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NON-NORMAL MODAL LOGICS, QUANTIFICATION, AND DEONTIC DILEMMAS. A STUDY IN MULTI-RELATIONAL SEMANTICS

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A Study in Multi-relational Semantics

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I can't go back to yesterday because I was a different person then.

Lewis Carroll, *Alice in Wonderland*

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Contents

Introduction

AI, Law and Logic

Art and music are used within medical protocols, while giant closeups of colourful viruses and bacteria hang on the walls of art galleries all over the world. Brand new ethical issues arise from any new technological achievement. It has become quite hard to find two distinct, not interacting disciplines. Being interdisciplinary is the new imperative. Sometimes the contamination of fields creates a juxtaposition of distinct flavours. Other times, however, genuinely new disciplines come up. This happened after the fortunate encounter of Artificial Intelligence (AI) and Law.

The use of technology is pervasive, allowing Artificial Intelligence and Computer Science to play a central role. Whether used as a mere tool, or as the focus of research, computers and the likes, along with the whole new world they generated in the past decades, are an essential part of almost any current scenario.

AI and Law look so distant, and they actually are very far away from each other under many aspects. For instance, Law was born long ago and it has always concerned moral antinomies as *good* and *bad*, *right* and *wrong*. Law is related to the social aspects of mankind, the need and wish of people to interact within organised societies. Artificial Intelligence and Computer Science, on the other hand, look contemporary, young, *scientific*, and technology related, far away from humanities, arts, and culture. However, after a closer look, their origins date as far back as Law's. AI and CS were born within the dream of controlling and mastering large quantities of information in a mechanical way.

It is possible to see a common path emerging. Both Law and AI share something, as they both aim at engaging themselves in “flexible problem-solving activities in complex domains” (Sartor and Rotolo, 2013, 199).

As pointed out by Sartor and Rotolo (2013), AI & Law research covers many different topics, such as

- formal theories of norms and normative systems,
- computational legal logic,
- argumentation and argumentation systems,
- ontologies for the law,
- game theory as applied to the law,
- formal models of institutions and MAS,
- simulations in legal and social norms,
- rule-interchange languages for the legal domain,
- legal e-discovery and information retrieval,
- NLP in the legal domain,
- machine learning in the law.

AI and Computer Science should not be seen as mere tools to be applied within the legal field. Actually, the discipline now called AI & Law is genuinely interactive and interdisciplinary, involving different fields as deontic logics, normative multi-agent systems, game theory, norms and trust, and norms and argumentation.

The interaction of Law and Logic has great relevance in this framework. Since the dawn of the research on modal logics, the deontic interpretation of the modal operators has been present, helping to refine the logical analysis of legal concepts such as the semantic difference between obligation and norms, the interaction among different normative systems, the arising of conflicts within legal systems. We find this latter issue of great interest and relevance. Handling and accommodating conflicts between different norms is essential for any system aiming at modeling legal concepts. It is a goal that can be achieved with different tools and strategies. That of choosing non-normal deontic logics is one of those and definitely one of the most promising.

This dissertation is devoted to the study of non-normal systems of deontic logics, both on the propositional level, and on the first order one. We shall present new completeness results concerning the semantic setting of several systems which are able to handle normative dilemmas and conflicts. Although primarily driven by issues related to the legal and moral field, these results are also relevant for the more theoretical field of Modal Logic itself, as we propose a syntactical, and semantic study of intermediate systems between the classical propositional calculus CPC and the minimal normal modal logic K.

Thesis Summary

Chapter 1 introduces the main philosophical topics related to the deontic interpretation of modal logics. Standard tools are known to generate several paradoxes. Some of these originate within the syntax of given systems, such as *deontic explosion*. There are different ways to deduce problematic schemata, and several solutions have been proposed. We chose to adopt the multi-relational semantics approach, providing a conceptual interpretation suitable to interpret deontic operators.

- in Section 1.1 we touch upon the Standard Paradigm SDL. After presenting normal Kripke Semantics, we analyse the standard deontic interpretation and some of the problems it raises. In particular, in Section 1.1.1 we introduce some first preliminary modal schemata as well as their deontic interpretation.
- In Section 1.2 we analyse one of the most problematic schemata related to the deontic interpretation of modal logics, namely, the formula called *Deontic Explosion*. Section 1.2.1 is a preliminary syntactic analysis of the systems generating such schema, underlying the reasons behind the choice of working with non normal systems. There are several syntactic solutions to prevent the derivation of deontic explosion formulae, as we see in Section 1.2.2, although

we chose to analyse in detail the ones that are more conservative with respect to normal modal logics, i.e., those systems that are weak enough to prevent deontic explosions, yet powerful enough to express juridical sentences. These systems are the so called *non normal* Modal logics.

- Non normal systems can be treated semantically using relational structures. In Section 1.3 we introduce a semantic interpretation of the modal deontic operators based on Kripke Semantics, or *Possible Worlds Framework*, as defined in Section 1.3.1. Kripke-frames are very intuitive and useful and provide an excellent tool to treat normal systems. However, they are known to be sound and complete with respect to normal systems. In order to keep the intuitive appeal of the possible worlds framework, and still using non normal systems, we decided to study further the so called *multi-relational* frames. In Section 1.3.2 and 1.3.3 we introduce this type of semantics, which is nothing else than a direct generalisation of standard Kripke Frames and models.

Chapter 2 is mainly technical and it presents some new results concerning non normal modal systems.

- In Section 2.1 we present some well known non normal calculi, namely the systems $\mathsf{E}, \mathsf{M}, \mathsf{NM}, \mathsf{R}$ as well as a syntactical analysis of the relations between well known schemata (see Chellas, 1980). In particular we shall see which systems count **DEX** among their theorems and which premisses entail schema **DEX**.
- Section 2.2 is the core of the Chapter and it is devoted to the semantic analysis of well known modal schemata within different scenarios. After a brief technical introduction to both strong, and weak multi-relational semantics in Section 2.2.1, we proceed to show how some well known Kripke-valid schemata, are no longer valid within the broader semantics described. Then,

in Section 2.2.3 we carry on a comparative analysis of modal schemata and semantic properties, i.e., we prove characterisation theorems for some well known schemata, namely, $\mathbf{M}, \mathbf{C}, \mathbf{N}, \mathbf{B}, \mathbf{T}$. Section 2.2.4 presents a semantic analysis of some formulae that are are particular relevant within deontic logics, namely, **CON** and **D**. It is well known that both schemata characterise precisely the same property in Kripke semantics, namely, seriality. However, this ceases to be true in multi-relational semantics and, as we shall see, these schemata define different readings of seriality within weak semantics.

– Section 2.3 is focused on proving semantic completeness for several systems using both strong, and weak semantic tools. We propose direct completeness proofs via canonical models for both classical systems (Section 2.3.1), and **N**-Monotonic systems (Section 2.3.2). Finally, in Section 2.3.3 we prove completeness theorems with respect to specific classes of frames for a few systems extending MN with well known schemata, namely, $\mathsf{MN} \oplus \mathbf{T}$, $\mathsf{MN} \oplus \mathbf{D}$, and $\mathsf{MN} \oplus \mathbf{CON}$.

Chapter 3 presents free first order extensions of some **N**-monotonic systems and above as well as completeness results with respect to multi-relational first order frames.

– Section 3.1 is an introduction to *Barcan Formulae* and their role within judicial syllogisms. There are several philosophical as well as related technical issues.

– Section 3.2 presents some well known results concerning quantified non normal modal logics and Neighborhood frames, as well as a first technical introduction to Barcan formulae and the related problems. We shall see the attempts made to accommodate Barcan schemata within both *constant domain*, and *varying domain* neighborhood frames.

– Section 3.3 is rather technical and presents multi-relational first order frames.

We chose to analyse frames with varying domains, in order to perform a finer distinction between actual individuals and *possibilia*.

- Section 3.4 The traditional distinction between *de dicto* and *de re* sentences is here seen under a new light, in terms of contextual obligation and the role of quantification within deontic contexts.
- Section 3.5 is the core of this Chapter. We shall present alternative semantic characterisations for the schema **CBF**. We compare our results with the standard ones in Kripke Semantics and we shall see different ways to generalise the concept of *increasing inner domains*.
- Section 3.6 is the technical core of the Chapter. Here we provide Henkin-style completeness theorems for several systems, namely, the smallest *free* quantified non normal **N**-monotonic logic $\mathsf{Q}^{\circ}_{=}.\mathsf{NM}$ and some extensions, including $\mathsf{Q}^{\circ}_{=}.\mathsf{NM} \oplus \mathbf{CBF}$.
- In Section 3.7 we provide characterisation results for **BF** and we compare it the case of neighborhood models.
- Finally, in Section 3.8 we discuss the role of identity and we present a completeness theorem for a system without the identity relation and some extensions.

Chapter 4 presents both a summary of all the results achieved, and some possible applications within new fields:

- Section 4.1 presents a summary of the technical results, both for the propositional case (4.1.1), and for the predicative one (4.1.2).
- Section 4.2 is focused on some of the main open problems related to non normal modal logics, both for the propositional, and the predicative case.

– In Section 4.3, finally, we present a possible application within epistemic logics. In fact non normal modal systems have been proposed as a possible solution to deal with the problem of logical omniscience. Here we define the concept of omniscience from the perspective of propositional modal logic. We start from the classical propositional calculus, adding and analysing epistemically many of the schemata seen throughout the dissertation.

Chapter 1

Deontic Logics and Dilemmas

At the beginning of the article soon to become a milestone on the path of deontic logic, von Wright (1951a) states that "the deontic modes [have] hardly at all been treated by logicians." More than sixty years have passed and this statement has seen its truth value changing radically (see, e.g., Åqvist, 2001; Carmo and Jones, 2002). Actually there has been a large amount of research on formal models of normative concepts. Being essentially an interdisciplinary domain, both logicians (like the already cited von Wright, 1951a), and legal theorists (such as Alchourrón, 1969), and computer scientists (like McCarty, 1986) have merged their efforts. In this Chapter we shall attempt to provide a brief introduction to some basic aspects of modal deontic logic. In particular, we shall present the so-called Standard Deontic Logic (SDL hereafter). Although it is nowadays heavily criticised, SDL has long been considered a reference for deontic logicians. We shall discuss both standard syntax, and semantics for SDL as this system, known as it is to generate several deontic paradoxes and problems (Åqvist, 2001; Carmo and Jones, 2002), looks like a good touchstone to compare new systems and ideas.

1.1 Once Upon a Time: The Standard Paradigm

Most logical investigations of the main normative legal concepts require a formal account of deontic notions, such as obligation (duty) and permission. These ideas have been characterized by using different logical tools, most frequently related to the possible-worlds semantics of modal logic (for an overview of , see Åqvist, 2001). Thus, expressions such as $\Box A$ and $\Box\neg A$ mean intuitively that A is obligatory and prohibited, respectively. Here, however, we shall not investigate specific and subtle aspects of deontic logic such as the distinction between obligation and permission. Rather, we assume that the operator $\Diamond$, representing permission, is simply the dual of $\Box$, i.e., it is defined logically as $\neg\Box\neg$, whose meaning is *not obligatory not.* Any deeper philosophical investigation of the meaning of permission is outside the scope of this work and can be found, for example, in (Makinson and van der Torre, 2003; Brown, 2000; Stolpe, 2010; Governatori et al., 2013).

1.1.1 Some Deontic Schemata

A (modal) deontic language can be easily obtained by extending that of classical propositional calculus (CPC hereafter) with one unary deontic operator $\Box$. The system SDL, built on this language, is a first, naive, axiomatic attempt at modeling deontic concepts. Although it is known to be quite a weak candidate, it is worth a closer look, as our effort towards this dissertation shall be that of providing an alternative to normal systems.

Let us consider the following schemata:

$$\mathbf{K} := \Box(A \to B) \to (\Box A \to \Box B) \tag{1.1}$$

$$\mathbf{D} := \Box A \to \neg\Box\neg A \tag{1.2}$$

Axiom schema (1.1) looks intuitively acceptable: If it is obligatory that buying a car implies paying for it, then if one is obliged to buy a car, he must also pay for it.

Axiom schema (1.2) seems to be reasonable: Indeed, if it is obligatory to compensate damages, then it is permitted to do so.

Other axiom schemata which are usually adopted in other modal logics, though not acceptable in deontic contexts, are, for instance:

$$\mathbf{T} \coloneqq \Box A \to A \tag{1.3}$$

The schema (1.3), in fact, sounds problematic: Can we say that the mere fact that it is obligatory to compensate damages implies that we actually do so? Assuming the validity of such schema would force us to admit that all obligations are indeed fulfilled.

Hence, afteer a first superficial analysis, schemata (1.1) and (1.2) sound reasonable and might be adopted as axioms of a system, whereas schemata such as (1.3) should be definitely dropped. Actually, the system SDL is obtained by adding both **K**, and **D** (and a necessitation rule, as we shall see later) to the classical propositional calculus.

1.2 Deontic Dilemmas: A Logical Point of View

The modal system SDL is simple, elegant and enjoys several desirable semantic properties. However, it raises more problems than it solves. Among the many issues related to it, one of the most problematic is that SDL is strong enough to generate *deontic explosion* whenever we can derive conflicting obligations (Goble, 2005). Roughly speaking, this means that it cannot accommodate deontic dilemmas, although, as we shall see, this is precisely one of the features a system willing to model deontic concepts should have. Let us see what this all means.

The traditional alethic intepretation of the modal operators is meant to model philosophical concepts like "necessity" and "possibility." According to a naif and intuitive reading of "necessity," one would deny that both a fact, and its contrary can be necessary at the same time and under a univocal reading of necessity. This can be formally expressed by the schema

$$\mathbf{DEX} \coloneqq \Box A \wedge \Box\neg A \to \Box B$$

This says that if such a situation arises, then anything is modally derivable, even a contradiction. Intuitively, in deontic logic this means that, whenever we derive, e.g., that it is obligatory to pay taxes and it is forbidden to do so, then we obtain any other obligation, such the obligation to drink water, to fly, ... Alethically, the antecedent of **DEX** is never fulfilled in a non trivial way and hence it is a perfectly acceptable schema. Traditionally, necessity implies possiblity. This latter concept is expressed by the schema

$$\mathbf{D} := \Box A \rightarrow \neg \Box \neg A$$

which is logically equivalent to the negation of the antecedent of **DEX**, since $\Box A \rightarrow \neg\Box\neg A$ is logically equivalent to $\neg(\Box A \wedge \Box\neg A)$. Any system counting **D** among its theorems has **DEX** as a theorem too and hence **DEX** itself collapses into the schema

$$\mathbf{EFQ} := \bot \rightarrow B$$

the classic "ex falso quodlibet sequitur," also known as the "principle of logical explosion."

However, the power of modal languages is that the operators can be read in different ways, in order to deal with several scenarios. For instance, if read epistemically, the situation is analogous to the alethic one. In fact, let us interpret the $\Box$ operator as meaning "it is known that ..." It is highly against our common intuition to assume that someone may know both a fact, and its negation... If I know that today, in Bologna, it is sunny and hot, I cannot know that today, in Bologna, it is snowy and cold. Moreover, it is usually argued that one should keep the principle that "knowledge implies truth":

$$\mathbf{T} := \Box A \rightarrow A$$

If **T** is in the system, **DEX** is again a theorem and it collapses on **EFQ**.

Things change, though, when dealing with deontic situations. It happens quite often to experience conflicts between norms. There might be situations in which the same state of affairs A both *ought* to be and *ought not* to be at the same time. Logically this means that the antecedent of the entailment **DEX** may actually be true in a non trivial way. This is precisely when *deontic dilemmas* occur. As Goble (2004b) points out[1]

> by a *deontic dilemma* I mean a situation in which, in a univocal sense of *ought*, some state of affairs, A, both ought to be and ought not to be, in which, that is, both OA and $O\neg A$ are true. More broadly, a deontic dilemma would be a situation in which there are inconsistent states of affairs, A and B, both of which ought to be, that is, a case where $\vdash A \to \neg B$ and yet OA and OB are true. More broadly still, a deontic dilemma would be a situation in which it is impossible for both A and B to be realized even though both ought to be, where the sense of impossibility could be anything appropriate to the context of discourse, from some metaphysical impossibility to the most mundane practical incompatibility. (Goble, 2004b, 75)

Therefore any logic aiming at modeling deontic concepts and norms should be able to accommodate, rather then preventing, *deontic dilemmas*.

1.2.1 The Logic behind Explosions

On the logical side, it is crucial for any system aiming at modeling deontic concepts to be able to avoid the problem of *deontic explosion*. In any simple modal propositional system this means that the logic itself should not be powerful enough to generate schema **DEX**, nor **D** (and, obviously **T**, although it is very counterintuitive and nobody would argue in favour of a schema stating that anything that ought to be is also the case). The reason is clear. Let us interpret the operator $\Box$ in terms of obligation to do something.

[1] OA reads "it is obligatory that A", which is standard notation in deontic logic. In this work, we prefer sticking to the general notation of modal logic, thus using $\Box$ instead.

Since in the *real world* conflicts among different norms may (and actually do) arise often, any system intended to model deontic behavior in a realistic way should not be able to generate the schema **DEX**, otherwise the logic itself could not accommodate such conflicts.

Schema **DEX** may be syntactically inferred in a variety of ways. In this preliminary analysis of the problem, we shall try to keep technicalities to a minimum, in order to focus on the main philosophical problems related to deontic logics. A proper syntactical study will be carried out in Chapter 2. Here, however, it is necessary to introduce at least a couple of inference rules which shall play a central role trhoughout this whole dissertation:

RE:= $\vdash A \leftrightarrow B \;\Rightarrow\; \vdash \Box A \leftrightarrow \Box B$

RM:= $\vdash A \rightarrow B \;\Rightarrow\; \vdash \Box A \rightarrow \Box B$

and a few well known modal schemata:

EFQ:= $A \wedge \neg A \rightarrow B$ (Ex Falso Quodlibet)

M:= $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$ (Distribution)

C:= $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$ (Aggregation)

DEX:= $\Box A \wedge \Box\neg A \rightarrow \Box B$ (Deontic Explosion)

A few simple deductions can show how to infer **DEX** within a system containing either **RM**, **M**, **C**, **EFQ** and closed under Modus Ponens. Actually it may be easily inferred the following:

$$(\mathbf{RM} \oplus \mathbf{EFQ} \oplus \mathbf{C}) = (\mathbf{M} \oplus \mathbf{C} \oplus \mathbf{EFQ} \oplus \mathbf{RE}) \Rightarrow \mathbf{DEX}$$

1.2.2 Possible Solutions: An Overview

One thing is now quite clear: Any (modal) deontic logic must not allow schema **DEX**. There are several strategies to prevent **DEX** from being derivable within a system. For instance, one can (see Goble, 2005):

- Drop the classical *ex falso quodlibet* **EFQ**, modeling deontic concepts on paraconsistent propositional logics (see, for instance Da Costa and Carnielli, 1986);
- Handle deontic concepts in a defeasible reasoning setting (see, e.g., Nute, 1997);
- Restrict schema **M**, or **C** (or both) (cf. Goble, 2005; Meheus et al., 2010);
- Drop either **M**, or **C**.

The last approach leads to the study of the so called *non normal modal logics* (see Goble, 2004b,a, 2001, 2005; Chellas, 1980; Schotch and Jennings, 1981). In our opinion this is the simplest and (logically) most elegant solution and it is, therefore, the approach we decided to follow. Roughly speaking, non normal modal logics are nothing but modal theories strictly smaller than the normal logic K. They are obtained by weakening normal systems by dropping one or more of those axioms which enable the inferential machine to generate deontic explosions. This may be done by dropping **M**, **C** or both while keeping the classical propositional calculus untouched. As Goble (2004b, 75ff.) points out, there are several other ways to solve the problem, for instance by applying hybrid approaches (cf. Van der Torre and Tan, 2000). A complete analysis of such ways, however, is outside the scope of our current research. We have rather focused our attention on the analysis of non normal logics.

Non normal modal logics are, we said, systems weaker than *normal* ones, i.e., systems weaker than K, the minimal normal modal logic. System K classically amounts to having schema **K** plus necessitation, which is equivalent, for example, to adding **C** to a system consisting of the closure of $\Box$ under logical equivalence, and the schemata **M** and **N** (i.e., $\Box\top$) (Chellas, 1980)[2]. Since K is complete with respect to Kripke frames, the first obvious observation is that Kripke semantics cannot be applied to these logics. This would be a strong deterrent, as Kripke semantics is very intuitive and easy to deal with. This is probably the main reason that lead modal logics to be so successful. However, this turns

[2]Details are also given in Chapter 2.

out to be only partially true and our dissertation aims at showing how a generalised version of Kripke semantics is actually suitable for a small class of non normal modal systems. Here, we shall introduce both Kripke semantics, and generalised relational semantics.

1.3 A Semantic Reading of Deontic Operators

What do we mean when we say that $\Box A$ is true under a deontic interpretation of the modal operator? Several philosophers argued that this question is meaningless because norms and obligations are not susceptible of any truth evaluation (see, e.g., von Wright, 1963; Makinson, 1999; Broersen and van der Torre, 2012). However, let us ignore here this objection (which is outside the scope of this research, as well as debatable) and try to check when sentences like $\Box A$ are true. For instance, suppose that it is obligatory to pay taxes. If this is true, this means that in all (e.g., legally) ideal situations we do actually pay taxes.

How can we formally express this intuition? It is standard to use in deontic logic the concept of *possible worlds* (Åqvist, 2001): Any possible world is a sort of description of how things are in the current situation (the actual world) or how they could be (alternatives). Worlds can thus be analysed in terms of possible truth assignments to all the atomic propositional letters describing how things are in a given situation: In our world logicians are smart, while we can conceive an alternative situation where they are not smart at all.

Notice that not all worlds are legally or morally ideal: We can imagine situations where all individuals massively commit atrocities. However, we can isolate a subset of possible worlds that are inherently good, where we always pay taxes, compensate damages, and do not commit any atrocity. At this point, it should clear what we mean by saying that $\Box A$ is true: it means that A is true in all (legally, morally, etc...) ideal worlds.

1.3.1 Kripke Frames and Models

To provide a formal method for checking the truth value of obligations, we mush find a way to identify, given a world w, a class of worlds that are *ideal* with respect to w itself. This can be done using the following formal structures (called Kripke frames and Kripke models):

Definition 1.3.1 (Kripke frames and models) *A* Kripke frame $\mathcal{F}$ *is a structure*

$$\langle W, R \rangle$$

where

- W *is the set of all possible worlds;*
- R *is a binary relation over* W *that determines the ideal worlds in* W *for each world in* W*.*

A Kripke model $\mathcal{M}$ *based on the frame* $\mathcal{F}$ *is a structure*

$$\langle W, R, V \rangle$$

where

- $\mathcal{F} = \langle W, R \rangle$*;*
- V *assigns the truth values* true *or* false *to any atomic sentence in any given world (i.e., it states what atomic sentences are true or false in each world). Hence for any proposition* p*,* $V(p)$ *is a set of possible worlds, i.e., all the possible situations in which* p *holds true.*

First, it should be noticed that logical sentences are evaluated locally: A formula A (e.g., "we pay taxes") can be true at world w and false at world v. Second the relation

R selects for each world w those states of affairs that are ideal with respect to w, and it plays a central role when evaluating the local truth value of modal formulae.

Formulae of classical propositional logic are evaluated as usual: Each possible world can be seen as a line within a truth table. Given any Kripke model $\mathcal{M}$ and any world w in it, we write $\models_w^V A$ to say that A is true at w in $\mathcal{M}$. Hence, if we consider for instance the propositional connectives $\neg$ and $\to$, the procedure to check the truth value of formulae is as follows: gien any Kripke model $\mathcal{M}$ and any world w in it

- if A is an atomic formula, $\models_w^V A$ if and only if $w \in V(A)$;
- if $A = \neg B$ then $\models_w^V A$ if and only if $\mathcal{M} \not\models_w^V B$
- if $A = B \to C$ then $\models_w^V A$ if and only if, $\mathcal{M} \not\models_w^V B$ or $\models_w^V C$.

What we have informally said before on the semantic meaning of $\Box$ should make now clear how to evaluate any sentence of the form $\Box A$: given any Kripke model $\mathcal{M}$ and any world w in it

- $\models_w^V \Box A$ if and only if, for each world w', if w' is ideal with respect to w according to R, then $\models_{w'} A$.

As usual in any modal logic, notice that this semantics requires to define different perspectives where a formula can be evaluated:

Definition 1.3.2 *A formula A is* true *in a world w of a model $\mathcal{M}$ iff $\models_w^V A$. A formula A is* true *in a model $\mathcal{M}$, $\models^V A$, iff for all w in $\mathcal{M}$, $\models_w^V A$. A formula A is* valid *on a frame $\mathcal{F} = \langle W, R \rangle$ iff for any model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ we have $\models^V A$. Given a class of frames X, a formula A is X-*valid*, $X \models A$, iff for any frame $\mathcal{F} \in X$, $\mathcal{F} \models A$.*

In Chapter 2 we shall see semantics features in detail. Here, however, it is enough to notice that within Kripke Semantics, schema **K** is valid, unlike **T** and **D**. Schema **T**, however, is known to be valid in all the reflexive frames, i.e., where the property

$\forall w \in W$, (wRw) holds, whereas **D** is valid in the class of serial ones, i.e., $\forall w \in W, \exists v \in W \ (wRv)$.

Kripke Semantics (see Kripke, 1959, 1963, 1980), we said, has a very intuitive interpretation which confers great appeal. The idea behind it is very simple. It takes its origins in the work of Leibniz, who stated that there is a plurality of possible worlds, and the actual one is nothing but one of the many possibilities. According to Leibniz, nevertheless, the actual world is definitely the best one among all the possibilities, chosen by God who has the capability of searching and choosing the perfect solution. Nowadays, however, researchers in modal logic tend to bypass these theoretical and metaphysical aspects while keeping the main idea of Leibniz's approach. For instance, let us suppose that we want to describe any situation which sees several agents interacting one with each other. Let us suppose that such agents are, for instance, playing dice. Then whenever the pair of dice is cast, there are several possible outputs. We can consider each of the possible outputs as a different world. This may be of use for instance if we want to make considerations on probability and so on. Moreover, we may turn our attention to the analysis of agents' knowledge. Any fact p is then *known* by an agent whenever he cannot consider as possible a state of affairs in which p does not hold. Thus, in epistemic contexts,

> the intuitive idea behind the possible-worlds model is that besides the true state of affairs, there are a number of other possible states of affairs or *worlds*. Given his current information, an agent may not be able to tell which of a number of possible worlds describes the actual state of affairs. An agent is then said to *know* a fact ϕ if ϕ is true at all the worlds he considers possible (given his current information). For example, agent 1 may be walking on the streets of San Francisco. Thus, in all the worlds that the agent considers possible, it is sunny in San Francisco. (We are implicitly assuming here that the agent does not consider it possible that he is hallucinating and in fact it

> is raining heavily in San Francisco.) On the other hand, since the agent has no information about the weather in London, there are worlds he considers possible in which it is sunny in London, and others in which it is raining in London. Thus, this agent knows that it is sunny in San Francisco but he does not know whether it is sunny in London. Intuitively, the fewer worlds an agent considers possible, the less his uncertainty, and the more he knows. If the agent acquires additional information – such as hearing from a reliable source that it is currently sunny in London – then he would no longer consider possible any of the worlds in which it is raining in London.(Fagin et al., 1995, 16)

The standard result is that the class of Kripke frames generates all and only the theorems of K, the smallest normal modal system (Chellas, 1980; Blackburn et al., 2001). Hence it is straightforward that Kripke frames are not a tool to be used if we aim at modeling non normal systems, as they generate sets of formulae that are strictly smaller than K. On the other hand, Kripke frames and models offer a highly intuitive tool to interpret modal operators using the possible worlds metaphor, whereas other tools like neighborhood semantics appeal less to intuition and look rather technical. However, there are ways to overcome this issue. The semantics we shall present in the following sections is precisely a generalization of Kripke semantics suitable to model non normal systems.

1.3.2 Semantics for Non Normal Systems

A number of significant contributions in the last four decades show that non-normal modal logics can be fruitfully employed in several applied fields. One well-known domain is epistemic logic, where non-normal systems are a solution to alleviate the so-called omniscience problem that affects stronger (normal) modal systems (Fagin et al., 1995) (see Chapter 4). Deontic logic is, as we said, another field where non-normal systems have been traditionally proposed to avoid many drawbacks of standard deontic logic, which does not tolerate deontic conflicts and gives rise to a number of paradoxes (Goble, 2005; Jones and Carmo, 2002). Other important applications are those systems that aim at capturing different aspects of the concepts of action and agency: the modal logic of agency (Segerberg, 1992; Elgesem, 1997; Governatori and Rotolo, 2005), concurrent propositional dynamic logic (Goldblatt, 1992), game logic (Parikh, 1985), and coalition logic (Pauly, 2002), among others, are all examples where some modal operators are axiomatized in logics weaker than K.

Semantics for non-normal systems have a long and distinguished tradition (Scott, 1970; Montague, 1970; Segerberg, 1971). This tradition goes beyond standard Kripke semantics and thus interprets modal systems in the so called neighborhood semantics, also known as Scott-Montague semantics, or minimal-models[3]. If compared to standard Kripke frames, neighborhood semantics considers a set of collections of worlds related to w instead of connecting worlds via an accessibility relation. These collections are the *neighborhoods* of w. Formally, a frame is a pair $\langle W, N\rangle$ where W is a set of possible worlds and N is a function assigning to each w in W a set of subsets of W (the neighborhoods of w). A model is thus a triple $\langle W, N, V\rangle$ where $\langle W, N\rangle$ is a frame and V is a valuation function defined as for Kripke models, except for $\Box\phi$, which is true at w iff the set of elements of W where ϕ is true is one of the sets in $N(w)$; i.e., iff it is a neighborhood of w.

Model-theoretic investigations on non-normal modal logics and neighborhood semantics have reached significant results (Hansen, 2003, for an overview) with respect, for for instance, to completeness (Segerberg, 1971; Chellas and McKinney, 1975; Chellas, 1980), and incompleteness (starting from Gerson, 1975), decidability (Chellas, 1980), bisimulation (Pauly, 2002; Hansen, 2003), and simulation in multi-modal normal modal logics (Gasquet and Herzig, 1996; Kracht and Wolter, 1999).

[3]Technical details are given in Chapter 3.

However, other two model-theoretic semantic settings can be used for non-normal modal logics: selection-function and multi-relational semantics. A selection-function model is a structure $\langle W, f, V \rangle$, where W is a set of possible worlds, f is a selection function with signature $\mathcal{P}(W) \times W \longrightarrow \mathcal{P}(W)$, and V assigns to each propositional letter a subset of W. A formula $\Box\phi$ is true at any world w iff w belongs to the set of worlds that f assigns to w and the truth set of ϕ. As is well-known, this semantics, if applied to standard non-normal modal logics, turns out to be a reformulation of neighborhood semantics (Hansson and Gärdenfors, 1973; Governatori and Rotolo, 2005).

The other alternative is multi-relational semantics.

1.3.3 Multi-relational Frames: Intuition and Technique

Among the alternatives proposed, that of using multi-relational frames can be seen as the most conservative with respect to Kripke Semantics. Indeed it can be seen as a direct generalization of Kripke frames. The only technical difference is that frames are allowed to have more than one relation. Hence multi-relational frames are nothing but Kripke frames with a set at most countable of binary relations over the base set:

Definition 1.3.3 *A multi-relational n-frame is a $n+1$-tuple $\mathcal{F}_n := \langle W, R_1, \ldots, R_n \rangle$ where W is a non empty set and any R_i $(1 \leq i \leq n)$ is a binary relation on W.*

Notice that the set of relations can be infinite. Within this broader scenario, Kripke frames are, therefore, a limit case of multi-relational semantics. The key questions here are two:

a. how can these frames be interpreted from a deontic perspective?

b. can they fix some of the problems related to the deontic reading of the modal operators?

Goble (2001) provides a partial answer to the first question:

> [...] in the multi-relational semantics, we may regard each relation in the set $\mathcal{R}$ [i.e., any relation R_i] as representing a partucular standard of value and picking out those worlds, b, that are best with respect to a from the perspective of that standard, while recognizing that there could be other standards according to which b is not ideal. [...] when both $\Box A$ and $\Box\neg A$ are true it is because A is prescribed by one set of norms or regulations while $\neg A$ is prescribed by another, distinct set. [...] Each set of norms or regulations is presumed to be internally consistent, and conflicts only emerge as a result of rivalry between sets of norms. (Goble, 2001)

Thus, Kripke frames interpret a univocal notion of *obligation*, one that cannot be contradictory. On the other hand, multi-relational frames provide a model closer to real legal systems: There are different norms, each norm carries obligations, such obligations may be in conflict with each other although internally consistent. Any agent has some obligations due to the application of certain norms. However, dilemmas may arise without generating logical paradoxes.

This view is mirrored by the semantic conditions we impose to interpret modalised propositions, thus defining a model (being nothing but an interpreted frame):

Definition 1.3.4 *A multi-relational n-model is a $n+2$-tuple $\mathcal{M}_n := \langle W, R_1, \dots, R_n, V\rangle$ where $\langle W, R_1, \dots, R_n\rangle$ is a multi-relational n-frame and V is a function (assignment) $V : Prop \longrightarrow \mathcal{P}(W)$.*

A valuation is hence a function which assigns to each proposition p a set of worlds, intuitively those worlds in which p itself is true. It is now possible to define truth values of modal formulae. Truth is, as usual, a *local* concept, meaning that it depends on the *place* we chose to evaluate a formula. A formula $\Box A$ can be true in a world w and false in another. Multi relational frames aim to capture the notion that a fact A is *obligatory* in a world w if there is a norm imposing A. This happens when there is at least one relation R_i connecting to w only worlds in which A itself is true. Thus, if all the worlds that are ideal with respect to w under an R_i norm, then A itself *ought to be the case* at w. Formally this is expressed by:

Truth conditions The truth conditions for all boolean operations are standard. Let us turn our attention on the ones intended to evaluate boxed formulae.

For any $w \in W$:

A. $\vDash_w \Box A$ iff $\exists R_i \forall v(wR_iv \Rightarrow \vDash_v A)$

Given these conditions, it is straightforward to see that several theorems belonging to K are no longer valid schemta in this broader scenario. We wondered, however, which axiomatic system, if any, is sound and complete with respect to multi-relational frames (with the truth conditions provided above). It turned out that the set of formulae which are valid on this class of frames is precisely that of N-monotonic logics. In Chapter 2 we shall indeed prove a completeness theorem for this system and some extensions. The interesting result is that the lowest non normal level one can achieve without dropping relational semantics as we know it is that of **N**-monotonic logics, i.e., one must keep certain schemata, like **N** and **M**. If such schemata are to be dropped, multi relational tools (at least given the truth conditions provided) must be either abandoned, or deeply modified.

If we evaluate box formulae with the conditions provided, it can be argued that our logic would be equivalent to a multi-modal system based on a language containing as many operators as the arity of the relations. Thus, if we have an n-relational frame, it would be enough to have a *normal* n-modal system without interaction among operators. In this case a formula $\Box A$ in our language would be translated as $\Box_1 A \vee \ldots \vee \Box_n A$. However, two problems arise. First of all our frames might contain a countable number of relations and thus it would be necessary to use an infinite number of box operators. However, for technical reasons, infinitary logical languages are very hard to handle, while it is much easier to work with a language with a finite set of operators. But there are also philosophical reasons for our choice. By adding distinct modalities, one should accept different senses of *obligation*: how to conceptually distinguish between a potentially infinite set of obligation types? On the contrary, we think that the concept of *obligation* should be rather independent by the norm or set of norms generating it. Norms may be in conflict, there may arise different senses of *ought*, although the meaning of *obligation* is steady. As Goble (2001) points out

> The multiple relations of the multiplex [multi-relational, ndr] models may be thought to represent different normative standards; each defines a specific sense of *ought*. The language of our deontic logic could contain distinct deontic operators to express each of these senses, but that is not necessary, and we shall not pursue such a multi-modal logic here. Nonetheless, one might naturally think of the *ought* defined through the multiplex rules as ambiguous between these many specific senses determined by each normative standard. OA says that it ought to be the case that A, but it does not specify under which sense of *ought*. It says only that it ought to be the case that A under *some* system of norms. Appeals to ambiguity are often plausible ways to account for apparent inconsistencies, even deontic conflicts. The multiplex semantics is made for that kind of account. (Goble, 2001, 119)

In other words, in this dissertation we work on logics designed for just one type of obligation, one type of obligation that can however be generated by using many norms or standards distinguished at the semantic level[4].

[4]Notice that a similar view has been defended by Alchourrón and Bulygin (1984) in regard to the concept of permission: there is for these authors just one type of permission, while we may have different permissive norms. We are also in line with some general arguments proposed by (Boella et al., 2009; Governatori and Rotolo, 2010) in regard to the problem of norm change, which assumed to distinguish norms from obligations and permissions.

Indeed a multi-modal language would force us to refer to specific norms and sets of norms even when one does not want or does not know what to refer to. Keeping the distinction of one modal operator and many semantic relations mirrors a natural situation: something *ought to be* even if we do not explicitly indicate which norm or set of norms generate such obligation.

Chapter 2

Non Normal Propositional Systems for Deontic Logics

Within applied logics, non-normal modal systems are one of the most effective alternatives implemented in order to avoid interpretational and theoretical problems generated by certain logical schemata and rules, as we saw in Chapter 1. For instance, such problems are related to that of *logical omniscience* within the field of epistemic logics, or *deontic explosion* within that of deontic logics.

The focus of this Chapter is on the technical issues related to multi-relational frames. Such structures are nothing but Kripke frames with any countable number of binary relations. While the classical evaluation of boolean formulae is standard and steady, that of boxed formulae is rather more problematic. Indeed there are two ways of interpreting $\Box$-formulae: We can either chose a strong interpretation, or a weak one. How they differ on the technical level will be clear later in the chapter. Here, it is enough to say that strong semantics is a rephrasing of neighborhood semantics, whereas weak semantics are a direct generalisation of Kripke frames. Thus, those concerning strong semantics are mainly a rephrasing of other results (see Chellas, 1980) and are thus only partially original. On the other hand, those concerning weak semantics are indeed original and, as fare as we are concerned, have not been presented elsewhere so far. The choice of including a relational account of neighborhood semantics was driven by the will to compare and analyse relational structures and special subclasses. For instance, when we analysed the different meanings of the property of seriality within multi-relational frames, it was very interesting to compare the results within three different, though related scenarios. This is patent only if we translate neighborhood semantics within a multi-relational setting. Moreover, here we provide a direct completeness proof for multi-relational strong semantics.

Our main goal was, however, to carry on a semantic analysis of generalised Kripke-frames, or multi-relational weak frames. Our research started from the observation that although there are so many works within the field of normal logics, Kripke semantics, and neighborhood semantics, not much has been said about multi-relational weak semantics, or *multiplex semantics* as these structures are sometimes called. There are works that use this kind of semantics (see, for instance, Goble, 2001, 2004b; Schotch and Jennings, 1981; Jennings and Schotch, 1981; Meheus et al., 2010), and there is also a sketch of a completeness theorem for a specific system.[1] However, many prominent questions lay without an answer. For instance:

(a) which theory is valid in the class of multi-relational weak structures?

(b) how do they differ from multi-relational strong frames (Neighborhood semantics)? And from Kripke semantics?

(c) the set of formulae which are valid in the class of all multi-relational frames can be generated by a finite axiomatic system? If so, which one?

(d) how well known modal schemata (among those relevant to deontic logic, like $\mathbf{M}, \mathbf{C}, \mathbf{T}, \mathbf{D}$, $\mathbf{B}, \mathbf{CON}, \mathbf{DEX}, \ldots$ behave within multi-relational weak frames? Do they characterise classes of frames with specific properties?

(e) how can well known first order properties be characterised by propositional schemata, if we assume a plurality of relations?

[1] Namely, the logic P in Goble (2001).

Our work provides an answer, or sometimes a partial one, to all these questions.

Overview. In Section 2.1 we present some well known non normal calculi, namely the systems $\mathsf{E}, \mathsf{M}, \mathsf{NM}, \mathsf{R}$ as well as a syntactical analysis of the relations between well known schemata (see Chellas (1980)). In particular we shall see which systems count **DEX** among their theorems and which premisses entail schema **DEX**.

Section 2.2 is the core of the Chapter and it is devoted to the semantic analysis of well known modal schemata within different scenarios. After a brief technical introduction to both strong, and weak multi-relational semantics in Section 2.2.1, we proceed to show how some well known Kripke-valid schemata, are no longer valid within the broader semantics described. Then, in Section 2.2.3 we carry on a comparative analysis of modal schemata and semantic properties, i.e. we prove characterisation theorems for some well known schemata, namely, $\mathbf{M}, \mathbf{C}, \mathbf{N}, \mathbf{B}, \mathbf{T}, \mathbf{4}$. Of course, not all of these schemata are meaningful in deontic logic—some of them should be in fact avoided—but their investigation in multi-relational semantics is anyway instructive to illustrate the formal machinery. Section 2.2.4 presents a semantic analysis of some schemata that are particular relevant within deontic logics, namely, **CON** and **D**. Both enforce in normal modal logics deontic consistency, hence they deny deontic dilemmas. It is well known that both schemata characterise precisely the same property in Kripke semantics, namely, seriality. However, this ceases to be true in multi-relational semantics and, as we shall see, these schemata define different readings of seriality within weak semantics.

Section 2.3 is focused on proving semantic completeness for several systems using both strong, and weak semantic tools. We propose direct completeness proofs via canonical models for both classical systems (Section 2.3.1), and **N**-Monotonic systems (Section 2.3.2). Finally, in Section 2.3.3 we prove completeness theorems with respect to specific classes of frames for a few systems extending MN with well known schemata, namely, $\mathsf{MN} \oplus \mathbf{D}$ and $\mathsf{MN} \oplus \mathbf{CON}$. With the purpose of better illustrating the machinery, we will also present a completeness result for $\mathsf{MN} \oplus \mathbf{T}$, being obvious, however, that schema $\mathbf{T}$

is usually rejected in deontic contexts.

2.1 Syntax: Modal Schemata and Rules

Below we shall recall some technical definition we have already introduced in Chapter 1 in order to make the material presented self contained.

As usual, a propositional logical language has two components: an alphabet, or signature, which includes all the symbols one is allowed to use and a series of formation rules, which gives precise instructions to build grammatical sentences.

The alphabet of the language $\mathcal{L}$ includes a countable set of propositional letters $Prop := \{p_1, \ldots, p_n, \ldots\}$, round brackets $(,)$ and the boolean operations $\{\rightarrow, \bot\}$ as well a modal operator $\Box$. Well formed formulae (wff's henceforth) are defined as follows: each propositional letter $p \in P$ is a wff and if A is a wff, then so are $\Box A$. We assume $\Diamond_i$ to be abbreviations for $\neg \Box \neg$. The boolean operations $\neg, \wedge, \vee$ are defined in the usual way by means of $\rightarrow$ and $\bot$. In particular $\top := \bot \rightarrow \bot$ (see Rybakov, 1997; Blackburn et al., 2001).

Inference Rules:

RE:= $\vdash A \leftrightarrow B \;\Rightarrow\; \vdash \Box A \leftrightarrow \Box B$

RM:= $\vdash A \rightarrow B \;\Rightarrow\; \vdash \Box A \rightarrow \Box B$

RN:= $\vdash A \;\Rightarrow\; \vdash \Box A$

RR:= $\vdash A \wedge B \rightarrow \mathbf{C} \;\Rightarrow\; \vdash \Box A \wedge \Box B \rightarrow \Box \mathbf{C}$

RK:= $\vdash A_1 \wedge \ldots \wedge A_n \rightarrow B \;\Rightarrow\; \vdash \Box A_1 \wedge \ldots \wedge \Box A_n \rightarrow \Box B \quad n \geq 0$

Schemata:

EFQ:= $A \wedge \neg A \rightarrow B$

M:= $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

C:= $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

K:= $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

N:= $\Box \top$

CON:= $\neg \Box \bot$

D:= $\Box A \to \neg \Box \neg A$

T:= $\Box A \to A$

4:= $\Box A \to \Box \Box A$

B:= $A \to \Box \Diamond A$

DEX:= $\Box A \wedge \Box \neg A \to \Box B$

The major issues to be avoided is the possibility for a system to derive and validate the schema **DEX**, the deontic explosion. This schema is derivable in any system above K, the minimal normal modal logic obtained by adding **K** and **N** to the classical propositional calculus CPC. The way this schema can be inferred is actually quite straightforward. Nevertheless, we shall prove a few simple lemmas that highlight some logical relations. In what follows, we shall always assume $\vdash$ to be CPC usual deduction relation, namely:

Definition 2.1.1 (Derivation, Deduction, Theoremhood) *A* **derivation** *of a formula A from the premisses $A_1, \ldots, A_j$, in symbols $A_1, \ldots, A_j \vdash_{\mathcal{AS}} A$ in an axiomatic system $\mathcal{AS}$ is a finite sequence of formulae $A_1, \ldots, A_j, A$ s.t. each A_i is either a premiss, or an instance of an axiom schema from $\mathcal{AS}$ or it has been obtained from a sequence of formulae $A_{k_1}, \ldots, A_{k_m}$ occurring before A_i via application of an inference rule from $\mathcal{AS}$.*
A **deduction** *in $\mathcal{AS}$ is a derivation with the empty set of premisses.*
A formula A is a ***theorem*** *in $\mathcal{AS}$, denoted by $\vdash_{\mathcal{AS}} A$, if there is a deduction of A in $\mathcal{AS}$.*

Most of the syntactic results presented below concerning non normal modal systems and their characterisation can be found in Chellas (1980).

Lemma 2.1.2 RM $\oplus$ EFQ $\oplus$ C $\Rightarrow$ DEX

PROOF. $\vdash A \wedge \neg A \to B$ **EFQ**

$\vdash \Box(A \wedge \neg A) \to \Box B$ **RM**

$\vdash \Box A \wedge \Box \neg A \to \Box B$ **C $\oplus$ M** ∎

Lemma 2.1.3 *If* **RE** *is in the system, then* **RM** *and* **M** *are equivalent.*

PROOF. 1. $\vdash A \wedge B \to B$

$\vdash \Box(A \wedge B) \to \Box B$ **RM**

$\vdash A \wedge B \to A$

$\vdash \Box(A \wedge B) \to \Box A$

$\vdash \Box(A \wedge B) \to \Box A \wedge \Box B$

2. $\vdash A \to B$ assumption.

$\vdash (A \to B) \to (A \to A \wedge B)$ classical tautology

$\vdash A \to A \wedge B$ **MP**

$\vdash A \wedge B \to A$ classical tautology

$\vdash A \leftrightarrow A \wedge B$

$\vdash \Box A \leftrightarrow \Box(A \wedge B)$ **RE**

$\vdash \Box(A \wedge B) \to \Box A \wedge \Box B$ **M**

$\vdash \Box A \to \Box A \wedge \Box B$ substitution

$\vdash \Box A \to \Box B$ ∎

Corollary 2.1.4 *(***RM** $\oplus$ **EFQ** $\oplus$ **C***) = (***M** $\oplus$ **C** $\oplus$ **EFQ** $\oplus$ **RE***)* $\Rightarrow$ **DEX**

There are different systems of propositional modal logics built to model various situations. In the following table we list some simple systems which may be considered as a *base* for more complex systems (for further details, see Chellas, 1980).

		Rules	Axioms
E	classical	**RE**	
M	monotonic	**RM**	E $\oplus$**M**
MN	**N**-monotonic	**RM** $\oplus$ **RN**	E $\oplus$**M** $\oplus$ **N**
R	regular	**RR**	E $\oplus$**M** $\oplus$ **C**
K	normal	**RK**	E $\oplus$**K** $\oplus$ **N** E $\oplus$ **M** $\oplus$ **C** $\oplus$ **N**

The lattice depicted in Figure 2.1 illustrates the inclusion relations in non normal systems.

According to what we have observed above, any logic above R (and hence K) does not look like a good candidate to accommodate deontic dilemmas. This holds true for SDL (Standard Deontic Logic as well, as it is a *normal* logic, i.e., it is a proper superset of K.

2.2 Semantic tools for Non Normal Systems

2.2.1 Multi-relational Frames

We start by defining Multi-relational frames for modal logics. Below, we shall see how these frames fail to validate some deontically relevant schemata, that are well known to be valid in Standard Kripke Semantics.

Definition 2.2.1 *A multi-relational n-frame is a $n+1$-tuple $\mathcal{F} := \langle W, R_1, \dots, R_n \rangle$ where W is a non empty set, n is at most countable, and any R_i $(1 \leq i \leq n)$ is a binary relation on W.*

Notice that this definition allows multi-relational frames to have a (countable) infinite number of binary relations over the base set.

Definition 2.2.2 *A multi-relational n-model is a $n+2$-tuple $\mathcal{M}_n := \langle W, R_1, \dots, R_n, V \rangle$*

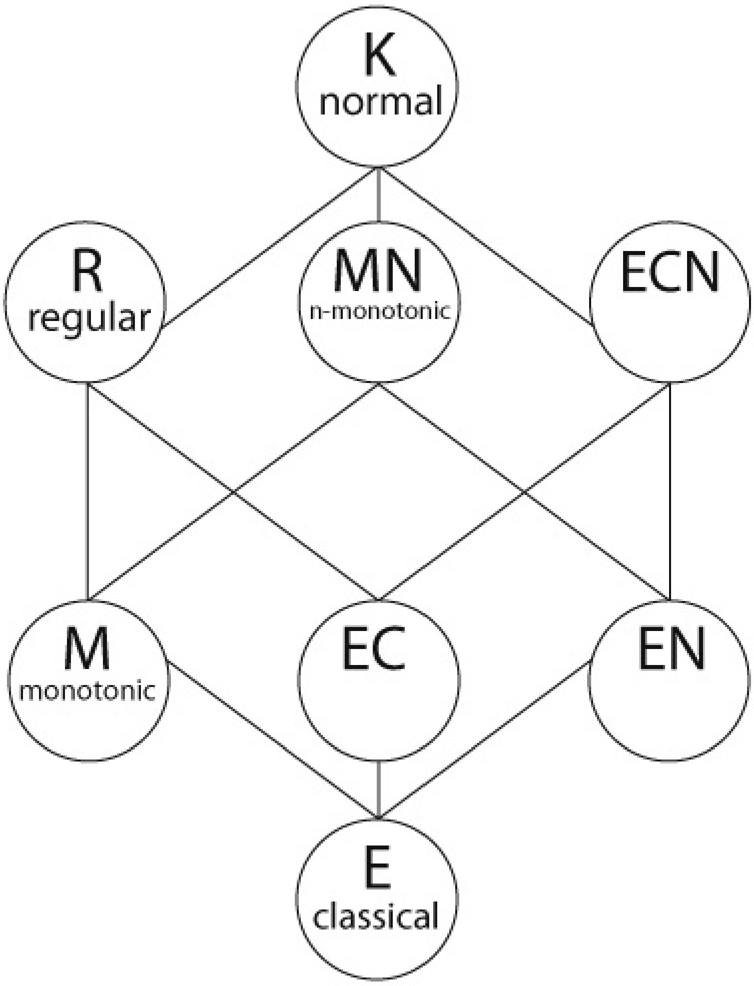

Figure 2.1: The lattice of non normal propositional systems (cf. Chellas, 1980, 237).

where $\langle W, R_1, \ldots, R_n\rangle$ *is a multi-relational n-frame and* V *is a function (assignment)* $V: \ Prop \longrightarrow \mathcal{P}(W)$.

Truth conditions. The truth conditions for all boolean operations are standard. Given a multi-relational frame $\mathcal{F} \coloneqq \langle W, R_1, \ldots, R_n\rangle$, a model $\mathcal{M} \coloneqq \langle \mathcal{F}, V\rangle$ and a world w from W:

$\vDash_w^V p$ if and only if $w \in \|p\|_V$ for any propositional letter p

$\nvDash_w^V \bot$

$\vDash_w^V A \to B$ if and only if either $\nvDash_w^V A$, or $\vDash_w^V B$

The clauses to evaluate modal formulae are a direct generalisation of the standard Kripke approach. Here there are two ways to evaluate $\Box$-formualae, namely by applying either *weak* or emphstrong conditions.

Weak modal conditions (A): $\vDash_w^V \Box A$ if and only if $\exists R_i \forall v\,(wR_iv \;\Rightarrow\; \vDash_v^V A)$

and of course:

(A.2) $\nvDash_w^V \Box A$ iff $\forall R_i \exists v(wR_iv \;\&\; \nvDash_v^V A)$

(A.3) $\vDash_w^V \Diamond A$ iff $\forall R_i \exists v(wR_iv \;\&\; \vDash_v^V A)$

(A.4) $\nvDash_w^V \Diamond A$ iff $\exists R_i \forall v(wR_iv \Rightarrow \nvDash_v^V A)$

Strong modal conditions (B): $\vDash_w^V \Box A$ iff $\exists R_i \forall v(wR_iv \;\Leftrightarrow\; \vDash_v^V A)$

and:

(B.2) $\nvDash_w^V \Box A$ iff $\forall R_i \exists v((wR_iv \;\&\; \nvDash_v^V A)\ Or\ (\neg(wR_iv) \;\&\; \vDash_v^V A))$

(B.3) $\vDash_w^V \Diamond A$ iff $\forall R_i \exists v((wR_iv \;\&\; \vDash_v^V A)\ Or\ (\neg(wR_iv) \;\&\; \nvDash_v^V A))$

(B.4) $\nvDash_w^V \Diamond A$ iff $\exists R_i \forall v(wR_iv \Leftrightarrow \nvDash_v^V A)$

A closer look reveals that multi-relational semantics with strong truth conditions is nothing but a rephrasing of neighborhood frames, though the translation is not always

obvious Governatori and Rotolo (2005). Moreover, Standard Kripke Semantics is nothing but a particular case of multi-relational semantics, namely it is the subset of multi-relational frames with truth conditions A and only one accessibility relation.
Concepts as *truth* and *validity* are defined as usual:

Truth in a world A formula A is *true* in w if and only if $\vDash_w^V A$.

Truth in a model A formula A is *true in a model* $\mathcal{M}$, in symbols $\mathcal{F} \vDash^V A$ (or $\mathcal{M} \vDash A$, or just $\vDash^V A$ if the contest is clear), if and only if $\vDash_w^V A$ for any world $w \in W$.

Validity A formula A is *valid on a frame* $\mathcal{F}$, in symbols $\mathcal{F} \vDash A$, if and only if $\mathcal{F} \vDash^V A$ for any valuation V for $\mathcal{F}$.

$\mathbb{F}$-Validity A formula A is *valid on a class of frames* $\mathbb{F}$, in symbols $\mathbb{F} \vDash A$, if and only if $\mathcal{F} \vDash A$ for any Frame $\mathcal{F}$ from $\mathbb{F}$.

Notation and abbreviations. Given a relation R_i and a world w, by the symbol $R_i(w)$ we refer to the set of all the worlds R_i-accessible from w, i.e.: $R_i(w) := \{x : wR_ix\}$. Given a model $\mathcal{M} := \langle W, R_1, \dots, R_j, V\rangle$ and a formula A, we define the *truth set* of A, in symbols $\|A\|_V$, as the set of all the worlds of the model in which A is true, i.e.: $\|A\|_V := \{w : \quad \vDash_w^V A\}$.

The choice of either strong, or weak clauses depends on the philosophical account one may want to give to *ought*. As we said, the intuition behind multi-relational semantics is that any relation can be seen as a set of norms, a standard. Thus, given a possible situation w, any standard of norms assigns to w a set of possible situations in which such norms are applied, i.e., those worlds which are ideal with respect to w from the standpoint of a given set of norms. If smoking is prohibited in public places, $\Box\neg A$, by some norms, the standard R_i, then in the current state of affair w, the relation R_i associates to w a set of worlds $R_i(w)$ that are ideal, namely, a set of worlds in which nobody actually smokes within public premises. The question, here, is how to chose such set of worlds. Technically, this means that $R_i(w)$ is a proper or improper subset of the truth set of the formula $\neg A$. According to *weak* evaluation clauses, it is enough to have a set containing *only* ideal worlds, i.e., $R_i(w) \subseteq \|\neg A\|_V$. On the other hand, B-clauses are said *strong* because they require a world to have access to *all and only* those possible situations in which people do not smoke within public places, i.e., $R_i(w) = \|\neg A\|_V$.

Another remark should be made concerning Nighborhood Semantics. While it is quite straightforward to see the link between strong frames and neighborhood semantics, weak frames look less similar. However, one may formulate truth evaluation clauses within Neighborhood Semantics in order to emulate multi-relational weak frames as follows:

$$\vDash_w^V \Box A \text{ if and only if } \exists X \in \mathcal{N}_w, \quad X \subseteq \|A\|_V$$

Thus, as one would expect, a formula as $\Box\top$ is valid in the class of these models, and it is indeed a MN-valid formula, as we shall see below.

2.2.2 Schemata and Validity: a Few Examples

It is well known that both neighborhood structures, and multi-relational weak frames fail to validate some very well known modal formulae that are theorems of normal systems. Here we shall see how it happens and, more important, we shall carry on a compared analysis between Kripke, weak, and strong multi-relational frames.

Let us start with the schema named after Kripke himself, namely $\mathbf{K} := \Box(A \to B) \to (\Box A \to \Box B)$. As we said, it is no longer valid.

Lemma 2.2.3 *The schema* $\mathbf{K} := \Box(A \to B) \to (\Box A \to \Box B)$ *is not valid in the class of all multi-relational frames (given either truth condition A or B), although it is valid in any normal modal system (henceforth,* K*-valid).*

Proof. Consider the model $\mathcal{M}_2 := \langle \{w, v, z\}, R_1, R_2, V\rangle$ where $R_1 := \{\langle w, v\rangle\}$, $R_2 :=$

$\{\langle w,w\rangle,\langle w,z\rangle\}$, $\|p\|_V := \{w,z\}$ and $V(q) := \varnothing$ as depicted in Figure 2.2. It is easy to see that there is a relation, namely R_1, such that for any x in the base set of the model, wR_1x if and only if $\vDash^V_x p \to q$ and hence $\vDash^V_w \Box(p \to q)$. Moreover $\vDash^V_w \Box p$, as for any x we have that wR_2x iff $\vDash^V_x p$. On the other hand, $\nvDash^V_w \Box q$, as for both R_1 and R_2 it holds that there is some x such that w is linked to it and $\nvDash^V_x q$. Therefore $\nvDash^V_w \Box(p \to q) \to (\Box p \to \Box q)$ and hence there is a multi-relational frame $\mathcal{F}$ that does not validate an instance of schema **K**.

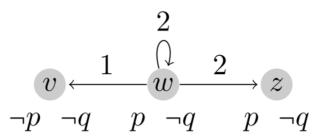

Figure 2.2: Countermodel for **K**.

∎

Another controversial schema within the deontic intepretation is, as we saw in Chapter 1, the aggregation of conjunction within the scope of the modal operator, i.e., schema $\mathbf{C} := (\Box A \wedge \Box B) \to \Box(A \wedge B)$. It is widely known as a K-valid schema and it plays a central role in the syntactic deduction of **DEX**. For this and other reasons, several authors argued to reject it, or at least to restrict it (for a discussion, see (Goble, 2005). See also (Hansen, 2005).). We will not commit here to any philosophical view on this schema. We show anyway that **C** is not a valid formula under both weak, and strong semantic conditions.

Lemma 2.2.4 *The schema* $\mathbf{C} := (\Box A \wedge \Box B) \to \Box(A \wedge B)$ *is not valid in the class of all multi-relational frames (given either truth condition A or B).*

Proof. Consider the model $\mathcal{M}_2 := \langle\{w,v,z\},R_1,R_2,V\rangle$ where $R_1 := \{\langle w,v\rangle\}$, $R_2 := \{\langle w,z\rangle\}$, $\|p\|_V := \{v\}$ and $V(q) := \{z\}$ a depicted in Figure 2.3. It is easy to see that there are two relations, namely R_1 and R_2, such that for any x in the base set of the model, wR_1x if and only if $\vDash_x^V p$ and wR_2x if and only if $\vDash_x^V q$ respectively, hence $w \vDash_V \Box p$ and $\vDash_w^V \Box q$. On the other hand, $w \nvDash_V \Box(p \wedge q)$, as for both R_1 and R_2 it holds that there is some x such that w is linked to it and $\nvDash_x^V p \wedge q$. Therefore $\nvDash_w^V \Box p \wedge \Box q \to \Box(p \wedge q)$ and hence there is a multi-relational frame $\mathcal{F}$ that does not validate an instance of schema **C**.

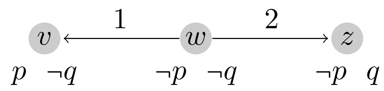

Figure 2.3: The model $\mathcal{M}_2$ does not validate schema **C**.

■

The main difference between semantic conditions A and B reveals itself when we look closely at schema $\mathbf{M} := \Box(A \wedge B) \to (\Box A \wedge \Box B)$. Indeed, here we see the first and most important difference: **M** is in fact a *valid* formula, if we assume semantic conditions A, whereas it is not if we assume conditions B.

Lemma 2.2.5 *The schema* **M** *is valid in the class of all multi-relational frames (given condition A).*

PROOF. Assume by reductio that there is a model on a frame $\mathcal{F}$ and a world in its base set such that (i) $\vDash_w \Box(p \wedge q)$ and either (ii) $\nvDash_w \Box p$ or (iii) $\nvDash_w \Box q$. By (i) there is a relation R_i such that for any x, if wR_ix then $\vDash_x p \wedge q$. If (ii) holds, then, assuming condition A, for any relation R_k there is a world x such that wR_kx and $\nvDash_x p$. Since this must also hold for R_i, there is a contradiction. Hence condition (iii) must be fulfilled: according to condition A, for any relation R_k there is a world x such that wR_kx and $\nvDash_x q$. Again, this must hold for R_i too and hence there is a contradiction. Therefore for any frame $\mathcal{F}$, $\mathcal{F} \vDash \mathbf{M}$, given condition A.

■

Lemma 2.2.6 *The schema* **M** *is not valid in the class of all multi-relational frames (given condition B).*

PROOF. Consider the model $\mathcal{M}_2 := \langle \{w, v, z\}, R_1, V \rangle$ where $R_1 := \{\langle w, v \rangle\}$, $\|p\|_V := \{v\}$ and $V(q) := \{v, w, z\}$ as depicted in Figure 2.4. Consider the relation R_1: it holds that for any x, wR_1x if and only if $\vDash_x p \wedge q$, hence $\vDash_w \Box(p \wedge q)$. However, according to condition B, $\nvDash_w \Box q$. Indeed, $\|q\|_V := \{w, v, z\}$ and for all i, $R_i \neq \{w, v, z\}$. Therefore, given condition B, schema **M** is not valid in the class of all multi-relational frames.

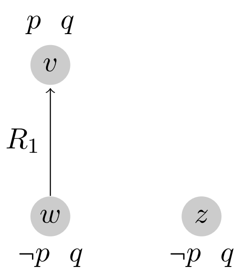

Figure 2.4: The model $\mathcal{M}_2$ does not validate schema **M** assuming condition B.

∎

In the light of what we have formerly pointed out, it is very easy to observe that **DEX** is no longer valid. Actually neither semantic conditions A, nor B allow the validity of the schema:

Lemma 2.2.7 *The schema* **DEX** $:= \Box A \wedge \Box\neg A \rightarrow \Box B$ *is not valid in the class of all multi-relational frames (given boh conditions A, and B).*

PROOF. Let $\mathcal{M} := \langle \{w, z\}, R_1, R_2, V \rangle$, where $\|p\|_V := \{w\}$ and $V(q) = \varnothing$; $R_1 := \{\langle w, w \rangle\}$, $R_2 := \{\langle w, z \rangle\}$, as shown in Figure 2.5. Then $\vDash^V_w \Box p$ by R_1, $\vDash^V_w \Box\neg p$ by R_2 and, since for all relations there is a world x such that wR_ix and $\nvDash^V_x q$, it holds that $\nvDash^V_w \Box q$.

∎

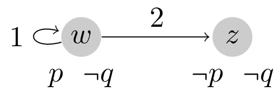

Figure 2.5: Falsification of the **DEX** schema

Let us consider schema **N**, i.e., $\Box\top$, which is not necessarily dangerous in deontic logic, although sometimes rejected for conceptual reasons: An obligation having as content the *truth* can never be violated (Sergot, 2001).

Lemma 2.2.8 (Schema N, Conditions A) *The schema* $\mathbf{N} := \Box\top$ *is valid in the class of all multi-relational frames given semantic conditions A.*

Proof. Assume by reductio ad absurdum that **N** is not valid, given A conditions. Then for some model $\mathcal{M} := \langle W, R_1, \dots R_n, V\rangle$ and some world w, it holds that $\nvDash_w^V \Box\top$. It follows that $\vDash_w^V \Diamond\bot$, i.e., for any i, there is a world v such that wR_iv and $\vDash_v^V \bot$. This leads a contradiction as, by definition, $\|\top\|_V := W$. ∎

Lemma 2.2.9 (Schema N– Conditions B) *The schema* $\mathbf{N} := \Box\top$ *is not valid in the class of all multi-relational frames, given semantic conditions B.*

Proof. Consider the simple model $\langle\{w\}, R_1, V\rangle$ where $R_1 := \varnothing$. Since $\top$ is true in any world, under any valuation, clearly $\|\top\|_V = W$. Hence $\vDash_w^V \Diamond\bot$ and thus $\nvDash_w^V \Box\top$. ∎

2.2.3 Frames and Properties: Comparing Results

According to what happens in Standard Kripke-semantics, different schemata define different characteristics a multi-relational frame should meet. In the following section we shall analyse a few well known schemata in order to clarify the differences between Kripke and Multi-relational semantics. Moreover these characteristics vary according to the truth conditions we decide to adopt.

Lemma 2.2.10 (Frame characterisation of M – Conditions B) *For any multi-relational frame $\mathcal{F}$, assuming condition B, the following holds:*
$\mathcal{F} \vDash \Box(A \wedge B) \rightarrow \Box A \wedge \Box B$ *iff $\mathcal{F}$ is **supplemented**, i.e., for any valuation V, for any world $w \in W$, for any R_i such that $R_i(w) = J \cap K$, there are two relations R_j and R_k such that $R_j(w) = J$ and $R_k(w) = K$.*

PROOF. ($\Rightarrow$) Given a multi-relational frame $\mathcal{F}$ assume there is a relation such that $R_i(w) = J \cap K$ and for any R_k, R_j either $R_k(w) \neq K$ or $R_j \neq J$. Define a valuation such that $\|p\|_V = J$ and $\|q\|_V = K$. Hence $J \cap K = \|p\|_V \cap \|q\|_V = \|p \wedge q\|_V$ and $w \vdash_V \Box(p \wedge q)$. Since for any relation either $R_j(w) \neq J$ or $R_k(w) \neq K$ it follows that $\nvDash_w^V \Box p \wedge \Box q$.

($\Leftarrow$) Assume there are a multi-relational frame $\mathcal{F}$, a valuation V and a world w such that $\vDash_w^V \Box(p \wedge q)$ and $\nvDash_w^V \Box p \wedge \Box q$. Then there is a relation R_i such that $R_i(w) = \|p \wedge q\|_V = \|p\|_V \cap \|q\|_V$. Since $\nvDash_w^V \Box p \wedge \Box q$, it follows that either for any relation R_k, $R_k(w) \neq \|p\|_V$ or for any relation R_k, $R_k(w) \neq \|q\|_V$. ∎

On the technical side, notice that in any model at any world w, if we assume as valid schema **M** and we have at least one boxed formula holding true at w, then we have $\mathbf{N} = \Box\top$ too. Indeed for any relation R_i and any world w, it holds trivially that $R_i(w) = R_(w) \cap W$ and hence, in such a frame, for any world w, there must always be a relation such that $R_j(w) = W$. By definition, for any frame and any valuation $W = \|\top\|_V$.

Lemma 2.2.11 (Frame characterisation of C – Conditions B) *For any multi-relational frame $\mathcal{F}$, assuming condition B, the following holds:*
$\mathcal{F} \vDash \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ *iff $\mathcal{F}$ is **closed under intersections**, i.e., for any valuation V, for any world $w \in W$, for any couple of relations R_j and R_k, there exists a relation R_i such that $R_i(w) = R_j(w) \cap R_k(w)$.*

PROOF. ($\Rightarrow$) Given a multi-relational frame $\mathcal{F}$ assume there are two relations R_j and R_k such that, given a world w, for any relation R_i, $R_i(w) \neq R_j(w) \cap R_k(w)$. Let

$\|p\|_V = R_j(w)$ and $\|q\|_V = R_k(w)$. Clearly $\vDash_w^V \Box p \wedge \Box q$. Since $R_i(w) \neq R_j(w) \cap R_k(w)$, i.e., $R_i(w) \neq \|p\|_V \cap \|q\|_V$ and hence $R_i(w) \neq \|p \wedge q\|_V$, it follows that $\mathcal{F}, \nvDash_w^V \Box(p \wedge q)$.

($\Leftarrow$) Assume there are a multi-relational frame $\mathcal{F}$, a valuation V and a world w such that $\vDash_w^V \Box A \wedge \Box B$ and $\nvDash_w^V \Box(A \wedge B)$. Then there are two relations R_j and R_k such that $R_j(w) = \|A\|_V$ and $R_k(w) = \|B\|_V$. Moreover, for any R_i it holds that $R_i(w) \neq \|A \wedge B\|_V$, i.e., $R_i(w) \neq \|A\|_V \cap \|B\|_V$. ∎

The frame characterisation for **C** with conditions A is very similar to the previous one, as one would expect:

Lemma 2.2.12 (Frame characterisation of C – Conditions A) *For any multi-relational frame $\mathcal{F}$ the following holds: $\mathcal{F} \vDash \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ iff for any world w, for any relation R_i, R_k there exists a relation R_j such that $R_j(w) \subseteq R_k(w) \cap R_i(w)$.*

Proof. ($\Rightarrow$) Assume there is a frame $\mathcal{F} := \langle W, R_1, \ldots, R_n \rangle$ and suppose that there are two relations R_i, R_k such that for all j, $R_j(w) \nsubseteq R_k(w) \cap R_i(w)$. Define the following valuation V for $\mathcal{F}$: $\|p\|_V := R_i(w)$ and $\|q\|_V := R_k(w)$. Clearly $\vDash_w^V \Box p$ and $\vDash_w^V \Box q$, hence $\vDash_w^V \Box p \wedge \Box q$. On the other hand $\|p \wedge q\|_V = \|p\|_V \cap \|q\|_v = R_i(w) \cap R_k(w)$. By assumption there is no relation R_j such that $R_j(w) \subseteq R_i(w) \cap R_k(w)$ and hence $\nvDash_w^V \Box(p \wedge q)$.

($\Leftarrow$) If **C** is not valid, then there are a frame e $\mathcal{F} := \langle W, R_1, \ldots, R_n \rangle$, a valuation V and a world w such that $\vDash_w^V \Box A$, $\vDash_w^V \Box B$, and $\nvDash \Box(A \wedge B)$. From this, trivially, it follows that there are two relations, namely R_i, R_j such that $R_i(w) \subseteq \|A\|_V$ and $R_j(w) \subseteq \|B\|_V$, whereas for any k, $R_k(w) \nsubseteq R_i(w) \cap R_j(w)$.

∎

Let us consider now schemata **T**, **B**, **4**: they are not usually adopted in deontic logic—sometimes, like with **T**—they are avoided. However, their study is instructive to illustrate multi-relational semantics.

Lemma 2.2.13 (T: general reflexivity – cond. A and B) *For any multi-relational frame $\mathcal{F}$, assuming either condition A or B, the following holds: $\mathcal{F} \vDash \Box A \rightarrow A$ iff for any world w, for any relation R_i, wR_iw, assuming either condition A, or B.*

Proof. ($\Rightarrow$) Assume that for some multi-relational frame there are a world w and a relation R_i such that $\neg(wR_iw)$. Let $\|p\|_V) = R_i(w)$ as shown in Figure 2.6, then clearly $\vDash_w \Box p$ and, since $w \notin R_i(w)$, $\nvDash_w p$.

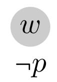

Figure 2.6: A simple non reflexive frame.

($\Leftarrow$) Assume that for all w of a given frame, for any i, wR_iw. Suppose that for some valuation V, $\vDash^V_w \Box A$. Hence for some j, $R_j(w) \subseteq \|A\|_V$. By assumption $w \in R_j(w)$, thus $w \in \|A\|_V$ and $\vDash^V_w A$.

∎

Lemma 2.2.14 (B: general symmetry – Conditions A) *For any multi-relational frame $\mathcal{F}$, assuming condition A, the following holds:*
$\mathcal{F} \vDash A \rightarrow \Box \Diamond A$ iff $\forall w \exists R_i \forall v(wR_iv \Rightarrow \forall R_k(vR_kw))$.

Proof. ($\Rightarrow$) Assume that in a frame $\mathcal{F}$, there is the following situation:
$\exists w \forall R_i \exists v_i(wR_iv_i \ \& \ \exists R_k \neg(v_iR_kw))$. Let V be a valuation such that $\|p\|_V = \{w\}$. Clearly $\vDash^V_w p$. By assumption for any relation R_i there is some v_i such that wR_iv_i and for some R_j, $\neg(v_iR_jw)$. Hence for any v_i it holds that $\nvDash^V_{v_i} \Diamond p$ (recall semantic condition A4). Therefore $\nvDash^V_w \Box \Diamond p$.

($\Leftarrow$) Assume (by reductio) that there is a frame $\mathcal{F}$ with the following characteristics:
(a) $\forall w \exists R_i \forall v(wR_iv \Rightarrow \forall R_k(vR_kw))$
(b) $\mathcal{F} \nvDash A \rightarrow \Box \Diamond A$. Then there is a valuation V on $\mathcal{F}$ and a world such that:

(b.1) $\vDash_w^V A$ and (b.2) $\nvDash_w^V \Box\Diamond A$, i.e., $\vDash_w^V \Diamond\Box\neg A$.

By (b.2) for any relation R_i there is a world v such that (b.3) wR_iv and $\vDash_v^V \Box\neg A$ and hence (b.4) there is a relation R_k such that for all worlds z, $vR_kz \Rightarrow \vDash_z^V \neg A$. Since by (a) vR_iw for any R_i and $\vDash_w^V A$, such R_k cannot exist. ■

The condition just described may indeed be called **general symmetry**. In fact, first of all, notice that this condition implies the following: Given a frame $\mathcal{F} := \langle W, R_1, \ldots, R_n\rangle$, for any world w there exists one relation R_i such that if $R_i(w) \neq \varnothing$, then all worlds $v \in R_i(w)$ are such that for all j, $w \in R_j(v)$, but $v \in R_j(w)$ (as the property holds for *all* worlds of the frame), so this generates a set of worlds mutually accessible under any relation. Notice that this is not a cluster, as general reflexivity is not granted here.

Whithin the framework of strong truth conditions, however, the class of frames characterised by Schema **B** is rather less intuitive.

Lemma 2.2.15 (Frame characterisation of B – Conditions B) *For any multi-relational frame $\mathcal{F}$, assuming condition B, the following holds:*
$\mathcal{F} \vDash A \to \Box\Diamond A$ iff for any valuation V, for any world $w \in W$, $w \in \|A\|_V$, then there is some relation R_j, such that $R_j(w) = \{v : \forall R_i\ (R_i(v) \neq \|\neg A\|_V\}$.

Proof. ($\Rightarrow$) Assume that there are a valuation V and a world $w \in W$ such that $w \in \|A\|_V$ and $\{v : \forall R_i\ (R_i(v) \neq \|\neg A\|_V)\} \neq R_j(w)$ for all R_j. The set $\{v : \forall R_i\ (R_i(v) \neq \|\neg A\|_V)\}$ is exactly the truth set of the formula $\Diamond A$. Indeed condition B3 says that $\vDash_w \Diamond A$ iff $\forall R_i \exists x((wR_ix\ \&\ \vDash_x A)\ Or\ (\neg(wR_ix)\ \&\ \nvDash_x A))$; given a world $z \in W$, $z \in \{v : \forall R_i\ (R_i(v) \neq \|\neg p\|_V)\}$ if and only if for any relation R_i, either there is some $u \in \|A\|_V$ such that zR_iu, or there is some world $u \in \|\neg A\|_V$ such that $\neg(zR_iu)$. By assumption for any R_j it holds that $R_j(w) \neq \|\Diamond A\|_V$ and hence $\nvDash_w^V \Box\Diamond A$ and $\vDash_w^V A$.

($\Leftarrow$) This is very straightforward. Suppose that there are a frame $\mathcal{F} := \langle W, R_1, \ldots, R_n\rangle$, a valuation V on $\mathcal{F}$ and a world w where an instance of **B** is false, i.e., for some formula

A, $\models_w^V A$ and $\not\models_w^V \Box \Diamond A$. This, by definition, implies both that $w \in \|A\|_V$, and that for all R_j, $R_j(w) \neq \| \Diamond A\|_V$, i.e., $R_j(w) \neq \{v : \forall R_i \ (R_i(v) \neq \|\neg A\|_V)\}$.

∎

Lemma 2.2.16 (Frame characterisation of 4 – Condition A) *For any multi-relational frame $\mathcal{F}$, assuming condition A, the following holds: $\mathcal{F} \models \Box A \rightarrow \Box\Box A$ iff*

$$\forall x \forall R_i \begin{cases} \forall y(xR_1y \Rightarrow (\forall z(yR_1z \Rightarrow (xR_iz)) \textit{ or} \ldots \\ \qquad \textit{or } \forall z(yR_nz \Rightarrow (xR_iz)))) \\ \textit{or} \\ \vdots \\ \textit{or} \\ \forall y(xR_ny \Rightarrow (\forall z(yR_1z \Rightarrow (xR_iz)) \textit{ or} \ldots \\ \qquad \textit{or } \forall z(yR_nz \Rightarrow (xR_iz)))) \end{cases}$$

Proof. ($\Rightarrow$) (contrapositive proposition) Given some multi relational frame $\mathcal{F}$, assume

$$\exists x \exists R_i \begin{cases} \exists y_1(xR_1y_1 \ \& \ (\exists z_1^1(y_1R_1z_1^1 \ \& \ \neg(xR_iz)) \ \& \ \ldots \\ \qquad \& \ \exists z_n^1(y_1R_nz_n^1 \ \& \ \neg(xR_iz_n^1)))) \\ \& \\ \vdots \\ \& \\ \exists y_n(xR_ny_n \ \& \ (\exists z_1^n(y_nR_1z_1^n \ \& \ \neg(xR_iz_1^n)) \ \& \ldots \\ \qquad \& \ \exists z_n^n(y_nR_nz_n^n \ \& \ \neg(xR_iz_n^n)))) \end{cases}$$

Let $V(p) = R_i(x)$. Since for any world z_m^l, $\neg(xR_iz_m^l)$, it holds that $x \models_V \Box p$ by R_i. Moreover it holds that for any world y_j and for any relation R_k, there is a world z_k^j such that $y_jR_kz_k^j$ and $z_k^j \not\models_V p$ and hence $y_j \not\models_V \Box p$. Since for any R_j there is such y_j and xR_jy_j, it holds that $x \not\models_V \Box\Box p$.

($\Leftarrow$) Assume by reductio that for some frame, valuation and world $w \nvDash_V \Box p \to \Box\Box p$ for some proposition p. Then $w \vDash_V \Box p$ and $w \nvDash_V \Box\Box p$. For some relation R_i, $R_i(w) \subseteq \|p\|$ and for any relation R_j there is a world y_j such that wR_jy_j and $y_j \nvDash_V \Box p$. Again, for any such y_j and any relation R_k there is some world z_k^j such that $y_jR_kz_k^j$ and $z_k^j \nvDash p$. By hypothesis, there is some world z_m^l such that for any relation R_k, $xR_kz_m^l$. Since this holds also for R_i there is a contradiction. ∎

Lemma 2.2.17 (Frame characterisation of 4 – Condition B) *For any multi-relational frame $\mathcal{F}$, assuming condition B, the following holds:*
$\mathcal{F} \vDash \Box A \to \Box\Box A$ iff for any $w \in W$, R_i and $X \subseteq W$ the following holds: if $X = R_i(w)$, then $\{y : \exists R_j(R_j(y) = X)\} = R_k(w)$ for some k.

Proof. ($\Rightarrow$) (contrapositive proposition) Given some multi relational frame $\mathcal{F}$, assume there are $w \in W$, R_i and $X \subseteq W$ such that $X = R_i(w)$ and $\{y : \exists R_j(R_j(y) = X)\} \neq R_k(w)$ for all k. Let $V(p) = X$, then $w \vDash_V \Box p$. Notice that $\{y : \exists R_j(R_j(y) = X)\} = \|\Box p\|$ and $\|\Box p\|$ is not empty as it contains at least w. Hence for any relation R_k there is a world x such that $\neg(wR_kx)$ and $x \vDash_V \Box p$ and therefore $w \nvDash_V \Box\Box p$.

($\Leftarrow$) Assume by reductio that for some frame, valuation and world $w \nvDash_V \Box p \to \Box\Box p$ for some proposition p. Then $w \vDash_V \Box p$ and $w \nvDash_V \Box\Box p$. For some relation R_i, $R_i(w) = \|p\|$. By assumption $\{y : \exists R_j(R_j(y) = \|p\|)\} = R_k(w)$ for some k. Since $\{y : \exists R_j(R_j(y) = \|p\|)\} = \|\Box p\|$ it follows that $w \vDash_V \Box\Box p$ which is contradictory.

∎

2.2.4 A Special Case: Schema CON and D

The semantic property of seriality can be decisive in deontic logic, as it usually imposes consistency of obligations. Of course, one may argue that, when deontic systems are weakened into non-normal ones in order to tolerate normative dilemmas, we no longer need to enforce consistency. However, the question is subtler than expected, since different ideas of seriality can be adopted. Again, we will not commit to any philosophical view, but simply offer technical alternatives.

Within the framework of Standard Kripke Semantics, it is well known that two famous schemata, namely $\mathbf{D} := \Box A \rightarrow \Diamond A$ and $\mathbf{CON} := \Diamond \top$, characterise the same class of Kripke-frames, i.e., the class of *serial* ones. A Kripke frame is said to be *serial* when any world is related to at least one other world:

Definition 2.2.18 (K-seriality) *A Kripke frame $\mathcal{F} := \langle W, R \rangle$ is* serial *if and only if*

$$\forall w\, \exists v\, (wRv) \tag{2.1}$$

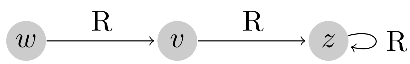

Figure 2.7: A case of K-seriality

Since there is only one relation, the notion expressed by 2.1 does not look problematic. But how can seriality be translated when the frame is broader and the number of relations is bigger than just one? There are several possible alternative answers to this question, for instance:

$$\forall w\, \exists i \forall v\, (wR_i v) \tag{2.2}$$

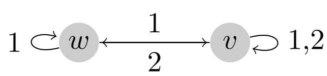

Figure 2.8: A case of Total seriality

This means that any world is connected to *all* the others by one relation and it can be named *Total Seriality*, in fact a formal alternative formulation of 2.2 is $\forall w\, \exists i$ such that $R_i(w) = W$.

Alternatively, one may want seriality be expressed by the fact that:

$$\forall w \; (\exists v_1 (w R_1 v_1) \; \& \ldots \& \; \exists v_n (w R_n v_n)) \tag{2.3}$$

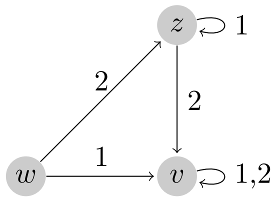

Figure 2.9: A case of General Seriality

The property expressed by 2.3 looks closer to the intuition behind seriality in Kripke Frames and it shall therefore be referred to as *General Seriality*. However, there are more ways to capture the idea of seriality, for instance:

$$\forall w \, \forall i \, \forall j \, (R_i(w) \cap R_j(w)) \neq \varnothing \tag{2.4}$$

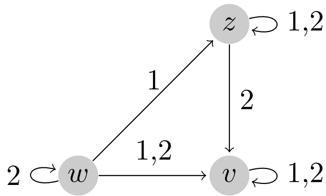

Figure 2.10: Seriality as described in 2.4

These are all different ways to model the intuitive concept of seriality within multi-relational frames. Kripke structures are not expressive enough to capture the difference. Syntactically this corresponds to the fact that some different schemata collapse on each other, namely **D** and **CON**. Since they characterise the same class of frames, not much attention is devoted to their deeply different syntactical structure. However this ceases to be true if we adopt the broader approach of multi-relational frames.

A first important observation is that **CON** and **D** are semantically distinct:

Lemma 2.2.19 *Given condition B, both the entailment* **CON** $\Rightarrow$ **D** *and* **D** $\Rightarrow$ **CON** *are not valid.*

Proof. (a) Consider the model $\mathcal{M}_2 := \langle \{w, z\}, R_1, R_2, V \rangle$ where $R_1 := \{\langle w, w \rangle\}$, $R_2 := \{\langle w, z \rangle\}$ and $\|p\|_V := \{w\}$ as depicted in Figure 2.11. In this frame, schema **CON** is valid. Indeed, for any world x in the base set of the model (and by definition, for any valuation), $\nvDash_x \bot$. Since for any R_i there is a world x such that wR_ix and $\nvDash_x \bot$, it follows that $\nvDash^V_w \Box\bot$, i.e., $\vDash^V_w \neg\Box\bot$. Moreover, the models disproves an instance of **D**. In fact for any x, wR_1x if and only if $\vDash_x p$ and wR_2x if and only if $\vDash_x \neg p$, hence $\vDash^V_w \Box p$ and $\vDash^V_w \Box\neg p$, i.e., $\nvDash^V_w \neg\Box\neg p$. Therefore $\nvDash^V_w \Box p \to \neg\Box\neg p$.

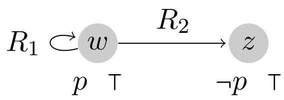

Figure 2.11: In $\mathcal{M}_2$ an instance of **CON** holds true, although an instance of **D** is not under semantic condition B. This is a countermodel for the entailment **CON** $\to$ **D**

(b) Consider a simple *dead end* frame, i.e., $\mathcal{F} := \langle \{w\}, R \rangle$, where $R = \varnothing$. Here, for any valuation V, schema **CON** is always false, as $R(w) = \varnothing = \|\bot\|$, which implies $\vDash_w \Box\bot$, i.e., $\nvDash \Diamond\top$. On the other hand, in this frame schema **D** is valid. Indeed suppose that for some formula A and for some valuation V, $\vDash^V_w \Box A$. Hence $R(w) = \|A\|_V = \varnothing = \|\bot\|_V$, so $\vDash^V_w \Box\bot$ (in such frames, any true boxed formula is equivalent to $\Box\bot$). Since there is no R_i, such that $R_i(w) = \{w\}$, $\nvDash_w \Box\top$, i.e., $\vDash_w \Diamond\bot$, and **D** is a valid schema.

∎

Lemma 2.2.20 *Given condition A, for any multi-relational frame $\mathcal{F}$, (a)* **D** $\Rightarrow$ **CON** *holds, whereas (b)* **CON** $\Rightarrow$ **D** *does not.*

PROOF. (a) It is enough to prove that any counter model for **CON** is based on a frame in which **D** is not valid. Indeed, if in a model $\mathcal{M} := \langle W, R_1, \ldots, R_n, V \rangle$, there is a world in which **CON** is false, we have that $\models_w^V \Box\bot$. If we assume **D** to be valid on the frame on which the model is based, we would get that $\models_w^V \Diamond\Box$ holds and hence for any relation of the frame, there must exists at list one connected world in which $\bot$ holds true, which is, of course, a contradiction. Hence any counter model for **CON** cannot be based on a **D**-frame.

(b) It follows from Lemma 2.2.19, item (i).

∎

Corollary 2.2.21 *The schema* $\Box A \to \neg \Box \neg A \leftrightarrow \neg \Box \bot$ *is not valid in the class of all multi-relational frames (assuming either condition A or B).*

A first analysis of the difference between schema **D** and **CON** is proposed by (Jennings and Schotch, 1981, 309) and Schotch and Jennings (1981) although, as far as we are concerned, no results of characterisation (as those we propose below) have yet been provided.

Returning to our analysis of the concept of seriality, the formulation expressed by 2.3 is captured by **CON**, a schema that characterises precisely the class of multi-relational frames with this condition (assuming either condition A or B).

Lemma 2.2.22 (CON: General Seriality – cond. A and B) *For any multi-relational frame $\mathcal{F}$, assuming either condition A or B, the following holds:*

$$\mathcal{F} \models \textbf{CON} \textit{ if and only if } \forall x \ (\exists y_1(xR_1y_1) \ \& \ldots \& \ \exists y_n(xR_ny_n))$$

PROOF. (⇒) If a frame has a relation R_i that is a *dead end* in a world w, i.e., $R_i(w) = \varnothing$, then, since given any valuation V, $\|\bot\|_V = \varnothing$, we have that $\models_w^V \Box\bot$, i.e., the frame falsifies **CON**.

($\Leftarrow$) This is also quite straightforward. Any frame falsifying **CON** must contain an R_i-dead-end world w for some R_i, i.e., a world w such that for some i, $R_i(w) = \varnothing$. ∎

A natural question is what (if any) class of frames is characterised by the validity of schema **D**. The Lemmas below provide an answer.

Lemma 2.2.23 (Axiom D - Conditions A) *For any multi-relational frame $\mathcal{F}$, assuming condition A, the following holds:*
$\mathcal{F} \vDash \mathbf{D}$ *iff for any $w \in W$ and any pair of relations R_i and R_j, $R_i(w) \cap R_j(w) \neq \varnothing$.*

PROOF. ($\Rightarrow$) Assume that in a frame $\mathcal{F} := \langle W, R_1, \ldots, R_n \rangle$ there are two relations R_i and R_j such that for some $w \in W$, $R_j(w) \cap R_i(w) = \varnothing$. There are two possible cases: either (a) $R_i(w) = \varnothing$ or (b) $R_i(w) \neq \varnothing$ and $R_j(w) \neq \varnothing$. If (a) holds, then trivially $\vDash_w^V \Box\bot$ and $\nvDash_w^V \Diamond\bot$ for any valuation V, falsifying **D**. If (b) holds, then let $\|p\|_V = R_i(w)$ for some propositional letter p. Hence $\vDash_w^V \Box p$. Since $R_j(w) \subseteq \|\neg p\|_V$, $\vDash_w^V \Box\neg p$, a countermodel for **D**.

($\Leftarrow$) Assume that $\mathcal{F} := \langle W, R_1, \ldots, R_n \rangle$ is a frame in which **D** is not valid. Then for some valuation V and some world w, $\vDash_w^V \Box A$ and $\vDash_w^V \Box\neg A$ for some A. Hence for some i, j, $R_i(w) \subseteq \|A\|_V$, whereas $R_j(w) \subseteq -\|A\|_V$, and their intersection is, of course, empty. ∎

Notice that this is yet another possible reading of the concept of seriality, namely what we labeled as 2.4.

It is interesting to notice that if we assume truth conditions B, axiom **D** does no longer capture any reading related to seriality, but it characterises those frames whose relations cannot be complementary (although they can actually be both empty):

Lemma 2.2.24 (Axiom D – Conditions B) *For any multi-relational frame $\mathcal{F}$, assuming condition B, the following holds:*
$\mathcal{F} \vDash \mathbf{D}$ *if and only if for any couple of relations R_i, R_j, $R_j(w) \neq -R_i(w)$ i.e., relations cannot be complementary.*

PROOF. ($\Rightarrow$) Assume there are two relations R_i and R_j such that for some $w \in W$, $R_j(w) = -R_i(w)$. Let $\|p\|_V = R_i(w)$ for some propositional letter p. Then $\vDash_w^V \Box p$. Since $R_i(w) \cup R_j(w) = W$ and $R_i(w) \cap R_j(w) = \varnothing$, the truth set of $\neg p$ is $-R_i(w)$, i.e., $R_j(w)$ and hence $\vDash_w^V \Box\neg p$, meaning that $\nvDash_w^V \Box p \to \neg \Box \neg p$.

($\Leftarrow$) Assume that $\mathcal{F} \coloneqq \langle W, R_1, \ldots, R_n \rangle$ is a frame in which **D** is not valid. Then for some valuation V and some world w, $\vDash_w^V \Box A$ and $\vDash_w^V \Box\neg A$ for some A. Hence for some i, j, $R_i(w) = \|A\|_V$, whereas $R_j(w) = -\|A\|_V$. ∎

One last observation should be made about schema **N**, which is valid on any frame if we assume conditions A, although it is not if we deal with conditions B. In this latter case, **N** captures those frames which are *serial* according the reading we gave in 2.2:

Lemma 2.2.25 (Axiom N– Conditions B) *For any multi-relational frame $\mathcal{F}$, assuming conditions B, the following holds: $\mathcal{F} \vDash \Box\top$ iff $\forall x \exists R_i \forall y (y \in R_i(x))$.*

PROOF. ($\Rightarrow$) Assume that for some multi-relational frame there is a world w such that for any relation R_i there is at least a world y, $y \notin R_i(w)$. Since $\|\top\|_V = W$ for any valuation, it follows that for any R_i there is a world y such that $\neg(wR_iy)$ and $y \nvDash_V \bot$ and hence $\vDash_w^V \Diamond\bot$ and $\nvDash_w^V \Box\top$.

($\Leftarrow$) Assume that $\mathcal{F} \coloneqq \langle W, R_1, \ldots, R_n \rangle$ is a frame in which **N** is not valid. Then for some valuation V and some world w, $\nvDash_w^V \Box\top$ and hence $\vDash_w^V \Diamond\bot$. This implies that any relation R_i is such that $R_i(w) \neq -\|\bot\|_V$ and clearly $-\|\bot\|_V = \|\top\|_V = W$ by definition.

∎

Tables below and Table 2.1 offer a comparative synoptical view of this extensive analysis of seriality and schemata.

Kripke Semantics

Kripke frames		**CON** $\Rightarrow$ **D**	**D** $\Rightarrow$ **CON**
Multi-relational frames	Conditions A	**CON** $\not\Rightarrow$ **D**	**D** $\Rightarrow$ **CON**
Multi-relational frames	Conditions B	**CON** $\not\Rightarrow$ **D**	**D** $\not\Rightarrow$ **CON**

Truth Conditions A – Weak Semantics

Schema	Class of Frames	First Order Property	
N	all		
CON	General Seriality	$\forall x\ (\exists y_1(xR_1y_1)\ \&\ldots\&\ \exists y_n(xR_ny_n))$	2.3
D	Closed under Intersection	$\forall w\, \forall i\, \forall j\, R_i(w) \cap R_j(w) \neq \varnothing$	2.4

Truth Conditions B – Strong Semantics

Schema	Class of Frames	First Order Property	
N	Total Seriality (contains the unit)	$\forall w\, \exists i \forall v\, (wR_iv)$	2.2
CON	General Seriality	$\forall x\ (\exists y_1(xR_1y_1)\ \&\ldots\&\ \exists y_n(xR_ny_n))$	2.3
D	Relations are not Complementary	$\forall w,\ i,\ j,\ R_j(w) \neq -R_i(w)$	

2.3 Completeness Results

The task of this final part is to provide several non normal modal propositional systems with completeness results with respect to certain class of frames. Although it is quite straightforward to check the validity of the schemata we presented (and hence to check the soundness of specific systems containing such axioms), when we focus on proving completeness of such systems, it is soon clear that standard techniques cannot be applied without twisting and adjusting them to the new task.

2.3.1 Soundness and Completeness of Classical Systems and above

The first thing to verify before proceeding to show completeness results is to check whether some specific axiomatic systems produce formulae which are actually valid within certain specific classes of multi-relational frames, assuming strong semantic conditions, namely what we labeled as B-conditions. Any axiomatic system enjoying such property with respect to a class of frames is said to be *sound* with respect to the specified class of structures. Thus we start by proving that the system E, the minimal modal system, is a sound system with respect to the class of *all* multi-relational frames, giving strong semantics. The theory E is defined as the smallest set of formulae containing all classical tautologies and closed under the rules **MP** and **RE**:

Theorem 2.3.1 (Classical logics - Soundness) *Let* $\mathsf{E}_{\vdash} := \{A \mid \mathsf{CPC} \oplus \mathbf{RE} \vdash A\}$ *and* $\mathsf{E}_{\vDash} := \{A \mid \quad \vDash A\}$*, given semantic conditions B. Then* $\mathsf{E}_{\vdash} \subseteq \mathsf{E}_{\vDash} j$*, i.e., all theorems are valid formulae in all multi-relational frames.*

Proof. The proof is standard and quite easy. It is carried out by induction on $lg(\mathcal{D})$, where $\mathcal{D} := \mathcal{D}_1, \ldots, \mathcal{D}_n$ is a deduction in the axiomatic system E with $A = \mathcal{D}_n$, i.e., $A \in \mathsf{E}_{\vdash}$) If $lg(\mathcal{D}) = 1$, then A is a classical tautology and the proof is trivial. Let us consider the case $lg(\mathcal{D}) = k + 1$. Then A has been obtained either via **MP** or via **RM**. Let us focus on the latter case. The formula A has the form $\Box B \leftrightarrow \Box C$ and it has been obtained via

Table 2.1: Frames and Properties: Summary.

SCHEMA	**Kripke Semantics**	**Multi-relational Weak Semantics**	**Multi-relational Strong Semantics**
M $:= \Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$	valid	valid	if $R_i(w) = J \cap K$, then there are R_j and R_k s.t. $R_j(w) = J$ and $R_k(w) = K$
C $:= (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$	valid	partial closure under intersection: for any pair R_i, R_k there is an R_j s.t. $R_j(w) \subseteq R_k(w) \cap R_i(w)$	closure under intersection: for any pair R_j and R_k, there is R_i s.t. $R_i(w) = R_j(w) \cap R_k(w)$
DEX $:= \Box A \wedge \Box\neg A \rightarrow \Box B$	valid	non valid	non valid
N $:= \Box\top$	valid	valid	Contains the unit: $\forall x \exists R_i \forall y (y \in R_i(x))$
CON $:= \Diamond\top$	seriality: $\forall w\ \exists v(wRv)$	general seriality 1: $\forall x \in W\ (\exists y_1(xR_1y_1) \,\&\ldots\&\, \exists y_n(xR_ny_n))$	general seriality: $\forall x \in W\ (\exists y_1(xR_1y_1) \,\&\ldots\&\, \exists y_n(xR_ny_n))$
D $:= \Box A \rightarrow \Diamond A$	seriality: $\forall w\ \exists v(wRv)$	general seriality 2: $\forall w \in W$ and any pair R_i and R_j, $R_i(w) \cap R_j(w) \neq \varnothing$	relations cannot be complementary: for any pair R_i, R_j, for any $w \in W$, $R_j(w) \neq -R_i(w)$
CON $\Leftrightarrow$ **D**	true	false	false
CON $\Rightarrow$ **D**	true	false	false
CON $\Leftarrow$ **D**	true	true	false
T $:= \mathcal{F} \vDash \Box A \rightarrow A$	reflexivity: $\forall w\ (wRw)$	general reflexivity: $\forall w \forall i\ (wR_iw)$	general reflexivity: $\forall w \forall i\ (wR_iw)$
B $:= \mathcal{F} \vDash A \rightarrow \Box \Diamond A$	symmetry: $\forall w \forall v\ (wRv \Rightarrow vRw)$	general symmetry: $\forall w\ \exists R_i\ \forall v(wR_iv \Rightarrow \forall R_k(vR_kw))$	for any valuation V, $\forall w \in W$, $w \in \lVert A \rVert_V$, then there is an R_j, s.t. $R_j(w) = \{v : \forall R_i\ (R_i(v) \neq \lVert\neg A\rVert_V\}$

the application of **RE** to a formula $\mathcal{D}_i = B \leftrightarrow C$, $i \le k$. By induction hypothesis (IH), it holds that $B \leftrightarrow C \in \mathsf{E}_{\vDash}$. Suppose by reductio that $\Box B \leftrightarrow \Box C \notin \mathsf{E}_{\vDash}$, hence there are a multi-relational frame $\mathcal{F}$, a valuation V and a world w such that either $\vDash_w \Box B$ and $\nvDash_w \Box C$, or $\nvDash_w \Box B$ and $\vDash_w \Box C$, given conditions B. Suppose $\vDash_w \Box B$ and $\nvDash_w \Box C$. By semantic condition B, $\vDash_w \Box B$ implies that there is a relation R_i such that for any world x, $wR_ix \quad \Leftrightarrow \quad \vDash_x B$. By IH, $\vDash B \leftrightarrow C$, hence $\|C\| = \|B\|$. Thus $wR_ix \quad \Leftrightarrow \quad \vDash_x C$ and hence $\vDash_w \Box C$, reaching a contradiction. ∎

Any formula generated by E is valid in the class of all multi relational frames with B semantic conditions. The goal is thus to show that if a formula A is valid in such class, then it ought to be generated by the axiomatic system as well. An achievement of this kind would tell us that the axiomatic system we have described generates *all* and *only* the theorems of E: In other words, the system is *sound and complete.* Although all the completeness results for neighborhood semantics can be found in (Chellas, 1980, 248ff.), we restate a few of them using multi-relational semantics (notice that, given strong conditions, multi-relational frames are equivalent to neighborhood models).

Definition 2.3.2 *Given an axiomatic system $\mathcal{AS}$ on a language $\mathcal{L}$, a set $\Delta \subset Fma(\mathcal{L})$ is:*

(a) $\mathcal{AS}$-consistent iff $\Delta \nvdash_{\mathcal{AS}} \bot$;

(b) $\mathcal{L}$-complete iff $\forall A \in Fma(\mathcal{L}) \quad A \in \Delta$ or $\neg A \in \Delta$;

(c) $\mathcal{AS}$-maximal iff Δ is $\mathcal{AS}$-consistent and $\mathcal{L}$-complete.

Definition 2.3.3 *Let $\mathcal{M} := \langle W, R_1, \ldots, R_n, V \rangle$ be a multi-relational model assuming semantic conditions B. $\mathcal{M}$ is an E-canonical model for E if and only if:*

(a) $W := \{w \mid w \text{ is } \mathsf{E}\text{-maximal}\}$

(b) For any formula $A \in Fma(\mathcal{L})$, for any $w \in W$, $\Box A \in w$ if and only if there is a relation R_i such that $R_i(w) = |A|_{\mathsf{E}}$, where $|A|_{\mathsf{E}} := \{v \in W \mid A \in v\}$

(c) for any propositional letter $p \in Prop$, $\|p\|_V := |p|_{\mathsf{E}}$, where $|p|_{\mathsf{E}} := \{w \in W \mid p \in w\}$

Lemma 2.3.4 *Given an* E*-canonical model* $\mathcal{M} := \langle W, R_1, \ldots, R_n, V \rangle$*, for any formula* $A, B \in Fma(\mathcal{L})$*, if* $|A|_{\mathsf{E}} = |B|_{\mathsf{E}}$*, then for any world* $w \in W$*,* $R_i(w) = |A|_{\mathsf{E}}$ *for some* R_i *if and only if* $R_k(w) = |B|_{\mathsf{E}}$ *for some* R_k*.*

PROOF. Suppose $|A|_{\mathsf{E}} = |B|_{\mathsf{E}}$, then $\vdash_{\mathsf{E}} A \leftrightarrow B$ and $\vdash \Box A \leftrightarrow \Box B$ by the **RE** rule. Hence for any world w in the E-canonical base set $\Box A \leftrightarrow \Box B \in w$. Suppose that, given any world w, there is some R_i such that $R_i(w) = |A|_{\mathsf{E}}$. Then, by Definition 2.3.3 it holds that $\Box A \in w$ and, since $\Box A \leftrightarrow \Box B \in w$, it follows that $\Box B \in w$ and thus, by Definition 2.3.3, there is some relation R_k such that $R_k(w) = |B|_{\mathsf{E}}$. ∎

Lemma 2.3.5 (Truth Lemma) *Given an* E*-canonical model* $\mathcal{M} := \langle W, R_1, R_2, \ldots, V \rangle$ *for the classical modal logic* E*, for any formula* $A \in Fma(\mathcal{L})$*, for any world* $w \in W$*, the following holds:* $\vDash^V_w A \quad \Leftrightarrow \quad A \in w$*, i.e.,* $|A|_{\mathsf{E}} = \|A\|_V$*.*

PROOF. The proof is given by induction on the length of a formula A. The basis of induction is trivial. Suppose $lg(A) = n + 1$. The only interesting case is when A has the form $\Box B$.

$\Rightarrow$) Suppose $\vDash^V_w \Box B$, then there is a relation R_i such that for any $v \in W$, $R_i(w) = \|B\|_V$. By the inductive hypothesis $R_i(w) = |B|_{\mathsf{E}}$, hence $\Box B \in w$ by Definition 2.3.3.

$\Leftarrow$) Suppose $\nvDash^V_w \Box B$, then for all relations R_i, $R_i(w) \neq \|B\|_V$ and by the inductive hypothesis $R_i(w) \neq |B|_{\mathsf{E}}$. Therefore $\Box B \notin w$ by Definition 2.3.3. ∎

Let $\mathcal{M}^C := \langle W, R_{A_1}, R_{A_2}, \ldots, V \rangle$ be a multi-relational model such that:

(a) $A_1, A_2, \ldots$ is an enumaration of all the formulae on $\mathcal{L}$

(b) $W := \{w \mid w \text{ is } \mathsf{E}\text{-maximal}\}$

(c) For any $w \in W$ and any formula $A \in Fma(\mathcal{L})$ let:

- $R_A(w) = |A|_{\mathsf{E}}$ if $\Box A \in w$
- $R_A(w) = X$ where X is such that $X \subseteq W$ and for any $B \in Fma(\mathcal{L})$, $X \neq |B|_{\mathsf{E}}$ otherwise

(d) for any propositional letter $p \in Prop$, $\|p\|_V := |p|_{\mathsf{E}}$, where $|p|_{\mathsf{E}} := \{w \in W \mid p \in w\}$

Since condition (b) of Definition 2.3.3 is fulfilled, the model $\mathcal{M}^C$ is E-canonical. Indeed suppose $\Box A \in w$ for some w. Then $R_A(w) = |A|_{\mathsf{E}}$. On the other hand assume by reductio that $\Box A \notin w$ and there is some B such that $R_B(w) = |A|_{\mathsf{E}}$. By definition of $\mathcal{M}^C$ it follows that $R_B(w) = |B|_{\mathsf{E}}$, $|A|_{\mathsf{E}} = |B|_{\mathsf{E}}$, $\vdash_{\mathsf{E}} A \leftrightarrow B$, $A \leftrightarrow B \in w$ for any w and hence $\Box A \leftrightarrow \Box B \in w$ (since any w si closed under **RE**) and by **MP** $\Box A \in w$, leading to a contradiction.

Lemma 2.3.6 (Completeness of E – Conditions B) *The logic E is complete with respect to the class of all multi-relational frames (semantic condition B).*

Proof. The proof follows from Lemma 2.3.5 and the existence of canonical models. ∎

Lemma 2.3.7 (Completeness of M – Conditions B) *The logic M is complete with respect to the class of multi-relational frames (semantic condition B) having the following property: for any valuation V, for any world $w \in W$, if there exists a relation R_i such that $R_i(w) = J \cap K$, then there are two relations R_j and R_k such that $R_j(w) = J$ and $R_k(w) = K$.*

Proof. Consider a canonical model for M where worlds are pairwise different with respect to the propositional letters they contain, i.e., a canonical model where there are no duplicates within the base set. Thus the formula $\bigwedge_{p \in w} p$ is characterising for w. Take any $w \in W$ and suppose that for some R_i, $R_i(w) = J \cap K$. Let $A := \bigvee_{v \in J} \bigwedge_{p \in v} p$ and $B := \bigvee_{v \in K} \bigwedge_{p \in v} p$; clearly $\vDash_v^V A$ iff $v \in J$ and $\vDash_v^V B$ iff $v \in K$ and hence $\|A\|_V = J$ and $\|B\|_V = K$. Then $R_i(w) = \|A\|_V \cap \|B\|_V = \|A \wedge B\|_V$. This implies that $\vDash_w^V \Box(A \wedge B)$, $\Box(A \wedge B) \in w$ and hence $\Box A \wedge \Box B \in w$ (schema **M** is in w). Thus $\vDash_w^V \Box A$ and there is a relation R_j such that $R_j(w) = \|A\|_V$ and $\vDash_w^V \Box B$ and there is a relation R_k such that $R_k(w) = \|B\|_V$. ∎

Lemma 2.3.8 (Completeness of R – Conditions B) *The logic R is complete with respect to the class of supplemented and closed under intersections multi-relational frames (semantic condition B), i.e., frames having the following properties:*

(a) for any valuation V, for any world $w \in W$, if there exists a relation R_i such that $R_i(w) = J \cap K$, then there are two relations R_j and R_k such that $R_j(w) = J$ and $R_k(w) = K$.

(b) for any valuation V, for any world $w \in W$, for any couple of relations R_j and R_k, there exists a relation R_i such that $R_i(w) = R_j(w) \cap R_k(w)$

Proof. (i) See proof of Lemma 2.3.7.

(ii) Assume that for some $w \in W$ there are two relations R_j and R_k such that $R_j(w) = J$ and $R_k(w) = K$. Let $A := \bigvee_{v \in J} \bigwedge_{p \in v} p$ and $B := \bigvee_{v \in K} \bigwedge_{p \in v} p$; clearly $\vDash_v^V A$ iff $v \in J$ and $\vDash_v^V B$ iff $v \in K$ and hence $\|A\|_V = J$ and $\|B\|_V = K$. Then $R_i(w) = \|A\|_V$ and $R_k(w) = \|B\|_V$. This implies that $\vDash_w^V \Box A \wedge \Box B$, $\Box A \wedge \Box B \in w$ and hence $\Box(A \wedge B) \in w$ (schema C is in w). Thus $\vDash_w^V \Box(A \wedge B)$ and therefore there is some relation R_i such that $R_i(w) = \|A \wedge B\|_V = \|A\|_V \cap \|B\|_V$. ∎

2.3.2 Weak Semantics and N-Monotonic Logics

We saw that multi-relational semantics with strong conditions is a good tool to treat a wide range of non normal systems, namely $\mathsf{E}, \mathsf{M}, \mathsf{C}, \mathsf{R}$ and more. However, if assuming strong semantics, we find that a great part of the intuition behind Kripke semantics is somehow lost. On the other hand, weak conditions allow for a a simpler picture which is certainly more appealing form the standpoint of intuition. Thus, a natural question arises: Which axiomatic system, if any, can be captured by these frames? In other words, which is the minimal system to be captured by multi-relational frames and weak semantics? This Section provides an answer to such questions. As far as multi-relational semantics is concerned, the literature provides only a completeness proof sketch for a specific deontic system, namely, P (Goble, 2001). A detailed completeness theorem for non normal systems and multi-relational semantics is presented here for the first time. Again, a first step is showing that a system is indeed sound with respect to a given class of frames:

Theorem 2.3.9 (N-Monotonic logics - Soundness) *Let* $\mathsf{MN}_{\vdash} := \{A \mid \mathsf{CPC} \oplus \mathbf{RE} \oplus \mathbf{M} \oplus \mathbf{N} \vdash A\}$ *and* $\mathsf{MN}_{\vDash} := \{A \mid \ \vDash A\}$*, given semantic conditions* A*. Then* $\mathsf{MN}_{\vdash} \subseteq \mathsf{MN}_{\vDash}$*.*

PROOF. (By induction on $lg(\mathcal{D})$, where $\mathcal{D} := \mathcal{D}_1, \ldots, \mathcal{D}_n$ is a deduction in the axiomatic system $\mathbf{M}$ with $A = \mathcal{D}_n$, i.e., $A \in \mathsf{E}_{\vdash}$) If $lg(\mathcal{D}) = 1$, then A is either a classical tautology and the proof is trivial, or an instance of the axiom schema $\mathbf{M}$, and it is valid by Lemma 2.2.5, or else an instance of the schema $\mathbf{N}$, and thus it holds by Lemma 2.2.8. Let us consider the case $lg(\mathcal{D}) = k + 1$. Then A has been obtained either via $\mathbf{MP}$ or via $\mathbf{RE}$ (please refer to Theorem 2.3.1). ∎

Completeness will be achieved by turning and changing some very well known standard technique, as the Lindenbaum's Lemma and Canonical Models.

Lemma 2.3.10 (Lindenbaum's Lemma) *Given a logic* L*, if* Δ *is a* L*-consistent set of formulae, then there is a* L*-maximal set* Δ^+ *such that* $\Delta \subseteq \Delta^+$*.*

Definition 2.3.11 (MN-Canonical Models) *Let* $\mathcal{M} := \langle W, \mathcal{R}, V\rangle$ *be a multi-relational model.* $\mathcal{M}$ *is a canonical model for* MN *if and only if:*

a. $W := \{w \mid w \text{ is } \mathsf{MN}\text{-}maximal\}$

b. For any formula $A \in Fma(\mathcal{L})$ *let* R_A *be a binary relation over* W*. For all* $w, v \in W$*,* wR_Av *iff* $\Box A \in w \Rightarrow A \in v$*.*

c. for any propositional letter $p \in Prop$*,* $\|p\|_V := |p|_{\mathsf{MN}}$*, where* $|p|_{\mathsf{MN}} := \{w \in W \mid p \in w\}$

Notice that the this definition implies that whenever a formula $\Box A$ does not belong to a state w, the relation associated to A for w would be $R_A(w) = W$, i.e., the whole universe. Otherwise if $\Box A \in w$, $R_A(w)$ would be exactly $|A|_{\mathsf{MN}}$, where $|A|_{\mathsf{MN}} := \{v \in W \mid A \in v\}$. Moreover, it is important to notice that the frame of a MN-canonical model is not always generally serial. Indeed it allows the presence of empty relations for any world, and hence the schema **CON**:=$\Diamond\top$ is not valid on the canonical frame.

Lemma 2.3.12 *Given a canonical model $\mathcal{M}$ for* MN*, for any $w \in W$, if $\Diamond\top \in w$, then w is locally serial, i.e., for any formula $B \in Fma(\mathcal{L})$ there is a state z such that wR_Bz.*

PROOF. Assume $\Diamond\top \in w$, then clearly $\Box\bot \notin w$. Consider any R_C. The formula C can be either (i) a theorem or (ii) a contradiction, or (iii) neither. Suppose that (i) C is a theorem. Then $\vdash_{\mathsf{MN}} C$ implies $\vdash_{\mathsf{MN}} \Box C$ by **RM**, **N** and **MP**, for any $z \in W$, $\Box C, C \in z$ and hence $\Box C \in w$ and $R_C(w) = W$, i.e., $R_C(w)$ is not empty. Suppose that (ii) C is a contradiction, then $\vdash_{\mathsf{MN}} \neg C$ and hence $\vdash_{\mathsf{MN}} \Box\bot \leftrightarrow \Box C$. Since by assumption $\Box\bot \notin w$, $\Box C \notin w$ and hence $R_C(w) = W$ (by definition of Canonical model) and it is not empty. Finally suppose that (iii) C is neither a theorem, nor a contradiction. Then $|C|_{\mathsf{MN}}$ and $|\neg C|_{\mathsf{MN}}$ are complementary and none of them is empty. Again, if $\Box C \notin w$, then $R_C(w) = W$ and if $\Box C \in w$ then $R_C(w) = |C|_{\mathsf{MN}}$ and none of these sets is empty. ■

Lemma 2.3.13 (Existence lemma) *Given a canonical model $\mathcal{M}$ for* MN*, for any $w \in W$, if $\Diamond A \in w$, then for any formula $B \in Fma(\mathcal{L})$ there is a state z such that wR_Bz and $A \in z$.*

PROOF. Assume that $\Diamond A \in w$. Since the schema $\Diamond A \to \Diamond\top$ is a theorem[2] of MN, it follows that $\Diamond\top \in w$ and by Lemma 2.3.12, w is locally serial. It remains to show that for any formula $B \in Fma(\mathcal{L})$, $R_B(w) \cap |A|_{\mathsf{MN}} \neq \emptyset$, i.e., for any B there is some z such that $zinR_B(w)$ and $A \in z$. (i) If A is a theorem, then $|A|_{NM} = W$ (A belongs to any maximal consistent set in W). (ii) A cannot be a contradiction, otherwise $\vdash_{\mathsf{MN}} \neg A$ efq, $\vdash_{\mathsf{MN}} \Box\neg A$, $\Box\neg A \in w$, $\neg\Box\neg A \notin w$, i.e., $\Diamond A \notin w$ which leads to a contradiction. (iii) If A is neither a theorem nor a contradiction, both $\{A\}$ and $\{\neg A\}$ are MN-consistent and both $|A|_{\mathsf{MN}}$ and $|\neg A|_{\mathsf{MN}}$ are not empty. Assume by reductio that for some $C \in Fma(\mathcal{L})$ $R_C(w) \subseteq |\neg A|_{\mathsf{MN}}$. This implies that $\Box C \in w$ (otherwise we would have $R_C(w) = W$ which is inconsistent with our assumption). Hence we have that in all MN-maximal sets z,

[2] Indeed $\vdash_{\mathsf{MN}} \bot \to \neg A$ *ex falso quodlibet*, $\vdash_{\mathsf{MN}} \Box\bot \to \Box\neg A$ by **RM**, $\vdash_{\mathsf{MN}} \Diamond A \to \Diamond\top$ by contraposition

$C \to \neg A \in z$ and therefore it is a theorem of MN. Thus $\vdash_{\mathsf{MN}} C \to \neg A$, $\vdash_{\mathsf{MN}} \Box C \to \Box\neg A$, $\Box\neg A \in w$ and hence $\Diamond A \notin w$ which is a contradiction. ∎

Lemma 2.3.14 (Truth Lemma) *Given a canonical model $\mathcal{M} := \langle W, \mathcal{R}, V \rangle$ for the* **N**-*Monotonic modal logic* MN*, for any formula $A \in Fma(\mathcal{L})$, for any world $w \in W$, the following holds: $w \vDash_V A \quad \Leftrightarrow \quad A \in w$.*

Proof. The proof is given by induction on the length of a formula A. The basis of induction is trivial. Suppose $lg(A) = n + 1$. The only interesting case is when A has the form $\Box B$. i. Suppose $w \vDash_V \Box B$, then there is a relation $R_C \in \mathcal{R}$ such that for any $v \in W$, $wR_C v \;\Rightarrow\; v \vDash_V B$. By the inductive hypothesis (IH henceforth) $B \in v$. Suppose $\nvdash_{\mathsf{MN}} C \to B$. Then the set $\{C, \neg B\}$ is consistent and it is contained in a maximal consistent set y such that $y \in W$. Hence $y \nvDash_V B$ by IH. This implies $\neg(wR_C y)$ and, by definition of R_C, $\Box C \in w$ and $C \notin y$ contradicting the hypothesis $C \in y$. Hence $\vdash_{\mathsf{MN}} C \to B$ and, by the **RM** rule, $\vdash_{\mathsf{MN}} \Box C \to \Box B$. Being w maximal, either (a) $\Box C \in w$ or (b) $\Box C \notin w$. If (a) holds, then $\Box B \in w$ by modus ponens. If $\Box C \notin w$, assume $\nvdash_{\mathsf{MN}} B$, then the set $\{\neg B\}$ is the subset of a maximal consistent set $z \in W$ and $z \nvDash_V B$. Since $\Box C \notin w$, it follows that $wR_C z$ and hence $z \vDash_V B$ and $B \in z$ by IH, a contradiction. Therefore $\vdash_{\mathsf{MN}} B$ and $\vdash_{\mathsf{MN}} \Box B$, i.e. $\Box B \in w$.[3] ii. Suppose $\Box B \in w$. By Definition 2.3.11 it holds that $wR_B v$ iff $\Box B \in w \;\Rightarrow\; B \in v$, i.e., $v \vDash_V B$ by IH. Hence $w \vDash_V \Box B$. ∎

Consider any formula B such that $\nvdash_{\mathsf{MN}} B$. Then $\neg B$ is consistent and there is some MN-maximal set w such that $\neg B \in w$. By Definition 2.3.11 the world w belongs to the base set of an MN-canonical model $\mathcal{M}$ and, by Lemma 2.3.14 $w \nvDash_V B$.

Corollary 2.3.15 (Completeness of MN**)** *The logic* MN *is complete with respect to the class of multi-relational frames.*

[3] Indeed assume $\vdash_{\mathsf{MN}} B$, then $\vdash_{\mathsf{MN}} \top \leftrightarrow B$, $\vdash_{\mathsf{MN}} \Box\top \leftrightarrow \Box B$ by **RE**, $\vdash_{\mathsf{MN}} \Box\top$ is an instance of the schema **N** and hence $\vdash_{\mathsf{MN}} \Box B$ by **MP**.

2.3.3 N-Monotonic Logics and Above

As usual, in order to show completeness results of specific systems L with respect to a restrict class of frames $\mathbb{F}$ enjoying a certain property, it is enough to show that if $\mathbb{F} \vDash A$ for some formula A, then A is also a theorem of L, i.e., $\vdash_{\mathsf{L}} A$. The canonical model techniques guarantees that if a formula is not a theorem of L, then *there exists* a model falsifying A itself. Hence, it is enough to show that such canonical model is based on a frame of the required kind. Keeping this in mind, we are going to show that the canonical frame of some specific systems does indeed enjoy certain properties.

Theorem 2.3.16 *The logic* $\mathsf{MN} \oplus \mathbf{T}$ *is complete with respect to the class of generally reflexive multi-relational frames, i.e., for any world* w*, for any relation* R_i*,* wR_iw*.*

Proof. It is enough to show that if $A \notin \mathsf{MN} \oplus \mathbf{T}$, then $\mathbb{F} \nvDash A$, where $\mathbb{F}$ is the class of reflexive multi-relational frames. Take any formula A such that $A \notin \mathsf{MN} \oplus \mathbf{T}$. By Theorem 2.3.14 there is a canonical model $\mathcal{M} := \langle W, R_1, \ldots, R_n, V\rangle$ for $\mathsf{MN} \oplus \mathbf{T}$ such that for some world w, $\nvDash_w^V A$. To show that $\mathcal{M}$ is based on a generally reflexive frame. Consider any formula $\Box B$; since $\mathbf{T}$ is a theorem, for any world v if $\Box B \in v$, then $B \in v$ by $\mathbf{T}$. Hence, by Definition 2.3.11, it holds that any v, for any relation R_B, vR_Bv, i.e., the frame is reflexive. ∎

Theorem 2.3.17 *The logic* $\mathsf{MN} \oplus \mathbf{CON}$ *is complete with respect to the class of generally serial multi-relational frames.*

Proof. Let L be the logic generated by $\mathsf{MN} \oplus \mathbf{CON}$. Let Γ be a L-consistent set of formulae. It is sufficient to find a model $\mathcal{M} := \langle W, \mathcal{R}, V\rangle$ such that (a) $w \vDash_V \Gamma$ for some world $w \in W$ and (b) $\mathcal{M}$ is based on a generally serial frame (see Blackburn et al. (2001)). Let $\mathcal{M}^{\mathsf{L}} := \langle W^{\mathsf{L}}, \mathcal{R}^{\mathsf{L}}, V^{\mathsf{L}}\rangle$ be a canonical model for L and let Γ^+ be a L-maximal consistent set extending Γ. Then $\Gamma^+ \vDash_{V^+} \Gamma$ by Lemma 2.3.14 and condition (a) is met. We have to show that the canonical frame is generally transitive and it is really quite straightforward. Indeed since any point z in W^{L} is L-maximal and consistent, it contains $\Diamond\top$ and hence by Lemma 2.3.14 $v \vDash_{V^{\mathsf{L}}} \Diamond\top$. By definition of $\Diamond$ this means that $\forall v \in W^{\mathsf{L}} \forall R_i^{\mathsf{L}} \in \mathcal{R}^{\mathsf{L}} \exists z(vR_i^{\mathsf{L}}z)$. ∎

Theorem 2.3.18 *The logic* $\mathsf{MN} \oplus \mathbf{D}$ *is complete with respect to the class of generally serial multi-relational frames.*

Proof. Again, it is enough to show that if $A \notin \mathsf{MN} \oplus \mathbf{T}$, then $\mathbb{F} \nvDash A$, where $\mathbb{F}$ is the class of generally serial frames. Take any formula A such that $A \notin \mathsf{MN} \oplus \mathbf{D}$. By Theorem 2.3.14 there is a canonical model $\mathcal{M} := \langle W, R_1, \ldots, R_n, V \rangle$ for $\mathsf{MN} \oplus \mathbf{D}$ such that for some world w, $\nvDash_w^V A$. To show that $\mathcal{M}$ is based on a generally serial frame. Take any relation R_B: it cannot be empty. Indeed, suppose $R_B(w) = \varnothing$, then $\Box\bot \in w$ and, by $\mathbf{D}$, $\Diamond\bot \in w$, a contradiction. ∎

When schema $\mathbf{D}$ is concerned, another stronger result can be proved:

Theorem 2.3.19 *The logic* $\mathsf{MN} \oplus \mathbf{D}$ *is complete with respect to the class of generally serial multi-relational frames fulfilling the following condition:*
for any $w \in W$ *and any pair of relations* R_i *and* R_j, $R_i(w) \cap R_j(w) \neq \varnothing$ *(see Lemma 2.2.23).*

Proof. Take any formula A such that $A \notin \mathsf{MN} \oplus \mathbf{D}$. By Theorem 2.3.14 there is a canonical model $\mathcal{M} := \langle W, R_1, \ldots, R_n, V \rangle$ for $\mathsf{MN} \oplus \mathbf{D}$ such that for some world w, $\nvDash_w^V A$. As it follows from Theorem 2.3.18, the frame of the canonical model is generally serial. Moreover, consider two relations R_A and R_B. Since no relation is empty, $R_A(w) = \|A\|_V$ and $R_B(w) = \|B\|_V$[4] and hence $\Box A \in w$ and $\Box B \in w$. By $\mathbf{D}$, $\Diamond A \in w$ and $\Diamond B \in B$ and, by $\mathbf{M}$[5] $\Diamond(A \wedge B) \in w$ and hence $\|A\|_V \cap \|B\|_V \neq \varnothing$.

∎

[4] Confer to the remarks after Definition 2.3.11.
[5] Clearly in its contrapositive version: $M^\star := \Diamond A \wedge \Diamond B \to \Diamond(A \wedge B)$.

2.4 Summary, Conclusions, Further Work

In the introduction of this Chapter we have listed a series of questions, namely, those who drove us to start this research. As we saw, we provided an answer to all of them:

(a) which theories are valid in the class of multi-relational weak structures? Any classical theory smaller or equal than **N**-monotonic logics;

(b) how do they differ from multi-relational strong frames (Neighborhood semantics)? Multi-relational strong frames validate a narrower set of formulae, namely, those theories smaller or equal than E;

(c) how well known modal schemata (among those relevant to deontic logic, like **M**, **C**, **T**, **D**, **B**, **CON**, **DEX**, ..., behave within multi-relational weak frames? Do they characterise classes of frames with specific properties? Yes, most of them characterise some specific classes of frames (please refer to Table 2.1);

(d) how can well known first order properties be characterised by propositional schemata, if we assume a plurality of relations? We have provided an answer to reflexivity, seriality, and symmetry;

(e) the set of formulae which are valid in the class of all multi-relational frames can be generated by a finite axiomatic system? If so, which one? Yes, by the system MN, which is then sound and complete with respect to the class of all multi-relational weak frames. Moreover, we saw that the systems $\mathsf{MN} \oplus \mathbf{T}$, $\mathsf{MN} \oplus \mathbf{D}$, $\mathsf{MN} \oplus \mathbf{CON}$ are sound and complete with respect to specific classes of frames.

In our opinion, the most interesting result we achieved is that generalised Kripke frames, i.e., the class of all weak multi-relational frames (with A semantics), can generate precisely MN, namely propositional **N**-Monotonic logic. Thus, this system enjoys two interesting characteristics: First of all, it is strictly smaller than the normal system K, and, most important, it does not generate Deontic Explosion. Although one may argue that non normal systems can be treated more efficiently with neighborhood semantics (see Chellas (1980)), we find quite interesting that the simple operation of generalising Kripke structures (while keeping the whole intuition behind them) is enough to generate systems which are actually finitely axiomatisable as well as non normal. In short, in order to avoid deontic dilemmas, one is not forced to drop relational semantics altogether. It is actually possible to keep relational semantics in a more general definition.

There are, however, some interesting problems yet to be addressed. One may wonder what class of structures, if any, is characterised by other modal schemata, for instance by those closer to other fields of applied logics, rather than deontic. One may wonder what class, if any, is captured by *positive introspection*, i.e., by schema $4 := \Box A \rightarrow \Box \Box A$, *negative introspection*, i.e. $\mathbf{5} := \neg \Box A \rightarrow \Box \neg \Box A$, or by other modal axioms.

On the technical side, there are other important issues to be addressed regarding the system MN:

- the finite model property;
- decidability and complexity;
- extending MN to the first order case.

A first preliminary answer to the latter question is provided in the next chapter. The others remain to be solved.

Chapter 3

Beyond Propositional Deontic Logics

Although the field of deontic logic is flourishing and getting increasing attention, the efforts devoted to the analysis of quantifiers within deontic modal logics are still rather limited. This may be due to the fact that influential authors as Von Wright and Castañeda have expressed sceptical views about extending deontic languages to include predicate logic. Moreover, those who have argued in favour of such extension have probably thought that quantified deontic logics (QDL henceforth) follow the same pattern as alethic modal logic. Although this sounds reasonable if we think about QDL as a mere extension of SDL (i.e., of the standard normal deontic system, see Chapter 1) the situation changes radically is the focus is on non normal systems. For instance, the role of Barcan schemata become significantly different and new patterns and problems of both philosophical, and technical relevance emerge.

Quantified modal logic has a long and distinguished tradition (Garson, 2001; Fitting and Mendelsohn, 1998), which is still lively and technically productive (see, among others, Corsi, 2002; Brauner and Ghilardi, 2007; Gabbay et al., 2009; Goldblatt, 2011). Nevertheless, almost all efforts have so far been devoted to the analysis of the normal case: Besides a few significant exceptions (Arló-Costa and Pacuit, 2006; Arló-Costa, 2002; Waagbø, 1992; Stolpe, 2003), which are based on neighbourhood semantics, the study of quantification in non-normal modal logics is still neglected. Despite that, quantified non-normal modal logics (QNML henceforth) exhibit a different behaviour with respect to normal modal logics. In particular, in contrast with quantified normal modal logics results in the literature show, e.g., that the Barcan and the Converse Barcan schemata (i) are not characterised by decreasing and increasing domains (ii) are tightly connected to the validity of propositional modal axiom schemata.

This Chapter provides a semantic analysis of quantification in a class of non-normal modal logics called **N**-Monotonic (as defined in Chapter 2). Again, instead of following the neighborhood semantics approach, we shall focus on multi-relational semantics.

As explained in Chapter 2, there are two ways to evaluate $\Box$-formulae in this framework. A first version (Goble, 2001, 2004b; Schotch and Jennings, 1981) simply extends the one for Kripke semantics, since it requires that the Kripke-style evaluation clause for $\Box$-formulae is satisfied for at at least one relation in the set of relations of the model. This is precisely the approach we called *weak semantics* in Chapter 2, Section 2.2. The other non-standard evaluation clause has been proposed by Governatori and Rotolo (2005) in order to cover more non-normal modal systems, including the classical ones. In fact, the weak semantic approach captures only stronger logics, like MN and above. The scope of this Chapter is to study quantification using weak semantics as well as considering frames with varying domains. The choice of weak semantics keeps the intuition behind Kripke semantics, while working with logics that are stronger than classical systems (E and above), yet strictly weaker than K. On the other hand, working with varying domains (i.e., with sets of individual existing in possible worlds that can vary from world to world) technically amounts to studying the most general case of quantified modal (and so, deontic) logics and, last but not least, means considering a case which is mostly neglected in the literature on logics weaker than **K** which are extended to the predicate case.

From the propositional modal standpoint, as we said in Chapter 1 and Chapter 2, as far as multi-relational semantics is concerned, the literature provides only a completeness proof sketch for a specific deontic system, namely the logic P introduced by Goble (2001, 2004b) and an indirect completeness result for Elgesem's modal logic of agency (Governatori and Rotolo, 2005) that exploits the equivalence between neighborhood and multi-relational models for classical modal logics. We have presented here, for the first time, a study about how several well known schemata behave within the framework of multi-relational weak semantics.

As far as we are concerned, from the predicative standpoint, this is the first study on quantification in multi-relational semantics, the second one investigating the case of varying domains in non-normal modal logics, and the first that provides a frame characterization of the Barcan schemata with varying domains.

Chapter Summary

Section 3.1 is an introduction to *Barcan Formulae* and their role within normative reasoning. There are several philosophical as well as technical issues related to such schemata.

Section 3.2 presents some well known results concerning quantified non normal modal logics and Neighborhood frames, as well as a first technical introduction to Barcan formulae and the problems related to such schemata. We shall see the attempts made to accommodate Barcan schemata within both *constant domain*, and *varying domain* neighborhood frames.

Section 3.3 is rather technical and presents multi-relational first order frames. We chose to analyse frames with varying domains, in order to perform a finer distinction between actual individuals and *possibilia*, namely, between the individuals that exist in each ideal world and those that are only possible from that viewpoint but that do not exist there (Garson, 2001; Fitting and Mendelsohn, 1998). Also, this choice technically amounts to studying the most general case of quantified modal logics.

Section 3.4 The traditional distinction between *de dicto* and *de re* sentences is here seen under a new light, in terms of contextual obligation and the role of quantification within deontic contexts.

Section 3.5 is the core of the Chapter. We shall present alternative semantic characterisations for the Converse Barcan schema (**CBF**). We compare our results with the standard ones in Kripke Semantics and we shall see different ways to generalise the concept of *increasing inner domains*.

Section 3.6 is the technical core of the Chapter. Here we provide Henkin-style completeness theorems for several systems, namely, the smallest *free* quantified non normal **N**-monotonic logic $\mathsf{Q}^{\circ}_{=}.\mathsf{NM}$ and some extensions, including $\mathsf{Q}^{\circ}_{=}.\mathsf{NM} \oplus \mathbf{CBF}$.

3.1 Quantification, Barcan Formulae, and Deontic Logics[1]

If a legal theorist were asked to formalise the basic structure of a judicial syllogism, he would very likely answer by providing the following inference schema (Alexy, 1989):

$$\begin{array}{l} \forall x(T(x) \rightarrow \Box R(x)) \\ T(s) \\ \hline \Box R(s) \end{array} \tag{3.1}$$

Indeed, suppose the major premise states that, for each individual x, if x commits theft, then it is obligatory that x gets punished. Hence, the fact that Schulze committed theft, entails that Schulze ought to be punished. Likewise, however, some legal theorists–being not so familiar with the complications arising in quantified modal logics[2]—would not probably appreciate why a judicial syllogism is correctly captured by (3.1) rather than

[1]This Section is partially based on an umpulished manuscript written by Antonino Rotolo, Guido Governatori, and myself.

[2]But notice that deontic logicians, too, sometimes recognise that the major premise of (3.1) is a good rendering of practical statements: cf., among others, Schurz (1994).

by inferences such as, for instance,

$$\frac{\begin{array}{l}\Box\forall x(T(x)\to R(x))\\ T(s)\end{array}}{\Box R(s)} \tag{3.2}$$

Technical reasons lead, of course, to reject schema (3.2) as incorrect. But, on the philosophical side, positive arguments—also based on the nature of judicial dynamics—may explain the preference of those who adopt (3.1). This holds in particular if we adopt the *actualist interpretation of quantification*, according to which quantifiers are interpreted existentially, as they range over individual domains depending on possible worlds, while parameters of a formula are evaluated as arbitrary individuals (see Fitting and Mendelsohn, 1998). Actually, the judicial application of law implies that one or more legal provisions are applied to a concrete case presented before the judge: For instance, *from the fact* that Schulze committed theft, it follows that Schulze ought to be punished. For it is often said that judges formulate a "decision rule" that makes the law applicable to the concrete case. Of course, (3.1), too, can be problematic. But, as far as the interplay between quantifiers and modalities is concerned, only the major premise of (3.1) allows to refer to concrete and existing individuals.

But there is more to say about such issues. And the core topic is precisely how and when the interplay between quantifiers over individuals and quantifiers over possible worlds should be allowed. This role is played by Barcan Schemata. However, a few technical details are needed before any precise analysis of such topics.

3.2 Neighbourhood Semantics for QNML

Neighborhood Semantics has always played a central role within quantified deontic non normal logics. As we have already observed, this is the best tool to deal with classical systems and above. Below we shall summarise the most influential results within this area. This approach will be useful to carry out a comparison between Neighborhood structures and multi-relational ones.

3.2.1 Syntax of Quantified Modal Logics

Let us extend the language $\mathcal{L}$ already presented with the universal quantifier $\forall$, a countable set of individual constants Const:=$\{a, b, c, \dots\}$, a set of individual variables, VAR:= $\{x, y, z, \dots\}$, the identity predicate $=$, and a set of n-ary predicate symbols (where $\omega > n \geq 1$). A *term* is either a variable, or an individual constant and $t_1, t_2, \dots$ are meta-variables for terms.

Well formed formuale (wff) are defined as usual:

(a) $\bot$ is wff;

(b) If P^n is an n-ary predicate symbol and $t_2, \dots, t_n$ are terms, then $P^n(t_1, \dots, t_n)$ is a wff;

(c) If A and B are wff, then $A \rightarrow B$, $\Box A$, and $\forall x A$ are wff;

(d) Nothing else is a wff.

Both boolean operators, and the existential quantifier $\exists$ is defined as usual: $\exists x A \leftrightarrow \neg \forall x \neg A$. As usual, $A(t/s)$ is the formula obtained by replacing in $A(s)$ all the free occurrences of s with t (cf. Corsi, 2002, 1484).

3.2.2 Neighborhood Models

Neighbourhood semantics for quantified modal logics has been introduced a long time ago (see Gabbay, 1976). Nevertheless, it received very little attention until the beginning of 1990s (cf. Waagbø, 1992). The study of QNML was afterwards the subject of a few works (cf., for instance, Arló-Costa, 2002; Arló-Costa and Pacuit, 2006; Stolpe, 2003). While Waagbø (1992), Arló-Costa (2002), and Arló-Costa and Pacuit (2006) study the case of structures with constant domains, Stolpe (2003) developed a preliminary investigation of varying domains. In this section we summarise their main results.

Let us consider the case of constant domains.

Definition 3.2.1 (Constant domain neighbourhood frames and models) *A* constant domain neighbourhood frame $\mathcal{F}$ *is a structure* $\langle W, \mathcal{N}, D\rangle$ *where*

- W *is a non-empty set of possible worlds;*

- $\mathcal{N}$ *is a function from* w *to* 2^{2^W}*;*

- D *is a non-empty set of individuals (the domain of the frame).*

For any $w \in W$*, a* w*-assignment* σ *is a function* $\sigma : Var(\mathcal{L}) \mapsto U_w$*.*
An x*-variant* τ *of a* w*-assignment* σ *is a* w*-assignment which may differ from* σ *for the value assigned to* x*. A* constant domain neighbourhood model $\mathcal{M}$ *is a structure* $\langle W, \mathcal{N}, D, I\rangle$ *where* $\langle W, \mathcal{N}, D\rangle$ *is a constant domain neighbourhood frame and* I *is an interpretation function such that, for any assignment* σ *and world* w*:*

- $I^\sigma_w(x) \in D$ *(global interpretation of variables/terms);*

- $\forall w, v \in W, I^\sigma_w(x) = I^\sigma_v(x)$ *(rigidity of variables/terms);*

- $I^\sigma_w(P(x_1, \dots, x_n)) \subseteq D^n$.

Notice that the notion of truth set has to take into account that the truth or falsity of open formulae depends on particular interpretations.

Definition 3.2.2 *Let* $\mathcal{M}$ *be a model with interpretation* I*,* σ *an assignment,* w *any world, and* A *any formula. The truth set of* A *wrt to* $\mathcal{M}$ *and* I^σ*,* $\|A\|^\sigma_I$ *is thus defined:*[3]

$$\|A\|^\sigma_I := \{w \in W : \mathcal{M} \vDash^\sigma_w A\}.$$

[3]When clear form the context, we also omit the reference to the model. The truth set of a closed formula does not depend on any interpretation and assignment.

The valuation conditions are as follows:

- $\mathcal{M} \vDash_w^\sigma P(x_1, \dots, x_n)$ iff $\langle I_w^\sigma(x_1), \dots, I_w^\sigma(x_n)\rangle \in I_w(P)$;

- Standard valuation conditions for negation and boolean connectives;

- $\mathcal{M} \vDash_w^\sigma \Box A$ iff $\|A\|_I^\sigma \in \mathcal{N}_w$;

- $\mathcal{M} \vDash_w^\sigma \forall x A(x)$ iff for every assignment τ that is like σ except for mapping x (i.e., τ is an x-variant of σ), $\mathcal{M} \vDash_w^\tau A(x)$.

Let us consider Barcan and Converse Barcan schemata

BF:= $\forall x \Box A \to \Box \forall x A$ **CBF:=** $\Box \forall x A \to \forall x \Box A$.

The choice of constant domains (i.e., that the resulting modal logics are extensions of standard First Order Logic, FOL henceforth), does not correspond to the validity of **BF** and **CBF**:

Theorem 3.2.3 *(Arló-Costa and Pacuit, 2006) The class of all constant domain neighbourhood frames is sound and complete for* $\mathsf{FOL} \oplus \mathsf{E}$.

BF and **CBF** were characterised in (Waagbø, 1992), but such results have been later made more precise in (Arló-Costa, 2002; Arló-Costa and Pacuit, 2006).

Let us consider the following frame properties (Arló-Costa, 2002; Arló-Costa and Pacuit, 2006):

Definition 3.2.4 (Frame properties) *A frame is* consistent *iff* $\forall w \in W : \mathcal{N}_w \neq \varnothing$ *and* $\{\varnothing\} \notin \mathcal{N}_w$.

A frame is closed under $\leq \kappa$ intersections *(where κ is a cardinal) iff*

$$\forall w \in W,\ \forall X = \{\mathcal{X}_i | i \in I\}\ \ where\ |I| \leq \kappa,\ \bigcap_{i \in I} \mathcal{X}_i \in \mathcal{N}_w.$$

A frame is trivial *iff* $|D| = 1$*, otherwise it is* non-trivial.

A frame is supplemented *iff* $\forall w \in W, X \cap Y \in \mathcal{N}_w \Rightarrow X \in \mathcal{N}_w$ *and* $Y \in \mathcal{N}_w$.

Theorem 3.2.5 ((Arló-Costa, 2002)) **BF** *is valid in the class of frames that are either (i) trivial, or (ii) closed under finite intersection, if* D *is finite, or (iii) closed under* $\leq k$ *intersections, if* D *is infinite and* $|D| = \kappa$.

Theorem 3.2.6 ((Waagbø, 1992; Arló-Costa and Pacuit, 2006)) **CBF** *is valid in the class of frames that are either supplemented or trivial.*

Theorem 3.2.6 establishes a strong relationship between **M** and **CBF**, since supplementation characterises **M**. Hence, for constant non-trivial domain neighbourhood frames **CBF** is valid whenever **M** is. Also, since the closure under $\leq k$ intersections implies the closure under intersection, it is not possible to falsify **C** when **BF** is valid (provided that the frame is non-trivial). However, from the above theorem we can build a countermodel for **BF** given **C** (Waagbø, 1992), although this is possible only for infinite frames (Arló-Costa, 2002).

An open problem of this semantics is that the system $\mathsf{FOL} \oplus \mathsf{E} \oplus$ **CBF** is strongly complete with respect to the class of frames that are either trivial or supplemented: Arló-Costa (2011) conjectured that **M** is thus derivable by adding in the logic a schema expressing non-triviality but no result is available.

Let us now move to the case of varying domains, which was explored only in Stolpe (2003). The peculiarity of Stolpe (2003)'s analysis is that it woks only with models and not with frames. Models are standardly defined as follows:

Definition 3.2.7 (Varying domain neighbourhood models) *A* varying domain neighbourhood frame $\mathcal{M}$ *is a structure* $\langle W, \mathcal{N}, D, \Sigma, I\rangle$ *where*

- W, $\mathcal{N}$, D, *and* I *are like in Definition 3.2.1 and*

- Σ *is a function assigning to each world* $w \in W$ *a set* D_w *of elements of* D.

The valuation condition for $\forall$-formulae is now as follows:

$$\mathcal{M} \vDash^{\sigma}_{w} \forall x A(x) \text{ iff } \mathcal{M} \vDash^{\tau}_{w} A(x) \text{ for every } x\text{-variant } \tau \text{ of } \sigma \text{ such that } \tau(x) \in D_w.$$

Stolpe (2003) defines two classes of varying neighbourhood models that characterize **BF** and **CBF**:

Theorem 3.2.8 (BF and CUPI models) *A varying neighbourhood model* $\mathcal{M} = \langle W, \mathcal{N}, D, I, \Sigma\rangle$ *is a CUPI model iff for any world* $w \in W$ *, if* $\|P(x)\|^{\sigma}_{I} \in \mathcal{N}_w$ *for every* σ *such that* $\sigma(x) \in D_w$*, then* $\|\forall x P(x)\|_I \in \mathcal{N}_w$*.*

BF *is valid in the class of CUPI models.*

Theorem 3.2.9 (CBF and CUPO models) *A varying neighbourhood model* $\mathcal{M} = \langle W, \mathcal{N}, D, I, \Sigma\rangle$ *is a CUPO model iff for any world* $w \in W$ *, if* $\|\forall x P(x)\|_I \in \mathcal{N}_w$*, then* $\|P(x)\|^{\sigma}_{I} \in \mathcal{N}_w$ *for every* σ *such that* $\sigma(x) \in D_w$*.*

CBF *is valid in the class of CUPO models.*

CUPI and CUPO models impose properties that trivially reflect the evaluation of **BF** and **CBF**. The main limit of this approach is that it does not appeal to frames. Thus, Arló-Costa (2011) rightly argues that Stolpe (2003) leaves open many questions, including the general characterization of **BF** and **CBF**. In this Chapter, we attempt to solve this problem for quantified **N**-Monotonic logics with varying domains.

3.3 Quantification in N-Monotonic Modal Logics

Several works carry on a semantic analysis of quantified deontic systems using neighborhood semantics. However, as far as we are concerned, very few lines are devoted to multi-relational semantics in a first order modal framework.

Let us define multi-relational structures for any quantified modal logic.

Definition 3.3.1 (Multi-relational frames) *A multi-relational frame is a tuple* $\mathcal{F} := \langle W, \mathcal{R}, D, U\rangle$ *where:*

- *W is a non empty set of worlds*

- *$\mathcal{R}$ is a (possibly infinite) set of binary relations over W*

- *D is a function associating to each world $w \in W$ a set D_w of individuals (the inner domain of w)*

- *U is a function associating to each world $w \in W$ a set U_w of individuals (the outer domain of w) such that for any $w \in W$, $U_w \neq \varnothing$ and $D_w \subseteq U_w$ and if wRv for some R, then $U_w \subseteq U_v$.*

The original definition given by Kripke (1963) states that for all worlds w, $U_w = \bigcup_{v \in W} D_v$, setting a unique outer domain for the whole frame. However, we decided to follow the broader approach proposed by Corsi (2002):

> The fact that $U_w \subseteq U_v$, if wRv, does not prevent D_w from being disjoint from D_v. Kripke (1963) stipulates that for all $v \in W$, $U_v = \bigcup_{w \in W} D_w$. We generalise Kripke's original semantics by allowing $U_w \subseteq U_v$, if wRv, and $\bigcup_{w \in W} U_w \supseteq \bigcup_{w \in W} D_w$. $\bigcup_{w \in W} U_w$ may contain individuals that never happen to come into existence. (Corsi, 2002, 1485)

Models, assignments and the concepts of *satisfaction, truth, validity* are defined in the standard way.

Definition 3.3.2 (Multi-relational models) *A multi-relational model is a tuple $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$ where $\langle W, \mathcal{R}, D, U \rangle$ is a multi-relational frame and I is a function $I : \mathcal{L} \mapsto U_w$ for any $w \in W$ such that:*

- $I_w(P^n) \subseteq (U_w)^n$

- $I_w(c) \in U_w$

- $I_w(=) = \{\langle d, d \rangle : d \in U_w\}$

Definition 3.3.3 (Assignments) *For any $w \in W$, a w-assignment σ is a function $\sigma : Var(\mathcal{L}) \mapsto U_w$.*

An x-variant τ of a w-assignment σ is a w-assignment which may differ from σ for the value assigned to x.

Notice that within the semantics framework proposed by Kripke (1963), since the outer domains are constant, any w-assignment σ is also a v-assignment for any couple of worlds. However, it should be noticed here that here the fact that $U_w \subseteq U_v$, if wRv for some R, still guarantees the fact that if two worlds w, v are related by some R, then any w-assignment is also a v-assignment, as all the variables of the language are still mapped on individuals without gaps.

Definition 3.3.4 (σ-interpretation) *Given a w-assignment σ*

(a) $I^\sigma_w(c) = I_w(c)$, *and*

(b) $I^\sigma_w(x) = \sigma(x)$.

Definition 3.3.5 (Truth conditions) *Truth evaluation clauses are as follows:*

- $\mathcal{M} \vDash^\sigma_w P^n(t_1, \dots, t_n)$ *iff* $\langle I^\sigma_w(t_1), \dots, I^\sigma_w(t_n)\rangle \in I_w(P^n)$
- $\mathcal{M} \nvDash^\sigma_w \bot$
- $\mathcal{M} \vDash^\sigma_w \exists x A$ *iff for some x-variant τ of σ such that* $\tau(x) \in D_w$, $\mathcal{M} \vDash^\tau_w A(x)$
- $\mathcal{M} \vDash^\sigma_w \forall x A$ *iff for every x-variant τ of σ such that* $\tau(x) \in D_w$, $\mathcal{M} \vDash^\tau_w A(x)$
- $\mathcal{M} \vDash^\sigma_w \Box A$ *iff* $\exists R_i \forall v(wR_iv \Rightarrow \mathcal{M} \vDash^\sigma_v A)$
- $\mathcal{M} \vDash^\sigma_w \Diamond A$ *iff* $\forall R_i \exists v(wR_iv \& \mathcal{M} \vDash^\sigma_v A)$

Satisfaction, Truth, Validity. A model $\mathcal{M}$ **satisfies** a set of formulae Δ iff for some world w and some w-assignment σ, $\mathcal{M} \vDash^\sigma_w D$ for all $D \in \Delta$. A formula A is **true** in a world w of a model $\mathcal{M}$, $\mathcal{M} \vDash_w A$, iff for any w-assignment σ, $\mathcal{M} \vDash^\sigma_w A$. A formula A is **true** in a model $\mathcal{M}$, $\mathcal{M} \vDash A$, iff for all w, $\mathcal{M} \vDash_w A$. A formula A is **valid** on a frame $\mathcal{F}$, $\mathcal{F} \vDash A$, iff for any model $\mathcal{M}$ on $\mathcal{F}$, $\mathcal{M} \vDash A$. Given a class of frames $\mathbb{F}$, a formula A is $\mathbb{F}$-**valid**, $\mathbb{F} \vDash A$, iff for any frame $\mathcal{F} \in \mathbb{F}$, $\mathcal{F} \vDash A$. $\mathcal{M}$ is a model for a logic L iff $\mathcal{M} \vDash A$ for all $A \in L$.

Abbreviations. Given a frame $\mathcal{F}$ and a model $\mathcal{M}$ on $\mathcal{F}$, for any formula A,

Satisfaction: $\|A\|^{\sigma}_{I} := \{w \mid \mathcal{M} \vDash^{\sigma}_{w} A\}$

Truth: $\|A\|_{I} := \{w \mid \mathcal{M} \vDash^{\sigma}_{w} A,$ for any assignment $\sigma\}$

Validity: $\|A\| := \{w \mid \mathcal{F} \vDash^{\sigma}_{w} A$ for any assignment σ and any interpretation $I\}$

Definition 3.3.6 *Given a multi-relational model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I\rangle$, an individual constant c is said to be a* rigid designator *iff $\forall w \forall v(\exists R_i(wR_iv) \Rightarrow I_w(c) = I_v(c))$.*

Lemma 3.3.7 *Given a multi-relational model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I\rangle$ and a w-assignment σ, if an individual constant c is a* rigid designator*, then $\vDash^{\sigma}_{w} A(c/x)$ iff $\vDash^{\tau}_{w} A(x)$ for any w-assignment τ which is an x-variant of σ such that $\tau(x) = I_w(c)$. (Cf. (Corsi, 2002, Lemma 1.1).)*

Proof. The proof is given by induction on the length of a formula A. Suppose A has the form $\Box B(x)$. If $\vDash^{\sigma}_{w} \Box B(c/x)$ then there is a relation R_i such that for any world v, if wR_iv, then $\vDash^{\sigma}_{v} B(c/x)$ and by induction hypothesis we have $\vDash^{\tau}_{v} B(x)$ where τ is an x-variant of σ such that $\tau(x) = I_v(c)$. Since c is a rigid designator by hypothesis, $I_v(c) = I_w(c)$, τ is a w-assignment and hence $\vDash^{\tau}_{w} \Box B(x)$.
If $\vDash^{\tau}_{w} \Box B(x)$ then there is a relation R_i such that for any world v, if wR_iv, then $\vDash^{\tau}_{v} B(x)$ and by induction hypothesis we have $\vDash^{\sigma}_{v} B(c/x)$ where τ is an x-variant of σ such that $\tau(x) = I_v(c)$. Since c is a rigid designator by hypothesis, $I_v(c) = I_w(c)$, σ is a w-assignment and hence $\vDash^{\sigma}_{w} \Box B(c/x)$. ∎

We assume all individual constants to be rigid designators.

Lemma 3.3.8 *Given a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I\rangle$ on it and a world w, if σ and τ are two w-assignments which coincide on any free variable occurring in a formula A, then it holds that $\mathcal{M} \vDash^{\sigma}_{w} A$ iff $\mathcal{M} \vDash^{\tau}_{w} A$.*

Proof. The proof is given by induction on the length of a formula A. Suppose A has the form $P^n(t_1, \dots, t_n)$, then $\vDash^{\sigma}_{w} P^n(t_1, \dots, t_n)$ if and only if $\langle I^{\sigma}_{w}(t_1), \dots, I^{\sigma}_{w}(t_n)\rangle \in I_w(P^n)$.

If t_i is an individual constant c, then $I^{\sigma}_{w}(c) = I_w(c) = I^{\tau}_{w}(c)$. Otherwise, if t_i is a variable x, $I^{\sigma}_{w}(x) = \sigma(x) = \tau(x)$ by hypothesis, hence $I^{\sigma}_{w}(x) = I^{\tau}_{w}(x)$. The other steps are straightforward.

■

3.4 The Deontic Role of Barcan Schemata

The issues presented in the Introduction of this Chapter are somehow discussed within deontic literature. Goble (1994, 1973, 1996), for example, provides interesting reasons to say that deontic operators are referentially transparent with respect to singular terms, since this assumption seems required to account for an intuitive analysis of the instantiation of general obligations into concrete cases. Goble pays special attention to the role of definite descriptions, an issue that is outside the scope of our work. However, a piece of his story should be mentioned here. His argument runs starting from semantical considerations. Suppose that Jones ought to give \$20 to the first homeless person who begs from him in 2006 and that Smith is such a homeless person. The question is: Is Jones obligated to give \$20 to Smith? The answer is, of course, yes, but the point is that, if deontic contexts are taken fully intensional, we may argue that Smith is not the individual corresponding to the first homeless begging from Jones in every ideal world. Goble's proposal is thus to change the standard truth-conditions of any formula $\Box F(t)$, where t is a singular term: Rather than checking whether, for every ideal world v related to the actual world w, the denotation of t at v is in the extension of F in v, the formula $\Box F(t)$ is true iff the denotation of t at w is in the extension of F at v.

Clearly, the foregoing is a roundabout way of considering in deontic logic the meaning of the distinction between *de dicto* and *de re* modal formulae, namely, between formulae with and without free occurrences of variables within the scope of the modal operator $\Box$.

The conceptual and technical distinction between *de dicto* and *de re* formulae has been widely investigated in alethic modal logic, i.e., the modal logic of necessity, possibility, impossibility, and contingency (see, e.g., Garson, 2001; Fitting and Mendelsohn, 1998; Gabbay, 1976; Fine, 1978; Cocchiarella, 2001). *De dicto* sentences occur whenever a modal property is associated to a *dictum* or sentence, as in the phrase *it is necessary that all men are mortal*, where the modal operator is applied to the sentence *all men are mortal* and thus necessity refers to the truth of that. On the other hand, we call a sentence a *de re* modality if the modal property is given to an object, as in the phrase *all men are necessarily mortal*, in which the property *being necessarily mortal* is applied to all mankind. It is clear that such a distinction is lost whenever we lose the expressive power of predicate logics in order to analyse the case of propositional calculus. A classical example to explain the necessity of possible worlds is provided by Thomas Aquinas.[4] In his *Summa contra Gentiles* Thomas Aquinas considers the problem of God's pre-knowledge. God can, according to the philosopher, see the action which is taking place. This is coherent with human freedom. In fact consider the truth value of the following sentence:

(1) *If I see someone sitting, he is necessarily sitting.*

This is clearly true if read in the *de dicto* way:

(2) *It is necessary that if I see someone sitting, that person is sitting.*

which is to say:

(2*) *In every possible world if I see someone sitting, that person is sitting.*

The sentence nevertheless ceases to hold true as soon as we apply the *de re* reading:

[4] Thomas Aquinas, *Summa de veritate catholicae fidei contra Gentiles* [1259-1264], Roma: edizione Leonina, 1918-1930, voll. XIII-XV.

(3) *If I see someone sitting, such person has the necessary property of being sat*

i.e., (3*) *If I see someone sitting, then in every possible world that person is sitting* which is clearly false.

It would not be possible, according to (Plantinga, 1974, chap. 1), to understand such a distinction if we cease to use the possible worlds framework. Just another example to understand such distinction is provided by (Fitting and Mendelsohn, 1998, 86) recalling one of Quine's famous discussions. Consider the sentence *The number of planets is necessarily odd*. A *de re* reading would suggest that the number of planets in the solar system is odd in every possible world. Any person without radically deterministic philosophical views would then disagree with it being true. On the other hand its *de re* interpretation proves to be true: *in every possible world it is true that in the actual world the number of planets is odd*.

Leaving planets and men sitting necessarily or not and moving to something more useful in our everyday life, Thomason (1968) shows how such a distinction may help in removing the ambiguity in some English expressions such as *any* and *some*. Consider the following couple of sentences:

(a) *Everyone* can come along with us.

(b) *Anyone* can come along with us.

In fact the sentence (a) could be read as *It is possible that all come with us*, i.e., $\Diamond\forall x Come(x, us)$, whereas (b) would be *All can possibly come with us*, i.e., $\forall x \Diamond Come(x, us)$. As soon as we formalise them, we realise how the syntactic difference of the two is actually linked to a different scope of the universal quantifier and this can help us understand the difference in the use of *any* and *some* in English (see Calardo, 2008).

But what about deontic contexts? According to Goble's story, it seems that *de re* formulae play a specific deontic role, as only $\forall x \Box F(x)$ can be reasonably instantiated into $\Box F(a)$ (where a is typically an individual constant symbol). However, one may argue that, unlike alethic and other kinds of modalities, in deontic logics it does not make any sense to distinguish between *de dicto* and *de re* modal formulae. In fact, despite what we said about (3.1) and (3.2), von Wright (1951b, 40) clearly maintains that "the operators 'P' and 'O' [...] yield sentences" and so "deontic modalities cannot be taken alternatively *de dicto* and *de re*." Hector-Neri Castañeda (1981), too, is sceptical in this regard, as he argues in favour of the complete extensionality of ordinary deontic concepts, thus making deontic *de re* and *de dicto* formulae virtually equivalent. Actually, $\forall x \Box P(x)$ and $\Box \forall x P(x)$ are intuitively different, as we have above recalled: the former is about existing individuals with respect to which we may say that P an essential property, whereas the latter modal statement is purely sentential. This does not hold with the following two sentences "There is someone for whom it is obligatory that he do A" and "It is obligatory for someone to do A"; according to semantical conventions in English, "deontic operators do not in any way affect the range of quantifiers" and so deontic logic is extensional (Castañeda, 1981, 67). Notice that Goble, too, defended the view that deontic logic is extensional, but, as we mentioned, his conclusions are not radical as those of von Wright and Castañeda.

There may be good reasons to subscribe to von Wright's and Castañeda's criticism, but still we have in our hands a formalism—quantified deontic logic based on possible-world semantics—which technically can embed the distinction bewteen *de re* and *de dicto* sentences. In general, if we adopt the actualist interpretation of quantifiers, it seems to us that *de dicto* and *de re* deontic sentences may correspond, respectively, to *non-contextual* (or generic) and *contextual* (or concrete, actual) obligations. This alternative reading, given Castañeda's criticism that essential deontic predication does not make any sense, seems roughly in line with our comments of schemata (3.1) and (3.2) and, also, with Goble's general intuitions. In fact, when we have formulae like $\Box \forall x F(x)$, obligations may leave the problem of reference (application) to existing individuals out of consideration, as the question of their concrete application is somehow put into brackets. In other words, we may state that something is obligatory for some individuals independently of any concern about concrete applicability. This is not absurd as we may argue that something is deontically correct, it ought to be case, for conceivable individuals that, as far as we know, may not exist. In the second case—when we have formulae like $\forall x \Box F(x)$—the focus is rather on the actual world with respect to which we want to state whether something is or is not obligatory[5].

On the other hand, the distinction between contextual and non-contextual (or *de re* and *de dicto*) deontic sentences is still far from being conceptually clear, as a lot depends on the philosophical role one wants to assign to deontically ideal worlds. This is evident if we just consider a formula such as $\exists x \Box (x = a)$. Actually, given this formula, Goble himself (Goble, 1973, 344) asks: "What would it be for a term to 'deontically' denote something"? Indeed, the question can be more generally reframed as follows: What does it mean that an individual exists in some deontically perfect worlds but does not in other perfect worlds? We think this is still an open philosophical question, which is outside the scope of this research. Despite the fact that we do not have general and conclusive insights about the meaning of the *de re*/*de dicto* distinction in quantified deontic logic, the role of the deontic versions of Barcan schemata may be anyway crucial. Consider again these schemata:

$$\forall x \Box \phi(x) \rightarrow \Box \forall x \phi(x) \tag{\textbf{BF}}$$

$$\Box \forall x \phi(x) \rightarrow \forall x \Box \phi(x) \tag{\textbf{CBF}}$$

[5] The thesis that the *de dicto*/*de re* distinction is significant in deontic logic is not new ((see, e.g., Hintikka, 1957, 1971; Kalinowski, 1973; Kutschera, 1982). Notably, Jakko Hintikka (1957), at least in some passages, seems to read *de re* deontic formulae as contextual statements, though he does not seem to be fully consistent in this regard. However, Hintikka's analysis is peculiar as, in his approach, quantifiers range over act-individuals and not over ordinary individuals. Makinson (1981) criticizes the choice of quantifying over act-individuals, but he seems still to ackowledge the meaningfulness of the *de dicto*/*de re* distinction.

In standard Kripke semantics, **BF** corresponds to the condition that the domains of quantification decrease across possible worlds, whereas **CBF** to the condition that they increase; their joint validity leads then to constant domains, which makes inessential the problem of the existence of individuals. A common philosophical interpretation of this choice is that quantifiers range over all (conceivable, i.e., existing and non-existing) individuals (see Cresswell, 1991). Thus, accepting **BF** and **CBF** entails weakening the existential reading of quantifiers, leading to the *possibilist interpretation of quantification* (see Fitting and Mendelsohn, 1998), which corresponds classically to the well known standard Kripke semantics with constant domains. This position makes some sense in deontic logic, as we have seen when we mentioned von Wright's and Castañeda's view. It is not by accident, we feel, that Castañeda (1981) himself accepts as theorems of his logic both **BF** and **CBF**[6]. Actually, besides different views such as the peculiar one proposed by Goble, subscribing to both **BF** and **CBF** is one of the most direct options to weaken the conceptual distinction between *de re* and *de dicto* sentences[7]. And this seems a good achievement, given the unclear nature of this distinction in deontic logic. In addition, the joint validity of **CBF** and **BF** also allows to keep standard first-order logic (FOL) untouched as it does not require any constraint on it.

What's the intuitive reading of **BF** and **CBF** in deontic logic? On the semantical side, once again, much depends on the philosophical interpretation one adopts in clarifying the notion of individuals' existence across deontically ideal worlds. At least provisionally, if we recall the ideas of contextual and non-contextual deontic sentences, **BF** states that a (universally quantified) contextual obligation implies a (universally quantified) non-contextual obligation. Semantically, this is guaranteed by the fact that **BF** determines decreasing domains. In other words, the price to pay, for moving from

[6] Notice that these principles are also adopted in Schurz (1994). Hintikka rejects **BF**, but, again, his scepticism mainly depends on the fact that quantifiers range over act-individuals.

[7] Barcan schemata alone are not, however, sufficient in general to eliminate *de re* modalities, namely, to prove that, given any modal logic S, for each formula ϕ, there exists a *de dicto* formula ϕ' such that $\mathsf{S} \vdash \phi \leftrightarrow \phi'$. This can be done only adding some extra-conditions and within strong modal systems such as $\mathsf{S5}$; see, e.g. Fine (1978); Kaminski (1997).

contextual obligations (for which, in the perspective of their instantiation, actual existence matters) to non-contextual obligations (for which actual instantation does not count anymore), is to assume that all individuals, existing in the ideal worlds, exist as well in the actual world. This intuitive, though partial reading of **BF** seems to be confirmed if we introduce the common and weak notion of permission $\Diamond$ corresponding to the dual of $\Box$. This permits to reframe **BF** as follows:

$$\Diamond \exists x\phi(x) \rightarrow \exists x \Diamond \phi(x) \tag{$\mathbf{BF'}$}$$

($\mathbf{BF'}$) permits to move from a non-contextual (existential) deontic statement to a contextual (existential) one. But this should not confuse the reader as we have to take into account the peculiar nature of the weak permission, which is nothing but the dual of an obligation, namely the negation of a prohibition. Analogous considerations may be reiterated for **CBF**, which is a principle stating that any non-contextual (generic) obligation implies that this obligation is applicable to all concrete cases.

It is clear that, under this intuitive but still partial reading, we do not have conclusive reasons to adopt in general **BF** and **CBF**. **BF**, in particular, can be highly problematic, both for its intuitive consequences and for the semantic conditions required to validate it in Kripke models. **CBF** seems less controversial: if a generic obligation holds, such an obligation must be applicable in the actual case, at least unless there is no contrary reason against this. (Also the corresponding condition that domains never decrease looks more reasonable, from the deontic point of view, as the range of a generic obligation may exceed the range of an obligation applying to a concrete case.)

On the other hand, as we have seen the joint validity of **BF** and **CBF** seems a first stept towards milding Castañeda's objections. In addition, independently of any philosophical reflection on the meaning of **BF** and **CBF**, their logical role is significantly that of making apparent possible deontic dilemmas. Suppose to have

$$\forall x \Box \phi(x) \qquad\qquad \Box\exists x \neg\phi(x)$$

Are these formulae in conflict with each other? Syntactically, it is clear that the appeal to **BF** is essential if we think that the formulae above are incompatible: $\forall x \Box \phi(x)$ implies $\Box\forall x\phi(x)$ by **BF**, while $\Box\exists x\neg\phi(x)$ is equivalent to $\Box\neg\forall x\phi(x)$. Analogously, **CBF** is essential, for example, if we want to make apparent the conflict between the following formulae:

$$\Box\forall x\phi(x) \qquad\qquad \exists x \Diamond \neg\phi(x)$$

In sum, **BF** and **CBF** can be accepted in deontic logic for at least the following two reasons:

- the joint validity of **BF** and **CBF** permits to keep standard FOL untouched and to weaken the logical distinction between *de re* and *de dicto* sentences, a distinction that can be problematic in deontic contexts;

- the joint validity of **BF** and **CBF** allows us to make apparent conflicts between deontic sentences expressible in quantified deontic logics.

So far, so good. But our question is to see what happens if we move from normal to non-normal modal logics. This is not only an issue in deontic logic but it concerns quantification in several intensional logics. But there are specific reasons to pose this question in deontic logic, as we have recalled in Chapters 1 and 2. As we shall see—unlike the case of normal deontic and modal logics—the joint validity of **BF** and **CBF** is not guaranteed by imposing constant domains of quantification. Additional conditions are required, but they pose indeed specific problems in deontic logic.

3.5 The Converse Barcan and the Ghilardi schemata

As we saw in the previous sections, only a few works have been devoted to the study of quantified non normal modal logics. Stolpe (2003) is certainly one of the most significant. However, Stolpe's aim is to find out "[...] what semantical restrictions must be imposed on a minimal *model* in order to validate the Barcan and the converse Barcan formulae" (Stolpe, 2003, 559, emphasis added). As Arló Costa and Pacuit point out:

> [...] unfortunately Stolpe does not appeal to frames in his semantics (he only uses models). So, it is obvious that there are many open questions not considered in Stolpe's paper. For example it would be nice to get frame conditions characterizing the Barcan and the Converse Barcan in this setting. (Arló-Costa and Pacuit, 2006, 21)

Our aim is rather broader. In this section we shall propose frame conditions for **CBF** in multi-relational weak semantics.

Standard results in Kripke Semantics (1-relational frames) state that both **BF**, and **CBF** are valid in constant domain frames (i.e., when D_w and U_w coincide for each w). Within first order Kripke frames with varying domains, however, these schemata cease to be valid. Let us recall below the proof of this well known result.

Lemma 3.5.1 **CBF** *is not valid in the class of multi-relational frames with varying inner domains.*

Proof. The proof is trivial. Let $\mathcal{M} = \langle W, \mathcal{R}, D, U, I\rangle$ be a 1-relational model such that: $W := \{w, v\}$, $\mathcal{R} := \{R\}$, $R := \{\langle w, v\rangle\}$, $D_w := \{a, b\}$, $D_v := \{a\}$, $U_w = U_v = \{a, b\}$, $I^\sigma_w(P^1) = \varnothing$ for all σ and $I^\sigma_v(P^1) = D_v$, for $\sigma(x) = a$ (see Figure 3.1). Then $\|\forall x(P)\|^\sigma_I = D_v$ and there is a relation, i.e., R, such that for any world t, wRt (if and) only if $\vDash^\sigma_t \forall xP$. Hence $\vDash^\sigma_w \Box\forall xP$. Moreover $\vDash^\sigma_w \exists x \Diamond \neg P$, as $\vDash^\tau_v \neg P(x)$ for $\tau(x) = b$. ∎

Recall a definition we gave in Chapter 2:

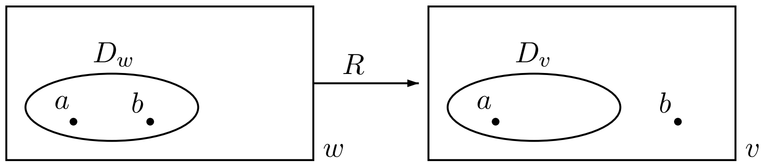

Figure 3.1: A frame to build a countermodel for **CBF**. Indeed set $I^\sigma_w(P^1) = \varnothing$ for all σ and $I^\sigma_v(P^1) = D_v$, for $\sigma(x) = a$. Hence $v \nvDash^\sigma_v \forall x A$.

Definition 3.5.2 (Local Seriality) *Given a multi-relational frame $\mathcal{F}$, a world w is* locally serial *iff for any relation R_i there is a world v such that wR_iv.*

Below is the characterising condition on frames (and the resulting lemma) for **CBF**.

Lemma 3.5.3 (CBF Characterisation result) *For any multi-relational frame $\mathcal{F} := \langle W, \mathcal{R}, D, U \rangle$, $\mathcal{F} \vDash \Box\forall x A \to \forall x \Box A$ iff $\forall w \in W$, If w is locally serial, then for any x such that $x \in D_w$, for any relation R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all t, $(wR_it \Rightarrow x \in D_t)$.*

Proof. According to the statement of the Lemma above, any counter model for **CBF** must be based on a frame $\mathcal{F}$ with the following three conditions:

there is a world $w \in W$, and a w-assignment σ such that:

(a) BOTH w is locally serial, i.e., for all k, $R_k(w) \neq \varnothing$

(b) AND there exists a w-assignment τ, which is an x-variant of σ such that $\tau(x) \in D_w$, there is a relation R_j

b.1 BOTH $\exists t(wR_jt \ \& \ \tau(x) \notin D_t)$, i.e., the inner domains are not increasing

b.2 AND for all the other relations R_i, IF $R_i(w) \subseteq R_j(w)$, THEN there exists some world t_i such that $(wR_it_i \ \& \ \tau(x) \notin D_{t_i})$.

Hence, these conditions are necessary and sufficient to build a countermodel for **CBF**. The idea is to provide an interpretation which makes the class $R_j(w)$ to be exactly the truth set of $\forall x A$ for some formula A. Then, we need an individual which *actually exists* in w and satisfies $\neg A(x)$ in some R_k-accessible state for all k. Indeed, let P be some unary predicate and let I be an interpretation defined in the following way: For any world z, $z \in \|P(x)\|^I_\vartheta$ if and only if wR_jz and ϑ is an x-variant of τ such that $\vartheta(x) \in D_z$.

Thus, $R_j(w) = \|\forall x P(x)\|^I_\tau$ and hence $\vDash^\tau_w \Box\forall xP$. Consider now any world z such that $z \notin R_j(w)$; We defined I in such a way that for any z-assignment ϑ, $\nvDash^\vartheta_z P(x)$ and thus this holds also for τ, $\nvDash^\tau_z P(x)$. Moreover our frame conditions guarantee that for any relation R_i, if $R_i(w) \subseteq R_j(w)$, then there is some $z \in R_i(w)$ such that $\tau(x) \notin D_z$ and, following our construction of I, $\nvDash^\tau_z P(x)$. Since $\tau(x) \in D_w$ by definition, it holds that $\vDash^\tau_w \exists x \Diamond \neg P(x)$.

It is easy to see that if **CBF** does not hold on a model, its frame fulfills the conditions stated above. Assume that for some frame $\mathcal{F}$, $\mathcal{F} \nvDash$ **CBF**. Then there are an assignment σ, a valuation I and a world w such that (a) $\vDash^\sigma_w \Box\forall xA$ and (b) $\vDash^\sigma_w \exists x \Diamond \neg A$ for some formula A. From (b) it follows that $\vDash^\tau_w \Diamond\neg A(x)$ for some $\tau(x) \in D_w$, thus there exists some n-tuple $\langle z_1, \dots, z_n\rangle$ such that wR_kz_k for any $R_k \in \mathcal{R}$ and $\nvDash^\tau_{z_k} A$. From (a) it follows that for some relation R_i, $R_i(w) \subseteq \|\forall xA\|^\sigma_I$, where $\|\forall xA\|^\sigma_I := \{t \mid \ \vDash^\tau_t A(x)$ for any τ such that $\tau(x) \in D_t\}$. Now, consider any relation R_k; if $R_k(w) \subseteq R_i(w)$, then by (b), $R_k \subseteq \|\forall xA\|^\sigma_I$ and by (a) there is some $z \in R_k(w)$ such that $z \notin \|A(x)\|^\tau_I$ for some $\tau(x) \in D_w$, thus we must conclude that $\tau(x) \notin D_z$.

∎

This Lemma leads to a result that is very close to that for Kripke semantics, as we shall see in what follows.

Some remarks on CBF. An immediate, yet interesting result is that the schemata **M** and **CBF** are independent, in contrast with Theorem 3.2.6, which establishes that, for *constant* non-trivial domain neighbourhood frames **CBF** is valid whenever **M** is. Indeed by Lemma 2.2.5 it holds that **M** is a valid schema, whereas it is possible to build a countermodel for **CBF** (see Lemma 3.5.1).

Corollary 3.5.4 *The validity of* **M** *does not imply the validity of* **CBF**.

Lemma 3.5.3 states some interesting facts about the class of **CBF**-frames. First of all it states an important property about individuals and their behaviour in alternative standards. It is well known that **CBF** imposes *increasing inner domains* on Kripke frames:

Definition 3.5.5 (Increasing inner domains - Kripke semantics) *A relational frame $\mathcal{F}$ has* increasing inner domains *iff for all worlds w, v, if wRv then $D_w \subseteq D_v$.*

Theorem 3.5.6 (CBF characterisation in Kripke frames) **CBF** *characterises the class of Kripke frames with increasing inner domains.*

Consider any Kripke-frame $\mathcal{F} := \langle W, R, D, U \rangle$ (which is nothing but an 1-relational frame with varying domains). From the condition stated above in Lemma 3.5.3 we have that $\mathcal{F} \vDash \Box \forall x A \to \forall x \Box A$ iff $\forall w \in W$, for any world z, if wRz then for any w-assignment σ such that $\sigma(x) \in D_w$, $\sigma(x) \in D_z$, i.e., $D_w \subseteq D_z$. Thus the conditions imposed by the schema **CBF** are the usual ones: **CBF** is valid in those frames whose connected worlds have increasing inner domains.

The situation within **N**-monotonic logics is rather different, yet connected. The first question that comes to mind is how to generalise the concept of *increasing inner domains*. There are a couple of alternatives at hand, which look very close to the definition given in Kripke-semantics. We might either define increasing domains in a very strong sense, by asking that *all* relation enjoy such property:

Definition 3.5.7 (General increasing inner domains) *A multi-relational frame $\mathcal{F}$ has* increasing inner domains *iff for any couple of worlds w, v for* any *relation R_i, if wR_iv then $D_w \subseteq D_v$.*

or we might keep such property lighter, asking for only one relation to fulfill it:

Definition 3.5.8 (Restricted increasing inner domains) *A multi-relational frame $\mathcal{F}$ has* increasing inner domains *iff for any couple of worlds w, v there is at least* one *relation R_i, such that if wR_iv then $D_w \subseteq D_v$.*

Generally, in the multi-relational weak semantics scenario, **CBF** does not capture any of these properties. Indeed Lemma 3.5.3 states that if a world w is locally serial, i.e., if there is access to other worlds under *any* standard, then, each *actual* individual (i.e., any individual belonging to the inner domain of w) keeps being actual in a subset of every standard. So it says, somehow, that individuals are bound to 'survive' under alternative relations, although not necessarily altogether. If Mary and Alex are both alive now (actual) the presence of **CBF** guarantees that for any standard, there is a subset of alternative ideal worlds in which Mary keeps being actual, and another one in which Alex is still actual.

Logically, if we consider the big union of all the inner domains connected to w by some relation i, the inner domain of w is a subset of it, i.e.:

Lemma 3.5.9 *If the schema* **CBF** *is valid on a given frame, then on any world w, if w is locally serial, i.e., for any i, $R_i(w) \neq \varnothing$, then for any relation R_i*

$$D_w \subseteq \bigcup \{D_v \mid v \in R_i(w)\}$$

Proof. Assume that the property stated in this Lemma does not hold. Then there are a locally serial world w and a relation R_i such that for some individual $d \in D_w$, for all $v \in R_i(w)$, $d \notin D_v$, i.e., $D_w \nsubseteq \bigcup\{D_v \mid v \in R_i(w)\}$. Consider an assignment σ on any world belonging to $R_i(w)$ such that $\sigma(x) = d$, then for all $v \in R_i(w)$, $\vDash_v^\sigma \neg\exists x(x = d)$ and then $\vDash_w^\sigma \Box\forall x(x \neq d)$. By **CBF** we get $\vDash_w^\sigma \forall x \Box (x \neq d)$, i.e., for any w-assignment τ such that $\tau(x) \in D_w$, $\vDash_w^\tau \Box(x \neq d)$. Since $d \in D_w$, this holds true for σ as well, where $\sigma(x) = d$. Hence we get that the formula $\Box(d \neq d)$ holds true at w. This would impose that for some j, $R_j(w) = \varnothing$ which is contradictory with the assumptions.

■

Hence, within the same set of alternative possible worlds, there is one world in which Mary lives and another in which Alex lives, although, again, they are not bound to necessarily coincide. Moreover, this property stated below turns out to be canonical for a free system with identity which includes the **CBF** schema (see Lemma 3.6.13).

A natural question is then how to force individuals to survive altogether under some standard. Well, intuitively, at first sight, it would be enough to add partial closure under intersection, i.e., the semantic property stated in Lemma 2.2.12 and characterised by Schema $\mathbf{C} \coloneqq \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$. However, this would generate a normal system (please refer to the lattice in Figure 2.1). Thus, in the presence of Axiom **C**, the situation becomes closer to the Kripkean case. This said, Definition 3.5.8 seems to capture the concept of 'increasing domains' better than the stronger alternative proposed. However, it turns out that the presence of **C** is not required to achieve the formerly stated property, which is actually (partially) granted by the presence of **CBF** alone:

Theorem 3.5.10 (CBF and Restricted Increasing Domains) *For any multi-relational frame $\mathcal{F} \coloneqq \langle W, \mathcal{R}, D, U \rangle$, $\mathcal{F} \vDash \Box \forall x A \rightarrow \forall x \Box A$ iff $\forall w \in W$, if w is locally serial, then for all R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all worlds v in $R_i(w)$, $D_w \subseteq D_v$.*

Proof. This is very straightforward. In fact, consider the formulation of Lemma 3.5.3, which states that if w is locally serial, then for any x such that $x \in D_w$, for any relation R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all t, $(wR_it \Rightarrow x \in D_t)$. Suppose the inner domain of w is the set $\{a, b\}$. Then, for any R_j there must be an R_i such that $R_i(w) \subseteq R_j(w)$ and a belongs to the inner domain of *all* the worlds from $R_i(w)$. However, since the statement of Lemma 3.5.3 refers to *all* relations and *all individuals*, we must consider also R_i and b. Hence there is an R_m such that $R_m(w) \subseteq R_i(w)$ and b belongs to the inner domain of any world from $R_m(w)$. Thus, all the worlds from

$R_m(w)$ have an inner domain containg both a, and b and their inner domain is, therefore, a superset of D_w. ∎

By all means, these results are much more specific than those proposed by Stolpe (2003), as nothing is known about inner domains of related worlds. In fact, by translating in terms of multi-relational semantics Stolpe (2003)'s CUPO condition (see Theorem 3.2.9) it is quite straightforward to prove a weaker correspondence result for **CBF** in multi-relational models.

Lemma 3.5.11 (CUPO models) *For any multi-relational model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$, $\mathcal{M} \models \Box\forall x A \to \forall x \Box A$ if and only if for any world w the following holds: given a w-assignment σ, if there is a relation R_i such that $R_i(w) \subseteq \|\forall x A\|^{\sigma}_I$, then for all x-variant τ of σ such that $\tau(x) \in D_w$ there is some j such that $R_j(w) \subseteq \|A(x)\|^{\tau}_I$.*

Proof. This proof is very simple and straightforward. For the left arrow, suppose there are a model $\mathcal{M} := \langle W, R_1, \dots, R_n, V \rangle$ and a σ such that $\models^{\sigma}_w \forall x A$. Hence, for some i, $R_i(w) \subseteq \|\forall x A\|^{\sigma}_w$. By hypothesis for all x-variant τ of σ such that $\tau(x) \in D_w$ there is some j such that $R_j(w) \subseteq \|A(x)\|^{\tau}_I$, thus ensuring $\models^{\tau}_w \Box A(x)$ and hence $\models^{\sigma}_w \forall x \Box A(x)$.The other version can be proved accordingly by contraposition. ∎

Mirroring Kripke semantics, **CBF** and the Ghilardi schema $\exists x \Box A \to \Box \exists x A$ (**GF**) are semantically equivalent:

Lemma 3.5.12 CBF *and* **GF** *are semantically equivalent*

Proof. To show this result it is enough to check that **GF** characterises the class of **CBF**-frames.

($\Rightarrow$) According to the statement of Lemma 3.5.3, any counter-model for **CBF** must be based on a frame $\mathcal{F}$ with the following three conditions:

there is a world $w \in W$, and a w-assignment σ such that:

(a) BOTH w is locally serial, i.e., for all k, $R_k(w) \neq \varnothing$

(b) AND there exists an individual a, $a \in D_w$, and there is a relation R_j

(b.1) BOTH $\exists t(wR_jt \ \& \ a \notin D_t)$, i.e., the inner domains are not increasing

(b.2) AND for all the other relations R_i, IF $R_i(w) \subseteq R_j(w)$, THEN there exists some world t_i such that $(wR_it_i \ \& \ a \notin D_{t_i})$.

Let us sketch a valuation to build a counter-model for **GF**. Let A be a unary predicate, take any w-assignment τ and set I as follows:

(a) IF $\tau(x) \neq a$, then $\|A(x)\|_I^\tau = W$;

(b) OTHERWISE if $\tau(x) = a$, then $\|A(x)\|_I^\tau = W - R_i(w)$.

From (b) it follows that $R_i(w) \subseteq \|\neg A(a)\|_I$, hence $\vDash_w^\sigma \exists x \Box \neg A(x)$.

Let us turn our attention to (a). Any world v which is not R_i-seen by w is such that for any assignment τ, $\vDash_v^\tau A(x)$ and hence $\vDash_\sigma^I \forall x A(x)$ for all σ. Concerning R_i the situation is the following: the fact that $\exists t(wR_jt \ \& \ a \notin D_t)$ guarantees that $\vDash_t^\sigma \forall x A(x)$, whereas the fact that for all the other relations R_i, IF $R_i(w) \subseteq R_j(w)$, THEN there exists some world t_i such that $(wR_it_i \ \& \ a \notin D_{t_i})$, guarantees the fact that there is always a world z in $R_j(w)$ such that $\vDash_z^\sigma \forall x A(x)$. This observation, together with the fact that w is locally serial by assumption, makes sure that $\vDash_w^\sigma \Diamond \forall x A(x)$, thus disproving an instance of **GF**.

∎

3.6 Completeness Results[8]

3.6.1 The system $\mathsf{Q}^\circ_=.\mathsf{MN}$

Here we present an axiomatic system extending MN with predicate logic, which is based on free quantified modal logic (see Corsi, 2002, 1498). The system $\mathsf{Q}^\circ_=.\mathsf{MN}$ (Free Quantified **N**-monotonic modal logic) contains the following axioms and inference rules:

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[8] I wish to thank Gabriele Tassi for the fruitful discussions and the useful insights he provided while I was working on this part of my dissertation.

- Propositional tautologies;
- **UI**$^\circ$:= $\forall y(\forall x A(x) \to A(y/x))$
- $\forall x \forall y A \leftrightarrow \forall y \forall x A$
- $A \to \forall x A$, x not free in A
- $\forall x(A \to B) \to (\forall x A \to \forall x B)$
- **I**:= $t = t$
- $(s = t) \to (A(s//x) \to A(t//x))$
- **ND**:= $\Box A \wedge s \neq t \to \Box(A \wedge s \neq t)$
- **NI**:= $\Box A \wedge s = t \to \Box(A \wedge s = t)$
- **M** := $\Box(A \wedge B) \to (\Box A \wedge \Box B)$
- **N** := $\Box\top$ to $\mathsf{Q}^\circ_=.\mathsf{E}$
- **MP** := $A \to B, A/B$
- **RE** := $A \leftrightarrow B/\Box A \leftrightarrow \Box B$
- **UG** := $A/\forall x A$

Some remarks on $\mathsf{Q}^\circ_=.\mathsf{MN}$. Besides the propositional part,[9] axiom schemata for the basic predicate part (not considering identity) are those originally proposed by Kripke (1963) (see Corsi, 2002) plus $\forall x \forall y A \leftrightarrow \forall y \forall x A$, which is conceptually harmless but needed to ensure completeness results (see Goldblatt, 2011). The language includes the identity symbol =, which makes the logic very expressive and able, for example, to handle definite descriptions, such as 'the first homeless person who begs from Jones in 2006' (Goble, 1996), which is usually represented with an expression like $\lambda x \mathit{Homeless\_Begging\_2006}(x)$ (using lambda notation) and which is typically taken to be equivalent to $\exists x(\mathit{Homeless\_Begging\_Jones\_2006}(x) \wedge \forall y(\mathit{Homeless\_Begging\_Jones\_2006}(y) \to y = x))$. As argued by Goble (1996), expressions such as 'It is obligatory to help the first homeless person who begs from you in 2006' are quite significant in normative contexts, even though an analysis of the role of definite descriptions in deontic logic is outside the scope of this work: anyway, $\mathsf{Q}^\circ_=.\mathsf{MN}$ can handle those expressions. In quantified alethic modal logic with the identity symbols logicians usually consider whether the following schemata are to be valid (for a philosophical discussion, see Kripke, 1980):

$$t = s \to \Box(t = s) \qquad\qquad t \neq s \to \Box(t \neq s)$$

[9] For alternative axiomatisations of the propositional part, please refer to Section 2.1.

Here, we consider a version of them that can also capture some *quite restricted* versions of **C**: indeed, **ND** and **NI** state that it is possible modally aggregate two formulae when one is any A (and thus A *can* also be a $\Box$ formula) and the other is either $t = s$ or $t \neq s$; this does *not* in general entail **C**. Notice that the schema $\mathbf{NI}^* := t = s \rightarrow \Box(t = s)$ is derivable within this system. Finally, it is worth noting that we will make an essential use of the identity symbol and schemata **NI** and $\mathbf{NI}^*$ to ensure completeness of the system $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus \mathbf{CBF}$, namely, when we add **CBF**. We will show later in the Chapter why this is technically needed.

Given a language $\mathcal{L}$, we shall henceforth refer to the set of its individual constants with the notation $C(\mathcal{L})$.

Theorem 3.6.1 (Soundness) *The system* $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ *is sound with respect to the class of all multi-relational frames with varying inner domains.*

PROOF. The proof is standard and it is carried out by induction on $lg(\mathcal{D})$, where $\mathcal{D} := \mathcal{D}_1, \ldots, \mathcal{D}_n$ is a deduction in the axiomatic system $Q^{\circ}_{=}.\mathsf{MN}$ with $A = \mathcal{D}_n$, i.e., $A \in Q^{\circ}_{=}.\mathsf{MN}_{\vdash}$).

(a) if $k = 1$, A is an axiom. Let us consider just a few cases. For the propositional schemata, please refer to Theorem 2.3.9. If A has the form $\forall y(\forall x B(x) \rightarrow B(y/x))$, then assume by reductio that there are a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$ on $\mathcal{F}$, a world w and a w-assignment σ such that $\mathcal{M} \nvDash^{\sigma}_{w} \forall y(\forall x B(x) \rightarrow B(y/x))$. Then there is a w-assignment τ which is an y-variant of σ such that $\mathcal{M} \nvDash^{\tau}_{w} \forall x B(x) \rightarrow B(y/x)$ and $\tau(y) \in D_w$. Hence $\mathcal{M} \vDash^{\tau}_{w} \forall x B(x)$ and $\mathcal{M} \nvDash^{\tau}_{w} B(y/x)$ and $\tau(y) \in D_w$. It follows that for any w-assignment ϑ, where ϑ is an x-variant of τ such that $\vartheta(x) \in D_w$, $\mathcal{M} \vDash^{\vartheta}_{w} B(x)$ and $\mathcal{M} \nvDash^{\tau}_{w} B(y/x)$ for $\tau(y) \in D_w$. Since one must consider all the x-variants of τ which map x to an element of the inner domain of w and since $\tau(y) \in D_w$, there is an assignment $\vartheta^{\star}$ such that it is an x-variant of τ, $\vartheta^\star(x) = \tau(y)$, $\mathcal{M} \vDash_w^{\vartheta^\star} B(x)$ and $\mathcal{M} \nvDash_w^\tau B(y/x)$.

If A has the form $B \to \forall x B$ for x not free in B, then suppose by reductio that there are a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I\rangle$ on $\mathcal{F}$, a world w and a w-assignment σ such that $\mathcal{M} \nvDash_w^\sigma B \to \forall x B$, thus $\mathcal{M} \vDash_w^\sigma B$ and $\mathcal{M} \nvDash_w^\sigma \forall x B$, i.e., $\mathcal{M} \vDash_w^\sigma \exists x \neg B$. From this it follows that $\mathcal{M} \nvDash_w^\tau B$ for some x-variant τ of σ such that $\tau(x) \in D_w$. Since x is not free in B, σ and τ coincide on any free variable occurring in B, hence, by Lemma 3.3.8 it holds that $\mathcal{M} \vDash_w^\sigma B$ and $\mathcal{M} \nvDash_w^\sigma B$.

If A has the form $\forall x(B \to C) \to (\forall x B \to \forall x C)$, then suppose by reductio that there are a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I\rangle$ on $\mathcal{F}$, a world w and a w-assignment σ such that $\vDash_w^\sigma \forall x(B \to C)$, $\vDash_w^\sigma \forall x B$ and $\vDash_w^\sigma \exists x \neg C$. Thus (i) $\mathcal{M} \vDash_w^\tau B \to C$ for any x-variant τ of σ such that $\tau(x) \in D_w$; (ii) $\mathcal{M} \vDash_w^\vartheta B$ for any x-variant ϑ of σ such that $\vartheta(x) \in D_w$; (iii) $\mathcal{M} \nvDash_w^\upsilon C$ for some x-variant υ of σ such that $\upsilon(x) \in D_w$. From (i) it follows that for any x-variant τ of σ such that $\tau(x) \in D_w$, either $\mathcal{M} \nvDash_w^\tau B$, contradicting (ii), or $\mathcal{M} \vDash_w^\tau C$, contradicting (iii).

If A has the form $\Box B \wedge s = t \to \Box(B \wedge s = t)$, then assume that there are a model, a world and an assignment such that $\vDash_w^\sigma \Box B \wedge s = t$. This entails the existence of a relation R_i such that $R_i(w) \subseteq \|B\|_I^\sigma$. Take any $v \in R_i(w)$. If t (or s) is an individual constant, then $I_w^\sigma(t) = I_v^\sigma(t)$, since constants are rigid designators; on the other hand if t (or s) is a variable, than, since $U_w \subseteq U_v$, it holds that $I_w^\sigma(t) = \sigma(t) = I_v^\sigma(t)$. Thus $\vDash_w^\sigma \Box(B \wedge (t = s))$.

(b) If $k = n+1$, then A is an axiom (see previous cases) or it has been obtained either via **MP**, or via **RM**, or else via **UG**. For the first two cases, please refer to Theorem 2.3.9. Let us focus on the latter case. If A has been obtained via the rule **UG**, it has the form $\forall x B$ and it has been derived applying **UG** to a formula B. By IH, $\vDash_w^\sigma B$ for any valuation I, any world w and any assignment σ. In particular this holds true for any assignment τ which is an x-variant of σ such that $\tau(x) \in D_w$ and hence $\vDash_w^\sigma \forall xB$.

∎

3.6.2 Some Auxiliary Results

Following Corsi (2002), let us establish some auxiliary results.

Definition 3.6.2 *Let* L *be a logic on the language* $\mathcal{L}$, $\Delta \subseteq \mathsf{L}$ *and* $Q \subseteq C(\mathcal{L})$.

- Δ *is* L*-consistent iff* $\Delta \nvdash_{\mathsf{L}} \bot$.
- Δ *is* L*-deductively closed iff for any sentence* A *of* $\mathcal{L}$, $\Delta \vdash_{\mathsf{L}} A$ *iff* $A \in \Delta$.
- Δ *is* L*-complete iff for any sentence* A *of* $\mathcal{L}$, *either* $A \in \Delta$ *or* $\neg A \in \Delta$.
- Δ *is* L*-maximal iff* Δ *is* L*-consistent and* L*-complete.*
- Δ *is* Q*-universal iff if* $\forall x A(x) \in \Delta$, *then* $A(c/x) \in \Delta$, *for all individual constants* $c \in Q$.
- Δ *is* Q*-existential iff if* $A(c/x) \in \Delta$ *for some individual constant* $c \in Q$, *then* $\exists x A(x) \in \Delta$.
- Δ *is* Q*-inductive iff if* $A(c/x) \in \Delta$ *for all individual constants* $c \in Q$, *then* $\forall x A(x) \in \Delta$.
- Δ *is* Q*-rich iff if* $\exists x A(x) \in \Delta$ *for some individual constant* $c \in Q$, *then* $A(c/x) \in \Delta$.
- Δ *is* L*-saturated iff if* Δ *is* L*-maximal and for some* $Q \subseteq Const(\mathcal{L})$, Δ *is* Q*-universal and* Q*-rich.*

Lemma 3.6.3 *Let* $L \subseteq \mathsf{Q}^\circ_=.\mathsf{MN}$ *be a logic on* $\mathcal{L}$. *Let* C *be a denumerable set of individual constants not occurring in* $\mathcal{L}$, *let* $\mathcal{L}^C$ *be the language obtained adding* C *to* $C(\mathcal{L})$ *and* L^C *be the logic on* $\mathcal{L}^C$.

i. If $\vdash_{\mathsf{L}^C} A(c_1, \ldots, c_n)$, *then* $\vdash_{\mathsf{L}^C} A(x_1/c_1, \ldots, x_n/c_n)$ *where* $x_1, \ldots, x_n$ *are variables not occurring in* $A(c_1, \ldots, c_n)$.

ii. If $\vdash_{\mathsf{L}^C} A$ *and no constants of* C *occur in* A, *then* $\vdash_{\mathsf{L}} A$.

iii. If Δ is an L*-consistent set of sentences and no constant from C occurs in Δ, then Δ is* L^C*-consistent. (Cf. (Corsi, 2002, Lemma 1.6))*

Lemma 3.6.4 *Given an* L*-maximal set Δ of sentences in $\mathcal{L}$ and $Q \subseteq C(\mathcal{L})$, if Δ is Q-universal then Δ is Q-existential.(Cf. (Corsi, 2002, Lemma 1.11))*

PROOF. Assume $A(c/x) \in \Delta$ for some individual constant $c \in C(\mathcal{L})$ and $\exists x A \notin \Delta$. Since Δ is L-maximal, $\neg\exists x A(x) \in \Delta$ and hence $\forall x \neg A(x) \in \Delta$. Thus, since Δ is Q-universal by definition, $\neg A(c/x) \in \Delta$ and hence $\perp \in \Delta$, contradicting the consistency of Δ. ∎

Lemma 3.6.5 (Lindenbaum's Lemma) *Given a logic* L*, if Δ is a* L*-consistent set of formulae, then there is a* L*-maximal set Δ^+ such that $\Delta \subseteq \Delta^+$.*

Lemma 3.6.6 *Let Δ be an* L*-consistent set of sentences of $\mathcal{L}$. Then for some not-empty denumerable set C of new constants, there is a set Π of sentences of $\mathcal{L}^C$ such that $\Delta \subseteq \Pi$, Π is* L^C*-maximal, Π is Q-universal and Q-rich for some set $Q \subseteq C(\mathcal{L}^C)$. (cf. (Corsi, 2002, Lemma 1.16))*

3.6.3 Canonical models

In order to define canonical models for a language with identity, we need to introduce a binary equivalence relation on individual constants:

$$a \sim b \text{ if and only if } (a = b) \in w$$

Any individual constant c may be interpreted on its equivalence class $[c]$, where $[c] := \{a \mid c \sim a\}$.

Definition 3.6.7 (Non normal canonical model for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$) *Let $\mathsf{L} \supseteq \mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ be a logic based on $\mathcal{L}$. Let V be a set of constants with cardinality $\aleph_0$ such that $V \supset Const(\mathcal{L})$ e $\mid V - Const(\mathcal{L}) \mid = \aleph_0$. A non normal canonical model for* L *is a tuple $\mathcal{M} = \langle W, R_1, \ldots, R_n, D, I \rangle$ such that:*

- *W is the class of all L_w-saturated sets of sentences w, where $\mathcal{L}_w = \mathcal{L}^S$ for some set S of constants such that $C(\mathcal{L}^S) \neq \varnothing$, $S \subset Q$ and $\| Q - C(\mathcal{L}^S) \| = \aleph_0$.*

- *For any formula $A \in Fma(\mathcal{L})$ let R_A be a binary relation over W. For all $w, v \in W$, $wR_A v$ iff $\Box A \in w \Rightarrow A \in v$ and for any constant $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$, where $[c]_v = \{b \in Const(\mathcal{L}_v) : (b = c) \in v\}$. The set of relations $\mathcal{R}$ is the collections of all such relations.*

- $D_w = \{[c]_w : \exists x(x = c) \in w\}$.

- $U_w = \{[c]_w : c \in const(\mathcal{L}_w)\}$

- $I_w(c) = [c]_w$.

- $I_w(P^n) = \{\langle [c_1]_w, ..., [c_n]_w \rangle : P^n(c_1, ..., c_n) \in w\}$.

Lemma 3.6.8 (Existence lemma) *Given a canonical model $\mathcal{M}$ for $\mathsf{Q}^\circ_=.\mathsf{MN}$, for any $w \in W$, if $\Diamond A \in w$, then for any formula $B \in Fma(\mathcal{L})$ there is a set v such that:*

1. *v belongs to the base set of $\mathcal{M}$*
2. *$A \in v$*
3. *If $\Box B \in w$ then $B \in v$*
4. *$Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$*
5. *for any individual constant $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$.*

Proof. Below we shall define a procedure to construct any such v for any formula B.

Let C be a denumerable set of constants which do not belong to $\mathcal{L}_w$ and let $\mathcal{L}_w^C = \mathcal{L}_w \cup C$. Let $H_1, H_2, H_3, ...$ be an enumeration of all the existential formulae of $\mathcal{L}_w^C$ with infinite repetitions.

Let Γ be a chain of sets as defined below:

1. Γ_0:

   (a) If $\Box B \in w$, then $\Gamma_0 := \{B\} \cup \{A\}$

   (b) otherwise, $\Gamma_0 := \{A\}$.

2. $\Gamma_1 := \Gamma_0 \cup \{(a = b) \mid (a = b) \in w\}$

3. Let Γ_n be already defined for $1 \leq n$ and let $H_{n+1} = \exists x F(x)$. The set Γ_{n+1} is defined as:

   (a) If $\exists x F(x)$ contains at least one constant c such that $c \notin Const(\Gamma_n)$, then $\Gamma_{n+1} := \Gamma_n$.

   (b) otherwise, if any constant occurring within $\exists x F(x)$ is already in Γ_n, then there are a few cases:

   (b.1) If $\Gamma_n \cup \{\exists x F(x)\}$ is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent, then

   (b.1.1) $\Gamma_{n+1} := \Gamma_n \cup \{\exists x F(x)\} \cup \{F(b/x)\}$, where $b \in Const(\Gamma_n)$ and $\Gamma_n \cup \{F(b/x)\}$ is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent;

   (b.1.2) $\Gamma_{n+1} := \Gamma_n \cup \{\exists x F(x)\} \cup \{F(c/x)\} \cup \{(c \neq b) \mid b \in Const(\Gamma_n)\}$, if $\Gamma_n \cup \{F(b/x)\}$ is not $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent for any $b \in Const(\Gamma_n)$, and $c \in C$ is a constant not occurring in Γ_n.

   (b.2) Otherwise if $\Gamma_n \cup \{\exists x F(x)\}$ is not $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent, then $\Gamma_{n+1} = \Gamma_n$.

4. $\Gamma = \bigcup_{n \in \mathcal{N}}(\Gamma_n)$.

In order to show that $\Gamma$is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent, we have to check that for any n, Γ_n is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent.

1. Γ_0 is consistent. Indeed:

   (a) If $\Gamma_0 := \{A, B\}$ and $\Box B \in w$, then suppose $\vdash_{\mathsf{MN}} A \wedge B \rightarrow \bot$, then $\vdash_{\mathsf{MN}} B \rightarrow (A \rightarrow \bot)$, $\vdash_{\mathsf{MN}} B \rightarrow \neg A$ and $\vdash_{\mathsf{MN}} \Box B \rightarrow \Box \neg A$ by the **RM** rule. By definition, it follows that $w \vdash_{\mathbf{MN}} \Box \neg A$, $\neg \Diamond A \in w$, leading to a contradiction as w is consistent by assumption.

(b) If $\Gamma_0 := \{A\}$, then assume it is not consistent and hence $\vdash_{\mathsf{MN}} A \to \bot$, i.e., $\vdash_{\mathsf{MN}} \neg A$. Thus by the necessitation rule we have $\vdash_{\mathsf{MN}} \Box\neg A$ and this implies that $\neg \Diamond A \in w$ and hence $\Diamond A \notin w$, which is in contradiction with our hypothesis.

2. Γ_1 is consistent. Indeed assume it is not. There are two cases:

(a) $\Gamma_0 := \{A, B\}$ and $\Box B \in w$, then $\vdash_{\mathsf{MN}} B \to (a = b \to (A \to \bot))$ for some $a = b \in w$. Then $\vdash_{\mathsf{MN}} B \to (a = b \to \neg A)$, and $\vdash_{\mathsf{MN}} \Box B \to \Box((a = b) \to \neg A)$ by the **RM** rule. By definition of w and **MP**, it follows that $w \vdash_{\mathbf{MN}} \Box((a = b) \to \neg A)$. Thus by the **RM** rule $\vdash_{\mathsf{MN}} \Box(a = b) \to \Box\neg A$. But $\Box(a = b) \in w$ by the **NI*** schema, hence $\neg \Diamond A \in w$, $\Diamond A \notin w$, which contradicts the hypothesis.

(b) If $\Gamma_0 := \{A\}$, then $\vdash_{\mathsf{MN}} (a = b) \to A \to \bot$, i.e., $\vdash_{\mathsf{MN}} (a = b) \to \neg A$ for some $(a = b) \in w$. Thus by the **RM** rule $\vdash_{\mathsf{MN}} \Box(a = b) \to \Box\neg A$. But $\Box(a = b) \in w$ by the **NI** schema, hence $\neg \Diamond A \in w$, $\Diamond A \notin w$, which contradicts the hypothesis.

3. Γ_{n+1} is also consistent, in fact:

(a) Γ_{n+1} is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent by Inductive Hypothesis (IH henceforth).

(b.1.1) Γ_{n+1} is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent by construction.

(b.1.2) $\Gamma_n \cup \{\exists x F(x)\}$ is consistent by hypothesis. First of all, let us show that $\Gamma_n \cup \{F(c/x)\}$ is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent. Assume by reductio that $\Gamma_n \cup \{F(c/x)\}$ is not $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent. Then there are sentences $\{D_1, ..., D_k\} \in \Gamma_n$ such that

$\vdash_{\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}} D_1 \wedge ... \wedge D_k \wedge F(c/x) \to \bot$

$\Gamma_n \vdash_{\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}} \neg(F(c/x)$

$\Gamma_n \vdash_{\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}} \neg F(y/c)$

$\Gamma_n \vdash_{\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}} \forall y \neg F(y/c)$

$\Gamma_n \vdash_{\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}} \neg\exists y F(y)$

contrary to the fact that $\Gamma_n \cup \{\exists x F(x)\}$ is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent by hypothesis, thus $\Gamma_n \cup \{F(c/x)\}$ is $\mathsf{L}^{\mathsf{C}}_{\mathsf{w}}$-consistent.

Assume by reductio that Γ_{n+1} is not $\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}$-consistent. Hence for some finite set of individual constants $\{b_1, ..., b_h\} \subseteq Const(\Gamma_n)$,

$\Gamma_n \vdash_{\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}} F(c/x) \wedge (c \neq b_1 \wedge ... \wedge c \neq b_h) \rightarrow \bot$

$\Gamma_n \vdash_{\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}} F(c/x) \rightarrow \neg(c \neq b_1 \wedge ... \wedge c \neq b_h)$

$\Gamma_n \vdash_{\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}} F(c/x) \rightarrow (c = b_1 \vee ... \vee c = b_h)$

$\Gamma_n \cup \{F(c/x)\} \vdash_{\mathsf{L}_{w}^{C}} c = b_1 \vee ... \vee c = b_h$

hence for some i, $1 \leq i \leq h$, $\Gamma_n \cup \{F(b_i/x)\}$ is $\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}$-consistent, in contradiction with the assumption that there is no constant $b \in Const(\Gamma_n)$ such that $\Gamma_n \cup \{F(b/x)\}$ is $\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}$-consistent. Therefore Γ_{n+1} is $\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}$-consistent.

(b) (b.2) Γ_{n+1} is $\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}$-consistent by IH.

4. Γ è $\mathsf{L}_{\mathsf{w}}^{\mathsf{C}}$-consistent by the Chain Lemma.

Let $Q = Const(\Gamma)$. We start by showing that

($*$) For any existential formula $\exists x F(x)$ of $\mathcal{L}_w^Q$ there is some Γ_k such that either $\exists x F(x) \in \Gamma_{k+1}$ or $\Gamma_k \cup \{\exists x F(x)\}$ is $\mathcal{L}_w^C$-inconsistent.

Let $c_1, \ldots, c_j$ be all the constants occurring in $\exists x F(x)$. Since $\exists x F(x) \in \mathcal{L}_w^Q$, $\{c_1, \ldots, c_j\} \subseteq Q$, hence for some j, $\{c_1, \ldots, c_j\} \subseteq Const(\Gamma_j)$. Since $\exists x F(x)$ occurs infinitely many times within $H_1, H_2, H_3, ...$, then $\exists x F(x) = H_k$ for some $k > j$. Therefore the (b) step of our construction is applied to $\exists x F(x)$ and hence ($*$) is proved. It follows that Γ è $\mathcal{L}_w^Q$-rich.

The set Γ can be extended to some v which is L_w^Q-consistent and L_w^Q- maximal. Let $\mathcal{L}_v = \mathcal{L}_w^Q$. The extension v does not compromise richness. Indeed, if an existential formula of $\mathcal{L}_w^Q$ belongs to v, by ($*$) it is also in Γ, and hence some exemplification of it is also in Γ and sinde $\Gamma \subseteq v$, it is also in v. Therefore v is $L_v^{=}$-saturated and it hence belong to the canonical base set. Therefore:

1. $v \in W$.

2. Since $A \in \Gamma_0$ and $v \supseteq \bigcup_{n \in \mathcal{N}}(\Gamma_n)$, it follows that $A \in v$.

3. Moreover it holds true that if $\Box B \in w$, then $B \in v$

4. Since $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_w^Q)$ it holds that $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$.

5. The set v is L_w^Q-maximal: the only constants occurring in v are those already present in Γ. Since by construction $\{(a = b) \mid (a = b) \in w\} \subseteq \Gamma$, it follows that for any b, $[b]_w \subseteq [b]_v$. On the other hand, suppose that there is some c from $Const(\mathcal{L}_v)$ which does not occur in $Const(\mathcal{L}_w)$. Then c has been added at some point in the construction of the Γ_j sets. The step (iii.b.1.2) is the only possible way to add the new constant c to Γ_{j+1} and it guarantees that for any $d \in Const(\mathcal{L}(\Gamma_j))$, $c \neq d \in \Gamma_{j+1}$, so $c \notin [b]_v$ for any $b \in Const(\mathcal{L}_w)$.

∎

Lemma 3.6.9 (Truth Lemma) *Given a canonical model $\mathcal{M}^{\mathsf{L}} = \langle W, \mathcal{R}, D, U, I\rangle$ for a quantified* $\mathbf{N}$*-Monotonic modal logic* L *extending* $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$*, for any formula $A \in Fma(\mathcal{L})$, for any world $w \in W$, the following holds: $\vDash_w^\sigma A(x_i) \quad \Leftrightarrow \quad A(\sigma(x_i)/x_i) \in w$*

Proof. By induction on the length of a formula A. We omit details of the induction base.

Suppose $lg(A) = n + 1$ and A has the form $\exists x B(x, y_1, \ldots, y_m)$.
(i) $\vDash_w^\sigma \exists x B(x, y_1, \ldots, y_m)$ iff for some x-variant τ of σ such that $\tau(x) \in D_w$, $\vDash_w^\tau B(x, y_1, \ldots, y_m)$. Suppose $\tau(x) = c$. Since by assumption all constants are rigid designators, by Lemma 3.3.7 it holds that $\vDash_w^\sigma B(c/x, y_1, \ldots, y_m)$. Hence $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ by IH. Since w is D_w-universal, w is also D_w-existential by Lemma 3.6.4 and hence $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$. (ii) Assume $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$. Since w is D_w-rich, $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ for some $c \in D_w$. Thus, by IH, $\vDash_w^\sigma B(c/x, y_1, \ldots, y_m)$ and, by Lemma 3.3.7 it holds that $\vDash_w^\tau B(x, y_1, \ldots, y_m)$ for some w-assignment τ which is an x-variant of σ such that

$\tau(x) = I_w(c) = c$. Therefore $\vDash^\sigma_w \exists x B(x, y_1, \ldots, y_m)$. If $lg(A) = n + 1$ and A has the form $\Box B$, please refer to Lemma 2.3.14. ∎

Lemma 3.6.10 *Let $\mathcal{M}^{\mathsf{L}} := \langle W, \mathcal{R}, D, U, I\rangle$ be a canonical model for a logic $\mathsf{L} \supseteq \mathsf{Q}^{\circ}_{=}.\mathsf{MN}$. If Δ is an* L*-consistent set of formulae, then for some $w \in W$ and some w-assignment σ, $\mathcal{M}^{\mathsf{L}} \vDash^\sigma_w D$ for any $D \in \Delta$. (cf. (Corsi, 2002, Lemma 1.19))*

Let L be any logic $L \supseteq \mathsf{Q}^{\circ}_{=}.\mathsf{MN}$. Consider any formula A such that $\nvdash_{\mathsf{L}} A$. Then $\{\neg A\}$ is L-consistent. By Lemma 3.8.6 there is a world w of a canonical model $\mathcal{M}^{\mathsf{L}}$ for L and a w-assignment σ such that $\mathcal{M}^{\mathsf{L}} \nvDash^\sigma_w A$ and hence $\mathcal{M}^{\mathsf{L}} \nvDash A$.

Corollary 3.6.11 (Completeness of $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$) *The logic $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ is strongly complete with respect to the class of all multi-relational frames.*

Recalling the results given in Chapter 2, we can state the following:

Corollary 3.6.12 *The logic $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}\oplus$*

- **T** *is complete with respect to the class of generally reflexive multi-relational frames, i.e., for any world w, for any relation R_i, wR_iw;*
- **CON** *is complete with respect to the class of generally serial multi-relational frames;*
- **D** *is complete with respect to the class of generally serial multi-relational frames;*
- **D** *is complete with respect to the class of generally serial multi-relational frames fulfilling the following condition:*
  for any $w \in W$ and any pair of relations R_i and R_j, $R_i(w) \cap R_j(w) \neq \varnothing$.

3.6.4 Completeness of $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus$ CBF

Turning our attention to the **CBF** schema, we can show that the logic $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus$ **CBF** is sound and complete with respect to an interesting class of frames. Again, this is but a generalisation of the Kripkean case. Recall that in Kripke semantics the logic $\mathsf{Q}^{\circ}_{=}.\mathsf{K} \oplus \mathbf{CBF}$ is sound and complete with respect to the frames with increasing inner domains, i.e., for any worlds w, v, if wRv, then $D_w \subseteq D_v$. In the multi-relational case, the result is, again, more general. Adding **CBF** to class of axioms gives the following result. If we take any world which is connected to at least one point for any relation (i.e., is locally serial) and we consider any actual individual (i.e., belonging to the inner domain of the world), we can show that thank to the schema **CBF**, such individual *survives* somewhere, namely each each relation R_i contains at least one point in which such individual belongs to the inner domain. This is stated formally in the following theorem:

Theorem 3.6.13 (Completeness of $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus \mathbf{CBF}$) *The logic $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus \mathbf{CBF}$ is complete with respect to the class of multi-relational frames with the following property. For any world w, if w is locally serial, i.e., for any i, $R_i(w) \neq \varnothing$, then for any relation R_i*

$$D_w \subseteq \bigcup\{D_v \mid v \in R_i(w)\}$$

Proof. Let $\mathcal{M}$ be a canonical model for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus \mathbf{CBF}$. Take any world w and assume it is locally serial. Then the schema $\Diamond\top$ belongs to w. By lemma 3.6.8, it follows that for any formula $B_i \in Fma(\mathcal{L})$ there is a world v_i such that $wR_{B_i}v_i$ and $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_{v_i})$ and for any individual constant $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_{v_i}$. This implies that $U_w \subseteq U_{v_i}$. Take any $[c] \in D_w$; by definition it holds that $\exists x(x = c) \in w$. From these facts it follows that $\vDash^{\sigma}_{w} x = c$ for some σ such that $\sigma(x) \in D_w$ and, moreover, for each i, $\vDash^{\sigma}_{v_i} x = c$. Hence $\vDash^{\sigma}_{w} \Diamond(x = c)$ and since $\sigma(x) \in D_w$, $\exists x \Diamond (x = c) \in w$. By CBF $\Diamond\exists x(x = c) \in w$. This means that for every *existing* individual b from D_w, for any relation R_B, there is a world $t \in R_B(w)$ such that $b \in D_t$ and therefore for any relation R_i, $D_w \subseteq \bigcup\{D_v \mid v \in R_i(w)\}$. ∎

3.7 The Barcan Formula: A Work in Progress

3.7.1 BF and CGF: Semantic Considerations

We have previously argued that, if we read *de re* and *de dicto* deontic statements as contextual and non-contextual obligations, respectively, then **CBF** seems less controversial than **BF**. Additional technical difficulties make **BF** harder to handle. The remainder of this Section provides some first preliminary results.

As for the case of **CBF**, the non validity of the **BF** schema follows immediately from its analogous result in Kripke semantics.

Lemma 3.7.1 **BF** *is not valid in the class of multi-relational frames.*

PROOF. Consider the following model: $\mathcal{M} := \langle W, R_1, D, U, I\rangle$ where $W := \{w, z\}$, $R_1 := \{\langle w, z\rangle\}$, $D_w = \{a\}$, $D_z = \{b\}$, $U_w = U_z = \{a, b\}$, $P(a) \in I_z(P)$ and $P(b) \notin I_z(P)$. Then it holds that for any $x - variant$ of a σ-assignment τ such that $\tau(x) \in D_w$, $\vDash^{\tau}_{w} \Box P(x)$ and hence $\vDash^{\sigma}_{w} \forall x \Box P(x)$. Moreover there is an x-variant θ of σ such that $\theta(x) \in D_z$ and $P(x) \in I^{\theta}_{z}(P)$, i.e. $\theta(x) = b$ and since wR_1z it holds that $\vDash^{\sigma}_{w} \Diamond\exists x \neg P(x)$. (See Figure 3.2) ∎

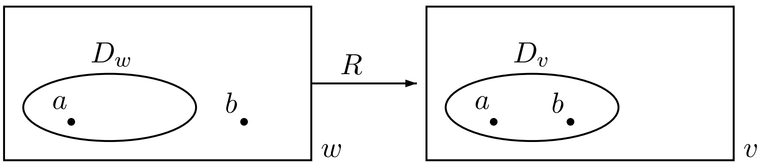

Figure 3.2: A Kripke frame to build a countermodel for BF.

Lemma 3.7.2 (BF Characterisation Result) *For any multi-relational frame* $\mathcal{F} := \langle W, \mathcal{R}, D, U\rangle$, $\mathcal{F} \vDash \forall x \Box A \to \Box\forall x A$ *iff* $\forall w \in W$, *either there is an* $R_i \in \mathcal{R}$ *such that* $R_i(w) = \varnothing$ *or for any* n*-tuple* $\langle z_1, \ldots, z_n\rangle$ *such that* n *is the cardinality of* $\mathcal{R}$ *and* wR_kz_k *for any* $R_k \in \mathcal{R}$, *for any* m*-tuple* $\langle R_{j_1}, \ldots, R_{j_m}\rangle$ *where* $\|D_w\| = m$, *i.e.* m *is the number of* *individuals belonging to the inner domain of* w*, there is a world* $t \in \{z_1, z_2, \ldots\} \cap R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$ *such that* $D_w \supseteq D_t$*.*

Proof. ($\Rightarrow$) Assume that $\exists w \in W$ such that for all $R_i \in \mathcal{R}$, $R_i(w) \neq \varnothing$ and for some $\langle z_1, \ldots, z_n \rangle$ such that $wR_k z_k$ for any $R_k \in \mathcal{R}$, for some m-tuple of relations $\langle R_{j_1}, \ldots, R_{j_m} \rangle$ the following holds: for all $t \in W$, if $t \in \{z_1, z_2, \ldots\} \cap R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$ then $D_w \subset D_t$. We shall now define an interpretation I in order to build a countermodel for **BF** in this frame. Let $D_w := \{d_1, \ldots, d_m\}$ and consider the w-assignments $\sigma_1, \ldots, \sigma_m$ such that for any i, $1 \leq i \leq m$, $\sigma_i(x) = d_i$. For some unary predicate P, for any i, let $\|P(x)\|_I^{\sigma_i} := R_{j_i}(w)$. Then clearly $\vDash_w^{\sigma_i} \Box P(x)$ for each i and hence, since $\sigma_1, \ldots, \sigma_m$ are all the x-variant of a w-assignment σ such that $\sigma_i \in D_w$ for each i, $\vDash_w^{\sigma} \forall x \Box P(x)$. Consider now any z_i from $\langle z_1, \ldots, z_n \rangle$ and define a z_i-assignment ϑ_i such that $\vartheta_i(x) \notin I_{z_i}^{\vartheta_i}(P(x))$ and $\vartheta_i(x) \in D_{z_i}$. Notice that it is always possible to define such an assignment. In fact even if $z_i \in R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$, by assumption we have that $D_{z_i} \supset D_w$. Hence $\vDash_{z_i}^{\vartheta_i} \neg P(x)$ and hence $\vDash_{z_i}^{\sigma} \neg \exists x P(x)$ which implies that $\nvDash_w^{\sigma} \Box \forall x P(x)$.

($\Leftarrow$) Assume that for some frame $\mathcal{F}$, $\mathcal{F} \nvDash \forall x \Box A \rightarrow \Box \forall x A$ for some formula A. Then there are a world w, an interpretation I and an assignment σ such that (a) $\vDash_w^{\sigma} \forall x \Box A(x)$ and (b) $\vDash_w^{\sigma} \Diamond \exists x \neg A(x)$. Given that $D_w := \{d_1, \ldots, d_m\}$, from (a) it follows that there are $\sigma_1, \ldots, \sigma_m$ w-assignments which are all the x-variants of σ such that $\sigma_i(x) \in D_w$ and $\vDash_w^{\sigma_i} \Box A(x)$. Thus there is a set of relations $R_1, \ldots, R_m$ such that for any R_i, $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$.

From (b) it follows that there is an n-tuple $\langle z_1, \ldots, z_n \rangle$ of worlds such that for each R_i, $wR_i z_i$ (hence for each R_i, $R_i(w) \neq \varnothing$) and for each i, $\vDash_{z_i}^{\sigma} \exists x \neg A(x)$. Thus for each z_i there is some z_i-assignment ϑ_i such that $\vartheta_i(x) \in D_{z_i}$ and $\vDash_{z_i}^{\vartheta_i} \exists x \neg A(x)$.

Clearly all the worlds t belonging to the intersection $R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$ are such that $\vDash_t^{\sigma_i} A(x)$ for all the x-variants of σ such that $\sigma_i(x) \in D_w$ and hence if a world z_i belongs to such intersection, the z_i-assignment ϑ_i must be such that $\vartheta_i(x) \in D_{z_i}$ but $\vartheta_i(x) \notin D_w$ and therefore for any such world $D_w \subset D_{z_i}$. ∎

Again, mirroring Kripke semantics, the schemata **BF** and **CGF** $:= \Box\exists xA \to \exists x \Box A$ are equivalent.

Lemma 3.7.3 *The schemata* **BF** *and* **CGF** *characterise the same class of multi-relational frames.*

PROOF. ($\Rightarrow$) Assume that $\exists w \in W$ such that for all $R_i \in \mathcal{R}$, $R_i(w) \neq \varnothing$ and for some $\langle z_1, \ldots, z_n \rangle$ such that $wR_k z_k$ for any $R_k \in \mathcal{R}$, for some m-tuple of relations $\langle R_{j_1}, \ldots, R_{j_m} \rangle$, where m is the number of individuals belonging to D_w, the following holds: for all $t \in W$, if $t \in \{z_1, z_2, \ldots\} \cap R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$ then $D_w \subset D_t$.

We shall now define an interpretation I in order to build a countermodel for CGF in this frame. Let $D_w := \{d_1, \ldots, d_m\}$ and consider the w-assignments $\sigma_1, \ldots, \sigma_m$ such that for any i, $1 \leq i \leq m$, $\sigma_i(x) = d_i$. For some unary predicate P, for any i, let $\|P(x)\|_I^{\sigma_i} := R_{j_i}(w)$. Then clearly $\vDash_w^{\sigma_i} \Box P(x)$ for each i and hence, since $\sigma_1, \ldots, \sigma_m$ are all the x-variant of a w-assignment σ such that $\sigma_i \in D_w$ for each i, $\vDash_w^{\sigma} \forall x \Box P(x)$.

Consider now any z_i from $\langle z_1, \ldots, z_n \rangle$ and define a z_i-assignment ϑ_i such that $\vartheta_i(x) \notin I_{z_i}^{\vartheta_i}(P(x))$ and $\vartheta_i(x) \in D_{z_i}$. Notice that it is always possible to define such an assignment. In fact even if $z_i \in R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$, by assumption we have that $D_{z_i} \supset D_w$. Hence $\vDash_{z_i}^{\vartheta_i} \neg P(x)$ and hence $\vDash_{z_i}^{\sigma} \neg\exists xP(x)$ which implies that $\nvDash_w^{\sigma} \Box\forall xP(x)$.

($\Leftarrow$) Assume that for some frame $\mathcal{F}$, $\mathcal{F} \nvDash \forall x \Box A \to \Box\forall xA$ for some formula A. Then there are a world w, an interpretation I and an assignment σ such that (a) $\vDash_w^{\sigma} \forall x \Box A(x)$ and (b) $\vDash_w^{\sigma} \Diamond\exists x\neg A(x)$. Given that $D_w := \{d_1, \ldots, d_m\}$, from (a) it follows that there are $\sigma_1, \ldots, \sigma_m$ w-assignments which are all the x-variants of σ such that $\sigma_i(x) \in D_w$ and $\vDash_w^{\sigma_i} \Box A(x)$. Thus there is a set of relations $R_1, \ldots, R_m$ such that for any R_i, $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$.

From (b) it follows that there is an n-tuple $\langle z_1, \ldots, z_n \rangle$ of worlds such that for each R_i, wR_iz_i (hence for each R_i, $R_i(w) \neq \varnothing$) and for each i, $\vDash_{z_i}^{\sigma} \exists x\neg A(x)$. Thus for each z_i there is some z_i-assignment ϑ_i such that $\vartheta_i(x) \in D_{z_i}$ and $\vDash_{z_i}^{\vartheta_i} \exists x\neg A(x)$.

Clearly all the worlds t belonging to the intersection $R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$ are such that $\vDash_t^{\sigma_i} A(x)$ for all the x-variants of σ such that $\sigma_i(x) \in D_w$ and hence if a world z_i belongs to such intersection, the z_i-assignment ϑ_i must be such that $\vartheta_i(x) \in D_{z_i}$ but $\vartheta_i(x) \notin D_w$ and therefore for any such world $D_w \subset D_{z_i}$. ∎

By translating in terms of multi-relational semantics also Stolpe (2003)'s CUPI condition (see Theorem 3.2.8) it is quite straightforward to prove a weaker correspondence result for **BF** in multi-relational models.

A weaker correspondence result for the schema BF can be obtained by rephrasing the CUPI schema introduced by Stolpe Stolpe (2003). However this implies conditions on models instead of frames.

Lemma 3.7.4 (CUPI models) *For any multi-relational model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$, $\mathcal{M} \vDash \forall x \Box A \rightarrow \Box \forall x A$ if and only if for any world w the following holds: given a w-assignment σ, if for any x-variant σ_i such that $\sigma_i(x) \in D_w$ there exists a relation R_i such that $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$, then there is a relation R_k such that $R_k(w) \subseteq \|\forall x A\|_I^{\sigma}$.*

Proof. The left to right arrow is trivial. For the other direction consider the contrapositive proposition. Assume that for some world $w \in W$ it holds true that, for all w-assignments σ, (a) for any x-variant σ_i of σ, $\sigma_i(x) \in D_w$, there exists a relation R_i such that $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$ and (b) for any relation R_k, $R_k(w) \nsubseteq \|\forall x A\|_I^{\sigma}$. From (a) it follows that for any σ_i, $\vDash_w^{\sigma_i} \Box A(x)$ and hence $\vDash_w^{\sigma} \forall x \Box A(x)$ whereas from (b) it follows that $\nvDash_w^{\sigma} \Box \forall x A$, thus $\nvDash_w^{\sigma} \forall x \Box A \rightarrow \Box \forall x A$. ∎

3.7.2 The independence of the schemata C and BF

Recall that the schema **C** imposes on frames the following property: $\mathcal{F} \vDash \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ iff for any world w, for any relation R_i, R_k there exists a relation R_j such that $R_j(w) \subseteq R_k(w) \cap R_i(w)$. Hence for any **C**-frame, the characterisation result for **BF** is the following:

Lemma 3.7.5 (BF on C-frames) *Let $\mathcal{F}$ be any* **C***-frame. Then for any world $w \in W$, given that $\|D_w\| = k$, if w is locally serial, then for all sets of relations with cardinality up to k $\{R^i_1, \ldots, R^i_k\}$, for all worlds $v \in R^i_1 \cap \ldots \cap R^i_k$ it holds that $D_w \supseteq D_v$.*

Proof. Indeed if we assume that the **BF**-condition does not hold on a **C**-frame, we would get the following situation. For some world w such that w is locally serial and $\|D_w\| = k$, there is some set $R_1, \ldots, R_k$ of relations such that for some world $v \in R_1(w) \cap \ldots \cap R_k(w)$ it holds that $D_w \not\supseteq D_v$. It is easy to build a counter-model in such situation. In Kripke semantics this is equivalent to deny decreasing domains. ∎

It is easy to find a frame validating **C** but not **BF**: It is enough to consider the frame described in Lemma 3.7.1, which is Kripkean and hence validates **C**.

It is not hard to show that the schema **BF** does not imply **C**. Below we describe a countermodel for **C** based on a frame for **BF**. Consider the model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I\rangle$ where $W = \{w, z_1, z_2\}$, $\mathcal{R} := \{R_1, R_2\}$, $R_1 := \{\langle w, z_1\rangle\}$, $R_2 := \{\langle w, z_2\rangle\}$, $D_w = D_{z_1} = D_{z_2} = \{d\}$, σ is a w-assignment such that $\sigma(x) = d$. Let P and Q be two unary predicates such that $\|P(x)\|^\sigma_I = \{z_1\}$ and $\|Q(x)\|^\sigma_I = \{z_2\}$. Then $\vDash^\sigma_w \Box P(x) \wedge \Box Q(x)$ but $\nvDash^\sigma_w \Box(P(x) \wedge Q(x))$. Moreover $\mathcal{M}$ is built on a frame which fulfills the conditions imposed by **BF**. Indeed w is locally serial but there is only one tuple $\langle z_1, z_2\rangle$ such that wR_1z_1 and wR_2z_2. The number of individuals from D_w is 1, thus: $z_1 \in \{z_1, z_2\} \cap R_1(w)$ and $D_w \subseteq D_{z_1}$ and $z_2 \in \{z_1, z_2\} \cap R_2(w)$ and $D_w \subseteq D_{z_2}$. Notice that any countermodel for **C** based on a **BF**-frame must contain a locally serial world w whose inner domain contains at most one individual and the frame of such model should have more than one relation.

Corollary 3.7.6 *The schemata* **BF** *and* **C** *are semantically mutually independent.*

The importance of this result is limited at the moment, as it lays only on semantic grounds. In fact, without a completeness result, one cannot infer that adding **BF** to a system does not generate normal systems, i.e., it is not clear wether **BF** does or does not syntactically imply **C**.

3.8 The Role of Identity

The choice of a language with the identity relation was driven by different reasons. First of all, the expressive power increases greatly. Nevertheless, there is also a technical reason behind it. Working with the identity symbol allowed us to build canonical models for **CBF**-frames, a very difficult goal to be achieved without it. It is actually much easier to prove completeness for systems without identity, at least for those systems which do not include any form of Barcan schema.

3.8.1 Completeness without Identity

Below we present a completeness theorem for an analogous system built on a language without identity. The axiomatic system $\mathsf{Q}°.\mathsf{MN}$ is obtained by deleting all the schemata concerning identity:

The system $\mathsf{Q}°.\mathsf{MN}$ contains the following axioms and inference rules:

- Propositional tautologies;
- $\mathbf{UI}° := \forall y(\forall x A(x) \to A(y/x))$
- $\forall x \forall y A \leftrightarrow \forall y \forall x A$
- $A \to \forall x A$, x not free in A
- $\forall x(A \to B) \to (\forall x A \to \forall x B)$
- $\mathbf{M} := \Box(A \wedge B) \to (\Box A \wedge \Box B)$
- $\mathbf{N} := \Box\top$ to $\mathsf{Q}^{\circ}_{=}.\mathsf{E}$
- $\mathbf{MP} := A \to B, A/B$
- $\mathbf{RE} := A \leftrightarrow B/\Box A \leftrightarrow \Box B$
- $\mathbf{UG} := A/\forall x A$

Theorem 3.8.1 (Soundness) *The system* $\mathsf{Q}°.\mathsf{MN}$ *is sound with respect to the class of all multi-relational frames with varying inner domains.*

PROOF. It follows from Theorem 3.6.1. ∎

Lemma 3.8.2 *Let* Δ *be an* L*-consistent set of sentences of* $\mathcal{L}$*. Then for some not-empty denumerable set* C *of new constants, there is a set* Π *of sentences of* $\mathcal{L}^C$ *such that* $\Delta \subseteq \Pi$*,* Π *is* L^C*-maximal,* Π *is* Q*-universal and* Q*-rich for some set* $Q \subseteq C(\mathcal{L}^C)$*. (cf. (Corsi, 2002, Lemma 1.16))*

Definition 3.8.3 (Non normal Canonical Models) *Let* L *be a non normal quantified modal logic on the language* $\mathcal{L}$ *such that* $L \supseteq \mathsf{Q}^\circ.\mathsf{MN}$*. Let* Q *be a set of constants of cardinality* $\aleph_0$ *such that* $Q \supset C(\mathcal{L})$ *and* $\|Q - C(\mathcal{L})\| = \aleph_0$*. A non normal canonical model* $\mathcal{M}^{\mathsf{L}} = \langle W, \mathcal{R}, D, U, I \rangle$ *for* L *is defined as follows:*

- *W is the class of all L_w-saturated sets of sentences w, where $\mathcal{L}_w = \mathcal{L}^S$ for some set S of constants such that $C(\mathcal{L}^S) \neq \varnothing$, $S \subset Q$ and $\|Q - C(\mathcal{L}^S)\| = \aleph_0$.*
- *For any formula $A \in Fma(\mathcal{L})$ let R_A be a binary relation over W. For all $w, v \in W$, wR_Av iff $\Box A \in w \Rightarrow A \in v$. The set of relations $\mathcal{R}$ is the collections of all such relations.*
- $D_w = \{c \in C(\mathcal{L}_w) \mid \forall x A \to A(c/x) \in w$, *for all sentences* $\forall x A$ *of* $\mathcal{L}_w\}$.
- $U_w = C(\mathcal{L}_w)$
- $I_w(c) = c$
- $I_w(P^n) = \{\langle c_1, \ldots, c_n \rangle \mid P^n(c_1, \ldots, c_n) \in w\}$

Lemma 3.8.4 (Existence lemma) *Given a canonical model* $\mathcal{M}$ *for* $\mathsf{Q}^\circ.\mathsf{MN}$*, for any* $w \in W$*, if* $\Diamond A \in w$*, then for any formula* $B \in Fma(\mathcal{L})$ *there is a state* z *such that* wR_Bz *and* $A \in z$*.*

The proof follows directly from Lemma 2.3.13.

Lemma 3.8.5 (Truth Lemma) *Given a canonical model* $\mathcal{M}^{\mathsf{L}} = \langle W, \mathcal{R}, D, U, I \rangle$ *for a quantified* **N**-*Monotonic modal logic* L *extending* $\mathsf{Q}^\circ.\mathsf{MN}$*, for any formula* $A \in Fma(\mathcal{L})$*, for any world* $w \in W$*, the following holds:* $\vDash^\sigma_w A(x_i) \quad \Leftrightarrow \quad A(\sigma(x_i)/x_i) \in w$

Proof. By induction on the length of a formula A. We omit details of the induction base. Suppose $lg(A) = n + 1$ and A has the form $\exists x B(x, y_1, \ldots, y_m)$.
(i) $\vDash^\sigma_w \exists x B(x, y_1, \ldots, y_m)$ iff for some x-variant τ of σ such that $\tau(x) \in D_w$, $\vDash^\tau_w B(x, y_1, \ldots, y_m)$. Suppose $\tau(x) = c$. Since by assumption all constants are rigid designators, by Lemma 3.3.7 it holds that $\vDash^\sigma_w B(c/x, y_1, \ldots, y_m)$. Hence $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ by IH. Since w is D_w-universal, w is also D_w-existential by Lemma 3.6.4 and hence $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$.

(ii) Assume $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$. Since w is D_w-rich, $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ for some $c \in D_w$. Thus, by IH, $\vDash^{\sigma}_{w} B(c/x, y_1, \ldots, y_m)$ and, by Lemma 3.3.7 it holds that $\vDash^{\tau}_{w} B(x, y_1, \ldots, y_m)$ for some w-assignment τ which is an x-variant of σ such that $\tau(x) = I_w(c) = c$. Therefore $\vDash^{\sigma}_{w} \exists x B(x, y_1, \ldots, y_m)$. If $lg(A) = n + 1$ and A has the form $\Box B$, please refer to Lemma 2.3.14. ∎

Lemma 3.8.6 *Let $\mathcal{M}^{\mathsf{L}} := \langle W, \mathcal{R}, D, U, I \rangle$ be a canonical model for a logic $L \supseteq \mathsf{Q}^{\circ}.\mathsf{MN}$. If Δ is an L-consistent set of formulae, then for some $w \in W$ and some w-assignment σ, $\mathcal{M}^{\mathsf{L}} \vDash^{\sigma}_{w} D$ for any $D \in \Delta$. (cf. (Corsi, 2002, Lemma 1.19))*

Let L be any logic $L \supseteq \mathsf{Q}^{\circ}.\mathsf{MN}$. Consider any formula A such that $\nvdash_L A$. Then $\{\neg A\}$ is L-consistent. By Lemma 3.8.6 there is a world w of a canonical model $\mathcal{M}^{\mathsf{L}}$ for L and a w-assignment σ such that $\mathcal{M}^{\mathsf{L}} \nvDash^{\sigma}_{w} A$ and hence $\mathcal{M}^{\mathsf{L}} \nvDash A$.

Theorem 3.8.7 (Completeness of $\mathsf{Q}^{\circ}.\mathsf{MN}$) *The logic $\mathsf{Q}^{\circ}.\mathsf{MN}$ is strongly complete with respect to the class of all multi-relational frames.*

Corollary 3.8.8 *The logic $\mathsf{Q}^{\circ}.\mathsf{MN}\oplus$*

- **T** *is complete with respect to the class of generally reflexive multi-relational frames, i.e., for any world w, for any relation R_i, wR_iw;*

- **CON** *is complete with respect to the class of generally serial multi-relational frames;*

- **D** *is complete with respect to the class of generally serial multi-relational frames;*

- **D** *is complete with respect to the class of generally serial multi-relational frames fulfilling the following condition:*
  for any $w \in W$ and any pair of relations R_i and R_j, $R_i(w) \cap R_j(w) \neq \varnothing$.

3.8.2 Some Remarks on Identity

The completeness proof for a system without identity follows from the propositional results given in Chapter 2, and from those stated by Corsi (2002). However, things change radically when the identity symbol is included. As Corsi (2002) observes:

> In a language with identity, the fact that constants are rigid designators can be expressed by the schema:
>
> $$(x = a) \to \Box(x = a)$$
>
> Therefore (...) all the systems of Q.M.L. with identity we are going to discuss are bound to be systems with rigid terms. (Corsi, 2002, 1499)

However this ceases to be true in the broader framework of multi-relational semantics. In fact if there is only one relation within a frame (the Kripke case), the problem of rigidity can be easily solved by stating that if two names denote the same individual in a world, then this is bound to be the case in *all* accessible worlds. On the other hand, if interpreted in the multi-relational case, this would only state that *under some standard* this couple of names denote the same individual. This is obviously not enough. Our version of the necessity of identity schema

$$\mathbf{NI} := \Box A \wedge (a = b) \to \Box(A \wedge (a = b))$$

says something more. It states that *for any formula*, i.e., semantically, for any relation, this must hold. It is a restricted form of axiom **C**, which holds only for specific formulae, namely identities. The same holds for the necessity of diversity, **ND**. The resulting system is still a proper subset of K even though it includes a restricted type of aggregation.

This may explain what happened in Lemma 3.6.8, in the construction of the set Γ_1. In the proof of Lemma 3.6.8, the set Γ_0 is built to ensure the truth of modal formulae:

$$\Gamma_0 :=$$

1. If $\Box B \in w$, then $\Gamma_0 := \{B\} \cup \{A\}$
2. otherwise, $\Gamma_0 := \{A\}$.

This is not sufficient to preserve rigidity of denotation. However, it can be amended by adding a further step in the construction:

$$\Gamma_1 := \Gamma_0 \cup \{(a = b) \mid (a = b) \in w\}$$

This further step is necessary to keep rigidity of designation as well as the validity of axiom **NI**. Moreover, the consistency of Γ_1 is guaranteed by the presence of **NI** in the system.

This is even more obvious within the proof of Theorem 3.6.13, where we make an essential use of identity formulae. In fact, when it comes to prove completeness results for systems including **CBF**, if identity is not present, it is very difficult to denote the same individual across different worlds. One may think about adding an existence predicate, but, besides the definitive philosophical objection that existence is not a predicate, this would be just a definite description in disguise. The standard way to introduce an existence predicate is to add to the language a unary predicate symbol E whose extension, for any world, is equal to the inner domain:

$$\vDash_w^\sigma E(x) \text{ if and only if } \sigma(x) \in D_w$$

This is clearly equivalent to state that $E(a)$ is satisfied in a world if and only if the formula $\exists x(x = a)$ holds. Hence, adding an existence predicate would make very little difference, if not at all, on the technical level and would not take us any closer to proof completeness for systems including **CBF**, which is still an open problem.

3.9 Conclusions and Further Work

This Chapter provided a semantic study in multi-relational semantics of quantified **N**-Monotonic modal logics with varying domains ($\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$):

- We defined multi-relational weak structures and proved soundness results for **N**-Monotonic systems;
- We provided Completeness results for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ and extensions;
- We provided Completeness results for $\mathsf{Q}^{\circ}.\mathsf{MN}$ and extensions;
- We proved semantic equivalence for the couples **CBF** - **GF**, and **BF** - **CGF**;
- We provided frame characterisation results for both **CBF**, and **BF**;
- We provided Completeness results for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus$ **CBF** and extensions;
- We showed that Schema **M** does not entail **CBF**;
- We proved that schema **C** and **BF** are semantically independent.

Problems related to the completeness of non normal systems, with and without identity, which include **BF** are still open and require a deeper analysis. Concerning systems without identity, this problem concerns both **CBF**, and **BF**. Moreover, we see other important directions for future research.

A preliminary question—somehow broader than the scope of this dissertation—regards the general relation between multi-relational and Neighborhood semantics. For propositional non-normal modal logics the two semantics are equivalent Governatori and Rotolo (2005). However, it is not obvious if they are still equivalent (in regard, e.g., to completeness and incompleteness results) above K: Normal systems do not necessarily make Governatori and Rotolo (2005)'s semantics collapse on Kripke's. Regarding quantified modal logics, a full comparison of the two semantics is anyway needed.

An immediate extension of the present work is to consider constant domains ($\mathsf{FOL} \oplus \mathsf{MN}$). It is definitely less trivial to adopt Governatori and Rotolo (2005)'s semantics and study the open problem that Arló-Costa (2011) mentioned for $\mathsf{FOL} \oplus \mathsf{E} \oplus \mathbf{CBF}$, as well as to investigate other quantified (classical, monotonic, and regular) systems with constant and varying domains.

Finally, a question concerning Kracht and Wolter (1999)'s proof that non-normal modal logics can be simulated by a normal modal logic with three modalities. This result, which holds for MN, is of great interest in the context of multi-relational semantics, whose structures can in fact recall Kripke frames for multi-modal logics. However, when predicate calculi are added to the propositional modal base we can obtain unexpected interactions between Barcan schemata and modal axioms. Hence, extending Kracht and Wolter (1999)'s case to quantified non normal modal logics is an open question.

Chapter 4

Conclusions, Applications, Future Work

4.1 Conclusions

The problems and paradoxes generating within the standard modal approach to deontic logics are relevant and numerous. Nevertheless, there are several possible solutions to amend them. We investigated further one of those, namely we studied a semantic framework to deal with non normal systems. Far from being the definitive answer to most problems related to deontic modal logics, we think, however, that the modal approach is definitely worthy of further attention and investigation. We decided to follow the multi-relational semantics paradigm, in order to limit deontic paradoxes while keeping the intuition and techniques associated to Kripke Semantics for normal modal systems. Even though the logical systems that can be treated with these tools are reacher and more powerful (and hence more problematic) than classical ones (traditionally associated to neighborhood semantics), they provide the possibility to use intuitive semantic tools.

As we saw in Chapter 1, the choice of studying non normal deontic systems was driven by the consideration that several philosophical issues related to deontic paradoxes originate within the syntax of given systems, such as *deontic explosion*. There are different ways to deduce problematic schemata, and several solutions have been proposed. In Section 1.1 we touched upon the Standard Paradigm SDL and presented normal Kripke Semantics, in order to focus on some of the problems it raises. In particular, in Section

1.1.1 we introduced some first preliminary modal schemata as well as their deontic interpretation. One of the most controversial dontic schemata, namely the formula called *Deontic Explosion* has been introduced in Section 1.2.1, where we proposed a preliminary syntactic analysis of the systems generating such schema, underlying the reasons behind the choice of working with non normal systems. There are several syntactic solutions to prevent the derivation of deontic explosion formulae, as we saw in Section 1.2.2, although we chose to analyse in detail the ones that are more conservative with respect to normal modal logics, i.e., those systems that are weak enough to prevent deontic explosions, yet powerful enough to express juridical sentences. These systems are the so called *non normal* modal logics. We also provided an informal introduction to non normal systems, their schemata and rules. Several different systems can be built on such syntactic foundations. Non normal systems can be treated semantically using relational structures. Moreover, we introduced a semantic interpretation of the modal deontic operators based on Kripke Semantics, or *Possible Worlds Framework*, as defined in Section 1.3.1. Kripke-frames are very intuitive and useful and provide an excellent tool to treat normal systems. However, they are known to be sound and complete with respect to normal systems. In order to keep the intuitive appeal of the possible worlds framework, while still using non normal systems, we decided to study further the so called *multi-relational* frames. In Section 1.3.2 and 1.3.3 we introduced this type of semantics, which is nothing but a direct generalisation of standard Kripke Frames and models.

Given this scenario, we decided to carry out a semantic analysis of multi-relational frames to provide tools to deal with non normal **N**-monotonic systems. Technical details of systems as well as several original completeness theorems were proposed in Chapter 2 and 3. In particular the first proposes an analysis of the propositional case, whereas the latter of the predicative one.

4.1.1 Propositional Results

In Chapter 2, for the first time, we carried out a semantic and syntactic analysis of several systems and modal schemata. In Section 2.1 we presented some well known non normal calculi, namely the systems $\mathsf{E}, \mathsf{M}, \mathsf{NM}, \mathsf{R}$ as well as a syntactical analysis of the relations between well known schemata. The most interesting results, however, can be found in Section 2.2, where we provided frame-characterisation of several modal formulae, namely, $\mathbf{M}, \mathbf{C}, \mathbf{N}, \mathbf{B}, \mathbf{T}, \mathbf{4}$. Section 2.2.4 presented a semantic analysis of some schemata that are relevant within deontic logics, namely, **CON** and **D**. It is well known that both schemata characterise precisely the same property in Kripke semantics, namely, seriality. However, this ceases to be true in multi-relational semantics and, as we shall see, these schemata define different readings of seriality within weak semantics.

Section 2.3 is focused on proving semantic completeness for several systems using both strong, and weak semantic tools. We proposed direct completeness proofs via canonical models for both classical systems (Section 2.3.1), and **N**-Monotonic systems (Section 2.3.2). Finally, in Section 2.3.3 we prove completeness theorems with respect to specific classes of frames for a few systems extending MN with well known schemata, namely, $\mathsf{MN} \oplus \mathbf{T}$, $\mathsf{MN} \oplus \mathbf{D}$, and $\mathsf{MN} \oplus \mathbf{CON}$.

Summarising, we provided an answer to several technical questions, namely:

(a) which theories are valid in the class of multi-relational weak structures? Any classical theory smaller or equal than **N**-monotonic logics;

(b) how do they differ from multi-relational strong frames (Neighborhood semantics)? Multi-relational strong frames validate a narrower set of formulae, namely, those theories smaller or equal than E;

(c) how well known modal schemata (among those relevant to deontic logic, like $\mathbf{M}, \mathbf{C}, \mathbf{T}, \mathbf{D}$, $\mathbf{B}, \mathbf{CON}, \mathbf{DEX}, \ldots$, behave within multi-relational weak frames? Do they characterise classes of frames with specific properties? Yes, most of them characterise some specific classes of frames (please refer to Table 2.1);

(d) how can well known first order properties be characterised by propositional schemata, if we assume a plurality of relations? We have provided an answer to reflexivity, seriality, and symmetry;

(e) the set of formulae which are valid in the class of all multi-relational frames can be generated by a finite axiomatic system? If so, which one? Yes, by the system MN, which is then sound and complete with respect to the class of all multi-relational weak frames. Moreover, we saw that the systems $\mathsf{MN} \oplus \mathbf{T}$, $\mathsf{MN} \oplus \mathbf{D}$, $\mathsf{MN} \oplus \mathbf{CON}$ are sound and complete with respect to specific classes of frames.

4.1.2 Predicative Results

Although quantified modal logic has a long and distinguished tradition, almost all efforts have so far been devoted to the analysis of the normal case: Besides a few significant exceptions based on neighbourhood semantics, the study of quantification in non-normal modal logics is still neglected. Despite that, quantified non-normal modal logics (QNML henceforth) exhibit a different behaviour with respect to normal modal logics. In particular, results in the literature show, e.g., that the Barcan and the Converse Barcan schemata (i) are not characterised by decreasing and increasing domains (ii) are tightly connected with the validity of propositional modal axiom schemata. In Chapter 3 we provided a semantic analysis of quantification in a class of non-normal modal logics called **N**-Monotonic (as defined in Chapter 2). Again, instead of following the neighborhood semantics approach, we shall focus on multi-relational semantics. presents free first order extensions of some **N**-monotonic systems and above as well as completeness results with respect to multi-relational first order frames. As far as we are concerned, from the predicative standpoint, this is the first study on quantification in multi-relational semantics, the second one investigating the case of varying domains in non-normal modal logics, and the first that provides a frame characterization of the Barcan schemata with varying domains.

Section 3.1 is an introduction to *Barcan Formulae* and their role within judicial syllogisms. There are several philosophical as well as technical issues related to such schemata. Section 3.2 presents some well known results concerning quantified non normal modal logics and Neighborhood frames, as well as a first technical introduction to Barcan formulae and the problems related to such schemata. We shall see the attempts made to accommodate Barcan schemata within both *constant domain*, and *varying domain* neighborhood frames. Section 3.3 is rather technical and presents multi-relational first order frames. We chose to analyse frames with varying domains, in order to perform a finer distinction between actual individuals and *possibilia*. Section 3.4 The traditional distinction between *de dicto* and *de re* sentences is here seen under a new light, in terms of contextual obligation and the role of quantification within deontic contexts. Section 3.5 is the core of this Chapter. We shall present alternative semantic characterisations for the schema **CBF**. We compare our results with the standard ones in Kripke Semantics and we shall see different ways to generalise the concept of *increasing inner domains*. Section 3.6 is the technical core of the Chapter. Here we provide Henkin-style completeness theorems for several systems, namely, the smallest *free* quantified non normal **N**-monotonic logic $\mathsf{Q}^{\circ}_{=}.\mathsf{NM}$ and some extensions, including $\mathsf{Q}^{\circ}_{=}.\mathsf{NM} \oplus \mathbf{CBF}$.

In this chapter, we provided an answer to several open questions, namely:

(a) We defined multi-relational weak structures and proved soundness results for **N**-Monotonic systems;

(b) We provided Completeness results for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ and extensions;

(c) We proved semantic equivalence for the couples **CBF** - **GF**, and **BF** - **CGF**;

(d) We provided frame characterisation results for both **CBF**, and **BF**;

(e) We provided Completeness results for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus \mathbf{CBF}$ and extensions;

(f) We showed that Schema **M** does not entail **CBF**;

(g) We proved that schema **C** and **BF** are semantically independent.

4.2 Open Problems and Future Work

The field of **N**-monotonic logics and multi-relational semantics is wide and, in our opinion, worthy of further investigation. Moreover, the area of interest is broader than deontic logics, since it is concerned with the whole field of modal logics. There are many open problems to be addressed, especially within the first order non normal systems.

Concerning the propositional case, one may wonder what class of structures, if any, is characterised by other modal schemata, for instance by those closer to other fields of applied logics, rather than deontic. One may wonder what class, if any, is captured by *positive introspection*, i.e., by schema $4 := \Box A \to \Box\Box A$, *negative introspection*, i.e. $\mathbf{5} := \neg\Box A \to \Box\neg\Box A$, or by other modal axioms.

On the technical side, there are other important open issues to be addressed in regard to the system MN:

- the finite model property;

- decidability and complexity.

From the predicative standpoint, there are even more open questions, for instance those related to the completeness of non normal systems which include **BF** are still open and require a deeper analysis. Moreover, we see other important directions for future research.

A preliminary question—somehow broader than the scope of this dissertation—regards the general relation between multi-relational and Neighborhood semantics. For propositional non-normal modal logics the two semantics are equivalent Governatori and

Rotolo (2005). However, it is not obvious if they are still equivalent (in regard, e.g., to completeness and incompleteness results) above K: Normal systems do not necessarily make Governatori and Rotolo (2005)'s semantics collapse on Kripke's. Regarding quantified modal logics, a full comparison of the two semantics is anyway needed.

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Finally, a question concerning Kracht and Wolter (1999)'s proof that non-normal modal logics can be simulated by a normal modal logic with three modalities. This result, which holds for MN, is of great interest in the context of multi-relational semantics, whose structures can in fact recall Kripke frames for multi-modal logics. However, when predicate calculi are added to the propositional modal base we can obtain unexpected interactions between Barcan schemata and modal axioms. Hence, extending Kracht and Wolter (1999)'s case to QNML is an open question.

4.3 Applications: Beyond Deontic Logics

As we said, that of non normal modal systems is a rather broad field and it concerns modal and applied logics besides deontic logics. There may be several applications within epistemic logics, for instance, as these systems can prevent one or more forms of omniscience. We have decided to conclude this dissertation by discussing a possible application of non normal systems in epistemic contexts, although the perspective taken is upside down. In fact, we decided to apply non normal concepts to *define* omniscience, rather than to block it. This final section does not provide any new result, nor it goes into great technical details. It can be seen as a technically light practical example of the flexiblity of both modal languages, and non normal systems.

Even though it can be argued that they do not represent a perfect solution for the problems we touched upon in Chapter 1, Non Normal modal logics are, however, a far better tool than normal systems to deal with deontic concepts. Their expressive power, moreover, is not limited to applications within legal and moral field. Indeed, non normal systems are a natural candidate within the field of epistemic logics when the goal is that of avoiding logical omniscience (Fagin et al., 1995, 335). In fact non normal logics may solve several problems related to epistemic concepts. Below we shall analyse the concept of omniscience form the modal standpoint.[1]

4.3.1 Defining Omniscience: a Logical Perspective

One of the major problems the logician willing to model knowledge and belief has to face is that of avoiding, or at least alleviating, the problem of omniscience. The efforts are usually focused on creating models for agents (either human or artificial) with bounded rationality and finite cognitive capabilities: such agents, thus, do not possess complete information about how the world is. Logical omniscience is often seen, therefore, as a problem to be solved and the solutions proposed so far are numerous (Fagin et al., 1995; Meyer, 2001).

Nevertheless, if the issue to be addressed is that of understanding and studying the concept of omniscience, such perspective should be reversed, in order to push the concept of knowledge to its most extreme possibilities. This stated, one may think that

[1] In a recent work, namely (Rotolo and Calardo, 2013), we actually have argued that such systems are a good tool even for the logical and philosophical analysis of traditional concepts like the problem of defining God's omniscience. Instead of trying to avoid, as it usually the case, agents' omniscience, in our paper we decided to build logically such concept starting from the epistemic reading of modal schemata. The usual approach is that of starting from a system like K and to weaken it until one reaches the desired level of *ignorance*. In our work, however, we have adopted the opposite perspective, namely we started from a system with ignorant agents and we kept adding axioms in order to achieve a perfect level of omniscience. We argued that one must start from the Classical Propositional Calculus CPC adding to it one by one all those rules and axioms useful to define total omniscience. However, and it is not surprising, we argued that in order to achieve the goal one must assume the modal system to be normal. In this Section, we shall summarize our work up to the point when adding normal schemata is necessary, in order to show with a practical example how any modal system weaker than K is actually better than a normal one to treat ignorance. This analysis will also provide a deeper analysis of the epistemic reading of several modal schemata.

omniscience is quite an easy property to get and hence to formalise. However, this is only partially true. If on the one hand it is quite easy to define *logical* omniscience in terms of knowledge of the logical truths (Fagin et al., 1995), on the other it turns out to be rather difficult to formally capture the insight of *factual* omniscience, which has to do with propositions having a different status (Girle, 2000).

A first informal definition may be that omniscient is the agent that has complete or maximal knowledge. What this informal definition means precisely is the research issue we address in (Rotolo and Calardo, 2013).

Even though we are exploring a different scenario, we shall use the same modal schemata we have already seen throughout this dissertation. We list them below just to make this Section somehow self contained. Here we used $\mathbf{K}_i$ (used instead of $\Box$) is an epistemic unary operator indexed with the label of an agent operating in the system and a formula as $\mathbf{K}_i A$ is to be read *agent i knows A*.

4.3.2 Principle of Co-extensionality: The Minimal Epistemic Logic E

Any system of epistemic logic, if based on the standard modal-logic paradigm (Meyer, 2001), should assume to enjoy some minimal formal properties. In particular, it is well-known that any modal logic should at least be closed under logical equivalence (Chellas, 1980). This will be our starting point to formally analyse the notion of omniscience.

When dealing with CPC, one standard and well-known option is adopting a Fregean approach to semantics. In a given state of affairs, propositions are taken to be different names of the only two semantical objects populating the universe: Truth and Falsehood. A tautology is a proposition which is true only in virtue of its logical form: the truth values of its components do not influence the truth of the whole in the slightest. The set of tautologies can hence be described as the class of all the *true* names of Truth: those propositions whose truth is certain and unchangeable. A most famous result in formal logic states that all theorems of CPC are tautologies and vice versa.

Two propositions that share *always* the same extension, can be regarded as logically equivalent and, in a logical sense, identical. This can be expressed symbolically as $A \equiv B$: whichever truth value A is given, it would be identical to B's and vice versa.

A basic requirement the knowledge base of any omniscient agent should meet is the principle that for any *known* sentence A, all its equivalents are also known. What we are stating is merely that if something is known, then all those facts which 'look' different but are actually the same (logically, extensionally) must also be known. This is a well-known modal principle and it can be compared to Leibniz's Law, here applied to propositional logic. What this principle states is that if two propositions are logically equivalent, they are epistemically interchangeable and, as we saw in Chapter 1, it is captured formally by the rule

$$\mathbf{RE} := A \equiv B \;/\; \Box A \equiv \Box B$$

As we explained in Chapter 2, it can be added to CPC to generate the minimal system of Classical Propositional Modal Logic E.

When knowledge and belief are modeled in epistemic logics like E, which are much weaker than K, then the epistemic logics can have a peculiar semantic reading, which is suitable to provide a fine-grained interpretation of logical omniscience. Modal logics weaker than K—which are called generically *non-normal*, in contrast with any *normal* logic that is stated to be as strong as, or stronger than, K (Chellas, 1980)—can be interpreted, as we have seen in Chapter 2 on multi relational models with strong truth conditions. The introduction of a plurality of worlds connected via a given accessibility relation R stems in epistemic logics from the need to represent agents' relative ignorance (i.e., partial knowledge) about the world. Given a state w, all the R-associated worlds t are seen as epistemic alternatives to w itself (Fagin et al., 1995; Meyer, 2001): When we have such a relation R which connects a world w with all alternatives where A is true, then we can say that $\Box A$ is true in a world w—and $\Box A$ is meant to say that an agent knows/believes that A is true. The plurality of worlds captures the notion of partial knowledge as follows. Suppose an agent i lives in Paris and does not know if today it is raining in London (p:= 'It is raining in London'). If i does not have access to any reliable source of information, he simply ignores all about the weather in London; hence he has at least two epistemic alternatives: for i in the perspective of Paris, (1) p is true, (2) p is false. However, as soon as the agent gains access to new pieces of information concerning the meteorological situation of London, the number of alternatives that he considers possible drops. If, for instance, he reads that it is currently raining in London, the epistemic alternatives he considers are only those which reflect the real situation, i.e., only those in which the proposition p is true. Her knowledge base would then change accordingly. However, the plurality of worlds expresses only one aspect of agents' relative ignorance. As we said, we also assume to work with a plurality of accessibility relations as discussed in Chapter 2.

Originally, as we have already stated, multi-relational semantics was developed in the field of deontic logic. In deontic logic the Kripke accessibility relation selects for each world those states of affairs that are (morally, legally, etc.) ideal with respect to it: hence, if $\Box A$ is true in a world w, this simply means that A is the case in all ideal alternatives to w. The interpretation of multi-relational models, as given for example in deontic logics, is thus that each accessibility relation corresponds to a particular "standard of value" or a norm that selects those ideal worlds; however, it is not guaranteed that such worlds are still ideal according to different standards of value or norms, namely, according to different accessibility relations. In this perspective, different relations correspond to different deontic standards or that conflicting norms are obtained from otherwise consistent different systems of norms.

If we import this intuition in the domain of epistemic logics, the multiplicity of relations may express the idea that there exist many epistemic standards and that the truth-conditions for knowledge assertions can vary across contexts as a result of shifting epistemic standards. The idea of plurality of epistemic standards (Pollock, 1986, 190–3) was defended within different philosophical theories of knowledge (Malcolm, 1952; Goldman, 1976; Rorty, 1979), none of which should be necessarily assumed to confer a minimal philosophical meaning to epistemic multi-relational models. Let us just consider how Hector-Neri Castañeda (Castañeda, 1980, 217) illustrates what a plurality of epistemic standards means and how it may affect the truth conditions of knowledge assertions:

Example 4.3.1 (Discovering America example adapted from (Castañeda, 1980)) *"What counts as knowing" that Cristoforo Colombo discovered America on October 12, 1492 might change depending on whether we are considering (i) a television quiz show, (ii) a high school student's essay, or (iii) a defense of the traditional dates of America's culture from some famous Harvard historian. Hence, we have in this example three epistemic standards. The fact that*

$$\mathbf{K}_i(\textit{Colombo discovered America on October 12, 1492}) \tag{4.1}$$

is true according, for example, to standard (i) does not entail that it is also true according to standard (iii), which is somehow more demanding.

Hence, in general, we could tolerate epistemic expressions such as

$$\mathbf{K}_i A \wedge \mathbf{K}_i \neg A \tag{4.2}$$

because different standards can lead to know that Colombo discovered America or to know that this was plainly false. This can said to be true even when the epistemic agent is the same. For instance imagine a modern scientist, who believes in darwinian evolution, who happens to be also a fervent Catholic. According to his scientific paradigm, he believes $A :=$ *living beings evolved through the ages from very simple and different forms* *to take their actual shape* and $\neg A$:= *living beings were created by God precisely as they are now and evolution is a lie.* Well these statements, although contradictory, can be known by the same agent, at the same time under different paradigms. Clearly one has to add a further condition here, namely that knowledge does not entail truth. Indeed, if we do not impose any special condition on multi-relational frames, then we have the modal system E. In this setting, it is easy to check that formula (4.2) is not contradictory.

Hence, if we interpret relations as different epistemic standards, it is not required that the truth of (4.2) corresponds to a genuine *cognitive dissonance* (Aronson, 1969), because there is no real epistemic conflict between $\mathbf{K}_i A$ and $\mathbf{K}_i \neg A$: Each formula refers to a different standard. A true cognitive dissonance rather occurs when $\mathbf{K}_i(A \wedge \neg A)$ is true, because this sentence means that there is a logical conflict within a same standard.

Assume now to formalize Example 4.3.1 following the above semantic intuitions.

Example 4.3.2 (Discovering America (cont'd)) *Let us denote 'Colombo discovered America on October 12, 1492' with* A *and represent standards as follows:*

(i)	*a television quiz show*	=	R_1
(ii)	*a high school student's essay*	=	R_2
(iii)	*a defense of the traditional dates of America's culture from some famous Harvard historian*	=	R_3

For formula

$$\mathbf{K}_i A \tag{4.3}$$

it is sufficient that A *is true in all worlds selected by one standard, as the model in Figure 4.1 shows.*

This analysis suggests that the agents' ignorance is not only captured by having more alternatives for a given world, but also by having more standards. In fact, the standards (i), (ii), and (iii) of Example 4.3.1 and 4.3.2 represent different contexts as well as "per-

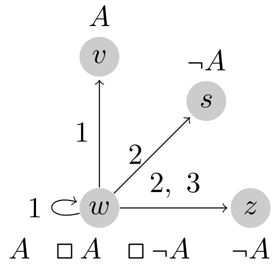

Figure 4.1: A simple model illustrating Castaẽda's example.

spectives" of knowledge, which overall express the fact of a structural bounded epistemic capability with regard to the time when America was discovered: for an omniscient agent g it would be odd to argue that in a certain perspective g knows that P is false while in another perspective he knows that P is true, because an omniscient being is supposed to know precisely what is objectively true: hence, a multiplicity of epistemic standards reflects a certain degree of ignorance, at least insofar as the absence of ignorance is taken to correspond to omniscience. Indeed, reducing the number of relations lessens the structural degree of ignorance of agents and leads to a higher degree of agents' omniscience.

4.3.3 An Easy Step after E

As already stated several times, the first step in the path that leads from E to full logical omniscience, is adding schema $\mathbf{M}$, i.e.,

$$\mathbf{M} := \mathbf{K}_i(A \wedge B) \rightarrow (\mathbf{K}_i A \wedge \mathbf{K}_i B)$$

This schema seems relatively acceptable in epistemic logic. First of all, its validity is assumed in most non-normal modal systems—it is actually discarded only by the system E. Second, the schema looks conceptually harmless: if I know/believe both sentences together, at the same time, then it must be also true that I know/believe that America was discovered by Colombo on October 12, 1492 and that I know/believe that Betsy

Ross reported in May of 1776 that she sewed the first American flag. Semantically, **M** corresponds to the property stated by Lemma 2.2.6. In other words, if there is one epistemic standard according to which A and B are jointly true, there are two standards that validate respectively A and B.

Consider the following example:

Example 4.3.3 (Colombo and Betsy Ross) *Let us denote 'Colombo discovered America on October 12, 1492' with P and 'Ross reported in May of 1776 that she sewed the first American flag' with Q. Again, suppose to work with the mentioned epistemic standards:*

(i)	*a television quiz show*	=	R_1
(ii)	*a high school student's essay*	=	R_2
(iii)	*a defense of the traditional dates of America's culture from some famous Harvard historian*	=	R_3

For formula

$$\mathbf{K}_i(P \wedge Q) \to (\mathbf{K}_i P \wedge \mathbf{K}_i Q) \tag{4.4}$$

it is sufficient to have supplemented models (see Lemma2.2.10) such as in Figure 4.2.

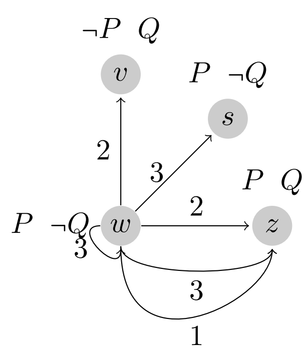

Figure 4.2: A simple model $\mathcal{M}$ illustrating Example 4.3.3.

In the model $\mathcal{M}$ represented in the figure, for any $x \in W$, we have that $x \vDash \mathbf{K}_i(P \wedge Q)$ because there is a relation R_1 such that $R_1(w) = \{z\} = ||P|| \cap ||Q||$. All epistemic alternatives that make both P and Q true are related to w via the perspective of the standard "(i) a television quiz show"; hence, it is true in w that the given agent knows/believes that $P \wedge Q$ is the case. Also, the other two standards, "(ii) a high school student's essay" and "(iii) a defense of the traditional dates of America's…" connect w respectively to precisely those worlds that make true the sentences P (via R_3) and Q (via R_2), hence

- *$\mathbf{K}_i(P \wedge Q)$ is true in w (via standard (i), i.e., relation R_1), $\mathbf{K}_i P$ is true in w (via standard (iii), i.e., relation R_3) and $\mathbf{K}_i Q$ is true in w (via standard (ii), i.e., relation R_2);*
- *for any other world $x \in \{v, s, z\}$, we have that $x \nvDash \mathbf{K}_i(P \wedge Q)$; therefore*
- *formula (4.4) is true in $\mathcal{M}$.*

4.3.4 Conflicts, Coherence and Epistemic Paradigms

There is a further important schema that from the perspective of epistemic systems plays a central role in our quest for a logical definition of the concept of omniscience, namely, the schema **C**:

$$\mathbf{C} := (\mathbf{K}_i A \wedge \mathbf{K}_i B) \to \mathbf{K}_i(A \wedge B)$$

Adding **C** to the formerly defined system M generates the system R, the smallest regular modal logic. This system shows very interesting properties. Let us focus on **C**. If there are two standards guaranteeing respectively that $\mathbf{K}_i P$ and $\mathbf{K}_i Q$ are true, then there is possibly a third standard that selects all the epistemic alternatives in which $P \wedge Q$ is true, namely, $\mathbf{K}_i(P \wedge Q)$ holds. In general, the result for **C** is the following:

Example 4.3.4 (Colombo and Betsy Ross (cont'd)) *For formula*

$$(\mathbf{K}_i P \wedge \mathbf{K}_i Q) \to \mathbf{K}_i(P \wedge Q) \tag{4.5}$$

it is sufficient to have structures closed under intersections. Notice that the model in Figure 4.2 also validates (4.5). However, consider a subtle variation, as depicted in Figure 4.3.

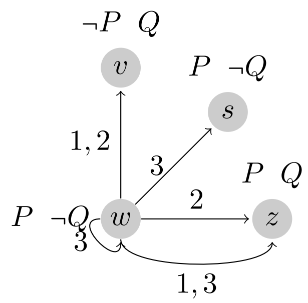

Figure 4.3: A variation $\mathcal{M}'$ of the model $\mathcal{M}$ of Figure 4 that validates (4.4) but falsifies (4.5).

The model $\mathcal{M}'$ in Figure 4.3 still validates (4.4). However,

- $\mathbf{K}_i P$ *is true in* w *(via standard (iii), i.e., relation* R_3*) and* $\mathbf{K}_i Q$ *is true in* w *(via standard (ii), i.e., relation* R_2*);*

- $\mathbf{K}_i(P \wedge Q)$ *is false in* w *because* $\|P\| \cap \|Q\| = \{z\}$ *but there is no accessibility relation* R_j *such that* $R_j(w) = \{z\}$.

- *formula (4.5) is false in* w *and not valid in* $\mathcal{M}'$.

Here, we may have indeed two epistemic standards that individually support the agent's knowledge/belief that 'Colombo discovered America on October 12, 1492' and 'Ross reported in May of 1776 that she sewed the first American flag' are true, but it is far from obvious that there is a standard that support them jointly. On the other hand, the difficulty in saying that there is such a standard for $P \wedge Q$ *does not undermine the truth of (4.4), since, if there is no such relation, then the formula is trivially true in* w *(its antecedent is false).*

Notice that schema **C** plays a crucial role in enforcing cognitive dissonances and in making explicit epistemic conflicts. Indeed, let us take Example 4.3.4 and replace Q with $\neg P$. Hence, we can simply consider the following instance of **C**:

$$(\mathbf{K}_i P \wedge \mathbf{K}_i \neg P) \to \mathbf{K}_i (P \wedge \neg P) \tag{4.6}$$

Since (4.6) is true for example in w, then there is at least an epistemic standard (represented in the example and in Figure 4.2 by R_1) that connects w to all epistemic alternatives that make $P \wedge \neg P$ true. However, $P \wedge \neg P \equiv \bot$, hence the standard refers to a contradiction, which makes void R_1 and hence we should have that $R_1(w) = \varnothing$. In a different but related perspective, since the modal system R makes valid the inference rule **RR**, i.e., $\vdash A \wedge B \to C \;\; \Rightarrow \;\; \vdash \mathbf{K}_i A \wedge \mathbf{K}_i B \to \mathbf{K}_i C$ (Chellas, 1980, chap. 2), then, if we have $\mathbf{K}_i P$ and $\mathbf{K}_i \neg P$, we obtain $\mathbf{K}_i X$ for any sentence X: $(P \wedge \neg P) \to X$ is in fact a tautology of CPC. Hence, suppose we know/believe that P and know/believe that $\neg P$. We could obtain $\mathbf{K}_i Q$, $\mathbf{K}_i \neg Q$, $\mathbf{K}_i (Q \wedge \neg Q)$, $\mathbf{K}_i$(Bologna is in the UK), and so forth.

4.3.5 A Different Path: Truth and Logical Omniscience

We have discussed some very weak epistemic logics. However, we highlighted that combining schemata **M** and **C** results in the well-known modal system R, where a much stronger version of logical omniscience emerges: Here, we can easily include any tautology and logical truth in an omniscient agent's knowledge base as well as making explicit any cognitive dissonance.

A different (and not equivalent) path can be taken to capture full understanding by assuming **M** and state that an agent knows the Truth. This last statement is expressed by the axiom schema known in the alethic tradition as the Necessity of Truth:

$$\mathbf{N} := \Box \top$$

As formerly observed, the propositional constant $\top$ is taken to mean the Truth and its truth value is, accordingly, always true. Notice that the schema **N** is enough to include any tautology and logical truth in an omniscient agent's knowledge base. Knowing only one theorem, only one logical Truth would be enough to know all the classical theorems. Indeed for any theorem A it holds that $A \equiv \top$ and it is enough to apply **RE** and **MP** to derive $\Box A$. Hence, it is sufficient to add the schema **N** to the system E to state that any omniscient agent knows all the truths of logic, i.e., all the theorems generated within the system. This intuition is usually captured by the rule **RN**:

$$\mathbf{RN} := A \;/\; \Box\, A$$

As we saw in Chapter 2, NM-systems is that it enables the switch from multi-relational strong semantics to weak frames.

Notice that **N** states something rather strong. It says, in fact, the any agent operating within the system knows all the truths of logic, all the theorems. However this type of omniscience still concerns the abstract truths of mathematics rather than contingent facts. This difference turns rather more evident if looked at from a semantic perspective. What the schema claims, in fact, is that an agent knows all *valid* propositions, i.e. those formulae which are true *everywhere*, in all possible worlds of all possible frames and under all possible valuations. On the other hand, if a fact happens to be true in a specific state of a model, under a specific valuation (but it can still be false under other conditions) there is no way—yet—to infer that an agent knows it.

So far we have presented a semantic scenario designed to accommodate different epistemic perspectives and paradigms. In fact, given the laws of CPC, we are bound to accept that any proposition has one and, even more important, only one truth value: the law of excluded middle $A \vee \neg A$ is a classical tautology. Semantically this is mirrored by the fact that the intersection of two complementary sets of epistemic alternatives is always empty. Hence no genuine epistemic standard (i.e., a relation which is not empty) can accommodate both A and $\neg A$. That known facts should be coherent is suggested, as we already said, by the schema **C**; it states is that if an agent knows two distinct facts and such facts are contradictory, then he must also use a further epistemic standard which is trivial, i.e. a standard which makes him believe everything (semantically: an empty binary relation). On the other hand, if the two facts are indeed consistent with each other (semantically: the intersection of $\|A\|$ and $\|B\|$ is not empty), then, by **C**, the agent must possess another epistemic standard to accommodate both propositions. In general for any couple of genuine epistemic paradigms, there must exist a third one which takes into account those facts that are common to both. This means that *true* knowledge is consistent and cannot handle contradictions, i.e., all non trivial epistemic paradigms are coherent with each other. Intuition would suggest that this is equivalent to possessing only one epistemic standard and this is perfectly consistent with our idea of perfect knowledge. Recall that, semantically, a multi-relational weak frame with only one binary relation is called a Kripke-frame.

It is often argued that from the epistemic perspective normal systems are too strong to model human agents and in this section we understood why: Despite the manageability of such logics for AI applications and their low computational complexity, normal epistemic logics raise a number of difficulties if employed to philosophically clarify the nature of human knowledge and belief. One of the most well-known problems is that normal epistemic logics are affected by various forms of logical omniscience, which looks mostly unsuitable for modeling human epistemic capabilities (Fagin et al., 1995; Meyer, 2001).

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