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On the GKP Vacuum in Gauge/Gravity Correspondences

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Introduction

Sometimes, certain peculiar behaviours hint a connection might exist to link theories apparently disjointed. Several examples have been provided by features which are encountered both in gauge theories, such as QCD, and string theories set on curved manifolds. An intriguing instance comes from the large spin properties of certain classes of gauge-invariant operators in QCD [4], the Wilson operators, made out of elementary quarks and gluons, and in addiction an arbitrary number of covariant derivatives. Obviously QCD is not scale independent, confinement being a self-evident symptom of that: nonetheless, due to asymptotic freedom, it can be treated as a conformal field theory in high energy processes, such as deep inelastic scattering. Hence the computation of conformal dimensions Δ for Wilson operators reveals a characteristic logarithmic scaling behaviour when their Lorentz spin s takes a very large value (and correspondingly so the number of derivatives does) [5]:

$$\Delta - s = 2\,\Gamma_{cusp}(\alpha_s)\,\ln s + \dots \quad ,$$

where the cusp anomalous dimension $\Gamma_{cusp}(\alpha_s)$, encoding the dependence on the coupling, emerges from calculations about polygonal light-like Wilson loops with cusps [6]. The same feature also arises in supersymmetric gauge theories and toy models for QCD: maybe the most renowned and studied example is provided by the $\mathcal{N} = 4$ Super Yang Mills.
In view of the AdS/CFT correspondence [2][3], Gubser, Klebanov and Polyakov [21] proposed a classical folded string configuration, laid on a subspace of AdS_5, which might be able to account for the logarithmic scaling. The GKP string, named after them, is then intended to offer a gravity theory counterpart to the above mentioned gauge theories, at least when studying some selected aspects such as the large spin scaling behaviour, as, in fact, the energy of the GKP string is governed by the leading order

$$E - S \sim \ln S \quad .$$

Anyway, the AdS/CFT duality claims that aspects of the gauge theory, which are suitable to a perturbative analysis, turn out to match strongly coupled regimes in the string theory, and vice-versa, so that, as a consequence, there is lack of overlaps to verify and improve the conjectured correspondence. Non-pertubative methods are thus needed to throw insight into regions of the theories, otherwise inaccessible.

The pursuit of this script mainly concerns the exploration of some aspects of the supposed duality, connecting a string theory set on the $AdS_5 \times S^5$ background and

the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory, endowed with the gauge group $SU(N)$. Remarkably, there is strong evidence that both these theories are integrable [11][28], so that a wide set of computation technologies are available to shed light also to non-perturbative aspects. In particular, when the number of colours N is extremely large, so that only planar Feynman diagrams are considered, the dilatation operator in $\mathcal{N} = 4$ SYM has been proven [28] (at least under some restrictions) to coincide with the hamiltonian of the spin chain (indeed, an integrable model), first introduced by Heisenberg in order to explain phenomena of ferromagnetism: gauge-invariant operators are then set to correspond to excitations over the ferromagnetic vacuum, whereas their anomalous dimensions shall match the eigenvalues of the spin chain hamiltonian, and therefore arise from its diagonalization by making use of Bethe Ansatz and related computation techniques. This layout has been extensively and deeply studied in literature, and, moreover, the role of its stringy counterpart, via AdS/CFT, is embodied by the string configuration proposed by Berenstein, Maldacena and Nastase (BMN) [20].
In what follows instead, the GKP classical solution will be taken into account, as a ground state, over which excitations are risen. This different choice of the stringy vacuum must thus reflect into a novel ground state for the spin chain: as the interest lies on the large spin limit for the GKP string, the more suitable option is provided, as it will be explained, by an antiferromagnetic vacuum. The powerful methods furnished by integrability allow to determinate the dynamics of excitations over the antiferromagnetic vacuum, and, since they do correspond to actual particles in $\mathcal{N} = 4$ SYM, at the same time the scattering matrices and dispersion laws of the latters are obtained.
While the energies of the particles promptly lead to the spectrum of anomalous dimension for the gauge invariant operators, the scattering matrices represent the starting point to compute the polygonal Wilson loops in $\mathcal{N} = 4$ SYM, upon calculating the pentagonal amplitudes in which they may be decomposed [46].

A further gauge/string duality will be examined in this text, namely the correspondence relating a three dimensional supersymmetric conformal gauge field theory, that is $\mathcal{N} = 6$ Chern-Simons Matter theory (else ABJM) [26], and a string theory living on the $AdS_4 \times \mathbb{CP}^3$ background. More precisely, the role of GKP string in this (conjectured) correspondence will be highlighted, by exploiting again the chances offered by the integrability dwelling in $\mathcal{N} = 6$ Chern-Simons [51][52]. In the large spin limit, the whole gauge theory is led to a low-energy reduction (corresponding to the Bykov model [50], on the string side), which reveals astonishing resemblances with the O(6) non-linear σ-model: remarkably, the O(6) σ-model governs the dynamics of another known gauge theory, namely the $\mathcal{N} = 4$ SYM, in the (low-energy, high spin) Alday-Maldacena decoupling limit [38]. Actually, this fact should be not quite a surprise, as it simply joins several further hints [62][58] that a relation between $\mathcal{N} = 6$ Chern-Simons and $\mathcal{N} = 4$ SYM may exist.

The text is organized as follows. The first chapter provides an essential introduction to spin chains and Bethe Ansatz and moreover a short summary of tools, stemming from integrability, which will be employed in subsequent chapters. Strictly speaking, those computational tools only represent a simplified version of the techniques actually used, nevertheless even a sketchy outline might be helpful in highlighting some interesting physical aspects underneath them.

The second chapter contains a very brief review of topics about the AdS_5/CFT_4 correspondence and the role assumed by integrability: this quick survey is just aimed to contextualize the next two chapters within the AdS/CFT framework.

The core of the work consists in the third and fourth chapters, which are written after [47], along with some (up to now) unpublished material. In the third chapter the excitations over the antiferromagnetic (GKP) vacuum will be introduced and examined, so to find the all loop expressions for the S-matrices governing their scattering [1]: the task is achieved upon turning an infinite set of Asymptotic Bethe Ansatz equations [30] to a finite number on non-linear integral equations (NLIE), according to [34][37]. Remarkably, all the processes can be written in terms of the fundamental scalar-scalar scattering phase.

In the fourth chapter, the scattering matrices, previously found, are employed as building blocks to construct the set of complete Asymptotic Bethe Ansatz (ABA) equations at all loops: from those equations, the dispersion laws of the particles are achieved.

The fifth chapter traces [27], in order to show a few results about the AdS_4/CFT_3 duality studied on the GKP vacuum, and in particular a low-energy reduction is taken into account: moreover, the low-energy reductions of $\mathcal{N} = 6$ Chern-Simons and $\mathcal{N} = 4$ SYM are compared, so that some peculiar common features can be displayed (even pictorially).

Finally, two appendices gather some useful formulae employed throughout the maintext.

[1]More precisely, the scalar factor in front of the matricial structure is computed.

Chapter 1

Ideas and tools of integrability

1 Integrability in a nutshell

This introductory chapter is intended to display a set of computational technologies, exploited to yield most of the results exposed throughout the rest of the text. Although these methods are outlined only in a sketchy form, far from being exhaustive of the subject, and moreover the achievements in the following rely on much more involved refinements on these basic tools, nevertheless it turns out instructive to portray the main steps and point out a few interesting aspects of the techniques hereafter.

Almost all the statements contained in the present chapter are referred to (quantum) integrable models set in two dimensions (precisely one time and one space direction). Quantum integrability means that the model is endowed with an infinite number of conserved charges in involution: say $\hat{H}$ the hamiltonian operator and be $\{Q_i\}$ an infinite set of Hermitian operators associated to Noether charges, *i.e.* $\left[\hat{H}, \hat{Q}_i\right] = 0$, integrability arises whenever it holds true

$$\left[\hat{Q}_i, \hat{Q}_j\right] = 0 \qquad \forall\ i, j \quad .$$

When referring this definition to two dimensional scattering models, several astonishing results stand out. First, the number of the particles, included in the system, do not change after collisions, hence creation or annihilation events being forbidden.
Moreover, the scattering is factorizable, meaning that every process, regardless of the number of particles involved, decomposes into a sequence of two body scatterings. However, the initial state of a process may evolve to final in many different ways, that is, there exists more than one sequence leading to the same final result: to ensure the consistency of the theory, an equivalence between those different scattering patterns ought to be stated. To this aim, the Yang-Baxter relation has been introduced:

$$S_{23}(u-v) \otimes S_{12}(u) \otimes S_{13}(v) = S_{13}(v) \otimes S_{12}(u) \otimes S_{12}(u-v) \quad .$$

These and several other implications of integrability often underlie formulae and methods employed in this text.

2 Spin chains and Coordinate Bethe Ansatz

A pivotal example of integrable system comes from the Heisenberg spin chain, introduced to study phenomena of ferromagnetism in metals. The spin chain is a one dimensional lattice made of L sites, and each of them is endowed with a spin, whose projection along a fixed axis can assume just two values, say either $+\frac{1}{2}$ or $-\frac{1}{2}$. The system is then governed by the hamiltonian

$$\hat{H} = J \sum_{l=1}^{L} \left(1 - 4\vec{S}_l \cdot \vec{S}_{l+1}\right) \quad : \tag{1.1}$$

each site hosts a two dimensional representation of algebra $su(2)$, so that $\vec{S}$ is proportional to Pauli matrices $\vec{S} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$. The interest lie on closed chains, hence a periodicity constraint must be imposed, with period L. The interaction in (1.1) is of nearest-neighbour type, in that it involves only pairwise adjacent sites, therefore, upon restricting to positive values of the coupling J (ferromagnetic case), the minimal eigenvalue of the hamiltonian occurs when all the spin are aligned: this configuration corresponds to the ferromagnetic vacuum. The eigenvectors of the 2×2 matrix $\vec{S}$ are

$$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \tag{1.2}$$

associated respectively to eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$: for definiteness, the choose of the ground state is now set to fall to the tensor product state

$$|0\,\rangle = \underbrace{e_1 \otimes \cdots \otimes e_1}_{L \text{ times}} \quad ; \tag{1.3}$$

the whole space state of the system is then built as a tensor product of single site $\mathbb{C}^2$ factors

$$\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L \text{ times}} \quad . \tag{1.4}$$

The lowest-lying excitation is achieved instead by flipping one spin from $-\frac{1}{2}$ to $+\frac{1}{2}$ and on the same footing one e_1 turns to e_2, whereas in general the inversion of M spins accounts for M excitations: for several purposes, reverted spins behave like pseudoparticles, customarily named magnons. The hamiltonian (1.1) may be recast into another useful form

$$\hat{H} = 2J \sum_{l=1}^{L} \left(\hat{I}_{l,l+1} - \hat{P}_{l,l+1}\right) \quad , \tag{1.5}$$

in term of the operator $\hat{P}_{l,l+1}$ which exchanges adjacent sites l, $l+1$, and $\hat{I}_{l,l+1}$, acting trivially on the chain. Along with the hamiltonian, a further example of observable is provided by the magnetization

$$\mathcal{M} = \sum_{l=1}^{L} S_l^{(z)} \quad , \tag{1.6}$$

which is found to commute with the hamiltonian $\left[\hat{H}, \mathcal{M}\right] = 0$: since each spin flip lowers the magnetization by one unit, the commutation relation entails that the number of excitations is a conserved quantity. In addition to that, the shift operator is introduced as a product of exchange operators

$$\hat{U} = \hat{P}_{1,2} \cdots \hat{P}_{L-1,L} \tag{1.7}$$

whose action consists in shifting all the sites in the chain by one position (properly, both left and right operators should be defined). Since the periodicity is imposed, the application of $\hat{U}$ repeated L times must not affect the system, $\hat{U}^L = 1$, thus the eigenvalues of the shift operator take the form

$$U = e^{2\pi i \frac{n}{L}}$$

for $n = 0, 1, \ldots, L-1$; what is more, $\left[\hat{H}, \hat{U}\right] = 0$, so that the eigenstates of hamiltonian $|\psi\,\rangle$ are simultaneously eigenstates of $\hat{U}$. These considerations then helps in quickly retrieving the eigenstate of (1.1) relative to a single excitation on the chain, as a superposition of plane waves (a Fourier transform):

$$|\psi(p)\,\rangle = \sum_{l=1}^{L} \psi(l)\,|l\,\rangle = \sum_{l=1}^{L} e^{ipl}\,|l\,\rangle \quad , \tag{1.8}$$

where the state is labelled by the momentum of the magnon, and $\psi(l)$ describes the amplitude for a pseudoparticle created over the site $|l\,\rangle = S_l^+|0\,\rangle$ ($S^+ = \frac{\sigma^{(x)} + i\sigma^{(y)}}{2}$). The energy eigenvalue E_1 for a single-excitation state is deduced by applying the hamiltonian (1.5) to the state (1.8):

$$\hat{H} \sum_{l=1}^{L} \psi(l)\,|l\,\rangle = E_1 \sum_{l=1}^{L} \psi(l)|l\,\rangle \quad . \tag{1.9}$$

$\hat{H}$ takes the form of a discretized laplacian

$$\hat{H}|\psi(p)\,\rangle = -2J \sum_{l=1}^{L} \left[(\psi(l+1) - \psi(l)) - (\psi(l) - \psi(l-1))\right]|l\,\rangle \quad , \tag{1.10}$$

and, since $\psi(l) = e^{ipl}$, it is promptly obtained:

$$E_1 = 8J \sin^2\left(\frac{p}{2}\right) \equiv \epsilon(p) \quad . \tag{1.11}$$

When the number of magnons is increased to two, the computations turn more delicated. The two-magnon state is described by

$$|\psi(p_1, p_2)\,\rangle = \sum_{1 \le l_1 < l_2 \le L} \psi(l_1, l_2)|l_1, l_2\,\rangle \quad ; \tag{1.12}$$

some attention should be paid to the summation in the formula above: the sequence of inequalities is needed in order to avoid the double counting of the addends, while the inequality $l_1 < l_2$ holds strictly (*i.e.* $l_1 \leq l_2$ is not allowed), as there must not appear two magnons on the same site, because of the spin $\frac{1}{2}$ representation of the $su(2)$ underlying algebra. Now, when considering the eigenvalue equation

$$\hat{H} \sum_{1 \leq l_1 < l_2 \leq L} \psi(l_1, l_2)|l_1, l_2\,\rangle = E_2 \sum_{1 \leq l_1 < l_2 \leq L} \psi(l_1, l_2)|l_1, l_2\,\rangle \tag{1.13}$$

two sets of equations arise. Whether the two magnons are far at least two sites away $l_2 - l_1 > 1$, the energy simply reads:

$$\begin{aligned} E_2\,\psi(l_1, l_2) &= [2\psi(l_1, l_2) - \psi(l_1 - 1, l_2) - \psi(l_1 + 1, l_2)] + \\ &+ [2\psi(l_1, l_2) - \psi(l_1, l_2 - 1) - \psi(l_1, l_2 + 1)] \quad , \end{aligned} \tag{1.14}$$

and that entails $E_2 = \epsilon(p_1) + \epsilon(p_2)$. Otherwise, when $l_2 = l_1 + 1$ a subtlety occurs: to discard the possibility of a double occupation of a site, a different kind of equations must be taken into account:

$$E_2\,\psi(l_1, l_1 + 1) = 2\psi(l_1, l_1 + 1) - \psi(l_1 - 1, l_1 + 1) - \psi(l_1, l_1 + 2) \quad . \tag{1.15}$$

To assure the consistency of (1.14)and (1.15) simultaneously and restore (1.12) as an eigenstate, Hans Bethe [7] (for review and applications, see [10][9][8] for instance) formulated his celebrated *ansatz*: the state (1.12) shall be the superposition of an incoming and an outgoing wave, provided the latter experience a phase shift $\delta(p_1, p_2)$, due to scattering of the two magnons (assumed to be reflectionless)

$$|\psi(p_1, p_2)\,\rangle = \sum_{1 \leq l_1 < l_2 \leq L} \left(e^{ip_1\, l_1 + ip_2\, l_2} + e^{ip_2\, l_1 + ip_1\, l_2 + i\delta}\right) |l_1, l_2\,\rangle \quad . \tag{1.16}$$

The request that (1.16) be an eigenfunction of the hamiltonian (1.5) leads to an expression for the phase $\delta(p_1, p_2)$

$$e^{i\delta} = -\frac{e^{ip_1 + ip_2} - 2e^{ip_2} + 1}{e^{ip_1 + ip_2} - 2e^{ip_1} + 1} \quad : \tag{1.17}$$

the phase delay thus enjoys a clear physical interpretation as the S-matrix between two magnons

$$e^{i\delta(p_1, p_2)} = \mathcal{S}(p_1, p_2) \quad . \tag{1.18}$$

A handy substitution could be performed in terms of the rapidity of a magnon

$$u \equiv \frac{1}{2} \cot \frac{p}{2} \quad ,$$

so that momentum and energy of a magnon read

$$e^{ip} = \frac{u + \frac{1}{2}}{u - \frac{1}{2}} \qquad\qquad \epsilon(u) = \frac{1}{u^2 + \frac{1}{4}} \tag{1.19}$$

and, what is more, the scattering phase (1.18) gets simpler, as, for instance, the unitarity is now self-evident:

$$\mathcal{S}(u_1, u_2) = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \quad . \tag{1.20}$$

The periodicity condition $\psi(l_1, l_2) = \psi(l_2, l_1 + L)$ entails

$$e^{ip_1 L} = e^{i\delta} \qquad\qquad e^{ip_2 L} = e^{-i\delta} \quad ,$$

which also implies the quantization condition of the momenta $p_1 + p_2 = 2\pi \frac{m}{L}$, some $m = 0, 1, \dots, L - 1$.

Aiming now to cope with the problem of an arbitrary number, it is useful to write the general eigenstate for M magnons

$$|\psi(p_1, \dots, p_M)\,\rangle = \sum_{1 \le l_1 < \dots < l_M \le L} \psi(l_1, \dots, l_M) S^+_{l_1} \cdots S^+_{l_M} \,|0\,\rangle \quad : \tag{1.21}$$

the ansatz by Bethe claims the wave function $\psi(l_1, \dots, l_M)$ takes the form

$$\psi(l_1, \dots, l_M) = \sum_\pi \exp\left[i \sum_{k=1}^{M} p_{\pi(k)} l_k + \frac{i}{2} \sum_{k<j} \delta(p_{\pi(k)}, p_{\pi(j)}) \right] \quad . \tag{1.22}$$

Upon identifying again the phase shift with a scattering matrix between magnons $\mathcal{S}(p_j, p_k) = \exp[i\delta(p_j, p_k)]$, the periodicity condition leads to the set of Bethe equations

$$e^{ip_k L} = \prod_{j \ne k}^{M} \mathcal{S}(p_k, p_j) \quad . \tag{1.23}$$

Actually the Bethe equations (1.23) are but a the quantization condition for the momentum of a particle, travelling across closed circuit: the particle is not free, for other particles are encountered within the path, thus a phase delay stems from the scatterings. Once reformulated with the aid of rapidities, the (1.23) become:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j \ne k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \quad . \tag{1.24}$$

Whether, in addiction to periodicity condition $\sum_{k=1}^{M} p_k = 2\pi \frac{m}{L}$ ($m = 0, \dots, L - 1$), the invariance of the system under the shift operator $\hat{U}$ is demanded, a further constraint arises

$$\prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1 \quad , \tag{1.25}$$

or, equivalently, $\sum_{k=1}^{M} p_k = 2\pi n$: the consistency of the two statements turns to a zero momentum condition

$$\sum_{k=1}^{M} p_k = 0 \quad . \tag{1.26}$$

As a final remark, it should be noted that in the $L \longrightarrow \infty$ limit, bound states may occur [12]. The Bethe equations (1.24) allow for complex solutions. Let for instance a complex valued rapidity u_1 be taken, and be its imaginary part positive (whether negative, the reasoning below would apply equivalently): the modulus below is larger than one

$$\left| \frac{Re\, u_1 + i\left(Im\, u_1 + \frac{1}{2}\right)}{Re\, u_1 + i\left(Im\, u_1 - \frac{1}{2}\right)} \right| > 1$$

hence, when $L \longrightarrow \infty$ the *l.h.s.* of (1.24) blows, hence this implies there must exist another root, say u_2, such that $u_2 = u_1 - i$ to compensate the divergence in the *r.h.s.*. Three alternatives may now occur. If the imaginary part of u_2 is lowered to a negative value, u_1 and u_2 couple to form a complex of roots, called 2-string, which arranges according to

$$\begin{cases} u_1 = \tilde{u} + \frac{i}{2} \\ u_2 = \tilde{u} - \frac{i}{2} \quad ; \end{cases}$$

after the requirement that the total momentum be real $p(u_1) + p(u_2) \in \mathbb{R}$, it follows that $u_1\,, u_2$ are complex conjugated, hence $\tilde{u}$ must be real. Otherwise, if u_2 lies on the real axis, a further root u_3 needs to be involved in the complex to ensure the reality of the momentum, thus a string of length three is assembled:

$$\begin{cases} u_1 = \tilde{u} + i \\ u_2 = \tilde{u} \\ u_3 = \tilde{u} - i \quad , \end{cases}$$

this time u_2 coincides with the real centre. Finally, the last case concerns a root u_2 whose imaginary part still remains positive: therefore the *l.h.s.* of (1.24) diverges again, thus another root is needed to parallel the effect on the *r.h.s.*, and so on: this procedure iterates until a root with non-positive imaginary part is encountered, so that the string gathers more entries and its length increases. Eventually, the reality condition of the momentum shapes a l-string bound state of solutions according to the arrangement

$$u_a = \tilde{u} + \frac{i}{2}(l + 1 - 2a) \qquad a = 1, \ldots, l \quad . \tag{1.27}$$

3 Nested Bethe Ansatz

Whenever the particles are endowed with some internal degree of freedom, the procedure outlined above needs to get refined [13][14]. Let the simplest example of this sort be considered: the particles do carry a colour index with two values $a \in \{1, 2\}$, on a closed spin chain made out of L sites. Sticking for a while to the two particle case, just to fix the notations, an ansatz might be proposed to describe the state:

$$\begin{aligned} |\psi(p_1, p_2)\,\rangle = \sum_{l_1 < l_2}^{L} \sum_{a_1, a_2 \in \{1,2\}} \Big[& B^{12}_{12}\, e^{ip_1\, l_1 + ip_2\, l_2}\, \phi^\dagger_{a_1}(l_1)\, \phi^\dagger_{a_2}(l_2)\, |0\,\rangle + \\ & + B^{12}_{21}\, e^{ip_1\, l_1 + ip_2\, l_2}\, \phi^\dagger_{a_2}(l_1)\, \phi^\dagger_{a_1}(l_2)\, |0\,\rangle + B^{21}_{12}\, e^{ip_1\, l_2 + ip_2\, l_1}\, \phi^\dagger_{a_1}(l_2)\, \phi^\dagger_{a_2}(l_1)\, |0\,\rangle + \\ & + B^{21}_{21}\, e^{ip_1\, l_2 + ip_2\, l_1}\, \phi^\dagger_{a_2}(l_2)\, \phi^\dagger_{a_1}(l_1)\, |0\,\rangle \Big] \end{aligned} \tag{1.28}$$

where the operator $\phi_a^\dagger(l)$ creates a particle on the site l of the chain. The amplitudes $B_{a_1,a_2}^{l_1,l_2}$ (the subscripts account for colour configuration, whereas the superscripts refer to the positions) are not independent, as the requirement that (1.28) be an eigenstate of the hamiltonian relates each other $B_{a_1a_2}^{21} = \mathbb{S}_{12}\, B_{a_1a_2}^{12}$ under the action of the matrix $\mathbb{S}$ defined on $\mathbb{C}^2 \otimes \mathbb{C}^2$ according to

$$\mathbb{S}_{kj}(u_k, u_j) = \frac{u_k - u_j - i\hat{P}_{kj}}{u_k - u_j + i} \tag{1.29}$$

$\mathbb{S}$ (here written by means of rapidities and exchange operator $\hat{P}_{kj}$) works as an S-matrix, and it allows to express all the amplitudes in terms of, say, $B_{a_1a_2}^{12}$. Now, coping with the general M particle state, the ansatz (1.28) generalizes to:

$$|\psi(p_1,\ldots,p_M)\,\rangle = \sum_{\{\pi\}}\sum_{\{\tau\}} B_\tau^\pi \exp[i\sum_{j=1}^{M} p_j\, l_{\pi(j)}]\, \phi_{\tau(1)}^\dagger(\pi(1))\phi_{\tau(2)}^\dagger(\pi(2))\cdots\phi_{\tau(M)}^\dagger(\pi(M))|0\,\rangle \tag{1.30}$$

where the summations are meant over the $M!$ permutations of the position indices π and over the permutations of τ colour indices. The imposing (1.30) as an eigenstate of the hamiltonian results in a set of 'scattering' relations connecting the amplitudes:

$$B_{a_1\ldots a_M}^{\pi} = \mathbb{S}_{j,j+1}\, B_{a_1\ldots a_M}^{[j,j+1]\circ\,\pi} \quad , \tag{1.31}$$

the permutation $[j, j+1]$ transposes the j-th and $j+1$-th sites. The periodicity condition then translates to

$$e^{ip_k\,L}\xi_0 = \mathbb{S}_{k+1,k}\mathbb{S}_{k+2,k}\cdots\mathbb{S}_{M,k}\mathbb{S}_{1,k}\mathbb{S}_{2,k}\cdots\mathbb{S}_{k-1,k}\xi_0 \tag{1.32}$$

(the k-th particle goes round the chain, colliding with all the other particles); the column vector ξ_0 is defined as $\xi_0 = \left(B_{a_1\ldots a_M}^{\mathbb{I}}\right)$, the identity permutation $\mathbb{I}$ acts trivially on the sequence of labels $12\cdots M$. In order to diagonalize the product of $\mathbb{S}$ matrices in (1.32), a nested structure of spin chains needs to be constructed. Since the numbers of type-1 particles and type-2 particles (the latter named Q) are conserved, the space of eigenstates of $\mathbb{S}_{k+1,k}\mathbb{S}_{k+2,k}\cdots\mathbb{S}_{M,k}\mathbb{S}_{1,k}\mathbb{S}_{2,k}\cdots\mathbb{S}_{k-1,k}$ is left characterized by two natural numbers, Q and the total number of particles M . A novel, auxiliary, $su(2)$ Heisenberg spin chain can thus be built over the original one: the M magnons now represent the sites of the new chain and the 'auxiliary' vacuum state is achieved when all the M colour labels are set to 1, for instance; every excitation is obtained by flipping a colour index to the entry 2, so that Q counts the number of such novel pseudoparticles. The vectors ξ_0 are identified then as the states of the auxiliary spin chain.
The case $Q = 0$ plainly coincides with the original $su(2)$ spin chain, discussed in previous section. The first non trivial instance occurs for $Q = 1$: the single auxiliary particle state is left determined by the ansatz (recalling(1.2))

$$\xi(M,1) = \sum_j \mathcal{A}_j(y)\, \underbrace{e_1\otimes\cdots\otimes e_1}_{j-1}\otimes e_2\otimes \underbrace{e_1\otimes\cdots\otimes e_1}_{M-j} \quad , \tag{1.33}$$

with the amplitude

$$\mathcal{A}_j(y) = f(y, u_j) \prod_{n=1}^{j-1} \tilde{S}(y, u_n) \tag{1.34}$$

accounting for an auxiliary particle with rapidity y over the vacuum formed by the M magnons, whose rapidities are u_n. In (1.34) above, the form conjectured for $\mathcal{A}_j(y)$ reminds of coloured-2 particle, born on site 1 then transported to the j-site, engaging scatterings $\tilde{S}(y, u_n)$ with the $j-1$ particles encountered while getting there. The claim that (1.33) be an eigenvalue is useful to fix the functions

$$\begin{aligned} f(y,u) &= \frac{y}{u - y + \frac{i}{2}} \\ \tilde{S}(y,u) &= \frac{u - y - \frac{i}{2}}{u - y + \frac{i}{2}} \quad , \end{aligned} \tag{1.35}$$

therefore the equation (1.32) turns to

$$e^{ip_k L} = \frac{u_k - y - \frac{i}{2}}{u_k - y + \frac{i}{2}} \prod_{j \neq k}^{M} \mathcal{S}(u_k, u_j) \tag{1.36}$$

with the usual magnon S-matrix (1.20).
Facing now the $Q = 2$ case, the ansatz can be formulated

$$\begin{aligned} \xi(M,2) = \sum_{j<k} \mathcal{A}_k(y_1)\mathcal{A}_j(y_2) \underbrace{e_1 \otimes \cdots \otimes e_1}_{j-1} \otimes e_2 \otimes \underbrace{e_1 \otimes \cdots \otimes e_1}_{k-j-1} \otimes e_2 \otimes \underbrace{e_1 \otimes \cdots \otimes e_1}_{M-k} + \\ + \mathbb{S}^{II} \sum_{j<k} \mathcal{A}_k(y_1)\mathcal{A}_j(y_2) \underbrace{e_1 \otimes \cdots \otimes e_1}_{j-1} \otimes e_2 \otimes \underbrace{e_1 \otimes \cdots \otimes e_1}_{k-j-1} \otimes e_2 \otimes \underbrace{e_1 \otimes \cdots \otimes e_1}_{M-k} \quad , \end{aligned} \tag{1.37}$$

the only novelty, with respect to $Q = 1$, coming from the scattering phase $\mathbb{S}^{II}$ weighting the outgoing-wave-like term (the second line, in the equation above): $\mathbb{S}^{II}(y_1, y_2)$ thus governs the scattering between two auxiliary particles

$$\mathbb{S}^{II}(y_1, y_2) = \frac{y_1 - y_2 - i}{y_1 - y_2 + i} \quad . \tag{1.38}$$

No new feature arises when dealing with the general Q case: the form of the states is similar to (1.37), the only difference consists in a more involved structure, due to the permutation of all the particles. For a system with arbitrary Q type-2 particles, the diagonalization of (1.32) leads to the set of equations:

$$e^{ip(u_k)L} = \prod_{j \neq k}^{M} \mathcal{S}(u_k, u_j) \prod_{\alpha=1}^{Q} \frac{u_k - y_\alpha - \frac{i}{2}}{u_k - y_\alpha + \frac{i}{2}} \tag{1.39}$$

$$1 = \prod_{j=1}^{M} \frac{y_\alpha - u_j + \frac{i}{2}}{y_\alpha - u_j - \frac{i}{2}} \prod_{\beta=1}^{Q} \frac{y_\alpha - y_\beta - i}{y_\alpha - y_\beta + i} \quad . \tag{1.40}$$

As a remark, in the equations above a momentum term appears in (1.39) alone, for only the magnons (u_k) do carry momentum: the auxiliary particles, in the shape of their rapidities y_α , are just artifice products, stemming from a diagonalization procedure.

Of course, the number of internal degrees of freedom can be raised, so the number of nested spin chains accordingly increases and the nested structure gets more involved [13][14]. In following chapters, numerous examples of this sort will be provided.

4 Thermodynamic Bethe Ansatz

A fruitful technique someway related to Bethe Ansatz, though actually independent, is provided by the so-called Thermodynamic Bethe Ansatz (TBA): this method has been introduced by Al.B.Zamolodchikov [15] (although its origins should be traced back to other works [16]) to solve a purely elastic scattering two dimensional theory by connecting its thermodynamics in the large size limit to the solution of a set of coupled non linear integral equations. For simplicity's sake, the theory taken into account will be assumed to enjoy two dimensional relativistic invariance. First, let the theory lie on a two dimensional torus, arising from the direct product of the generators $\mathcal{C}$ and $\mathcal{G}$, two circumferences respectively of length R and L : the latter, for future purposes, will be considered very large $L \gg R$. The theory studied can be laid on the surface of the torus in two different manners: in the first embedding, the momenta of the particles are quantized along $\mathcal{C}$ whereas the time direction is chosen parallel to $\mathcal{G}$, so that in the $L \longrightarrow \infty$ limit the torus is equivalent to an infinite high cylinder; the second option concerns instead a theory set on the long circumference $\mathcal{G}$ with a compact time with period R , or, on the same footing, with a finite temperature $T = \frac{1}{R}$.
In the latter setup, the partition function is computed by taking the trace on the eigenstates of the hamiltonian $\hat{H}_{II}$:

$$Z_{II}(L, R) = Tr\left[e^{-RH_{II}}\right] = e^{-RL\,f(R)} \quad , \tag{1.41}$$

where $f(R)$ is the density of free energy $f(R) = F(R)/L$. In addition to that, a sort of wave function can be built to characterize the system. Aiming just to highlight some interesting aspects, for the time being the system is set to contain only one species of particles. Since the length L of the space circumference is extremely long, the particles can be thought to be very distant each other, the space in between them being much wider than they correlation length (*i.e.* the inverse of their mass) $|x_i - x_j| \gg 1/m$: therefore the particles propagate almost freely, so that relativistic effects, such as particles creation, may be neglected, and the introduction of a wave function thus makes sense. Whenever two particles, with rapidities θ_1 and θ_2 , get near and eventually collide, the scattering information may be encoded into the scattering matrix $S(\theta_1, \theta_2)$, built after the requirements of being reflectionless, unitary, real analytic and crossing-symmetric. A wave function for M particles can be then written, made out of a superposition of plane waves, according to the proposal

$$\psi(x_1, \ldots, x_M) = \sum_{\pi} \exp\left[i \sum_{k=1}^{M} p_{\pi(k)} x_k + \frac{i}{2} \sum_{k<j} \chi(p_{\pi(k)}, p_{\pi(j)})\right] \tag{1.42}$$

(defining $S = e^{i\chi}$): this is exactly the same as the Bethe ansatz (1.22), up to the explicit form of the scattering matrix S. Correspondingly, the very same form of equations constraining the rapidities arise (1.23):

$$e^{ip_k L} = \prod_{j \neq k}^{M} S(p_k, p_j) \quad . \tag{1.43}$$

Turning back to the first embedding instead, the behaviour of partition function is dominated in the large L regime by the lowest energy state

$$Z_I(L, R) \sim e^{-E_0(R)L} \quad . \tag{1.44}$$

Since the theory is relativistic [1], (1.44) and (1.41) must coincide, $Z_I = Z_{II}$, as the two layouts are connected through a two dimensional rotation on the torus, therefore

$$E_0(R) = Rf(R) \quad : \tag{1.45}$$

this fact entails that the ground state energy E_0 can be computed by means macroscopic thermodynamic quantities, mainly related to dominant microscopic configurations. The knowledge of the relevant thermodynamic functions stems from equations (1.43):

$$p(\theta_k)L + \sum_{j \neq k}^{M} \chi(\theta_k - \theta_j) = 2\pi J_k \quad , \tag{1.46}$$

where the numbers J_k are integers, and roughly speaking they counts the number of available states with rapidity θ_k. It turns out useful to define the density of accessible states σ

$$\sigma(u_k) \equiv \frac{J_{k+1} - J_k}{L\,(u_{k+1} - u_k)} \tag{1.47}$$

(so that the number of available states in the interval $[u_k, u_{k+1}]$ is given by $N_k = L(u_{k+1} - u_k)\sigma(u_k)$) and the density ρ of states actually occupied

$$\rho(u_k)\, L(u_{k+1} - u_k) \equiv n_k \tag{1.48}$$

(n_k being the number of occupied states around u_k), and thus the number of equivalent combinations of particles inside the interval $[u_k, u_{k+1}]$ corresponds to

$$\mathcal{N}_k \equiv \frac{N_k!}{n_k!\,(N_k - n_k)!} \quad . \tag{1.49}$$

In fact, the thermodynamic limit is performed by sending the length of the system L and the number of particles M to infinity, while keeping their ratio finite: the rapidity

[1] When the theory is not relativistic, a two dimensional Lorentz transform is not allowed anymore. Nevertheless, a similar result can be achieved by passing to an Euclidean two dimensional space-time, then performing a double Wick rotation [19]: in general, the hamiltonians of the original and the Wick-rotated (mirror) theory need not to coincide.

becomes a continuous parameter, so that density take the form of derivatives and summations turn to integrals (ranging from $-\infty$ to $+\infty$, unless explicitly stated). Therefore the logarithm of (1.43) can be more suitably recast into an integral equation

$$2\pi\sigma(\theta) = m\,L\,\frac{dp(\theta)}{d\theta} + \int \varphi(\theta-\theta')\,\rho(\theta')d\theta' \tag{1.50}$$

whose kernel is defined as $\varphi(\theta) \equiv \frac{d\chi(\theta)}{d\theta}$, and relies only on differences of rapidities, as the theory is relativistic; moreover, energy and momentum of the particles may be parametrized respectively as

$$e(u) = m\,\cosh\theta \qquad\qquad p(u) = m\,\sinh\theta \quad .$$

In terms of densities, the thermodynamic the hamiltonian reads

$$H_{II}[\rho] = \int d\theta\, m\cosh\theta\,\rho(\theta) \tag{1.51}$$

and, upon performing the Stirling approximation to (1.49), the entropy is

$$\mathcal{S}[\sigma,\rho] = \int d\theta\,[\sigma\ln\sigma - \rho\ln\rho - (\sigma-\rho)\ln(\sigma-\rho)] \quad . \tag{1.52}$$

Now, according (1.45), the ground state energy E_0 equals the free energy, which, in turn, arises from the minimization of

$$RL\,f[\sigma,\rho] = R\,H_{II}[\rho] - \mathcal{S}[\sigma,\rho] \quad . \tag{1.53}$$

The pseudoenergy $\varepsilon(\theta)$ is suitably introduced

$$e^{-\varepsilon} \equiv \frac{\rho}{\sigma-\rho}$$

and alongside it $L(\theta) \equiv \ln\left[1+e^{-\varepsilon(\theta)}\right]$, so that the minimization condition yields the non linear integral equation for the Thermodynamic Bethe Ansatz [15][17]:

$$\varepsilon(\theta) = mR\,\cosh\theta - \int \frac{d\theta'}{2\pi}\,\varphi(\theta-\theta')\,L(\theta') \quad . \tag{1.54}$$

Roughly speaking, the pseudoenergy $\varepsilon(\theta)$ assumes the physical meaning of a 'renormalized' energy of a particle [17], taking a different value from the free case for it receives quantum corrections due to the interactions with the other real particles and virtual as well: its expression, obtained as a solution of (1.54), enters the formula for the ground state energy:

$$E_0 = R\,f(R) = -m\int \frac{d\theta}{2\pi}\,\cosh\theta\,L(\theta) \quad . \tag{1.55}$$

The model can be improved by adding different species of particles: be Q the number of different species and let M_a denote the number of type a particles with mass m_a

$(a = 1, \dots Q)$, so that $\sum_{a=1}^{Q} M_a = M$. Imposing that the scattering still be reflectionless and bound states do not occur, the integral kernel is defined as:

$$\varphi_{ab}(\theta) = -i \frac{d}{d\theta} \ln S_{ab}(\theta) \quad .$$

In the thermodynamic limit $L,\, M_a \longrightarrow \infty$ for all a, provided the ratios $\frac{M_a}{L} \equiv D_a$ stay fixed, the densities of accessible and occupied states, σ_a and ρ_a respectively, are introduced in the usual way, as well as the pseudoenergies ε_a; hence (1.50) generalizes

$$2\pi\sigma_a(\theta) = m_a \, L \, \frac{dp(\theta)}{d\theta} + \sum_b \int \varphi_{ab}(\theta - \theta') \, \rho_b(\theta') d\theta' \quad . \tag{1.56}$$

The same step followed to find (1.54) now lead to the set of coupled non linear integral equations

$$\varepsilon_a(\theta) = m_a R \cosh\theta - \sum_{ab} \int \frac{d\theta'}{2\pi} \varphi_b(\theta - \theta') \, L_a(\theta') \qquad a = 1, \dots, Q \quad , \tag{1.57}$$

the solutions of whom give the ground state energy

$$E_0 = -\sum_{a=1}^{Q} \frac{m_a}{2\pi} \int d\theta \, \cosh\theta \, L_a(\theta) \quad . \tag{1.58}$$

A further improvement [17] is needed when the conservation of the number of particles

$$M_a = L \, D_a[\rho] = \int d\theta \rho_a(\theta)$$

is imposed: such requirement yields a constraint while minimizing the free energy

$$L\, f[\sigma, \rho] = H_{II}[\rho] - T\mathcal{S}[\sigma, \rho] - \sum_a \mu_a D_a \quad , \tag{1.59}$$

where the lagrangian multipliers μ_a assume the meaning of chemical potentials. Therefore, the novel constraints modify the TBA equations by adding a constant term

$$\varepsilon_a(\theta) = -\mu_a R + m_a R \cosh\theta - \sum_{ab} \int \frac{d\theta'}{2\pi} \varphi_b(\theta - \theta') \, L_a(\theta') \qquad a = 1, \dots, Q \quad , \tag{1.60}$$

and this results in the new formula for E_0:

$$E_0 = -\sum_{a=1}^{Q} \frac{m_a}{2\pi} \int d\theta \, \cosh\theta \, L_a(\theta) + R \sum_{a=1}^{Q} \mu_a \, D_a \quad . \tag{1.61}$$

Chapter 2

AdS/CFT duality and integrability

1 Basics of the AdS_5/CFT_4 duality

This chapter is devoted to a very concise summary of the basic concepts in the AdS/CFT correspondence [2][3], mainly highlighting the role of integrability in attempting to unravel several complicated issues. The aim consists in providing a very essential prospect of the notions, handled hereafter. According to the mentioned duality, a relation is conjectured to exist between two theories in principle very different, a supersymmetric conformal gauge field theory and a gravitational theory, concerning supersymmetric strings living on a curved background. Actually, in this text the interest will stick primarily to one of such correspondences, namely the AdS_5/CFT_4.

The AdS_5/CFT_4 correspondence states a duality involving a string theory in a ten dimensional layout and a four dimensional gauge theory. On one side then, there is a type II B string theory, set up in a ten dimensional gravitational background formed by a five dimensional Anti de Sitter space per a five dimensional sphere, $AdS_5 \times S^5$, both factor spaces with radius R; the theory is governed by two coupling constants: a world-sheet coupling g_σ, corresponding to the inverse of the effective string tension

$$T = \frac{R^2}{2\pi\alpha'} \equiv \frac{1}{g_\sigma^2}$$

(α' stands for the Regge slope), and g_{str}, regulating the interactions between strings.
On the other side of the correspondence the $\mathcal{N} = 4$ Super Yang-Mills (SYM) dwells, as the gauge field theory dual to $AdS_5 \times S^5$: the information contained in the $AdS_5 \times S^5$ background (inside its bulk) is holographically projected onto its boundary, a flat Minkowski space-time in $3 + 1$ dimensions where $\mathcal{N} = 4$ SYM is set to live in. $\mathcal{N} = 4$ SYM is endowed with a gauge symmetry group $SU(N)$, associated to the coupling g_{ym}.

The claim of the AdS/CFT correspondence concerns the identification of string and gauge theories, in the sense that there exists a stringent link between the characteristic observables from one theory and those from the other. Moreover, the couplings from the two sides of the correspondence must be connected via a duality relation:

$$g_\sigma^{-4} = g_{ym}^2 N \quad , \tag{2.1}$$

being N the number of colours given by the $SU(N)$ gauge group. This identification implies that the strong coupling regime for a theory matches the weak coupling results in the dual model: therefore the correspondence allows to access sector of the gauge theory otherwise unapproachable with standard perturbative methods, simply by studying the dual string theory -and, clearly, viceversa. On the other hand, perturbative regimes in the two theories have no overlaps, so to prove the conjectured correspondence: the role of integrability thus stands out, since it provides alternative non perturbative ways to explore regions else inaccessible by usual methods[1].
Following sections are meant to outline the main features of $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ string theories, in order to state with slightly more precision how the correspondence works and how to exploit the opportunities offered by integrability.

1.1 $\mathcal{N} = 4$ SYM gauge theory

$\mathcal{N} = 4$ SYM is a supersymmetric conformal theory, endowed with a non-abelian gauge group $SU(N)$, living on a space with three space-like dimensions and one time. The renormalization group beta function has been proven to equal zero at all loops $\beta(g_{ym}) = \mu \frac{\partial g_{ym}}{\partial \mu} = 0$ (μ represents the renormalization scale): the theory is thus conformal, so that it enjoys the four dimensional conformal group symmetry (with Minkowski signature) $SO(2,4) \sim SU(2,2)$. The designation $\mathcal{N} = 4$ denotes that there are four generator of supersymmetry, resulting in 16 odd supercharges; the four generators can be rotated into each other by means of a further bosonic rigid symmetry (R-symmetry) $SU(4) \sim SO(6)$. The even and odd generators together do not suffice to close the overall graded algebra: to accomplish this request 16 more supercharges are needed. To sum up, the $\mathcal{N} = 4$ SYM possesses a $SO(2,4) \times SU(4)$ bosonic symmetry, enhanced to the whole $PSU(2,2|4)$ group upon adding the 32 odd supercharges.

The theory contains the vector supermultiplet $(A_\mu, \psi^A_\alpha, \bar{\psi}^{\bar{A}}_{\dot{\alpha}}, \Phi^i)$. the gauge field A_μ is a singlet under the R-symmetry (it is in the **1** of $SU(4)$), and the Lorentz vector index μ runs from 0 to 3; from A_μ, the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ and the field strength $F_{\mu\nu} = [D_\mu, D_\nu]$ are defined. The six real scalars Φ^i ($i = 1, \ldots, 6$) transform according the **6** of $SU(4)$; for later purposes, the scalars could be recast into complex combinations

$$Z = \frac{1}{\sqrt{2}}(\Phi^1 + i\Phi^2) \qquad W = \frac{1}{\sqrt{2}}(\Phi^3 + i\Phi^4) \qquad Y = \frac{1}{\sqrt{2}}(\Phi^5 + i\Phi^6) \quad ,$$

along with their complex conjugated $\bar{Z}$, $\bar{W}$, $\bar{Y}$. Finally, the left and right Weyl spinors ψ^A_α, $\bar{\psi}^{\bar{A}}_{\dot{\alpha}}$ are respectively in the **4** and $\bar{\mathbf{4}}$ of $SU(4)$ ($A, \bar{A} = 1,2,3,4$), with the Lorentz spindor indices α, $\dot{\alpha} = 1,2$. Since all these fields belong to the same supermultiplet, they must transform under the gauge group $SU(N)$ according to the same representation,

[1] Since the original work [28] about integrability on the gauge side, many valuable reviews appeared, such as [9][29]; on the string theory side, integrability has been discovered by [11]

precisely the adjoint. They represent the ingredients to write the lagrangian of $\mathcal{N} = 4$ SYM:

$$\mathcal{L} = \frac{1}{g_\sigma^2} Tr_N \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^i D^\mu \Phi_i + \sum_{j>k} [\Phi^j, \Phi^k]^2 + \{Fermionic\ Part\} \right] \quad (2.2)$$

where the fermionic part has been left indicated; the trace $Tr_N[\dots]$ is performed on the $SU(N)$ matrix indices.

The gauge invariant local operators, specifically the key objects of the theory, are then constructed starting from the fields listed above, in the form of the trace of products of operator, generically expressed as

$$O \sim Tr_N[\varphi_1 \varphi_2 \dots] \quad (2.3)$$

where φ_k stands for any field $\varphi \in \{D_\mu, A_\mu, \psi_\alpha^A, \bar{\psi}_{\dot{\alpha}}^{\bar{A}}, \Phi^i\}$; besides single trace operators, multi-trace should be taken into account, too

$$O' \sim Tr_N[\varphi_1 \varphi_2 \dots] \cdot Tr_N[\varphi_a \varphi_b \dots] \cdot \dots \quad (2.4)$$

Since $\mathcal{N} = 4$ SYM is a conformal theory, it turns out described by the spectrum of dimensions relative to the operators, which can be found by taking the two point correlators of one operator with itself; at classical level, when quantum corrections are overlooked (or with the coupling constant g_{ym} set to zero), it is found for an arbitrary local operator $O(x)$:

$$\langle\, O(x)\, O(y)\, \rangle \sim \frac{1}{|x-y|^{2\Delta_0}} \quad , \quad (2.5)$$

the number Δ_0 being the bare (classical) dimension of $O(x)$, so that a space-time variable dilatation (a rescaling on the argument) $x \longrightarrow \lambda x$ entails a rescaling of th operator

$$O(\lambda x) \longrightarrow \lambda^{\Delta_0} O(x) \quad .$$

When performing a perturbative expansion in g_{ym}^2, already at one-loop the correlator above blows, owing to an UV divergence, to be treated, for instance, by introducing an UV cut-off Λ. The same effect could be achieved by introducing a renormalized version $O_{ren}(x)$ of the operator $O(x)$:

$$O_{ren}(x) = \left(\frac{\mu}{\Lambda}\right)^{\gamma(g_{ym})} O(x) \quad ,$$

so that a rescaling of the renormalization group scale (of distances) $\mu \longrightarrow \lambda\mu$ leads to

$$O_{ren}(\lambda x) \longrightarrow \lambda^{\Delta_0 + \gamma(g_{ym})} \left(\frac{\mu}{\Lambda}\right)^{\gamma(g_{ym})} O_{ren}(x) \quad . \quad (2.6)$$

The scaling dimension of the operator $O_{ren}(x)$ is then $\Delta = \Delta_0 + \gamma(g_{ym})$, the contribution stemming from quantum corrections $\gamma(g_{ym})$ being told the anomalous dimension. A fundamental remark concerns a special class of operators, the so-called chiral primary operators, as supersymmetry prevents them from acquiring correction from quantum processes, leaving thus $\gamma(g_{ym}) = 0$ at all values of the coupling: objects of this kind are said to saturate the BPS bound. An example of chiral primaries could be offered by operators with the form:

$$O \sim Tr_N[\Phi^{i_1}\Phi^{i_2}\dots\Phi^{i_L}]\,\sigma_{i_1,i_2,\dots i_L} \tag{2.7}$$

when $\sigma_{i_1,i_2,\dots i_L}$ is a completely symmetric tensor of $SO(6)$, and the trace of any two indices is null. To conclude, since the bosonic subalgebra $so(2,4) \oplus su(4) \subset psu(2,2|4)$ is rank six, the single trace operators can be labelled by means of a sextuplet of charges $(\Delta, S_1, S_2;\, J_1, J_2, J_3)$. The first three charges referr to conformal group: Δ is the eigenvalue of dilatation operator, the scaling dimension, while $S_1,\ S_2$ are the Lorentz spins (the charges $SO(1,3)$); on the R-symmetry side, $J_1,\ J_2,\ J_3$ represent the angular momentum components, namely the eigenvalues of the three generators in the Cartan subalgebra of $SO(6)$. For instance, the complex scalars Z carry the sextuplet of charges $(1,0,0;1,0,0)$, whereas the W fields are associated to $(1,0,0;0,1,0)$ and the Y's to $(1,0,0;0,0,1)$. The fermionic fields ψ^A_α hold the values $(\frac{3}{2}, \pm\frac{1}{2}, 0; \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ where the number of negative signs in the $SO(6)$ part is even; the antifermions $\bar{\psi}^{\bar{A}}_{\dot{\alpha}}$ instead carry the charges $(\frac{3}{2}, 0, \pm\frac{1}{2}; \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with an odd number of negative signs for the last three values.

1.2 $AdS_5 \times S^5$ string theory

A supersymmetric string theory on a curved manifold is described by the Green-Schwarz action

$$\mathcal{S} = \frac{1}{g_\sigma^2}\int d^2\sigma\,\sqrt{-g}\,g^{ab}\,G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \mathcal{S}_F \quad , \tag{2.8}$$

where $g_{ab}(\sigma)$ is the metric of the two dimensional world-sheet, spanned by the coordinates $\sigma = (\sigma_0, \sigma_1)$, while $X^\mu = X^\mu(\sigma)$ ($\mu = 1, 2, \dots, 10$) are the coordinate fields on the ten dimensional target space whose metric is $G_{\mu\nu}(X)$; the fermionic part of the action $\mathcal{S}_F$ has not been written explicitly.
When dealing with the type II B string theory set on the target space $AdS_5 \times S^5$, it is more suitable to divide the X coordinates in two subsets, referred to the component manifolds AdS_5 and S^5, $X^\mu = (x^1, x^2, \dots, x^5; y^1, \dots, y^5)$, so to easily describe the five dimensional Anti de Sitter space by means of the constant negative curvature surface, embedded in the $\mathbb{R}^{2,4}$ space:

$$-(x^o)^2 - (x^1)^2 + \sum_{a=2}^{5}(x^a)^2 = -R^2 \tag{2.9}$$

whereas the five dimensional sphere S^5 is embedded in $\mathbb{R}^6$ as

$$\sum_{b=0}^{5}(y^b)^2 = R^2 \tag{2.10}$$

(the radius R is the same for both AdS_5 and S^5). The metric of the Anti de Sitter space can be expressed in several (almost) equivalent way: a special choice which turns out particularly enlightening consists in adopting the Poincaré coordinates (z_0, z_1, z_2, z_3, t), so that the line element of AdS_5 reads:

$$ds^2 = \frac{R^2}{z_0^2}\left(-dt^2 + \sum_{i=1}^{3} dz_i^2 + dz_0^2\right) \quad , \tag{2.11}$$

though this choice does not cover the whole space, but instead only the patch $-\infty < t, z_i < +\infty$ and $0 < z_0 < \infty$ is properly described. Remarkably, up to a conformal factor the line element (2.11) coincides with a $3+1$ dimensional flat Minkowsi space, enriched with an additional coordinate z_0, which assumes the meaning of kind of 'warp parameter': this is, in fact, the Minkowski space where the dual $\mathcal{N} = 4$ SYM gauge theory lives in.

Once (2.8) is specialized to type II B $AdS \times S^5$ background, the Metsaev-Tseytlin string action results, in the shape of a $PSU(2,2|4)$ non-linear σ-model, therefore corresponding to the $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ coset model. The bosonic restriction consists in the direct product $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$, so that the bosonic part of Metsaev-Tseytlin action is the sum of the actions of $SO(2,4)$ (AdS_5) and $SO(6)$ (S^5) non-linear σ-models, the only interactions between the two models arising from coupling with fermions; nonetheless, within a special dynamical regime the two bosonic σ-models decouple, and the $O(6)$ model takes over.

Turning back to AdS/CFT duality's claims, now that the key objects of gauge ant string theories (namely local operators and strings, respectively) have been sketchily introduced, the correspondences between them may be stated more clearly. The features of a single-trace gauge invariant operator are asked to match those of single string states: in particular the spectrum of anomalous conformal dimensions shall be equivalent to the spectrum of string excitations (over a chosen vacuum reference state). Moreover, the remaining five charges labelling a gauge theory operator turn to quantum number carried by a single string: S_1 and S_2 become angular momentum numbers for a string moving across the AdS_5 space, while J_1, J_2, J_3 describe its motion on S^5.

2 Integrability in AdS_5/CFT_4

There exists an interesting limit to look at, when studying $\mathcal{N} = 4$ SYM, for it allows to reveal several important properties of such gauge theory: it consists in pushing the number of colours N (relative to $SU(N)$ gauge symmetry) to infinity, while, on the same time, the parameter $\lambda \equiv g_{ym}^2 N$ is kept finite; λ is said the 't Hooft coupling. For instance, the $N \longrightarrow \infty$ limit allows to show the theory is integrable. To get an idea about the simplifications brought by the large N limit and how integrability emerges, it useful to consider the sector of operators made out of scalars only, as an example (however, the results are far more general). In a small the small coupling g_{ym}^2 regime, the tree-level correlator (2.5) could be generalized as

$$\langle\, O(x)O(y)\,\rangle \simeq \frac{1}{|x-y|^{2\Delta_0}}\left(1 - \gamma(g_{ym}) \ln \Lambda^2 |x-y|^2\right) \quad ; \tag{2.12}$$

the anomalous dimension $\gamma(g_{ym})$ may be thought as the eigenvalue of the higher-than-tree-level dilatation operator Γ, else said the mixing matrix: the search for anomalous dimensions amounts to diagonalizing Γ. When evaluating the two-point correlator (2.12), an expansion can be performed in the the small parameter $1/N$, so to gain then the chance to overlook non planar Feynman diagrams, as they are associated to subleading contributions. In fact, the Feynman graphs arising from the contractions involved in the correlator (2.12) can be represented by means of simplicial complexes: each propagator corresponds to an edge of a simplex, and in terms of 't Hooft coupling it is left associated to a weight $g_{ym}^2 = \lambda/N$, while a vertex brings a factor $\frac{1}{g_{ym}^2} = \frac{N}{\lambda}$, finally each face of a simplex is weighted by N. Eventually, calling E,V and F the number of edges, vertices and faces, a diagram is weighted in a large N power expansion by a factor

$$N^{V+F-E}\lambda^{E-V} = N^{\chi}\lambda^{E-V} \quad , \tag{2.13}$$

where the Euler character is $\chi = V + F - E$. The Euler character is also written as $\chi = 2-2G$, with G denoting the genus of the topology over which the graph can be laid, so that the largest is the genus the less the Feynman diagram contributes: hence planar diagrams (*i.e.* those which can lie entirely on a plane) carry the leading contributions in computing observables, such as (2.12), non planar ones being suppressed at least for a factor $1/N^2$ in comparison.
Turning back now to the evaluation of (2.12), an arbitrary scalar-made operator reads, up to a symmetry normalization factor:

$$O_{i_1 \dots i_L}(x) \simeq \left(\frac{4\pi^2}{N}\right)^{\frac{L}{2}} Tr\left[\Phi_{i_1}(x)\cdots\Phi_{i_L}(x)\right] \tag{2.14}$$

(from now on the subscript in Tr_N, to denote the trace is taken on $N\times N$ matrices, will be dropped). Then, the two-point correlation function of the operator $O_{i_1 \dots i_L}(x)$ with itself can be expressed at tree-level as:

$$\langle\, O_{i_1 \dots i_L}(x)\, O^{j_1 \dots j_L}(y)\,\rangle_{tree} \simeq \frac{1}{|x-y|^{2L}}\left(\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\dots\delta_{i_L}^{j_L} + \{cycles\}\right) \tag{2.15}$$

where $\{cycles\}$ stands for the summation over cyclic permutations of superscripts j_k. Recalling the discussion above, when the $N \gg 1$ regime is considered, the computation of the one loop correction to (2.15) does not involve the contributions stemming from non-planar Feynman diagrams; this simplification allows to retrieve the one loop result:

$$\begin{aligned}\langle\, O_{i_1 \dots i_L}(x)\, O^{j_1 \dots j_L}(y)\,\rangle_{1-loop} &= \frac{\lambda}{16\pi^2}\frac{\ln(\Lambda^2|x-y|^2)}{|x-y|^{2L}} \times \\ &\times \sum_{l=1}^{L}\left[2\hat{P}_{l,l+1} - \hat{K}_{l,l+1} - 2\right]\left(\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\dots\delta_{i_L}^{j_L} + \{cycles\}\right) \quad .\end{aligned} \tag{2.16}$$

The operator $\hat{P}_{l,l+1}$ exchanges the flavour indices for the positions l and $l+1$, so applied to a Kronecker δ, it produces the effect:

$$\hat{P}_{l,l+1}(\delta_{i_1}^{j_1}\dots\delta_{i_l}^{j_l}\delta_{i_{l+1}}^{j_{l+1}}\dots\delta_{i_L}^{j_L}) = (\delta_{i_1}^{j_1}\dots\delta_{i_l}^{j_{l+1}}\delta_{i_{l+1}}^{j_l}\dots\delta_{i_L}^{j_L}) \quad ; \tag{2.17}$$

the trace operator $\hat{K}_{l,l+1}$ instead acts on a δ by contracting flavour indices for neighbour positions:

$$\hat{K}_{l,l+1}(\delta^{j_1}_{i_1}\dots\delta^{j_l}_{i_l}\delta^{j_{l+1}}_{i_{l+1}}\dots\delta^{j_L}_{i_L}) = (\delta^{j_1}_{i_1}\dots\delta_{i_l\, i_{l+1}}\delta^{j_l\, j_{l+1}}\dots\delta^{j_L}_{i_L}) \quad . \tag{2.18}$$

Upon summing together tree-level (2.15) and one loop correlators (2.16), a comparison with the general form (2.12) reveals that the mixing matrix Γ assumes the expression:

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left[1 - \hat{P}_{l,l+1} + \frac{1}{2}\,\hat{K}_{l,l+1}\right] + O(\lambda^2) \quad . \tag{2.19}$$

The salient remark is that the one loop anomalous dimension operator (2.19) turns out to share the very same form of the hamiltonian of a spin chain, an integrable system.
The spin-spin interactions stand out more clearly after defining the spin operators

$$\sigma^{ab}_{ij} = \delta^a_i\delta^b_j - \delta^a_j\delta^b_i \quad ,$$

so that (2.19) reads:

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left[\sigma_l\,\sigma_{l+1} - \frac{1}{16}(\sigma_l\,\sigma_{l+1})^2 + \frac{9}{4}\right] \tag{2.20}$$

(the subscripts l, $l+1$ label the site).

2.1 The $SU(2)$ sector

In general, it happens that operators mix under renormalization, so that, for example, operators made out of scalars only usually mingle with operators involving non-scalar fields too[2]. Nonetheless there exist sectors composed of a restricted variety of field species. One of such example, within scalar sector, is provided by the so-called $SU(2)$ sector, involving just two kinds of scalars, Z and W (without their complex conjugates), associated respectively to charges $(1,0,0;1,0,0)$ and $(1,0,0;0,1,0)$. A trace operator, with bare dimension L and composed of M fields of type W and $L-M$ of type Z, is described by the charges $(L,0,0;L-M,M,0)$: since the operator mixing preserves the total Lorentz and R-symmetry charges, this operator can combine only to other ones with the same number of Z's and W's, rearranged according to any possible permutation; hence, the sector stays closed.
Upon sticking to $SU(2)$ sector, the mixing matrix (2.19) becomes

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left[\hat{I}_{l,l+1} - \hat{P}_{l,l+1}\right] \quad : \tag{2.21}$$

interestingly it coincides with the hamiltonian (1.1) of the Heisenberg model for ferromagnetism ($XXX_{\frac{1}{2}}$ spin chain with L sites):

$$\hat{H} = \sum_{l=1}^{L}\left[\frac{1}{2} - 2\vec{S}_l\cdot\vec{S}_{l+1}\right] \tag{2.22}$$

[2]Anyway, the operator mixing for scalar sector does not take place at one loop

with magnetization $\mathcal{M} = \sum_{l=1}^{L} S_l^{(z)}$.

When attempting to diagonalize Γ, first of all it takes to fix a vacuum reference state for the spin chain, associated to (2.21). A suitable choice concerns the operator $\mathcal{V} = Tr[Z^L]$, for it is a chiral primary and thus its anomalous dimension equals zero, taking so the minimal eigenvalue available for Γ. In exploiting the analogy with a spin chain, each Z field can be thought as, say, a down spin, whereas W behaves like an up spin (only two values are available for the spin, in the present case): the vacuum $\mathcal{V}$ therefore corresponds to the ferromagnetic state, with all the spins aligned, so to minimize the amount of energy (read anomalous dimension) of the system. The replacement of a Z with a W accounts for a spin flip, as a down spin turns to up: the operation increases the eigenvalue of the 'energy' (2.21), so it relates to the bearing of an excitation, customarily called magnon, which propagates along the chain. Each time a spin is reverted, an excitation rises. The periodicity condition is imposed by taking trace in the construction of gauge invariant operators. The solution of such a model has been discussed in section 2 of chapter 1.

2.2 The BMN vacuum

The $SU(2)$ sector finds a natural correspondence with a particular dynamical configuration for strings in the $AdS_5 \times S^5$ background [20].

According to a theorem due to Penrose, the space-time in the neighbourhood of a null geodesic takes the form of a plane fronted gravitational wave (pp wave), regardless of the original metric. Let the focus stick to a string travelling at the speed of light across a circumference S^2, subspace of S^5, at rest in the AdS_5: it carries no S_1, S_2 charges, and relatively to angular momentum in S^5, let only one quantum number be non null, say $J_1 = L$. Under these assumptions, the pp-wave limit arises when taking the radius $R \longrightarrow \infty$ by properly rescaling the coordinates. As a result, in light-cone coordinates the string has got the following momentum components:

$$p^- = \frac{\Delta - L}{2}$$
$$p^+ = \frac{\Delta + L}{2R^2}$$

(in view of the gauge-string correspondence, Δ here denotes the string energy). Under the requirement that $p^\pm$ are non-negative[3] and finite, it must hold $L \sim R^2 \sim \sqrt{N}$, and in addition the difference $\Delta - L$ stays fixed. Keeping an eye to the $\mathcal{N} = 4$ SYM side of the duality, the lowest value $\Delta - L = 0$ associates uniquely to the chiral primary operator $\mathcal{V} = Tr[Z^L]$, which hence assumes the role of a string ground state, and a spin chain vacuum as well. Sticking to $SU(2)$ sector, the only way to rise $\Delta - L$ to the value 1 consists in substituting one Z field with differently flavoured scalar W: such operations then bears the lowest string excited state, and correspondingly a magnon in the (gauge) spin chain. The following excitations/magnons arise in the same manner, so that the parallel between BMN (classical) string solution and the $SU(2)$ operator sector is accomplished.

[3]As a matter of fact, the supersymmetry imposes the BPS condition $\Delta \geq |L|$

2.3 $sl(2)$ sector and GKP string

The $SU(2)$ sector is not the only one closed under renormalization. Indeed, a further example is provided by the $sl(2)$ sector (also called $SU(1,1)$ sector): it stems from combining together, into trace operators, Z fields along with light-cone coordinate covariant derivarives D_+ , associated to the sextuplet of charges $(1, \frac{1}{2}, \frac{1}{2}; 0, 0, 0)$. Single trace operators of this kind arise by starting from a reference state, again $\mathcal{V} = Tr[Z^L]$, which behaves like a vacuum, over whom the application of covariant derivatives produces the rising of excitations (one for each derivative): a generic operator

$$O \sim Tr[(D_+)^s Z^L] + \dots$$

describes a state with s excitations on the vacuum (s also represents the Lorentz spin of the operator); the dots account for operator mixing, whose action consists in redistributing the covariant derivatives over the Z fields in all possible combinations. A peculiarity of $sl(2)$ sector, against $SU(2)$, resides in that an arbitrary number of derivatives can be applied to each Z due to the non-compactness of algebra $sl(2)$: in the spin chain language, any number of excitations may arise on a single site.

Lot of the interest has been devoted to $sl(2)$ sector of $\mathcal{N} = 4$ SYM ([41],[31]-[32] for instance), since it shares several features with other important fields of research, such as the study of QCD and deep inelastic scattering, and Wilson loops as well. For instance, the computations of anomalous dimension for $sl(2)$ operators in the large s limit leads to the meaningful Sudakov logarithmic scaling:

$$\Delta = s + f(\lambda) \ln s + O(s^0) \quad , \tag{2.23}$$

where $f(\lambda)$ is the so-called universal scaling function, and does not depend on L: it is remarkable that (2.23) does not involve supersymmetry. The scaling behaviour (2.23) also occurs in computations concerning the above-mentioned theories: for instance, the *cusp anomalous dimension* for light-like cusped Wilson loops coincides with the $\frac{f(\lambda)}{2}$. For $\mathcal{N} = 4$ SYM $sl(2)$ sector the scaling behaviour (2.23) can be further refined, so that the expression for anomalous dimension, in large s, arises [44]:

$$\gamma(\lambda, s, L) = f(\lambda) \ln s + f_{sl}(\lambda, L) + \sum_{n=1}^{\infty} \gamma^{(n)}(\lambda, L) (\ln s)^{-n} + O\left(\frac{\ln s}{s}\right) \quad . \tag{2.24}$$

There exists a special classical string solution that works successfully in properly describing the results from $sl(2)$ sector and, most of all, nicely fits the Sudakov form (2.23): such string configuration has been retrieved by Gubser, Klebanov and Polyakov [21], customarily called GKP string. It consists in a closed folded string, moving in the AdS_3 subspace of AdS_5. Let the AdS_3 line element be considered, first:

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho \, d\phi^2\right] \quad ;$$

After imposing a gauge for which it holds $\tau = t$, $\phi = \omega t$, along with the periodicity condition $\rho(\sigma) = \rho(\sigma + 2\pi)$ (τ, σ being the world-sheet coordinates), the string lagrangian reads:

$$\mathcal{L} = -4\,\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\,\sqrt{\cosh^2\rho - \left(\frac{d\phi}{d\tau}\right)^2 \sinh^2\rho} \quad ; \tag{2.25}$$

the maximum radial coordinate ρ_0 is left determined by

$$\coth^2\rho_0 = \omega$$

and the factor 4 appears since there are four segments of string stretching from 0 to ρ_0. Energy and spin of the string thus follow from (2.25):

$$E = 4\,\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\,\frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} \tag{2.26}$$

$$S = 4\,\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\,\frac{\omega\sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} \tag{2.27}$$

When considering the large spin limit $S \longrightarrow \infty$, the maximal value for the AdS_3 radial coordinate ρ_0 grows indefinitely too, since $\rho_0 \sim \frac{1}{2}\ln S$ and, moreover, the length of the folded string can be estimated to $l \sim 4\rho_0 \sim 2\ln S$. Carrying a more careful analysis of the $S \gg \sqrt{\lambda}$ regime, it could be inferred that $\omega \longrightarrow 1$ from above, so that $\omega = 1 + 2\eta$, where $\eta \ll 1$. Hence at leading order in η, it holds

$$\rho_0 \simeq \frac{1}{2}\ln\left(\frac{1}{\eta}\right) \quad ,$$

therefore energy and spin enjoy the following expansion in η:

$$E = \frac{R^2}{2\pi\alpha'}\left(\frac{1}{\eta} + \ln\frac{1}{\eta} + \dots\right) \tag{2.28}$$

$$S = \frac{R^2}{2\pi\alpha'}\left(\frac{1}{\eta} - \ln\frac{1}{\eta} + \dots\right) \quad .$$

Hence, in the regime $S \gg \sqrt{\lambda}$ the parametric form (2.28) turns to an asymptotic expression for energy as a function of the spin $E = E(S)$, so that the Sudakov logarithmic scaling behaviour shows up (see (2.23)):

$$E = S + \frac{\sqrt{\lambda}}{\pi}\ln(S/\sqrt{\lambda}) + \dots \quad . \tag{2.29}$$

3 Beisert-Staudacher equations

The parallel between $\mathcal{N} = 4$ SYM and a spin chain, exposed in section 2, is demonstrated only up to first order within a perturbative expansion in λ. This result can be extended rigorously to a few orders further, a complete non-perturbative approach being missing so far. Nevertheless, Beisert and Staudacher [30] proposed a long-range (*i.e.* the range of interactions increases with the loop order) spin chain and a set of all-order (nested) Bethe ansatz equations, so to fully account the $\mathcal{N} = 4$ SYM gauge theory. Strictly speaking, they should be taken as Asymptotic Bethe Ansatz equations, in the sense they are valid pertubatively in λ and solve the underlying spin chain only up to λ^{2L} order. Nonetheless these equations passed numerous non trivial tests of consistency, so that their reliability is now considered out of doubts, therefore they will represent the foundation of the work exposed in the next two chapters. The Beisert-Staudacher Asymptotic Bethe Ansatz equations are then displayed below [4]:

$$\begin{aligned}
1 &= \prod_{j\neq k}^{K_2} \frac{u_{1,k}-u_{2,j}-\frac{i}{2}}{u_{1,k}-u_{2,j}+\frac{i}{2}} \prod_{j=1}^{s} \frac{1-\frac{g^2}{2x_{1,k}\,x^-_{4,j}}}{1-\frac{g^2}{2x_{1,k}\,x^+_{4,j}}} \qquad (2.30)\\
1 &= \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}+i}{u_{2,k}-u_{2,j}-i} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}-\frac{i}{2}}{u_{2,k}-u_{1,j}+\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}-\frac{i}{2}}{u_{2,k}-u_{3,j}+\frac{i}{2}}\\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}-\frac{i}{2}}{u_{3,k}-u_{2,j}+\frac{i}{2}} \prod_{j=1}^{s} \frac{x_{3,k}-x^-_{4,j}}{x_{3,k}-x^+_{4,j}}\\
1 &= \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{j\neq k}^{s} \frac{x^-_{4,k}-x^+_{4,j}}{x^+_{4,k}-x^-_{4,j}} \frac{1-\frac{g^2}{2x^+_{4,k}\,x^-_{4,j}}}{1-\frac{g^2}{2x^-_{4,k}\,x^+_{4,j}}}\,\sigma^2(u_{4,k},u_{4,j})\times\\
&\times \prod_{j=1}^{K_3} \frac{x^+_{4,k}-x_{3,j}}{x^-_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^+_{4,k}-x_{5,j}}{x^-_{4,k}-x_{5,j}} \prod_{j=1}^{K_1} \frac{1-\frac{g^2}{2x^+_{4,k}\,x_{1,j}}}{1-\frac{g^2}{2x^-_{4,k}\,x_{1,j}}} \prod_{j=1}^{K_7} \frac{1-\frac{g^2}{2x^+_{4,k}\,x_{7,j}}}{1-\frac{g^2}{2x^-_{4,k}\,x_{7,j}}}\\
1 &= \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}-\frac{i}{2}}{u_{5,k}-u_{6,j}+\frac{i}{2}} \prod_{j=1}^{s} \frac{x_{5,k}-x^-_{4,j}}{x_{5,k}-x^+_{4,j}}\\
1 &= \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i} \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}-\frac{i}{2}}{u_{6,k}-u_{7,j}+\frac{i}{2}} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}-\frac{i}{2}}{u_{6,k}-u_{5,j}+\frac{i}{2}}\\
1 &= \prod_{j\neq k}^{K_6} \frac{u_{7,k}-u_{6,j}-\frac{i}{2}}{u_{7,k}-u_{6,j}+\frac{i}{2}} \prod_{j=1}^{s} \frac{1-\frac{g^2}{2x_{7,k}\,x^-_{4,j}}}{1-\frac{g^2}{2x_{7,k}\,x^+_{4,j}}}
\end{aligned}$$

where it has been introduced the rescaled coupling constant $g^2 = \frac{\lambda^2}{8\pi}$, and the spectral parameters x are related to rapidities u through the Jukovsky map

$$x(u) = \frac{u}{2}\left[1+\sqrt{1-\frac{2g^2}{u^2}}\right] \qquad u(x) = x+\frac{g^2}{2x} \qquad x^{\pm}(u) \equiv x(u\pm\frac{i}{2}) \quad ; \quad (2.31)$$

for compactness' sake, the shorthand notation has been adopted $x_{i,j} \equiv x(u_{i,j})$. Since the momentum $p(u)$ is expressed via the Jukovski variables as

$$e^{ip(u)} = \frac{x^+(u)}{x^-(u)} \quad , \qquad (2.32)$$

[4]The original article [30] furnishes four variations (four gradings) of the Beisert-Staudacher equations, corresponding to four distinct Dynkin diagrams encoding the underlying $su(2,2|4)$ algebra: here the grading $\eta_1 = \eta_2 = -1$ has been chosen

the zero momentum condition, to be imposed to solutions of the equations (2.30), translates to the constraint

$$1 = \prod_{j=1}^{s} \frac{x_{4,j}^{+}}{x_{4,j}^{-}} \quad . \tag{2.33}$$

Please note that only the fourth (the central) equation amongst (2.30) contains a momentum term: the meaning relies on the fact that only the roots of type 4, corresponding to rapidities $u_{4,k}$, do indeed carry momentum and energy, and all the other charges as well, being then usually referred to as the main roots. Therefore the total amount of a charge Q_r , belonged to the whole system, can be computed as:

$$Q_r = \sum_{k=1}^{s} q_r(u_{4,k}) = \frac{i}{r-1} \sum_{k=1}^{s} \left[\frac{1}{(x_{4,k}^{+})^{r-1}} - \frac{1}{(x_{4,k}^{-})^{r-1}} \right] \tag{2.34}$$

and, in particular, the anomalous dimension (proportional to total energy) is

$$\delta D = g^2\, Q_2 = i g^2 \sum_{k=1}^{s} \left[\frac{1}{x_{4,k}^{+}} - \frac{1}{x_{4,k}^{-}} \right] \quad . \tag{2.35}$$

In the fourth of (2.30), a peculiar factor deserves further attention: the dressing factor $\sigma^2(u,v)$. The asymptotic scattering matrices for gauge and string theories are seen to differ for an overall flavour-independent factor

$$S^{string} = \sigma^2 \cdot S^{gauge} \quad :$$

the presence of σ^2 is aimed at taking into account the crossing symmetry in string theory [24], so to properly match the gauge and string S-matrices. The dressing factor $\sigma^2(u,v) = e^{2i\theta(u,v)}$ is explicitly given by [25]

$$\theta(u,v) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r,r+2\nu+1}(g) \left[q_r(u) q_{r+2\nu+1}(v) - q_r(v) q_{r+2\nu+1}(u) \right] \tag{2.36}$$

where $q_r(u)$ stands for the density of the r-th charge (refer to (2.34)), while for the functions $\beta_{r,r+2\nu+1}(g)$ it holds

$$\begin{aligned} \beta_{r,r+2\nu+1}(g) &= 2 \sum_{\mu=\nu}^{\infty} (-1)^{r+\mu+1} \frac{g^{2(r+\nu+\mu)}}{2^{r+\nu+\mu}} \frac{(r-1)(r+2\nu)}{2\mu+1} \times \\ &\times \ \zeta(2\mu+1) \begin{pmatrix} 2\mu+1 \\ \mu-r-\nu+1 \end{pmatrix} \begin{pmatrix} 2\mu+1 \\ \mu-\nu \end{pmatrix} = \\ &= \sum_{\mu=\nu}^{\infty} \frac{g^{2r+2\nu+2\mu}}{2^{r+\nu+\mu}} \beta_{r,r+2\nu+1}^{(r+\nu+\mu)} \end{aligned} \tag{2.37}$$

The phase σ^2 does not affect the solution of (2.30) until the fourth perturbative order at least, becoming relevant insteadd in the strong coupling regime.

A further remark is now appropriate, for later reference. After imposing the system (2.30) contains no root of any type, but s of type 4, let the logarithm of the fourth equation be taken into account: in the large s limit, the summations turn to integral and the main roots are more suitably described by means of a continuous density function. When dealing with the $sl(2)$ sector, this density encodes the properties of the vacuum, and most noticeably its higher-than-one-loop component σ_{BES} reveals the behaviour of quantum fluctuations. The Fourier transform of σ_{BES} is governed by the so-called BES equation (after Beisert, Eden and Staudacher [23], who refined the results of [22] by adding the contribution of the dressing factor):

$$\hat{\sigma}_{BES}(k) = \frac{2\pi g^2 k}{e^k - 1}\left[\hat{K}(\sqrt{2}gk, 0) - \int_0^\infty \frac{dt}{\pi}\,\hat{K}(\sqrt{2}gk, \sqrt{2}gt)\hat{\sigma}_{BES}(t)\right] \tag{2.38}$$

where the kernel splits into two components $\hat{K}(k,t) = \hat{K}_m(k,t) + \hat{K}_d(k,t)$, the first due to 'main part' of the scattering of type-4 roots

$$\hat{K}_m(k,t) = \frac{J_1(k)J_0(t) - J_0(k)J_1(t)}{k-t} \tag{2.39}$$

and the second stemming from the dressing phase

$$\begin{aligned}\hat{K}_d(k,t) \;&=\; \frac{2\sqrt{2}}{kt}\sum_{\tau=1}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=\nu}^{\infty}\frac{(-1)^\nu}{2^\mu}g^{2\mu+1}\left[\beta^{(2\tau+\nu+\mu)}_{2\tau,2\tau+2\nu+1}\,J_{2\tau+2\nu}(k)J_{2\tau-1}(t)\, +\right.\\ &+\; \left.\beta^{(2\tau+\nu+\mu+1)}_{2\tau+1,2\tau+2\nu+2}\,J_{2\tau}(k)J_{2\tau+2\nu+1}(t)\right] \quad .\end{aligned} \tag{2.40}$$

Eventually, the universal scaling function in (2.23) results from the solution of (2.38):

$$f(g) = 4g^2 - 16g^4\int_0^\infty dt\,\hat{\sigma}_{BES}(t)\,\frac{J_1(\sqrt{2}gt)}{\sqrt{2}gt} \quad . \tag{2.41}$$

Chapter 3

Scattering matrices

1 Non-Linear Integral Equations

This paragraph is intended to outline the main steps of the techniques, used throughout the chapter, to extract information from the Bethe ansatz equations (2.30). Namely, by means of the procedure introduced by Destri and De Vega [34] (further applications are in [35], see instead [36] for a review), an infinite set of equations are encoded into a single Non-Linear Integral Equation (NLIE), or, more generally, into a limited number of them.
In order to illustrate the method, a valuable and interesting example is provided the Beisert-Staudacher equations [30] for a system made up only of type-4 roots, which are associated with covariant derivatives in the twist operator standpoint. In setting up such a system, all Bethe roots but u_4 are turned off, so that only the fourth set of equations from (2.30) survives:

$$1 = \left(\frac{x^-(u_{4,k})}{x^+(u_{4,k})}\right)^L \prod_{j\neq k}^{s} \frac{x^-(u_{4,k}) - x^+(u_{4,j})}{x^+(u_{4,k}) - x^-(u_{4,j})} \frac{1 - \frac{g^2}{2\, x^+(u_{4,k})x^-(u_{4,j})}}{1 - \frac{g^2}{2\, x^-(u_{4,k})x^+(u_{4,j})}} \sigma^2(u_{4,k}, u_{4,j}) \quad (3.1)$$

(that corresponds to $sl(2)$ sector, see section 2.3 of chapter 2).
The first step consists in introducing the counting function $Z_4(u)$, which behaves, in a way to be explained in the following, as a counter of roots; in the all loops case, the counting function $Z_4(u)$ reads:

$$Z_4(u) = L\Phi(u) - \sum_{k=1}^{s} \phi(u, u_{4,k}) \quad , \quad (3.2)$$

where it reveals fruitful to operate a splitting of the Φ and ϕ functions in terms of the one-loop and higher-loop components

$$\Phi(u) = \Phi_0(u) + \Phi_H(u)\,, \quad \phi(u,v) = \phi_0(u-v) + \phi_H(u,v) \qquad , \quad (3.3)$$

once the following definitions are introduced

$$\Phi_0(u) = -2\arctan 2u\,, \quad \Phi_H(u) = -i\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right)\,, \tag{3.4}$$

$$\phi_0(u-v) = 2\arctan(u-v)\,, \quad \phi_H(u,v) = -2i\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right) + i\theta(u,v)\right]$$

the spectral parameter being defined via the Jukovski map as usual

$$x(u) = \frac{u}{2}\left[1+\sqrt{1-\frac{2g^2}{u^2}}\right] \quad x^\pm(u) = x(u \pm \frac{i}{2}) \quad . \tag{3.5}$$

The counting function manifests its importance as it allows to recast the set of equations 3.1 as a single one, explicitly:

$$(-1)^{L+s+1} = e^{iZ_4(u)} \quad ; \tag{3.6}$$

indeed, the Bethe roots satisfy the equation above. At this time being, a pivotal remark is in order. By definition, the counting function is monotonously decreasing: as u gets bigger, every time it reaches a value corresponding to a root of (3.6), $Z_4(u)$ is lowered by an amount of π. Finally, when u grows to infinity, a glance to (3.2) reveals that

$$\lim_{u\to\pm\infty} Z_4(u) = \mp(L+s)\pi \quad . \tag{3.7}$$

Since $Z_4(u)$ is a continuous and monotonous function, there exist $L+s$ values of u for which it becomes a integer multiple of π, and that implies the equation (3.6) enjoys $L+s$ solutions: remarkably, only s of them are actually Bethe roots, the remaining L 'fake solutions' u_h, satisfying the condition

$$Z_4(u_h) = \pi(2h-1+L) \qquad h = 1,\ldots,L \tag{3.8}$$

are commonly referred to as holes, meaning that they are missing roots in the root distribution.

Now, with the analytic structure of the counting function $Z_4(u)$ in mind, let us consider an observable (real analytic function) $O(v)$, integrated over the rectangular closed path in the complex plane Γ, made up of two horizontal lines stretching on an arbitrary interval $[-B,B]$ (the lower line lying on the real axis, the other shifted upward by a positive possibly infinitesimal value ϵ), connected by two vertical segments, while $L+s$ half circles (γ_k and γ_h) surround roots and holes. From the Cauchy theorem, the following identity arises:

$$0 = \oint_\Gamma \frac{dv}{\pi}\, O(v)\, \frac{i\frac{Z_4(v)}{dv}}{1+(-1)^{L+s}e^{-iZ_4(v)}} \tag{3.9}$$

For clarity's sake, the integration over the single parts composing Γ have been explicitly

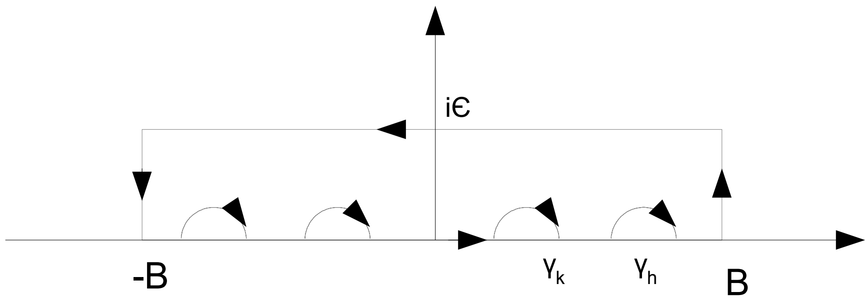

Figure 3.1: The path of integration, in the complex plane

singled out:

$$0 = \left(\int_{-B}^{B} + \int_{B}^{B+i\varepsilon} + \int_{B+i\varepsilon}^{-B+i\varepsilon} + \int_{-B+i\varepsilon}^{-B} + \sum_{k=1}^{s} \int_{\gamma_k} + \sum_{h=1}^{L} \int_{\gamma_h}\right) \frac{dv}{\pi}\, O(v)\, \frac{i\frac{Z_4(v)}{dv}}{1+(-1)^{L+s} e^{-iZ_4(v)}} \tag{3.10}$$

After the residues over the hal circles have been taken, the relation (3.10) is rearranged as

$$\begin{aligned} 0 \quad &= \int_{-B}^{B} \frac{dv}{\pi}\, O(v)\, \frac{i\frac{Z_4(v)}{dv}}{1+(-1)^{L+s} e^{-iZ_4(v)}} + i\sum_{k=1}^{s} O(u_{4,k}) + i\sum_{h=1}^{L} O(u_h) + \\ &- \int_{-B}^{B} \frac{dv}{\pi}\, O(v)\, \frac{d}{dv} \ln\left[1+(-1)^{L+s} e^{iZ_4(v+i\varepsilon)}\right] + \\ &+ \int_{0}^{-\varepsilon} \frac{dv}{\pi}\, O(B-iv) \ln\left[1+(-1)^{L+s} e^{iZ_4(B-iv)}\right] - \\ &- \int_{0}^{-\varepsilon} \frac{dv}{\pi}\, O(-B-iv) \ln\left[1+(-1)^{L+s} e^{iZ_4(-B-iv)}\right] \end{aligned} \tag{3.11}$$

Next, the logarithm can be expanded in powers of $e^{iZ_4(v)}$

$$\ln\left[1+(-1)^{L+s} e^{iZ_4(v)}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-1)^{n(L+s)}\, e^{inZ_4(v)} \quad , \tag{3.12}$$

suggesting the usage of the following relations, obtained by means of repeated integrations by parts:

$$\begin{aligned} \int dv\, O(v+i\varepsilon) \frac{d}{dv} e^{inZ_4(v+i\varepsilon)} &= e^{inx} \sum_{k=0}^{\infty} \left(\frac{i}{n}\right)^k \left(\frac{d}{dx}\right)^k O(Z_4^{-1}(x))\Big|_{x=Z_4(v+i\varepsilon)} \\ \int dv\, O(\pm B-iv) \frac{d}{dv} e^{inZ_4(\pm B-iv)} &= e^{inx} \sum_{k=0}^{\infty} \left(\frac{i}{n}\right)^k \left(\frac{d}{dx}\right)^k O(Z_4^{-1}(x))\Big|_{x=Z_4(\pm B-iv)} \end{aligned} \tag{3.13}$$

Plugging the above useful formulæ into (3.11) and taking the imaginary part, an explicit expression is provided for the summation of the observable $O(v)$ evaluated in correspondence of the Bethe roots:

$$\sum_{k=1}^{s} O(u_{4,k}) = -\int_{-B}^{B} \frac{dv}{2\pi} O(v) \frac{d}{dv} Z_4(v) - \sum_{h=1}^{L} O(u_h) + \quad (3.14)$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{(-1)^{n(L+s)}}{2\pi} e^{inx} \sum_{k=0}^{\infty} \left(\frac{i}{n}\right)^k \left(\frac{d}{dx}\right)^k O(Z_4^{-1}(x)) \Bigg|_{x=Z_4(B)}^{x=Z_4(-B)}$$

B may be chosen such that $e^{iZ_4(\pm B)} = (-1)^{L+s}$, so that the worthy relation follows:

$$\sum_{k=1}^{s} O(u_{4,k}) = -\int_{-B}^{B} \frac{dv}{2\pi} O(v) \frac{d}{dv} Z_4(v) - \sum_{h=1}^{L} O(u_h) -$$
$$- \sum_{k=0}^{\infty} \frac{(2\pi)^{2k+1}}{(2k+2)!} B_{2k+2}\left(\frac{1}{2}\right) \left[\left(\frac{\partial}{\partial x}\right)^{2k+1} O(Z_4^{-1}(x))\right] \Bigg|_{x=Z_4(-B)}^{x=Z_4(B)} \quad (3.15)$$

where $B_k(x)$ are the Bernoulli (even) polynomials

$$B_{2k}(x) = \frac{(-1)^{k-1} 2(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{n^{2k}}$$

Eventually, most of our interest will dwell in the large B limit. First of all, when $B \longrightarrow \infty$, the functions $\phi(v)$ and $Z(v)$, along with their derivatives, can be estimated:

$$\frac{d^n}{dv^n}\phi(v)\Big|_{\pm B} = \mathcal{O}\left(\frac{1}{B^{n+1}}\right)$$
$$\frac{d^n}{dv^n} Z_4(v)\Big|_{\pm B} = \mathcal{O}\left(\frac{1}{B^n}\right) \quad (3.16)$$

It is now worthwhile to examinate in slightly more detail the position of Bethe roots and holes. The interval $[-B, B]$ gathers all the s roots, and $L-2$ holes as well (named 'internal') [36]; besides them, two more holes fall outside that interval, gaining the designation of 'external' or 'large'. The $B \longrightarrow \infty$ limit matches the large spin s regime for a twist operator, leading to the estimate $B = \frac{s}{2}\left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right)$. Now, it is customary to define

$$L_4(v) \equiv Im \ln \left[1 + (-1)^{L+s} e^{iZ_4(v+i\varepsilon)}\right] \quad .$$

When s grows to infinity, $L_4(v) = \mathcal{O}\left(\frac{1}{s^2}\right) = \mathcal{O}\left((\ln s)^{-\infty}\right)$, and, accordingly, (3.15) catches the form, extensively employed throughout this chapter:

$$\sum_{k=1}^{s} O(u_{4,k}) = -\sum_{h=1}^{L} O(u_h) - \int_{-\infty}^{\infty} \frac{dv}{2\pi} O(v) \frac{d}{dv}[Z_4(v) - 2L(v)] =$$
$$= -\sum_{h=1}^{L} O(u_h) - \int_{-\infty}^{\infty} \frac{dv}{2\pi} O(v) \frac{d}{dv} Z_4(v) + \mathcal{O}\left((\ln s)^{-\infty}\right) \quad (3.17)$$

The formula (3.17) just achieved, can be fruitfully applied to the function $\phi(u, v)$, summed over the Bethe roots in the definition of $Z_4(u)$ (3.2), so that the operation results in a non linear integration for $Z_4(u)$:

$$Z_4(u) = f(u) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \phi(u, v) \frac{d}{dv}[Z_4(v) - 2L_4(v)] \quad , \quad (3.18)$$

where the forcing term has been defined as

$$f(u) \equiv L\Phi(u) + \sum_{h=1}^{L} \phi(u, u_h) \qquad .$$

Actually, the equation (3.18) proves not so handy, and it would be convenient to recast it in a more suitable form. For instance, it turns out useful to plug $Z_4(u)$ (3.18) back into itself, so to get the relation

$$\begin{aligned} Z_4(u) &= f(u) - \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{d\phi(u,v)}{dv} Z_4(v) + 2 \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{d\phi(u,v)}{dv} L_4(v) = \\ &= f(u) - \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{d\phi(u,v)}{dv} f(v) + 2 \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{d\phi(u,v)}{dv} L_4(v) - \\ &- 2 \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{dw}{2\pi} \frac{d\phi(u,v)}{dv} \frac{d\phi(v,w)}{dw} L_4(w) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{dw}{2\pi} \frac{d\phi(u,v)}{dv} \frac{d\phi(v,w)}{dw} Z_4(w) \ , \end{aligned} \tag{3.19}$$

then iterating this same procedure for an arbitrary large number of times [41]. Eventually, a new integral equation for the counting function arises:

$$Z_4(u) = F(u) + 2 \int_{-\infty}^{+\infty} dv\, G(u,v) L_4(v) \quad . \tag{3.20}$$

Though, in the equation above the forcing $F(u)$ term and the kernel $G(u,v)$ look nasty and uncomfortable to computation; explicitly they read:

$$F(u) = f(u) + \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} dw_1 \varphi(u, w_1) \int_{-\infty}^{+\infty} dw_2 \varphi(w_1, w_2) \dots \int_{-\infty}^{+\infty} dw_k \varphi(w_{k-1}, w_k) f(w_k) \tag{3.21}$$

$$G(u,v) = \varphi(u,v) + \sum_{k=1}^{\infty} \int_{-\infty}^{+\infty} dw_1 \varphi(u, w_1) \int dw_2 \varphi(w_1, w_2) \dots \int dw_k \varphi(w_{k-1}, w_k) \varphi(w_k, v) \tag{3.22}$$

making use of the shorthand notation

$$\varphi(u,v) = \frac{1}{2\pi} \frac{d}{dv} \phi(u,v) \quad . \tag{3.23}$$

Nevertheless, $F(u)$ and $G(u,v)$ can be regarded as solutions to two novel linear integral equations, namely:

$$F(u) = f(u) - \int_{-\infty}^{+\infty} dv\, \varphi(u,v) F(v) \tag{3.24}$$

$$G(u,v) = \varphi(u,v) - \int_{-\infty}^{+\infty} dw\, \varphi(u,w) G(w,v) \qquad ; \tag{3.25}$$

in addition to that, the functions $F(u)$ and $G(u,v)$ combine into the relation:

$$F(u) = f(u) - \int_{-\infty}^{\infty} dv\, G(u,v) f(v) \qquad . \tag{3.26}$$

The inspection of (3.24) and (3.25), together with the definition of $f(u)$ and $\varphi(u,v)$ (3.23), suggests the function $F(u)$ can be decomposed as the sum of two part: one of them depends explicitly from the hole rapidities u_h, whereas the other takes into account just the variable u. Thus, $F(u)$ can be expressed as $F(u) = L\tilde{P}(u) + \sum_{h=1}^{L} R(u,u_h)$, provided $R(u,v)$ satisfies the equation

$$R(u,v) = \phi(u,v) - \int_{-\infty}^{+\infty} dw\varphi(u,w)R(w,v)\,, \quad \frac{1}{2\pi}\frac{d}{dv}R(u,v) = G(u,v) \tag{3.27}$$

while instead the function $\tilde{P}(u)$ results from the linear integral equation

$$\tilde{P}(u) = \Phi(u) - \int_{-\infty}^{+\infty} dw\varphi(u,w)\tilde{P}(w) \qquad . \tag{3.28}$$

2 The GKP vacuum from the spin chain perspective

Clearly, the analysis of excitations shall start from the description of the ground state. It is therefore appropriate to recall the construction of the BMN vacuum; then, the recipe to switch to GKP vacuum will be given. The BMN (half BPS, ferromagnetic) vacuum corresponds, within the gauge theory, to the single trace operator made out of two Z fields, thus carrying spin zero and twist two:

$$\mathcal{O}_{BMN} \sim Tr[ZZ] \qquad .$$

Moving out of the ground state, the insertion of covariant (light cone) derivatives D_+ adds no twist value to the operator, whereas the total spin is increased by one unit for each D_+; then, an operator composed of the two vacuum Z fields over which s derivatives act shows up as

$$\mathcal{O} \sim Tr[Z\,D_+^s Z] + \dots \qquad :$$

objects of this sort constitute the $sl(2)$ sector.
Remarkably, the number of derivatives s, and correspondingly the total spin, needs not to stay finite; conversely, s may be pushed to infinity. The large spin limit turns out to correspond to the GKP, long (*i.e.* fast spinning) string, and, from now on, such a state will be settled as the actual vacuum of the theory. In this framework, the excitations arise as insertions of fields, let us generically say ξ, in between the sea of D_+'s: more precisely, the raising of a single particle over the vacuum gets associated to the operator

$$\mathcal{O}' \sim Tr[Z\,D_+^{s-s_1}\xi\,D_+^{s_1}Z] + \dots \qquad . \tag{3.29}$$

For the GKP vacuum breaks the entire symmetry $PSU(2,2|4)$ down to $SU(4) \simeq SO(6)$, the excitation fields do belong to multiplets under the residual $SU(4)$, or, rather, the insertions ξ stand for the highest weights of some representation of $su(4)$, associated to each species of particle: for instance, the introduction of additional Z fields, such as $\xi = Z$, accounts for the excitation of a scalar particle transforming under the **6** of $su(4) \simeq so(6)$, whilst $\xi = \psi_+$, $\bar{\psi}_+$ involves, respectively, the bearing of a fermion in the

4 of $su(4)$ or an antifermion in the $\bar{\mathbf{4}}$, then finally $\xi = F_{+\perp}$, $\bar{F}_{+\perp}$ casts the corresponding two components of the gluon field, behaving as a singlet to $su(4)$. The double excitation state reflects in the operator

$$\mathcal{O}" \sim Tr[Z\, D_+^{s-s_1-s_2}\xi_1\, D_+^{s_1}\xi_2\, D_+^{s_2}Z] + \dots \qquad , \tag{3.30}$$

and so on, increasing the number of particles for every ξ_k inserted.

This picture must be then transposed onto the set-up of spin chain. While the ferromagnetic vacuum comfortably mirrors the BMN regime, the match between the GKP string and the antiferromagnetic vacuum is alluring at least. The large spin limit suggests that the covariant derivatives D_+, which are embedded as u_4 roots in the Beisert-Staudacher equations (2.30), are filling every state they are allowed to occupy, in that mimicking a Fermi sea: to bear any excitation, a u_4 root has to be 'pulled' away from the distribution of main roots, so that a hole peers in the sea. Eventually, the hole remains, taking so the role of a 'fake Bethe root' previously described in (3.8), and the circumstance results in the raising of a scalar excitation (in the operator standpoint, a Z insertion). Else, the hole might be 'filled up' with another type of Bethe root, generating a fermion (u_1 or u_3, depending on the cinematical regime) or a gluon (real centre of a gluonic stack of roots). The picture outlined above accounts for the particles which carry momentum and energy on the GKP string. In addition to them, in the next chapter it will turn out necessary to introduce three more kind of roots, lacking of momentum and energy as well: their task strictly relates to the residual $SU(4)$ residual symmetry of the vacuum just chosen.

3 Scalar excitations: one loop

The one loop problem deserves a little attention, for it provides a first simple application of the tools achieved in the present chapter, whereas remarkably all the relevant features of the all-loop problem are already present. To begin with, the study sticks on the sector made up of scalars only. Then, in the following sections, this restriction will drop, allowing the introduction of every different kind of particle belonging to the spectrum of the theory.

As a first instance, in the one loop case, the counting function for the twist sector reads

$$Z_{4,0}(u) = L\Phi_0(u) - \sum_{k=1}^{s} \phi_0(u - u_{4,k})\,, \tag{3.31}$$

The L holes are characterised by rapidities u_h, $h = 1, ..., L$: the two external (or 'large') holes, specifically u_1, u_L, lie outside Bethe root interval (*i.e.* $u_1 < u_k < u_L$), while internal (or 'small') ones are labelled as $u_2, ...u_{L-1}$.
By sticking to one loop [1] (*i.e.* fixing $g = 0$), the equation (3.18) becomes:

$$Z_{4,0}(u) = L\Phi_0(u) + \sum_{h=1}^{L} \phi_0(u - u_h) + \int_{-\infty}^{\infty} \frac{dv}{2\pi}\, \phi_0(u-v) \frac{d}{dv}[Z_{4,0}(v) - 2L_{4,0}(v)] \quad , \tag{3.32}$$

[1] Throughout this text, the subscript 0 means that the function considered is restricted to the form it assumes in the limit $g = 0$

where

$$L_{4,0}(u) = \mathrm{Im}\ln[1 + (-1)^{L+s}\, e^{iZ_{4,0}(u+i0^+)}] \,. \tag{3.33}$$

It turns out useful to remark that in the one loop limit the function $\phi(u, v)$ does not depend on the two entries separately, but rather on the difference of them, so that the integral equation above can be easily solved by Fourier transforming [2] it and using the *faltung* theorem [3]. Since

$$\hat{\Phi}_0(k) = -\frac{2\pi e^{-\frac{|k|}{2}}}{ik} \,, \quad \hat{\phi}_0(k) = \frac{2\pi e^{-|k|}}{ik} \,, \tag{3.34}$$

the equation for $Z_{4,0}$ in the Fourier space simplifies to

$$\hat{Z}_{4,0}(k) = -\frac{2\pi L e^{-\frac{|k|}{2}}}{ik(1-e^{-|k|})} + \frac{2\pi e^{-|k|}}{ik(1-e^{-|k|})}\sum_{h=1}^{L} e^{-ikx_h} - 2\frac{e^{-|k|}}{1-e^{-|k|}}\hat{L}_{4,0}(k) \quad . \tag{3.35}$$

Antitransforming back to direct (coordinate) space, the relation (3.18) comes to

$$\begin{aligned} Z_{4,0}(u) \;&=\; -iL\ln\frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} + i\sum_{h=1}^{L}\ln\frac{\Gamma(1+iu-ix_h)}{\Gamma(1-iu+ix_h)} + \\ &+ \int_{-\infty}^{+\infty}\frac{dv}{\pi}[\psi(1+iu-iv)+\psi(1-iu+iv)]L_{4,0}(v) \,, \end{aligned} \tag{3.36}$$

Since the GKP string requires $s \longrightarrow \infty$, the non linear term can be safely approximated [43]

$$\int_{-\infty}^{+\infty}\frac{dv}{\pi}[\psi(1+iu-iv)+\psi(1-iu+iv)]L_{4,0}(v) = -2u\ln 2 + O\left(\frac{1}{s^2}\right) \quad , \tag{3.37}$$

so that in the large spin limit the non-linear equation (3.36) gets linearised. In addition to that, the regime just chosen allows to extract pieces of information (for an accurate analysis please refer to [44][41] for instance, and references therein) about the position of holes and roots on the real rapidity axis, by studying equations (3.6) for Bethe roots and (3.8). First of all, when considering the minimal anomalous dimension state (as it is the case of interest), roots and holes arrange themselves symmetrically around the origin, and moreover they all lie inside a closed interval, say $[-B, B]$, the only exception being the two external holes forming the vacuum, as they both fall outside. Furthermore, the minimal anomalous dimension is achieved when the internal holes satisfy the equations $Z(u_h) = \pi(2h - 1 - L)$ involving the set of integers $h \in \{2, \ldots, L-1\}$, as any different choice of occupation numbers leads to higher values of the dimension. As a result the

[2] The Fourier transform of a function $f(u)$ is the function $\hat{f}(k) = \int_{-\infty}^{+\infty} du\, e^{-iku} f(u)$

[3] The Fourier transform $\mathcal{F}_k$ of the convolution between two functions amounts to the product of the transforms of both the functions separately; explicitly:

$$\mathcal{F}_k[f * g] = \mathcal{F}_k[\int_{-\infty}^{\infty} dv\, f(u-v)g(v)] = \hat{f}(k)\hat{g}(k)$$

internal holes gather around the origin, concentrating within the interval $[-C, C]$ with $C \sim \frac{1}{\ln s}$ [32], where no Bethe root is present at all: the latter instead belong to either the interval $[-C, -B]$ or $[C, B]$. At last, in the large spin limit the two external holes place at

$$x_L = -x_1 = \frac{s}{\sqrt{2}} \left(1 + \frac{L - 1 + f(g)}{2s} \right) + O\left(\frac{1}{s}\right) ,$$

where $f(g)$ is the universal scaling function.

With these considerations in mind, the next task will be then recognizing the scattering matrix involving the physical particles of the theory, that is the holes. On the BMN vacuum the equation (3.6) means nothing but the momentum quantization on the spin chain: a probe particle travels along the circuit and gets its propagation phase shifted every time it scatters another particle, eventually coming back to the initial value when a round is complete. So, in the one-loop approximation, the Bethe equations read[4]

$$(-1)^{L-1} = e^{-iZ_{4,0}(u_k)} = e^{iL\mathcal{P}_0(u_k)} \prod_{j \neq k}^{s} \mathcal{S}_0(u_k, u_j) , \tag{3.38}$$

where $\mathcal{P}_0(u)$ and $\mathcal{S}_0(u, v)$ stand for, respectively, the one-loop momentum and scattering phase over the BMN vacuum. By means of the formula (3.17), the counting function drops the explicit dependence on Bethe roots and becomes expressed in terms solely of the holes which are the physical excitations, while, thanks to the large spin limit, the roots saturate the number of available states except for the L holes: in doing so the scattering theory dwells no more on the ferrromagnetic vacuum, but over the antiferromagnetic (GKP), instead. Once rewritten on the GKP vacuum, and involving the holes uniquely, the equation (3.38) now turns to

$$\begin{aligned}(-1)^{L-1} = e^{-iZ_{4,0}(u_h)} = e^{i\Lambda_{s,0}(u_h)} \prod_{h' \neq h \ , \ h'=2}^{L-1} (-S_0(u_h, u_{h'})) \Rightarrow \\ \Rightarrow 1 = e^{i\Lambda_{s,0}(u_h)} \prod_{h' \neq h \ , \ h'=2}^{L-1} S_0(u_h, u_{h'}) \quad .\end{aligned} \tag{3.39}$$

Interpreting anew the equation (3.39) as a quantization condition, this time starting from the antiferromagnetic vacuum, the scattering matrices among two holes are now described -at one loop- by $S_0(u, v)$, whereas the propagation phase of the h-th hole $\Lambda_0(u_h)$ contains the product between the momentum $P_0(u_h)$ and the effective length R associated to this novel spin chain. In order to properly identify these features, it is worth observing that the total phase $Z_{4,0}(u)$ (3.36) includes one part accounting for the changes the probe particle experience whenever it collides with a hole, together with a part which does not depend on the particle in the chain: so while the former vanishes when no hole is present, the latter still remains. Reminding that in the setup chosen, the number of scalar excitations is $L - 2$, for the two external holes do form the vacuum

[4]The integer s is taken to be even, from now on

and thus can never be removed, the one loop counting function $Z_{4,0}(u)$ (3.36) splits accordingly:

$$\begin{aligned} Z_{4,0}(u) &= \left[-2i \ln \frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} + i \ln \frac{\Gamma(1+iu-iu_L)\Gamma(1+iu-iu_1)}{\Gamma(1-iu+iu_L)\Gamma(1-iu+iu_1)} - 2u \ln 2 \right] + \\ &+ \sum_{h=2}^{L-1} \left[-i \ln \frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} + i \ln \frac{\Gamma(1+iu-iu_h)}{\Gamma(1-iu+iu_h)} \right] + O\left(\frac{1}{\ln s^2}\right) \end{aligned} \tag{3.40}$$

Aiming to find the scattering matrix between two scalars (two internal holes), the first suitable case consists in $L = 4$: such a state is implemented by adapting the expression for $Z_{4,0}(u)$ above to the present configuration, and plugging the rapidity u_h of one the (actual) internal hole as an entry of the counting function. It would be then tantalizing to identify the content of the bracket in the second line of (3.40) as the scattering phase:

$$i \ln \frac{\Gamma\left(\frac{1}{2}-iu_h\right)}{\Gamma\left(\frac{1}{2}+iu_h\right)} \frac{\Gamma(1+iu_h-iu_{h'})}{\Gamma(1-iu_h+iu_{h'})}$$

Nevertheless, this guess does not fit, since the expression is not unitary. It is thus needed to reinstate the unitarity 'by hand': this task can be accomplished by adding $i \ln \frac{\Gamma\left(\frac{1}{2}+iu_{h'}\right)}{\Gamma\left(\frac{1}{2}-iu_{h'}\right)}$ to the expression above, and, moreover, the result enjoys good asymptotic properties, *i.e.* it goes to zero when one of the rapidities goes to infinity. To sum up, after subtracting in the first line the quantity just summed in the second, the counting function for $L = 4$ becomes

$$\begin{aligned} Z_{4,0}(u) = i &\left[-2 \ln \frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} - \sum_{h=2}^{3} \ln \frac{\Gamma\left(\frac{1}{2}+iu_h\right)}{\Gamma\left(\frac{1}{2}-iu_h\right)} + \right. \\ &\left. + \ln \frac{\Gamma(1+i(u-u_L))\Gamma(1+i(u-u_1))}{\Gamma(1-i(u-u_L))\Gamma(1-i(u-u_1))} + 2iu \ln 2 \right] + \\ +i &\left[-2 \ln \frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} + \sum_{h=2}^{3} \ln \frac{\Gamma\left(\frac{1}{2}+iu_h\right)}{\Gamma\left(\frac{1}{2}-iu_h\right)} + \sum_{h=2}^{3} \ln \frac{\Gamma(1+iu-iu_h)}{\Gamma(1-iu+iu_h)} \right] + O\left(\frac{1}{\ln s^2}\right) \end{aligned} \tag{3.41}$$

while the one loop scattering matrix involving two holes is

$$S_0(u_h, u_{h'}) = -\frac{\Gamma\left(\frac{1}{2}-iu_h\right)\Gamma\left(\frac{1}{2}+iu_{h'}\right)\Gamma(1+iu_h-iu_{h'})}{\Gamma\left(\frac{1}{2}+iu_h\right)\Gamma\left(\frac{1}{2}-iu_{h'}\right)\Gamma(1-iu_h+iu_{h'})} \quad , \tag{3.42}$$

in agreement with the results by Basso-Belitsky [33] and Dorey-Zhao [45].

On the other hand, when $L = 3$ formula (3.40) provides a way to recover the momentum of a hole. Indeed, it is straightforward to read the propagation phase:

$$\begin{aligned} \Lambda_{s,0}(u) &= i \ln \frac{\Gamma(1+i(u-u_L))\Gamma(1+i(u-u_1))}{\Gamma(1-i(u-u_L))\Gamma(1-i(u-u_1))} - 2u \ln 2 - \\ &- 2i \ln \frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} - i \sum_{h=2}^{L-1} \ln \frac{\Gamma\left(\frac{1}{2}+iu_h\right)}{\Gamma\left(\frac{1}{2}-iu_h\right)} + O\left(\frac{1}{\ln s^2}\right) \end{aligned} \tag{3.43}$$

The (one loop) momentum $P_0(u)$ can be extracted from $\Lambda(u)$ in the large spin limit, as it appears in the form of a product per the effective length R of the chain, so more explicitly:

$$\Lambda_{s,0}(u) = R \cdot P_{s,0}(u) + D_{s,0}(u) \quad , \tag{3.44}$$

where $D_{s,0}(u)$ is a function, subleading in $\ln s$, yet to be discussed, while R also corresponds to the length of the GKP string, $R \sim 2\ln s$ (the factor 2 owing to the fact that the GKP is a folded string), and then

$$P_{s,0}(u) = 2u \tag{3.45}$$

For arbitrary L, the propagation phase in the $s \longrightarrow \infty$ limit reduces to

$$\Lambda_{s,0}(u) = 4u\ln s - 2i\ln\frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} - i\sum_{h=2}^{L-1}\ln\frac{\Gamma\left(\frac{1}{2}+iu_h\right)}{\Gamma\left(\frac{1}{2}-iu_h\right)} + O\left(\frac{1}{\ln s^2}\right) \quad . \tag{3.46}$$

Now, since for the internal holes $u_h \sim \frac{1}{\ln s}$, the sum $\sum_{h=2}^{L-1}\ln\frac{\Gamma\left(\frac{1}{2}+iu_h\right)}{\Gamma\left(\frac{1}{2}-iu_h\right)}$ can be neglected, hence the function $D(u)$ is identified:

$$D_{s,0}(u) = -2i\ln\frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} \quad . \tag{3.47}$$

The explanation is that whenever a particle goes round the chain, it experiences an additional phase shift $D_{s,0}(u)$ amenable to the presence of two static defects (*i.e.* they are not associated to any proper rapidity), each one corresponding to one tip of the folded GKP string.

In the end, all the features of the one loop spin chain have been displayed in the large spin limit, so the Bethe equations (3.39), computed over the antiferromagnetic vacuum, can be recovered from (3.42),(3.45),(3.47):

$$\begin{aligned}(-1)^{L-1} &= e^{i(2u_h)\,2\ln s}\left(\frac{\Gamma\left(\frac{1}{2}-iu_h\right)}{\Gamma\left(\frac{1}{2}+iu_h\right)}\right)^2 \times \\ &\times \prod_{h'\neq h\ ,\ h'=2}^{L-1}\frac{\Gamma\left(\frac{1}{2}-iu_h\right)\Gamma\left(\frac{1}{2}+iu_{h'}\right)\Gamma(1+iu_h-iu_{h'})}{\Gamma\left(\frac{1}{2}+iu_h\right)\Gamma\left(\frac{1}{2}-iu_{h'}\right)\Gamma(1-iu_h+iu_{h'})} \quad .\end{aligned} \tag{3.48}$$

4 Scalar excitations: all loops

The analysis carried out in the previous section is now going to be extended to the all loop problem. The calculations are obviously more involved, nevertheless the starting point remains the Bethe quantization condition:

$$1 = (-1)^{L-1}\exp(-iZ_4(u_h)) = e^{i\Lambda_s(u_h)}\prod_{\{h'=2,\ h'\neq h\}}^{L-1} S(u_h, u_{h'}) \quad . \tag{3.49}$$

It now useful to recall from (3.20) that the all loop counting function reads:

$$Z_4(u) = L\tilde{P}(u) + \sum_{h=1}^{L} R(u,u_h) + 2\int_{-\infty}^{\infty} dv\, G(u,v)L(v) \quad . \tag{3.50}$$

Repeating the reasoning adopted for the one loop case, the counting function should be slit into a part existing even when there is no internal holes in the system, plus another strictly dependent on the presence of those scalar excitations; hence explicitly:

$$Z_4(u) = \left\{ 2\tilde{P}(u) + R(u,u_1) + R(u,u_L) + 2\int_{-\infty}^{\infty} dv\, G(u,v)L_4(v) \right\} + \sum_{h=2}^{L-1} \{R(u,u_h) + \tilde{P}(u)\} \quad . \tag{3.51}$$

The quantity in the last bracket cannot be identified with the scattering matrix between holes, for it lacks unitarity and so it needs to be completed with an additional term. From the equations (3.27) and (3.28) the properties of the functions $R(u,v)$ and $\tilde{P}(u)$ can be easily deduced

$$\tilde{P}(u) = -\tilde{P}(-u)\,, \quad R(u,v) = -R(-u,-v)\,, \quad R(u,v) = -R(v,u)\,, \tag{3.52}$$

therefore the unitarity can be restored in a simple way, by properly adding and subtracting $\tilde{P}(u_h)$ in the counting function above, so to get:

$$i\ln(-S(u,v)) = R(u,v) + \tilde{P}(u) - \tilde{P}(v) \equiv \Theta(u,v) \quad , \tag{3.53}$$

and, owing to (3.52), $\Theta(u,v)$ turns out to be antisymmetric

$$\Theta(u,v) = -\Theta(v,u) \quad ,$$

as it should, in order to satisfy the requests on $S(u,v)$.

The propagation phase $\Lambda_s(u)$ may thus be read from (3.50):

$$\Lambda_s(u) = -2\tilde{P}(u) - R(u,u_1) - R(u,u_L) - 2\int_{-\infty}^{\infty} dv\, G(u,v)L(v) + \sum_{h=2}^{L-1} \tilde{P}(u_h) \quad . \tag{3.54}$$

As already learnt from the one loop case, $\Lambda_s(u_h)$ consists of the sum of the momentum of the h-th internal root plus a phase shift associated to the presence of defects on the spin chain. Anyway, all these features will be discussed in more detail in the next chapter. A further consideration is in order about the nonlinear term

$$2\int dv\, G(u,v)L_s(v) \equiv NL(u) \quad ,$$

which in the high spin limit reveals to be subdominant ($O(1)$) with respect to the momentum. In fact, starting from the equation (3.25) for the function $G(u,v)$ in Fourier space

$$\hat{G}(k,t) = -\frac{e^{-|k|}}{1-e^{-|k|}} 2\pi\delta(t+k) + \frac{\hat{\varphi}_H(k,t)}{1-e^{-|k|}} - \frac{1}{1-e^{-|k|}} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \hat{\varphi}_H(k,p)\hat{G}(-p,t) \quad , \tag{3.55}$$

it follows from the definition of $NL(k)$ that

$$\hat{N}L(k) = -2\frac{e^{-|k|}}{1-e^{-|k|}}\hat{L}_4(k)+2\int \frac{dt}{2\pi}\frac{\hat{\varphi}_H(k,t)}{1-e^{-|k|}}\hat{L}_4(-t)-\frac{1}{1-e^{-|k|}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\varphi}_H(k,p)\hat{N}L(-p)\,. \tag{3.56}$$

The equation above gets linear when s grows to infinity [44]

$$\hat{N}L(k) = -\frac{4\pi\ln 2}{ik}\delta(k) - \frac{1}{1-e^{-|k|}}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\hat{\varphi}_H(k,p)\hat{N}L(-p) + O\left(\frac{1}{s^2}\right) \tag{3.57}$$

and it is hence apparent that the non linear term contributes from from order $O\left((\ln s)^0\right)$ onward.

4.1 Scattering phase between holes

In order to study the scalar scattering phase $\Theta(u,v)$, it is useful to split it into its even and odd (with respect to the second variable) part:

$$\Theta(u,v) = M(u,v) + N(u,v) \quad ,$$

where $M(u,v)$ and $N(u,v)$ has been introduced according to:

$$\begin{aligned} M(u,v) &= \frac{\Theta(u,v)+\Theta(u,-v)}{2} = M(u,-v) \\ N(u,v) &= \frac{\Theta(u,v)-\Theta(u,-v)}{2} = -N(u,-v) \quad . \end{aligned} \tag{3.58}$$

It is now convenient to recast the functions above in term of $R(u,v)$ and $\tilde{P}(u)$ for the reason that they can be determined as solutions of already known integral equations; then he definition of $\Theta(u,v)$ (3.53) allows to write:

$$\begin{aligned} M(u,v) &= \frac{R(u,v)+\tilde{P}(u)}{2} + \frac{R(u,-v)+\tilde{P}(u)}{2} \\ N(u,v) &= \frac{R(u,v)-R(u,-v)}{2} - \tilde{P}(v) \end{aligned} \tag{3.59}$$

A more careful glance at the odd part of the scattering phase reveals that $N(u,v)$ may be actually related to $M(u,v)$, by means of the properties of $R(u,v)$ and $\tilde{P}(u)$ (3.52):

$$M(v,u) = \frac{R(v,u)+R(v,-u)}{2} + \tilde{P}(v) = \frac{-R(u,v)+R(u,-v)}{2} + \tilde{P}(v) = -N(u,v) \quad . \tag{3.60}$$

To sum up, the computation of the even component $M(u,v)$ brings to the knowledge of the scattering phase $\Theta(u,v)$ as a whole

$$\Theta(u,v) = M(u,v) - M(v,u) \quad , \tag{3.61}$$

and, in addition to that, $M(u,v)$ is completely determined by the sum $R(u,v)+\tilde{P}(u)$ by virtue of its definition (3.58). Aiming to find a description for the scalar scattering phase, it is useful to add together the equations (3.27),(3.28), once turned to Fourier space:

$$\hat{R}(k,t)+\hat{\tilde{P}}(k)=\hat{\phi}(k,t)+\hat{\Phi}(k)2\pi\delta(t)-\int_{-\infty}^{+\infty}\frac{dp}{4\pi^2}ip\hat{\phi}(k,p)\left[\hat{R}(-p,t)+\hat{\tilde{P}}(-p)\right] \quad (3.62)$$

which more explicitly becomes

$$\hat{R}(k,t)+\hat{\tilde{P}}(k)=4\pi^2\delta(k+t)\frac{e^{-|k|}}{ik}-\frac{4\pi^2e^{-\frac{|k|}{2}}}{ik}\delta(t)+e^{-|k|}\left[\hat{R}(k,t)+\hat{\tilde{P}}(k)\right]+ \quad (3.63)$$
$$+\frac{4\pi^2e^{-\frac{|k|}{2}}}{ik}[1-J_0(\sqrt{2}gk)]\delta(t)+\hat{\phi}_H(k,t)-\int_{-\infty}^{+\infty}\frac{dp}{4\pi^2}ip\hat{\phi}_H(k,p)\left[\hat{R}(-p,t)+\hat{\tilde{P}}(-p)\right]$$

It turns out convenient to distinguish in (3.63) above the one loop and higher-than-one loop contributions, so that such operation results in two distinct equations, respectively:

$$\hat{R}_0(k,t)+\hat{\tilde{P}}_0(k) = \frac{4\pi^2}{ik}\frac{e^{-|k|}}{1-e^{-|k|}}\delta(k+t)-\frac{4\pi^2}{ik}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}}\delta(t) \quad (3.64)$$

$$\hat{R}_H(k,t)+\hat{\tilde{P}}_H(k) = \frac{\hat{\phi}_H(k,t)}{(1-e^{-|k|})(1-e^{-|t|})}+2\pi\delta(t)\,A_H(k)-$$
$$-\int_{-\infty}^{+\infty}\frac{dp}{4\pi^2}\frac{ip\hat{\phi}_H(k,p)}{1-e^{-|k|}}\left[\hat{R}_H(-p,t)+\hat{\tilde{P}}_H(-p)\right] \quad , \quad (3.65)$$

with the shorthand notation

$$A_H(k)=\frac{2\pi e^{-\frac{|k|}{2}}}{ik(1-e^{-|k|})}[1-J_0(\sqrt{2}gk)]-\frac{1}{1-e^{-|k|}}\int_{-\infty}^{+\infty}\frac{dp}{4\pi}\frac{\hat{\phi}_H(k,p)}{\sinh\frac{|p|}{2}} \quad .$$

Thanks to (3.59), it is possible to get the equations for the one loop and higher loop components of $\hat{M}(k,t)$:

$$\hat{M}_0(k,t) = \frac{2\pi^2}{ik}\frac{e^{-|k|}}{1-e^{-|k|}}[\delta(k+t)+\delta(k-t)]-\frac{4\pi^2}{ik}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}}\delta(t) \quad (3.66)$$

$$\hat{M}_H(k,t) = \frac{\hat{\phi}_H(k,t)+\hat{\phi}_H(k,-t)}{2(1-e^{-|k|})(1-e^{-|t|})}+2\pi\delta(t)A_H(k)-\int_{-\infty}^{+\infty}\frac{dp}{4\pi^2}\frac{ip\hat{\phi}_H(k,p)}{1-e^{-|k|}}\hat{M}_H(-p,t)$$

Those two equations join together into a novel one associated to $\hat{M}(k,t)=\hat{M}_0(k,t)+\hat{M}_H(k,t)$

$$\hat{M}(k,t) = \frac{\hat{\phi}_H(k,t)+\hat{\phi}_H(k,-t)}{2(1-e^{-|k|})}-2\pi^2\delta(t)\frac{J_0(\sqrt{2}gk)}{ik\sinh\frac{|k|}{2}}+ \quad (3.67)$$
$$+\frac{2\pi^2}{ik}\frac{e^{-|k|}}{1-e^{-|k|}}[\delta(k+t)+\delta(k-t)]-\int_{-\infty}^{+\infty}\frac{dp}{4\pi^2}\frac{ip\hat{\phi}_H(k,p)}{1-e^{-|k|}}\hat{M}(-p,t)\ ;$$

as a consequence the following properties of $\hat{M}(k,t)$ hold true:

$$\hat{M}(k,t)=\hat{M}(k,-t)\,,\quad \hat{M}(k,t)=-\hat{M}(-k,t) \quad .$$

Thanks to this behaviour of $\hat{M}(k,t)$ under parity, the equation (3.67) may be restricted to the sector $t>0$, $k>0$ without loss of generality. Hence (3.67) should be handfully reformulated by introducing the function $K(k,t)$, commonly referred to as 'magic kernel' in the study of twist sector:

$$K(k,t) = \frac{2}{kt}\left[\sum_{n=1}^{\infty} nJ_n(k)J_n(t) + 2\sum_{m=1}^{\infty}\sum_{l=0}^{\infty}(-1)^{m+l}c_{2m+1,2l+2}(g)J_{2m}(k)J_{2l+1}(t)\right] \quad (3.68)$$

where $J_n(t)$ stands for the n-th Bessel function of the first kind [1]; indeed the function $\phi_H(k,t)$ relates to $K(k,t)$ in a pretty simple way

$$\hat{\phi}_H(k,t) + \hat{\phi}_H(k,-t) = 8i\pi^2 g^2 e^{-\frac{t+k}{2}} K(\sqrt{2}gk, \sqrt{2}gt)\,, \quad t,k>0\,, \quad (3.69)$$

and that thus allows to recast the equation (3.67) for $\hat{M}(k,t)$ (for $k>0$, $t>0$) in a fruitful fashion, since it recalls known results achieved in works about high spin twist sector:

$$\begin{aligned}\hat{M}(k,t) &= \frac{2i\pi^2 g^2}{\sinh\frac{k}{2}} e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt) - 2\pi^2\delta(t)\frac{J_0(\sqrt{2}gk)}{ik\sinh\frac{k}{2}} + \quad (3.70)\\ &+ \frac{2\pi^2}{ik}\frac{e^{-k}}{1-e^{-k}}\delta(k-t) + \frac{ig^2}{\sinh\frac{k}{2}}\int_0^{+\infty} dp\,\hat{K}(\sqrt{2}gk,\sqrt{2}gp)e^{-\frac{p}{2}}ip\hat{M}(p,t) \quad .\end{aligned}$$

More specifically, let us introduce the density as the derivative of the counting function: $\sigma(u) \equiv \frac{dZ(u)}{du}$, satisfying the equation (upon switching to Fourier space and restricting to $k>0$):

$$\begin{aligned}\hat{\sigma}(k) &= \frac{\pi L}{\sinh\frac{k}{2}}\left[e^{-\frac{k}{2}} - J_0(\sqrt{2}gk)\right] + \frac{2\pi e^{-k}}{1-e^{-k}}\sum_{h=1}^{L}(\cos ku_h - 1) - (2\ln 2)2\pi\delta(k) - \\ &- \frac{g^2 k}{\sinh\frac{k}{2}}\int_0^{\infty} dt\, e^{-\frac{t}{2}}\, K(\sqrt{2}gk,\sqrt{2}gt)\left[2\pi\sum_{h=1}^{L}\cos ku_h + \hat{\sigma}(t)\right] + O\left(\frac{1}{s^2}\right) \ .(3.71)\end{aligned}$$

The density $\sigma(u)$ could be decomposed into a part proportional to $\ln s\,\frac{L-2}{\ln s}$ (customarily called first generalised scaling function, see for instance [31] for more details), $(L-2)\sigma^{(1)}(u)$, plus a part depending on the holes. Moreover, in the latter the contribution coming from the external holes $\sigma(u)|_{NIH}$ could be distinguished from the one stemming from all the internal holes $\sigma(u)|_{AIH}$:

$$\begin{aligned}\sigma(u) &= (L-2)\sigma^{(1)}(u) + \sigma(u)|_{AIH} + \sigma(u)|_{NIH} = \\ &= (L-2)\sigma^{(1)}(u) + \sum_{h=2}^{L-1}\sigma(u,u_h)|_{IH} + \sigma(u)|_{NIH} \quad . \quad (3.72)\end{aligned}$$

where the Fourier transformed function $\hat{\sigma}^{(1)}(k)$ solves the equation

$$\hat{\sigma}^{(1)}(k) = \frac{\pi}{\sinh\frac{k}{2}}[e^{-\frac{k}{2}} - J_0(\sqrt{2}gk)] - \frac{g^2 k}{\sinh\frac{k}{2}}\int_0^{\infty} dt\,^{-\frac{t}{2}}\, K(\sqrt{2}gk,\sqrt{2}gt)[2\pi + \hat{\sigma}^{(1)}(t)] \quad (3.73)$$

whereas $\hat{\sigma}(k,u_h)|_{IH}$, which is related to a single internal hole (whose rapidity is u_h), is associated to the equation:

$$\begin{aligned}\hat{\sigma}(k,u_h)|_{IH} &= \frac{2\pi e^{-k}}{1-e^{-k}}(\cos ku_h - 1) + \quad (3.74)\\ &- \frac{g^2 k}{\sinh\frac{k}{2}}\int_0^{\infty} dt\,^{-\frac{t}{2}}\, K(\sqrt{2}gk,\sqrt{2}gt)\left[2\pi(\cos tu_h - 1) + \hat{\sigma}(k,u_h)|_{IH}\right] \ ;\end{aligned}$$

at last, about the part relative to the external holes, it holds:

$$\begin{aligned}\hat{\sigma}(k)|_{NIH} &= -\frac{2\pi}{\sinh\frac{k}{2}}J_0(\sqrt{2}gk) + \frac{4\pi e^{-k}}{1-e^{-k}}\cos ku_L - (2\ln 2)2\pi\delta(k) - \\ &- \frac{g^2 k}{\sinh\frac{k}{2}}\int_0^{\infty} dt\, e^{-\frac{t}{2}}\, K(\sqrt{2}gk, \sqrt{2}gt)\, [4\pi\cos ku_L + \hat{\sigma}(t)|_{NIH}] + O\left(\frac{1}{s^2}\right) \quad .\end{aligned} \tag{3.75}$$

Finally, the sum of (3.73) and (3.74) gives the total contribution a single hole brings to the counting function, and a comparison with (3.70) suggests the relations linking $M(u,v)$ to these densities :

$$ik\hat{M}(k,t) = \int_{-\infty}^{+\infty} du\, e^{-itu}[\hat{\sigma}^{(1)}(k) + \hat{\sigma}(k,u)|_{IH}] \tag{3.76}$$

$$\frac{d}{du}M(u,v) = \sigma^{(1)}(u) + \sigma(u,v)|_{IH} \quad . \tag{3.77}$$

Now, reminding that $\Theta(u,v) = M(u,v) - M(v,u)$, it clearly follows that the scattering phase between scalar excitations is determined by the densities $\sigma^{(1)}$, $\sigma|_{IH}$, which have been widely and deeply studied in literature (see for instance ([44])).

5 Gluonic excitations

The excitation of gluon fields on the GKP string corresponds on the gauge theory side to the insertion of the field strength components $F_{+\perp}$, $\bar{F}_{+\perp}$ inside the trace operator(3.29). In the framework of the spin chain, the fields $F_{+\perp}$ and $\bar{F}_{+\perp}$ are not associated to a single kind of root, solving anyone among the Beisert-Staudacher equations, instead they are embedded as complexes of roots of different types. Precisely, a gluon $F_{+\perp}$ is represented as a stack composed by two complex u_3 roots, which are conjugated each other and symmetrically placed with respect to a u_2 roots laying on the real axis, then referred to as the centre of the stack u^g [42]:

$$\begin{aligned} u_{3\pm} &= u^g \pm \frac{i}{2} \\ u_{2j} &= u^g \quad . \end{aligned} \tag{3.78}$$

On the other hand, $\bar{F}_{+\perp}$ associates to:

$$\begin{aligned} u_{5\pm} &= u^{\bar{g}} \pm \frac{i}{2} \\ u_{6j} &= u^{\bar{g}} \quad . \end{aligned} \tag{3.79}$$

In addition to those excitations, which increase the twist of the operator by one unit, the field content of the theory also includes bound states obtained from the reiterate application of covariant derivatives on the gluons, namely partons of the type $D_{\perp}^{l-1}F_{+\perp}$ or $\bar{D}_{\perp}^{l-1}\bar{F}_{+\perp}$. The field $D_{\perp}^{l-1}F_{+\perp}$ is then represented as a stack composed by $l+1$ roots

u_3, l u_2 and $l-1$ u_1-root solutions [42]:

$$\begin{aligned} u_{1j_1} &= u^{g[l]} + i(j_1 - \frac{l}{2}) && j_1 = 1,\ldots,l-1 \\ u_{2j_2} &= u^{g[l]} + i(j_2 - \frac{1}{2} - \frac{l}{2}) && j_2 = 1,\ldots,l \\ u_{3j_3} &= u^{g[l]} + i(j_3 - 1 - \frac{l}{2}) && j_3 = 1,\ldots,l+1 \end{aligned} \tag{3.80}$$

with the real centres $u^{g[l]}$, whereas similarly $\bar{D}_\perp^{l-1}\bar{F}_{+\perp}$ comes from the replacement of the roots u_1, u_2, u_3 with u_7, u_6, u_5, respectively (obviously fixing $l=1$ leads to $F_{+\perp}$, $\bar{F}_{+\perp}$).

Let the system be composed of N_g gluonic bound states $D_\perp^{l-1}F_{+\perp}$ (with different l) moving over the vacuum: $n^{(m)}$ of them are associated to length-m stacks with real centres $u_j^{g[m]}$ for every value of m from one to infinity, such that $\sum_{m=1}^{\infty} n^{(m)} = N_g$. These gauge fields live on a sea of covariant derivatives D_+ (embedded as u_4 roots): an infinite number of main roots fills every accessible state except for $L=2$ holes forming the vacuum along with the N_g stacks. As a matter of fact, in order to have the N_g gluonic bound states excited, a u_4 root has to be removed from the sea for each gauge field and then substituted by the centrer of the stack. The system built this way matches to a spin chain of length $L + N_g \equiv L'$ (the chain lengthens owing to the presence of the excitations), the latter being described by the Bethe equations:

- $$1 = e^{-ip_{4,k}L'} \prod_{j=1}^{s} \mathcal{S}^{(44)}(u_{4,k}, u_{4,j}) \prod_m \prod_{i=1}^{n^{(m)}} \mathcal{S}_m^{(4g)}(u_{4,k}, u_i^{g[m]}) \tag{3.81}$$
- $$1 = \prod_m \prod_{i=1}^{n^{(m)}} \mathcal{S}_{lm}^{(gg)}(u_k^{g[l]}, u_i^{g[m]}) \prod_{j=1}^{s} \mathcal{S}_l^{(g4)}(u_k^{g[l]}, u_{4,j}) \quad . \tag{3.82}$$

The last one (3.82) is obtained by multiplying among themselves the Beisert-Staudacher equations associated to every root involved in the construction of a l-stack: the fusion of $l+1$ type-3, l type-2 and $l-1$ type-1 equations, properly shifted in order to admit the roots itemized in (3.80) as their solutions, leads to the assembly of the scattering matrices for the excitations over the ferromagnetic (BMN) vacuum: $\mathcal{S}^{(44)}(u_k, u_j)$ describes the scattering involving two type-4 Bethe roots

$$\mathcal{S}^{(44)}(u_k, u_j) = \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - \frac{g^2}{2x^+(u_k)x^-(u_j)}}{1 - \frac{g^2}{2x^-(u_k)x^+(u_j)}} \right)^2 \sigma^2(u_k, u_j) \quad , \tag{3.83}$$

while, for a u_4 root colliding with a gluonic stack (with length l and real centre u_j^l) it is to be considered

$$\begin{aligned} \mathcal{S}_l^{(4g)}(u_k, u_j^{g[l]}) &= \frac{u_k - u_j^{g[l]} + i\frac{l+1}{2}}{u_k - u_j^{g[l]} - i\frac{l+1}{2}} \frac{1 - \frac{g^2}{2x^-(u_k)x(u_j^{g[l]} - i\frac{l}{2})}}{1 - \frac{g^2}{2x^+(u_k)x(u_j^{g[l]} - i\frac{l}{2})}} \frac{1 - \frac{g^2}{2x^-(u_k)x(u_j^{g[l]} + i\frac{l}{2})}}{1 - \frac{g^2}{2x^+(u_k)x(u_j^{g[l]} + i\frac{l}{2})}} = \\ &= [\mathcal{S}_l^{(g4)}(u_j^{g[l]}, u_k)]^{-1} \quad ; \end{aligned} \tag{3.84}$$

finally, the matrix for the scattering of gluonic bound states, in terms of real centres, reads

$$\mathcal{S}^{(gg)}_{lm}(u_k^{g[l]}, u_j^{g[m]}) = \frac{u_k^{g[l]} - u_j^{g[m]} - \frac{i}{2}(l+m)}{u_k^{g[l]} - u_j^{g[m]} + \frac{i}{2}(l+m)} \frac{u_k^{g[l]} - u_j^{g[m]} + \frac{i}{2}(l-m)}{u_k^{g[l]} - u_j^{g[m]} - \frac{i}{2}(l-m)} \times \times \prod_{\gamma}^{l-1} \left(\frac{u_k^{g[l]} - u_j^{g[m]} - \frac{i}{2}(l+m-2\gamma)}{u_k^{g[l]} - u_j^{g[m]} + \frac{i}{2}(l+m-2\gamma)} \right)^2 \quad (3.85)$$

Aiming to formulate a set of Bethe equations portraying the excitations over the antiferromagnetic (GKP) vacuum, the path previously outlined for the scalars will be adapted for gauge fields and counting functions will be introduced. In addition to definitions (3.4), (3.4), it is handful to define

$$\chi(v, u|l) \equiv \chi_0(v - u|l+1) + \chi_H(v, u - \frac{il}{2}) + \chi_H(v, u + \frac{il}{2}), \quad (3.86)$$

where the function χ has been torn into its one-loop and higher-loop parts:

$$\chi_0(u|l) \equiv 2 \ \arctan \frac{2u}{l} = i \ln \frac{il + 2u}{il - 2u} \quad (3.87)$$

$$\chi_H(u, v) \equiv i \ln \left(\frac{1 - \frac{g^2}{2x^-(u)x(v)}}{1 - \frac{g^2}{2x^+(u)x(v)}} \right) \quad ; \quad (3.88)$$

moreover, for later use,

$$\tilde{\chi}(u, v|l, m) \equiv \chi_0(u - v|l+m) - \chi_0(u - v|l-m) + 2 \sum_{\gamma=1}^{l-1} \chi_0(u - v|l+m-2\gamma) \quad (3.89)$$

A slight change of notations has been implicitly performed, by dropping in $u^{g[l]}$ the superscript referring to the stack length, according to $\chi(v, u^g|l) \equiv \chi(v, u^{g[l]})$. It is now straightforwrd to define the counting function pertinent to the equation (3.81), by adapting (3.31) to the case at hand, in order to include the N_g gluonic stacks:

$$Z_4(u) = (L + N_g)\Phi(u) - \sum_{j=1}^{s} \phi(u, u_j) + \sum_{k=1}^{N_g} \chi(u, u_k^g|m_k). \quad (3.90)$$

Alongside it, another counting function shall be introduced for a probe particle consisting in a gluonic stack, with real centre u and length l, which collides only with roots and gluonic excitations (centre's rapidity u_k^g, length m_k):

$$Z_g(u|l) = \sum_{k=1}^{N_g} \tilde{\chi}(u, u_k^g|l, m_k) + \sum_{j=1}^{s} \chi(u_j, u|l) \quad . \quad (3.91)$$

The Bethe equations (3.81), (3.82) thus turns into a more compact fashion:

$$(-1)^{L+s-1} = e^{iZ(u_k)} \quad (3.92)$$

$$1 = e^{iZ_g(u^g|l)} \quad . \quad (3.93)$$

When considering the high spin limit, the relation (3.17) allows to recast the counting functions into a new form, more suitable for the antiferromagnetic vacuum as the explicit dependence on s is concealed while the holes in the D_+-distribution are now manifest. In fact, the renewed counting function (3.90) arises as the solution of an integral equation:

$$\begin{aligned} Z_4(v) &= (L+N_g)\Phi(v) + \int_{-\infty}^{+\infty} \frac{dw}{2\pi}\phi(v,w)\frac{d}{dw}[Z_4(w)-2L_4(w)] + \\ &+ \sum_{h=1}^{L}\phi(v,u_h) + \sum_{k=1}^{N_g}\chi(v,u_k^g|m_k) \end{aligned} \tag{3.94}$$

Formula (3.94) suggests that $Z_4(v)$ could also be gained by solving the nonlinear integral equation

$$Z_4(v) = F_4(v) + 2\int_{-\infty}^{+\infty} dw\, G(v,w)L_4(w) \quad , \tag{3.95}$$

whose forcing term $F_4(v)$ may be decomposed according to

$$F_4(v) = (L+N_g)\tilde{P}(v) + \sum_{h=1}^{L} R(v,u_h) + \sum_{k=1}^{N_g} T(v,u_k^g|m_k) \quad , \tag{3.96}$$

where $\tilde{P}(v)$, $R(v,u)$, and $T(v,u|m)$ are the solutions respectively of (3.28), (3.27) and

$$T(v,u|m) = \chi(v,u|m) - \int_{-\infty}^{+\infty}\frac{dw}{2\pi}\left[\frac{d}{dw}\phi(v,w)\right]T(w,u|m) \quad . \tag{3.97}$$

For later purposes, it is worth plugging the expression (3.97) back into itself and iterating the procedure, so to get the formal solution:

$$T(v,u|m) = \chi(v,u|m) - \int_{-\infty}^{+\infty} dw\, G(v,w)\,\chi(w,u|m) \quad , \tag{3.98}$$

Turning now to (3.91), the relation (3.17) leads to the formula

$$Z_g(u|l) = \sum_{k=1}^{N_g}\tilde{\chi}(u,u_k^g|l,m_k) - \int_{-\infty}^{+\infty}\frac{dv}{2\pi}\chi(v,u|l)\,\frac{d}{dv}[Z_4(v)-2L_4(v)] - \sum_{h=1}^{L}\chi(u_h,u|l) \tag{3.99}$$

and eventually, after substituting the expression (3.95) for $Z_4(v)$, the gluonic counting function becomes:

$$\begin{aligned} Z_g(u|l) &= \int_{-\infty}^{\infty}\frac{dv}{\pi}\,\frac{dL_4}{dv}(v)\,T(v,u|l) - \sum_{h=1}^{L}\left[T(u_h,u|l) + \int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\tilde{P}(v)\right] + \\ &+ \sum_{k=1}^{N_g}\left[\tilde{\chi}(u,u_k^g|l,m_k) - \int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\left(T(v,u_k^g|m_k)+\tilde{P}(v)\right)\right] \end{aligned} \tag{3.100}$$

Once the numbers of excitations L and N_g are properly fixed, the scattering matrices simply stand out from (3.100), it only takes paying attention to correctly impose the unitarity.

For instance, when just two gauge fields $D_\perp^{l-1} F_{+\perp}$ (so $N_g = 2$) live over the vacuum ($L = 2$), it is straightforward to read the scattering matrix between the gluonic stacks (lengths l, m and centres u, v, respectively), taking as reference the antiferromagnetic vacuum:

$$\begin{aligned} i \ln(-S_{lm}^{(gg)}(u,u')) &= \tilde{\chi}(u,u'|l,m) - \int_{-\infty}^{\infty} \frac{dv}{2\pi}\, \chi(v,u|l)\, \frac{d}{dv} T(v,u'|m) - \\ &- \int_{-\infty}^{\infty} \frac{dv}{2\pi} \chi(v,u|l) \frac{d}{dv} \tilde{P}(v) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \chi(v,u'|m) \frac{d}{dv} \tilde{P}(v) \end{aligned} \quad (3.101)$$

Now, since the relations hold (following from (3.53) and 3.28))

$$G(u,v) = \frac{1}{2\pi} \frac{d}{dv} R(u,v) = \frac{1}{2\pi} \frac{d}{dv} \Theta(u,v) + \frac{1}{2\pi} \frac{d}{dv} \tilde{P}(v) \quad , \quad (3.102)$$

$$\frac{d}{dv} \tilde{P}(v) = \frac{d}{dv} \Phi(v) - \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left[\frac{d}{dv} \frac{d}{dw} \Theta(v,w) \right] \Phi(w) \quad , \quad (3.103)$$

with the aid of formula (3.98) the scattering phase between gluons rearranges:

$$\begin{aligned} & i \ln(-S_{lm}^{(gg)}(u,u')) = \tilde{\chi}(u,u'|l,m) + \quad (3.104) \\ & + \int \frac{dv}{2\pi} \frac{dw}{2\pi} \left[\chi(v,u|l) + \Phi(v) \right] \frac{d}{dv} \left[\frac{d}{dw} \Theta(v,w) - 2\pi \delta(v-w) \right] \left[\chi(w,u'|m) + \Phi(w) \right] \end{aligned}$$

A system with $L = 3$ and $N_g = 1$ instead is suitable for studying the scattering involving one gauge field and one internal hole. From gluonic counting function (3.100) the gluon-scalar scattering phase follows, again taking care of unitarity:

$$\begin{aligned} i \ln S^{(gs)}(u,u_h|l) &= -T(u_h,u|l) - \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \chi(w,u|l) \frac{d}{dw} \tilde{P}(w) - \tilde{P}(u_h) = \\ &= -\chi(u_h,u|l) - \Phi(u_h) + \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left[\frac{d}{dw} \Theta(u_h,w) \right] (\chi(w,u|l) + \Phi(w)) = \\ &= \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left[\frac{d}{dw} \Theta(u_h,w) - 2\pi \delta(u_h - w) \right] (\chi(w,u|l) + \Phi(w)) \quad . \end{aligned} \quad (3.105)$$

On the other hand, from the scalar counting function (3.94) the scalar-gluon scattering phase is found, in the form expected:

$$\begin{aligned} i \ln S^{(sg)}(u_h,u|l) &= T(u_h,u|l) + \tilde{P}(u_h) + \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \chi(w,u|l) \frac{d}{dw} \tilde{P}(w) = \quad (3.106) \\ &= -\int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left[\frac{d}{dw} \Theta(u_h,w) - 2\pi \delta(u_h - w) \right] (\chi(w,u|l) + \Phi(w)) \ . \end{aligned}$$

Once the scattering phases between gluons and between gluons and holes have been singled out, the remaining terms in (3.99):

$$\begin{aligned} \Lambda_g(u) &= T(u_1,u|l) + T(u_L,u|l) - \int_{-\infty}^{\infty} \frac{dv}{\pi} \frac{dL_4}{dv}(v)\, T(v,u|l) + \quad (3.107) \\ &+ 2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \chi(v,u|l) \frac{d}{dv} \tilde{P}(v) + \sum_{k=1}^{N_g} \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \chi(v,u_k^g|m_k) \frac{d}{dv} \tilde{P}(v) - \sum_{h=2}^{L-1} \tilde{P}(u_h) \end{aligned}$$

In order to complete the bestiary of matrices which involve gauge fields, the $\bar{D}_\perp \bar{F}_{+\perp}$ fields have to be taken into account, too. A system composed of L (internal and external) holes, N_g bound states $D_\perp F_{+\perp}$ and $N_{\bar{g}}$ bound states $\bar{D}_\perp \bar{F}_{+\perp}$ is determined in the large spin regime by three counting functions:

$$\begin{aligned}
Z_4(v) &= (L+N_g+N_{\bar{g}})\Phi(v) + \int_{-\infty}^{+\infty} \frac{dw}{2\pi}\phi(v,w)\frac{d}{dw}[Z_4(w)-2L_4(w)] + \\
&+ \sum_{h=1}^{L}\phi(v,u_h) + \sum_{k=1}^{N_g}\chi(v,u_k^g|m_k) + \sum_{j=1}^{N_{\bar{g}}}\chi(v,u_j^{\bar{g}}|q_j) \\
Z_g(u|l) &= \int_{-\infty}^{\infty}\frac{dv}{\pi}\frac{dL_4}{dv}(v)\,T(v,u|l) - \sum_{h=1}^{L}\left[T(u_h,u|l) + \int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\tilde{P}(v)\right] + \\
&+ \sum_{k=1}^{N_g}\left[\tilde{\chi}(u,u_k^g|l,m_k) - \int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\Big(T(v,u_k^g|m_k)+\tilde{P}(v)\Big)\right] - \\
&- \sum_{j=1}^{N_{\bar{g}}}\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\Big(T(v,u_j^{\bar{g}}|q_j)+\tilde{P}(v)\Big) \qquad (3.108) \\
Z_{\bar{g}}(u|l) &= \int_{-\infty}^{\infty}\frac{dv}{\pi}\frac{dL_4}{dv}(v)\,T(v,u|l) - \sum_{h=1}^{L}\left[T(u_h,u|l) + \int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\tilde{P}(v)\right] + \\
&+ \sum_{j=1}^{N_{\bar{g}}}\left[\tilde{\chi}(u,u_j^{\bar{g}}|l,q_j) - \int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\Big(T(v,u_j^{\bar{g}}|q_j)+\tilde{P}(v)\Big)\right] - \\
&- \sum_{k=1}^{N_g}\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\chi(v,u|l)\,\frac{d}{dv}\Big(T(v,u_k^g|m_k)+\tilde{P}(v)\Big) \quad .
\end{aligned}$$

By restricting the system above to just one $D_\perp F_{+\perp}$ and one $\bar{D}_\perp \bar{F}_{+\perp}$ over the vacuum (so $N_g = N_{\bar{g}} =$ and $L=2$), the matrix $S_{lm}^{(g\bar{g})}$ is thus obtained:

$$\begin{aligned}
i\ln S_{lm}^{(g\bar{g})}(u^g,t^{\bar{g}}) &= \int_{-\infty}^{+\infty}\frac{dv\,dw}{4\pi^2}[\chi(v,u^g|l)+\Phi(v)]\frac{d}{dv}\left[\frac{d}{dw}\Theta(v,w)-2\pi\delta(v-w)\right]\times \\
&\times\ [\chi(w,t^{\bar{g}}|m)+\Phi(w)] \qquad (3.109)
\end{aligned}$$

No surprises in discovering that missing matrices to be computed are $S^{(s\bar{g})}(u,v|l) = S^{(sg)}(u,v|l)$ and $S_{lm}^{(\bar{g}\bar{g})}(u,v) = S_{lm}^{(gg)}(u,v)$, owing to the symmetry $(u_1,u_2,u_3) \longrightarrow (u_7,u_6,u_5)$.

Actually, since most of the interest of this work addresses to gluons rather than to bound states, the length $l=1$ stacks (gluons, as already explained) deserve to be focused on and displayed all together. The previously retrieved scattering matrices (3.104), (3.105), (3.109) then specialise to the case at hand (with a slight change of notations about stack-length subscripts):

$$i\ln S^{(gg)}(u,u') \equiv i\ln S_{11}^{(gg)}(u,u') = i\ln\frac{u-u'+i}{u-u'-i} + \qquad (3.110)$$

$$+\int_{-\infty}^{+\infty}\frac{dv}{2\pi}\frac{dw}{2\pi}\,[\chi(v,u|l)+\Phi(v)]\frac{d}{dv}\left[\frac{d}{dw}\Theta(v,w)-2\pi\delta(v-w)\right][\chi(w,u'|m)+\Phi(w)]$$

$$i\ln S^{(gs)}(u,u_h) \equiv i\ln S^{(gs)}(u,u_h|1) = i\ln S^{(\bar{g}s)}(u,u_h|1) =$$
$$= \int_{-\infty}^{+\infty}\frac{dw}{2\pi}\left[\frac{d}{dw}\Theta(u_h,w) - 2\pi\delta(u_h-w)\right](\chi(w,u|1)+\Phi(w)) \tag{3.111}$$
$$S^{(g\bar{g})}(u,u') = S^{(gg)}(u,u')\,\frac{u-u'-i}{u-u'+i} \tag{3.112}$$
$$S^{(\bar{g}g)}(u,u') = [S^{(g\bar{g})}(u',u)]^{-1} \quad . \tag{3.113}$$

5.1 One-loop

When $g=0$, the system (3.108), in Fourier transform, simplifies to:

$$\hat{Z}_{4,0}(k) = -\frac{2\pi(L+N_g+N_{\bar{g}})}{ik}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}} - \frac{2\,e^{-|k|}}{1-e^{-|k|}}\hat{L}_{4,0}(k) + \tag{3.114}$$
$$+\sum_{h=1}^{L}\frac{2\pi}{ik}\,e^{-iku_h}\,\frac{e^{-|k|}}{1-e^{-|k|}} + 2\pi\sum_{i=1}^{N_g}\frac{e^{-iku_i^g}}{ik}\frac{e^{-|k|\frac{m_i+1}{2}}}{1-e^{-|k|}} + 2\pi\sum_{j=1}^{N_{\bar{g}}}\frac{e^{-iku_j^{\bar{g}}}}{ik}\frac{e^{-|k|\frac{q_j+1}{2}}}{1-e^{-|k|}}$$

for the type-4 counting function,

$$\frac{\hat{Z}_{0,g}(k|l)}{2\pi} = -\frac{1}{ik}\frac{(L+N_g+N_{\bar{g}})\,e^{-\frac{|k|}{2}(l+2)}}{1-e^{-|k|}} + \sum_{h=1}^{L}e^{-iku_h}\frac{1}{ik}\frac{e^{-\frac{|k|}{2}(l+1)}}{1-e^{-|k|}} + \tag{3.115}$$
$$+\sum_{i=1}^{N_g}\frac{e^{-iku_i^g}}{ik}\left[e^{-\frac{|k|}{2}(l+m_i)} - e^{-\frac{|k|}{2}(l-m_i)} + 2\sum_{\gamma=1}^{l-1}e^{-\frac{|k|}{2}(l+m_i-2\gamma)} + \frac{e^{-\frac{|k|}{2}(l+m_i+2)}}{1-e^{-|k|}}\right] +$$
$$+\sum_{j=1}^{N_{\bar{g}}}\frac{e^{-iku_j^{\bar{g}}}}{ik}\frac{e^{-\frac{|k|}{2}(l+q_j+2)}}{1-e^{-|k|}} - \frac{\hat{L}_{4,0}(k)}{\pi}\frac{e^{-\frac{|k|}{2}(l+1)}}{1-e^{-|k|}}$$

for the gauge fields, while for the barred-gauge fields

$$\frac{\hat{Z}_{0,\bar{g}}(k|l)}{2\pi} = -\frac{1}{ik}\frac{(L+N_g+N_{\bar{g}})\,e^{-\frac{|k|}{2}(l+2)}}{1-e^{-|k|}} + \sum_{h=1}^{L}e^{-iku_h}\frac{1}{ik}\frac{e^{-\frac{|k|}{2}(l+1)}}{1-e^{-|k|}} + \tag{3.116}$$
$$- \frac{\hat{L}_{4,0}(k)}{\pi}\frac{e^{-\frac{|k|}{2}(l+1)}}{1-e^{-|k|}} + \sum_{i=1}^{N_g}\frac{e^{-iku_i^g}}{ik}\frac{e^{-\frac{|k|}{2}(l+m_i+2)}}{1-e^{-|k|}} +$$
$$+ \sum_{j=1}^{N_{\bar{g}}}\frac{e^{-iku_j^{\bar{g}}}}{ik}\left[e^{-\frac{|k|}{2}(l+q_j)} - e^{-\frac{|k|}{2}(l-q_j)} + 2\sum_{\gamma=1}^{l-1}e^{-\frac{|k|}{2}(l+q_j-2\gamma)} + \frac{e^{-\frac{|k|}{2}(l+q_j+2)}}{1-e^{-|k|}}\right]$$

It is now very simple to distinguish the one loop scattering matrices. For instance, the exponentiation of the formula (3.115) results in

$$e^{-iZ_g(u|l)} = \left[\frac{\Gamma\left(\frac{l+2}{2}+iu\right)}{\Gamma\left(\frac{l+2}{2}-iu\right)}\right]^{(L+N_g+N_{\bar{g}})} e^{\frac{1}{\pi}\int dv L_{4,0}(v)\left[\psi\left(\frac{l+1}{2}+i(u-v)\right)+\psi\left(\frac{l+1}{2}-i(u-v)\right)\right]} \times$$
$$\times\prod_{h=1}^{L}\left(\frac{\Gamma\left(\frac{l+1}{2}-i(u-x_h)\right)}{\Gamma\left(\frac{l+1}{2}+i(u-x_h)\right)}\right)\prod_{j=1}^{N_{\bar{g}}}\left(\frac{\Gamma\left(1+\frac{l+q_j}{2}-i(u-u_j^{\bar{g}})\right)}{\Gamma\left(1+\frac{l+q_j}{2}+i(u-u_j^{\bar{g}})\right)}\right)\times$$
$$+\prod_{k=1}^{N_g}\left[\frac{1+i\frac{2(u-u_k^g)}{l+m_k}}{1-i\frac{2(u-u_k^g)}{l+m_k}}\,\frac{1-i\frac{2(u-u_k^g)}{l-m_k}}{1+i\frac{2(u-u_k^g)}{l-m_k}}\prod_{\gamma=1}^{l-1}\left(\frac{1+i\frac{2(u-u_k^g)}{l+m_k-2\gamma}}{1-i\frac{2(u-u_k^g)}{l+m_k-2\gamma}}\right)^2\frac{\Gamma\left(\frac{2+l+m_k}{2}-i(u-u_k^g)\right)}{\Gamma\left(\frac{2+l+m_k}{2}+i(u-u_k^g)\right)}\right]$$

so that for the scattering between a gluon stack (rapidity u and length l) and a hole with rapidity u_h, it holds:

$$S_l^{(gs)}(u,u_h)=\frac{\Gamma\left(\frac{l+1}{2}+i(u-u_h)\right)}{\Gamma\left(\frac{l+1}{2}-i(u-u_h)\right)}\frac{\Gamma\left(\frac{l+2}{2}-iu\right)}{\Gamma\left(\frac{l+2}{2}+iu\right)}\frac{\Gamma\left(\frac{1}{2}+iu_h\right)}{\Gamma\left(\frac{1}{2}-iu_h\right)} \tag{3.117}$$

In the same way, a process of scattering involving two gluons, with rapidities respectively $u,\ u'$ and lengths $l,\ m$ is associated to:

$$S_{lm}^{(gg)}(u,u')=-\frac{u-u'+\frac{i}{2}(l+m)}{u-u'-\frac{i}{2}(l+m)}\frac{u-u'+\frac{i}{2}|l-m|}{u-u'-\frac{i}{2}|l-m|}\frac{\Gamma\left(1+\frac{l+m}{2}+i(u-u')\right)}{\Gamma\left(1+\frac{l+m}{2}-i(u-u')\right)}\times$$
$$\times\frac{\Gamma\left(\frac{l+2}{2}-iu\right)}{\Gamma\left(\frac{l+2}{2}+iu\right)}\frac{\Gamma\left(\frac{m+2}{2}+iu'\right)}{\Gamma\left(\frac{m+2}{2}-iu'\right)}\prod_{\gamma=1}^{min(l,m)-1}\left(\frac{u-u'+\frac{i}{2}(|l-m|+2\gamma)}{u-u'-\frac{i}{2}(|l-m|+2\gamma)}\right)^2 \tag{3.118}$$

For the missing matrices involving barred-fields, the very same relations (3.111) stay untouched.

Although the computation of energy and momentum of an excitation will be treated with more detail and generality in the next chapter, the one loop case offers the chance to face the method to be used with a heuristic sight, so to highlight a physical interpretation underneath. Let us take into exam, for instance, a system containing only one gluonic stack, with rapidity u^g and length l. The introduction of such an excitation thus induces an alteration on the density of the sea of covariant derivatives D_+ forming the vacuum. As first step towards estimating the shuffle occurred to the covariant derivative distribution, it is useful to adapt the the main root counting function (in Fourier space (3.114)) to the case considered, that is by setting $N_g=1$, $L=2$ (accounting for the two external holes, with rapidities u_1 and u_2) and $N_{\bar{g}}=0$:

$$\begin{aligned}\hat{Z}_{4,0}^{(1g)}(k;u^g) &= -\frac{6\pi}{ik}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}}-\frac{2\,e^{-|k|}}{1-e^{-|k|}}\hat{L}_{4,0}^{(1g)}(k)+\\ &+\frac{2\pi}{ik}\frac{e^{-|k|}}{1-e^{-|k|}}(e^{-iku_1}+e^{-iku_2})+e^{-iku^g}\frac{2\pi}{ik}\frac{e^{-|k|\frac{l+1}{2}}}{1-e^{-|k|}}\end{aligned} \tag{3.119}$$

with the superscript $(1g)$ to distinguish the 'one gluon' case function from the vacuum (v) $(N_g=0)$:

$$\hat{Z}_{4,0}^{(v)}(k)=-\frac{4\pi}{ik}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}}-\frac{2\,e^{-|k|}}{1-e^{-|k|}}\hat{L}_{4,0}^{(v)}(k)+\frac{2\pi}{ik}\frac{e^{-|k|}}{1-e^{-|k|}}(e^{-iku_1}+e^{-iku_2}) \tag{3.120}$$

The introduction of the gluonic stack affects the root distribution, so that the root density turns from a vacuum (one-loop) value $\sigma_0^{(v)}(u)\equiv\frac{dZ_{4,0}^{(v)}(u)}{du}$ to a new one $\sigma_0^{(1g)}(u;u^g)\equiv\frac{dZ_{4,0}^{(1g)}(u;u^g)}{du}$. We can estimate then such a variation by means of the one-loop fluctuation density:

$$\begin{aligned}\tau_0(u,u^g) &\equiv \frac{\sigma_0^{(1g)}(u;u^g)-\sigma_0^{(v)}(u)}{2\pi}=\\ &= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{n+\frac{l+1}{2}}{(u-u^g)^2+(n+\frac{l+1}{2})^2}-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{u^2+(n+\frac{1}{2})^2}\end{aligned} \tag{3.121}$$

recalling that in the $s \to \infty$ limit, non linear terms are negligeable. As a first instance, the density fluctuation allows to compute the (one-loop) energy of the excitation. The observable $e(u) = \dfrac{1}{u^2 + \frac{1}{4}}$ measures the energy associated to a Bethe root with rapidity u, and it is now appropriate to remark that only the type-4 roots carry energy and momentum: all the particles acquire some amount of these charges after interacting with u_4 roots. Therefore, in order to compute the total energy of the system, it takes to evaluate $e(u)$ by summing upon every $u_{4,k}$, and this operation, under the large s limit, turns to an integration on the rapidity. Hence, a way to measure the energy of an excitation consists in computing the variation the u_4 roots experience after the introduction of the particle. Explicitly, for the single gauge field system, it holds:

$$\begin{aligned} E_0(u^g) &= \int_{-\infty}^{\infty} \frac{\sigma_0^{(1g)}(u;u^g)}{u^2+\frac{1}{4}} du - \int_{-\infty}^{\infty} \frac{\sigma_0^{(v)}(u)}{u^2+\frac{1}{4}} du = \int_{-\infty}^{\infty} \frac{\tau_0(u,u^g)}{u^2+\frac{1}{4}} du = \\ &= 2\Psi(1) - \Psi(\frac{l}{2}+1-iu^g) - \Psi(\frac{l}{2}+1+iu^g) \quad , \end{aligned} \tag{3.122}$$

and that matches the result by Basso [42].
Moreover, the density $\tau_0(u, u^g)$ also provides a manner to calculate the momentum carried by the excitation: upon singling out its odd part, $\tau_0^{odd}(u, u^g) \equiv \frac{1}{2}\Big[\tau_0(u, u^g) - \tau_0(-u, u^g)\Big]$, it can be found:

$$\begin{aligned} \tau_0^{odd}(u,u^g) = \frac{1}{4\pi}\Bigg[&\Psi\left(\frac{l+1}{2}+i(u+u^g)\right) + \Psi\left(\frac{l+1}{2}-i(u+u^g)\right) - \\ &- \Psi\left(\frac{l+1}{2}+i(u-u^g)\right) - \Psi\left(\frac{l+1}{2}-i(u-u^g)\right)\Bigg] \end{aligned} \tag{3.123}$$

The digamma function ($\Psi(z) \equiv \frac{d}{dz}\ln\Gamma(z)$) for large z behaves according to following asymptotic series:

$$\Psi(z) \sim \ln\, z - \frac{1}{2z} + \mathcal{O}(z^{-2}) \tag{3.124}$$

So, for large rapidity u, it is verified that

$$\tau_0^{odd}(u,u^g) \sim \frac{u^g}{\pi u} \quad : \tag{3.125}$$

that is exactly the relation (4.3) of [42] (at one-loop), relating the momentum of the excitation to the odd part of density fluctuation

$$P(u^g) = 2\pi u \tau_0^{odd}(u,u^g) = 2u^g \tag{3.126}$$

6 Fermionic Excitations

To parametrise the dynamics of a fermionic excitation, the rapidity to look at is actually x, which is related to the Bethe rapidity u via the Jukovski map $u(x) = x + \frac{g^2}{2x}$. To properly invert the Jukovski map, in order to cover the whole range of values of x, two u-planes must be glued together, each one of them being a Riemann sheet. The two sheets are related to two distinct regimes of the fermionic excitations [42]: large fermions, embedded in Beisert-Staudacher equations [30] as u_3 roots, which do carry energy and momentum even at one-loop; small fermions, corresponding to u_1 roots, which couple to main root equations just at higher loops. The function $x(u)$ can be analytically continued from the u_3 Riemann sheet to the u_1-sheet by means of the map $x(u_3) \longrightarrow (g^2)/(2x(u_1))$: remarkably the Beisert-Staudacher equations are invariant under this exchange $u_3 \leftrightarrow u_1$, provided the spin-chain length increases (see [30] for details). Exactly the same reasoning holds for anti-fermions, after the replacements $u_3 \to u_5$ and $u_1 \to u_7$: turning on u_3 (u_1) roots means exciting fermionic fields Ψ_+, while u_5 (u_7) corresponds to $\bar{\Psi}_+$. After pointing out these remarks, the concern addresses to a system made of N_F large fermions $u_F = u_3$, of physical rapidities $x_{F,j} = x(u_{F,j})$ with the arithmetic square root for $x(u) = (u/2)\left[1 + \sqrt{1-(2g^2)/u^2}\right]$, and N_f small fermions $u_f = u_1$, of rapidities $x_{f,j} = (g^2)/(2x(u_{f,j}))$, over the sea of u_4 roots; this system is then portrayed by the following Bethe equations:

- $$1 = e^{-ip_k L} \prod_{j\neq k}^{s} \mathcal{S}^{(44)}(u_{4,k}, u_{4,j}) \prod_{j=1}^{N_F} \mathcal{S}^{(4F)}(u_{4,k}, u_{F,j}) \prod_{j=1}^{N_f} \mathcal{S}^{(4f)}(u_{4,k}, u_{f,j}) \tag{3.127}$$
- $$1 = \prod_{j=1}^{s} \mathcal{S}^{(F4)}(u_{F,k}, u_{4,j}) \tag{3.128}$$
- $$1 = \prod_{j=1}^{s} \mathcal{S}^{(f4)}(u_{f,k}, u_{4,j}) \tag{3.129}$$

where, in addition to previously defined matrices, the scalar-fermion scattering phases have been introduced:

$$\mathcal{S}^{(F4)}(u_{F,k}, u_{4,j}) = \frac{u_{F,k} - u_{4,j} + \frac{i}{2}}{u_{F,k} - u_{4,j} - \frac{i}{2}} \left(\frac{1 - \frac{g^2}{2x_{F,k}x_j^+}}{1 - \frac{g^2}{2x_{F,k}x_j^-}} \right) = \left[\mathcal{S}^{(4F)}(u_{4,j}, u_{F,k})\right]^{-1} \tag{3.130}$$

for large fermions, while small fermions are associated to

$$\mathcal{S}^{(f4)}(u_{f,k}, u_{4,j}) = \frac{1 - \frac{g^2}{2x_{f,k}x_j^-}}{1 - \frac{g^2}{2x_{f,k}x_j^+}} = \left[\mathcal{S}^{(4f)}(u_{4,j}, u_{f,k})\right]^{-1} \quad . \tag{3.131}$$

Now that all elements have been set up, the search for scattering matrices follows the usual path. The first step consists in writing the counting functions, for u_4 roots, large and small fermions, namely:

- $$Z_4(u) = (L + N_F + N_f)\,\Phi(u) - \sum_{j=1}^{s} \phi(u, u_{4,j}) + \sum_{j=1}^{N_F} \chi_F(u, u_{F,j}) - \sum_{j=1}^{N_f} \chi_H(u, u_{f,j})$$
- $$Z_F(u) = \sum_{j=1}^{s} \chi_F(u_{4,j}, u) \tag{3.132}$$
- $$Z_f(u) = -\sum_{j=1}^{s} \chi_H(u_{4,j}, u)$$

where for concision's sake it has been defined $\chi_F(v,u) \equiv \chi_0(v-u|1) + \chi_H(v,u)$. The very same procedure, adopted so far for the other kinds of particles, allows to formulate integral equations whose solutions are the counting functions (3.132). To begin with, the first one of (3.132) becomes the solution of

$$Z_4(u) = (L + N_F + N_f)\,\Phi(u) + \sum_{j=1}^{N_F} \chi_F(u, u_{F,j}) - \sum_{j=1}^{N_f} \chi_H(u, u_{f,j}) + \\ + \int \frac{dv}{2\pi}\,\phi(u,v)\,\frac{d}{dv}\,[Z_4(v) - 2L_4(v)] + \sum_{h=1}^{L} \phi(u, x_h) \qquad : \tag{3.133}$$

the shape of this equation entails that $Z_4(v)$ takes should read as

$$Z_4(v) = (L+N_F+N_f)\tilde{P}(v) + \sum_{h=1}^{L} R(v,u_h) + \sum_{k=1}^{N_F} F_F(v,u_{F,k}) + \sum_{k=1}^{N_f} F_f(v,u_{f,k}) \tag{3.134}$$

where the functions $F_F(v,u)$ and $F_f(v,u)$ solve the equations

$$\begin{aligned} F_F(u,u^F) &= \chi_F(u,u^F) - \int dv\,\varphi(u,v)\,F_F(v,u^F) \\ F_f(u,u^f) &= -\chi_H(u,u^f) - \int dv\,\varphi(u,v)\,F_f(v,u^f) \qquad , \end{aligned} \tag{3.135}$$

which, in addition, admit as formal solutions

$$\begin{aligned} F_F(u,u^F) &= \chi_F(u,u^F) - \int dv\,G(u,v)\,\chi_F(u,u^F) \\ F_f(u,u^f) &= -\chi_H(u,u^f) + \int dv\,G(u,v)\,\chi_H(v,u^f) \qquad . \end{aligned} \tag{3.136}$$

As usual, after the high spin limit has been taken and the formula (3.17) applied, the remaining two relations in (3.132) are left associated to two more equations, suitable for describing fermionic counting functions on the antiferromagnetic vacuum:

$$\begin{aligned} Z_F(u) &= -\int \frac{dv}{2\pi}\,\chi_F(v,u)\,\frac{d}{dv}[Z_4(v) - 2L_4(v)] - \sum_{h=1}^{L} \chi_F(u_h,u) \qquad (3.137) \\ Z_f(u) &= \int \frac{dv}{2\pi}\,\chi_H(v,u)\,\frac{d}{dv}[Z_4(v) - 2L_4(v)] + \sum_{h=1}^{L} \chi_H(u_h,u) \end{aligned}$$

The experience gained in earlier study on scalars and gauge fields, this time helps in surveying the behave of both large and small fermionic excitations. Hence the matrices regulating the scattering processes which involve fermions are thus gathered below:

- large fermion-large fermion

$$\begin{aligned} i\log S^{(FF)}(u,u') &= \int \frac{dvdw}{(2\pi)^2}\,[\chi_F(v,u) + \Phi(v)]\frac{d}{dv}\left(\frac{d\Theta(v,w)}{dw} - 2\pi\delta(v-w)\right) \\ &\times\ [\chi_F(w,u') + \Phi(w)] \end{aligned} \tag{3.138}$$

• small fermion-small fermion

$$i \log S^{(ff)}(u,u') = \int \frac{dv\, dw}{(2\pi)^2}\, \chi_H(v,u) \frac{d}{dv} \left(\frac{d}{dw}\Theta(v,w) - 2\pi\delta(v-w) \right) \chi_H(w,u') \tag{3.139}$$

• small fermion-large fermion

$$\begin{aligned} & i \log S^{(fF)}(u,u') = -i \log S^{(Ff)}(u',u) = \\ & = -\int \frac{dv}{2\pi}\frac{dw}{2\pi}\, \chi_H(v,u)\, \frac{d}{dv} \left(\frac{d}{dw}\Theta(v,w) - 2\pi\delta(v-w) \right) \big[\chi_F(w,u') + \Phi(w)\big] \end{aligned} \tag{3.140}$$

• scalar-small fermion

$$\begin{aligned} & i \log S^{(sf)}(u,u') = F_f(u,u') + \tilde{P}(u) - \int \frac{dv}{2\pi}\, \chi_H(v,u')\, \frac{d}{dv}\tilde{P}(v) = \\ & = \int \frac{dv}{2\pi} \left(\frac{d}{dv}\Theta(u,v) - 2\pi\delta(u-v) \right) \chi_H(v,u') = -i \log S^{(fs)}(u',u) \end{aligned} \tag{3.141}$$

• scalar-large fermion

$$\begin{aligned} & i \log S^{(sF)}(u,u') = F_F(u,u') + \tilde{P}(u) + \int \frac{dv}{2\pi}\, \big(\chi_0(v-u') + \chi_H(v,u')\big)\, \frac{d}{dv}\tilde{P}(v) = \\ & = -\int \frac{dv}{2\pi} \left(\frac{d}{dv}\Theta(u,v) - 2\pi\delta(u-v) \right) \big[\chi_F(v,u') + \Phi(v)\big] = -i \log S^{(Fs)}(u',u) \end{aligned} \tag{3.142}$$

As a general feature, the S-matrices involving small fermions are achieved from the corresponding ones for large fermions by means of the plain replacement

$$\chi_F(v,u) + \Phi(v) \longrightarrow -\chi_H(v,u) \quad ,$$

due to the $u_3 \leftrightarrow u_1$ duality residing in the Beisert-Staudacher equations, as already discussed in the beginning of the present section.

Anyway the set of scattering phases for fermionic excitations has not been accomplished yet: in fact, the matrices accounting of antifermions, both large and small, are still missing. Nevertheless it is not complicate to enhance the system (3.132) so that with the antifermion terms, because of the duality amongst $u_3 \leftrightarrow u_5$ and $u_1 \leftrightarrow u_7$ roots: it just takes to supplement (3.132) with two more counting functions, with the purpose of taking care of antifermions, namely:

$$\bullet \quad Z_{\bar{F}}(u) = \sum_{j=1}^{s} \chi_F(u_{4,j},u) \tag{3.143}$$

$$\bullet \quad Z_{\bar{f}}(u) = -\sum_{j=1}^{s} \chi_H(u_{4,j},u)$$

To be noticed: the counting functions $Z_{\bar{F}}(u)$ and $Z_{\bar{f}}(u)$ for antifermions do not differ from $Z_F(u)$, $Z_f(u)$, and so neither the corresponding NLIE do diversify (in shape) from (3.137). Since at the level of Beisert-Staudacher equations the u_1, u_3, u_5, u_7 roots do not interact among themselves, the only discrepancy, arising from the addition of antifermions, occurs in the u_4 counting function $Z_4(u)$, which gets modified by the presence of new terms which explicitly recall the (pseudo) rapidities of those particles:

$$\begin{aligned} Z_4(u) &= (L + N_F + N_f)\,\Phi(u) - \sum_{j=1}^{s} \phi(u, u_{4,j}) + \sum_{j=1}^{N_F} \chi_F(u, u_{F,j}) - \sum_{j=1}^{N_f} \chi_H(u, u_{f,j}) \\ &+ \sum_{j=1}^{N_{\bar F}} \chi_F(u, u_{\bar F,j}) - \sum_{j=1}^{N_{\bar f}} \chi_H(u, u_{\bar f,j}) \end{aligned} \tag{3.144}$$

Therefore it is a straightforward task to show that the there is no distinction between fermions and antifermions, as long as the scattering phases are considered, so that:

$$\begin{aligned} S^{(\bar F\bar F)}(u,u') = S^{(\bar F F)}(u,u') &= S^{(F\bar F)}(u,u') = S^{(FF)}(u,u') \\ S^{(s\bar F)}(u,u') &= S^{(sF)}(u,u') \quad ; \end{aligned} \tag{3.145}$$

also, the same relations hold for small fermions too. Though, that 'conjugation independence' of the scattering theory stays no longer valid when the system also includes gauge and barred-gauge fields, as a novel term appears both in gluonic and (anti)fermionic counting functions in order to take into account the gluon-(anti)fermion interaction, already present in the Beisert-Staudacher equations. For sake of clarity, it is useful to report the fermionic counting functions for a system composed of fermions, antifermions and gauge bound states:

$$Z_F(u) = \sum_{j=1}^{s} \chi_F(u_{4,j}, u) + \sum_{j=1}^{N_g} \chi_0(u - u_j^g | m_j) \tag{3.146}$$

$$Z_f(u) = -\sum_{j=1}^{s} \chi_H(u_{4,j}, u) + \sum_{j=1}^{N_g} \chi_0(u - u_j^g | m_j)$$

$$Z_{\bar F}(u) = \sum_{j=1}^{s} \chi_F(u_{4,j}, u) + \sum_{j=1}^{N_{\bar g}} \chi_0(u - u_j^{\bar g} | \bar m_j)$$

$$Z_{\bar f}(u) = -\sum_{j=1}^{s} \chi_H(u_{4,j}, u) + \sum_{j=1}^{N_{\bar g}} \chi_0(u - u_j^{\bar g} | \bar m_j) \quad ;$$

please mind that gluonic bound states do not mix with antifermions, nor barred-gluonic bound states do with fermions. In the end, the usual procedure leads to the following matrices:

$$S^{(gF)}(u,u') = [S^{(Fg)}(u',u)]^{-1} = \frac{u-u'+\frac{il}{2}}{u-u'-\frac{il}{2}} \exp\left\{ i \int \frac{dv\,dw}{4\pi^2} [\chi(v,u|l)+\Phi(v)] \times \right.$$
$$\left. \times \frac{d}{dv}\left(2\pi\delta(v-w) - \frac{d\Theta(v,w)}{dw}\right) [\chi_F(w,u')+\Phi(w)] \right\} \tag{3.147}$$
$$S^{(gF)}(u,v) = S^{(\bar{g}\bar{F})}(u,v) \tag{3.148}$$
$$S^{(\bar{g}F)}(u,v) = [S^{(F\bar{g})}(v,u)]^{-1} = \exp\left\{ i \int \frac{dv\,dw}{4\pi^2} [\chi(v,u|l)+\Phi(v)] \times \right. \tag{3.149}$$
$$\left. \times \frac{d}{dv}\left(2\pi\delta(v-w) - \frac{d\Theta(v,w)}{dw}\right) [\chi_F(w,u')+\Phi(w)] \right\}$$
$$S^{(g\bar{F})}(u,v) = [S^{(\bar{F}g)}(v,u)]^{-1} = S^{(\bar{g}F)}(u,v) \tag{3.150}$$

6.1 One-loop

At one loop, the counting function for a system composed of L holes (internal and external)and N_F large fermions (recalling that small fermions count just at higher loops) reads, upon exponentiating:

$$e^{-iZ_{0,F}(u)} = \left[\frac{\Gamma(1+iu)}{\Gamma(1-iu)}\right]^{L+N_F} e^{\frac{i}{\pi}\int L_{F,0}\left[\psi(\frac{1}{2}+i(u-v))+\psi(\frac{1}{2}-i(u-v))\right]dv} \times$$
$$\times \prod_{h=1}^{L}\left(\frac{\Gamma(\frac{1}{2}-i(u-u_h))}{\Gamma(\frac{1}{2}+i(u-u_h))}\right)\prod_{j=1}^{N_F}\left(\frac{\Gamma(1-i(u-u_{F,j}))}{\Gamma(1+i(u-u_{F,j}))}\right) \tag{3.151}$$

From (3.151), it is simple to single out the matrices describing the (one-loop) scattering involving large fermions

$$S^{(FF)}(u,u_F) = \frac{\Gamma(1+i(u-u_F))}{\Gamma(1-i(u-u_F))}\,\frac{\Gamma(1-iu)}{\Gamma(1+iu)}\,\frac{\Gamma(1+iu_F)}{\Gamma(1-iu_F)} \tag{3.152}$$

and between a fermion and a scalar excitation (hole)

$$S^{(Fs)}(u,u_h) = \frac{\Gamma(\frac{1}{2}+i(u-u_h))}{\Gamma(\frac{1}{2}-i(u-u_h))}\,\frac{\Gamma(1-iu)}{\Gamma(1+iu)}\,\frac{\Gamma(\frac{1}{2}+iu_h))}{\Gamma(\frac{1}{2}-iu_h))} \tag{3.153}$$

The scattering phases for the antifermions can be deduced from the matrices above, according to relations (3.145).

Paralleling what has already been carried out for gauge fields, the energy and momentum of a (large) fermion can be computed by studying the variation of the u_4 distribution. As already seen, formula (3.120) for the vacuum gets modified by the insertion of a large fermion (associated to u_F):

$$\hat{Z}^{(1F)}_{4,0}(k) = -\frac{6\pi}{ik}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}} + \frac{2\pi}{ik}e^{-iku_F}\frac{e^{-\frac{|k|}{2}}}{1-e^{-|k|}} - 2\hat{L}_{4,0}(k)\frac{e^{-|k|}}{1-e^{-|k|}} + \sum_{h=1}^{2}\frac{2\pi}{ik}e^{-iku_h}\frac{e^{-|k|}}{1-e^{-|k|}} \quad ; \tag{3.154}$$

As already pointed out, the fermion may be thought as a missing root in the covariant derivative sea distribution. Such a variation reflects in a change of root density ($\sigma_0^{(1F)}(u;u_F) = \frac{dZ_{4,0}^{(1F)}(u;u_F)}{du}$), which involves the density fluctuation

$$\begin{aligned} \tau_0^{(1F)}(u,u_F) &= \frac{\sigma_0^{(1F)}(u,u_F) - \sigma_0^{(v)}(u)}{2\pi} = \\ &= -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{u^2+(n+\frac{1}{2})^2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{(u-u_F)^2+(n+\frac{1}{2})^2} \quad , \end{aligned} \tag{3.155}$$

The value of the momentum at one loop for the fermion stems from the exam of the odd part of the density fluctuation [42]

$$\begin{aligned} \tau_0^{odd}(u,u_F) = \frac{1}{4\pi}\Big[&\Psi(\frac{1}{2}+i(u+u_F)) + \Psi(\frac{1}{2}-i(u+u_F)) - \\ &-\Psi(\frac{1}{2}+i(u-u_F)) - \Psi(\frac{1}{2}-i(u-u_F))\Big] \end{aligned} \tag{3.156}$$

which, by means of the $u \gg 1$ expansion $\tau_0^{odd}(u,u_F) \sim \frac{u_F}{\pi u} + \mathcal{O}\left(\frac{1}{u^2}\right)$, results in:

$$p(u_F) = 2\pi u \tau_0^{odd}(u,u_F) = 2u_F \tag{3.157}$$

Furthermore the introduction of the large fermionic excitation brings an additional energy into the system which, at one-loop, amounts to

$$\begin{aligned} E(u_F) &= \int_{-\infty}^{\infty}\frac{\tau_0^{(1F)}(u;u_F)}{u^2+\frac{1}{4}}du = \\ &= 2\Psi(1) - \Psi(1-iu_F) - \Psi(1+iu_F) \quad . \end{aligned} \tag{3.158}$$

7 Strong Coupling

When considering the strong coupling regime of the scattering matrices just retrieved, three possible ways of performing the $g \gg 1$ limit could be outlined. In first instance, the perturbative drives energy and momentum to taking values in the order of unity: $E, p \sim g^0$. In the giant hole (or semiclassical) regime the dispersion relations, relative to the excitation taken into exam, do not depend on the kind of particle considered; energy and momentum share the same order as the coupling constant, $E, p \sim g$. Moreover, the near flat-space regime interpolates between the previos two: it leads to $E, p \sim g^{\frac{1}{4}}$ and all the particles behave as if massless since, upon rescaling $\tilde{E} \equiv \frac{E}{g^{1/4}}$ and $\tilde{p} \equiv \frac{p}{g^{1/4}}$, the dispersion laws read $\tilde{E} = \tilde{p}$ at leading order.
In the following, several strong coupling limit for excitation scatterings will be displayed.

7.1 Scalars

The $g \longrightarrow \infty$ limit should be provided with any description of the dynamics of the particle, in order to properly reconstruct the different strong coupling regimes just outlined. In the scalar scattering for instance, the giant hole regime requires a large value of the rapidity $u \gg \sqrt{2}g$ to be performed. When rapidity and coupling constant share the same order $u - \sqrt{2}g \sim 1$ the near flat-space limit could be implemented. The perturbative regime instead is achieved when $1 \ll u \ll \sqrt{2}g$. Moreover, in addition to the common three, the scalar scattering contemplates one further strong coupling regime, the non-perturbative one [38], occurring for $g \gg 1$ while $u = O(1)$. In this limit the scalar modes decouple from the rest and consequently behave according to an $O(6)$ non-linear σ model. Energy and momentum are exponentially suppressed: $E, p \sim e^{-\pi g}$.

Non-perturbative regime:
Let a scalar-scalar scattering process be considered. When the $g \to \infty$ limit is taken while keeping the two hole rapidities u, v fixed [44], the Fourier-transformed single particle densities $\hat{\sigma}^{(1)}(k)$ and $\hat{\sigma}(k)|_{IH}$ tend toward the limit values:

$$\hat{\sigma}^{(1)}(k) \quad \longrightarrow \quad \hat{\sigma}^{(1)}_{lim}(k) = 2\pi \left[\frac{e^{-|k|}}{1-e^{-|k|}} - \frac{e^{\frac{|k|}{2}}}{2\sinh\frac{|k|}{2}\cosh k} \right] \tag{3.159}$$

$$\hat{\sigma}(k,v)|_{IH} \quad \longrightarrow \quad \hat{\sigma}^{(1)}_{lim}(k)(\cos kv - 1) = 2\pi \left[\frac{e^{-|k|}}{1-e^{-|k|}} - \frac{e^{\frac{|k|}{2}}}{2\sinh\frac{|k|}{2}\cosh k} \right](\cos ku - 1)$$

Remarkably the antitransform of the limit function $\hat{\sigma}^{(1)}_{lim}(k)$ is

$$\sigma^{(1)}_{lim}(u) = -\frac{1}{4}\left[\psi\left(1 - i\frac{u}{4}\right) + \psi\left(1 + i\frac{u}{4}\right) - \psi\left(\frac{1}{2} - i\frac{u}{4}\right) - \psi\left(\frac{1}{2} + i\frac{u}{4}\right) + \frac{2\pi}{\cosh\frac{\pi}{2}u}\right] \tag{3.160}$$

Then, upon recalling the relation (3.76) connecting the even part of the scalar scattering phase to the densities studied above, it can be seen:

$$\lim_{g\to\infty} \quad \frac{d}{du}M(u,v) = \frac{1}{2}[\sigma^{(1)}_{lim}(u-v) + \sigma^{(1)}_{lim}(u+v)] \tag{3.161}$$

and hence that leads to

$$\begin{aligned} M(u,v) \quad &= \quad -\frac{i}{2}\ln\frac{\Gamma\left(1 - i\frac{u-v}{4}\right)\Gamma\left(\frac{1}{2} + i\frac{u-v}{4}\right)\Gamma\left(1 - i\frac{u+v}{4}\right)\Gamma\left(\frac{1}{2} + i\frac{u+v}{4}\right)}{\Gamma\left(1 + i\frac{u-v}{4}\right)\Gamma\left(\frac{1}{2} - i\frac{u-v}{4}\right)\Gamma\left(1 + i\frac{u+v}{4}\right)\Gamma\left(\frac{1}{2} - i\frac{u+v}{4}\right)} - \\ &- \quad \frac{1}{2}\mathrm{gd}\left(\frac{\pi(u-v)}{2}\right) - \frac{1}{2}\mathrm{gd}\left(\frac{\pi(u+v)}{2}\right) \end{aligned} \tag{3.162}$$

where the Gudermannian function is defined as

$$\mathrm{gd}(u) \equiv \arctan(\sinh u) \quad .$$

Since $\Theta(u,v) = M(u,v) - M(v,u)$, the scattering phase between holes $\Theta(u,v)$, in the non-perturbative strong coupling regime, takes the form:

$$\lim_{g\to\infty} \quad \Theta(u,v) = -i\ln\frac{\Gamma\left(1 - i\frac{u-v}{4}\right)\Gamma\left(\frac{1}{2} + i\frac{u-v}{4}\right)}{\Gamma\left(1 + i\frac{u-v}{4}\right)\Gamma\left(\frac{1}{2} - i\frac{u-v}{4}\right)} - \mathrm{gd}\left(\frac{\pi(u-v)}{2}\right) \quad , \tag{3.163}$$

notably (3.163) signals that in this regime the scattering process described is relativistic, meaning that $\Theta(u,v)$ depends not on the two rapidities separately, but rather on their difference alone.

Perturbative regime:
The perturbative regime can be obtained by sending $g \to \infty$ while keeping the rescaled rapidity $\bar{u} \equiv \frac{u}{\sqrt{2}g}$ fixed and such that $|\bar{u}| < 1$. Then, the interesting object to be computed is the double derivative of the scattering factor $\Theta(u, v)$, which clearly depends only on the density $\sigma(u, v)|_{IH}$:

$$\frac{d}{du}\frac{d}{dv}\Theta(u, v) = \frac{d}{dv}\sigma(u, v)|_{IH} - \frac{d}{du}\sigma(v, u)|_{IH} \quad . \tag{3.164}$$

In the large g limit $\frac{d}{d\bar{v}}\sigma(u, v)|_{IH}$ can be expressed as

$$\frac{d}{d\bar{v}}\sigma(u, v)|_{IH} = \int_0^\infty \frac{d\bar{t}}{\sqrt{2}g}\cos\bar{t}\bar{u}\left[\frac{d}{d\bar{v}}\Gamma_-(\bar{t}, \bar{v}) - \frac{d}{d\bar{v}}\Gamma_+(\bar{t}, \bar{v}) - \frac{2e^{-\frac{\bar{t}}{\sqrt{2}g}}}{1 - e^{-\frac{\bar{t}}{\sqrt{2}g}}}\bar{t}\sin\bar{t}\bar{v}\right] + O\left(\frac{1}{g}\right) \tag{3.165}$$

where $\Gamma_+(\bar{t}, \bar{v})$, $\Gamma_-(\bar{t}, \bar{v})$ are an even and an odd function with respect to $\bar{t}$, satisfying the integral equation (at leading order g^1):

$$\int_0^\infty d\bar{t}\left[e^{i\bar{t}\bar{u}}\frac{d}{d\bar{v}}\Gamma_-(\bar{t}; \bar{v}) - e^{-i\bar{t}\bar{u}}\frac{d}{d\bar{v}}\Gamma_+(\bar{t}; \bar{v})\right] = \int_0^\infty d\bar{t}e^{i\bar{t}\bar{u}}\frac{\bar{t}\sin\bar{t}\bar{v}}{\sinh\frac{\bar{t}}{2\sqrt{2}g}} \cong 2\sqrt{2}g\frac{\bar{v}}{\bar{v}^2 - \bar{u}^2} \, , \tag{3.166}$$

The solution of the equation (3.166) can be inserted into (3.165) to give:

$$\begin{aligned} \frac{d}{d\bar{v}}\sigma(\sqrt{2}g\bar{u}, \sqrt{2}g\bar{v})|_{IH} &= -\frac{1}{2}H(\bar{u}^2 - 1)H(\bar{v}^2 - 1)\Bigg[\frac{\left(\frac{\bar{v}-1}{\bar{v}+1}\right)^{\frac{1}{4}}\left(\frac{\bar{u}-1}{\bar{u}+1}\right)^{\frac{1}{4}} + \left(\frac{\bar{v}+1}{\bar{v}-1}\right)^{\frac{1}{4}}\left(\frac{\bar{u}+1}{\bar{u}-1}\right)^{\frac{1}{4}}}{\bar{u} + \bar{v}} \\ &+ \frac{\left(\frac{\bar{v}-1}{\bar{v}+1}\right)^{\frac{1}{4}}\left(\frac{\bar{u}+1}{\bar{u}-1}\right)^{\frac{1}{4}} + \left(\frac{\bar{v}+1}{\bar{v}-1}\right)^{\frac{1}{4}}\left(\frac{\bar{u}-1}{\bar{u}+1}\right)^{\frac{1}{4}}}{\bar{v} - \bar{u}}\Bigg] \end{aligned} \tag{3.167}$$

($H(x)$ is the Heaviside function). Eventually, from (3.164) it easy to find (up to $O(g^0)$ order)

$$\frac{d}{d\bar{u}}\frac{d}{d\bar{v}}\Theta(\sqrt{2}g\bar{u}, \sqrt{2}g\bar{v}) \simeq \sqrt{2}gH(\bar{u}^2 - 1)H(\bar{v}^2 - 1)\frac{\left(\frac{\bar{u}+1}{\bar{u}-1}\right)^{\frac{1}{4}}\left(\frac{\bar{v}-1}{\bar{v}+1}\right)^{\frac{1}{4}} + \left(\frac{\bar{u}-1}{\bar{u}+1}\right)^{\frac{1}{4}}\left(\frac{\bar{v}+1}{\bar{v}-1}\right)^{\frac{1}{4}}}{\bar{u} - \bar{v}} \tag{3.168}$$

The phase delay that two giant holes (solitonic excitations on the GKP string) experience upon scattering amounts to [45]:

$$\begin{aligned} \Theta_{sol}(\nu_1, \nu_2) &= \frac{1}{\sqrt{2}g}(p_1E_2 - p_2E_1) + \sqrt{2}g\Bigg[\left(\nu_1 - \nu_2 + \frac{1}{\nu_1} - \frac{1}{\nu_2}\right)\ln\nu_{cm} + \\ &+ \frac{1}{\gamma(\nu_1)\,\gamma(\nu_2)}\left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right)\Bigg] \end{aligned} \tag{3.169}$$

where ν_k, E_k, p_k represent respectively speed, energy and momentum of the k-th soliton, $\gamma(\nu) = \frac{1}{\sqrt{1-\nu^2}}$, while ν_{cm} stands for the speed in the centre of mass frame. The calculation (3.168) is seen to confirm the analogous result obtained from (3.169) [45].

7.2 Gluons

Perturbative regime:
The perturbative regime can be implemented for gluons by taking the limit $g \to \infty$ while keeping $u = \bar{u}\sqrt{2}g$ and $\tilde{u} = \bar{\tilde{u}}\sqrt{2}g$ fixed and bounded by the condition $\bar{u}^2 < 1$, $\bar{\tilde{u}}^2 < 1$. In this limit, the shifted Jukovsky variable $x^{\pm}(u)$ can substituted with the expression:

$$x^{\pm}(u) \to \frac{g}{\sqrt{2}}[\bar{u} \pm i\text{sgn}\bar{u}\sqrt{1-\bar{u}^2}] \tag{3.170}$$

so that it may be used the simplification

$$\lim_{g\to\infty}[\chi(v,u|1) + \Phi(v)] = 2\arctan(v - \sqrt{2}g\bar{u}) - 2\arctan(2v - 2\sqrt{2}g\bar{u}) \tag{3.171}$$

Looking back at (3.104), the scattering matrix decomposes into three terms:

$$i\ln\left(-S_{11}^{(gg)}(u,\tilde{u})\right) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \tag{3.172}$$

The first part in this limit becomes

$$\begin{aligned} \mathcal{I}_1 &= \tilde{\chi}(u,\tilde{u}|1,1) = -2\arctan\sqrt{2}g(\bar{\tilde{u}} - \bar{u}) = \\ &= -\pi\text{sgn}(\bar{\tilde{u}} - \bar{u}) + \frac{\sqrt{2}}{g(\bar{\tilde{u}} - \bar{u})} + O(1/g^3) \end{aligned} \tag{3.173}$$

The second part $\mathcal{I}_2$, instead, turns out to be negligible n the strong coupling limit, since

$$\mathcal{I}_2 = -\int_{-\infty}^{+\infty}\frac{dv}{2\pi}\left[\chi(v,u|1) + \Phi(v)\right]\frac{d}{dv}[\chi(v,\tilde{u}|1) + \Phi(v)] = \tag{3.174}$$

$$= -2\arctan\left[\frac{g(\bar{\tilde{u}} - \bar{u})}{\sqrt{2}}\right] - 2\arctan[g(\bar{\tilde{u}} - \bar{u})\sqrt{2}] + 4\arctan\left[\frac{2\sqrt{2}g(\bar{\tilde{u}} - \bar{u})}{3}\right] = O(\frac{1}{g^3})$$

The integral that constitutes the last term $\mathcal{I}_3$ can be more suitably treated after the change of variables $v = \sqrt{2}g\bar{v}$, $w = \sqrt{2}g\bar{w}$, so that it gets simplified, assuming the form:

$$\begin{aligned} \mathcal{I}_3 &= \int_{-\infty}^{+\infty}\frac{d\bar{v}}{2\pi}\frac{d\bar{w}}{2\pi}\left[\chi(v,u|1) + \Phi(v)\right]\left[\frac{d}{d\bar{v}}\frac{d}{d\bar{w}}\Theta(\sqrt{2}g\bar{v},\sqrt{2}g\bar{w})\right]\left[\chi(w,\tilde{u}|1) + \Phi(w)\right] = \\ &\cong \int_{-\infty}^{+\infty}\frac{d\bar{v}}{2\pi}\int_{-\infty}^{+\infty}\frac{d\bar{w}}{2\pi}\frac{1}{\sqrt{2}g\bar{v} - \sqrt{2}g\bar{u}}\frac{1}{\sqrt{2}g\bar{w} - \sqrt{2}g\bar{\tilde{u}}}\frac{d}{d\bar{v}}\frac{d}{d\bar{w}}\Theta(\sqrt{2}g\bar{v},\sqrt{2}g\bar{w}) \end{aligned} \tag{3.175}$$

By making use of the expression (3.168), the integration above can be performed, so that for $\mathcal{I}_3$ the relation holds:

$$\mathcal{I}_3 = \frac{1}{2\sqrt{2}g(\bar{u} - \bar{\tilde{u}})}\left[2 - \left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{1/4}\left(\frac{1-\bar{\tilde{u}}}{1+\bar{\tilde{u}}}\right)^{1/4} - \left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{1/4}\left(\frac{1+\bar{\tilde{u}}}{1-\bar{\tilde{u}}}\right)^{1/4}\right] \tag{3.176}$$

Finally, the three parts (3.173), (3.174), (3.176) sum up, so to give the gluon-gluon scattering phase in the perturbative regime:

$$\begin{aligned} S_{11}^{(gg)}(u,\tilde{u}) &= \exp\left[\frac{i}{\sqrt{2}g(\bar{u} - \bar{\tilde{u}})}\left(1 + \frac{1}{2}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{1/4}\left(\frac{1-\bar{\tilde{u}}}{1+\bar{\tilde{u}}}\right)^{1/4} + \right.\right. \\ &+ \left.\left.\frac{1}{2}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{1/4}\left(\frac{1+\bar{\tilde{u}}}{1-\bar{\tilde{u}}}\right)^{1/4} + O(1/g^2)\right)\right] \quad , \end{aligned} \tag{3.177}$$

thus conferming results previously known in literature [46].

Giant hole regime:

This time the attention lie on the gluonic scattering matrix (3.104) in the giant limit. Such a regime is achieved by taking the $g \to \infty$ limit, provided the rescaled rapidities $\bar{u}$ and $\bar{\tilde{u}}$ stay fixed and above the threshold $\bar{u}^2 > 1$, $\bar{\tilde{u}}^2 > 1$. Referring again to the decomposition (3.172), the $\mathcal{I}_1$ and $\mathcal{I}_2$ terms do not vary from the perturbative regime. The difference from the previous case instead resides in the last part: in fact, in the desired regime $\mathcal{I}_3$ assumes the form

$$\mathcal{I}_3 \cong \frac{1}{2g^2}\int_{-\infty}^{+\infty}\frac{d\bar{v}}{2\pi}\int_{-\infty}^{+\infty}\frac{d\bar{w}}{2\pi}\,P\frac{1}{\bar{v}-\bar{u}}\,P\frac{1}{\bar{w}-\bar{\tilde{u}}}\,\frac{d}{d\bar{v}}\frac{d}{d\bar{w}}\Theta(\sqrt{2}g\bar{v},\sqrt{2}g\bar{w}) \tag{3.178}$$

with P meaning principal value. The integration then results in

$$\mathcal{I}_3 = \frac{1}{2\sqrt{2}g(\bar{u}-\bar{\tilde{u}})}\left[1-\frac{1}{2}\left(\frac{\bar{u}+1}{\bar{u}-1}\right)^{1/4}\left(\frac{\bar{\tilde{u}}-1}{\bar{\tilde{u}}+1}\right)^{1/4}-\frac{1}{2}\left(\frac{\bar{u}-1}{\bar{u}+1}\right)^{1/4}\left(\frac{\bar{\tilde{u}}+1}{\bar{\tilde{u}}-1}\right)^{1/4}\right] \tag{3.179}$$

In the end, the gluon-gluon scattering matrix in the giant hole regime thus reads:

$$\begin{aligned} S_{11}^{(gg)}(u,\tilde{u}) &= \exp\Big[\frac{i}{2\sqrt{2}g(\bar{u}-\bar{\tilde{u}})}\Big(3+\frac{1}{2}\left(\frac{\bar{u}+1}{\bar{u}-1}\right)^{1/4}\left(\frac{\bar{\tilde{u}}-1}{\bar{\tilde{u}}+1}\right)^{1/4}+ \\ &+ \frac{1}{2}\left(\frac{\bar{u}-1}{\bar{u}+1}\right)^{1/4}\left(\frac{\bar{\tilde{u}}+1}{\bar{\tilde{u}}-1}\right)^{1/4}+O(1/g^2)\Big)\Big] \end{aligned} \tag{3.180}$$

7.3 Fermions

Perturbative regime:

Before studying the strong coupling limit of the fermion-fermion scattering, it takes to remark a few facts. As already explained, the rapidity that must actually be handled when talking about fermions is a x variable, rather than an u; the two variables are related each other in different ways, according the issue is about a small or a large fermion: for the former, it holds $x_f = (g^2)/(2x(u_f)) = (u_f/2)\left[1-\sqrt{1-(2g^2)/u_f^2}\right]$, whereas the latter requires $x_F = x(u_{F,j}) = (u_F/2)\left[1+\sqrt{1-(2g^2)/u_F^2}\right]$. When dealing with strong coupling expansions, the perturbative regime needs to be connected to the state of a fermion at rest, that means the rapidity set equal to zero $x = 0$: as a matter of facts, such a condition could be obtained just in the small fermion dynamical regime.

The starting point to examine the strong coupling perturbative regime is obviously the small fermion scattering matrix (3.139), where the function $\chi_H(v,u)$ makes its appearance: for the considerations just displayed, it may be expressed as

$$\chi_H(v,u) = -i\ln\frac{1-\frac{x_f(u)}{x^+(v)}}{1-\frac{x_f(u)}{x^-(v)}} \quad , \tag{3.181}$$

where both fermionic x_f and 'scalar' rapidities $x^{\pm}(v) = \frac{v \pm i/2}{2}\left[1+\sqrt{1-\frac{2g^2}{(v\pm i/2)^2}}\right]$ are involved. The perturbative regime demands the rescaled fermion rapidity $\bar{x}_f(\bar{u}) \equiv \frac{x_f(u)}{\sqrt{2}g}$ obey the condition

$$|\bar{x}_f(\bar{u})| \leq \frac{1}{2} \quad . \tag{3.182}$$

When taking into account the scalar rapidity, it reveals of any use to rescale $v = \sqrt{2}g\bar{v}$, then to perform an expansion at large g:

$$x^{\pm}(v) = \sqrt{2}g\bar{x}(\bar{v}) \pm \frac{i}{4}\frac{1+\sqrt{1-\frac{1}{\bar{v}^2}}}{\sqrt{1-\frac{1}{\bar{v}^2}}} + O(1/g)\,, \quad \bar{x}(\bar{v}) = \frac{\bar{v}}{2}\left[1+\sqrt{1-\frac{1}{\bar{v}^2}}\right] \quad . \tag{3.183}$$

To sum up, the function $\chi_H(w,u)$ can be approximated as

$$\chi_H(v,u) = -i\ln\frac{1-\frac{x_f(u)}{x^+(v)}}{1-\frac{x_f(u)}{x^-(v)}} \cong -\frac{\bar{x}_f(\bar{u})}{\sqrt{2}g}\frac{1}{\bar{v}\sqrt{1-\frac{1}{\bar{v}^2}}}\frac{1}{\bar{x}_f(\bar{u})-\bar{x}(\bar{v})} \quad . \tag{3.184}$$

Then the small fermion scattering matrix (3.139) is written as the sum of two parts, $-i\ln S^{(ff)}(u,v) = J_1 + J_2$; the first term is subleading with respect to the second, as

$$J_1 = \int_{-\infty}^{+\infty}\frac{dw}{2\pi}\chi_H(w,u)\frac{d}{dw}\chi_H(w,v) = O(1/g^2) \tag{3.185}$$

while instead, after plugging in the expression (3.168) for the scalar phase,

$$\begin{aligned} J_2 &= -\int\frac{dw}{2\pi}\frac{dz}{2\pi}\chi_H(w,u)\frac{d^2}{dwdz}\Theta(w,z)\chi_H(z,v) = \\ &\cong -\frac{1}{\sqrt{2}g}\int_{|\bar{w}|\geq 1}\frac{d\bar{w}}{2\pi}\int_{|\bar{z}|\geq 1}\frac{d\bar{z}}{2\pi}\frac{1}{\bar{w}\sqrt{1-\frac{1}{\bar{w}^2}}}\frac{\bar{x}_f(\bar{u})}{\bar{x}_f(\bar{u})-\bar{x}(\bar{w})}\cdot \\ &\cdot \quad \frac{\left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}}\left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}} + \left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}}\left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}}}{\bar{w}-\bar{z}}\frac{1}{\bar{z}\sqrt{1-\frac{1}{\bar{z}^2}}}\frac{\bar{x}_f(\bar{v})}{\bar{x}_f(\bar{v})-\bar{x}(\bar{z})} \end{aligned} \tag{3.186}$$

Thanks to the identity

$$\frac{\left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}}\left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}} + \left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}}\left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}}}{\bar{w}-\bar{z}} = \frac{1}{\bar{x}(\bar{w})-\bar{x}(\bar{z})}\frac{\sqrt{1+\sqrt{1-\frac{1}{\bar{w}^2}}}\sqrt{1+\sqrt{1-\frac{1}{\bar{z}^2}}}}{\left(1-\frac{1}{\bar{w}^2}\right)^{\frac{1}{4}}\left(1-\frac{1}{\bar{z}^2}\right)^{\frac{1}{4}}} \tag{3.187}$$

the integral in the second line above can be factorised and then exactly computed, so to get

$$J_2 \cong -\frac{1}{2\sqrt{2}g}\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})[\bar{x}_f(\bar{v})-\bar{x}_f(\bar{u})]\mathcal{J}(\bar{u},\bar{v})^2 \tag{3.188}$$

where the function $\mathcal{J}(\bar{u},\bar{v})$, after some calculation, can be expressed as

$$\mathcal{J}(\bar{u},\bar{v}) = \frac{\sqrt{2}}{\bar{x}_f(\bar{u}) - \bar{x}_f(\bar{v})}\left[\frac{1}{\sqrt{1-4\bar{x}_f(\bar{u})^2}} - \frac{1}{\sqrt{1-4\bar{x}_f(\bar{v})^2}}\right] \tag{3.189}$$

At last, in the large g perturbative regime, the fermion scattering phase (3.139) takes the form:

$$-i\ln S^{(ff)}(u,v) \cong -\frac{1}{\sqrt{2}g}\frac{\bar{x}_f(\bar{u})\bar{x}_f(\bar{v})}{\bar{x}_f(\bar{v}) - \bar{x}_f(\bar{u})}\left[\frac{1}{\sqrt{1-4\bar{x}_f(\bar{v})^2}} - \frac{1}{\sqrt{1-4\bar{x}_f(\bar{u})^2}}\right]^2 \tag{3.190}$$

7.4 Mixed matrices

Now a few cases of scattering involving particles of different kinds will be taken into exam in the strong coupling limit.

Gluons-scalars:
Let the $g \to \infty$ limit be considered for the scattering between a gluon with rapidity $u = \bar{u}\sqrt{2}g$ and a hole, whose rapidity is $u_h = \bar{u}_h\sqrt{2}g$: the process is thus described by the formula (3.105) for $i\ln S^{(gs)}(u,u_h)$. The regime $|\bar{u}| \leq 1$ has been chosen. The phase delay splits into two parts $i\ln S^{(gs)}(\sqrt{2}g\bar{u},\sqrt{2}g\bar{u}_h) = I_1^{gs} + I_2^{gs}$: the first one corresponds to

$$\begin{aligned} I_1^{gs} &= -\chi(\sqrt{2}g\bar{u}_h,\sqrt{2}g\bar{u}|1) - \Phi(\sqrt{2}g\bar{u}_h) \cong -2\arctan(\sqrt{2}g\bar{u}_h - \sqrt{2}g\bar{u}) + \\ &+ 2\arctan(2\sqrt{2}g\bar{u}_h - 2\sqrt{2}g\bar{u}) \cong \frac{2}{\sqrt{2}g(\bar{u}_h - \bar{u})} - \frac{1}{\sqrt{2}g(\bar{u}_h - \bar{u})} \end{aligned} \tag{3.191}$$

whereas the second is given by

$$\frac{d}{du_h}I_2^{gs} = \frac{1}{\sqrt{2}g}\int \frac{d\bar{w}}{2\pi}\frac{d}{d\bar{u}_h}\frac{d}{d\bar{w}}\Theta(\sqrt{2}g\bar{u}_h,\sqrt{2}g\bar{w})\left[\chi(\sqrt{2}g\bar{w},\sqrt{2}g\bar{u}|1) + \Phi(\sqrt{2}g\bar{w})\right] . \tag{3.192}$$

With the help of the scalar-scalar phase (3.168) and formulae (A.15), (A.16), the result can be achieved, in the regime $|\bar{u}| \leq 1$, $|\bar{u}_h| \geq 1$:

$$\begin{aligned} \frac{d}{du_h}I_2^{gs} &= -\frac{1}{\sqrt{2}g}\int_{|w|\geq 1}\frac{d\bar{w}}{2\pi}\frac{1}{\bar{w}-\bar{u}}P\frac{1}{\bar{u}_h - \bar{w}}\left[\left(\frac{\bar{u}_h+1}{\bar{u}_h-1}\right)^{\frac{1}{4}}\left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}} + \right. \\ &+ \left.\left(\frac{\bar{u}_h-1}{\bar{u}_h+1}\right)^{\frac{1}{4}}\left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}}\right]H(\bar{u}_h^2 - 1) = \\ &= \frac{1}{2\sqrt{2}g}\frac{1}{\bar{u}-\bar{u}_h}\left[\left(\frac{\bar{u}_h-1}{\bar{u}_h+1}\right)^{\frac{1}{4}} - \left(\frac{\bar{u}_h+1}{\bar{u}_h-1}\right)^{\frac{1}{4}}\right] - \frac{1}{2g}\frac{1}{\bar{u}-\bar{u}_h}\left[\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}} - \left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}}\right] \end{aligned} \tag{3.193}$$

Gluons-fermions in the perturbative regime:
This time the focus lie on the scattering matrix involving a gluon (rapidity $u = \sqrt{2}g\bar{u}$)

and a small fermions (rapidity $x_f(v) = \sqrt{2}g\bar{x}_f(\bar{v})$), studied in the strong coupling perturbative regime, which is implemented by assuming the conditions $|\bar{u}| \leq 1$ and $|\bar{x}_f(\bar{v})| \leq 1/2$ (with $v\bar{v} = \frac{v}{\sqrt{2}g} \geq 1$). The scattering phase previously found is composed of three parts

$$i \ln\left(-S^{(gf)}(u,v)\right) = I_1^{gf} + I_2^{gf} + I_3^{gf} \tag{3.194}$$

where

$$I_1^{gf} = 2\arctan 2(u-v) \tag{3.195}$$

$$I_2^{gf} = \int_{-\infty}^{+\infty} \frac{dw}{2\pi}[\chi(w,u|1) + \Phi(w)]\frac{d}{dw}\chi_H(w,v) + \Phi(w) \tag{3.196}$$

$$I_3^{gf} = -\int \frac{dw}{2\pi}\frac{dz}{2\pi}[\chi(w,u|1) + \Phi(w)]\frac{d^2}{dwdz}\Theta(w,x)\chi_H(z,v) \tag{3.197}$$

In the strong coupling $g \to \infty$ limit, the first term takes the form

$$I_1^{gf} = \pi \mathrm{sgn}(\bar{u} - \bar{v}) - \frac{1}{\sqrt{2}g(\bar{u}-\bar{v})} + O(1/g^2) \tag{3.198}$$

whereas I_2^{gf} is subleading, while the third term may be recast into

$$\begin{aligned} I_3^{gf} &= \int_{|\bar{w}|,|\bar{z}|\geq 1} \frac{d\bar{w}}{2\pi}\frac{d\bar{z}}{2\pi}\frac{1}{\bar{u}-\bar{w}} \frac{\left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}}\left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}} + \left(\frac{\bar{w}-1}{\bar{w}+1}\right)^{\frac{1}{4}}\left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}}}{\bar{w}-\bar{z}} \times \\ &\times \frac{\bar{x}_f(\bar{v})}{\sqrt{2}g\bar{z}\sqrt{1-\frac{1}{\bar{z}^2}}}\frac{1}{\bar{x}_f(\bar{v}) - \bar{x}(\bar{z})} \end{aligned} \tag{3.199}$$

Formulæ(A.17) and (A.18) allow to retrieve the result:

$$I_3^{gf} = \frac{1}{4g}\frac{\sqrt{\frac{1-2\bar{x}_f(\bar{v})}{1+2\bar{x}_f(\bar{v})}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} + \sqrt{\frac{1+2\bar{x}_f(\bar{v})}{1-2\bar{x}_f(\bar{v})}}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}} - \sqrt{2}}{\bar{v}-\bar{u}} \tag{3.200}$$

The sum of I_1^{gf} and I_3^{gf} eventually leads to the strong coupling expression for the gluon-fermion scattering phase in the perturbative regime:

$$S^{(gF)}(u,v) = \exp\left[\frac{i}{4g}\frac{\sqrt{2} + \sqrt{\frac{1-2\bar{x}_f}{1+2\bar{x}_f}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} + \sqrt{\frac{1+2\bar{x}_f}{1-2\bar{x}_f}}\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}}}{\bar{u}-\bar{v}} + O(1/g^2)\right] \quad . \tag{3.201}$$

Chapter 4

Dispersion Relations

1 Isotopic Roots

The set of excitations studied in the previous chapter does not actually cover the whole particle content of the theory, since the total number of degrees of freedom includes eight bosonic plus eight fermionic modes. They all fall into multiplets under the $SU(4)$ residual symmetry of the GKP vacuum: six scalars transform according to the **6** of $su(4)$, two more bosonic degrees of freedom correspond to two components of the gluon field strength, namely $F_{+\perp}$ (previously referred to as 'gluon') and $\bar{F}_{+\perp}$ (so far called 'barred-gluon'), while the 4 components of the left Weyl spinor ψ (in the **4** of $su(4)$) together with the 4 components of the right Weyl spinor $\bar{\psi}$ (in the $\bar{\mathbf{4}}$ of $su(4)$) account for the fermionic modes. In fact, the excitations considered up to now are related to the highest weights of each of the $su(4)$ representations just cited: moving from them is nevertheless possible to reconstruct the complete multiplets. Indeed three more solutions of the Bethe equations deserve an accurate exam, as they are responsible for 'rotating' the multiplets under the $SO(6)$ ($\sim SU(4)$) symmetry. Two of them match to the u_2 and u_6 Bethe roots taken as single (not belonging to any stack), whereas the third type actually is a composite object, since it is composed of two (complex conjugate) u_4 main roots, together with one u_3 and one u_5 on the top of each other, corresponding to the real centre of the stack, *i.e.*:

$$
\begin{aligned}
u_{4\pm} &= u_s \pm \frac{i}{2} \\
u_s &= u_3 = u_5 \quad .
\end{aligned}
\tag{4.1}
$$

They all are isotopic, as they do not carry energy nor momentum, not even any other charge [42]: this behaviour is apparent for the u_2 and u_6 roots, for they do not interact with main roots u_4 within the Beisert-Staudacher equations; otherwise, for the isotopic stack the issue is more involved to cope with.

Remarkably, the role of these roots pertains to the very structure of the vacuum. In fact, when the choice of the ground state falls onto the GKP vacuum, the overall symmetry $PSU(2,2|4)$ breaks down to a residual $SU(4)$, whose Goldstone bosons are related to the isotopic roots themselves: indeed u_2 and u_6 are associated to $SU(2) \times$

$SU(2) \subset SU(4)$, while the stack accounts for the remainder of $SU(4)$ [1].

Within a system composed of H holes, $N_F\,(N_f)$ large (small) fermions, $N_{\bar{F}}\,(N_{\bar{f}})$ large (small) antifermions and $N_g\,(N_{\bar{g}})$ (barred) gluons, the isotopic roots are properly described by the following equations:

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}+i}{u_{2,k}-u_{2,j}-i} \prod_{j=1}^{K_s} \frac{u_{2,k}-u_{s,j}-\frac{i}{2}}{u_{2,k}-u_{s,j}+\frac{i}{2}} \prod_{j=1}^{N_F} \frac{u_{2,k}-u_{F,j}-\frac{i}{2}}{u_{2,k}-u_{F,j}+\frac{i}{2}} \prod_{j=1}^{N_f} \frac{u_{2,k}-u_{f,j}-\frac{i}{2}}{u_{2,k}-u_{f,j}+\frac{i}{2}} \tag{4.2}$$

$$1 = \prod_{j\neq k}^{K_s} \frac{u_{s,k}-u_{s,j}+i}{u_{s,k}-u_{s,j}-i} \prod_{j=1}^{K_2} \frac{u_{s,k}-u_{2,j}-\frac{i}{2}}{u_{s,k}-u_{2,j}+\frac{i}{2}} \prod_{j=1}^{K_6} \frac{u_{s,k}-u_{6,j}-\frac{i}{2}}{u_{s,k}-u_{6,j}+\frac{i}{2}} \prod_{h=1}^{H} \frac{u_{s,k}-u_{h}-\frac{i}{2}}{u_{s,k}-u_{h}+\frac{i}{2}} \tag{4.3}$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i} \prod_{j=1}^{K_s} \frac{u_{6,k}-u_{s,j}-\frac{i}{2}}{u_{6,k}-u_{s,j}+\frac{i}{2}} \prod_{j=1}^{N_{\bar{F}}} \frac{u_{6,k}-u_{\bar{F},j}-\frac{i}{2}}{u_{6,k}-u_{\bar{F},j}+\frac{i}{2}} \prod_{j=1}^{N_{\bar{f}}} \frac{u_{6,k}-u_{\bar{f},j}-\frac{i}{2}}{u_{6,k}-u_{\bar{f},j}+\frac{i}{2}} \tag{4.4}$$

The equations (4.2),(4.4) for u_2 and u_6 come directly from the second and the sixth of the Beisert-Staudacher equations: it is worth observe the cancellation of the contributions coming from gluonic stacks. Differently, the derivation of (4.3) is less straightforward. As a matter of fact, the definition of the stack requests it should be taken the product of the third equation for $u_{3,k} = u_{s,k}$ per the fifth with $u_{5,k} = u_{s,k}$, then also for the fourth, shifted above and below the real axis by a displacement $\pm\frac{i}{2}$, *i.e.* it takes a further multiplication per both the fourth for $u_{4,k} = u_{s,k}+i/2$ and the fourth for $u_{4,k} = u_{b,k}-i/2$. Such a procedure results in:

$$\begin{aligned}
1 = & \prod_{j=1}^{K_2} \frac{u_{s,k}-u_{2,j}-i/2}{u_{s,k}-u_{2,j}+i/2} \prod_{j=1}^{K_6} \frac{u_{s,k}-u_{6,j}-i/2}{u_{s,k}-u_{6,j}+i/2} \prod_{j=1}^{K_4} \frac{u_{s,k}-u_{4,j}+i/2}{u_{s,k}-u_{4,j}-i/2} \times \\
& \times \prod_{j=1}^{N_g} \frac{u_{s,k}-u_j^g-i/2}{u_{s,k}-u_j^g+i/2} \prod_{j=1}^{N_{\bar{g}}} \frac{u_{s,k}-u_j^{\bar{g}}-i/2}{u_{s,k}-u_j^{\bar{g}}+i/2} \left[\left(\frac{x_{s,k}^{--}}{x_{s,k}^{++}} \right)^L \times \right. \\
& \times \prod_{j\neq k}^{K_4} \frac{x_{s,k}^{--}-x_{4,j}^{+}}{x_{s,k}^{++}-x_{4,j}^{-}} \frac{1-\frac{g^2}{2x_{s,k}^{++}x_{4,j}^{-}}}{1-\frac{g^2}{2x_{s,k}^{--}x_{4,j}^{+}}} \sigma^2(u_{s,k}+\frac{i}{2},u_{4,j})\sigma^2(u_{s,k}-\frac{i}{2},u_{4,j}) \times \\
& \times \prod_{j=1}^{K_s} \left(\frac{x_{s,k}^{++}-x_{s,j}}{x_{s,k}^{--}-x_{s,j}} \right)^2 \prod_{j=1}^{N_F} \frac{x_{s,k}^{++}-x_{F,j}}{x_{s,k}^{--}-x_{F,j}} \prod_{j=1}^{N_{\bar{F}}} \frac{x_{s,k}^{++}-x_{\bar{F},j}}{x_{s,k}^{--}-x_{\bar{F},j}} \prod_{j=1}^{N_f} \frac{1-\frac{x_{f,j}}{x_{s,k}^{++}}}{1-\frac{x_{f,j}}{x_{s,k}^{--}}} \prod_{j=1}^{N_{\bar{f}}} \frac{1-\frac{x_{\bar{f},j}}{x_{s,k}^{++}}}{1-\frac{x_{\bar{f},j}}{x_{s,k}^{--}}} \times \\
& \times \left. \prod_{j=1}^{N_g} \frac{x_{s,k}^{++}-x_j^{g+}}{x_{s,k}^{--}-x_j^{g+}} \frac{x_{s,k}^{++}-x_j^{g-}}{x_{s,k}^{--}-x_j^{g-}} \prod_{j=1}^{N_{\bar{g}}} \frac{x_{s,k}^{++}-x_j^{\bar{g}+}}{x_{s,k}^{--}-x_j^{\bar{g}+}} \frac{x_{s,k}^{++}-x_j^{\bar{g}-}}{x_{s,k}^{--}-x_j^{\bar{g}-}} \right] \quad ,
\end{aligned} \tag{4.5}$$

[1]Actually in the one loop limit two more Bethe roots behave as if isotopic, namely u_1 and u_7, associated to small fermions, thus enhancing the vacuum symmetry up to $SL(2|4)$; the small fermions then acquire energy and momentum at higher loops

where the double-shifted Jukovsky variable explicitly means $x^{\pm\pm}(u) = x(u \pm i)$ and

$$L = 2 + H + N_F + N_{\bar{F}} + N_g + N_{\bar{g}} \quad . \tag{4.6}$$

The product in the first line may be substituted with the expression obtained by means of the NLIE:

$$\prod_{j=1}^{K_4} \frac{u_{s,k} - u_{4,j} + i/2}{u_{s,k} - u_{4,j} - i/2} = \prod_{j=1}^{K_s} \frac{u_{s,k} - u_{s,j} + i}{u_{s,k} - u_{s,j} - i} \prod_{h=1}^{H} \frac{u_{s,k} - u_h - i/2}{u_{s,k} - u_h + i/2} \left(1 + O(1/s^2)\right) \times$$
$$\times \quad \exp\left[-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{u_{s,k} - v + i/2}{u_{s,k} - v - i/2} (Z_4'(v) - 2L_4'(v))\right] \tag{4.7}$$

in wich $Z_4(v)$ stands for the counting function for scalars including every possible type of root:

$$\begin{aligned} Z_4(u) \;\; = \;\; & L\Phi(u) - \sum_{j=1}^{K_4} \phi(u, u_j) + 2i \sum_{j=1}^{K_s} \ln \frac{x^+(u) - x_{s,j}}{x_{s,j} - x^-(u)} + \\ & + \sum_{j=1}^{N_g} \chi(u, u_j^g|1) + \sum_{j=1}^{N_{\bar{g}}} \chi(u, u_j^{\bar{g}}|1) + \sum_{j=1}^{N_F} \chi_F(u, u_{F,j}) + \sum_{j=1}^{N_{\bar{F}}} \chi_F(u, u_{\bar{F},j}) - \\ & - \sum_{j=1}^{N_f} \chi_H(u, u_{f,j}) - \sum_{j=1}^{N_{\bar{f}}} \chi_H(u, u_{\bar{f},j}) \quad . \end{aligned} \tag{4.8}$$

Once formula (4.8) is plugged into (4.7), it is therefore found[2] that the term

$$\exp\left[-\int_{-\infty}^{+\infty} \frac{dv}{2\pi} \ln \frac{u_{s,k} - v + i/2}{u_{s,k} - v - i/2} Z_4'(v)\right] \tag{4.9}$$

coincides with the inverse of the expression inside the square brackets in (4.5); moreover the nonlinear term containing $L_4'(v)$ is recalled to be subleading in the large spin limit, amounting to an $O(1/s^2)$ contribution. In the end, the equation (4.3) for the isotopic stack u_s follows.

1.1 SU(4) symmetry

In a celebrated paper [48], Ogievetsky and Wiegmann wrote the set of Bethe equations describing any spin chain built up starting from scattering matrices, symmetrical under the action of a simple Lie algebra in some representation. A simple Lie algebra, say g, turns out to be completely determined by the set of simple roots $\{\alpha_q\}$ (normalized to unity); when diagonalizing the transfer matrix of a spin chain with g as a symmetry, each α_q should be associated to an auxiliary root u_q, which may be intended as the rapidity of a pseudo-particle with no energy nor momentum, carrying instead just colour indices (relative to g). As a byproduct of the diagonalization, the set of Bethe equations arises:

$$\left(\frac{u_{q,k} + i\vec{\alpha}_q \cdot \vec{w}_R}{u_{q,k} - i\vec{\alpha}_q \cdot \vec{w}_R}\right)^N = \prod_{j \neq k}^{K_q} \frac{u_{q,k} - u_{q,j} + i\vec{\alpha}_q \cdot \vec{\alpha}_q}{u_{q,k} - u_{q,j} - i\vec{\alpha}_q \cdot \vec{\alpha}_q} \prod_{q' \neq q} \prod_{j=1}^{K_{q'}} \frac{u_{q,k} - u_{q',j} + i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}{u_{q,k} - u_{q',j} - i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}} \quad ;$$

[2] This fact has been originally observed by Basso [42].

(4.10)

it should be noticed that whilst the *r.h.s.* of (4.10) does rely on any representation of g in particular, on the other hand the *l.h.s.* exhibits the highest weight $\vec{w}$, belonging to the representation chosen for the algebra.

The present interest focuses on the algebra $g = su(4) \simeq so(6)$ [3] (in fact, the group $SU(4)$ is the double cover of $SO(6)$). The $su(4)$ is a rank three algebra, therefore its properties are fully stated by the choice of three simple roots, for instance:

$$\begin{aligned} \vec{\alpha}_1 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \\ \vec{\alpha}_2 &= \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right) \\ \vec{\alpha}_3 &= \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{6}}\right) \quad , \end{aligned} \tag{4.11}$$

whose norm is set equal to one; the Cartan matrix then follows:

$$A_{ij} = \left|\frac{2\vec{\alpha}_i \cdot \vec{\alpha}_j}{(\vec{\alpha}_j)^2}\right| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \quad . \tag{4.12}$$

At the same time, the simple roots fix the fundamental weights, that is the three vectors obeying to the definitory condition $\frac{2\vec{\alpha}_j \cdot \vec{\varphi}_k}{(\vec{\alpha}_j)^2} = \delta_{kj}$:

$$\begin{aligned} \vec{\varphi}_1 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \\ \vec{\varphi}_2 &= \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right) \\ \vec{\varphi}_3 &= \left(0, 0, \frac{3}{2\sqrt{6}}\right) \quad . \end{aligned} \tag{4.13}$$

One element is still missing in order to suitably adapt the equations (4.10) to a spin chain endowed with the $SU(4)$ symmetry: a representation of the algebra $su(4)$ needs to be chosen, and to this purpose the highest weight $\vec{w}_R$ suffices. Every representation is then uniquely associated to a tern of positive integers $(\lambda_1, \lambda_2, \lambda_3)$, the so-called Dynkin labels, so that the highest weight results easily as $\vec{w}_R = \sum_k^3 \lambda_k \vec{\varphi}_k$, the remaining weights being obtained by properly acting on it by means of the simple roots.

[3]Prior to [48], the correct Bethe equations for this case, and in general for $so(2n)$ spin chains as well, was found by [49]; then the results grew in generality and widened to simple algebras thanks to [48]

To sum up, the Bethe equations in (4.10) get specialized for the $su(4)$ algebra:

$$\begin{aligned}
\left(\frac{u_{a,k}+i\vec{\alpha}_1\cdot\vec{w}_R}{u_{a,k}-i\vec{\alpha}_1\cdot\vec{w}_R}\right)^N &= \prod_{j\neq k}^{K_a}\frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i}\prod_{j=1}^{K_b}\frac{u_{a,k}-u_{b,j}-i/2}{u_{a,k}-u_{b,j}+i/2} \\
\left(\frac{u_{b,k}+i\vec{\alpha}_2\cdot\vec{w}_R}{u_{b,k}-i\vec{\alpha}_2\cdot\vec{w}_R}\right)^N &= \prod_{j\neq k}^{K_b}\frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i}\prod_{j=1}^{K_a}\frac{u_{b,k}-u_{a,j}-i/2}{u_{b,k}-u_{a,j}+i/2}\prod_{j=1}^{K_c}\frac{u_{b,k}-u_{c,j}-i/2}{u_{b,k}-u_{c,j}+i/2} \\
\left(\frac{u_{c,k}+i\vec{\alpha}_3\cdot\vec{w}_R}{u_{c,k}-i\vec{\alpha}_3\cdot\vec{w}_R}\right)^N &= \prod_{j\neq k}^{K_c}\frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i}\prod_{j=1}^{K_b}\frac{u_{c,k}-u_{b,j}-i/2}{u_{c,k}-u_{b,j}+i/2}
\end{aligned} \tag{4.14}$$

Upon setting to zero the number of any kind of momentum-carrier excitation, a comparison between the Bethe equations (4.2)-(4.4) and (4.14) hints the identification of the isotopic variables u_2, u_s, u_6 with the colour-carrying auxiliary roots u_a, u_b, u_c, according to the statements

$$u_2 = u_a \qquad u_s = u_b \qquad u_6 = u_c \quad , \tag{4.15}$$

and that finally sheds light onto the role of the isotopic roots, making more stringent the relation with the residual $SU(4)$ symmetry of the GKP vacuum.

In the remainder of the paragraph, a few examples of representations of $su(4)$ will be displayed, as they will turn out useful in the following, when the endeavour of extending the results from the previous chapter will be faced.

Fundamental representation:
The tern of Dynkin labels $(1,0,0)$ associates to the fundamental representation of $su(4)$, else referred to as the 4-dimensional (shortly, the **4**), thus the highest weight is

$$\vec{w}_4 = \vec{\varphi}_1 = \left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right) \quad .$$

By the way, the other weights of the representation descend from $\vec{w}_4$ under the action of the simple roots (4.11), eventually arranging themselves into a tetrahedron in the three dimensional root space:

$$\begin{aligned}
&\vec{w}_4 = \vec{\varphi}_1 && \vec{\varphi}_1-\vec{\alpha}_1 = \left(0,-\frac{1}{\sqrt{3}},\frac{1}{2\sqrt{6}}\right) \\
&\vec{\varphi}_1-\vec{\alpha}_1-\vec{\alpha}_2 = \left(-\frac{1}{2},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{6}}\right) && \vec{\varphi}_1-\vec{\alpha}_1-\vec{\alpha}_2-\vec{\alpha}_3 = \left(0,0,-\frac{3}{2\sqrt{6}}\right) = -\vec{\varphi}_3 \ .
\end{aligned}$$

The Bethe equations (4.14) for the fundamental representation thus read:

$$\begin{aligned}
\left(\frac{u_{a,k}+\frac{i}{2}}{u_{a,k}-\frac{i}{2}}\right)^N &= \prod_{j\neq k}^{K_a}\frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i}\prod_{j=1}^{K_b}\frac{u_{a,k}-u_{b,j}-i/2}{u_{a,k}-u_{b,j}+i/2} \\
1 &= \prod_{j\neq k}^{K_b}\frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i}\prod_{j=1}^{K_a}\frac{u_{b,k}-u_{a,j}-i/2}{u_{b,k}-u_{a,j}+i/2}\prod_{j=1}^{K_c}\frac{u_{b,k}-u_{c,j}-i/2}{u_{b,k}-u_{c,j}+i/2} \\
1 &= \prod_{j\neq k}^{K_c}\frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i}\prod_{j=1}^{K_b}\frac{u_{c,k}-u_{b,j}-i/2}{u_{c,k}-u_{b,j}+i/2}
\end{aligned} \tag{4.16}$$

Antifundamental representation:

The antifundamental representation of $su(4)$ (also denoted $\bar{\mathbf{4}}$) is easily obtained from the fundamental, since the highest weight $\vec{w}_{\bar{4}}$ simply coincides with the opposite of the lowest weight of the $\mathbf{4}$, that means $\vec{w}_{\bar{4}} = \vec{\varphi}_3$ (Dynkin labels $(0,0,1)$): in fact, the $\bar{\mathbf{4}}$ is the complex conjugate representation of the $\mathbf{4}$. Hence the Bethe equations (4.14) become:

$$\begin{aligned}
1 &= \prod_{j\neq k}^{K_a} \frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i} \prod_{j=1}^{K_b} \frac{u_{a,k}-u_{b,j}-i/2}{u_{a,k}-u_{b,j}+i/2} \\
1 &= \prod_{j\neq k}^{K_b} \frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i} \prod_{j=1}^{K_a} \frac{u_{b,k}-u_{a,j}-i/2}{u_{b,k}-u_{a,j}+i/2} \prod_{j=1}^{K_c} \frac{u_{b,k}-u_{c,j}-i/2}{u_{b,k}-u_{c,j}+i/2} \\
\left(\frac{u_{c,k}+\frac{i}{2}}{u_{c,k}-\frac{i}{2}}\right)^N &= \prod_{j\neq k}^{K_c} \frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i} \prod_{j=1}^{K_b} \frac{u_{c,k}-u_{b,j}-i/2}{u_{c,k}-u_{b,j}+i/2} \qquad (4.17)
\end{aligned}$$

Antisymmetric representation:

Since the tern $(0,1,0)$ corresponds to the set of Dynkin labels relative to the antisymmetric $\mathbf{6}$ of $su(4)$, this time $\vec{w}_6 = \vec{\varphi}_2$ plays the role of the highest weight. For later purpose, it turns out handful to display the complete set of weights in the representation:

$$\begin{aligned}
\vec{w}_6 &= \vec{\varphi}_2 & \vec{\varphi}_2 - \vec{\alpha}_2 &= \left(0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \qquad (4.18) \\
\vec{\varphi}_2 - \vec{\alpha}_2 - \vec{\alpha}_1 &= \left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{6}}\right) & \vec{\varphi}_2 - \vec{\alpha}_2 - \vec{\alpha}_3 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{6}}\right) \\
\vec{\varphi}_2 - \vec{\alpha}_2 - \vec{\alpha}_1 - \vec{\alpha}_3 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{6}}\right) & \vec{\varphi}_2 - 2\vec{\alpha}_2 - \vec{\alpha}_1 - \vec{\alpha}_3 &= -\vec{\varphi}_2 \quad .
\end{aligned}$$

The weights listed above coincide with the vertices of an octahedron in the three dimensional root space and, remarkably, for each one of them the opposite still belongs to the weight system: this fact signals that the representation is real (self-conjugate). At last, the $\mathbf{6}$ of $su(4)$ is described by the set of Bethe equations:

$$\begin{aligned}
1 &= \prod_{j\neq k}^{K_a} \frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i} \prod_{j=1}^{K_b} \frac{u_{a,k}-u_{b,j}-i/2}{u_{a,k}-u_{b,j}+i/2} \\
\left(\frac{u_{b,k}+\frac{i}{2}}{u_{b,k}-\frac{i}{2}}\right)^N &= \prod_{j\neq k}^{K_b} \frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i} \prod_{j=1}^{K_a} \frac{u_{b,k}-u_{a,j}-i/2}{u_{b,k}-u_{a,j}+i/2} \prod_{j=1}^{K_c} \frac{u_{b,k}-u_{c,j}-i/2}{u_{b,k}-u_{c,j}+i/2} \\
1 &= \prod_{j\neq k}^{K_c} \frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i} \prod_{j=1}^{K_b} \frac{u_{c,k}-u_{b,j}-i/2}{u_{c,k}-u_{b,j}+i/2} \qquad (4.19)
\end{aligned}$$

(again obtained by adapting (4.14)).

2 Towards the Asymptotic Bethe Ansatz

The present paragraph is intended to provide an explanatory derivation of the complete set of Asymptotic Bethe Ansatz (ABA) over the GKP (antiferromagnetic) vacuum. To begin with, a summary will be supplied of all the scattering matrices involving the momentum-carrier excitations. These matrices indeed play the role of building blocks towards the assembly of the ABA, the aim pursued so far. Nevertheless, the next step will not consist in directly displaying the equations for the most general and complete system. Instead, several sectors will be studied before, each containing only a narrower variety of excitations, so to highlight some peculiar features of the particles considered. After that, the widest and richest system will be finally taken into account.

2.1 Dramatis personae

In the previous chapter, the complete set of formulae was retrieved to describe the scattering matrices for all the excitations dwelling on the antiferromagnetic vacuum. Below, all those matrices will be resumed, in order to give the chance to survey them all in a glance: as a matter of fact, they represent the main characters in writing the Asymptotic Bethe Ansatz equations, therefore a concise recap will make them more suitable to be handled.

First of all, for sake of compactness of the following expressions, it turns out appropriate to introduce the kernel

$$\mathcal{K}(v,w) \equiv 2\pi\delta(v-w) - \frac{d\Theta}{dw}(v,w) \quad : \tag{4.20}$$

remarkably, the apparent ubiquitousness of such a function enables to appreciate that all the scattering processes rely on the scalar-scalar phase $\Theta(v,w)$, so that its fundamental role is left clearly pointed out.
The scattering phases are gathered in clusters, according to the types of particle involved. The features of the scattering processes (unitarity, crossing, charge conjugation) help in relating the matrices among themselves. Scalars, to begin with:

- **Scalar:**

$$\begin{aligned}
S^{(ss)}(u,u') &= -\exp[-i\Theta(u,u')] \qquad (4.21)\\
S^{(sg)}(u,u') &= \exp\Big\{-i\int\frac{dv}{2\pi}\,\mathcal{K}(u,v)\,[\chi(v,u'|1)+\Phi(v)]\Big\}\\
S^{(sF)}(u,u') &= \exp\Big\{-i\int\frac{dv}{2\pi}\,\mathcal{K}(u,v)\,[\chi_F(v,u')+\Phi(v)]\Big\}\\
S^{(s\bar{g})}(u,u') &= S^{(sg)}(u,u')\\
S^{(s\bar{F})}(u,u') &= S^{(sF)}(u,u')
\end{aligned}$$

Henceforth only gluons (and barred-gluons, as well) will be taken into account, leaving out gauge field bound states, instead:

• **Gluons:**

$$
\begin{aligned}
S^{(gs)}(u,u') &= [S^{(sg)}(u',u)]^{-1} \qquad (4.22)\\
S^{(gg)}(u,u') &= -\exp\Big\{-i\chi_0(u-u'|2)+\\
&+ i\int\frac{dv}{2\pi}\frac{dw}{2\pi}\,[\chi(v,u|1)+\Phi(w)]\,\frac{d\mathcal{K}}{dv}(v,w)\,[\chi(w,u'|1)+\Phi(w)]\Big\}\\
S^{(gF)}(u,u') &= -\exp\Big\{-i\chi_0(u-u'|1)+\\
&+ i\int\frac{dv}{2\pi}\frac{dw}{2\pi}\,[\chi(v,u|1)+\Phi(v)]\,\frac{d\mathcal{K}}{dv}(v,w)\,[\chi_F(w,u')+\Phi(w)]\Big\}\\
S^{(\bar{g}s)}(u,u') &= [S^{(s\bar{g})}(u',u)]^{-1}=S^{(gs)}(u,u')=[S^{(sg)}(u',u)]^{-1}\\
S^{(\bar{g}\bar{g})}(u,u') &= S^{(gg)}(u,u')=\\
&= S^{g\bar{g}}(u,u')\frac{u-u'+i}{u-u'-i}=[S^{\bar{g}g}(u',u)]^{-1}\frac{u-u'+i}{u-u'-i}\\
S^{(\bar{g}\bar{F})}(u,u') &= S^{(gF)}(u,u')\\
S^{(g\bar{F})}(u,u') &= S^{(\bar{g}F)}(u,u')=\\
&= \exp\Big\{i\int\frac{dv}{2\pi}\frac{dw}{2\pi}\,[\chi(v,u|1)+\Phi(v)]\frac{d\mathcal{K}}{dv}(v,w)\,[\chi_F(w,u')+\Phi(w)]\Big\}
\end{aligned}
$$

Below, just large fermion scattering phases are listed:

• **Large fermions:**

$$
\begin{aligned}
S^{(Fs)}(u,u') &= [S^{(sF)}(u',u)]^{-1} \qquad (4.23)\\
S^{(Fg)}(u,u') &= [S^{(gF)}(u',u)]^{-1}\\
S^{(FF)}(u,u') &= \exp\Big\{i\int\frac{dv}{2\pi}\frac{dw}{2\pi}\,[\chi_F(v,u)+\Phi(v)]\,\frac{d\mathcal{K}}{dv}(v,w)\,[\chi_F(w,u')+\Phi(w)]\Big\}\\
S^{(\bar{F}s)}(u,u') &= S^{(Fs)}(u,u')\\
S^{(\bar{g}\bar{F})}(u,u') &= S^{(gF)}(u,u')\\
S^{(F\bar{g})}(u,u') &= [S^{(\bar{g}F)}(u',u)]^{-1}=\\
&= \exp\Big\{-i\int\frac{dv}{2\pi}\frac{dw}{2\pi}\,[\chi(v,u'|1)+\Phi(v)]\,\frac{d\mathcal{K}}{dv}(v,w)\,[\chi_F(w,u)+\Phi(w)]\Big\}\\
S^{(\bar{F}g)}(u,u') &= [S^{(g\bar{F})}(u',u)]^{-1}\\
S^{(F\bar{F})}(u,u') &= S^{(FF)}(u,u')=S^{(\bar{F}F)}(u,u')=S^{(\bar{F}\bar{F})}(u,u')
\end{aligned}
$$

The expressions for the scattering matrices involving small fermions may be reconstructed directly from the analogous ones for large fermions, after replacing:

$$
\chi_F(v,u)+\Phi(v)\;\longrightarrow\;-\chi_H(v,u)\quad . \qquad (4.24)
$$

Finally, it is worth recollecting from (4.2)-(4.4) the interaction of the the isotopic roots with the momentum-carrying particles, just aiming to point out which excitations the formers act on:

• u_a roots - fermions (both large and small):

$$
i\ln S^{(aF)}(u_a,u)=\frac{u_a-u-i/2}{u_a-u+i/2}=i\ln S^{(af)}(u_a,u)
$$

• u_b roots - holes:

$$i \ln S^{(bs)}(u_b, u) = \frac{u_b - u - i/2}{u_b - u + i/2}$$

• u_c roots - antifermions (both large and small):

$$i \ln S^{(c\bar{F})}(u_c, u) = \frac{u_c - u - i/2}{u_c - u + i/2} = i \ln S^{(c\bar{f})}(u_c, u) \quad .$$

Strictly speaking, the expressions above are not scattering matrices, since the auxiliary roots u_a, u_b, u_c are not actual particles, but just artifice products stemming from solving a nested Bethe ansatz (see section 3 of chapter 1).

2.2 Restricted sectors

As already explained, the excitations studied throughout this text (scalars, gluons, fermions) belong to some multiplet under the $SU(4)$ symmetry, in dependence of the kind of particle. Starting from the scattering matrices retrieved in the previous chapter, the Bethe equations may be assembled for every sort of excitation; anyway, they are actually able to catch only a single state in each multiplet, precisely the one corresponding to the highest weight state of the representation. In this paragraph the focus moves to a few sectors of the complete theory, which include just one type (or two at most) of excitations, along with the set of isotopic roots: the purpose is to show the behaviour of the different kinds of particle under $SU(4)$ and also to explain how to include all the states forming the multiplets into the Bethe equations.

• Scalar sector:
A system composed only of scalar excitations and isotopic roots will now be considered. Since u_a and u_c do not couple to main roots u_4, the counting function (3.50) experience a modification stemming from u_b roots alone, hence taking the form:

$$Z_4(u) = \sum_{h=2}^{L-1} \Theta(u, u_h) - R\Lambda_s(u) + \delta Z_4(u) \quad . \tag{4.25}$$

The variation $\delta Z_4(u)$ due to the introduction of type-b roots reads

$$\delta Z_4(u) = \delta\tilde{P}(u) - \sum_{j=1}^{K_b} [R(u, u_{b,j} + i/2) + R(u, u_{b,j} - i/2)] \tag{4.26}$$

where the function $R(u, v)$ is the solution of (3.27), whereas $\delta\tilde{P}(u)$ is found to solve the equation:

$$\delta\tilde{P}(u) = 2i \sum_{j=1}^{K_b} \ln\left(-\frac{x^+(u) - x_{b,j}}{x^-(u) - x_{b,j}}\right) - \int dv\, \varphi(u, v)\delta\tilde{P}(v) \quad . \tag{4.27}$$

While the one loop contribution to the variation amounts to

$$\delta Z_{4,0}(u) = i \sum_{j=1}^{K_b} \ln \frac{i/2 + u - u_{b,j}}{i/2 - u + u_{b,j}} \quad . \tag{4.28}$$

the higher-than-one-loop part $\delta Z_{4,0}$ satisfies the equation

$$\begin{aligned} \delta Z_{4,H}(u) &= 2i \sum_{j=1}^{K_b} \left[\ln \frac{1 - \frac{g^2}{2x^+(u)x^{--}_{b,j}}}{1 - \frac{g^2}{2x^-(u)x^{++}_{b,j}}} + i\theta(u, u_{b,j} + i/2) + i\theta(u, u_{b,j} - i/2) \right] + \\ &+ \int \frac{dv}{2\pi} \phi_H(u,v) \frac{d}{dv} \delta Z_{4,0}(v) - \int dv \, \varphi(u,v) \delta Z_{4,H}(v) \quad , \end{aligned}$$

and, eventually, it can be proved that $\delta Z_{4,H}(u) = 0$, therefore the scalar counting function (4.25) becomes:

$$Z_4(u) = \sum_{h=2}^{L-1} \Theta(u, u_h) - R\Lambda_s(u) + i \sum_{j=1}^{K_b} \ln \frac{i/2 + u - u_{b,j}}{i/2 - u + u_{b,j}} \tag{4.29}$$

The Bethe equations $e^{iZ_4(u_h)} = (-1)^{L-1}$ at last assume the form:

$$1 = e^{-iR\Lambda_s(u_h)} \prod_{h'=2,\ h' \neq = h}^{L-1} S(u_h, u_{h'}) \prod_{j=1}^{K_b} \frac{u_h - u_{b,j} + i/2}{u_h - u_{b,j} - i/2} \tag{4.30}$$

When, in addition to the $L-2$ scalar excitations, the system includes K_a roots of type u_a, K_b of type u_b and K_c type-c roots, the set (4.30) must be supplemented with the equations for auxiliary roots, then:

$$\begin{aligned} 1 &= \prod_{j \neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}} \\ \prod_{h=2}^{L-1} \left(\frac{u_{b,k} - u_h + \frac{i}{2}}{u_{b,k} - u_h - \frac{i}{2}} \right) &= \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}} \\ 1 &= \prod_{j \neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}} \end{aligned} \tag{4.31}$$

A comparison with (4.19) promptly reveals that the equations (4.31) describe a spin chain associated to the **6** of $su(4)$, whose length is $L-2$ (matching the number of internal holes); the *r.h.s.* of the second in (4.31) denotes that the scalar excitations behave like $L-2$ inhomogeneities (with rapidities u_h, $h = 2, \dots, L-1$), whose dynamics is regulated by the equations (4.30). Finally, (4.19) and (4.31), together, entail that scalars transform according to the antisymmetric representation (the **6**) of $su(4)$.

In the gauge theory standpoint, the structure of (4.31) enables to infer the recipe to obtain all the states of the **6** multiplet, descending from the highest weight. The $\mathcal{N} = 4$ SYM gauge theory contains six scalar degrees of freedom: for convenience, they could be arranged into three complex fields, say Z, W, Y, along with their complex conjugated

$\bar{Z}$, $\bar{W}$, $\bar{Y}$. Let Z be associated to the highest weight state of the **6**. Recalling (3.29), the bearing of a single Z excitation matches to the gauge-invariant operator

$$\mathcal{O}' \sim Tr[Z\, D_+^{s-s_1}\, Z\, D_+^{s_1} Z] + \dots \quad ,$$

which will be denoted $|Z\,\rangle$ hereafter; likewise $|ZZ\,\rangle$ stands for the two Z state, while the insertion of any other scalar field correspond to $|W\,\rangle$, $|Y\,\rangle$, and so on. In order to reconstruct the complete **6** from $|Z\,\rangle$, it could be af any use to recollect a few notions about the $su(4)$ algebra: the Cartan subalgebra includes three generators $(H_1,\, H_2,\, H_3) = \vec{H}$, while the others arrange into couples of raising (lowering) generators $E^+_{\alpha_j}$ $(E^-_{\alpha_j})$ where α_j stands for a simple root[4]. Let an arbitrary state of the representation be considered, whose weight is $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$

$$(H_1, H_2, H_3)|\mu\,\rangle = \vec{\mu}|\mu\,\rangle \quad :$$

$|\mu\,\rangle$ can be deduced from the highest weight state $|w_6\,\rangle \equiv |Z\,\rangle$ (such that $\vec{H}|w_6\,\rangle = \vec{w}_6|w_6\,\rangle$) by properly acting on it with lowering generators

$$\begin{aligned} E^-_{\alpha_{j_k}} \cdots E^-_{\alpha_{j_1}}|w_6\,\rangle &= |\mu\,\rangle \\ \vec{H}\, E^-_{\alpha_{j_k}} \cdots E^-_{\alpha_{j_1}}|w_6\,\rangle &= (\vec{w}_6 - \vec{\alpha}_{j_k} \cdots \vec{\alpha}_{j_1})|\mu\,\rangle = \vec{\mu}|\mu\,\rangle \quad . \end{aligned}$$

Upon comparing to (4.14), equations (4.31) tell the only way to act on the highest weight state consists in applying once the lowering generator related to the root $\vec{\alpha}_2$ (4.11), as all the inner products (in the three dimensional root space) $2\vec{\alpha}_j \cdot \vec{w}_6 = 0$, except for $j = 2$ (indeed $2\vec{\alpha}_2 \cdot \vec{w}_6 = 1/2$) . Since the fundamental weights of $su(4)$ are known (4.13), a glance at (4.31) shows that all the right sides are equal to one, but the second, and it is thus easy to infer the Dynkin labels of the representation are $(0, 1, 0)$: once the Dynkin labels and the highest weight of the **6**, and also the simple roots are known, the whole representation is achieved, as those features fix the sequences of lowering generators to retrieve all the states. In the end, all the multiplet can be reconstructed:

$$\begin{array}{ll} |Z\,\rangle & \vec{H}|Z\,\rangle = \vec{w}_6 \\ |W\,\rangle = E^-_{\alpha_2}|Z\,\rangle & \vec{H}\, E^-_{\alpha_2}|Z\,\rangle = (\vec{w}_6 - \vec{\alpha}_2)|W\,\rangle \\ |Y\,\rangle = E^-_{\alpha_1}|W\,\rangle = E^-_{\alpha_1}E^-_{\alpha_2}|Z\,\rangle & \vec{H}\, E^-_{\alpha_1}E^-_{\alpha_2}|Z\,\rangle = (\vec{w}_6 - \vec{\alpha}_1 - \vec{\alpha}_2)|Y\,\rangle \\ |\bar{Y}\,\rangle = E^-_{\alpha_3}|W\,\rangle = E^-_{\alpha_3}E^-_{\alpha_2}|Z\,\rangle & \vec{H}\, E^-_{\alpha_3}E^-_{\alpha_2}|Z\,\rangle = (\vec{w}_6 - \vec{\alpha}_3 - \vec{\alpha}_2)|\bar{Y}\,\rangle \\ |\bar{W}\,\rangle = E^-_{\alpha_3}|Y\,\rangle = E^-_{\alpha_1}|\bar{Y}\,\rangle & \vec{H}|\bar{W}\,\rangle = (\vec{w}_6 - \vec{\alpha}_3 - \vec{\alpha}_2 - \vec{\alpha}_1)|\bar{W}\,\rangle \\ |\bar{Z}\,\rangle = E^-_{\alpha_2}|\bar{W}\,\rangle & \vec{H}|\bar{Z}\,\rangle = (\vec{w}_6 - \vec{\alpha}_3 - 2\vec{\alpha}_2 - \vec{\alpha}_1)|\bar{Z}\,\rangle \ ; \end{array} \tag{4.32}$$

the derivation of these states parallels the computation of the weights of the **6** (4.18).

• (Large) Fermionic sector

In order to study the properties of large fermions, it is appropriate to stick to a system composed of N_F large fermions $u_{F,j}$, $j = 1, ..., N_F$, together with K_a roots of type u_a,

[4]For the present purpose, the explicit form of the generators is not interesting, for only the root system (4.11) matters

K_b of type and K_b u_c roots. As previously shown, fermions do interact only with the u_a roots, hence only that kind of auxiliary particles affects the fermionic counting function; anyway, reasoning analogous to the scalar case prove that just the one loop contributions survive, the higher loop effects being erased. Therefore for the system at hand the Bethe equations read:

$$1 = \exp\left[i\Lambda_F(u_{F,k})\right] \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + \frac{i}{2}}{u_{F,k} - u_{a,j} - \frac{i}{2}} \prod_{j=1}^{N_F} S^{(FF)}(u_{F,k}, u_{F,j}) \quad . \tag{4.33}$$

Alongside them, the equations for auxiliary roots must be added:

$$\begin{aligned} \prod_{j=1}^{N_F} \left(\frac{u_{a,k} - u_{F,j} + \frac{i}{2}}{u_{a,k} - u_{F,j} - \frac{i}{2}} \right) &= \prod_{j\neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}} \\ 1 &= \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}} \\ 1 &= \prod_{j\neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}} \end{aligned} \tag{4.34}$$

A look to (4.16) suggests the equations (4.34) should be associated to a spin chain related to the **4** of $su(4)$; the number of fermions N_F represents the chain length, while the fermion themselves are treated as inhomogeneities, whose rapidities $u_{F,j}$, $j = 1, \ldots, N_F$ should be deduced from equations (4.33).

Otherwise, when only large antifermions (in number of $N_{\bar{F}}$) appears on the vacuum, again accompanied by isotopic roots u_a (K_a), u_b (K_b) and K_c (K_c), the system is described by the set of Bethe equations

$$1 = \exp\left[i\Lambda_{\bar{F}}(u_{\bar{F},k})\right] \prod_{j=1}^{K_c} \frac{u_{\bar{F},k} - u_{c,j} + \frac{i}{2}}{u_{\bar{F},k} - u_{c,j} - \frac{i}{2}} \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{F}\bar{F})}(u_{\bar{F},k}, u_{\bar{F},j}) \quad . \tag{4.35}$$

together with the auxiliary root equations

$$\begin{aligned} 1 &= \prod_{j\neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}} \\ 1 &= \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}} \\ \prod_{j=1}^{N_{\bar{F}}} \left(\frac{u_{c,k} - u_{\bar{F},j} + \frac{i}{2}}{u_{c,k} - u_{\bar{F},j} - \frac{i}{2}} \right) &= \prod_{j\neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}} \quad . \end{aligned} \tag{4.36}$$

The (4.36) are in fact the equations for $\bar{\mathbf{4}}$ spin chain, as may be read from (4.17). The inhomogeneity rapidities (else, the antifermions) $u_{\bar{F},j}$ can be obtained from the Bethe equations (4.35).

Some interest should be paid to a system including both (large) fermions and (large) antifermions, jointly to auxiliary roots. In order to highlight a peculiar feature of these kinds of particles, let fermions and antifermions be excited, setting the number of them so to coincide, that is $N_F = N_{\bar{F}} = N$; they are described by the Bethe equations

$$\begin{aligned} 1 &= \exp\left[i\Lambda_F(u_{F,k})\right] \prod_{j=1}^{N} S^{(FF)}(u_{F,k}, u_{F,j}) \prod_{j=1}^{N} S^{(F\bar{F})}(u_{F,k}, u_{\bar{F},j}) \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + \frac{i}{2}}{u_{F,k} - u_{a,j} - \frac{i}{2}} \\ 1 &= \exp\left[i\Lambda_{\bar{F}}(u_{\bar{F},k})\right] \prod_{j=1}^{N} S^{(\bar{F}\bar{F})}(u_{\bar{F},k}, u_{\bar{F},j}) \prod_{j=1}^{N} S^{(\bar{F}F)}(u_{\bar{F},k}, u_{F,j}) \prod_{j=1}^{K_c} \frac{u_{\bar{F},k} - u_{c,j} + \frac{i}{2}}{u_{\bar{F},k} - u_{c,j} - \frac{i}{2}} \end{aligned} \tag{4.37}$$

whereas for the auxiliary roots the relations hold:

$$\begin{aligned} \prod_{j=1}^{N} \left(\frac{u_{a,k} - u_{F,j} + \frac{i}{2}}{u_{a,k} - u_{F,j} - \frac{i}{2}} \right) &= \prod_{j\neq k}^{K_a} \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i} \prod_{j=1}^{K_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2}}{u_{a,k} - u_{b,j} + \frac{i}{2}} \\ 1 &= \prod_{j=1}^{K_b} \frac{u_{b,k} - u_{b,j} + i}{u_{b,k} - u_{b,j} - i} \prod_{j=1}^{K_a} \frac{u_{b,k} - u_{a,j} - \frac{i}{2}}{u_{b,k} - u_{a,j} + \frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k} - u_{c,j} - \frac{i}{2}}{u_{b,k} - u_{c,j} + \frac{i}{2}} \\ \prod_{j=1}^{N} \left(\frac{u_{c,k} - u_{\bar{F},j} + \frac{i}{2}}{u_{c,k} - u_{\bar{F},j} - \frac{i}{2}} \right) &= \prod_{j\neq k}^{K_c} \frac{u_{c,k} - u_{c,j} + i}{u_{c,k} - u_{c,j} - i} \prod_{j=1}^{K_b} \frac{u_{c,k} - u_{b,j} - \frac{i}{2}}{u_{c,k} - u_{b,j} + \frac{i}{2}} \quad . \end{aligned} \tag{4.38}$$

The knowledge of $su(4)$ simple roots (4.11) and fundamental weights (4.13) enable to claim the equations (4.38) are associated to a spin chain, related to the representation of $su(4)$ whose Dynkin labels are $(1, 0, 1)$, namely the **15**. The reason stems from the physical interpretation of the event described. Fermions and antifermions are intended to collide: since they transform according respectively to the **4** and the $\bar{\mathbf{4}}$ of $su(4)$, their scattering is expected to mirror the direct product of representations $\mathbf{4} \otimes \bar{\mathbf{4}}$. The Clebsch-Gordan rule entails that the process decomposes into two channels:

$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1} \oplus \mathbf{15} \quad ; \tag{4.39}$$

the singlet **1** channel is concealed in (4.38), so that the **15** stands out.

• **Gluonic sector**

The case now considered concerns a system where N_g gluons (with rapidities u_j^g) arise over the vacuum, together with isotopic roots. The gluon rapidities are constrained by the Bethe equations

$$1 = \exp\left[i\Lambda_g(u_k^g)\right] \prod_{j\neq k}^{N_g} S^{(gg)}(u_k^g, u_j^g) \tag{4.40}$$

while the isotopic roots decouple from them, since

$$\begin{aligned}
1 &= \prod_{j\neq k}^{K_a} \frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i} \prod_{j=1}^{K_b} \frac{u_{a,k}-u_{b,j}-\frac{i}{2}}{u_{a,k}-u_{b,j}+\frac{i}{2}} \qquad (4.41)\\
1 &= \prod_{j=1}^{K_b} \frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i} \prod_{j=1}^{K_a} \frac{u_{b,k}-u_{a,j}-\frac{i}{2}}{u_{b,k}-u_{a,j}+\frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k}-u_{c,j}-\frac{i}{2}}{u_{b,k}-u_{c,j}+\frac{i}{2}}\\
1 &= \prod_{j\neq k}^{K_c} \frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i} \prod_{j=1}^{K_b} \frac{u_{c,k}-u_{b,j}-\frac{i}{2}}{u_{c,k}-u_{b,j}+\frac{i}{2}} \quad :
\end{aligned}$$

therefore, gluon excitations behave like singlets (**1**) under $SU(4)$. The very same reasoning applies to barred-gluons.

3 General equations

This paragraph is meant to provide the complete Asymptotic Bethe Ansatz equations, intended to describe the most general system set up with every sort of excitation over the antiferromagnetic vacuum. The scattering matrices[5] appearing in the following equations have been listed in paragraph 2.1. Finally, the explicit expressions for the free-particle propagation phases $\Lambda_*(u)$ will be provided at the end of this section: each one of these functions stands for the phase owned by a particle, when no further excitations are present in the spin chain, so that the Bethe ansatz equations simply reduce to the momentum quantization conditions in a periodic system. As hinted when studying the one loop scalar case in section 3 of the previous chapter, the scattering phases are composed of a part amenable to the particle momentum, plus a further phase shift coming from defects in the spin chain.
Below, the ABA equations are displayed per type of excitation:

- **Scalars:**

$$\begin{aligned}
1 &= e^{i\Lambda_s(u_h)} \prod_{\{h'=1,\, h'\neq h\}}^{H} S^{(ss)}(u_h, u_{h'}) \prod_{j=1}^{K_b} \frac{u_h-u_{b,j}+\frac{i}{2}}{u_h-u_{b,j}-\frac{i}{2}} \times \qquad (4.42)\\
&\times \prod_{j=1}^{N_g} S^{(sg)}(u_h, u_j^g) \prod_{j=1}^{N_{\bar g}} S^{(s\bar g)}(u_h, u_j^{\bar g}) \prod_{j=1}^{N_F} S^{(sF)}(u_h, u_{F,j}) \prod_{j=1}^{N_{\bar F}} S^{(s\bar F)}(u_h, u_{\bar F,j})\\
&\times \prod_{j=1}^{N_f} S^{(sf)}(u_h, u_{f,j}) \prod_{j=1}^{N_{\bar f}} S^{(s\bar f)}(u_h, u_{\bar f,j})
\end{aligned}$$

[5] More precisely, they represent the scalar overall factor multiplying the pure matrix part

• **Gluons:**

$$1 = \exp\left[i\Lambda_g(u_k^g)\right] \prod_{j\neq k}^{N_g} S^{(gg)}(u_k^g, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(g\bar{g})}(u_k^g, u_j^{\bar{g}}) \times \tag{4.43}$$
$$\times \prod_{h=1}^{H} S^{(gs)}(u_k^g, u_h) \prod_{j=1}^{N_F} S^{(gF)}(u_k^g, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(g\bar{F})}(u_k^g, u_{\bar{F},j}) \times$$
$$\times \prod_{j=1}^{N_f} S^{(gf)}(u_k^g, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(g\bar{f})}(u_k^g, u_{\bar{f},j})$$

• **Barred-gluons:**

$$1 = \exp\left[i\Lambda_g(u_k^{\bar{g}})\right] \prod_{j\neq k}^{N_{\bar{g}}} S^{(\bar{g}\bar{g})}(u_k^{\bar{g}}, u_j^{\bar{g}}) \prod_{j=1}^{N_g} S^{(\bar{g}g)}(u_k^{\bar{g}}, u_j^g) \times \tag{4.44}$$
$$\times \prod_{h=1}^{H} S^{(\bar{g}s)}(u_k^{\bar{g}}, u_h) \prod_{j=1}^{N_F} S^{(\bar{g}F)}(u_k^{\bar{g}}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{g}\bar{F})}(u_k^{\bar{g}}, u_{\bar{F},j}) \times$$
$$\times \prod_{j=1}^{N_f} S^{(\bar{g}f)}(u_k^{\bar{g}}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{g}\bar{f})}(u_k^{\bar{g}}, u_{\bar{f},j}) \tag{4.45}$$

• **Large fermions:**

$$1 = \exp\left[i\Lambda_F(u_{F,k})\right] \prod_{j=1}^{N_F} S^{(FF)}(u_{F,k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(F\bar{F})}(u_{F,k}, u_{\bar{F},j}) \times \tag{4.46}$$
$$\times \prod_{j=1}^{N_f} S^{(Ff)}(u_{F,k}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(F\bar{f})}(u_{F,k}, u_{\bar{f},j}) \prod_{h=1}^{H} S^{(Fs)}(u_{F,k}, u_h) \times$$
$$\times \prod_{j=1}^{N_g} S^{(Fg)}(u_{F,k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(F\bar{g})}(u_{F,k}, u_j^{\bar{g}}) \prod_{j=1}^{K_a} \frac{u_{F,k} - u_{a,j} + \frac{i}{2}}{u_{F,k} - u_{a,j} - \frac{i}{2}}$$

• **Large antifermions:**

$$
\begin{aligned}
1 &= \exp\left[i\Lambda_F(u_{\bar{F},k})\right] \prod_{j=1}^{N_F} S^{(\bar{F}F)}(u_{\bar{F},k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{F}\bar{F})}(u_{\bar{F},k}, u_{\bar{F},j}) \times \\
&\times \prod_{j=1}^{N_f} S^{(\bar{F}f)}(u_{\bar{F},k}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{F}\bar{f})}(u_{\bar{F},k}, u_{\bar{f},j}) \prod_{h=1}^{H} S^{(\bar{F}s)}(u_{\bar{F},k}, u_h) \times \\
&\times \prod_{j=1}^{N_g} S^{(\bar{F}g)}(u_{\bar{F},k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(\bar{F}\bar{g})}(u_{\bar{F},k}, u_j^{\bar{g}}) \prod_{j=1}^{K_c} \frac{u_{\bar{F},k} - u_{c,j} + \frac{i}{2}}{u_{\bar{F},k} - u_{c,j} - \frac{i}{2}}
\end{aligned} \tag{4.47}
$$

- **Small fermions:**

$$
\begin{aligned}
1 &= \exp\left[i\Lambda_f(u_{f,k})\right] \prod_{j=1}^{N_f} S^{(ff)}(u_{f,k}, u_{f,j}) \prod_{j=1}^{N_{\bar{f}}} S^{(f\bar{f})}(u_{f,k}, u_{\bar{f},j}) \times \\
&\times \prod_{j=1}^{N_F} S^{(fF)}(u_{f,k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(f\bar{F})}(u_{f,k}, u_{\bar{F},j}) \prod_{h=1}^{H} S^{(fs)}(u_{f,k}, u_h) \times \\
&\times \prod_{j=1}^{N_g} S^{(fg)}(u_{f,k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(f\bar{g})}(u_{f,k}, u_j^{\bar{g}}) \prod_{j=1}^{K_a} \frac{u_{f,k} - u_{a,j} + \frac{i}{2}}{u_{f,k} - u_{a,j} - \frac{i}{2}}
\end{aligned} \tag{4.48}
$$

- **Small antifermions:**

$$
\begin{aligned}
1 &= \exp\left[i\Lambda_f(u_{\bar{f},k})\right] \prod_{j=1}^{N_f} S^{(\bar{f}f)}(u_{\bar{f},k}, u_{\bar{f},j}) \prod_{j=1}^{N_{\bar{f}}} S^{(\bar{f}\bar{f})}(u_{\bar{f},k}, u_{\bar{f},j}) \times \\
&\times \prod_{j=1}^{N_F} S^{(\bar{f}F)}(u_{\bar{f},k}, u_{F,j}) \prod_{j=1}^{N_{\bar{F}}} S^{(\bar{f}\bar{F})}(u_{\bar{f},k}, u_{\bar{F},j}) \prod_{h=1}^{H} S^{(\bar{f}s)}(u_{\bar{f},k}, u_h) \times \\
&\times \prod_{j=1}^{N_g} S^{(\bar{f}g)}(u_{\bar{f},k}, u_j^g) \prod_{j=1}^{N_{\bar{g}}} S^{(\bar{f}\bar{g})}(u_{\bar{f},k}, u_j^{\bar{g}}) \prod_{j=1}^{K_c} \frac{u_{\bar{f},k} - u_{c,j} + \frac{i}{2}}{u_{\bar{f},k} - u_{c,j} - \frac{i}{2}}
\end{aligned} \tag{4.49}
$$

- **Isotopic roots:**

$$1 = \prod_{j\neq k}^{K_a} \frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i} \prod_{j=1}^{K_b} \frac{u_{a,k}-u_{b,j}-\frac{i}{2}}{u_{a,k}-u_{b,j}+\frac{i}{2}} \prod_{j=1}^{N_F} \frac{u_{a,k}-u_{F,j}-\frac{i}{2}}{u_{a,k}-u_{F,j}+\frac{i}{2}} \prod_{j=1}^{N_f} \frac{u_{a,k}-u_{f,j}-\frac{i}{2}}{u_{a,k}-u_{f,j}+\frac{i}{2}} \tag{4.50}$$

$$1 = \prod_{j\neq k}^{K_b} \frac{u_{b,k}-u_{b,j}+i}{u_{b,k}-u_{b,j}-i} \prod_{j=1}^{K_a} \frac{u_{b,k}-u_{a,j}-\frac{i}{2}}{u_{b,k}-u_{a,j}+\frac{i}{2}} \prod_{j=1}^{K_c} \frac{u_{b,k}-u_{c,j}-\frac{i}{2}}{u_{b,k}-u_{c,j}+\frac{i}{2}} \prod_{h=1}^{H} \frac{u_{b,k}-u_{h}-\frac{i}{2}}{u_{b,k}-u_{h}+\frac{i}{2}} \tag{4.51}$$

$$1 = \prod_{j\neq k}^{K_c} \frac{u_{c,k}-u_{c,j}+i}{u_{c,k}-u_{c,j}-i} \prod_{j=1}^{K_b} \frac{u_{c,k}-u_{b,j}-\frac{i}{2}}{u_{c,k}-u_{b,j}+\frac{i}{2}} \prod_{j=1}^{N_{\bar F}} \frac{u_{c,k}-u_{\bar F,j}-\frac{i}{2}}{u_{c,k}-u_{\bar F,j}+\frac{i}{2}} \prod_{j=1}^{N_{\bar f}} \frac{u_{c,k}-u_{\bar f,j}-\frac{i}{2}}{u_{c,k}-u_{\bar f,j}+\frac{i}{2}} \tag{4.52}$$

Propagation phases:

First, it is convenient to define the number ΔP, whose meaning will get clearer later:

$$\begin{aligned} \Delta P \equiv\ & \sum_{h=1}^{H} \tilde P(u_h) - \sum_{j=1}^{N_g} \int \frac{dv}{2\pi} \chi(v,u_j^g|1) \frac{d}{dv}\tilde P(v) - \sum_{j=1}^{N_{\bar g}} \int \frac{dv}{2\pi} \chi(v,u_j^{\bar g}|1) \frac{d}{dv}\tilde P(v) - \\ - & \sum_{j=1}^{N_F} \int \frac{dv}{2\pi} \chi_F(v,u_{F,j}) \frac{d}{dv}\tilde P(v) - \sum_{j=1}^{N_{\bar F}} \int \frac{dv}{2\pi} \chi_F(v,u_{\bar F,j}) \frac{d}{dv}\tilde P(v) + \\ + & \sum_{j=1}^{N_f} \int \frac{dv}{2\pi} \chi_H(v,u_{f,j}) \frac{d}{dv}\tilde P(v) + \sum_{j=1}^{N_{\bar f}} \int \frac{dv}{2\pi} \chi_H(v,u_{\bar f,j}) \frac{d}{dv}\tilde P(v) \end{aligned} \tag{4.53}$$

The propagation phase appearing in the scalar Bethe equations (4.42) corresponds to:

$$-\Lambda_s(u) = 2\tilde P(u) + R(u,s/\sqrt2) + R(u,-s/\sqrt2) + 2\int dv\, G(u,v)L_4(v) + \Delta P \tag{4.54}$$

while for a gluon (4.43) it holds:

$$\begin{aligned} -\Lambda_g(u) =\ & -\int \frac{dv}{2\pi}\chi(v,u|1)\frac{d}{dv}[R(v,s/\sqrt2)+R(v,-s/\sqrt2)] + \int \frac{dv}{\pi} F_G(v,u)L_4'(v) - \\ - & 2\int \frac{dv}{2\pi}\chi(v,u|1)\frac{d}{dv}\tilde P(v) + \Delta P \end{aligned} \tag{4.55}$$

(the function stays the same also for barred-gluons (4.44)). Both large fermions (4.47) and large antfermions (4.47), the propagation phase reads

$$\begin{aligned} -\Lambda_F(u) =\ & -\int \frac{dv}{2\pi}\chi_F(v,u)\frac{d}{dv}[R(v,s/\sqrt2)+R(v,-s/\sqrt2)] + \int \frac{dv}{\pi} F_F(v,u)L_4'(v) - \\ - & 2\int \frac{dv}{2\pi}\chi_F(v,u)\frac{d}{dv}\tilde P(v) + \Delta P \end{aligned} \tag{4.56}$$

and finally for small fermions (4.48) and antifermions (4.49), it is found that

$$\begin{aligned}
-\Lambda_f(u) &= \int \frac{dv}{2\pi}\chi_H(v,u)\frac{d}{dv}[R(v,s/\sqrt{2})+R(v,-s/\sqrt{2})]+\int \frac{dv}{\pi}F_f(v,u)L_4'(v) \\
&+ 2\int \frac{dv}{2\pi}\chi_H(v,u)\frac{d}{dv}\tilde{P}(v) = \qquad\qquad (4.57)\\
&= -\int \frac{dw}{2\pi}\frac{d}{dw}\left[\Theta(s/\sqrt{2},w)+\Theta(-s/\sqrt{2},w)\right]\chi_H(w,u)+ \\
&+ \int \frac{dv}{\pi}\left[\frac{d}{dv}\chi_H(v,u)-\int \frac{dw}{2\pi}\frac{d^2}{dvdw}\Theta(v,w)\chi_H(w,u)\right]L_4(v) \quad .
\end{aligned}$$

Complete scalar counting function:
Since the scalar counting function appears in the formula (3.17), massively exploited throughout this text can, it might be considered the most fundamental one. Therefore it is worth writing down the non-linear integral equation , in its most general form:

$$\begin{aligned}
Z_4(u) &= (L+N_g+N_{\bar{g}}+N_F+N_{\bar{F}})\tilde{P}(u)+\sum_{h=2}^{L-2}R(u,u_h)+\sum_{j=1}^{N_g}T(u,u_j^g|1)+ \\
&+ \sum_{j=1}^{N_{\bar{g}}}T(u,u_j^{\bar{g}}|1)+\sum_{j=1}^{N_F}F_F(u,u_{F,j})+\sum_{j=1}^{N_{\bar{F}}}F_F(u,u_{\bar{F},j})+ \\
&+ \sum_{j=1}^{N_f}F_f(u,u_{f,j})+\sum_{j=1}^{N_{\bar{f}}}F_f(u,u_{\bar{f},j})+\sum_{j=1}^{K_b}+2\int dv\, G(u,v)L_4(v) = \\
= \quad & \sum_{h=1}^{L}[\Theta(u,u_h)+\tilde{P}(u,h)]+\sum_{j=1}^{N_g}[T(u,u_j^g|1)+\tilde{P}(u)]+ \qquad\qquad (4.58)\\
&+ \sum_{j=1}^{N_{\bar{g}}}[T(u,u_j^{\bar{g}}|1)+\tilde{P}(u)]+\sum_{j=1}^{N_F}[F_F(u,u_{F,j})+\tilde{P}(u)]+\sum_{j=1}^{N_{\bar{F}}}[F_F(u,u_{\bar{F},j})+\tilde{P}(u)] \\
&+ \sum_{j=1}^{N_f}[F_f(u,u_{f,j})+\tilde{P}(u)]+\sum_{j=1}^{N_{\bar{f}}}[F_f(u,u_{\bar{f},j})+\tilde{P}(u)]+\sum_{j=1}^{K_b} \\
&+ 2\int dv\, G(u,v)L_4(v) \quad .
\end{aligned}$$

It is now appropriate to point out a few remarks about the scalar counting function when no excitations scalar are present, *i.e.* referring to formula (4.58) when all the excitations numbers are set to zero while $L=2$:

$$Z_4(u)=2\tilde{P}(u)+R(u,u_1)+R(u,u_2) \quad ,$$

where u_1, u_2 are the external holes, $u_1=-u_2=\dfrac{s}{\sqrt{2}}+O(s^0)$. By recalling (3.72), it is found that[6]:

$$R(u,u_1)+R(u,-u_1)+2\tilde{P}(u)=\int^u dv\,\sigma(v)|_{NIH} \qquad (4.59)$$

[6]In the $s\longrightarrow\infty$ limit, $\tilde{P}(u)=O(1)$

Much interest lies in the function Z_{BES}, defined as the higher-than-one-loop part of $Z_4(u)$, proportional to $\ln s$ in the large spin limit:

$$Z_4(u) = \ln s\,(-4u + Z_{BES}(u)) + O(s^0) \tag{4.60}$$

($Z_4(u)$ is an odd function $Z_{BES}(u) = -Z_{BES}(-u)$); from $Z_{BES}(u)$, the density $\sigma_{BES}(u)$ can be defined as its derivative

$$\frac{d}{du} Z_{BES}(u) = \sigma_{BES}(u) \quad .$$

The Fourier transform of $\sigma_{BES}(u)$ enjoys several remarkable properties: it equals half of the value of the universal scaling function $f(g) = 2\hat{\sigma}_{BES}(0)$ and it can be obtained upon solving the linear integral equation (2.38), previously encountered at the end of section 3 in chapter 2:

$$\hat{\sigma}_{BES}(k) = \frac{2\pi g^2 k}{e^k - 1}\left[\hat{K}(\sqrt{2}gk, 0) - \int_0^\infty \frac{dt}{\pi}\,\hat{K}(\sqrt{2}gk, \sqrt{2}gt)\hat{\sigma}_{BES}(t)\right] \quad k > 0 \tag{4.61}$$

The transformed density could be handy reformulated in terms of the function $\gamma^{\o}_{\pm}(t)$ (whose importance will get clearer in computing energy and momentum of the excitations):

$$\hat{\sigma}_{BES}(t) = \frac{2\pi e^{\frac{t}{2}}}{e^t - 1}\left[\gamma^{\o}_{-}(t) + \gamma^{\o}_{+}(t)\right] \tag{4.62}$$

The functions behave under parity according to $\gamma^{\o}_{\pm}(-t) = \pm\gamma^{\o}_{\pm}(t)$ and enjoy the Neumann expansion, in terms of the Bessel functions $J_n(t)$ of the first kind:

$$\begin{aligned} \gamma^{\o}_{-}(t) &= 2\sum_{n=1}^{\infty}(2n-1)\,J_{2n-1}(t)\,\gamma^{\o}_{2n-1} \\ \gamma^{\o}_{+}(t) &= 2\sum_{n=1}^{\infty}(2n)\,J_{2n}(t)\,\gamma^{\o}_{2n} \end{aligned} \tag{4.63}$$

where the Neumann modes satisfies the integral equations

$$\gamma^{\o}_{n}(t) + \int_0^\infty \frac{dt}{t}\,J_n(\sqrt{2}gt)\,\frac{\gamma^{\o}_{+}(\sqrt{2}gt) - (-1)^n\gamma^{\o}_{-}(\sqrt{2}gt)}{e^t - 1} = \sqrt{2}g\,\delta_{n1} \quad . \tag{4.64}$$

4 On the energies

This section is devoted to the computation of the energy owned by excitations on the antiferromagnetic vacuum. First, a few general considerations are provided. The general expression for the eigenvalue of the r-th conserved charge of the motion reads, as a function of rapidity u:

$$q_r(u) = \frac{ig^2}{r-1}\left[\left(\frac{1}{x^+(u)}\right)^{r-1} - \left(\frac{1}{x^(u)}\right)^{r-1}\right] \quad . \tag{4.65}$$

On the ferromagnetic (BMN) vacuum, only main roots u_4 carry a non zero value of the charge, so that the total value of any Q_r associated to whole system is given by

$$Q_r = \sum_{j=1}^{s} q_r(u_j) = \frac{ig^2}{r-1} \sum_{j=1}^{s} \left[\left(\frac{1}{x^+(u_j)} \right)^{r-1} - \left(\frac{1}{x^(u_j)} \right)^{r-1} \right] \quad . \tag{4.66}$$

When switching to the antiferromagnetic (GKP) vacuum, taking s to infinity, formula (3.17) should be applied to (4.66):

$$\begin{aligned} Q_r &= ig^2 \sum_{h=1}^{L} \left[\frac{1}{(x^+(u_h))^{r-1}} - \frac{1}{(x^(u_h))^{r-1}} \right] + \sum_{j=1}^{K_b} \left[\frac{ig^2}{(x^{++}(u_{b,j}))^{r-1}} - \frac{ig^2}{(x^{--}(u_{b,j}))^{r-1}} \right] \\ &- ig^2 \int \frac{dv}{2\pi} \left[\left(\frac{1}{x^+(v)} \right)^{r-1} - \left(\frac{1}{x^-(v)} \right)^{r-1} \right] \frac{d}{dv} [Z_4(v) - 2L_4(v)] \quad . \end{aligned} \tag{4.67}$$

The summation of terms depending on b-roots in the first line of (4.67) gets erased by contribution ascribable to that kind of isotopic root included in $Z_4(v)$: the type-b are then showed not to carry charges. Henceforth the focus will stick to the $r = 2$ charge, corresponding to energy

$$q_2(u) = ig^2 \left[\frac{1}{x^+(u)} - \frac{1}{x^-(u)} \right] \quad ,$$

so that the total energy of the system is achieved by substituting the counting function (4.58) in (4.67) and setting r to 2 ($Q_2 = E$):

$$E = ig^2 \sum_{h=1}^{H} \left(\frac{1}{x_h^-} - \frac{1}{x_h^+} \right) - ig^2 \int \frac{dv}{2\pi} \left(\frac{1}{x^+(v)} - \frac{1}{x^-(v)} \right) \frac{d}{dv} [Z_4(v) - 2L_4(v)] \, . \tag{4.68}$$

In (4.68), all nonlinear terms could be dropped, taking into account the estimate [44]

$$\int \frac{dv}{2\pi} \left(\frac{1}{x^+(v)} - \frac{1}{x^-(v)} \right) \frac{d}{dv} L_4(v) \sim \int_0^{+\infty} dt \hat{K}(0, \sqrt{2} gt) ite^{-\frac{t}{2}} \hat{L}_4(t) = O\left(\frac{1}{s^2} \right) \quad . \tag{4.69}$$

Therefore the energy (4.68) becomes, in a more explicit form:

$$\begin{aligned} E &= -ig^2 \int \frac{dv}{2\pi} \left(\frac{1}{x^+(v)} - \frac{1}{x^-(v)} \right) \frac{d}{dv} \left[\sum_{h=1}^{L} (\Theta(u, u_h) + \tilde{P}(u_h)) + \right. \\ &+ \left. \sum_{*} \sum_{j=1}^{N_*} (F_*(v, u_{*,j}) + \tilde{P}(v)) \right] + ig^2 \sum_{h=1}^{H} \left(\frac{1}{x(u_h)^-} - \frac{1}{x(u_h)^+} \right) + O\left(\frac{1}{s^2} \right) \, , \end{aligned} \tag{4.70}$$

where the formal summation $\sum_*$ is taken over the particle types $* \in \{g, \bar{g}, F, \bar{F}, f, \bar{f}\}$, with the identifications $F_g(v, u) = F_{\bar{g}}(v, u) = T(v, u|1)$. The energy belonging to each type of excitation is obtained by splitting formula (4.70) into the contributions amenable to the different kinds of particles.

• **Scalars:**

First, the energy of an internal hole will be computed, by extrapolating the corresponding contribution in the first line of (4.70):

$$E_s(u) = ig^2 \left(\frac{1}{x^-(u)} - \frac{1}{x^+(u)} \right) - ig^2 \int \frac{dv}{2\pi} \left(\frac{1}{x^+(v)} - \frac{1}{x^-(v)} \right) \frac{d}{dv} \Theta(v,u) \quad . \quad (4.71)$$

Formula (4.71) can be recast in terms of integrals deeply studied in literature about $sl(2)$ sector ([31], [44] for details):

$$E_s(u) = -q_2(u) - \int \frac{dk}{4\pi^2}\, \hat{q}_2(k)\, [\hat{\sigma}^{(1)}(k) + \hat{\sigma}(k;u)|_{IH}\,] \quad . \quad (4.72)$$

Eventually the energy for a scalar excitation assumes a form which recalls the solution of BES equation (4.61), namely involving the functions $\gamma^{\o}_{\pm}(t)$ introduced at the end of section 3:

$$E_s(u) = \int_0^{+\infty} \frac{dt}{t} \left[\frac{e^{-\frac{t}{2}} - \cos tu}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \gamma^{\o}_{-}(\sqrt{2}gt) + \frac{\cos tu - e^{\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \gamma^{\o}_{+}(\sqrt{2}gt) \right] \quad . \quad (4.73)$$

The expression above matches the analogous result by Basso [42].

• **Gluons:**

The energy of a gluon can be extracted from (4.70), and, thanks to(4.71) and (3.105), its value relates to the energy of scalar excitations:

$$\begin{aligned} E_g(u) &= -\int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{dE_s(w)}{dw} [\chi(w,u|1) + \Phi(w)] = \qquad (4.74)\\ &= \int_{-\infty}^{\infty} \frac{dk}{4\pi^2} \frac{i\pi}{e^{\frac{k}{2}} - e^{-\frac{k}{2}}} (\gamma^{\o}_{+}(\sqrt{2}gk) - sgn(k)\gamma^{\o}_{-}(\sqrt{2}gk)) \left[\frac{2\pi}{ik} e^{-|k|} e^{-iku} \right.\\ &- \frac{2\pi}{ik} e^{-\frac{|k|}{2}} \sum_{n=1}^{\infty} \left(\left(\frac{g}{\sqrt{2}ix(u+\frac{il}{2})} \right)^n + \left(\frac{g}{\sqrt{2}ix(u-\frac{il}{2})} \right)^n \right) J_n(\sqrt{2}gk)\\ &- \left. \frac{2\pi}{ik} J_0(\sqrt{2}gk) e^{-\frac{|k|}{2}} \right] \quad ; \end{aligned}$$

By means of equation (4.64), and by making use of the identity

$$\int_0^{\infty} \frac{dk}{k}\, e^{-k(\frac{l}{2} \pm iu)}\, J_n(\sqrt{2}gk) = \frac{(\pm 1)^n}{n} \left(\frac{g}{\sqrt{2}ix(u \mp \frac{il}{2})} \right)^n \quad , \quad (4.75)$$

the expression above is rewritten as [42]:

$$E_g(u) = \int_0^{\infty} \frac{dk}{k} \frac{\gamma^{\o}_{+}(\sqrt{2}gk)}{1-e^{-k}} [\cos ku\, e^{-\frac{k}{2}} - 1] - \int_0^{\infty} \frac{dk}{k} \frac{\gamma^{\o}_{-}(\sqrt{2}gk)}{e^k - 1} [\cos ku\, e^{-\frac{k}{2}} - 1] \, . \quad (4.76)$$

A very similar result holds for gluonic bound states too:

$$E_{gbs}(u|l) = \int_0^{\infty} \frac{dk}{k} \frac{\gamma^{\o}_{+}(\sqrt{2}gk)}{1-e^{-k}} [\cos ku\, e^{-k\frac{l}{2}} - 1] - \int_0^{\infty} \frac{dk}{k} \frac{\gamma^{\o}_{-}(\sqrt{2}gk)}{e^k - 1} [\cos ku\, e^{-k\frac{l}{2}} - 1] \, .$$

(4.77)

• **Fermions**

The computation of energy for a large fermion parallels the very same reasoning outlined before for gluons, so that:

$$E_F(u) = -\int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{dE_s(w)}{dw} \left[\chi_F(w,u|l) + \Phi(w)\right] = \tag{4.78}$$
$$= \int_0^{\infty} \frac{dk}{k} \frac{\gamma_+^{\o}(\sqrt{2}gk) - \gamma_-^{\o}(\sqrt{2}gk)}{e^k - 1} \left[\cos ku - 1\right] + \int_0^{\infty} \frac{dk}{k} \gamma_+^{\o}(\sqrt{2}gk) \left[\frac{1}{2}\cos ku - 1\right]$$

For small fermions, instead, the same steps lead to:

$$\begin{aligned} E_f(u) &= \int_{-\infty}^{\infty} \frac{dw}{2\pi} \frac{dE_s(w)}{dw} \chi_H(w,u|l) = \\ &= -\frac{1}{2} \int_0^{\infty} \frac{dk}{k} \gamma_+^{\o}(\sqrt{2}gk) \cos ku \end{aligned} \tag{4.79}$$

5 On the propagation phase

The eigenvalue of the momentum, owned by a root, is formulated as a function of rapidity:

$$p(u) = i \ln\left(\frac{x^+(u)}{x^-(u)}\right) \quad . \tag{4.80}$$

Since only u_4 roots carry momentum, the total value associated to the spin chain, as computed on the ferromagnetic vacuum, amounts to the summation

$$P = \sum_{j=1}^{s} p(u_{4,j}) = i \sum_{j=1}^{s} \ln\left(\frac{x^+(u)}{x^-(u)}\right)$$

Large spin limit and formula (3.17) allow to switch to antiferromagnetic vacuum, so that the total momentum the excitations bring to the spin chain equals:

$$P = -i \sum_{h=1}^{L} \ln\frac{x^+(u_h)}{x^-(u_h)} + i \sum_{j=1}^{K_b} \ln\frac{x^{++}(u_{b,j})}{x^{--}(u_{b,j})} - i \int \frac{dv}{2\pi} \ln\frac{x^+(v)}{x^-(v)} \frac{d}{dv}[Z_4(v) - 2L_4(v)] \; . \tag{4.81}$$

As it happened for (4.67), the sum on the b-roots in (4.81) cancels out the contribution from isotopic roots inside the scalar counting function $Z_4(v)$ (4.58). Moreover, the terms $\frac{d\tilde{P}(v)}{dv}$ and $\frac{dR}{dv}(v, \pm s/\sqrt{2})$ disappear, for they are integrated with the odd function $\ln \frac{x^+(v)}{x^-(v)}$. Hence, recollecting the quantity ΔP from (4.53), the total momentum reads:

$$P = \Delta P + \int \frac{dv}{\pi} \frac{d\tilde{P}}{dv} L_4(v) \quad . \tag{4.82}$$

Since the spin chain is supposed not to have momentum with respect its centre of mass frame, the zero-momentum condition $P = 0$ (equivalent to $\prod_{j=1}^{s} \frac{x^+(u_{4,j})}{x^-(u_{4,j})} = 1$ in Beisert-Staudacher equations) should be imposed as a constraint, thus the expressions for propagation phases in section 3 get simplified. On the antiferromagnetic vacuum, the propagation phase is assumed to take the form:

$$-\Lambda_*(u) = R \cdot P_*(u) + D_*(u) \quad ,$$

hence hereafter the particle momentum $P_*(u)$ will be inferred, by extracting the part proportional to the spin chain length $R = 2\ln s$ (GKP string length) in a $s \gg 1$ expansion, the subleading remainder $D_*(u)$ being amenable to a defect.

• **Scalars:**

The propagation phase for scalars equals:

$$\begin{aligned} -\Lambda_s(u) &= R(u, s/\sqrt{2}) + R(u, -s/\sqrt{2}) + 2\tilde{P}(u) + \int \frac{dv}{\pi} \frac{d}{dv} [R(u,v) - \tilde{P}(v)] L_4(v) = \\ &= i \ln[-S^{(ss)}(u, s/\sqrt{2})] + i \ln[-S^{(ss)}(u, -s/\sqrt{2})] + \\ &+ \ i \int \frac{dv}{\pi} \frac{d}{dv} \ln[-S^{(ss)}(u,v)] L_4(v) \end{aligned} \tag{4.83}$$

Now, recollecting the notions about the scalar counting function provided at the end of section 3, it is found that

$$R(u, \frac{s}{\sqrt{2}}) + R(u, -\frac{s}{\sqrt{2}}) = 2 \ln \frac{s}{\sqrt{2}} \left(-2u + \frac{1}{2} Z_{BES}(u) \right) + O\left(\frac{1}{s^2}\right) \quad , \tag{4.84}$$

reminding that $\tilde{P}(u) = O(s^0)$. Talking about the nonlinear terms, the large spin limit of $L_4(u)$ is controlled by the leading order behaviour of $Z_4(u)$ (not depending on excitations in this regime, as they start contributing at s^0 order)

$$Z_4(u) = 2 \ln s \left(-2u + \frac{1}{2} Z_{BES}(u) \right) + O(s^0) \quad , \tag{4.85}$$

hence, it is found that:

$$\begin{aligned} &\int \frac{dv}{\pi} \frac{d}{dv} R(u,v) L_4(v) = \ln 2 \left(-2u + \frac{1}{2} Z_{BES}(u) \right) + O\left(\frac{1}{s^2}\right) \\ &\int \frac{dv}{\pi} \frac{d\tilde{P}}{dv} L_4(v) = O\left(\frac{1}{s^2}\right) \end{aligned} \tag{4.86}$$

These contributions altogether lead to the propagation phase in high spin limit:

$$\begin{aligned}\Lambda_s(u) &= 2\ln s\left(2u - \frac{1}{2}Z_{BES}(u)\right) - 2\tilde{P}(u) + O\left(\frac{1}{s^2}\right) \\ &= R \cdot P_s(u) + D_s(u)\end{aligned} \tag{4.87}$$

Therefore, the momentum of a scalar excitation is left identified:

$$P_s(u) = 2u - \frac{1}{2}Z_{BES}(u) \tag{4.88}$$

The results achieved from the study of $sl(2)$ sector allow to make more explicit the expression (4.88) for the momentum of a scalar [42]:

$$P_s(u) = 2u - \int_0^{+\infty} \frac{dk}{k}\sin(ku)\ e^{\frac{k}{2}}\left[\frac{\gamma_-^{\o}(\sqrt{2}gk) + \gamma_+^{\o}(\sqrt{2}gk)}{e^k - 1}\right] \tag{4.89}$$

At last, the interaction between scalar excitations (internal holes) and the two external holes entails a subleading phase shift, assimilable to a defect:

$$D_{scal}(u) = -2\tilde{P}(u) \quad . \tag{4.90}$$

- **Gluons:**

For gluons

$$\begin{aligned}-\Lambda_g(u) &= \int \frac{dv}{2\pi}\frac{d}{dv}\left[R(s/\sqrt{2},v) + R(-s/\sqrt{2},v)\right]\chi(v,u|1) + \\ &+ \ i\int \frac{dv}{\pi}\frac{d}{dv}\ln[S^{(gs)}(u,v)]L_4(v) - 2\int \frac{dv}{2\pi}\frac{d}{dv}\tilde{P}(v)\chi(v,u|1)\end{aligned} \tag{4.91}$$

Upon restrict to order $\ln s$, it is found:

$$\begin{aligned}&\int \frac{dv}{2\pi}\chi(v,u|1)\frac{d}{dv}[R(v,\frac{s}{\sqrt{2}}) + R(v,-\frac{s}{\sqrt{2}})] = \\ &= 2\ln\frac{s}{\sqrt{2}}\Big\{2u - \left(\frac{g^2}{x^-} + \frac{g^2}{x^+}\right) + \int \frac{dk}{8\pi^2}\Big[-2\pi\frac{\sin ku}{k}e^{-|k|} + \\ &+ \ i\sum_{n=1,\,n\ \mathrm{odd}}\left(\frac{g}{i\sqrt{2}x^-(u)}\right)^n \frac{2\pi}{k}e^{-\frac{|k|}{2}}J_n(\sqrt{2}gk) + \\ &+ \ i\sum_{n=1,\,n\ \mathrm{odd}}\left(\frac{g}{i\sqrt{2}x^+(u)}\right)^n \frac{2\pi}{k}e^{-\frac{|k|}{2}}J_n(\sqrt{2}gk)\Big]\hat{\sigma}_{BES}(k)\Big\}\end{aligned} \tag{4.92}$$

Now, the equalities

$$\int \frac{dk}{k}e^{-\frac{|k|}{2}}J_n(\sqrt{2}gk)\hat{\sigma}_{BES}(k) = 4\pi[\sqrt{2}g\delta_{n,1} - \gamma_n^{\o}] \tag{4.93}$$

$$\sum_{\substack{n=1 \\ n\ \mathrm{odd}}}^{\infty}\left[\left(\frac{g}{i\sqrt{2}x^-}\right)^n + \left(\frac{g}{i\sqrt{2}x^+}\right)^n\right]\gamma_n^{\o} = -i\int_0^{+\infty}\frac{dk}{k}\sin ku\ e^{-\frac{k}{2}}\ \gamma_-^{\o}(\sqrt{2}gk)$$

turn out useful to eventually get

$$2\ln\frac{s}{\sqrt{2}}\Big\{2u-\int_0^{+\infty}\frac{dk}{k}\sin ku\ e^{-\frac{k}{2}}\left[\frac{\gamma_-^{\o}(\sqrt{2}gk)}{1-e^{-k}}+\frac{\gamma_+^{\o}(\sqrt{2}gk)}{e^k-1}\right]\Big\} \tag{4.94}$$

and, at the same time, the nonlinear terms give a contribution

$$\ln 2\Big\{2u-\int_0^{+\infty}\frac{dk}{k}\sin ku\ e^{-\frac{k}{2}}\left[\frac{\gamma_-^{\o}(\sqrt{2}gk)}{1-e^{-k}}+\frac{\gamma_+^{\o}(\sqrt{2}gk)}{e^k-1}\right]\Big\}+O(1/s^2) \quad . \tag{4.95}$$

Given the usual form for the propagation phase

$$\Lambda_g(u)=R\cdot P_g(u)+D_g(u) \quad , \tag{4.96}$$

the momentum of a gluon with rapidity u is recognized to be

$$P_{glu}(u)=2u-\int_0^{+\infty}\frac{dk}{k}\sin ku\ e^{-\frac{k}{2}}\left[\frac{\gamma_-^{\o}(\sqrt{2}gk)}{1-e^{-k}}+\frac{\gamma_+^{\o}(\sqrt{2}gk)}{e^k-1}\right] \quad , \tag{4.97}$$

which agrees with [42], while the subleading interaction between a gluon and the two external holes produces the defect phase shift

$$D_g(u)=2\int\frac{dv}{2\pi}\chi(v,u|1)\frac{d}{dv}\tilde{P}(v) \quad . \tag{4.98}$$

- **Large fermions:**

The propagation phase associated to a large fermion with rapidity u reads

$$\begin{aligned}-\Lambda_F(u) \ &= \ \int\frac{dv}{2\pi}\chi_F(v,u)\frac{d}{dv}[R(v,x_L)+R(v,-x_L)]+2\int\frac{dv}{2\pi}\chi_F(v,u)\frac{d}{dv}\tilde{P}(v)+\\ &+ \ i\int\frac{dv}{\pi}\frac{d}{dv}\ln[S^{(Fs)}(u,v)]L_4(v)\end{aligned} \tag{4.99}$$

THe attention now sticks to order $\ln s$, so to get:

$$\begin{aligned}&\int\frac{dv}{2\pi}\chi_F(v,u)\frac{d}{dv}[R(v,x_L)+R(v,-x_L)]= \\ &= \ 2\ln\frac{s}{\sqrt{2}}\Big\{2u-\int_0^{+\infty}\frac{dk}{k}\sin ku\ \frac{\gamma_+^{\o}(\sqrt{2}gk)+\gamma_-^{\o}(\sqrt{2}gk)}{e^k-1}- \\ &- \ \frac{1}{2}\int_0^{\infty}\frac{dk}{k}\sin ku\,\gamma_-^{\o}(\sqrt{2}gk)\Big\} \quad ;\end{aligned} \tag{4.100}$$

in order to obtain the equation, the following relation has been used:

$$\int_0^{\infty}\frac{dk}{2k}\sin ku\,\gamma_-^{\o}(\sqrt{2}gk)=i\sum_{n=1}^{\infty}\left(\frac{g}{\sqrt{2}ix(u)}\right)^{2n-1}\gamma_{2n-1}^{\o} \quad . \tag{4.101}$$

The contribution of the nonlinear terms can be estimated (up to $O(1/s^2)$ terms)

$$\ln 2\Big\{2u-\int_0^{+\infty}\frac{dk}{k}\sin ku\ \frac{\gamma_+^{\o}(\sqrt{2}gk)+\gamma_-^{\o}(\sqrt{2}gk)}{e^k-1}-\frac{1}{2}\int_0^{\infty}\frac{dk}{k}\sin ku\,\gamma_-^{\o}(\sqrt{2}gk)\Big\}$$

To sum up, the propagation phase becomes

$$\Lambda_F(u) = RP_F(u) + D_F(u) \tag{4.102}$$

provided identification of the momentum of a large fermion (with rapidity u) as

$$P_F(u) = 2u - \int_0^\infty \frac{dk}{k}\sin(ku)\,\frac{\gamma_+^{\o}(\sqrt{2}gk)+\gamma_-^{\o}(\sqrt{2}gk)}{e^k-1} - \frac{1}{2}\int_0^\infty \frac{dk}{k}\sin(ku)\,\gamma_-^{\o}(\sqrt{2}gk) \tag{4.103}$$

(recalling [42]), so that the defect term stands out:

$$D_F(u) = 2\int \frac{dv}{2\pi}\chi_F(v,u)\frac{d}{dv}\tilde{P}(v) \quad . \tag{4.104}$$

• **Small fermions:**

The reasoning for the propagation phase of a small fermion (rapidity u) mimic very closely the large fermion case

$$\begin{aligned} -\Lambda_f(u) &= -\int \frac{dv}{2\pi}\,\chi_H(v,u)\,\frac{d}{dv}[R(v,\frac{s}{\sqrt{2}}) + R(v,-\frac{s}{\sqrt{2}})] + & (4.105)\\ &+ \int \frac{dv}{\pi}\frac{d}{dv}\ln[S^{(fs)}(u,v)]\,L_4(v) + 2\int\frac{dv}{2\pi}\chi_H(v,u)\frac{d}{dv}\tilde{P}(v) = \\ &= RP_f(u) + D_f(u) \qquad , & (4.106)\end{aligned}$$

where $P_f(u)$ stands for the momentum of a small femion with rapidity u [42]

$$P_f(u) = \frac{1}{2}\int_0^\infty \frac{dk}{k}\sin(ku)\,\gamma_-^{\o}(\sqrt{2}gk) \tag{4.107}$$

while the defect might be identified as

$$D_f(u) = -2\int \frac{dv}{2\pi}\chi_H(v,u)\frac{d}{dv}\tilde{P}(v) \tag{4.108}$$

5.1 Weak coupling expansion

Now a very short summary, gathering all the propagation phases in large spin limit, will be provided at one loop approximation (or at lowest order in g^2, when talking about small fermions). More precisely, in getting the following expressions a $s \gg 1$ expansion has been performed, then the $g \longrightarrow 0$ limit.

• Scalars:

$$\Lambda_s(u) = 4u\ln x_L + 2i\ln\frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} + 2u\ln 2 = 4u\ln s + 2i\ln\frac{\Gamma\left(\frac{1}{2}+iu\right)}{\Gamma\left(\frac{1}{2}-iu\right)} + O\left(\frac{1}{\ln s}\right) \tag{4.109}$$

• Gluons:

$$\Lambda_g(u) = 4u\ln s + 2i\ln\frac{\Gamma\left(\frac{3}{2}+iu\right)}{\Gamma\left(\frac{3}{2}-iu\right)} + O\left(\frac{1}{\ln s}\right) \tag{4.110}$$

• Large fermions:

$$\Lambda_F(u) = 4u \ln s + 2i \ln \frac{\Gamma(1+iu)}{\Gamma(1-iu)} + O\left(\frac{1}{\ln s}\right) \tag{4.111}$$

• Small fermions:

$$\Lambda_f(u) = (2 \ln s + O(1)) \frac{g^2}{u} + O(g^4) \quad . \tag{4.112}$$

Chapter 5

$AdS_4 \times \mathbb{CP}^3$ and GKP vacuum

1 Integrability in AdS_4/CFT_3 and large spin limit

This chapter will be devoted to the study of some issues on the AdS_4/CFT_3 gravity-gauge theory dualities. In particular, the interest will address to the low-energy reduction of the $AdS_4 \times \mathbb{CP}^3$ type II A string theory in the Alday-Maldacena decoupling limit [38] and its counterpart in the dual gauge theory $\mathcal{N} = 6$ Chern-Simons, both considered on a vacuum state associated to the GKP string.
First, the $AdS_4 \times \mathbb{CP}^3$ type II A string σ-model has got $SO(2,3) \times SU(4)$ as its isometry group, furthermore the theory is found to be integrable at classical level. Let the Green-Schwarz action of the theory be quantized upon the GKP classical solution, corresponding to a string spinning in the AdS_3 subspace of AdS_4 with spin S, moreover let this vacuum solution be generalized by taking into account another non zero Noether charge, say J, associated to the rotation on the circumference S^1 in $\mathbb{CP}^3$. The Alday-Maldacena limit is achieved by taking the ratio $\frac{J}{\ln S} \longrightarrow 0$ (with both J and S large): in this regime the massless modes are found to dominate over the massive particles. Therefore six massless bosons associated to $\mathbb{CP}^3$ and one massless Dirac fermion decouple from the whole string σ-model, thus giving rise to an effective low-energy $\mathbb{CP}^3$ model, coupled to a fermion via a Thirring term. This is the so-called Bykov model [50], and it is described by the Lagrangian:

$$\mathcal{L} = \kappa(\partial_\alpha - i\mathcal{A}_\alpha)\bar{z}^j\,(\partial^\alpha + i\mathcal{A}^\alpha)z^j + i\bar{\psi}\gamma^\alpha(\partial_\alpha - 2i\mathcal{A}_\alpha)\psi + \frac{1}{4\kappa}(\bar{\psi}\gamma^\alpha\psi)^2 \tag{5.1}$$

where $\mathcal{A}_\alpha$ is non-dynamical $U(1)$ gauge field, whereas the z^j bosonic fields $j \in \{1,\ldots,4\}$, along with their complex conjugated $\bar{z}^j$, are left constrained to $S^7 \subset \mathbb{C}^4$ by the condition

$$\sum_{j=1}^{4} z^j\,\bar{z}^j = 1 \quad ; \tag{5.2}$$

the $SU(4)$ symmetry joins to a further $U(1)$ related to the conservation of fermion number. The lagrangian (5.1) has been shown to lead to an integrable model by [58].

Turning now to the gauge theory side of the AdS/CFT correspondence, the sting theory on $AdS_4 \times \mathbb{CP}^3$ is conjectured to be dual to a three dimensional supersymmetric

$\mathcal{N}=6$ Chern-Simons endowed with a $SU(N) \times SU(N)$ gauge group (ABJM [26]). The $SO(2,3)$ conformal group on the three dimensional Minkowski is juxtaposed to the R-symmetry $SU(2)_R \times SU(2)_L \times U(1)$, enhanced to a full $SU(4)$ by the Chern-Simons term. The theory contemplates the presence of two sets of scalars, transforming in the bifundamental of $SU(N)$: the scalars from the first set behave as doublets under $SU(2)_R$ (say A_a) in the $(\mathbf{N}, \bar{\mathbf{N}})$ of $SU(N) \times SU(N)$, else the second is composed of doublets of $SU(2)_L$ (say $B_{\dot{a}}$) in the $(\bar{\mathbf{N}}, \mathbf{N},)$. Moreover, under the complete R-symmetry, they arrange themselves into two multiplets, namely $Y^a = (A_1, A_2, B^\dagger_{\dot{1}}, B^\dagger_{\dot{2}})$ in the fundamental $\mathbf{4}$ representation of $SU(4)$ and $Y^\dagger_a = (A^\dagger_1, A^\dagger_2, B_{\dot{1}}, B_{\dot{2}})$ in the antifundamental $\bar{\mathbf{4}}$ of $SU(4)$. These multiplets are the building blocks to construct a class of gauge invariant operators, in the shape of single trace such as:

$$\mathcal{O} \sim Tr\left[Y^{a_1} Y^\dagger_{b_1} Y^{a_2} Y^\dagger_{b_2} \cdots Y^{a_L} Y^\dagger_{b_L}\right] \eta^{b_1,\ldots,b_L}_{a_1,\ldots,a_L} \quad . \tag{5.3}$$

The classical dimension of (5.3) is L: when $\eta^{b_1,\ldots,b_L}_{a_1,\ldots,a_L}$ is symmetric with respect to both a_i and b_j indices, and all the traces are zero, then (5.3) is a chiral primary operator, whose dimension gets fixed by the supersymmetry to the bare value L, otherwise $\mathcal{O}$ has got a non zero anomalous dimension. The quantum part of the dilatation operator for the set of objects (5.3) turns out to get associated to an integrable hamiltonian [51]. This fact allows to relate the class of operators to a spin chain of length $2L$, where the sites alternate between fundamental and antifundamental representations of $SU(4)$, and the interactions connect next-to-nearest neighbour sites (nearest neighbour interaction contributions cancel out [51]).

Now, the attention will turn to the issue of identifying a vacuum state which might correspond to the GKP string, then finding upon it the relative excitations and their dynamics, paralleling what has been done for the $sl(2)$ sector in $\mathcal{N}=4$ SYM. A suitable choice for the vacuum could be provided by the chiral primary

$$\mathcal{O} \sim Tr\left[(Y^1 Y^\dagger_4)^L\right] \tag{5.4}$$

whose dimension stays untouched by quantum corrections. Differently from the $\mathcal{N}=4$ SYM, there is not a closed sector built out of scalars and covariant derivatives alone. Instead, the smallest non-compact sector available consists in the $sl(2|1)$ [52], where the field content restricts to Y^1, ψ^1_+ and covariant derivatives (on the $(N,\bar{N})$ sites) whose insertions on the vacuum accounts for magnonic excitations, or $Y^\dagger$, $\psi^\dagger_{+4}$ and covariant derivatives (on the $(\bar{N}, N)$ sites) standing for antimagnons. A generic excited state reads then

$$\mathcal{O}' \sim Tr\left[D \ldots Y^1 D \ldots D\psi^\dagger_{+4} D \ldots D\psi^1_+ D \ldots D Y^\dagger_4\right] \quad : \tag{5.5}$$

the twist of an operator equals L, while the Lorentz spin s depends on the number of magnons, say M, and antimagnons, $\bar{M}$, so that

$$s = \frac{1}{2}(M + \bar{M}) \quad . \tag{5.6}$$

Let the equations retrieved by Minahan and Zarembo [51] turn to the $sl(2)$ grading; for simplicity's sake, let the weak coupling limit be considered and, for now, only momentum carrying roots are taken into account. Therefore, a system with M magnons (with rapidities u_k) and $\bar{M}$ antimagnons (rapidities $\bar{u}_k$) is described by the (one loop) Bethe equations:

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{j=1}^{\bar{M}} \frac{u_k-\bar{u}_j-i}{u_k-\bar{u}_j+i} \tag{5.7}$$
$$\left(\frac{\bar{u}_k+\frac{i}{2}}{\bar{u}_k-\frac{i}{2}}\right)^L = \prod_{j=1}^{M} \frac{\bar{u}_k-u_j-i}{\bar{u}_k-u_j+i} \quad .$$

By mimicking what has been done for the $sl(2)$ sector in $N=4$ SYM, the Bethe equations can be rewritten in terms of the (monotonously increasing) counting functions:

$$Z(u) = 2L\arctan 2u + \sum_{j=1}^{\bar{M}} 2\arctan(u-\bar{u}_j) \tag{5.8}$$
$$\bar{Z}(u) = 2L\arctan 2u + \sum_{j=1}^{M} 2\arctan(u-u_j) \quad ,$$

so that (5.7) become

$$(-1)^{L+\bar{M}} = e^{iZ(u_k)} \tag{5.9}$$
$$(-1)^{L+M} = e^{i\bar{Z}(\bar{u}_k)} \quad .$$

Since $\lim_{u\to\pm\infty} Z(u) = \pm\pi(L+\bar{M})$, the overall number of available staes is $L+\bar{M}-1$, only M of whom being actually occupied by magnons: the remaining $H = L+\bar{M}-M-1$ unoccupied states are the holes, then. The same reasoning holds for the $\bar{H} = L+M-\bar{M}-1$ antiholes. The system is left therefore endowed with two selection rules for choosing the solutions: both sum and difference of holes and antiholes must be even numbers, as

$$H+\bar{H} = 2(L-1) \qquad H-\bar{H} = 2(\bar{M}-M) \quad ,$$

moreover the zero-momentum has to be imposed:

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}} \prod_{j=1}^{\bar{M}} \frac{\bar{u}_j+\frac{i}{2}}{\bar{u}_j-\frac{i}{2}} = 1 \quad . \tag{5.10}$$

Once that this setup has been supplied, the spin chain (gauge theory) counterpart of the quantization over the GKP string can be easily portrayed. In the large spin limit $s\longrightarrow\infty$ [1], the state (5.4) accounts for the antiferromagnetic vacuum, whereas the excitations to actually look at are the holes and antiholes, appearing on the sea formed by an infinite number of magnons and antimagnons (refer to (5.6)). These claims thus accomplish the parallel with the antiferromagnetic spin chain for $\mathcal{N}=4$ SYM.

[1] The large Lorentz spin regime is, in fact, the required condition for both GKP string and Bykov's model to be realized

The next step to fully outline the theory to be examined (henceforth) consists in discussing the excitations for ABJM on this antiferromagnetic vacuum. The lowest-lying twist-1/2 excitations are the holes and antiholes, commented so far: they respectively transform under the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$, and, due to their nature of fake solutions of some Bethe equations, they someway relate to scalar excitations in $\mathcal{N}=4$ SYM. The fermions, instead, are found to be associated to the $\mathbf{6}$ of $SU(4)$ in ABJM, differently from $\mathcal{N}=4$ SYM which owns two types of fermionic excitations (fermions and antifermions), in the $\mathbf{4}$ and $\bar{\mathbf{4}}$. Finally, the gauge fields are $SU(4)$ singlets in both the theories, and moreover they appear in bound states $D_\perp^{l-1}F_{+\perp}$ ($l \geq 1$): anyway, while in $\mathcal{N}=6$ there is only one type of gauge fields, the $\mathcal{N}=4$ SYM theory manifests two kind of them, related to two components of the field strength, the $D_\perp^{l-1}F_{+\perp}$ and the $\bar{D}_\perp^{l-1}\bar{F}_{+\perp}$.

This amusing 'quarrelling' about excitations over the antiferromagnetic vacuum, namely either the doubling or halving the number of varieties for the different kinds of particles, is not a lone hint about an interesting relation between $\mathcal{N}=6$ Chern-Simons and $\mathcal{N}=4$ SYM. An analogous behaviour will be encountered hereafter, when dealing with Y-systems and folding process (on Bykov model). Moreover, several similarities intertwine the two theories. First, the scaling dimension of the vacuum, in the large spin regime: in $\mathcal{N}=4$ SYM, talking about the scaling dimension for twist-two operators, it holds [40]:

$$\Delta_{vacuum}^{\mathcal{N}=4} = s + f(g)(\ln s + \gamma_E) + B_2(g) + o(s^0) \tag{5.11}$$

where $B_2(g)$ stands for the virtual scaling function and $g = \frac{\sqrt{\lambda}}{4\pi}$; on the other side, for $\mathcal{N}=6$ Chern-Simons it is found [54]:

$$\Delta_{vacuum}^{\mathcal{N}=6} = s + \frac{f(h)}{2}(\ln(2s) + \gamma_E) + \frac{1}{2}B_2(h) + o(s^0) \quad , \tag{5.12}$$

this time $h = h(\lambda)$ is some function of the t'Hooft coupling λ.
Finally, further resemblances between these two theories come from the dispersion relations [58]. Indeed the holes, belonging either to $\mathcal{N}=6$ Chern-Simons or to $\mathcal{N}=4$ SYM, share almost the same formula for energies and momenta, as their expressions are equal up to a factor two, at all couplings:

$$\begin{aligned} E_{hole}^{\mathcal{N}=4}(u) &= 2\,E_{hole}^{\mathcal{N}=6}(u) \\ p_{hole}^{\mathcal{N}=4}(u) &= 2\,p_{hole}^{\mathcal{N}=6}(u) \quad . \end{aligned} \tag{5.13}$$

2 ABA/TBA for Bykov model

Henceforth, the focus will be put upon the Bykov low-energy effective model, quantized over the GKP string: more precisely, the sector of operators dual to Bykov model on the antiferromagnetic vacuum will be taken into exam. Moreover, the analysis will stick on lowest-lying excitations, namely holes and antiholes. The starting point consists in the set of Asymptotic Bethe Ansatz (ABA) equations proposed by [51] for $\mathcal{N} = 6$ Chern-Simons, upon restricting to hole-antihole sector in the large spin limit. In fact the equations, the interest will be addressed to, do coincide with the ABA proposed by [57] for the $SU(4) \times U(1)$ symmetric.

Before studying them, however, a few remarks are appropriated. In this chapter, up to now, the auxiliary (isotopic) roots have been disregarded. From now on, instead, their role will become pivotal in properly describe the system chosen. As the symmetry of the antiferromagnetic vacuum (GKP string) is $SU(4)$, again there exist three kinds of auxiliary roots v_1, v_2, v_3, even though they are embedded in general equations [51], in a way different from the $\mathcal{N} = 4$ SYM case.
The second remark concerns a novel feature of the ABA, that is, the equations are twisted: the twists q, $\bar{q}$ are introduced so to satisfy the conditions

$$q\,\bar{q} = e^{iP} \qquad q/\bar{q} = (-1)^F \quad , \tag{5.14}$$

where the number F stands for $F = \frac{1}{2}(H - \bar{H})$, while P is the total momentum of the spin chain, resulting from the sum of momenta belonging to the single excitations. The ciclicity condition forces the identification $q = 1/\bar{q}$; anyway, q is not left completely determined, as it remains twice valued $q = \pm 1$ for even F, else $q = \pm i$ for odd F.

Hence, now the Bethe ansatz equations for $\mathcal{N} = 6$ Chern-Simons in the lowest-lying excitation sector can be displayed: considering H holes u_k and $\bar{H}$ antiholes $\bar{u}_k$, along with M_i auxiliary type-k roots $v_{i,j}$ (i=1,2,3), the system is left described by

$$\begin{aligned}
e^{-ip(u_k)R} &= q \prod_{j\neq k}^{H} S(u_k - u_j) \prod_{j=1}^{\bar{H}} \bar{S}(u_k - \bar{u}_j) \prod_{j=1}^{M_1} \left(\frac{u_k - v_{1,j} + \frac{i}{2}}{u_k - v_{1,j} - \frac{i}{2}} \right) \qquad (5.15) \\
1 &= \prod_{j\neq k}^{M_1} \left(\frac{v_{1,k} - v_{1,j} + i}{v_{1,k} - v_{1,j} - i} \right) \prod_{j=1}^{M_2} \left(\frac{v_{1,k} - v_{2,j} - \frac{i}{2}}{v_{1,k} - v_{2,j} + \frac{i}{2}} \right) \prod_{j=1}^{H} \left(\frac{v_{1,k} - u_j - \frac{i}{2}}{v_{1,k} - u_j + \frac{i}{2}} \right) \\
1 &= \prod_{j\neq k}^{M_2} \left(\frac{v_{2,k} - v_{2,j} + i}{v_{2,k} - v_{2,j} - i} \right) \prod_{j=1}^{M_1} \left(\frac{v_{2,k} - v_{1,j} - \frac{i}{2}}{v_{2,k} - v_{2,j} + \frac{i}{2}} \right) \prod_{j=1}^{M_3} \left(\frac{v_{2,k} - v_{3,j} - \frac{i}{2}}{v_{2,k} - v_{3,j} + \frac{i}{2}} \right) \\
1 &= \prod_{j\neq k}^{M_3} \left(\frac{v_{3,k} - v_{3,j} + i}{v_{3,k} - v_{3,j} - i} \right) \prod_{j=1}^{M_2} \left(\frac{v_{3,k} - v_{2,j} - \frac{i}{2}}{v_{3,k} - v_{2,j} + \frac{i}{2}} \right) \prod_{j=1}^{\bar{H}} \left(\frac{v_{3,k} - \bar{u}_j - \frac{i}{2}}{v_{3,k} - \bar{u}_j + \frac{i}{2}} \right) \\
e^{-ip(\bar{u}_k)R} &= \bar{q} \prod_{j\neq k}^{\bar{H}} S(\bar{u}_k - \bar{u}_j) \prod_{j=1}^{H} \bar{S}(\bar{u}_k - u_j) \prod_{j=1}^{M_3} \left(\frac{\bar{u}_k - v_{3,j} + \frac{i}{2}}{\bar{u}_k - v_{3,j} - \frac{i}{2}} \right)
\end{aligned}$$

provided the spin chain length is identified with $R = 2 \ln 2s$ (the GKP string length), while the momentum of a hole (the antihole shares the very same formula) $p(u)$ is half the value of that associated to a scalar excitation in $\mathcal{N} = 4$ SYM (4.89), so that $p(u) = \frac{1}{2} P_s(u)$, according to (5.13); explicitly, at weak coupling it holds $p(u) = u + o(g^2)$.

The scattering matrix for hole-hole processes, and for antihole-antihole ass well, reads

$$S(u) = -\frac{\Gamma\left(1 + i\frac{u}{4}\right)\Gamma\left(\frac{1}{4} - i\frac{u}{4}\right)}{\Gamma\left(1 - i\frac{u}{4}\right)\Gamma\left(\frac{1}{4} + i\frac{u}{4}\right)} \quad , \tag{5.16}$$

whereas the mixed scattering matrix, involving both holes and antiholes, is expressed by

$$\bar{S}(u) = \frac{\Gamma\left(\frac{1}{2} - i\frac{u}{4}\right)\Gamma\left(\frac{3}{4} + i\frac{u}{4}\right)}{\Gamma\left(\frac{1}{2} + i\frac{u}{4}\right)\Gamma\left(\frac{3}{4} - i\frac{u}{4}\right)} \quad . \tag{5.17}$$

It should be remarked that, apart from the twists q, $\bar{q}$ and the different expressions for the momentum and the spin chain length (though not as far, actually) in the propagation phase, the matrices $S(u)$ and $\bar{S}(u)$ represent the only discrepancies from the fermion-antifermion sector equations (4.37), (4.38) seen in the previous chapter for $\mathcal{N} = 4$ SYM.

In further analysing equations (5.15), it is worth considering the thermodynamic limit, which is performed by taking the number of particles M_k to infinity, beside of the limit $R \to \infty$ (which now coincides with the large spin limit), provided the ratios M_k/R stay fixed. Indeed, in this regime the auxiliary roots are conjectured to dispose into strings [12], which are described by referring to their (real) centres:

$$v^{(l)}_{1,ka} = \lambda^{(l)}_k + \frac{i}{2}(l + 1 - 2a), \quad (a = 1, \dots, l), \tag{5.18}$$

$$v^{(m)}_{2,kb} = \mu^{(m)}_k + \frac{i}{2}(m + 1 - 2b), \quad (b = 1, \dots, m), \tag{5.19}$$

$$v^{(n)}_{3,kc} = \nu^{(n)}_k + \frac{i}{2}(n + 1 - 2c), \quad (c = 1, \dots, n). \tag{5.20}$$

The ABA equations turn out easier to be handled by introducing the matrices for the scattering between two strings (whose lengths are l, m), written in terms of the relative rapidity of their centres; these matrices are obtained by fusing together the scattering of every single root included in the strings, so to find:

$$S_{l,m}(u) = \prod_{a=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}} \left(\frac{u - ia}{u + ia}\right) = \prod_{a=1}^{l} \left(\frac{u - \frac{i}{2}(l + m + 1 - 2a)}{u + \frac{i}{2}(l + m + 1 - 2a)}\right). \tag{5.21}$$

It might be useful to notice that all the scattering matrices involved depend only on differences of rapidities (*i.e.* not on two rapidities separately), as the Bykov model, whose gauge theory counterpart is outlined here, is relativistic. Therefore in the strong coupling regime dual to Bykov model reduction, momentum and energy of a hole (antihole) are found to take the simple expression $p(u) = m \sinh\left(\frac{\pi u}{2}\right)$ and $E(u) = m \cosh\left(\frac{\pi u}{2}\right)$, where m stands for the mass gap of Bykov model, and equals half the value of the $O(6)$ non-linear σ-model mass gap [58].

Hence the ABA equations (5.15) are rewritten in term of strings as:

$$
\begin{aligned}
e^{-ip(u_k)R} &= \prod_{j\neq k}^{H} S(u_k-u_j)\prod_{j=1}^{\bar H}\bar S(u_k-\bar u_j)\prod_{l=1}^{\infty}\prod_{j=1}^{M^{(l)}}\left[S_{1,l}\left(u_k-\lambda_j^{(l)}\right)\right]^{-1} \qquad (5.22)\\
1 &= \prod_{j=1}^{H} S_{l,1}\left(\lambda_k^{(l)}-u_j\right)\prod_{m=1}^{\infty}\prod_{j=1}^{M^{(m)}} S_{l,m}\left(\lambda_k^{(l)}-\mu_j^{(m)}\right)\\
&\quad\times\prod_{l'=1}^{\infty}\prod_{j=1}^{M^{(l')}}\left[S_{l,l'+1}\left(\lambda_k^{(l)}-\lambda_j^{(l')}\right)\right]^{-1}\left[S_{l,l'-1}\left(\lambda_k^{(l)}-\lambda_j^{(l')}\right)\right]^{-1}\\
1 &= \prod_{m'=1}^{\infty}\prod_{j=1}^{M^{(m')}}\left[S_{m,m'+1}\left(\mu_k^{(m)}-\mu_j^{(m')}\right)\right]^{-1}\left[S_{m,m'-1}\left(\mu_k^{(m)}-\mu_j^{(m')}\right)\right]^{-1}\\
&\quad\times\prod_{n=1}^{\infty}\prod_{j=1}^{M^{(n)}} S_{m,n}\left(\mu_k^{(m)}-\nu_j^{(n)}\right)\prod_{l=1}^{\infty}\prod_{j=1}^{M^{(l)}} S_{m,l}\left(\mu_k^{(m)}-\lambda_j^{(l)}\right)\\
1 &= \prod_{j=1}^{\bar H} S_{n,1}\left(\nu_k^{(n)}-\bar u_j\right)\prod_{m=1}^{\infty}\prod_{j=1}^{M^{(m)}} S_{n,m}\left(\nu_k^{(n)}-\mu_j^{(m)}\right)\\
&\quad\times\prod_{n'=1}^{\infty}\prod_{j=1}^{M^{(n')}}\left[S_{n,n'+1}\left(\nu_k^{(n)}-\nu_j^{(n')}\right)\right]^{-1}\left[S_{n,n'-1}\left(\nu_k^{(n)}-\nu_j^{(n')}\right)\right]^{-1}\\
e^{-ip(\bar u_k)R} &= \prod_{j\neq k}^{\bar H} S(\bar u_k-\bar u_j)\prod_{j=1}^{M}\bar S(\bar u_k-u_j)\prod_{n=1}^{\infty}\prod_{j=1}^{M^{(n)}}\left[S_{1,n}\left(\bar\theta_k-\nu_j^{(l)}\right)\right]^{-1} \quad ,
\end{aligned}
$$

where $M^{(q)}$ counts the number of strings of length q. In the thermodynamic limit, equations (5.22) are more suitably recast by means of the densities of states. Hence, the densities of states accessible to holes and antiholes, respectively $\sigma(u)$ and $\bar\sigma(u)$, are introduced, and likewise the densities for auxiliary root strings $\sigma_n^{(1)}(u)$, $\sigma_n^{(2)}(u)$, $\sigma_n^{(3)}(u)$, labelled according to their lengths $n=1,2,\dots$. In the same manner, the states actually occupied by holes, antiholes a the three different types of strings are described by ρ, $\bar\rho$, $\rho_n^{(1)}$, $\rho_n^{(2)}$, $\rho_n^{(3)}$. The equations (5.22) thus become:

$$
\begin{aligned}
\sigma(u) &= m\cosh\left(\frac{\pi u}{2}\right)+\mathcal{K}*\rho(u)+G*\bar\rho(u)-\sum_{l=1}^{\infty}K_{1,l}*\rho_l^{(1)}(u)\\
\sigma_n^{(1)}(u) &= K_{n,1}*\rho(u)+\sum_{l=1}^{\infty}\left(K_{n,l}*\rho_l^{(2)}(u)-(K_{n,l+1}+K_{n,l-1})*\rho_l^{(1)}(u)\right)\\
\sigma_n^{(2)}(u) &= \sum_{l=1}^{\infty}\left(K_{n,l}*\rho_l^{(3)}(u)+K_{n,l}*\rho_l^{(1)}(u)-(K_{n,l+1}+K_{n,l-1})*\rho_l^{(2)}(u)\right) \qquad (5.23)\\
\sigma_n^{(3)}(u) &= K_{n,1}*\bar\rho(u)+\sum_{l=1}^{\infty}\left(K_{n,l}*\rho_l^{(2)}(u)-(K_{n,l+1}+K_{n,l-1})*\rho_l^{(3)}(u)\right)\\
\bar\sigma(u) &= m\cosh\left(\frac{\pi u}{2}\right)+\mathcal{K}*\bar\rho(u)+G*\rho(u)-\sum_{l=1}^{\infty}K_{1,l}*\rho_l^{(3)}(u) \quad ,
\end{aligned}
$$

where the convolution operation $*$ has been defined as $f*g(u) = \int_{-\infty}^{+\infty} f(u-u')\, g(u')du'$. The kernels in the integral equations above are the logarithmic derivatives of the scattering matrices, that is:

$$\begin{aligned} \mathcal{K}(u) &= \frac{1}{2\pi i}\frac{\partial}{\partial u}\ln S(u) && (5.24)\\ G(u) &= \frac{1}{2\pi i}\frac{\partial}{\partial \theta}\ln \bar{S}(u) \\ K_{l,m}(u) &= \frac{1}{2\pi i}\frac{\partial}{\partial u}\ln S_{lm}(u) \quad ; && (5.25) \end{aligned}$$

the properties of $\mathcal{K}(u)$, $G(u)$ and $K_{l,m}(u)$ are listed and described in Appendix B.

As already sketched in section 4 of chapter 1 when outlining the main steps of the Thermodynamic Bethe Ansatz, the theory at hand can be put on a torus of radii R (the spin chain length) and $\mathcal{R}$ (related to the temperature or compact time by $T = 1/\mathcal{R}$): since the theory is relativistic, a rotation of the axis can be flawlessly performed on the surface of the torus, leaving thus the physical content unaltered.
Going through the ordinary steps of TBA [15][17], the system of non linear integral equations then follows:

$$\begin{aligned} \epsilon(u) &= i\alpha + m\mathcal{R}\cosh\left(\frac{\pi u}{2}\right) - \mathcal{K} * L(u) - G * \bar{L}(u) - \sum_{l=1}^{\infty} K_{1,l} * L_{1,l}(u) \\ \epsilon_{1,n}(u) &= K_{n,1} * L(u) - \sum_{l=1}^{\infty}\left(K_{n,l} * L_{2,l}(u) - (K_{n,l+1} + K_{n,l-1}) * L_{1,l}(u)\right) \\ \epsilon_{2,n}(u) &= \sum_{l=1}^{\infty}\left((K_{n,l+1} + K_{n,l-1}) * L_{2,l}(u) - K_{n,l} * L_{1,l}(u) - K_{n,l} * L_{3,l}(u)\right) && (5.26) \\ \epsilon_{3,n}(u) &= K_{n,1} * \bar{L}(u) - \sum_{l=1}^{\infty}\left(K_{n,l} * L_{2,m}(u) - (K_{n,l+1} + K_{n,l-1}) * L_{3,l}(u)\right) \\ \bar{\epsilon}(u) &= -i\alpha + m\mathcal{R}\cosh\left(\frac{\pi u}{2}\right) - \mathcal{K} * \bar{L}(u) - G * L_0(u) - \sum_{l=1}^{\infty} K_{1,l} * L_{3,l}(u) \quad ; \end{aligned}$$

the equations above have been formulated in terms of the pseudoenergies
$\epsilon(u)$, $\bar{\epsilon}(u)$, $\epsilon_{l,n}(u)$, introduced by means of the densities:

$$\frac{\rho(u)}{\sigma(u)-\rho(u)} = e^{-\epsilon(u)}\,, \quad \frac{\bar{\rho}(u)}{\bar{\sigma}(u)-\bar{\rho}(u)} = e^{-\bar{\epsilon}(u)}\,, \quad \frac{\rho_m^{(i)}(u)}{\sigma_m^{(i)}(u)-\rho_m^{(i)}(u)} = e^{-\epsilon_{i,m}(u)}\,, \qquad (5.27)$$

and the non-linear terms read

$$L(u) = \ln\left(1+e^{-\epsilon(u)}\right)\,, \quad \bar{L}(u) = \ln\left(1+e^{-\bar{\epsilon}(u)}\right), \quad L_{i,m}(u) = \ln\left(1+e^{-\epsilon_{i,m}(u)}\right).$$

In (5.26), the chemical potential α has been included, after [17]. For the ground state, achieved by imposing $\lambda = e^{i\alpha} = 1$, the free-energy can be expressed as:

$$E_\lambda(m,\mathcal{R}) = -\frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta\, \cosh\theta\,(L_0(\theta) + \bar{L}_0(\theta)) \quad . \qquad (5.28)$$

3 Non-crossed, crossed Y-systems and universal TBA

The TBA integral equations enjoy a further, more elegant and powerful form: indeed, thanks to identities (B.11) listed in Appendix B, the (5.26) can be recast into a set of functional equations, the so-called Y-system [60]:

$$Y_0^{++}\, Y_0^{--} = \frac{\bar{Y}_0}{Y_0}\left(1+Y_{11}^{+}\right)\left(1+Y_{11}^{-}\right)\left(1+Y_{21}\right) \tag{5.29}$$

$$Y_{1,l}^{+}\, Y_{1,l}^{-} = \Big(1+\delta_{l1}Y_0\Big)\frac{\left(1+Y_{1,l-1}\right)\left(1+Y_{1,l+1}\right)}{\left(1+\dfrac{1}{Y_{2,l}}\right)} \tag{5.30}$$

$$Y_{2,l}^{+}\, Y_{2,l}^{-} = \frac{\left(1+Y_{2,l-1}\right)\left(1+Y_{2,l+1}\right)}{\left(1+\dfrac{1}{Y_{1,l}}\right)\left(1+\dfrac{1}{Y_{3,l}}\right)} \tag{5.31}$$

$$Y_{3,l}^{+}\, Y_{3,l}^{-} = \Big(1+\delta_{l1}\bar{Y}_0\Big)\frac{\left(1+Y_{3,l-1}\right)\left(1+Y_{3,l+1}\right)}{\left(1+\dfrac{1}{Y_{2,l}}\right)} \tag{5.32}$$

$$\bar{Y}_0^{++}\, \bar{Y}_0^{--} = \frac{Y_0}{\bar{Y}_0}\left(1+Y_{31}^{+}\right)\left(1+Y_{31}^{-}\right)\left(1+Y_{21}\right) \quad . \tag{5.33}$$

In the system above, each formula links to a kind of particle: equations (5.29) and (5.33) are related to holes and antiholes (the only types carrying mass and momentum), provided the definition of the Y-functions

$$Y_0(u) = e^{-\epsilon(u)} \qquad\qquad \bar{Y}_0(u) = e^{-\bar{\epsilon}(u)} \quad ,$$

whereas $Y_{i,l}(u) = e^{\epsilon_{i,l}(u)}$ are the relevant Y-functions for the equations (5.30)-(5.32) (henceforward called 'magnonic'), which refer to strings of type-i auxiliary roots, with length l (so i runs from 1 to 3, while $l = 1,2,3,\dots$); with (zero column) boundary condition

$$Y_{i,0}(\theta) \equiv 0 \tag{5.34}$$

the superscript $\pm$ corresponds to a shift $u \to u \pm \frac{i}{2}$, while the double sign $\pm\pm$ accounts for a double shift $u \to u \pm i$ (otherwise the argument of the Y-functions is meant to be u).

The magnonic equations (5.30)-(5.32) enjoy the usual pictorial interpretation. Each Y-function gets associated to a node in a Dynkin-like graph[2], so that the equation whose *lhs* contains $Y_{i,l}^{+}\, Y_{i,l}^{-}$ is meant to indicate the node labelled by (i,l), while the *rhs* describe

[2] In fact, the diagram depicted from the Y-system is customarily (though not strictly speaking) called Dynkin diagram. In this text, this custom will be maintained

the surrounding environment: every time a factor $(1+Y_{i',l'})$ is present, a horizontal link is drawn from the node (i,l) to (i',l'), whereas each factor $(1+Y^{-1}_{i",l"})^{-1}$ corresponds to a vertical link from (i,l) to $(i",l")$. In different words, the complex shifts in the *lhs* somehow get mirrored into shifts of discrete indices in the *rhs*. Finally, the equations (5.30)-(5.32) get encoded into a rectangular lattice, whose (say) left corners links to the nodes corresponding to massive particles. The massive nodes (those related to massive particle equations (5.29),(5.33)) prevent, in fact, from a net graphical interpretation of the Y-system at hand. A sort of 'double tadpole' $\frac{\bar{Y}_0}{Y_0}$ appears in the *rhs*, left without a clear pictorial counterpart; moreover, the complex shift also in the *r.h.s.* are a quite uncommon (though not novel at all, see [70]) feature. Anyway, the massive node equations suggest kind of conservation of the sum of the *absolute* value of displacements: the double (complex) shifts applied to Y_0 or $\bar{Y}_0$ in the *lhs* of (5.29) and (5.33) reflect into double shift on the Y-functions in the *rhs*, be vertical displacements or a single complex shift ($\pm$ as a superscript), the tadpole terms being overlooked for a while.

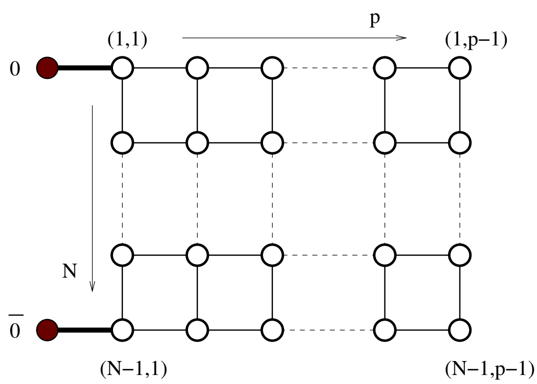

Figure 5.1: The $(\mathbb{CP}^{N-1})_p \times U(1)$ diagram.

This peculiar behaviour and, in particular, the missing interpretation for the double tadpole hints a more transparent form of the Y-system (5.29)-(5.33) might exist. Indeed the TBA equations entail, via identities (B.11) in Appendix B, a further functional relation involving 'massive' Y's, referred to in what follows key relation:

$$\frac{Y_0^+ \, Y_0^-}{\bar{Y}_0^+ \, \bar{Y}_0^-} = e^{4i\alpha} \, \frac{1+Y_{1,1}}{1+Y_{3,1}} \quad . \tag{5.35}$$

This key relation helps in finding, upon being shifted alternely by $u_{\rightarrow} u \pm \frac{i}{2}$, two novel equations for massive particles, so to replace the tricky (5.29) and (5.33), and thus they join (5.30)-(5.32) to implement a new Y-system, lacking of tadpoles though featuring a quite unusual appearance:

$$Y_0^{++} \, \bar{Y}_0^{--} = (1+Y_{11}^+) \, (1+Y_{21}) \, (1+Y_{31}^-) \tag{5.36}$$

$$Y_0^{--} \, \bar{Y}_0^{++} = (1+Y_{11}^-) \, (1+Y_{21}) \, (1+Y_{31}^+) \quad . \tag{5.37}$$

In fact, the *lhs* of (5.36) and (5.37) above does not involve either (shifted) Y_0 or $\bar{Y}_0$ alone, instead they both appear at the same time, with opposite shifts. From now on, this novel Y-system, where the magnonic (5.30)-(5.32) are joined by (5.36),(5.37) will be named 'crossed', as in the latters Y_0 and $\bar{Y}_0$ intertwine (in the *l.h.s.*), in opposition to the 'uncrossed system' made of (5.29) and (5.33) along with the magnonic equations (seen previously).

This peculiar form of the Y-system suggests a new pictorial interpretation The crossed equations (5.36) and (5.37), somehow, are drawing trajectories (flows) across the Dynkin diagram. The *l.h.s.* sets the endpoints of the pattern, while the *r.h.s.* plots it: the distance between two consecutive nodes touched by the path strictly amounts to steps of two units, which may be achieved by means of horizontal or vertical displacement (*i.e.* a decrement/increment on one index of the Y-function), or even a purely imaginary shifts. Let (5.36) be considered, as an example: the flow starts from $Y_0 \sim Y_{1,0}$ (with a pretty formal meaning), endowed with a double complex shift $++$; then one shift is dropped, while the second index increases, so to get $Y_{1,1}^{+}$, and later the $+$ disappears whereas the first index is raised, to touch $Y_{2,1}$; the pattern walks further, this time with negative complex shifts, so to pass through $Y_{3,1}^{-}$ and finally reach $Y_{3,0}^{--} \sim Y_0^{--}$. Besides that, it is tempting to claim the path's length gets fixed by the ratio between $2i$ and the imaginary unitary shift $\frac{i}{2}$ (or, someway, the inverse of the Coxeter number, when this quantity is defined). The key relation (5.35) seems to establish an equivalence class between closed patterns, passing either through Y_0 or through $\bar{Y}_0$.
A further comment: the crossed appearance of the two nodes $\mathbf{0}$ and $\bar{\mathbf{0}}$ in the *l.h.s.* hints that they are not to be treated as completely distinguished nodes. Indeed, they could be thought as originating from a sort of folding procedure of the Dynkin diagram, by splitting a unique massive node; this issue will be considered in more detail later.

3.1 Crossing Algebra

Actually, the Y-system (5.36),(5.37), (5.30)-(5.32) is not the only one known to possess a crossed form: indeed, before it two Y-system have been found to share this feature [3]. The first example has been provided by the Y-system for $\mathcal{N} = 6$ super Chern-Simons (AdS_4/CFT_3) [62], which, remarkably, is the model whose low-energy reduction is now under exam. The other crossed system has been discovered to describe the strong coupling of gluon scattering amplitude in $\mathcal{N} = 4$ SYM [64].
In the following, a few considerations are provided, which refer to system (5.36),(5.37),(5.30)-(5.32) but can easily extend to other crossed Y-systems as well.
First, the *key relation* (5.35) will be briefly indicated as χ; by shifting it, the equations χ^+ and χ^- are promptly obtained:

$$\chi^+ : \qquad \frac{Y_0^{++}\, Y_0}{\bar{Y}_0^{++}\bar{Y}_0} = \frac{1+Y_{1,1}^{+}}{1+Y_{3,1}^{+}} \tag{5.38}$$

$$\chi^- : \qquad \frac{Y_0^{--}\, Y_0}{\bar{Y}_0^{--}\bar{Y}_0} = \frac{1+Y_{1,1}^{-}}{1+Y_{3,1}^{-}} \tag{5.39}$$

[3] Moreover, there is an intriguing common aspect shared by all these three cases, that is, the intertwine of Y-functions is observed to occur for massive nodes only

In the same way, the crossed equations (5.36),(5.37) will be called respectively C_1 and C_2, while the uncrossed ones (5.29),(5.33) will be designated as U_1 and U_2. Actually, the equations now recollected are not quite independent: as a matter of fact, it suffices to know the key relation χ (more precisely its shifted versions χ^+ and χ^-) together with only one among C_1, C_2, U_1 or U_2 to recover all the others and describe the whole system[4]. The (quite general) rules to accomplish this task are summarized below, after introducing the formal operation $\times$ as a side by side multiplication between equations (on the same footing, the quotient corresponds to a division).

Getting an Uncrossed equation from another Uncrossed:

$$\begin{aligned} U_2 \times \chi^+ \times \chi^- &= U_1 \\ \frac{U_1}{\chi^+ \times \chi^-} &= U_2 \end{aligned} \tag{5.40}$$

Getting a Crossed equation from another Crossed:

$$\begin{aligned} C_2 \times \frac{\chi^+}{\chi^-} &= C_1 \\ C_1 \times \frac{\chi^-}{\chi^+} &= C_2 \end{aligned} \tag{5.41}$$

Getting an Uncrossed equation from Crossed:

$$\begin{aligned} C_1 \times C_2 \times \chi^+ \times \chi^- &= (U_1)^2 \\ \frac{C_1 \times C_2}{\chi^+ \times \chi^-} &= (U_2)^2 \end{aligned} \tag{5.42}$$

Getting a Crossed equation from an Uncrossed:

$$\begin{aligned} \frac{U_1}{\chi^+} &= C_2 \\ \frac{U_1}{\chi^-} &= C_1 \\ U_2 \times \chi^+ &= C_1 \\ U_2 \times \chi^- &= C_2 \end{aligned} \tag{5.43}$$

To be more explicit, the first rule in (5.40), for instance, means that equation U_1 (5.29) may be obtained upon multiplying side by side the equation U_2 (5.33) per the shifted-key relations (5.38) and (5.39).

The rules stated above do not apply to crossed system (5.36),(5.37) only: otherwise, with minor modifications, they extend to Y-system in [62] and [64], as well. Let the first example be considered. The Y-system for gluon scattering amplitudes in $\mathcal{N} = 4$ SYM

[4] In fact, the key relation holds a crucial role, which should be someway related to the implementation of Z_2 symmetry on the Y-system

reads:

$$\begin{aligned} Y^+_{1,m}\,Y^-_{3,m} &= \frac{(1+Y_{1,m-1})(1+Y_{3,m+1})}{(1+\frac{1}{Y_{2,m}})} \\ Y^+_{2,m}\,Y^-_{2,m} &= \frac{(1+Y_{2,m-1})(1+Y_{2,m+1})}{(1+\frac{1}{Y_{1,m}})(1+\frac{1}{Y_{3,m}})} \\ Y^+_{3,m}\,Y^-_{1,m} &= \frac{(1+Y_{3,m-1})(1+Y_{1,m+1})}{(1+\frac{1}{Y_{2,m}})} \quad ; \end{aligned} \tag{5.44}$$

this time, the superscript $\pm$ denotes a shift $\theta \to \theta \pm \frac{i\pi}{4}$, while the double sign stands for a double shift ($\pm\frac{i\pi}{2}$). Also in this case, a key relation exists, namely the ratio between the first and the third equations (5.44)

$$\chi: \qquad \frac{X^+_{1,m}\,X^-_{3,m}}{X^+_{3,m}\,X^-_{1,m}} = \frac{(1+X_{1,m-1})(1+X_{3,m+1})}{(1+X_{3,m-1})(1+X_{1,m+1})} \tag{5.45}$$

There comes now a slight difference from system (5.36),(5.37): while χ^+ is simply the up-shift ($\theta \to \theta + \frac{i\pi}{4}$) of χ, to correctly recover χ^- it takes to down-shift χ ($\theta \to \theta - \frac{i\pi}{4}$), then invert both sides of the equation obtained this way:

$$\begin{aligned} \chi^+: \qquad & \frac{X^{++}_{1,m}\,X_{3,m}}{X^{++}_{3,m}\,X_{1,m}} = \frac{(1+X^+_{1,m-1})(1+X^+_{3,m+1})}{(1+X^+_{3,m-1})(1+X^+_{1,m+1})} \\ \chi^-: \qquad & \frac{X_{3,m}\,X^{--}_{1,m}}{X_{1,m}\,X^{--}_{3,m}} = \frac{(1+X^-_{3,m-1})(1+X^-_{1,m+1})}{(1+X^-_{1,m-1})(1+X^-_{3,m+1})} \quad . \end{aligned} \tag{5.46}$$

The equations to be named C_1 is obtained by taking the product between the first equation in (5.44) up-shifted per the down-shifted version; in a the analogous way, C_2 is provided:

$$\begin{aligned} C_1: \quad & X^{++}_{1,m}\,X^{--}_{3,m}\,X_{1,m}\,X_{3,m} = \frac{(1+X^+_{1,m-1})(1+X^-_{1,m-1})(1+X^+_{3,m+1})(1+X^-_{3,m+1})}{(1+\frac{1}{X^+_{2,m}})(1+\frac{1}{X^-_{2,m}})} \\ C_2: \quad & X^{++}_{3,m}\,X^{--}_{1,m}\,X_{1,m}\,X_{3,m} = \frac{(1+X^+_{1,m+1})(1+X^-_{1,m+1})(1+X^+_{3,m-1})(1+X^-_{3,m-1})}{(1+\frac{1}{X^+_{2,m}})(1+\frac{1}{X^-_{2,m}})} \end{aligned} \tag{5.47}$$

In order to get U_1 and U_2, it takes a little more time. First of all, it is useful to write down the kernels employed in universal TBA system from [64]:

$$\begin{aligned} K_1(\theta) &\equiv \frac{1}{2\pi\,\cosh\theta} \\ K_2(\theta) &\equiv \frac{\sqrt{2}\,\cosh\theta}{\pi\,\cosh 2\theta} \\ K_3(\theta) &\equiv \frac{i}{\pi}\tanh 2\theta \quad ; \end{aligned} \tag{5.48}$$

they enjoy the following identities, involving shifts of $\frac{i\pi}{4}$:

$$\begin{aligned} & K^+_1 + K^-_1 - K_2 = 0 \\ & K^+_2 + K^-_2 - 2K_1 = \delta(\theta) \\ & K^-_3 - K^+_3 = \delta(\theta) \end{aligned} \tag{5.49}$$

and, in addition to the those, the identities for double shifts ($\frac{i\pi}{2}$):

$$\begin{aligned}
K_1^{++} + K_1^{--} &= \delta(\theta) \\
K_2^{++} + K_2^{--} &= \delta(\theta + \frac{i\pi}{4}) + \delta(\theta - \frac{i\pi}{4}) \\
K_3^{++} + K_3^{--} &= 2K_3 - \delta(\theta + \frac{i\pi}{4}) + \delta(\theta - \frac{i\pi}{4}) \quad .
\end{aligned}$$

By means of the relations above, from (5.44) the uncrossed Y-system follows:

$$\begin{aligned}
Y_{1,m}^{++} Y_{1,m}^{--} &= \frac{(1+Y_{1,m-1}^{+})(1+Y_{1,m+1}^{-})(1+Y_{3,m+1}^{+})(1+Y_{3,m-1}^{-})}{(Y_{3,m})^2 \; (1+\frac{1}{Y_{2,m}^{+}})(1+\frac{1}{Y_{2,m}^{-}})} \\
Y_{3,m}^{++} Y_{3,m}^{--} &= \frac{(1+Y_{3,m-1}^{+})(1+Y_{3,m+1}^{-})(1+Y_{1,m+1}^{+})(1+Y_{1,m-1}^{-})}{(Y_{1,m})^2 \; (1+\frac{1}{Y_{2,m}^{+}})(1+\frac{1}{Y_{2,m}^{-}})}
\end{aligned} \tag{5.50}$$

Hence, the first and second equations in (5.50) are respectively labelled U_1 and U_2. With those identifications in mind, the crossing algebra, claimed above, does indeed apply. In a very similar fashion, the Y-system from [62] is found to satisfy the same rules.

4 UV limit: central charge and perturbing operator

Picking a suggestion by Fendley [68], the Bykov model could be studied by conjecturing it as a two-dimensional conformal field theory (CFT) perturbed by a (marginal) relevant operator. In particular, a CFT gets characterised by the value of its conformal anomaly, c, which appears in the expression for vacuum free energy (5.28), when the Ultra-Violet (UV) limit $mR \ll 1$ is taken into account [66]:

$$E_{UV} = -\frac{\pi}{6R}\,(c - 12d) \tag{5.51}$$

where d is the conformal dimension of the ground state of the theory, and it is equal to zero when unitary theories are considered (as it is the present case).In order to get the value of the central charge c, the TBA equations should be analysed hence in the limit $r = mR \to 0$, and that reveals the solutions of the system flatten to develop a central plateau, whose width increases as r approaches zero [15][17]. Following [67], the central charge splits can be computed by splitting it into two contributions:

$$c = c^{(UV)} - c^{(IR)} \tag{5.52}$$

The first part explicitly reads

$$c^{(UV)} = \frac{6}{\pi^2}\left[\mathcal{L}\left(\frac{y_0}{1+y_0}\right) + \mathcal{L}\left(\frac{\bar{y}_0}{1+\bar{y}_0}\right) + \sum_{i=1}^{3}\sum_{l=1}^{\infty}\mathcal{L}\left(\frac{y_{i,l}}{1+y_{i,l}}\right)\right] \tag{5.53}$$

where the function $\mathcal{L}(x)$ is the Rogers Dilogarithm

$$\mathcal{L}(x) = -\frac{1}{2}\int_0^x \left[\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right] dt \quad , \quad 0 < x < 1 \quad : \tag{5.54}$$

the constants y stem from the stationary solution of the Y-system (5.36),(5.37), (5.30)-(5.32), in other words, not dependent on u:

$$\begin{aligned} & y_0\,\bar{y}_0 = (1+y_{1,1})\,(1+y_{2,1})\,(1+y_{3,1}) \\ & y_{i,l} = (1+\delta_{l,1}\delta_{i,1}\,y_0+\delta_{l,1}\delta_{i,3}\,\bar{y}_0)\,\frac{(1+y_{i,l+1})\,(1+y_{i,l-1})}{\left(1+\dfrac{1}{y_{i+1,l}}\right)\left(1+\dfrac{1}{y_{i-1,l}}\right)} \\ & \quad i=1,2,3 \quad l=1,2,3,\dots \end{aligned} \tag{5.55}$$

supplemented by the boundary conditions $y_{i,0} = (y_{4,l})^{-1} = (y_{0,l})^{-1} = 0$, (equations (5.36) and (5.37) obviously coincide in the stationary-solution regime); the equations above can be interpreted as an Ultra-Violet regime, where all the equations become massless (magnonic). The second contribution, instead, could be thought as an Infra-Red regime, in which the non-magnonic equations decouple from the rest, as the mass of the associated particles grows to infinity; therefore, the massive nodes disappear, so to leave the simpler system:

$$z_{i,l} = \frac{(1+z_{i,l+1})\,(1+z_{i,l-1})}{\left(1+\dfrac{1}{z_{i+1,l}}\right)\left(1+\dfrac{1}{z_{i-1,l}}\right)} \qquad i=1,2,3 \quad l=1,2,3,\dots \tag{5.56}$$

whose solutions enter the second part of the chentral charge

$$c^{(IR)} = \frac{6}{\pi^2}\sum_{i=1}^{3}\sum_{l=1}^{\infty}\mathcal{L}\left(\frac{z_{i,l}}{1+z_{i,l}}\right) \tag{5.57}$$

(imposing $z_{i,0} = (z_{4,l})^{-1} = (z_{0,l})^{-1} = 0$).
Since (5.55) and (5.56) are systems made out of an infinite number of non-linear equations, they result uncomfortable to cope with. Hence a fruitful strategy to handle those two systems consists in studying a truncated model, by reducing the number of magnonic variables to a finite value (*i.e.* the label l runs from 1 to n); later, the original model will be recovered after performing the $n \to \infty$ limit. The IR contribution can be analytically computed, after [61],[65], upon evaluating the sum of dilogarithms in (5.57):

$$c_N^{(IR)} = \frac{3n(n+1)}{n+5} \quad . \tag{5.58}$$

The UV contribution instead turns out unfit to be approached analytically, the numerical analysis revealing more suitable [5]; the latter option helps in conjecturing for (5.53) the form:

$$c_n^{(UV)} = \frac{(4+3n)(n+1)}{n+4} \quad . \tag{5.59}$$

Composing then the two contributions, the central charge of the truncated model results:

$$c_n = c_n^{(UV)} - c_n^{(IR)} = \frac{7n^2+27n+20}{n^2+9n+20} \quad : \tag{5.60}$$

[5]The author of this text wishes to acknowledge Alessandro Fabbri and Roberto Tateo for their precious numerical work when writing [27]

the $n \to \infty$ limit takes back to the original model, so that the central charge is discovered to assume the value $c = 7$, as expected from the plain counting of degrees of freedom.

What is more, the result (5.60) deserves interest even at finite n, for this expression represents the central charge of the $\frac{SU(4)_{n+1}}{SU(3)_{n+1} \otimes U(1)}$ coset conformal field theory, perturbed by some (relevant) operator [68]. The anomalous dimension $\Delta_{n,pert}$ of such perturbing operator can be recognized upon retrieving the purely imaginary period [60] P_n of the Y-system obtained after the truncation of (5.29-5.33). Indeed, for the Y-functions the following periodicity relation holds:

$$Y\,(u + 2i\,P_n) = Y(\theta) \tag{5.61}$$

and, according to [60], the period P_n relates to the dimension of the perturbative operator:

$$\Delta_{n,pert} = 1 - \frac{1}{P_n} \quad . \tag{5.62}$$

Numerical analysis shows that the Y-system (5.36),(5.37),(5.30)-(5.32) is endowed with a period

$$P_n = \frac{n+4}{2} \tag{5.63}$$

and thus the conformal dimension of the adjoint operator, perturbing the conformal field theory at finite n, is revealed:

$$\Delta_{n,pert} = 1 - \frac{2}{n+4} \quad . \tag{5.64}$$

As a byproduct of the considerations above, the CP^3 sigma model, coupled to one massless Dirac fermion [50], is equivalent (up to the features taken into account) to the large level limit $n \to \infty$ of the $\frac{SU(4)_{n+1}}{SU(3)_{n+1} \otimes U(1)}$ coset conformal field theory, perturbed by the (relevant) operator whose dimension is read from (5.64).

Before concluding this section, a last remark is worth being mentioned. The truncated version of Y-system (5.36),(5.37),(5.30)-(5.32) suggests a natural generalization to a novel family of systems (labelled by two positive integers N and p), which are left associated to any $SU(N)$ algebra with coset level p. Models belonging to this family are conjectured to exhibit, for what concerns the massive nodes, the generalized equations:

$$\begin{aligned} Y_0(u+i)\,\bar{Y}_0(u-i) &= \prod_{l=1}^{N-1} \left(1 + Y_{l,1}(u + i - i\frac{2l}{N})\right) \\ Y_0(u-i)\,\bar{Y}_0(u+i) &= \prod_{l=1}^{N-1} \left(1 + Y_{l,1}(u - i + i\frac{2l}{N})\right) \quad , \end{aligned} \tag{5.65}$$

while for the magnonic nodes the relations hold:

$$\begin{aligned} &Y_{i,j}(u + \frac{2i}{N})\,Y_{i,j}(u - \frac{2i}{N}) = \left(1 + \delta_{i,1}\delta_{j,1}\,Y_0(u) + \delta_{i,N-1}\delta_{j,1}\,\bar{Y}_0(u)\right) \times \\ &\times \prod_{l=1}^{p-1} (1 + Y_{i,l}(u))^{A^{(p-1)}_{l,j}} \prod_{l'=1}^{N-1} \left(1 + \frac{1}{Y_{l',j}(u)}\right)^{-A^{(N-1)}_{l',i}} \quad ; \end{aligned} \tag{5.66}$$

where $A^{(q)}_{l,j}$ stands for the incidence matrix associated to the A_q Dynkin diagram, namely

$$A^{(q)}_{l,j} = \delta_{l,\,j+1} + \delta_{l,\,j-1} \tag{5.67}$$

with $l, j \in \{1, 2, \ldots, q\}$. The system studied so far corresponds to the case $N = 4$ and $p \to \infty$. The TBA equations corresponding to (5.65) are displayed in the universal form [60], by making use of the Fourier integrals in (B.13):

$$\begin{aligned} \epsilon_0(u) \quad &+ \quad \bar{\epsilon}_0(u) = 2m\mathcal{R}\cosh\left(\frac{\pi u}{2}\right) - \sum_{l=1}^{N-1} \chi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(u) && (5.68) \\ \epsilon_0(u) \quad &- \quad \bar{\epsilon}_0(u) = i2\alpha - \sum_{l=1}^{N-1} \psi_{(1-\frac{2l}{N})} * \Lambda_{(l,1)}(u) \\ \epsilon_{i,j}(u) \quad &= \quad \delta_{i,1}\delta_{j,1}\phi_{\frac{N}{2}} * L_0(u) + \delta_{i,N-1}\delta_{j,1}\phi_{\frac{N}{2}} * \bar{L}_0(u) + \sum_{l=1}^{p-1} A^{(p-1)}_{l,j}\phi_{\frac{N}{2}} * \Lambda_{(i,l)}(u) - \\ &- \quad \sum_{l=1}^{N-1} A^{(N-1)}_{l,i}\phi_{\frac{N}{2}} * L_{(l,j)}(u) \quad , && (5.69) \end{aligned}$$

with chemical potential $\alpha \in \{0, \pi\}$, $L_A(u) = \ln(1 + e^{-\epsilon_A(u)})$ and $\Lambda_A(u) = \ln(1 + e^{\epsilon_A(u)})$. A parallel with (5.60), for $N = 4$ and $p \to \infty$, leads to conjecture (after an accurate numerical analysis) the following formula for the central charge of the arbitrary (N, p)-model:

$$c_{(N,p)} = \frac{p(1 - p - N + N^2 + 2Np)}{(N + p)(N + p - 1)} = \frac{p \dim[SU(N)]}{p + N} - \frac{p \dim[SU(N - 1)])}{p + N - 1} \tag{5.70}$$

where it holds $\dim[SU(N)] = N^2 - 1$. Eventually, the central charge (5.70), obtained from equations (5.65), is seen to exactly match the charge of the coset model:

$$(\mathbb{CP}^{N-1})_p \times U(1) = \frac{SU(N)_p}{SU(N-1)_p \times U(1)} \times U(1) \quad . \tag{5.71}$$

Finally the large level limit $p \to \infty$ of (5.70) leads to the central charge of the $SU(N) \times U(1)$ sigma model:

$$c_{(N,\infty)} = \dim[SU(N)] - \dim[SU(N - 1)] = 2N - 1. \tag{5.72}$$

Strictly speaking, a mere equality of conformal charges does not suffice to uniquely determine the identification (5.71); nevertheless further evidence of this claim, such as the dimension of the perturbing operator, is provided in [27].

5 Folding diagrams

This section is devoted to the discussion of some peculiar features of the Y-system of Bykov model, studied so far: in particular, it will be compared to another interesting class of Y-systems, those related to non-linear $O(2n)$ σ-models. The focus will turn mainly to the $n = 3$ case, the $O(6)$ σ-model: in fact, the Dynkin diagram for the $O(6)$ stunningly resembles the $\mathbb{CP}^3 \times U(1)$ diagram, retrieved in this chapter, once a sort of pictorial folding has been performed.

The $O(2n)$ Non-Linear σ Model TBA and Y-system:
According to [69][70][68], the TBA system for the $O(2n)$ ($n \geq 2$) non-linear σ models is obtained from the TBA describing a certain perturbed conformal model, in the limit of infinitely many equations present (the number also corresponds to Kac-Moody level):

$$\epsilon_0(u) = m\mathcal{R}\cosh\frac{\pi u}{2} - \sum_{j=1}^{n-2} \chi_{\frac{2}{g}(n-1-j)} * L_{j,1}(u) - \phi_1 * [L_{n-1,1} + L_{n,1}] \tag{5.73}$$

$$\begin{aligned}\epsilon_{a,m}(u) = {} & -\delta_{m1}[\delta_{a1} + \delta_{a2}\delta_{n2}]\,\phi_{\frac{g}{2}} * L_0(u) - \phi_{\frac{g}{2}} * [L_{a,m-1} + L_{a,m+1}] + \\ & + \sum_{b=1}^{n} A_{ab}\,\phi_{\frac{g}{2}} * \Lambda_{b,m}(u)\end{aligned} \tag{5.74}$$

where the label a runs from 1 to n and $m = 1, \ldots, p-1$ (then, the limit $p \to \infty$ should be performed). In addition to that, $g = 2(n-1)$ represents the Coxeter number associated to the D_n Lie algebra, while $A_{ab} = \delta_{a,\,b+1} + \delta_{a,\,b-1}$ stands for its incidence matrix; moreover, it is useful to recollect the definitions:

$$\begin{aligned} L_0(u) &= \ln\left(1 + e^{-\epsilon_0(u)}\right) \qquad L_{a,m}(u) = \ln\left(1 + e^{-\epsilon_{a,m}(u)}\right) \\ \Lambda_{a,m}(u) &= \ln\left(1 + e^{\epsilon_{a,m}(u)}\right) \quad . \end{aligned} \tag{5.75}$$

Thanks to the relation for the kernel $\chi_{\frac{2}{g}(n-1-j)}(u)$

$$\chi_{\frac{2}{g}(n-1-j)}(u+i) + \chi_{\frac{2}{g}(n-1-j)}(u-i) = \delta(u+\frac{2i(n-1-a)}{g}) + \delta(u-\frac{2i(n-1-a)}{g}) \tag{5.76}$$

and after the definitions

$$\begin{aligned} X_{a,\,m}(u) &\equiv e^{-\epsilon_{a,m}(u)} \\ X_0(u) &\equiv e^{-\epsilon_0(u)} \quad . \end{aligned} \tag{5.77}$$

the Y-system for $O(2n)$ non-linear σ-models stands out [70]:

$$\begin{aligned} X_0\,(u+i)\;X_0\,(u-i) = {} & \prod_{a=1}^{n-2}\left[\left(1 + X_{a,1}(u - \frac{2i(n-1-a)}{g})\right) \times \right. \\ & \left. \times \left(1 + X_{a,\,1}(u + \frac{2i(n-1-a)}{g})\right)\right] (1 + X_{n-1,\,1}(u))\,(1 + X_{n,\,1}(u)) \\ X_{a,m}\left(u + \frac{2i}{g}\right) X_{a,m}\left(u - \frac{2i}{g}\right) = {} & [1 + \delta_{1m}(\delta_{a1} + \delta_{n2}\delta_{a2})X_0(u)] \times \\ & \times \frac{(1 + X_{a,m+1}(u))(1 + X_{a,m-1}(u))}{\prod_{b=1}^{n}\left(1 + \frac{1}{X_{b,m}(u)}\right)^{A_{ab}}} \quad ; \end{aligned} \tag{5.78}$$

by following the prescriptions stated above, this Y-system may be encoded in the diagram of *Fig.*(5.2). It should be noted that (5.78) exhibits complex shifts even in the *r.h.s.* of the massive node equations, analogously to $(\mathbb{CP}^{N-1})_p \times U(1)$ models.

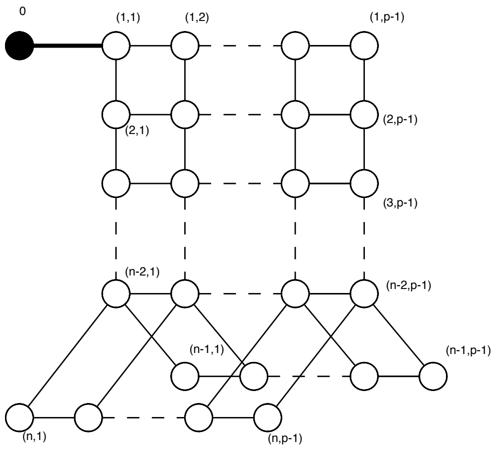

Figure 5.2: The $O(2n)$ diagram. The labels of each node are associated to the functions Y in (5.78)

Folding diagrams:
For present purposes, the Y-system for the $O(6)$ non-linear sigma model reveals reasons of interest:

$$\begin{aligned} X_0(u+i)\,X_0(u-i) &= \left(1+X_{2,1}(u+\frac{i}{2})\right)\left(1+X_{2,1}(u-\frac{i}{2})\right)(1+X_{1,1})\,(1+X_{3,1}) \\ X_{a,m}(u+\frac{i}{2})\,X_{a,m}(u-\frac{i}{2}) &= (1+\delta_{m1}\delta_{a2}X_0)\,\frac{(1+X_{a,m+1})\,(1+X_{a,m-1})}{\left(1+\dfrac{1}{X_{a+1,m}}\right)\left(1+\dfrac{1}{X_{a-1,m}}\right)} \end{aligned} \tag{5.79}$$

($a \in \{1,2,3\}$ and $m = 1,2,3,...,p-1$), where the boundary conditions $X_{a,0} = X_{a,p} = (X_{0,m})^{-1} = (X_{(4,m)})^{-1} = 0$ are required and the limit $p \to \infty$ should be taken. The Y-system (5.79) enjoys the customary uncrossed form, and the usual pictorial rules allow to encode it into the diagram in *Fig.*(5.3). Interestingly, the $O(6)$ diagram *Fig.*(5.3) can

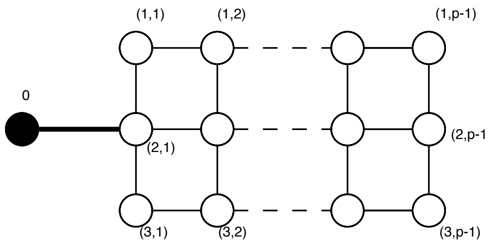

Figure 5.3: The $O(6)$ diagram. The labels of each node are to be intended as the subscripts of the functions X appearing in (5.79).

be obtained directly from the $\mathbb{CP}^3 \times U(1)$ by means of kind of folding, performed on the

diagram in *Fig.*(5.1): row 1 and 3 merge together, and so do the massive nodes, which are identified into a unique one; on the other hand, each node along the symmetry row 2 is 'torn' into two, distinct. This process can be reverted, so to retrieve this time *Fig.*(5.1) upon folding *Fig.*(5.3): again the middle row 2 in *Fig.*(5.3) gets doubled, whereas rows 1 and 3 overlap and glue together; the central massive node splits into two, which in turn become the two massive nodes of $\mathbb{CP}^3 \times U(1)$, associated to the holes and the antiholes of the theory.
Remarkably, the very same procedure could be applied to the diagram encoding the AdS_5 Y-system [59],resulting in the AdS_4 diagram [62][63]. This fact catches even more interest, for $O(6)$ σ-model and $\mathbb{CP}^3 \times U(1)$ are found to represent the low-energy reductions precisely of AdS_5 and AdS_4 indeed, therefore hinting some deeper relation between those two theories might exist.

Conclusions

The main achievements of this work pertain to the finding of the scalar factor of the scattering matrices involving excitations over the antiferromagnetic vacuum (associated to the GKP string), in the $\mathcal{N}=4$ SYM gauge theory. Those S-matrices have then been employed as building blocks towards the construction of the all loop Asymptotic Bethe Ansatz equations, that is to say, the analogous of the Beisert-Staudacher equations [30] in the $sl(2)$ grading, written instead upon taking the GKP vacuum as the reference state. The analysis of the Bethe equations led to the formulation of the dispersion laws, at any coupling, for the excitations considered, thus confirming the results by Basso [42]: the main difference from [42] concerns a different approach, based on the method introduced by Destri and De Vega [34]. In fact, as a starting point, the set of infinite Beisert-Staudacher equations has been turned, in the large spin limit, to a limited number of integral equations, easier to be handled.
In addition to that, the $\mathcal{N}=6$ Chern-Simons Matter has been considered, when choosing a vacuum corresponding to GKP string. Remarkably, the low-energy (large spin) reduction achieved this way reveals intriguing connections to the O(6) non-linear σ-model, which governs the dynamics of $\mathcal{N}=4$ SYM in a special large spin limit [38].

The results achieved may be contextualized. The method used to obtain energies and momenta of the different species of excitations turned out more handy than the ones previously adopted to get the same results, hence, hopefully, it could bring precious contributions in formulating new tests in verifying the AdS/CFT conjecture.
The (two dimensional) S-matrices help in computing the scattering amplitudes for the four dimensional $\mathcal{N}=4$ SYM gauge theory. In fact, when a quark and an antiquark (at speed of light) interact via the exchange of a gluon, under the $\mathcal{N}=4$ SYM $SU(N)$ gauge theory, the process can be intended in two equivalent ways: a GKP string could be thought to stretch among the two particles, spanning a two dimensional world-sheet as the time runs, else the flux tube in between the quark and the antiquark may be framed by a null Wilson loop. As a general feature of $\mathcal{N}=4$ SYM, a polygonal null Wilson loop can be decomposed into a superposition of pentagonal and square transition amplitudes [46]. The pentagonal amplitude $P(u|v)$, in turn, can be computed by means of the scattering matrices (say generically $S(u,v)$) portrayed in chapter 3; indeed [46],

$$P(u|v)^2 = \frac{\mathcal{F}(u,v)}{g^2\,(u-v)(u-v-i)}\,\frac{S(u,v)}{S(u^\gamma,v)} \quad , \tag{5.80}$$

where $\mathcal{F}(u,v)$ is a known function, while the superscript γ denotes a mirror transform. Hence, as the formula above suggests, at least some of the results described in this text find a direct application in meaningful open issues.

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Appendix A

Useful formulæ

1 Fourier transforms

In this appendix a collection is provided of the Fourier transforms

$$\hat{f}(k) = \int_{-\infty}^{+\infty} due^{-iku} f(u) \tag{A.1}$$

for several functions $f(u)$ used in the maintext.
For scalars it has been used

$$\Phi_0(u) = -2\arctan 2u \quad \Rightarrow \quad \hat{\Phi}_0(k) = \int_{-\infty}^{+\infty} due^{-iku}\Phi_0(u) = -\frac{2\pi}{ik}e^{-\frac{|k|}{2}} \tag{A.2}$$

$$\Phi_H(u) = -i\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right) \quad \Rightarrow \quad \hat{\Phi}_H(k) = \frac{2\pi}{ik}e^{-\frac{|k|}{2}}[1 - J_0(\sqrt{2}gk)] \tag{A.3}$$

and also

$$\phi_0(u-v) = 2\arctan(u-v) \quad \Rightarrow \quad \hat{\phi}_0(k) = \frac{2\pi e^{-|k|}}{ik}, \tag{A.4}$$

$$\varphi_0(u-v) = \frac{1}{2\pi}\frac{d}{dv}\phi_0(u-v) = -\frac{1}{\pi}\frac{1}{1+(u-v)^2} \quad \Rightarrow \quad \hat{\varphi}_0(k) = -e^{-|k|}, \tag{A.5}$$

$$\begin{aligned}
\phi_H(u,v) &= -2i\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right) + i\theta(u,v)\right] \\
\hat{\phi}_H(k,t) &= -8i\pi^2\frac{e^{-\frac{|t|+|k|}{2}}}{k|t|}\Big[\sum_{r=1}^{\infty} r(-1)^{r+1}J_r(\sqrt{2}gk)J_r(\sqrt{2}gt)\frac{1-\text{sgn}(kt)}{2} + \\
&+ \text{sgn}(t)\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} c_{r,r+1+2\nu}(g)(-1)^{r+\nu}\Big(J_{r-1}(\sqrt{2}gk)J_{r+2\nu}(\sqrt{2}gt) - \\
&- J_{r-1}(\sqrt{2}gt)J_{r+2\nu}(\sqrt{2}gk)\Big)\Big]
\end{aligned} \tag{A.6}$$

For what concerns gluonic stacks, it has been introduced

$$\chi_0(u|l) = 2\arctan\frac{2u}{l} = i\ln\frac{il+2u}{il-2u} \quad \Rightarrow \hat{\chi}_0(k|l) = \int_{-\infty}^{+\infty} due^{-iku}\chi_0(u|l) = \frac{2\pi}{ik}e^{-|k|\frac{l}{2}}$$

(A.7)

and for higher loops the function

$$\chi(v,u|l) = \chi_0(v-u|l+1) + \chi_H\left(v, u-\frac{il}{2}\right) + \chi_H\left(v, u+\frac{il}{2}\right)$$

where

$$\chi_H(v,u) = i\ln\left(\frac{1-\frac{g^2}{2x^-(v)x(u)}}{1-\frac{g^2}{2x^+(v)x(u)}}\right) \tag{A.8}$$

whose Fourier transform reads

$$\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv e^{-ikv} e^{-itu}\chi(v,u|l) = 2\pi\delta(t+k)\frac{2\pi}{ik}e^{-|k|\frac{l+1}{2}} +$$
$$+ \quad i\sum_{n=1}^{+\infty} n(-1)^n \frac{2\pi}{k}\frac{2\pi}{|t|}e^{-\frac{|k|}{2}}e^{-\frac{|t|l}{2}} J_n(\sqrt{2}gk)J_n(\sqrt{2}gt) \tag{A.9}$$

In getting (A.9) the Fourier tranforms has been employed

$$\int du e^{-iku}\frac{1}{x\left(u\pm i\frac{l}{2}\right)^n} = \pm n\left(\frac{\sqrt{2}}{ig}\right)^n \theta(\pm k)\frac{2\pi}{k}e^{\mp\frac{l}{2}k}J_n(\sqrt{2}gk) \tag{A.10}$$

The Fourier transform of $\chi(v,u|l)$ and $\chi_H(v,u)$, with respect to the variable v only, read:

$$\int_{-\infty}^{+\infty} dv e^{-ikv}\chi(v,u|l) = e^{-iku}\frac{2\pi}{ik}e^{-|k|\frac{l+1}{2}} - \tag{A.11}$$
$$-\left[\sum_{n=1}^{\infty}\left(\frac{g}{\sqrt{2}i\; x\left(u-\frac{il}{2}\right)}\right)^n + \left(\frac{g}{\sqrt{2}i\; x\left(u+\frac{il}{2}\right)}\right)^n\right]\frac{2\pi}{ik}\, e^{-\frac{|k|}{2}} J_n(\sqrt{2}gk)$$

$$\int_{-\infty}^{+\infty} dv e^{-ikv}\chi_H(v,u) = i\frac{2\pi}{k}e^{-\frac{|k|}{2}}\sum_{n=1}^{+\infty}\left(\frac{g}{\sqrt{2}i\; x(u)}\right)^n J_n(\sqrt{2}gk) \tag{A.12}$$

When $l=1$ the identity arises

$$\chi(v,u|1) = -i\ln\left(\frac{x^+(u)-x^-(v)}{x^+(u)-x^+(v)}\frac{x^-(u)-x^-(v)}{x^+(v)-x^-(u)}\right) \tag{A.13}$$

Finally, for large fermions the function is introduced

$$\chi_F(u,v) = \chi_0(u-v|1) + \chi_H(u,v) = i\ln\frac{x^+(u)-x(v)}{x(v)-x^-(u)} \tag{A.14}$$

2 Integrals

In various calculations throughout the text the following integrals have been employed:

$$\int_{-1}^{1} dk\frac{1}{u-k}\left(\frac{1+k}{1-k}\right)^{\frac{1}{4}} = -\pi\sqrt{2}\left[1-\left(\frac{u+1}{u-1}\right)^{\frac{1}{4}}\right] \;, \quad |u|>1 \tag{A.15}$$

$$\int_{-1}^{1} dk PV\frac{1}{u-k}\left(\frac{1+k}{1-k}\right)^{\frac{1}{4}} = -\pi\sqrt{2} + \pi\left(\frac{1+u}{1-u}\right)^{\frac{1}{4}} \;, \quad |u|<1 \tag{A.16}$$

$$\int_{|\bar{w}|\geq 1} \frac{d\bar{w}}{2\pi} \frac{1}{\bar{w}-\bar{u}} PV \frac{1}{\bar{w}-\bar{z}} \left(\frac{\bar{w}+1}{\bar{w}-1}\right)^{\frac{1}{4}} =$$
$$= \frac{1}{2}\left(\frac{\bar{z}+1}{\bar{z}-1}\right)^{\frac{1}{4}} \frac{1}{\bar{z}-\bar{u}} + \frac{1}{\sqrt{2}}\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} \frac{1}{\bar{u}-\bar{z}}, \qquad |\bar{u}| \leq 1\ ,\ |\bar{z}| \geq 1\,. \quad \text{(A.17)}$$

$$\int_{|\bar{z}|\geq 1} \frac{d\bar{z}}{2\pi} \frac{1}{\bar{z}\sqrt{1-\frac{1}{\bar{z}^2}}} \frac{1}{\bar{x}_f(\bar{v})-\bar{x}(\bar{z})} \left(\frac{\bar{z}-1}{\bar{z}+1}\right)^{\frac{1}{4}} \frac{1}{\bar{u}-\bar{z}} = \quad \text{(A.18)}$$
$$= \frac{1}{2\bar{x}_f(\bar{u})(\bar{u}-\bar{v})} \left\{ \sqrt{\frac{1-2\bar{x}_f(\bar{v})}{1+2\bar{x}_f(\bar{v})}} + \frac{1}{\sqrt{2}}\left(\bar{x}_f(\bar{v})-\frac{1}{2}\right)\left[\left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}} + \left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}}\right] + \right.$$
$$\left. + \frac{1}{\sqrt{2}}\left(\bar{x}_f(\bar{v})+\frac{1}{2}\right)\sqrt{\frac{1-\bar{u}}{1+\bar{u}}}\left[\left(\frac{1-\bar{u}}{1+\bar{u}}\right)^{\frac{1}{4}} - \left(\frac{1+\bar{u}}{1-\bar{u}}\right)^{\frac{1}{4}}\right]\right\}$$

Appendix B

Scattering amplitudes and TBA kernels

This appendix contains the explicit expressions for scattering amplitudes and the corresponding TBA kernels appearing in the main text.

Hole-hole scattering

The hole-hole S-matrix amplitude [57] is

$$S(u) = -\frac{\Gamma\left(1+i\frac{u}{4}\right)\Gamma\left(\frac{1}{4}-i\frac{u}{4}\right)}{\Gamma\left(1-i\frac{u}{4}\right)\Gamma\left(\frac{1}{4}+i\frac{u}{4}\right)}, \tag{B.1}$$

and the corresponding kernel $\mathcal{K}(u)$

$$\mathcal{K}(u) = \frac{1}{2\pi i}\frac{\partial}{\partial u}\ln S(u), \tag{B.2}$$

which may be represented in several alternative ways as [1]

$$\begin{aligned}\mathcal{K}(u) &= \frac{1}{8\pi}\left(\Psi\left(1+i\frac{u}{4}\right)+\Psi\left(1-i\frac{u}{4}\right)-\Psi\left(\frac{1}{4}+i\frac{u}{4}\right)-\Psi\left(\frac{1}{4}-i\frac{u}{4}\right)\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\frac{4n+1}{u^2+(4n+1)^2}-\frac{1}{\pi}\frac{4n+4}{u^2+(4n+4)^2}\right)\end{aligned} \tag{B.4}$$

Hole-antihole scattering

The S-matrix amplitude associated to the hole-antihole scattering is

$$\bar{S}(u) = \frac{\Gamma\left(\frac{1}{2}-i\frac{u}{4}\right)\Gamma\left(\frac{3}{4}+i\frac{u}{4}\right)}{\Gamma\left(\frac{1}{2}+i\frac{u}{4}\right)\Gamma\left(\frac{3}{4}-i\frac{u}{4}\right)}. \tag{B.5}$$

[1] It could be useful to remind that

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma_E - \sum_{n=0}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right), \tag{B.3}$$

where γ_E stands for the Euler constant.

Consequently the kernel $G(u)$ is

$$G(u) = \frac{1}{2\pi i}\frac{\partial}{\partial u}\ln \bar{S}(u), \tag{B.6}$$

explicitly

$$\begin{aligned} G(u) &= \frac{1}{8\pi}\left(\Psi\left(\frac{3}{4}+i\frac{u}{4}\right)+\Psi\left(\frac{3}{4}-i\frac{u}{4}\right)-\Psi\left(\frac{1}{2}+i\frac{u}{4}\right)-\Psi\left(\frac{1}{2}-i\frac{u}{4}\right)\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{\pi}\frac{4n+2}{u^2+(4n+2)^2}-\frac{1}{\pi}\frac{4n+3}{u^2+(4n+3)^2}\right) \end{aligned} \tag{B.7}$$

Magnon bound state scattering

Magnonic string solutions scatter according to the amplitudes

$$S_{l,m}(u) = \prod_{a=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}}\left(\frac{u-ia}{u+ia}\right), \tag{B.8}$$

from which

$$K_{l,m}(u) = \frac{1}{2\pi i}\frac{\partial}{\partial u}\ln S_{lm}(u) = \sum_{a=\frac{|l-m|+1}{2}}^{\frac{l+m-1}{2}}\frac{1}{\pi}\frac{a}{u^2+a^2}. \tag{B.9}$$

1 Helpful Relations in Bootstrapping Matrices and Kernels

Below, a list is offered of identities between scattering matrices (*cfr* [60][61]): they reveal necessary to write the Y-system and universal form TBA

$$\begin{aligned} &S_{lm}\left(\theta+\frac{i}{2}\right)\,S_{lm}\left(\theta-\frac{i}{2}\right) = S_{l-1,m}\,(u)\;S_{l+1,m}\,(u)\;e^{2\pi i\Theta(u)\,\delta_{lm}} \\ &\bar{S}\left(\theta+\frac{i}{2}\right)\,\bar{S}\left(u-\frac{i}{2}\right) = -S\left(u+\frac{i}{2}\right)\,S\left(u-\frac{i}{2}\right)\,[S_{11}(u)]^{-1} \\ &S\,(u+i)\;S\,(u-i) = -\frac{\bar{S}(u)}{S(u)}S_{12}(u)\,e^{2\pi i\Theta(u)} \\ &\bar{S}\,(u+i)\;\bar{S}\,(u-1) = -\frac{S(u)}{\bar{S}(u)} \\ &S_{lm}\,(u+i)\;S_{lm}\,(u-i) = S_{l-2,m}\,(u)\;S_{l+2,m}\,(u)\;e^{2\pi i\Theta(u)\,I_{lm}} \end{aligned} \tag{B.10}$$

($\Theta(x)$ stands for the Heaviside step function, while $I_{lm} = \delta_{l-1,m} + \delta_{l+1,m}$). These relations, in terms of the kernels, turn to:

$$
\begin{aligned}
&K_{lm}\left(u+\frac{i}{2}\right)+K_{lm}\left(u-\frac{i}{2}\right)=K_{l-1,m}\,(u)+K_{l+1,m}\,(u)+\delta(u)\,\delta_{lm}\\
&G\left(u+\frac{i}{2}\right)+G\left(u-\frac{i}{2}\right)=\mathcal{K}\left(u+\frac{i}{2}\right)+\mathcal{K}\left(u-\frac{i}{2}\right)-K_{11}(u)\\
&\mathcal{K}\,(u+i)+\mathcal{K}\,(u-i)=-\mathcal{K}(u)+G(u)+K_{12}(u)+\delta(u)\\
&G\,(u+i)+G\,(u-i)=\mathcal{K}(u)-G(u)\\
&K_{lm}\,(u+i)+K_{lm}\,(u-i)=K_{l-2,m}\,(u)+K_{l+2,m}\,(u)+\delta(u)\,I_{lm}+\\
&\qquad\qquad+\delta_{l1}\,\delta_{m1}\left[\delta(u+\frac{i}{2})+\delta(u-\frac{i}{2})\right]
\end{aligned}
\tag{B.11}
$$

(the last relation makes sense [2] provided we define $K_{l,0}=0$, $K_{l,-1}=-K_{l,1}$). Moreover, we find:

$$
\begin{aligned}
\mathcal{K}(u+i)+G(u-i)-K_{11}(u+\frac{i}{2}) &= 0\\
\mathcal{K}(u-i)+G(u+i)-K_{11}(u-\frac{i}{2}) &= 0\\
\mathcal{K}(u+i)+G(u-i)+K_{11}(u-\frac{i}{2}) &= K_{12}(u)+\delta(u)\\
\mathcal{K}(u-i)+G(u+i)+K_{11}(u+\frac{i}{2}) &= K_{12}(u)+\delta(u)
\end{aligned}
\tag{B.12}
$$

The universal kernels

The kernels appearing in the Zamolodchikov's universal form of the TBA equations (5.68) are

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\cosh(\frac{\pi}{2}a\omega)}{\cosh(\frac{\pi\omega}{2})}\,e^{i\omega\theta} &= \frac{2}{\pi}\frac{\cos(a\pi/2)\cosh\theta}{\cos(a\pi)+\cosh(2\theta)}=\chi_a(\theta),\\
\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\sinh(\frac{\pi}{2}a\omega)}{\sinh(\frac{\pi\omega}{2})}\,e^{i\omega\theta} &= \frac{1}{\pi}\frac{\sin(a\pi)}{\cos(a\pi)+\cosh(2\theta)}=\psi_a(\theta),\\
\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{2\cosh(\frac{\pi\omega}{2a})}\,e^{i\omega\theta} &= \frac{a}{2\pi\,\cosh(a\theta)}=\phi_a(\theta).
\end{aligned}
\tag{B.13}
$$

[2]Actually, the contact terms $\delta(u\pm\frac{i}{2})$ are but a pretty formal scripture: relations (B.11) always appear in integrals and it is to be taken into account a residue calculation, whose net result is equivalent to the effect of some kind of complex-argument defined delta function.

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