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Stability and Computation in Multidimensional Size Theory

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Introduction

Shape comparison is probably one of the most challenging issues in shape recognition, classification and retrieval, which are very lively research topics for the disciplines of Cognitive Sciences, Pattern Recognition, Computer Vision and Computer Graphics. Shape models carry a high value of information with them, and search engines able to match, classify and retrieve such information would be useful to speed-up content design, processing and re-use. Keyword-based annotation is not sufficient to achieve the necessary capability of resource exploration for shape models. Therefore, a variety of methods has been proposed in the literature to tackle the problem of *content-based* shape analysis and retrieval.

The theoretical aspects underlying these questions have captured the attention of several research groups: In past years many papers have been devoted to these subjects and new mathematical techniques have been developed to deal with these problems. Recently, there has been an increasing interest towards geometrical-topological methods for shape comparison, whose main idea is to perform a topological exploration of the shape according to some quantitative geometric properties provided by a real function defined on the shape [5, 7, 15, 28]: That is to say, the real function plays the role of a *lens* through which we look at the properties of the shape.

In this context, Size Theory was proposed in the early 90's as a geometrical/topological approach to shape discrimination: The main idea is that the comparison of two objects in a dataset (e.g. 3D-models, images or sounds) can be translated into the comparison of two suitable topological spaces $\mathcal{M}$ and $\mathcal{N}$, endowed with two continuous real functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$. These functions are called *measuring functions* and are chosen according to the application. In other words, they can be seen as descriptors of the properties considered relevant for the comparison. The pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ are said to be *size pairs*. In this setting, the definition of the natural pseudo-distance d between the size pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ was introduced (cf. [21]), setting $d\left((\mathcal{M}, \varphi), (\mathcal{N}, \psi)\right)$ equal to the infimum of the change of the measuring function, induced by composition with all the homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

However, a common scenario in applications is to have two or more functions defined on the same shape, carrying information on different features of the phenomenon under study: Indeed, the shape of an object can be more thoroughly characterized by means of a set of real functions, each one investigating specific features of the shape under study. Examples arise in the context of computational biology, in medical environments, as well as in scientific simulations of natural phenomena. Therefore, a great challenge is to develop and define tools to extract knowledge from high-dimensional data, by means of the concurrent analysis of different properties conveyed by different real functions.

These considerations have quite early led, in [31], to the extension of the natural pseudo-distance, in order to approach the problem of shape discrimination by considering topological spaces endowed with continuous functions taking values in $\mathbb{R}^k$, modeling the shapes under study. According to this multidimensional setting, the natural pseudo distance d between two size pairs $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{N}, \vec{\psi})$, where $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k$, $\vec{\psi} : \mathcal{N} \to \mathbb{R}^k$, is defined as

$$d((\mathcal{M}, \vec{\varphi}), (\mathcal{N}, \vec{\psi})) = \inf_{f} \max_{P \in \mathcal{M}} \|\vec{\varphi}(P) - \vec{\psi}(f(P))\|_\infty,$$

where $\|\vec{\varphi}(P)-\vec{\psi}(f(P))\|_\infty = \max_{1\le i\le k}|\varphi_i(P)-\psi_i(f(P))|$ and f varies among all the homeomorphisms between $\mathcal{M}$ and $\mathcal{N}$.

Unfortunately, the study of d is quite difficult, even for $k = 1$, although strong properties can be proved in this case. Size Theory tackles this problem by introducing a mathematical tool that allows us to easily obtain lower bounds for d, named *k-dimensional size function*. The idea is to study the pairs $(\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\rangle, \mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\rangle)$, where $\mathcal{M}\langle\vec{\varphi} \preceq \vec{t}\,\rangle$ is defined by setting $\mathcal{M}\langle\vec{\varphi} \preceq \vec{t}\,\rangle = \{P \in \mathcal{M} : \varphi_i(P) \le t_i, i = 1,\ldots,k\}$ for $\vec{t} = (t_1,\ldots,t_k) \in \mathbb{R}^k$: k-dimensional size functions count the number of connected components in $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\rangle$ that meet $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\rangle$.

More recently, similar research lines led Edelsbrunner et al. to the definition of *Persistent Homology* (cf. [22]), and Allili et al. to the definition of the *Morse Homology Descriptor* (cf. [1]), presenting some links with the concepts expressed by Size Theory.

From the beginning of the 90's, size functions have been studied and applied in the case of measuring functions taking values in $\mathbb{R}$ (namely in the case of 1-dimensional size functions) (cf., e.g., [20, 23, 24, 25, 26, 33, 36, 37, 38, 39]). The multidimensional case presented more severe difficulties, since a concise, complete and stable description of k-dimensional size functions was not available before the results reported in this thesis, differently from what happens for the 1-dimensional case (cf., e.g., [17, 29]).

The first result of this thesis is the proof that in Size Theory the comparison of multidimensional size functions can be reduced to the 1-dimensional case by a suitable change of variables (Theorem 2.1.2). The key idea is to show that a foliation in half-planes can be given, such that the restriction of a k-dimensional size function to these half-planes turns out to be a 1-dimensional size function. Our approach implies that each size function, with respect to a measuring function taking values in $\mathbb{R}^k$, can be completely and compactly described by a parameterized family of discrete descriptors (Remark 9). This follows from the results reported in [19, 27, 29] about the representation of 1-dimensional size functions by means of formal series of points and lines, applied to each plane in our foliation. An important consequence is that we can easily prove the stability of this new descriptor (and hence of the corresponding k-dimensional size function) also in higher dimensions (Proposition 2.2.2 and Proposition 2.2.3), by using a recent result of stability proved for 1-dimensional size functions (cf. [17, 18]). Moreover, we show that a matching distance between size functions, with respect to measuring functions taking values in $\mathbb{R}^k$, can easily be introduced (Definition 2.2), providing a lower bound for the natural pseudo-distance (Theorem 2.2.4). All these results open the way to the use of Multidimensional Size Theory in real applications.

Outline. This thesis contains the results of our research activity carried out in the last years in order to face the problem of the extension to the multidimensional case of Size Theory (cf. [12, 13]). In Chapter 1 we give preliminary definitions for k-dimensional size functions, and the main results for the particular case $k = 1$. In Chapter 2 the foliation we use is presented, and the reduction to the 1-dimensional case is proved. Moreover, we show the stability of our computational method, implying a lower bound for the natural pseudo-distance. Additionally, a new distance between multidimensional size functions is introduced. Chapter 3 is devoted to present some computational techniques for computing k-dimensional size functions in applications. Some discussion and proposals for future research conclude the thesis.

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Chapter 1

Preliminary definitions and results

This chapter is devoted to the introduction of preliminary definitions and results about natural pseudo-distance and k-dimensional size functions. Particular attention will be given to the special case of 1-dimensional size functions: Indeed, this is the most well-studied instance, showing a certain number of interesting features. For more details about properties of natural pseudo-distance and 1-dimensional size functions, the reader is referred to [17, 18, 28, 29, 33, 39].

1.1 Preliminary definitions

Let $\mathcal{M}$ be a non-empty, compact, connected and locally connected Hausdorff space.

The assumption on the connectedness of $\mathcal{M}$ can easily be weakened to any finite number of connected components without much affecting the following results. More serious problems would derive from considering an infinite number of connected components.

Let also $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k) : \mathcal{M} \to \mathbb{R}^k$ be a continuous function. We shall call any pair $(\mathcal{M}, \vec{\varphi})$ a *size pair*. The function $\vec{\varphi}$ is said to be a *k-dimensional measuring function*, and can be seen like a descriptor of those features that are considered to be relevant in comparing $(\mathcal{M}, \vec{\varphi})$ with other size pairs. We shall call *Size* the collection of all size pairs $(\mathcal{M}, \vec{\varphi})$.

The following relations $\preceq$, $\prec$ are defined in $\mathbb{R}^k$: for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, we shall say $\vec{x} \preceq \vec{y}$ (resp $\vec{x} \prec \vec{y}$) if and only if $x_i \leq y_i$ (resp. $x_i < y_i$) for every index $i = 1, \ldots, k$. Moreover, $\mathbb{R}^k$ is endowed with the usual max-norm: $||(x_1, \ldots, x_k)||_\infty = \max_{1 \leq i \leq k} |x_i|$.

Given two size pairs $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{N}, \vec{\psi})$, let $H(\mathcal{M}, \mathcal{N})$ be the set of all homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

Definition 1.1. *We shall call* natural pseudo-distance *the pseudo-distance* $d : Size \times Size \to \mathbb{R} \cup \{+\infty\}$ *defined as*

$$d\Big((\mathcal{M}, \vec{\varphi}), (\mathcal{N}, \vec{\psi})\Big) = \inf_{f \in H(\mathcal{M}, \mathcal{N})} \max_{P \in \mathcal{M}} ||\vec{\varphi}(P) - \vec{\psi}(f(P))||_\infty \tag{1.1}$$

if $H(\mathcal{M}, \mathcal{N}) \neq \emptyset$, and $+\infty$ otherwise.

Remark 1. We observe that the term pseudo-distance means that d can vanish even if the size pairs do not coincide.

Remark 2. In other words, the natural pseudo-distance d between two size pairs $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{N}, \vec{\psi})$ is equal to the infimum of the change of the measuring functions $\vec{\varphi}$, $\vec{\psi}$, induced by each homeomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Unfortunately, the computation of the natural pseudo-distance d is quite difficult, since it involves the study of all the homeomorphisms between two topological spaces. Moreover, the infimum in (1.1) is not always a minimum, i.e. there does not always exist a homeomorphism h realizing the equality $d\Big((\mathcal{M}, \vec{\varphi}), (\mathcal{N}, \vec{\psi})\Big) = \max_{P \in \mathcal{M}} ||\vec{\varphi}(P) - \vec{\psi}(h(P))||_\infty$. These problems justify the introduction of the concept of *k-dimensional size functions*, a mathematical tool simpler to deal than natural pseudo-distance, allowing us to obtain information about d without computing it.

In what follows, $\mathbb{R}^k \times \mathbb{R}^k$ and $\mathbb{R}^{2k}$ will be identified. We shall use the following notations: Δ^+ will be the open set $\{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$, while $\Delta = \partial\Delta^+$. For every k-tuple $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, let $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\,\rangle$ be the set $\{P \in \mathcal{M} : \varphi_i(P) \leq x_i,\ i = 1, \ldots, k\}$.

Definition 1.2. *For every k-tuple $\vec{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, we shall say that two points $P, Q \in \mathcal{M}$ are $\langle\vec{\varphi} \preceq \vec{y}\,\rangle$-*connected *if and only if a connected subset of $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\,\rangle$ exists, containing P and Q.*

Remark 3. Obviously, the relation of $\langle\vec{\varphi} \preceq \vec{y}\,\rangle$-connectedness is an equivalence relation.

Definition 1.3. *We shall call k-*dimensional size function *associated with the size pair $(\mathcal{M}, \vec{\varphi})$ the function $\ell_{(\mathcal{M},\vec{\varphi})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y})$ equal to the number of equivalence classes in which the set $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\,\rangle$ is divided by the $\langle\vec{\varphi} \preceq \vec{y}\,\rangle$-connectedness relation.*

Remark 4. In other words, $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y})$ counts the connected components in $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\,\rangle$ containing at least one point of $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\,\rangle$.

Remark 5. In definition 1.3, the case $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y}) = +\infty$ is not considered: Indeed, under our assumption on $\mathcal{M}$ and $\vec{\varphi}$ it is possible to prove that $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y})$ is finite for every $(\vec{x}, \vec{y}) \in \Delta^+$.

1.2 Special case: 1-dimensional size functions

In this section we introduce 1-dimensional size functions, in order to make the previous definitions and concepts clearer. Furthermore, the case $k = 1$ has been extensively studied in recent past years (cf., e.g., [20, 23, 24, 25, 26, 29, 33, 36, 37, 38, 39]) and presents a certain number of interesting properties that turn out to be useful in our approach to the k-dimensional problem.

As the term “1-dimensional” suggests, in this case we have to deal with topological spaces endowed with continuous functions taking value in $\mathbb{R}$: This is why, in the sequel of the chapter, the symbol $\vec{\varphi}$, representing a k-dimensional measuring function, will be replaced, when $k = 1$, by the more natural φ.

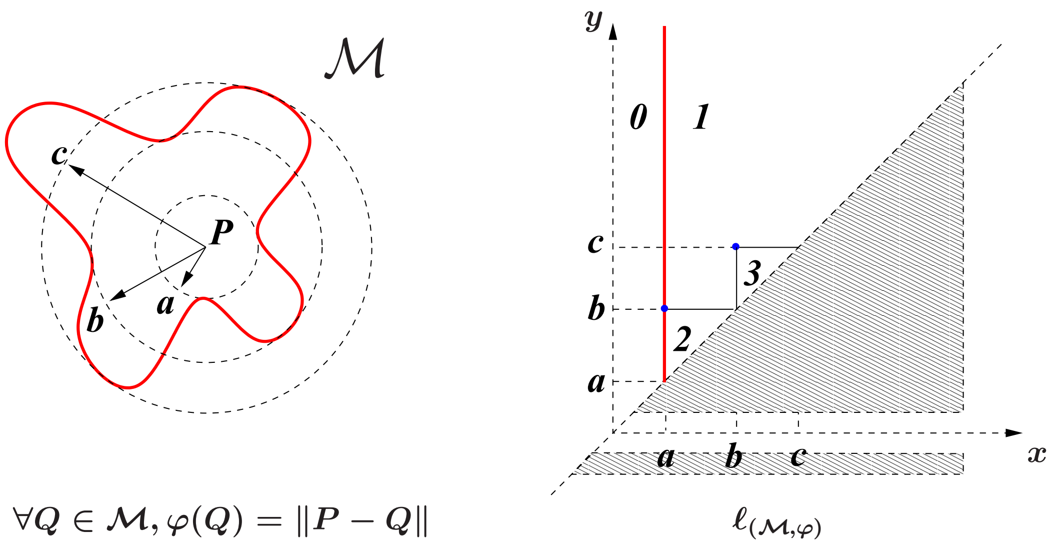

Figure 1.1: A size pair and the associated 1-dimensional size function.

Figure 1.1(left) shows an example of a size pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a closed curve and the chosen measuring function φ is defined as the Euclidean distance from the point P. Figure 1.1(right) represents the 1-dimensional size function associated to $(\mathcal{M}, \varphi)$. Since φ takes value in $\mathbb{R}$, the domain Δ^+ of $\ell_{(\mathcal{M},\varphi)}$ is the subset of the real plane defined as $\{(x, y) \in \mathbb{R}^2 : x < y\}$. For $k = 1$, Δ^+ is divided by solid lines, representing the discontinuity points of the 1-dimensional size function, into triangular regions. In all these regions the value of $\ell_{(\mathcal{M},\varphi)}$ is constant, and equal to the numbers displayed in figure. Two of the most interesting features of 1-dimensional size functions are their resistance to noise (useful especially in applications) and their modularity: in particular, 1-dimensional size functions inherit their invariance properties

directly from the chosen measuring functions. As an example, we observe that it would be possible to apply rotations around P to the closed curve in figure 1.1, being sure that no changing occurs in the related 1-dimensional size function.
Other interesting properties, showing that 1-dimensional size function have a very simple structure, are the following: if $(\mathcal{M}, \varphi)$ is a size pair, with $\varphi : \mathcal{M} \to \mathbb{R}$, then (i) $\ell_{(\mathcal{M},\varphi)}(x, y)$ is non-decreasing in x and non-increasing in y, (ii) $\ell_{(\mathcal{M},\varphi)}(x, y)$ is finite for every $(x, y) \in \Delta^+$, (iii) $\ell_{(\mathcal{M},\varphi)}(x, y) = 0$ for every $x < \min_{P\in\mathcal{M}} \varphi(P)$.

In what follows some results about 1-dimensional size functions will be shown, useful to describe their capability in discriminating size pairs. For the sake of conciseness, proofs will be omitted. For more details, the reader is referred to [17, 18, 26, 28, 29].

1.3 Algebraic representation of 1-dimensional size functions

In order to compare size pairs, discontinuity points of 1-dimensional size functions play an important role: indeed, they divide the domain Δ^+ into triangular regions, in which the value of the 1-dimensional size function is constant. More precisely, these triangular regions may overlap, with a side on the diagonal of the real plane, and may have finite or infinite area. As an example, in figure 1.1 one can see three overlapping triangles: Two of them are bounded, and each one has a unique vertex lying in Δ^+, with coordinates respectively (a, b) and (b, c). The last triangle is unbounded, with a side on the diagonal and another vertical side on the line $x = a$.

The key idea is to make use of this property of 1-dimensional size functions in order to describe them in a simpler way. This can be done by identifying a bounded triangular region with its vertex not lying onto the diagonal, and a triangular region of infinite area with its unbounded vertical side.

The following definitions formalize this idea:

Definition 1.4. (Proper cornerpoint) *For every point $p = (x, y) \in \Delta^+$ and for every positive real number ϵ with $x + \epsilon < y - \epsilon$, let us define the number $\mu_\epsilon(p)$ as*

$$\ell_{(\mathcal{M},\varphi)}(x+\epsilon, y-\epsilon) - \ell_{(\mathcal{M},\varphi)}(x-\epsilon, y-\epsilon) - \ell_{(\mathcal{M},\varphi)}(x+\epsilon, y+\epsilon) + \ell_{(\mathcal{M},\varphi)}(x-\epsilon, y+\epsilon).$$

The finite number $\mu(p) := \min\{\mu_\epsilon(p) : \epsilon > 0, x + \epsilon < y - \epsilon\}$ will be called multiplicity of p *for $\ell_{(\mathcal{M},\varphi)}$. Moreover, we shall call* proper cornerpoint *for $\ell_{(\mathcal{M},\varphi)}$ any point $p \in \Delta^+$ such that the number $\mu(p)$ is strictly positive.*

Definition 1.5. (Cornerpoint at infinity) *For every vertical line r, with equation $x = k$, and for every positive real number ϵ with $k + \epsilon < 1/\epsilon$, let us identify r with the pair (k, ∞), and define the number $\mu_\epsilon(r)$ as*

$$\ell_{(\mathcal{M},\varphi)}(k + \epsilon, 1/\epsilon) - \ell_{(\mathcal{M},\varphi)}(k - \epsilon, 1/\epsilon).$$

When the finite number $\mu(r) := \min\{\mu_\epsilon(r) : \epsilon > 0, k + \epsilon < 1/\epsilon\}$, called multiplicity of r *for $\ell_{(\mathcal{M},\varphi)}$, is strictly positive, we shall call the line r a* cornerpoint at infinity *for the 1-dimensional size function.*

Remark 6. Obviously, the numbers $\mu_\epsilon(p)$, $\mu_\epsilon(r)$ in definitions 1.4 and 1.5 are integer numbers. Moreover, it is possible to prove that they are always non-negative. Hence, it follows that the multiplicity of points and vertical lines is well-defined and non-negative.

Remark 7. Under our assumptions on $\mathcal{M}$, $\mu(r)$ can only take the values 0 and 1, but the definition can easily be extended to spaces with any finite number of connected components, so that $\mu(r)$ can equal any natural number. Moreover, the connectedness assumption also implies that there is exactly one cornerpoint at infinity.

As an example, in Figure 1.1 the only proper cornerpoints for $\ell_{(\mathcal{M},\varphi)}$ are the points with coordinates (a, b) and (b, c), with $\mu((a, b)) = 1$ and

$\mu((b,c)) = 2$, while the unique cornerpoint at infinity is the line $r : x = a$ and it has multiplicity $\mu(r) = \mu((a,\infty)) = 1$.

Therefore, proper cornerpoints and cornerpoints at infinity of a 1-dimensional size function allow us to univocally identify the (bounded and unbounded) overlapping triangular regions splitting the domain Δ^+. The sides lying on Δ^+ of such triangles represent the discontinuity point of the 1-dimensional size function, "created" by cornerpoints and spreading downwards and rightwards to $\Delta = \{(x,y) \in \mathbb{R}^2 : x = y\}$, as the next proposition states.

Proposition 1.3.1. (Propagation of discontinuities) *If $\bar{p} = (\bar{x}, \bar{y})$ is a proper cornerpoint for $\ell_{(\mathcal{M},\varphi)}$, then the following statements hold:*

i) *If $\bar{x} \leq x < \bar{y}$, then $\bar{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(x,\cdot)$;*

ii) *If $\bar{x} < y < \bar{y}$, then $\bar{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot,y)$.*

If $\bar{r} = (\bar{x},\infty)$ is the cornerpoint at infinity for $\ell_{(\mathcal{M},\varphi)}$, then the following statement holds:

iii) *If $\bar{x} < y$, then $\bar{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot,y)$.*

The position of cornerpoints in Δ^+ is related to the extrema of the measuring function as the next proposition states, immediately following from the definitions.

Proposition 1.3.2. (Localization of cornerpoints) *If $\bar{p} = (\bar{x}, \bar{y})$ is a proper cornerpoint for $\ell_{(\mathcal{M},\varphi)}$, then*

$$\bar{p} \in \{(x,y) \in \mathbb{R}^2 : \min\varphi \leq x < y \leq \max\varphi\}.$$

If $\bar{r} = (\bar{x},\infty)$ is the cornerpoint at infinity for $\ell_{(\mathcal{M},\varphi)}$, then $\bar{x} = \min\varphi$.

Proposition 1.3.1 and Proposition 1.3.2 imply that the number of cornerpoints is either finite or countably infinite. In fact, the following result can be proved.

Proposition 1.3.3. (Local finiteness of cornerpoints) *For each strictly positive real number ϵ, 1-dimensional size functions have, at most, a finite number of cornerpoints in* $\{(x,y) \in \mathbb{R}^2 : x + \epsilon < y\}$.

Therefore, if the set of cornerpoints of a 1-dimensional size function has an accumulation point, it necessarily belongs to the diagonal Δ. The next result shows that cornerpoints, with their multiplicities, uniquely determine 1-dimensional size functions. The open (resp. closed) half-plane Δ^+ (resp. $\bar{\Delta}^+$) extended by the points at infinity of the kind (k, ∞) will be denoted by Δ^* (resp. $\bar{\Delta}^*$), i.e. $\Delta^* := \Delta^+ \cup \{(k,\infty) : k \in \mathbb{R}\}$, $\bar{\Delta}^* := \bar{\Delta}^+ \cup \{(k,\infty) : k \in \mathbb{R}\}$.

Theorem 1.3.4. (Representation Theorem) *For every* $(\bar{x},\bar{y}) \in \Delta^+$ *we have*

$$\ell_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}) = \sum_{\substack{(x,y)\in\Delta^* \\ x\leq\bar{x},y>\bar{y}}} \mu((x,y)).$$

In the previous summation, only finitely many terms are different from zero because of Proposition 1.3.3 (Local finiteness of cornerpoints).

The Representation Theorem 1.3.4 shows the importance of proper cornerpoints and cornerpoints at infinity, since the value of a 1-dimensional size function at a point p of its domain can be obtained as the sum of the multiplicities of proper cornerpoints and cornerpoints at infinity representing those triangular regions containing p. As an example, consider once more Figure 1.1: The value taken by the 1-dimensional size function at a point (x,y) of Δ^+, with $b < x < y < c$, is equal to 3, and it is given by the sum $\mu((b,c)) + \mu((a,\infty))$.

According to these last considerations, it follows that it is possible to represent any 1-dimensional size function by a formal series of points and lines of the real plane, i.e. by means of its proper cornerpoints and cornerpoints at infinity, counted with their multiplicities. In this way, the hard problem of comparing topological spaces endowed with continuous real functions can be translated, by means of 1-dimensional size functions and their compact representation based on proper cornerpoints and cornerpoints at infinity, into the much simpler problem of comparing formal series.

1.4 Matching distance between 1-dimensional size functions

In this section a matching distance between 1-dimensional size functions will be introduced. The idea is to compare 1-dimensional size functions by measuring the cost of transporting the cornerpoints of a 1-dimensional size function to those of the other one, with the property that the longest of the transportations should be as short as possible. Since, in general, the number of cornerpoints of two 1-dimensional size functions is different, it will be possible to transport the cornerpoints onto the points of Δ: In other words, it will be possible to "destroy them".

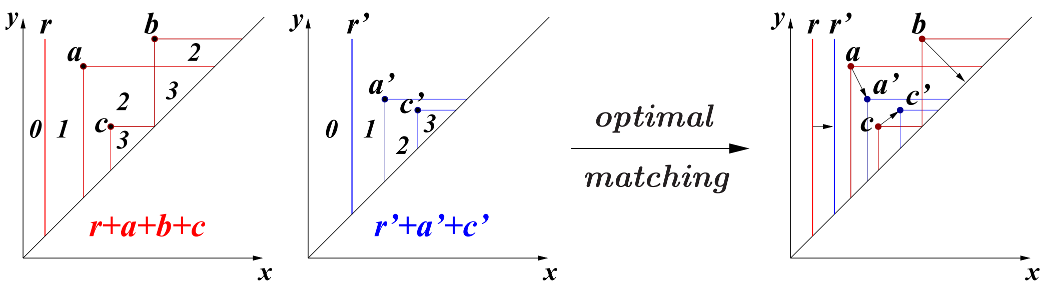

Figure 1.2: An example of optimal matching between two 1-dimensional size functions.

It is important to underline that the number of cornerpoints representing a 1-dimensional size function may be finite or countably infinite: Nevertheless, it is possible to prove that there always exists an optimal matching between two 1-dimensional size functions. Figure 1.2 shows an example of optimal matching between 1-dimensional size functions.

The matching distance is not the only metric between 1-dimensional size functions: Other metrics for size functions have been considered in the past ([19, 27]). However, the matching distance is of particular interest since it allows for a connection with the natural pseudo-distance between size pairs: In particular, it has been proved that the matching distance provides the best estimate for the natural pseudo-distance. Moreover, it has already been successfully tested in [6]. For a detailed treatment about matching distance, see [17, 18].

In order to introduce the matching distance between 1-dimensional size functions we need some new definitions.

Definition 1.6. (Representative sequence) *Let ℓ be a 1-dimensional size function. We shall call representative sequence for ℓ any sequence of points $a : \mathbb{N} \to \bar{\Delta}^*$, (briefly denoted by (a_i)), with the following properties:*

(1) *a_0 is the cornerpoint at infinity for ℓ;*

(2) *For each $i > 0$, either a_i is a proper cornerpoint for ℓ, or a_i belongs to Δ;*

(3) *If p is a proper cornerpoint for ℓ with multiplicity $\mu(p)$, then the cardinality of the set $\{i \in \mathbb{N} : a_i = p\}$ is equal to $\mu(p)$;*

(4) *The set of indexes for which a_i is in Δ is countably infinite.*

We now consider the following pseudo-distance d_M on $\bar{\Delta}^*$ in order to assign a cost to each deformation of 1-dimensional size functions:

$$d_M((x,y),(x',y')) := \min\left\{\max\{|x-x'|,|y-y'|\}, \max\left\{\frac{y-x}{2}, \frac{y'-x'}{2}\right\}\right\},$$

with the convention about ∞ that $\infty - y = y - \infty = \infty$ for $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{\infty, c\} = c$, $\max\{\infty, c\} = \infty$. In other words, the pseudo-distance d_M between two points p and p' compares the cost of moving p to p' and the cost of moving p and p' onto the diagonal and takes the smaller. Costs are computed using the distance induced by the max-norm. In particular, the pseudo-distance d_M between two points p and p' on the diagonal is always 0; the pseudo-distance d_M between two points p and p', with p above the diagonal and p' on the diagonal, is equal to the distance, induced by the max-norm, between p and the diagonal. Points at infinity have a finite distance only to other points at infinity, and their distance depends on their abscissas. Therefore, $d_M(p, p')$ can be considered a measure of the minimum of the costs of moving p to p' along two different paths (i.e. the path that takes p directly to p' and the path that passes through Δ). This observation easily yields that d_M is actually a pseudo-distance.

Definition 1.7. (Matching distance) *Let ℓ_1 and ℓ_2 be two 1-dimensional size functions. If (a_i) and (b_i) are two representative sequences for ℓ_1 and ℓ_2 respectively, then the matching distance between ℓ_1 and ℓ_2 is the number*

$$d_{match}(\ell_1, \ell_2) := \inf_{\sigma} \sup_{i} d_M(a_i, b_{\sigma(i)}),$$

where i varies in $\mathbb{N}$ and σ varies among all the bijections from $\mathbb{N}$ to $\mathbb{N}$.

Proposition 1.4.1. *d_{match} is a distance between 1-dimensional size functions.*

1.4.1 Stability of the matching distance

This subsection is devoted to show that the matching distance between 1-dimensional size functions is stable, i.e. that if φ and ψ are two 1-dimensional measuring functions on $\mathcal{M}$ whose difference at the points of $\mathcal{M}$ is controlled by ϵ (namely $\max_{P\in\mathcal{M}} ||\varphi(P) - \psi(P)||_\infty = \max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon$), then the matching distance between $\ell_{(\mathcal{M},\varphi)}$ and $\ell_{(\mathcal{M},\psi)}$ is also controlled by ϵ (namely $d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{M},\psi)}) \leq \epsilon$). Moreover, two results stating that the matching distance between 1-dimensional size functions furnishes the best lower bound for the natural pseudo-distance between size pairs will be provided.

Theorem 1.4.2. (Matching Stability Theorem) *Let $(\mathcal{M},\varphi)$ be a size pair. For every real number $\epsilon \geq 0$ and for every measuring function $\psi : \mathcal{M} \to \mathbb{R}$, such that* $\max_{P\in\mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon$*, the matching distance between $\ell_{(M,\varphi)}$ and $\ell_{(M,\psi)}$ is smaller than or equal to ϵ.*

The next theorem states that the inf and the sup in the definition of matching distance are actually attained, that is to say, a matching σ exists for which $d_{match}(\ell_1, \ell_2) = \min_\sigma \max_i d_M(a_i, b_{\sigma(i)})$. Every such matching will henceforth be called *optimal*.

Theorem 1.4.3. (Optimal Matching Theorem) *Let (a_i) and (b_i) be two representative sequences of points for the 1-dimensional size functions ℓ_1 and ℓ_2 respectively. Then the matching distance between ℓ_1 and ℓ_2 is equal to the number* $\min_\sigma \max_i d_M(a_i, b_{\sigma(i)})$*, where i varies in $\mathbb{N}$ and σ varies among all the bijections from $\mathbb{N}$ to $\mathbb{N}$.*

The next theorem shows that the matching distance between the 1-dimensional size functions associated to two size pairs $(\mathcal{M},\varphi)$, $(\mathcal{N},\psi)$ is a lower bound for the natural pseudo-distance $d((\mathcal{M},\varphi),(\mathcal{N},\psi))$.

Theorem 1.4.4. *Let $\epsilon \geq 0$ be a real number and let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ be two size pairs with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic. Then*

$$d_{match}\left(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{N},\psi)}\right) \leq \inf_f \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f(P))| = d((\mathcal{M},\varphi),(\mathcal{N},\psi)),$$

where f ranges among all the homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$ and d is the natural pseudo-distance between $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$.

Thanks to this last result, we are able to by-pass the hard problem of comparing size pairs by means of the natural pseudo-distance, introducing and dealing with 1-dimensional size functions and their compact representations by formal series, that are simpler tools: Indeed, theorem 1.4.4 states that it is possible to obtain information about the dissimilarity of two size pairs without computing the related natural pseudo-distance d, but directly from the matching distance between the associated 1-dimensional size functions, that provides a lower bound for d.

As mentioned at the beginning of this section, the matching distance is not the only metric between 1-dimensional size functions. However, it would be not so useful considering other distances, since the lower bound for natural pseudo-distance provided by d_{match} is the best we can obtain, as the next proposition shows.

Proposition 1.4.5. *Let δ be a distance between 1-dimensional size functions, such that*

$$\delta(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{N},\psi)}) \leq \inf_{f \in H(\mathcal{M},\mathcal{N})} \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|,$$

for any two size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic. Then,

$$\delta(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{N},\psi)}) \leq d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{N},\psi)}).$$

Chapter 2

Stability in Multidimensional Size Theory

In this chapter, the extension of size functions to the multidimensional case will be faced.

Figure 2.1 shows an example aimed to introduce the problem we want to tackle by generalizing the concepts expressed in Chapter 1: In $\mathbb{R}^3$ consider the set $\mathcal{Q} = [1,1]\times[1,1]\times[1,1]$ and the sphere $\mathcal{S}$ of equation $x^2+y^2+z^2 = 1$. Let also $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be the continuous function defined as $\Phi(x,y,z) = |x|$. In this setting, consider the size pairs $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ where $\mathcal{M} = \partial\mathcal{Q}$, $\mathcal{N} = \mathcal{S}$, and φ and ψ are respectively the restrictions of Φ to $\mathcal{M}$ and $\mathcal{N}$. The 1-dimensional size functions associated to $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ are displayed: as we can see, it results that $\ell_{(\mathcal{M},\varphi)} \equiv \ell_{(\mathcal{N},\psi)}$.

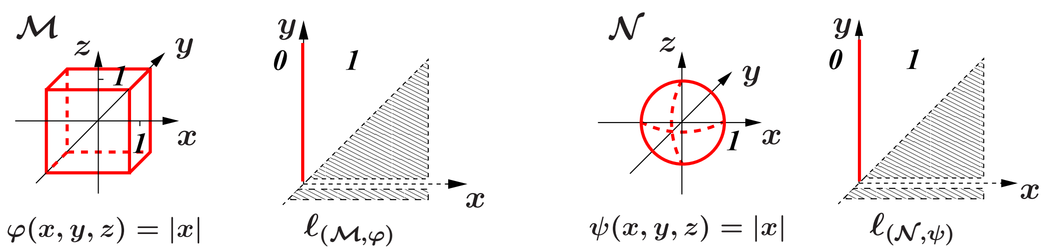

Figure 2.1: A critical example.

In other words, the 1-dimensional size functions, with respect to φ, ψ, are not able to discriminate the cube and the sphere, that is to say, we cannot deduce any information about the natural pseudo-distance $d((\mathcal{M},\varphi),(\mathcal{N},\psi))$.

The example shown in Figure 2.1 is more than a particular case due to a critical choice of φ and ψ: In general, the 1-dimensional size functions associated to different size pairs may coincide. In previous works ([10, 11, 20, 32, 34, 37]), this problem has been faced by increasing the number of 1-dimensional measuring functions describing specific features of the topological spaces under study: According to this approach, in order to evaluate the dissimilarity between two topological spaces $\mathcal{M}$, $\mathcal{N}$, we could consider two *families* of 1-dimensional measuring functions $\{\varphi_i : i = 1,\ldots,k\}$, $\{\psi_i : i = 1,\ldots,k\}$, and merge the information arising from the associated 1-dimensional size functions by computing, e.g, the distance given by $\frac{1}{k}\sum_{i=1}^{k} d_{match}(\ell_{(\mathcal{M},\varphi_i)},\ell_{(\mathcal{M},\psi_i)})$.

However, this technique is not always an optimal solution: As an example, consider once more the spaces $\mathcal{M}$ and $\mathcal{N}$ displayed in Figure 2.1, and the continuous functions $\Phi_1,\Phi_2:\mathbb{R}^3\to\mathbb{R}$ defined as $\Phi_1(x,y,z)=|x|$, $\Phi_2(x,y,z)=|z|$. In this setting, consider the size pairs $(\mathcal{M},\varphi_1)$, $(\mathcal{M},\varphi_2)$, $(\mathcal{N},\psi_1)$, $(\mathcal{N},\psi_2)$, where φ_1, φ_2, ψ_1, ψ_2 are respectively the restrictions of Φ_1 and Φ_2 to $\mathcal{M}$ and $\mathcal{N}$. As previously shown it holds that $\ell_{(\mathcal{M},\varphi_1)}\equiv\ell_{(\mathcal{N},\psi_1)}$. Moreover, it is easy to verify that $\ell_{(\mathcal{M},\varphi_2)}\equiv\ell_{(\mathcal{N},\psi_2)}$. Hence $\frac{1}{2}(d_{match}(\ell_{(\mathcal{M},\varphi_1)},\ell_{(\mathcal{M},\psi_1)})+d_{match}(\ell_{(\mathcal{M},\varphi_2)},\ell_{(\mathcal{M},\psi_2)}))=0$, i.e. this approach is not useful to distinguish the cube and the sphere when considering the two families of 1-dimensional measuring functions $\{\varphi_1,\varphi_2\}$, $\{\psi_1,\psi_2\}$.

According to all these consideration, the introduction of the concept of k-dimensional size function seems to represent the right direction to follow, since it allows us to consider, *at the same time*, different features that turn out to be useful in the comparison of size pairs. Moreover, in general the size function with respect to a k-dimensional measuring function $\varphi=(\varphi_1,\ldots,\varphi_k)$

contains more information than the set of all 1-dimensional size functions with respect to $\varphi_1, \ldots, \varphi_k$, considered independently.

As a confirmation of this last statement, in Example 2.1 we will show that the problem of discriminating the spaces $\mathcal{M}$ and $\mathcal{N}$ displayed in Figure 2.1 can be solved by comparing the 2-dimensional size functions associated to the size pairs $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{N}, \vec{\psi})$, where $\vec{\varphi}$ and $\vec{\psi}$ are respectively the restrictions to $\mathcal{M}$ and $\mathcal{N}$ of the function $\vec{\Phi} = (\Phi_1, \Phi_2) : \mathbb{R}^3 \to \mathbb{R}^2$, defined as $\vec{\Phi}(x, y, z) = (|x|, |z|)$.

2.1 Reduction to the 1-dimensional case

In dealing with k-dimensional size functions, we have to face some problems: (i) a satisfactory representation by formal series seems not to exist: In particular, the most natural extension of the definition of multiplicity to points, lines, planes and hyperplanes of $\mathbb{R}^{2k}$, leads to objects with negative multiplicities; moreover, these structures, together with the ones with positive multiplicity, are not localized in the domain Δ^+, differently from what happens for cornerpoints and cornerpoints at infinity of 1-dimensional size functions (see prop. 1.3.2); (ii) a direct approach to the multidimensional case enforces us to work in subsets of $\mathbb{R}^{2k}$, implying higher computational costs in evaluating k-dimensional size functions and, due to the absence of a representation by means of formal series, in comparing them.

All these problems can be by-passed by means of a suitable change of variables, that allows us to reduce k-dimensional size functions to the 1-dimensional case: Indeed, we have proved that a foliation of Δ^+ in half-planes can be given, such that the restriction of a k-dimensional size function to these half-planes turns out to be a classical size function in two scalar variables. Our approach implies that each size function, with respect to a k-dimensional measuring function, can be completely and compactly described by a parameterized family of discrete descriptors. This follows from the results of chapter 1 about the representation of classical size functions by means of formal series of points and lines, applied to each plane in our foliation.

The following definition fixes in a formal way the concept of foliation in half-planes of Δ^+:

Definition 2.1. *For every unit vector $\vec{l} = (l_1, \ldots, l_k)$ of $\mathbb{R}^k$ such that $l_i > 0$ for every $i = 1, \ldots, k$, and for every vector $\vec{b} = (b_1, \ldots, b_k)$ of $\mathbb{R}^k$ such that $\sum_{i=1}^k b_i = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is* admissible. *We shall denote the set of all admissible pairs in $\mathbb{R}^k \times \mathbb{R}^k$ by Adm_k. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^k \times \mathbb{R}^k$ by the following parametric equations:*

$$\pi_{(\vec{l},\vec{b})} : \begin{cases} \vec{x} = s\vec{l} + \vec{b} \\ \vec{y} = t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

Remark 8. The restriction on the choice of the vectors $\vec{l}$ and $\vec{b}$ guarantees a unique linear parametric representation for each half-plane $\pi_{(\vec{l},\vec{b})}$.

Proposition 2.1.1. *For every $(\vec{x}, \vec{y}) \in \Delta^+$ there exists one and only one admissible pair $(\vec{l}, \vec{b})$ such that $(\vec{x}, \vec{y}) \in \pi_{(\vec{l},\vec{b})}$.*

Proof. The claim immediately follows by taking, for $i = 1, \ldots, k$,

$$l_i = \frac{y_i - x_i}{\sqrt{\sum_{j=1}^k (y_j - x_j)^2}}, \quad b_i = \frac{x_i \sum_{j=1}^k y_j - y_i \sum_{j=1}^k x_j}{\sum_{j=1}^k (y_j - x_j)}.$$

Then, $\vec{x} = s\vec{l} + \vec{b}$, $\vec{y} = t\vec{l} + \vec{b}$, with

$$s = \frac{\sum_{j=1}^k x_j}{\sum_{j=1}^k l_j} = \frac{\sum_{j=1}^k x_j \sqrt{\sum_{j=1}^k (y_j - x_j)^2}}{\sum_{j=1}^k (y_j - x_j)}$$

$$t = \frac{\sum_{j=1}^k y_j}{\sum_{j=1}^k l_j} = \frac{\sum_{j=1}^k y_j \sqrt{\sum_{j=1}^k (y_j - x_j)^2}}{\sum_{j=1}^k (y_j - x_j)}.$$

□

Proposition 2.1.1 imply that the family $\{\pi_{(\vec{l},\vec{b})} : (\vec{l},\vec{b}) \in Adm_k\}$ is actually a foliation.

Now we can prove the reduction to the 1-dimensional case.

Theorem 2.1.2. *Let $(\vec{l},\vec{b})$ be an admissible pair, and $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} : \mathcal{M} \to \mathbb{R}$ be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) = \max_{i=1,\dots,k} \left\{ \frac{\varphi_i(P) - b_i}{l_i} \right\} .$$

Then, for every $(\vec{x},\vec{y}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y}) = \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t) .$$

Proof. For every $\vec{x} = (x_1,\dots,x_k) \in \mathbb{R}^k$, with $x_i = sl_i + b_i,\ i = 1,\dots,k$, it holds that $\mathcal{M}\langle \vec{\varphi} \preceq \vec{x}\,\rangle = \mathcal{M}\langle F^{\vec{\varphi}}_{(\vec{l},\vec{b})} \le s\rangle$. This is true because

$$\begin{aligned}
\mathcal{M}\langle \vec{\varphi} \preceq \vec{x}\,\rangle &= \{P \in \mathcal{M} : \varphi_i(P) \le x_i,\ i = 1,\dots,k\} = \\
&= \{P \in \mathcal{M} : \varphi_i(P) \le sl_i + b_i,\ i = 1,\dots,k\} = \\
&= \left\{ P \in \mathcal{M} : \frac{\varphi_i(P) - b_i}{l_i} \le s,\ i = 1,\dots,k \right\} = \mathcal{M}\langle F^{\vec{\varphi}}_{(\vec{l},\vec{b})} \le s\rangle.
\end{aligned}$$

Analogously, for every $\vec{y} = (y_1,\dots,y_k) \in \mathbb{R}^k$, with $y_i = tl_i + b_i,\ i = 1,\dots,k$, it holds that $\mathcal{M}\langle \vec{\varphi} \preceq \vec{y}\,\rangle = \mathcal{M}\langle F^{\vec{\varphi}}_{(\vec{l},\vec{b})} \le t\rangle$. Therefore Remark 4 implies the claim. □

In the following, we shall use the symbol $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ in the sense of Theorem 2.1.2.

Remark 9. Theorem 2.1.2 allows us to represent each k-dimensional size function as a parameterized family of formal series of points and lines, on the basis of the description introduced in Chapter 1 for the 1-dimensional case. Indeed, we can associate a formal series $\sigma_{(\vec{l},\vec{b})}$ with each admissible pair $(\vec{l},\vec{b})$, with $\sigma_{(\vec{l},\vec{b})}$ describing the 1-dimensional size function $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$. The family $\{\sigma_{(\vec{l},\vec{b})} : (\vec{l},\vec{b}) \in Adm_k\}$ is a new complete descriptor for $\ell_{(\mathcal{M},\vec{\varphi})}$, in the sense that two k-dimensional size functions coincide if and only if the corresponding parameterized families of formal series coincide.

2.2 Lower bounds for the k-dimensional natural pseudo-distance

In Chapter 1, it has been shown that 1-dimensional size functions can be compared by means of the matching distance. We recall that this distance is based on the observation that each 1-dimensional size function is the sum of characteristic functions of triangles. The matching distance is computed by finding an optimal matching between the sets of triangles describing two size functions. In the sequel, we shall denote by $d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ the matching distance between the 1-dimensional size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$.

The following result is an immediate consequence of Theorem 2.1.2 and Remark 9.

Corollary 2.2.1. *Let us consider the size pairs $(\mathcal{M},\vec{\varphi})$, $(\mathcal{N},\vec{\psi})$. Then, the identity $\ell_{(\mathcal{M},\vec{\varphi})}\equiv\ell_{(\mathcal{N},\vec{\psi})}$ holds if and only if $d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})=0$, for every admissible pair $(\vec{l},\vec{b})$.*

The next result proves that small enough changes in $\vec{\varphi}$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance.

Proposition 2.2.2. *If $(\mathcal{M},\vec{\varphi})$, $(\mathcal{M},\vec{\chi})$ are size pairs and $\max_{P\in\mathcal{M}}\|\vec{\varphi}(P)-\vec{\chi}(P)\|_\infty\le\epsilon$, then for each admissible pair $(\vec{l},\vec{b})$, it holds that*

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathcal{M},F^{\vec{\chi}}_{(\vec{l},\vec{b})})})\le\frac{\epsilon}{\min_{i=1,\dots,k}l_i}.$$

Proof. From the Matching Stability Theorem 1.4.2, we obtain that

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathcal{M},F^{\vec{\chi}}_{(\vec{l},\vec{b})})}) \;\le\; \max_{P\in\mathcal{M}}|F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P)-F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P)|.$$

Let us now fix $P\in\mathcal{M}$. Then, denoting by $\hat{\imath}$ the index for which $\max_i\frac{\varphi_i(P)-b_i}{l_i}$

is attained, by the definition of $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ and $F^{\vec{\chi}}_{(\vec{l},\vec{b})}$ we have that

$$\begin{aligned}
& F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P) = \max_i \frac{\varphi_i(P) - b_i}{l_i} - \max_i \frac{\chi_i(P) - b_i}{l_i} = \\
= & \frac{\varphi_{\hat{\imath}}(P) - b_{\hat{\imath}}}{l_{\hat{\imath}}} - \max_i \frac{\chi_i(P) - b_i}{l_i} \leq \frac{\varphi_{\hat{\imath}}(P) - b_{\hat{\imath}}}{l_{\hat{\imath}}} - \frac{\chi_{\hat{\imath}}(P) - b_{\hat{\imath}}}{l_{\hat{\imath}}} = \\
= & \frac{\varphi_{\hat{\imath}}(P) - \chi_{\hat{\imath}}(P)}{l_{\hat{\imath}}} \leq \frac{\|\vec{\varphi}(P) - \vec{\chi}(P)\|_\infty}{\min_{i=1,\dots,k} l_i}.
\end{aligned}$$

In the same way, we obtain $F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) \leq \frac{\|\vec{\varphi}(P)-\vec{\chi}(P)\|_\infty}{\min_{i=1,\dots,k} l_i}$. Therefore, if $\max_{P\in\mathcal{M}} \|\vec{\varphi}(P) - \vec{\chi}(P)\|_\infty \leq \epsilon$,

$$\max_{P\in\mathcal{M}} \left| F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P) \right| \leq \max_{P\in\mathcal{M}} \frac{\|\vec{\varphi}(P) - \vec{\chi}(P)\|_\infty}{\min_{i=1,\dots,k} l_i} \leq \frac{\epsilon}{\min_{i=1,\dots,k} l_i}.$$

□

Analogously, small enough changes in $(\vec{l},\vec{b})$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance, as the next proposition states.

Proposition 2.2.3. *If $(\mathcal{M},\vec{\varphi})$ is a size pair and $(\vec{l},\vec{b})$, $(\vec{l}',\vec{b}')$ are admissible pairs with $\|\vec{l}-\vec{l}'\|_\infty \leq \epsilon$, $\|\vec{b}-\vec{b}'\|_\infty \leq \epsilon$ and $\epsilon < \min_{i=1,\dots,k}\{l_i\}$, it holds that*

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')})}) \leq \epsilon \cdot \frac{\max_{P\in\mathcal{M}} \|\vec{\varphi}(P)\|_\infty + \|\vec{l}\|_\infty + \|\vec{b}\|_\infty}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}.$$

Proof. From the Matching Stability Theorem 1.4.2, we obtain that

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')})}) \quad \leq \quad \max_{P\in\mathcal{M}} |F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')}(P)|.$$

Let us now fix $P \in \mathcal{M}$. Then, denoting by $\hat{\imath}$ the index for which $\max_i \frac{\varphi_i(P)-b_i}{l_i}$

is attained, by the definition of $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ and $F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')}$ we have that

$$\begin{aligned}
& F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')}(P) = \max_i \frac{\varphi_i(P)-b_i}{l_i} - \max_i \frac{\varphi_i(P)-b'_i}{l'_i} = \\
= \quad & \frac{\varphi_{\hat{\imath}}(P)-b_{\hat{\imath}}}{l_{\hat{\imath}}} - \max_i \frac{\varphi_i(P)-b'_i}{l'_i} \leq \frac{\varphi_{\hat{\imath}}(P)-b_{\hat{\imath}}}{l_{\hat{\imath}}} - \frac{\varphi_{\hat{\imath}}(P)-b'_{\hat{\imath}}}{l'_{\hat{\imath}}} = \\
= \quad & \frac{(l'_{\hat{\imath}}-l_{\hat{\imath}})\varphi_{\hat{\imath}}(P) - l'_{\hat{\imath}}b_{\hat{\imath}} + l_{\hat{\imath}}b'_{\hat{\imath}}}{l_{\hat{\imath}}l'_{\hat{\imath}}} = \frac{(l'_{\hat{\imath}}-l_{\hat{\imath}})\varphi_{\hat{\imath}}(P) + l_{\hat{\imath}}(b'_{\hat{\imath}}-b_{\hat{\imath}}) + b_{\hat{\imath}}(l_{\hat{\imath}}-l'_{\hat{\imath}})}{l_{\hat{\imath}}l'_{\hat{\imath}}} \leq \\
\leq \quad & \frac{|l'_{\hat{\imath}}-l_{\hat{\imath}}||\varphi_{\hat{\imath}}(P)| + |l_{\hat{\imath}}||b'_{\hat{\imath}}-b_{\hat{\imath}}| + |b_{\hat{\imath}}||l_{\hat{\imath}}-l'_{\hat{\imath}}|}{l_{\hat{\imath}}l'_{\hat{\imath}}} \leq \frac{\epsilon(\|\vec{\varphi}(P)\|_\infty + \|\vec{l}\|_\infty + \|\vec{b}\|_\infty)}{l_{\hat{\imath}}(l_{\hat{\imath}}-\epsilon)} \\
\leq \quad & \frac{\epsilon(\|\vec{\varphi}(P)\|_\infty + \|\vec{l}\|_\infty + \|\vec{b}\|_\infty)}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}.
\end{aligned}$$

Analogously, we can prove that $F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')}(P) - F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) \leq \frac{\epsilon(\|\vec{\varphi}(P)\|_\infty + \|\vec{l}\|_\infty + \|\vec{b}\|_\infty)}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}$. Therefore,

$$\max_{P\in\mathcal{M}} \left| F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')}(P) \right| \leq \epsilon \cdot \frac{\max_{P\in\mathcal{M}} \|\vec{\varphi}(P)\|_\infty + \|\vec{l}\|_\infty + \|\vec{b}\|_\infty}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}.$$

□

Proposition 2.2.2 and Proposition 2.2.3 prove the stability of our computational approach.

Now we are able to prove our next result, showing that a lower bound for the natural pseudo-distance exists, provided by the restrictions of k-dimensional size functions to the half-planes of the foliation.

Theorem 2.2.4. *Let $(\mathcal{M},\vec{\varphi})$ and $(\mathcal{N},\vec{\psi})$ be two size pairs, with $\mathcal{M}$, $\mathcal{N}$ homeomorphic. Setting $d((\mathcal{M},\vec{\varphi}),(\mathcal{N},\vec{\psi})) = \inf_f \max_{P\in\mathcal{M}} \|\vec{\varphi}(P) - \vec{\psi}(f(P))\|_\infty$, where f varies among all the homeomorphisms between $\mathcal{M}$ and $\mathcal{N}$, it holds that*

$$\sup_{(\vec{l},\vec{b})\in Adm_k} \min_{i=1,\dots,k} l_i \cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) \leq d((\mathcal{M},\vec{\varphi}),(\mathcal{N},\vec{\psi})).$$

Proof. For any homeomorphism f between $\mathcal{M}$ and $\mathcal{N}$, it holds that $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})} \equiv \ell_{(\mathcal{M},F^{\vec{\psi}}_{(\vec{l},\vec{b})}\circ f)}$. Moreover, by applying Proposition 2.2.2 with $\epsilon = \max_{P\in\mathcal{M}} \|\vec{\varphi}(P) - \vec{\psi}(f(P))\|_\infty$ and $\vec{\chi} = \vec{\psi}\circ f$, and observing that $F^{\vec{\psi}}_{(\vec{l},\vec{b})} \circ f \equiv F^{\vec{\psi}\circ f}_{(\vec{l},\vec{b})}$, we have

$$\min_{i=1,\dots,k} l_i \cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) \le \max_{P\in\mathcal{M}} \|\vec{\varphi}(P) - \vec{\psi}(f(P))\|_\infty$$

for every admissible $(\vec{l},\vec{b})$. Since this is true for each homeomorphism f between $\mathcal{M}$ and $\mathcal{N}$, the claim immediately follows. □

Remark 10. We observe that the left side of the inequality in Theorem 2.2.4 defines a distance between k-dimensional size functions associated with homeomorphic spaces. When the spaces are not assumed to be homeomorphic, it still verifies all the properties of a distance, except for the fact that it may take the value $+\infty$. In other words, it defines an extended distance.

Definition 2.2. *Let* $(\mathcal{M},\vec{\varphi})$ *and* $(\mathcal{N},\vec{\psi})$ *be two size pairs. We shall call* multidimensional matching distance *the extended distance defined by setting*

$$D_{match}(\ell_{(\mathcal{M},\vec{\varphi})}, \ell_{(\mathcal{N},\vec{\psi})}) = \sup_{(\vec{l},\vec{b})\in Adm_k} \min_{i=1,\dots,k} l_i \cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}).$$

Remark 11. If we choose a non-empty subset $A \subseteq Adm_k$ and we substitute $\sup_{(\vec{l},\vec{b})\in Adm_k}$ with $\sup_{(\vec{l},\vec{b})\in A}$ in Definition 2.2, we obtain an (extended) pseudo-distance between k-dimensional size functions. If $A = \{(\vec{l^j},\vec{b^j}) : j = 1,\dots,n\}$ is finite, this pseudo-distance appears to be particularly suitable for applications, from a computational point of view; another interesting choice for applications seems to be the weighted mean pseudo-distance defined as $\sum_{j=1}^n w^j \cdot \min_{i=1,\dots,k} l_i^j \cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})})})$ (assuming that w^j are real numbers with $w^j > 0$, for every $j = 1,\dots,n$, and $\sum_{j=1}^n w^j = 1$), that takes into account the information conveyed from each leaf $(\vec{l^j},\vec{b^j}) \in A$. It is easy to verify that also this pseudo-distance furnishes a lower bound for the natural pseudo-distance.

Example 2.1. *We want to show that the critical example introduced at the beginning of this chapter, i.e. the discrimination of the cube and the sphere,* *can be successfully approached by means of k-dimensional size functions.*

Consider once more the topological spaces $\mathcal{M} = \partial\mathcal{Q}$ *and* $\mathcal{N} = \mathcal{S}$*, where* $\mathcal{Q} = [-1,1]\times[-1,1]\times[-1,1]$ *and* $\mathcal{S}$ *is the sphere of equation* $x^2+y^2+z^2=1$*. Let also* $\vec{\Phi} = (\Phi_1, \Phi_2) : \mathbb{R}^3 \to \mathbb{R}^2$ *be the continuous function defined as* $\vec{\Phi}(x,y,z) = (|x|,|z|)$*. Our goal is to obtain a meaningful lower bound for the natural pseudo-distance between the size pairs* $(\mathcal{M},\vec{\varphi})$ *and* $(\mathcal{N},\vec{\psi})$*, where* $\vec{\varphi}$ *and* $\vec{\psi}$ *are respectively the restrictions of* $\vec{\Phi}$ *to* $\mathcal{M}$ *and* $\mathcal{N}$*.*

In order to compare the k-dimensional size functions $\ell_{(\mathcal{M},\vec{\varphi})}$ *and* $\ell_{(\mathcal{N},\vec{\psi})}$*, we are interested in studying the foliation in half-planes* $\pi_{(\vec{l},\vec{b})}$*, where* $\vec{l} = (\cos\theta, \sin\theta)$ *with* $\theta \in (0, \frac{\pi}{2})$*, and* $\vec{b} = (a,-a)$ *with* $a \in \mathbb{R}$*. Any such half-plane is represented by*

$$\begin{cases} x_1 = s\cos\theta + a \\ x_2 = s\sin\theta - a \\ y_1 = t\cos\theta + a \\ y_2 = t\sin\theta - a \end{cases},$$

with $s,t \in \mathbb{R}$*,* $s < t$*. Figure 2.2 shows the* 1*-dimensional size functions* $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ *and* $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$*, for* $\theta = \frac{\pi}{4}$ *and* $a = 0$*, i.e.* $\vec{l} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ *and* $\vec{b} = (0,0)$*. With this choice, we obtain that* $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} = \sqrt{2}\max\{\varphi_1,\varphi_2\} = \sqrt{2}\max\{|x|,|z|\}$ *and* $F^{\vec{\psi}}_{(\vec{l},\vec{b})} = \sqrt{2}\max\{\psi_1,\psi_2\} = \sqrt{2}\max\{|x|,|z|\}$*. Therefore, Theorem 2.1.2 implies that, for every* $(x_1,x_2,y_1,y_2) \in \pi_{(\vec{l},\vec{b})}$

$$\ell_{(\mathcal{M},\vec{\varphi})}(x_1,x_2,y_1,y_2) = \ell_{(\mathcal{M},\vec{\varphi})}\left(\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}},\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}\right) = \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t)$$

$$\ell_{(\mathcal{N},\vec{\psi})}(x_1,x_2,y_1,y_2) = \ell_{(\mathcal{N},\vec{\psi})}\left(\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}},\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}\right) = \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}(s,t)\ .$$

In this case, by Theorem 2.2.4 and Remark 11 (applied for A containing just the admissible pair that we have chosen), a lower bound for the natural pseudo-distance $d((\mathcal{M},\vec{\varphi}),(\mathcal{N},\vec{\psi}))$ *is given by*

$$\frac{\sqrt{2}}{2} d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) = \frac{\sqrt{2}}{2}(\sqrt{2}-1) = 1 - \frac{\sqrt{2}}{2}.$$

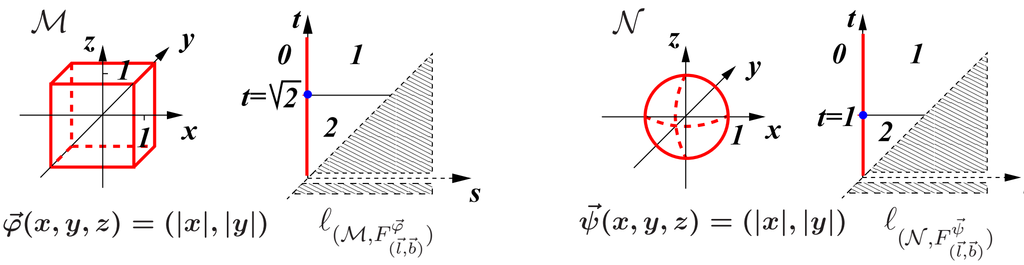

Figure 2.2: The topological spaces $\mathcal{M}$ and $\mathcal{N}$ and the size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$, $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$ associated with the half-plane $\pi_{(\vec{l},\vec{b})}$, for $\vec{l} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\vec{b} = (0,0)$.

Indeed, the matching distance $d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ *is equal to the cost of moving the point of coordinates* $(0,\sqrt{2})$ *onto the point of coordinates* $(0,1)$*, computed with respect to the* max*-norm. The points* $(0,\sqrt{2})$ *and* $(0,1)$ *are representative of the characteristic triangles of the size functions* $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ *and* $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$*, respectively.*

We want to underline that $\ell_{(\mathcal{M},\varphi_1)} \equiv \ell_{(\mathcal{N},\psi_1)}$*, and* $\ell_{(\mathcal{M},\varphi_2)} \equiv \ell_{(\mathcal{N},\psi_2)}$ *(as shown in the example at the beginning of this chapter). In other words, the* k*-dimensional size functions, with respect to* $\vec{\varphi}, \vec{\psi}$*, are able to discriminate the cube and the sphere, while both the* 1*-dimensional size functions, with respect to* φ_1, φ_2 *and* ψ_1, ψ_2*, cannot do that. The higher discriminatory power of* k*-dimensional size functions motivates their definition and use.*

Remark 12. In a recent paper [14], Cohen-Steiner et al. have introduced the concept of *vineyard*, that is a 1-parameter family of persistence diagrams associated with the homotopy f_t, interpolating between f_0 and f_1. We recall that dimension p persistence diagrams are a concise representation of the function *rank* $H_p^{x,y}$, where $H_p^{x,y}$ denotes the dimension p persistent homology group computed at point (x,y) (cf. [14]). These authors assume that the topological space under study is homeomorphic to the body of a simplicial complex, and that the measuring functions are *tame*, i.e. they have only finitely many homological critical values and the homology groups of their sublevel sets all have finite rank. In [13] we have proved that, under the same assumptions, there exist some links between k-dimensional size functions and dimension 0 vineyards, i.e. vineyards associated to 0 persistence diagrams.

However, although some links exist, the concept of k-dimensional size function has quite different purposes than that of vineyard: In particular, [14] does not aim to identify distances for the comparison of vineyards, while we are interested in quantitative methods for comparing k-dimensional size functions.

Chapter 3

Computation in Multidimensional Size Theory

The approach suggested by Multidimensional Size Theory in studying and comparing topological spaces endowed with continuous functions has been developed also from a computational point of view, in order to use k-dimensional size functions in shape discrimination applied problems. Our efforts are encouraged by the promising results obtained for the particular case $k = 1$: Indeed, it has been extensively studied, and applying 1-dimensional size functions has revealed to be particularly useful for quite a lot of applications, especially in the field of Computer Vision, where the considered shapes are images ([8, 9, 10, 11]), and in Computer Graphics, comparing, e.g., 3D-models ([2, 4]). This is due to the fact that size functions show resistance to noise and invariance properties ([28]).

Obviously, dealing with applications involves the development of a discrete counterpart of the theory. From this point of view, a shape is modeled by a graph $G = (V(G), E(G))$ whose vertices are labeled by a function $\vec{\varphi} : V(G) \to \mathbb{R}^k$, representing the feature considered relevant for shape characterization ([28]). This leads to considering pairs of the type $(G, \vec{\varphi})$, called *size graphs*. In this mathematical setting, *discrete k-dimensional size func-*

tions count the number of connected components in $G\langle\vec{\varphi} \preceq \vec{y}\,\rangle$ containing at least one vertex of $G\langle\vec{\varphi} \preceq \vec{x}\,\rangle$ where, for $\vec{t} \in \mathbb{R}^k$, $G\langle\vec{\varphi} \preceq \vec{t}\,\rangle$ is defined as the subgraph of G obtained by erasing all vertices of G at which φ_i takes a value strictly greater than t_i, for at least one index $i \in \{1, \ldots, k\}$, and all the edges connecting those vertices to others.

Therefore, in computing discrete k-dimensional size functions, we have to count the connected components of particular subgraphs of a size graph. As stressed in Chapter 2, the greater the dimension k, the higher the discriminatory power of k-dimensional size functions. On the other hand, the smaller the graph, the faster the computation. Moreover, in applications we often have to deal with big graphs, implying high computational costs. According to these considerations, it follows that the problem of reducing a size graph without changing the associated discrete k-dimensional size function is important for our purposes.

In previous works ([16, 30]), it has been proved that, in the case $k = 1$, a size graph can be reduced by means of a global method (its application requires the knowledge of all the size graph) and a local method (it requires only a local knowledge of a size graph), obtaining a very simple structure. In this chapter, the extension of the global reduction method to the multidimensional case will be shown, together with a theorem stating that discrete k-dimensional size functions are invariant with respect to this reduction method (cf. [12]).

3.1 Basic definitions

In this section we provide some basic definitions about discrete k-dimensional size functions. According to the notations introduced in Section 1.1, for every $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, we shall say $\vec{x} \preceq \vec{y}$ (resp.

$\vec{x} \succeq \vec{y}$) if and only if $x_i \leq y_i$ (resp. $x_i \geq y_i$) for every index $i = 1, \ldots, k$. Moreover, we shall write $\vec{x} \npreceq \vec{y}$ (resp. $\vec{x} \nsucceq \vec{y}$) when the relation between $\vec{x}$ and $\vec{y}$ expressed by the operator $\preceq$ (resp. $\succeq$) is not verified. Finally, we recall that Δ^+ is defined as the open set $\{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$.

Definition 3.1. *Let $G = (V(G), E(G))$ be a finite ordered graph with $V(G)$ set of vertices and $E(G)$ set of edges. Assume that a function $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k) : V(G) \to \mathbb{R}^k$ is given. Then, the pair $(G, \vec{\varphi})$ will be called a* size graph.

Definition 3.2. *For every $\vec{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, we denote by $G\langle \vec{\varphi} \preceq \vec{y} \,\rangle$ the subgraph of G obtained by erasing all vertices $v \in V(G)$ such that $\vec{\varphi}(v) \npreceq \vec{y}$, and all the edges connecting those vertices to others. If $v_a, v_b \in V(G)$ belong to the same connected component of $G\langle \vec{\varphi} \preceq \vec{y} \,\rangle$, we shall write $v_a \cong_{G\langle \vec{\varphi} \preceq \vec{y} \rangle} v_b$.*

Definition 3.3. *We shall call* discrete k-dimensional size function *of the size graph $(G, \vec{\varphi})$ the function $\ell_{(G,\vec{\varphi})} : \Delta^+ \to \mathbb{N}$ defined by setting $\ell_{(G,\vec{\varphi})}(\vec{x}, \vec{y})$ equal to the number of connected components in $G\langle \vec{\varphi} \preceq \vec{y} \,\rangle$ containing at least one vertex of $G\langle \vec{\varphi} \preceq \vec{x} \,\rangle$.*

Example 3.1. *Figure 3.1 represents a possible discretization of the size pair shown in Figure 1.1, together with the related discrete 1-dimensional size function: We recall that in the case $k = 1$ the symbols $\vec{\varphi}, \vec{x}, \vec{y}$ are replaced by φ, x, y respectively. For example, for $a \leq x < b$, the subgraph $G\langle \varphi \preceq x \,\rangle$ consists of two connected components, contained in different connected components of the subgraph $G\langle \varphi \preceq y \,\rangle$ when $x < y < b$. Therefore, $\ell_{(G,\varphi)}(x, y) = 2$ for $a \leq x < y < b$. When $a \leq x < b$ and $y \geq b$, the two connected components in $G\langle \varphi \preceq x \,\rangle$ are contained in the same connected component of $G\langle \varphi \preceq y \,\rangle$, so $\ell_{(G,\varphi)}(x, y) = 1$.*

In what follows, we will assume that the set of vertices $V(G)$ of the graph G is a subset of a Euclidean space.

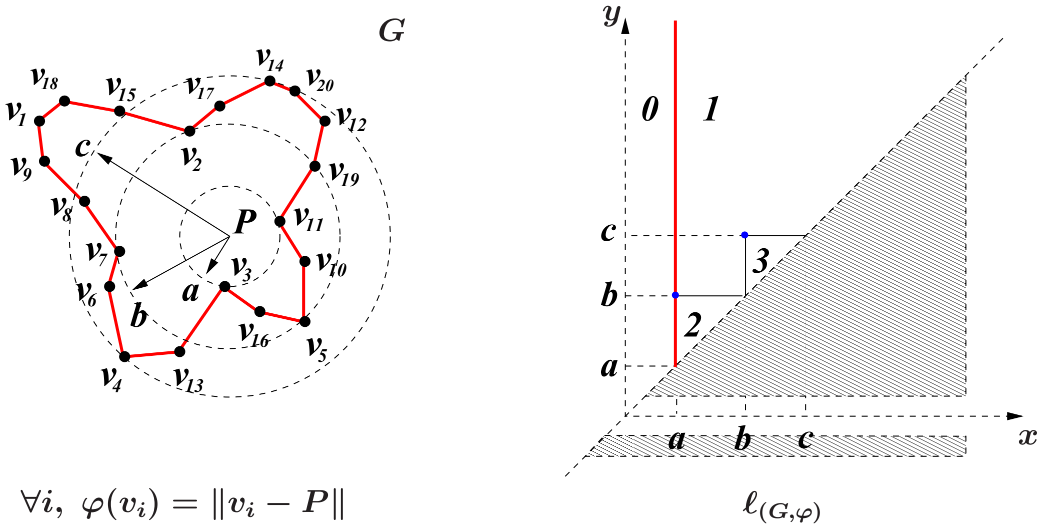

Figure 3.1: A size graph and the associated discrete size function.

3.2 A global method for reducing $(G, \vec{\varphi})$: the $\mathcal{L}$-reduction

Now we are ready to introduce the global method for reducing a size graph.

As stressed at the beginning of this chapter, our goal is to reduce a size graph $(G, \vec{\varphi})$ without changing the related discrete k-dimensional size function: This can be done by erasing all those vertices of G that do not contain, in terms of discrete k-dimensional size functions, "meaningful information". Indeed, in order to compute the discrete k-dimensional size function of $(G, \vec{\varphi})$, we are only interested in capturing the "birth" of new connected components and the "death", i.e. the merging, of the existing ones: As will be shown, these events are strongly related to particular vertices of G, that can be seen, in some sense, as "critical points" of the function $\vec{\varphi}$ with respect to the relation $\preceq$. According to these considerations, we first need to detect such vertices.

In what follows, we assume that a size graph $(G, \vec{\varphi})$ is given. Moreover, for every $v_i \in V(G)$ we define A_i as the set $\{v_j : (v_i, v_j) \in E(G), \vec{\varphi}(v_j) \preceq \vec{\varphi}(v_i)\} \cup \{v_i\}$.

Definition 3.4. *Let $L : V(G) \to V(G)$ be a function defined in this way: for every $v_i \in V(G)$ let $B_i \subseteq A_i$ be the set whose elements are the vertices $w \in A_i$ for which the Euclidean norm $\|\vec{\varphi}(w) - \vec{\varphi}(v_i)\|$ takes the largest value. Finally, we choose the vertex $v_k \in B_i$ for which the index k is minimum. Then, we set $L(v_i) = v_k$. We shall call L the* single step descent flow operator*.*

Remark 13. From the definition of L and the finiteness of $V(G)$, if follows that for every $v \in V(G)$ there must exist a minimum index $m(v) \leq |V(G)|$ such that $L^{m(v)}(v) = L^{m(v)+1}(v)$ (if $L(v) = v$ we will set $m(v) = 0$).

Definition 3.5. *For every $v \in V(G)$ we set $\mathcal{L}(v) = L^{m(v)}(v)$. We shall call the function $\mathcal{L} : V(G) \to V(G)$ the* descent flow operator*.*

Remark 14. In other words, the descent flow operator takes each vertex $v_i \in V(G)$ to a sort of "local minimum" $v_j = \mathcal{L}(v_i)$ of the function $\vec{\varphi}$, with respect to the relation $\preceq$. This implies that, starting from v_j we are not able to reach a vertex w adjacent to it with $\vec{\varphi}(w) \preceq \vec{\varphi}(v_j)$, strictly decreasing the value of at least one component of $\vec{\varphi}$.

During the descent, indexes are used to univocally decide the path in case the set B_i contains more than one vertex.

Example 3.2. *Figure 3.2 shows some possible cases arising from the action of the operators L and $\mathcal{L}$ when $\vec{\varphi} = (\varphi_1, \varphi_2)$. As can be seen, the vertex v_1 is taken by the operator L to $v_4 = L^5(v_1)$. Since it is not possible to reach another vertex from v_4 decreasing the values of both φ_1 and φ_2, we shall set $v_4 = \mathcal{L}(v_1)$. Analogously, we have $v_5 = \mathcal{L}(v_2)$. The last considered case is represented by the vertex v_3: it can be seen as a fixed point with respect to the operator L, i.e. it holds that $L(v_3) = v_3$, so we shall set $\mathcal{L}(v_3) = v_3$.*

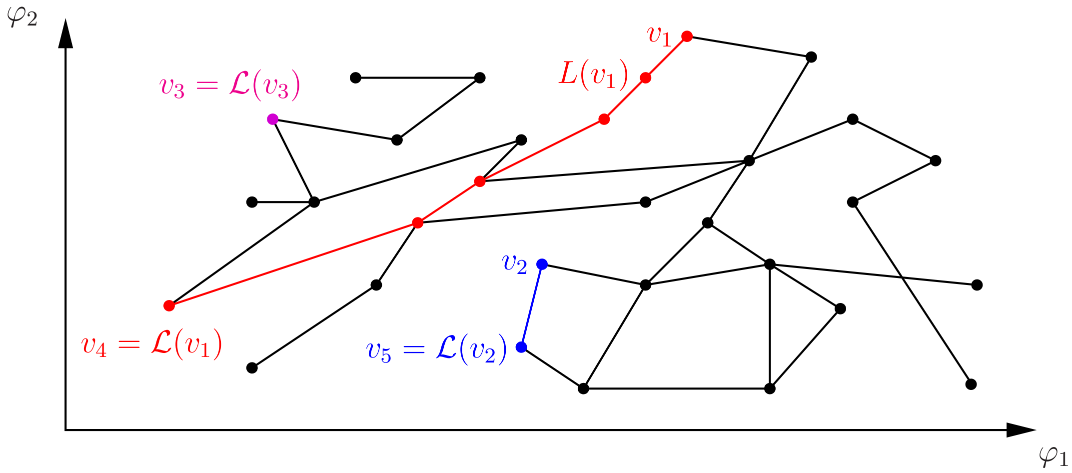

Figure 3.2: Some possible cases arising from the action of the operators L and $\mathcal{L}$ when $\vec{\varphi} = (\varphi_1, \varphi_2)$.

Definition 3.6. *Each vertex v for which $\mathcal{L}(v) = v$ will be called a* minimum vertex *of $(G, \vec{\varphi})$. Call M the set of minimum vertices of $(G, \vec{\varphi})$.*

We point out that M is the set of all those vertices representing the "birth" of new connected components in $(G, \vec{\varphi})$: Indeed, by increasing the values of $\varphi_1, \ldots, \varphi_k$, such an event occurs only when the values labeling a minimum vertex are reached.

The following two definitions characterize the "death-points" of existing connected components of $(G, \vec{\varphi})$.

Definition 3.7. *Assume that $v_{j_1}, v_{j_2} \in V(G)$ are two distinct minimum vertices of $(G, \vec{\varphi})$. Suppose v_{i_1}, v_{i_2} are two adjacent vertices of G, such that $\{\mathcal{L}(v_{i_1}), \mathcal{L}(v_{i_2})\} = \{v_{j_1}, v_{j_2}\}$; we shall call $\{v_{i_1}, v_{i_2}\}$ a* ridge pair adjacent to the minimum vertices v_{j_1} and v_{j_2}.

Definition 3.8. *Assume that $v_{j_1}, v_{j_2} \in V(G)$ are two distinct minimum vertices of $(G, \vec{\varphi})$. Suppose $\{v_{i_1}, v_{i_2}\}$ is a ridge pair adjacent to the minimum vertices v_{j_1}, v_{j_2} such that the following statements hold:*

1. *there does not exists another ridge pair $\{v_{i_3}, v_{i_4}\}$ adjacent to the minimum vertices v_{j_1}, v_{j_2} with*

$$\begin{cases} \max\{\varphi_h(v_{i_3}), \varphi_h(v_{i_4})\} \leq \max\{\varphi_h(v_{i_1}), \varphi_h(v_{i_2})\}, \ h = 1, \ldots, k \\ \exists \bar{h} : \max\{\varphi_{\bar{h}}(v_{i_3}), \varphi_{\bar{h}}(v_{i_4})\} < \max\{\varphi_{\bar{h}}(v_{i_1}), \varphi_{\bar{h}}(v_{i_2})\}, \ \bar{h} \in \{1, \ldots, k\}; \end{cases}$$

2. *if $\{v_{i_3}, v_{i_4}\}$ is another ridge pair adjacent to the minimum vertices v_{j_1}, v_{j_2} with*

$$\max\{\varphi_h(v_{i_3}), \varphi_h(v_{i_4})\} = \max\{\varphi_h(v_{i_1}), \varphi_h(v_{i_2})\}, \ h = 1, \ldots, k,$$

then (i_1, i_2) precedes (i_3, i_4) in the lexicographic order.

We shall call the set $\{v_{i_1}, v_{i_2}\}$ the main saddle adjacent to the minimum vertices v_{j_1}, v_{j_2}. *Call S the set of main saddles of $(G, \vec{\varphi})$.*

Remark 15. In other words, the set of ridge pairs of $(G, \vec{\varphi})$ can be partially ordered by means of the relation $\preceq$. In this sense, the main saddles will be the lowest ridge pairs.

Example 3.3. *Figure 3.3(a), 3.3(b), 3.3(c) shows some examples of ridge pairs and main saddles, when function $\vec{\varphi}$ takes values in $\mathbb{R}^2$.*

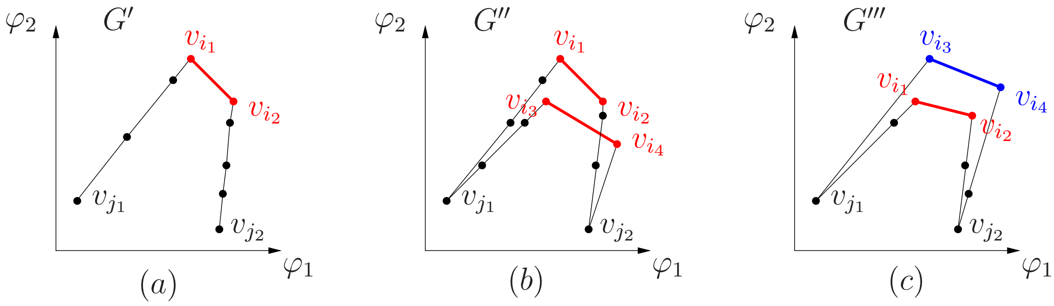

Figure 3.3: Some examples of ridge pairs and main saddles.

In order to clarify the role of main saddles, we are interested in studying the changing in the number of connected components of the subgraphs $G'\langle\vec{\varphi} \preceq \vec{y}\rangle$, $G''\langle\vec{\varphi} \preceq \vec{y}\rangle$ and $G'''\langle\vec{\varphi} \preceq \vec{y}\rangle$, with $\vec{y} \in \mathbb{R}^2$, just for $\vec{y} \succeq$

$(\max\{\varphi_1(v_{j_1}),\varphi_1(v_{j_2})\},\max\{\varphi_2(v_{j_1}),\varphi_2(v_{j_2})\})$: *Indeed, we want to capture the merging of the connected components arising from the minimum vertices* v_{j_1} *and* v_{j_2} *in the three instances. According to this consideration, by means of the chosen assumption on* $\vec{y}$ *we ensure that both* v_{j_1} *and* v_{j_2} *belong to the subgraphs* $G'\langle\vec{\varphi}\preceq\vec{y}\rangle$, $G''\langle\vec{\varphi}\preceq\vec{y}\rangle$ *and* $G'''\langle\vec{\varphi}\preceq\vec{y}\rangle$.

In figure 3.3(a) a main saddle adjacent to the minimum vertices v_{j_1} *and* v_{j_2} *is displayed. In this setting, by varying the values taken by* $\vec{y}$ *under the assumption* $\vec{y}\succeq(\max\{\varphi_1(v_{j_1}),\varphi_1(v_{j_2})\},\max\{\varphi_2(v_{j_1}),\varphi_2(v_{j_2})\})$, *it holds that for* $\vec{y}\not\succeq(\max\{\varphi_1(v_{i_1}),\varphi_1(v_{i_2})\},\max\{\varphi_2(v_{i_1}),\varphi_2(v_{i_2})\})$ *the subgraph* $G'\langle\vec{\varphi}\preceq\vec{y}\rangle$ *consists of the two connected components arising from* v_{j_1} *and* v_{j_2}, *reducing to a unique one when* $\vec{y}\succeq(\max\{\varphi_1(v_{i_1}),\varphi_1(v_{i_2})\},\max\{\varphi_2(v_{i_1}),\varphi_2(v_{i_2})\})$.

Figure 3.3(b) represents two ridge pairs that can be considered "uncomparable", due to the fact that $\max\{\varphi_1(v_{i_1}),\varphi_1(v_{i_2})\}<\max\{\varphi_1(v_{i_3}),\varphi_1(v_{i_4})\}$, *while* $\max\{\varphi_2(v_{i_1}),\varphi_2(v_{i_2})\}>\max\{\varphi_2(v_{i_3}),\varphi_2(v_{i_4})\}$. *Thus, both* $\{v_{i_1},v_{i_2}\}$ *and* $\{v_{i_3},v_{i_4}\}$ *will be main saddles. In this case, when* $\vec{y}$ *varies under the assumption* $\vec{y}\succeq(\max\{\varphi_1(v_{j_1}),\varphi_1(v_{j_2})\},\max\{\varphi_2(v_{j_1}),\varphi_2(v_{j_2})\})$, *the number of the connected components in the subgraph* $G''\langle\vec{\varphi}\preceq\vec{y}\rangle$ *decreases (from* 2 *to* 1*) when the relation* $\vec{y}\succeq(\max\{\varphi_1(v_{i_1}),\varphi_1(v_{i_2})\},\max\{\varphi_2(v_{i_1}),\varphi_2(v_{i_2})\})$ *(or, alternatively, the relation* $\vec{y}\succeq(\max\{\varphi_1(v_{i_3}),\varphi_1(v_{i_4})\},\max\{\varphi_2(v_{i_3}),\varphi_2(v_{i_4})\})$*) becomes true.*

Finally, figure 3.3(c) shows two comparable ridge pairs, hence the "lower" one, that is $\{v_{i_1},v_{i_2}\}$, *will be a main saddle, while the other will be not. Consider* $G'''\langle\vec{\varphi}\preceq\vec{y}\rangle$, *assuming that* $\vec{y}$ *varies under the restriction* $\vec{y}\succeq(\max\{\varphi_1(v_{j_1}),\varphi_1(v_{j_2})\},\max\{\varphi_2(v_{j_1}),\varphi_2(v_{j_2})\})$: *It consists of two connected components arising from* v_{j_1} *and* v_{j_2}, *merging into a unique one when the relation* $\vec{y}\succeq(\max\{\varphi_1(v_{i_1}),\varphi_1(v_{i_2})\},\max\{\varphi_2(v_{i_1}),\varphi_2(v_{i_2})\})$ *becomes true. In particular, for* $\vec{y}=(\max\{\varphi_1(v_{i_3}),\varphi_1(v_{i_4})\},\max\{\varphi_2(v_{i_3}),\varphi_2(v_{i_4})\})$ *the number of the connected components in* $G'''\langle\vec{\varphi}\preceq\vec{y}\rangle$ *is equal to* 1, *since the merging* *of the existing ones has already occurred.*

As example 3.3 suggests, S is the set of all those couples of vertices representing the "death", i.e. the merging, of existing connected components in the given size graph $(G, \vec{\varphi})$.

We are now ready to introduce the concept of $\mathcal{L}$-reduced size graph:

Definition 3.9. *Let $G_{\mathcal{L}} = (V(G_{\mathcal{L}}), E(G_{\mathcal{L}}))$ be the graph with $V(G_{\mathcal{L}}) = M \cup S$ and $E(G_{\mathcal{L}})$ defined as follows: $u, v \in V(G_{\mathcal{L}})$ are adjacent if and only if one of them is a minimum vertex and the other is a main saddle adjacent to it (in the sense of Definition 3.8). Let also $\vec{\varphi}_{\mathcal{L}} : V(G_{\mathcal{L}}) \to \mathbb{R}^k$ be a function defined in this way: $\vec{\varphi}_{\mathcal{L}}(v) = \vec{\varphi}(v)$ if $v \in M$ and $\vec{\varphi}_{\mathcal{L}}(u) = (\max\{\varphi_1(v_{i_1}), \varphi_1(v_{i_2})\}, \ldots, \max\{\varphi_k(v_{i_1}), \varphi_k(v_{i_2})\})$ if $u = \{v_{i_1}, v_{i_2}\} \in S$. The size graph $(G_{\mathcal{L}}, \vec{\varphi}_{\mathcal{L}})$ will be called the* $\mathcal{L}$-reduction of $(G, \vec{\varphi})$.

Remark 16. We stress that each main saddle $\{v, w\}$ of a size graph $(G, \vec{\varphi})$ will be represented, in the $\mathcal{L}$-reduced size graph, by a *unique* vertex labeled by the k-tuple $(\max\{\varphi_1(v), \varphi_1(w)\}, \ldots, \max\{\varphi_k(v), \varphi_k(w)\})$.

Remark 17. The global reduction method we have just defined is strongly related to the concept of Pareto-Optimality, a well-known topic in Economy, especially in the field of Multi-Objective Optimization. Anyway, we think that this thesis is not the suitable context to deepen this subject. For a detailed treatment about Pareto-Optimality, the reader is referred to [35].

The importance of the $\mathcal{L}$-reduction is shown by the main result of this chapter, stating that discrete k-dimensional size functions are invariant with respect to this global reduction method.

Theorem 3.2.1. *For every $(\vec{x}, \vec{y}) \in \Delta^+$, the equality $\ell_{(G,\vec{\varphi})}(\vec{x}, \vec{y}) = \ell_{(G_{\mathcal{L}},\vec{\varphi}_{\mathcal{L}})}(\vec{x}, \vec{y})$ holds.*

In order to prove Theorem 3.2.1, we need the following lemma.

Lemma 3.2.2. *Let v_1, v_2 be two minimum vertices of $(G, \vec{\varphi})$. Then, for every $\vec{y} \in \mathbb{R}^k$, it holds that $v_1 \cong_{G\langle \vec{\varphi} \preceq \vec{y} \rangle} v_2$ if and only if $v_1 \cong_{G_{\mathcal{L}}\langle \vec{\varphi}_{\mathcal{L}} \preceq \vec{y} \rangle} v_2$.*

Proof. Suppose that $v_1 \cong_{G\langle\vec{\varphi}\preceq\vec{y}\rangle} v_2$. Then, by definition there exists a sequence $(v_1 = v_{j_1}, v_{j_2}, \ldots, v_{j_{m-1}}, v_{j_m} = v_2)$ such that $(v_{j_k}, v_{j_{k+1}}) \in E(G)$ for every $k = 1, \ldots, m-1$, and $v_{j_k} \in G\langle\vec{\varphi} \preceq \vec{y}\rangle$ for every $k = 1, \ldots, m$. Consider the sequence $(\mathcal{L}(v_1) = v_1, \mathcal{L}(v_{j_2}), \ldots, \mathcal{L}(v_{j_{m-1}}), \mathcal{L}(v_2) = v_2)$ of minimum vertices. By substituting each subsequence of equal consecutive vertices by a unique vertex representing such a subsequence, we obtain a new sequence $(v_1 = w_1, w_2, \ldots, w_{s-1}, w_s = v_2)$ (In other words, we substitute the sequence $(u_1, u_1, \ldots, u_1, u_2, u_2 \ldots, u_2, \ldots, u_k, u_k, \ldots, u_k)$ with $(u_1, u_2, \ldots, u_k)$). It is easy to prove that, for every index $j < s$, there exists at least one main saddle σ_j adjacent to w_j and w_{j+1}, such that $\sigma_j \in G\langle\vec{\varphi} \preceq \vec{y}\rangle$. Then, consider the sequence $(w_1, \sigma_1, v_2, \sigma_2, \ldots, w_{s-1}, \sigma_{s-1}, w_s)$: such a sequence proves that $v_1 \cong_{G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}}\preceq\vec{y}\rangle} v_2$.

On the other side, suppose that $v_1 \cong_{G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}}\preceq\vec{y}\rangle} v_2$. By definition there exists a sequence $(v_1 = w_1, \sigma_1, w_2, \sigma_2, \ldots, w_{s-1}, \sigma_{s-1}, w_s = v_2)$ of vertices of $G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}} \preceq \vec{y}\rangle$ such that every vertex w_j is a minimum vertex and every σ_j is a main saddle adjacent to w_j and w_{j+1}. Therefore, we can modify such a sequence in order to obtain the following one: for every index $j < s$, between w_j and $\sigma_j = \{v_{i_j}, v_{k_j}\}$ insert the sequence $(L^{m(v_{i_j})-1}(v_{i_j}), L^{m(v_{i_j})-2}(v_{i_j}), \ldots, L^2(v_{i_j}), L(v_{i_j}))$, while between σ_j e w_{j+1} insert the sequence $(L(v_{k_j}), L^2(v_{k_j}), \ldots, L^{m(v_{k_j})-2}(v_{k_j}), L^{m(v_{k_j})-1}(v_{k_j}))$ (we are assuming $w_j = \mathcal{L}(v_{i_j})$ and $w_{j+1} = \mathcal{L}(v_{k_j})$). Finally, by substituting the vertices v_{i_j} e v_{k_j} (taken in this order) for every main saddle σ_j, we obtain a new sequence proving that $v_1 \cong_{G\langle\vec{\varphi}\preceq\vec{y}\rangle} v_2$. $\square$

Now we are ready to prove Theorem 3.2.1.

Proof. Let $(\vec{x}, \vec{y}) \in \Delta^+$. We have to prove that there exists a bijection $F : G\langle\vec{\varphi} \preceq \vec{x}\rangle / \cong_{G\langle\vec{\varphi}\preceq\vec{y}\rangle} \rightarrow G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}} \preceq \vec{x}\rangle / \cong_{G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}}\preceq\vec{y}\rangle}$. For every equivalence class $C \in G\langle\vec{\varphi} \preceq \vec{x}\rangle / \cong_{G\langle\vec{\varphi}\preceq\vec{y}\rangle}$ we choose a minimum vertex $v_C \in C$. Obviously, v_C is also a vertex of $G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}} \preceq \vec{x}\rangle$. Therefore in $G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}} \preceq \vec{x}\rangle / \cong_{G_{\mathcal{L}}\langle\vec{\varphi}_{\mathcal{L}}\preceq\vec{y}\rangle}$ there exists an equivalence class D containing v_C. We shall set $F(C) \stackrel{def}{=} D$. From Lemma 3.2.2 it follows that F is injective. The surjectivity of F is trivial, since each equivalence class in $G_{\mathcal{L}}\langle \vec{\varphi}_{\mathcal{L}} \preceq \vec{x} \rangle / \cong_{G_{\mathcal{L}}\langle \vec{\varphi}_{\mathcal{L}} \preceq \vec{y} \rangle}$ contains at least one minimum vertex of $G\langle \vec{\varphi} \preceq \vec{x} \rangle$. $\square$

Remark 18. The $\mathcal{L}$-reduction of a size graph $(G, \vec{\varphi})$ is not unique. In particular, changing the ordering of the set $V(G)$ can produce different, non-isomorphic $\mathcal{L}$-reduced size graphs. On the other hand, Theorem 3.2.1 shows that we will always obtain $\mathcal{L}$-reductions of $(G, \vec{\varphi})$ with the same associated discrete k-dimensional size function.

Therefore, Theorem 3.2.1 allows us to evaluate the discrete k-dimensional size function of a size graph $(G, \vec{\varphi})$ directly dealing with one of its $\mathcal{L}$-reductions. This implies a faster and easier computation for $\ell_{(G,\vec{\varphi})}$, since a $\mathcal{L}$-reduction of $(G, \vec{\varphi})$ offers a decrease in the number of vertices, preserving the same information in terms of discrete k-dimensional size functions.

Example 3.4. *In figure 3.4 an example of $\mathcal{L}$-reduction is displayed. In $\mathbb{R}^3$*

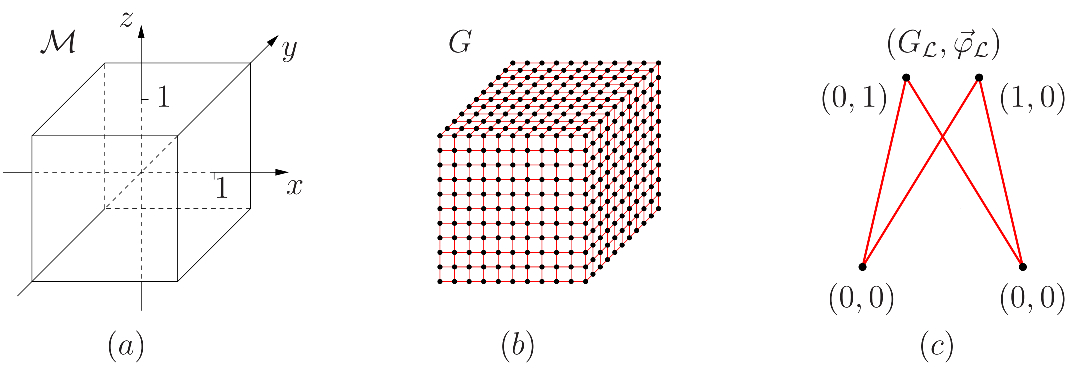

Figure 3.4: An example of $\mathcal{L}$-reduction.

consider the set $\mathcal{Q} = [-1,1] \times [-1,1] \times [-1,1]$. Let also $\vec{\Phi} = (\Phi_1, \Phi_2) : \mathbb{R}^3 \to \mathbb{R}^2$ be the continuous function defined as $\vec{\Phi}(x,y,z) = (|x|, |z|)$. In this setting, we define $\mathcal{M} = \partial\mathcal{Q}$ (figure 3.4(a)) and $\vec{\varphi} = \vec{\Phi}_{|\mathcal{M}}$. Figure 3.4(b) shows a possible discretization G of $\mathcal{M}$ (the chosen ordering of vertices is not displayed for obvious graphical reasons). We are interested in reducing the size graph $(G, \vec{\varphi})$ (we use the symbol "$\vec{\varphi}$" also to denote the restriction of the function $\vec{\varphi}$ to the set $V(G)$). In figure 3.4(c) the $\mathcal{L}$-reduction of $(G, \vec{\varphi})$

is displayed, showing a strong reduction in the number of vertices and a very simple structure.

As a final remark, we point out that the reduction of k-dimensional size functions to the 1-dimensional case, described in Section 2.1, allows us to exploit the existing computational methods for calculating discrete 1-dimensional size functions also in the multidimensional setting (cf. [3]). However, since the $\mathcal{L}$-reduction allows us to decrease the numbers of vertices of a given size graph, it could be used for a fruitful merge with the available techniques in the 1-dimensional case, in order to easily and fast compute discrete k-dimensional size functions for applications.

Conclusions and future work

In this thesis we have proved that the theory concerning k-dimensional size functions can be reduced to the 1-dimensional case by a suitable change of variables. This equivalence implies that multidimensional size functions are stable, with respect to the new distance D_{match}. Moreover, we have proposed a global method for reducing size graphs, i.e. particular graphs modeling size pairs in the discrete counterpart of the theory, and a theorem, stating that discrete k-dimensional size functions are invariant with respect to this reduction method, allowing us to easily and fast compute k-dimensional size functions in applications.

Many theoretical problems deserve further investigation, among them we list a few here.

- **Choice of the foliation.** Other foliations, different from the one we propose are possible. In general, we can choose a family Γ of continuous curves $\vec{\gamma}_{\vec{\alpha}} : \mathbb{R} \to \mathbb{R}^k$ such that (i) for $s < t$, $\vec{\gamma}_{\vec{\alpha}}(s) \prec \vec{\gamma}_{\vec{\alpha}}(t)$, (ii) for every $(\vec{x}, \vec{y}) \in \Delta^+$ there is one and only one $\vec{\gamma}_{\vec{\alpha}} \in \Gamma$ through $\vec{x}$, $\vec{y}$ and (iii) the curve $\gamma_{\vec{\alpha}}$ depends continuously on the parameter $\vec{\alpha}$ (this last hypothesis is important in computation for stability reasons). It would be interesting to study the dependence of our results on the choice of the foliation.

- **Choice of the planes inside the foliation.** The comparison technique expressed by Remark 11 requires the choice of a finite set of foliation leaves, on which we compute the reduction from multidimensional to 1-dimensional size functions. It would be interesting to determine a method to make this choice optimal.

- **Existence of size pairs having assigned $\boldsymbol{k}$-dimensional size functions.** At this time we do not know if any link exists between the 1-dimensional size functions associated with the planes $\pi_{(\vec{l},\vec{b})}$, apart from continuity. A question naturally arises about the conditions of existence of size pairs having an assigned continuous family of size functions on the planes of our foliation.

- **Existence of a local reduction method for size graphs in the $\boldsymbol{k}$-dimensional case.** In previous works ([16, 30]), it has been proved that, in the 1-dimensional case, a size graph can be reduced, without changing the associate discrete size function, by means of a global and a local method. Presently we do not have developed a meaningful local method for reducing size graphs in the multidimensional case, and we think it would be useful looking for such a reduction technique, in order to improve, from a computational point of view, the evaluation of discrete k-dimensional size functions.

- **Merging of available methods for reducing size graphs.** At the end of Chapter 2.1.2 it has been mentioned the chance of merging the reduction method developed for the multidimensional case with the available techniques for reducing size graphs in the 1-dimensional setting. However, an exhaustive analysis of such an approach has not yet occurred.

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