, H)-2–gauge transformations.*

2–gauge transformations are gauge for gauge transformations: they act on 1–gauge transformations.

Proposition 33. *Let (f, g, W) be a (G, H)–cocycle, (κ, Ψ, Φ) be an (f, g, W)–1–gauge transformation and A be a (G, H)–2–gauge transformation. Then, the maps $^A\kappa \in \mathrm{Map}(\mathbb{R} \times \mathbb{R}, G)$, $^A\Psi \in \mathrm{Map}(\mathbb{R}^2 \times \mathbb{R}, H)$, $^A\Phi \in \mathrm{Map}(\mathbb{R} \times \mathbb{R}^2, H)$ defined by the expressions*

$$^A\kappa(x; y) = \kappa(x; y) t(A(x; y)), \tag{4.2.68a}$$

$$^A\Psi_{|y}(x', x) = A_{|y}(x)^{-1} \Psi_{|y}(x', x) m(f_{|y}(x', x)^{-1})(A_{|y}(x')), \tag{4.2.68b}$$

$$^A\Phi_{|x}(y', y) = A_{|x}(y)^{-1} \Phi_{|x}(y', y) m(g_{|x}(y', y)^{-1})(A_{|x}(y')), \tag{4.2.68c}$$

where we have set $A_{|y}(x) = A_{|x}(y) = A(x; y)$ for clarity, constitute an (f, g, W)–1–gauge transformation $(^A\kappa, {}^A\Psi, {}^A\Phi)$, the 2–gauge transform of (κ, Ψ, Φ) by A.

Proof. Using the defining relations (4.2.68), one verifies that $(^A\kappa, {}^A\Psi, {}^A\Phi)$ satisfies 1–gauge cocycle conditions (4.2.53) whenever (κ, Ψ, Φ) does. □

2–gauge equivalent 1–gauge transformations yield the the same gauge transform of the underlying cocycle.

Proposition 34. *Let (f, g, W) be a (G, H)–cocycle, (κ, Ψ, Φ) be an (f, g, W)–1–gauge transformation and A be a (G, H)–2–gauge transformation. Then the transformed cocycles $({}^{\kappa,\Psi,\Phi}f, {}^{\kappa,\Psi,\Phi}g, {}^{\kappa,\Psi,\Phi}W)$, $({}^{A_\kappa,{}^A\Psi,{}^A\Phi}f, {}^{A_\kappa,{}^A\Psi,{}^A\Phi}g, {}^{A_\kappa,{}^A\Psi,{}^A\Phi}W)$ are equal.*

Proof. This is readily checked by computing $({}^{{}^A\kappa,{}^A\Psi,{}^A\Phi}f, {}^{{}^A\kappa,{}^A\Psi,{}^A\Phi}g, {}^{{}^A\kappa,{}^A\Psi,{}^A\Phi}W)$ inserting the expressions (4.2.68) into the (4.2.55) and using the target matching condition (4.2.11). □

As we proved in subsects. 4.2.1, 4.2.2, every Lie crossed module cocycle can be regarded as a smooth functor form the rectangle double groupoid to the delooping double groupoid of the Lie crossed module and any 1–gauge transformation as a double natural transformation between a Lie crossed module cocycle and its gauge transform. In the same spirit, a 2–gauge transformation can be viewed as a double modification between a 1–gauge transformation and its 2–gauge transform (cf. subsect. 1.4.6). We warn the reader that our definition of double modification hinges on that of double natural transformation (cf. subsect. 1.4.5), which, as we have recalled above, differs from the one customarily provided in the literature.

Proposition 35. *If (f, g, W) is (G, H)–cocycle, (κ, Ψ, Φ) is a (f, g, W)–1–gauge transformation and A is a (G, H)–2–gauge transformation. Then, A is equivalent to a double modification $(\kappa, \Psi, \Phi) \Rrightarrow ({}^A\kappa, {}^A\Psi, {}^A\Phi)$ of the double natural transformations (κ, Ψ, Φ), $({}^A\kappa, {}^A\Psi, {}^A\Phi)$.*

Proof. The data of a 2–gauge transformation A are equivalent to a mapping of the set of object of $\mathbb{GR}^2$ into the set of arrow square of $B(G, H)$,

$$
\begin{array}{ccc}
& * \xleftarrow{\quad 1_G \quad} * & \\
(x, y) \longmapsto \quad {}^A\kappa(x;y)\Big\uparrow \quad & \underset{A(x;y)}{\Swarrow} \quad & \Big\uparrow\kappa(x;y) \\
& * \xleftarrow[\quad 1_G \quad]{} * &
\end{array}
\tag{4.2.69}
$$

(cf. eqs. (1.4.23)). The fulfillment of the target matching condition (1.4.36) is guaranteed by relation (4.2.68a). (4.2.69) are precisely the data required for a double modification from the first to the second of the double natural transformations

$$(\kappa, \Psi, \Phi) : (f, g, W) \Rightarrow ({}^{\kappa,\Psi,\Phi}f, {}^{\kappa,\Psi,\Phi}g, {}^{\kappa,\Psi,\Phi}W),$$

$$({}^A\kappa, {}^A\Psi, {}^A\Phi) : (f, g, W) \Rightarrow ({}^{{}^A\kappa,{}^A\Psi,{}^A\Phi}f, {}^{{}^A\kappa,{}^A\Psi,{}^A\Phi}g, {}^{{}^A\kappa,{}^A\Psi,{}^A\Phi}W) = ({}^{\kappa,\Psi,\Phi}f, {}^{\kappa,\Psi,\Phi}g, {}^{\kappa,\Psi,\Phi}W).$$

The only thing left to check is the double modification conditions (1.4.25), (1.4.27) Using the expressions of the operations of the double groupoid $B(G, H)$ of subsect. 1.4.8, it is easily checked that these are equivalent to relations (4.2.55c) written in the form

$$A(x; y){}^A\Psi(x', x; y) = \Psi(x', x; y)m(f(x', x; y)^{-1})(A(x'; y)), \tag{4.2.70a}$$

$$A(x; y){}^A\Phi(x; y', y) = \Phi(x; y', y)m(g(x; y', y)^{-1})(A(x; y')), \tag{4.2.70b}$$

Intuitively, the double modification condition can be interpreted as the requirement that, for any horizontal and vertical arrow of $\mathbb{GR}^2$

$$
\begin{array}{cc}
& (x, y') \qquad\qquad\qquad\qquad (4.2.71) \\
(x', y) \longleftarrow (x, y) & \Big\uparrow \\
& (x, y)
\end{array}
$$

the cylinder diagrams

$$(4.2.72a)$$

$$(4.2.72b)$$

both commute, where all double arrows have been dropped for clarity and the identity morphisms of the modification arrow squares have been collapsed (cf. eqs. (1.4.28a), (1.4.28b)). The precise meaning of this statement is given by the diagrammatic identities (1.4.25), (1.4.27) adapted to the edge symmetric folded groupoid $B(G, H)$.   □

By theor. 6, there exists one–to–one correspondence between integral (f, g, W)–1–gauge transformations (κ, Ψ, Φ) and differential (G, H)–1–gauge transformations $(\varkappa, \Gamma)$. So, the action of a (G, H)–2–gauge transformation A must translate into one on the data $(\varkappa_{\kappa,\Psi,\Phi}, \Gamma_{\kappa,\Psi,\Phi})$.

Theorem 8. *Let (f, g, W) be a (G, H)–cocycle, (κ, Ψ, Φ) be an (f, g, W)–1–gauge transformation and A a (G, H)–2–gauge transformation. Then,*

$$\varkappa_{{}^A\kappa, {}^A\Psi, {}^A\Phi} = t(\tilde{A})\varkappa_{\kappa,\Psi,\Phi}, \tag{4.2.73a}$$

$$\Gamma_{{}^A\kappa, {}^A\Psi, {}^A\Phi} = \tilde{A}\Gamma_{\kappa,\Psi,\Phi}\tilde{A}^{-1} - d\tilde{A}\tilde{A}^{-1} - Q(a_{\kappa,\Psi,\Phi f, \kappa,\Psi,\Phi g, \kappa,\Psi,\Phi W}, \tilde{A}) \tag{4.2.73b}$$

(cf. eq. (2.4.49)), where we have set

$$\tilde{A} = m(\kappa)(A) \tag{4.2.74}$$

with $\tilde{A}$ viewed as an element of $\mathrm{Map}(\mathbb{R}^2, H)$.

Proof. These relations follow from substituting the (4.2.68) into the (4.2.61) through a relatively straightforward calculation. See also ref. [39].   □

If we take the (G, H)–connection doublets and the differential (G, H)–1–gauge transformations as basic cocycle and gauge transformation data relying on theors. 4, 6, then the (4.2.73) define an action of 2–gauge transformations on differential 1–gauge transformations for any assigned connection doublet.

Definition 93. *Let (a, B) be a (G, H)–connection doublet and $(\varkappa, \Gamma)$ be a differential (G, H)–1–gauge transformation. For any (G, H)–1–gauge transformation $\tilde{A}$, one sets*

$$^{\tilde{A}}\varkappa_{|a,B} = t(\tilde{A})\varkappa, \tag{4.2.75a}$$

$$^{\tilde{A}}\Gamma_{|a,B} = \tilde{A}\Gamma\tilde{A}^{-1} - d\tilde{A}\tilde{A}^{-1} - Q(^{\varkappa,\Gamma}a, \tilde{A}). \tag{4.2.75b}$$

By theor. 4 and def. 91, the action of the integral (G, H)–1–gauge transformation on the (G, H)–cocycles translates into an action of the differential (G, H)–1–gauge transformations corresponding to the integral ones onto the (G, H)–connection doublets corresponding to the cocycles, as given by eqs. (4.2.67). 2–gauge equivalent differential 1–gauge transformations yield the the same gauge transformed connection doublet.

Proposition 36. *Let (a, B) be a (G, H)–connection doublet, $(\varkappa, \Gamma)$ be a differential (G, H)–1–gauge transformation and A be (G, H)–2–gauge transformation. Then, one has*

$$^{\tilde{A}}\varkappa_{|a,B}, {}^{\tilde{A}}\Gamma_{|a,B}a = {}^{\varkappa,\Gamma}a, \tag{4.2.76a}$$

$$^{\tilde{A}}\varkappa_{|a,B}, {}^{\tilde{A}}\Gamma_{|a,B}B = {}^{\varkappa,\Gamma}B. \tag{4.2.76b}$$

Proof. Let (f, g, W) be a cocycle, (κ, Ψ, Φ) be a (f, g, W)–1–gauge transformation and A be a 2–gauge transformation. By prop. 33, $(^A\kappa, {}^A\Psi, {}^A\Phi)$ is also a (f, g, W)–1–gauge transformation. By (4.2.66), (4.2.67) combined, we have

$$\left(a_{\kappa,\Psi,\Phi f,\kappa,\Psi,\Phi g,\kappa,\Psi,\Phi W}, B_{\kappa,\Psi,\Phi f,\kappa,\Psi,\Phi g,\kappa,\Psi,\Phi W}\right)$$
$$= \left(^{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}}a_{f,g,W}, {}^{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}}B_{f,g,W}\right)$$

and similarly with (κ, Ψ, Φ) replaced by $(^A\kappa, {}^A\Psi, {}^A\Phi)$. By (4.2.75), (4.2.76), we have further

$$\left(\varkappa_{^A\kappa,{}^A\Psi,{}^A\Phi}, \Gamma_{^A\kappa,{}^A\Psi,{}^A\Phi}\right) = \left(^{\tilde{A}}\varkappa_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}, {}^{\tilde{A}}\Gamma_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}\right)$$

. By prop. 34, we have then that

$$\left(^{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}}a_{f,g,W}, {}^{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}}B_{f,g,W}\right) \tag{4.2.77}$$
$$= \left(^{\tilde{A}\varkappa_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}, {}^{\tilde{A}}\Gamma_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}}a_{f,g,W}, \right.$$
$$\left. {}^{\tilde{A}\varkappa_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}, {}^{\tilde{A}}\Gamma_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}}B_{f,g,W}\right).$$

By theors. 4, 6, (f, g, W) and (κ, Ψ, Φ) being arbitrary, (4.2.76a), (4.2.76b) hold true. $\square$

4.3 Higher parallel transport theory

In this section, we rederive the higher parallel transport theory worked out in refs. [38, 39, 40] and [41, 42, 43] relying on the theory of Lie crossed module cocycles

and their gauge transformation developed in sect. 4.2. We review first the theory of the path and fundamental 2–groupoids of a manifold to recall the reader the basic properties of these which are most relevant in the following. Next, we show how the 1– and 2–parallel transport induced by a connection doublet can be defined in terms of an associated cocycle. Then, we exhibit how 1–gauge transformation of the connection doublet affects the associated parallel transport by inducing an integral 1–gauge transformation of the underlying cocycle. The role of 2–gauge transformation is also highlighted. The 2–categorical interpretation of parallel transport and 1– and 2–gauge transformation thereof is recovered. We also touch the issue of smoothness of the parallel transport. Finally we make explicit the equivalence of our approach to the earlier ones recalled above. Again, to help intuition, we present our construction stressing its being an extension of the ordinary parallel transport theory.

4.3.1 Path and fundamental 2–groupoid

In this subsection, we review the basic notions of smooth thin homotopy and homotopy aiming to the definition of the path 2–groupoid of a manifold, one of the essential elements of higher parallel transport theory. As this material is not original, we provide no proof of the basic results.

We begin by considering the ordinary path and fundamental groupoids of a manifold M. Roughly, these are groupoids having points and curves joining pairs of points as its 0– and 1–cells. We make this more precise next.

Definition 94. *Let p_1, p_2 be points. A curve $\gamma : p_0 \to p_1$ with sitting instants is a mapping $\gamma \in \mathrm{Map}(\mathbb{R}, M)$ such that*

$$\gamma(x) = p_0 \qquad \text{for } x < \epsilon, \tag{4.3.1a}$$

$$\gamma(x) = p_1 \qquad \text{for } x > 1 - \epsilon \tag{4.3.1b}$$

for some $\epsilon > 0$ with $\epsilon < 1/2$ depending on γ. All curves will have sitting instants unless otherwise stated. We denote the set of all curves of M by $\Pi_1 M$.

Definition 95. *Let p be a point. The unit curve $\iota_p : p \to p$ of p is defined by*

$$\iota_p(x) = p. \tag{4.3.2}$$

Let p_0, p_1 be points and $\gamma : p_0 \to p_1$ be a curve. The inverse curve of γ is the curve $\gamma^{-1\circ} : p_1 \to p_0$ defined by

$$\gamma^{-1\circ}(x) = \gamma(1 - x). \tag{4.3.3}$$

Let p_0, p_1, p_2 be points and $\gamma_1 : p_0 \to p_1$, $\gamma_2 : p_1 \to p_2$ be curves. The composition of γ_1, γ_2 is the curve $\gamma_2 \circ \gamma_1 : p_0 \to p_2$ defined by

$$\gamma_2 \circ \gamma_1(x) = \gamma_1(2x) \qquad \text{for } x \leq 1/2, \tag{4.3.4a}$$

$$\gamma_2 \circ \gamma_1(x) = \gamma_2(2x - 1) \qquad \text{for } x \geq 1/2. \tag{4.3.4b}$$

The above are the type of operations which would be required for $(M, \Pi_1 M)$ to be a groupoid, but $(M, \Pi_1 M)$ is not, as is well–known, as invertibility and associativity do not hold. To construct a groupoid out of $(M, \Pi_1 M)$, one has to quotient out by the relation of either thin homotopy or homotopy.

Definition 96. *Let p_1, p_2 be points and $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves. A thin homotopy of γ_0, γ_1 is a mapping $h \in \mathrm{Map}(\mathbb{R}^2, M)$ such that*

$$h(x, y) = p_0 \qquad for\ x < \epsilon, \tag{4.3.5a}$$

$$h(x, y) = p_1 \qquad for\ x > 1 - \epsilon, \tag{4.3.5b}$$

$$h(x, y) = \gamma_0(x) \qquad for\ y < \epsilon, \tag{4.3.5c}$$

$$h(x, y) = \gamma_1(x) \qquad for\ y > 1 - \epsilon \tag{4.3.5d}$$

for some $\epsilon > 0$ with $\epsilon < 1/2$ and that

$$\mathrm{rank}(dh(x, y)) \leq 1. \tag{4.3.6}$$

γ_0, γ_1 are thin homotopy equivalent, a property denoted as $\gamma_1 \sim_1 \gamma_0$, if there is thin homotopy h of γ_0, γ_1. If condition (4.3.6) is not imposed, then h is a homotopy of γ_0, γ_1 and γ_0, γ_1 are homotopy equivalent, $\gamma_0 \sim^0_1 \gamma_1$.

$\sim_1$, $\sim^0_1$ are both equivalence relations. We denote by $P_1 M$ and $P^0_1 M$ the set of all thin homotopy and homotopy classes of curves of M.

Theorem 9. *$(M, P_1 M)$ and $(M, P^0_1 M)$ are both groupoids, the path groupoid and the fundamental groupoid of M.*

By modding out thin homotopy equivalence, the algebraic structure we have defined on $\Pi_1 M$ induces one of the same form on $P_1 M$ satisfying the axioms of invertibility and associativity, rendering $(M, P_1 M)$ a true groupoid. Similarly, by modding out homotopy equivalence, $(M, P^0_1 M)$ also turns out to be a groupoid. Diagrammatically, the content of these groupoids can be represented as

$$p_1 \xleftarrow{\quad \gamma \quad} p_0 \,. \tag{4.3.7}$$

where γ is understood as a (thin) homotopy class of curves.

Let M be a manifold. The path and fundamental 2–groupoids of M are 2–groupoids roughly having points, curves joining pairs of points and surfaces joining pairs of curves with common endpoints as its 0–, 1– and 2–cells. They are the simplest higher extensions of path and fundamental groupoids.

Definition 97. *For points p_0, p_1, a curve $\gamma : p_0 \to p_1$ is defined as before. The set of all curves is denoted again by $\Pi_1 M$.*

Let p_1, p_2 be points and $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves. A surface $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a map $\Sigma \in \mathrm{Map}(\mathbb{R}^2, M)$ such that

$$\Sigma(x,y) = p_0 \qquad for\ x < \epsilon, \tag{4.3.8a}$$

$$\Sigma(x,y) = p_1 \qquad for\ x > 1 - \epsilon, \tag{4.3.8b}$$

$$\Sigma(x,y) = \gamma_0(x) \qquad for\ y < \epsilon, \tag{4.3.8c}$$

$$\Sigma(x,y) = \gamma_1(x) \qquad for\ y > 1 - \epsilon \tag{4.3.8d}$$

for some $\epsilon > 0$ with $\epsilon < 1/2$ depending on γ_0, γ_1, Σ. All surfaces will be assumed to have sitting instants unless otherwise stated. The set of all surfaces is denoted by $\Pi_2 M$.

Definition 98. *For a point p, the unit curve $\iota_p : p \to p$ of p is defined as before. For points p_0, p_1 and a curve $\gamma : p_0 \to p_1$, the inverse curve $\gamma^{-1\circ}$ is also defined as before. For points p_0, p_1, p_2 and curves $\gamma_1 : p_0 \to p_1$, $\gamma_2 : p_1 \to p_2$, the composed curve $\gamma_2 \circ \gamma_1 : p_0 \to p_2$ is again defined as before.*

Let p_0, p_1 be points and $\gamma : p_0 \to p_1$ be a curve. The unit surface $I_\gamma : \gamma \Rightarrow \gamma$ of γ is the surface defined by

$$I_\gamma(x,y) = \gamma(x). \tag{4.3.9}$$

Let p_0, p_1 be points and $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ be a surface. The vertical inverse of Σ is the surface $\Sigma^{-1\bullet} : \gamma_1 \Rightarrow \gamma_0$

$$\Sigma^{-1\bullet}(x,y) = \Sigma(x, 1-y). \tag{4.3.10}$$

Let p_0, p_1 be points and $\gamma_0, \gamma_1, \gamma_2 : p_0 \to p_1$ be curves and $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_2 : \gamma_1 \Rightarrow \gamma_2$ be surfaces. The vertical composition of Σ_1, Σ_2 is the surface $\Sigma_2 \bullet \Sigma_1 : \gamma_0 \Rightarrow \gamma_2$ defined by

$$\Sigma_2 \bullet \Sigma_1(x,y) = \Sigma_1(x, 2y) \qquad for\ y \leq 1/2, \tag{4.3.11a}$$

$$\Sigma_2 \bullet \Sigma_1(x,y) = \Sigma_2(x, 2y-1) \qquad for\ y \geq 1/2. \tag{4.3.11b}$$

Let p_0, p_1 be points and $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ be a surface. The horizontal inverse of Σ is the surface $\Sigma^{-1\circ} : \gamma_0^{-1\circ} \Rightarrow \gamma_1^{-1\circ}$

$$\Sigma^{-1\circ}(x,y) = \Sigma(1-x, y). \tag{4.3.12}$$

Let p_0, p_1, p_2 be points and $\gamma_0, \gamma_1 : p_0 \to p_1$, $\gamma_2, \gamma_3 : p_1 \to p_2$ be curves and $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_2 : \gamma_2 \Rightarrow \gamma_3$ be surfaces. The horizontal composition of Σ_1, Σ_2 is the surface $\Sigma_2 \circ \Sigma_1 : \gamma_2 \circ \gamma_0 \Rightarrow \gamma_3 \circ \gamma_1$ defined by

$$\Sigma_2 \circ \Sigma_1(x,y) = \Sigma_1(2x, y) \qquad for\ x \leq 1/2, \tag{4.3.13a}$$

$$\Sigma_2 \circ \Sigma_1(x,y) = \Sigma_2(2x-1, y) \qquad for\ x \geq 1/2. \tag{4.3.13b}$$

The above are the type of operations which would be required for $(M, \Pi_1 M, \Pi_2 M)$ to be a 2–groupoid, but $(M, \Pi_1 M, \Pi_2 M)$ fails to be one as invertibility and associativity do not hold both for curves and surfaces. To construct a 2–groupoid out of $(M, \Pi_1 M, \Pi_2 M)$, one has to quotient out by a suitable higher version of the relation of either thin homotopy or homotopy.

Definition 99. *For points p_0, p_1 and curves $\gamma_0, \gamma_1 : p_0 \to p_1$ the notions of thin homotopy h and thin homotopy equivalence of γ_0, γ_1 are defined exactly as before. We denote again by $\sim_1$ thin homotopy equivalence and by $P_1 M$ the set of all thin homotopy classes of curves of M.*

Let p_0, p_1 be points, $\gamma_0, \gamma_1, \gamma_2, \gamma_3 : p_0 \to p_1$ be curves and $\Sigma_0 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_1 : \gamma_2 \Rightarrow \gamma_3$ be surfaces. A thin homotopy of Σ_0, Σ_1 is a mapping $H \in \mathrm{Map}(\mathbb{R}^3, M)$ with the property that

$$H(x, y, z) = p_0 \qquad \text{for } x < \epsilon, \tag{4.3.14a}$$

$$H(x, y, z) = p_1 \qquad \text{for } x > 1 - \epsilon, \tag{4.3.14b}$$

$$H(x, y, z) = H(x, 0, z) \qquad \text{for } y < \epsilon, \tag{4.3.14c}$$

$$H(x, y, z) = H(x, 1, z) \qquad \text{for } y > 1 - \epsilon, \tag{4.3.14d}$$

$$H(x, y, z) = \Sigma_0(x, y) \qquad \text{for } z < \epsilon, \tag{4.3.14e}$$

$$H(x, y, z) = \Sigma_1(x, y) \qquad \text{for } z > 1 - \epsilon \tag{4.3.14f}$$

for some $\epsilon > 0$ and that

$$\mathrm{rank}(dH(x, 0, z)), \ \mathrm{rank}(dH(x, 1, z)) \leq 1, \tag{4.3.15a}$$

$$\mathrm{rank}(dH(x, y, z)) \leq 2. \tag{4.3.15b}$$

Σ_0, Σ_1 are thin homotopy equivalent, which fact we write as $\Sigma_1 \sim_2 \Sigma_0$, if there is thin homotopy H of Σ_0, Σ_1. If condition (4.3.15b) is not imposed, then H is a homotopy of Σ_0, Σ_1 and Σ_0, Σ_1 are homotopy equivalent, $\Sigma_0 \sim^0{}_2 \Sigma_1$.

$\sim_2$, $\sim^0{}_2$ are both equivalence relations by conditions (4.3.14a)–(4.3.14f). Condition (4.3.15a) implies that the source and target curves of of Σ_0, Σ_1 are thin homotopy equivalent, $\gamma_0 \sim_1 \gamma_2$, $\gamma_1 \sim_1 \gamma_3$. We denote by $P_2 M$ and $P^0{}_2 M$ the set of all thin homotopy and homotopy classes of surfaces of M.

Theorem 10. $(M, P_1 M, P_2 M)$ *and* $(M, P_1 M, P^0{}_2 M)$ *are bot 2–groupoids, the path 2–groupoid and the fundamental 2–groupoid of M, respectively.*

By modding out thin homotopy equivalence, the algebraic structure we have defined on $\Pi_1 M$, $\Pi_2 M$ induces one of the same form on $P_1 M$, $P_2 M$ satisfying the axioms of invertibility and associativity, rendering $(M, P_1 M, P_2 M)$ a true 2–groupoid. Similarly, modding out homotopy equivalence, $(M, P_1 M, P^0{}_2 M)$ also turns out to be a 2–groupoid. Diagrammatically, the content of these 2–groupoids can be represented as

$$p_1 \underset{\gamma_1}{\overset{\gamma_0}{\rightleftarrows}} \ \Sigma \Downarrow \ p_0 \tag{4.3.16}$$

where γ_0, γ_1 is understood as thin homotopy class of curves and Σ as a (thin) homotopy class of surfaces.

Now we are ready to formulate our parallel transport theory.

4.3.2   2–parallel transport

In this subsection, we shall define and study higher parallel transport. Our approach is inspired by that of ref. [39], but relies systematically on the cocycle set–up developed in sect. 4.2. We assume throughout a trivial principal bundle background.

We begin by reviewing parallel transport in ordinary gauge theory. Let M be a manifold and G be a Lie group. The basic datum required to define parallel transport is a G–connection.

Definition 100. *A G–connection on M, or simply a G–connection, is a form $\theta \in \Omega^1(M, \mathfrak{g})$. We denote the set of G–connections by $\mathrm{Conn}(M, G)$.*

If γ is a curve and θ is a G–connection on M, $\gamma^*\theta$ is a G–connection in the sense of def. 100. By theor. 3, to $\gamma^*\theta$ there then corresponds a G–cocycle $f_{\gamma^*\theta}$.

Definition 101. *Let θ be a G–connection. Let further p_0, p_1 be points and $\gamma : p_0 \to p_1$ be a curve. The parallel transport along γ induced by θ is*

$$F_\theta(\gamma) = f_{\gamma^*\theta}(1, 0). \tag{4.3.17}$$

Let us fix a G–connection θ. We have then a mapping $F_\theta : \Pi_1 M \to G$.

Proposition 37. *For any point p, one has*

$$F_\theta(\iota_p) = 1_G. \tag{4.3.18}$$

For any two points p_0, p_1 and curve $\gamma : p_0 \to p_1$, one has

$$F_\theta(\gamma^{-1\circ}) = F_\theta(\gamma)^{-1}. \tag{4.3.19}$$

For any three points p_0, p_1, p_2 and two curves $\gamma_1 : p_0 \to p_1$, $\gamma_2 : p_1 \to p_2$, i

$$F_\theta(\gamma_2 \circ \gamma_1) = F_\theta(\gamma_2) F_\theta(\gamma_1). \tag{4.3.20}$$

Proof. If f is a G–cocycle and $\phi : \mathbb{R} \to \mathbb{R}$ is a map, then the mapping $\phi^* f : \mathbb{R}^2 \to G$ defined by the expression

$$\phi^* f(x', x) = f(\phi(x'), \phi(x)) \tag{4.3.21}$$

satisfies (4.2.2) and, so, is also a G–cocycle, the pull–back $\phi^* f$ of f by ϕ. The one–to–one correspondence between G–connections a and G–cocycles $f \in \mathrm{Cyc}(G)$ established by theor. 3 is natural with respect to pull-back, as $f_{\phi^*a} = \phi^* f_a$ and $a_{\phi^*f} = \phi^* a_f$.

For illustration, we show (4.3.20). Define $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ by $\phi_1(x) = x/2$ and $\phi_2(x) = x/2 + 1/2$. It follows from (4.3.4) that $(\gamma_2 \circ \gamma_1) \circ \phi_1(x) = \gamma_1(x)$ for $x \leq 1$ and

$(\gamma_2 \circ \gamma_1) \circ \phi_2(x) = \gamma_2(x)$ for $x \geq 0$. Then,

$$F_\theta(\gamma_2 \circ \gamma_1) = f_{\gamma_2 \circ \gamma_1 * \theta}(1, 0) \tag{4.3.22}$$

$$= f_{\gamma_2 \circ \gamma_1 * \theta}(1, 1/2) f_{\gamma_2 \circ \gamma_1 * \theta}(1/2, 0)$$

$$= f_{\gamma_2 \circ \gamma_1 * \theta}(\phi_2(1), \phi_2(0)) f_{\gamma_2 \circ \gamma_1 * \theta}(\phi_1(1), \phi_1(0))$$

$$= \phi_2{}^* f_{\gamma_2 \circ \gamma_1 * \theta}(1, 0) \phi_1{}^* f_{\gamma_2 \circ \gamma_1 * \theta}(1, 0)$$

$$= f_{\phi_2{}^* \gamma_2 \circ \gamma_1 * \theta}(1, 0) f_{\phi_1{}^* \gamma_2 \circ \gamma_1 * \theta}(1, 0)$$

$$= f_{(\gamma_2 \circ \gamma_1) \circ \phi_2 * \theta}(1, 0) f_{(\gamma_2 \circ \gamma_1) \circ \phi_1 * \theta}(1, 0)$$

$$= f_{\gamma_2 * \theta}(1, 0) f_{\gamma_1 * \theta}(1, 0) = F_\theta(\gamma_2) F_\theta(\gamma_1).$$

(4.3.18), (4.3.19) are proven by similar techniques.   □

F_θ has the fundamental property of homotopy invariance as stated by the following proposition.

Theorem 11. *Let p_0, p_1 be points and $\gamma_y : p_0 \to p_1$, $y \in \mathbb{R}$, be a smooth 1–parameter family of curves such that the mapping $h : \mathbb{R}^2 \to M$ defined by $h(x, y) = \gamma_y(x)$ is a thin homotopy of γ_0, γ_1. Then,*

$$F_\theta(\gamma_1) = F_\theta(\gamma_0). \tag{4.3.23}$$

Proof. The proof is based on the variational formula

$$f_{\gamma_y * \theta}(x, x_0)^{-1} \partial_y f_{\gamma_y * \theta}(x, x_0) \tag{4.3.24}$$

$$= - \int_{x_0}^x d\xi \, f_{\gamma_y * \theta}(\xi, x_0)^{-1} h^*(d\theta + [\theta, \theta]/2)_{yx}(\xi, y) f_{\gamma_y * \theta}(\xi, x_0)$$

$$- f_{\gamma_y * \theta}(x, x_0)^{-1} h^* \theta_y(x, y) f_{\gamma_y * \theta}(x, x_0) + h^* \theta_y(x_0, y),$$

which is straightforward though lengthy to derive. Since h is a thin homotopy, $h^*(d\theta + [\theta, \theta]/2)_{xy}(x, y) = 0$, by (4.3.6), and $h^* \theta_y(1, y) = h^* \theta_y(0, y) = 0$, by (4.3.5a), (4.3.5b). Hence, by (4.3.17), in virtue of (4.3.24),

$$F_\theta(\gamma_y)^{-1} \partial_y F_\theta(\gamma_y) = f_{\gamma_y * \theta}(1, 0)^{-1} \partial_y f_{\gamma_y * \theta}(1, 0) = 0, \tag{4.3.25}$$

from which (4.3.23) follows.   □

The map $F_\theta : \Pi_1 M \to G$ factors so through one $\bar{F}_\theta : P_1 M \to G$ from the path groupoid 1–cell set $P_1 M$ into G, giving a categorical map $\bar{F}_\theta : (M, P_1 M) \to BG$

$$p_1 \xleftarrow{\;\gamma\;} p_0 \qquad \longmapsto \qquad * \xleftarrow{\;\bar{F}_\theta(\gamma)\;} *, \tag{4.3.26}$$

from the path groupoid $(M, P_1 M)$ into the delooping groupoid BG of the group G (cf. subsects. 4.2.1 and 4.3.1).

Proposition 38. *$\bar{F}_\theta$ is a groupoid functor.*

Proof. The statement follows from combining props. 37, 11. Functoriality results from relations (4.3.18)–(4.3.20). □

Definition 102. *The G–connection θ is said flat if*

$$d\theta + \frac{1}{2}[\theta, \theta] = 0. \tag{4.3.27}$$

Theorem 12. *Let θ be flat. Let p_0, p_1 be points and $\gamma_y : p_0 \to p_1$, $y \in \mathbb{R}$, be a smooth 1–parameter family of curves such that the mapping $h : \mathbb{R}^2 \to M$ defined by $h(x, y) = \gamma_y(x)$ is a homotopy of γ_0, γ_1. Then,*

$$F_\theta(\gamma_1) = F_\theta(\gamma_0). \tag{4.3.28}$$

Proof. The proof is based on relation (4.3.24) and follows the same lines as that of theor. 11 except for the vanishing of the integral term in the right hand side of (4.3.24) which is now due to the flatness of θ instead of the thinness of H. □

Hence, the map $F_\theta : \Pi_1 M \to G$ factors through one $\bar{F}^0{}_\theta : P^0{}_1 M \to G$ from the fundamental groupoid 1–cell set $P^0{}_1 M$ into G yielding a categorical map $\bar{F}_\theta : (M, P^0{}_1 M) \to BG$ of the fundamental groupoid $(M, P^0{}_1 M)$ into the delooping groupoid BG.

Proposition 39. *When the connection θ is flat, $\bar{F}^0{}_\theta : (M, P^0{}_1 M) \to BG$ is a groupoid functor.*

Proof. The statement follows from combining prop. 37 and theor. 12 with functoriality resulting again from relations (4.3.18)–(4.3.20). □

We consider now the higher case. Let M be a manifold and (G, H) be a Lie crossed module. The basic datum required to define parallel transport is a (G, H)–connection doublet.

Definition 103. *A (G, H)–connection doublet on M, or simply a (G, H)–connection doublet, is a pair of forms $(\theta, \Upsilon) \in \Omega^1(M, \mathfrak{g}) \times \Omega^2(M, \mathfrak{h})$ satisfying the zero fake curvature condition*

$$d\theta + \frac{1}{2}[\theta, \theta] - \dot{t}(\Upsilon) = 0. \tag{4.3.29}$$

We denote the set of (G, H)–connection doublets by $\mathrm{Conn}(M, G, H)$.

If Σ is a surface and (θ, Υ) is a (G, H)–connection doublet on M, then $(\Sigma^*\theta, \Sigma^*\Upsilon)$ is a (G, H)–connection in the sense of def. 103. By theor. 4, with $(\Sigma^*\theta, \Sigma^*\Upsilon)$ there is then associated a (G, H)–cocycle $(f_{\Sigma^*\theta, \Sigma^*\Upsilon|0}, g_{\Sigma^*\theta, \Sigma^*\Upsilon|0}, W_{\Sigma^*\theta, \Sigma^*\Upsilon|0})$.

Definition 104. *Let (θ, Υ) be a (G, H)–connection. Let further p_0, p_1 be points, $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ be a surface. The 1–parallel transport along γ_0, γ_1 and 2–parallel transport along Σ induced by (θ, Υ) are*

$$F_{\theta, \Upsilon}(\gamma_0) = f_{\Sigma^*\theta, \Sigma^*\Upsilon|0}(1, 0) \tag{4.3.30a}$$

$$F_{\theta, \Upsilon}(\gamma_1) = f_{\Sigma^*\theta, \Sigma^*\Upsilon|1}(1, 0) \tag{4.3.30b}$$

$$F_{\theta, \Upsilon}(\Sigma) = W_{\Sigma^*\theta, \Sigma^*\Upsilon}(0, 1; 1, 0) \tag{4.3.30c}$$

From the target matching condition $(f_{\Sigma^*\theta,\Sigma^*\Upsilon|0}, g_{\Sigma^*\theta,\Sigma^*\Upsilon|0}, W_{\Sigma^*\theta,\Sigma^*\Upsilon|0})$ obeys (cf. eq. (4.2.11)), one has the following result.

Proposition 40. *Let p_0, p_1 and let $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ be surfaces. Then, one has*

$$F_{\theta,\Upsilon}(\gamma_1) = t(F_{\theta,\Upsilon}(\Sigma))F_{\theta,\Upsilon}(\gamma_0) \tag{4.3.31}$$

Proof. To begin with, we observe that there is $\epsilon > 0$ with $\epsilon < 1/2$ such that

$$g_{\Sigma^*\theta,\Sigma^*\Upsilon|x}(y', y) = 1_G \tag{4.3.32}$$

for $x < \epsilon$ or $x > 1 - \epsilon$ and arbitrary y, y'. This follows from the fact that, by theor. 4, $g_{\Sigma^*\theta,\Sigma^*\Upsilon|x}(y', y)$ is the solution of the differential problem (4.2.24b), (4.2.25b) with $a_y(x, y)$ replaced by $\Sigma^*\theta_y(x, y)$ and that $\Sigma^*\theta_y(x, y) = 0$ identically for the values of x indicated on account of (4.3.5a), (4.3.5b).

By (4.3.30a)–(4.3.30c), using the properties (4.2.11), (4.2.13b) and taking (4.3.32) into account, we find

$$t(F_{\theta,\Upsilon}(\Sigma)) = t(W_{\Sigma^*\theta,\Sigma^*\Upsilon}(0, 1; 1, 0)) \tag{4.3.33}$$

$$= g_{\Sigma^*\theta,\Sigma^*\Upsilon|1}(1, 0)^{-1} f_{\Sigma^*\theta,\Sigma^*\Upsilon|1}(0, 1)^{-1} g_{\Sigma^*\theta,\Sigma^*\Upsilon|0}(1, 0) f_{\Sigma^*\theta,\Sigma^*\Upsilon|0}(0, 1)$$

$$= f_{\Sigma^*\theta,\Sigma^*\Upsilon|1}(1, 0) f_{\Sigma^*\theta,\Sigma^*\Upsilon|0}(1, 0)^{-1}$$

$$= F_{\theta,\Upsilon}(\gamma_1) F_{\theta,\Upsilon}(\gamma_0)^{-1},$$

which leads immediately to (4.3.31). □

Physical intuition suggests that it should be possible to express the 1–parallel transport $F_{\theta,\Upsilon}(\gamma)$ along a curve γ independently from any other curve γ' with the same endpoints and surface Σ connecting γ to γ'. This is indeed the case, as we shall show next.

Lemma 1. *Let p_0, p_1 be points and $\gamma : p_0 \to p_1$ be a curve. Then, $f_{I_\gamma^*\theta,I_\gamma^*\Upsilon|y}$, where $I_\gamma : \gamma \Rightarrow \gamma$ is the unit surface of γ (cf. eq. (4.3.9)), is independent from the value of y.*

Proof. By theor. 4, $f_{I_\gamma^*\theta,I_\gamma^*\Upsilon|y}(x, x_0)$ is the solution of the differential problem (4.2.24a), (4.2.25a) with $a_x(x, y) = I_\gamma^*\theta_x(x, y)$. Since $I_\gamma^*\theta_x(x, y) = \gamma^*a_x(x)$ is independent from y, so is $f_{I_\gamma^*\theta,I_\gamma^*\Upsilon|y}(x, x_0)$. □

Definition 105. *If p_0, p_1 are points and $\gamma : p_0 \to p_1$ is a curve, one sets*

$$F_{\theta,\Upsilon}(\gamma) = f_{I_\gamma^*\theta,I_\gamma^*\Upsilon|y}(1, 0). \tag{4.3.34}$$

Proposition 41. *If p_0, p_1 are points, $\gamma_0, \gamma_1 : p_0 \to p_1$ are curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a surface, then the value of $F_{\theta,\Upsilon}(\gamma_i)$ computed using (4.3.30a), (4.3.30b) equals that obtained using (4.3.34).*

Proof. By theor. 4, $f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(x, x_0)$ is the solution of the differential problem (4.2.24a), (4.2.25a) with $a_x(x, y) = \Sigma^*\theta_x(x, y)$. Likewise, $f_{I_{\gamma_i}{}^*\theta,I_{\gamma_i}{}^*\Upsilon|y}(x, x_0)$ solves the differential problem (4.2.24a), (4.2.25a) with $a_x(x, y) = I_{\gamma_i}{}^*\theta_x(x, y)$. Since $I_{\gamma_i}{}^*\theta_x(x, y) = \Sigma^*\theta_x(x, i)$ for $i = 1, 2$ and any y, we have $f_{I_{\gamma_i}{}^*\theta,I_{\gamma_i}{}^*\Upsilon|y}(x, x_0) = f_{\Sigma^*\theta,\Sigma^*\Upsilon|i}(x, x_0)$. Hence, (4.3.30a), (4.3.30b) and (4.3.34) furnish the same value of $F_{\theta,\Upsilon|y}(\gamma_i)$. $\qquad\square$

Let us fix a (G, H)–connection doublet (θ, Υ). We have then two mappings $F_{\theta,\Upsilon} : \Pi_1 M \to G$ and $F_{\theta,\Upsilon} : \Pi_2 M \to H$.

Proposition 42. *For any point p, one has*

$$F_{\theta,\Upsilon}(\iota_p) = 1_G. \tag{4.3.35}$$

For any two points p_0, p_1 and curve $\gamma : p_0 \to p_1$, one has

$$F_{\theta,\Upsilon}(\gamma^{-1\circ}) = F_{\theta,\Upsilon}(\gamma)^{-1}. \tag{4.3.36}$$

For any three p_0, p_1, p_2 and two curves $\gamma_1 : p_0 \to p_1$, $\gamma_2 : p_1 \to p_2$,

$$F_{\theta,\Upsilon}(\gamma_2 \circ \gamma_1) = F_{\theta,\Upsilon}(\gamma_2)F_{\theta,\Upsilon}(\gamma_1). \tag{4.3.37}$$

For any two points p_0, p_1 and curve $\gamma : p_0 \to p_1$,

$$F_{\theta,\Upsilon}(I_\gamma) = 1_H. \tag{4.3.38}$$

If p_0, p_1 are points, $\gamma_0, \gamma_1 : p_0 \to p_1$ are curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a surface, then

$$F_{\theta,\Upsilon}(\Sigma^{-1\bullet}) = F_{\theta,\Upsilon}(\Sigma)^{-1}. \tag{4.3.39}$$

If p_0, p_1 are points, $\gamma_0, \gamma_1, \gamma_2 : p_0 \to p_1$ are curves and $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_2 : \gamma_1 \Rightarrow \gamma_2$ are surfaces, then

$$F_{\theta,\Upsilon}(\Sigma_2 \bullet \Sigma_1) = F_{\theta,\Upsilon}(\Sigma_2)F_{\theta,\Upsilon}(\Sigma_1). \tag{4.3.40}$$

If p_0, p_1 are points, $\gamma_0, \gamma_1 : p_0 \to p_1$ are curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a surface, then

$$F_{\theta,\Upsilon}(\Sigma^{-1\circ}) = m(F_{\theta,\Upsilon}(\gamma_0)^{-1})(F_{\theta,\Upsilon}(\Sigma)^{-1}). \tag{4.3.41}$$

If p_0, p_1, p_2 are points, $\gamma_0, \gamma_1 : p_0 \to p_1$, $\gamma_2, \gamma_3 : p_1 \to p_2$ are curves and $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$, $\Sigma_2 : \gamma_2 \Rightarrow \gamma_3$ are surfaces, then

$$F_{\theta,\Upsilon}(\Sigma_2 \circ \Sigma_1) = F_{\theta,\Upsilon}(\Sigma_2)m(F_{\theta,\Upsilon}(\gamma_2))(F_{\theta,\Upsilon}(\Sigma_1)). \tag{4.3.42}$$

Proof. For any map $\phi : \mathbb{R} \to \mathbb{R}$, we define two maps $l_\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $r_\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by setting $l_\phi(x, y) = (\phi(x), y)$, $r_\phi(x, y) = (x, \phi(y))$. If (f, g, W) is a (G, H)–cocycle, the maps $l_\phi{}^*f : \mathbb{R}^2 \times \mathbb{R} \to G$, $l_\phi{}^*g : \mathbb{R} \times \mathbb{R}^2 \to G$, $l_\phi{}^*W : \mathbb{R}^2 \times \mathbb{R}^2 \to H$ given by

$$l_\phi{}^*f(x', x; y) = f(\phi(x'), \phi(x); y), \tag{4.3.43a}$$

$$l_\phi{}^*g(x; y', y) = g(\phi(x); y', y), \tag{4.3.43b}$$

$$l_\phi{}^*W(x', x; y', y) = W(\phi(x'), \phi(x); y', y) \tag{4.3.43c}$$

and those $r_\phi{}^*f : \mathbb{R}^2 \times \mathbb{R} \to G$, $r_\phi{}^*g : \mathbb{R} \times \mathbb{R}^2 \to G$, $r_\phi{}^*W : \mathbb{R}^2 \times \mathbb{R}^2 \to H$ by

$$r_\phi{}^*f(x', x; y) = f(x', x; \phi(y)), \tag{4.3.44a}$$

$$r_\phi{}^*g(x; y', y) = g(x; \phi(y'), \phi(y)), \tag{4.3.44b}$$

$$r_\phi{}^*W(x', x; y', y) = W(x', x; \phi(y'), \phi(y)) \tag{4.3.44c}$$

satisfy (4.2.11) and (4.2.12) and, consequently, constitute two (G, H)–cocycles, the left and right pull-back $(l_\phi{}^*f, l_\phi{}^*g, l_\phi{}^*W)$, $(r_\phi{}^*f, r_\phi{}^*g, r_\phi{}^*W)$ of (f, g, W) by ϕ.

The one–to–one correspondence between (G, H)–connections (a, B) and (G, H)–cocycles (f, g, W) stated by theor. 4 is natural with respect to left/right pull-back, as one has

$$(f_{l_\phi{}^*a, l_\phi{}^*B}, g_{l_\phi{}^*a, l_\phi{}^*B}, W_{l_\phi{}^*a, l_\phi{}^*B}) = (l_\phi{}^*f_{a,B}, l_\phi{}^*g_{a,B}, l_\phi{}^*W_{a,B})$$

and

$$(a_{l_\phi{}^*f, l_\phi{}^*g, l_\phi{}^*W}, B_{l_\phi{}^*f, l_\phi{}^*g, l_\phi{}^*W}) = (l_\phi{}^*a_{f,g,W}, l_\phi{}^*B_{f,g,W})$$

for left pull-back and

$$(f_{r_\phi{}^*a, r_\phi{}^*B}, g_{r_\phi{}^*a, r_\phi{}^*B}, W_{r_\phi{}^*a, r_\phi{}^*B}) = (r_\phi{}^*f_{a,B}, r_\phi{}^*g_{a,B}, r_\phi{}^*W_{a,B})$$

and

$$(a_{r_\phi{}^*f, r_\phi{}^*g, r_\phi{}^*W}, B_{r_\phi{}^*f, r_\phi{}^*g, r_\phi{}^*W}) = (r_\phi{}^*a_{f,g,W}, r_\phi{}^*B_{f,g,W})$$

for right pull-back.

As an illustration, we prove (4.3.40). Define $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ by $\phi_1(x) = x/2$ and $\phi_2(x) = x/2 + 1/2$. It follows from (4.3.13) that $(I_{\gamma_2} \circ I_{\gamma_1}) \circ l_{\phi_1}(x, y) = I_{\gamma_1}(x, y)$ for $x \leq 1$ and $(I_{\gamma_2} \circ I_{\gamma_1}) \circ l_{\phi_2}(x, y) = I_{\gamma_2}(x, y)$ for $x \geq 0$. Then, by (4.2.12c) and (4.2.13b), we have

$$\begin{aligned}
F_{\theta, \Upsilon}(\Sigma_2 \circ \Sigma_1) &= W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(0, 1) \tag{4.3.45} \\[4pt]
&= W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(1/2, 1) \\[4pt]
&\qquad \times m(f_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 0}(1/2, 1)^{-1})(W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(0, 1/2)) \\[4pt]
&= W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(\phi_2(0), \phi_2(1)) \\[4pt]
&\qquad \times m(f_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 0}(\phi_2(0), \phi_2(1))^{-1})(W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(\phi_1(0), \phi_1(1))) \\[4pt]
&= l_{\phi_2}{}^*W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(0, 1) \\[4pt]
&\qquad \times m(l_{\phi_2}{}^*f_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 0}(0, 1)^{-1})(l_{\phi_1}{}^*W_{\Sigma_2 \circ \Sigma_1{}^*\theta, \Sigma_2 \circ \Sigma_1{}^*\Upsilon | 1, 0}(0, 1))
\end{aligned}$$

$$= W_{l_{\phi_2}*\Sigma_2\circ\Sigma_1*\theta,l_{\phi_2}*\Sigma_2\circ\Sigma_1*\Upsilon|1,0}(0,1)$$

$$\times\, m\big(f_{l_{\phi_2}*\Sigma_2\circ\Sigma_1*\theta,l_{\phi_2}*\Sigma_2\circ\Sigma_1*\Upsilon|0}(0,1)^{-1}\big)\big(W_{l_{\phi_1}*\Sigma_2\circ\Sigma_1*\theta,l_{\phi_1}*\Sigma_2\circ\Sigma_1*\Upsilon|1,0}(0,1)\big)$$

$$= W_{(\Sigma_2\circ\Sigma_1)\circ l_{\phi_2}*\theta,(\Sigma_2\circ\Sigma_1)\circ l_{\phi_2}*\Upsilon|1,0}(0,1)$$

$$\times\, m\big(f_{(\Sigma_2\circ\Sigma_1)\circ l_{\phi_2}*\theta,(\Sigma_2\circ\Sigma_1)\circ l_{\phi_2}*\Upsilon|0}(0,1)^{-1}\big)\big(W_{(\Sigma_2\circ\Sigma_1)\circ l_{\phi_1}*\theta,(\Sigma_2\circ\Sigma_1)\circ l_{\phi_1}*\Upsilon|1,0}(0,1)\big)$$

$$= W_{\Sigma_2*\theta,\Sigma_2*\Upsilon|1,0}(0,1)\, m\big(f_{\Sigma_2*\theta,\Sigma_2*\Upsilon|0}(0,1)^{-1}\big)\big(W_{\Sigma_1*\theta,\Sigma_1*\Upsilon|1,0}(0,1)\big)$$

$$= F_{\theta,\Upsilon}(\Sigma_2)\, m\big(F_{\theta,\Upsilon}(\gamma_2)\big)\big(F_{\theta,\Upsilon}(\Sigma_1)\big)$$

(4.3.40) is proven by a similar procedure involving this time right pull-back. The other relations are shown by using similar techniques. □

Analogously to the ordinary case, $F_{\theta,\Upsilon}$ is thin homotopy invariant as established by the following theorem.

Theorem 13. *Let p_0, p_1 be points and $\gamma_{0z}, \gamma_{1z} : p_0 \to p_1$ and $\Sigma_z : \gamma_{0z} \Rightarrow \gamma_{1z}$, $z \in \mathbb{R}$ be 1–parameter families of curves and surfaces such that the mapping $H : \mathbb{R}^3 \to M$ defined by $H(x,y,z) = \Sigma_z(x,y)$ is a thin homotopy of Σ_0, Σ_1. Then, one has the identities*

$$F_{\theta,\Upsilon}(\gamma_{01}) = F_{\theta,\Upsilon}(\gamma_{00}), \tag{4.3.46a}$$

$$F_{\theta,\Upsilon}(\gamma_{11}) = F_{\theta,\Upsilon}(\gamma_{10}), \tag{4.3.46b}$$

$$F_{\theta,\Upsilon}(\Sigma_1) = F_{\theta,\Upsilon}(\Sigma_0). \tag{4.3.46c}$$

Proof. The proof is based on the variational formulae

$$f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(x,x_0)^{-1}\partial_z f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(x,x_0) \tag{4.3.47}$$

$$= -\int_{x_0}^{x} d\xi\, f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(\xi,x_0)^{-1}\dot{t}(H^*\Upsilon_{zx}(\xi,y,z))f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(\xi,x_0)$$

$$- f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(x,x_0)^{-1}H^*\theta_z(x,y,z)f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(x,x_0) + H^*\theta_z(x_0,y,z),$$

$$g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(y,y_0)^{-1}\partial_z g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(y,y_0) \tag{4.3.48}$$

$$= -\int_{y_0}^{y} d\eta\, g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(\eta,y_0)^{-1}\dot{t}(H^*\Upsilon_{zy}(x,\eta,z))g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(\eta,y_0)$$

$$- g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(y,y_0)^{-1}H^*\theta_z(x,y,z)g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(y,y_0) + H^*\theta_z(x,y_0,z),$$

$$W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0)^{-1}\partial_z W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0) \tag{4.3.49}$$

$$= -\int_{x_0}^{x} d\xi \int_{y_0}^{y} d\eta\, W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;\eta,y_0)^{-1}\dot{m}\big(g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x_0}(\eta,y_0)^{-1}$$

$$\times\, f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|\eta}(\xi,x_0)^{-1}\big)(H^*(d\Upsilon + [\theta,\Upsilon])_{xyz}(\xi,\eta,z))W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;\eta,y_0)$$

$$-\int_{x_0}^{x} d\xi \Big[W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0)^{-1}\dot{m}(g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x_0}(y,y_0)^{-1}$$

$$\times f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y}(\xi,x_0)^{-1})(H^*\Upsilon_{xz}(\xi,y,z))W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0)$$

$$-\dot{m}(f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y_0}(\xi,x_0)^{-1})(H^*\Upsilon_{xz}(\xi,y_0,z))\Big]$$

$$+\int_{y_0}^{y} d\eta \Big[W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0)^{-1}$$

$$\times \dot{m}(g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x_0}(\eta,y_0)^{-1})(H^*\Upsilon_{zy}(x_0,\eta,z))W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0)$$

$$-\dot{m}(f_{\Sigma_z*\theta,\Sigma_z*\Upsilon|y_0}(x,x_0)^{-1}g_{\Sigma_z*\theta,\Sigma_z*\Upsilon|x}(\eta,y_0)^{-1})(H^*\Upsilon_{zy}(x,\eta,z))\Big]$$

$$+Q(H^*\theta_z(x_0,y_0,z),W_{\Sigma_z*\theta,\Sigma_z*\Upsilon}(x,x_0;y,y_0)^{-1}),$$

which are straightforward albeit very lengthy to obtain. Since H is a thin homotopy, $H^*\Upsilon_{zx}(x,i,z)=0$ for $i=0,1$, by (4.3.15a), $H^*(d\Upsilon+[\theta,\Upsilon])_{xyz}(x,y,z)=0$, by (4.3.15b), and $H^*\theta_z(i,j,z)=0$ and $H^*\Upsilon_{yx}(i,y,z)=0$ for $i,j=0,1$, by (4.3.14a), (4.3.14b). Therefore, by (4.3.30a)–(4.3.30c), in virtue of (4.3.47), (4.3.49), we have

$$F_{\theta,\Upsilon}(\gamma_{0z})^{-1}\partial_z F_{\theta,\Upsilon}(\gamma_{0z})=0, \tag{4.3.50a}$$

$$F_{\theta,\Upsilon}(\gamma_{1z})^{-1}\partial_z F_{\theta,\Upsilon}(\gamma_{1z})=0, \tag{4.3.50b}$$

$$F_{\theta,\Upsilon}(\Sigma_z)^{-1}\partial_z F_{\theta,\Upsilon}(\Sigma_z)=0, \tag{4.3.50c}$$

from which (4.3.46a)–(4.3.46c) follow. $\qquad\square$

The thin homotopy invariance of 1–parallel transport holds also if the latter is defined autonomously according to def. 105.

Theorem 14. *Let p_0,p_1 be points and $\gamma_y:p_0\to p_1$, $y\in\mathbb{R}$, be a smooth 1–parameter family of curves such that the mapping $h:\mathbb{R}^2\to M$ defined by $h(x,y)=\gamma_y(x)$ is a thin homotopy of γ_0,γ_1. Then,*

$$F_{\theta,\Upsilon}(\gamma_1)=F_{\theta,\Upsilon}(\gamma_0). \tag{4.3.51}$$

Proof. Under the assumptions made, the 1–parameter family of surfaces $I_{\gamma_z}:\gamma_z\Rightarrow\gamma_z$ is such that $H(x,y,z)=I_{\gamma_z}(x,y)=\gamma_z(x)$ is a thin homotopy of $I_{\gamma_0},I_{\gamma_1}$. The statement then follows from theor. 13 with $\gamma_{0z}=\gamma_{1z}=\gamma_z$ and $\Sigma_z=I_{\gamma_z}$. $\qquad\square$

The maps $\bar{F}_{\theta,\Upsilon}:\Pi_1 M\to G$, $\bar{F}_{\theta,\Upsilon}:\Pi_2 M\to H$ factor therefore through others $\bar{F}_{\theta,\Upsilon}:P_1 M\to G$, $\bar{F}_{\theta,\Upsilon}:P_2 M\to H$ from the path groupoid 1– and 2–cell sets $P_1 M$, $P_2 M$ into G, H, respectively, and, so, it induces a categorical map $\bar{F}_{\theta,\Upsilon}:(M,P_1 M,P_2 M)\to B_0(G,H)$

$$p_0 \underset{\gamma_1}{\overset{\gamma_0}{\Longrightarrow}}{}_{\Sigma\Downarrow} p_1 \longmapsto * \underset{\bar{F}_{\theta,\Upsilon}(\gamma_1)}{\overset{\bar{F}_{\theta,\Upsilon}(\gamma_0)}{\Longleftarrow}}{}_{\bar{F}_{\theta,\Upsilon}(\Sigma)\Downarrow} * \tag{4.3.52}$$

of the path 2–groupoid $(M,P_1 M,,P_2 M)$ into the delooping 2–groupoid $B_0(G,H)$ of the Lie crossed module (G,H). (cf. subsects. 4.2.1 and 4.3.1).

Proposition 43. *$\bar{F}_{\theta,\Upsilon}$ is a 2–groupoid 2–functor.*

Proof. The statement follows from combining props. 42, 13. Functoriality results from relations (4.3.35)–(4.3.42). $\square$

Definition 106. *The (G, H)–connection (θ, Υ) is said flat if*

$$d\Upsilon + [\theta, \Upsilon] = 0 \tag{4.3.53}$$

Theorem 15. *Let (θ, Υ) be flat. Let p_0, p_1 be points and $\gamma_{0z}, \gamma_{1z} : p_0 \to p_1$ and $\Sigma_z : \gamma_{0z} \Rightarrow \gamma_{1z}$, $z \in \mathbb{R}$ be 1–parameter families of curves and surfaces such that the mapping $H : \mathbb{R}^3 \to M$ defined by $H(x, y, z) = \Sigma_z(x, y)$ is a homotopy of Σ_0, Σ_1. Then, one has the identities*

$$F_{\theta,\Upsilon}(\gamma_{01}) = F_{\theta,\Upsilon}(\gamma_{00}), \tag{4.3.54a}$$

$$F_{\theta,\Upsilon}(\gamma_{11}) = F_{\theta,\Upsilon}(\gamma_{10}), \tag{4.3.54b}$$

$$F_{\theta,\Upsilon}(\Sigma_1) = F_{\theta,\Upsilon}(\Sigma_0). \tag{4.3.54c}$$

Proof. The proof is based on the variational formulae (4.3.47), (4.3.49) and follows the same lines as that of theor. 13 except for the vanishing of the double integral term in the right hand side of (4.3.49) which is now due to the flatness of (θ, Υ) instead of the thinness of H. $\square$

Theor. 14 of course keeps holding unchanged.

In this way, the maps $\bar{F}_{\theta,\Upsilon} : \Pi_1 M \to G$, $\bar{F}_{\theta,\Upsilon} : \Pi_2 M \to H$ factor through others $\bar{F}^0{}_{\theta,\Upsilon} : P_1 M \to G$, $\bar{F}^0{}_{\theta,\Upsilon} : P^0{}_2 M \to H$ from the fundamental groupoid 1– and 2–cell sets $P_1 M$, $P^0{}_2 M$ into G, H, respectively, yielding so a a categorical map $\bar{F}^0{}_{\theta,\Upsilon} : (M, P_1 M, P^0{}_2 M) \to B_0(G, H)$ of the fundamental 2–groupoid $(M, P_1 M, , P^0{}_2 M)$ into the delooping 2–groupoid $B_0(G, H)$.

Proposition 44. *When the connection doublet (θ, Υ) is flat, $\bar{F}^0{}_{\theta,\Upsilon} : (M, P_1 M, P^0{}_2 M) \to B_0(G, H)$ is a 2–groupoid 2–functor.*

Proof. The statement follows from combining prop. 42 and theor. 15 with functoriality resulting again from relations (4.3.35)–(4.3.42). $\square$

We now turn to the analysis of 1–gauge transformation of parallel transport.

4.3.3   2–parallel transport and 1–gauge transformation

In this subsection, we shall analyze 1–gauge transformation of higher parallel transport relying on the cocycle 1–gauge transformation set–up of sect. 4.3.

We begin by reviewing parallel transport in ordinary gauge theory. Let M be a manifold and G be a Lie group.

Definition 107. *A G–gauge transformation is a map $g \in \mathrm{Map}(M, G)$. We denote by $\mathrm{Gau}(M, G)$ the set of all gauge transformations.*

G–gauge transformations act on G–connections (cf. def. 100).

Definition 108. *Let a be a G–connection and g be a G–gauge transformation. The gauge transformed G–connection $^g\theta$ is*

$$^g\theta = \mathrm{Ad}\, g(a) - dgg^{-1}. \tag{4.3.55}$$

Proposition 45. *If θ is a flat G–connection, then, for any G–gauge transformation g, $^g\theta$ is also a flat G–connection (cf. def. 102).*

Proof. Indeed, using (4.3.55), one computes

$$d^g\theta + \frac{1}{2}[^g\theta, {}^g\theta] = \mathrm{Ad}\, g\Big(d\theta + \frac{1}{2}[\theta, \theta]\Big) = 0, \tag{4.3.56}$$

which shows the flatness of $^g\theta$. $\square$

The following theorem is a classic result.

Theorem 16. *Let θ be a G–connection and g be a G–gauge transformation. Let further p_0, p_1 be points and $\gamma : p_0 \to p_1$ be a curve. Then, the parallel transports $F_\theta(\gamma)$ and $F_{g\theta}(\gamma)$ along γ are related as*

$$F_{g\theta}(\gamma) = g(p_1)F_\theta(\gamma)g(p_0)^{-1}. \tag{4.3.57}$$

Proof. According to theor. 3, there exists a one–to–one correspondence between $\mathfrak{g}$–valued 1–forms a on $\mathbb{R}$ and G–cocycles f. By (4.2.51), (4.2.52), the action of a gauge transformation $\varkappa$ on a cocycle f is such that $a_{\varkappa f} = {}^\varkappa a_f$. Then,

$$^\varkappa a = a_{\varkappa f}\big|_{f=f_a}. \tag{4.3.58}$$

From this relation, it follows so that

$$f_{\varkappa a} = {}^\varkappa f_a. \tag{4.3.59}$$

Setting $a = \gamma^*\theta$ and $\varkappa = \gamma^* g$ in the above relation, we obtain

$$f_{\gamma^* g \gamma^*\theta} = {}^{\gamma^* g} f_{\gamma^*\theta}. \tag{4.3.60}$$

From here, noting that $\gamma^{*g}\theta = {}^{\gamma^* g}\gamma^*\theta$, we find

$$F_{g\theta}(\gamma) = f_{\gamma^{*g}\theta}(1,0) = f_{\gamma^* g \gamma^*\theta}(1,0) = {}^{\gamma^* g} f_{\gamma^*\theta}(1,0) \tag{4.3.61}$$

$$= \gamma^* g(1) f_{\gamma^*\theta}(1,0) \gamma^* g(0)^{-1} = g(p_1)F_\theta(\gamma)g(p_0)^{-1}$$

as was to be shown. $\square$

Recall that, for a G–connection θ, the mapping $F_\theta : \Pi_1 M \to G$ induces a groupoid functor $\bar{F}_\theta : (M, P_1 M) \to BG$ of the path groupoid $(M, P_1 M)$ of M in the delooping BG of G in virtue of its thin homotopy invariance (cf. prop. 38). Likewise, when the G–connection θ is flat, by its homotopy invariance, F_θ induces a groupoid functor $\bar{F}^0{}_\theta : (M, P^0{}_1 M) \to BG$ of the fundamental groupoid $(M, P^0{}_1 M)$ of M into BG (cf. prop. 39).

Proposition 46. *For any G–connection θ, a G–gauge transformation g encodes a natural transformation $\bar{F}_\theta \Rightarrow \bar{F}_{g\theta}$ of functors. If θ is flat, then g yields a natural transformation $\bar{F}^0{}_\theta \Rightarrow \bar{F}^0{}_{g\theta}$.*

Proof. By (4.3.57), the diagram

$$
\begin{array}{ccc}
* & \xleftarrow{\ F_\theta(\gamma)\ } & * \\
{\scriptstyle g(p_1)}\Big\downarrow & & \Big\downarrow{\scriptstyle g(p_0)} \\
* & \xleftarrow{\ F_{g\theta}(\gamma)\ } & *
\end{array}
\qquad (4.3.62)
$$

commutes, identifying g as a natural transformation $\bar{F}_\theta \Rightarrow \bar{F}_{g\theta}$ or $\bar{F}^0{}_\theta \Rightarrow \bar{F}^0{}_{g\theta}$. $\qquad\square$

We now shift to higher gauge theory, introduce the notion of 1–gauge transformation and study its action on connection doublets and 2–parallel transport.

Let M be a manifold and (G, H) be a Lie crossed module.

Definition 109. *A differential (G, H)–1–gauge transformation is a pair of a map $g \in \mathrm{Map}(M, G)$ and a form $J \in \Omega^1(M, \mathfrak{h})$. We denote by $\mathrm{Gau}_1(M, G, H)$ the set of all 1–gauge transformations.*

Differential (G, H)–1–gauge transformations act on (G, H)–connections doublets (cf. def. 103).

Definition 110. *Let (θ, Υ) be a (G, H)–connection doublet and (g, J) be a (G, H)–1–gauge transformation. The gauge transformed (G, H)–connection doublet $({}^{g,J}\theta, {}^{g,J}\Upsilon)$ is*

$$
{}^{g,J}\theta = \mathrm{Ad}\, g(\theta) - dg g^{-1} - \dot{t}(J), \tag{4.3.63a}
$$

$$
{}^{g,J}\Upsilon = \dot{m}(g)(\Upsilon) - dJ - \frac{1}{2}[J, J] - \widehat{m}(\mathrm{Ad}\, g(\theta) - dg g^{-1} - \dot{t}(J), J). \tag{4.3.63b}
$$

It can be checked that this gauge transformation is compatible with the zero fake curvature condition (4.3.29).

Proposition 47. *If (θ, Υ) is a flat (G, H)–connection doublet, then, for any (G, H)–1–gauge transformation (g, J), $({}^{g,J}\theta, {}^{g,J}\Upsilon)$ is also a flat (G, H)–connection doublet (cf. def. 106).*

Proof. Indeed, using (4.3.63), taking (4.3.29) into account, one finds

$$
d\,{}^{g,J}\Upsilon + [{}^{g,J}\theta, {}^{g,J}\Upsilon] = \dot{m}(g)(d\Upsilon + [\theta, \Upsilon]) = 0, \tag{4.3.64}
$$

which shows the flatness of $({}^{g,J}\theta, {}^{g,J}J)$. $\qquad\square$

Recall that, by theor. 4, with a (G, H)–connection doublet (a, B) in the sense of

def. 86 there is associated a (G, H)–cocycle $(f_{a,B}, g_{a,B}, W_{a,B})$. Further, by theor. 6, with a differential (G, H)–1–gauge transformation $(\varkappa, \Gamma)$ in the sense of def. 90, there is associated an $(f_{a,B}, g_{a,B}, W_{a,B})$–1–gauge transformation $(\kappa_{\varkappa,\Gamma;a,B}, \Psi_{\varkappa,\Gamma;a,B}, \Phi_{\varkappa,\Gamma;a,B})$. This depends not only on the differential transformation $(\varkappa, \Gamma)$ but also on the connection doublet (a, B), when this latter is allowed to vary. The following basic result extends theor. 16 to higher gauge theory in a non trivial manner.

Theorem 17. *Let (θ, Υ) be a (G, H)–connection doublet and (g, J) be a (G, H)–1–gauge transformation. Let further p_0, p_1 be points, $\gamma_0, \gamma_1 : p_0 \to p_1$ be curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ be a surface. Then, we have*

$$F_{g,J\theta,g,J\Upsilon}(\gamma_0) = g(p_1)t(G_{g,J;\theta,\Upsilon}(\gamma_0))F_{\theta,\Upsilon}(\gamma_0)g(p_0)^{-1}, \tag{4.3.65a}$$

$$F_{g,J\theta,g,J\Upsilon}(\gamma_1) = g(p_1)t(G_{g,J;\theta,\Upsilon}(\gamma_1))F_{\theta,\Upsilon}(\gamma_1)g(p_0)^{-1}, \tag{4.3.65b}$$

$$F_{g,J\theta,g,J\Upsilon}(\Sigma) = m(g(p_1))\big(G_{g,J;\theta,\Upsilon}(\gamma_1)F_{\theta,\Upsilon}(\Sigma)G_{g,J;\theta,\Upsilon}(\gamma_0)^{-1}\big), \tag{4.3.65c}$$

where $G_{g,J;\theta,\Upsilon}(\gamma_0)$, $G_{g,J;\theta,\Upsilon}(\gamma_1)$ are given by (g, J) are

$$G_{g,J;\theta,\Upsilon}(\gamma_0) = \Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|0}(0, 1), \tag{4.3.66a}$$

$$G_{g,J;\theta,\Upsilon}(\gamma_1) = \Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|1}(0, 1). \tag{4.3.66b}$$

Proof.. By (4.2.66), (4.2.67), the one–to–one correspondence between (G, H)–cocycles (f, g, W) and (G, H) connections (a, B) (in the sense of def. 103) on one hand and integral (f, g, W)–1–gauge transformations (κ, Ψ, Φ) and differential (G, H)–1–gauge transformations (in the sense of def. 90) on the other is such that $a_{\kappa,\Psi,\Phi f,\kappa,\Psi,\Phi g,\kappa,\Psi,\Phi W} = {}^{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}}a_{f,g,W}$, $B_{\kappa,\Psi,\Phi f,\kappa,\Psi,\Phi g,\kappa,\Psi,\Phi W} = {}^{\varkappa_{\kappa,\Psi,\Phi},\Gamma_{\kappa,\Psi,\Phi}}B_{f,g,W}$. Using these results, it is readily checked that

$${}^{\varkappa,\Gamma}a = a_{\kappa,\Psi,\Phi f,\kappa,\Psi,\Phi g,\kappa,\Psi,\Phi W} \tag{4.3.67a}$$

$$\Big|_{\kappa=\kappa_{\varkappa,\Gamma;a,B},\Psi=\Psi_{\varkappa,\Gamma;a,B},\Phi=\Phi_{\varkappa,\Gamma;a,B};f=f_{a,B},g=g_{a,B},W=W_{a,B}},$$

$${}^{\varkappa,\Gamma}B = B_{\kappa,\Psi,\Phi f,\kappa,\Psi,\Phi g,\kappa,\Psi,\Phi W} \tag{4.3.67b}$$

$$\Big|_{\kappa=\kappa_{\varkappa,\Gamma;a,B},\Psi=\Psi_{\varkappa,\Gamma;a,B},\Phi=\Phi_{\varkappa,\Gamma;a,B};f=f_{a,B},g=g_{a,B},W=W_{a,B}}.$$

From these relation, it follows immediately that

$$f_{\varkappa,\Gamma a,\varkappa,\Gamma B} = {}^{\kappa_{\varkappa,\Gamma;a,B},\Psi_{\varkappa,\Gamma;a,B},\Phi_{\varkappa,\Gamma;a,B}}f_{a,B}, \tag{4.3.68a}$$

$$g_{\varkappa,\Gamma a,\varkappa,\Gamma B} = {}^{\kappa_{\varkappa,\Gamma;a,B},\Psi_{\varkappa,\Gamma;a,B},\Phi_{\varkappa,\Gamma;a,B}}g_{a,B}, \tag{4.3.68b}$$

$$W_{\varkappa,\Gamma a,\varkappa,\Gamma B} = {}^{\kappa_{\varkappa,\Gamma;a,B},\Psi_{\varkappa,\Gamma;a,B},\Phi_{\varkappa,\Gamma;a,B}}W_{a,B}. \tag{4.3.68c}$$

Setting $a = \Sigma^*\theta$, $B = \Sigma^*\Upsilon$ and $\varkappa = \Sigma^*g$, $\Gamma = \Sigma^*J$ in the (4.3.68), we obtain

$$f_{\Sigma^*g,\Sigma^*J\,\Sigma^*\theta,\Sigma^*g,\Sigma^*J\,\Sigma^*\Upsilon} \tag{4.3.69a}$$

$$= {}^{\kappa_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon}} f_{\Sigma^*\theta,\Sigma^*\Upsilon},$$

$$g_{\Sigma^*g,\Sigma^*J\,\Sigma^*\theta,\Sigma^*g,\Sigma^*J\,\Sigma^*\Upsilon} \tag{4.3.69b}$$

$$= {}^{\kappa_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon}} g_{\Sigma^*\theta,\Sigma^*\Upsilon},$$

$$W_{\Sigma^*g,\Sigma^*J\,\Sigma^*\theta,\Sigma^*g,\Sigma^*J\,\Sigma^*\Upsilon} \tag{4.3.69c}$$

$$= {}^{\kappa_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon}} W_{\Sigma^*\theta,\Sigma^*\Upsilon}.$$

We can now complete the proof of the theorem. We show relation (4.3.65c) only, the proof of (4.3.65a), (4.3.65b) being analogous. We showed earlier that $g_{\Sigma^*\theta,\Sigma^*\Upsilon|x}(y',y) = 1_G$ for $x < \epsilon$ or $x > 1 - \epsilon$ and arbitrary y,y' (cf. eq. (4.3.32)). Similarly, we can show that $\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|x}(y',y) = 1_H$ for the same range of x and y,y' values, by considering the differential problem (4.2.63b), (4.2.64b) with $\varkappa$, Γ replaced by Σ^*g, Σ^*J and observing that $\Sigma^*\Gamma_y(x,y) = 0$ identically for the values of x indicated on account of (4.3.5a), (4.3.5b). Then, from (4.3.69c), using (4.2.55c) and noting that by (4.3.63) $\Sigma^{*g,J}\theta = {}^{\Sigma^*g,\Sigma^*J}\Sigma^*\theta$, $\Sigma^{*g,J}\Upsilon = {}^{\Sigma^*g,\Sigma^*J}\Sigma^*\Upsilon$, we find

$$F_{g,J\,\theta,g,J\Upsilon}(\Sigma) = W_{\Sigma^*g,J\theta,\Sigma^*g,J\Upsilon}(0,1;1,0) \tag{4.3.70}$$

$$= W_{\Sigma^*g,\Sigma^*J\,\Sigma^*\theta,\Sigma^*g,\Sigma^*J\,\Sigma^*\Upsilon}(0,1;1,0)$$

$$= {}^{\kappa_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon},\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon}} W_{\Sigma^*\theta,\Sigma^*\Upsilon}(0,1;1,0)$$

$$= m(\kappa_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon}(1;0))\big(\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|1}(1,0)$$

$$\times m(g_{\Sigma^*\theta,\Sigma^*\Upsilon|1}(1,0)^{-1})(\Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|1}(0,1))$$

$$\times W_{\Sigma^*\theta,\Sigma^*\Upsilon}(0,1;1,0)$$

$$\times m(f_{\Sigma^*\theta,\Sigma^*\Upsilon|0}(0,1)^{-1})(\Phi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|0}(1,0)^{-1})$$

$$\times \Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|0}(0,1)^{-1}\big)$$

$$= m(g(p_1))\big(G_{g,J;\theta,\Upsilon}(\gamma_1)F_{\theta,\Upsilon}(\Sigma)G_{g,J;\theta,\Upsilon}(\gamma_0)^{-1}\big),$$

showing (4.3.65c). □

In theor. 17, a new object appears, $G_{g,J;\theta,\Upsilon}(\gamma)$. As it turns out, it has a number o relevant properties which are the topic of the rest of this subsection.

$G_{g,J;\theta,\Upsilon}(\gamma)$ can be defined for any curve γ independently from any other curve γ' with the same endpoints and surface Σ connecting γ to γ'.

Lemma 2. *Suppose that p_0, p_1 are points and $\gamma : p_0 \to p_1$ is a curve. Then, $\Psi_{I_\gamma^*g,I_\gamma^*J;I_\gamma^*\theta,I_\gamma^*\Upsilon|y}$, where $I_\gamma : \gamma \Rightarrow \gamma$ is the unit surface of γ (cf. eq. (4.3.9)), is independent from y.*

Proof. By theor. 6, $\Psi_{I_\gamma^*g,I_\gamma^*J;I_\gamma^*\theta,I_\gamma^*\Upsilon|y}$ is the solution of the differential problem (4.2.63a), (4.2.64a) with $f(x,x_0;y) = f_{I_\gamma^*\theta,I_\gamma^*\Upsilon}(x,x_0;y)$, $\varkappa(x,y) = I_\gamma^*g(x,y)$ and $\Gamma_x(x,y) = I_\gamma^*J_x(x,y)$. Now, by lemma 1, $f_{I_\gamma^*\theta,I_\gamma^*\Upsilon}(x,x_0;y)$ is independent from y. Further, $I_\gamma^*g(x,y) = \gamma^*g(x)$, $I_\gamma^*J_x(x,y) = \gamma^*J_x(x)$ are also independent from y. So, $\Psi_{I_\gamma^*g,I_\gamma^*J;I_\gamma^*\theta,I_\gamma^*\Upsilon|y}$ is y independent.

Definition 111. *If p_0,p_1 are points and $\gamma : p_0 \to p_1$ is a curve, one sets*

$$G_{g,J;\theta,\Upsilon}(\gamma) = \Psi_{I_\gamma^*g,I_\gamma^*J;I_\gamma^*\theta,I_\gamma^*\Upsilon|y}(0,1). \tag{4.3.71}$$

(4.3.71) gives the same result as (4.3.66a), (4.3.66b).

Proposition 48. *If p_0,p_1 are points, $\gamma_0,\gamma_1 : p_0 \to p_1$ are curves and $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a surface, then then the value of $G_{g,J;\theta,\Upsilon}(\gamma_i)$ computed using (4.3.66a), (4.3.66b) equals that obtained using (4.3.71).*

Proof. By theor. 4, $\Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|y}(x',x)$ is the solution of the differential problem (4.2.63a), (4.2.64a) with $f(x,x_0;y) = f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(x,x_0)$, $\varkappa(x,y) = \Sigma^*g(x,y)$, $\Gamma_x(x,y) = \Sigma^*J_x(x,y)$. Likewise, $\Psi_{I_{\gamma_i}^*g,I_{\gamma_i}^*J;I_{\gamma_i}^*\theta,I_{\gamma_i}^*\Upsilon|y}(x',x)$ solves the differential problem (4.2.63a), (4.2.64a) with $f(x,x_0;y) = f_{I_{\gamma_i}^*\theta,I_{\gamma_i}^*\Upsilon|y}(x,x_0)$, $\varkappa(x,y) = I_{\gamma_i}^*g(x,y)$, $\Gamma_x(x,y) = I_{\gamma_i}^*J_x(x,y)$. Now, we have $f_{I_{\gamma_i}^*\theta,I_{\gamma_i}^*\Upsilon|y}(x,x_0) = f_{\Sigma^*\theta,\Sigma^*\Upsilon|i}(x,x_0)$ (see the proof of prop. 41) and also $I_{\gamma_i}^*g(x,y) = \Sigma^*g(x,i)$, $I_{\gamma_i}^*J_x(x,y) = \Sigma^*J_x(x,i)$ for $i = 1,2$ and any y. So, $\Psi_{I_{\gamma_i}^*g,I_{\gamma_i}^*J;I_{\gamma_i}^*\theta,I_{\gamma_i}^*\Upsilon|y}(x,x_0) = \Psi_{\Sigma^*g,\Sigma^*J;\Sigma^*\theta,\Sigma^*\Upsilon|i}(x,x_0)$. From this relation, recalling (4.3.66a), (4.3.66b) and (4.3.71), the statement follows. □

Let us fix a (G,H)–connection doublet (θ,Υ) and a (G,H)–1–gauge transformation (g,J). We have then a mapping $G_{g;J;\theta,\Upsilon} : \Pi_1 M \to H$.

Proposition 49. *For any two points p_0,p_1 and curve $\gamma : p_0 \to p_1$, one has*

$$F_{g,J\theta,g,J\Upsilon}(\gamma) = g(p_1)t(G_{g,J;\theta,\Upsilon}(\gamma))F_{\theta,\Upsilon}(\gamma)g(p)^{-1}, \tag{4.3.72}$$

Proof. This follows from (4.3.65a), (4.3.65a), setting $\Sigma = I_\gamma$ and using (4.3.34) and (4.3.71). □

Proposition 50. *For any point p, one has*

$$G_{g,J;\theta,\Upsilon}(\iota_p) = 1_H. \tag{4.3.73}$$

For any two points p_0,p_1 and curve $\gamma : p_0 \to p_1$, one has

$$G_{g,J;\theta,\Upsilon}(\gamma^{-1\circ}) = m(F_{\theta,\Upsilon}(\gamma)^{-1})(G_{g,J;\theta,\Upsilon}(\gamma)^{-1}). \tag{4.3.74}$$

For any three p_0,p_1,p_2 and two curves $\gamma_1 : p_0 \to p_1$, $\gamma_2 : p_1 \to p_2$,

$$G_{g,J;\theta,\Upsilon}(\gamma_2 \circ \gamma_1) = G_{g,J;\theta,\Upsilon}(\gamma_2)m(F_{\theta,\Upsilon}(\gamma_2))(G_{g,J;\theta,\Upsilon}(\gamma_1)). \tag{4.3.75}$$

Proof. The proof is analogous to that of prop. 42, relying on the pull–back action of the map $l_\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $l_\phi(x,y) = (\phi(x),y)$, induced by a function $\phi : \mathbb{R} \to$

$\mathbb{R}$. The left pull–back $(l_\phi{}^*f, l_\phi{}^*g, l_\phi{}^*W)$ of a (G, H)–cocycle (f, g, W) is the (G, H)–cocycle defined by eqs. (4.3.43). The left pull–back $(l_\phi{}^*\kappa, l_\phi{}^*\Psi, l_\phi{}^*\Phi)$ of an (f, g, W)–1–gauge transformation (κ, Ψ, Φ) is the $(l_\phi{}^*f, l_\phi{}^*g, l_\phi{}^*W)$–gauge transformation given by

$$l_\phi{}^*\kappa(x; y) = \kappa(\phi(x); y), \tag{4.3.76a}$$

$$l_\phi{}^*\Psi(x', x; y) = \Psi(\phi(x'), \phi(x); y), \tag{4.3.76b}$$

$$l_\phi{}^*\Phi(x; y', y) = \Phi(\phi(x); y', y). \tag{4.3.76c}$$

The verification of the validity of the cocycle relations (4.2.53) is straightforward.

The one–to–one correspondence between form pairs $(a, B) \in \Omega^1(\mathbb{R}^2, \mathfrak{g}) \times \Omega^2(\mathbb{R}^2, \mathfrak{h})$ and (G, H)–cocycles $(f, g, W) \in \mathrm{Cyc}(G, H)$ established by theor. 4 is natural with respect to left pull-back. Likewise, the one–to–one correspondence between pairs $(\varkappa, \Gamma) \in \mathrm{Map}(M, G) \times \Omega^1(\mathbb{R}^2, \mathfrak{h})$ and $(f_{a,B}, g_{a,B}, W_{a,B})$–gauge transformation is natural, meaning that the relations

$$(\kappa_{l_\phi{}^*\varkappa, l_\phi{}^*\Gamma; l_\phi{}^*a, l_\phi{}^*B}, \Psi_{l_\phi{}^*\varkappa, l_\phi{}^*\Gamma; l_\phi{}^*a, l_\phi{}^*B}, \Phi_{l_\phi{}^*\varkappa, l_\phi{}^*\Gamma; l_\phi{}^*a, l_\phi{}^*B}) = (l_\phi{}^*\kappa_{\varkappa, \Gamma; a, B}, l_\phi{}^*\Psi_{\varkappa, \Gamma; a, B}, l_\phi{}^*\Phi_{\varkappa, \Gamma; a, B})$$

as well as

$$(\varkappa_{l_\phi{}^*\kappa, l_\phi{}^*\Psi, l_\phi{}^*\Phi}, \Gamma_{l_\phi{}^*\kappa, l_\phi{}^*\Psi, l_\phi{}^*\Phi}) = (l_\phi{}^*\varkappa_{\kappa, \Psi, \Phi}, l_\phi{}^*\Gamma_{\kappa, \Psi, \Phi})$$

hold.

Given these results, the proof of relations (4.3.73), (4.3.74), (4.3.75) is totally analogous to that of (4.3.38), (4.3.41), (4.3.42). For instance, the verification of (4.3.75) proceeds along the same lines as that of (4.3.42) as indicated in (4.3.45): replace Σ_i by I_{γ_i} and $W_{\Sigma_i{}^*\theta, \Sigma_i{}^*\Upsilon}$ by $\Psi_{I_{\gamma_i}{}^*g, I_{\gamma_i}{}^*J; I_{\gamma_i}{}^*\theta, I_{\gamma_i}{}^*\Upsilon}$ and use (4.2.53a). $\qquad \square$

Naturally, thin homotopy invariance holds for gauge transformation along a curve.

Theorem 18. *Let p_0, p_1 be points and $\gamma_y : p_0 \to p_1$, $y \in \mathbb{R}$, be a smooth 1–parameter family of curves such that the mapping $h : \mathbb{R}^2 \to M$ defined by $h(x, y) = \gamma_y(x)$ is a thin homotopy of γ_0, γ_1. Then,*

$$G_{g, J; \theta, \Upsilon}(\gamma_1) = G_{g, J; \theta, \Upsilon}(\gamma_0). \tag{4.3.77}$$

Proof. The proof is based on the variational formula

$$\partial_z \Psi_{I_{\gamma_z}{}^*g, I_{\gamma_z}{}^*J; I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(x, x_0) \Psi_{I_{\gamma_z}{}^*g, I_{\gamma_z}{}^*J; I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(x, x_0)^{-1} \tag{4.3.78}$$

$$= -\int_{x_0}^x d\xi \, \Psi_{I_{\gamma_z}{}^*g, I_{\gamma_z}{}^*J; I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(\xi, x_0) \dot m(f_{I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(\xi, x_0)^{-1})$$

$$\left(H^*(\dot m(g^{-1})(dJ + [J, J]/2 + \widehat m(\mathrm{Ad}\, g(\theta) - dgg^{-1} - \dot t(J), J)))_{zx}(\xi, y, z) \right.$$

$$+ \widehat m \Big(\int_{x_0}^\xi d\xi_0 \, f_{I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(\xi, \xi_0) \dot t(H^*\Upsilon_{zx}(\xi_0, y, z)) f_{I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(\xi, \xi_0)^{-1}$$

$$- f_{I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(\xi, x_0) H^*\theta_z(x_0, y, z)) f_{I_{\gamma_z}{}^*\theta, I_{\gamma_z}{}^*\Upsilon|y}(\xi, x_0)^{-1},$$

$$H^*(\dot{m}(g^{-1})(J))_x(\xi, y, z)\Big)\Big)\Psi_{I_{\gamma_z}*g, I_{\gamma_z}*J; I_{\gamma_z}*\theta, I_{\gamma_z}*\Upsilon|y}(\xi, x_0)^{-1}$$

$$- \Psi_{I_{\gamma_z}*g, I_{\gamma_z}*J; I_{\gamma_z}*\theta, I_{\gamma_z}*\Upsilon|y}(x, x_0)\dot{m}(f_{I_{\gamma_z}*\theta, I_{\gamma_z}*\Upsilon|y}(x, x_0)^{-1})$$

$$(H^*(\dot{m}(g^{-1})(J))_z(x, y, z))\Psi_{I_{\gamma_z}*g, I_{\gamma_z}*J; I_{\gamma_z}*\theta, I_{\gamma_z}*\Upsilon|y}(x, x_0)^{-1}$$

$$+ H^*(\dot{m}(g^{-1})(J))_z(x_0, y, z),$$

where $H : \mathbb{R}^3 \to M$ is the mapping defined by $H(x, y, z) = I_{\gamma_z}(x, y) = \gamma_z(x)$. Under the assumptions made, the 1–parameter family of surfaces $I_{\gamma_z} : \gamma_z \Rightarrow \gamma_z$ is such that H is a thin homotopy of I_{γ_0}, I_{γ_1} with the property that $\text{rank}(dH(x, y, z)) \leq 1$. So, $H^*(dJ + [J, J]/2 + \widehat{m}(\text{Ad}\, g(\theta) - dgg^{-1} - \dot{t}(J), J)) = 0$ and $H^*\Upsilon_{zx}(x, y, z) = 0$. Further, $H^*\theta_z(i, y, z) = 0$ and $H^*J_z(i, y, z) = 0$ for $i = 0, 1$, by (4.3.14a), (4.3.14b). So, by (4.3.71) and (4.3.78), we have

$$\partial_z G_{g,J;\theta,\Upsilon}(\gamma_z)G_{g,J;\theta,\Upsilon}(\gamma_z)^{-1} = 0 \tag{4.3.79}$$

from which (4.3.77) follows immediately. □

Recall that, for a (G, H)–connection doublet (θ, Υ), the mappings $\bar{F}_{\theta,\Upsilon} : \Pi_1 M \to G$, $\bar{F}_{\theta,\Upsilon} : \Pi_2 M \to H$ induce a 2–groupoid functor $\bar{F}_{\theta,\Upsilon} : (M, P_1 M, P_2 M) \to B_0(G, H)$ of the path 2–groupoid $(M, P_1 M, P_2 M)$ of M into the delooping 2–groupoid $B_0(G, H)$ of the Lie crossed module (G, H) by their thin homotopy invariance (cf. prop. 43). Furthermore, when the (G, H)–connection doublet (θ, Υ) is flat, the $\bar{F}_{\theta,\Upsilon}$ induce a 2–groupoid functor $\bar{F}^0_{\theta,\Upsilon} : (M, P_1 M, P^0_2 M) \to B_0(G, H)$ of the fundamental 2–groupoid $(M, P_1 M, P^0_2 M)$ of M into $B_0(G, H)$ by their homotopy invariance (cf. prop. 44). By what found above, the map $G_{g,J;\theta,\Upsilon} : \Pi_1 M \to H$ factors through one $\bar{G}_{g,J;\theta,\Upsilon} : P_1 M \to H$ from the path groupoid 1–cell set $P_1 M$ into H.

Proposition 51. *For any (G, H)–connection doublet (θ, Υ), a (G, H)–1–gauge transformation (g, J) encodes a pseudonatural transformation $\bar{G}_{g,J;\theta,\Upsilon} : \bar{F}_{\theta,\Upsilon} \Rightarrow \bar{F}_{g, J\theta, g, J\Upsilon}$ of 2–functors. If (θ, Υ) is flat, then (g, J) yields a pseudonatural transformation $\bar{G}^0_{g,J;\theta,\Upsilon} : \bar{F}^0_{\theta,\Upsilon} \Rightarrow \bar{F}^0_{g, J\theta, g, J\Upsilon}$.*

Proof. By (4.3.72), for any curve $\gamma : p_0 \to p_1$ we have a 2–cell of $B_0(G, H)$

$$\tag{4.3.80}$$

where $\tilde{G}_{g,J;\theta,\Upsilon}(\gamma)$ is given by

$$\tilde{G}_{g,J;\theta,\Upsilon} = m(g(p_1))(G_{g,J;\theta,\Upsilon}) \tag{4.3.81}$$

The 2–cells (4.3.80) define a pseudonatural transformation $\bar{F}_{\theta,\Upsilon} \Rightarrow \bar{F}_{g, J\theta, g, J\Upsilon}$ if

$$\tag{4.3.82}$$

for any pair of curves $\gamma_1 : p_0 \to p_1$, $\gamma_2 : p_1 \to p_2$ and

$$
\begin{array}{c}
\\
\end{array}
\tag{4.3.83}
$$

for any surface $\Sigma : \gamma_0 \to \gamma_1$ hold, where the diagrams are composed by the usual pasting algorithm. These conditions are in fact satisfied. (4.3.82) holds as a consequence of (4.3.75). (4.3.83) follows from relation (4.3.65c). The first part of the proposition follows. The proof of the second half is essentially identical. □

4.3.4   2–parallel transport and 2–gauge transformation

In this subsection, we shall study 2–gauge transformation in higher parallel transport theory. This has no analogue in ordinary gauge theory.

Let M be a manifold and (G, H) be a Lie crossed module.

Definition 112. *A (G, H)–2–gauge transformation is a mapping $\tilde{\Omega} \in \mathrm{Map}(M, H)$. We denote by $\mathrm{Gau}_2(M, G, H)$ the set of all 2–gauge transformations.*

(G, H)–2–gauge transformations act on (G, H)–2–gauge transformations, the action depending on an assigned (G, H)–connection doublet.

Definition 113. *Let (θ, Υ) be a (G, H)–connection doublet, (g, J) be a (G, H)–1–gauge transformation and $\tilde{\Omega}$ a (G, H)–2–gauge transformation. The 2–gauge transformed 1–gauge transformation $(^{\tilde{\Omega}}g_{|\theta,\Upsilon}, {}^{\tilde{\Omega}}J_{|\theta,\Upsilon})$ is*

$$
{}^{\tilde{\Omega}}g_{|\theta,\Upsilon} = t(\tilde{\Omega})g, \tag{4.3.84a}
$$

$$
{}^{\tilde{\Omega}}J_{|\theta,\Upsilon} = \tilde{\Omega}J\tilde{\Omega}^{-1} - d\tilde{\Omega}\tilde{\Omega}^{-1} - Q(^{g,J}\theta, \tilde{\Omega}). \tag{4.3.84b}
$$

where $^{g,J}\theta$ is given by (4.3.63a) and $\tilde{\Omega}$ is defined by

$$
\tilde{\Omega} = m(g)(\Omega). \tag{4.3.85}
$$

2–gauge equivalent 1–gauge transformations yield the the same gauge transformed connection doublet.

Proposition 52. *Let (θ, Υ) be a (G, H)–connection doublet, (g, J) be a (G, H)–1–gauge transformation and Ω be a (G, H)–2–gauge transformation. Then,*

$$
{}^{\tilde{\Omega}}g_{|\theta,\Upsilon}, {}^{\tilde{\Omega}}J_{|\theta,\Upsilon}\theta = {}^{g,J}\theta, \tag{4.3.86a}
$$

$$
{}^{\tilde{\Omega}}g_{|\theta,\Upsilon}, {}^{\tilde{\Omega}}J_{|\theta,\Upsilon}\Upsilon = {}^{g,J}\Upsilon. \tag{4.3.86b}
$$

Proof. This is straightforwardly verified evaluating (4.3.63a), (4.3.63b) for the 1–gauge transformation $({}^{\tilde{\Omega}}g_{|\theta,\Upsilon}, {}^{\tilde{\Omega}}J_{|\theta,\Upsilon})$ and using the zero fake curvature condition (4.3.29). $\square$

The action of 2–gauge transformations on 1–gauge transformations translates into one on the map $G_{g,J;\theta,\Upsilon} : \Pi_1 M \to H$.

Proposition 53. *Let (θ, Υ) be a (G,H)–connection doublet, (g,J) be a (G,H)–1–gauge transformation and Ω be a (G,H)–2–gauge transformation. Then, for any curve $\gamma : p_0 \to p_1$, one has*

$$G_{\tilde{\Omega}g_{|\theta,\Upsilon}, \tilde{\Omega}J_{|\theta,\Upsilon};\theta,\Upsilon}(\gamma) = \Omega(p_1)^{-1} G_{g,J;\theta,\Upsilon}(\gamma) m(F_{\theta,\Upsilon}(\gamma))(\Omega(p_0)) \tag{4.3.87}$$

where Ω is related to $\tilde{\Omega}$ by

$$\Omega = m(g^{-1})(\tilde{\Omega}). \tag{4.3.88}$$

Proof. In the course of the proof of prop. 36, it was found that $(\varkappa_{A_\kappa, {}^A\Psi, {}^A\Phi}, \Gamma_{A_\kappa, {}^A\Psi, {}^A\Phi}) = ({}^{\tilde{A}}\varkappa_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}}, {}^{\tilde{A}}\Gamma_{\kappa,\Psi,\Phi|a_{f,g,W}B_{f,g,W}})$ for any (G,H)–cocycle (f, g, W), (f, g, W)–1–gauge transformation (κ, Ψ, Φ) and (G,H)–2–gauge transformation A, where A and $\tilde{A}$ are related by (4.2.74). Setting $(f, g, W) = (f_{a,B}, g_{a,B}, W_{a,B})$ and $(\kappa, \Psi, \Phi) = (\kappa_{\varkappa,\Gamma,a,B}, \Psi_{\varkappa,\Gamma,a,B}, \Phi_{\varkappa,\Gamma,a,B},))$ in this relation, where (a, B) and $(\varkappa, \Gamma)$ are a (G,H)–connection doublet and a differential (G,H)–1–gauge transformation in the sense of defs. 90 and 90, respectively, we find that

$$\left(\kappa_{\tilde{A}\varkappa_{|a,B}, \tilde{A}\Gamma_{|a,B};a,B}, \Psi_{\tilde{A}\varkappa_{|a,B}, \tilde{A}\Gamma_{|a,B};a,B}, \Phi_{\tilde{A}\varkappa_{|a,B}, \tilde{A}\Gamma_{|a,B};a,B}\right) \tag{4.3.89}$$

$$= ({}^A\kappa_{\varkappa,\Gamma;a,B}, {}^A\Psi_{\varkappa,\Gamma;a,B}, {}^A\Phi_{\varkappa,\Gamma;a,B})$$

Using the mid component of (4.3.89) and the cocycle relation (4.2.68b) and the definitions (4.3.34) and (4.3.66), we find

$$G_{\tilde{\Omega}g_{|\theta,\Upsilon}, \tilde{\Omega}J_{|\theta,\Upsilon};\theta,\Upsilon}(\gamma) = \Psi_{I_\gamma{}^*\tilde{\Omega}g_{|\theta,\Upsilon}, I_\gamma{}^*\tilde{\Omega}J_{|\theta,\Upsilon};I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon|y}(0,1) \tag{4.3.90}$$

$$= \Psi_{I_\gamma{}^*\tilde{\Omega}I_\gamma{}^*g_{|I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon}, I_\gamma{}^*\tilde{\Omega}I_\gamma{}^*J_{|I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon};I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon|y}(0,1)$$

$$= {}^{I_\gamma{}^*\Omega}\Psi_{I_\gamma{}^*g, I_\gamma{}^*J;I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon|y}(0,1)$$

$$= I_\gamma{}^*\Omega_{|y}(1)^{-1}\Psi_{I_\gamma{}^*g, I_\gamma{}^*J;I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon|y}(0,1)$$

$$\times m(f_{I_\gamma{}^*\theta, I_\gamma{}^*\Upsilon|y}(0,1)^{-1})(I_\gamma{}^*\Omega_{|y}(0))$$

$$= \Omega(p_1)^{-1} G_{g,J;\theta,\Upsilon}(\gamma) m(F_{\theta,\Upsilon}(\gamma))(\Omega(p_0)).$$

(4.3.87) is so proven $\square$

Recall that, for a (G,H)–connection doublet (θ, Υ) and a (G,H)–1–gauge transformation (g, J), the map $G_{g,J;\theta,\Upsilon} : \Pi_1 M \to H$ furnishes the data of a pseudonatural transformation $\bar{G}_{g,J;\theta,\Upsilon} : \bar{F}_{\theta,\Upsilon} \Rightarrow \bar{F}_{g,J\theta,g,J\Upsilon}$ of the parallel transport 2-functor $\bar{F}_{\theta,\Upsilon}$ of (θ, Υ) to that $\bar{F}_{g,J\theta,g,J\Upsilon}$ of $({}^{g,J}\theta, {}^{g,J}\Upsilon)$ and likewise one $\bar{G}^0{}_{g,J;\theta,\Upsilon} : \bar{F}^0{}_{\theta,\Upsilon} \Rightarrow \bar{F}^0{}_{g,J\theta,g,J\Upsilon}$ when (θ, Υ) is flat (cf. prop. 51).

Proposition 54. *For every (G,H)–connection doublet (θ,Υ) and (G,H)–1– gauge transformation (g,J), a (G,H)–2–gauge transformation $\tilde{\Omega}$ encodes a modification $\bar{H}_{g,J;\theta,\Upsilon;\tilde{\Omega}} : \bar{G}_{g,J;\theta,\Upsilon} \Rightarrow \bar{G}_{\tilde{\Omega}_{g_{|\theta,\Upsilon}},\tilde{\Omega}_{J_{|\theta,\Upsilon}};\theta,\Upsilon}$ of pseudonatural transformations. If (θ,Υ) is flat, then $\tilde{\Omega}$ yields a pseudonatural transformation modification $\bar{H}^0{}_{g,J;\theta,\Upsilon;\tilde{\Omega}} : \bar{G}^0{}_{g,J;\theta,\Upsilon} \Rightarrow \bar{G}^0{}_{\tilde{\Omega}_{g_{|\theta,\Upsilon}},\tilde{\Omega}_{J_{|\theta,\Upsilon}};\theta,\Upsilon}$.*

Proof. By (4.3.84a), for any point p we have a 2–cell of $B_0(G,H)$,

$$\tag{4.3.91}$$

$\tilde{\Omega}$ defines a modification $\bar{H}_{g,J;\theta,\Upsilon;\tilde{\Omega}} : \bar{G}_{g,J;\theta,\Upsilon} \Rightarrow G_{\tilde{\Omega}_{g_{|\theta,\Upsilon}},\tilde{\Omega}_{J_{|\theta,\Upsilon}};\theta,\Upsilon}$ if

$$\tag{4.3.92}$$

for every curve $\gamma : p_0 \to p_1$, where $\tilde{G}_{g,J;\theta,\Upsilon}$ is given in (4.3.81) and similarly $\tilde{G}_{\tilde{\Omega}_{g_{|\theta,\Upsilon}},\tilde{\Omega}_{J_{|\theta,\Upsilon}};\theta,\Upsilon}$ and the diagrams are composed by the usual pasting algorithm. This conditions is indeed fulfilled as a consequence of (4.3.87). The first part of the proposition follows. The proof of the second half is essentially identical. $\square$

4.3.5 Smoothness properties of parallel transport

In this subsection, we shall examine the smoothness properties of the parallel transport functors constructed in the preceding sections.

Let M be a manifold and G be a Lie group.

Proposition 55. *Let θ be a G–connection. Then, the parallel transport functor $\bar{F}_\theta : (M,P_1M) \to BG$ is smooth in the diffeological sense: if γ_α is a family of curves depending smoothly on a set of parameters α varying in a bounded closed domain A of $\mathbb{R}^d$ for some d, then the mapping $\alpha \in A \to F_\theta(\gamma_\alpha) \in G$ is smooth. When the connection θ is flat, the same property holds for the parallel transport functor $\bar{F}^0{}_\theta : (M,P^0{}_1M) \to BG$.*

Proof. Let a_α be a G–connection in the sense of def. 84 depending smoothly on a set of parameters α varying in a bounded closed domain A of $\mathbb{R}^d$ for some d. Then the G–cocycle f_{a_α} given by (4.2.7) solving the differential problem (4.2.8), (4.2.9) with a

replaced by a_α depends smoothly on α meaning that the mapping $\alpha \in A \to f_{a_\alpha}(x', x) \in G$ is smooth for any fixed $x, x' \in \mathbb{R}$.

Let now θ be a G–connection and γ_α be a family of curves depending smoothly on $\alpha \in A$. Then, $\gamma_\alpha^*\theta$ is a G–connection in the sense of def. 84 depending smoothly on α. By (4.3.17), then, $\alpha \to F_\theta(\gamma_\alpha) = f_{\gamma_\alpha^*\theta}(1, 0)$ is smooth. The statement follows. The flat case is treated similarly. $\qquad\square$

The above results extend straightforwardly to higher parallel transport. Let M be a manifold and (G, H) be a Lie crossed module.

Proposition 56. *Let (θ, Υ) be a (G, H)–connection doublet. Then, the parallel transport 2–functor $\bar{F}_{\theta,\Upsilon} : (M, P_1M, P_2M) \to B_0(G, H)$ is smooth in the diffeological sense: if $\Sigma_\alpha : \gamma_{0\alpha} \Rightarrow \gamma_{1\alpha}$ is a family of surfaces depending smoothly on a set of parameters α varying in a bounded closed domain A of $\mathbb{R}^d$ for some d, then the mappings $\alpha \in A \to F_{\theta,\Upsilon}(\gamma_{0\alpha}) \in G$, $\alpha \in A \to F_{\theta,\Upsilon}(\gamma_{1\alpha}) \in G$, $\alpha \in A \to F_{\theta,\Upsilon}(\Sigma_\alpha) \in H$ are smooth. When the connection doublet (θ, Υ) is flat, the same property holds for the parallel transport functor $\bar{F}^0{}_{\theta,\Upsilon} : (M, P_1M, P^0{}_2M) \to B_0(G, H)$.*

Proof. Let (a_α, B_α) be a (G, H)–connection doublet in the sense of def. 86 depending smoothly on a set of parameters α varying in a bounded closed domain A of $\mathbb{R}^d$ for some d. Then, the (G, H)–cocycle $(f_{a_\alpha,B_\alpha}, g_{a_\alpha,B_\alpha}, W_{a_\alpha,B_\alpha})$ given by (4.2.23) solving the differential problem (4.2.24), (4.2.25) with a, B replaced by a_α, B_α depends smoothly on α meaning that the mapping $\alpha \in A \to f_{a_\alpha,B_\alpha}(x', x; y) \in G$, $\alpha \in A \to g_{a_\alpha,B_\alpha}(x; y', y) \in G$, $\alpha \in A \to W_{a_\alpha,B_\alpha}(x', x; y', y) \in H$ are all smooth for any fixed $x, x', y, y' \in \mathbb{R}$.

Let now (θ, Υ) be a (G, H)–connection doublet and $\Sigma_\alpha : \gamma_{0\alpha} \Rightarrow \gamma_{1\alpha}$ be a family of surfaces depending smoothly on $\alpha \in A$. Then, $(\Sigma_\alpha^*\theta, \Sigma_\alpha^*\Upsilon)$ is a (G, H)–connection doublet in the sense of def. 86 depending smoothly on α. By (4.3.30), then, $\alpha \to F_{\theta,\Upsilon}(\gamma_{0\alpha}) = f_{\Sigma_\alpha^*\theta,\Sigma_\alpha^*\Upsilon|0}(1, 0)$, $\alpha \to F_{\theta,\Upsilon}(\gamma_{1\alpha}) = f_{\Sigma_\alpha^*\theta,\Sigma_\alpha^*\Upsilon|1}(1, 0)$ and $\alpha \to F_{\theta,\Upsilon}(\Sigma_\alpha) = W_{\Sigma_\alpha^*\theta,\Sigma_\alpha^*\Upsilon}(0, 1; 1, 0)$ are smooth. The statement follows. The flat case is treated similarly. $\qquad\square$

The above proposition has a counterpart at the level of 1–gauge transformations.

Proposition 57. *Let (θ, Υ) be a (G, H)–connection doublet and (g, J) a (G, H)–1–gauge transformation. Then, the gauge pseudonatural transformation $\bar{G}_{g,J;\theta,\Upsilon} : \bar{F}_{\theta,\Upsilon} \Rightarrow \bar{F}_{g,J\theta,g,J\Upsilon}$ is smooth in the diffeological sense: if γ_α is a family of curves depending smoothly on a set of parameters α varying in a bounded closed domain A of $\mathbb{R}^d$ for some d, then the mapping $\alpha \in A \to G_{g,J;\theta,\Upsilon}(\gamma_{0\alpha}) \in H$ is smooth. When the connection doublet (θ, Υ) is flat, the same property holds for the gauge pseudonatural transformation $\bar{G}^0{}_{g,J;\theta,\Upsilon} : \bar{F}^0{}_{\theta,\Upsilon} \Rightarrow \bar{F}^0{}_{g,J\theta,g,J\Upsilon}$.*

Proof. The statement is proven by a reasoning analogous to that showing prop. 56 relying on the smoothness properties of the solution of the differential problem (4.2.63a), (4.2.64a) and using (4.3.71). $\qquad\square$

4.3.6   Relation to other formulations

In this subsect, we shall analyze the relation between our formulation of higher parallel transport and other formulations appeared in the literature. This is an important point.

Let M be a manifold and (G, H) be a Lie crossed module. According to Schreiber and Waldorf [38, 39, 40], higher parallel transport is constructed as follows.

Definition 114. *Let (θ, Υ) be a (G, H)–connection. For a curve γ, the 1–parallel transport along γ is given by*

$$F_{SW\theta, \Upsilon}(\gamma) = f_{SW\theta, \Upsilon; \gamma}(1), \tag{4.3.93}$$

where $f_{SW\theta, \Upsilon; \gamma}(x)$ is the solution of the differential problem

$$d_x u(x) u(x)^{-1} = -\gamma^* \theta_x(x), \tag{4.3.94}$$

$$u(0) = 1_G \tag{4.3.95}$$

with $u : \mathbb{R} \to G$ a smooth mapping. For a surface Σ, the 2–parallel transport along Σ is given by

$$F_{SW\theta, \Upsilon}(\Sigma) = W_{SW\theta, \Upsilon; \Sigma}(1), \tag{4.3.96}$$

where $W_{SW\theta, \Upsilon; \Sigma}(y)$ is the solution of the differential problem

$$\partial_y E(y) E(y)^{-1} = \int_0^1 d\xi \, \dot{m}(F_{SW\theta, \Upsilon}(\gamma_{\Sigma\xi, y})) \Sigma^* \Upsilon_{xy}(\xi, y), \tag{4.3.97}$$

$$E(0) = 1_H \tag{4.3.98}$$

with $E : \mathbb{R} \to H$ a smooth mapping. Here, $\gamma_{\Sigma\xi, y} : \Sigma(\xi, y) \to \Sigma(1, y)$ is the curve defined by the expression

$$\gamma_{\Sigma\xi, y}(x) = \Sigma(\xi + (1 - \xi)\varphi(x), y), \tag{4.3.99}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\varphi(x) = 0$ for $x < \epsilon$ and $\varphi(x) = 1$ for $x > 1 - \epsilon$ for some small $\epsilon > 0$.

The function φ is introduced to ensure that $\gamma_{\Sigma\xi, y}$ has sitting instants. Its choice is immaterial, as a change of it amounts to a thin homotopy that leaves $F_{SW\theta, \Upsilon}(\gamma_{\Sigma\xi, y})$ invariant. The following proposition holds.

Proposition 58. *For any curve γ,*

$$F_{SW\theta, \Upsilon}(\gamma) = F_{\theta, \Upsilon}(\gamma). \tag{4.3.100}$$

Similarly, for any surface Σ,

$$F_{SW\theta, \Upsilon}(\Sigma) = F_{\theta, \Upsilon}(\Sigma). \tag{4.3.101}$$

Proof. We show first (4.3.100). As $\gamma^* \theta_x(x) = I_\gamma^* \theta_x(x, y)$ for any y, the differential problem (4.3.94), (4.3.95) is identical to that (4.2.24a), (4.2.25a) with $a_x(x, y) = I_\gamma^* \theta_x(x, y)$ and $x_0 = 0$, which is solved precisely by $f_{I_\gamma^* \theta, I_\gamma^* \Upsilon | y}(x, 0)$. So,

$$f_{SW\theta, \Upsilon; \gamma}(x) = f_{I_\gamma^* \theta, I_\gamma^* \Upsilon | y}(x, 0). \tag{4.3.102}$$

(4.3.100) then follows from (4.3.93) and (4.3.34).

The proof of (4.3.101) requires more work but follows a similar route. We begin with noticing that $f_{SW\theta,\Upsilon;\gamma_{\Sigma\xi,y}}(x)$ is the solution of the differential problem (4.3.94), (4.3.95) with $\gamma = \gamma_{\Sigma\xi,y}$. Since

$$\gamma_{\Sigma\xi,y}{}^*\theta_x(x) = (1 - \xi)d_x\varphi(x)\Sigma^*\theta_x(\xi + (1 - \xi)\varphi(x), y) \qquad (4.3.103)$$

by (4.3.99), the differential problem can thus more explicitly be stated as

$$d_x u(x)u(x)^{-1} = -(1 - \xi)d_x\varphi(x)\Sigma^*\theta_x(\xi + (1 - \xi)\varphi(x), y), \qquad (4.3.104)$$

$$u(0) = 1_G. \qquad (4.3.105)$$

Comparing this with the differential problem (4.2.24a), (4.2.25a) with $a_x(x, y) = \Sigma^*\theta_x(x, y)$ and $x_0 = \xi$, solved by $f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(x, \xi)$, we find that

$$f_{SW\theta,\Upsilon;\gamma_{\Sigma\xi,y}}(x) = f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(\xi + (1 - \xi)\varphi(x), \xi). \qquad (4.3.106)$$

From (4.3.93) with $\gamma = \gamma_{\Sigma\xi,y}$, it follows that

$$F_{SW\theta,\Upsilon}(\gamma_{\Sigma\xi,y}) = f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(\xi, 1)^{-1}. \qquad (4.3.107)$$

Recalling (4.3.32), we also have that

$$1_G = g_{\Sigma^*\theta,\Sigma^*\Upsilon|1}(y, 0)^{-1}. \qquad (4.3.108)$$

Taking (4.3.107), (4.3.108) into account, we can recast the differential problem (4.3.97), (4.3.98) in the form

$$\partial_y E(y)E(y)^{-1} \qquad (4.3.109)$$

$$= \int_0^1 d\xi \, \dot{m}(g_{\Sigma^*\theta,\Sigma^*\Upsilon|1}(y, 0)^{-1}f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(\xi, 1)^{-1})\Sigma^*\Upsilon_{xy}(\xi, y),$$

$$E(0) = 1_H. \qquad (4.3.110)$$

This is equivalent to the first form of the differential problem (4.2.24c), (4.2.25c) with $v_{|x_0,y_0}(y) = g_{\Sigma^*\theta,\Sigma^*\Upsilon|x_0}(y, y_0)$, $u_{|y,x_0}(x) = f_{\Sigma^*\theta,\Sigma^*\Upsilon|y}(x, x_0)$ and $B_{xy}(x, y) = \Sigma^*\Upsilon_{xy}(x, y)$ after integrating with respect to x and setting $x = 0$, $x_0 = 1$ and $y_0 = 0$. From here, it follows that

$$W_{SW\theta,\Upsilon;\Sigma}(y) = W_{\Sigma^*\theta,\Sigma^*\Upsilon|0,1}(y, 0). \qquad (4.3.111)$$

(4.3.101) then follows from (4.3.96) and (4.3.30c). $\qquad\qquad\qquad\qquad\qquad\square$

The prescription given by Martins and Picken in [41, 42] for the computation of higher parallel transport is essentially equivalent to that of Schreiber and Waldorf and, consequently, to ours.

Part III

Higher Chern-Simons theory

Chapter 5

Chern-Simons theory

We will now review ordinary Chern-Simons theory, to pave the way for the definition of the 2-term L_∞ Chern-Simons model which will be the subject of the next chapter and the main task of this thesis.

Chern-Simons is a topological field theory of the Schwarz kind. The classical action is the integral of the Chern-Simons form associated with a connection on a principal bundle [76], and although it had been considered as a field theory before [77],[78], it was with a famous paper by Witten [64] that it gained his great popularity. Witten found out that Chern-Simons theory is quantizable and solvable, and showed that it has remarkable connections with topology and knot theory. Thereafter, much effort have been spent studying this theory [79],[80], which has been analyzed and understood very deeply.

5.1 Classical action and gauge invariance

We start with a principal G bundle P on a three manifold M. Usually G is a compact semisimple Lie group, we will restrict ourselves to $G = SU(N)$ for simplicity. For our purposes the principal bundle can well be taken to be trivial, $P = M \times G$. The dynamic fields are connections $\omega \in \Omega^1(M; \mathfrak{g})$, and the Chern-Simons action for ω is defined to be

$$S_{CS}(\omega) = \kappa \int_M \left(\omega, d\omega + \frac{1}{3}[\omega, \omega] \right), \tag{5.1.1}$$

where κ is a real number and $(\cdot, \cdot)$ is an invariant bilinear form on $\mathfrak{g}$. If we take this form to be realized by the trace over some representation of the Lie algebra $\mathfrak{g}$ this action can be rewritten in the more explicit form

$$S_{CS}(\omega) = \kappa \int_M \mathrm{tr}(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega). \tag{5.1.2}$$

This action is topological because it doesn't depend on a choice of metric on M. The equation of motion is the condition of flatness for the connection:

$$\frac{\delta S_{CS}(\omega)}{\delta \omega} = 2\kappa F_\omega = 0. \tag{5.1.3}$$

In order to be integrated over the base manifold and to give a well defined functional on the space of connections on the principal bundle $M \times G$, the Chern-Simons

lagrangian needs to enjoy gauge invariance with respect to gauge transformations of the conncetion forms. Unfortunately this invariance doesn't hold completely. In fact, under a gauge transformation $g \in \mathrm{Map}(M, G)$ acting on the connection as

$$\omega \to \omega' = g\omega g^{-1} - dg g^{-1}, \tag{5.1.4}$$

the Chern-Simons lagrangian varies according to

$$\mathcal{L}_{CS}(\omega') = \mathcal{L}_{CS}(\omega) - \frac{\kappa}{3} \mathrm{tr}(g^{-1}dg g^{-1}dg g^{-1}dg) - \kappa d\,\mathrm{tr}(\omega g^{-1}dg). \tag{5.1.5}$$

The last term is exact and thus can be neglected when integrating on the compact manifold M, therefore is doesn't affect the action. On the other hand there is no reason for the second term to vanish somehow, and this leads to a gauge non-invariance of the theory. Luckily, this term can be scaled in such a way that the invariance holds only up to integers. In this way the action S_{CS} can be defined as a functional taking values in $\mathbb{R}/\mathbb{Z}$ instead of simply $\mathbb{R}$, and with the appropriate choice of κ it's possible to enforce that the variation of the action due to a gauge transformation is $2\pi k$ with $k \in \mathbb{Z}$. Therefore the quantity

$$e^{iS_{CS}(\omega)} \tag{5.1.6}$$

will be well defined, leading to a sensible quantum theory.

Things work as follows. The following integral

$$w(g) := \frac{1}{24\pi^2} \int_M \mathrm{tr}(g^{-1}dg g^{-1}dg g^{-1}dg) \tag{5.1.7}$$

for $g : M \to G$ is called the *winding number of the map g* in topology. It is a classical result that $w(g)$ is an integer. Thus, it is enough to restrict κ to be

$$\kappa = \frac{k}{4\pi} \tag{5.1.8}$$

with $k \in \mathbb{Z}$ to restrict the gauge anomaly of the Chern-Simons action generated by the term $\frac{\kappa}{3} \mathrm{tr}(g^{-1}dg g^{-1}dg g^{-1}dg)$ to 2π times an integer. k remains a free parameter which is called the *level* of the theory.

Another, more elegant way of seeing this is as follows. Take a 4-manifold N which has M as its boundary, extend the trivial bundle P to a new trivial bundle P' as well as the connection ω to a new connection ω' on the bulk of N, which is always possible. The differential of the Chern-Simons lagrangian extended to the whole N reads:

$$d\,\mathrm{tr}(\omega' \wedge d\omega' + \frac{2}{3}\omega' \wedge \omega' \wedge \omega') = \mathrm{tr}(F_{\omega'} \wedge F_{\omega'}), \tag{5.1.9}$$

which is a representative of the second Chern class of P' $\mathrm{ch}_2(P')$ up to a factor $1/(8\pi)$. In fact, this is the defining property of the Chern-Simons 3-form, which is the lagrangian of the Chern-Simons action and from which the theory takes its name [76]. Stokes theorem makes it possible to redefine Chern-Simons action in this way:

$$S_{CS}(\omega) := \kappa \int_N \mathrm{tr}(F_{\omega'} \wedge F_{\omega'}). \tag{5.1.10}$$

This is well defined and gauge invariant. Nevertheless there is an ambiguity in this definition, since it looks like (5.1.10) depends on the manifold N we chose and on how we extended our original data P and ω to P' and ω'. Actually, there is no ambiguity relying on P' and ω': since we are dealing with trivial bundles we are forced to the obvious definition $P' = N \times G$ and it is known that the Chern classes are independent of the choice of connection and are determined by the principal bundle itself. Instead, the arbitrariness of the choice of N truly affects the uniqueness of $S_{CS}(\omega)$ as defined in (5.1.10). Nevertheless, if we choose two different manifolds N and L that have M as boundary, the difference

$$\int_N \text{tr}(F_{\omega'} \wedge F_{\omega'}) - \int_L \text{tr}(F_{\omega'} \wedge F_{\omega'}) = \int_{N \cup \bar{L}} \text{tr}(F_{\omega'} \wedge F_{\omega'}) \qquad (5.1.11)$$

is the integral of $\text{tr}(F_{\omega'} \wedge F_{\omega'})$ on a closed manifold, as we glued N and L along their common boundary M. (Here with an abuse of notation we indicated by ω' any extension of ω to the manifold on which we integrate.) By a well-known result in topology stating that $\text{ch}_2(P')$ is an integral cohomology group of the base manifold, this integral gives 8π times an integer. Again, if we put $\kappa = k/4\pi$ we see that (5.1.10) defines $S_{CS}(\omega)$ up to 2π times an integer.

Now we can write the Chern-Simons action more precisely as

$$S_{CS}(\omega) = \frac{k}{4\pi} \int_M \text{tr}(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega), \qquad (5.1.12)$$

with $k \in \mathbb{Z}$. This phenomenon is called *quantization of the level.*

5.2 Perturbative quantization

We will now look at the large k limit of the partition function

$$Z = \int \mathcal{D}\omega \exp\left(\frac{ik}{4\pi} \int_M \text{tr}(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega)\right). \qquad (5.2.1)$$

Our treatment will follow closely Witten [64].

For $k \to \infty$ the phase of the integrand oscillates uncontrollably and the integral must be evaluated by a stationary phase method. Namely, the fast oscillations will cancel out upon integration and the only contribution to Z will come from the points where the phase is zero. These stationary points are the connections ω such that

$$\frac{\delta S_{CS}(\omega)}{\delta \omega} = 0, \qquad (5.2.2)$$

i.e. flat connections on M modulo gauge transformations. For simplicity, we assume that the topology of M is such that the set of all flat G connections on M modulo gauge transformations is discrete and finite. The partition function can then be written as the sum

$$Z = \sum_{i \in J} \mu(\omega^{(i)}), \qquad (5.2.3)$$

where J is the set of indexes labeling the flat connections and $\mu(\omega^{(i)})$ is the evaluation of the integral (5.2.1) by expansion around the flat connection $\omega^{(i)}$. The $\mu(\omega^{(i)})$ can be computed by changing the integration variable. We make the shift

$$\omega = \omega^{(i)} + \beta, \tag{5.2.4}$$

and we use the 1-form β as new integration variable, leaving the flat connection $\omega^{(i)}$ as a background field. Since this is a linear shift, the measure doesn't change:

$$\mathcal{D}\omega = \mathcal{D}\beta. \tag{5.2.5}$$

We now expand $S_{CS}(\omega^{(i)}+\beta)$ in a power series of β around $\omega^{(i)}$. The first non vanishing order in β is the quadratic term, due to the fact that $\omega^{(i)}$ is a critical point of S_{CS} being flat. We neglect all terms of third order in β or superior as we consider just the 1-loop approximation. A straightforward computation shows that

$$S_{CS}(\omega) = S_{CS}(\omega^{(i)}) + \frac{k}{4\pi}\int_M \operatorname{tr}(\beta \wedge D\beta), \tag{5.2.6}$$

where

$$D\beta = d\beta + [\omega^{(i)}, \beta] \tag{5.2.7}$$

is the covariant derivative with respect to $\omega^{(i)}$. Notice that β transforms covariantly under gauge transformations, being a 1-form in the adjoint representation of $\mathfrak{g}$, and this renders the β dependent part of the action fully gauge invariant.

The very first term $S_{CS}(\omega^{(i)})$ doesn't depend on β and can thus be factored out of the integral. To perform the gaussian integral over β of the second term we need a gauge fixing. We choose to adopt the Lorenz gauge fixing

$$D * \beta = 0, \tag{5.2.8}$$

where $*$ is the Hodge star. This relies on a choice of metric on M, therefore it spoils the topological flavor of the Chern-Simons action. This would happen with every choice of gauge fixing, but remarkably at the end of the computation the result will be topologically invariant, regardless of the choice of gauge fixing.

We implement the gauge fixing by means of the Faddeev-Popov mechanism. First of all, we add to the lagrangian the term

$$\frac{k}{4\pi}\operatorname{tr}(\phi \, D * \beta), \tag{5.2.9}$$

where ϕ is a 0-form in the adjoint representation of $\mathfrak{g}$, which acts as a Lagrange multiplier enforcing the gauge fixing condition (5.2.8). We then need to adjust the integration measure by inserting a functional determinant which is realized as a gaussian integration over anticommuting ghost. In our case the term to be added is

$$\frac{k}{4\pi}\operatorname{tr}(\bar{c} \, D * Dc), \tag{5.2.10}$$

where c and $\bar{c}$ are the scalar ghosts. The quantity we wish now to compute is

$$\mu(\omega^{(i)}) = e^{iS_{CS}(\omega^{(i)})}\int \mathcal{D}\beta\mathcal{D}\phi\mathcal{D}\bar{c}\mathcal{D}c \exp\left(\frac{ik}{4\pi}\int_M \operatorname{tr}(\beta \wedge D\beta + \phi \, D * \beta + \bar{c} \, D * Dc)\right). \tag{5.2.11}$$

To carry this out, we use the fact that the operator

$$L := *D + D* \tag{5.2.12}$$

is self adjoint on the space of differential forms on M in the adjoint representation of $\mathfrak{g}$ with respect to the usual inner product

$$\langle \xi, \eta \rangle = \int_M \mathrm{tr}(\xi \wedge *\eta). \tag{5.2.13}$$

Using the duality between 0-forms and 3-forms induced by the Hodge star, we can think of ϕ as being a 3-form on M instead of a scalar, and rewrite the gauge fixing term as

$$\frac{k}{4\pi} \, \mathrm{tr}(*\phi \, D *\beta). \tag{5.2.14}$$

A simple computation shows that

$$\langle \beta + \phi, L(\beta + \phi) \rangle = \int_M \mathrm{tr}(\beta \wedge D\beta + 2 *\phi \, D *\beta), \tag{5.2.15}$$

which can be made equivalent to the quadratic terms in ϕ and β in the exponent of (5.2.11) up to an irrelevant rescaling of the variables which can cancel the relative factor. Since L sends forms of odd degree to forms of odd degree, the integral over β and ϕ gives

$$(\det(L_-))^{-\frac{1}{2}}, \tag{5.2.16}$$

where by L_- we denote precisely the restriction of L to odd forms. The ghost sector is even more immediate: the operator $\Delta := D*D$ is self adjoint on 0-forms, and the result of the integral is just $\det(\Delta)$. Altogether these computations lead to the result

$$\mu(\omega^{(i)}) = e^{iS_{CS}(\omega^{(i)})} \frac{\det(\Delta)}{\sqrt{\det(L_-)}}. \tag{5.2.17}$$

As was shown in [81], the modulus of this ratio of determinants is nothing else than the Ray-Singer analytic torsion of $\omega^{(i)}$, which is a topological invariant and doesn't depend on the particular metric chosen on M for the gauge fixing. However, the phase of this determinants needs a deeper analysis to be determined. We will now summarize some of the main steps in the study of this phase, whose technical details can be found in [64].

The Δ operator has real determinant, so we only need to compute the phase of $\det(L_-)$. It can be shown that this phase is

$$\exp\left(i \frac{\pi}{2} \eta(\omega^{(i)}) \right), \tag{5.2.18}$$

where $\eta(\omega^{(i)})$ is the eta invariant defined as

$$\eta(\omega^{(i)}) := \frac{1}{2} \lim_{s \to 0} \sum_k \mathrm{sign}\lambda_k |\lambda_k|^{-s}. \tag{5.2.19}$$

In the previous formula $\{\lambda_k\}$ is the full set of eigenvalues of L_-. A hint of how the phase (5.2.18) arises from the determinant of L_- can be given by looking at the integral

$$\int \mathcal{D}\beta\mathcal{D}\phi \exp\left(\langle\beta + \phi, L_-(\beta + \phi)\rangle\right) \tag{5.2.20}$$

after expanding $\beta + \phi = \sum_k x_k v_k$, for $\{v_k\}$ a complete set of eigenvectors of L_- with eigenvalues λ_k. The functional integral (5.2.20) defining $\det(L_-)$ becomes then

$$\prod_i \int_{-\infty}^{+\infty} dx\, e^{-i\lambda_i x^2}, \tag{5.2.21}$$

which, after a sensible regularization method is adopted, gives the phase

$$e^{i\frac{\pi}{4}\sum_k \operatorname{sign}\lambda_k}. \tag{5.2.22}$$

The eta invariant obeys the rule

$$\eta(\omega^{(i)}) - \eta(0) = \frac{c_2(G)}{\pi} S_{CS}^1(\omega^{(i)}), \tag{5.2.23}$$

where S_{CS}^0 is the Chern-Simons action with level $k = 1$, and $c_2(G)$ denotes the value of the quadratic Casimir operator of the gauge group in the adjoint representation. For $G = SU(N)$ we have $c_2(SU(N)) = 2N$. This expression can be put into the computation of Z, but still we haven't reached manifest topological invariance because $\eta(0)$, the eta invariant associated with the trivial gauge field $\omega = 0$, is not a topological invariant, depending on the choice of a metric for the gauge fixing. To bypass this inconvenience, a counterterm is required in the Chern-Simons action. The right choice is a term proportional to a gravitational Chern-Simons:

$$I(g) = \frac{1}{4\pi} \int_M \operatorname{tr}(\omega_g \wedge d\omega_g + \frac{2}{3}\omega_g \wedge \omega_g \wedge \omega_g), \tag{5.2.24}$$

with ω_g the Levi-Civita connection associated with the metric g, because the Atiyah-Patodi-Singer theorem implies that the quantity

$$\frac{\eta(0)}{\dim G} + \frac{I(g)}{12\pi} \tag{5.2.25}$$

is a topological invariant. Putting all the pieces together we finally come then to the answer to our computation:

$$Z = \exp\left(i\frac{\pi}{2}(\eta(0) + \frac{\dim G\, I(g)}{12\pi})\right) \sum_i e^{i(k+\frac{c_2(G)}{2})S_{CS}^1(\omega^{(i)})} T_i, \tag{5.2.26}$$

being T_i the Ray-Singer torsion of the flat connection field $\omega^{(i)}$. In truth, this is not a honest topological invariant, because the term $I(g)$ depends intimately on a choice of trivialization of the tangent bundle of M. Nevertheless, the behavior of $I(g)$ under a change in the choice of framing is fully determined in terms of the number n of relative twists between the new framing and the old one, and it is

$$I(g) \to I(g) + 2\pi n, \tag{5.2.27}$$

leading to the following change in the partition function:

$$Z \to Z e^{i\frac{n\pi \dim G}{12}}. \tag{5.2.28}$$

5.3   Canonical quantization

The canonical quantization of the Chern-Simons theory can be performed if we take the base manifold to be of the form $M = \Sigma \times \mathbb{R}$, with Σ an arbitrary closed 2-manifold. The $\mathbb{R}$ direction coordinate will be denoted by t and it will play the role of time throughout the quantization process.

The connection 1-form can be split as the sum of two distinguished components:

$$\omega = \omega_\Sigma + \omega_0, \tag{5.3.1}$$

where ω_Σ contains only the form degrees in the Σ directions while ω_0 is a 1-form in the time direction, but both of them vary on the whole $\Sigma \times \mathbb{R}$. Choosing local coordinates x, y on Σ we can thus write

$$\omega_\Sigma = \omega_x dx + \omega_y dy \ , \ \omega_0 = \omega_t dt. \tag{5.3.2}$$

Also the de Rham differential can be divided explicitly:

$$d = dt\, \partial_t + d_\Sigma \ , \ d_\Sigma = dx\, \partial_x + dy\, \partial_y. \tag{5.3.3}$$

The Chern-Simons action is cast in the following form:

$$S_{CS}(\omega) = \frac{k}{4\pi} \int_{-\infty}^{+\infty} dt \int_\Sigma \operatorname{tr}\left(-\omega_\Sigma \wedge \partial_t \omega_\Sigma + 2\omega_t \wedge F_{\omega_\Sigma}\right). \tag{5.3.4}$$

Here $F_{\omega_\Sigma} = d_\Sigma \omega_\Sigma + \frac{1}{2}[\omega_\Sigma, \omega_\Sigma]$ is the curvature of ω_Σ, computed as if the latter was a connection form on Σ at fixed time. Notice that F_{ω_Σ} as well as ω_Σ depend on t. In this expression it is immediately clear that ω_t plays the role of a Lagrange multiplier instead of a dynamic variable. We can thus integrate it away and take into account the constraint that its equation of motion imposes, namely:

$$F_{\omega_\Sigma} = 0. \tag{5.3.5}$$

From now on the dynamic fields will be those ω_Σ fulfilling (5.3.5). The effective action reduces to

$$S_{CS}(\omega) = -\frac{k}{4\pi} \int_{-\infty}^{+\infty} dt \int_\Sigma \operatorname{tr}\left(\omega_\Sigma \wedge \partial_t \omega_\Sigma\right). \tag{5.3.6}$$

Since this action is linear in the time derivative, the phase space will come out to be constrained. Indeed, the momentum conjugated to the field ω_Σ is ω_Σ itself times a constant:

$$\frac{\delta \mathcal{L}}{\delta(\partial_t \omega_\Sigma)} = \frac{k}{4\pi} \omega_\Sigma, \tag{5.3.7}$$

and the Poisson brackets accordingly are

$$\{\omega_{\Sigma,\mu}^a(x, y), \omega_{\Sigma,\nu}^b(x', y')\} = \frac{4\pi}{k} \epsilon_{\mu\nu} \delta^{ab} \delta(x - x') \delta(y - y'), \tag{5.3.8}$$

where $\mu, \nu = x, y$, a, b are indexes in the Lie algebra $\mathfrak{g}$ and $\epsilon_{\mu\nu}$ is the antisymmetric tensor in two dimensions (regarding x, y as $0, 1$).

This means that the phase space to be quantized is the space of $\mathfrak{g}$ connections on Σ. To quantize this space, we have to choose a polarization, that is we have to formally separate variables into coordinates and momenta, and the Hilbert space of the states of the theory will be the space of functionals of the coordinates. Both coordinates and momenta will act on these states as operators, in such a way that the classical Poisson brackets become commutators.

We also have to take into account that the constraint (5.3.5) needs to be imposed. We can do this before or after the quantization process. Doing it before leads to the quantization of a different space, the space of flat $\mathfrak{g}$-connections modulo gauge transformations. This brings the advantage that this space is finite dimensional, unlike the full space of $\mathfrak{g}$-connection on Σ we start with. Using this approach Witten [64] showed a very important result which relates the quantization of Chern-Simons theory to two dimensional conformal field theory: he proved that the space of states obtained by the quantization of Chern-Simons theory on a two dimensional surface Σ is exactly the space of conformal blocks of the Wess-Zumino-Witten model on that same Σ. Furthermore this means that if Σ is compact the space of quantum states of the Chern-Simons theory is finite-dimensional. The Wess-Zumino-Witten model is a two-dimensional CFT on which we will say more in the next section.

Here we will follow the other path, namely we will first quantize our phase space and then we will impose the constraint on the states at the quantum level.

To define a polarization on the phase space we pick a complex structure J on Σ, $J : T\Sigma \to T\Sigma$, $J^2 = -1$. This makes it possible to work with holomorphic and antiholomorphic coordinates on Σ. Locally the picture is as follows: we introduce complex variables

$$z = x + iy \ , \ \bar{z} = x - iy. \tag{5.3.9}$$

The de Rham differential is

$$d = dz\,\partial_z + d\bar{z}\,\partial_{\bar{z}}. \tag{5.3.10}$$

The connection also naturally splits into holomorphic and antiholomorphic component:

$$\omega_\Sigma = \omega_z dz + \omega_{\bar{z}} d\bar{z}, \tag{5.3.11}$$

defined as

$$\omega_z = \frac{1}{2}(\omega_x - i\omega_y) \ , \ \omega_{\bar{z}} = \frac{1}{2}(\omega_x + i\omega_y). \tag{5.3.12}$$

The Poisson bracket expressed in these variables are

$$\{\omega_z^a(x,y), \omega_{\bar{z}}^b(x',y')\} = \frac{2\pi i}{k}\delta^{ab}\delta(x-x')\delta(y-y'), \tag{5.3.13}$$

$$\{\omega_z^a(x,y), \omega_z^b(x',y')\} = \{\omega_{\bar{z}}^a(x,y), \omega_{\bar{z}}^b(x',y')\} = 0. \tag{5.3.14}$$

The holomorphic and the antiholomorphic components of the connection are thus mutually conjugate. We choose to define our physical states as the functionals of the holomorphic component ω_z. ω_z will then act as a multiplication operator on the states, while the Poisson brackets imply that the antiholomorphic part $\omega_{\bar{z}}$ has to be realized as a functional derivative operator:

$$\hat{\omega}_{\bar{z}} = -\frac{2\pi i}{k}\frac{\delta}{\delta\omega_z}. \tag{5.3.15}$$

The scalar product on the Hilbert space of physical states will be defined in the following way. Given two states Ψ and Φ, their product is

$$\langle \Psi, \Phi \rangle := \int \mathcal{D}\omega_z \mathcal{D}\omega_{\bar{z}} \exp\left(-\frac{k}{4\pi} \int_\Sigma \mathrm{tr}(\omega_z^* \omega_z) dz \wedge d\bar{z} \right) \Psi(\omega_z)^* \Phi(\omega_z). \qquad (5.3.16)$$

Now we have to impose the constraint arising from the classical equation of motion of ω_0. This is done by requiring

$$\hat{F}_{\omega_\Sigma} \Psi(\omega_z) = 0 \qquad (5.3.17)$$

which, in holomorphic coordinates, becomes

$$\left(\partial_{\bar{z}}\omega_z - \frac{2\pi i}{k} \partial_z \frac{\delta}{\delta\omega_z} - \frac{2\pi i}{k} [\omega_z, \frac{\delta}{\delta\omega_z}] \right) \Psi(\omega_z) = 0. \qquad (5.3.18)$$

This is a Ward identity that must be fulfilled by the physical states of the theory.

What we have done so far depends on the particular choice of complex structure J we made on Σ. This is a bit unpleasant, because the topological flavor of Chern-Simons theory is spoiled and we would have preferred a quantum theory determined only by the topological datum, which is the 2-manifold Σ. Nevertheless, we can regard our set of Hilbert spaces arising from the canonical quantization for different choices of J as a fiber bundle on the space of all complex structures on Σ. This bundle admits a flat connection, which allows us to go from one Hilbert space to another and therefore to link the spaces of states arising from different complex structures J in a unique way.

5.4 Wess-Zumino-Witten model

In this section we will explain some of the multiple connections that exist between Chern-Simons theory and the Wess-Zumino-Witten model (WZW). This is a conformal two dimensional sigma model which rises naturally when studying Chern-Simons theory. In particular, the space of conformal blocks in WZW theory is also the Hilbert space of states of the canonically quantized Chern-Simons. We will not attempt to go deeper in the details behind this statement, instead we will show some direct appearances of the WZW functional in the quantization of the Chern-Simons theory.

We begin with the definition of the WZW model. Given a closed two dimensional manifold Σ and a compact semisimple Lie group G, the field content of the WZW model is a smooth map $g : \Sigma \to G$. The action is defined as:

$$S_{WZW}(g) = -\frac{ik}{8\pi} \int_\Sigma \mathrm{tr}(g^{-1}d_z g \wedge g^{-1}d_{\bar{z}} g) - \frac{ik}{24\pi} \int_B \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg), \quad (5.4.1)$$

with $k \in \mathbb{Z}$. The first of the summands is the kinetic term, while the second is a topological term. Here it is understood that a complex structure has been picked on Σ, analogously to what we did in the last section in the canonical quantization of Chern-Simons theory, so that we can work with complex variables z and $\bar{z}$. The topological term of this action is defined on a compact three dimensional manifold which contains Σ as its boundary. We denoted this 3-manifold B. The g entering in

the second integral is an extension of the original g from Σ to the whole bulk of B. This extension is not unique, therefore there is an ambiguity in this action. Again, this ambiguity is only up to integers, because, as the reader may have noticed, the three dimensional term in this action is proportional to the winding number $w(g)$ we introduced in section 5.1, and which is quantized. For instance, if G is simply connected and $\Sigma = S^2$ then everything is well defined because $\pi_2(G)$ vanishes. Disregarding this ambiguity, this action only depends on the two dimensional data g and Σ even if its definition is intrinsically three dimensional.

If we put a G-connection 1-form ω on Σ, it is possible to define the so-called *gauged* WZW model, whose action reads

$$S_{WZW}(g;\omega) = -\frac{ik}{8\pi} \int_\Sigma \mathrm{tr}\left((g^{-1}d_z g - 2\omega_z \, dz) \wedge g^{-1}d_{\bar{z}}g\right) -$$

$$-\frac{ik}{24\pi} \int_B \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \tag{5.4.2}$$

The holomorphic part of the kinetic term is shifted by $-2\omega_z \, dz$. We could have shifted the antiholomorphic part and leave the holomorphic one unchanged as well. Here the connection ω is treated as a background field and the only dynamic variable is g as in the ordinary WZW model. The importance of this gauged model will be shown soon.

Under an infinitesimal variation δg of the field the variation of the action is

$$\delta S_{WZW}(g) = \frac{ik}{4\pi} \int_\Sigma \mathrm{tr}\left(g^{-1}\delta g d_z(g^{-1}d_{\bar{z}}g)\right), \tag{5.4.3}$$

and the variation of the gauged action reads

$$\delta S_{WZW}(g;\omega) = \frac{ik}{4\pi} \int_\Sigma \mathrm{tr}\left(g^{-1}\delta g d_z(g^{-1}d_{\bar{z}}g) + \delta g g^{-1}d_{\bar{z}}(g\omega_z dz g^{-1})\right). \tag{5.4.4}$$

From these formulas we can derive the classical equation of motion which is

$$\partial_z(g^{-1}\partial_{\bar{z}}g) - \partial_{\bar{z}}\omega_z - [g^{-1}\partial_{\bar{z}}g, \omega_z] = 0 \tag{5.4.5}$$

for the gauged model, and the equation of motion for the action (5.4.1) is recovered from this by putting $\omega_z = 0$.

Let us make the following remark which will be of use later: if the infinitesimal variation of S_{WZW} is taken near the identity map $g = 1_G$, expression (5.4.4) takes the form

$$\delta S_{WZW}(g,\omega)|_{g=1_G} = -\frac{k}{2\pi} \int_\Sigma \mathrm{tr}(\alpha \partial_{\bar{z}}\omega_z), \tag{5.4.6}$$

where the integration measure $idzd\bar{z}/2$ has been made understood. In the last formula $\alpha = \delta g g^{-1} : \Sigma \to \mathfrak{g}$ is the element of the Lie algebra of G which parametrizes the variation around $g = 1_G$.

A remarkable property of the WZW model is the Polyakov-Wiegmann formula, which describes how the action behaves under the composition of maps in G. For the simple model it is

$$S_{WZW}(hg) = S_{WZW}(h) + S_{WZW}(g) - \frac{ik}{4\pi} \int_\Sigma \mathrm{tr}(h^{-1}d_{\bar{z}}h \wedge d_z g g^{-1}). \tag{5.4.7}$$

For the gauged model, this formula becomes simpler:

$$S_{WZW}(hg; \omega) = S_{WZW}(h; {}^g\omega) + S_{WZW}(g; \omega). \tag{5.4.8}$$

The first formula can be recovered from the second one by putting $\omega = 0$. The extra term in (5.4.7) is due to the dgg^{-1} shift in the gauge transformed connection ${}^g\omega$.

In the last section we quantized Chern-Simons theory to obtain the Hilbert space of physical quantum states of the theory, which are functionals of ω_z that satisfy a Ward identity (5.3.18). We need that these functionals behave under a gauge transformation of the connection in such a way that the physics is not affected. Namely, we want that $\Psi(\omega_z)$ and $\Psi({}^g\omega_z)$ change only by a phase, so that every quantity computed using the inner product of the Hilbert space is unchanged:

$$\Psi({}^g\omega_z) = \exp\left(i\Xi(g, \omega_z)\right)\Psi(\omega_z). \tag{5.4.9}$$

This phase cannot be arbitrary but it must fulfill some coherence conditions.

The first one derives from the Ward identity (5.3.18). Consider an infinitesimal gauge transformation governed by $\alpha : \Sigma \to \mathfrak{g}$:

$$\omega \to \omega + \delta\omega = \omega + D_\omega\alpha. \tag{5.4.10}$$

The variation of the state Ψ is

$$\Psi(\omega_z + \delta\omega_z) = \Psi(\omega_z) + \int_\Sigma \mathrm{tr}\left(\frac{\delta\Psi(\omega_z)}{\delta\omega_z}D_{\omega_z}\alpha\right) =$$

$$= \Psi(\omega_z) + \int_\Sigma \mathrm{tr}\left(D_{\omega_z}\frac{\delta\Psi(\omega_z)}{\delta\omega_z}\alpha\right) = \Psi(\omega_z) - \frac{ik}{2\pi}\int_\Sigma \mathrm{tr}(\partial_{\bar{z}}\omega_z\alpha). \tag{5.4.11}$$

We used (5.3.18) in the last equality. If we now expand (5.4.9) in the first order in α for the infinitesimal gauge transformation $g = 1_g + \alpha$ we have:

$$\Psi(\omega_z + \delta\omega_z) = \exp\left(i\Xi(1_g + \alpha, \omega_z)\right)\Psi(\omega_z) = \Psi(\omega_z) + i\delta\Xi(g, \omega_z)|_{g=1_G}\Psi(\omega_z). \tag{5.4.12}$$

Comparing the two computation we come to the constraint

$$\delta\Xi(g, \omega_z)|_{g=1_G} = -\frac{k}{2\pi}\int_\Sigma \mathrm{tr}(\partial_{\bar{z}}\omega_z\alpha). \tag{5.4.13}$$

The second coherence condition follows from the gauge action being a group action on the space of connections. To implement the relation

$$\Psi\left({}^h({}^g\omega_z)\right) = \Psi\left({}^{hg}\omega_g\right) \tag{5.4.14}$$

we need that

$$\exp\left(i\Xi(h, {}^g\omega_z)\right)\exp\left(i\Xi(g, \omega_z)\right)\Psi(\omega_z) = \exp\left(i\Xi(hg, \omega_z)\right)\Psi(\omega_z). \tag{5.4.15}$$

Relation (5.4.13) is identical to the infinitesimal variation of the gauged WZW model (5.4.6), while formula (5.4.15) is equivalent to the Polyakov-Wiegmann formula (5.4.8). These two conditions are enough to determine that the functional Ξ must be

the WZW gauged model. The states Ψ therefore transform under a gauge transformation as

$$\Psi(^g\omega_z) = \exp\left(iS_{WZW}(g;\omega_z)\right)\Psi(\omega_z). \tag{5.4.16}$$

That's how WZW model naturally rises in the quantization of the Chern-Simons theory. As we mentioned, the WZW action as an ambiguity in its definition which is contained in the winding number $w(g)$ of the extension g of the gauge transformation to the bulk of a three manifold having Σ as its boundary. Looking at formula (5.4.16) is appears that we need the well-definiteness only of the imaginary exponential of the WZW action functional, and this happens provided that the constant k in the action is integer. This is another way to obtain level quantization in Chern-Simons theory.

Let us now consider the canonical quantization of Chern-Simons theory on a manifold M with boundary [79]. Again, in order to employ canonical quantization we take $M = \mathbb{R} \times \Sigma$ with $\mathbb{R}$ regarded as the time direction and Σ this time is a compact 2-manifold with non trivial boundary $\partial\Sigma$. The presence of a boundary in the base manifold spoils the classical equation of motion and the gauge invariance of the model: the variation of the action reads

$$\delta S_{CS}(\omega) = \frac{k}{2\pi}\int_M \mathrm{tr}(\delta\omega \wedge F_\omega) + \frac{k}{4\pi}\int_{\partial M}\mathrm{tr}(\delta\omega \wedge \omega), \tag{5.4.17}$$

with a boundary term arising from integration by parts; gauge invariance suffers the non-vanishing of the total differential in (5.1.5), which gives a contribution on the boundary to the gauge transformed action. Both these problems can be circumvented by choosing a suitable gauge fixing on the boundary $\partial M = \mathbb{R} \times \partial\Sigma$. For example we set $\omega_0|_{\partial M} = 0$.

We can proceed as we did in the last section to obtain the action

$$S_{CS}(\omega) = -\frac{k}{4\pi}\int dt \int_\Sigma \mathrm{tr}(\omega_\Sigma \wedge \partial_t \omega_\Sigma). \tag{5.4.18}$$

with the constraint $F_{\omega_\Sigma} = 0$. Now suppose that the topology of Σ is trivial with no non-contractible loops, $\pi_1(\Sigma) = 0$, such as for the disc $\Sigma = D^2$, $\partial\Sigma = S^1$. In this case it is far more convenient to first impose the constraint and then to quantize, because with no nontrivial loops all the flat connections are gauge equivalent to the trivial one and the constraint $F_{\omega_\Sigma} = 0$ has the only solution:

$$\omega_\Sigma = -d_\Sigma U U^{-1}, \tag{5.4.19}$$

with $U : M \to G$. We can now make a change of variable in the functional integral of the partition function. Fortunately there is no Jacobian involved because it cancels out with the change of variable in the delta function $\delta(F_{\omega_\Sigma})$:

$$\int \mathcal{D}\omega_\Sigma\, \delta(F_{\omega_\Sigma}) = \int \mathcal{D}U\, \frac{\delta\omega_\Sigma}{\delta U}\, \frac{\delta(F_{-d_\Sigma U U^{-1}})}{(\delta\omega_\Sigma/\delta U)} = \int \mathcal{D}U. \tag{5.4.20}$$

Rewriting the action in terms of U leads straightforwardly to

$$S_{CS}(U) = S_{CS}(\omega_\Sigma = -d_\Sigma U U^{-1}) =$$

$$= \frac{k}{4\pi}\int_{\partial M}\mathrm{tr}\left(U^{-1}d_t U \wedge U^{-1}d_\theta U\right) + \frac{k}{12\pi}\int_M \mathrm{tr}\left(U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU\right). \tag{5.4.21}$$

which is $S_{WZW}(U)$ up to a constant. This shows that the Chern-Simons theory canonically quantized on a disc is the same as a WZW model on the disc.
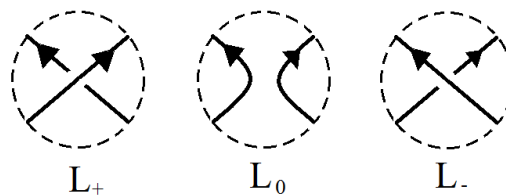
Figure 5.1: How the knots in the skein relation differ

5.5 Wilson Loops and Knot invariants

Chern-Simons theory can be successfully used to compute several kinds of topological invariants of the base manifold. The most remarkable results in this direction are obtained in the computation of knot invariants.

Let M be a three dimensional manifold. Given two smooth maps $\phi, \psi : S^1 \to M$, we say that ϕ and ψ are *ambient isotopy equivalent* if there is a smooth map $F : [0, 1] \times S^1 \to M$ such that $\phi = F(0, \cdot)$ and $\psi = F(1, \cdot)$. A knot in M is defined as an ambient isotopy equivalence class of smooth maps from the circle S^1 to M. Concretely, two embeddings of the circle in M are identified if their images can be stretched and moved without breaking them in such a way that they coincide. Usually $M = S^3$ is chosen, but other manifolds can be used. The most elementary knot is the *unknot*, which is intuitively an embedded circle which is not knotted on itself. More rigorously, is can be defined as a knot which is isotopic to a circle in M.

A *link* is instead a collection of knots in M which do not intersect. It can also be viewed as an ambient isotopy class of maps from n-fold products of S^1 to M. These knots can be both knotted one with each other or disconnected. A knot can be seen as a link with just one component.

The classification of knots and links is a very complicated task. A main tool are the so called knot invariants, which are topological invariants of the embeddings of the circle which are preserved by ambient isotopy. Among these there are the knot polynomials. These are polynomials in one or more formal variables which can be associated with knots. Here we are interested in the Jones polynomial. It is defined by means of a particular recursive formula called *skein relation* as follows. First of all the value of the polynomial for the unknot, denoted O, has to be normalized. It is usually normalized to 1:

$$P(O) = 1. \tag{5.5.1}$$

Then, for every other knot (or link) the polynomial can be computed using the formula

$$-tP(L_+) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})P(L_0) + t^{-1}P(L_-) = 0. \tag{5.5.2}$$

Here L_+, L_0 and L_- are three links that are identical but in a small region where they are as in figure 5.1.

Relation (5.5.2) is called skein relation. It's easy to see that (5.5.2) together with (5.5.1) makes it possible to compute the Jones polynomial for every link.

This definition relies on a projection of the knot on a two dimensional plane. This plane is arbitrary and there is no way to prefer a choice over another, nevertheless it

has been proved that the polynomial doesn't depend on the particular two dimensional projection.

The Jones polynomial is an important tool in the classification of knots and links, but there are still some open questions: it's known that different Jones polynomials can belong only to non equivalent knots, but it is not sure that different knots always generate different polynomials.

In [64] Witten showed that the Chern-Simons theory can be used to compute Jones-like polynomial invariants of knots and links. This is done by the evaluation of the Wilson loop

$$W_R(C) = \text{tr}_R \mathcal{P} \exp \left(\int_C \omega \right), \tag{5.5.3}$$

associated with a knot C and a representation R of the gauge group. This Wilson loop is gauge invariant and is a quantum observable of the theory. Its expectation value is

$$Z(C) = \int \mathcal{D}\omega e^{iS_{CS}(\omega)} W_R(C). \tag{5.5.4}$$

We can also take a link with several knots $C_1, \ldots, C_k$ and compute the product of their Wilson loops:

$$Z(C_1, \cdots, C_k) = \int \mathcal{D}\omega e^{iS_{CS}(\omega)} \prod_{i=1}^{k} W_{R_i}(C_i). \tag{5.5.5}$$

Let us take $G = SU(N)$. Furthermore all the representations R_i associated with the Wilson loops are taken to be the defining N dimensional ones. Now suppose that the partition function of the Chern-Simons theory is calculated on a base manifold M with boundary ∂M. Let us denote it with $Z(M)$. By canonical quantization an Hilbert space $\mathcal{H}$ is associated with the two dimensional closed manifold ∂M. $Z(M)$ then defines a vector in $\mathcal{H}$. If the orientation of the boundary is reversed then the Hilbert space is the canonical dual $\mathcal{H}^\vee$. Take now two three manifold $\tilde{M}_1$ and $\tilde{M}_2$ which have a two sphere S^2 as boundary, with opposite orientation. $Z(\tilde{M}_1)$ and $Z(\tilde{M}_2)$ are vectors in reciprocally dual Hilbert spaces. If we consider the three manifold M obtained by gluing $\tilde{M}_1$ and $\tilde{M}_2$ along their boundaries, then we have that $Z(M) = \langle Z(\tilde{M}_1), Z(\tilde{M}_2) \rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical dual pairing. Now take two three balls B_1 and B_2 whose S^2 boundaries have opposite orientation. Then also $Z(B_1)$ and $Z(B_2)$ are vectors in dual spaces. These balls can be glued together to get a three sphere S^3, or they can be glued to $\tilde{M}_1$ and $\tilde{M}_2$ to obtain new closed three manifolds M_1 and M_2. This is illustrated in figure 5.2. Now we have that $Z(S^3) = \langle Z(B_1), Z(B_2) \rangle$, $Z(M_1) = \langle Z(\tilde{M}_1), Z(B_2) \rangle$, $Z(M_2) = \langle Z(B_1), Z(\tilde{M}_2) \rangle$. The Hilbert spaces associated to some two manifolds via the Feynman path integral of the Chern-Simons theory can be found with CFT on the same manifolds as we saw. The interesting point now is that for S^2 this space comes out to be one dimensional. Thus it follows from simple linear algebra observations that

$$Z(M)Z(S^3) = Z(M_1)Z(M_2) \tag{5.5.6}$$

or alternatively

$$\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \frac{Z(M_2)}{Z(S^3)}. \tag{5.5.7}$$
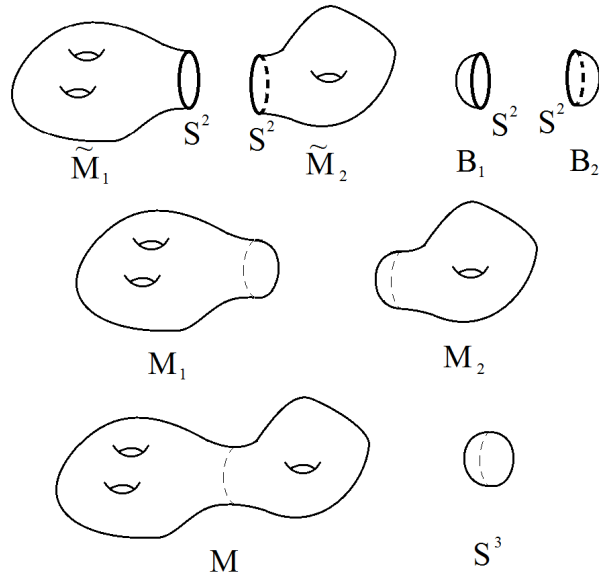
Figure 5.2: Manifolds $\tilde{M}_1$, $\tilde{M}_2$, B_1 and B_2 can be glued together to form M_1, M_2, M and S^3.

Suppose that we have S^3 with several mutually unknotted knots $C_1, \ldots, C_k$ embedded in it. Denote by $Z(S^3, C_1, \ldots, C_k)$ the evaluation of (5.5.5) in this case. Define

$$\langle C_1 \ldots C_k \rangle := Z(S^3; C^1, \ldots, C_k). \tag{5.5.8}$$

Then from (5.5.7) it follows that

$$\langle C_1 \ldots C_k \rangle = \prod_{i=1}^{k} \langle C_i \rangle. \tag{5.5.9}$$

This is obtained by cutting the original S^3 into several pieces which contain only one knot.

If we have S^3 with a link C embedded in it, we can cut out a region M which includes two segments of the link, and view the three sphere with the link as the union of M and the remainder N. We have that $Z(S^3, C) = \langle Z(M), Z(N) \rangle$. The pairing is the natural pairing on the mutually dual Hilbert spaces arising from the boundaries of M and N, which are identical but with opposite orientation. These boundaries are 2-spheres with four marked points - the points where the two segments of the link C which were included in M were cut and separated from the rest of C, and each point inherits a representation of $SU(N)$ from C: two of them carry the defining N-dimensional representation, the other two its dual version. Exploiting the correlation with CFT, one finds that the quantum Hilbert space $\mathcal{H}$ associated by the Chern-Simons theory with a marked 2-sphere is two- dimensional with this choice of representations for the points. This means that every two vectors ϕ, ψ in $\mathcal{H}$ will fulfill the relation

$$aZ(M) + b\phi + c\psi = 0 \tag{5.5.10}$$

B:
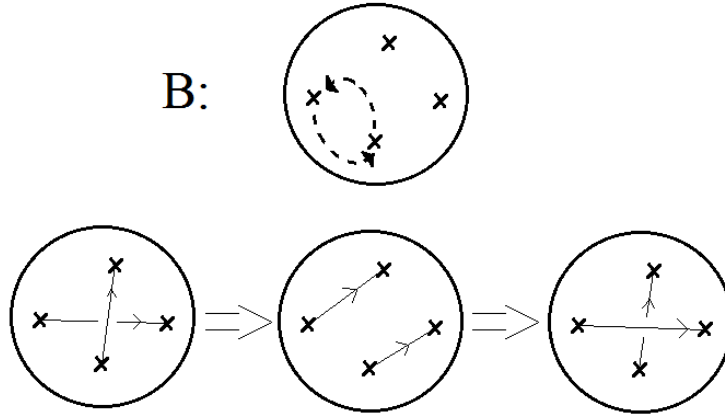
Figure 5.3: The geometric action of the operator B on the marked points of the 2-sphere and its effect on the segments in M

for some complex numbers a, b, c. Two such vectors can be found just by changing the braid of the lines in M when computing the path integral. This can be done by using the operator B, which realizes on $\mathcal{H}$ the geometric swapping of two of the marked points by making a half turn of one point around the other. Applying B once and twice we obtain two different braids as shown in fig. 5.3. So we have

$$\phi = BZ(M) \quad , \quad \psi = B^2 Z(M). \tag{5.5.11}$$

Notice that if we take the two balls with the lines rearranged as in the second and third images of fig. 5.3 and then we glue it with N in place of M, we obtain a 3-sphere S^3 with a link that is identical to C but for a small region (the one that was cut) where the lines differ from C as L_0 and L_- differ from L_+ in fig. 5.1. Let us call the new links C_1 and C_2. Now we have that (5.5.10) implies

$$a\langle Z(M), Z(N)\rangle + b\langle \phi, Z(N)\rangle + c\langle \psi, Z(N)\rangle = 0 \tag{5.5.12}$$

which means

$$a\langle C\rangle + b\langle C_1\rangle + c\langle C_2\rangle = 0. \tag{5.5.13}$$

The last equation is formally a skein relation for our invariant $\langle C\rangle$ defined in (5.5.8). To make it explicit we have to determine the numbers a, b, c. To do this we notice that since B acts on a two dimensional vector space, it obeys

$$B^2 - \operatorname{tr} B + \det B = 0, \tag{5.5.14}$$

which means that relation (5.5.10) is satisfied by

$$a = \det B \quad , \quad b = -\operatorname{tr} B \quad , \quad c = 1. \tag{5.5.15}$$

Thus, to compute a, b, c we just need to find the eigenvalues of B which are functions of N and k [64][82]. The final result is:

$$a = -e^{i\pi \frac{2}{N(N+k)}}, \tag{5.5.16}$$

$$b = -e^{i\pi \frac{2-N-N^2}{N(N+k)}} + e^{i\pi \frac{2+N-N^2}{N(N+k)}}, \tag{5.5.17}$$

$$c = e^{i\pi \frac{2-2N^2}{N(N+k)}}. \tag{5.5.18}$$

After multiplying by an overall factor of $\exp(i\pi\frac{N^2-2}{N(N+k)})$ and introducing the notation

$$t = e^{\frac{2\pi i}{N+k}}, \tag{5.5.19}$$

the final result is

$$a = -t^{N/2}, \tag{5.5.20}$$
$$b = t^{1/2} - t^{-1/2}, \tag{5.5.21}$$
$$c = t^{-N/2}. \tag{5.5.22}$$

It appears immediately that relation (5.5.13) with these coefficients for $N = 2$ is exactly the skein relation for the Jones polynomials (5.5.2). It remains to determine the value of $\langle O \rangle$. This can't be fixed arbitrarily, as a simple use of the skein relation (5.5.13) shows that

$$\langle O \rangle = -\frac{a+c}{b} \tag{5.5.23}$$

which with the explicit expressions of the coefficients reads

$$\langle O \rangle = \frac{t^{N/2} - t^{-N/2}}{t^{1/2} - t^{-1/2}}, \tag{5.5.24}$$

which for the case of $N = 2$ becomes

$$\langle O \rangle = t^{1/2} + t^{-1/2}. \tag{5.5.25}$$

This is different from the usual normalization that one chooses for the Jones polynomial, which is $\langle O \rangle = 1$ for the unknot, and this is due to the fact that our Wilson lines observable respect the multiplicativity rule (5.5.9), which forces this normalization for the unknot and does not hold under other normalizations.

Chapter 6

2-Term L_∞ Chern-Simons

In this chapter we shall construct and analyse a 4–dimensional semistrict analog of the standard Chern–Simons theory [64]. Beside providing a potentially interesting example of higher gauge theory, our construction, if it turns out successful, may furnish a basic field theoretic framework for the study of 4–dimensional topology. This chapter is taken from [19].

Our model was already introduced in lesser generality in ref. [21], where it was analysed mainly employing the Batalin–Vilkovisky quantization algorithm [59, 60] in the geometric AKSZ formulation [61]. Generalized Chern-Simons theory were studied in [11] and in [62] in an AKSZ framework. See also [63].

6.1   Semistrict higher Chern–Simons theory

In this section, we shall describe in detail Lie 2–algebra Chern–Simons theory. To highlight the way in which the model generalizes ordinary Chern–Simons theory [64], we first review this latter using the gauge theoretic framework developed in chapter 3.

Ordinary Chern–Simons Theory

The basic algebraic datum of ordinary Chern–Simons theory is a Lie algebra $\mathfrak{g}$ equipped with an invariant symmetric form $(\cdot,\cdot)$. The topological background is a compact oriented 3–fold N. The field content consists in a $\mathfrak{g}$–connection ω on N. The classical action functional reads

$$\mathrm{CS}_1(\omega) = \kappa_1 \int_N \left[(\omega, f) - \frac{1}{6}(\omega, [\omega, \omega]) \right], \tag{6.1.1}$$

where the curvature f is given by (3.1.1). The classical field equations are

$$f = 0, \tag{6.1.2}$$

(cf. eq. (3.1.1)) and entail that the connection ω is flat. We shall denote this classical field theory by $\mathrm{CS}_1(N, \mathfrak{g})$ or simply CS_1.

Let X be any manifold. In gauge theory, the de Rham complex $\Omega^*(X)$ contains the special subcomplex $\Omega_{\mathfrak{g}}{}^*(X)$ formed by those forms that are polynomials in one or more connections ω_a and their differentials $d\omega_a$. In turn, $\Omega_{\mathfrak{g}}{}^*(X)$ includes the subcomplex

$\Omega_{\mathfrak{g}\text{inv}}{}^*(X)$ of the elements invariant under the action (3.1.29) of the orthogonal gauge transformation group $\mathrm{OGau}(X, \mathfrak{g})$. For any $\mathfrak{g}$–connection ω on X, a form $\mathcal{L}_1 \in \Omega^3(X)$,

$$\mathcal{L}_1 = (\omega, f) - \frac{1}{6}(\omega, [\omega, \omega]), \tag{6.1.3}$$

formally identical to the Lagrangian density of the CS_1 action is defined. While $\mathcal{L}_1 \in \Omega_{\mathfrak{g}}{}^3(X)$, one has $\mathcal{L}_1 \notin \Omega_{\mathfrak{g}\text{inv}}{}^3(X)$, since, as is well–known,

$$^g\mathcal{L}_1 = \mathcal{L}_1 - \frac{1}{3}(\sigma_g, d\sigma_g) + d(\sigma_g, \omega) \tag{6.1.4}$$

for $g \in \mathrm{OGau}(X, \mathfrak{g})$. It is a standard result of gauge theory that

$$d\mathcal{L}_1 = \mathcal{C}_1, \tag{6.1.5}$$

where $\mathcal{C}_1 \in \Omega^4(X)$ is the curvature bilinear

$$\mathcal{C}_1 = (f, f). \tag{6.1.6}$$

Clearly, $\mathcal{C}_1 \in \Omega_{\mathfrak{g}}{}^4(X)$. Unlike $\mathcal{L}_1$, however, $\mathcal{C}_1$ is invariant under $\mathrm{OGau}(X, \mathfrak{g})$,

$$^g\mathcal{C}_1 = \mathcal{C}_1. \tag{6.1.7}$$

Thus, $\mathcal{C}_1 \in \Omega_{\mathfrak{g}\text{inv}}{}^4(X)$ as well. By (6.1.4) and (6.1.5), $\mathcal{C}_1$, while exact in the complex $\Omega_{\mathfrak{g}}{}^*(X)$, is generally only closed in the $\mathrm{OGau}(X, \mathfrak{g})$–invariant complex $\Omega_{\mathfrak{g}\text{inv}}{}^*(X)$. It thus defines a class $[\mathcal{C}_1]_{\text{inv}} \in H_{\mathfrak{g}\text{inv}}{}^4(X)$. More can be said. The variation $\delta\mathcal{C}_1$ of $\mathcal{C}_1$ under arbitrary variations of $\delta\omega$ of ω is given by

$$\delta\mathcal{C}_1 = 2d(\delta\omega, f). \tag{6.1.8}$$

where the 3–form in the right hand side is $\mathrm{OGau}(X, \mathfrak{g})$ invariant

$$(^g\delta\omega, {}^g f) = (\delta\omega, f). \tag{6.1.9}$$

It follows that, albeit $\mathcal{C}_1$ is not necessarily exact in $\Omega_{\mathfrak{g}\text{inv}}{}^*(X)$, its variation $\delta\mathcal{C}_1$ always is. This property characterizes $\mathcal{L}_1$ as the Chern–Simons form of a characteristic class $[\mathcal{C}_1]_{\text{inv}}$, in fact the 2nd Chern class.

    The CS_1 action is not invariant under the $\mathrm{OGau}(N, \mathfrak{g})$ action (3.1.29). In fact, from (6.1.4), one has

$$\mathrm{CS}_1(^g\omega) = \mathrm{CS}_1(\omega) - \kappa_1 Q_1(g) \tag{6.1.10}$$

for $g \in \mathrm{OGau}(N, \mathfrak{g})$, where the anomaly $Q_1(g)$ is given by

$$Q_1(g) = \frac{1}{3} \int_N (\sigma_g, d\sigma_g). \tag{6.1.11}$$

$Q_1(g)$ is in fact simply related to the CS_1 functional itself,

$$Q_1(g) = \kappa_1{}^{-1} \mathrm{CS}_1(\sigma_g). \tag{6.1.12}$$

The independence of $Q_1(g)$ from the connection ω implies so that the field equations (6.1.2) are gauge invariant. Indeed this follows directly and independently from eq. (3.1.30).

From (6.1.11), the anomaly density is the form $q_1 \in \Omega^3(N)$

$$q_1 = \frac{1}{3}(\sigma_g, d\sigma_g). \tag{6.1.13}$$

Note that, since σ_g is a connection, $q_1 \in \Omega_{\mathfrak{g}}{}^3(N)$. From (6.1.4), (6.1.5) and (6.1.7), it is readily seen that q_1 is closed. The variation of q_1 under continuous deformations of the gauge transformation g is instead exact

$$\delta q_1 = d(\delta\sigma_g, \sigma_g). \tag{6.1.14}$$

$Q_1(g)$ is so a topological invariant of g. Another way of showing this is by using relation (6.1.12): since flat connections ω are the ones solving the classical field equations (6.1.2), and σ_g is a flat connection for any g (cf. eq. (3.1.27)), the variation of $Q_1(g) = \kappa_1{}^{-1} \mathrm{CS}_1(\sigma_g)$ under an infinitesimal variation of g necessarily vanishes. $Q_1(g)$ reduces in fact up to a factor to the customary winding number of the gauge transformation g when $g = \mathrm{Ad}\,\gamma$, $\sigma_g = \gamma^{-1}d\gamma$ for a map $\gamma \in \mathrm{Map}(N, G)$, G being a Lie group integrating $\mathfrak{g}$.

By (3.1.27), the anomaly density q_1 can be cast as

$$q_1 = -\frac{1}{6}(\sigma_g, [\sigma_g, \sigma_g]). \tag{6.1.15}$$

This relation indicates that with q_1 there is associated a special Chevalley–Eilenberg cochain $\chi_1 \in \mathrm{CE}^3(\mathfrak{g})$,

$$\chi_1 = -\frac{1}{6}(\pi, [\pi, \pi]), \tag{6.1.16}$$

which is in fact a cocycle. By (3.1.27) and (2.1.14), if χ_1 is exact in $\mathrm{CE}(\mathfrak{g})$, then q_1 is exact in $\Omega_{\mathfrak{g}}{}^*(N)$. In order the anomaly $Q_1(g)$ to be non vanishing, so, it is necessary that $H_{CE}{}^3(\mathfrak{g}) \neq 0$. This is the case if $\mathfrak{g}$ is semisimple.

Since $Q_1(g)$ vanishes for any gauge transformation g continuously connected with the identity i, CS_1 is annihilated by the BRST operator s (cf. eq. (3.1.14)),

$$s\mathrm{CS}_1(\omega) = 0, \tag{6.1.17}$$

as can be directly verified from (6.1.1). This property opens the way to the gauge invariant perturbative quantization of the model.

Due to the $\mathrm{OGau}(N, \mathfrak{g})$ gauge non invariance of the CS_1 action functional, the gauge invariant path integral quantization of the CS_1 field theory is possible only if the value of κ_1 is such that $\kappa_1 Q_1(g) \in 2\pi\mathbb{Z}$ for all $g \in \mathrm{OGau}(N, \mathfrak{g})$. For $\mathfrak{g} = \mathfrak{u}(n)$ and $(\cdot, \cdot) = -\mathrm{tr}_{\mathrm{fund}}(\cdot\,\cdot)$ this is achieved if

$$\kappa_1 = -\frac{k}{4\pi}, \tag{6.1.18}$$

where $k \in \mathbb{Z}$ is an integer called level.

Semistrict higher Chern–Simons theory

After reviewing ordinary Chern–Simons theory, we introduce the semistrict higher Chern–Simons theory, which is the main topic of this paper. The basic algebraic datum of the model is a balanced Lie 2–algebra $\mathfrak{v}$ equipped with an invariant form $(\cdot,\cdot)$ (cf. sect. 2.4.6). The topological background is a compact oriented 4–fold N. The field content consists in a $\mathfrak{v}$–connection doublet (ω, Ω_ω) on N. The classical action functional is

$$\mathrm{CS}_2(\omega, \Omega_\omega) = \kappa_2 \int_N \left[\frac{1}{2}(2f + \partial\Omega_\omega, \Omega_\omega) - \frac{1}{24}(\omega, [\omega, \omega, \omega]) \right], \qquad (6.1.19)$$

where f is given by (3.2.7). The classical field equations of $\mathrm{CS}_2(N, \mathfrak{v})$ are

$$f = 0, \qquad (6.1.20a)$$

$$F_f = 0 \qquad (6.1.20b)$$

(cf. eqs. (3.2.7), (3.2.8)). They imply that the connection doublet (ω, Ω_ω) is flat, analogously to standard CS theory. We shall denote this classical field theory by $\mathrm{CS}_2(N, \mathfrak{v})$ or simply CS_2.

Let X be any manifold. In semistrict gauge theory, in analogy to ordinary gauge theory, the de Rham complex $\Omega^*(X)$ contains the special subcomplex $\Omega_\mathfrak{v}^*(X)$ formed by those forms that are polynomials in the components of one or more connection doublets (ω_a, Ω_a) and their differentials $(d\omega_a, d\Omega_a)$. In turn, $\Omega_\mathfrak{v}^*(X)$ includes the subcomplex $\Omega_{\mathfrak{v}\mathrm{inv}}^*(X)$ of the elements invariant under the action (3.2.42) of the orthogonal 1–gauge transformation group $\mathrm{OGau}_1(X, \mathfrak{v})$. For any $\mathfrak{v}$–connection doublet (ω, Ω_ω) on X, a form $\mathcal{L}_2 \in \Omega^4(X)$

$$\mathcal{L}_2 = \frac{1}{2}(2f + \partial\Omega_\omega, \Omega_\omega) - \frac{1}{24}(\omega, [\omega, \omega, \omega]). \qquad (6.1.21)$$

formally identical to the Lagrangian density of the CS_2 action is defined. While $\mathcal{L}_2 \in \Omega_\mathfrak{v}^4(X)$, one has $\mathcal{L}_2 \notin \Omega_{\mathfrak{v}\mathrm{inv}}^4(X)$, since

$$^g\mathcal{L}_2 = \mathcal{L}_2 - \frac{1}{4}(\sigma_g, d\Sigma_g) - d\left[\frac{1}{2}(\sigma_g, \Sigma_g) \right. \qquad (6.1.22)$$

$$\left. + \frac{1}{6}(\omega - \sigma_g, g_1^{-1}g_2(\omega - \sigma_g, \omega - \sigma_g) + 6\Sigma_g - 3\tau_g(\omega - \sigma_g)) \right].$$

for $g \in \mathrm{OGau}_1(X, \mathfrak{v})$. Similarly to standard gauge theory, one has

$$d\mathcal{L}_2 = \mathcal{C}_2, \qquad (6.1.23)$$

where $\mathcal{C}_2 \in \Omega^5(X)$ is the curvature bilinear

$$\mathcal{C}_2 = (f, F_f). \qquad (6.1.24)$$

Clearly, $\mathcal{C}_2 \in \Omega_\mathfrak{v}^5(X)$. Unlike $\mathcal{L}_2$, however, $\mathcal{C}_2$ is invariant under $\mathrm{OGau}_1(X, \mathfrak{v})$,

$$^g\mathcal{C}_2 = \mathcal{C}_2, \qquad (6.1.25)$$

implying that $\mathcal{C}_2 \in \Omega_{\mathfrak{v}\mathrm{inv}}{}^5(X)$. By (6.1.22) and (6.1.23), $\mathcal{C}_2$, while exact in the complex $\Omega_{\mathfrak{v}}{}^*(X)$, is generally only closed in the $\mathrm{OGau}_1(X, \mathfrak{v})$–invariant complex $\Omega_{\mathfrak{v}\mathrm{inv}}{}^*(X)$. It thus defines a class $[\mathcal{C}_2]_{\mathrm{inv}} \in H_{\mathfrak{v}\mathrm{inv}}{}^5(X)$. Further, the variation $\delta\mathcal{C}_2$ of $\mathcal{C}_2$ under arbitrary variations variations $\delta\omega, \delta\Omega_\omega$ of ω, Ω_ω is given by

$$\delta\mathcal{C}_2 = d\big[(\delta\omega, F_f) + (f, \delta\Omega_\omega)\big]. \tag{6.1.26}$$

where the 5–form in the right hand side is $\mathrm{OGau}_1(X, \mathfrak{v})$ invariant

$$({}^g\delta\omega, {}^g F_f) + ({}^g f, {}^g\delta\Omega_\omega) = (\delta\omega, F_f) + (f, \delta\Omega_\omega). \tag{6.1.27}$$

It follows that, although $\mathcal{C}_2$ is not necessarily exact in $\Omega_{\mathfrak{v}\mathrm{inv}}{}^*(X)$, its variation $\delta\mathcal{C}_2$ always is. This property characterizes then $\mathcal{L}_2$ as the Chern–Simons form of a higher characteristic class $[\mathcal{C}_2]_{\mathrm{inv}}$.

The CS_2 action is not invariant under the $\mathrm{OGau}_1(N, \mathfrak{v})$ action (3.2.42). In fact, from (6.1.22), analogously to ordinary Chern–Simons theory, one has

$$\mathrm{CS}_2({}^g\omega, {}^g\Omega_\omega) = \mathrm{CS}_2(\omega, \Omega_\omega) - \kappa_2 Q_2(g) \tag{6.1.28}$$

for $g \in \mathrm{OGau}_1(N, \mathfrak{v})$, where the anomaly $Q_2(g)$ is given by

$$Q_2(g) = \frac{1}{4} \int_N \big[2(d\sigma_g, \Sigma_g) - (\sigma_g, d\Sigma_g)\big]. \tag{6.1.29}$$

$Q_2(g)$ is in fact simply related to the CS_2 action itself,

$$Q_2(g) = \kappa_2{}^{-1} \mathrm{CS}_2(\sigma_g, \Sigma_g). \tag{6.1.30}$$

Again, the independence of $Q_2(g)$ from the connection doublet (ω, Ω_ω) implies that the field equations (6.1.20) are gauge invariant, a property that follows also directly and independently from eqs. (3.2.43).

From (6.1.29), the anomaly density is the form $q_2 \in \Omega^4(N)$

$$q_2 = \frac{1}{4}\big[2(d\sigma_g, \Sigma_g) - (\sigma_g, d\Sigma_g)\big]. \tag{6.1.31}$$

Note that, since (σ_g, Σ_g) is a connection doublet, $q_2 \in \Omega_{\mathfrak{v}}{}^4(N)$. From (6.1.22), (6.1.23) and (6.1.25), it is readily seen that q_2 is closed. The variation of q_2 under continuous deformations of the gauge transformation g is instead exact

$$\delta q_2 = d(\delta\sigma_g, \Sigma_g). \tag{6.1.32}$$

In CS_2 too, $Q_2(g)$ is so a topological invariant of g. Another way of showing this is by using relation (6.1.30): since flat connections (ω, Ω_ω) are the ones solving the classical field equations (6.1.20) and (σ_g, Σ_g) is a flat connection doublet for any g (cf. eqs. (3.2.15)), the variation of $Q_2(g) = \kappa_2{}^{-1} \mathrm{CS}_2(\sigma_g, \Sigma_g)$ under an infinitesimal variation of g necessarily vanishes. In analogy to ordinary Chern–Simons theory, $Q_2(g)$ represents a higher winding number of the higher gauge transformation g.

By using (3.2.15b), the anomaly density q_2 can be cast as

$$q_2 = -\frac{1}{24}(\sigma_g, [\sigma_g, \sigma_g, \sigma_g]) + \frac{1}{2}(\partial\Sigma_g, \Sigma_g). \tag{6.1.33}$$

With q_2 there is therefore associated a special higher Chevalley–Eilenberg cochain $\chi_2 \in \mathrm{CE}^4(\mathfrak{v})$,

$$\chi_2 = -\frac{1}{24}(\pi, [\pi, \pi, \pi]) + \frac{1}{2}(\partial\Pi, \Pi), \tag{6.1.34}$$

which is in fact a cocycle. By (3.2.15) and (2.4.40), if χ_2 is exact in $\mathrm{CE}(\mathfrak{v})$, then q_2 is exact in $\Omega_{\mathfrak{v}}{}^*(N)$. In this way, in order the anomaly $Q_2(g)$ to be non trivial, it is necessary that $H_{CE}{}^4(\mathfrak{v}) \neq 0$.

Since $Q_2(g)$ vanishes for any 1–gauge transformation g continuously connected with the identity i, CS_2 is invariant under the BRST operator (3.2.84),

$$s\mathrm{CS}_2(\omega, \Omega_\omega) = 0, \tag{6.1.35}$$

a property that can be directly verified from (6.1.19). As shown in subsect. 3.2.6, defining the BRST variations of the ghost fields c, C_c, Γ according to (3.2.89a), (3.2.89b) (3.2.91), the BRST operator s turns out to be nilpotent provided the vanishing fake curvature condition $f = 0$ is satisfied, since $s^2\mathcal{F} = 0$ for all fields and ghost fields $\mathcal{F}$ except for Ω_ω, in which case one has

$$s^2\Omega_\omega = -[f, \Gamma] + \frac{1}{2}[f, c, c]. \tag{6.1.36}$$

Being $f = 0$ one of the field equations, s is nilpotent on shell. Perturbative quantization of the model is still possible, but it requires the Batalin–Vilkovisky quantization algorithm [21].

As in ordinary Chern–Simons theory, the fact that the CS_2 action is not $\mathrm{OGau}_1(N, \mathfrak{v})$ invariant makes the gauge invariant path integral quantization of the CS field theory impossible unless certain conditions are met. The pair of the 4–fold N and the balanced Lie 2–algebra $\mathfrak{v}$ with invariant form is said *admissible* if there exists a positive value of κ_2 such that $\kappa_2 Q_2(g) \in 2\pi\mathbb{Z}$ for all $g \in \mathrm{OGau}_1(N, \mathfrak{v})$. Letting $\kappa_{2N\mathfrak{v}}$ be the smallest value of κ_2 with such property, the gauge invariant path integral quantization of the $\mathrm{CS}_2(N, \mathfrak{v})$ theory is possible, at least in principle, provided that

$$\kappa_2 = k\kappa_{2N\mathfrak{v}}, \tag{6.1.37}$$

where $k \in \mathbb{Z}$ is an integer, which we shall call level as in the ordinary theory.

An important issue of the theory is the classification of the admissible pairs $(N, \mathfrak{v})$. We cannot provide any solution of it presently. This is also related to the fact that the integrability of a semistrict Lie 2–algebra $\mathfrak{v}$ to a semistrict Lie 2–group V is not guaranteed in general. In the canonical quantization of semistrict higher Chern–Simons theory carried out in the next subsections, we assume as a working hypothesis that $\mathfrak{v}$ is a balanced Lie 2–algebra with invariant form such that $(N, \mathfrak{v})$ is admissible for a sufficiently ample class of closed 4–folds N.

6.2   Canonical quantization

In this section, we shall briefly review the canonical quantization of ordinary Chern–Simons theory and then pass to that of the semistrict higher Chern–Simons theory.

To carry out the canonical quantization of a field theory, we restrict to the case where the base manifold N is of the form $N = \mathbb{R} \times M$ with M a compact oriented manifold. Let t denote the standard coordinate of $\mathbb{R}$. Then, the derivation operator d_t is a globally defined nowhere vanishing vector field on $\mathbb{R} \times M$. We denote by $\Omega_{\mathrm{h}}{}^p(\mathbb{R} \times M)$ the subspace of $\Omega^p(\mathbb{R} \times M)$ consisting of those p–forms α such that $i_{d_t}\alpha = 0$. Every p–form $\alpha \in \Omega^p(\mathbb{R} \times M)$ decomposes uniquely as $\alpha = dt\alpha_t + \alpha_s$, where $\alpha_t \in \Omega_{\mathrm{h}}{}^{p-1}(\mathbb{R} \times M)$, $\alpha_s \in \Omega_{\mathrm{h}}{}^p(\mathbb{R} \times M)$. Analogously, the differential d of $\mathbb{R} \times M$ decomposes as $d = dtd_t + d_s$, d_s being the differential along M in $\mathbb{R} \times M$.

Ordinary Chern–Simons theory

In the $\mathrm{CS}_1(\mathbb{R} \times M, \mathfrak{g})$ theory, the $\mathfrak{g}$–connection ω decomposes as

$$\omega = dt\omega_t + \omega_s, \tag{6.2.1}$$

where $\omega_t \in \Omega_{\mathrm{h}}{}^0(\mathbb{R} \times M, \mathfrak{g})$, $\omega_s \in \Omega_{\mathrm{h}}{}^1(\mathbb{R} \times M, \mathfrak{g})$. The curvature f of ω splits as

$$f = dtf_t + f_s, \tag{6.2.2}$$

where $f_t \in \Omega_{\mathrm{h}}{}^1(\mathbb{R} \times M, \mathfrak{g})$, $f_s \in \Omega_{\mathrm{h}}{}^2(\mathbb{R} \times M, \mathfrak{g})$, in similar fashion (cf. eqs. (3.1.1)). ω_s is itself a $\mathfrak{g}$–connection and f_s is the associated curvature. The CS_1 action (6.1.1) reads then as

$$\mathrm{CS}_1(\omega) = \kappa_1 \int_{\mathbb{R} \times M} dt \left[-(\omega_s, d_t\omega_s) + 2(\omega_t, f_s) \right]. \tag{6.2.3}$$

The field equations read then as

$$f_s = 0, \tag{6.2.4a}$$

$$d_t\omega_s - D_s\omega_t = 0, \tag{6.2.4b}$$

where D_s denotes the covariant differentiation operator associated with the connection ω_s defined according to (3.1.7) and ω_t is treated as a bidegree $(0,0)$ field.

The momenta ξ_t, ξ_s canonically conjugate to ω_t, ω_s can easily be read off from (6.2.3). In virtue of the linear isomorphisms $\mathfrak{g}^\vee \simeq \mathfrak{g}$ induced by the bilinear form $(\cdot, \cdot)$, we have $\xi_t \in \Omega_{\mathrm{h}}{}^2(\mathbb{R} \times M, \mathfrak{g})$, $\xi_s \in \Omega_{\mathrm{h}}{}^1(\mathbb{R} \times M, \mathfrak{g})$,

$$\xi_t = 0, \tag{6.2.5a}$$

$$\xi_s = -\kappa_1\omega_s. \tag{6.2.5b}$$

Ordinary Chern–Simons theory is therefore constrained. This requires the application of Dirac's quantization algorithm.

To this end, we set below

$$\langle g, g' \rangle = \int_M (g, g') \tag{6.2.6}$$

for $g \in \Omega^p(M, \mathfrak{g})$, $g' \in \Omega^{2-p}(M, \mathfrak{g})$, for notational convenience. Further, for any $\Omega^p(M, \mathfrak{g})$–valued phase function ψ, we denote by g_ψ a $\Omega^{2-p}(M, \mathfrak{g})$–valued phase constant.

In the Hamiltonian formulation of $CS_1(\mathbb{R} \times M, \mathfrak{g})$, the canonical field coordinates are $\omega_t \in \Omega^0(M, \mathfrak{g})$, $\omega_s \in \Omega^1(M, \mathfrak{g})$ and their canonically conjugate momenta are respectively $\xi_t \in \Omega^2(M, \mathfrak{g})$, $\xi_s \in \Omega^1(M, \mathfrak{g})$. The basic Poisson brackets are

$$\{\langle g_{\omega_t}, \omega_t \rangle, \langle \xi_t, g_{\xi_t} \rangle\}_P = \langle g_{\omega_t}, g_{\xi_t} \rangle, \tag{6.2.7a}$$

$$\{\langle g_{\omega_s}, \omega_s \rangle, \langle \xi_s, g_{\xi_s} \rangle\}_P = \langle g_{\omega_s}, g_{\xi_s} \rangle, \tag{6.2.7b}$$

The canonical Hamiltonian drawn from (6.2.3) is

$$H = -2\kappa_1 \langle \omega_t, f_s \rangle. \tag{6.2.8}$$

The primary constraints corresponding to the relations (6.2.5a), (6.2.5b) are

$$\xi_t \approx 0, \tag{6.2.9a}$$

$$\kappa_1 \omega_s + \xi_s \approx 0. \tag{6.2.9b}$$

Implementation of the Dirac's algorithm leads to the secondary constraints

$$f_s \approx 0, \tag{6.2.10}$$

and no higher order constraints. Further, the phase functions ξ_t and f_s are identified as generators of gauge symmetries. Gauge fixing is thus required. A complete fixing of the symmetry, however, leads to unwanted non locality in the resulting gauge fixed theory. To remain in the framework of local field theory, we fix only the gauge symmetry associated with ξ_t leaving that corresponding to f_s unfixed. The gauge fixing condition we choose to impose is

$$\omega_t \approx 0, \tag{6.2.11}$$

The constraints (6.2.9a), (6.2.9b), (6.2.11) form a second class set and, so, they can be used to construct the Dirac brackets on the associated constrained phase space. The only independent phase variable remaining after the constraints are taken into account is ω_s, whose Dirac brackets are

$$\{\langle g_{\omega_s}, \omega_s \rangle, \langle \omega_s, g_{\omega_s}{}' \rangle\}_D = -\frac{1}{2\kappa_1} \langle g_{\omega_s}, g_{\omega_s}{}' \rangle. \tag{6.2.12}$$

The constraint (6.2.10) remains pending. f_s generates now the constrained phase space BRST transformations. Introducing a ghost field $c_s \in \Omega^0(M, \mathfrak{g}[1])$, we have

$$\{\langle f_s, c_s \rangle, \langle \omega_s, g_{\omega_s} \rangle\}_D = \frac{1}{2\kappa_1} \langle s_s \omega_s, g_{\omega_s} \rangle, \tag{6.2.13}$$

where $s_s \omega_s$ is given by

$$s_s \omega_s = -D_s c_s, \tag{6.2.14}$$

in agreement with (3.1.14).

We quantize $CS_1(\mathbb{R} \times M, \mathfrak{g})$ by replacing the classical field ω_s satisfying the Dirac brackets (6.2.12) with a corresponding quantum field $\widehat{\omega}_s$ satisfying the commutation relations

$$[\langle g_{\omega_s}, \widehat{\omega}_s \rangle, \langle \widehat{\omega}_s, g_{\omega_s}{}' \rangle] = -\frac{i}{2\kappa_1} \langle g_{\omega_s}, g_{\omega_s}{}' \rangle. \tag{6.2.15}$$

The constraint (6.2.10), which we left pending in the classical theory, becomes a condition obeyed by the state vectors Ψ of the theory,

$$\langle \widehat{f}_s, g_{f_s} \rangle \Psi = 0. \tag{6.2.16}$$

Semistrict higher Chern–Simons theory

The canonical quantization of semistrict higher Chern–Simons theory proceeds on the same lines as the ordinary case. The structural similarities and differences of the two models should be evident to the reader.

In the $CS_2(\mathbb{R} \times M, \mathfrak{v})$ theory, the $\mathfrak{v}$–connection doublet (ω, Ω_ω) splits as

$$\omega = dt\omega_t + \omega_s, \tag{6.2.17a}$$

$$\Omega_\omega = dt\Omega_{\omega t} + \Omega_{\omega s}, \tag{6.2.17b}$$

where $\omega_t \in \Omega_h{}^0(\mathbb{R} \times M, \mathfrak{v}_0)$, $\omega_s \in \Omega_h{}^1(\mathbb{R} \times M, \mathfrak{v}_0)$, $\Omega_{\omega t} \in \Omega_h{}^1(\mathbb{R} \times M, \mathfrak{v}_1)$, $\Omega_{\omega s} \in \Omega_h{}^2(\mathbb{R} \times M, \mathfrak{v}_1)$. Similarly, the curvature doublet (f, F_f) of (ω, Ω_ω) splits as

$$f = dtf_t + f_s, \tag{6.2.18a}$$

$$F_f = dtF_{ft} + F_{fs} \tag{6.2.18b}$$

(cf. eqs. (3.2.7), (3.2.8)), where $f_t \in \Omega_h{}^1(\mathbb{R} \times M, \mathfrak{v}_0)$, $f_s \in \Omega_h{}^2(\mathbb{R} \times M, \mathfrak{v}_0)$, $F_{ft} \in \Omega_h{}^2(\mathbb{R} \times M, \mathfrak{v}_1)$, $F_{fs} \in \Omega_h{}^3(\mathbb{R} \times M, \mathfrak{v}_1)$. Here, $(\omega_s, \Omega_{\omega s})$ is itself a $\mathfrak{v}$–connection doublet and (f_s, F_{fs}) is the associated curvature doublet. The CS_2 action (6.1.19) reads then as

$$CS_2(\omega, \Omega_\omega) = \kappa_2 \int_{\mathbb{R} \times M} dt \left[\frac{1}{2}(d_t\omega_s, \Omega_{\omega s}) \right. \tag{6.2.19}$$
$$\left. - \frac{1}{2}(\omega_s, d_t\Omega_{\omega s}) + (\omega_t, F_{fs}) + (f_s, \Omega_t) \right].$$

The field equations read then as

$$f_s = 0, \tag{6.2.20a}$$

$$F_{fs} = 0, \tag{6.2.20b}$$

$$d_t\omega_s - D_s\omega_t = 0, \tag{6.2.20c}$$

$$d_t\Omega_{\omega s} - D_s\Omega_{\omega t} = 0, \tag{6.2.20d}$$

where D_s denotes the covariant differentiation operator associated with the connection doublet $(\omega_s, \Omega_{\omega s})$ defined according to (3.2.13a), (3.2.13b) and $(\omega_t, \Omega_{\omega t})$ is treated as a bidegree $(0,0)$ field doublet.

The expressions of momenta $\Xi_{\xi t}$, $\Xi_{\xi s}$, ξ_t, ξ_s canonically conjugate to ω_t, ω_s, $\Omega_{\omega t}$, $\Omega_{\omega s}$ can easily be read off from (6.2.19). In virtue of the linear isomorphisms $\mathfrak{v}_0{}^\vee \simeq \mathfrak{v}_1$, $\mathfrak{v}_1{}^\vee \simeq \mathfrak{v}_0$ induced by the non singular bilinear pairing $(\cdot, \cdot)$ of $\mathfrak{v}_0$ and $\mathfrak{v}_1$, we have $\Xi_{\xi t} \in \Omega_{\mathrm{h}}{}^3(\mathbb{R} \times M, \mathfrak{v}_1)$, $\Xi_{\xi s} \in \Omega_{\mathrm{h}}{}^2(\mathbb{R} \times M, \mathfrak{v}_1)$, $\xi_t \in \Omega_{\mathrm{h}}{}^2(\mathbb{R} \times M, \mathfrak{v}_0)$, $\xi_s \in \Omega_{\mathrm{h}}{}^1(\mathbb{R} \times M, \mathfrak{v}_0)$ and

$$\Xi_{\xi t} = 0, \tag{6.2.21a}$$

$$\Xi_{\xi s} = \frac{\kappa_2}{2} \Omega_{\omega s}, \tag{6.2.21b}$$

$$\xi_t = 0, \tag{6.2.21c}$$

$$\xi_s = -\frac{\kappa_2}{2} \omega_s. \tag{6.2.21d}$$

Higher semistrict Chern–Simons theory, as ordinary one, is therefore constrained. This requires once more the application of Dirac's quantization algorithm. Its implementation turns out to be straightforward.

For notational convenience, below we set

$$\langle g, G \rangle = \int_M (g, G) \tag{6.2.22}$$

for $g \in \Omega^p(M, \mathfrak{v}_0)$, $G \in \Omega^{3-p}(M, \mathfrak{v}_1)$. Further, for any $\Omega^p(M, \mathfrak{v}_0)$–valued phase function ψ, we denote by G_ψ a $\Omega^{3-p}(M, \mathfrak{v}_1)$–valued phase constant and, for any $\Omega^p(M, \mathfrak{v}_1)$–valued phase function Ψ, we denote by g_Ψ a $\Omega^{3-p}(M, \mathfrak{v}_0)$–valued phase constant.

In the Hamiltonian formulation of $\mathrm{CS}_2(\mathbb{R} \times M, \mathfrak{v})$, the canonical field coordinates are $\omega_t \in \Omega^0(M, \mathfrak{v}_0)$, $\omega_s \in \Omega^1(M, \mathfrak{v}_0)$, $\Omega_{\omega t} \in \Omega^1(M, \mathfrak{v}_1)$, $\Omega_{\omega s} \in \Omega^2(M, \mathfrak{v}_1)$ and their canonically conjugate momenta are respectively $\Xi_{\xi t} \in \Omega^3(M, \mathfrak{v}_1)$, $\Xi_{\xi s} \in \Omega^2(M, \mathfrak{v}_1)$, $\xi_t \in \Omega^2(M, \mathfrak{v}_0)$, $\xi_s \in \Omega^1(M, \mathfrak{v}_0)$. The basic Poisson brackets are

$$\{\langle \omega_t, G_{\omega t} \rangle, \langle g_{\Xi_{\xi t}}, \Xi_{\xi t} \rangle\}_P = \langle g_{\Xi_{\xi t}}, G_{\omega t} \rangle, \tag{6.2.23a}$$

$$\{\langle \omega_s, G_{\omega s} \rangle, \langle g_{\Xi_{\xi s}}, \Xi_{\xi s} \rangle\}_P = \langle g_{\Xi_{\xi s}}, G_{\omega s} \rangle, \tag{6.2.23b}$$

$$\{\langle g_{\Omega_{\omega t}}, \Omega_{\omega t} \rangle, \langle \xi_t, G_{\xi t} \rangle\}_P = \langle g_{\Omega_{\omega t}}, G_{\xi t} \rangle, \tag{6.2.23c}$$

$$\{\langle g_{\Omega_{\omega s}}, \Omega_{\omega s} \rangle, \langle \xi_s, G_{\xi s} \rangle\}_P = \langle g_{\Omega_{\omega s}}, G_{\xi s} \rangle. \tag{6.2.23d}$$

The canonical Hamiltonian implied by (6.2.19) is

$$H = -\kappa_2 [\langle \omega_t, F_{fs} \rangle + \langle f_s, \Omega_{\omega t} \rangle]. \tag{6.2.24}$$

The primary constraints stemming from relations (6.2.21a)–(6.2.21d) are

$$\Xi_{\xi t} \approx 0, \tag{6.2.25a}$$

$$\frac{\kappa_2}{2}\Omega_{\omega s} - \Xi_{\xi s} \approx 0, \tag{6.2.25b}$$

$$\xi_t \approx 0, \tag{6.2.25c}$$

$$\frac{\kappa_2}{2}\omega_s + \xi_s \approx 0. \tag{6.2.25d}$$

Implementation of the Dirac's algorithm leads to the secondary constraints

$$f_s \approx 0, \tag{6.2.26a}$$

$$F_{fs} \approx 0 \tag{6.2.26b}$$

and no higher order constraints. Further, the phase functions ξ_t, $\Xi_{\xi t}$, f_s and F_{fs} are identified as generators of gauge symmetries. Gauge fixing is thus required. A complete fixing of the symmetry, however, leads to a problematic non local gauge fixed theory as in the ordinary case. To remain in the framework of local field theory, we fix only the gauge symmetry associated with ξ_t, $\Xi_{\xi t}$ leaving that corresponding to f_s and F_{fs} unfixed. The gauge fixing conditions we impose are

$$\omega_t \approx 0, \tag{6.2.27a}$$

$$\Omega_{\omega t} \approx 0. \tag{6.2.27b}$$

The constraints (6.2.25a)–(6.2.25d), (6.2.27a), (6.2.27b) form a second class set and, so they can be used to construct the Dirac brackets on the associated constrained phase space. The only independent phase variables remaining after the constraints are taken into account are ω_s, $\Omega_{\omega s}$ and their Dirac brackets are

$$\{\langle \omega_s, G_{\omega_s} \rangle, \langle g_{\Omega_{\omega s}}, \Omega_{\omega s} \rangle\}_D = \frac{1}{\kappa_2}\langle g_{\Omega_{\omega s}}, G_{\omega_s} \rangle. \tag{6.2.28}$$

The constraints (6.2.26a), (6.2.26b) are left pending. As it is immediate to see, f_s, F_{fs} generate constrained phase space BRST transformations. Introducing ghost fields $c_s \in \Omega^0(M, \mathfrak{v}_0[1])$ and $C_{cs} \in \Omega^1(M, \mathfrak{v}_1[1])$, we have

$$\{\langle f_s, C_{cs} \rangle + \langle c_s, F_{fs} \rangle, \langle \omega_s, G_{\omega_s} \rangle\}_D = \frac{1}{\kappa_2}\langle s_s\omega_s, G_{\omega_s} \rangle, \tag{6.2.29a}$$

$$\{\langle f_s, C_{cs} \rangle + \langle c_s, F_{fs} \rangle, \langle g_{\Omega_{\omega s}}, \Omega_{\omega s}\}_D = -\frac{1}{\kappa_2}\langle g_{\Omega_{\omega s}}, s_s\Omega_{\omega s} \rangle. \tag{6.2.29b}$$

where $s_s\omega_s$, $s_s\Omega_{\omega s}$ are given by

$$s_s\omega_s = -D_s c_s, \tag{6.2.30a}$$

$$s_s\Omega_{\omega s} = -D_s C_{cs}, \tag{6.2.30b}$$

in agreement with (3.2.84a), (3.2.84b).

We quantize $\mathrm{CS}_2(\mathbb{R} \times M, \mathfrak{v})$ by replacing the classical fields ω_s, $\Omega_{\omega s}$ satisfying the Dirac brackets (6.2.28) with corresponding quantum fields $\widehat{\omega}_s$, $\widehat{\Omega}_{\omega s}$ satisfying the commutation relations

$$[\langle \widehat{\omega}_s, G_{\omega_s} \rangle, \langle g_{\Omega_{\omega s}}, \widehat{\Omega}_{\omega s} \rangle] = \frac{i}{\kappa_2} \langle g_{\Omega_{\omega s}}, G_{\omega_s} \rangle. \tag{6.2.31}$$

The constraints (6.2.26a), (6.2.26b), which we left pending in the classical theory, translate into conditions obeyed by the state vectors Ψ of the theory

$$\langle \widehat{f}_s, G_{f_s} \rangle \Psi = 0, \tag{6.2.32a}$$

$$\langle g_{F_{fs}}, \widehat{F}_{fs} \rangle \Psi = 0. \tag{6.2.32b}$$

6.3 Choice of polarization and Ward identities

To build a representation of the operator algebra yielded by canonical quantization, we must choose a polarization, a maximal integrable distribution on the classical phase space, the restriction of the Dirac symplectic form to which vanishes. The polarization must be gauge invariant by consistency.

Henceforth, we shall make reference exclusively to the space manifold M. We shall thus suppress the index s throughout as it is no longer necessary lightening in this way the notation.

Ordinary Chern–Simons theory

In the canonically quantized $\mathrm{CS}_1(\mathbb{R} \times M, \mathfrak{g})$ theory reviewed in subsect. 6.2, the space manifold M is a 2–dimensional surface. The conventionally normalized Dirac symplectic form is in this case

$$\langle \delta \omega, \delta \omega \rangle = -2\kappa_1 \int_M (\delta \omega, \delta \omega). \tag{6.3.1}$$

This can be checked to be invariant under any gauge transformation $g \in \mathrm{OGau}\,(M, \mathfrak{g})$ acting by (3.1.29).

A generic phase space vector field is of the form

$$\left\langle g_{\frac{\delta}{\delta \omega}}, \frac{\delta}{\delta \omega} \right\rangle F = \int_M \left(g_{\frac{\delta}{\delta \omega}}, \frac{\delta F}{\delta \omega} \right) \tag{6.3.2}$$

where $\delta/\delta\omega$ is a $\Omega^1(M, \mathfrak{g})$–valued vector field. A standard polarization of the phase space ω is built as follows. One picks a complex structure on the surface M and uses the marks 10, 01 to denote the holomorphic and antiholomorphic components of a 1–form. Setting $\delta/\delta\omega^{10} = -i(\delta/\delta\omega)^{01}$, $\delta/\delta\omega^{01} = i(\delta/\delta\omega)^{10}$, the polarization is defined by the integrable distribution of the vector fields

$$\left\langle v_{\frac{\delta}{\delta \omega}}{}^{10}, \frac{\delta}{\delta \omega^{10}} \right\rangle, \tag{6.3.3}$$

where $v_{\delta/\delta\omega}{}^{10}(\omega)$ is a phase function. The distribution is gauge invariant, since one has ${}^g\delta/\delta\omega^{10} = g(\delta/\delta\omega^{10})$ for $g \in \mathrm{OGau}(M, \mathfrak{g})$.

With the above choice of polarization, the quantum Hilbert space $\mathcal{H}$ of the CS_1 theory consists of phase space functionals $\Psi(\omega)$ satisfying

$$\left\langle v_{\frac{\delta}{\delta\omega}}{}^{10}, \frac{\delta\Psi}{\delta\omega^{10}} \right\rangle = 0, \tag{6.3.4}$$

that is of holomorphic wave functionals $\Psi(\omega^{01})$. The Hilbert structure appropriate for $\mathcal{H}$, as realized in [80], is thus of the Bargmann type. The Ψ belonging to $\mathcal{H}$ must satisfy the formal square integrability condition

$$\int \mathcal{D}\omega^{01}\mathcal{D}\omega^{10} \exp\left(2i\kappa_1\langle\omega^{10},\omega^{01}\rangle\right)|\Psi(\omega^{01})|^2 < \infty, \tag{6.3.5}$$

where $\mathcal{D}\omega^{01}\mathcal{D}\omega^{10}$ is a formal functional measure. Note that a restriction on the sign of κ_1 is implied by the convergence of (6.3.5). The Hilbert inner product is correspondingly given by Bargmann expression

$$\langle\Psi_1,\Psi_2\rangle = \int \mathcal{D}\omega^{01}\mathcal{D}\omega^{10} \exp\left(2i\kappa_1\langle\omega^{10},\omega^{01}\rangle\right)\Psi_1(\omega^{01})^*\,\Psi_2(\omega^{01}). \tag{6.3.6}$$

The field operators $\widehat{\omega}^{01}$, $\widehat{\omega}^{10}$ satisfying (6.2.31) are represented by

$$\langle g_\omega{}^{10}, \widehat{\omega}^{01}\rangle = \langle g_\omega^{10}, \omega^{01}\cdot\rangle, \tag{6.3.7a}$$

$$\langle\widehat{\omega}^{10}, g_\omega{}^{01}\rangle = \left\langle -\frac{1}{2\kappa_1}\frac{\delta}{\delta\omega^{01}}, g_\omega{}^{01}\right\rangle. \tag{6.3.7b}$$

In virtue of the exponential factor in the inner product, one has $\widehat{\omega}^{01+} = \widehat{\omega}^{10}$ as required.

In the representation (6.3.7), the vanishing curvature constraint (6.2.16) takes the form

$$\left\langle d^{10}\omega^{01} - \frac{1}{2\kappa_1}\left(d^{01}\frac{\delta}{\delta\omega^{01}} + \left[\omega^{01},\frac{\delta}{\delta\omega^{01}}\right]\right), g_f \right\rangle\Psi(\omega^{01}) = 0, \tag{6.3.8}$$

This is a WZW type Ward identity determining the variation of $\Psi(\omega^{01})$ under an infinitesimal gauge transformation $u \in \mathfrak{oaut}(M,\mathfrak{g})$ with $u = \operatorname{ad}\theta$, $\dot{\sigma}_u = d\theta$ with θ being a bidegree $(0,0)$ field. Noting that the resulting variation of ω is

$$\delta_u\omega^{01} = D^{01}\theta \tag{6.3.9}$$

by (6.2.14), we have

$$\delta_u\Psi(\omega^{01}) = 2i\kappa_1\langle d^{10}\omega^{01}, \theta\rangle\Psi(\omega^{01}). \tag{6.3.10}$$

Therefore, the gauge variation of $\Psi(\omega^{01})$ under a finite gauge transformation $g \in \mathrm{OGau}(M,\mathfrak{g})$ is given by a universal multiplicative factor

$$\Psi({}^g\omega^{01}) = \exp(iS_{WZW1}(g,\omega^{01}))\Psi(\omega^{01}), \tag{6.3.11}$$

where $S_{WZW1}(g,\omega^{01})$ is the gauged WZW action. By consistency with the group action property of gauge transformation on connections, $S_{WZW1}(g,\omega^{01})$ obeys the Polyakov-Wiegmann identity

$$S_{WZW1}(h \diamond g, \omega^{01}) = S_{WZW1}(h, {}^g\omega^{01}) + S_{WZW1}(g, \omega^{01}) \quad \mathrm{mod}\ \ 2\pi. \tag{6.3.12}$$

To reproduce the infinitesimal variation (6.3.24), $S_{WZW1}(g, \omega)$ must satisfy the normalization condition

$$\delta_u S_{WZW1}(g, \widetilde{\omega}^{01})|_{g=i} = 2\kappa_1 \langle d^{10}\omega^{01}, \theta \rangle, \tag{6.3.13}$$

where the tilde notation indicates that δ_u is inert on ω^{01}. (6.3.12), (6.3.13) essentially determine the expression of $S_{WZW1}(g, \omega)$. When M is the boundary of a 3–fold B and g can be extended to an element of $\mathrm{OGau}(B, \mathfrak{g})$, we have

$$S_{WZW1}(g, \omega^{01}) = \kappa_1 \int_M \left[(\sigma_g{}^{10}, \sigma_g{}^{01}) - 2(\sigma_g{}^{10}, \omega^{01}) \right] \tag{6.3.14}$$

$$+ \frac{\kappa_1}{3} \int_B (\sigma_g, d\sigma_g) \mod 2\pi,$$

a classic result [83]. The independence of $\exp(iS_{WZW1}(g, \omega^{01}))$ from the choice of B requires that the CS_1 anomaly density 3–form $\kappa_1 q_1$ (cf. eq. (6.1.13)) integrates to an integer multiple of 2π on any closed 3–fold of the form $N = B \cup -B'$ with $\partial B = \partial B' = M$. This is how the quantization condition of κ_1 emerges in the canonical quantization of the CS_1 theory.

Semistrict Chern–Simons theory

In the canonically quantized $\mathrm{CS}_2(\mathbb{R} \times M, \mathfrak{v})$ theory worked out in subsect. 6.2, the space manifold M is a 3–dimensional space. The associated normalized Dirac symplectic form is in this case

$$\langle \delta\omega, \delta\Omega_\omega \rangle = \kappa_2 \int_M (\delta\omega, \delta\Omega_\omega). \tag{6.3.15}$$

The form is invariant under any 1–gauge transformation $g \in \mathrm{OGau}_1(M, \mathfrak{v})$ acting via (3.2.42). In 3 dimensions, 1– and 2–forms have the same number of functional degrees of freedom. The phase space has thus the usual Hamiltonian form.

The vector fields $\delta/\delta\omega$, $\delta/\delta\Omega_\omega$ are specified by the relation

$$\left[\left\langle g_{\frac{\delta}{\delta\omega}}, \frac{\delta}{\delta\omega} \right\rangle + \left\langle \frac{\delta}{\delta\Omega_\omega}, G_{\frac{\delta}{\delta\Omega_\omega}} \right\rangle \right] F \tag{6.3.16}$$

$$= \int_M \left[\left(g_{\frac{\delta}{\delta\omega}}, \frac{\delta F}{\delta\omega} \right) + \left(\frac{\delta F}{\delta\Omega_\omega}, G_{\frac{\delta}{\delta\Omega_\omega}} \right) \right],$$

for any phase function $F(\omega, \Omega_\omega)$. A canonical polarization in the phase space (ω, Ω_ω) is defined as follows. It is spanned by the vector fields of the form

$$\left\langle \frac{\delta}{\delta\Omega_\omega}, V_{\frac{\delta}{\delta\Omega_\omega}} \right\rangle, \tag{6.3.17}$$

where $V_{\delta/\delta\Omega_\omega}(\omega, \Omega_\omega)$ is a phase function and it is understood that $\delta/\delta\Omega_\omega$ does not act on $V_{\delta/\delta\Omega_\omega}$. The distribution (6.3.17) is clearly integrable. It is also checked that it is gauge invariant by noting that ${}^g\delta/\delta\omega = g_1(\delta/\delta\omega) +$ terms linear in $\delta/\delta\Omega_\omega$, ${}^g\delta/\delta\Omega_\omega = g_0(\delta/\delta\Omega_\omega)$ under a gauge transformation $g \in \mathrm{OGau}_1(M, \mathfrak{v})$.

With the above choice of polarization, the quantum Hilbert space $\mathcal{H}$ consists of phase space functionals $\Psi(\omega, \Omega_\omega)$ satisfying

$$\left\langle \frac{\delta\Psi}{\delta\Omega_\omega}, V_{\frac{\delta}{\delta\Omega_\omega}} \right\rangle = 0, \tag{6.3.18}$$

that is of wave functionals $\Psi(\omega)$ depending on ω only. The Ψ belonging to $\mathcal{H}$ must satisfy a square integrability condition of the form

$$\int \mathcal{D}\omega \, |\Psi(\omega)|^2 < \infty. \tag{6.3.19}$$

where $\mathcal{D}\omega$ is a suitable formal functional measure. The Hilbert inner product has then the familiar form

$$\langle \Psi_1, \Psi_2 \rangle = \int \mathcal{D}\omega \, \Psi_1(\omega)^* \Psi_2(\omega). \tag{6.3.20}$$

The field operators $\widehat{\omega}$, $\widehat{\Omega}_\omega$ satisfying (6.2.31) are represented by

$$\langle \widehat{\omega}, G_\omega \rangle = \langle \omega \cdot , G_\omega \rangle, \tag{6.3.21a}$$

$$\langle g_{\Omega_\omega}, \widehat{\Omega}_\omega \rangle = \Big\langle g_{\Omega_\omega}, -\frac{i}{\kappa_2} \frac{\delta}{\delta\omega} \Big\rangle. \tag{6.3.21b}$$

They are manifestly formally selfadjoint with respect to the Hilbert product (6.3.20): $\widehat{\omega}^+ = \widehat{\omega}$ and $\widehat{\Omega}_\omega{}^+ = \widehat{\Omega}_\omega$.

By (6.3.21), the constraints (6.2.32) take the form

$$\Big\langle d\omega + \frac{1}{2}[\omega, \omega] + \frac{i}{\kappa_2} \partial \frac{\delta}{\delta\omega}, G_f \Big\rangle \Psi(\omega) = 0, \tag{6.3.22a}$$

$$\Big\langle g_F, -\frac{i}{\kappa_2} \Big(d\frac{\delta}{\delta\omega} + \Big[\omega, \frac{\delta}{\delta\omega}\Big] \Big) - \frac{1}{6}[\omega, \omega, \omega] \Big\rangle \Psi(\omega) = 0. \tag{6.3.22b}$$

These are the Ward identities obeyed by Ψ. They determine the variation of $\Psi(\omega)$ under an infinitesimal gauge transformation $u \in \mathfrak{oaut}_0(M, \mathfrak{v})$ with $u = \operatorname{ad}\theta$, $\dot{\sigma}_u = d\theta + \partial\Theta_\theta$, $\dot{\Sigma}_u = d\Theta_\theta$, $\dot{\tau}_u(\pi) = -[\pi, \Theta_\theta]$, (θ, Θ_θ) being a bidegree $(0,0)$ field doublet. Noting that the resulting variation of ω is

$$\delta_u \omega = D\theta \tag{6.3.23}$$

(cf. eq. (6.2.30a)), we have

$$\delta_u \Psi(\omega) = i\kappa_2 \Big[\Big\langle d\omega + \frac{1}{2}[\omega, \omega], \Theta_\theta \Big\rangle - \frac{1}{6} \langle \theta, [\omega, \omega, \omega] \rangle \Big] \Psi(\omega). \tag{6.3.24}$$

Therefore, the gauge variation of $\Psi(\omega)$ under a finite gauge transformation $g \in \mathrm{OGau}_1(M, \mathfrak{v})$ is given by a universal multiplicative factor

$$\Psi({}^g\omega) = \exp(iS_{WZW2}(g, \omega))\Psi(\omega), \tag{6.3.25}$$

where $S_{WZW2}(g, \omega)$ is a higher analog of the gauged WZW action. In analogy to its ordinary counterpart, $S_{WZW2}(g, \omega)$ obeys a higher version of the Polyakov-Wiegmann identity

$$S_{WZW2}(h \diamond g, \omega) = S_{WZW2}(h, {}^g\omega) + S_{WZW2}(g, \omega) \mod 2\pi. \tag{6.3.26}$$

To reproduce the infinitesimal variation (6.3.24), $S_{WZW2}(g,\omega)$ must satisfy further the normalization condition

$$\delta_u S_{WZW2}(g,\widetilde{\omega})|_{g=i} = \kappa_2 \left[\left\langle d\omega + \frac{1}{2}[\omega,\omega], \Theta_\theta \right\rangle - \frac{1}{6}\langle \theta, [\omega,\omega,\omega]\rangle \right], \qquad (6.3.27)$$

where the tilde indicates that δ_u is inert on ω. An expression of $S_{WZW2}(g,\omega)$ fulfilling relations (6.3.26), (6.3.27) holding when M is the boundary of a 4–fold B and g can be extended to and element of $\mathrm{OGau}_1(B,\mathfrak{v})$ is

$$S_{WZW2}(g,\omega) = -\frac{\kappa_2}{2} \int_M \left[(\sigma_g - \omega, \tau_g(\sigma_g - \omega)) - 2(\omega - \sigma_g, \Sigma_g) \right. \qquad (6.3.28)$$

$$\left. + \frac{1}{3}(\sigma_g - \omega, g_1^{-1}g_2(\sigma_g - \omega, \sigma_g - \omega)) \right]$$

$$+ \frac{\kappa_2}{4} \int_B \left[2(d\sigma_g, \Sigma_g) - (\sigma_g, d\Sigma_g) \right] \mod 2\pi.$$

As in the ordinary case, the independence of $\exp(iS_{WZW2}(g,\omega))$ from the choice of B requires that the CS_2 anomaly density 4–form $\kappa_2 q_2$ (cf. eq. (6.1.31)) integrates to an integer multiple of 2π on any closed 4–fold of the form $N = B \cup -B'$ with $\partial B = \partial B' = M$. This will be the case if the pair $(N,\mathfrak{v})$ is admissible for a sufficiently broad class of closed 4-folds N, as we assumed earlier at the end of subsect. 6.1

The polarization we have constructed above is fully topological in the sense that its definition does not require the choice of any auxiliary structure on the threefold M. In this respect, the associated semistrict Chern–Simons theory is manifestly topological in a way ordinary Chern–Simons theory is not. There is however another choice of polarization more similar in flavour to standard Chern–Simons' in that it assumes the assignment of a strictly pseudoconvex CR structure on M.

We review briefly a few basic facts about CR structures to the reader's benefit. (See refs. [84, 85]for background material.) In a CR 3–fold M, the complexified cotangent bundle $T^*M \otimes \mathbb{C}$ has a direct sum decomposition $T^{*100}M \oplus T^{*010}M \oplus T^{*001}M$, where $T^{*100}M$, $T^{*010}M$, $T^{*001}M$ are line subbundles of $T^*M \otimes \mathbb{C}$, $T^{*001}M = \overline{T^{*100}M}$ and $T^{*010}M$ is the complexification of a trivial line subbundle E of T^*M, the one fiberwise generated by the underlying contact form. Forms of M are graded accordingly. For instance, a 1–form $\alpha \in \Omega^1(M)$ has three components, $\alpha = \alpha^{100} + \alpha^{010} + \alpha^{001}$. A 2–form $\beta \in \Omega^2(M)$ has also three components, $\beta = \beta^{110} + \beta^{101} + \beta^{011}$. Strictly pseudoconvex CR spaces are the closest 3–dimensional analog of Riemann surfaces. In particular, with the strictly pseudoconvex CR structure of a space there is associated a class of metrics, called Webster metrics, related to each other by a change of the normalization of the contact form, much as with a conformal structure of a surface there is associated a conformal class of metrics.

A second polarization of the phase space (ω, Ω_ω) is built as follows. One picks a strictly pseudoconvex CR structure on M. Setting $\delta/\delta\omega^{100} = -i(\delta/\delta\omega)^{011}$, $\delta/\delta\omega^{010} = -i(\delta/\delta\omega)^{101}$, $\delta/\delta\omega^{001} = -i(\delta/\delta\omega)^{110}$ and $\delta/\delta\Omega_\omega^{011} = -i(\delta/\delta\Omega_\omega)^{100}$, $\delta/\delta\Omega_\omega^{101} = -i(\delta/\delta\Omega_\omega)^{010}$, $\delta/\delta\Omega_\omega^{110} = -i(\delta/\delta\Omega_\omega)^{001}$, the polarization is spanned by the vector fields of the form

$$\left\langle \frac{\delta}{\delta\Omega_\omega^{110}}, V_{\frac{\delta}{\delta\Omega_\omega}}^{110} \right\rangle + \left\langle \frac{\delta}{\delta\Omega_\omega^{011}}, V_{\frac{\delta}{\delta\Omega_\omega}}^{011} \right\rangle + \left\langle v_{\frac{\delta}{\delta\omega}}^{010}, \frac{\delta}{\delta\omega^{010}} \right\rangle, \qquad (6.3.29)$$

where $V_{\delta/\delta\Omega_\omega}(\omega,\Omega_\omega)^{110}$, $V_{\delta/\delta\Omega_\omega}(\omega,\Omega_\omega)^{011}$, $v_{\delta/\delta\omega}(\omega,\Omega_\omega)^{010}$ are phase functions and again it is understood that $\delta/\delta\Omega_\omega{}^{110}$, $\delta/\delta\Omega_\omega{}^{011}$ does not act on $V_{\delta/\delta\Omega_\omega}{}^{110}$, $V_{\delta/\delta\Omega_\omega}{}^{110}$. It is easily checked that the distribution (6.3.29) is integrable. It is also checked that it is gauge invariant by noting that ${}^g\delta/\delta\omega^{010} = g_1(\delta/\delta\omega^{010}) +$ terms linear in $\delta/\delta\Omega_\omega{}^{110}$, $\delta/\delta\Omega_\omega{}^{011}$ and ${}^g\delta/\delta\Omega_\omega{}^{110} = g_0(\delta/\delta\Omega_\omega{}^{110})$, ${}^g\delta/\delta\Omega_\omega{}^{011} = g_0(\delta/\delta\Omega_\omega{}^{011})$ under a gauge transformation $g \in \mathrm{OGau}_1(M,\mathfrak{v})$.

With the above choice of polarization, the quantum Hilbert space $\mathcal{H}$ consists of phase space functionals $\Psi(\omega,\Omega_\omega)$ satisfying

$$\left\langle \frac{\delta\Psi}{\delta\Omega_\omega{}^{110}}, V_{\frac{\delta}{\delta\Omega_\omega}}{}^{110} \right\rangle + \left\langle \frac{\delta\Psi}{\delta\Omega_\omega{}^{011}}, V_{\frac{\delta}{\delta\Omega_\omega}}{}^{011} \right\rangle + \left\langle v_{\frac{\delta}{\delta\omega}}{}^{010}, \frac{\delta\Psi}{\delta\omega^{010}} \right\rangle = 0 \qquad (6.3.30)$$

that is of wave functionals $\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101})$. The Ψ must satisfy a square integrability condition of the form

$$\int \mathcal{D}\omega^{100}\mathcal{D}\omega^{001}\mathcal{D}\Omega_\omega{}^{101}\, |\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101})|^2 < \infty. \qquad (6.3.31)$$

where $\mathcal{D}\omega^{100}\mathcal{D}\omega^{001}\mathcal{D}\Omega_\omega{}^{101}$ is a suitable functional measure. The Hilbert inner product is then

$$\langle \Psi_1, \Psi_2 \rangle = \int \mathcal{D}\omega^{100}\mathcal{D}\omega^{001}\mathcal{D}\Omega_\omega{}^{101} \qquad (6.3.32)$$

$$\times\, \Psi_1(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101})^* \Psi_2(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}).$$

The field operators $\widehat{\omega}$, $\widehat{\Omega}_\omega$ satisfying (6.2.31) are realized as

$$\langle \widehat{\omega}^{100}, G_\omega{}^{011} \rangle = \langle \omega^{100} \cdot, G_\omega{}^{011} \rangle, \qquad (6.3.33\mathrm{a})$$

$$\langle \widehat{\omega}^{010}, G_\omega{}^{101} \rangle = \left\langle -\frac{1}{\kappa_2}\frac{\delta}{\delta\Omega_\omega{}^{101}}, G_\omega{}^{101} \right\rangle,$$

$$\langle \widehat{\omega}^{001}, G_\omega{}^{110} \rangle = \langle \omega^{001} \cdot, G_\omega{}^{110} \rangle,$$

$$\langle g_{\Omega_\omega}{}^{100}, \widehat{\Omega}_\omega{}^{011} \rangle = \left\langle g_{\Omega_\omega}{}^{100}, \frac{1}{\kappa_2}\frac{\delta}{\delta\omega^{100}} \right\rangle, \qquad (6.3.33\mathrm{b})$$

$$\langle g_{\Omega_\omega}{}^{010}, \widehat{\Omega}_\omega{}^{101} \rangle = \langle g_{\Omega_\omega}{}^{010}, \Omega_\omega{}^{101} \cdot \rangle,$$

$$\langle g_{\Omega_\omega}{}^{001}, \widehat{\Omega}_\omega{}^{110} \rangle = \left\langle g_{\Omega_\omega}{}^{001}, \frac{1}{\kappa_2}\frac{\delta}{\delta\omega^{001}} \right\rangle.$$

They satisfy the natural adjunction relations $\widehat{\omega}^{100+} = \widehat{\omega}^{001}$, $\widehat{\omega}^{010+} = \widehat{\omega}^{010}$ and $\widehat{\Omega}_\omega{}^{011+} = \widehat{\Omega}_\omega{}^{110}$, $\widehat{\Omega}_\omega{}^{101+} = \widehat{\Omega}_\omega{}^{101}$.

By (6.3.33), the constraints (6.2.32) presently read

$$\left\langle \frac{1}{\kappa_2}\left(d^{100}\frac{\delta}{\delta\Omega_\omega{}^{101}} + \left[\omega^{100}, \frac{\delta}{\delta\Omega_\omega{}^{101}}\right]\right.\right. \qquad (6.3.34\mathrm{a})$$

$$\left. + \partial\frac{\delta}{\delta\omega^{001}}\right) - d^{010}\omega^{100}, G_f{}^{001}\Big\rangle\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) = 0,$$

$$\langle d^{100}\omega^{001} + d^{001}\omega^{100} + [\omega^{100}, \omega^{001}] - \partial\Omega_\omega{}^{101}, G_f{}^{010}\rangle\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) = 0,$$

$$\Big\langle\frac{1}{\kappa_2}\Big(d^{001}\frac{\delta}{\delta\Omega_\omega{}^{101}} + \Big[\omega^{001}, \frac{\delta}{\delta\Omega_\omega{}^{101}}\Big]$$

$$\left. + \partial\frac{\delta}{\delta\omega^{100}}\right) - d^{010}\omega^{001}, G_f{}^{100}\Big\rangle\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) = 0,$$

$$\Big\langle g_F, \frac{1}{\kappa_2}\Big(d^{100}\frac{\delta}{\delta\omega^{100}} + d^{001}\frac{\delta}{\delta\omega^{001}} + \Big[\omega^{100}, \frac{\delta}{\delta\omega^{100}}\Big] + \Big[\omega^{001}, \frac{\delta}{\delta\omega^{001}}\Big] \qquad (6.3.34\mathrm{b})$$

$$-\Big[\frac{\delta}{\delta\Omega_\omega{}^{101}}, \Omega_\omega{}^{101}\Big] + \Big[\omega^{100}, \frac{\delta}{\delta\Omega_\omega{}^{101}}, \omega^{001}\Big]\Big) + d^{010}\Omega_\omega{}^{101}\Big\rangle\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) = 0.$$

In the fifth term of (6.3.34b), it is understood that $\delta/\delta\Omega_\omega{}^{101}$ is inert on $\Omega_\omega{}^{101}$. These are the Ward identities obeyed by Ψ in this CR canonical quantization scheme. They determine the variation of a $\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101})$ under an infinitesimal gauge transformation $u \in \mathfrak{oaut}_0(M, \mathfrak{v})$ of the form $u = \mathrm{ad}\,\theta$, $\dot{\sigma}_u = d\theta + \partial\Theta_\theta$, $\dot{\Sigma}_u = d\Theta_\theta$, $\dot{\tau}_u(\pi) = -[\pi, \Theta_\theta]$, (θ, Θ_θ) as earlier. The resulting variations of ω^{100}, ω^{001}, $\Omega_\omega{}^{101}$ are given by

$$\delta_u\omega^{100} = (D\theta)^{100} = d^{100}\theta + [\omega^{100}, \theta] + \partial\Theta_\theta{}^{100}, \qquad (6.3.35\mathrm{a})$$

$$\delta_u\omega^{001} = (D\theta)^{001} = d^{001}\theta + [\omega^{001}, \theta] + \partial\Theta_\theta{}^{001},$$

$$\delta_u\Omega_\omega{}^{101} = (D\Theta_\theta)^{101} = d^{100}\Theta_\theta{}^{001} + [\omega^{100}, \Theta_\theta{}^{001}] \qquad (6.3.35\mathrm{b})$$

$$+ d^{001}\Theta_\theta{}^{100} + [\omega^{001}, \Theta_\theta{}^{100}] - [z, \Omega_\omega{}^{101}] + [\omega^{100}, \omega^{001}, z]$$

(cf. eq. (6.2.30a)). On account of (6.3.35), we have

$$\delta_u\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) = i\kappa_2\big[\langle\theta, d^{010}\Omega_\omega{}^{101}\rangle \qquad (6.3.36)$$

$$+ \langle d^{010}\omega^{100}, \Theta_\theta{}^{001}\rangle + \langle d^{010}\omega^{001}, \Theta_\theta{}^{100}\rangle\big]\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}).$$

Therefore, the gauge variation of $\Psi(\omega)$ under a finite gauge transformation $g \in \mathrm{OGau}_1(M, \mathfrak{v})$ is given by a universal multiplicative factor

$$\Psi\big({}^g\omega^{100}, {}^g\omega^{001}, {}^g\Omega_\omega{}^{101}\big) \qquad (6.3.37)$$

$$= \exp(iS_{WZW2}(g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101}))\Psi(\omega^{100}, \omega^{001}, \Omega_\omega{}^{101}),$$

where $S_{WZW2}(g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101})$ is another higher analog of the gauged WZW action. Again, as its ordinary counterpart, it obeys a higher Polyakov-Wiegmann identity

$$S_{WZW2}(h \diamond g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) \qquad (6.3.38)$$

$$= S_{WZW2}(h, {}^g\omega^{100}, {}^g\omega^{001}, {}^g\Omega_\omega{}^{101}) + S_{WZW2}(g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) \quad \mathrm{mod}\ 2\pi$$

To reproduce the infinitesimal variation (6.3.36), $S_{WZW2}(g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101})$ must satisfy the normalization condition

$$\delta_u S_{WZW2}(g, \widetilde{\omega}^{100}, \widetilde{\omega}^{001}, \widetilde{\Omega}_\omega{}^{101})|_{g=i} \tag{6.3.39}$$

$$= \kappa_2 \big[\langle \theta, d^{010} \Omega_\omega{}^{101} \rangle + \langle d^{010} \omega^{100}, \Theta_\theta{}^{001} \rangle + \langle d^{010} \omega^{001}, \Theta_\theta{}^{100} \rangle \big]$$

where again the tilde notation indicates that δ_u is inert on ω^{100}, ω^{001}, $\Omega_\omega{}^{101}$. An expression of $S_{WZW2}(g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101})$ fulfilling relation (6.3.38) holding when M is the boundary of a 4–fold B and g can be extended to an element of $\mathrm{Gau}_1(B, \mathfrak{v})$ is

$$S_{WZW2}(g, \omega^{100}, \omega^{001}, \Omega_\omega{}^{101}) \tag{6.3.40}$$

$$= -\frac{\kappa_2}{2} \int_M \Big[2(\sigma_g{}^{100} - \omega^{100}, \tau_g{}^{010}(\sigma_g{}^{001} - \omega^{001}))$$

$$- 2(\omega^{100} - \sigma_g{}^{100}, \Sigma_g{}^{011}) - 2(\omega^{001} - \sigma_g{}^{001}, \Sigma_g{}^{110}) + 2(\sigma_g{}^{010}, \Omega_\omega{}^{101}) \Big]$$

$$+ \frac{\kappa_2}{4} \int_B \big[2(d\sigma_g, \Sigma_g) - (\sigma_g, d\Sigma_g) \big] \mod 2\pi,$$

where for the last term the same considerations as before hold. This action does not fulfill (6.3.39) however, but a weaker version of it,

$$\delta_u S_{WZW2}(g, \widetilde{\omega}^{100}, \widetilde{\omega}^{001}, \widetilde{\Omega}_\omega{}^{101})|_{g=i} \tag{6.3.41}$$

$$= \kappa_2 \big[\langle \theta, d^{010} \Omega_\omega{}^{101} \rangle + \langle d^{010} \omega^{100}, \Theta_\theta{}^{001} \rangle + \langle d^{010} \omega^{001}, \Theta_\theta{}^{100} \rangle$$

$$+ \langle d^{100} \omega^{001} + d^{001} \omega^{100} + [\omega^{100}, \omega^{001}] - \partial \Omega_\omega{}^{101}, \Theta_\theta{}^{010} \rangle \big].$$

This however poses no problem. By the second Ward identity (6.3.34a), the field functionals $\Psi(\omega^{001}, \Omega_\omega{}^{101})$ are supported precisely on the functional hypersurface $d^{100} \omega^{001} + d^{001} \omega^{100} + [\omega^{100}, \omega^{001}] - \partial \Omega_\omega{}^{101} = 0$. Thus the last offending term in (6.3.41) vanishes identically upon insertion in (6.3.37).

To summarize, we have found that, when certain conditions are met, semistrict higher Chern–Simons theory admits two distinct canonical quantizations and correspondingly two sets of higher WZW Ward identities each characterized by a gauged WZW action.

The first canonical quantization is manifestly topological, as it does not necessitate a choice of any additional structure on the spacial 3–fold. The second one requires instead a choice of a CR structure on the latter. The unitary equivalence of the quantizations associated with distinct CR structures is an open problem. A solution of it on the same lines as that presented in ref. [80] for the ordinary case requires a full fledged deformation theory of CR structure, which to the best of our knowledge is missing presently. Furthermore, the relationship between the the topological and CR quantizations remains mysterious.

It would be interesting to investigate the properties of the solutions of the Ward identities for both canonical quantizations. Here, we limit ourselves to observe that the solutions are generically functional distributions. For instance, the second Ward

identity (6.3.34a) entails that the wave functional is supported on connections with vanishing 101 curvature component and thus exhibits a corresponding functional Dirac delta singularity.

6.4   Examples

We present a few examples to illustrate the higher Chern–Simons theory developed in subsect. 6.1.

Balanced differential Lie crossed modules

A differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$ is balanced if it is so when viewed as a strict Lie 2–algebra. Thus, $(\mathfrak{g}, \mathfrak{h})$ is balanced if it is equipped with a non singular bilinear pairing $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$ such that

$$(\tau(X), Y) - (\tau(Y), X) = 0, \tag{6.4.1a}$$

$$([\pi, x], X) + (x, \mu(\pi)(X)) = 0 \tag{6.4.1b}$$

(cf. eqs. (2.4.122), (2.4.123)). Below, we assume that $(\mathfrak{g}, \mathfrak{h})$ is the differential Lie crossed module of a Lie crossed module (G, H).

By (6.1.19), since the three argument bracket vanishes in the present case, the higher Chern–Simons theory $\mathrm{CS}_2(N, \mathfrak{g}, \mathfrak{h})$ is formally a BF theory, with the 2 form connection component playing the role of the B field. This conclusion is however unwarranted, because the symmetry structure of $\mathrm{CS}_2(N, \mathfrak{g}, \mathfrak{h})$ is basically different from that of an ordinary BF model.

There exists a distinguished 2–subgroup $\overline{\mathrm{Gau}}(N, G, H)$ of the gauge transformation strict 2–group $\mathrm{Gau}(N, \mathfrak{g}, \mathfrak{h})$ [21]. The 1–gauge transformations belonging to $\overline{\mathrm{Gau}}(N, G, H)$ are of the form

$$g_\gamma = \phi_\gamma, \tag{6.4.2a}$$

$$\sigma_{g_\gamma} = \gamma^{-1} d\gamma + \mathrm{Ad}\, \gamma^{-1}(\tau(\chi_\gamma)), \tag{6.4.2b}$$

$$\Sigma_{g_\gamma} = \dot{m}(\gamma^{-1})\Big(d\chi_\gamma + \frac{1}{2}[\chi_\gamma, \chi_\gamma]\Big), \tag{6.4.2c}$$

$$\tau_{g_\gamma}(x) = \mu(x)(\dot{m}(\gamma^{-1})(\chi_\gamma)), \tag{6.4.2d}$$

where $\gamma \in \mathrm{Map}(N, G)$, $\chi_\gamma \in \Omega^1(N, \mathfrak{h})$. Here, for $a \in G$, $\phi_a \in \mathrm{Aut}_1(\mathfrak{v})$ is defined by $\phi_{a0}(\pi) = \mathrm{Ad}\, a(\pi)$, $\phi_{a1}(\Pi) = \dot{m}(a)(\Pi)$ and $\phi_{a2}(\pi, \pi) = 0$ and (6.4.2a) is understood to hold pointwise on N. τ, μ, t and m are related by (2.4.46), (2.4.47) and $\dot{m}$ is given by (2.4.48). For two 1–gauge transformations g_ζ, g_η associated with the data $\zeta, \eta \in \mathrm{Map}(N, G)$ and $\chi_\zeta, \chi_\eta \in \Omega^1(N, \mathfrak{h})$, the 2–gauge transformations of $\overline{\mathrm{Gau}}(N, G, H)$ with source g_ζ and target g_η are those of the form

$$F_\Lambda(x) = \Phi_{\zeta,\Lambda}(x), \tag{6.4.3a}$$

$$A_{F_\Lambda} = \dot{m}(\zeta^{-1})(-\Lambda^{-1}d\Lambda + \chi_\zeta + \mathrm{Ad}\,\Lambda^{-1}(B_\Lambda - \chi_\zeta)), \tag{6.4.3b}$$

where $\Lambda \in \mathrm{Map}(N, H)$ and $B_\Lambda \in \Omega^1(N, \mathfrak{h})$ with

$$\eta = t(\Theta)\zeta, \tag{6.4.4a}$$

$$\chi_\zeta - \chi_\eta = B_\Lambda. \tag{6.4.4b}$$

Here, for $a \in G$ and $A \in H$, $\Phi_{a,A}$ is defined by $\Phi_{a,A}(\pi) = Q(\mathrm{Ad}\,a(\pi), A)$ and (6.4.3a) is understood to hold pointwise on N. Q is given by (2.4.49).

Let (ω, Ω_ω) be a connection doublet and (f, F_f) be its curvature doublet. Inserting eqs. (6.4.2b)–(6.4.2d) into the relations (3.2.42), we obtain

$$^{g_\gamma}\omega = \mathrm{Ad}\,\gamma(\omega) - d\gamma\gamma^{-1} - \tau(\chi_\gamma), \tag{6.4.5a}$$

$$^{g_\gamma}\Omega_\omega = \dot{m}(\gamma)(\Omega_\omega) - d\chi_\gamma - \frac{1}{2}[\chi_\gamma, \chi_\gamma]. \tag{6.4.5b}$$
$$- \mu(\mathrm{Ad}\,\gamma(\omega) - d\gamma\gamma^{-1} - \tau(\chi_\gamma))(\chi_\gamma)$$

Inserting eqs. (6.4.2b)–(6.4.2d) into (3.2.43), we find further

$$^{g_\gamma}f = \mathrm{Ad}\,\gamma(f), \tag{6.4.6a}$$

$$^{g_\gamma}F_f = \dot{m}(\gamma)(F_f) - \mu(\mathrm{Ad}\,\gamma(f))(\chi_\gamma). \tag{6.4.6b}$$

These expressions are identical to those obtained originally in refs. [47, 48].

The anomaly $Q_2(g_\gamma)$ turns out to vanish for all 1–gauge transformations g_γ of $\overline{\mathrm{Gau}}(N, G, H)$. Indeed, the anomaly density q_2 is exact

$$q_2 = \frac{1}{2}(\tau(\Sigma_{g_\gamma}), \Sigma_{g_\gamma}) = \frac{1}{2}d\left(\tau(\chi_\gamma), d\chi_\gamma + \frac{1}{3}[\chi_\gamma, \chi_\gamma]\right). \tag{6.4.7}$$

Therefore the higher Chern–Simons theory $\mathrm{CS}_2(N, \mathfrak{g}, \mathfrak{h})$ is non anomalous, at least when restricting to the 1–gauge transformations drawn from $\overline{\mathrm{Gau}}(N, G, H)$, and there is no level quantization.

Balanced Lie 2–algebra $\mathfrak{v}$ with invertible ∂

Let $\mathfrak{v}$ be a balanced Lie 2–algebra with invariant form such that ∂ is invertible. Then, the gauge anomaly $Q_2(g)$ of the classical action of the Chern–Simons theory $\mathrm{CS}_2(N, \mathfrak{v})$ vanishes identically. Indeed, the Chevalley–Eilenberg cocycle $\chi_2 \in \mathrm{CE}^4(\mathfrak{v})$ of eq. (6.1.34) turns out to be exact in this case, being

$$\chi_2 = \mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}\frac{1}{2}\left(\pi, \Pi - \frac{1}{6}\partial^{-1}[\pi, \pi]\right) \tag{6.4.8}$$

and, as we have shown in sect 6.1, this implies that $Q_2(g) = 0$. Consequently, in this case too the higher Chern–Simons theory $\mathrm{CS}_2(N, \mathfrak{v})$ is non anomalous and there is no level quantization.

Balanced Lie 2–algebra $\mathfrak{v}$ with vanishing ∂

In the category of Lie 2–algebras, seen as 2–term L_∞ algebras, every Lie 2–algebra $\mathfrak{v}$ is equivalent to one with vanishing boundary map ∂. We are thus led to consider a balanced Lie 2–algebra $\mathfrak{v}$ with invariant form such that $\partial = 0$. By (2.4.37), $\mathfrak{v}_0 = \mathfrak{g}$ is a Lie algebra with brackets $[\cdot, \cdot]$. Since the invariant form $(\cdot, \cdot)$ is non singular, $\mathfrak{v}_1 = \mathfrak{g}^*$ with duality pairing $\langle \cdot, \cdot \rangle = (\cdot, \cdot)$. By the invariance of the pairing $(\cdot, \cdot)$, eq. (2.4.123), $\mathfrak{v}_1$ is just the coadjoint $\mathfrak{g}$–module. The property (2.4.39) is equivalent to the three argument bracket $[\cdot, \cdot, \cdot]$ defining a $\mathfrak{g}^*$–valued Chevalley–Eilenberg cocycle $\phi \in \mathrm{CE}^3(\mathfrak{g}, \mathfrak{g}^*)$

Recall that the Chevalley–Eilenberg complex $\mathrm{CE}^*(\mathfrak{g}, \mathfrak{g}^*)$ of $\mathfrak{g}$ with values in $\mathfrak{g}^*$ is the graded vector space $\mathrm{Fun}(\mathfrak{g}[1], \mathfrak{g}^*)$ equipped with the coboundary operator $\mathcal{Q}_{\mathrm{CE}(\mathfrak{g}, \mathfrak{g}^*)}$ defined by

$$\mathcal{Q}_{\mathrm{CE}(\mathfrak{g}, \mathfrak{g}^*)}\phi(\pi, \ldots, \pi) = [\pi, \phi(\pi, \ldots, \pi)] - \frac{p}{2}\phi([\pi, \pi], \pi, \ldots, \pi),$$

for a p–cochain $\phi \in \mathrm{CE}^p(\mathfrak{g}, \mathfrak{g}^*)$ seen as as a linear map $\phi \in \mathrm{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g}^*)$. The associated cohomology is $H_{\mathrm{CE}}^*(\mathfrak{g}, \mathfrak{g}^*)$. A p–cochain $\phi \in \mathrm{CE}^p(\mathfrak{g}, \mathfrak{g}^*)$ is cyclic if

$$\langle x, \phi(y, \pi, \ldots, \pi)\rangle + \langle y, \phi(x, \pi, \ldots, \pi)\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of $\mathfrak{g}$. The cyclic cochain form a subcomplex $\mathrm{CCE}^*(\mathfrak{g}, \mathfrak{g}^*)$ of $\mathrm{CE}^*(\mathfrak{g}, \mathfrak{g}^*)$ with cohomology $H_{\mathrm{CCE}}^*(\mathfrak{g}, \mathfrak{g}^*)$ isomorphic to $H_{\mathrm{CE}}^*(\mathfrak{g})[-1]$, the -1 degree shifted real valued cohomology of $\mathfrak{g}$ [52]. The correspondence is defined by

$$\hat{\phi}(\pi, \ldots, \pi) = \frac{1}{p+1}\langle \pi, \phi(\pi, \ldots, \pi)\rangle$$

at the level of representatives. (See also [86] for reference.) On account of the cyclicity property (2.4.124), ϕ is cyclic and, so,

$$\hat{\phi} = \frac{1}{4}\langle \pi, [\pi, \pi, \pi]\rangle, \tag{6.4.9}$$

is a Chevalley–Eilenberg cocycle $\phi \in \mathrm{CE}^4(\mathfrak{g})$. $\hat{\phi}$ is in fact simply related to the Chevalley–Eilenberg cocycle $\chi_2 \in \mathrm{CE}^4(\mathfrak{v})$ of eq. (6.1.34).

$$\chi_2 = -\hat{\phi}/6 \tag{6.4.10}$$

Since $\mathrm{CE}^*(\mathfrak{g})$ is a subcomplex of $\mathrm{CE}^*(\mathfrak{v})$ when $\partial = 0$ by (2.1.14) and (2.4.40a), χ_2 is exact in $\mathrm{CE}^*(\mathfrak{v})$ if $\hat{\phi}$ is in $\mathrm{CE}^*(\mathfrak{g})$. In that case, we have $Q_2(g) = 0$ and there is no level quantization in the associated $\mathrm{CS}_2(N, \mathfrak{v})$ Chern–Simons model. If the 4–cocycle $\hat{\phi}$ is not a coboundary, then $Q_2(g)$ may be non trivial and level quantization may obtain. Now $H_{\mathrm{CE}}^4(\mathfrak{g}) = 0$ for all simple Lie algebras $\mathfrak{g}$. $H_{\mathrm{CE}}^4(\mathfrak{g}) \neq 0$, e. g. $\mathfrak{g} = \mathfrak{u}(n)$ with $n \geq 2$. Below, we assume tacitly that manifold on which fields are defined is oriented and that the fields satisfy asymptotic or boundary conditions allowing for the convergence of the integration and integration by parts.

Chapter 7

Outlook and open problems

Our study on the higher Chern-Simons theory is divided roughly in two parts. The first part is devoted to the analysis of the gauge invariance of higher Chern–Simons theory. We find that, analogously to ordinary Chern–Simons theory, the higher Chern–Simons action is invariant under a higher gauge transformation up to a higher winding number only. Full gauge invariance of the quantum theory requires that the winding number be quantized in appropriate units. In all the examples which we have been able to work out in detail, the winding number actually vanishes, but we cannot prove its quantization in general and we are forced to assume it as a working hypothesis. This is a first aspect of the theory that requires further investigation.

The second part deals with quantization. Several approaches to the problem of quantization are possible in principle. Perturbative quantization based on a straightforward extension of Lorenz gauge fixing involves the choice of a background metric on the base manifold as well as the introduction of Faddeev–Popov ghost and ghost for ghost fields. In the presence of a metric we cannot maintain gauge covariance without resorting to gauge rectifiers whose existence and interpretation is still problematic [21]. We are left with canonical quantization. We find that the theory admits two apparently inequivalent canonical quantizations. We obtain correspondingly two sets of higher WZW Ward identities and we find the explicit expressions of two higher versions of the gauged WZW action.

The canonical quantization of the first kind is manifestly topological in that it does not require a choice of any additional structure on the spacial 3–fold. That of the second kind involves fixing a CR structure on the latter. This is more akin to ordinary Chern–Simons theory's canonical quantization. CR spaces are in fact in many ways the closest 3–dimensional analog of Riemann surfaces. The unitary equivalence of the quantization associated with distinct CR structures is an open problem necessitating a non trivial extension of the analysis of ref. [80]. Furthermore, the relationship between the the topological and CR quantizations remains elusive.

It is necessary to clarify a point on the higher WZW actions emerging in the process of canonically quantizing our higher Chern–Simons theory. They encode the gauge covariance of the relevant wave functionals and, so, are determined by the Ward identities these obey and by a cocycle conditions extending the familiar Polyakov–Wiegmann relation. Presently, however, we have no evidence that they are related to some kind of 3–dimensional sigma model as the ordinary gauged WZW action, al-

though this remains a distinct possibility. In this respect it may be more useful to consider the restriction of the higher Chern–Simons action to flat connection configurations expressed as gauge transformation of the trivial connection on the same lines as [79].

The solution of the questions raised in the preceding paragraphs requires a more fundamental theory of higher gauge transformation than that employed in the present paper. Until recently, this was available only for the strict case [47, 48]. Promising new results in this direction can be found in ref. [74] .

Using the results of the present work and restricting to the flat case, we plan to reconsider in the companion paper [87] the theory of higher holonomy, already studied in [41, 42, 43] and reanalyzed recently in a very general setting in [45, 46], and tackle the problem of the proper definition of higher holonomy invariants. The quest for the latter is particularly important for the applications they may have in a study of 2–knots in 4–folds based on the higher Chern–Simons theory. (See ref. [88] for a related endeavour.)

Our 2-term L_∞ Chern-Simons theory has non-strict higher gauge structure, while higher parallel transport works only for strict 2-groups. Nevertheless, to any semistrict connection doublet we can associate a strict one by suitably applying the adjoint functor of the 2-term L_∞ algebra (see prop. 20). Even more, it is possible to make this association a strict 2-functor. Recall that given an orientable smooth manifold M and a weak 2-term L_∞-algebra $\mathfrak{v}$ we can define the strict 2-groupoid of $\mathfrak{v}$-connection doublets on M with vanishing fake curvature, which we call $\texttt{Conn}_f(M, \mathfrak{v})$ (see subsect. 3.2.3).

It is also possible to define the strict 2-groupoid $\texttt{Conn}_f(M, (\mathfrak{g}, \mathfrak{h}))$ of $(\mathfrak{g}, \mathfrak{h})$-connection doublets with vanishing fake curvature on a smooth manifold M for $(\mathfrak{g}, \mathfrak{h})$ a differential Lie crossed module:

Objects Objects are, as before, connection doublets $(A, B) \in (\Omega^1(M) \otimes \mathfrak{g}) \oplus (\Omega^2(M) \otimes \mathfrak{h})$ with vanishing fake curvature, $dA + \frac{1}{2}[A, A] - \dot{t}(B) = 0$.

1-Morphisms Given two connection doublets (A, B) and (A', B'), a 1-morphism $\gamma : (A, B) \to (A', B')$ is a couple (γ, χ_γ) made of a map $\gamma : M \to G$ and a 1-form $\chi_\gamma \in \Omega^1(M) \otimes \mathfrak{h}$ such that

$$A' = \mathrm{Ad}_\gamma A - \dot{t}(\chi_\gamma) - \gamma^* \mu_G, \tag{7.0.1}$$

$$B' = \dot{m}(\gamma, B) - d\chi_\gamma + \frac{1}{2}[\chi_\gamma, \chi_\gamma]_\mathfrak{h} - [\mathrm{Ad}_\gamma A, \chi_\gamma] + [\gamma^* \mu_G, \chi_\gamma], \tag{7.0.2}$$

where μ_G is the Maurer-Cartan 1-form on G.

2-Morphisms Given two 1-morphisms γ, ξ with the same source (A, B) and target (A', B'), a 2-morphism $\theta : \gamma \Rightarrow \xi$ is a map $\theta : M \to H$ such that

$$\xi = t(\theta)\gamma, \tag{7.0.3}$$

$$\chi_\xi = -Q(A', \theta) + \mathrm{Ad}_\theta \chi_\gamma - \theta^* \mu_H. \tag{7.0.4}$$

We are now going to define a strict 2-functor Φ from the strict 2-groupoid $\texttt{Conn}_f(M, \mathfrak{v})$ to the strict 2-groupoid $\texttt{Conn}_f(M, (\mathfrak{aut}_0(\mathfrak{v}), \mathfrak{aut}_1(\mathfrak{v})))$, where $\mathfrak{v}$ is an arbitrary weak 2-term L_∞ algebra and $\mathfrak{aut}(\mathfrak{v})$ is the strict 2-term L_∞ algebra associated to the strict 2-group $\text{Aut}(\mathfrak{v})$:

Objects On objects the functor Φ is:

$$\Phi(\omega) = (\text{ad}_0(\omega), \text{ad}_1(\Omega_\omega) - \frac{1}{2} \text{ad}_2(\omega, \omega)) =: (\tilde{\omega}, \tilde{\Omega}). \qquad (7.0.5)$$

1-Morphisms To a 1-morphism $g = (g, \sigma_g, \Sigma_g, \tau_g) : \omega \to \omega'$ in $\texttt{Conn}_f(M, \mathfrak{v})$ the functor associates a 1-morphism $\Phi(g) : (\tilde{\omega}, \tilde{\Omega}) \to (\tilde{\omega}', \tilde{\Omega}')$ which is the couple consisting of the map g itself and the 1-form

$$\phi_g(x) = g_1 \tau_g g_0^{-1} x - g_2(\omega - \sigma_g, g_0^{-1} x). \qquad (7.0.6)$$

From the definition of $(\tilde{\omega}, \tilde{\Omega})$ and from the target matching condition for g it follows that $\Phi(g)$ also fulfills the target matching condition. The units are preserved, and we have that

$$\Phi(h \circ g) = \Phi(h) \circ \Phi(g). \qquad (7.0.7)$$

2-Morphisms To a 2-morphism $F : g \Rightarrow h$ in $\texttt{Conn}_f(M, \mathfrak{v})$ the functor associates the 2-morphism in $\texttt{Conn}_f(M, (\mathfrak{aut}_0(fv), \mathfrak{aut}_1(\mathfrak{v})))$ defined by $\Phi(F) = F g_0^{-1}$. Target matching condition, units and both compositions are preserved.

This result makes it possible to employ semistrict connection doublet in the computation of surface holonomies, and this in turn may be exploited to build gauge invariant observables for the higher Chern-Simons alike the knot invariants that play a role in ordinary Chern-Simons theory. Indeed, if a $\mathfrak{v}$ connection doublet ω is flat, $F = 0$, then the strict doublet $\Phi(\omega)$ also is flat. This means that the critical points of the higher Chern-Simons action would generate homotopy invariant Wilson surfaces, as happens in the ordinary Chern-Simons regarding Wilson loops. But for all this to work, many steps still have to be climbed and many holes have to be filled. The biggest gap is the definition of trace: this is essential in ordinary gauge theory to extrapolate a real number from a Wilson loops and to ensure full gauge invariance. At the state of the art a suitable notion of representation for Lie crossed modules is lacking, and therefore a working notion of trace is still missing. Wilson surfaces behave in a very particular manner under gauge 1-transformations, and although for closed surfaces this behaviour simplifies it still non trivial to build some kind of map from a Lie crossed module to the real numbers that renders Wilson surfaces gauge invariant.

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