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**Harnack inequalities
in sub-Riemannian settings**

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Introduction

In the theory of fully nonlinear elliptic equations a crucial role is played by the Krylov-Safonov's Harnack inequality for nonnegative solutions to the linear equations in non-divergence form and rough coefficients. The key point of this celebrated result is that the Harnack's constant is independent of the regularity of the operator coefficients, but it depends just on the bounds for the eigenvalues of the coefficient matrix. After the proof of this profound result in [24], the analysis developed by Caffarelli in [7] about fully nonlinear operators pointed out a deep relation between the Krylov-Safonov-Harnack inequality and the Alexandrov-Bakelman-Pucci maximum principle: nowadays the importance of this maximum principle for proving the result in [24] is well-recognized (see e.g [17], Section 9.7-9.8).

On the other hand, in several research areas such as Complex or CR Geometry, there are fully nonlinear equations which are characterized by an underlying sub-Riemannian structure and are not elliptic at any point, see e.g. [29],[33],[30],[31],[9],[11],[28]. The existence theory for viscosity solutions to such equations is well settled, mainly thanks to the papers [33],[30],[11]. On the contrary, the problem of the solutions regularity is still widely open. This is mainly due to the lack of pointwise estimates for solutions to linear sub-elliptic equations with rough coefficients. In this context, a long-standing open problem is an invariant Harnack inequality of Krylov-Safonov type for positive solutions to horizontally elliptic equations on Lie groups, in non-divergence form, and rough coefficients. Similarly, an analogous of the Alexandrov-Bakelman-Pucci estimate in these settings is still unknown.

However, Di Fazio, Gutiérrez, and Lanconelli in [15] developed an axiomatic procedure to establish a scale-invariant Harnack inequality in very general settings like doubling Hölder quasi-metric spaces. This approach allows to handle both divergence and non-divergence linear equations. They proved that the *double-ball property* and the ε -*critical density* are sufficient conditions for the Harnack inequality to hold. What are these notions? These properties arose just from the techniques in [7] for uniformly elliptic fully non-linear equations. They were then extended to the linearized Monge-Ampère equation in [8], where Caffarelli and Gutiérrez proved an invariant Harnack inequality on some suitable sets. In [18] (Chapter 2) these notions have been treated in depth for the classical case of linear uniformly elliptic equations, and the Krylov-Safonov's result have been there proved using this alternative approach. In [15] the double ball and the critical density found an abstract statement and the techniques for proving an Harnack result have been generalized for the purpose of being used in general settings. A key role is played in particular by a Besicovitch-type covering lemma which yields a crucial power decay property. In [19] this general approach has been exploited in the setting of the Heisenberg group \mathbb{H} : Gutiérrez and Tournier proved, for second order linear operators which are elliptic with respect to the vector fields generating \mathbb{H} , the double ball property and, under an extra-assumption on the coefficients of the operator, the critical density. Recently we have investigated the double ball property in step two Carnot groups and the critical density in H-type groups (respectively in [34] and [35]). In the present thesis we want to show the results of these studies, develop them further and give also a general presentation of the problem by trying to make this manuscript as much self-contained as we can.

At the beginning of any chapter there is a very short introduction about the topics analyzed therein. Here we want to give the general outline of the thesis and to exhibit the main results.

In Chapter 1 we present the powerful approach adopted in [15]. The double ball property and the critical density are displayed as the main notions. We

will try to give a special attention to their close relationship and to their differences even by discussing and changing the definitions. Moreover we treat the power decay property and how it implies an abstract Harnack inequality. We show also two possible proofs for the Hölder regularity result in this context: one resulting from Harnack and one from a critical density estimate.

In Chapter 2 we talk about the uniformly elliptic operators and the application to this case of the axiomatic approach, that is the Krylov-Safonov's result. To this aim we mainly follow [18], most of all for the presentation of the critical density theorem. Regarding the double ball property, we show a new proof of this fact based on the weak maximum principle and on the barrier functions of the classical Hopf's Lemma.

In Chapter 3, we investigate the application of the axiomatic approach to the particular context of the homogeneous Carnot groups and for general stratified groups. The horizontally elliptic operators are central for this discussion: they are peculiar second order degenerate elliptic operators which are elliptic only with respect to the vector fields generating the first layer of the Lie algebra. We will see how much the validity of the double ball and of the critical density are intrinsic in these settings.

In Chapter 4 we highlight as the double ball property is related to the solvability of a kind of exterior Dirichlet problem for horizontally elliptic operators in homogeneous Carnot groups. More precisely, it is a consequence of the existence of some suitable interior barrier functions of Bouligand-type. By following these ideas, we prove the double ball property for a generic step two Carnot group, which is in fact our main result in [34]. If the step of nilpotence is 2 we have indeed an explicit characterization of the vector fields defining the operator which allows us to construct explicit barriers. We also give a different proof for the particular case of the Métivier groups.

Finally, in Chapter 5, we generalize to the setting of H-type groups some arguments regarding the critical density adopted in [19]. We recognize that the critical density holds true in these peculiar contexts by assuming a Cordes-Landis estimate for the coefficient matrix A . As a matter of fact we assume

that A satisfies the following condition

$$\sup_x \left(\frac{\operatorname{Tr}(A(x)) + (Q + 2 - m) \max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} \right) < Q + 4,$$

where m and Q are some characteristic constants of the setting. We follow the main points of the powerful arguments by Gutiérrez and Tournier in the Heisenberg group, even if the condition on A they found is different. A condition similar to ours was exploited by Landis in [26] in order to prove an invariant Harnack inequality of Cordes type. We will use it for the same purposes at the end of the thesis by showing our main result in [35], that is an invariant Harnack in H-type groups (of Cordes-Landis type). The constants appearing in such inequality will be uniform in the class of A with prescribed bounds for the eigenvalues and satisfying a Cordes-Landis condition. They will be thus independent of the regularity of the coefficients of the matrices A in that class. This final proof will put together most of the notions discussed through the thesis.

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Chapter 1

An axiomatic approach

We want to show here the axiomatic approach presented by Di Fazio, Gutiérrez, and Lanconelli in [15]. As they did, we are going to present it in the abstract setting of doubling quasi-metric Hölder spaces. We will highlight the notions of *double ball*, *critical density*, and *power decay* and their crucial role for obtaining an Harnack inequality.

1.1 The main notions

In order to clarify the setting we want to fix, we need some definitions.

Definition 1.1.1. *Let Y be a non empty set. We say that Y is a quasi-metric space if there exists a function $d : Y \times Y \rightarrow [0, +\infty)$ which is symmetric, strictly positive away from $\{(x, y) \in Y \times Y : x = y\}$ and such that, for some constant $K \geq 1$, we have*

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for all $x, y, z \in Y$. We will call d a quasi-distance. The d -ball with center $x_0 \in Y$ and radius $R > 0$ is given by

$$B_R(x_0) := \{y \in Y : d(x_0, y) < R\}.$$

Definition 1.1.2. Let (Y, d) be a quasi-metric space and μ be a positive measure on a σ -algebra of subsets of Y containing the d -balls. We say that μ satisfies the doubling property if there exists a positive constant C_d such that

$$0 < \mu(B_{2R}(x_0)) \leq C_d \mu(B_R(x_0))$$

for all $x_0 \in Y$ and $R > 0$. In particular, this implies

$$\mu(B_{R_2}(x_0)) \leq C_d \left(\frac{R_2}{R_1} \right)^Q \mu(B_{R_1}(x_0)) \quad (1.1)$$

for any $0 < R_1 < R_2$, where $Q := \log_2(C_d)$.

The previous definitions clarify what we mean for a *doubling quasi-metric space*. Any doubling quasi-metric space is in particular of homogeneous type (see [15], Definition 2.1, and the references therein).

Definition 1.1.3. Let (Y, d) be a quasi-metric space. The quasi distance d is said to be Hölder continuous if there exist positive constants β and $0 < \alpha \leq 1$ such that

$$|d(x, y) - d(x, z)| \leq \beta d(y, z)^\alpha (d(x, y) + d(x, z))^{1-\alpha}$$

for all $x, y, z \in Y$.

A space satisfying all the three previous definitions is said to be a doubling quasi-metric Hölder space. Of course, \mathbb{R}^n with the Euclidean distance and the Lebesgue measure is the first example to be recalled. Another remarkable example is given by the homogeneous Lie groups. This case will be discussed in Chapter 3 and it is pivotal for the main results of the thesis. Even the Carnot-Carathéodory spaces identify a setting where the approach might be applied: we will mention how at the end of the next Section.

Thus we fix a doubling quasi-metric Hölder space (Y, d, μ) . In such a space, let Ω be an open subset of Y . Let us give two more definitions in order to complete all the structural assumptions we need.

Definition 1.1.4. We say that (Y, d, μ) has the reverse doubling condition in Ω if there exists $0 < \delta < 1$ such that

$$\mu(B_R(x_0)) \leq \delta \mu(B_{2R}(x_0))$$

for every $B_{2R}(x_0) \subset \Omega$.

Definition 1.1.5. We say that (Y, d, μ) satisfies a log-ring condition if there exists a nonnegative function $\omega(\varepsilon)$, with $\omega(\varepsilon) = o((\log(\frac{1}{\varepsilon}))^{-2})$ as $\varepsilon \rightarrow 0^+$, such that

$$\mu(B_R(x_0) \setminus B_{(1-\varepsilon)R}(x_0)) \leq \omega(\varepsilon) \mu(B_R(x_0))$$

for every ball $B_R(x_0)$ and all ε sufficiently small.

Following the notations in [15], we denote by \mathcal{K}_Ω a family of μ -measurable functions with domain contained in Ω . If $u \in \mathcal{K}_\Omega$ and its domain contains a set $A \subset \Omega$, we will write $u \in \mathcal{K}_\Omega(A)$. We suppose that

(I) \mathcal{K}_Ω is closed under multiplications by positive constants.

We are ready to give the statements of the double-ball property and the ε -critical density.

Definition 1.1.6. (Abstract Double Ball Property) We say that \mathcal{K}_Ω satisfies the double ball property if there exist $\eta_D > 2$ and $\gamma > 0$ such that, for every $B_{\eta_D R}(x_0) \subset \Omega$ and every $u \in \mathcal{K}_\Omega(B_{\eta_D R}(x_0))$ with $\inf_{B_R(x_0)} u \geq 1$, we have

$$\inf_{B_{2R}(x_0)} u \geq \gamma.$$

Definition 1.1.7. (Abstract ε -Critical Density) Let $0 < \varepsilon < 1$. We say that \mathcal{K}_Ω satisfies the ε -critical density property if there exist $\eta_C > 2$ and $c > 0$ such that, for every $B_{\eta_C R}(x_0) \subset \Omega$ and for every $u \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$ with

$$\mu(\{x \in B_{2R}(x_0) : u(x) \geq 1\}) \geq \varepsilon \mu(B_{2R}(x_0)),$$

we have

$$\inf_{B_R(x_0)} u \geq c.$$

Actually, with respect to [15] we have changed a little bit these definitions. There the constants η_D and η_C are fixed to be 4, whereas here we allow them to change. Nonetheless, as we will see, this fact does not infer the results they proved and the very meaning of these notions.

Remark 1.1.8. We stress that the ε -critical density property implies the ε' -critical density for any $\varepsilon' \geq \varepsilon$ (with the same constants η_C and c). Moreover, if \mathcal{K}_Ω satisfies the double ball property for some η_D and γ (respectively, the ε -critical density property for some η_C and c), then it satisfies the double ball property also for any $\tilde{\eta}_D \geq \eta_D$ and the same γ (resp., the ε -critical density for any $\tilde{\eta}_C \geq \eta_C$ and the same c): in our notations we have in fact that $\mathcal{K}_\Omega(A') \subseteq \mathcal{K}_\Omega(A)$ if $A \subseteq A' \subset \Omega$.

The two properties under investigation refer to an abstract family of functions \mathcal{K}_Ω . We will see in Chapter 2 and in Chapter 3 that we should interpret \mathcal{K}_Ω as the family of the nonnegative functions u satisfying $\mathcal{L}u \leq 0$ for an elliptic or sub-elliptic operator \mathcal{L} (see respectively (2.2) and (3.4)). In the following sections we are going to show how the critical density and the double ball property imply an abstract invariant Harnack inequality for the *non-negative solutions*: this is in fact the main result in [15]. From such Harnack inequality, an Hölder regularity theorem follows by standard arguments. On the other hand, it is also known that the ε -critical density property with $\varepsilon \leq \frac{1}{2}$ is sufficient to get the Hölder regularity result (see [20], Theorem 4.10). We now give a complete proof of this fact in our notations and settings.

Theorem 1.1.9. *Let us suppose \mathcal{K}_Ω satisfies the ε -critical density with $\varepsilon \leq \frac{1}{2}$ (and for some η_C, c). Assume also that the following conditions hold true:*

- $u \in \mathcal{K}_\Omega(B_R(x_0))$ and $u \leq M$ in $B_R(x_0) \Rightarrow M - u \in \mathcal{K}_\Omega(B_R(x_0))$;
- $u \in \mathcal{K}_\Omega(B_R(x_0))$ and $u \geq m$ in $B_R(x_0) \Rightarrow u - m \in \mathcal{K}_\Omega(B_R(x_0))$.

There exists $\nu \in (0, 1)$ depending just on c such that, for any locally bounded function $u \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$, we have

$$\sup_{B_R(x_0)} u - \inf_{B_R(x_0)} u \leq \nu \left(\sup_{B_{2R}(x_0)} u - \inf_{B_{2R}(x_0)} u \right)$$

Proof. Fix a locally bounded function $u \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$. Let us denote $\omega(R) := \sup_{B_R(x_0)} u - \inf_{B_R(x_0)} u$. By the local boundedness we can also put

$$\alpha_2 = \sup_{B_{2R}(x_0)} u, \quad \text{and} \quad \beta_2 = \inf_{B_{2R}(x_0)} u.$$

By the assumption (I) and the additional hypotheses in the statement, we have

$$\frac{u - \beta_2}{\alpha_2 - \beta_2} \quad \text{and} \quad \frac{\alpha_2 - u}{\alpha_2 - \beta_2} \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0)).$$

It is simple to verify the following equivalences

$$u \geq \frac{1}{2}(\alpha_2 + \beta_2) \Leftrightarrow \frac{u - \beta_2}{\alpha_2 - \beta_2} \geq \frac{1}{2},$$

$$u \leq \frac{1}{2}(\alpha_2 + \beta_2) \Leftrightarrow \frac{\alpha_2 - u}{\alpha_2 - \beta_2} \geq \frac{1}{2}.$$

Thus, it holds true at least one of the following two cases:

$$\mu \left(\left\{ x \in B_{2R}(x_0) : \frac{2(u(x) - \beta_2)}{\alpha_2 - \beta_2} \geq 1 \right\} \right) \geq \frac{1}{2} \mu(B_{2R}(x_0)) \quad \text{or}$$

$$\mu \left(\left\{ x \in B_{2R}(x_0) : \frac{2(\alpha_2 - u(x))}{\alpha_2 - \beta_2} \geq 1 \right\} \right) \geq \frac{1}{2} \mu(B_{2R}(x_0)).$$

In both cases, we can exploit the critical density property by reminding Remark 1.1.8. We get respectively

$$\inf_{B_R(x_0)} u \geq \beta_2 + \frac{c}{2}(\alpha_2 - \beta_2) \quad \text{or} \quad \sup_{B_R(x_0)} u \leq \alpha_2 - \frac{c}{2}(\alpha_2 - \beta_2).$$

Let us set now

$$\alpha_1 = \sup_{B_R(x_0)} u \quad \text{and} \quad \beta_1 = \inf_{B_R(x_0)} u.$$

Since $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, we have in any case that

$$\omega(R) = \alpha_1 - \beta_1 \leq \left(1 - \frac{c}{2}\right) (\alpha_2 - \beta_2) =: \nu(\alpha_2 - \beta_2) = \nu\omega(2R).$$

□

With very standard arguments (see [17], Section 8.9, for the elliptic case) we can deduce the Hölder regularity result from the last theorem.

Corollary 1.1.10. *Under the hypotheses and the notations of the previous theorem, there exist $C, k_0 > 0$ and $\alpha_0 \in (0, 1]$ such that, for any locally bounded function $u \in \mathcal{K}_\Omega(B_{\frac{\eta_C}{2}R}(x_0))$, we have*

$$|u(x) - u(y)| \leq C \left(\frac{d(x, y)}{R} \right)^{\alpha_0} \sup_{B_R(x_0)} |u| \quad \forall x, y \in B_{\frac{R}{k_0}}(x_0).$$

The constants C, k_0 and α_0 depend just on η_C, c and the fact that $\varepsilon \leq \frac{1}{2}$.

Proof. Fix $k_0 = K(1+2K)$ and put $r = \frac{2KR}{k_0}$. Thus we have $B_r(y) \subset B_R(x_0)$ for any $y \in B_{\frac{R}{k_0}}(x_0)$. Take u as in the statement. For any fixed $y \in B_{\frac{R}{k_0}}(x_0)$, let us denote by $\omega(\rho)$ the oscillation of u in the ball $B_\rho(y)$ as before. The previous Theorem tells us that $\omega(\frac{1}{2}\rho) \leq \nu\omega(\rho)$ if $0 < \rho \leq r$. Hence, for any positive integer m , we get $\omega(\frac{1}{2^m}r) \leq \nu^m\omega(r)$. For any $0 < \rho < r$ there exists a positive integer m such that $\frac{1}{2^m}r < \rho \leq \frac{1}{2^{m-1}}r$. Since the function ω is non-decreasing, we have

$$\omega(\rho) \leq \omega\left(\frac{1}{2^{m-1}}r\right) \leq \nu^{m-1}\omega(r) = \frac{\nu^m}{\nu}\omega(r) \leq \frac{1}{\nu} \left(\frac{\rho}{r}\right)^{\frac{\log \nu}{\log \frac{1}{2}}} \omega(r)$$

where the last inequality is justified by the fact that

$$\nu^m \leq \left(\frac{\rho}{r}\right)^{\frac{\log \nu}{\log \frac{1}{2}}} \Leftrightarrow \frac{1}{2^m} \leq \frac{\rho}{r}.$$

Fix $\alpha_0 = \frac{\log \frac{1}{\nu}}{\log 2}$ (which is less than 1, since $c \leq 1$). By recalling the definition of r and the fact that $\omega(r) \leq 2 \sup_{B_R(x_0)} |u|$, we get

$$\omega(\rho) \leq C \left(\frac{\rho}{R}\right)^{\alpha_0} \sup_{B_R(x_0)} |u|$$

for a suitable positive constant C depending on K, ν . For any $x \in B_{\frac{R}{k_0}}(x_0)$ we have $d(x, y) < \frac{2KR}{k_0} = r$. If we put $\rho = d(x, y)$, the last inequality implies

$$u(x) - u(y) \leq \omega(\rho) \leq C \left(\frac{d(x, y)}{R}\right)^{\alpha_0} \sup_{B_R(x_0)} |u|.$$

By the symmetry in x and y , the proof is complete. \square

Critical density and double ball property are in general independent but, as already noticed in [15] (Proposition 4.3), if the ε -critical density holds true for ε sufficiently small, then this implies the double ball property.

Proposition 1.1.11. *Suppose \mathcal{K}_Ω satisfies the ε -critical density property for some $0 < \varepsilon < \frac{1}{C_d^2}$. Then \mathcal{K}_Ω satisfies the double ball property.*

Proof. We are going to prove the double ball property with $\eta_D = 2\eta_C$ and $\gamma = c$. Let $B_{\eta_D R}(x_0) \subset \Omega$ and $u \in \mathcal{K}_\Omega(B_{2\eta_C R}(x_0))$ such that $\inf_{B_R(x_0)} u \geq 1$. Suppose by contradiction that $\inf_{B_{2R}(x_0)} u < c$. Then we get

$$\mu(\{x \in B_{4R}(x_0) : u(x) \geq 1\}) < \varepsilon \mu(B_{4R}(x_0)).$$

Since $B_R(x_0) \subseteq \{x \in B_{4R}(x_0) : u(x) \geq 1\}$, by the doubling condition we would have

$$\mu(B_R(x_0)) \leq \varepsilon \mu(B_{4R}(x_0)) \leq \varepsilon C_d^2 \mu(B_R(x_0)),$$

which is a contradiction if $\varepsilon < \frac{1}{C_d^2}$. \square

This is not the only relation between critical density and double ball property. In Section 1.4 we will single out other aspects of this relationship. Here and in the next Section we want to show how they can work together (see [15], Proposition 4.4 and Lemma 4.5).

Note 1.1.12. In the fixed setting (Y, d, μ) , if the family of functions \mathcal{K}_Ω satisfies some properties, we will call *structural* a constant depending just on the setting and the constants appearing in the definition of such a property. For example, if \mathcal{K}_Ω satisfies Definition 1.1.6 and Definition 1.1.7, the constant may depend on ε , η_C , c , η_D , γ , the constant K of the triangle-inequality, and the doubling constants C_d and δ . Analogously for future properties.

Proposition 1.1.13. *Assume that \mathcal{K}_Ω satisfies the double ball property and the ε -critical density property for some $0 < \varepsilon < 1$. Fix $\eta = \frac{1}{2} \max\{\eta_C, \eta_D\}$. Then, for any $\alpha > 0$ and any $u \in \mathcal{K}_\Omega(B_{\eta R}(x_0))$ with*

$$\mu(\{x \in B_R(x_0) : u(x) \geq \alpha\}) \geq \varepsilon \mu(B_R(x_0)),$$

we have

$$\inf_{B_R(x_0)} u \geq \alpha c \gamma.$$

Proof. It is straightforward by applying first the critical density to the function $\frac{u}{\alpha}$ and then the double ball property to the function $\frac{u}{c\alpha}$. This can be done thanks to (I) and the definition of η . \square

Lemma 1.1.14. *Let \mathcal{K}_Ω satisfy the double ball property and the ε -critical density for some $0 < \varepsilon < 1$. There exist structural positive constants σ , M , θ satisfying the following condition. If $u \in \mathcal{K}_\Omega(B_{\theta R}(x_0))$ with $\inf_{B_R(x_0)} u \leq 1$ and $\alpha > 0$, $\rho < 2KR$, $y \in B_R(x_0)$ are such that*

$$\mu(\{x \in B_\rho(y) : u(x) > \alpha\}) \geq \varepsilon\mu(B_\rho(y)),$$

then $\rho \leq (\frac{M}{\alpha})^\sigma R$.

Proof. In order to make the arguments easier, we can assume that $\eta_C = \eta_D =: \eta$ by Remark 1.1.8. We are going to prove this lemma with

$$\theta = K(1 + 2K\eta), \quad \sigma = \frac{\log 2}{\log \gamma^{-1}}, \quad M = \frac{1}{c\gamma}(4K)^{\frac{1}{\sigma}}.$$

Fix u , α , ρ , and y as in the statement. By the last Proposition we get

$$\inf_{B_\rho(y)} u \geq \alpha c\gamma$$

since $B_{\frac{\eta}{2}\rho}(y) \subset B_{\theta R}(x_0)$. Since also $B_{\eta\rho}(y) \subset B_{\theta R}(x_0)$, by the double ball property we have

$$\inf_{B_{2\rho}(y)} u \geq \alpha c\gamma^2.$$

We could iterate this argument if $B_{2^{p-1}\eta\rho}(y) \subset B_{\theta R}(x_0)$: in this case we would have

$$\inf_{B_{2^p\rho}(y)} u \geq \alpha c\gamma^{p+1}.$$

Let us now choose an integer number $p \geq 1$ such that $2^{p-1} \leq \frac{2KR}{\rho} \leq 2^p$.

With this choice the following inclusions hold true

$$B_R(x_0) \subset B_{2^p\rho}(y), \quad B_{2^{p-1}\eta\rho}(y) \subset B_{\theta R}(x_0).$$

Thus, we have just proved that

$$1 \geq \inf_{B_R(x_0)} u \geq \inf_{B_{2^p\rho}(y)} u \geq \alpha c\gamma^{p+1}.$$

Since $2 = \gamma^{-\sigma}$, we get

$$\rho \leq 4KR2^{-p} = 4KR\gamma^{\sigma p} \leq \frac{4KR}{(\alpha c \gamma)^\sigma} = R \left(\frac{M}{\alpha} \right)^\sigma.$$

□

1.2 Power decay

In the previous section we have presented the critical density and the double ball property as the main notions of this thesis. Their importance is due to the fact that they can imply a *power decay* property for the functions in \mathcal{K}_Ω . This last property is crucial to get an Harnack inequality result. Let us start with the definition.

Definition 1.2.1. *The family \mathcal{K}_Ω satisfies the power decay property if there exist constants $\eta_0, M_0 > 1$, and $0 < \gamma_0 < 1$ such that, for any $u \in \mathcal{K}_\Omega(B_{\eta_0 R}(x_0))$ with $\inf_{B_R(x_0)} u \leq 1$, we have*

$$\mu \left(\left\{ x \in B_{\frac{R}{2}}(x_0) : u(x) > M_0^k \right\} \right) \leq \gamma_0^k \mu \left(B_{\frac{R}{2}}(x_0) \right) \quad \text{for } k = 1, 2, \dots$$

To prove such a property, Di Fazio, Gutiérrez, and Lanconelli established a result of Besicovitch type for the metric balls in (Y, d, μ) (see [15], Lemma 3.1). From this result, by exploiting the *log-ring* condition of Definition 1.1.5, they proved the following theorem.

Theorem 1.2.2. *Assume that (Y, d, μ) satisfies the log-ring condition. Suppose also that $\mu(B_{R_0}(\xi_0)) < \delta \mu(B_{2R_0}(\xi_0))$. If $E \subset B_{R_0}(\xi_0)$ is a μ -measurable set with $\mu(E) > 0$, then there exists a constant $c(\delta) \in (0, 1)$ (depending just on δ) and a family of balls $\{B_j := B_{r_j}(x_j)\}_{j=1}^\infty$ with $r_j \leq 3KR_0$ satisfying*

(i) *for any j the points x_j are density points for E , i.e. $\frac{\mu(B_r(x_j) \cap E)}{\mu(B_r(x_j))} \rightarrow 1$ as $r \rightarrow 0^+$;*

(ii) *$E \subset \cup_{j=1}^\infty B_j$ μ -almost everywhere;*

(iii) *$\frac{\mu(B_j \cap E)}{\mu(B_j)} = \delta$ for $j = 1, 2, \dots$;*

$$(iv) \quad \mu(E) \leq c(\delta)\mu\left(\bigcup_{j=1}^{\infty} B_j\right).$$

We say that the family $\{B_j := B_{r_j}(x_j)\}_{j=1}^{\infty}$ is a covering of E at the level δ .

We refer the reader to [15] (Theorem 3.3) for the proof. Here, we are mainly interested in the properties of the functions. That is why we want to show the statement and the complete proof of the power decay for \mathcal{K}_{Ω} starting from the double ball and the critical density (Theorem 4.7 in [15], set of assumptions (A)).

Theorem 1.2.3. *Let us assume that the log-ring condition and the reverse doubling condition in Ω hold true in (Y, d, μ) . Suppose that \mathcal{K}_{Ω} satisfies the double ball property and the ε -critical density for some $0 < \varepsilon < 1$. Then, the family \mathcal{K}_{Ω} satisfies the power decay property for some suitable structural constants.*

Proof. By Remark 1.1.8 we can assume that $\varepsilon \geq \delta$, where δ is the constant in the reverse doubling condition. We are also going to keep the notations of θ and M used in Proposition 1.1.13 and Lemma 1.1.14. Let η_0 and M_1 be positive numbers we will determine later. Fix $u \in \mathcal{K}_{\Omega}(B_{\eta_0 R}(x_0))$ with $\inf_{B_R(x_0)} u \leq 1$ and denote $\eta = \max\{\eta_C, \eta_D\}$. Let us put

$$E_k = \{x \in B_{\eta_0 R}(x_0) : u(x) \geq M_1^k\}, \quad \text{for } k = 1, 2, \dots$$

We claim that we can build up a family of balls $B_k := B_{t_k}(x_0)$ for a suitable sequence $R =: t_0 > t_1 > t_2 > \dots > \frac{1}{2}R$ such that

$$\mu(B_{k+1} \cap E_{k+2}) \leq c(\varepsilon)\mu(B_k \cap E_{k+1}), \quad \text{for } k = 0, 1, \dots \quad (1.2)$$

The constant $c(\varepsilon) \in (0, 1)$ is the one appearing in Theorem 1.2.2. Once we have proved it, we are done. As a matter of fact, we would have $\mu(\{x \in B_{\frac{R}{2}}(x_0) : u(x) > M_1^{k+2}\}) \leq \mu(B_{k+1} \cap E_{k+2})$ for any k and thus we get

$$\begin{aligned} \mu\left(\left\{x \in B_{\frac{R}{2}}(x_0) : u(x) > M_1^{k+2}\right\}\right) &\leq (c(\varepsilon))^{k+1}\mu(B_R(x_0)) \\ &\leq (c(\varepsilon))^{k+1}C_d\mu\left(B_{\frac{R}{2}}(x_0)\right) \end{aligned}$$

by the doubling property. Let us now choose a positive integer k_0 such that $c(\varepsilon)^{k_0} C_d < 1$ and set $\gamma_0 = c(\varepsilon)$. If $M_0 = M_1^{k_0+2}$, we deduce

$$\begin{aligned} \mu \left(\left\{ x \in B_{\frac{R}{2}}(x_0) : u(x) > M_0^k \right\} \right) &\leq \mu \left(\left\{ x \in B_{\frac{R}{2}}(x_0) : u(x) > M_1^{k+k_0+1} \right\} \right) \\ &\leq \gamma_0^k \mu \left(B_{\frac{R}{2}}(x_0) \right). \end{aligned}$$

That is why it is enough to prove (1.2) for any $k = 0, 1, \dots$

Case $k = 0$. Let us apply Theorem 1.2.2 to the set $B_1 \cap E_2$ at the level ε : we will fix t_1 later. The theorem gives us the existence of the covering $\{B_{r_j}(x_j)\}$, where the x_j 's are density points for $B_1 \cap E_2$ and

$$\varepsilon = \frac{\mu(B_{r_j}(x_j) \cap B_1 \cap E_2)}{\mu(B_{r_j}(x_j))} \leq \frac{\mu(B_{r_j}(x_j) \cap E_2)}{\mu(B_{r_j}(x_j))}. \quad (1.3)$$

Moreover we have $r_j \leq 3Kt_1 \leq 3KR$ for any j . We want to choose t_1 such that

$$B_{r_j}(x_j) \subset B_0 \cap E_1 \quad \forall j = 1, 2, \dots$$

Thus the condition (iv) of Theorem 1.2.2 would imply (1.2) for $k = 0$. We first prove that $B_{r_j}(x_j) \subset E_1$ for any j . If $\eta_0 > K(1 + \frac{3}{2}K\eta)$, then $B_{\frac{\eta}{2}r_j}(x_j) \subset B_{\eta_0 R}(x_0)$ since $x_j \in \overline{B_1}$. Hence $u \in \mathcal{K}_\Omega(B_{\frac{\eta}{2}r_j}(x_j))$ and the equation (1.3) says that

$$\mu(\{x \in B_{r_j}(x_j) : u(x) \geq M_1^2\}) \geq \varepsilon \mu(B_{r_j}(x_j)).$$

By Proposition 1.1.13 we get $u \geq M_1^2 c\gamma$ in $B_{r_j}(x_j)$. If we pick $M_1 > \frac{1}{c\gamma}$, we thus have $B_{r_j}(x_j) \subset E_1$. For the second inclusion, we want to show that $r_j \leq 2KR$ for any j . By contradiction, suppose this is not true for some fixed j . Then $B_{2KR}(x_j) \subset B_{r_j}(x_j)$. Since $x_j \in \overline{B_1}$ and $t_1 < R$, we get

$$\inf_{B_R(x_0)} u \geq \inf_{B_{2KR}(x_j)} u \geq \inf_{B_{r_j}(x_j)} u \geq M_1^2 c\gamma > M_1 > 1$$

which is a contradiction. We are now ready to prove that $B_{r_j}(x_j) \subset B_0$ for all j by exploiting Lemma 1.1.14. In our notations $y = x_j$, $\rho = r_j$, and $\alpha = M_1^2$. If $\eta_0 \geq \theta$ we obtain $r_j \leq \left(\frac{M}{M_1^2}\right)^\sigma R$. For $\xi \in B_{r_j}(x_j)$, the Hölder property of

Definition 1.1.3 yields

$$\begin{aligned} d(\xi, x_0) &\leq d(x_j, x_0) + \beta (d(x_j, \xi))^\alpha (d(x_j, x_0) + d(x_j, \xi))^{1-\alpha} \\ &\leq t_1 + \beta \left(\frac{M}{M_1^2} \right)^{\sigma\alpha} R^\alpha \left(t_1 + \left(\frac{M}{M_1^2} \right)^\sigma R \right)^{1-\alpha}. \end{aligned}$$

Let us write $t_1 = T_1 R$. By denoting $\beta_1 = \beta M^{\sigma\alpha}$, we get

$$d(\xi, x_0) \leq \left(T_1 + \beta_1 \frac{1}{M_1^{2\sigma\alpha}} \left(T_1 + \left(\frac{M}{M_1^2} \right)^\sigma \right)^{1-\alpha} \right) R.$$

Choose $T_1 = \frac{3}{4}$. If $M_1^2 > 4^{\frac{1}{\sigma}} M$, the last inequality implies

$$d(\xi, x_0) \leq \left(\frac{3}{4} + \beta_1 \frac{1}{M_1^{2\sigma\alpha}} \right) R.$$

If M_1 is big enough such that $\beta_1 \frac{1}{M_1^{2\sigma\alpha}} < \frac{1}{4}$, we have $B_{r_j}(x_j) \subset B_R(x_0)$ and the case $k = 0$ is done.

Case $k = 1$. We argue similarly. For some t_2 , we apply Theorem 1.2.2 to the set $B_2 \cap E_3$ at the level ε . Thus we have a covering $\{B_{r_j}(x_j)\}$ for suitable $r_j \leq 3Kt_2 \leq 3KR$ such that the inequality corresponding to (1.3) holds true. In order to conclude, we want as before a $t_2 = T_2 R < T_1 = \frac{3}{4} R$ so that $B_{r_j}(x_j) \subset B_1 \cap E_2$ for any j . The inclusion $B_{r_j}(x_j) \subset E_2$ is implied by Proposition 1.1.13 for the same choice of η_0 of the case $k = 0$: we have indeed that

$$\inf_{B_{r_j}(x_j)} u \geq M_1^3 c\gamma > M_1^2.$$

The arguments of the previous case yield also that $r_j \leq 2KR$ (since $x_j \in \overline{B_2}$).

Hence we can apply Lemma 1.1.14 and obtain

$$r_j \leq \left(\frac{M}{M_1^3} \right)^\sigma R \quad \text{for any } j.$$

If $\xi \in B_{r_j}(x_j)$, from the Hölder property of d we get

$$\begin{aligned} d(\xi, x_0) &\leq d(x_j, x_0) + \beta (d(x_j, \xi))^\alpha (d(x_j, x_0) + d(x_j, \xi))^{1-\alpha} \\ &\leq t_2 + \beta \left(\frac{M}{M_1^3} \right)^{\sigma\alpha} R^\alpha \left(t_2 + \left(\frac{M}{M_1^3} \right)^\sigma R \right)^{1-\alpha} \\ &= \left(T_2 + \beta_1 \frac{1}{M_1^{3\sigma\alpha}} \left(T_2 + \left(\frac{M}{M_1^3} \right)^\sigma \right)^{1-\alpha} \right) R. \end{aligned}$$

We note that $T_2 + \left(\frac{M}{M_1^3}\right)^\sigma < T_1 + \left(\frac{M}{M_1^2}\right)^\sigma < 1$ with the choice $M_1^2 > 4^{\frac{1}{\sigma}} M$.

This implies

$$d(\xi, x_0) \leq \left(T_2 + \beta_1 \frac{1}{M_1^{3\sigma\alpha}}\right) R.$$

The choice $T_2 = T_1 - \beta_1 \frac{1}{M_1^{3\sigma\alpha}}$ gives us the desired inclusion and concludes the case $k = 1$.

For the sake of clearness, let us try to sum up the proof for the general k . We can fix $\eta_0 = \theta > K(1 + \frac{3}{2}K\eta)$. We are going to fix also a positive number $M_1 > \max\{\frac{1}{c^\gamma}, 2^{\frac{1}{\sigma}}\sqrt{M}\}$. Let us denote $q = \frac{1}{M^{\alpha\sigma}}$. We choose the sequence $t_k = T_k R$, for $k = 0, 1, \dots$, with

$$T_0 = R, \quad T_1 = \frac{3}{4}, \quad T_2 = T_1 - \beta_1 q^3, \quad \dots, \quad T_k = T_1 - \beta_1 q^3 \sum_{l=0}^{k-2} q^l.$$

We fix M_1 big enough such that

$$\beta_1 q^3 \sum_{l=0}^{+\infty} q^l < \frac{1}{4}, \quad \text{in particular we have } T_k > \frac{1}{2} \quad \forall k.$$

No more choices are needed. Theorem 1.2.2 provides us a covering $\{B_{r_j}(x_j)\}$ for the set $B_{k+1} \cap E_{k+2}$ at the level ε . The property

$$\varepsilon = \frac{\mu(B_{r_j}(x_j) \cap B_{k+1} \cap E_{k+2})}{\mu(B_{r_j}(x_j))} \leq \frac{\mu(B_{r_j}(x_j) \cap E_{k+2})}{\mu(B_{r_j}(x_j))}$$

allows us to say that $u \geq M_1^{k+2} c^\gamma > M_1^{k+1}$ in $B_{r_j}(x_j)$. Hence $B_{r_j}(x_j) \subset E_{k+1}$. Moreover, arguing as before, we get $r_j \leq \left(\frac{M}{M_1^{k+2}}\right)^\sigma R$. Thus, for $\xi \in B_{r_j}(x_j)$, we have

$$\begin{aligned} d(\xi, x_0) &\leq t_{k+1} + \beta \left(\frac{M}{M_1^{k+2}}\right)^{\sigma\alpha} R^\alpha \left(t_{k+1} + \left(\frac{M}{M_1^{k+2}}\right)^\sigma R\right)^{1-\alpha} \\ &= \left(T_{k+1} + \beta_1 q^{k+2} \left(T_{k+1} + \left(\frac{M}{M_1^{k+2}}\right)^\sigma\right)^{1-\alpha}\right) R. \end{aligned}$$

We note that $T_{k+1} + \left(\frac{M}{M_1^{k+2}}\right)^\sigma < T_1 + \left(\frac{M}{M_1^2}\right)^\sigma < 1$. Therefore we deduce

$$d(\xi, x_0) < (T_{k+1} + \beta_1 q^{k+2}) R = T_k R$$

and $B_{r_j}(x_j) \subset B_k$. The last condition in Theorem 1.2.2 yields

$$\mu(B_{k+1} \cap E_{k+2}) \leq c(\varepsilon)\mu\left(\bigcup_{j=1}^{\infty} B_{r_j}(x_j)\right) \leq c(\varepsilon)\mu(B_k \cap E_{k+1})$$

and the equation (1.2) is finally proved. We have already showed how to conclude with the right choice of M_0 . \square

In [15] it has been recognized another set of assumptions under which the power decay property holds true. The general setting is always the one of doubling Hölder quasi-metric spaces and the reverse doubling condition is still assumed. The assumption which has been dropped is the log-ring condition (Definition 1.1.5). We will see in Section 3.1 that this condition is easily satisfied in homogeneous Lie groups, which is the setting we will fix from Chapter 3 up to the end. Nonetheless, it might be difficult to verify this condition without knowing explicitly the measure of the balls. The weaker condition they took in consideration in [15] is the continuity of the metric balls, i.e. they assumed

$$r \mapsto \mu(B_r(x_0)) \quad \text{is continuous for any } x_0 \in Y. \quad (1.4)$$

Any doubling metric space satisfying the segment property verifies this continuity condition ([15], Lemma 2.8). The Carnot-Carathéodory spaces related to families of vector fields in \mathbb{R}^N (with the Carnot-Carathéodory distance, or control distance) do satisfy the segment property. Thus, this new set of structural assumptions is enough general for being applied to the whole class of Carnot-Carathéodory spaces. Without further comments, we just report the statement of this alternative approach to the power decay proved in [15] (Theorem 4.7, set of assumptions (B)).

Theorem 1.2.4. *Let (Y, d, μ) be a doubling Hölder quasi-metric space where the reverse doubling condition hold true in an open set $\Omega \subset Y$. Suppose that the condition (1.4) is satisfied. Then,*

$$\text{if } \mathcal{K}_\Omega \text{ satisfies the } \varepsilon\text{-critical density for some } 0 < \varepsilon < \frac{1}{C_d^2},$$

the family \mathcal{K}_Ω satisfies also the power decay property for some suitable structural constants.

1.3 An abstract Harnack inequality

We have mentioned everywhere throughout the thesis that the aim of this approach is to get an Harnack inequality. For the sake of completeness, we want to show the details of the last step in this direction: the power decay property implies an abstract Harnack inequality result. We verbatim proceed as in [15] (Theorem 5.1 and Theorem 5.2).

Theorem 1.3.1. *Let (Y, d, μ) be a doubling quasi-metric Hölder space and let $\Omega \subset Y$ be an open set. Suppose that \mathcal{K}_Ω satisfies the power decay property. Let us assume in addition that*

$$\text{if } u \in \mathcal{K}_\Omega(B_R(x_0)) \text{ and } u \leq M \text{ in } B_R(x_0), \text{ then } M - u \in \mathcal{K}_\Omega(B_R(x_0)).$$

Then, there exist positive constants η, C independent of u, R , and x_0 such that, if $u \in \mathcal{K}_\Omega(B_{\eta R}(x_0))$ is nonnegative and locally bounded, we have

$$\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u.$$

We start with the proof of the following lemma.

Lemma 1.3.2. *We assume the same hypotheses of Theorem 1.3.1. Let η_0, M_0, γ_0 be the constants of the power decay and let Q be the constant appearing in (1.1). Let $u \in \mathcal{K}_\Omega(B_{2\eta_0 R}(z_0))$ be such that $\inf_{B_{2R}(x_0)} u \leq 1$. Then, there exists a structural constant c such that,*

$$\text{if } x_0 \in B_R(z_0) \text{ and } k \geq 2 \text{ are such that } u(x_0) \geq M^k$$

$$\text{and } B_\rho(x_0) \subset B_R(z_0) \text{ with } \rho = c\gamma_0^{\frac{k}{Q}} R,$$

then

$$\sup_{B_\rho(x_0)} u \geq \left(1 + \frac{1}{M_0}\right) u(x_0).$$

Proof. Fix u, x_0, k , and ρ as in the statement. By the power decay property we have

$$\mu(A_1) := \mu(\{x \in B_R(z_0) : u(x) \geq M_0^{k-1}\}) \leq \gamma_0^{k-1} \mu(B_R(z_0)).$$

Let us argue by contradiction. In particular, by the assumptions on \mathcal{K}_Ω , we suppose that the function

$$w(x) = M_0 + 1 - \frac{M_0}{u(x_0)}u(x)$$

belongs to $\mathcal{K}_\Omega(B_\rho(x_0))$. By definition we have also $w(x_0) = 1$ and then $\inf_{B_{\frac{\rho}{\eta_0}}(x_0)} w \leq 1$. We thus can apply again the power decay and obtain

$$\mu(A_2) := \mu\left(\{x \in B_{\frac{\rho}{2\eta_0}}(x_0) : w(x) \geq M_0\}\right) \leq \gamma_0 \mu\left(B_{\frac{\rho}{2\eta_0}}(x_0)\right).$$

In our notations we have $B_{\frac{\rho}{2\eta_0}}(x_0) \subset A_1 \cup A_2$. As a matter of fact, since $B_{\frac{\rho}{2\eta_0}}(x_0) \subset B_\rho(x_0) \subset B_R(z_0)$, if $x \in B_{\frac{\rho}{2\eta_0}}(x_0) \setminus A_1$ then $u(x) < M_0^{k-1}$ and so $w(x) > M_0 + 1 - \frac{M_0^k}{u(x_0)} \geq M_0$ which means $x \in A_2$. Therefore we get

$$\mu\left(B_{\frac{\rho}{2\eta_0}}(x_0)\right) \leq \mu(A_1) + \mu(A_2) \leq \gamma_0^{k-1} \mu(B_R(z_0)) + \gamma_0 \mu\left(B_{\frac{\rho}{2\eta_0}}(x_0)\right).$$

Since $B_R(z_0) \subset B_{2KR}(x_0)$, the doubling property (1.1) gives us

$$\mu(B_R(z_0)) \leq \mu(B_{2KR}(x_0)) \leq C_d \left(\frac{4K\eta_0 R}{\rho}\right)^Q \mu\left(B_{\frac{\rho}{2\eta_0}}(x_0)\right).$$

Hence we have

$$\mu\left(B_{\frac{\rho}{2\eta_0}}(x_0)\right) \leq \left(\gamma_0^{k-1} C_d \left(\frac{4K\eta_0 R}{\rho}\right)^Q + \gamma_0\right) \mu\left(B_{\frac{\rho}{2\eta_0}}(x_0)\right).$$

Since $\mu(B_{\frac{\rho}{2\eta_0}}(x_0)) > 0$, this implies

$$1 - \gamma_0 \leq \gamma_0^{k-1} C_d \left(\frac{4K\eta_0 R}{\rho}\right)^Q = \frac{C_d}{\gamma_0} \left(\frac{4K\eta_0}{c}\right)^Q.$$

By choosing c big enough we have a contradiction. \square

Proof of Theorem 1.3.1. Fix $\eta = K(1 + 2K\eta_0)$. We want to prove that, for any fixed u as in the statement with in addition $\inf_{B_R(x_0)} u < 1$, we have $\sup_{B_R(x_0)} u \leq C$. To this aim, we will prove that, for any ball $B_R(z)$ with $z \in B_R(x_0)$, we have

$$u(x) \leq C \left(\frac{R}{R - d(x, z)}\right)^{\frac{\nu}{\alpha}} \quad \forall x \in B_R(z)$$

for a structural constant ν , where α is the exponent in the Hölder property of d . Once we have proved this, by taking $x = z$ we get the desired inequality. Fix $\nu > 0$ such that $\gamma_0^{\frac{\nu}{Q}} = \frac{1}{M_0}$, where here and in what follows we keep the notations of the power decay property. Let us denote

$$f(x, R) = \left(\frac{R - d(x, z)}{R} \right)^{\frac{\nu}{\alpha}}$$

and define

$$D := \sup_{x \in B_R(z)} u(x) f(x, R).$$

Since for $D = 0$ there is nothing to prove, let us assume $D > 0$ and let $0 < D^* < D$. We want to bound from above D^* by a structural constant C . Take $x^* \in B_R(z)$ such that $D^* < u(x^*) f(x^*, R)$ and let k be an integer such that

$$M_0^k \leq u(x^*) \leq M_0^{k+1}.$$

Let \bar{k} be a structural constant we will fix later on. If $k \leq \bar{k}$, then

$$D^* < M_0^{\bar{k}+1} f(x^*, R) \leq M_0^{\bar{k}+1}$$

and we are done. Thus, suppose that $k > \bar{k}$. We have

$$f(x^*, R) > \frac{D^*}{M_0^{k+1}} = \frac{D^*}{M_0} \gamma_0^{\frac{\nu k}{Q}} = \frac{D^*}{M_0} \left(\frac{\rho}{cKR} \right)^\nu$$

with $\rho = c\gamma_0^{\frac{k}{Q}} KR$ and c as in Lemma 1.3.2. We now assume by contradiction that, for any possible structural constant β_* , we would have always

$$\frac{D^*}{M_0 c^\nu K^\nu} \geq \beta_*^{\frac{\nu}{\alpha}}.$$

By the definition of f we would get

$$R - d(x^*, z) > \beta_* R^{1-\alpha} \rho^\alpha \quad \text{that is} \quad d(x^*, z) < R - \beta_* R^{1-\alpha} \rho^\alpha. \quad (1.5)$$

For any $y \in B_\rho(x^*)$, the Hölder property of d implies that

$$\begin{aligned} d(y, z) &\leq d(z, x^*) + \beta (d(x^*, y))^\alpha (d(x^*, y) + d(z, x^*))^{1-\alpha} \\ &\leq R - \beta_* R^{1-\alpha} \rho^\alpha + \beta \rho^\alpha (\rho + R)^{1-\alpha}. \end{aligned}$$

Let us fix \bar{k} (bigger than 2) such that, for any $k \geq \bar{k}$,

$$\left(1 + \frac{\rho}{R}\right)^{1-\alpha} = \left(1 + cK\gamma_0^{\frac{k}{Q}}\right)^{1-\alpha} \leq \left(1 + cK\gamma_0^{\frac{\bar{k}}{Q}}\right)^{1-\alpha} < 2. \quad (1.6)$$

With this choice, if in addition $\beta_* > 2\beta$, we get

$$d(y, z) < R - \beta_* R^{1-\alpha} \rho^\alpha + 2\beta \rho^\alpha R^{1-\alpha} < R$$

which means $B_\rho(x^*) \subset B_R(z)$. Since $z \in B_R(x_0)$, we know also $B_{2KR}(z) \supset B_R(x_0)$ and $\inf_{B_{2KR}(z)} u < 1$. We recall that $u(x^*) \geq M_0^k$ and we note that $u \in \mathcal{K}_\Omega(B_{2KR\eta_0}(z))$. Therefore we are in the position to apply Lemma 1.3.2.

This gives

$$\sup_{B_\rho(x^*)} u \geq u(x^*) \left(1 + \frac{1}{M_0}\right) > \frac{D^*}{f(x^*, R)} \left(1 + \frac{1}{M_0}\right).$$

On the other hand, since $B_\rho(x^*) \subset B_R(z)$, we have

$$\sup_{B_\rho(x^*)} u \leq D \sup_{y \in B_\rho(x^*)} \frac{1}{f(y, R)} = \frac{D}{f(x^*, R)} \sup_{y \in B_\rho(x^*)} \frac{f(x^*, R)}{f(y, R)}.$$

Using again the Holder property, we deduce

$$\begin{aligned} \left(\frac{f(x^*, R)}{f(y, R)}\right)^{\frac{\alpha}{\nu}} &= \frac{R - d(z, x^*)}{R - d(y, z)} \leq \frac{R - d(z, x^*)}{R - (d(z, x^*) + \beta \rho^\alpha (d(z, x^*) + \rho)^{1-\alpha})} \\ &\leq \frac{1}{1 - \frac{\beta \rho^\alpha (d(z, x^*) + \rho)^{1-\alpha}}{\beta_* \rho^\alpha R^{1-\alpha}}} \leq \frac{1}{1 - \frac{2\beta}{\beta_*}} \end{aligned}$$

where we have exploited the relations (1.5) and (1.6). By putting together the two bounds for $\sup_{B_\rho(x^*)} u$ we obtain

$$1 + \frac{1}{M_0} \leq \frac{D}{D^*} \left(\frac{\beta_*}{\beta_* - 2\beta}\right)^{\frac{\nu}{\alpha}}.$$

Letting $D^* \rightarrow D$, this leads to a contradiction since, for β_* big enough (in particular bigger than 2β), it is not possible that

$$1 + \frac{1}{M_0} \leq \left(\frac{\beta_*}{\beta_* - 2\beta}\right)^{\frac{\nu}{\alpha}}.$$

Hence, for a suitable choice of β_* , we have $\frac{D^*}{M_0 c^\nu K^\nu} \leq \beta_*^{\frac{\nu}{\alpha}}$. For what we said at the beginning of the proof, this completes the argument. \square

The Hölder regularity result readily follows.

Corollary 1.3.3. *In addition to the hypotheses of Theorem 1.3.1 we add*

if $u \in \mathcal{K}_\Omega(B_R(x_0))$ and $u \geq m$ in $B_R(x_0)$, then $u - m \in \mathcal{K}_\Omega(B_R(x_0))$.

Then there exist $\eta \geq 2$, $\tilde{C}, k_0 > 0$ and $\alpha_0 \in (0, 1]$ such that, for any locally bounded function $u \in \mathcal{K}_\Omega(B_{\eta R}(x_0))$, we have

$$|u(x) - u(y)| \leq C \left(\frac{d(x, y)}{R} \right)^{\alpha_0} \sup_{B_R(x_0)} |u| \quad \forall x, y \in B_{\frac{R}{k_0}}(x_0).$$

Proof. We could say that the Harnack inequality implies the ε -critical density for any ε (see Remark 1.3.5 below) and then conclude via Theorem 1.1.9 and Corollary 1.1.10. Since the implication ‘‘Harnack \Rightarrow Hölder’’ is classical in the literature, we want to use the standard arguments (as in [15], Theorem 5.3). Let η, C be the constants in Theorem 1.3.1. Fix $k_0 = K(1 + 2K)$ and put $r = \frac{2KR}{k_0}$. Thus we have $B_r(y) \subset B_R(x_0)$ for any $y \in B_{\frac{R}{k_0}}(x_0)$. Take a locally bounded function $u \in \mathcal{K}_\Omega(B_{\eta R}(x_0))$, with \mathcal{K}_Ω as in the statement. For any fixed $y \in B_{\frac{R}{k_0}}(x_0)$, let us denote by

$$M(\rho) := \sup_{B_\rho(y)} u, \quad m(\rho) := \inf_{B_\rho(y)} u, \quad \omega(\rho) := M(\rho) - m(\rho)$$

for $0 < \rho \leq r$. We can apply the Harnack inequality to the nonnegative functions $M(\rho) - u$ and $u - m(\rho)$. This gives respectively

$$M(\rho) - m\left(\frac{1}{2}\rho\right) \leq \sup_{B_\rho(y)} (M(\rho) - u) \leq C \inf_{B_\rho(y)} (M(\rho) - u) \leq M(\rho) - M\left(\frac{1}{2}\rho\right),$$

$$M\left(\frac{1}{2}\rho\right) - m(\rho) \leq \sup_{B_\rho(y)} (u - m(\rho)) \leq C \inf_{B_\rho(y)} (u - m(\rho)) \leq m\left(\frac{1}{2}\rho\right) - m(\rho).$$

By summing up these inequalities we get

$$\omega(\rho) + \omega\left(\frac{1}{2}\rho\right) \leq C \left(\omega(\rho) - \omega\left(\frac{1}{2}\rho\right) \right),$$

which means

$$\omega\left(\frac{1}{2}\rho\right) \leq \frac{C-1}{C+1}\omega(\rho).$$

Therefore we have obtained the result of Theorem 1.1.9 by using the Harnack inequality. The standard arguments in the proof of Corollary 1.1.10 allows us to conclude the proof. \square

Note 1.3.4. We have already said that, if we want \mathcal{K}_Ω to satisfy the critical density and the double ball property (and then the power decay), we should interpret it as the family of the nonnegative functions u satisfying $\mathcal{L}u \leq 0$ for some specific operators. The additional hypotheses we have just seen regarding

$$\begin{aligned} \cdot u \in \mathcal{K}_\Omega(B_R(x_0)) \text{ and } u \leq M \text{ in } B_R(x_0) &\Rightarrow M - u \in \mathcal{K}_\Omega(B_R(x_0)) \\ \cdot u \in \mathcal{K}_\Omega(B_R(x_0)) \text{ and } u \geq m \text{ in } B_R(x_0) &\Rightarrow u - m \in \mathcal{K}_\Omega(B_R(x_0)), \end{aligned}$$

have to be interpreted as the request for \mathcal{K}_Ω to be the family of the nonnegative solutions to the equation $\mathcal{L}u = 0$. As a matter of fact the classical Harnack inequality and the Hölder regularity result hold true for that family. See also Theorem 1.1.9 and Corollary 1.1.10. In Chapter 2 all these notions and notations will be clearer.

Remark 1.3.5. Theorems 1.2.3 and 1.3.1 tell us that the critical density and the double ball property (with additional structural hypotheses) imply Harnack inequality. We want to finish this Section by showing a kind of reverse. To this aim, let us assume the Harnack inequality for \mathcal{K}_Ω as in the statement of Theorem 1.3.1. If $u \in \mathcal{K}_\Omega(B_{2\eta R}(x_0))$ is a nonnegative locally bounded function with $\inf_{B_R(x_0)} u \geq 1$, then we have

$$\inf_{B_{2R}(x_0)} u \geq \frac{1}{C} \sup_{B_{2R}(x_0)} u \geq \frac{1}{C}$$

and the double ball property is satisfied. On the other hand, for a nonnegative locally bounded function $u \in \mathcal{K}_\Omega(B_{2\eta R}(x_0))$ with $\inf_{B_R(x_0)} u < \frac{1}{C}$, we have

$$\sup_{B_{2R}(x_0)} u \leq C \inf_{B_{2R}(x_0)} u \leq \inf_{B_R(x_0)} u < 1$$

and thus we get

$$\mu(\{x \in B_{2R}(x_0) : u(x) < 1\}) = \mu(B_{2R}(x_0)) \geq \varepsilon \mu(B_{2R}(x_0))$$

for any $0 < \varepsilon < 1$. Therefore also the ε -critical density is satisfied for any ε .

1.4 Inbetween definitions

We have already mentioned that we changed a little bit Definition 1.1.6-1.1.7 with respect to [15]. We have introduced the constants η_D and η_C , instead of fixing them to 4. Here we want to consider other little variations. We are going to allow the constant 2 to change: is the double radius "special" in some sense?

Definition 1.4.1. ("More" Abstract Double Ball Property) *We could say that \mathcal{K}_Ω satisfies the double ball property if there exist $\eta_D'' > \eta_D' > 1$ and $\gamma > 0$ such that, for every $B_{\eta_D'' R}(x_0) \subset \Omega$ and every $u \in \mathcal{K}_\Omega(B_{\eta_D'' R}(x_0))$ with $\inf_{B_R(x_0)} u \geq 1$, we have*

$$\inf_{B_{\eta_D'' R}(x_0)} u \geq \gamma.$$

Definition 1.4.2. ("More" Abstract ε -Critical Density) *Let $0 < \varepsilon < 1$. We could say that \mathcal{K}_Ω satisfies the ε -critical density property if there exist $\eta_C'' > \eta_C' > 1$ and $c > 0$ such that, for every $B_{\eta_C'' R}(x_0) \subset \Omega$ and for every $u \in \mathcal{K}_\Omega(B_{\eta_C'' R}(x_0))$ with*

$$\mu(\{x \in B_{\eta_C'' R}(x_0) : u(x) \geq 1\}) \geq \varepsilon \mu(B_{\eta_C'' R}(x_0)),$$

we have

$$\inf_{B_R(x_0)} u \geq c.$$

Definition 1.1.6 and 1.1.7 seem to be less general than these ones. We are going to show why and in which sense they are actually not. These modifications are not made just for speculative reasons. If we look at the proof of the critical density and the double ball property in [18] (respectively Theorem

2.1.1 and Theorem 2.1.2), we may see that the definitions are satisfied in this latter and *more abstract* sense: we can also refer the reader to our Chapter 2 (Theorem 2.1.2, 2.1.3, and 2.1.4). On the other hand, by investigating the little gap, we take the opportunity to discuss more in details the definitions we have given and their relationships.

Remark 1.4.3. If the double ball property with respect to Definition 1.4.1 is satisfied with $\eta'_D \geq 2$, then it is trivial that the double ball property (w.r.t Definition 1.1.6) holds true with $\eta_D = \eta''_D$ and the same γ .

It is easy also the case of the ε -critical density w.r.t Definition 1.4.2 with $\eta'_C \leq 2$. As a matter of fact, if this is satisfied, we put $\eta_C = 2\eta''_C$ (any number greater than $\frac{2\eta''_C}{\eta'_C}$ will be fine). If $u \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$ is such that $\mu(\{x \in B_{2R}(x_0) : u(x) \geq 1\}) \geq \varepsilon\mu(B_{2R}(x_0))$, then we get $u \geq c$ in $B_{\frac{2}{\eta'_C}R}(x_0) \supseteq B_R(x_0)$. Thus, the ε -critical density w.r.t Definition 1.1.7 is satisfied too.

Let us give a definitive answer to the problem of defining the double ball property. We are going to exploit just that \mathcal{K}_Ω is closed under multiplications by positive constants (assumption (I)).

Proposition 1.4.4. *Definition 1.4.1 "implies" Definition 1.1.6. Hence, these definitions are equivalent.*

Proof. Suppose the double ball property w.r.t Definition 1.4.1 is satisfied (with constants $\eta''_D, \eta'_D, \gamma$). By the last Remark, we are left with the case $1 < \eta'_D < 2$. Let n_0 be the first positive integer such that $(\eta'_D)^{n_0} \geq 2$. Put η_D any real number greater than $\eta''_D(\eta'_D)^{n_0-1}$, for example let us fix $\eta_D = 2\eta''_D$. If $u \in \mathcal{K}_\Omega(B_{\eta_D R}(x_0))$ with $u \geq 1$ in $B_R(x_0)$, then by hypothesis we get $\frac{u}{\gamma} \geq 1$ in $B_{\eta'_D R}(x_0)$. Moreover, by (I) we have $\frac{u}{\gamma} \in \mathcal{K}_\Omega(B_{\eta_D R}(x_0))$. Thus, $u \geq \gamma^2$ in $B_{(\eta'_D)^2 R}(x_0)$ since $\eta''_D \eta'_D \leq 2$. We can apply this argument $n_0 - 1$ times because we supposed $\eta_D \geq \eta''_D(\eta'_D)^{n_0-1}$. At the end we will get

$$u \geq \gamma^{n_0} \quad \text{in } B_{(\eta'_D)^{n_0} R}(x_0) \supseteq B_{2R}(x_0).$$

□

We can thus stop taking care of Definition 1.4.1: from now on, we refer to Definition 1.1.6 when we talk about double ball property. The critical density property is more delicate. The following proposition shows that Definition 1.1.7 and Definition 1.4.2 are equivalent if we assume the double ball property.

Proposition 1.4.5. *Suppose \mathcal{K}_Ω satisfies the double ball property. Then, Definition 1.4.2 "implies" Definition 1.1.7 (with the same ε).*

Proof. Suppose the ε -critical density w.r.t Definition 1.4.2 is satisfied (with constants η_C'', η_C'). By Remark 1.4.3, we are left with the case $\eta_C' > 2$. We denote as usual by η_D, γ the constants of the double ball property. Fix $\eta_C = \max\{\eta_C'', \eta_D\}$. Let n_0 be the first positive integer such that $\frac{1}{2^{n_0}} \leq \frac{2}{\eta_C}$. If $u \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$ with $\mu(\{x \in B_{2R}(x_0) : u(x) \geq 1\}) \geq \varepsilon \mu(B_{2R}(x_0))$, then by hypothesis we get $\frac{u}{c} \geq 1$ in $B_{\frac{2}{\eta_C} R}(x_0)$ since $\eta_C > \frac{2\eta_C''}{\eta_C}$. Now we can argue similarly to the proof of the last proposition. By using (I), we have also $\frac{u}{c} \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$. By the double ball property, $u \geq c\gamma$ in $B_{\frac{4}{\eta_C} R}(x_0)$ since $\eta_C > \frac{2\eta_D}{\eta_C}$. We can apply this argument n_0 times because $\eta_C \geq \eta_D \frac{2^{n_0}}{\eta_C}$. At the end we will get

$$u \geq c\gamma^{n_0} \quad \text{in } B_{\frac{2^{n_0+1}}{\eta_C} R}(x_0) \supseteq B_R(x_0).$$

□

Roughly speaking, in Definition 1.4.2 we can always allow η_C' to be larger; but, if we want to make η_C' smaller, we have to assume for example the double ball property.

Remark 1.4.6. By applying the same arguments of Proposition 1.1.11, we can see how the ε -critical density w.r.t Definition 1.4.2 implies the double ball property if ε is small enough. To be precise, it has to be $0 < \varepsilon < \frac{1}{C_a^2(\eta_C')^Q}$ (by exploiting (1.1)). Therefore, by the last Proposition, Definition 1.4.2 and 1.1.7 are equivalent if the first one is satisfied for small ε .

On the other hand, allowing ε to be bigger, we can see the same fact in a different way. Suppose the ε -critical density w.r.t Definition 1.4.2 satisfied

(with constants η''_C, η'_C, c and $\eta'_C > 2$) for some $\varepsilon < \frac{1}{(\eta'_C)^Q}$. Then, put $\eta_C = \eta''_C$ and $\varepsilon' = \varepsilon(\eta'_C)^Q$. Take $u \in \mathcal{K}_\Omega(B_{\eta_C R}(x_0))$ with

$$\mu(\{x \in B_{2R}(x_0) : u(x) \geq 1\}) \geq \varepsilon' \mu(B_{2R}(x_0)).$$

Since $B_{\eta'_C R}(x_0) \supset B_{2R}(x_0)$, by using (1.1) we get

$$\mu(\{x \in B_{\eta'_C R}(x_0) : u(x) \geq 1\}) \geq \frac{\varepsilon'}{(\eta'_C)^Q} \mu(B_{\eta'_C R}(x_0)).$$

Thus, $u \geq c$ in $B_R(x_0)$ and the ε' -critical density w.r.t Definition 1.1.7 is satisfied.

Summing up, we have seen (beyond the problems of the definitions) how the critical density and the double ball property are connected and once more how they can work together. We would like to remark that the arguments we used here are somehow similar to the ones in Proposition 1.1.13 and Lemma 1.1.14 and they will reappear in the next chapters.

Chapter 2

The elliptic case

Despite the shortness, this chapter is the core of all the investigations pursued through the whole thesis. The notions studied and the arguments developed in Chapter 1 find here a true justification. Furthermore, this is the starting point for most of the discussions in the following chapters. We are going to give complete proofs for the double ball property and the critical density for uniformly elliptic operators: this gives us the Krylov-Safonov's Harnack inequality with a method different from [24]. To this aim, our main reference is [18].

2.1 Uniformly elliptic operators

Let us consider a second order linear operator in non-divergence form

$$\mathcal{L}_A = \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i x_j}^2, \quad \text{for } x \in \Omega \subset \mathbb{R}^n. \quad (2.1)$$

We suppose that the matrix $A(x) = (a_{ij}(x))_{i,j=1}^n$ is symmetric and uniformly positive definite, i.e. there exist $0 < \lambda \leq \Lambda$ such that, for any x , we have

$$\lambda \|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda \|\xi\|^2$$

for every $\xi \in \mathbb{R}^n$. We are going to denote by $M_n(\lambda, \Lambda)$ the set of the $n \times n$ symmetric matrices satisfying these bounds. We will write simply $A \in$

$M_n(\lambda, \Lambda)$ instead of writing $A(x) \in M_n(\lambda, \Lambda)$ for every x in the open set Ω . With respect to the regularity of the coefficients of $A(x)$, we can assume they are smooth but we do not want that the estimates we are interested in depend on this regularity.

We would like here to clarify the notations and the abstract language of Chapter 1. The properties introduced in Section 1.1 will be showed for the following families of functions

$$\mathcal{K}_\Omega^A := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \geq 0 \text{ and } \mathcal{L}_A u \leq 0 \text{ in } V\}. \quad (2.2)$$

with $A \in M_n(\lambda, \Lambda)$. We stress that the family \mathcal{K}_Ω^A clearly satisfies the invariance property (I). In this situation, the doubling quasi-metric Hölder space is \mathbb{R}^n with the euclidean distance and the Lebesgue measure: without further comments we can say that it easily satisfies any structural assumption needed in the exposition of Chapter 1. In particular any ball $B_R(x_0)$ in this Chapter will be euclidean and the euclidean norm will be denoted by $\|\cdot\|$.

Remark 2.1.1. Keeping in mind the Krylov-Safonov's result, we want something which depends on A just through the ellipticity constants λ and Λ . To this aim, the family \mathcal{K}_Ω^A has to satisfy the double ball property and the critical density uniformly for A belonging to the class $M_n(\lambda, \Lambda)$.

Let us start with the double ball property. In [18] (Theorem 2.1.2) it is proved the following theorem.

Theorem 2.1.2. *Fix $0 < \lambda \leq \Lambda$. There exists a positive constant γ depending on n, λ, Λ such that for any $A \in M_n(\lambda, \Lambda)$, if*

$$u \geq 0 \text{ classically satisfies } \mathcal{L}_A u \leq 0 \text{ in } \Omega \supset \overline{B_{2R}(x_0)} \quad \text{with } \inf_{B_R(x_0)} u \geq 1,$$

we have

$$u \geq \gamma \quad \text{in } B_{\frac{3}{2}R}(x_0).$$

The proof presented in [18] compares some powers of the function u with a suitable barrier function. Those arguments are taken from [8] (Theorem 2), where Caffarelli and Gutiérrez proved a doubling property for some peculiar

sections in their study of the linearized Monge-Ampère equation. Here we want to give a different proof. Our arguments rely on some barriers introduced by Hopf in his celebrated Hopf's Lemma (see [21]). Our proof suggests a kind of link between the double ball property and the Dirichlet problem in the exterior of the euclidean balls. The statement reads now as follows.

Theorem 2.1.3. *Fix $0 < \lambda \leq \Lambda$. There exists a positive constant $\nu \in (1, 2)$ depending on n, λ, Λ such that for any $A \in M_n(\lambda, \Lambda)$, if*

$$u \geq 0 \text{ classically satisfies } \mathcal{L}_A u \leq 0 \text{ in } \Omega \supset \overline{B_{2R}(x_0)} \quad \text{with } \inf_{B_R(x_0)} u \geq 1,$$

we have

$$u \geq \frac{1}{2} \quad \text{in } B_{\nu R}(x_0).$$

Proof. We are going to prove the statement for $R = 1$ and $x_0 = 0$. The arguments hold true for any R and x_0 with the same constant ν by rescaling and translating the problem (we refer the mistrustful reader to our Remark 3.2.8). Fix a point $\xi \in \partial B_1(0)$ and take a ball $B_\rho(\xi_0)$ which is tangent to $\partial B_1(0)$ at ξ and strictly contained in $B_1(0)$, that is

$$\overline{B_\rho(\xi_0)} \setminus \{\xi\} \subset B_1(0).$$

Let's say $\xi_0 = \frac{1}{2}\xi$ and $\rho = \frac{1}{2}$. For some positive constant α to be fixed, consider the Hopf's barrier function

$$h(x) := e^{-\alpha\rho^2} - e^{-\alpha\|x-\xi_0\|^2}.$$

We remark that $h(\xi) = 0$ and

$$\{x \in \mathbb{R}^n : h(x) \leq 0\} \setminus \{\xi\} \subseteq B_1(0).$$

Moreover a simple calculation shows

$$\partial_{x_i x_j}^2 h(x) = 2\alpha e^{-\alpha\|x-\xi_0\|^2} (\delta_{ij} - 2\alpha(x - \xi_0)_i(x - \xi_0)_j)$$

for $i, j = 1, \dots, n$. For any $A \in M_n(\lambda, \Lambda)$ we thus get

$$\begin{aligned} \mathcal{L}_A h(x) &= 2\alpha e^{-\alpha\|x-\xi_0\|^2} \left(\text{Tr}(A(x)) - 2\alpha \langle A(x)(x - \xi_0), (x - \xi_0) \rangle \right) \\ &\leq 2\alpha e^{-\alpha\|x-\xi_0\|^2} \left(n\Lambda - 2\alpha\lambda\|x - \xi_0\|^2 \right) =: H(x). \end{aligned}$$

The function H depends just on n, λ, Λ : it is uniform for $A \in M_n(\lambda, \Lambda)$. We note that

$$H(\xi) = 2\alpha e^{-\alpha\rho^2} (n\Lambda - 2\alpha\rho^2\lambda) = 2\alpha e^{-\frac{\alpha}{4}} \left(n\Lambda - \frac{1}{2}\alpha\lambda \right).$$

Hence, if we fix $\alpha = 4n\frac{\Lambda}{\lambda}$, we get $H(\xi) < 0$ and there exists an open neighborhood U_ξ of ξ (depending just on H) where H is negative: with our choices we can take $U_\xi = B_{\frac{1}{12}}(\xi)$. Let us put $V_\xi = (U_\xi \cap B_2(0)) \setminus \overline{B_1(0)}$, we have that $h \geq 0$ and $\mathcal{L}_A h \leq 0$ in V_ξ . Let us now consider the boundary $\partial V_\xi = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial V_\xi \cap \partial B_1(0)$ and $\Gamma_2 = \partial V_\xi \setminus \Gamma_1$. The number $m = \inf_{\Gamma_2} h$ is strictly positive since $\{x \in \partial V : h(x) = 0\} = \{\xi\}$. The function $w = 1 - \frac{1}{m}h$ is thus well defined. We get

$$\mathcal{L}_A w = -\frac{1}{m}\mathcal{L}_A h \geq 0 \text{ in } V_\xi, \quad w \leq 1 \text{ on } \Gamma_1 \text{ and } w \leq 0 \text{ on } \Gamma_2.$$

Take now a function u as in the statement. We have

$$\mathcal{L}_A u \leq \mathcal{L}_A w \text{ in } V_\xi, \quad u \geq w \text{ on } \partial V_\xi.$$

By the weak maximum principle for elliptic operators (see e.g. [17], Theorem 3.1), $u \geq w$ in V_ξ . Since $w(\xi) = 1$, there exists an open neighborhood W_ξ of ξ contained in $U_\xi \cap B_2(0)$ such that $w \geq \frac{1}{2}$ in $W_\xi \cap V_\xi$. The set W_ξ depends only on the barrier function and on U_ξ . The compact set $\overline{B_1(0)}$ is contained in the open set $O := B_1(0) \cup (\cup_{\xi \in \partial B_1(0)} W_\xi)$. Hence there exists $\nu > 1$ such that $B_\nu(0) \subset O$. Therefore, we deduce

$$u \geq \frac{1}{2} \text{ in } B_\nu(0)$$

since $u \geq 1$ in $B_1(0)$ and $u \geq w = w_\xi$ in $W_\xi \cap V_\xi$. \square

Theorem 2.1.2 and Theorem 2.1.3 are saying the same by Proposition 1.4.4: \mathcal{K}_Ω^A satisfies (uniformly for $A \in M_n(\lambda, \Lambda)$) the double ball property. For an identical reason (remind Proposition 1.4.5 or Remark 1.4.3) the following theorem is the ε -critical density for \mathcal{K}_Ω^A . We report here the very elegant proof which can be found in [18] (Theorem 2.1.1). This proof does not

make use of the convex envelope, nor any explicit use of the Alexandrov-Bakelman-Pucci estimate. The arguments follow instead an idea by Cabré in [6] (Lemma 4.1), where he established a critical density estimate for some non-divergent elliptic equations on Riemannian manifolds with nonnegative sectional curvature. For any fixed ellipticity constants λ and Λ we are going to see that \mathcal{K}_Ω^A satisfies (uniformly for $A \in M_n(\lambda, \Lambda)$) the ε -critical density property for every $1 > \varepsilon \geq 1 - \left(\frac{\lambda}{7\Lambda}\right)^n$. We denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$.

Theorem 2.1.4. *Fix $0 < \lambda \leq \Lambda$. Then, for any $A \in M_n(\lambda, \Lambda)$, if $u \geq 0$ is a classical solution of $\mathcal{L}_A u \leq 0$ in $B_{2R}(x_0)$ such that*

$$\left| \left\{ x \in B_{\frac{7}{4}R}(x_0) : u(x) \geq 1 \right\} \right| \geq \left(1 - \left(\frac{\lambda}{7\Lambda} \right)^n \right) \left| B_{\frac{7}{4}R}(x_0) \right|,$$

we have

$$u \geq \frac{8}{33} \quad \text{in } B_R(x_0).$$

Proof. Fix $A \in M_n(\lambda, \Lambda)$. Suppose $u \geq 0$, $\mathcal{L}_A u \leq 0$ in $B_{2R}(x_0)$, and $\inf_{B_R(x_0)} u < 1$. We want to prove that

$$\left| \left\{ x \in B_{\frac{7}{4}R}(x_0) : u(x) < \frac{33}{8} \right\} \right| \geq \left(\frac{\lambda}{\Lambda} \right)^n \left| B_{\frac{R}{4}}(x_0) \right| = \left(\frac{\lambda}{7\Lambda} \right)^n \left| B_{\frac{7}{4}R}(x_0) \right|.$$

For any $y \in B_{\frac{R}{4}}(x_0)$, we put

$$\phi_y(x) = \frac{R^2}{4}u(x) + \frac{1}{2}\|x - y\|^2.$$

We look for the minimum of ϕ_y : it cannot be very far from x_0 . For $x \in B_R(x_0)$, we have $\|x - y\| \leq \frac{5}{4}R$ and thus

$$\inf_{B_R(x_0)} \phi_y < \frac{R^2}{4} + \frac{1}{2} \left(\frac{5}{4}R \right)^2 = \frac{33}{32}R^2.$$

On the other hand, if $x \in B_{2R}(x_0) \setminus B_{\frac{7}{4}R}(x_0)$, we have

$$\|x - y\| \geq \|x - x_0\| - \|y - x_0\| \geq \frac{3}{2}R \quad \text{and} \quad \phi_y(x) \geq \frac{9}{8}R^2 = \frac{36}{32}R^2.$$

Hence we get

$$\inf_{B_{2R}(x_0)} \phi_y = \inf_{B_{\frac{7}{4}R}(x_0)} \phi_y$$

and there exists $z \in B_{\frac{7}{4}R}(x_0)$ such that $\phi_y(z) = \inf_{B_{2R}(x_0)} \phi_y$. Consider the set

$$H := \left\{ z \in B_{\frac{7}{4}R}(x_0) : \exists y \in B_{\frac{R}{4}}(x_0) \text{ such that } \phi_y(z) = \inf_{B_{2R}(x_0)} \phi_y \right\}.$$

We remark that, if $z \in H$ and y is a related point as in the definition of H , then we have

$$\begin{aligned} \frac{R^2}{4} u(z) &\leq \frac{R^2}{4} u(z) + \frac{1}{2} \|z - y\|^2 = \phi_y(z) \\ &= \inf_{B_{2R}(x_0)} \phi_y \leq \inf_{B_{BR}(x_0)} \phi_y < \frac{33}{32} R^2. \end{aligned}$$

This implies

$$H \subset \left\{ x \in B_{\frac{7}{4}R}(x_0) : u(x) < \frac{33}{8} \right\}. \quad (2.3)$$

Since the points $z \in H$ are minimum points for some ϕ_y , we have

$$\nabla \phi_y(z) = 0 \quad \text{and} \quad \mathcal{H}_{\phi_y}(z) \geq 0,$$

where we denote by $\mathcal{H}_{\phi_y}(z)$ the Hessian matrix of ϕ_y at z . Hence we get there is just one point y related to z : we have indeed

$$0 = \nabla \phi_y(z) = \frac{R^2}{4} \nabla u(z) + z - y, \quad \text{i.e.} \quad y = \frac{R^2}{4} \nabla u(z) + z.$$

Let us define the map

$$\varphi(z) = \frac{R^2}{4} \nabla u(z) + z.$$

By construction we have $B_{\frac{R}{4}}(x_0) \subset \varphi(H)$. Then, by changing the variables inside the integral, we have

$$\left| B_{\frac{R}{4}}(x_0) \right| \leq \int_H |\mathcal{J}_\varphi(x)| dx,$$

where $\mathcal{J}_\varphi(x)$ is the determinant of the Jacobian matrix of φ at x , that is

$$\mathcal{J}_\varphi(x) = \det \left(\frac{R^2}{4} \mathcal{H}_u(x) + \mathbb{I}_n \right) = \det \left(\mathcal{H}_{\phi_{(\cdot)}}(x) \right).$$

In particular, $\mathcal{J}_\varphi(x) \geq 0$ for $x \in H$ since $\mathcal{H}_{\phi_{(\cdot)}}(x)$ is nonnegative definite. Now we exploit the following fact generalizing the arithmetic-geometric inequality:

if M is a symmetric and nonnegative definite $n \times n$ matrix we have

$$\det(AM) \leq \left(\frac{\operatorname{Tr}(AM)}{n} \right)^n$$

for any symmetric $n \times n$ matrix $A \geq 0$.

In our situation $M = \frac{R^2}{4} \mathcal{H}_u(x) + \mathbb{I}_n$ and $A \in M_n(\lambda, \Lambda)$. We get

$$\begin{aligned} \left| B_{\frac{R}{4}}(x_0) \right| &\leq \frac{1}{n^n} \int_H \frac{1}{\det(A(x))} \left(\operatorname{Tr} \left(A(x) \left(\frac{R^2}{4} \mathcal{H}_u(x) + \mathbb{I}_n \right) \right) \right)^n dx \\ &= \frac{1}{n^n} \int_H \frac{1}{\det(A(x))} \left(\frac{R^2}{4} \mathcal{L}_A u(x) + \operatorname{Tr}(A(x)) \right)^n dx \\ &\leq \frac{1}{n^n} \int_H \frac{1}{\det(A(x))} (\operatorname{Tr}(A(x)))^n dx \end{aligned}$$

by reminding that $\operatorname{Tr}(A(x)\mathcal{H}_u(x)) = \mathcal{L}_A u(x) \leq 0$ in $B_{2R}(x_0) \supset H$. Since $A \in M_n(\lambda, \Lambda)$ we have

$$\operatorname{Tr}(A(x)) \leq n\Lambda \quad \text{and} \quad \det(A(x)) \geq \lambda^n$$

and therefore

$$\left| B_{\frac{R}{4}}(x_0) \right| \leq \frac{\Lambda^n}{\lambda^n} |H|.$$

By recalling (2.3) we deduce the inequality

$$\left| B_{\frac{R}{4}}(x_0) \right| \leq \frac{\Lambda^n}{\lambda^n} \left| \left\{ x \in B_{\frac{7}{4}R}(x_0) : u(x) < \frac{33}{8} \right\} \right|$$

and we conclude the proof. \square

2.2 Krylov-Safonov's Harnack inequality

Since the double ball property and the critical density are satisfied for the family \mathcal{K}_Ω^A in (2.2), the machinery of Chapter 1 can be triggered. In particular, by Theorem 1.2.3, \mathcal{K}_Ω^A satisfies the power decay property. The constants involved in the power decay are what we called *structural*, i.e. they depend just on the setting $(\mathbb{R}^n, \|\cdot\|, |\cdot|)$ and on the constants of the main properties $(\varepsilon, c, \eta_C, \eta_D, \gamma)$. Since we have seen that \mathcal{K}_Ω^A satisfies these

properties uniformly for $A \in M_n(\lambda, \Lambda)$, we have uniformity even for the power decay.

Regarding the Harnack inequality and the Hölder regularity result, we have seen that \mathcal{K}_Ω has to satisfy some extra assumptions (see Remark 1.3.4). These are trivially not satisfied by \mathcal{K}_Ω^A : after all, the nonnegative solutions (not the supersolutions) to an elliptic equation are the candidates for satisfying such a Harnack inequality. That is why we consider the following subset of \mathcal{K}_Ω^A

$$\overline{\mathcal{K}_\Omega^A} := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \geq 0 \text{ and } \mathcal{L}_A u = 0 \text{ in } V\}.$$

Theorem 1.3.1 gives us the following Harnack inequality for $\overline{\mathcal{K}_\Omega^A}$ which is uniform for $A \in M_n(\lambda, \Lambda)$.

Theorem 2.2.1. *Fix $0 < \lambda \leq \Lambda$. Then there exist a positive constants C, η depending just on n, λ, Λ such that, for any $A \in M_n(\lambda, \Lambda)$, if u is a nonnegative solution of $\mathcal{L}_A u = 0$ in $\Omega \supset B_{\eta R}(x_0)$, we have*

$$\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u.$$

This is the celebrated result by Krylov and Safonov in [24]. Their original method relied on some probabilistic techniques (see also [23]). The importance of this result is given by the fact that C and η are independent of the regularity of the coefficient matrix A . This independence transfers directly to the Hölder regularity result (see Corollary 1.3.3). We already mentioned in the Introduction the impact it had on the theory of fully nonlinear elliptic equations. We have made the proof descend from the axiomatic approach of Chapter 1 in abstract settings. We refer the reader to [18] for an exhaustive study of these arguments in the euclidean setting and for the connections of this problem with the theory of the Monge-Ampère operator.

Chapter 3

The sub-elliptic case

A remarkable example of doubling Hölder quasi-metric space is given by the homogeneous Lie groups. In this chapter we investigate the horizontally elliptic operators as possible direction of application for the abstract theory of Chapter 1.

3.1 An application

Beyond the double-ball property and the ε -critical density, there are five assumptions (from Definition 1.1.1 up to Definition 1.1.5) in Chapter 1 concerning just the structure of the setting we deal with. It had been already stressed in [15] that an example where these structural assumptions are satisfied is the setting of homogeneous Lie groups. Let us show the reasons.

Definition 3.1.1. *Let $\mathbb{G} = (\mathbb{R}^N, \circ)$ a Lie group on \mathbb{R}^N , where we denote by \circ the group operation. We say that \mathbb{G} is an homogeneous Lie group if there exist N real numbers $1 \leq \sigma_1 \leq \dots \leq \sigma_N$ such that the dilation defined by*

$$\delta_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad \delta_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N)$$

is an automorphism of \mathbb{G} for any $\lambda > 0$.

We thus fix an homogeneous Lie group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$. We will denote always by \mathfrak{g} the Lie algebra of \mathbb{G} . We refer the reader to [3] for any unclear

notion we are going to exploit in these settings: where it is possible we try to define everything we need in the exposition or to keep the notations adopted there.

In such groups, the identity element is 0 and the exponential map $Exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a globally defined diffeomorphism with inverse denoted by Log (see e.g. [3], Theorem 1.3.28). We want to define also the Jacobian basis and some peculiar norms.

Definition 3.1.2. *For any $x \in \mathbb{G}$, consider the Jacobian matrix \mathcal{J}_{τ_x} at the origin of the left translation τ_x ($\tau_x(y) := x \circ y$ for $y \in \mathbb{G}$). For any $j = 1, \dots, N$, consider the vector field Z_j whose coefficients are given by the j -th column of \mathcal{J}_{τ_x} . The set $\{Z_1, \dots, Z_N\}$ is a basis of \mathfrak{g} and it is called the Jacobian basis (see [3], Proposition 1.2.4-1.2.7 and Definition 1.2.15).*

Definition 3.1.3. *We call homogeneous symmetric norm on \mathbb{G} any continuous function $d : \mathbb{G} \rightarrow [0, +\infty)$ such that*

- (i) *d is δ_λ -homogeneous of degree one, i.e. $d(\delta_\lambda(x)) = \lambda d(x)$ for any $\lambda > 0$ and $x \in \mathbb{G}$;*
- (ii) *$d(x) > 0$ if and only if $x \neq 0$;*
- (iii) *$d(x^{-1}) = d(x)$ for every $x \in \mathbb{G}$;*
- (iv) *d is smooth away from 0.*

In a relevant subclass of homogeneous Lie groups, homogeneous symmetric norms do exist.

Remark 3.1.4. Let us build up a good candidate for being an homogeneous symmetric norm. For any $x \in \mathbb{G}$, we can consider $Log(x) = \sum_{j=1}^N y_j Z_j$ where $\{Z_1, \dots, Z_N\}$ is the Jacobian basis. We may define

$$d(x) = (|y_1|^{2\sigma_2 \cdots \sigma_N} + |y_2|^{2\sigma_1 \sigma_3 \cdots \sigma_N} + |y_N|^{2\sigma_1 \cdots \sigma_{N-1}})^{\frac{1}{2\sigma_1 \cdots \sigma_N}}.$$

By exploiting the properties proved in [3] (Theorem 1.3.28 and Corollary 1.3.29) we can see that this function satisfies the conditions (i), (ii), and

(iii) in Definition 3.1.3. Regarding the smoothness, it depends on the σ_j 's. In the homogeneous Carnot groups (which is the setting we fix from the next Section until the end), we can always take the σ_j 's as positive (consecutive) integers and d is thus smooth. Therefore, in homogeneous Carnot groups d is an example of homogeneous symmetric norm (see [3], Example 5.1.2).

Every homogeneous symmetric norm induces in a natural way a quasi-distance in \mathbb{G} . As a matter of fact, we can denote

$$d(x, y) := d(y^{-1} \circ x) \quad \text{for } x, y \in \mathbb{R}^N.$$

By verbatim proceeding as in [3] (Proposition 5.1.6), we can say that there exists $K \geq 1$ such that

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for all $x, y, z \in \mathbb{R}^N$. Thus (\mathbb{R}^N, d) is a quasi-metric space.

It is possible to prove also an improved version of this pseudo-triangle inequality. In [15] (Remark 2.5) it is indeed proved that there exist β such that

$$d(x, y) \leq d(x, z) + \beta d(y, z) \quad \forall x, y, z \in \mathbb{R}^N$$

(see also [3], Proposition 5.14.1). This inequality readily implies the Hölder property for d with respect to Definition 1.1.3 with $\alpha = 1$.

Furthermore, let us denote by $|\cdot|$ the Lebesgue measure in \mathbb{R}^N . By the δ_λ -homogeneity of the d -balls and the Proposition 1.3.21 in [3], we get

$$|B_R(x_0)| = |B_R(0)| = R^Q |B_1(0)| =: c_Q R^Q, \quad (3.1)$$

where $Q := \sum_{i=1}^N \sigma_i$ is the homogeneous dimension of \mathbb{G} . Therefore, the doubling property holds true with constant $C_d = 2^Q$. This proves that $(\mathbb{R}^N, d, |\cdot|)$ is a doubling Hölder quasi-metric space.

The equation (3.1) will be largely used in what follows. For example, it implies also the other two structural assumptions involved in the approach of Chapter 1. The reverse doubling property is easily satisfied with constant $\delta = \frac{1}{2^Q}$. Finally, also the log-ring condition holds true since we have

$$|B_R(x_0) \setminus B_{(1-\varepsilon)R}(x_0)| = c_Q R^Q (1 - (1 - \varepsilon)^Q) \leq Q\varepsilon |B_R(x_0)|.$$

Hence, this section justifies the fact that it is worthwhile to investigate of the double ball property and ε -critical density in homogeneous Lie groups.

3.2 Horizontally elliptic operators

Suppose in addition \mathbb{G} is stratified, i.e. \mathbb{G} is an homogeneous Carnot group (see e.g. [3], Definition 1.4.1 and Remark 1.4.2).

Definition 3.2.1. *An homogeneous Lie group \mathbb{G} is said to be an homogeneous Carnot group if the following properties hold true*

- \mathbb{R}^N can be split in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \dots \times \mathbb{R}^{N_r}$ and the dilations take the form

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}) \quad \text{for } x^{(i)} \in \mathbb{R}^{N_i};$$

- let Z_1, \dots, Z_{N_1} be the left invariant vector fields on \mathbb{G} such that $Z_j(0) = \frac{\partial}{\partial x_j}$, then the smallest Lie algebra containing Z_1, \dots, Z_{N_1} has rank N (i.e. it is the whole \mathfrak{g}).

We also says that \mathbb{G} has step (of nilpotence) r and $N_1 =: m$ generators.

We denote by \mathfrak{g}_1 the linear subspace of \mathfrak{g} composed by the left-invariant vector fields δ_λ -homogeneous of degree one: at any x it is spanned by the vectors $Z_1(x), \dots, Z_m(x)$ of the definition. Fix a generic basis $\{X_1, \dots, X_m\}$ for \mathfrak{g}_1 .

Remark 3.2.2. For $j = 1, \dots, m$, let us write explicitly the vector fields X_j 's as

$$X_j(x) = \sum_{l=1}^N c_{jl}(x) \frac{\partial}{\partial x_l}.$$

By Proposition 1.3.5 in [3], the coefficients $c_{jl}(x)$ are polynomials for any $l = 1, \dots, N$ and any $j = 1, \dots, m$: in particular $c_{jl}(x)$ are constants c_{jl} for any $1 \leq l \leq m$. Also the vectors Z_1, \dots, Z_m are a basis for \mathfrak{g}_1 and we have $Z_j(0) = \partial_{x_j}$. The $m \times m$ constant matrix $C = (c_{ij})_{i,j=1}^m$ is thus the

one bringing the basis of \mathfrak{g}_1 $\{Z_1, \dots, Z_m\}$ into the basis $\{X_1, \dots, X_m\}$. In particular, C is a non-singular matrix.

Fix also an open set $\Omega \subseteq \mathbb{R}^N$. We want to consider the linear second order operator in non-divergence form

$$\mathcal{L}_A = \sum_{i,j=1}^m a_{ij}(x) X_i X_j \quad \text{for } x \in \Omega. \quad (3.2)$$

The symmetric matrix $A(x) = (a_{ij}(x))_{i,j=1}^m$ is supposed to be uniformly elliptic: we recall it means that there exist $0 < \lambda \leq \Lambda$ such that, for every x , we have

$$\lambda \|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda \|\xi\|^2$$

for every $\xi \in \mathbb{R}^m$. As in the previous chapter, we denote by $M_m(\lambda, \Lambda)$ the set of the $m \times m$ symmetric matrices satisfying these bounds.

Remark 3.2.3. If $A \in M_m(\lambda, \Lambda)$, then it is easy to see that the coefficients of the matrices $A(x)$ are uniformly bounded in x . For the diagonal elements we get $\lambda \leq a_{ii}(x) = \langle A(x)e_i, e_i \rangle \leq \Lambda$ for any i . For $i \neq j$ we have

$$2\lambda \leq a_{ii}(x) + a_{jj}(x) + 2a_{ij}(x) = \langle A(x)(e_i + e_j), (e_i + e_j) \rangle \leq 2\Lambda$$

and then we deduce $|a_{ij}(x)| \leq \Lambda - \lambda$ by the estimates for the diagonal elements.

Definition 3.2.4. If $A \in M_m(\lambda, \Lambda)$ for some $0 < \lambda \leq \Lambda$, the second order linear operator in non-divergence form \mathcal{L}_A defined in (3.2) is called *horizontally elliptic operator*.

It is well known that some Maximum Principles for this kind of operators hold true. Since we are going to exploit a Weak Maximum Principle for \mathcal{L}_A , we report here the statement and a sketch of the proof.

Theorem 3.2.5. (Weak Maximum Principle) Let \mathcal{L}_A be an horizontally elliptic operator, for some $A \in M_m(\lambda, \Lambda)$. Let O be an open bounded subset of \mathbb{R}^N . Suppose $u, v \in C(\overline{O}) \cap C^2(O)$ satisfy $u \geq v$ on ∂O and $\mathcal{L}_A u \leq \mathcal{L}_A v$ in O . Then, we have $u \geq v$ in O .

Proof. In the notations of Remark 3.2.2, the operator \mathcal{L}_A take the form

$$\mathcal{L}_A = \sum_{k,l=1}^N (C^t(x)A(x)C(x))_{kl} \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{l=1}^N b_l(x) \frac{\partial}{\partial x_l}, \quad (3.3)$$

where $b_l(x) = \sum_{i,j=1}^m \sum_{k=1}^N c_{ik}(x) a_{ij}(x) \frac{\partial c_{jl}(x)}{\partial x_k}$ and $C(x) = (c_{jl}(x))$ is an $m \times N$ matrix. Moreover, denoting $C_1 = (c_{11}, \dots, c_{m1})$, we have C_1 is a constant non-zero vector of \mathbb{R}^m by Remark 3.2.2. Thus, we get the uniform bound $(C^t(x)A(x)C(x))_{11} = \langle A(x)C_1, C_1 \rangle \geq \lambda \|C_1\|^2$. Furthermore, in the bounded set O the functions b_l are uniformly bounded since $c_{jl}(x)$ are smooth functions and $a_{ij}(x)$ are uniformly bounded by Remark 3.2.3. Therefore, under these hypotheses the Weak Maximum Principle can be proved as in [25] (Corollary 1.3). \square

In [4] it is proved an invariant Harnack inequality for horizontally elliptic operators \mathcal{L}_A assuming an Hölder continuity for the coefficients of A . Here we are interested in an Harnack inequality for horizontally elliptic operators which is independent of the regularity of $A(x)$. To this aim, we try to imitate the case of uniformly elliptic operators (see (2.2)) and set

$$\mathcal{K}_\Omega^A := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \geq 0 \text{ and } \mathcal{L}_A u \leq 0 \text{ in } V\}. \quad (3.4)$$

Once more we stress that \mathcal{K}_Ω^A is closed under multiplications by positive constants. We now fix the definitions of the double ball property and the ε -critical density for \mathcal{K}_Ω^A in this specific context with the additional uniformity condition with respect to the class $M_m(\lambda, \Lambda)$.

Definition 3.2.6. (*Double Ball Property for horizontally elliptic operators*) Fix an homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ with m generators. In \mathbb{G} , fix an homogeneous symmetric norm d and the vector fields X_1, \dots, X_m generating \mathfrak{g}_1 . In the doubling quasi-metric Hölder space $(\mathbb{G}, d, |\cdot|)$, let Ω be an open set. We say that the double ball property for horizontally elliptic operators is satisfied if, for every $0 < \lambda \leq \Lambda$, \mathcal{K}_Ω^A satisfies the double ball property with respect to Definition 1.1.6 for any $A \in M_m(\lambda, \Lambda)$.

The constants γ and η_D have to depend on A just through the ellipticity constants λ, Λ .

Definition 3.2.7. (*Critical Density for horizontally elliptic operators*) Fix an homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ with m generators. In \mathbb{G} , fix an homogeneous symmetric norm d and the vector fields X_1, \dots, X_m generating \mathfrak{g}_1 . In the doubling quasi-metric Hölder space $(\mathbb{G}, d, |\cdot|)$, let Ω be an open set. We say that the critical density property for horizontally elliptic operators is satisfied if, for every $0 < \lambda \leq \Lambda$, there exists $0 < \varepsilon < 1$ such that \mathcal{K}_Ω^A satisfies the ε -critical density property with respect to Definition 1.1.7 for any $A \in M_m(\lambda, \Lambda)$. The constants c and η_C have to depend on A just through the ellipticity constants λ, Λ .

Gutiérrez and Tournier considered in [19] the case of the Heisenberg group $\mathbb{H} = \mathbb{H}^1$ with generators

$$X_1 = \partial_{x_1} - \frac{x_2}{2} \partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + \frac{x_1}{2} \partial_{x_3}.$$

They chose the homogeneous norm

$$d(x_1, x_2, x_3) = ((x_1^2 + x_2^2)^2 + \mu x_3^2)^{\frac{1}{4}}$$

for some fixed constant μ . They worked in \mathbb{R}^3 , but all the arguments and the results work in \mathbb{R}^{2n+1} (i.e. in \mathbb{H}^n). In that context, they proved the double ball property for horizontally elliptic operators as we have just defined. They proved also a ε -critical density estimate by assuming a bound for the ratio $\frac{\Lambda}{\lambda}$ (so the property is not verified for every $\lambda \leq \Lambda$): they proved that \mathcal{K}_Ω^A satisfies the critical density in \mathbb{H} uniformly in the class of the matrices A belonging to $M_m(\lambda, \Lambda)$ and satisfying a prescribed bound for $\frac{\Lambda}{\lambda}$.

Remark 3.2.8. If we look at the very definition of the double ball and critical density properties, some relations have to be satisfied for all the balls $B_{\eta R}(x_0) \subset \Omega$. The properties of our setting and the choice of \mathcal{K}_Ω^A allows us to check the relations just for some fixed R and x_0 . As a matter of fact, suppose to have proved the property for a ball $B_\eta(0) \subset \Omega$ and every function

$u \in \mathcal{K}_\Omega^A(B_\eta(0))$, for any given $A \in M_m(\lambda, \Lambda)$. Let us now take another ball $B_{\eta R}(x_0) \subset \Omega$ and, for some fixed $A \in M_m(\lambda, \Lambda)$, a function $v \in \mathcal{K}_\Omega^A(B_{\eta R}(x_0))$. We put

$$u(x) := v(x_0 \circ \delta_R(x)).$$

Since the X_j 's are left-invariant and δ_λ -homogeneous of degree one, we get

$$\mathcal{L}_A u(x) = R^2 \sum_{i,j=1}^m a_{ij}(x) (X_i X_j v)(x_0 \circ \delta_R(x)) = R^2 \mathcal{L}_{\tilde{A}} v(x_0 \circ \delta_R(x)) \quad (3.5)$$

where $\tilde{A}(x) = A(\delta_{\frac{1}{R}}(x_0^{-1} \circ x))$. Thus, $u \in \mathcal{K}_\Omega^{\tilde{A}}(B_\eta(0))$ and we have assumed to know it satisfies some properties. We note that the matrices A and \tilde{A} have the same bounds of ellipticity, i.e. A and \tilde{A} belongs to the same class $M_m(\lambda, \Lambda)$. Hence, the properties under investigation transfer to v with the same constants. In fact, we have

$$\begin{aligned} |B_{2R}(x_0)| &= R^Q |B_2(0)|, \\ |\{x \in B_{2R}(x_0) : v(x) \geq 1\}| &= R^Q |\{x \in B_2(0) : u(x) \geq 1\}|, \\ \text{and also } \inf_{B_1(0)} u &= \inf_{B_R(x_0)} v. \end{aligned}$$

Remark 3.2.9. The hypothesis that our homogeneous Lie group is stratified is crucial. The problem is not the one described in Remark 3.1.4 about the existence of a homogeneous symmetric norm. The problem concerns the properties we are asking to be satisfied by \mathcal{K}_Ω^A . Suppose \mathbb{G} is an homogeneous Lie group with the dilations given by consecutive integers from 1 up to r , but assume G is not Carnot. Take a basis $\{X_1, \dots, X_m\}$ of the vector fields δ_λ -homogeneous of degree one and denote by $\text{Lie}\{X_1, \dots, X_m\}$ the smallest Lie algebra containing $\{X_1, \dots, X_m\}$. Then the dimension as a vector space of the set

$$\{Z(x) : Z \in \text{Lie}\{X_1, \dots, X_m\}\}$$

is a constant (by left-invariance, see e.g. [3], Proposition 1.2.13) strictly less than N . By Frobenius Theorem (see e.g. [32], 6-19, Theorem 5) there is a local change of variables $(y_1, \dots, y_N) = y = \varphi(x)$ such that y_N is not seen by

the vector fields X_j 's, i.e.

$$X_j \varphi_N = 0 \quad \text{for any } j \quad (\text{at least locally}).$$

In this case an Harnack inequality for \mathcal{L}_A cannot be satisfied: in fact Bony pointed out in [5] (Remarque 3.2) that for such operators we cannot hope in a Strong Maximum Principle. We would like to stress here that in this situation both the double ball property and the critical density for \mathcal{L}_A cannot be satisfied too. The reasons are similar. Let us first consider the double ball property. For any fixed intervals $I_1 \subset \bar{I}_1 \subset I_2$, it is not difficult to build up, for every $\gamma > 0$, a positive function ψ_γ greater than 1 in I_1 and less than γ somewhere in I_2 . The functions defined by

$$u(x) = \psi_\gamma(\varphi_N(x))$$

falsify the double ball property w.r.t. Definition 3.2.6 since $\mathcal{L}_A u = 0$: we can indeed consider

$$I_1 := \varphi_N(B_R(x_0)) \quad \text{and} \quad I_2 := \varphi_N(B_{2R}(x_0))$$

for some $B_R(x_0) \in \mathbb{G}$. About the critical density, for the same choice of I_1, I_2 and for any $c > 0$, we can construct a positive function ψ_c less than c in I_1 such that

$$|\{t \in I_2 : \psi_c(t) < 1\}|$$

is as small as we want. The functions $u(x) = \psi_c(\varphi_N(x))$ show that the ε -critical density cannot be satisfied for any $0 < \varepsilon < 1$.

3.3 Independence of choices

Parallel to Section 1.4, we are going to discuss here Definition 3.2.6 and Definition 3.2.7. They seem to depend on many possible choices we can do. We would like to show why they actually do not and in which sense they are somehow intrinsic.

First of all, the definition of horizontally elliptic operator in (3.2) depends on the choice of the basis $\{X_1, \dots, X_m\}$ of the first layer \mathfrak{g}_1 . Let $\{Y_1, \dots, Y_m\}$ be another basis of \mathfrak{g}_1 : this means there exists an $m \times m$ invertible matrix $D = (d_{ij})_{i,j=1}^m$ such that $Y_i = \sum_{j=1}^m d_{ij} X_j$. We can consider, for any $A \in M_m(\lambda, \Lambda)$, the operator $\tilde{\mathcal{L}}_A = \sum_{i,j=1}^m a_{i,j}(x) Y_i Y_j$. By putting $\tilde{A}(x) = D^t A(x) D$, we have

$$\tilde{\mathcal{L}}_A = \sum_{i,j=1}^m (D^t A(x) D)_{ij} X_i X_j = \mathcal{L}_{\tilde{A}}.$$

The symmetric matrix $\tilde{A}(x)$ is positive definite and we get $\tilde{A} \in M_m(\tilde{\lambda}, \tilde{\Lambda})$ for some $\tilde{\lambda} \leq \tilde{\Lambda}$ which are in general different from λ and Λ (see the Remark below). Since the conditions in Definition 3.2.6 and 3.2.7 have to be satisfied for every $\lambda \leq \Lambda$, we deduce that, if they hold true for a particular choice of $\{X_1, \dots, X_m\}$, then they hold true for every choice. Thus, these definitions are independent of the choice of the basis of \mathfrak{g}_1 .

Remark 3.3.1. We explicitly stress that the ratio between the maximum and the minimum eigenvalue increases by passing from A to \tilde{A} without knowing anything a priori about A and D . The best we can expect is, indeed, $\tilde{\lambda} = \lambda \sigma_D$ and $\tilde{\Lambda} = \Lambda \Sigma_D$, where σ_D and Σ_D are respectively the smallest and the biggest eigenvalue of $D^t D$. As a matter of fact, for any x , we get

$$\Lambda \Sigma_D \|\xi\|^2 \geq \Lambda \|D\xi\|^2 \geq \langle \tilde{A}(x)\xi, \xi \rangle = \langle A(x)D\xi, D\xi \rangle \geq \lambda \|D\xi\|^2 \geq \lambda \sigma_D \|\xi\|^2$$

for every $\xi \in \mathbb{R}^m$. Of course we have $\frac{\tilde{\Lambda}}{\tilde{\lambda}} \geq \frac{\Lambda}{\lambda}$ and they are equal if and only if D is a multiple of an orthogonal matrix.

Note 3.3.2. By the last considerations, if we allow the ellipticity constants λ and Λ just to have a bounded ratio in the Definition 3.2.6 and 3.2.7 (and not to be any possible $\lambda \leq \Lambda$), the double ball property and the critical density would not be stable under changes of basis. As we mentioned, this is the case for the critical density in \mathbb{H} proved by Gutiérrez and Tournier.

A very natural choice is the one regarding the Lebesgue measure. This is the Haar measure for \mathbb{G} : it is translation invariant and it is well behaved also with the dilations of the group (see [3], Proposition 1.3.21). Hence, we are

not going to change it. On the contrary, we would like to consider the choice of the homogeneous symmetric norm. Let us take two different ones: let us say d_1 and d_2 . In [3] (Proposition 5.1.3), it is proved that the homogeneous norms on \mathbb{G} are all equivalent, i.e. there exists a constant $K \geq 1$ such that

$$\frac{1}{K}d_2(x) \leq d_1(x) \leq Kd_2(x) \quad \text{for every } x \in \mathbb{G}.$$

This means that, for the d_1 -balls $B_R^1(x_0)$ and the d_2 -balls $B_R^2(x_0)$, there are the following relations

$$B_R^1(x_0) \subseteq B_{KR}^2(x_0), \quad B_R^2(x_0) \subseteq B_{KR}^1(x_0) \quad \text{for every } R > 0, x_0 \in \mathbb{G}.$$

By exploiting the considerations we did in Section 1.4, we can prove the following.

Proposition 3.3.3. *The double d_1 -ball property is equivalent to the double d_2 -ball property.*

Proof. Suppose the double ball property is satisfied with respect to the d_1 -balls. By Proposition 1.4.4, this is equivalent to suppose that, for any $\lambda \leq \Lambda$, there exist $\gamma > 0$ and $\eta > 2K^2$ such that, if

$$u \in \mathcal{K}_\Omega^A(B_{\eta R}^1(x_0)) \quad \text{with } u \geq 1 \text{ in } B_R^1(x_0), \quad \text{we have } u \geq \gamma \text{ in } B_{2K^2 R}^1(x_0)$$

(for every $A \in M_m(\lambda, \Lambda)$). Let us take a function $u \in \mathcal{K}_\Omega^A(B_{\eta R}^2(x_0))$ with $u \geq 1$ in $B_R^2(x_0)$. In particular we have $u \in \mathcal{K}_\Omega^A(B_{\frac{\eta R}{K}}^1(x_0))$ with $u \geq 1$ in $B_{\frac{R}{K}}^1(x_0)$. Thus, we get

$$u \geq \gamma \text{ in } B_{2KR}^1(x_0) \supseteq B_{2R}^2(x_0).$$

□

With regard to the critical density, its independence of the homogeneous norm is more delicate. This is related to the discussions we had about Definition 1.4.2.

Proposition 3.3.4. *The critical density property for the d_1 -balls implies the critical density for the d_2 -balls w.r.t. Definition 1.4.2.*

Proof. Suppose the critical density is satisfied with respect to the d_1 -balls. This means that, for any $\lambda \leq \Lambda$, there exist $0 < \varepsilon < 1$, $c > 0$ and $\eta_C > 2$ such that, if $u \in \mathcal{K}_\Omega^A(B_{\eta_C R}^1(x_0))$ with $\inf_{B_R^1(x_0)} u < c$, we have

$$|\{x \in B_{2R}^1(x_0) : u(x) < 1\}| > (1 - \varepsilon) |B_{2R}^1(x_0)|.$$

Put $\eta_C'' = K^2 \eta_C$ and $\eta_C' = 2K^2$. Take $u \in \mathcal{K}_\Omega^A(B_{\eta_C'' R}^2(x_0)) \subseteq \mathcal{K}_\Omega^A(B_{K \eta_C R}^2(x_0))$ with $\inf_{B_R^2(x_0)} u < c$. Since $B_R^2(x_0) \subseteq B_{KR}^1(x_0)$, we have $\inf_{B_{KR}^1(x_0)} u < c$. Thus, we get

$$\begin{aligned} |\{x \in B_{2K^2 R}^2(x_0) : u(x) < 1\}| &\geq |\{x \in B_{2KR}^1(x_0) : u(x) < 1\}| > \\ &> (1 - \varepsilon) |B_{2KR}^1(x_0)| \geq (1 - \varepsilon) |B_{2R}^2(x_0)| = \frac{1 - \varepsilon}{K^{2Q}} |B_{2K^2 R}^2(x_0)|. \end{aligned}$$

By noting that $0 < \frac{1 - \varepsilon}{K^{2Q}} < 1$, we can say that the $(1 - \frac{1 - \varepsilon}{K^{2Q}})$ -critical density w.r.t Definition 1.4.2 is satisfied. \square

From what we have seen before and in Section 1.4, we can state that the critical density is independent of the choice of the homogeneous norm if the double ball property is satisfied for some (and then for any) norm.

Finally, we want to discuss what is going to happen if we change the Lie group under isomorphism. With isomorphism we mean a diffeomorphism with respect to the differential structure and an isomorphism with respect to the group structure. It is known that homogeneous Carnot groups “are not left in” homogeneous Carnot groups by isomorphisms (see [3], Remark 2.2.4). Since we know how to apply the axiomatic approach in the setting of homogeneous Carnot groups, we could think of transferring this machinery by isomorphism. Let us say that $\phi : \mathbb{G} \rightarrow \tilde{\mathbb{G}}$ is an isomorphism where $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is an homogeneous Carnot group. By fixing an homogeneous symmetric norm d in \mathbb{G} , we could define

$$\tilde{d}(y_1, y_2) := d(\phi^{-1}(y_1), \phi^{-1}(y_2)) = d((\phi^{-1}(y_2))^{-1} \circ \phi^{-1}(y_1))$$

for any $y_1, y_2 \in \tilde{\mathbb{G}}$ and $\mu(E) = |\phi^{-1}(E)|$ for any set $E \subseteq \tilde{\mathbb{G}}$ such that $\phi^{-1}(E)$ is Lebesgue-measurable. Then, \tilde{d} is an Hölder continuous quasi-distance in

$\tilde{\mathbb{G}}$. Moreover, by definition we have $\tilde{B}_R(y_0) = \phi(B_R(\phi^{-1}(y_0)))$ and so we get $\mu(\tilde{B}_R(y_0)) = R^Q |B_1(0)|$. In $(\tilde{\mathbb{G}}, \tilde{d}, \mu)$ all the structural assumptions of Chapter 1 are satisfied. For an open set $\tilde{\Omega} = \phi(\Omega)$ in $\tilde{\mathbb{G}}$ we can define by isomorphism also

$$K_{\tilde{\Omega}} := \{u : u \circ \phi \in \mathcal{K}_{\Omega}\}$$

(we apologize for denoting by \circ also the composition of functions): thus, the double ball property (or the critical density) for $K_{\tilde{\Omega}}$ in this new setting is equivalent by definition to the same property in the old one. What about Definition 3.2.6 and 3.2.7? Do they have their own meaning in this new setting? “To be a Carnot group” (not necessarily homogeneous) is an invariant property under isomorphisms (see [3], Proposition 2.2.10). Moreover, if $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ is a stratification for the Lie algebra of \mathbb{G} , then $d\phi(\mathfrak{g}_1) \oplus \dots \oplus d\phi(\mathfrak{g}_r)$ is a stratification for the Lie algebra of $\tilde{\mathbb{G}}$, where $d\phi$ is the differential of the isomorphism ϕ evaluated at the origin. Thus, fixing a basis X_1, \dots, X_m of \mathfrak{g}_1 , we have $\{Y_j = d\phi(X_j)\}_{j=1}^m$ is a basis of the first layer of the $\tilde{\mathbb{G}}$ Lie algebra. Since $Y_j(\phi(x)) = d_x\phi(X_j(x))$ for any $x \in \mathbb{G}$, we get

$$\begin{aligned} K_{\tilde{\Omega}}^A &:= \{u : u \circ \phi \in \mathcal{K}_{\Omega}^A\} = \\ &= \left\{ u \in C^2(\phi(V), \mathbb{R}) : V \subset \Omega, u(\phi(x)) \geq 0 \text{ and} \right. \\ &\quad \left. \sum_{i,j=1}^m a_{ij}(x) X_i X_j (u \circ \phi)(x) \leq 0 \text{ for every } x \in V \right\} = \\ &= \left\{ u \in C^2(\phi(V), \mathbb{R}) : \phi(V) \subset \tilde{\Omega}, u(y) \geq 0 \text{ and} \right. \\ &\quad \left. \sum_{i,j=1}^m a_{ij}(\phi^{-1}(y)) Y_i Y_j (u)(y) \leq 0 \text{ for every } y \in \phi(V) \right\} \end{aligned}$$

for every $A \in M_m(\lambda, \Lambda)$. Putting $\tilde{A}(y) = A(\phi^{-1}(y))$, of course we have $\tilde{A} \in M_m(\lambda, \Lambda)$. Hence, $K_{\tilde{\Omega}}^A$ is exactly what we would have naturally denoted by $K_{\tilde{\Omega}}^{\tilde{A}}$ in order to define a double ball (respectively, a critical density) property for horizontally elliptic operators with respect to $\{Y_1, \dots, Y_m\}$ in $(\tilde{\mathbb{G}}, \tilde{d}, \mu)$.

Remark 3.3.5. Since we have already remarked that another choice for the basis of the first layer does not affect the validity of our properties, we could state that the double ball (critical density) property for horizontally elliptic operators w.r.t \mathfrak{g}_1 in $(G, d, |\cdot|)$ implies the double ball (critical density) property for horizontally elliptic operators w.r.t $d\phi(\mathfrak{g}_1)$ in $(\tilde{G}, \tilde{d}, \mu)$.

In [3] (Theorem 2.2.18) it is also proved that there is a “canonical” way to build up an isomorphism from a Carnot group to an homogeneous one: via the inverse of the exponential map we pass from the Carnot group to its algebra and then we identify the algebra with \mathbb{R}^N by choosing a basis for the algebra adapted to the stratification. Why is it “canonical” if it depends on the choice of the adapted basis? After having fixed a stratification $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$, if we choose two different bases adapted to it, we get two different homogeneous Carnot groups $\mathbb{G}_1 = (\mathbb{R}^N, \circ_1, \delta_\lambda)$ and $\mathbb{G}_2 = (\mathbb{R}^N, \circ_2, \delta_\lambda)$. But there is a linear isomorphism between them given by an invertible diagonal blocks matrix C (see [3], Remark 2.2.20), where the dimension of the i -th blocks corresponds to the dimension of \mathfrak{g}_i . By exploiting the block-form of C , we can transfer any homogeneous symmetric norm in \mathbb{G}_1 to any homogeneous symmetric norm in \mathbb{G}_2 . Since the norms in an homogeneous Carnot group are all equivalent, we have just seen how to handle them. That’s why we are going to give the following definitions.

Definition 3.3.6. (Double Ball Property in \mathbb{G}) Let \mathbb{G} be an N -dimensional Carnot group of step r with m generators. We say that the double ball property holds true in \mathbb{G} if there exists a stratification of \mathfrak{g} such that the double ball property for horizontally elliptic operators is satisfied in $(\tilde{G}, d, |\cdot|)$ with respect to X_j ’s (Definition 3.2.6) for every homogeneous Carnot group $\tilde{\mathbb{G}} = (\mathbb{R}^N, \circ, \delta_\lambda)$ canonically isomorphic to \mathbb{G} , for every homogeneous symmetric norm d in $\tilde{\mathbb{G}}$ and for every choice of the generators X_1, \dots, X_m .

Definition 3.3.7. (Critical Density in \mathbb{G}) Let \mathbb{G} be an N -dimensional Carnot group of step r with m generators. We say that the critical density property holds true in \mathbb{G} if there exists a stratification of \mathfrak{g} such that the critical density property for horizontally elliptic operators is satisfied in $(\tilde{G}, d, |\cdot|)$

with respect to X_j 's (Definition 3.2.7) for every homogeneous Carnot group $\tilde{\mathbb{G}} = (\mathbb{R}^N, \circ, \delta_\lambda)$ canonically isomorphic to \mathbb{G} , for every homogeneous symmetric norm d in $\tilde{\mathbb{G}}$ and for every choice of the generators X_1, \dots, X_m .

We have seen in this section it is enough to prove the double ball property for one fixed homogeneous group $\tilde{\mathbb{G}}$, for one fixed homogeneous norm d in $\tilde{\mathbb{G}}$ and for a fixed set of generators. On the other hand, if the double ball properties holds true in \mathbb{G} for a certain stratification, in order to prove the critical density it is enough to do it for some fixed $\tilde{\mathbb{G}}$, d and $\{X_1, \dots, X_m\}$ (w.r.t. the same stratification).

Note 3.3.8. Definition 3.3.6 and 3.3.7 are the last (but hopefully not the least) definitions we give about the double ball and the critical density properties.

Chapter 4

Double Ball Property

The starting point of this chapter is our proof of Theorem 2.1.3. The idea that the double ball property is related to a kind of solvability for an exterior Dirichlet problem is transferred here in the setting of homogeneous Carnot groups. In the particular case of step two Carnot groups we prove the validity of the double ball property: this result can also be found in our work [34].

4.1 Interior barriers in homogeneous settings

We fix an homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$, an homogeneous symmetric norm d and a system of generators X_1, \dots, X_m . The main tool of the approach we want to outline is the existence of some interior barrier functions. They have to play the role of the Hopf-type barrier in the proof of Theorem 2.1.3. The important feature of these barriers for \mathcal{L}_A is that they are uniform for $A \in M_m(\lambda, \Lambda)$: they have to be independent of the coefficients of the matrix $A(x)$ and of their regularity. Let us give the definition.

Definition 4.1.1. *Let O be an open set of \mathbb{R}^N with non-empty boundary. Fix $x_0 \in \partial O$ and $0 < \lambda \leq \Lambda$. A function h is an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for O at x_0 if*

- *h is a C^2 function defined on an open bounded neighborhood U of x_0 ,*

- h and U depend just on O, x_0, Λ, λ (and on \mathbb{G}, d and the X_j 's),
- $\mathcal{L}_A h \leq 0$ in U for any $A \in M_m(\lambda, \Lambda)$,
- $h(x_0) = 0$,
- $\{x \in U : h \leq 0\} \setminus \{x_0\} \subseteq O$.

Looking at the definition, we can recognize that this is a kind of Bouligand type barrier for the complement of O . In [34] we considered the case of step two Carnot groups and we proved that the existence of an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier for $B_1(0)$ at every point of its boundary implies the double ball property. Here we want to generalize this fact to every homogeneous Carnot group and every bounded open neighborhood of the origin.

Lemma 4.1.2. *Let T be a compact subset of a bounded open set $O \subset \mathbb{R}^N$. There exists $\nu_0 > 1$ such that*

$$\delta_\nu T \subset O$$

for all $\nu \in [1, \nu_0]$.

Proof. The sets T and $\mathbb{R}^N \setminus O$ are close and disjoint. Thus, their distance δ is a positive number. Since T is bounded, there exists $M > 0$ such that, if $x = (x_1, \dots, x_N) \in T$, we have $|x_j| \leq M$. Therefore, for $x \in T$ and $\nu \geq 1$, we get

$$\text{dist}(\delta_\nu(x), T) \leq \|\delta_\nu(x) - x\| \leq M \sum_{j=1}^N (\nu^{\sigma_j} - 1).$$

It is easy to choose $\nu_0 > 1$ such that $\sup_{x \in T} \text{dist}(\delta_\nu(x), T) < \delta$ for all $\nu \in [1, \nu_0]$. \square

Let us fix a bounded open set $B \subset \mathbb{R}^N$ such that $0 \in B$. For any $r > 0$, we denote by B_r the set $\delta_r B$. By the boundedness of B and the structure of the dilations, there exist $R_0 \geq r_0 > 0$ such that

$$B_{r_0} \subseteq B_1(0) \subseteq B_{R_0}. \quad (4.1)$$

Keeping in mind (3.4), we denote

$$K_0^A = \{u \in C^2(V, \mathbb{R}) : B_2 \subseteq V, u \geq 0 \text{ and } \mathcal{L}_A u \leq 0 \text{ in } B_2\} = \mathcal{K}_\Omega^A(B_2).$$

The following lemma generalizes what we have seen for Theorem 2.1.3 and is the key fact: it is an application of the Weak Maximum Principle (Theorem 3.2.5).

Lemma 4.1.3. *Suppose that, for some $0 < \lambda \leq \Lambda$, there exists an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for B at every $x_0 \in \partial B$. Then there exists $1 < \nu < 2$ such that, for any $A \in M_m(\lambda, \Lambda)$, if $u \in K_0^A$ with $u \geq 1$ in B , we have*

$$u \geq \frac{1}{2} \text{ in } B_\nu.$$

Proof. Fix $A \in M_m(\lambda, \Lambda)$ and $x_0 \in \partial B$. Take the barrier function $h = h_{x_0}$, which is defined in $U = U_{x_0}$. If we put $V = (U \cap B_2) \setminus \bar{B}$, we have that $h \geq 0$ and $\mathcal{L}_A h \leq 0$ in V . Let us now consider the boundary $\partial V = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \partial V \cap \partial B$ and $\Gamma_2 = \partial V \setminus \Gamma_1$. The number $m = \inf_{\Gamma_2} h$ is strictly positive since $\{x \in \partial V : h(x) = 0\} = \{x_0\}$. Thus, the function $w = 1 - \frac{1}{m}h$ is well defined. We get

$$\mathcal{L}_A w = -\frac{1}{m} \mathcal{L}_A h \geq 0 \text{ in } V, \quad w \leq 1 \text{ on } \Gamma_1 \text{ and } w \leq 0 \text{ on } \Gamma_2.$$

Hence, if $u \in K_0^A$ with $\inf_B u \geq 1$, we have

$$\mathcal{L}_A u \leq \mathcal{L}_A w \text{ in } V, \quad u \geq w \text{ on } \partial V.$$

By Theorem 3.2.5, $u \geq w$ in V . Since $w(x_0) = 1$, there exists an open neighborhood W_{x_0} of x_0 contained in $U \cap B_2$ where $w \geq \frac{1}{2}$. The sets W_{x_0} depend only on the barrier functions and on B : they are in fact independent of the matrix A . The compact set ∂B is contained in the open set $O = \cup_{x_0 \in \partial B} W_{x_0}$. By Lemma 4.1.2, there exists $\nu > 1$ such that $(B_\nu \setminus B) \subset O$. Therefore, we deduce

$$u \geq \frac{1}{2} \text{ on } B_\nu$$

for all $u \in K_0^A$. □

Now, in order to get the double ball property, we gather some considerations present in Remark 3.2.8 and Proposition 3.3.3.

Proposition 4.1.4. *Suppose that, for any $0 < \lambda \leq \Lambda$, there exists an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for B at every point of ∂B . Then, the double ball property holds true in \mathbb{G} .*

Proof. The condition (4.1) says that the sets $x_0 \circ B_R$ are somehow equivalent balls with respect to the d -balls $B_R(x_0)$. Proposition 1.4.5 and Proposition 3.3.3 suggest us how to handle equivalent balls. Let us give the details. Fix $0 < \lambda \leq \Lambda$ and $A \in M_m(\lambda, \Lambda)$. By putting $\eta_D = 4\frac{R_0}{r_0}$, take a d -ball $B_{\eta_D}(0) \subset \Omega$ and a function $u \in \mathcal{K}_\Omega^A(B_{\eta_D}(0))$ with $u \geq 1$ in $B_1(0)$. Consider the function $v_1 = u \circ \delta_{r_0}$. By definition (see also (3.5)), $v_1 \in \mathcal{K}_\Omega^{A_1}(B_{\frac{\eta_D}{r_0}}(0))$ with $v_1 \geq 1$ in $B_{\frac{1}{r_0}}(0)$, where $A_1(x) = A(\delta_{r_0}(x))$. Since we have $B_{\frac{1}{r_0}}(0) \supseteq B$, $B_{\frac{\eta_D}{r_0}}(0) \supseteq B_{\eta_D}$ and $\eta_D > 2$, we get $v_1 \in K_0^{A_1}$ and $v_1 \geq 1$ in B . By the last lemma, the existence of a $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for ∂B (and the fact that $A_1 \in M_m(\lambda, \Lambda)$) implies $v_1 \geq \frac{1}{2}$ in B_ν for some fixed $1 < \nu < 2$. Now we put $v_2 = 2v_1 \circ \delta_\nu = 2u \circ \delta_{r_0\nu}$. Thus, since $\eta_D \geq 4 > 2\nu$, we have $v_2 \in K_0^{A_2}$ where $A_2(x) = A_1(\delta_\nu(x)) = A(\delta_{r_0\nu}(x)) \in M_m(\lambda, \Lambda)$. Moreover, $v_2 \geq 1$ in B . Hence, Lemma 4.1.3 implies again $v_2 \geq \frac{1}{2}$ in B_ν , i.e. $v_1 \geq \frac{1}{4}$ in B_{ν^2} . If $\nu^2 \geq 2\frac{R_0}{r_0}$ we are done because in this case we get

$$u \geq \frac{1}{4} \quad \text{in } B_{r_0\nu^2} \supseteq B_{2R_0} \supseteq B_2(0).$$

If $\nu^2 < 2\frac{R_0}{r_0}$, the argument can be reapplied. As a matter of fact, let n_0 be the first integer such that $\nu^{n_0} \geq 2\frac{R_0}{r_0}$: the existence of n_0 is provided by $\nu > 1$. For any positive integer $n < n_0$, we put

$$v_{n+1} = 2v_n \circ \delta_\nu = 2^n v_1 \circ \delta_{\nu^n} = 2^n u \circ \delta_{r_0\nu^n}.$$

We can iterate the procedure since at every step we have $v_{n+1} \geq 1$ in B and, ensured by $\eta_D \geq 2\nu^n$, we have also $v_{n+1} \in K_0^{A_{n+1}}$ where $A_{n+1}(x) = A_n(\delta_\nu(x))$. Therefore, at the last step we get $v_{n_0} \geq \frac{1}{2}$ in B_ν , that is

$$u \geq \frac{1}{2^{n_0}} =: \gamma \quad \text{in } B_{r_0\nu^{n_0}} \supseteq B_{2R_0} \supseteq B_2(0).$$

In Remark 3.2.8, we have seen how to handle the case of generic d -balls $B_{\eta_D R}(x_0) \subset \Omega$. Thus, if $u \in \mathcal{K}_\Omega^A(B_{\eta_D R}(x_0))$ with $u \geq 1$ in $B_R(x_0)$, we have $u \geq \gamma$ in $B_{2R}(x_0)$ (with the same constants η_D, γ). We stress that η_D depends just on r_0, R_0 (i.e. on B, d, \mathbb{G}) and γ depends just on ν (i.e. on λ, Λ and the barriers on ∂B). \square

Thus, the double ball problem is “reduced” to finding the barrier functions we have described. The Hopf-type functions we have used for the euclidean case works in our more general setting for a large class of points.

Lemma 4.1.5. *Let B_0 be a bounded open set defined by*

$$B_0 = \{x \in \mathbb{R}^N : F(x) < 0\},$$

where F is a real-valued function. Fix $x_0 \in \partial B_0$. Suppose that F is smooth near x_0 and

$$\nabla_X F := (X_1 F, \dots, X_m F) \neq 0$$

at x_0 . Then, for any $0 < \lambda \leq \Lambda$, there exists an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for B_0 at x_0 .

Proof. Since F is smooth near x_0 and $\nabla F(x_0) \neq 0$, we can consider an euclidean ball $B_\rho^e(\xi_0)$ tangent to ∂B_0 at x_0 such that $\overline{B_\rho^e(\xi_0)} \setminus \{x_0\} \subset B_0$. To this aim, let us fix

$$\xi_0 = x_0 - \rho \frac{\nabla F(x_0)}{\|\nabla F(x_0)\|}$$

with ρ small enough (depending on x_0, F). As in the proof of Theorem 2.1.3, let us consider the function

$$h(x) = e^{-\alpha \rho^2} - e^{-\alpha \|x - \xi_0\|^2}.$$

The positive constant α will be fixed later on. This function is strictly positive out of $B_\rho^e(\xi_0)$ and it vanishes on the sphere. By using the notations of Remark 3.2.2, for $j = 1, \dots, m$ we can compute

$$X_j h(x) = 2\alpha e^{-\alpha \|x - \xi_0\|^2} \sum_{l=1}^N c_{jl}(x) (x - \xi_0)_l = 2\alpha e^{-\alpha \|x - \xi_0\|^2} \left(C(x)(x - \xi_0) \right)_j.$$

For every $\lambda \leq \Lambda$ and $A \in M_m(\lambda, \Lambda)$, by the formula (3.3) we get

$$\begin{aligned} \mathcal{L}_A h(x) &= 2\alpha e^{-\alpha\|x-\xi_0\|^2} \left(\text{Tr}(C^t(x)A(x)C(x)) + \sum_{l=1}^N b_l(x)(x-\xi_0)_l + \right. \\ &\quad \left. - 2\alpha \left\langle A(x) \left(C(x)(x-\xi_0) \right), \left(C(x)(x-\xi_0) \right) \right\rangle \right) \leq \\ &\leq 2\alpha e^{-\alpha\|x-\xi_0\|^2} (N\Lambda\Sigma_C + M_{\lambda,\Lambda} - 2\alpha\lambda\|C(x)(x-\xi_0)\|^2) \\ &=: H(x), \end{aligned}$$

where we denoted by

$$\Sigma_C = \max_{x \in \bar{B}_0} \{\lambda_C(x) : \lambda_C(x) \text{ is eigenvalue of } C^t(x)C(x)\}$$

(see Remark 3.3.1) and by $M_{\lambda,\Lambda} = \max_{x \in \bar{B}_0} \{\sum_{l=1}^N |b_l(x)| |(x-\xi_0)_l|\}$. The fact that we can take a bound for b_l 's which is uniform for $A \in M_m(\lambda, \Lambda)$ is justified in the proof of Theorem 3.2.5. That's why we can state that the function H depends on $\lambda, \Lambda, x_0, F, X_j$, but it does not depend on the coefficients of the matrix A . We also remark that

$$C(x_0)(x_0 - \xi_0) = \frac{\rho}{\|\nabla F(x_0)\|} C(x_0) \nabla F(x_0) = \frac{\rho}{\|\nabla F(x_0)\|} \nabla_X F(x_0) \neq 0$$

by our key hypothesis. Hence, if we choose

$$\alpha > \frac{\|\nabla F(x_0)\|^2}{\rho^2 \|\nabla_X F(x_0)\|^2} \frac{N\Lambda\Sigma_C + M_{\lambda,\Lambda}}{2\lambda},$$

we get $H(x_0) < 0$. Therefore, there exists an open bounded neighborhood U of x_0 (depending just on the function H) where $\mathcal{L}_A h \leq H < 0$. The function h satisfies all the properties required to be an interior $\mathcal{L}_{\lambda,\Lambda}$ -barrier function for B_0 at x_0 . \square

Definition 4.1.6. Let B_0 be a bounded open subset of \mathbb{R}^N with a smooth boundary. We say that a point $x_0 \in \partial B_0$ is characteristic for ∂B_0 (with respect to the vector fields X_1, \dots, X_m) if all the vectors $X_1(x_0), \dots, X_m(x_0)$ are tangent to ∂B_0 at x_0 .

Suppose B_0 is given by $B_0 = \{x \in \mathbb{R}^N : F(x) < 0\}$ for some smooth real-valued function F such that ∇F does not vanish at any point of ∂B_0 . Then, since the normal direction to ∂B_0 at x_0 is given by $\nabla F(x_0)$, x_0 is characteristic for ∂B_0 iff the horizontal gradient $\nabla_X F(x_0) = 0$.

Since the vector fields X_1, \dots, X_m satisfy the Hörmander condition, a result by Derridj ([13], Théorème 1) tells us that almost every point of ∂B_0 (with respect to the surface measure on ∂B_0) is non-characteristic for ∂B_0 , provided that B_0 is a bounded open set with smooth boundary. Thus, by Lemma 4.1.5, we are able to build a barrier at almost every point of any open bounded neighborhood B of the origin with smooth boundary. The hope of finding such a B totally without characteristic points is frustrated by the following example.

Example 4.1.7. *In the Heisenberg group $\mathbb{H} = \mathbb{H}^1$, fix the generators $X_1 = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}$ and $X_2 = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3}$. Take a bounded open neighborhood B of $0 = (0, 0, 0)$ with a smooth boundary such that B is defined by a smooth function F as in Lemma 4.1.5. Suppose B is homeomorphic to the sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. Then, there exists at least one characteristic point for ∂B_0 . As a matter of fact, for any $x \in \partial B$ we can define*

$$V(x) = V(x_1, x_2, x_3) = (x_2, -x_1, 2) - \left\langle (x_2, -x_1, 2), \frac{\nabla F(x)}{\|\nabla F(x)\|} \right\rangle \frac{\nabla F(x)}{\|\nabla F(x)\|}.$$

By the regularity of the boundary, ∇F is always different from 0 at the boundary points and so V defines a continuous vector field. Moreover, for any $x \in \partial B$, $V(x)$ is tangent to ∂B at x (it is a projection on the tangent bundle of the vector field $(x_2, -x_1, 2)$). Since we cannot comb ∂B (i.e. for \mathbb{S}^2 we have the hairy ball theorem), it has to exist $x_0 \in \partial B$ such that $V(x_0) = 0$, that is the non-null vector $(x_2^0, -x_1^0, 2)$ has to be parallel to $\nabla F(x_0)$. We have chosen the vector field $(x_2, -x_1, 2)$ just because it is orthogonal at every point to both the vectors $X_1(x) = (1, 0, -\frac{x_2}{2})$ and $X_2(x) = (0, 1, \frac{x_1}{2})$. Hence, $\nabla F(x_0)$ is orthogonal to both the vectors, i.e.

$$\nabla_X F(x_0) = 0$$

4.2 The case of step two Carnot groups

In this section we want to give a conclusive answer to the double ball problem in the case of an N -dimensional Carnot group of step two with m generators. Up to fixing a stratification and applying a canonical isomorphism (see [3], Theorem 3.2.2, and our Definition 3.3.6), we can thus consider an homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ such that the composition law \circ is defined by

$$(x, t) \circ (x_1, t_1) = \left(x + x_1, t + t_1 + \frac{1}{2} \langle Bx, x_1 \rangle \right), \quad (4.2)$$

for $(x, t), (x_1, t_1) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^N$. Here we have denoted by $\langle Bx, x_1 \rangle$ the vector of \mathbb{R}^n whose components are $\langle B^k x, x_1 \rangle$ (for $k = 1, \dots, n$) and B^1, \dots, B^n are $m \times m$ linearly independent skew-symmetric matrices. The group of dilations is defined as

$$\delta_\lambda((x, t)) = (\lambda x, \lambda^2 t)$$

and the inverse of (x, t) is $(-x, -t)$. We can choose as homogeneous symmetric norm the function $d : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$d((x, t)) = (\|x\|^4 + \|t\|^2)^{\frac{1}{4}};$$

from here on we denote by $\|\cdot\|$ both the euclidean norms in \mathbb{R}^m and in \mathbb{R}^n . Hence, we have $B_R(x_0) = x_0 \circ B_R(0)$ where

$$B_R(0) = \{(x, t) \in \mathbb{R}^N : \|x\|^4 + \|t\|^2 < R^4\}.$$

Let us fix m vector fields generating the Lie algebra of \mathbb{G} , for example the ones of the Jacobian basis: they are given by

$$X_i(x, t) = \partial_{x_i} + \frac{1}{2} \sum_{k=1}^n (B^k x)_i \partial_{t_k} \quad \text{for } i = 1, \dots, m. \quad (4.3)$$

By exploiting the approach we have drawn, we want to prove the double ball property for horizontally elliptic operators in this setting. In particular, in

order to apply Proposition 4.1.4, we are going to find, for any $\lambda \leq \Lambda$, explicit interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier functions for $B_1(0)$ at every point of $\partial B_1(0)$. At the non-characteristic points of $\partial B_1(0)$, we know how to do by Lemma 4.1.5.

Remark 4.2.1. We have already reminded that the set of the characteristic points of $\partial B_1(0)$ has surface measure zero. Actually, we can explicitly say which points are characteristic. If we denote with F the defining function of $B_1(0)$, i.e.

$$F(x, t) = d^4((x, t)) = \|x\|^4 + \|t\|^2 - 1,$$

we have

$$\nabla_X F(x, t) = (X_1 F, \dots, X_m F)(x, t) = 4 \|x\|^2 x + \sum_{k=1}^n t_k B^k x.$$

Since the matrices B^k 's are skewsymmetric, the vectors x and $B^k x$ are orthogonal for every $k = 1, \dots, n$. This implies that

$$\nabla_X F(x, t) = 0 \quad \Leftrightarrow \quad x = 0.$$

Thus, the characteristic set is $\{(0, t) : \|t\| = 1\}$, which is an $n-1$ -dimensional sphere.

For any $t_0 = (t_1^0, \dots, t_n^0) \neq 0$, we know that the matrix $\sum_{k=1}^n t_k^0 B^k$ is not the null matrix. Actually, for an homogeneous Lie group with a composition law as in (4.2), the linear independence of the matrices B^k 's is equivalent to the Carnot property (see [3], Section 3.2).

Lemma 4.2.2. *Let $t_0 = (t_1^0, \dots, t_n^0)$ be a unit vector such that the matrix $\sum_{k=1}^n t_k^0 B^k$ is non singular. Then, for any $\lambda \leq \Lambda$, we can find an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier for $B_1(0)$ at $(0, t_0)$.*

Proof. Consider the function

$$h_M(x, t) = e^{-\beta} - e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0 \rangle)},$$

where $t' = t - \langle t, t_0 \rangle t_0$ is the projection of t on the orthogonal of t_0 and β is a positive constant to be fixed. If we define $U_0 = \{(x, t) \in \mathbb{R}^N : \langle t, t_0 \rangle > 0\}$,

we get that the set $\{(x, t) \in U_0 : h_M(x, t) \leq 0\} \setminus \{(0, t_0)\}$ is contained in $B_1(0)$. Moreover, by denoting $t' = (t'_1, \dots, t'_n)$, we have $\partial_{t_k} \|t'\|^2 = 2t'_k$. Thus, a straightforward calculation shows that

$$\begin{aligned} X_j h_M(x, t) &= \beta e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0 \rangle)} \left(4 \|x\|^2 x_j + \sum_{k=1}^n t'_k (B^k x)_j + \right. \\ &\quad \left. + \frac{1}{2} \sum_{k=1}^n t_k^0 (B^k x)_j \right) =: \beta e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0 \rangle)} v_j(x, t). \end{aligned}$$

For any $A \in M_m(\lambda, \Lambda)$, by using that the product of a symmetric matrix and a skew-symmetric matrix has zero trace, we get

$$\begin{aligned} \mathcal{L}_A h_M(x, t) &= \beta e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0 \rangle)} \left(4 \|x\|^2 \operatorname{Tr}(A(x, t)) + \right. \\ &\quad + 8 \langle A(x, t)x, x \rangle + \sum_{k=1}^n \left(t'_k + \frac{1}{2} t_k^0 \right) \operatorname{Tr}(A(x, t) B^k) + \\ &\quad - \frac{1}{2} \left\langle A(x, t) \sum_{k=1}^n t_k^0 B^k x, \sum_{k=1}^n t_k^0 B^k x \right\rangle \\ &\quad + \frac{1}{2} \sum_{k=1}^n \langle A(x, t) B^k x, B^k x \rangle - \beta \langle A(x, t)v(x, t), v(x, t) \rangle \Big) \\ &\leq \beta e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0 \rangle)} \left(4\Lambda \|x\|^2 (m+2) + \frac{\Lambda}{2} \sum_{k=1}^n \|B^k x\|^2 + \right. \\ &\quad \left. - \frac{\lambda}{2} \left\| \sum_{k=1}^n t_k^0 B^k x \right\|^2 - \beta \lambda \|v(x, t)\|^2 \right). \end{aligned}$$

Put $M = \max_k \{\|B^k\|\}$. Since $\sum_{k=1}^n t_k^0 B^k$ is non singular, we have

$$\left\| \sum_{k=1}^n t_k^0 B^k x \right\| \geq \sigma \|x\| \quad \text{for every } x \in \mathbb{R}^m,$$

where $\sigma = \|(\sum_{k=1}^n t_k^0 B^k)^{-1}\|$. This fact and the orthogonality of x and $B^k x$

imply that

$$\begin{aligned} \|v(x, t)\| &\geq \left\| 4 \|x\|^2 x + \frac{1}{2} \sum_{k=1}^n t_k^0 B^k x \right\| - \left\| \sum_{k=1}^n t'_k B^k x \right\| \\ &\geq \frac{1}{2} \left\| \sum_{k=1}^n t_k^0 B^k x \right\| - \|t'\| nM \|x\| \geq \left(\frac{\sigma}{2} - nM \|t'\| \right) \|x\|. \end{aligned}$$

If $\|t'\| < \frac{\sigma}{4nM}$, we have $\|v(x, t)\| \geq \frac{\sigma}{4} \|x\|$. Hence, for $\|t'\| < \frac{\sigma}{4nM}$, we get

$$\mathcal{L}_A h_M(x, t) \leq \frac{\beta}{2} \|x\|^2 e^{-\beta(\|x\|^4 + \|t'\|^2 + \langle t, t_0 \rangle)} \left((8m + 16 + nM^2)\Lambda - (8 + \beta)\frac{\sigma^2}{8}\lambda \right)$$

which is not positive if β is big enough. Therefore, the function h_M , defined in the domain $U = \{(x, t) \in U_0 : \|t'\| < \frac{\sigma}{4nM}\}$, is an $\mathcal{L}_{\lambda, \Lambda}$ -barrier. \square

Remark 4.2.3. In the Heisenberg group $\mathbb{H} = \mathbb{H}^l$ with $m = 2l$ generators in \mathbb{R}^{2l+1} , there is just one $(2l) \times (2l)$ matrix B which is

$$B = \begin{pmatrix} 0 & -\mathbb{I}_l \\ \mathbb{I}_l & 0 \end{pmatrix}.$$

Such a matrix is non singular: in fact it is an orthogonal matrix. By the last lemma, we can find a barrier for $B_1(0)$ at $(0, \pm 1)$. Thus, we have found a different proof for the result by Gutiérrez and Tournier ([19], Theorem 4.1) in \mathbb{H}^l . We note that, despite the differences in our techniques, their approach exploits some kind of barriers which have a paraboloidal shape similar to h_M .

In a generic step two Carnot group as in (4.2), the non-singularity of the matrices $\sum_{k=1}^n t_k^0 B^k$ is not provided. It is easy, indeed, to build some examples: if m is odd, the skew-symmetry implies the singularity. In the following proposition, we overcome this difficulty.

Proposition 4.2.4. *For any $\lambda \leq \Lambda$, there exists an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for $B_1(0)$ at every point of $\partial B_1(0)$.*

Proof. By Lemma 4.1.5 and Remark 4.2.1, it is left the case of the points $(0, t_0) \in \partial B_1(0)$. Thus, fix $t_0 = (t_1^0, \dots, t_n^0)$ with $\|t_0\| = 1$. Since we have

proved Lemma 4.2.2, we assume that the matrix $\sum_{k=1}^n t_k^0 B^k$ has a non-trivial kernel. Let us denote by Q the orthogonal projector on $\text{Ker}(\sum_{k=1}^n t_k^0 B^k)$ and by P the orthogonal projector on $\text{Range}(\sum_{k=1}^n t_k^0 B^k) = \text{Ker}(\sum_{k=1}^n t_k^0 B^k)^\perp$ (also P is non-null because of the linear independence of the B^k 's). We remind that $x = Px + Qx$ and

$$\left\| \sum_{k=1}^n t_k^0 B^k x \right\| \geq \sigma \|Px\| \quad \text{for every } x \in \mathbb{R}^m,$$

where $\sigma > 0$ is the smallest positive singular value of $\sum_{k=1}^n t_k^0 B^k$. Denote by N_1 the rank of the matrix P : we know that $0 < N_1 \leq m$. We put also $M = \max_k \|B^k\|$. For a fixed

$$\gamma > \frac{\Lambda}{\lambda} \left(\frac{5m}{2N_1} + \frac{15 + m - N_1}{N_1} + \frac{5nM^2}{16N_1} \right)$$

(in particular we note that $\gamma > 2$ and $\gamma > \frac{\Lambda}{\lambda} \frac{m-N_1}{N_1}$), we set

$$f(x, t) = \|x\|^4 + (\|Qx\|^2 - \gamma \|Px\|^2)^2 + \|t'\|^2 + \langle t, t_0 \rangle,$$

where $t' = t - \langle t, t_0 \rangle t_0$ as in Lemma 4.2.2. For a positive constant β to be fixed later on, we consider

$$h(x, t) = e^{-\beta} - e^{-\beta f(x, t)}.$$

The function h vanishes at $(0, t_0)$ and it is negative if and only if $f < 1$. Thus, we have

$$\{(x, t) \in \mathbb{R}^n : h(x, t) \leq 0, \langle t, t_0 \rangle > 0\} \setminus \{(0, t_0)\} \subset B_1(0).$$

A straightforward calculation shows that

$$\begin{aligned} X_j h(x, t) &= \beta e^{-\beta f(x, t)} \left(4 \|x\|^2 x_j + 4 (\|Qx\|^2 - \gamma \|Px\|^2) (Qx - \gamma Px)_j \right. \\ &\quad \left. + \sum_{k=1}^n t'_k (B^k x)_j + \frac{1}{2} \sum_{k=1}^n t_k^0 (B^k x)_j \right) = \beta e^{-\beta f(x, t)} X_j f(x, t). \end{aligned}$$

For every $\lambda \leq \Lambda$ and $A \in M_m(\lambda, \Lambda)$, we get

$$\begin{aligned}
\mathcal{L}_A h(x, t) &= \beta e^{-\beta f(x, t)} \left(4 \|x\|^2 \operatorname{Tr}(A(x, t)) + 8 \langle A(x, t)x, x \rangle + \right. \\
&+ 4(\|Qx\|^2 - \gamma \|Px\|^2) \left(\operatorname{Tr}(A(x, t)Q) - \gamma \operatorname{Tr}(A(x, t)P) \right) + \\
&+ 8 \langle A(x, t)(Qx - \gamma Px), Qx - \gamma Px \rangle + \\
&- \frac{1}{2} \left\langle A(x, t) \sum_{k=1}^n t_k^0 B^k x, \sum_{k=1}^n t_k^0 B^k x \right\rangle + \\
&+ \left. \frac{1}{2} \sum_{k=1}^n \langle A(x, t) B^k x, B^k x \rangle - \beta \langle A(x, t) \nabla_X f(x, t), \nabla_X f(x, t) \rangle \right) \\
&\leq \beta e^{-\beta f(x, t)} \left(4\Lambda \|x\|^2 (m+2) + 8\Lambda(\|Qx\|^2 + \gamma^2 \|Px\|^2) + \right. \\
&+ 4(\|Qx\|^2 - \gamma \|Px\|^2) \left(\operatorname{Tr}(A(x, t)Q) - \gamma \operatorname{Tr}(A(x, t)P) \right) + \\
&+ \left. \frac{\Lambda}{2} \sum_{k=1}^n \|B^k x\|^2 - \frac{\lambda}{2} \left\| \sum_{k=1}^n t_k^0 B^k x \right\|^2 - \beta \lambda \|\nabla_X f(x, t)\|^2 \right).
\end{aligned}$$

Since $\gamma > \frac{\Lambda}{\lambda} \frac{m-N_1}{N_1}$, we have

$$\operatorname{Tr}(A(x, t)Q) - \gamma \operatorname{Tr}(A(x, t)P) \leq (m - N_1)\Lambda - \gamma N_1 \lambda < 0.$$

We are going to consider two cases. If $\|Px\|^2 \leq \frac{1}{\gamma^2} \|Qx\|^2$, then in particular $\|Qx\|^2 - \gamma \|Px\|^2 \geq \frac{2}{5} \|x\|^2$ (since $\gamma > 2$). Hence, we get

$$\begin{aligned}
\mathcal{L}_A h(x, t) &\leq \beta e^{-\beta f(x, t)} \|x\|^2 \left(4m\Lambda + 24\Lambda + \frac{8}{5}((m - N_1)\Lambda - \gamma N_1 \lambda) + \right. \\
&+ \left. \frac{nM^2}{2} \Lambda \right) < 0
\end{aligned}$$

because of our choice of γ . Otherwise, if $\|Px\|^2 > \frac{1}{\gamma^2} \|Qx\|^2$, then $\|Px\|^2 \geq \frac{1}{1+\gamma^2} \|x\|^2$ and we have

$$\begin{aligned}
\|\nabla_X f(x, t)\| &\geq \left\| 4 \|x\|^2 x + 4(\|Qx\|^2 - \gamma \|Px\|^2)(Qx - \gamma Px) + \right. \\
&+ \left. \frac{1}{2} \sum_{k=1}^n t_k^0 B^k x \right\| - \left\| \sum_{k=1}^n t_k^0 B^k x \right\| \geq
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \left\| \sum_{k=1}^n t_k^0 B^k x \right\| - \|t'\| \sum_{k=1}^n \|B^k x\| \\
&\geq \frac{\sigma}{2} \|Px\| - \|t'\| nM \|x\| \geq \left(\frac{\sigma}{2\sqrt{1+\gamma^2}} - \|t'\| nM \right) \|x\|.
\end{aligned}$$

Here we have exploited the orthogonality of the vectors $\sum_{k=1}^n t_k^0 B^k x$, Px and Qx . Then, if in addition $\|t'\| < \frac{\sigma}{4nM\sqrt{1+\gamma^2}}$, we have

$$\|\nabla_X f(x, t)\| \geq \frac{\sigma}{4\sqrt{1+\gamma^2}} \|x\|.$$

Therefore we get

$$\begin{aligned}
\mathcal{L}_A h(x, t) &\leq \beta e^{-\beta f(x, t)} \|x\|^2 \left(4\Lambda(m+2) + 4\gamma(\gamma N_1 \Lambda - (m - N_1)\lambda) + \right. \\
&\quad \left. + 16\Lambda\gamma^2 + \Lambda n \frac{M^2}{2} - \frac{\lambda}{2} \frac{\sigma^2}{1+\gamma^2} - \beta\lambda \frac{\sigma^2}{16(1+\gamma^2)} \right).
\end{aligned}$$

By choosing β big enough, we obtain $\mathcal{L}_A h < 0$. Thus, the function h is an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier for $B_1(0)$ at $(0, t_0)$ if we consider it on the domain $U = \left\{ (x, t) : \langle t, t_0 \rangle > 0, \|t'\| < \frac{\sigma}{4nM\sqrt{1+\gamma^2}} \right\}$. \square

Putting together the last proposition and Proposition 4.1.4, we have proved the following theorem, which is the main result of [34].

Theorem 4.2.5. *The double ball property holds true in every Carnot group of step two.*

4.3 A naive proof in Métivier groups

The title of this Section forces us to start with a definition.

Definition 4.3.1. *Let \mathfrak{g} be an N -dimensional Lie algebra and let us denote by \mathfrak{z} its center. We say that \mathfrak{g} is a Métivier algebra (or an H -type algebra in the sense of Métivier) if it admits a vector space decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with*

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2 \quad \text{and} \quad \mathfrak{g}_2 \subseteq \mathfrak{z}$$

such that, for every $\eta \in \mathfrak{g}_2^*$, the skew-symmetric bilinear form on \mathfrak{g}_1 defined by

$$B_\eta : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathbb{R} \quad B_\eta(X, X') = \eta([X, X'])$$

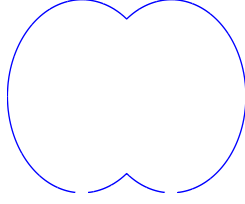
is non degenerate, whenever $\eta \neq 0$.

We say that a Lie group is a Métivier group (or an H -type group in the sense of Métivier) if its Lie algebra is a Métivier algebra.

This definition implies that every Métivier group is a particular stratified group of step two (see [3], Remark 3.7.2). Moreover, if we look it in coordinates through a canonical isomorphism, we get an homogeneous Carnot group with composition law as in (4.2), for which every non-vanishing linear combination of the matrices B^k 's is non singular (see [3], Proposition 3.7.4). In particular, m has to be even. Every group of Heisenberg type is a Métivier group ([3], Remark 3.7.5).

Remark 4.3.2. Hence, in an homogeneous Métivier group we have that, for any unit vector $t_0 = (t_1^0, \dots, t_n^0) \in \mathbb{R}^n$, the matrix $\sum_{k=1}^n t_k^0 B^k$ is non singular. By Lemma 4.2.2, we know that the paraboloidal shaped functions denoted by h_M work as $\mathcal{L}_{\lambda, \Lambda}$ -barrier for $B_1(0)$ at every characteristic point of $\partial B_1(0)$. Therefore, the proof of the double ball we have presented above is simpler in the setting of Métivier groups.

We are going to give a different proof for the double ball property in this setting. The approach we have described allows us to choose a basis of the neighborhoods (i.e. to choose one bounded open neighborhood B of the origin) which is different from the d -balls $B_R(x_0)$ (i.e. from $B_1(0)$). How can we choose such a B ? Looking at the proof of Lemma 4.1.5, we recognize that, in order to find a barrier for B at x_0 , we exploited the existence of an interior euclidean ball centered at ξ_0 such that $v = x_0 - \xi_0$ is a non-characteristic direction at x_0 (that is v is not orthogonal to each $X_j(x_0)$). Since the points $(0, t_0)$ are characteristic for $B_1(0)$, it seems reasonable to us to consider a set B as the one described by the following figure (the horizontal “axis” refers to $\|x\|$ and the vertical one to $\|t\|$)



Let us fix a smooth and convex function $g = [0, +\infty) \rightarrow \mathbb{R}$ such that

$$g(0) = 0, \quad g'(0) < 0, \quad g(R) = 1, \quad g'(R) > 0$$

for some $R > 0$. Thus, there exists by convexity one point $R_0 \in (0, R)$ such that $g'(R_0) = 0$ and $g(R_0) = \min g < 0$. Even if it is not needed, let us fix $g(\rho) = 4\rho^2 - 3\rho$: for such a function $R = 1, R_0 = \frac{3}{8}$ and $g(R_0) = -\frac{9}{16}$. We define

$$B := \{(x, t) \in \mathbb{R}^N : g(\|x\|) + \|t\|^2 < 1\}.$$

This set is an open neighborhood of the origin. Moreover it is bounded, since we have

$$(x, t) \in B \Rightarrow \|x\| < R = 1, \quad \|t\| < \sqrt{1 - g(R_0)} = \frac{5}{4}.$$

Remark 4.3.3. The boundary $\partial B = \{(x, t) \in \mathbb{R}^N : g(\|x\|) + \|t\|^2 = 1\}$ is a smooth hypersurface of \mathbb{R}^N except from the points $(0, t_0)$ with $\|t_0\| = 1$. Thus, the set of non-regular points is an $n - 1$ -dimensional sphere and it has surface measure zero. The defining function of ∂B is

$$F(x, t) = g(\|x\|) + \|t\|^2 - 1.$$

It turns out that the horizontal gradient of F is non-vanishing at the regular points. As a matter of fact, we have

$$\nabla_X F(x, t) = \frac{g'(\|x\|)}{\|x\|} x + \sum_{k=1}^n t_k B^k x.$$

By the skew-symmetry, the vectors x and $\sum_{k=1}^n t_k B^k x$ are orthogonal vectors of \mathbb{R}^m . Since $x \neq 0$ at a regular point, we get

$$\nabla_X F(x_0, t_0) = 0 \Leftrightarrow g'(\|x_0\|) = 0 \text{ and } \sum_{k=1}^n t_k^0 B^k x_0 = 0$$

for a regular point $(x_0, t_0) \in \partial B$ where $t_0 = (t_1^0, \dots, t_n^0)$. Our hypotheses on g imply that $g'(\|x_0\|) = 0$ iff $\|x_0\| = R_0$. On the other hand, if $\sum_{k=1}^n t_k^0 B^k x_0 = 0$, the fact that the group is Métivier implies that $t_0 = 0$ and so we have $g(\|x_0\|) = 1$. Since $g(R_0) \neq 1$, it is not possible that $\nabla_X F(x_0, t_0) = 0$ at a regular point. In other terms, each regular point of ∂B is non-characteristic.

Hence, we know by Lemma 4.1.5 how to build a barrier at every point $(x_0, t_0) \in \partial B$ with $x \neq 0$. The other points are going to be considered in the following proposition.

Proposition 4.3.4. *For any $0 < \lambda \leq \Lambda$, there exists an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier function for B at every point of ∂B .*

Proof. Fix $0 < \lambda \leq \Lambda$. By the last remark, we are left with the case of the boundary points $(0, t_0)$ for any fixed unit vector $t_0 = (t_1^0, \dots, t_n^0)$. We are going to build a conic shaped barrier. To this aim, we fix a number $\gamma > -\frac{2}{g'(0)} = \frac{2}{3}$. If the point (x, t) satisfies $\gamma \|t - t_0\| \leq \|x\|$, we have

$$g(\|x\|) + \|t\|^2 \leq g(\|x\|) + (\|t - t_0\| + 1)^2 \leq g(\|x\|) + \left(\frac{1}{\gamma} \|x\| + 1\right)^2 =: G(\|x\|).$$

Since $G(0) = 1$ and $G'(0) = g'(0) + \frac{2}{\gamma} < 0$, there exists $\delta > 0$ such that $G < 1$ in the interval $(0, \delta)$. Thus we get

$$\{(x, t) \in \mathbb{R}^N : \gamma \|t - t_0\| \leq \|x\| < \delta\} \setminus \{(0, t_0)\} \subseteq B.$$

Let us define the function

$$h(x, t) = 1 - e^{\|x\|^2 - \gamma^2 \|t - t_0\|^2}.$$

We have just seen that $\{(x, t) : \|x\| < \delta, h(x, t) \leq 0\} \setminus \{(0, t_0)\} \subseteq B$. For $j = 1, \dots, m$, we can compute

$$\begin{aligned} X_j h(x, t) &= -e^{\|x\|^2 - \gamma^2 \|t - t_0\|^2} \left(2x_j - \gamma^2 \sum_{k=1}^n (t - t_0)_k (B^k x)_j \right) \\ &=: -e^{\|x\|^2 - \gamma^2 \|t - t_0\|^2} v_j(x, t). \end{aligned}$$

For $A \in M_m(\lambda, \Lambda)$, a straightforward calculation shows that

$$\begin{aligned}
\mathcal{L}_A h(x, t) &= e^{\|x\|^2 - \gamma^2 \|t - t_0\|^2} \left(-2\text{Tr}(A(x, t)) + \frac{\gamma^2}{2} \sum_{k=1}^n \langle A(x, t) B^k x, B^k x \rangle + \right. \\
&\quad \left. - \langle A(x, t) v(x, t), v(x, t) \rangle \right) \leq \\
&\leq e^{\|x\|^2 - \gamma^2 \|t - t_0\|^2} \left(-2m\lambda + \frac{\gamma^2}{2} \Lambda \sum_{k=1}^n \|B^k x\|^2 - \lambda \|v(x, t)\|^2 \right) = \\
&=: H(x, t).
\end{aligned}$$

By definition, we have $H(0, t_0) = -2m\lambda < 0$. Hence, there exists an open neighborhood U_0 of $(0, t_0)$ where $\mathcal{L}_A h \leq 0$ for any $A \in M_m(\lambda, \Lambda)$. The set U_0 depends just on the function H and thus it depends on A just through λ, Λ . Therefore the function h , defined in $U = \{(x, t) \in U_0 : \|x\| < \delta\}$, is an interior $\mathcal{L}_{\lambda, \Lambda}$ -barrier for B at $(0, t_0)$. \square

Let us recap we have just showed a different proof for the double ball property in the Métivier case (and in particular for the Heisenberg case): the approach is the same of Section 4.1, but there are different barriers and “balls” with respect to Section 4.2.

Chapter 5

Critical density property

In this last chapter we generalize a result in [19] by Gutiérrez and Tournier for the Heisenberg group by identifying a class for which the critical density property is uniformly satisfied. Our approach works in any H-type groups. The class we identify is different from the one in [19] and it is related to a Landis condition. The resulting invariant Harnack inequality we report here is also our main result in [35].

5.1 The case of H-type groups

We have to start with a definition.

Definition 5.1.1. *An H-type algebra is a finite-dimensional real Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ which can be endowed with an inner product $\langle \cdot, \cdot \rangle$ such that*

$$[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z},$$

where \mathfrak{z} is the center of \mathfrak{g} . Moreover, for any fixed $z \in \mathfrak{z}$, the map $J_z : \mathfrak{z}^\perp \longrightarrow \mathfrak{z}^\perp$ defined by

$$\langle J_z(v), w \rangle = \langle z, [v, w] \rangle \quad \forall w \in \mathfrak{z}^\perp$$

is an orthogonal map whenever $\langle z, z \rangle = 1$. We say that a simply connected Lie group is an H-type group if its Lie algebra is an H-type algebra.

The H-type groups are particular Carnot groups of step two: a stratification is just given by

$$\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z}.$$

We are going to denote $\mathfrak{b} = \mathfrak{z}^\perp$ and $\|q\| = \langle q, q \rangle$ for $q \in \mathfrak{g}$. Moreover, we put $m = \dim(\mathfrak{b})$ and $n = \dim(\mathfrak{z})$. The associated homogeneity in the Lie group is thus given by the dilations $\delta_\lambda((x_{(1)}, x_{(2)})) = (\lambda x_{(1)}, \lambda^2 x_{(2)})$ (for $(x_{(1)}, x_{(2)}) \in \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$) and the homogeneous dimension is equal to $Q := m + 2n$.

We now fix an orthonormal (with respect to $\langle \cdot, \cdot \rangle$) basis X_1, \dots, X_m for \mathfrak{b} and an orthonormal basis Z_1, \dots, Z_n for \mathfrak{z} . Then $X_1, \dots, X_m, Z_1, \dots, Z_n$ is an orthonormal basis for \mathfrak{g} and we have

$$v + z = \sum_{j=1}^m \langle v, X_j \rangle X_j + \sum_{k=1}^n \langle z, Z_k \rangle Z_k \quad \forall v \in \mathfrak{b}, \quad \forall z \in \mathfrak{z}.$$

For any $z \in \mathfrak{z}$, the map J_z satisfies, among the others, the following properties

$$\langle J_z(v), v \rangle = 0 \quad \text{and} \quad \|J_z(v)\| = \|z\| \|v\| \quad \forall v \in \mathfrak{b}. \quad (5.1)$$

A proof of these facts and other nice properties of H-type groups can also be found in [12] (Section 6) and in [3] (Chapter 18).

Example 5.1.2. *As it is well-known, the Heisenberg-Weyl group is a particular H-type group. If the generic point of \mathbb{R}^{2k+1} is given by (x_1, \dots, x_{2k}, z) and the vector fields*

$$\begin{aligned} X_j &= \partial_{x_j} - \frac{x_{j+k}}{2} \partial_z, & X_{j+k} &= \partial_{x_{j+k}} + \frac{x_j}{2} \partial_z \quad (\text{for } j \in \{1, \dots, k\}), \\ Z &= Z_1 = \partial_z \end{aligned}$$

form the usual basis of the Lie algebra, then the standard inner product induced by the basis $\{X_1, \dots, X_{2k}, Z\}$ is the one needed for satisfying Definition 5.1.1. Furthermore, in this case the map $J_{(\cdot)}$ is given by

$$J_{cZ} \left(\sum_{j=1}^{2k} a_j X_j \right) = c \sum_{j=1}^k (-a_{j+k} X_j + a_j X_{j+k}).$$

Following Kaplan's notations, we define the functions $v : \mathbb{G} \longrightarrow \mathfrak{b}$ and $z : \mathbb{G} \longrightarrow \mathfrak{z}$ by the following relation

$$x = \text{Exp}(v(x) + z(x)), \quad \text{for } x \in \mathbb{G},$$

where Exp denotes the exponential map. We remind that Exp is a globally defined diffeomorphism with inverse denoted by Log . Thus, for any $x \in \mathbb{G}$ we have

$$v(x) := \sum_{j=1}^m \langle \text{Log}(x), X_j \rangle X_j, \quad \text{and} \quad z(x) := \sum_{k=1}^m \langle \text{Log}(x), Z_k \rangle Z_k.$$

The approach of Chapter 1 requires the choice of an homogeneous symmetric norm in order to define a quasi distance d and the d -balls. In the H-type groups there are some preferable choices. As a matter of fact, let us consider the function

$$\varphi(x) = (\|v(x)\|^4 + 16 \|z(x)\|^2)^{\frac{2-Q}{4}}.$$

Kaplan proved in [22] (Theorem 2) that there exists a positive constant k such that $k\varphi$ is the fundamental solution at the origin (in the sense of Definition 5.3.1 in [3]) of the sub-Laplacian $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$.

Definition 5.1.3. *A $\Delta_{\mathbb{G}}$ -gauge on \mathbb{G} is an homogeneous symmetric norm d , smooth out of the origin and satisfying*

$$\Delta_{\mathbb{G}}(d^{2-Q}) = 0 \quad \text{in } \mathbb{G} \setminus \{0\}.$$

If Γ is a fundamental solution at the origin for $\Delta_{\mathbb{G}}$, then

$$d(x) := \begin{cases} (\Gamma(x))^{\frac{1}{2-Q}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a $\Delta_{\mathbb{G}}$ -gauge on \mathbb{G} (see [3], Proposition 5.4.2).

Thus, as homogeneous symmetric norm we choose the following $\Delta_{\mathbb{G}}$ -gauge function

$$d(x) := (\|v(x)\|^4 + 16 \|z(x)\|^2)^{\frac{1}{4}}. \quad (5.2)$$

Danielli, Garofalo and Nhieu proved in [12] (Theorem 6.8) that d has also the remarkable property to be horizontally-convex. A direct proof of the fact that d is an homogeneous symmetric norm can be found in [3] (Remark 18.3.2). Hence, the d -ball of radius R centered at x_0 has the following form

$$B_R(x_0) = x_0 \circ B_R(0) = x_0 \circ \{x \in \mathbb{G} : \|v(x)\|^4 + 16 \|z(x)\|^2 < R^4\}.$$

Note 5.1.4. In every homogeneous Carnot group the balls of the gauge have a great importance in the analysis of the sub-Laplacian and more in general in the geometry of the group. As a matter of fact, it is very well-known that some mean-value representation formulas hold true on such balls (see e.g. [3], Theorem 5.5.4). In what follows we will use the kernel of the *surface* mean-value formula at 0. We recall it is given by

$$\psi_0(\xi) = \frac{\|\nabla_X d(\xi)\|^2}{\|\nabla d(\xi)\|} \quad \text{and} \quad \frac{\beta}{R^{Q-1}} \int_{\partial B_R(0)} \psi_0(\xi) d\sigma(\xi) = 1 \quad (5.3)$$

for some positive constant β . Here we have denoted by $d\sigma$ the $(N-1)$ -dimensional Hausdorff measure and, with an abuse of notations, by $\|\cdot\|$ even the euclidean norms in \mathbb{R}^m and in \mathbb{R}^N .

For $A \in M_m(\lambda, \Lambda)$ we are interested in the horizontally elliptic operators \mathcal{L}_A as in (3.2), where the X_j 's are the orthonormal vector fields we have fixed. We want to prove a critical density estimate for \mathcal{K}_Ω^A . To this aim, we follow the main steps of the proof adopted in [19]. The crucial point is the existence of a very specific barrier function. To show this fact, let us compute explicitly the horizontal gradient $\nabla_X d$ and the horizontal Hessian matrix $(X_i X_j d)_{i,j=1}^m$. We put

$$\phi = d^4 = \|v(x)\|^4 + 16 \|z(x)\|^2,$$

which is a smooth function in the whole \mathbb{G} . If we define for any fixed $x \in \mathbb{G}$ the functions

$$\phi_j(t) = \phi(x \circ \text{Exp}(tX_j)) \quad \text{and} \quad \phi_{i,j}(s, t) = \phi(x \circ \text{Exp}(sX_i) \circ \text{Exp}(tX_j))$$

for $s, t \in \mathbb{R}$ and $i, j \in \{1, \dots, m\}$, by definition we have

$$X_j \phi(x) = \phi'_j(0) \quad \text{and} \quad X_i X_j \phi(x) = \frac{\partial^2}{\partial s \partial t} \phi_{i,j}(0, 0).$$

We remind that the Campbell-Hausdorff formula for step two Lie algebras state that

$$\text{Exp}(A) \circ \text{Exp}(B) = \text{Exp}\left(A + B + \frac{1}{2}[A, B]\right) \quad \forall A, B \in \mathfrak{g}.$$

Thus, since $z(x) \in \mathfrak{z}$, we get

$$\begin{aligned} & \text{Exp}\left(v(x \circ \text{Exp}(sX_i) \circ \text{Exp}(tX_j)) + z(x \circ \text{Exp}(sX_i) \circ \text{Exp}(tX_j))\right) = \\ &= x \circ \text{Exp}(sX_i) \circ \text{Exp}(tX_j) = \\ &= \text{Exp}(v(x) + z(x)) \circ \text{Exp}\left(sX_i + tX_j + \frac{st}{2}[X_i, X_j]\right) = \\ &= \text{Exp}\left(v(x) + z(x) + sX_i + tX_j + \frac{st}{2}[X_i, X_j] + \frac{s}{2}[v(x), X_i] + \frac{t}{2}[v(x), X_j]\right). \end{aligned}$$

Since we have

$$\begin{aligned} v(x) + sX_i + tX_j &\in \mathfrak{b} \quad \text{and} \\ z(x) + \frac{st}{2}[X_i, X_j] + \frac{s}{2}[v(x), X_i] + \frac{t}{2}[v(x), X_j] &\in \mathfrak{z}, \end{aligned}$$

we deduce

$$\begin{aligned} v(x \circ \text{Exp}(sX_i) \circ \text{Exp}(tX_j)) &= v(x) + sX_i + tX_j \quad \text{and} \\ z(x \circ \text{Exp}(sX_i) \circ \text{Exp}(tX_j)) &= z(x) + \frac{st}{2}[X_i, X_j] + \frac{s}{2}[v(x), X_i] + \frac{t}{2}[v(x), X_j]. \end{aligned}$$

For $s = 0$, this gives also

$$v(x \circ \text{Exp}(tX_j)) = v(x) + tX_j \quad \text{and} \quad z(x \circ \text{Exp}(tX_j)) = z(x) + \frac{t}{2}[v(x), X_j].$$

Now we have an explicit expression for $\phi_j(t)$ and $\phi_{i,j}(s, t)$ and we can perform explicit calculations.

Remark 5.1.5. By definition we have

$$\begin{aligned} \phi_j(t) &= \|v(x) + tX_j\|^4 + 16 \left\| z(x) + \frac{t}{2}[v(x), X_j] \right\|^2 = \\ &= (\|v(x)\|^2 + t^2 + 2t \langle v(x), X_j \rangle)^2 + \\ &+ 16 \left(\|z(x)\|^2 + t \langle z(x), [v(x), X_j] \rangle + \frac{t^2}{4} \|[v(x), X_j]\|^2 \right). \end{aligned}$$

Since $\langle z(x), [v(x), X_j] \rangle = \langle J_{z(x)}(v(x)), X_j \rangle$, by differentiating we get

$$X_j \phi(x) = 4 \langle \|v(x)\|^2 v(x) + 4J_{z(x)}(v(x)), X_j \rangle. \quad (5.4)$$

On the other hand we have

$$\begin{aligned} \phi_{i,j}(s, t) &= \|v(x) + sX_i + tX_j\|^4 + \\ &+ 16 \left\| z(x) + \frac{st}{2}[X_i, X_j] + \frac{s}{2}[v(x), X_i] + \frac{t}{2}[v(x), X_j] \right\|^2 = \\ &= (\|v(x)\|^2 + s^2 + t^2 + 2s \langle v(x), X_i \rangle + 2t \langle v(x), X_j \rangle + st\delta_{ij})^2 + \\ &+ 16 \left(\|z(x)\|^2 + \frac{s^2 t^2}{4} \|[X_i, X_j]\|^2 + \frac{s^2}{4} \|[v(x), X_i]\|^2 + \right. \\ &+ \frac{t^2}{4} \|[v(x), X_j]\|^2 + st \langle z(x), [X_i, X_j] \rangle + s \langle z(x), [v(x), X_i] \rangle + \\ &+ t \langle z(x), [v(x), X_j] \rangle + \frac{s^2 t}{2} \langle [X_i, X_j], [v(x), X_i] \rangle + \\ &\left. + \frac{st^2}{2} \langle [X_i, X_j], [v(x), X_j] \rangle + \frac{st}{2} \langle [v(x), X_i], [v(x), X_j] \rangle \right). \end{aligned}$$

By differentiating this formula and observing that

$$\langle [v(x), X_i], [v(x), X_j] \rangle = \sum_{k=1}^n \langle J_{Z_k}(v(x)), X_i \rangle \langle J_{Z_k}(v(x)), X_j \rangle,$$

we finally get

$$\begin{aligned} X_i X_j \phi(x) &= 4 \|v(x)\|^2 \delta_{ij} + 8 \langle v(x), X_i \rangle \langle v(x), X_j \rangle + 16 \langle z(x), [X_i, X_j] \rangle + \\ &+ 8 \sum_{k=1}^n \langle J_{Z_k}(v(x)), X_i \rangle \langle J_{Z_k}(v(x)), X_j \rangle. \end{aligned} \quad (5.5)$$

By the equality (5.4) and the properties in (5.1), we deduce the relation

$$\|\nabla_X \phi(x)\|^2 = 16 \|v(x)\|^2 \phi(x),$$

which implies

$$\|\nabla_X d(x)\|^2 = \frac{\|v(x)\|^2}{d^2(x)}. \quad (5.6)$$

This relation had already been remarked in [12] (Lemma 6.3). On the other hand, for $i, j \in \{1, \dots, m\}$ we get

$$\begin{aligned} X_i X_j d(x) &= -\frac{3}{d(x)} X_i d(x) X_j d(x) + \frac{1}{4d^3(x)} X_i X_j \phi(x) = \\ &= \frac{\|\nabla_X d(x)\|^2 \delta_{ij} - 3X_i d(x) X_j d(x)}{d(x)} + \\ &+ \frac{2}{d^3(x)} \left(\langle v(x), X_i \rangle \langle v(x), X_j \rangle + 2 \langle z(x), [X_i, X_j] \rangle + \right. \\ &\left. + \sum_{k=1}^n \langle J_{Z_k}(v(x)), X_i \rangle \langle J_{Z_k}(v(x)), X_j \rangle \right). \end{aligned}$$

If $A \in M_m(\lambda, \Lambda)$, since the matrix $(\langle z(x), [X_i, X_j] \rangle)_{i,j=1}^m$ is skew-symmetric and the product of a symmetric matrix with a skew-symmetric one has zero trace, we have

$$\begin{aligned} \mathcal{L}_A d(x) &= \frac{1}{d(x)} \langle (\text{Tr}(A(x)) \mathbb{I}_m - 3A(x)) \nabla_X d(x), \nabla_X d(x) \rangle + \\ &+ \frac{2}{d^3(x)} \left(\langle A(x) V(x), V(x) \rangle + \sum_{k=1}^n \langle A(x) J_k V(x), J_k V(x) \rangle \right) \end{aligned} \quad (5.7)$$

where we have denoted the two vectors of \mathbb{R}^m

$$V(x) := (\langle v(x), X_j \rangle)_{j=1}^m \quad \text{and} \quad J_k V(x) := (\langle J_{Z_k}(v(x)), X_j \rangle)_{j=1}^m.$$

We are almost ready to prove our main Lemma, which is the counterpart of Lemma 3.1 in [19]. Before doing it, let us state the following condition: we say that a positive definite coefficient matrix A satisfies the δ -Landis condition in Ω if there exists $\delta \in (0, 2]$ such that

$$\text{Tr}(A(x)) + (Q+2-m) \max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle \leq (Q+4-\delta) \min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle \quad \forall x \in \Omega. \quad (5.8)$$

We are going to fully discuss the meaning of this condition in the next section.

Lemma 5.1.6. *Fix $0 < \lambda \leq \Lambda$ and $0 < \delta \leq 2$. For any open set O such that $O \subseteq B_1(0)$, we consider the function*

$$h(x) = -\frac{1}{Q-\delta} \int_O (d(x^{-1} \circ \xi))^{-Q+\delta} d\xi.$$

For $\varepsilon > 0$, let $\eta_\varepsilon \in C^\infty([0, +\infty))$ such that $0 \leq \eta_\varepsilon \leq 1$, $\eta_\varepsilon(\rho) = 0$ for $0 \leq \rho < \varepsilon$ and $\eta_\varepsilon(\rho) = 1$ for $\rho \geq 2\varepsilon$. Consider the C^∞ function

$$h_\varepsilon(x) = -\frac{1}{Q - \delta} \int_O \frac{\eta_\varepsilon(d(x^{-1} \circ \xi))}{d^{Q-\delta}(x^{-1} \circ \xi)} d\xi$$

which converges uniformly to h as $\varepsilon \rightarrow 0^+$. Then, for any compact set $O' \subset O$, we have

$$\mathcal{L}_A h_\varepsilon(x) \geq C\lambda \quad \forall x \in O', \quad (5.9)$$

for every $A \in M_m(\lambda, \Lambda)$ satisfying the δ -Landis condition,

and for every $0 < 2\varepsilon < d(O', \partial O) := \inf \{d(\xi^{-1} \circ x) : x \in O', \xi \in \partial O\}$.

The positive constant C has to depend just on δ , d , Q , and the X_j 's.

Proof. Fix $\alpha = Q - \delta$. Put also $g(\xi) = -\frac{1}{\alpha} d^{-\alpha}(\xi)$ and $g_\varepsilon(\xi) = g(\xi)\eta_\varepsilon(d(\xi))$. By the symmetry of d , these functions are symmetric, i.e. $g(\xi^{-1}) = g(\xi)$ and $g_\varepsilon(\xi^{-1}) = g_\varepsilon(\xi)$. Thus, we have

$$h_\varepsilon(x) = \int_O g_\varepsilon(x^{-1} \circ \xi) d\xi = \int_O g_\varepsilon(\xi^{-1} \circ x) d\xi.$$

We note that, for $x \in B_1(0)$, we have $B_1(0) \subseteq B_{2K}(x)$. The smoothness of g_ε and the left-invariance of the vector fields imply, for every $i, j = 1, \dots, m$, that

$$\begin{aligned} X_i X_j h_\varepsilon(x) &= \int_O (X_i X_j g_\varepsilon(\xi^{-1} \circ \cdot))(x) d\xi = \int_O (X_i X_j g_\varepsilon)(\xi^{-1} \circ x) d\xi = \\ &= \int_{B_{2K}(x)} X_i X_j g_\varepsilon(\xi^{-1} \circ x) d\xi - \int_{B_{2K}(x) \setminus O} X_i X_j g_\varepsilon(\xi^{-1} \circ x) d\xi = \\ &= \int_{B_{2K}(0)} X_i X_j g_\varepsilon(\xi^{-1}) d\xi - \int_{B_{2K}(x) \setminus O} X_i X_j g_\varepsilon(\xi^{-1} \circ x) d\xi. \end{aligned}$$

Since $d\xi$ is also inversely invariant and the balls are symmetric, we get

$$\begin{aligned} X_i X_j h_\varepsilon(x) &= \int_{B_{2K}(0)} X_i X_j g_\varepsilon(\xi) d\xi - \int_{B_{2K}(x) \setminus O} X_i X_j g_\varepsilon(\xi^{-1} \circ x) d\xi = \\ &= \int_{\partial B_{2K}(0)} X_j g_\varepsilon(\xi) \frac{X_i d(\xi)}{\|\nabla d(\xi)\|} d\sigma(\xi) - \int_{B_{2K}(x) \setminus O} X_i X_j g_\varepsilon(\xi^{-1} \circ x) d\xi, \end{aligned}$$

where the last equality is justified by the divergence theorem: the vector fields X_i 's are indeed divergence-free because of the δ_λ -homogeneity (see [3], Remark 1.3.7). Assume now $0 < \varepsilon < 1$. Then, if ξ belongs to a small open neighborhood of $\partial B_{2K}(0)$, $g_\varepsilon(\xi) = g(\xi)$. Moreover, let O' be a compact subset of O such that $0 < 2\varepsilon < d(O', \partial O)$. If $x \in O'$ and $\xi \in B_{2K}(x) \setminus O$, then $g_\varepsilon(\xi^{-1} \circ x) = g(\xi^{-1} \circ x)$. Thus, for $x \in O'$, we get

$$\begin{aligned} & X_i X_j h_\varepsilon(x) = \\ &= \int_{\partial B_{2K}(0)} X_j g(\xi) \frac{X_i d(\xi)}{\|\nabla d(\xi)\|} d\sigma(\xi) - \int_{B_{2K}(x) \setminus O} X_i X_j g(\xi^{-1} \circ x) d\xi = \\ &= \int_{\partial B_{2K}(0)} \frac{X_j d(\xi) X_i d(\xi)}{(2K)^{\alpha+1} \|\nabla d(\xi)\|} d\sigma(\xi) - \int_{B_{2K}(x) \setminus O} \frac{X_i X_j d(\xi^{-1} \circ x)}{(d(\xi^{-1} \circ x))^{\alpha+1}} d\xi + \\ &+ (\alpha + 1) \int_{B_{2K}(x) \setminus O} \frac{X_i d(\xi^{-1} \circ x) X_j d(\xi^{-1} \circ x)}{(d(\xi^{-1} \circ x))^{\alpha+2}} d\xi. \end{aligned}$$

Hence, for $A \in M_m(\lambda, \Lambda)$ and for $x \in O'$, we have

$$\begin{aligned} & \mathcal{L}_A h_\varepsilon(x) = \\ &= \int_{\partial B_{2K}(0)} \frac{\langle A(x) \nabla_X d(\xi), \nabla_X d(\xi) \rangle}{(2K)^{\alpha+1} \|\nabla d(\xi)\|} d\sigma(\xi) + \\ &- \int_{B_{2K}(x) \setminus O} \frac{\sum_{i,j=1}^m a_{ij}(x) X_i X_j d(\xi^{-1} \circ x)}{(d(\xi^{-1} \circ x))^{\alpha+1}} d\xi + \\ &+ (\alpha + 1) \int_{B_{2K}(x) \setminus O} \frac{\langle A(x) \nabla_X d(\xi^{-1} \circ x), \nabla_X d(\xi^{-1} \circ x) \rangle}{(d(\xi^{-1} \circ x))^{\alpha+2}} d\xi \geq \\ &\geq \frac{\lambda}{(2K)^{\alpha+1}} \int_{\partial B_{2K}(0)} \frac{\|\nabla_X d(\xi)\|^2}{\|\nabla d(\xi)\|} d\sigma(\xi) - \int_{B_{2K}(x) \setminus O} \frac{(\mathcal{L}_{A_\varepsilon} d)(\xi^{-1} \circ x)}{(d(\xi^{-1} \circ x))^{\alpha+1}} d\xi + \\ &+ (\alpha + 1) \int_{B_{2K}(x) \setminus O} \frac{\langle A(x) \nabla_X d(\xi^{-1} \circ x), \nabla_X d(\xi^{-1} \circ x) \rangle}{(d(\xi^{-1} \circ x))^{\alpha+2}} d\xi, \end{aligned}$$

where $A_\varepsilon(x) = A(\xi \circ x)$. By recognizing that the term $\frac{\|\nabla_X d(\xi)\|^2}{\|\nabla d(\xi)\|} = \psi_0(\xi)$ and using (5.3), we have

$$\begin{aligned} \mathcal{L}_A h_\varepsilon(x) &\geq \frac{\lambda}{\beta} (2K)^{Q-2-\alpha} - \int_{B_{2K}(x) \setminus O} \frac{(\mathcal{L}_{A_\varepsilon} d)(\xi^{-1} \circ x)}{(d(\xi^{-1} \circ x))^{\alpha+1}} d\xi + \\ &+ (\alpha + 1) \int_{B_{2K}(x) \setminus O} \frac{\langle A(x) \nabla_X d(\xi^{-1} \circ x), \nabla_X d(\xi^{-1} \circ x) \rangle}{(d(\xi^{-1} \circ x))^{\alpha+2}} d\xi. \end{aligned}$$

Exploiting (5.7) and denoting by $\Lambda_{A_x} = \max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle$, we deduce that

$$\begin{aligned}
& \mathcal{L}_A h_\varepsilon(x) \geq \\
& \geq \frac{\lambda}{\beta} (2K)^{Q-2-\alpha} - 2\Lambda_{A_x} \int_{B_{2K}(x) \setminus O} \frac{\|V(\xi^{-1} \circ x)\|^2 + \sum_{k=1}^n \|J_k V(\xi^{-1} \circ x)\|^2}{(d(\xi^{-1} \circ x))^{\alpha+4}} d\xi \\
& + \int_{B_{2K}(x) \setminus O} \frac{\langle ((\alpha+4)A(x) - \text{Tr}(A(x))\mathbb{I}_m) \nabla_X d(\xi^{-1} \circ x), \nabla_X d(\xi^{-1} \circ x) \rangle}{(d(\xi^{-1} \circ x))^{\alpha+2}} d\xi \\
& = \frac{\lambda}{\beta} (2K)^{Q-2-\alpha} - \int_{B_{2K}(x) \setminus O} \frac{\left((2+2n)\Lambda_{A_x} + \text{Tr}(A(x)) \right) \|v(\xi^{-1} \circ x)\|^2}{(d(\xi^{-1} \circ x))^{\alpha+4}} d\xi + \\
& + (\alpha+4) \int_{B_{2K}(x) \setminus O} \frac{\left\langle A(x) \frac{\nabla_X d(\xi^{-1} \circ x)}{\|\nabla_X d(\xi^{-1} \circ x)\|}, \frac{\nabla_X d(\xi^{-1} \circ x)}{\|\nabla_X d(\xi^{-1} \circ x)\|} \right\rangle \|v(\xi^{-1} \circ x)\|^2}{(d(\xi^{-1} \circ x))^{\alpha+4}} d\xi,
\end{aligned}$$

where in the last equality we have used the second property in (5.1) and the orthonormality of the basis $X_1, \dots, X_m, Z_1, \dots, Z_n$. Assuming that A satisfies the condition (5.8), then for any unit vector ζ we have

$$(\alpha+4) \langle A(x)\zeta, \zeta \rangle = (Q+4-\delta) \langle A(x)\zeta, \zeta \rangle \geq \text{Tr}(A(x)) + (2+2n)\Lambda_{A_x}$$

uniformly in x . Therefore we finally get

$$\mathcal{L}_A h_\varepsilon(x) \geq \frac{\lambda}{\beta(2K)^{\alpha+2-Q}} = \frac{\lambda}{\beta(2K)^{2-\delta}}$$

for any $x \in O'$. \square

Note 5.1.7. The proof of this Lemma is the only part of the arguments where the condition (5.8) is needed. In [19] Gutiérrez and Tournier made different estimates for $\mathcal{L}_A h$ and thus they found a different condition written in terms of the maximum and minimum eigenvalue of A . See the next Section for further comments between (5.8) and other conditions.

We stress that the uniform convergence of h_ε (and the resulting continuity of h) is given by the condition $Q - \delta < Q$, that is $\delta > 0$ (see [3], Corollary 5.4.5). Moreover, in [19] it has been remarked the following nice fact.

Remark 5.1.8. Fix $x \in \mathbb{G}$. Among all the possible sets O with a fixed measure, the one who maximizes the function

$$\int_O \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi$$

is the ball centered at x . As a matter of fact, consider the ball $B_\rho(x)$ where $\rho > 0$ is such that $|B_\rho| = |O|$, i.e. $|B_1| \rho^Q = |O|$. We have

$$|O \setminus B_\rho(x)| = |B_\rho(x) \setminus O| \quad \text{and}$$

$$\begin{aligned} \int_O \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi &= \int_{O \cap B_\rho(x)} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi + \int_{O \setminus B_\rho(x)} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi \\ &\leq \int_{O \cap B_\rho(x)} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi + \frac{1}{\rho^\alpha} |O \setminus B_\rho(x)| \\ &= \int_{O \cap B_\rho(x)} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi + \frac{1}{\rho^\alpha} |B_\rho(x) \setminus O| \\ &\leq \int_{B_\rho(x) \cap O} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi + \int_{B_\rho(x) \setminus O} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi \\ &= \int_{B_\rho(x)} \frac{1}{(d(x^{-1} \circ \xi))^\alpha} d\xi \end{aligned}$$

This means that

$$0 \geq h(x) \geq -\frac{1}{\alpha} \int_{B_\rho(x)} (d(x^{-1} \circ \xi))^{-\alpha} d\xi.$$

By the behavior of the Lebesgue measure under translations and dilations, for such ρ we get

$$0 \geq h(x) \geq -\frac{\rho^{Q-\alpha}}{\alpha} \int_{B_1(0)} (d(\xi))^{-\alpha} d\xi =: -\gamma |O|^{\frac{\delta}{Q}}. \quad (5.10)$$

By keeping in mind the arguments in [19] (Theorem 3.2-3.3), we can now prove the following theorem.

Theorem 5.1.9. *Fix $0 < \lambda \leq \Lambda$ and $0 < \delta \leq 2$. The family \mathcal{K}_Ω^A satisfies the critical density property for any $A \in M_m(\lambda, \Lambda)$ satisfying the δ -Landis condition. The constants ε, η_C , and c depend just on λ, Λ, δ , and the setting we have fixed.*

Proof. Fix $A \in M_m(\lambda, \Lambda)$ as in the statement. By Remark 3.2.8 we can prove the property for $R = \frac{1}{2}$ and $x_0 = 0$. Let us prove the critical density

with $\eta_C = 4$ and $c = \frac{1}{2}$. Take $u \in \mathcal{K}_\Omega^A(B_2(0))$ and suppose there exists a point \bar{x} in $B_{\frac{1}{2}}(0)$ where u is less than $\frac{1}{2}$. We want to prove that

$$|\{x \in B_1(0) : u(x) < 1\}| \geq \varepsilon |B_1(0)|$$

for some $0 < \varepsilon < 1$ depending just on λ, Λ, δ and the structural constants of \mathbb{G} .

In order to prove it, we use the barrier of the Lemma 5.1.6 and an auxiliary function involving $\phi = d^4$. In our notations, by (5.5) we get

$$\begin{aligned} \mathcal{L}_A \phi(x) &= 4 \|v(x)\|^2 \operatorname{Tr}(A(x)) + \\ &+ 8 \langle A(x)V(x), V(x) \rangle + 8 \sum_{k=1}^n \langle A(x)J_k V(x), J_k V(x) \rangle \leq \\ &\leq 4\Lambda(m+2+2n) \|v(x)\|^2 \leq 4(Q+2)\Lambda \end{aligned}$$

for any $x \in B_1(0)$. If C is the positive constant in (5.9), we set

$$w(x) = \frac{C\lambda}{8(Q+2)\Lambda} (u(x) + \phi(x) - 1).$$

By the hypothesis $\mathcal{L}_A u \leq 0$, hence we have $\mathcal{L}_A w \leq \frac{C}{2}\lambda$ in $B_1(0)$. Moreover, w is nonnegative on $\partial B_1(0)$ and

$$w(\bar{x}) \leq \frac{C\lambda}{8(Q+2)\Lambda} \left(\frac{1}{2} + \frac{1}{2^4} - 1 \right) = -\frac{7C}{128(Q+2)} \frac{\lambda}{\Lambda}.$$

We put $O := \{x \in B_1(0) : w(x) < 0\}$. We remark that O is an open set such that

$$O \subseteq \{x \in B_1(0) : u(x) < 1\}.$$

This set is non-empty since $\bar{x} \in O$. With this choice of O , we build the barrier h of Lemma 5.1.6 and we consider the continuous function $h - w$. We claim that $h - w \leq 0$ in O . We already know that this inequality holds true on ∂O since $w \geq 0$ on $\partial B_1(0)$. Suppose by contradiction that there exists $\xi_0 \in O$ such that $h(\xi_0) - w(\xi_0) = 2\delta > 0$. Of course, this implies the existence of $\varepsilon_0 > 0$ such that $h_\varepsilon(\xi_0) - w(\xi_0) \geq \delta$ if $\varepsilon \leq \varepsilon_0$. Now, for any compact set $O' \subset O$ containing ξ_0 , by the weak maximum principle (Theorem

3.2.5) we would get $\max_{\partial O'}(h_\varepsilon - w) \geq \delta$ if $\varepsilon < \min\{\frac{1}{2}d(O', \partial O), \varepsilon_0\}$ since $\mathcal{L}_A(h_\varepsilon - w) \geq \frac{C}{2}\lambda$ in O' . Letting $\varepsilon \rightarrow 0^+$, we deduce $\max_{\partial O'}(h - w) \geq \delta$ for any O' which is a contradiction since $h - w \leq 0$ on ∂O . Thus we have proved the claim. In particular we get

$$-\frac{7C}{128(Q+2)}\frac{\lambda}{\Lambda} \geq w(\bar{x}) \geq h(\bar{x}) \geq -\gamma|O|^{\frac{\delta}{Q}}$$

by the relation (5.10). Therefore we have

$$|\{x \in B_1(0) : u(x) < 1\}| \geq |O| \geq \left(\frac{7C}{128\gamma(Q+2)}\frac{\lambda}{\Lambda}\right)^{\frac{Q}{\delta}} =: \varepsilon|B_1(0)|$$

and the theorem is proved. □

5.2 An invariant Harnack inequality under a Landis condition

In this last section of the thesis we want to sum up and discuss the results achieved. First of all we go back to the δ -Landis condition (5.8). What does it mean? Let us state it again in the following equivalent form

$$\sup_{x \in \Omega} \left(\frac{\text{Tr}(A(x)) + (Q+2-m) \max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} \right) < Q+4. \quad (5.11)$$

This is a Cordes-type condition, in the sense that it imposes a limitation on the spreading of the eigenvalues of the coefficient matrix A .

Remark 5.2.1. Suppose $A \in M_m(\lambda, \Lambda)$ with

$$\frac{\Lambda}{\lambda} < \frac{Q+3}{Q+1},$$

then the condition (5.11) is satisfied. Moreover, the constant δ in the δ -Landis condition can be taken as

$$\delta = (Q+1) \left(\frac{Q+3}{Q+1} - \frac{\Lambda}{\lambda} \right).$$

As a matter of fact we have

$$\begin{aligned}
& \frac{\operatorname{Tr}(A(x)) + (Q + 2 - m) \max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} = \\
&= \frac{\operatorname{Tr}(A(x)) - \min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} + (Q + 2 - m) \frac{\max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} + 1 \\
&\leq (m - 1) \frac{\Lambda}{\lambda} + (Q + 2 - m) \frac{\Lambda}{\lambda} + 1 = (Q + 1) \frac{\Lambda}{\lambda} + 1
\end{aligned}$$

which is less than $Q + 4$ if $\frac{\Lambda}{\lambda} < \frac{Q+3}{Q+1}$. Moreover δ can be chosen as we said since

$$(Q + 1) \frac{\Lambda}{\lambda} + 1 = Q + 4 - (Q + 1) \left(\frac{Q + 3}{Q + 1} - \frac{\Lambda}{\lambda} \right) = Q + 4 - \delta.$$

Estimates of Cordes-type in subelliptic settings for operators in non-divergence form are already present in the literature. They have been considered for the problem of interior regularity of p -harmonic functions in the Heisenberg group in [16] and in the Grušin plane in [14].

Remark 5.2.2. The original Cordes' condition introduced in [10] is actually not very similar to (5.11). For a symmetric positive definite $m \times m$ matrix A , the Cordes' condition involves the Frobenius norm of A (which is $\sqrt{\operatorname{Tr}(A^2)}$) and the trace: it is equivalent to asking that

$$\sup_{x \in \Omega} \frac{\operatorname{Tr}((A(x))^2)}{(\operatorname{Tr}(A(x)))^2} < \frac{1}{m - 1}.$$

Our condition involves the trace and the operator norms of A and A^{-1} , i.e. the maximum and the minimum eigenvalue. It is closer in the aspect and in the purposes to the one used by Landis in [26] (see also [27], Chapter 1, Section 7). Landis' condition reads indeed as follows

$$\sup_{x \in \Omega} \frac{\operatorname{Tr}(A(x))}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} < m + 2.$$

Before the work [24] by Krylov and Safonov, Landis proved in [25] an invariant Harnack inequality for non-divergence elliptic operator under this additional condition. In this way Landis obtained the same result by Cordes

but exploiting different techniques. In particular Landis used his extra condition for a reason which is very similar to our needs inside Lemma 5.1.6. That is why we have referred to (5.8) as the δ -Landis condition.

For $0 < \lambda \leq \Lambda$, we denote by $M_m(\lambda, \Lambda, \delta)$ the class of all the $A \in M_m(\lambda, \Lambda)$ satisfying the δ -Landis condition for some $\delta \in (0, 2]$. Both here and in the previous section we have mentioned that δ has to be between 0 and 2. The reason is simple and we explain it now.

Remark 5.2.3. There are no matrices satisfying the δ -Landis condition for δ bigger than 2. We have indeed that

$$\begin{aligned} & \frac{\text{Tr}(A(x))}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} + (Q + 2 - m) \frac{\max_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle}{\min_{\|\xi\|=1} \langle A(x)\xi, \xi \rangle} \geq \\ & \geq m + Q + 2 - m = Q + 2 \end{aligned}$$

The same inequality shows us that

$$A \in M_m(\lambda, \Lambda, 2) \iff \Lambda = \lambda \quad \text{and} \quad A = \lambda \mathbb{I}_m$$

Furthermore we note that $A \in M_m(\lambda, \Lambda, \delta)$ implies that $A \in M_m(\lambda, \Lambda, \delta')$ for any $0 < \delta' \leq \delta$.

With these new notations, let us summarize and state again the main result obtained in the previous section (Theorem 5.1.9).

Theorem 5.2.4. *In an H-type group \mathbb{G} , let $\{X_1, \dots, X_m\}$ be an orthonormal basis of the first layer of \mathfrak{g} and let d be as in (5.2). Consider the horizontal elliptic operators as in (3.2) built with such a basis. The family \mathcal{K}_Ω^A in (3.4) satisfies the critical density property uniformly in the class $M_m(\lambda, \Lambda, \delta)$.*

Note 5.2.5. By Remark 5.2.1 this critical density property is uniform in the whole class $M_m(\lambda, \Lambda)$ if $\frac{\Lambda}{\lambda} < \frac{Q+3}{Q+1}$.

Analogously to Remark 3.3.1 and to the classical Cordes result, we stress that also the class $M_m(\lambda, \Lambda, \delta)$ is not stable under change of variables or generators. That is why it is very important the *right* choice of the basis

$\{X_1, \dots, X_m\}$. For a better understanding, let us make a digression and some concrete examples.

As remarked by Kaplan in [22] (Corollary 1), there exist H-type algebras with center of any given dimension. We want to show here a representative class. Since these algebras are nilpotent of step 2, we look at the model described in Section 4.2. We recall that the composition law \circ in $\mathbb{R}^N = \mathbb{R}^{m+n}$ is given by

$$(x, t) \circ (x_1, t_1) = \left(x + x_1, t + t_1 + \frac{1}{2} \langle Bx, x_1 \rangle \right) \quad \text{for } (x, t), (x_1, t_1) \in \mathbb{R}^N,$$

for some suitable $m \times m$ matrices B^1, \dots, B^n . According to [3] (Definition 3.6.1), such a group is called prototype group of H-type if the following properties are satisfied:

- B^j is skew-symmetric and orthogonal for any $j \leq n$;
- $B^i B^j = -B^j B^i$ for every $i, j \in \{1, \dots, n\}$ with $i \neq j$.

This class of homogeneous Lie groups belongs to the class of H-type groups and any H-type group is isomorphic to one of these ([3], Theorem 18.2.1). Consider the vector fields X_1, \dots, X_m of the Jacobian basis as in (4.3). The standard inner product on \mathfrak{g} with respect to the basis

$$X_1, \dots, X_m, \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$$

induces on \mathfrak{g} the structure of H-type algebra. Moreover, in these groups the exponential map is the identity map on \mathbb{R}^N and the gauge function d we have exploited is

$$d(x, t) = \|x\|^4 + 16 \|t\|^2$$

(see e.g. [3], Remark 18.3.3).

Example 5.2.6. *The Heisenberg-Weyl group is also a particular prototype group of H-type. It is easy to see that the matrix B showed in Remark 4.2.3 satisfies the assumptions of skew-symmetry and orthogonality. The vector fields X_j 's of the Jacobian matrix are just the usual generators in Example*

5.1.2.

Let us take other examples from [3] (Remark 3.6.6). In $\mathbb{R}^7 = \mathbb{R}^4 \times \mathbb{R}^3$, we can consider the matrices

$$B^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

These matrices satisfy the prototype H-type group conditions. This gives us an example of H-type group with $m = 4$ and a center of dimension $n = 3$. The jacobian vector fields can be easily constructed through (4.3).

Despite the fact that any H-type group is isomorphic to a prototype one, we have to be careful and we do think it is useful to give another example. The problem has been already mentioned and it will be clear in a moment: our result is not stable under a change of the basis $\{X_1, \dots, X_m\}$. Let us consider the Lie group on \mathbb{R}^5 with the usual composition law as in (4.2) and

$$B = B^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Here $m = 4$ and $n = 1$. The matrix B is not orthonormal and so it is not a prototype H-type group, but it is an H-type group since it is isomorphic to the Heisenberg group \mathbb{H}^2 . This group is well-studied in the literature. In [1] (Example 6.6) Balogh and Tyson gave an explicit expression for the fundamental solution of the canonical sublaplacian $\Delta_{\mathbb{G}} = \sum_{j=1}^4 X_j^2$, where $X_j = \partial_{x_j} + \frac{1}{2}(Bx)_j \partial_t$ ($j = 1, \dots, 4$) are the horizontal fields of the Jacobian

basis. If we look at that formula, we can recognize it is different from being a power of $\|v(x)\|^4 + 16 \|z(x)\|^2$ (or to a power of $\|x\|^4 + 16 \|t\|^2$ since also here the exponential map is the identity). In [2] Bonfiglioli proved that the gauge function associated to $\Delta_{\mathbb{C}}$ is not even horizontally convex. The problem is that the Jacobian basis is not orthonormal with respect to the scalar product inducing in this group the structure of H-type group. Thus, if we want to apply our result on the horizontally elliptic operator in this group $\mathcal{L}_A = \sum_{i,j=1}^m a_{ij}(x,t)X_iX_j$ we need that our Cordes-Landis condition is satisfied not for the matrix A but for the matrix $D^tA(x,t)D$ where D brings an orthonormal basis in $\{X_1, \dots, X_m\}$. It is easy to verify that, in this situation, the basis $X_1, X_2, \frac{1}{\sqrt{2}}X_3, \frac{1}{\sqrt{2}}X_4$ is orthonormal. Hence, the *right* choice of the vector fields is crucial: this allowed us in particular to make the *right* choice of the gauge function related to the sub-Laplacian.

We can now put together the results obtained in the last two chapters and the approach showed in Chapter 1. We thus deduce an invariant Harnack inequality in H-type groups for horizontally elliptic operators \mathcal{L}_A with $A \in M_m(\lambda, \Lambda, \delta)$. The constants appearing in the Harnack inequality will depend on $A \in M_m(\lambda, \Lambda, \delta)$ just through the constants λ, Λ, δ , the structure of the group, the orthonormal vector fields X_j 's and the norm d . In particular, they are independent of the regularity of $A(x)$'s coefficients. Once more we stress that, if we suppose $\frac{\Lambda}{\lambda} < \frac{Q+3}{Q+1}$, we have an invariant Harnack inequality which is uniform in the class of $A \in M_m(\lambda, \Lambda)$.

Theorem 5.2.7. *Let G be an homogeneous Lie group whose algebra is of H-type. Suppose $0 < \lambda \leq \Lambda$ and $0 < \delta \leq 2$. There exist constants C and η depending just on Λ, λ, δ such that, for any $A \in M_m(\lambda, \Lambda, \delta)$, if we have a function u with*

$$u \geq 0 \quad \text{and} \quad \mathcal{L}_A u = 0 \quad \text{in} \quad \Omega \supset B_{\eta R}(x_0),$$

then it has to be

$$\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u.$$

Proof. Consider $(\mathbb{G}, d, |\cdot|)$, where $|\cdot|$ is the Lebesgue measure and d is the gauge function defined in (5.2). By what we showed in Section 3.1, this is a doubling quasi metric Hölder space satisfying the reverse doubling condition and the log-ring condition. Consider the horizontally elliptic operators \mathcal{L}_A as in (3.2) and the family of functions \mathcal{K}_Ω^A defined in (3.4). By Theorem 4.2.5, the family \mathcal{K}_Ω^A satisfies the double ball property uniformly for $A \in M_m(\lambda, \Lambda)$ for any $0 < \lambda \leq \Lambda$. Furthermore, for a fixed δ as in the statement, \mathcal{K}_Ω^A satisfies the critical density property uniformly for $A \in M_m(\lambda, \Lambda, \delta)$. By Theorem 1.2.3, \mathcal{K}_Ω^A satisfies also the power decay property uniformly for $A \in M_m(\lambda, \Lambda, \delta)$. By keeping in mind Section 2.2, we define the following subset of \mathcal{K}_Ω^A

$$\overline{\mathcal{K}_\Omega^A} := \{u \in C^2(V, \mathbb{R}) : V \subset \Omega, u \geq 0 \text{ and } \mathcal{L}_A u = 0 \text{ in } V\}.$$

The family $\overline{\mathcal{K}_\Omega^A}$ verifies all the assumptions of Theorem 1.3.1. Therefore that theorem gives us the desired Harnack inequality. We thus complete the proof and the thesis. □

Acknowledgments

‘Would you tell me, please, which way I ought to go from here?’

‘That depends a good deal on where you want to get to,’ said the Cat.

‘I don’t much care where..’ said Alice.

‘Then it doesn’t matter which way you go,’ said the Cat.

‘..so long as I get somewhere,’ Alice added as an explanation.

‘Oh, you’re sure to do that,’ said the Cat, ‘if you only walk long enough.’

This is not about Alice. In fact, Alice is not my favorite character. But, at the end of the day, the people you have bumped into make your journey. The characters Alice met were the true *Alice’s Adventures in Wonderland*. This is the reason why I need to thank some people: because they have made my journey.

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Ok. I think I am done here. The other people I have to thank (and there are others), I really can do it by voice. And I am sure I will.

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