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Max Abraham's and Tullio Levi-Civita's approach to Einstein Theory of Relativity

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Introduction

This thesis deals with the theory of relativity and its diffusion in Italy in the first decades of the XX century. Albert Einstein's theory of Special and General relativity is deeply linked with Italy and Italian scientists.
Not many scientists really involved themselves in that theory understanding, but two of them, Max Abraham and Tullio Levi-Civita left a deep mark in the theory development. Max Abraham engaged a real battle against Einstein between 1912 and 1914 about electromagnetic theories and gravitation theories, while Levi-Civita played a fundamental role in giving Einstein the correct mathematical instruments for the general relativity formulation since 1915.

Many studies have already been done to explain their role in the development of Einstein theory from both a historical and a scientific point of view. This work, which doesn't have the aim of a mere historical chronicle of the events, wants to highlight two particular perspectives.

1. Abraham's objections against Einstein focused on three important conceptual kernels of theory of relativity: the constancy of light speed, the relativity principle and the equivalence hypothesis. Einstein was forced by Abraham to explain scientific and epistemological reasons of the formulation of his theory. In that occasion Einstein gave also rigorous logical structure to the reasoning sequence appeared in different instants in the papers published in those years. The possibility of taking la careful look at the basics of the theory explained by the author

himself makes the contents of this contrast an exceptional resource for understanding Einstein's thinking.

2. Tullio Levi-Civita, after some years of waiting and seeing, accepted to be involved by Abraham himself in the discussion about Einstein theory of relativity since 1915. As known, Levi-Civita's involvement gave Einstein the possibility to correct some errors and come to a definite version of his theory using the absolute differential calculus by Ricci and Levi-Civita himself.

   Levi-Civita's contribution to the theory of Relativity has already been analyzed by many authors from both a scientific and a historical point of view. The aim of this work is to underline that Levi-Civita's particular approach to relativity was a significant interpretation of the same theory, a little different from the original one in some aspects.

   Levi-Civita began to involve himself in Relativity, when Einstein had already published the first *Entwurf* of General Relativity. In that period the research of an invariant form for the gravitational field equations led physicists and mathematicians to propose a modification of Hamilton variational principle in order to make it invariant with respect to every coordinates transformation. We are going to show that this kind of approach, shared by many authors in facing General Relativity, is historically proposed by Levi-Civita as key even to Special Relativity.

The first chapter presents Relativity spread in Italy just from a historical point of view, highlighting its main characters and events.These historical remarks will allow us to have a correct background in order to understand the meaning of the scientific themes discussed in the following chapters.

The second chapter presents the analysis of the three conceptual kernels of Max Abraham's objections against Einstein's theory of Relativity mentioned before.

The scientific nodes of the debate are analyzed from 1905 to 1914, through an exam of the original articles. The aim of this part of the work is mainly to compare authors ideas as they were originally explained.

The third chapter focuses the attention on Levi-Civita's interpretation of Special Relativity. Levi-Civita played a fundamental role in giving a correct form to General Relativity. This special kind of approach made him formulate Special Relativity in a very original way, as it arises from his lecture entitled *How can a conservative get to the threshold of new mechanics.* In this work part a deep analysis of the contents of this lecture is made together with the attempt to reconstruct the mathematical steps that were neglected.

In the conclusion we outline some general reflections which could be future research perspectives.

Contents

Chapter 1

Historical Remarks

It's useful to give some historical background of the Italian Relativity reception before entering into details of Max Abraham and Levi-Civita role in it.[1]

One of the first traces of Italian diffusion of Relativity is 1906 Roberto Marcolongo's paper about Lorentz transformations [?]. It witnesses that the mathematicians and not the physicists were the first to give attention to the theory of Relativity. In 1907 Physics Review entitled "*Le recenti teorie elettromagnetiche e il moto assoluto*" by Orso Mario Corbino [5], Einstein theory is not even mentioned among the contributions about moving electrons behavior.
According to Corbino, the experiments which had revealed the existence of the corpuscles called *electrons*, led, as consequence, to suppose electrons having "una massa apparente di natura elettromagnetica, variabile col cambiare della velocità".[2] Lorentz theory of deformable electron and Abraham theory of rigid electron were the two theories which contended these phenomena explanation.

[1]For a more complete chronicle of the Relativity events in Italy see [1], [2] and [3]. About the same arguments see also [4]

[2]" apparent electromagnetic mass, changing with speed"

In 1911, Tullio Levi-Civita, describing Italian mathematical physics evolution [6] between 1860 and 1910, touched on mechanics basics critique by Einstein, writing:

> c'è anche in meccanica la tendenza rivoluzionaria. Tale apparisce agli ortodossi seguaci di Newton e Lagrange quella che, in nome del principio di relatività di Lorentz-Einstein, è condotta a fondere i concetti di spazio e di tempo e a negare l'invariabilità della massa. Si renderebbe in conseguenza necessaria una ricostruzione *ab imis* di tutta la filosofia naturale. Attendiamo per giudicare. Basti intanto riconoscere l'importanza dell'attuale movimento relativista e l'influsso innovatore che esso va suscitando[3]

Levi-Civita is taking time, but recognizes Einstein relativity principle to be inherently untouchable, even if it brings usual ways of thinking in question. He stays in such a waiting attitude until Max Abraham, become Rational Mechanics professor at the Milan Politecnico in 1909, started a real controversy against Albert Einstein about Special Relativity contents.
Even if with criticism, Max Abraham's objections against Einstein theory contribute to spread Special Relativity contents on Italian journals. However the number of Italian scientists really interested in debate's contents is very small. The debate mentioned, developed between 1912 and 1914, will be analyzed in the next chapter with big attention, because all the conceptual kernels of Einstein theory are discussed, clarified and developed in it.

During those years, Einstein has published, among the other important papers[4], the famous *Entwurf* [12], written in collaboration with Marcel Gross-

[3]There's a revolutionary tendency in mechanics too. That is the one which leads to merge space and time concepts together and to refuse mass invariance, in Lorentz-Einstein relativity principle's name. This idea appears to be revolutionary to Newton and Lagrange's disciples.As consequence all the natural philosophy has to be built *ab imis* again. Let's wait and see. It suffices, for now, to acknowledge the relativistic movement importance and the innovator influence it's raising.

[4]See [7], [8], [9], [10], [11]

man. That is the nearest model to the General Einstein theory, in which he defined the mathematical structure and the physical basics almost completely.
In this paper Einstein and Grossman considered the possibility of using the Ricci curvature scalar as the Hamilton function inside a suitable variational principle and obtaining field equations generally covariant. Even in the expositions and explanations published in 1914[5] Einstein kept this variational approach, extending the invariance group of gravitational equations to the maximum.

The choice of the correct Hamilton function to be put as the basis of the variational principle was the reason of the last Abraham attack against Einstein. In 1915, in fact, Abraham was sent off from Italy because of his German nationality.
Levi-Civita was involved by Abraham in this controversy. So he decided to devote in person in the theory and started a correspondence with Einstein which became of fundamental historical importance.

Only thanks to Levi-Civita's help, Einstein managed to solve the problems of his General theory. Einstein and Levi-Civita's collaboration continued for some years, even with moments of controversy and independent work. Many studies have already been done about this collaboration and its importance, so it won't be examined in depth here.

It's important to underline, instead, that, since 1917, while Levi-Civita took part as a protagonist at the most important international debates about Einstein theories, in Italy, with the war ongoing, every German expression is totally refused. That is the case of theory of Relativity.
The mathematical physicists, Roberto Marcolongo, Gian Antonio Maggi e Tommaso Boggio were the only ones interested in it, even if with very differ-

[5] See [**?**] and [**?**]

ent attitudes and approaches. All of them had Levi-Civita's works as point of reference and approached the theory of Relativity directly as a general gravitation theory.
Despite of hate for Germans, the theory of Relativity spread among people scientifically closest to Levi-Civita. Attilio Palatini was one of the most brilliant of them. In 1919 Palatini, influenced by Levi-Civita's work, published a paper[13] about the deduction of the field equations from an invariant variational principle. He criticized Einstein methods, judging them as unsatisfactory because they weren't invariant at every step of reasoning and proposed his own method.

In the same year (1919) a series of treatments was organized in Rome, at the Mathematical Seminar of the Science Faculty of Rome University, directed by Vito Volterra, in order to explain the basics of Relativity and of the mathematical instrument it used, that is the absolute differential calculus by Ricci and Levi-Civita.
The series of treatments was mainly organized by Castelnuovo, who invited Levi-Civita and Marcolongo to speak. Levi-Civita was ordered to do the introductory lecture in order to attract even the new mechanics opponents' attention, while the two following lectures, which presented respectively the contents of Special and General Relativity were committed to Marcolongo.

The particular aim of Levi-Civita's lecture makes it extremely interesting from both a scientific and historical point of view. It's an introduction to the theory of Relativity in which the author tried to omit the revolutionary nature of Einstein physics postulates, deciding to obtain Relativity results through quantitative modifications of classic Hamilton principle.

This lecture will be analyzed in details in the third chapter in order to highlight its innovative and, in some aspects, unique nature.
This series of treatments is very important even because it's the act which

gives diffusion to an italian version of Einstein Relativity, that is a theory with a strong mathematical nature, based on the classic Ricci and Levi-Civita formulation of the absolute differential calculus.

When Einstein himself came to Italy and hold a series of divulgative lectures in Bologna in 1921, the basics of Relativity became discussion objects on the main italian newspapers.
The italian involvement in Relativistic field was glorified with particular mention to Castelnuovo's, Marcolongo's, Maggi's, Palatini's and expecially Ricci and Levi-Civita's contribution.
In particular the last one was considered to be one of the direct author of the theory because of his correspondence with Einstein during the years of conepts development.
Levi-Civita underlines many times how necessary is to

> cominciare col distinguere fra la rivoluzione nella rappresentazione matematica dei concetti fisici e la speculazione puramente filosofica. Che una rivoluzione cosi profonda come quella dell'Einstein non abbia dei riflessi filosofici nessuno vuole escludere, ma l'importanza positiva, come teoria di sintesi e mezzo di previsioni, resta enorme all'infuori di ogni controversia filosofica.[14][6]

In the moment of Einstein's coming, Levi-Civita had completely removed every possible objections against the new theory, even if he continued to admit the difficulty in understanding Einstein theory and to witness his position of actual isolation due to lacking interest in Relativity by the italian mathematicians.

[6] "start to distinguish between the revolutionary mathematical representation of the physics concepts and the pure phylosophical speculation. No one could neglect the phylosophical influences of such a deep revolution as Einstein's one, but its positive importance, as a synthetic theory and as a prevision instrument, remains very deep other than every controversy"

In conclusion Relativity's birth in Italy can be situated in the years between 1906 and 1921 thanks to the work of the mathematicians close to Levi-Civita. Some fundamentals phases of its spread were:

1. the divulgative work about electromagnetic and gravitation theories done by Max Abraham since 1909 and his contrast against Einstein between 1912 and 1914;

2. Levi-Civita's involvement in the theory discussion and his collaboration with Einstein between 1915 and 1917;

3. the 1919 fundamental contribution for vulgarization by the Mathematical Seminar of Rome University with a series of lectures;

4. the 1921 divulgative lectures by Einstein in Bologna.

Chapter 2

Max Abraham's objections against relativity

Many works have already been written about Abraham's controversy with Einstein. In particular, a great historical reconstruction of the facts is contained in the paper *"Max Abraham and the Reception of Relativity in Italy: His 1912 and 1914 Controversies with Einstein"* by C. Cattani and M. De Maria [2] [1].

It would be useless to repeat here the chronicle of events, therefore the intention of this paper will be the discussion about the contents of the debate between Abraham and Einstein from an epistemological point of view. In particular the attention will be focused on the scientific nodes of the debate, as they arise from the original works of the authors.

We will highlight that many important epistemological basic ideas of the theory of Relativity were discussed inside Einstein and Abraham's papers of that time. Finally it will be interesting to notice that Einstein's ideas became

[1] Other papers about Einstein and Abraham controversy are [15], [16]. For a detailed historical report about the impact of Relativity in Italy, included Abraham and Einstein contrast, see also the recent book by S. Linguerri and R. Simili [1]

clearer and clearer over the years, trying to answer Abraham's objections.

2.1 Scientific preliminary remarks

In order to correctly observe the contrast between Max Abraham and Albert Einstein, which developed from 1912 to 1914, we need to have a quick look at their previous scientific works.

The first Abraham's work was about electromagnetic theories. He began with some studies on electrical phenomena and the nature of the electron published in the papers "Prinzipien der Dynamik des Elektrons" of 1903 and "Die Grundhypotesen der Elektronentheorie" of 1904. His studies are recollected in "Theorie der Elektrizitat" [17].

In order to place Abraham's work in his time scientific context, let's follow the historical narration by Orso Mario Corbino, one of the most important Italian physicists of that time, in [5].
Max Abraham was the author of the *Theory of rigid electron*, whose fondamental idea was the fact that the electron, in its motion, at whatever velocity, mantained a spherical shape and an unchanged volume. His theory was confirmed by the fact that the values he calculated for the apparent mass of the electron were in good agreement with the values found by Walter Kaufmann in his experiments.

As the opposing Lorentz's "*Theory of deformable electron*", even Abraham's theory was based on the hypothesis of the existence of a privileged system of reference in absolute quiet, the aether. Both theories had to face the problem of revealing Earth's absolute motion. According to both the theories, in fact, the simple observation of electrical phenomena should make possible to deduce the motion of the Earth referred to the aether. But, according to O. M. Corbino, "*finora, ogni esperimento condotto per osservare questo effetto ha prodotto un risultato negativo*" [2] [5].

Only afterwards, Lorentz managed to demonstrate the observable electrical phenomena inside moving bodies to be completely indipendent from the absolute bodies' velocity. In a following development of his theory, in fact, assuming the hypothesis that all the bodies, including the electrons, in translation with constant, nonzero, velocity contract themselves along the direction of the motion, while the other dimensions remain unchanged, he answers the question raised by the negative result of Michelson and Morley's experiment. [3]

The debate between the two theories was solved only by the gradual affirmation of the principle of relativity of Einstein and the consequential abandonment of the aether hypothesis. It's interesting quoting what Corbino argued in 1907, when Einstein's theory wasn't already affirmed:

> [...] abbiamo già visto che il contrasto tra le due teorie è sempre subordinato all'ipotesi fondamentale di un etere in quiete assoluta, senza del quale è possibile affermare insieme il principio di relatività e una legge di variazione della massa elettromagnetica con la velocità di tipo diverso rispetto a quella prevista da Lorentz. [...][L'ipotesi di un etere assoluto], *per la sua semplicità continua ad essere, oggi, l'unica base accettabile per qualsiasi teoria elettromagnetica concretamente sviluppata* [4]

[2] "up to now, every experiment made to see this effect yielded a negative result"

[3] The ideas expressed so far are a those of O. M. Corbino in [5]

[4] "[...]we've already seen that the contrast between the theory in discussion is always subordered to the fundamental hypothesis of an aether in absolute quiet, without which it's possible to state together the principle of relativity and a law of variation of electromagnetic mass with velocity of a different kind with respect to that previewed by Lorentz. [...] [The hypothesis of an absolute aether], *because of its simplicity, continues to be today the only acceptable basis of any electromagnetic theory concretely developed.*"

In order to understand how the contrast between Abraham and Einstein raised, it's useful to read the introduction of the famous 1905 Einstein's paper, "*Electrodynamics of moving bodies*" [18]:

> [...] in all coordinate systems in which the mechanical equations are valid, also the same electrodynamic and optical laws are valid. [...] We shall raise this conjecture to the status of a postulate and shall introduce, in addition, the postulate, *only seemingly incompatible with the former one*, that in empty space light is always propagated with a definite velocity V which is independent of the state of motion of the emitting body. These two postulates suffice for arriving at a simple and consistent electrodynamics of moving bodies on the basis of Maxwell's theory for bodies at rest. *The introduction of a "luminiferous aether" will prove to be superfluous*

Einstein assumes a clear position. aether hypothesis is superfluous in his theory and so erased. In Abraham's theory aether was a fundamental hypothesis, in Einstein's one was only a useless thing. The different view about the existence of the aether can be considered as the starting point of Abraham's opposition to the theory of special relativity.

In this work the fundamental ideas presented by Einstein in his three famous papers published on the "*Annalen der Physik*" from 1905 ahead [5] will be considered as known, and the attention will be immediately focused on the crucial points of the contrast between Einstein and Abraham.
There were two different moments of open contrast between the two scientists. The first is in 1912, with a series of papers appeared on "*Annalen der Physik*". The second is in 1914, with a series of three papers, two of them by Abraham, with the reply by Einstein, published on two volumes of the journal "*Scientia*"[6].

[5] "On the electrodynamics of the moving bodies" [18] and "Does the inertia of a body depend upon its energy-content?" [19] in 1905 and "On the influence of gravitation on the propagation of light" [7] in 1911

[6] See in the bibliography [20], [10], [21], [11], [22], [23], [24]

It's not possible here to watch the contents of every single paper[7] so only those conceptual kernels that made Einstein and Abraham positions divergent are highlighted. They are

1. *the postulate of the constancy of the light speed;*
2. *the principle of relativity and the existence of aether;*
3. *the equivalence hypothesis and the gravitation problem.*

2.2 The constancy of the velocity of light

Two, actually, are Einstein's statements about the velocity of light.

The first is the postulate he poses as basis of special relativity in the 1905 paper. Looking carefully at this paper two assumptions about the speed of light can be found in it.

One is that *light is always propagated in empty space with a definite velocity V which is independent of the state of motion of the emitting body.* This means that the value of light speed has to be considered as a constant V, whatever be the state of motion of the source and whatever be the state of motion of the system of reference with respect to which you get the measure. In fact, if the source is at rest and you perform the measurement from a system in relative motion, it's however possible to consider the case of a relative movement of the source.

The other assumption is that *the time needed for the light to travel from A to B is equal to the time it needs to travel from B to A.* We can look at this as the hypothesis of the *isotropy* of propagation of light.

The second statement about the velocity of light is given in 1911. Ein-

[7] Looking carefully at every single article is particularly interesting because it's possible to recognize, following debate's evolution, a growing consciousness of the author own ideas, together with their clearer and clearer formulation.

stein, thanks to the equivalence hypothesis[8], achieves a relationship that **ties together** the **value of the light speed** and the value of the **gravitation potential**.
According to his theory, if we measure time in presence of a gravitational field, in a point of potential ϕ, we must use a clock which goes $(1 + \phi/c^2)$ times more slowly than the clock used in a point of potential $\phi = 0$.
If we call c_0, the velocity of light in the point of potential $\phi = 0$, then the velocity c in a point of gravitation potential ϕ will be given by the relation

$$c = c_0(1 + \phi/c^2)$$

This is the relation by which the velocity of light results linked to the value of gravitation potential. How does this fact conciliate with the second postulate of the relativity?
Trying to involve the gravitation among his considerations, Einsten has to check his previous assumptions. I want to underline that he will never decide to renounce at them, but he will apply himself to clarify their meaning and their validity.

It's no more possible to speak about *the constancy of the value of light speed*, because it does not hold anymore in the same terms if we measure the velocity of light in places of different gravitation potential.
So it's of fundamental importance trying to understand Einstein's statement in conclusion of the 1911 paper [7] according to which the second postulate of relativity continues to hold good, but "*in a different form from that which usually underlies the ordinary theory of relativity*" [9].

We must look more carefully at what is changed with respect to the 1905 theory. With the development of the theory, in 1911, light does not acquire a new property at all. Instead, it is the device we use to measure light velocity

[8] A stationary system of coordinates in a gravitational field is equivalent to a system of coordinates uniformly accelerated in a space free of gravitational fields

[9] See [7]

that is recognized to have a new property.
It is any single clock that varies its rhythm, in agreement with the variation of the gravitation potential
Surely the *constancy* of light velocity is lost, as consequence, but the *invariance* of velocity of light respectively to any motion of the emitting body in any system of coordinates keeps unmodified.
In this sense, Einstein is allowed to state that he has not to reject the second postulate of his theory completely, but he only has to clarify its meaning and validity.

The first Abraham's attack against Einstein about the problem of the constancy of the velocity of light is contained into the declaration of his "New theory of gravitation" before the Italian Society for Science Progress (SIPS).
From 1911, Abraham begins to develop his own theory of gravitation. He's already an accomplished lecturer at the top of his scientific activity when he, getting inspiration from Einstein's results, begins to present his own version of the gravitation problem.
Encouraged by Levi-Civita, who largely appreciated him, publishes two notes that appear on the *Rendiconti dell'Accademia dei Lincei* in 1911 and in 1912 [25], in which he drafts some first results. A more complete form at his theory is given in occasion of the lecture he delivered at the *SIPS* in Genoa on 19^{th} October 1912, which was, later, published on *Il Nuovo Cimento*, journal of the Physics Italian Society.

It's useful to look at some elements of this theory to understand in what point Abraham's position diverged from Einstein's one.
The beginning point of Abraham is, first of all, the realization that

> We have to renounce to the strict analogy between gravitation and electromagnetism, though holding good some essential elements of the Maxwell's theory, namely:

1. the fundamental laws have to be differential equations which describe the gravitational field's propagation and excitement
2. a positive energy density and a current of energy have to be assigned to the field

Besides Abraham considers urgent to understand the real importance of the Einstein relation between mass and energy. His idea is simple. If you move a body in a gravitational field, the gravitation potential variation produces body's potential energy variation and so the first member of the equation $E = mc^2$ varies. In this way one or both of the factors at the second member depend upon the gravitation potential.
As Einstein in 1911, also Abraham decides to make the hypothesis that the second factor, that is *the velocity of light, depends upon the gravitational potential.* In fact, this is the first postulate of his gravitation theory:

> The surfaces c =constant coincide with the equipotential surfaces of the gravitational field, namely, the negative gradient of c indicates the direction of gravity

So, the beginning point to Abraham's theory development is exactly the direct consequence of some Einstein's results, namely that the velocity of light could be considered as a constant only in correspondence of a constant potential. But, according to Abraham's point of view, the fact that the velocity of light was tied together with the gravitational potential value meant "*to give a deep cut to one of the two roots of Einstein theory of relativity*".

Einstein's position against the radical attack about the constancy of the velocity of light is twice expressed in 1912, both in the paper "The speed of light and the statics of the gravitational field" [8], and in "Relativity and gravitation" [10]. He thinks that the principle of the constancy of light velocity can be hold good only restricting to region of constant gravitational potential and realizes that, at that time, gravitation can't be incorporated into the scheme of his theory in a consistent manner.

But, in his view, this state of affair did not represent the failure of the method based on the principle of relativity.
The theory of relativity continued to retain its significance as the simplest theory for the important limiting case of spatio-temporal events in the presence of a constant gravitational potential.

2.3 The principle of relativity and the aether

Let's turn back on the reasoning contained in Abraham's gravitation theory. Starting from the hypothesis of a variable value of c and of gravitational mass proportional to energy, such as inertial mass, according to the equation $E = mc^2$, Abraham obtains an equation which links the field to the energy of matter in this way, according to his own notations:

$$\Box u = \Delta u - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial u}{\partial t}\right) = 2\alpha\mu \cdot u = 2\alpha \cdot \frac{\eta}{u} \qquad (2.1)$$

where $u = \sqrt{c}$, α is a universal constant of degree c^0, $\mu = \lim_{V\to 0} \frac{M}{V}$ is the "specific density" and η is energy density of the matter we are considering.

Abraham notes that the equation deduced by that one,

$$u\Box u = 2\alpha\eta$$

is not invariant with respect to the group of Lorentz transformations. The first member of the equation, in fact, is an invariant with respect to the group, but not the second, proportional to energy density.
In the empty space, instead, the equation (2.1), because $\mu = 0$, assumes the form $\Box u = 0$, an invariant form for Lorentz transformations.

It's important to remember that Einstein's *principle of relativity*, actually was an invariance of laws of nature referred to systems of coordinates in relative uniform translation motion warranted by a formal invariance of the laws with respect to the group of Lorentz transformations. In the paper "*On the gravitation problem*" [23] it's Einstein's own statement that:

> The heuristic value of the theory of relativity consists in the fact of giving a condition which every equation system that expresses general laws of nature must correspond to. Every system of equations must be made in such a way that can be transformed in a system of equations of the same form when a Lorentz transformation is applied.

The fundamental equation of Abraham's theory is, however, invariant only in the case of an empty space free of matter. When you consider the presence of matter that invariance vanishes.

From this contradiction between Einstein and his own theory, Abraham draws a clear conclusion: 1905 theory of relativity must be completely rejected. The hypothesis that c is variable contradicts the second postulate. Also, from the hypothesis that gravitational mass is proportional to energy we reach an equation, which conveys the connection between gravitational field and energy of matter, that's not invariant with respect to Lorentz transformations. The invariance holds good only in vacuum. Namely, when matter enters in the considerations of our physic system, the first requirement of the theory of relativity fails.

It's the revolutionary and fruitful hypothesis of a relation of proportionality between gravitational mass and energy, deduced by Einstein as a consequence of relativity, that lead us to abandon the Lorentz group.
In Abraham's opinion there's only one possible solution to all these objections i.e. rejecting the *principle of relativity* and coming back to consider the existence of a privileged reference system, the *aether*, which laws of nature have to be referred to.

Indeed, according to Abraham the aether acquires a new dignity thanks to his theory. It's no more only the medium of electric radiation, but also of

gravitational waves.
He states, in fact, in conclusion of the lecture at the SIPS:

> That so well-founded hypothesis that the attractive mass is proportional to energy, force us to abandon the Lorentz group even in the infinitesimal. So Einstein's theory of relativity (1905) declines. Will a new, more general, relativity principle raise up again, as the phoenix from the ash? Or will we come back to the absolute space? And recall that so despised aether, in order to give it the role of carrying the gravitational field as well as the electromagnetic field.

In the 1912 paper, "*The speed of light and the statics of the gravitational field*" [8], Einstein answers back to Abraham showing that his gravitation theory was wrong, but at the same time expresses self-criticism. The problem reported by Abraham, namely that considering the presence of matter led to loose the invariance of the fundamental equations of the theory respectively to Lorentz transformations, was, in his own opinion, even "harder".

In the conclusion of the paper, indeed, Einstein himself shows that as soon as you reject the universal constancy of c, the Lorentz transformations stop to hold even in the infinitely small.
If they hold good you would have the relations

$$dx' = \frac{dx - vdt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$dt' = \frac{-\frac{v}{c^2}dx + dt}{\sqrt{1-\frac{v^2}{c^2}}}$$

with dx' e dt' total differentials.

So it would even hold

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}\right)=\frac{\partial}{\partial x}\left(\frac{-v}{\sqrt{1-\dfrac{v^2}{c^2}}}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\dfrac{-v}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}\right)=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}\right)$$

Consider as static the gravitational field in the system not primed.
In that case c is an arbitrary function of x, but it's independent from t. If the primed system is in uniform motion, then v is independent from t for a fixed x. So the left member of both the equations must vanish and so the right member. But this is absurd, because if c is an arbitrary function of x, then the two right members can be taken not both null, by choosing correctly the function v of x. Besides this means that Lorentz transformations can't be considered good neither for infinitesimal regions of space as soon as we abandon the universal constancy of c.

So, in Einstein's idea, Abraham's theory cannot be correct, because, in opposition to what's just been shown, it states that in absence of matter it's possible to reach a fundamental equation for the gravitational field that is invariant. It's the differential equation he found to describe the field ($\Box u = 0$) that allows him to deduce this.
The problem still held for Einstein too. Admitting the hypothesis of equivalence carried the consequence that c was variable as well as the gravitational potential. Rejecting c as a universal constant implied loosing the invariance of the fundamental laws with respect to the Lorentz group. How to conciliate the 1905 theory with this new reasoning?
In the last lines of the 1912 paper [8] he gave a first answer.

> It seems to me that these are the terms of the problems. If you limit yourself in a region of constant gravitational potential, then the laws of nature will get a simple and invariant form if they are referred

> to spatio-temporal systems connected each other by Lorentz transformations with constant c.
> If you don't limit yourself in a region with constant c, then the manifolds of equivalent systems, as the manifold of transformations which leave unchanged the laws of nature will become more extensive, but for these systems the laws will become more complicated.

As we've already seen in the previous paragraph, Einstein explicitly confesses, answering to Abraham on the "*Annalen der Physik*" [10], he doesn't manage to incorporate gravitation into the scheme of the 1905 relativity.

In 1913 Einstein publishes the famous paper written together with Marcel Grossman [12] in which he exposes a new relativistic theory of gravitation. Having solved the contradiction between the two theories Einstein is ready to give a synthetic view of his principle of relativity. He does it in the 1914 paper "On the gravitation problem" published on the journal "*Scientia*" [23], with the goal of permitting the journal readers to consider a positive opinion about the theory of relativity after some papers of criticism by some valuable scientists being published on this journal. One of these papers, entitled "The new mechanics"[22], was written by Abraham.

The order in which Einstein exposes his results in this paper is not exactly as they historically came to his mind, but rather is logically the most rigorous way to look at the basics of his theory. Now we'll follow quite exactly the steps proposed by Einstein in the paper.

The first statement is that "the principle of relativity is as old as mechanics". He's talking about the Galilean principle of relativity, which is declared in the following way. If a motion is referred to a system K, in such a way that Newton equations hold good, that system of coordinates is not the only one with respect to which those laws of mechanics hold. Infinite other ones exist having the property that respectively to them the same laws of motion hold and precisely they are all the systems of coordinates K', arbitrarily oriented in the space, in uniform translation motion with respect to K.

> The identity of all the systems of coordinates K, K', etc. for the formulation of the laws of motion, *indeed of all the general laws of physics*, is called " principle of relativity" (in a narrower sense)

So, Einstein's principle of relativity is posed in complete continuity with the Galilean and Newtonian one.
Both his predecessors, in fact, state the principle of relativity as the impossibility of knowing if an experiment is referred to a system in rest or in straight uniform motion. Indeed, Galileo and Newton affirm that laws of motion have the same form in all the reference systems, either in rest or in uniform rectilinear motion.
Einstein simply extends the same principle to *every physical law.*

It's impossible, he thinks, to doubt about the validity of the principle. It's never been put in discussion as long as classical mechanics has been used as the basis for the theoretic description of all the processes. Also, it's difficult to be put in doubt even from the point of view given by the experiments. As a confirmation Einstein states that

> if it wasn't valid, the natural processes of a reference system at rest with respect to the Earth would seem influenced by the motion (velocity) of Earth annual revolution around the Sun; The spaces of observation on Earth would have to behave physically anisotropically because of the existence of a motion like that. Even with the strongest researches the physicists haven't managed to observe a similar anisotropy.

In Einstein's opinion the validity of the principle of relativity for all the phenomena has been threatened by the electrodynamics of Maxwell and Lorentz. Such a principle, in fact, is in apparent incompatibility with electrodynamics.

Let's follow Einstein in his simple reasoning. If you consider a system K with respect to which the equations of Maxwell and Lorentz theory hold, then every ray of light spreads in vacuum with a certain velocity c not dependent from propagation direction and from the state of motion of the light source. This statement contained in the theory of Maxwell and Lorentz is called by Einstein the "principle of the constancy of the speed of light". If a ray of light as the one just described is observed from a system in motion with respect to K, the velocity of propagation will appear generally different from c to this observer. In the most simple case, i.e. either the ray either the human observer to be in motion along the positive axis x of K , respectively with velocity c and v, the observer will conclude that the ray spread with velocity $c-v$. In this sense with respect to a system K' moving with the observer the principle of the constancy of the light velocity seems to be not valid anymore.

The problem, Einstein says, is that both Maxwell theory and the classical mechanics, with his principle of relativity, are theories too solid and too confirmed by experiments to reach an easy solution of this contradiction.

> Everyone, who's able to consider the compactness of the theory [of Maxwell and Lorentz], the low number of hypothesis at its basis and its good results in the theoretic describing of the experiments about electrodynamics and optics, can hardly reject the sensation the fundamental ideas of that theory to be considered as definitive as the equations of classical mechanics. [10]

To solve the problem Einstein states he did a precise analysis of the physical contents of the common affirmations about space and time.
In his opinion, the contradiction raises because two assumptions are hidden in the reasoning just made above.
The first is that the judgment of simultaneity of two events that happen in

[10] See [23]

two different places is independent from the choice of the reference system. The second implicit assumption is that the spatial distance of the places in which two simultaneous events happen is independent from the choice of the reference system.

These two hypothesis are arbitrarily chosen, in Einstein opinion. Their apparent evidence is just based on the fact that events happening far from each other are apparently instantaneously witnessed by light and that we commonly deal with bodies whose speeds are much smaller than speed of light c.
If those hypothesis are rejected the principle of relativity becomes compatible with the principle of the constancy of light, which results from Maxwell and Lorentz's electrodynamics.
In order to maintain the assumption that the same light ray is propagated in vacuum with speed c not only with respect to K, but also with respect to every other system K' in a uniform translation motion with respect to K, it's sufficient to choose an appropriate system of transformations between the spatio-temporal coordinates (x, y, z, t) of K and those (x', y', z', t') of K'. Following this way, the system of equations produced is the Lorentz's transformation one. Einstein's statement is that if these transformations are used instead of the traditional Newtonian mechanics' ones the contradiction is solved.

Trusting in the results obtained the year before with Grossman, Einstein, in the remaining part of the 1914 paper, wants to clarify why incorporating gravitation into the relativity scheme is so difficult. He tries to consider the question by a different point of view with respect to the previous years. First of all he declares that restricted relativity is not sufficient to formulate a complete theory of gravitation. Other hypothesis are necessary.

> It would be a great mistake to consider the theory of relativity as an universal method to formulate a totally correct theory for a range

> of phenomena arbitrarily few empirically investigated. [...] There's a field of fundamental importance that is so little empirically known, that the few acknowledgments we have, joined with relativity, are absolutely not sufficient to determine an univocal general theory. This is the field of gravitational phenomena.

For this reason, according to Einstein, some new physical hypothesis are necessary to be added to the empirical facts. These hypothesis have to be chosen in such a way that they result to be as *natural* as possible. After some reasoning he takes as the first hypothesis the *coincidence of inertial and gravitational mass*, because it's confirmed by people's common sense and most of all, by Eötvös' experiments. As we've already noticed, this is not the first hypothesis Einstein introduced in his previous formulations of theory of gravitation. In the usual formulation Einstein first states the *principle of equivalence.*
It's once again necessary to notice that Einstein's aim here, is to make even this principle appear as logically necessary. So, before stating it, it's convenient to dispel any other objections.

The coincidence between inertial and gravitational mass, added to the principle of light's inertia, logical consequence of restricted relativity, may appear as sufficient to formulate a theory of gravitation, but this is not true. Adding only this hypothesis, in fact, produces a deep lack both in classical mechanics and in the theory of relativity. A lack defined as *epistemological* by Einstein. This is the fact that a sort of privilege is assigned to the accelerated systems of reference. A privilege that has no sufficient reasons. Referred to systems in uniformly accelerated motion, laws of physics are not the same as referred to system in translation motion or in rest. But there's no acceptable reason for this to happen.

Here is the key to manage to incorporate the gravitation into a relativistic scheme. It's necessary to expand the principle of relativity, generalizing the theory. That is, it's necessary to postulate the equivalence of all the reference systems, even those uniformly accelerated, with respect to every physical process.

While Abraham asked to abandon the principle of relativity, Einstein "enpowers" it even further, expanding it, with the aim of plugging a conceptual gap. In Einstein's thought, extending the principle of relativity is nevertheless strictly linked with admitting the equivalence hypothesis.

2.4 The equivalence hypothesis and the gravitation

As already said, the first Einstein's formulation of the equivalence principle is contained inside the 1911 paper "On the influence of gravitation on the propagation of light " [7].

Abraham's accuse against this principle, stated inside "Relativity and gravitation " published on the *Annalen der Physik*, is that Einstein's *equivalence* is based on " fluctuating hypothesis". with respect to that formulation Einstein himself admits to have some concerns [11], so it will be best to avoid to focus our attention on it. Let's look again at the rigorous exposition of the 1914 paper " On the relativity problem", suspended in the previous paragraph.

As described before, Einstein wants to eliminate the epistemological error

[11] In his answer to Abraham, published on the *Annalen*, Einstein acknowledges he was able to carry through the conception that the static gravitational field is physically identical with an uniformly accelerated system in a consistent way only for infinitely small spaces. He admits furthermore that he cannot give any satisfactory reason for that fact. See [10]

contained both in classical mechanics and relativity. The error is to consider the accelerated systems as different from all the other systems without any sufficient reason.

So, his purpose is to generalize the theory of relativity in such a way that this problem will disappear. He exposes the reasoning which led him to the equivalence hypothesis as follows:

> To start with, I recognized that gravitation in general will have to be assigned a most fundamental role in such a theory. For it already follows from what has been said before that every physical process must also generate a gravitational field, because quantities of energy correspond to it. On the other hand, the empirical fact that all bodies fall equally fast in a gravitational field suggests the idea that physical processes occur in exactly the same way in a gravitational field as they do relative to an accelerated reference system (equivalence hypothesis).

With this new hypothesis Einstein states he has managed to replace the theory of relativity in the narrower sense by a more general theory that contains the former as a limiting case. He says that the details of this solution are described in the 1913 paper written in collaboration with Marcel Grossman and that, for the moment, he has obtained an equation of motion of a mass point in a gravitational filed in a form which is totally independent of the choice of place and time variables.

> By leaving the choice of these variables *a priori* totally arbitrary, and thus not privileging any specific spatio-temporal systems, one avoids the epistemological objection explained above.

But Abraham still finds a physically unacceptable statement in the theory formulated by Einstein and Grossman in 1913 [12].
The two scientists, Abraham explains, thanks to the help of the absolute differential calculus of Ricci and Levi-Civita, succeed to give their dynamic and

electromagnetic fundamental equations a form which, at least in infinitesimal, is coherent with what relativity asks.

However, it's the gravitational field itself which does not fit into the general scheme. The differential equations obtained by the two authors are not invariant with respect to the general space-time transformations from which they start. This means that a system of masses reciprocally attracting in a rotatory or not uniform motion is not identical, in general, with a system in rest.

According to Abraham, it's a fact known to everyone that a relativity with respect to rotatory motions is not possible. However Einstein and Grossmann try to move the question about rotatory motion from dynamics to gravitation replacing centrifugal forces with appropriate gravitational forces.
In this way they give relativistic form to the second law of dynamics even for rotational systems.

If the rotational forces are added to the gravitational forces, dynamics becomes relativistic, but the gravitational field of a system in rotatory motion is not the same as that of the non-rotational system, at least from a physical point of view according to Abraham.

If you want to add the centrifugal forces to the count of gravitational forces, it's necessary to contradict the principle of dynamics which identifies the attractive masses as the exclusive sources of the gravitational field. The new added component has a source which doesn't depends on the attractive mass, but on the revolution speed of Earth.
For this reason it's wrong to look for a gravitation theory which corresponds to the general relativistic scheme.

Contrasting Einstein's conviction of having obtained an equation of motion of a mass point in a gravitational filed in a form which is totally independent of the choice of place and time variables, Abraham concludes the last paper of the debate [24] writing:

> At first, basing upon his "equivalence hypothesis", Einstein has been probably led by the hope of achieving a general theory of which would comprehend both the rotatory and accelerated motions. He should nevertheless realize that he has not achieved this result and the balance of the theory of relativity presents here a deficit to cover.

2.5 Conclusive remarks

The "deficit" declared by Abraham was actually real. Einstein, indeed, had to work a lot before succeeding in correcting the mathematical mistakes in his theory, thanks also to the help of Tullio Levi-Civita.

Paradoxically, it was a letter dated 23th of February 1915 by Abraham himself [26], a strongest opponent of relativity, which drove Levi-Civita into considering Einstein's work. [12]

From March to May 1915 Einstein and Levi-Civita discussed the weak points of the theory of general relativity in a prolific correspondence. Even thanks to it, Einstein managed to eliminate the mistakes of his theory and in November 1915 reached the definitive formulation of the equations of the gravitational field with the correct properties.

Abraham, even far from Italy because of the World War I, continued to follow the development of the theory by Einstein and Levi-Civita and studied the first important works by Levi-Civita on general relativity. However, he probably did never abandon his opposition against Einstein. In a letter sent from Switzerland to Levi-Civita in August 1917 he wrote indeed [27]:

[12] See [15] for more details

> Dear friend and colleague,
>
> I really thank you for the last work you have sent to me; the remarks about the new Einstein's theory are very interesting. They should be pre-eventively mathematically submitted to the board of censors. Hilbert's words "Physics is too difficult for physicists" seem to be true.

In conclusion, what emerges from the analysis of the three nodes of the debate can be summarized as it follows:

1. In 1905 Einstein postulates the *constancy* of the value of speed of light and the *isotropy* of propagation of light. In 1911 he states the relationship between the value of the speed of light and the value of the gravitational potential. Abraham accepts this relationship, whereas he states that the postulate about the constancy of speed of light, and the whole 1905 theory, must be rejected. Einstein's reply is that nothing must be rejected: the isotropy of propagation and the relationship found in 1911 still hold good. Also, the speed of light cannot be considered as a universal constant, but continues to be *invariant* from any state of motion of the source.

2. Abraham says that the principle of relativity holds good only in vacuum. As soon as you consider the presence of matter the invariance of the equation of the field with respect to Lorentz transformations is lost. Einstein's reply is that *his* principle of relativity is the same as the one of Galilei and Newton. It's impossible to doubt about its validity. Only the Maxwell's theory has put it in doubt, but the contradiction is solved by the deep reflection on space and time Einstein made as the basis of his 1905 theory and by the use of Lorentz transformations instead of the traditional Newton mechanics'ones. Anyway, incorporating gravitation into the relativity scheme needs other hypothesis.

3. The equivalence hypothesis is introduced by Einstein since 1911. This hypothesis is considered by him as the most simple and natural solution

to the epistemological problem of assigning a sort of privilege to the accelerated systems of reference. Thanks to this new hypothesis he obtains a complete relativistic theory of gravitation. Abraham's last reply of 1914 highlights the mathematical and physical problems that general relativity still really has at that time.

At the end, Einstein emerged as *the victor* from the collision against the *only* real opponent of the theory of relativity.
Why Abraham was the *only real* opponent, almost from a pure scientific point of view? Here it will be given an hypothetical answer.

Abraham was one of the leading experts on electron theory of that time. He knew perfectly the theory of electromagnetic field and was a strong supporter of the aether.
For this reason, Einstein's ideas, which called into question the concept of aether and the electromagnetic theory and proposed to solve its internal incongruence, were a real challenge for Abraham.
Einstein wanted to call into question the matter in which Abraham was a acknowledged authority, so he had to face the challenge.

On the other hand, the general indifference of Abraham's colleagues was probably due to the fact that they couldn't completely understand the depth of debate's contents. Levi-Civita himself changed his position from a cautious indifference to an enthusiastic support to the theory only after being introduced by Abraham into the real contents of it.

However, it was a behavior of total open-mindedness and great confidence in human reason and in its creative ability which led Einstein over Abraham, one who might probably be even technically superior to him.

Chapter 3

Relativity in Levi-Civita's thinking

In 1919 Levi-Civita was asked to deliver the Relativity introductory lecture at the 1919 series of talks organized by Castelnuovo at the Mathematical Seminar of the Rome University.
The following two lectures, which presented respectively the contents of Einstein Special and General Relativity, were committed to Marcolongo. So Levi-Civita limits himself to a first introductory conference entitled "How could a conservative get to the threshold of new mechanics"[1] and leaves to Marcolongo the task to explain Special and General Relativity in two following lectures.

The title chosen for the first conference is very significant. Levi-Civita knows that his colleagues don't appreciate the "revolutionary" nature of Einstein's theory. Probably he doesn't appreciate it too. So he wants to avoid to state any kind of theoretical physical postulate and decides to show that the theory of relativity can be obtained as a result of very light modifications of the Hamilton principle.

[1]The original italian title is "Come potrebbe un conservatore giungere alle soglie della nuova meccanica" see [28]

Levi-Civita's aim is clear from the first words of his speech. He states that, while in politics people don't love to call themselves as conservative, many scientists

> possono, direi quasi debbono, essere conservatori per la stessa loro missione di custodire con gelosa cura un certo patrimonio intellettuale ben consolidato, e di vagliare con severo spirito critico tutto ciò che importa variazione od alienazione del patrimonio stesso.[2]

Levi-Civita knows that he's talking to many *conservatives* in that instant and he's proud of it. His task is to show how you can get to the threshold of Einstein mechanics *attraverso un paio di formule classiche*[3]. He underlines that the reason for formulating a new mechanics is:

> un legittimo desiderio di generalizzazione formale, da un lato, e di sintesi concettuale, dall'altro. [4]

together with the awareness that Einstein's new theoretical system allowed to

> fornire spiegazione esauriente di più esperienze, e specialmente di una celebre esperienza d'ottica, e di un fatto astronomico (lo spostamento del perielio di Mercurio) di fronte a cui restavano impotenti i vecchi e pur gloriosi schemi, nonostante i più vigorosi sforzi.[5]

So Levi-Civita presents Einstein's theory as an important upgrade from both a theoretical and an experimental point of view. In this way he manages

[2] "have to be conservative because of their mission of keeping the established cultural heritage. They also have to examine with strict critical sensibility everything which implies modifying or changing that heritage"

[3] "through a few classic formulas"

[4] "a legitimate desire of formal generalization, on one hand, and, on the other hand, of conceptual synthesis"

[5] "give a complete explanation to many experiences, and especially to a famous optical experiment and to an astronomical phenomenon (Mercury's perihelion motion) that were unexplained by the old and glorious schemes."

to obtain the attention of the audience. Considering how prejudice and skeptisim about Einstein's theory prevailed in Italy at that time, Levi-Civita's attitude turns out strong and cannot be taken for granted.

The basic ideas developed during the lecture can be itemized as follows:

1. classic equivalence between Newton's equations of motion, Hamilton variational principle and Lagrange equations

2. invariance of Hamilton principle and Lagrange equation under coordinates transformation or Lagrange parameter variation

3. equivalence between Hamilton principle for motions with synchronous and asyncrhonous variations[6] and equalization of the time coordinate to the other coordinates.

4. lack of invariance for Hamilton principle under general time-dependent coordinates transformation

5. Hamilton principle modification in order to obtain invariance under general time-dependent coordinates transformation

6. Lorentz transformations as a particular application of the previous method

7. basis of Einstein's Special and General theory of Relativity as a consequence of modified Hamilton principle and Lorentz transformations

[6] The original italian expression is *moti variati sincroni e asincroni* see [29] chap.

3.1 The contents of the lecture "How could a conservative get to the threshold of new mechanics"

Let's try, now, to make Levi-Civita's ideas clearer as he does during the lecture.
In order to recreate the mathematical steps made by Levi-Civita, all the definitions and the notations are referred to those contained into the "*Lezioni di meccanica razionale*" by Amaldi and Levi-Civita[7].

3.1.1 Hamilton principle

First of all it's necessary to make a brief recall about the classic Hamilton principle and about its meaning in mechanics.

Consider the equations of motion of a point particle in a conservative field. Given a unit potential U, the equations of motion in cartesian coordinates y_1, y_2, y_3 as referred to a fixed axis can be written as:

$$\ddot{y}_i = \frac{\partial U}{\partial y_i} \qquad i = 1, 2, 3 \tag{3.1}$$

where $\dot{y}_i = dy_i/dt$.

If

$$dl_0^2 = \sum_{i=1}^{3} dy_i^2$$

is the square of the linear displacement dy occurring in time dt and v the speed modulus of the point mass, there risults

$$v^2 = \frac{dl_0^2}{dt^2} = \sum_{i=1}^{3} \dot{y}_i^2$$

[7] "Lectures of rational mechanics" see [30], [31] and [29]

where $\frac{dl_0^2}{dt^2} = \left(\frac{dl_0}{dt}\right)^2$ indicates the square of the elements rather than a derivative.

Defining now

$$L = \frac{1}{2}v^2 + U$$

equations (3.1) can be resumed in the following variational formulation

$$\delta \int L dt = 0 \tag{3.2}$$

It is important to define precisely the use Levi-Civita has made of the symbol δ. It classically means *variation* but, for the sake of clarity, it is useful for our argumentation to distinguish variables from constants.
Following Levi-Civita's textbook [29], you can obtain Hamilton principle [8] in the following way.
Consider a natural motion M characterized by a certain law $y_i = y_i(t)$, $i = 1, 2, 3$. Consider, then, a motion with a synchronous variation (hereinafter indicated with SVM) M_s [9] with respect to M, characterized by the law $y_i^s = y_i(t) + \delta y_i(t)$, $i = 1, 2, 3$ where δy_i are virtual infinitesimal changes of y_i. At every instant t the virtual changes can be arbitrarily chosen, but once the choice is made they both become functions of time. So it makes sense of derivative with respect to t.

It's important to underline again that, according to the definition of virtual change, i.e. "quell'ipotetico spostamento atto a fare passare il sistema da una qualsiasi sua configurazione ad un'altra infinitamente vicina e relativa *al medesimo istante*"[10] the assumption is made that in the same instant t the system adopts two different configuration. The first one is given by y_i relative to the natural motion M and the other one is given by y_i^s relative to

[8] See [29] chap. XI

[9] The original italian for a motion with synchronous variation is *moto variato sincrono*

[10] "that hypothetical change which makes the system move from a given position to another infinitely nearby and relative to the same instant." See [30]

the motion M_s with synchronous variation.

These hypotheses allow Levi-Civita to prove the equivalence between the symbolic equation of dynamics, Newton equations (3.1) and the Hamilton principle expressed by (3.2), where the integral is done over an arbitrarily fixed interval $[t_0, t_1]$ and the variation is done with respect to a SVM with null changes δy_i in the endpoints of the integration interval. Levi-Civita notices that this is the simplest formulation of the Hamilton principle in which t is constant and consequentely $\delta t = 0$.

In order to approach Einstein mechanics a natural step is to involve time variation in the previous reasoning. Levi-Civita makes two important observations before trying this approach.

3.1.2 Hamilton principle invariance with respect to general transformation of coordinates

The first observation is the following result.
Changing the coordinates y_i with any of the three Lagrange parameters x_1, x_2, x_3 linked to the y_1, y_2, y_3 with regular and invertible time-dependent relations,

$$x_h = x_h(y_1, y_2, y_3, t) \qquad h = 1, 2, 3 \tag{3.3}$$

or

$$y_i = y_i(x_1, x_2, x_3, t) \qquad i = 1, 2, 3 \tag{3.4}$$

and using these expressions in L, it becomes a function $L(x, \dot{x}, t)$, quadratic in $\dot{x}$ (generally non homogeneous). Even using this transformed L equation (3.2) still holds good with respect to x.

If you give explicit form to the L and execute the variation you obtain the classic result [11]

$$\delta \int L dt = 0$$

$$\Longleftrightarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_h}\right) - \frac{\partial L}{\partial x_h} = 0 \quad h = 1, 2, 3 \tag{3.5}$$

Thanks to this equivalence invariance of Lagrange equations (3.5) with respect to every transformation of the form (3.3) as well as the invariance of equation (3.2) with respect to every choice of parameters x_h is obtained. Levi-Civita explains more clearly the meaning of the word *invariance* he's using. He states that

> la qualifica di invariante di fronte a qualsiasi scelta e quindi anche trasformazione delle x di tipo (3.3) testé attribuita alle equazioni del moto va presa in senso *relativo*, cioè di un invarianza subordinata ad una qualche funzione, base della trasformazione. La base delle trasformazioni (3.3) è evidentemente la L, unico elemento di cui occorre e basta procurarsi l'espressione esplicita $L(x, \dot{x}, t)$ nelle nuove variabili. Facendo intervenire questo elemento ausiliario la struttura delle equazioni (3.5) è sempre la stessa, qualunque siano le coordinate di riferimento.[12]

A second observation explains that making a time-indipendent coordinates transformation will result in an apparently changed system of equations both in physics and in mathematics. Assuming the square of the linear element dl_o^2 as the basis for the transformarion, it is necessary to represent it with respect to the x coordinates we want to refer to. However it will always

[11] See [29] chap. XI, p. 509 ff.

[12] "the invariance of the equations of motion with respect to every choice and transformation of the coordinates x of the kind (3.3) has a relative meaning. It's an invariance subordinated to a function, the basis of the transformation. The basis of the transformations (3.3) is, obviously, the L, the only element that has to be written in an explicit way $L(x, \dot{x}, t)$ with respect to the new variables. Thanks to this auxiliary element the structure of the Lagrange equations (3.5) is always the same referred to any system of coordinates."

appear like the following

$$dl_0^2 = \sum_{i,h=1}^{3} a_{ih} dx_i dx_h \tag{3.6}$$

The last observation shows, in an analytic way, that actually applying a generic trasformation (3.3) to the dynamic basis L you obtain a new formulation of it, in which the geometric basis dl_0^2 (3.6) is contained. So the conclusion is ,

> la base geometrica è in ogni caso inclusa nella base dinamica e non viceversa.[13]

3.1.3 Introducing time change into Hamilton principle

In this part of the lecture, Levi-Civita shows that Hamilton variation principle and its equivalence with Newton equations still hold good even if time coordinate is changing. Time is so equivalent to space coordinates.

> si possono pertanto, nel principio variazionale (3.2) trattare alla stessa stregua le coordinate di spazio x_1, x_2, x_3 e anche la t[14]

In this section we recall and explain the mathematical steps of Levi-Civita's reasoning, making reference to his own treatise in [29] [15].

It's necessary to introduce the definition of *motion with asynchronous variation*[16] (hereinafter indicated with AVM). Given a natural motion M, described by the law $y_i = y_i(t)$, $i = 1, 2, 3$, its corresponding AVM is a motion in which the y_i go through the change $y_i^a = y_i + \delta y_i$ in the instant of time $t + \delta t$ and not in the same instant t. As well as the space coordinates, also

[13] “the geometric basis is always included in the dynamic one, while the inverse is not true.”

[14] “the coordinate t can be handled as well as the space coordinates x_1, x_2, x_3 in the variation principle (3.2) ”

[15] See [29] pp. 491 ff.

[16] The original Levi-Civita's italian expression is “moto variato asincrono”

time changes can be arbitrarily chosen. Once chosen, the δt become function of time. For all AVM M_a there's a corresponding SVM M_s in which the y_i have the same law, but the time doesn't change.

Indicating with δ^* the asynchronous variation, in order to distinguish it from the synchronous one, you can observe that:

- $\delta^* y_i = \delta y_i$ by definition
- $\delta^* q = \delta q$ when q represents just a positional quantity.

Note. It's meaningless to distinguish between $\delta^* t$ and δt, because time variation can only be considered in the asynchronous case. This variation will always be indicated with δt.

Observation 1. The following relationship is given

$$d\delta t = \delta dt \tag{3.7}$$

where dt is the time differential in a natural motion, and δt is the time differential in the AVM.

Proof. Every instant t of the motion M corresponds to the instant $t_a = t + \delta t$ of the AVM, so dt corresponds to $dt_a = dt + d\delta t$. In conclusion the variation of time differential δdt is given by

$$\delta dt = dt_a - dt = d\delta t$$

□

Observation 2. δ^* doesn't commute with time derivative. So it results

$$\delta^* \dot{y}_i = \delta \dot{y}_i - \dot{y}_i \frac{d}{dt} \delta t$$

Proof. In M_a the system assumes the position $y_i + \delta y_i$ at the instant $t + \delta t$. So, applying the given definition, the speed of the system in the AVM is

$$\dot{y}_i + \delta^* \dot{y}_i = \frac{d(y_i + \delta y_i)}{d(t + \delta t)}$$

Dividing both numerator and denominator in the previous expression by dt it results

$$\dot{y}_i + \delta^* \dot{y}_i = \frac{\frac{dy_i}{dt} + \frac{d\delta y_i}{dt}}{\frac{dt}{dt} + \frac{d\delta t}{dt}} = \frac{\dot{y}_i + \frac{d\delta y_i}{dt}}{1 + \frac{d\delta t}{dt}}$$

Considering

$$\frac{d\delta y_i}{dt} = \delta \dot{y}_i$$

and assuming that, if second or higher order infinitesimals are neglected,

$$\left(1 + \frac{d\delta t}{dt}\right)^{-1} = \left(1 - \frac{d\delta t}{dt}\right)$$

you obtain

$$\dot{y}_i + \delta^* \dot{y}_i = (\dot{y}_i + \delta \dot{y}_i)\left(1 - \frac{d\delta t}{dt}\right) = \dot{y}_i - \dot{y}_i \frac{d\delta t}{dt} + \delta \dot{y}_i - \delta \dot{y}_i \frac{d\delta t}{dt}$$

Neglecting again second order infinitesimals like $\delta \dot{y}_i \delta t$ you obtain, at the end,

$$\delta^* \dot{y}_i = \delta \dot{y}_i - \dot{y}_i \frac{d\delta t}{dt}$$

□

Thanks to these observations it's possible to perform the asynchronous variation in analytic way.
It's useful to recall that if you actually solve the *synchronous* variation of the integral $\int_{t_0}^{t_1} L dt$, with the arbitrary interval $[t_0, t_1]$ you obtain

$$\delta \int_{t_0}^{t_1} L dt = \sum_i \left[\frac{\partial L}{\partial \dot{y}_i} dy_i\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i}\right] \delta y_i dt \qquad (3.8)$$

Note that, in the previous calculation, the time-indipendence of the variation δ plays a fundamental role as it implies differentiation is operated only on the L function. This is not true in the *asynchronous* differentiation δ^*. It results[17]

$$\begin{aligned}\delta^* \int_{t_0}^{t_1} L dt &= \int_{t_0}^{t_1} \delta^* L dt + \int_{t_0}^{t_1} L \delta dt = \int_{t_0}^{t_1} \delta^* L dt + \int_{t_0}^{t_1} L d\delta t \\ &= \int_{t_0}^{t_1} \delta^* L dt + \int_{t_0}^{t_1} L \frac{d\,\delta t}{dt}\, dt = \int_{t_0}^{t_1} \left[\delta^* L + L\frac{d\,\delta t}{dt}\right] dt \quad (3.9)\end{aligned}$$

On the other hand: [18]

$$\begin{aligned}\delta^* L &= \sum_h \frac{\partial L}{\partial y_i}\delta^* y_i + \sum_i \frac{\partial L}{\partial \dot{y}_i}\delta^* \dot{y}_i + \frac{\partial L}{\partial t}\delta t \\ &= \sum_i \frac{\partial L}{\partial y_i}\delta y_i + \sum_i \frac{\partial L}{\partial \dot{y}_i}\left(\delta \dot{y}_i - \dot{y}_i \frac{d\,\delta t}{dt}\right) + \frac{\partial L}{\partial t}\delta t \\ &= \sum_i \left(\frac{\partial L}{\partial y_i}\delta y_i + \frac{\partial L}{\partial \dot{y}_i}\delta \dot{y}_i\right) - \sum_i \frac{\partial L}{\partial \dot{y}_i}\dot{y}_i \frac{d\,\delta t}{dt} + \frac{\partial L}{\partial t}\delta t \\ &= \delta L + \frac{\partial L}{\partial t}\delta t - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\frac{d\,\delta t}{dt} \quad (3.10)\end{aligned}$$

So, using what obtained in (3.9):

$$\begin{aligned}\delta^* \int_{t_0}^{t_1} L dt &= \int_{t_0}^{t_1} \left[\delta L + \frac{\partial L}{\partial t}\delta t - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\frac{d\,\delta t}{dt} + L\frac{d\,\delta t}{dt}\right] dt \quad (3.11) \\ &= \delta \int_{t_0}^{t_1} L dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial t}\delta t\, dt + \int_{t_0}^{t_1} \left(L - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\right)\frac{d\,\delta t}{dt} dt\end{aligned}$$

Finally, integrating by parts,

$$\begin{aligned}\int_{t_0}^{t_1} \left(L - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\right)\frac{d\,\delta t}{dt} dt &= \left[\left(L - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\right)\delta t\right]_{t_0}^{t_1} \\ &\quad - \int_{t_0}^{t_1} \frac{d}{dt}\left(L - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\right)\delta t\, dt \quad (3.12)\end{aligned}$$

[17] See Obs. 3.7

[18] Remember that $\delta^* y_i = \delta y_i$ because y_i are only place coordinates. Remember also the result of the obs. 2

In conclusion, remembering the (3.8), the *asynchronous* variation assumes the form:

$$\begin{aligned}\delta^* \int_{t_0}^{t_1} L dt &= \sum_i \left[\frac{\partial L}{\partial \dot{y}_i} dy_i\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i}\right] \delta y_i dt \\ &+ \left[\left(L - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\right)\delta t\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i} - L\right)\delta t\, dt\end{aligned} \tag{3.13}$$

Choosing $\delta t, \delta y_i = 0$ at the endpoints of the integration you obtain the following expression:

$$\begin{aligned}\delta^* \int_{t_0}^{t_1} L dt &= -\int_{t_0}^{t_1} \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i}\right] \delta y_i dt \\ &+ \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial t} + \frac{d}{dt}\left(\sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i} - L\right)\right]\delta t\, dt\end{aligned} \tag{3.14}$$

Finding the derivative in the second integral of the previous expression, you can obtain a formulation in which some terms, similar to those in Lagrange equations, appear. Effectively,

$$\begin{aligned}\frac{d}{dt}\left(\sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i} - L\right) &= \sum_i \ddot{y}_i \frac{\partial L}{\partial \dot{y}_i} + \dot{y}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i}\dot{y}_i - \frac{\partial L}{\partial \dot{y}_i}\ddot{y}_i - \frac{\partial L}{\partial t} \\ &= \sum_i \dot{y}_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i}\right] - \frac{\partial L}{\partial t}\end{aligned}$$

So, the second integral of the (3.14) is:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i} - L\right)\delta t\, dt = \int_{t_0}^{t_1} \sum_i \dot{y}_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i}\right]\delta t\, dt$$

Thanks to this elucidation it's easy to conclude that the zeros of the asynchronous variation of the L integral (3.14) are equivalent to the zeros of the Lagrange equations (3.5).

Proposition 3.1.1. *For every natural motion the zeros of Lagrange equations are equivalent to the zeros in the variation of the L-function integral* *taken in a general interval* $[t_0, t_1]$, *with respect to any AVM with null variations* δy_i *and* δt *at the endpoints.*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0 \Longleftrightarrow \delta^* \int_{t_0}^{t_1} Ldt = 0 \tag{3.15}$$

In other words, in the previous proposition, Levi-Civita obtains to compare the time variable to the space variables.
For this reason, hereinafter Levi-Civita develops his resoning on a 4-dimensions manifold, indicated with y_1, y_2, y_3, t. This is a

> varietà a quattro dimensioni in cui vengono a trovarsi rappresentati simultaneamente lo spazio e il tempo.[19]

3.1.4 Loss of Hamilton principle invariance with respect to a general transformation of time-dependent coordinates

Levi-Civita shows, at this point, what kind of problem would be generated involving time coordinate in his reasonings.

If time doesn't change, it's possible to get two important results:

1. the equivalence between Lagrange equations and Hamilton principle
2. the invariance of the Hamilton principle (and of Lagrange equations, by equivalence) with respect to any transformations of space coordinates

If time changes, instead, you only obtain equivalence between Lagrange equations and Hamilton variation principle, as shown in the proposition 3.1.3. Levi-Civita shows how a general time-dependent transformation of coordinates makes the integral $\int Ldt$ invariance fall.

[19] "a 4-dimensions manifold in which space and time are represented together"

Similarly to (3.4), a general transformation of coordinates in V_4 consists in a series of relations like the following

$$\begin{cases} x_0 = x_0(y_1, y_2, y_3, t) \\ x_1 = x_1(y_1, y_2, y_3, t) \\ x_2 = x_2(y_1, y_2, y_3, t) \\ x_3 = x_3(y_1, y_2, y_3, t) \end{cases} \tag{3.16}$$

If you substitute the Cartesian coordinates y_1, y_2, y_3, t with four independent combinations x_0, x_1, x_2, x_3, involving time t, and you assume the L function as a basis, the shape of the integral $\int Ldt$ has no invariant character with respect to transformations (3.16). According to Levi-Civita, in fact, the dt in the integral will be substituted by a linear expression with the differentials of all the four variables x_i.

In order to mantain the Hamilton principle invariant, Levi-Civita presents two possible solutions.
The first solution would be the substitution of the L basis with something more general

> allora sarebbe possibile raggiungere l'intento, ma in modo complesso e infecondo, perdendo in semplicità concettuale e formale ben più di quanto si guadagni in generalità.[20]

The second solution would be, instead, the modification of Hamilton principle through very small quantitative corrections, so that

> non è avvertibile il divario fra esso e l'ipotetico principio rigoroso nelle applicazioni correnti, non solo tecniche, ma anche astronomiche. Una tale circostanza si presenta manifestamente, qualora i termini correttivi abbiano, rispetto agli omologhi della teoria ordinaria, un ordine

[20] "in this way you will reach the goal, but in a difficult and fruitless way, losing conceptual and formal simplicity much more than gaining generality

di grandezza non superiore al centomilionesimo (10^{-8}).[21]

This is the solution chosen by Levi-Civita and explained in the rest of the lecture.

3.1.5 Hamilton principle modification in order to obtain invariance with respect to a general transformation of coordinates involving time

Levi-Civita wants the modification of the Hamilton's principle to be quantitative, so he has to define the range of values he wants to consider. He estimates the quantity c, as a constant speed, much bigger than any other reachable one in the motions he wants to deal with.
In particular he wants to consider the case in which both the numbers $\frac{v^2}{c^2}$ and $\frac{U}{c^2}$ are negligible with respect to unity.
This circumstance happens, for example, when c is comparable with the speed of light and v is the speed of any motion in terrestrial or celestial mechanics.
If you consider v as the speed of Earth around Sun and U the relative Newtonian potential, then U has the same magnitude of v^2 and

$$\frac{v^2}{c^2} \sim \frac{U}{c^2} \sim 10^{-8} \tag{3.17}$$

Now the attempt is to give a rigorous mathematical structure to the reasoning of Levi-Civita, organizing it in observations and propositions.
The first step of the modification of the variational Hamilton principle (3.2) by Levi-Civita is the following obvious observation:

Observation 3. If $\delta t = 0$ in t_0 and t_1

$$\delta \int_{t_0}^{t_1} Ldt = 0 \quad \Longleftrightarrow \quad \delta \int_{t_0}^{t_1} c^2 - Ldt = 0 \tag{3.18}$$

[21] "it will not be possible to distinguish between the rigorous hypothetical principle and the modified one in the current tecnical and astronomical applications. A similar circumstance happens when the corrective terms aren't bigger than 10^{-8} order"

Let's look carefully at what Levi-Civita is doing at this point of his reasoning, because the following observation contains the key assumption of his modification.

Observation 4. If you neglige terms of higher order than $\frac{v^2}{c^2} \sim \frac{U}{c^2}$, according to the range of values considered for v, U and c the following identity holds good

$$c^2 - L = c\sqrt{c^2 - v^2 - 2U} \tag{3.19}$$

Proof. If $L = \frac{1}{2}v^2 + U$,

$$c^2 - L = c^2 - \frac{1}{2}v^2 - U = c^2\left(1 - \frac{1}{2}\left(\frac{v^2}{c^2} - \frac{2U}{c^2}\right)\right)$$

Thanks to the hypothesis, we're allowed to write

$$c^2\left(1 - \frac{1}{2}\left(\frac{v^2}{c^2} - \frac{2U}{c^2}\right)\right) = c^2\sqrt{1 - \frac{v^2}{c^2} - \frac{2U}{c^2}} = c\sqrt{c^2 - v^2 - 2U}$$

□

At this point Levi-Civita is ready to present his important result

Proposition 3.1.2. *In the same hypothesis of observations 3 and 4, writing* $v = \frac{dl_0}{dt}$*, the following equivalence holds good:*

$$\delta\int_{t_0}^{t_1} Ldt = 0 \Longleftrightarrow \delta\int_{t_0}^{t_1} ds = 0 \tag{3.20}$$

with $ds^2 = (c^2 - 2U)\,dt^2 - dl_0^2$

Proof. Thanks to the previous observations

$$\delta\int_{t_0}^{t_1} Ldt = 0 \Longleftrightarrow \delta\int_{t_0}^{t_1} \left(c^2 - L\right)dt = 0 \Longleftrightarrow \delta\int_{t_0}^{t_1} c\sqrt{c^2 - v^2 - 2U}dt = 0$$

If you write $\frac{dl_0^2}{dt^2}$ instead of v^2, the chain of equivalences above can be continued as follows

$$\delta \int_{t_0}^{t_1} c\sqrt{c^2 - v^2 - 2U}dt = 0 \Longleftrightarrow \delta \int_{t_0}^{t_1} \sqrt{c^2 - \frac{dl_0^2}{dt^2} - 2U}dt = 0$$

Writing

$$ds^2 = \left(c^2 - 2U\right) dt^2 - dl_0^2 \tag{3.21}$$

the integral $\delta \int_{t_0}^{t_1} \sqrt{c^2 - \frac{dl_0^2}{dt^2} - 2U}dt$ can be written as

$$\delta \int_{t_0}^{t_1} \sqrt{c^2 - \frac{dl_0^2}{dt^2} - 2U}dt = \delta \int_{t_0}^{t_1} \sqrt{\frac{(c^2 - 2U)dt^2}{dt^2} - \frac{dl_0^2}{dt^2}}dt = \delta \int_{t_0}^{t_1} \sqrt{ds^2}$$

So

$$\delta \int_{t_0}^{t_1} Ldt = 0 \Longleftrightarrow \delta \int_{t_0}^{t_1} ds = 0$$

□

Levi-Civita makes some observations:

Observation 5. In cartesian coordinates ds^2 is the quadratic differential indefinite form

$$ds^2 = (c^2 - 2U)dt^2 - \sum_{i=1}^{3} dy_i^2$$

Observation 6. In the range of both terrestrial and celestial mechanics it's always true that $ds^2 > 0$.

Proof. If you consider $ds^2 = c^2dt^2\left(1 - \frac{2U}{c^2} - \frac{v^2}{c^2}\right)$ and remember the considered range of values it's possible to estimate $\frac{2U}{c^2} << 1$ and $\frac{v^2}{c^2} << 1$, so that $\left(1 - \frac{2U}{c^2} - \frac{v^2}{c^2}\right) > 0$. □

Levi-Civita shows the main consequence of his modification of the Hamilton's principle.

Proposition 3.1.3. *Assuming* ds^2 *as the basis of the transformation,* $\delta \int_{t_0}^{t_1} ds = 0$ *is invariant with respect to every transformation of coordinates in* V_4.

Proof. If you substitute the variables t, y_1, y_2, y_3 in $ds^2 = (c^2 - 2U)dt^2 - \sum_{i=1}^{3} dy_i^2$ with any linear independent combinations of them x_0, x_1, x_2, x_3 through a transformation similar to (3.16), the ds^2 will be again a quadratic form in the differentials of the independent variables as follows:

$$ds^2 = \sum_{i,k=0}^{3} g_{ik} dx_i dx_k \tag{3.22}$$

In this sense the principle of Hamilton expressed by (3.20) is invariant with respect to every transformation of coordinates. □

Levi-Civita wants to give a geometric interpretation of his result. He states that, through an analytic extension to the imaginary, it's possible to see ds^2 as the square of the distance between two imaginary points in V_4.
In this way it's possible to state that the ds^2 establishes a metric determination in V_4 and that

> le geodetiche proprie di questa metrica (curve che minimizzano $\int ds$ senza annullare ds) altro non sono che le curve orarie dell'originario problema meccanico.[22]

3.1.6 Lorentz transformations as a particular application of the previous method

Levi-Civita tries, at this point, to express a sort of principle of relativity without a direct statement. Let's follow his reasoning.
He recalls the important result of the classic mechanics which states that if the net force is zero - or equivalently if U is zero - then uniform motion according to Newton's equations is the result. First of all he clarifies that, thanks to the rigorous equivalence between the Newton's equations of motion (3.1) and the Hamilton principle (3.2), the latter also defines uniform motions if $U = 0$. Levi-Civita states that this property still holds good using

[22] "the geodesic curves of this metrics (curves which minimize $\int ds$ without vanishing it") are nothing but the curves which solve the original mechanical problem.

the modified Hamilton principle (3.20), instead of the original one, even if (3.20) is not rigorously equivalent to the Newton's equations (3.1).
It's possible to reformulate Levi-Civita's reasoning with the following proposition

Proposition 3.1.4. *If* $U = 0$, $\int ds = 0$ *implies* $\dot{y}_i = const$

Proof. If $U = 0$ the ds^2 becomes

$$ds_0^2 = c^2 dt^2 - dl_0^2 \tag{3.23}$$

and the modified Hamilton principle (3.20) becomes

$$\delta \int_{t_0}^{t_1} ds_0 = 0 \tag{3.24}$$

which, writing dl_0^2 in cartesian coordinates and $ds_0 = \sqrt{\left(c^2 - \sum_{i=1}^{3} \frac{y_i^2}{dt^2}\right) dt^2}$, assumes the form

$$\delta \int_{t_0}^{t_1} L^* dt = 0$$

where $L^* = \sqrt{c^2 - \sum_{i=1}^{3} \dot{y}_i{}^2}$
For what has been shown in 3.1.2, whatever be the form of L^*, $\delta \int_{t_0}^{t_1} L^* dt = 0$ is equivalent to

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{y}_i}\right) - \frac{\partial L^*}{\partial y_i} = 0$$

The function L^* does not depend directly on y_i, so the previous equation gives

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{y}_i}\right) = 0$$

that is

$$\frac{\partial L^*}{\partial \dot{y}_i} = \text{const}$$

for $i = 1, 2, 3$. This means that all the $\dot{y}_i$ are constant for $i = 1, 2, 3$. □

If equations like (3.16), changing from (t, y_1, y_2, y_3) to (t', y_1', y_2', y_3') and preserving the invariance of ds_0^2 were available, consequently, thanks to the proposition 3.1.4 and the equation (3.24), a uniform motion with respect to

(t, y_1, y_2, y_3) coordinates, would have the same form in the new (t', y'_1, y'_2, y'_3) coordinates.
Levi-Civita states that the Lorentz transformations, denoted with Λ, are exactly this kind of transformation. Applying Lorentz transformations in order to change (t, y_1, y_2, y_3) into (t', y'_1, y'_2, y'_3), it results

$$ds_0^2 = c^2 dt^2 - \sum_{i=1}^{3} dy_i^2 = c^2 dt'^2 - \sum_{i=1}^{3} dy_i'^2$$

Such an invariance means that all the uniform motions can be considered as equivalent with respect to a change of coordinates of the type Λ. This is a mathematical reformulation of the principle of relativity for all the reference frame in uniform relative motion.

It's important to notice, according to Levi-Civita, that

Observation 7. Every transformation Λ changes a uniform motion into a uniform motion again, but the speed varies, in general. Only the motion with speed c holds unchanged.

Proof. If the motion has speed c, it means that

$$c^2 = \frac{dl_0^2}{dt^2}$$

So it's $ds_0^2 = c^2 dt^2 - dl_0^2 = c^2 dt^2 - c^2 dt^2 = 0$.
Thanks to the invariance of ds_0^2, and, in particular, of the form $c^2 dt^2 - \sum_{i=1}^{3} dy_i^2$, after using the transformation Λ, it results $cdt'^2 - \sum_{i=1}^{3} dy_i'^2 = 0$.
So, in the new coordinates (t', y'_1, y'_2, y'_3), it's

$$c^2 = \sum_{i=1}^{3} \frac{dy_i'^2}{dt'^2}$$

. □

3.1.7 The optics problem

Levi-Civita faces the problem about the speed of light and the principle of relativity. He presents the problem saying that in the classical modelization

of the propagation of light, as well as in Newtonian mechanics, an absolute frame of reference is admitted to exist.
This one is represented by an hypothetical medium at rest,i.e. the aether. It's also the support of the optical phenomena.
In the vacuum light travels along a right line with constant speed c with respect to the aether, or, which is the same, respect to a reference frame at rest in the aether. c is the speed of light measured by any observer O, at rest in the aether.
Let's consider a solid substance C in uniform translational motion with speed u and a light beam which travels in the same direction of the motion. In the O frame the light beam can be considered as moving uniformely along a straight line with constant speed c, but an observer O', moving together with C, should measure the speed $c - u$.
The fact that the speed of light is c also with respect to O' is a result confirmed, indeed, by the classic experience by Michelson, later repeated by Majorana[23].
Levi-Civita proposes to explain the experimental results in a simple way.

> [...] basta evidentemente che ciò che macroscopicamente appare come traslazione di un corpo C con velocità u, sia, in un più affinato stadio di misura, una trasformazione Λ[24]

This way of thinking is not so strange, according to Levi-Civita. Every ordinary uniform translation can actually be identified with a transformation Λ for less than 10^{-8}, if $\frac{u}{c} < 10^{-4}$.
So a synthetic solution for the optics problem can be given as follows:

Proposition 3.1.5. *Classic laws of geometric optics and Michelson's experience are both satisfied if light propagation is admitted to obey*

$$\delta \int ds_0 = 0 \qquad \textit{(i.e uniform motion)}$$

[23]Quirino Majorana confirmed the constancy of the speed of light emitted by a movin source in 1918, for more details see [32]

[24]"what macroscopically appears as a traslatory motion of a substance C with speed u has to be considered, thanks to an improved system of measure, a transformation Λ"

with

$$ds_0^2 = 0 \quad \textit{(i.e. motion with speed c)}$$

and you consider translation of solid substances as a transformation Λ *(just a very little different from ordinary).*

Levi-Civita insists to notice[25] that it's the physical experience itself which ascribes a fundamental importance to the form

$$ds_0^2 = c^2 dt^2 - dl_0^2$$

3.1.8 Confluence of optics and mechanics

According to what's shown so far, that special kind of motion corresponding to the propagation of light in the aether in absence of forces has $c^2dt^2 - dl_0^2$ as basis form, where the constant c has a well determined numeric value. All the other usual motions, with speed less than celestial, subjected to conservative forces, have the form

$$ds^2 = (c^2 - 2U)dt^2 - dl_0^2$$

as the basis form for transformations, where the constant c has nothing but the feature of being big enough.
It's important to focus the attention on the criterion declared by Levi-Civita to achieve the desired confluence of optics and mechanics.

> Se si aspira alla unità di concezione dei fenomeni fisici, si è ovviamente tratti ad ammettere che, *caeteris paribus*, una stessa forma differenziale ds^2 domini così il moto dei punti materiali come l'andamento dei raggi luminosi fungendo da base in entrambi i casi.[26]

The consequences in adopting this criterion of unity are very fruitful.

[25] See also [33]

[26] "If you aim for unity of conception of the physical phenomena, you must admit that, *as things stand*, the same differential form ds^2 dominates the planet's motion as well as the light propagation, being the basis for both the cases." See again [33]

Observation 8. The validity of the typical value c in optical phenomena should be accepted also in the general dynamic phenomena. So, in the absence of forces, with $U = 0$, the form ds^2 typical of mechanics is actually identified with the form ds_0^2 typical of optics.

Observation 9. As well as in the proposition 3.1.5 on page 50 optics laws have been obtained as limit of mechanics laws in the hypothesis $U = 0$, so the same criterion can be extended to the general case $U \neq 0$.

The previous observation is expressed by the following in a clearer way

Proposition 3.1.6. *These are the postulates of the propagation of light:*

1. $\delta \int ds = 0$ *that is a geodetic principle*

2. $ds^2 = 0$ *that is the square of speed*

$$\frac{dl^2}{dt^2} = c^2 - 2U = c^2\left(1 - \frac{2U}{c^2}\right)$$

The second postulate states a very important feature about the value of the speed of light. Considering the presence of a potential $U \neq 0$ the speed of light V is different from c and, neglecting some unimportant terms, is correctly expressed by the following

$$V = c\left(1 - \frac{U}{c^2}\right) \tag{3.25}$$

A more expressive form of the previous postulates is given by Levi-Civita in a geometric way:

Proposition 3.1.7. *In the metrics given by* $ds^2 = (c^2 - 2U)dt^2 - dl_0^2$ *light beams are geodetic curves of zero length.*

3.1.9 Light rays curving under the action of masses

It's very interesting to look at the way Levi-Civita explains the inevitable consequence of the presence of the potential U inside the form ds^2, that is the curvature of light beams.

On one hand the curves followed by light beams can be defined simplifying the differential equations equivalent to the variational principle and solving them for dt, through the equation $ds^2 = 0$. On the other hand, according to Levi-Civita, some physical remarks could be done to get an expressive confirmation of the passagge from straight to curved lines caused by a force field.
First of all it's known that radioactive elements have great amount of energy inside them. Even if radioactivity is not a general property of the substances, it proves that there's a lot of energy inside matter. A quantitative valuation of this energy gives the result that a mass m of matter possesses energy equals to mc^2. Because of the factor c^2 this energy is extremely bigger than kinetic and potential ones, but may be ignored in ordinary mechanics as it remains essentially unchanged during motion. This is due to its intrinsic nature.
What is important is that once established the proportionality between mass and energy these two entities become concurrent. When the first is present, the second is too. So, if you write down a *physical* theory of light, where light beams are considered as flux lines or energy trajectories, then you must say that along the light beams there's also matter flux. Levi-Civita says that this flux is so little that it's not necessary to recall the ancient corpuscolary theory, and the ondulatory theory with its consequences still holds good. However the flux of matter is sufficient to be affected by the Newtonian gravitational attraction and this is the cause of the qualitative effect of the curvature of light beams.

3.1.10 Equations of mathematical physics corrected by Einstein i.e. Restricted relativity

Levi-Civita exposes the basis of the theory of relativity by Einstein, introducing it as a possibility to extend ordinary physics laws to a space with any metrics. A space, indeed, where the square of the linear element is a given differential form

$$dl^2 = \sum_{i,k=1}^{3} a_{ik} dx_i dx_k$$

which is not euclidean, in general.
All physics laws, once translated into equations according to the classic theories, posses an invariant character with respect to every transformation of spatial coordinates if you assume dl_0^2 as the basis for the transformation. Levi-Civita notices that the time coordinate t appears in these equations, but it's a separate variable. Involving it inside the transformations would make the equations loose their invariant character.
If you consider

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad \dots \quad , \Omega_m = 0 \tag{3.26}$$

as the translation into equations of a certain physical theory, you will find some parameters $p_1, p_2, \dots, p_n$ specific of the theory and coordinates of space and time among them. Besides, refererring the system to general coordinates x_1, x_2, x_3, there will be also the coefficients a_{ik} of the square of the linear element

$$dl_0^2 = \sum_{i,k=1} 3a_{ik} dx_i dx_k \tag{3.27}$$

The system (3.26) expresses physical-geometrical relations, so it's invariant with respect to coordinates transformations.
However, the real goal is to substitute the equations (3.26) with other equations

$$R_1 = 0, \quad R_2 = 0, \quad \dots \quad , R_m = 0 \tag{3.28}$$

identical to (3.26) in static conditions and invariant with respect to any 4 independent variables transformation (not only coordinates changing).
For this aim it suffices to assume whatever 4-dimension form ds^2 (reducing to $-dl^2$ if $dt = 0$), instead of the space form dl^2.
Levi-Civita doesn't explain the details of this process, but shows only the conceptual steps.
He states that, considering the original equations (3.26) as being referred to

Euclidean space of classic physics, you can take the form

$$ds_0^2 = c^2 dt^2 - dl_o^2$$

, which dominates the geometric optics (in absence of forces), as the new basis.
This is the choice made by Einstein to develop his Theory of Restricted Relativity (RR). Great emphasis is given to the fact that the equations (3.28), based on the optic ds_0^2, completely identify themselves with the system (3.26), which describes the electro-magnetic phenomena in medium at rest.
According to Levi-Civita this is the reason why the first systematic expositions of the RR have had a strong electromagnetic character.

3.1.11 Influence of all the physics phenomena on the measures of space and time i.e General Relativity

Having assumed the 4-dimensions formula

$$ds^2 = c^2 \left(1 - \frac{2U}{c^2}\right) dt^2 - dl_0^2$$

instead of ds_0^2 gives the result that the matter influences all the phenomena through the potential U. This allows us to guess that every physical circumstance similarly[27] influences the structure of the ds^2
As a consequence the geometric structure of the space will be quite different from the rigorous Euclidean space which is usually assumed as the basis for every solid physics theory. According to Levi-Civita this relationship between the ds^2 (which includes into itself space and time measures) and the whole physics phenomena is the qualitative postulate of Einstein theory of General Relativity (GR). As a conclusion for his lecture Levi-Civita underlines the great conceptual result reached by the kind of view he proposed.

[27] in the same order of $\frac{U}{c^2}$

A deep interconnection between geometry, kinematic and the whole physics is definitely estabilished because space and time[28] aren't the frame of the physics phenomena anymore but they changes together with the phenomena and viceversa.

> Come la meccanica di Newton, introducendo la gravitazione universale, ha realizzato una interdipendenza generale fra il moto di tutti i corpi ponderabili, così, più generalmente, la nuova meccanica, mediante le equazioni delle singole teorie fisiche, lievemente modificate, e le equazioni gravitazionali, lega fra loro tutti i fenomeni naturali in uno schema unitario, il quale (assumendo per base lo specifico ds^2 Einsteniano che conviene al caso considerato) ha carattere invariantivo di fronte a tutte le trasformazioni dei quattro parametri indipendenti che complessivamente individuano posto e tempo[29]

[28] i.e. geometry and kinematic according to Levi-Civita

[29] "The new mechanics, through indivdual physics theories equations (lightly modified) and gravitational equations link together all the natural phenomena into a unitary scheme as well as Newton mechanics estabilished the general interconnection between all the moving bodies, introducing the universal gravitation. Besides, the new mechanics has invariant character with respect to every transformation of the four independent parameters which identify space and time, assuming the appropriate Einstein ds^2 according to the circumstances."

3.2 Comments

Summing up Levi-Civita's reasoning we can focus the attention on the contents of the paragraphs 3.1.5 and 3.1.6. They can be intemized as follows:

1. The range of values accepted for the quantities v, u and U are fixed in such a way that
$$0 < \frac{v^2}{c^2} \sim \frac{U}{c^2} << 1$$

2. The Hamilton principle is modified through the insertion of a zero-quantity [30]
$$\delta \int_{t_0}^{t_1} Ldt = 0 \Longleftrightarrow \delta \int_{t_0}^{t_1} c^2 - Ldt = 0$$

3. Neglecting terms of higher order than $\frac{v^2}{c^2} \sim \frac{U}{c^2}$ it results
$$c^2 - L = c\sqrt{c^2 - v^2 - 2U}$$

4. Writing $v = \frac{dl_0}{dt}$ and $ds^2 = (c^2 - 2U)dt^2 - dl_0^2$ it results
$$\delta \int_{t_0}^{t_1} Ldt = 0 \Longleftrightarrow \delta \int_{t_0}^{t_1} ds = 0$$

5. Assuming ds^2 as the basis of the trasformation. $\delta \int_{t_0}^{t_1} ds = 0$ is invariant with respect to every transformation of coordinates in V_4.

6. Assuming the form $ds^2 = (c^2 - 2U)dt^2 - dl_0^2$, even using the modified Hamilton principle $\delta \int_{t_0}^{t_1} ds = 0$, null forces - or equivalently $U = 0$ - correspond to uniform motions.

7. Lorentz transformations, denoted with Λ, change from (t, y_1, y_2, y_3) to (t', y_1', y_2', y_3') and preserve the exact form of $ds_0^2 = c^2dt^2 - dl_0^2$ therefore[31]
$$ds_0^2 = c^2dt^2 - \sum_{i=1}^{3} dy_i^2 = c^2dt'^2 - \sum_{i=1}^{3} dy_i'^2$$

[30] in the hypothesis that $\delta t = 0$ in t_0 and t_1

[31] ds_0^2 is ds^2 with $U = 0$

8. All the uniform motions can be considered as equivalent with respect to a change of coordinates of the type Λ

3.2.1 On the meaning of ds^2 invariance

If we look carefully at the item 5 of the previous enumeration, which sums up the propositions 3.1.2 on page 45 and 3.1.3 on page 46, we can note that the invariance attributed to the new formulation of the Hamilton principle is a formal invariance.
In proposition 3.1.2 Levi-Civita formulates the Hamilton principle in terms of a differential form variation with no particular hypothesis except for the values of v, c and U. In proposition 3.1.3 it's proved that changing from (t, y_1, y_2, y_3) to any linear independent combinations of them (x_0, x_1, x_2, x_3) the quadratic differential form ds^2 results again into a quadratic differential form. Therefore the expression $\delta \int ds = 0$ of the Hamilton principle can be considered as invariant respect to every transformation of coordinates, but this invariance doesn't have a corresponding physical meaning.

3.2.2 The hidden postulates of the Einstein relativity

The physical sense of this invariance which turns into a relativity principle is given by the steps 6 and 7. First of all Levi-Civita highlights the fundamental role of the particular form $ds^2 = (c^2 - 2U)dt^2 - dl_0^2$. This precise form, when $U = 0$, corresponds to state of uniform motion thanks to the result of the proposition 3.1.4 on page 48. Furthermore the particular property of Lorentz transformations which leave the form ds_0^2 perfectly unchanged is the key for the statement of a relativity principle.
This fact makes Levi-Civita explain all the cases of relative motion using Lorentz tranformations and the particular form of the ds^2 given by optics. This conception, well explained in the paragraphs 3.1.7 and 3.1.8 of Levi-Civita's lecture, gives to the variable c the particular value of the speed of

light, assumed in optics, even in mechanics problems.
The importance of the value of c, together with the assumption of its constancy, arises here without a direct statement.
The *aim for unity of conception* mentioned by Levi-Civita as a good reason for adopting the same differential form ds^2 both in mechanics and in optics can be considered as a real philosophical choice made by Levi-Civita, recalling the *simplicity of nature* proper of Einstein thought.

3.2.3 Differences from Einstein theory of relativity

Even if there are some similarities between Einstein's and Levi-Civita's thought, it's impossible to neglect the differences between their kind of approach. Replying Abraham's objections, Einstein presented his theory of Special Relativity as an upgrade of the postulates which have always been put as basis of physical theories. He proposed a new version of these postulates and they constituted the real revolutionary content of Special Relativity.

Einstein, starting from postulates he considered the clearest, the simplest and the most confirmed by experiments, developed a theory which solved some problems left unsolved by classic theories.
Levi-Civita, instead, wanted to formulate a mathematical theory, characterized by small quantitative modifications, where the postulates and the physical meaning of the results played a marginal role.

As already said, from 1915 on, thanks to the publication of the "*Entwurf*" by Einstein and Grossmann, Levi-Civita and some other Italian mathematicians focused their attention on General Relativity. They looked for a correct formulation of the gravitational field equations which would be invariant with respect to any coordinates changes.
Many studies about Levi-Civita's contribution to the development of this Relativity chapter revealed he was one of the main author of this result. As consequence, it's clear that his research for an invariant mathematical

principle became his specific key to interpret the whole Relativity theory.

3.2.4 Uniqueness of Levi-Civita's reasoning

It's historically important to underline Levi-Civita's choice of introducing Einstein theory of relativity through the modification of Hamilton principle. While the lecture is dated 1919, the mathematical details of the reasoning presented are well explained inside Levi-Civita and Amaldi textbook *Lezioni di meccanica razionale*[32] first printed in 1923. It's not possible to state that this kind of process was originally taken on by Levi-Civita for the first time. As explained in the previous chapter, in fact, from 1915 on, the idea of formulating general relativity through modifications of variational Hamilton principle spread out. Levi-Civita's contribution in this sense is clearly witnessed by this lecture of 1919.[33]

Levi-Civita really believed in this kind of approach. He actually proposed it the same way in his 1928 book "*Fondamenti di meccanica relativistica*"[34] A very similar process can be found in the more recent classic book "*The Classical Theory of fields*" by L.D. Landau and E.M. Lifshitz [33] first published in 1939.

[32] See [30], [31] and [29]

[33] Levi-Civita inspired another famous contribution to the diffusion of this way of explaining relativity by Attilio Palatini in the same year. See [13]

[34] See [34]

Conclusion

The essential scientific comments about Abraham's and Levi-Civita's approach to Einstein relativity have already been done at the end of both previous chapters.
As a conclusion we'd like to fix some general ideas which arise looking at this whole work.
They represent future perspectives of research much more than fully supported statements.

The difference between Abraham's and Levi-Civita's approach

Abraham's approach actually was a physical approach. He even presented exclusively mathematical nature objections, highlighting some formal mistakes in Einstein's formulation. However, he forced Einstein to explain the physical and epistemological basis of his theory.
The possibility to look at the logical structuring of the physical, sometimes philosophical, principles which constituted Einstein theory suppositions makes Abrahams's contribution exceptional.

Levi-Civita's approach was mainly mathematical. He focused his contribution on the research for a correct form of Einstein's ideas. Obviously, as already seen, the choice of a particular form revealed also Levi-Civita's physical assumptions, but they were left aside on purpose.

Levi-Civita's influence on Italian school of Relativity

Levi-Civita's particular approach, which tries voluntarily to move the physical basis of the theory to the background, influenced all the following Relativity introduction in the Italian mathematical and physical community. As highlighted by C. Cattani in [3], the Italian school of Relativity was totally linked to the mathematicians close to Levi-Civita. For this reason, after 1919, the Relativity continued to be introduced as a particular application of the absolute differential calculus.

An important proof of this fact could be found in the analysis of the contents of the course "Calcolo differenziale assoluto e Teoria della relatività" taught by Roberto Marcolongo at the Naples University in 1919/1920[35]. In this course the first part of lessons is dedicated to a complete and rigorous introduction of the absolute differential calculus. It is followed by the second part dedicated to the presentation of the Theory of Relativity as an application of the previous mathematical methods. There's a clear contrast between the rigorousness of the method based on the absolute differential calculus and the qualitative character given to the Relativity postulates[36].

Loss of Einstein own work of logical structuring inside the Italian scientific literature

The work made by Einstein, led in order to clarify the basis of the theory and explain them in a logical exemplary manner, produced as result the great exposition of the theory development contained in the paper "On the relativity problem" [23].
The same logical structure is given by Einstein to the popular translation of

[35] The original notes of the course have been published in the book "Calcolo differenziale assoluto e Teoria della relatività by Mario Merola in 2006 [35]

[36] For example the postulate of the constancy of light speed is commented with the following words by Marcolongo "Is this a postulate given completely arbitrarily? Or does it have its own basis? Positive, on one hand, but negative on the other." See [35] p. 112

his Special and General Relativity first published in 1916 [37].
Even if the aim of Einstein 1914 paper was very similar to the 1919 Levi-Civita lecture's one, that was to present Relativity as a natural upgrade of classic mechanics, Levi-Civita didn't consider the contents of Einstein's paper at all.
It would be very interesting to continue seeking in the Relativity literature, that came after Einstein, traces of the original logical structure he gave to his theory in the 1914 paper and in 1916 book, mentioned above.

[37] See the English translation [36] of the original 1916 book

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