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ON THE RENORMALIZATION FLOW REPRESENTATION OF FIELD THEORY AND SOME APPLICATIONS

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1 Introduction

A field theory (FT) is any physical theory based on the notion of fields, i.e. of a continuous infinity of degrees of freedom. Thus, FT is naturally formulated in terms of functionals, depending on some field variables.

Classical field theories are dynamical systems whose configuration space $\mathcal{Q}$ is infinite dimensional, being a set of histories, i.e. of maps from some set $\mathcal{S}$ to some target $\mathcal{M}$. In many cases of interest one can assume these maps to be smooth and $\mathcal{S}$ and $\mathcal{M}$ to be differentiable. Then one can simply interpret configuration space as a set of sections q of the bundle $\mathcal{M} \times \mathcal{S} \to \mathcal{S}$, in other words for each space point $x \in \mathcal{S}$ there is a corresponding point $q(x) \in \mathcal{M}$. Also the associated phase space $\mathcal{P}$ will be infinite dimensional. Since in ordinary classical dynamics it is useful to consider phase space as a contangent bundle, also in field theory one can think about the conjugate momentum p as a map from $\mathcal{S}$ to the the cotangent space of $\mathcal{M}$, such that $p(x) \in \mathcal{T}^*_{q(x)}\mathcal{M}$. In this sense we will consider phase space as the set of sections (p, q) of the extended cotangent bundle $\mathcal{T}^*\mathcal{M} \times \mathcal{S} \to \mathcal{S}$, where $\mathcal{T}^*\mathcal{M}$ denotes the cotangent bundle of $\mathcal{M}$. In classical field theory the equations of motion are functional differential equations, that can often be obtained from a variational principle, and any sum over a subset of phase or configuration space is a functional integral. Such sums appear for example in statistical field theory (SFT), due to the need of averaging over microscopical ensambles.

Also in quantum mechanics (QM) and quantum field theory (QFT) the equations of mo-

tion are functional differential equations that can be obtained from a variational principle, but these equations are either for operators or for expectation values, as required by the uncertainty principle. QM and QFT also allow for a representation in which quantum expectation values take the form of statistical averages, again functional integrals over a set of histories, provided one replaces the probability density, encoded in the statistical distribution, with a probability amplitude, as required by the superposition principle.

Therefore FT is a unifying theoretical framework for the description of a great variety of phenomena, classical and quantum, often also of systems that are known to be far from continuous, or to show only a finite number of degrees of freedom, such as few or many body quantum systems or discrete statistical systems. For the latter this is not surprising since approximating the dynamics of many degrees of freedom by an infinite continuum of them is a very old procedure which is computationally advantageous and close to many experimental situations. However, such an approximation always goes along with a challenging problem, that is at the base of the so-called renormalization group (RG), and that can be described as follows. If one assumes a system to be continuous, there is practically no way to specify or to measure its properties other than by putting a bound to the resolution with which one performs such operations. This introduces a scale in the theoretical framework, and almost every building piece of the theory will acquire a dependence on such a scale. In other words, at every different scale one has a different theory for the description of the system at that resolution. Therefrom the rising of the question: how to relate all these descriptions in a single framework? The answer to this question is of high priority both for phenomenological and for theoretical reasons: on the one hand we would like to be able to predict the result of measurements without restricting ourselves to a fixed scale, on the other hand the consistency of the field-theoretic-framework requires us to be able to connect the properties that this continuum shows at different resolutions. The flow connecting two theories referring to two different scales is called RG flow, and its study is therefore of central importance in both SFT and QFT. As a side remark, of course this idea of limiting the resolution with which one looks at some system and then to link the corresponding different pictures can be applied also to discrete systems and not only to FT. In fact it was first proposed by Kadanoff in the particular form of a blocking and scaling of spin systems [1], and it is often referred to as a "coarse-graining" procedure.

In a functional representation, the field variables themselves are defined by the symmetry requirements on the functionals (which groups and which representations). The set of all the possible functionals of these fields enjoying the corresponding symmetries can be interpreted as an infinite dimensional manifold called "theory space". The parameters let free by the symmetry requirements (the couplings) further specify the dynamics and can be considered as

local coordinates on such a space. The RG flow describes how these parameters change when the scale at which they are "measured" changes, and it is generated by a vector field on the theory space. Nonperturbative methods allow one to get an approximate knowledge of this vector field even in regions where the theory is strongly interacting. One of these methods, developed by K. Wilson, F.J. Wegner and A. Houghton [2, 3, 4, 5], is based on a functional equation for the quantum effective action of the system (in this case theory space is just the space of all the possible effective actions). This representation of QFT, known as functional RG (fRG), has been intensively studied and improved in the last four decades, especially its reformulations by J. Polchinski [6], for the generator of amputated connected Green's functions, and by C. Wetterich [7, 8], for the generator of one particle irreducible (1PI) vertices. In this thesis we will adopt the latter, that is most easily derived from the functional integral representation. The starting point of this construction is the introduction of an external parameter in the theory. This allows to reduce the task of computing the functional integral to the simpler task of computing its infinitesimal variation under changes of such a parameter. In quantum mechanics the external parameter can be dimensionless, since the number of degrees of freedom is finite and no regularization is needed. Instead the generalization of the construction to FT's requires the introduction of a dimensionful parameter k, such that its variation corresponds to a coarse graining operation (otherwise we meet infinities in the computation of the infinitesimal variation). This can be done by implementing Wilson's idea of shell-by-shell integration, i.e. interpreting k as a cutoff-scale for the Fourier modes of the fields. An alternative way is to assume that the theory has already been regularized, as for example by the introduction of a UV cutoff Λ, in which case it is possible to deal with a dimensionless parameter also in field theories (related to the ratio between the dimensionful k and Λ). Since by varying k we will get a one parameter flow of theories, we will need initial conditions in order to integrate it. A convenient way to deal with this issue is to choose the dependence on k in such a way that the flow interpolates between full functional integration (conventionally at $k = 0$) and no integration at all (conventionally at $k = \Lambda$, even if Λ might in some cases be displaced at $+\infty$). The no integration limit can also be realized considering k as a mathematical parameter unrelated to a physical coarse-graining procedure, and, in the presence of the physical UV cutoff Λ, taking the limit $k \to \infty$. From the k-dependent functional integral one can get, by means of a k-dependent Legendre transform, a corresponding 1PI generator Γ_k, called average effective action (AEA). This is in general a highly non local object which encodes all the quantum properties of the system. In Euclidean space, the AEA satisfies the flow equation

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \mathrm{STr}\{[\Gamma_k^{(2)}[\Phi] + R_k]^{-1}(\partial_t R_k)\} \,. \tag{1.1}$$

that can now be completely disentangled from its derivation in terms of functional integrals, and taken as the defining property of Γ_k and consequently of the dynamics. Here $\Gamma_k^{(2)}$ is the second functional derivative with respect to the field Φ (the latter representing a collective field variable for all bosonic or fermionic degrees of freedom), and R_k denotes a momentum-dependent regulator function that suppresses IR modes below a momentum scale k. The solution to this equation provides for an RG trajectory in theory space interpolating between the bare action S_Λ (Wilson's effective action) $\Gamma_{k\to\Lambda} \to S_\Lambda$ and the full quantum effective action $\Gamma = \Gamma_{k\to 0}$. The integration of this equation starting from an initial condition S_Λ at $k = \Lambda$ is equivalent to the computation of the functional integral based on this bare action.

Choosing some coordinate system in theory space, one can parameterize the AEA by a possibly infinite set of generalized dimensionless couplings g_i (that could be functions of $t = \log(k/k_0)$ only, or even of other variables built out of fields or coordinates or momenta). Then, the flow equation provides us with the corresponding beta functions $\partial_t g_i = \beta_{g_i}(g_1, g_2, \dots)$. In order to compute and analyze the RG flow, truncations of the theory space are helpful, in order to get from this infinite set of partial or ordinary differential equations, a solvable subset. In this thesis we are going to perform such truncations within the general scheme of the derivative expansion. An alternative scheme would be the vertex expansion, which we are not going to discuss. For general reviews about the fRG see [9, 10, 11, 12, 13, 14, 15].

Regardless the need for the introduction of a scale k, the application of this representation to gauge theories has also been successfully studied. Several frameworks for this application have been developed, comprehending both manifestly gauge-covariant and noncovariant methods [13, 14]. In this thesis we are going to take advantage of a formulation based on the background field method (BFM), adapted to the 1PI fRG in [16]. As in the usual BFM one introduces the background field $\bar{A}_\mu^a$ and a background gauge-fixing, such as for instance

$$\bar{D}_\mu^{ij} a_j^\mu = (\partial_\mu \delta^{ij} + \bar{g} f^{ilj} \bar{A}_\mu^l) a_j^\mu = 0$$

which breaks the invariance under the full gauge transformation of the fluctuation field a_μ^i but preserves invariance under a split transformation in which the background transforms inhomogeneously while the fluctuation transforms homogeneously

$$\delta_\varepsilon \bar{A}_\mu^i = \frac{1}{\bar{g}} \bar{D}_\mu^{ij} \varepsilon_j \,, \qquad \delta_\varepsilon a_\mu^i = f^{ilj} a_{l\mu} \varepsilon_j \,.$$

As a result the corresponding effective action can be defined and it is also invariant under this split transformation, in such a way that by setting the average $A_\mu^i = \bar{A}_\mu^i + \bar{a}_\mu^i$ equal to $\bar{A}_\mu^i$ one recovers invariance under the full inhomogeneous transformation of A. Therefore at any fixed

k the AEA can be split in two parts

$$\Gamma_k[\Phi, A, \bar{A}] = \bar{\Gamma}_k[\Phi, \bar{A}] + \Gamma_k^{\text{gauge}}[\Phi, A, \bar{A}]$$

the first one being gauge independent and the second one vanishing if one sets $A = \bar{A}$. If one keeps $A \neq \bar{A}$ the gauge invariance of the bare action must be imposed on the flow by means of modified Ward-Takahashi or Slavnov-Taylor identities. In this framework the r.h.s. of the flow equation is the same of (1.1) but with few caveats: first, the set of fields comprehends, apart for non-gauge fields, both the bosonic vector A_μ^i and the fermionic ghosts $\bar{c}^i, c^i$; second, the fluctuation matrix $\Gamma_k^{(2)}$ is to be interpreted, in the gauge sector, as resulting from differentiations w.r.t. A at fixed $\bar{A}$.

Generalizations of the flow equation (1.1) allowing for a manifest invariance under reparameterizations of the fields have also been succesfully developed [17], still relying on the introduction of a background field, within the general framework of the geometric effective action [18, 19].

By means of eq. (1.1) one can compute the RG flow of a theory and get a picture of how the dynamics of a system changes when the scale of observation changes. Since the symmetries exhibited by the system can change along the flow, because of spontaneous symmetry breaking or approximate emergence of new symmetries, even the parameterization of the effective action in terms of the same set of fields in the UV and IR might be inappropriate. Nevertheless, in this work we use the same bosonic and fermionic degrees of freedom on all scales.

In the derivation of eq. (1.1) from a functional integral, no assumption is made about the range of modes one is integrating on. Therefore this flow equation representation can be applied both to effective FT's and to fundamental (often called renormalizable) ones. The former are by construction limited to a bounded range of resolutions (i.e. of Fourier modes over which we integrate), while the latter can be used at every observational scale. Indeed, understanding if a FT can be fundamental is one of the hardest questions to be answered, because the global knowledge of the RG trajectories is required. The flow equation representation is particularly useful for addressing this task, because its truncations are not limited to weakly coupled regimes. In this representation the issue of renormalizability can be restated as follows: a theory is fundamental if no bound is imposed on the Fourier modes of the fields, and hence it is possible to safely move the cutoff scale k within the whole range $[0, +\infty)$. By "safely" one means that the ability to get meaningful predictions for the observables is unaltered, and this is usually associated to the fact that rescaled dimensionless couplings, propagators and vertices, i.e. Γ_k itself, stay finite.

A sufficient condition for this is that a suitable fixed point (FP) of the RG flow S^* exists in

theory space such that $S_{\Lambda\to\infty} \to S^*$. A FP is a point g_i^* in theory space such that

$$\beta_i(g_1^*, g_2^*, \ldots) = 0\,,\ \forall\, i \tag{1.2}$$

where g_i are couplings rescaled with respect to (w.r.t.) the floating scale in order to be pure numbers. This is non-Gaußian (NGFP) if at least one coupling is nonvanishing $g_j^* \neq 0$, and Gaußian (GFP) otherwise.

A well known scenario for the UV completion of a QFT, asymptotic safety (AS), requires the existence of a NGFP of the RG flow, with a finite number of UV-attractive directions. This possibility was proposed as a generalization of asymptotic freedom [20]. An asymptotically free theory lies on a particular subset of theory space (the so-called UV critical dimension of the GFP) defined by the requirement that in the extreme UV the RG flow drives it into the GFP. This simple request guarantees that the theory enjoy many good qualities: since at every finite scale the theory does not sit on the GFP, it is interacting and nontrivial; since the UV limit is free there is no upper bound on the energy range of applicability; since the points lying off (but close to) the UV critical surface are brought towards it when flowing to the IR, it is legitimate to give an approximate description of the system at any finite energy by means of a theory assumed to lie exactly on this surface and to enjoy asymptotic freedom. It is clear that all these properties are not specific of the GFP but descend from the only requirement that there is a FP of the RG with a nontrivial UV critical surface. The further requirement specified in the definition of AS, namely that this surface be finite dimensional, is intended to automatically provide asymptotically safe theories with another necessary good property: predictivity. In fact, in this case the only condition that the theory lie on the UV critical surface guarantees that there is only a finite number of free parameters in it.

How does one compute the dimensionality of the UV critical surface? Assuming the regularity of the flow, it is sufficient to study the linearized flow around the FP

$$\partial_t g_i = B_i{}^j (g_j - g_j^*) + \ldots\,, \qquad B_i{}^j = \left.\frac{\partial \beta_{g_i}}{\partial g_j}\right|_{g=g^*}. \tag{1.3}$$

The critical exponents Θ^I correspond to the negative of the eigenvalues of the stability matrix $B_i{}^j$. They allow for a classification of physical parameters. All eigendirections with $\Theta^I < 0$ are IR attractive and UV repulsive, the eigendirections with $\Theta^I > 0$ are IR repulsive and UV attractive. The former are called irrelevant, because in the case of the GFP these couplings die out towards the IR, while the latter are called relevant because in the case of the GFP they grow towards the IR and thus determine the macroscopic physics. The behavior of the marginal di-

rections, with $\Theta^I = 0$, depends on the higher-order terms in the expansion about the fixed point. Hence the number of relevant and marginally-relevant directions is the number of couplings parameterizing the UV critical surface. The theory is predictive if this number is finite. In the case of the GFP, the present construction corresponds to the standard perturbative power-counting analysis and the critical exponents are equal to the canonical dimensions of the couplings.

In conclusion, after one has found a "fundamental", or "microscopic", action S^* and has computed the RG flow around it, one can decide which one-parameter-family of bare actions S_Λ can be quantized thus leading to a renormalizable theory. Examples of asymptotically safe theories are provided by four-fermion models such as the Gross-Neveu model in $2 < d < 4$ dimensions [21]. Even though these models are perturbatively nonrenormalizable and thus seemingly trivial, they are nonperturbatively renormalizable at a NGFP and can be extended to arbitrarily high scales. A similar result was found for nonlinear sigma models in $d > 2$ [17, 22]. Even the possibility that quantum Einstein gravity could be renormalizable within the AS scenario has been supported by many studies [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]; these also include certain models with a nontrivial scalar and/or fermion sector [34, 35, 36]. AS scenarios have also been successfully developed for extra-dimensional gauge theories [37].

The whole construction just discussed in QFT has a very important analog in SFT. In the latter case S_Λ can be interpreted as the action for a statistical theory defined on a lattice of spacing $a \sim (1/\Lambda)$. Since it is known that phase transitions can occur only in the continuum limit $a \to 0$, and that at the transition the theory becomes conformal and hence RG-invariant, looking for FP of the RG that are attractive in the $k \to \Lambda \to \infty$ limit corresponds to looking for critical theories describing phase transitions. The critical exponents Θ^I of each FP are then the usual critical exponents describing the approach to the corresponding phase transition. The dimensionality of the critical surface thus corresponds to the universality class of the critical theory. Hence, in the unifying framework of FT, the search for FP's of the RG has the double interest of a research of possibly renormalizable QFT's and of critical points and universality classes in SFT.

In this thesis we are going to reconsider and extend the RG flow equation representation of FT in Chapters 2 and 3, and we are going to discuss some specific examples of its application in Chapters 3, 4, and 5.

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2

1PI functional renormalization flow equation

Starting from the basic path integral in phase space, we reconsider the functional approach to the RG flow of the one particle irreducible average effective action. On employing a balanced coarse-graining procedure for the canonical variables we obtain a functional integral with a non trivial measure which leads to a flow equation for the Lagrangian average effective action. We first address quantum mechanics for boson and fermion degrees of freedom and we then extend the construction to quantum field theories. For this flow equation we discuss the reconstruction of the bare action and the implications on the computation of the vacuum energy density.

2.1 Coarse-graining in phase space

In the introduction of this thesis we have reviewed how the Wilsonian idea of renormalization [2, 3, 4], which started from the analysis of Kadanoff's blocking and scaling of spin systems [1] (more generally coarse-graining), can be conveniently formulated analytically in a functional formulation. The idea of a step-by-step integration of the quantum fluctuations typically belonging to a momentum shell, followed by rescaling, can be implemented in a smooth way [5, 6, 7, 8] and leads to a differential equation for the effective action as a function of the

scale parameter.

The starting point of all these formalisms is the functional quantization procedure based on the path integral in phase space with a Liouville measure corresponding to the configuration variables "Q" and their conjugate canonical momenta "P". The integrand depends on an action which is built from an Hamiltonian and thus at this level the approach is not manifestly Lorentz covariant. For Hamiltonians quadratically dependent on the conjugate momenta, the integration in the "P" variables is trivially performed getting infinite factors contributing to the functional measure, which usually in flat spacetime is constant and thus can be neglected in the computation of correlation functions. Thus one is left with a covariant formulation with functional integrals in configuration space. At this point one usually implements a coarse-graining on the configuration variables "Q" only.

The insertion of a smooth cutoff must be performed in such a way to modify the Lagrangian action and the configuration space functional measure [7]. In order to understand that this is necessary to get a balanced regularization of the divergences of the path integral it is sufficient to recall that the path integral of quantum mechanics is finite, as can be seen performing a suitable skeletonization, thanks to the fact that singular contributions coming from the configuration space functional measure, i.e. from the integration in the canonical momenta, are canceled by others appearing in the integration of the Lagrangian action in configuration space.

The need for a consistent regularization of the configuration space functional measure is better understood from the more fundamental point of view of the phase space path integral, as we will try to explain in the following sections. As already stated, the idea of coarse-graining as "integrating out" quantum fluctuations in the UV region is usually implemented in a functional integral after the integration of the conjugate momenta is performed. In this way even if any degree of freedom is associated to a pair of "P" and "Q" variables, the integration of the "P" modes inside one shell is performed well before the integration of the "Q" modes of the same shell. Thus the whole procedure seems unbalanced.

In this chapter we therefore propose to implement a *balanced* coarse-graining procedure in phase space, by introducing a cutoff operator which affects both configuration variables and conjugate momenta, and we will show how this point of view singles out the proper way to regularize the configuration space path integral. In fact, after a regularization of the functional integral in phase space, one can choose to integrate out the conjugate momenta (for a quadratic dependence) and obtain a path integral in configuration space with a non trivial dependence of the measure on the cutoff operator. According to this regularization, we investigate the RG flow equation for the AEA. This flow equation, in contrast with the one neglecting the regularization of the functional measure, leads to the same results obtained by other quantization methods

under the Weyl ordering prescription.

We first analyze quantum mechanics of a system of bosonic degrees of freedom and then consider also the fermionic case. Later we move to QFT, where the choice of the operator implementing the coarse-graining procedure is guided by the requirement of Lorentz invariance. Nevertheless we briefly discuss also some consequences of Lorentz breaking choices.

Finally we illustrate in some cases the relation between the AEA and the bare action, defined as the one appearing inside the path integral, with or without a UV cutoff in the theory. In the UV region the approach of the AEA to the bare action is found to depend on the choice of the coarse-graining operator, and in particular on its singularity properties.

During our discussion we also analyze the flow of the constant term in the potential of the AEA, which is related to the vacuum energy. We anticipate that the Lorentz invariant coarse-graining leads naturally to an interesting fact: in presence of an UV cutoff Λ the vacuum energy, which is computed integrating the flow from such a UV scale down to the IR, is quadratically (and not quartically) divergent in Λ for a free massive theory, while it vanishes for a free massless theory. This property of the vacuum energy density was recently discussed [38, 39] on different grounds in a standard perturbative QFT framework, by performing subtractions justified by symmetry and reality conditions.

2.2 Flow equation in quantum mechanics

2.2.1 Bosonic degrees of freedom

Let's consider a classical system with one bosonic degree of freedom governed by the following Hamiltonian:

$$H(p,q) = \frac{1}{2}p^2 + V(q) \tag{2.1}$$

where q and p are canonically conjugate variables with Poisson bracket: $[q,p] = 1$. The quantization of such a system is performed via the following Euclidean phase-space path integral ($\hbar = 1$):

$$Z[J] = \int [dpdq]\, \mu[p,q] e^{\int dt\, [ip(t)\partial_t q(t) - H(p(t),q(t)) + J(t)q(t)]}\,. \tag{2.2}$$

Here we explicited the presence of an unspecified functional measure $\mu[p,q]$. In a skeletonized version of the path integral one usually considers N time-slices and at each instant of time integrates over the corresponding phase space. Since the correct measure for each of these phase-space integrations is the Liouville measure, that is the square root of the determinant of the symplectic form, in a time-slicing definition of the path integral the functional measure is the product of N Liouville measures, which is clearly ill-defined in the continuum limit $N \to \infty$.

Of course this is not the only possible source of infinities for Z: both the p and the subsequent q integration bring ill-defined factors. Since Z could be UV-finite, as e.g. in the free particle case, we must conclude that all these divergences can mutually cancel. This leads to think that if one regularizes one of the contributions, also the others should be regularized in a consistent way. This is not what is done in the manipulations of the functional integral leading to the many exact RG flow equations present in the literature. In the following we are going to explain why this is so, restricting ourselves to the one particle irreducible framework. The translation of our reasoning to the flow of Wilson's effective action should be straightforward.

The usual modified configuration-space path integral lying behind Wetterich equation [7] reads:

$$Z_k[J] = e^{W_k[J]} = \int [dq]\mu[q] e^{-(S[q]+\Delta S_k[q])+\int dt\, Jq}\,. \tag{2.3}$$

where $\mu[q]$ is a k-independent Lagrangian measure, for our bosonic system $S[q]$ is the time integral of:

$$L(q(t), \partial_t q(t)) = \frac{1}{2}\left(\partial_t q(t)\right)^2 + V(q(t)) \tag{2.4}$$

and in (2.3) one adds to it a cutoff term $\Delta S_k(q) = \frac{1}{2}\int q(t)\, R_k(-\partial_t^2) q(t)$ to allow for the integration of $(-\partial_t^2)$-modes only above some infrared cutoff k^2. In other words, one introduces a regulator in the kinetic term of the action, by means of the substitution:

$$\frac{1}{2}\int dt\ \left(\partial_t q(t)\right)^2 \to \frac{1}{2}\int dt\, \partial_t q(t)\left(1 + r_k(-\partial_t^2)\right)^2 \partial_t q(t) \tag{2.5}$$

where

$$R_k(-\partial_t^2) =: \left(\left(1 + r_k(-\partial_t^2)\right)^2 - 1\right)(-\partial_t^2)\,. \tag{2.6}$$

In this way one affects the divergences arising from the integration of the exponential factor $e^{-S[q]}$, but does not modify the infinite determinant implicit in the Lagrangian functional measure. Equivalently, the modified generating functional (2.3) can be obtained by the following phase-space path integral:

$$Z_k[J] = \int [dpdq]\, \mu[p,q] e^{\int dt \left[p(t)(1+r_k(-\partial_t^2))i\partial_t q(t) - H(p(t),q(t)) + J(t)q(t)\right]} \tag{2.7}$$

by completing the square in the exponential and then integrating in the momenta, thus getting an infinite factor changing the Liouville measure $\mu[p,q]$ into the Lagrangian measure $\mu[q]$. Later in this section we will write such a factor as:

$$\frac{\mu[q]}{\mu[p,q]} = \left(\mathrm{Det}(-\partial_t^2)\right)^{\frac{1}{2}} \tag{2.8}$$

because for the system under consideration we can express this ratio in terms of the functional integral for a free particle[1]. Therefore we see that the modification (2.5) affects the definition of the integral over $q(t)$ but it does nothing for the integral over $p(t)$ nor for the Hamiltonian functional measure. The last form (2.7) of the modified generating functional is suggestive because it allows the interpretation of the coarse graining procedure as a modified Legendre transform, i.e. a k-dependent definition of the bare Lagrangian corresponding to a fixed bare Hamiltonian.

This also suggests us a way to implement the previously described principle of regularizing on the same footing both the exponential factor and the measure: if we modify the Legendre transform, we should also consistently modify the symplectic structure because the two are strictly tied together. In fact, recall that the Legendre transform term $p(t)\partial_t q(t)dt$ is just the pull back of the Liouville 1-form $\lambda = pdq$ by means of the trajectory-parameterizing map $(p(t), q(t))$ and that the symplectic form is $\sigma = d\lambda$. Thus if we substitute $\lambda \to \lambda_k = (1 + r_k)\,\lambda$ we should also substitute $\sigma \to \sigma_k = (1 + r_k)\,\sigma$ and correspondingly $\mu = (\mathrm{Det}\,\sigma)^{\frac{1}{2}} \to \mu_k = (\mathrm{Det}\,\sigma_k)^{\frac{1}{2}}$. In the following, to simplify the notation, we will take advantage of the fact that for our system the Liouville measure is a constant, equal to one for canonical coordinates, and we will write the previous functional measure as $\mu_k = (\mathrm{Det}(1 + r_k))$.

To sum up, as a regularization we introduce a frequency-dependence in the symplectic structure, leaving unaltered the Hamiltonian and the phase space manifold. In the present case this leads to the following generating functional:

$$Z_k[J] = \int [dpdq]\,(\mathrm{Det}(1 + r_k))\; e^{\int dt\,[p(1+r_k)i\partial_t q - H(p,q) + Jq]}\,. \tag{2.9}$$

To see that this modification of the path integral affects all possible sources of divergences, i.e. the p-integration, the q-integration and the measure, it is sufficient to change the p-integration variable:

$$P(t) = (1 + r_k(-\partial_t^2))^{\frac{1}{2}} p(t) - i(1 + r_k(-\partial_t^2))^{\frac{3}{2}} \partial_t q(t) \tag{2.10}$$

and get:

$$\begin{aligned} Z_k[J] &= \int [dPdq]\,(\mathrm{Det}(1 + r_k))^{\frac{1}{2}}\; e^{-(S[q]+\Delta S_k[q]) + \int dt\left[-\frac{1}{2}(1+r_k)^{-1}P^2 + Jq\right]} \\ &= \int [dq]\,\left(\mathrm{Det}(-\partial_t^2)\mathrm{Det}(1 + r_k)^2\right)^{\frac{1}{2}}\; e^{-(S[q]+\Delta S_k[q]) + \int dt\, Jq}\,. \end{aligned} \tag{2.11}$$

[1]Recall [40] that in a time-slicing definition of the path integral for a Green function such that $T = \int dt = N\varepsilon$ we have $\mu[p, q] = 1^N$, $\mu[q] = \varepsilon^{-N/2}$. For a free particle $\langle q, T|0, 0\rangle = \mu[q]\,(\mathrm{Det}(-\partial_t^2))^{-\frac{1}{2}}\; e^{-q^2/2T} = (2\pi T)^{-\frac{1}{2}} e^{-q^2/2T}$ (in units $m = \hbar = 1$). On the rhs of eqn. (2.8) there is no factor $(2\pi T)^{-\frac{1}{2}}$ because we consider a measure $[dp, dq]$ for a partition function.

where we evaluated the integral over $P(t)$ as in eqn.(2.8) but this time with a k correction. In terms of the usual effective average action (AEA), defined by:

$$\Gamma_k[\bar{q}] = \min_J \left(\int dt J\bar{q} - W_k[J] \right) - \Delta S_k[\bar{q}] \tag{2.12}$$

eqn.(3.20) entails the following exact RG equation (ERGE)

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \dot{R}_k \right] - \frac{1}{2}\mathrm{Tr}\left[\left(-\partial_t^2 + R_k \right)^{-1} \dot{R}_k \right] \tag{2.13}$$

where the dot stands for differentiation with respect to $\log k$ and must not be confused with a time derivative, that in this chapter will always be denoted as ∂_t. The derivation of this equation in brief goes as follows: using eqs.(2.12,3.20,2.6) and recalling that the connected two point function is the inverse of the Hessian matrix for $(\Gamma_k + \Delta S_k)$, we can write:

$$\begin{aligned}
\dot{\Gamma}_k[\bar{q}] &= -\dot{W}_k \left[\frac{\delta \Gamma_k}{\delta \bar{q}} \right] - k\partial_k \Delta S_k \left[\bar{q}\right] \\
&= \frac{1}{2} \int dt \left\langle (q-\bar{q})\,(t) \dot{R}_k (-\partial_t^2)\,(q-\bar{q})\,(t) \right\rangle_{\frac{\delta \Gamma_k}{\delta \bar{q}}} - k\partial_k \log\left(\mathrm{Det}(-\partial_t^2 + R_k) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \iint dt dt' \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} - \left(-\partial_t^2 + R_k \right)^{-1} \right](t,t')\, \dot{R}_k(-\partial_t^2)\delta(t-t')
\end{aligned}$$

Notice that the naked differential operator in the additional subtraction term in eq.(2.13) seemingly breaks the invariance under constant field rescalings. This is not the case because the general form of this new term is a $(\log k)$-derivative of the logarithm of the regularized functional measure. Under rescalings of the fields inside the path integral, the functional measure correctly transforms and so does the subtraction term. For example, since in the present case $\mu_k[q] = (\mathrm{Det}(-\partial_t^2 + R_k))^{\frac{1}{2}}$, if $q = \sqrt{Z_k}\, q'$ then $\mu_k[q'] = (\mathrm{Det}\, Z_k(-\partial_t^2 + R_k))^{\frac{1}{2}}$ such that the subtraction term becomes $-\frac{1}{2}\mathrm{Tr}\left[(Z_k(-\partial_t^2 + R_k))^{-1} k\partial_k (Z_k R_k) \right]$.

Using eqn.(2.13) in the case of a single harmonic oscillator and integrating the flow from $k = \infty$ down to $k = 0$, one obtains $\omega/2$ for the energy of the vacuum. This result was already derived in [14] starting from the usual Wetterich equation and adding to it a subtraction term interpreted as corresponding to an UV counter-term in the bare action, required to guarantee that for zero potential and frequency the ground state energy of the oscillator be zero. Our point of view is different in that we would like to have a flow equation representation of quantum mechanics where the quantities are finite at all scales and counter-terms are unnecessary. We find that this can be achieved with our construction by requiring that the $k \to \Lambda$ limit of Γ_k be the classical action we are quantizing, free of any R_k dependence. For a general discussion on

this point we refer the reader to section 4. In the following we will briefly sketch how to derive this result in the present context.

Let us consider a truncation for the effective action with $V_k(q) = E_k + \frac{1}{2}\omega^2 q^2$, so that $\Gamma_k^{(2)} = (-\partial_t^2 + \omega^2)\delta(t-t')$. We shall consider the UV "initial conditions" such that $E_{k=\infty} = 0$ and look for $E_{k=0}$. We shall later comment on this choice when discussing the relation between the AEA and the bare action. On employing the so called optimized [41, 42] cutoff function $R_k(z) = (k^2 - z)\theta(k^2 - z)$ and switching to a Fourier representation of the operators, one obtains

$$\int dt\, \dot{V}_k = \frac{1}{2}\int dt \int \frac{dE}{2\pi}\theta(k^2 - E^2)2k^2 \left[\frac{1}{k^2 + V_k''} - \frac{1}{k^2}\right] \tag{2.14}$$

which, after removing the "volume" factor ($\int dt$) on both sides of the equation, leads to

$$\partial_k E_k = \frac{1}{\pi}\frac{-\omega^2}{k^2 + \omega^2} \Longrightarrow E_{k=0} = \frac{\omega}{2}\,. \tag{2.15}$$

Let us finally interpret this result analyzing directly the integro-differential equation which is satisfied by the AEA:

$$e^{-\Gamma_k[\bar{q}]} = \int [dq]\, \mu_k \exp\left(-S[q] + \int (q-\bar{q})\frac{\delta\Gamma_k[\bar{q}]}{\delta\bar{q}} - \frac{1}{2}\int (q-\bar{q})R_k(q-\bar{q})\right)\,. \tag{2.16}$$

Since for a free theory one has an AEA $\Gamma_k[\bar{q}] = S[\bar{q}] + E_k$, one finds, using a compact notation where "$\cdot$" stands for an integration,

$$\begin{aligned} e^{-\Gamma_k[\bar{q}]} &= \int [dq]\, \mu_k \exp\left(-\frac{1}{2}(q-\bar{q})\cdot(-\partial_t^2 + R_k + \omega^2)\cdot(q-\bar{q}) - \frac{1}{2}\bar{q}\cdot(-\partial_t^2 + \omega^2)\cdot\bar{q}\right) \\ &= \left(\frac{\mathrm{Det}\,(-\partial_t^2 + R_k)}{\mathrm{Det}\,(-\partial_t^2 + R_k + \omega^2)}\right)^{1/2} \exp\left(-\frac{1}{2}\bar{q}\cdot(-\partial_t^2 + \omega^2)\cdot\bar{q}\right) \end{aligned} \tag{2.17}$$

One then notes that the first factor in the last line of eqn. (2.17) becomes 1 in the $k \to \infty$ limit while for $k \to 0$ gives the expected zero energy contribution $e^{-\int dt \frac{\omega}{2}}$. We remark that in order to obtain these values in the UV and IR limit of k the cutoff operator R_k should probably satisfy some regularity conditions. These are fulfilled for the previously mentioned optimized cutoff and for the so called Callan-Symanzik cutoff ($R_k = k^2$), but not for discontinuous cutoffs such as $R_k(z) = k^2\theta(k^2 - z)$.

2.2.2 Fermionic degrees of freedom

In this section we will study a free system whose Lagrangian variables are n real Grassmann-valued functions of time: $\left\{\theta^i(t)\right\}_{i=1,\dots,n}$, evolving according to the following Lagrangian:

$$L(\theta(t), \partial_t\theta(t)) = \frac{1}{2}\theta^i(t) i\partial_t\theta^j(t)\delta_{ij}. \tag{2.18}$$

Just like in the previous section we consider as a starting point the quantization of this theory by means of a Hamiltonian path integral. In building a phase space out of (3.44) we find n second class primary constraints:

$$\chi_a(t) := \pi_a(t) + \frac{i}{2}\delta_{aj}\theta^j(t) = 0 \tag{2.19}$$

which cause the canonical Hamiltonian to vanish. The relevant phase space is the surface $\mathcal{S}$ defined by (3.45), a complete set of independent coordinates on it is given by θ^i and the functional integral is to be taken over all paths $\theta^i(t)$ lying on this surface. The appropriate measure for functional integration over $\mathcal{S}$ is again the square root of the superdeterminant of the symplectic form on $\mathcal{S}$. In presence of second class constraints and assuming that the whole phase space is endowed with a symplectic structure σ, we can define a nondegenerate symplectic form $\tilde{\sigma}$ on the reduced phase space, simply by restricting σ to $\mathcal{S}$. As the inverse of σ is the Poisson bracket $[\ ,\ ]$, the inverse of $\tilde{\sigma}$ is the Dirac bracket $[\ ,\ ]_\sim$, which in the reduced phase space coordinates θ^i has components: $[\theta^i, \theta^j]_\sim = -i\delta^{ij} = [\chi_i, \chi_j]$. (Everything we write about constrained systems is explained for example in [43].) Thus the functional integral over the reduced phase space reads:

$$Z = \int [d\theta]\mu[\theta] e^{-\frac{1}{2}\int dt\, \theta^i(t)\partial_t\theta^j(t) i\delta_{ij}}. \tag{2.20}$$

Here the Lagrangian (3.44) emerges from the $\partial_t\theta^j\pi_j$ term in phase space after having solved the second class constraints, or equivalently after having performed the following integration over momenta:

$$Z = \int [d\theta d\pi]\mu[\theta,\pi] \left(\prod_a \delta[\chi_a]\right) e^{-\int dt\,[\partial_t\theta^j\pi_j - H]}. \tag{2.21}$$

Following the same coarse graining scheme explained in the previous section we modify the symplectic structure of the reduced phase space replacing $\tilde{\sigma}$ with $\tilde{\sigma}_k = (1 + r_k)\tilde{\sigma}$, where the definition of r_k is still such that $(1 + r_k(-\partial_t^2))^2(-\partial_t^2)$ is an IR safe second order differential

operator. However eqn. (2.6) does not apply to the fermionic case, because in this case one usually chooses R_k in such a way that $|i\partial_t + R_k(i\partial_t)|^2$ is a regularized kinetic operator for a bosonic degree of freedom. Therefore in this case we can write $R_k(i\partial_t) = r_k(-\partial_t^2)i\partial_t$. Correspondingly the functional measure becomes: $\mu_k = (\mathrm{SDet}\,\tilde{\sigma}_k)^{\frac{1}{2}} = \mu\ (\mathrm{SDet}(1+r_k))^{\frac{n}{2}}$. Then the modified path integral reads:

$$\begin{aligned} Z_k &= \int [d\theta d\pi]\mu[\theta,\pi]\left(\prod_a \delta[\chi_a]\right)(\mathrm{SDet}(1+r_k))^{\frac{n}{2}}\, e^{-\int dt\,(1+r_k)\partial_t\theta^j\pi_j}\,. \\ &= \int [d\theta]\mu[\theta]\,(\mathrm{SDet}(1+r_k))^{\frac{n}{2}}\, e^{-\frac{1}{2}\int dt\,\theta^i(t)(1+r_k)\partial_t\theta^j(t)i\delta_{ij}}. \end{aligned} \tag{2.22}$$

Such a k-dependence can be translated in the following equation for the usual effective average action:

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\dot{R}_k\right] - \frac{1}{2}\mathrm{STr}\left[(i\partial_t + R_k)^{-1}\dot{R}_k\right] \tag{2.23}$$

where the traces as usual count also the number of Lagrangian variables. Note that the $\frac{1}{2}$ factors on the rhs are consistent with the traditional Wetterich equation for Fermi fields, since in our case we are dealing with *real* Grassmann variables.

The generalization of the previous discussion to the case of n complex Grassmann variables $\{\eta^i\}_{i=1,\dots,n}$ and to interacting systems is straightforward. As long as the kinetic term of the Lagrangian is of first order in time-derivatives and real, such as for example in: $L = \bar{\eta}^i i\partial_t\eta^j\delta_{ij} + V(\eta^i,\bar{\eta}^j)$, we find 2n second class primary constraints: $\{\chi_a,\bar{\chi}_a\}_{a=1,\dots,n}$. χ relates the conjugate momentum of η to $\bar{\eta}$, whilst $\bar{\chi}$ relates the conjugate momentum of $\bar{\eta}$ to η. In order to find the correct functional measure we can just compute the matrix of the Poisson brackets of these constraints. Since:

$$[\chi_a,\chi_\beta] = [\bar{\chi}_a,\bar{\chi}_\beta] = 0\,, \qquad [\chi_a,\bar{\chi}_\beta] = [\bar{\chi}_a,\chi_\beta] \tag{2.24}$$

then $\left|\mathrm{SDet}\,\tilde{\sigma}_{ij}\right|^{\frac{1}{2}} = |\mathrm{SDet}\,([\eta^i,\bar{\eta}^j]_\sim^{-1})|$ therefore, if we do not count the complex conjugate of a bracket as an independent bracket, the $\frac{1}{2}$ exponent of the superdeterminants gets simplified in all previous formulas. As a consequence, applying the same regularization scheme of eqn. (3.47) we are led to an equation for the effective average action which is identical to eqn. (2.23) but without the $\frac{1}{2}$ factors on the rhs.

Such an equation can be used to compute the vacuum energy of a fermionic oscillator in quantum mechanics. Pick a complex Grassmann variable η, and investigate the following trun-

cation ($\omega > 0$):

$$\Gamma_k[\eta] = \int dt \ (\eta^* i\partial_t \eta + \omega \eta^* \eta + E_k) \ . \tag{2.25}$$

Proceeding along the same lines as for the bosonic oscillator, one finds that the quantum energy of the vacuum is $E_0 = -\omega/2$, i.e. again what one would have computed by canonical quantization based on Weyl ordering prescription.

2.3 Flow equation in field theory

In this section we want to generalize the previous discussion to the case of field theory. Let's start with the example of a scalar field theory with (Euclidean) Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_0 \varphi)^2 + \frac{1}{2}|\nabla \varphi|^2 + V(\varphi) \tag{2.26}$$

or equivalently, defining the momentum conjugate variable π w.r.t. φ, with Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla \varphi|^2 + V(\varphi) \, . \tag{2.27}$$

As in section 2.1, our starting point is the quantization of such a system by the usual (Euclidean) Hamiltonian path integral

$$Z = \int [d\pi d\varphi] \, \mu[\pi, \varphi] e^{-\int d^d x \, (-i\pi \partial_0 \varphi + \mathcal{H})} \ . \tag{2.28}$$

Again, in a skeletonized version of the path integral one considers N time-slices and at each instant of time integrates over a corresponding phase space; therefore also in this case the functional measure is related to the Liouville measure. Hence let's just perform the same modification of the Liouville form we introduced before:

$$\begin{aligned} \int \pi \partial_0 \varphi \quad &\to \quad \int \pi \, (1 + r_k) \, \partial_0 \varphi \\ \mu[\pi, \varphi] \quad &\to \quad \mu_k[\pi, \varphi] = (\mathrm{Det} \, (1 + r_k)) \, \mu[\pi, \varphi] \end{aligned} \tag{2.29}$$

where we still do not specify which differential operator r_k depends on. Exactly the same manipulations we performed in equations (2.10,3.20) show that also in this case such a prescription is sufficient to affect simultaneously the measure and the following two quadratic forms in

the action:

$$\begin{aligned}\frac{1}{2}\int \partial_0\varphi\partial_0\varphi &\rightarrow \frac{1}{2}\int \partial_0\varphi\,(1+r_k)^2\,\partial_0\varphi \\ \frac{1}{2}\int \Pi^2 &\rightarrow \frac{1}{2}\int \Pi\,(1+r_k)^{-1}\,\Pi\end{aligned}$$

where Π is defined in analogy with eqn.(2.10). But the remaining quadratic form $\partial_i\varphi\partial^i\varphi$ is left unaffected by (2.29). Therefore we must supplement (2.29) with a second regularization:

$$\frac{1}{2}\int \partial_i\varphi\partial^i\varphi \rightarrow \frac{1}{2}\int \partial_i\varphi\,(1+\tilde{r}_k)^2\,\partial^i\varphi \tag{2.30}$$

for some $\tilde{r}_k$ a priori independent of r_k. In conclusion the final modified path integral for a generic theory of one scalar field reads:

$$\begin{aligned}Z_k &= \int [d\pi d\varphi]\,\mu[\pi,\varphi]\,(\mathrm{Det}(1+r_k))\,e^{\int d^dx\,(i\pi(1+r_k)\partial_0\varphi-\mathcal{H})-\Delta H_k} \\ &= \int [d\varphi]\,\mu[\varphi]\,(\mathrm{Det}(1+r_k))\,e^{-(S[\varphi]+\Delta S_k[\varphi])}\end{aligned} \tag{2.31}$$

where we denoted:

$$\begin{aligned}\Delta H_k &= \frac{1}{2}\int \partial_i\varphi\left[(1+\tilde{r}_k)^2-1\right]\partial^i\varphi \\ \Delta S_k &= \frac{1}{2}\int \partial_0\varphi\left[(1+r_k)^2-1\right]\partial^0\varphi+\Delta H_k\,.\end{aligned} \tag{2.32}$$

Regarding the freedom to independently choose r_k and $\tilde{r}_k$ we shall discuss in the next subsections several choices one can make.

There are also possible alternative approaches which are leading directly to covariant results. One could be to adopt the so called covariant Hamiltonian formalism for classical field theory, in which one introduces d conjugate momenta to φ, one for each partial derivative. In this formulation a regularization of the consequent polysymplectic structure would automatically provide a $\tilde{r}_k = r_k$. Such an approach deserves further analysis to define the appropriate functional measure which should be adopted in the corresponding Hamiltonian path integral.

Another could be to slightly modify our regularization prescription, starting not from the phase space but from the configuration space path integral. In this framework the introduction of a covariant k-dependent operator in the Lagrangian must be accompanied by a similar modification of the corresponding Lagrangian measure, which is the reciprocal of the square root of the determinant of the advanced Green function (see for example [18]). We shall not

discuss more on this here.

On the base of common sense we expect the case of fermionic fields to differ only by two aspects: the fields will be Grassmann-valued and there will be constraints. The first point only changes determinants in superdeterminants; the second one must be dealt with as in section 2.2, integrating over the reduced phase space and, since a Dirac fermion comprehends two complex Grassmann variables, using as a measure $(\mathrm{SDet}([\psi,\bar{\psi}]_{\sim}^{-1}))^2$. Following these lines one gets the following modified path integral:

$$Z_k = \int [d\bar{\psi}d\psi]\mu[\bar{\psi},\psi]\mathrm{SDet}(1+r_k)e^{-(S[\psi]+\Delta S_k[\psi])} \tag{2.33}$$

where:

$$\Delta S_k = \int \bar{\psi} r_k i\gamma^0\partial_0\psi + \int \bar{\psi}\tilde{r}_k i\gamma^i\partial_i\psi\,. \tag{2.34}$$

Now it is time to comment on the definition of the operators r_k and $\tilde{r}_k$. Different choices of them can be made, specifying their forms as functions of the modes and also the differential operator whose modes they depend on. Let us analyze few examples.

2.3.1 Lorentz invariant cutoff

The most natural choice is to take a Lorentz invariant cutoff. For a pure scalar theory it is sufficient to investigate the modes of the Lorentz scalar operator $-\Box \equiv -\partial_\mu\partial^\mu$, and in order to preserve Lorentz symmetry we are compelled to choose $\tilde{r}_k(-\Box) = r_k(-\Box)$. As already recalled in section 2.1 the usual notation is: $(1+r_k(-\Box))^2(-\Box) = -\Box + R_k(-\Box)$ with the cutoff function $R_k(z)$ enjoying all the properties required to suppress the functional integration for $z << k^2$. Thus in this case eqn. (2.32) reduces to the more traditional form:

$$\Delta S_k[\varphi] = \frac{1}{2}\int d^dx\,\varphi R_k(-\Box)\varphi\,. \tag{2.35}$$

Plugging this expression into (2.31), defining the effective average action as usual, and taking the k-derivative one gets:

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}+R_k\right)^{-1}\dot{R}_k\right] - \frac{1}{2}\mathrm{Tr}\left[(-\Box+R_k)^{-1}\dot{R}_k\right]\,. \tag{2.36}$$

The requirement of Lorentz invariance also leads to a similar equation for Dirac Fermions:

$$\dot{\Gamma}_k = \mathrm{STr}\left[\left(\Gamma_k^{(2)}+R_k\right)^{-1}\dot{R}_k\right] - \mathrm{STr}\left[(i\partial\!\!\!/+R_k)^{-1}\dot{R}_k\right]\,. \tag{2.37}$$

Let us analyze the flow equation for the scalar field case, using an optimized cutoff function $R_k(z) = (k^2 - z)\theta(k^2 - z)$. In the local potential approximation and neglecting anomalous dimensions for $d = 4$ one finds

$$\dot{V}_k(\varphi) = \frac{k^4}{2(4\pi)^2}\left[\frac{1}{1 + V_k''(\varphi)/k^2} - 1\right] \tag{2.38}$$

This is valid without approximation for a free massive field with $V_k''(\varphi) = m^2$. In this case the only non trivially running parameter is the field independent term v of the potential which we expect to contribute to the "vacuum energy", while the dimensionful mass m is constant along the flow.

One may consider the case in which Λ is the scale at which the bare action is defined. Then integrating the flow from $k = \Lambda$ down to $k = 0$ one obtains

$$v_{k=0} - v_{k=\Lambda} = \frac{1}{4(4\pi)^2}\left[m^2\Lambda^2 - m^4 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right] . \tag{2.39}$$

However this is not a quantitatively trustable estimate of the difference between the quantum vacuum energy and the bare one at the cutoff Λ, for reasons that we shall discuss in detail in section 4. Nevertheless it qualitatively agrees with the result we will compute in that section, because it correctly shows that the contribution to v from the quantum fluctuations, with the above Lorentz invariant prescription, is positive and diverges only quadratically as $\Lambda \to \infty$. Moreover it vanishes in the case of a massless field. This possibility was indeed discussed [38, 39] on different grounds in a standard QFT framework. We note that in the present approach this fact is indeed related to Lorentz invariance, but it is also the consequence of having treated carefully the measure in the path integral which is the starting point of the quantization procedure. Indeed if one employs a flow equation in which the subtraction term due to the regularization of the measure is dropped, then $v_{k=0} - v_{k=\Lambda}$ has an extra contribution $-\Lambda^4/\left(8\,(4\pi)^2\right)$ that leads to a negative value for $\Lambda \gg m$, is quartic divergent with the Lorentz invariant cutoff, and does not vanish in the massless case. As already discussed for the harmonic oscillator, this extra contribution can be interpreted as due to a deformation of the bare action at the scale $k = \Lambda$ by the regulator ΔS_Λ, and washed away by means of a controlled subtraction in the flow equation, which is tantamount to adding counter-terms to $\Gamma_{k=\Lambda}$. The fact that in the present approach these counter-terms are not needed leads to the expectation that something has changed in the relationship between the EAA and the bare action, as we will discuss in section 4.

2.3.2 Lorentz breaking pseudo cutoffs

We consider here a special case where the cutoff is such that it organizes the integration in the path integral according to the modes associated to the operator $-\partial_t^2$. More precisely we employ the same $r_k(-\partial_t^2)$ of eqn. (2.6) and we take a vanishing $\tilde{r}_k$. Therefore the quadratic form in the spatial derivatives of the field is not affected at all and any integration in the $(d-1)$-dimensional space is divergent and should be regularized by other means. The flow equation now reads

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\dot{R}_k\right] - \frac{1}{2}\mathrm{Tr}\left[\left(-\partial_t^2 + R_k\right)^{-1}\dot{R}_k\right] . \tag{2.40}$$

In the Fourier representation and employing the same optimized cutoff function $R_k(z)$ as before, one can write

$$\dot{V}_k(\varphi) = \frac{k^3}{\pi}\int \frac{d^{d-1}\bar{p}}{(2\pi)^{d-1}}\left[\frac{1}{k^2 + |\bar{p}|^2 + V_k''(\varphi)} - \frac{1}{k^2}\right] \tag{2.41}$$

Again for a free massive field, assuming an implicit regularization for the $\bar{p}$ integration one can perform the integration of the flow and, defining $\omega_{\bar{p}} = \sqrt{|\bar{p}|^2 + m^2}$, obtains

$$v_{k=0} - v_{k=\Lambda} = \int_{reg} \frac{d^{d-1}\bar{p}}{(2\pi)^{d-1}}\frac{\omega_{\bar{p}}}{\pi}\arctan\frac{\Lambda}{\omega_{\bar{p}}} \underset{\Lambda\to\infty}{\longrightarrow} \int_{reg}\frac{d^{d-1}\bar{p}}{(2\pi)^{d-1}}\frac{\omega_{\bar{p}}}{2} \tag{2.42}$$

which is the usual integrated vacuum energy of all the vacuum fluctuations, which is not a Lorentz scalar. Note that the flow equation without the subtraction term due to the measure, for such a cutoff, leads to an extra divergent negative contribution: $-\frac{\Lambda}{\pi}\int\frac{d^{d-1}\bar{p}}{(2\pi)^{d-1}}$.

Other kinds of Lorentz-breaking coarse-graining procedure can be implemented using more complicated cutoff operators. As a simple non trivial example consider $R_k(-\partial_t^2, -\Box) = (k^2 - (-\partial_t^2))\theta[k^2 - (-\Box)]$ for $(1 + r_k)^2 = 1 + R_k(-\partial_t^2, -\Box)/(-\partial_t^2)$, while $\tilde{r}_k = 0$. This cutoff organizes the integration of the modes according to the eigenvalues of the Laplacian, which is a Lorentz invariant operator, but modifies only the time derivative part in the action (and the conjugate momenta) leaving the spatial derivative term untouched.

Let us note that sometimes it may be useful from a phenomenological point of view to allow for a Lorentz-breaking coarse-graining procedure, depending on which observable one may be interested in and on the experimental setup.

2.4 Relation between the average effective action and the bare action

An interesting point to be discussed, already addressed in [44], is the relation between the AEA satisfying the flow equation and the bare action of the theory, appearing inside the functional integral. Traditionally physicists have been interested in investigating this relationship only in one direction, i.e. moving from the choice of a "classical" bare action to the computation of the corresponding "quantum" effective action, i.e. in flowing the RG towards the IR. Why should the other direction being interesting?

One answer could be that if we turn the previous point of view upside down, looking for the action to be plugged inside a path integral in order to get a previously chosen quantum effective action, we are just looking for a Wilson effective action, whose scale of reference depends on the scale we use to regularize the path integral. Thus, in the limit in which this regularization is removed, the bare action we are looking for becomes the UV limit of Wilson's effective action. This is why the RG flow of the AEA towards the UV has been used in many recent studies devoted to investigate the possible UV completion of several QFTs, in the sense of Weinberg's asymptotic safety [20], as already discussed in the introducing chapter of this thesis.

Another possible answer, as already stressed in [44], is that finding which classical system, once quantized, leads to the theory under investigation, is important to establish relationships with other theories that should describe the same system but that follow from different quantization schemes. For example, in asymptotic safety scenarios for gravity, knowledge of the bare action might lead to a better understanding of possible relations between the QFT defined by renormalizable trajectories in the theory space of Einstein gravity and other approaches to quantum gravity.

Thus we face the problem of computing the bare action from the AEA just by means of the flow equation and its solutions, without resorting to the path integral formulation of QFT. In this section for sake of simplicity we will address this problem only for a scalar theory, restricting ourselves to a Lorentz invariant cutoff $R_k(-\Box)$ (or $R_k(-\partial_t^2)$ in the QM case). We shall see that in the present case of a modified Wetterich equation we can push the analysis of the relation between the AEA and the bare action in a slightly different direction from the standard approach. We organize our discussion distinguishing between two qualitatively different cases, the one in which a sharp UV cutoff is introduced, and the one without it. No need to recall that the latter is allowed because of the ERGE being free of UV divergences.

2.4.1 In presence of a UV cutoff

So, let's assume that our theory has a Lagrangian bare action S^Λ defined in the presence of a configuration space measure μ^Λ, both dependent on a UV cutoff Λ. Of course one is free to redefine the bare action adding the $-\log\mu^\Lambda$ term to S^Λ and removing μ^Λ from the measure. The dependence on the UV cutoff Λ can also be seen as reflecting the fact that our bare action might follow from a coarse-graining procedure started with some other bare action defined on a larger space with a larger cutoff, in the Wilsonian sense. Therefore for different values of Λ one has a set of different Wilsonian actions S^Λ all referring to the same physical system. The removal of the UV cutoff is associated to the limiting procedure $\Lambda \to \infty$, and is possible only for fundamental, in contrast to effective, theories. Starting from a path integral of the kind

$$\int [d\chi]^\Lambda \, \mu^\Lambda e^{-S^\Lambda[\chi]} \tag{2.43}$$

for fundamental theories one should obtain finite meaningful matrix elements in the limit $\Lambda \to \infty$.

In presence of both the IR cutoff k and the UV cutoff Λ, the definition of the AEA Γ_k^Λ is formally the same of the case without any Λ. As before, under the requirement that R_k vanishes in the limit $k \to 0$, Γ_k^Λ approaches the standard effective action Γ^Λ in the same limit. A less simple problem is what happens to Γ_k^Λ as k grows. Regardless of the presence of any UV cutoff, since as k becomes bigger and bigger less and less modes of the fields are being integrated, the most reasonable requirement seems to be that when the functional integration is completely suppressed, the AEA approaches the bare action. The best way to understand how this can happen is to look at the integro-differential equation satisfied by the AEA, which in our case depends on a regulated configuration space measure μ_k^Λ:

$$e^{-\Gamma_k^\Lambda[\varphi]} = \int [d\chi]^\Lambda \, \mu_k^\Lambda \exp\left(-S^\Lambda[\chi] + \int^\Lambda (\chi-\varphi)\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi} - \frac{1}{2}\int^\Lambda (\chi-\varphi)\hat{R}_k(\chi-\varphi)\right). \tag{2.44}$$

If, as k grows and approaches some limiting value, the k-dependent part of the integrand on the rhs of eq.(2.44) converges to a representation of a functional delta

$$\mu_k^\Lambda \exp\left(\int^\Lambda (\chi-\varphi)\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi} - \frac{1}{2}\int^\Lambda (\chi-\varphi)\hat{R}_k(\chi-\varphi)\right) \longrightarrow \delta^\Lambda[\chi-\varphi]\,. \tag{2.45}$$

then at this limiting value of k the functional integral is completely suppressed and the AEA equals the bare action. To ensure that the k-dependent terms define a rising delta functional we need to make assumptions both on the UV asymptotics of the AEA and on the properties

of the regulator R_k. In fact the lhs of eqn. (2.45) can be rewritten, in condensed notation and by completing the quadratic form, as

$$\mu_k^\Lambda \exp\left\{\frac{1}{2}\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi}\cdot\hat{R}_k^{-1}\cdot\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi}-\frac{1}{2}\left(\chi-\varphi-\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi}\cdot\hat{R}_k^{-1}\right)\cdot\hat{R}_k\cdot\left(\chi-\varphi-\hat{R}_k^{-1}\cdot\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi}\right)\right\}. \tag{2.46}$$

From this last equation we see that if the first term in the exponent of eqn. (2.46) and the shift term $\hat{R}_k^{-1}\cdot\frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi}$ both vanish when k reaches its limiting UV value, and if the remaining functional

$$\mu_k^\Lambda \exp\left\{-\frac{1}{2}(\chi-\varphi)\cdot\hat{R}_k\cdot(\chi-\varphi)\right\} \tag{2.47}$$

behaves as a normalized Gaussian functional with vanishing variance in the same limit, this is enough to recover a delta functional. These conditions can on their turn be satisfied by assuming that physically allowed AEA are bounded in k and that the dimensionless version of R_k diverges as k reaches its limiting UV value.

Of course the details about which limiting UV value of k one could approach and of which properties the regulator R_k must enjoy in order to completely suppress the integration and furnish the required rising delta strongly depend on the presence or absence of a UV cutoff. In the following we shall comment on all these details and also on the assumption that the AEA be bounded in k, but before starting analyzing all possible scenarios let us stress that in order to have a chance to build a rising delta functional a crucial role is played by the regularized functional measure (corresponding to a regularized Liouville measure in phase space), without which we would lack the Gaussian normalization factor in (2.47). Later on we will explicitly work out in a simple specific case the proof that our regularization of the functional measure is exactly what is needed to normalize the Gaussian rising delta. Finally it could be useful to recall that in studying the UV asymptotics, and in other computations too, one should choose a unit of mass and work with dimensionless quantities, that is, one should perform a general rescaling with respect to some M. In presence of a UV cutoff Λ, $M = \Lambda$ is a possible choice related to the domain of definition of our theory. If the UV cutoff Λ is absent $M = k$ is also a natural choice.

So far only general arguments, so let us start getting more specific. Let us first deal with the case in which the presence of a UV cutoff Λ is explicitly assumed. In this case there seem to exist two main choices for the limiting value that k must approach to suppress the integration:

a) $k \to \Lambda$

b) $k \to \infty$

The former corresponds to the interpretation of k as the scale of an IR cutoff, that must therefore be smaller or equal to the UV cutoff. The latter is allowed since the k-dependent

operator is a smooth IR regulator and not a sharp cutoff, thus one might prefer to think about k just as an external parameter that could take any value, in which case (b) corresponds to having removed to infinity the arbitrary value of k at which the regulator is expected to kill the integration. The next question is which properties must R_k enjoy in cases (a) and (b) in order to realize the two following scenarios:

a) $\lim_{k\to\Lambda} \Gamma_k^\Lambda[\varphi] = S^\Lambda[\varphi]$

b) $\lim_{k\to\infty} \Gamma_k^\Lambda[\varphi] = S^\Lambda[\varphi]$

Since case (b) is more similar to the case most frequently addressed in the literature, i.e. the one in which no UV cutoff is assumed, we prefer to start with this case. Here we want the dimensionless regulator R_k to diverge as $k \to \infty$ and, because in this case the most natural choice of the unit is $M = k$ for $k \leq \Lambda$ and $M = \Lambda$ for $k > \Lambda$, this singular behavior is enjoyed by all regulators usually present in the literature, of the form

$$\langle x|\hat{R}_k|y\rangle = k^2 f(-\Box_x/k^2)\delta(x-y) \quad , \quad f(0) > 0 \tag{2.48}$$

(the optimized cutoff corresponds to $f(z) = (1-z)\theta(1-z)$). For such regulators and under the assumption of UV boundedness of Γ_k^Λ one can easily check that for $k > \Lambda$, rescaling all dimensionful quantities w.r.t. $M = \Lambda$, the first term in the exponent of eqn. (2.46) and the shift term $\hat{R}_k^{-1} \cdot \frac{\delta\Gamma_k^\Lambda[\varphi]}{\delta\varphi}$ both vanish in the limit $k \to \infty$. Also, by the same token, the remaining exponential has a vanishing variance. The last ingredient missing for a rising delta is the correct normalization, therefore it is time to give explicit arguments showing that this is provided by the regularized measure. To this end let us analyze briefly the QM case in the skeletonized version of the path integral (see for example [40]) with a time slicing such that $T = N\varepsilon$. We will not work out the exact discretized version of any cutoff differential operator, but we will just consider its asymptotic IR and UV behaviors. In the $k \to 0$ limit, corresponding to the standard un-coarse-grained path integral, R_k and Δ_k disappear and for a standard unit Liouville measure the momenta integration leads to the usual configuration space measure $\mu[q] = \mathcal{N} = (2\pi\varepsilon)^{-N/2}$, as already discussed in footnote 1. In the case of $k \to \infty$, $-\partial_t^2/M^2$ is negligible with respect to R_k/M^2 for all modes, because of the cutoff Λ. Also, having in mind a path integral toy-modeling a vacuum persistence amplitude, we consider the case in which we have to perform N integrals over phase space (p, q), so that the appropriate power of the regularization of the Liouville measure is also N. Therefore:

$$\mu_k[q] \underset{k\to\infty}{\sim} \mathcal{N}\left(\frac{\mathrm{Det}[R_k/M^2]}{\mathrm{Det}[-\partial_t^2/M^2]}\right)^{1/2} \sim (2\pi\varepsilon M)^{-N/2} \frac{\left(\frac{R_k}{M^2}\right)^{N/2}}{(\varepsilon M)^{-N}} = \left(\frac{\varepsilon R_k}{2\pi M}\right)^{N/2} . \tag{2.49}$$

Here the $\mathrm{Det}[-\partial_t^2]$ has been evaluated in terms of the finite difference operators ∇ and $\bar{\nabla}$ as in [40]. This measure and the satisfied requirement that the k-dependent exponent on the rhs of eqn. (2.44) approaches a Gaussian in the $k \to \infty$ limit, leads to the following discretized version of the l.h.s. of eqn. (2.45), or equivalently of (2.47)

$$\left(\frac{\varepsilon R_k}{2\pi M}\right)^{N/2} e^{-\frac{1}{2M}\varepsilon \sum_n R_k(q_n-\bar{q}_n)^2} \underset{k\to\infty}{\longrightarrow} \prod_n \delta(q_n - \bar{q}_n) \tag{2.50}$$

showing the correct normalization. Recall that in the last expression the rescaling has converted q_n and $\bar{q}_n$ into dimensionless quantities and that in this discretized approach ε is related to the inverse of the UV cutoff.

In conclusion, for the class of operators (2.48) and for any fixed bare action $S^\Lambda[\varphi]$ at a given scale Λ, one has $\lim_{k\to\infty} \Gamma_k^\Lambda[\varphi] = S^\Lambda[\varphi]$. Thus, in order for the bare action in the limit $\Lambda \to \infty$ to be the initial condition at $k \to \infty$ for the RG flow of the EEA, one must deal with the following order in the limits: $\lim_{\Lambda\to\infty} \lim_{k\to\infty} \Gamma_k^\Lambda[\varphi]$. The only hypothesis we still have to comment on, is the one regarding the boundedness in k of Γ_k^Λ. With such an aim in mind let us analyze the flow of the (modified) exact RG equation (ERGE) for truncations like the local potential approximation. The computation of the trace in Fourier space requires to integrate in p, over the domain $|p| < \Lambda$, a function $g(p, \Lambda, k, \varphi)$ depending on all the scales. Rescaling everything w.r.t. Λ ($z = p^2/\Lambda^2$ and $\tilde{\varphi} = \varphi/\Lambda$), and adopting again an optimized cutoff as an example, on has for the ERGE:

$$k\partial_k v_{k/\Lambda}(\tilde{\varphi}) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^{min\{\frac{k^2}{\Lambda^2},1\}} dz\, z^{\frac{d}{2}-1} \frac{-v''_{k/\Lambda}(\tilde{\varphi})}{\frac{k^2}{\Lambda^2} + v''_{k/\Lambda}(\tilde{\varphi})}\,, \tag{2.51}$$

whose r.h.s. for a generic potential is expected to vanish when Λ is fixed and $k \to \infty$. For example for a free massive theory $v''_{k/\Lambda}(\tilde{\varphi}) = m^2/\Lambda^2$, which is actually k-independent. Therefore in such a case the AEA really approaches the Λ dependent bare action. Integrating the flow for the massive free theory from $k = \infty$ (instead of from $k = \Lambda$) to $k = 0$ one finds for the dimensionful energy density:

$$\begin{aligned}
d=1: &\quad \frac{m \arctan\left(\frac{\Lambda}{m}\right)}{\pi} + \frac{\Lambda \log\left(1+\frac{m^2}{\Lambda^2}\right)}{2\pi} \underset{\Lambda\to\infty}{\longrightarrow} \frac{m}{2} \\
d=2: &\quad \frac{m^2 \log\left(\frac{\Lambda^2}{m^2}+1\right)}{8\pi} + \frac{\Lambda^2 \log\left(1+\frac{m^2}{\Lambda^2}\right)}{8\pi} \underset{\Lambda\to\infty}{\longrightarrow} \frac{m^2 \log\left(\frac{\Lambda}{m}\right)}{4\pi} + \frac{m^2}{8\pi} \\
d=4: &\quad \frac{m^2\Lambda^2 - m^4 \log\left(\frac{\Lambda^2}{m^2}+1\right)}{64\pi^2} + \frac{\Lambda^4 \log\left(1+\frac{m^2}{\Lambda^2}\right)}{64\pi^2} \underset{\Lambda\to\infty}{\longrightarrow} \frac{m^2\Lambda^2}{32\pi^2} - \frac{m^4 \log\left(\frac{\Lambda}{m}\right)}{32\pi^2} - \frac{m^4}{128\pi^2}
\end{aligned} \tag{2.52}$$

From the last expression on can see that in a four dimensional spacetime the term in the Wilsonian action associated to the vacuum energy density induced by free massive bosonic fields grows quadratically with the UV cutoff Λ and is zero for the massless case. A similar behavior (but opposite sign) is shown by fermion fields. We note that for a generic dimension d one can write an analytic form as the sum of two contributions: one comes from the flow in the region $\Lambda < k < \infty$ and is given by $\frac{\Lambda^d}{d\,\Gamma(d/2)(4\pi)^{d/2}} \log\left(1+\frac{m^2}{\Lambda^2}\right)$ while the other is obtained from the region $0 < k < \Lambda$ and can in general be written in terms of a regularized hypergeometric function. More general truncations with higher derivative terms or more complicated operators have to be studied to understand if such a behavior can be spoiled.

Let us now turn to case (a), again in presence of a finite UV cutoff. Here we want the dimensionless regulator R_k to diverge as $k \to \Lambda$ from below. Since the flow always stays in the region $k < \Lambda$ the choice of unit $M = k$ is allowed and is in fact to be preferred to $M = \Lambda$ because rescalings with respect to the running cutoff correspond to the Wilsonian procedure of iterated shell integration and subsequent rescaling. The cutoff function in general may be written as $R_k = k^2 f(z = \frac{p^2}{k^2}, x = \frac{k^2}{\Lambda^2})$. If $f(z,x)$ is *finite* at $x = 1$ one does not obtain a Gaussian representation of the delta and also both the first term and the shifts to $\chi - \varphi$ in the exponent of eqn. (2.46) do not vanish. In this case there is no simple relation between $\Gamma_{k\to\Lambda}$ and S. On the other hand if we ask that $f(z,x) \to \infty$ as $x \to 1$, and still we assume the boundedness of the AEA, then for any mode we recover a functional delta representation as in eqn. (2.45). In such a case one may write $\lim_{k\to\Lambda} \Gamma_k^\Lambda[\varphi] = S^\Lambda[\varphi]$.

One can check that for a free theory, independently on the choice of the functional form providing the singularity in the cutoff function one obtains the same results already illustrated in eq. (2.52). This is a confirmation that one has realized a representation of the functional delta inside the path integral. In particular we have considered the explicit cutoff $R_k(p^2) = f(\frac{k^2}{\Lambda^2})(k^2-p^2)\theta(k^2-p^2)$. In order to perform the analytic computation we used $f(x) = (1-x)^{-a}$ for positive a and numerically checked that other suitable choices of f lead to the same result. In this case therefore it should be possible to find a change of variable in order to make this property manifest for any f with the right singular behavior.

We stress again that the difference between the scenarios realized in cases (a) and (b) just depends on our freedom to choose cutoff operators with a different singular behavior. The AEA depends on them at all scales but in the UV and IR limit, provided such cutoffs enjoy regularity properties allowing to bring the $k \to 0$ limit inside the path integral, and that their singular behavior in the UV is what is needed to recover a functional delta.

2.4.2 In absence of any UV cutoff

Finally we want to study the case in which no UV cutoff is present. As already said this last case is the one which is usually considered in the AEA approach, with Γ_k assumed to be bounded in k because of renormalizability. In this framework the most reasonable request is that

$$\lim_{k\to\infty} \Gamma_k[\varphi] = S[\varphi]$$

and one should look for cutoff operators leading to a representation of the delta for $k \to \infty$. Since the only natural choice of unit in this case is $M = k$, we see that the traditional regulators like (2.48), after rescalings with respect to k, do not diverge as $k \to \infty$ unless $f(0)$ is infinite. Thus, an example of a suitable regulator could be $R_k(p^2) = g(\frac{p^2}{k^2})(k^2 - p^2)\theta(k^2 - p^2)$ with $g(z) \to \infty$ for $z \to 0$. Let us remark that with this kind of choice the singularity $k \to \infty$ is the same as the one for $p \to 0$ since k is the only available scale in the cutoff. This means that the zero mode is treated differently inside the path integral.

The simplest computation one can imagine is to check that this prescription leads to the expected vacuum energy in $d = 1$ for a free theory with frequency m (which we know to be finite starting from a bare action with no vacuum energy at $k = \infty$). Choosing $g(z) = z^{-1}$ one can compute analytically the beta function from the trace (its expression is more involved that for the simpler optimized cutoff) and numerically check that the flow leads to $E_{k=0} = \frac{m}{2}$. A full numerical analysis on a family of cutoff defined, for example, by $g(z) = z^{-a}$ for $a > 0$ gives correctly the same result.

Another suitable family of cutoff functions, far easier to deal with, is $R_k(zk^2) = k^2(z^{-a} - z)\theta(1 - z)$ for $a > 0$. These regulators simply replace (p^2/k^2) with $\left(p^2/k^2\right)^{-a}$, whenever $p^2 < k^2$. The choice $a = d/2$ allows a straightforward computation of the traces thanks to the advantageous change of variable $z' = z^{d/2}$. As an example, for a scalar field in the local potential approximation one finds:

$$\dot{V}_k(\varphi) = \frac{k^d}{(4\pi)^{\frac{d}{2}}} \frac{\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d}{2}+1\right)} \left[\frac{k^2}{V_k''(\varphi)} \log\left(1 + \frac{V_k''(\varphi)}{k^2}\right) - 1\right] . \tag{2.53}$$

Again we see the independence of the result from the choice of the cutoff function with the required singularity structure. Clearly without a regularizing UV cutoff the integrated vacuum energy for $d > 1$ is a divergent quantity so one should only deal with the expressions for the beta functions. Again further investigations are needed regarding more general truncations of the AEA. Finally let us remind that for the special case of a free theory we have shown in the end of section 2.1 (for the QM oscillator) that it is sufficient to employ even a non singular cutoff

operator to have $\lim_{k\to\infty} \Gamma_k[\varphi] = S[\varphi]$, because of the dimensionful mass being independent of k.

2.5 Remarks and summary

We have presented a RG flow equation for the effective average action based on the regularization of the functional integral in space space, wherein we perform a balanced coarse-graining procedure by means of the introduction of a scale dependence in the symplectic form. This procedure corresponds to the standard Weyl ordering prescription for quantization. Under this regularization both the action and the functional measure become dependent on a smooth cutoff. Such a non trivial measure implies the presence of a subtraction term in the flow equation, as given in eqns. (2.13) and (2.23) for boson and fermion d.o.f. in quantum mechanics. The corresponding RG flow equations in QFT, for a Lorentz invariant regularization, are given in eqns. (2.35) and (2.37). The subtraction between the two traces on the r.h.s. of these equations, induced by the non trivial measure, gives rise to a better convergence; that is, the r.h.s. could be finite even if the convergence of the single integrals is not provided by the cutoff operator.

In flat space, as long as one is not interested in the vacuum energy which is not observable, and for cutoffs that do not involve any coupling but the field strength, nor background fields, the subtraction term due to the measure can be dropped. If however one adopts cutoff schemes more complicated than the ones we discussed in this chapter, we cannot exclude some differences between the flow generated by the present equation, following from phase space coarse-graining, and the one based on the standard Wetterich equation. This is indeed what we expect in the case of the so called non-pure cutoff schemes that usually include more couplings than just the field strength. For example, for a free massive scalar field one could choose a cutoff scheme involving also the mass term, such as $R_k(-\Box + m^2)$. Then the flow equation would have a vanishing r.h.s. because of a complete cancellation between the two traces. Therefore in our framework such cutoff schemes should be avoided. Let us remark that in applications of the Wetterich equation to the study of matter fields interacting with gravity, scheme dependences in the beta function of the cosmological constant were indeed noticed [32, 35] also in the standard approach. Also, if the background field method is used, which is very commonly adopted in gauge theories in order to preserve gauge invariance, the cutoff operator becomes dependent on the background fields. Therefore the subtraction term induced by the non trivial measure also depends on them and this leads to an AEA whose background dependence differs from the usual one [2].

[2] A flow equation for scalar QED modified by a subtraction term dependent on the background gauge field was already used in [45] with the motivation of minimizing some quantities [46].

We have also discussed the relation which ties the modified AEA Γ_k^Λ to the Wilsonian (i.e. bare) action S^Λ in the presence of an UV cutoff Λ. The non trivial functional measure plays here a fundamental role, providing the correct normalization for a rising functional delta inside the path integral, which is realized when the IR cutoff scale of the AEA reaches its UV limit. We have analyzed how such a UV limit is defined according to the properties of the cutoff operator which implements the coarse-graining, and classified some possible ways in which the AEA can approach the Wilsonian action. Finally we have addressed the same problem in the case in which no UV cutoff is present.

In so doing we have computed the contribution to the vacuum energy density of free massive theories in arbitrary dimensions. In particular we have found that, under preservation of Lorentz invariance, the contributions of quantum fluctuations to the vacuum energy density grow only quadratically in the UV cutoff and vanish in the massless case. Such a scenario, i.e the absence of the quartic divergences, was already invoked recently [38, 39] in a standard perturbative QFT approach (where infinite constant contributions from the functional measure were neglected) by performing ad hoc regularizations and subtractions justified by symmetry and reality conditions. We think that our computation gives a straightforward and neat derivation of such a behavior. Also, a paradoxical effect about the contribution of the low energy modes to the cosmological constant was observed in [44] by using the flow equation without the subtraction induced by the measure. By taking the subtraction into account such an effect disappears; e.g. for bosonic fields such a contribution grows monotonically for decreasing k. The method of following the RG flow of the vacuum energy density, or of the cosmological constant term in a curved spacetime, is also suitable for studying the case of interacting theories and it would be interesting to address such a problem.

One of the possible areas in which a proper subtraction term in the ERGE could lead to important effects is the study of gravity plus matter systems. It would be interesting to trace the possible differences brought by this approach in the flow and fixed point structures of such interacting theories. In the absence of an UV cutoff in the theory the effect of the choice of cutoff operators with the right singularity structure to provide a convergence to the bare action, as discussed in the section 4.2, is also an important point to be investigated. Simpler questions related to QFT on curved background spacetimes can also be addressed in this framework.

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3

The effective Hamiltonian action

We generalize the phase-space analysis of the previous chapter to a fully Hamiltonian framework. We first review the definition and properties of the quantum effective Hamiltonian action and we describe its renormalization flow by a functional RG equation. This equation can be used for a non-perturbative quantization and study also of theories with bare Hamiltonians which are not quadratic in the momenta. As an example the vacuum energy and gap of quantum mechanical models are computed. Extensions of this framework to quantum field theories are discussed. In particular one possible Lorentz covariant approach for simple scalar field theories is developed. Fermionic degrees of freedom, being naturally described by a first order formulation, can be easily accommodated in this approach.

3.1 Outline

The effective action most commonly discussed in the literature is of the Lagrangian type, since it is derived from the second order Lagrangian formulation of the bare theory. There is a very good reason to do that, namely that people usually consider bare Hamiltonians which are quadratic in the momenta such that one can easily move to a Lagrangian description. The rationale for this is obtaining a manifestly Lorentz-covariant formulation in d space-time di-

mensions. Another advantage of passing to a second order formulation is that the number of fields in configuration space is half the one in phase space, since in the functional formulation the conjugated momenta have been integrated out.

On the other hand one may also consider the reasons to choose a first order Hamiltonian description on the phase space of a theory. Clearly this is unavoidable when dealing with the quantization of theories with bare Hamiltonians non quadratic in the momenta. In such a case the full phase space variables are needed for a quantum description of the system. Traditionally the main advantage attributed to the Hamiltonian formulation is that it makes unitarity manifest [47]. This is due to the strict relationship established by canonical quantization between the classical symplectic structure on phase space and the inner product on the Hilbert space. The Hamiltonian approach may be useful also when configuration space is not a vector space, since phase space can usually be interpreted as a cotangent bundle and it could be easier to deal with. In the functional integral representation this is translated in the possibility that the measure in phase space be field independent while the one in configuration space be not. This happens for instance in the case of non linear sigma models. Of course, even in this case whenever the bare theory is quadratic in the momenta the Lagrangian and the Hamiltonian formulations lead to the same results (Matthews Theorem), as proved by perturbative studies [48, 49]. In a functional integral representation, the Hamiltonian approach is based on quantum generating functionals obtained introducing sources in the phase space path integral [50]. From them, one can define a quantum effective Hamiltonian action which generates the proper vertices. This was recently studied in [51], on the wake of a renewed interest in Hamiltonian gauge theories such as QCD, in particular in the Coulomb gauge (see [52] and references therein).

The purpose of the present chapter is to present a non-perturbative framework which allows to compute, within specific approximation schemes, the quantum effective Hamiltonian action. This will be obtained by constructing a fRG equation from the functional integral representation. In the previous chapter we proposed the use of cutoff operators affecting the symplectic form of phase space and implementing a more balanced coarse-graining and regularization, with respect to the cases where the coarse-graining is performed on the fluctuations in configuration space only, but after this choice of regularization, we restricted our discussion to bare Hamiltonians quadratic in the momenta and we fully integrated out the momenta, obtaining a cutoff dependent functional measure in the Lagrangian path integral, which was leading to a subtraction term in the RG flow equation. Here instead we are interested in retaining the full dynamics in phase space, building a flow which realizes the idea of shell-by-shell simultaneous integration on both phase space variables. As a disclaimer let us add that other

non-perturbative RG flows called "Hamiltonian flows" already appeared in the literature, but they largely differ from our formulation. Examples are the similarity RG [53], which is generated by iterated unitary transformations within the operatorial representation, and the flows based on a variational solution of the Schrödinger equation [54].

In this chapter we start our discussion from quantum mechanical systems ($0+1$ dimensional QFT's) with scalar degrees of freedom, for which we review some of the properties of the Hamiltonian effective action in the first part of Section 2, and we prove some formula useful for the subsequent developments. Throughout this chapter we will adopt a real time (Minkowskian) formulation, since the Hamiltonian formulation relies on a preferred role played by time, but the Euclidean notations and a discussion of Wick rotation can be found in appendix A.4. In the second part of Section 2 we derive the main equations satisfied by the average effective Hamiltonian action (AEHA) of a quantum mechanical system. They depend on a cutoff operator which suppresses part of the functional integration generating a one-parameter flow from the UV to the IR. In particular we give the simpler equations associated to the so called local Hamiltonian approximation (LHA), which is the lowest order term of the derivative expansion of the full functional, for some specific cutoff operators. These are then used (Section 2.3) to study a family of exactly solvable Hamiltonians which are not quadratic in the momenta and indeed we show that one can easily extract informations like the ground state and the first energy gap of such systems. The same approach can be used to study general systems with arbitrary bare Hamiltonians. We conclude Section 2 discussing the extension of the formalism to quantum mechanical theories with fermionic degrees of freedom.

In section 3 we start to address quantum field theories. The extension to the non covariant version of QFT is straightforward and we first discuss it briefly for the case of scalar QFT. Since in the traditional Hamiltonian formulation of QFT one pays explicit unitarity with the disguising of Lorentz invariance, we discuss one possible way around this drawback, that is, we spend the last part of the chapter in discussing a manifestly Lorentz symmetric (but maybe not manifestly unitary) extension of the previous framework inside the realm of the covariant Hamiltonian formalism.

This is a subject which has a long history in classical physics [55, 56, 57, 58], but whose applications to quantum dynamics are pretty rare to be found in the literature. Even if under different names, the covariant Hamiltonian formulation of Yang Mills theory is one of the oldest examples. M.B. Halpern in 1977 addressed such a formalism for QCD, generically naming it "first order formalism" [59] but he immediately abandoned the full phase space formulation integrating out the gauge vector fields thus being left with a theory, containing only conjugate momenta, that he called "field strenght formulation", which was studied in the following years

(see [60] and references in it). More recently a slight variant of the "first order formalism" (still covariant) for Yang Mills theory has received fresh attentions from the perspective of topological BF theories [61]. In particular the reader can find in [62] an explicit one loop computation of what we call the effective covariant Hamiltonian action of pure Yang Mills, reproducing the expected asymptotic freedom result. Despite these successful examples, the main open question about covariant Hamiltonian QFT is still about its foundations, even if these have begun to be studied recently by some author [63, 64, 65]. These investigations can shed light on the issue of unitarity of this covariant formulation. Without a sound Lorentz covariant quantization prescription, covariant Hamiltonian formalism seems but a game, legitimate only in the special case of Hamiltonians quadratic in the momenta. On the other hand, only by studying this approach in more general cases and by looking for its applications to real physical systems one can hope to find a legitimation for the search of foundations.

In this chapter, for what concerns a covariant Hamiltonian formulation of QFT's, we restrict ourselves to defining the average effective covariant Hamiltonian action of a scalar field theory in a particularly simple case. This consists in assuming that the non trivial dependence on the covariant momenta is in the longitudinal (w.r.t. Fourier variable) subspace of the space of conjugate momenta. This definition is compatible with both QM in $0+1$ dimensions and with QFT's whose bare Hamiltonians are quadratic in the momenta, and it provides a particular dynamical extension outside this domain. For this simple case we present a framework for studying such a model by a non-perturbative RG flow equation. For completeness we also comment on the corresponding covariant Hamiltonian formulation for theories with Dirac fermions.

In the last section the reader will find a discussion about the physical motivations for the introduction of this formalism, as well as a proposal of some possible developments, extensions and future applications of this method. Some technical issues are described in more details in App. A.

3.2 The effective Hamiltonian action in quantum mechanics

In this section we shall work within quantum mechanics (QM), i.e. a 0+1 dimensional quantum field theory (QFT). As an example we will quantize a classical system with one bosonic degree of freedom governed by the following Hamiltonian action:

$$S[p,q] = \int dt \left[p(t)\partial_t q(t) - H\left(p(t), q(t)\right) \right] \tag{3.1}$$

where the (bare) Hamiltonian can have an arbitrary dependence in the momenta, departing therefore from the usual quadratic form

$$H(p,q) = \frac{1}{2}\,p^2 + V(q)\,. \tag{3.2}$$

Here and in the following p and q denote canonically conjugate variables. The quantization of such a system is performed via the following phase-space path integral:

$$e^{\frac{i}{\hbar}W[I,J]} = \int [dpdq]\,\mu[p,q]e^{\frac{i}{\hbar}\{S[p,q]+I\cdot p+J\cdot q\}} \tag{3.3}$$

where the dots stand for ordinary integrations. The functional measure on the physical phase space is usually assumed to be $\mu[p,q] = \mathrm{Det}\frac{1}{2\pi\hbar}$. Since we want to keep our discussion as general as possible we will not specify the precise space of functions on which the functional integral is defined.

It is possible to study the system by a functional which may be called the quantum effective Hamiltonian action, which is a trivial generalization of the more widely known effective Lagrangian action. The latter Γ^L is defined by introducing in the configuration-space path integral external sources J coupled to the Lagrangian variables, and by taking the Legendre transform of the generating functional of the connected green's functions $W[J]$ with respect to (w.r.t.) J. Similarly, in order to define the effective Hamiltonian action Γ^H, one starts from the phase-space path integral (3.3) and performs a Legendre transform:

$$\Gamma^H\left[\bar{p},\bar{q}\right] = \underset{I,J}{\mathrm{ext}}\left(W[I,J] - I\cdot\bar{p} - J\cdot\bar{q}\right)\,, \tag{3.4}$$

where

$$\bar{p} = \frac{\delta W}{\delta I} \quad , \quad \bar{q} = \frac{\delta W}{\delta J}\,.$$

The introduction of such a functional is not a novelty, as we have discussed in the previous section. There are several ways to convince ourselves that from this functional one can get every information about the quantum system.
First, by taking functional derivatives w.r.t. $\bar{q}(t)$ and $\bar{p}(t)$ one immediately gets

$$I = -\frac{\delta\Gamma^H}{\delta\bar{p}} \quad , \quad J = -\frac{\delta\Gamma^H}{\delta\bar{q}}\,. \tag{3.5}$$

For zero sources one has the equations for the vacuum configuration $(\bar{q},\bar{p})$. They appear as the classical equations of motion obtained from the quantum effective Hamiltonian action.

Second, Γ^H satisfies the following integro-differential equation

$$e^{\frac{i}{\hbar}\Gamma^H[\bar{p},\bar{q}]} = \int [dpdq]\, \mu[p,q] e^{\frac{i}{\hbar}\left\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\right\}} . \tag{3.6}$$

This is a central identity and it could also be promoted to the definition of Γ^H.
Third, from this equation one can get a different proof that the classical equations satisfied by the effective Hamiltonian action encode the full quantum dynamics, because they are equivalent to the Hamiltonian Dyson-Schwinger equations. In fact, the identities:

$$\begin{aligned} 0 &= \int [dpdq]\, \frac{\delta}{\delta p}\left(\mu[p,q] e^{\frac{i}{\hbar}\left\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\right\}}\right) \\ &= \int [dpdq]\, \frac{\delta}{\delta q}\left(\mu[p,q] e^{\frac{i}{\hbar}\left\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^H}{\delta\bar{q}}-(p-\bar{p})\cdot\frac{\delta\Gamma^H}{\delta\bar{p}}\right\}}\right) \end{aligned}$$

lead to:

$$\left\langle -i\hbar\frac{\delta}{\delta p}\log\mu[p,q]+\frac{\delta S}{\delta p}\right\rangle = \frac{\delta\Gamma^H}{\delta\bar{p}} \quad , \quad \left\langle -i\hbar\frac{\delta}{\delta q}\log\mu[p,q]+\frac{\delta S}{\delta q}\right\rangle = \frac{\delta\Gamma^H}{\delta\bar{q}} .$$

Forth, just like for the effective action, the effective Hamiltonian action has a similar interpretation as the generator of the one-particle-irreducible (1PI) proper vertices. For more details and a proof of this statement see Appendix A.1.
Fifth, by evaluating the effective Hamiltonian action on its stationarity $\bar{p}$ values one gets the effective Lagrangian action. In fact, defining

$$\Gamma^L[\bar{q}] = \operatorname*{ext}_{\bar{p}} \Gamma^H[\bar{p},\bar{q}]$$

and calling $\bar{p}_{\bar{q}}$ the extremal point, it is straightforward to show that

$$I = -\frac{\delta\Gamma^H}{\delta\bar{p}}[\bar{p}_{\bar{q}},\bar{q}] = 0 \quad , \quad J = -\frac{\delta\Gamma^H}{\delta\bar{q}}[\bar{p}_{\bar{q}},\bar{q}] = -\frac{\delta\Gamma^L}{\delta\bar{q}}[\bar{q}] .$$

Therefore $\Gamma^L[\bar{q}] = W\left[0,-\frac{\delta\Gamma^L}{\delta\bar{q}}\right] + \bar{q}\cdot\frac{\delta\Gamma^L}{\delta\bar{q}}$, wherefrom we learn that Γ^L satisfies the integro-differential equation:

$$e^{\frac{i}{\hbar}\Gamma^L[\bar{q}]} = \int [dpdq]\, \mu[p,q] e^{\frac{i}{\hbar}\left\{S[p,q]-(q-\bar{q})\cdot\frac{\delta\Gamma^L}{\delta\bar{q}}\right\}}$$

which is a generalization of the usual configuration space integro-differential equation satisfied by the effective action, since it does not require S to be quadratic in the momenta. Due to this

simple relation between the two effective actions, from here on and for the rest of this thesis we will use the same letter Γ for both, dropping the superscripts, since the reader will be able to distinguish them by their arguments ($\bar{p}$,$\bar{q}$ for the Hamiltonian one and $\bar{q}$ only for the Lagrangian one).

Sixth, the effective Hamiltonian action can be defined from the operatorial representation by means of a time-dependent variational principle, in a way which is the direct generalization of the usual construction in configuration space [66]. Let $\hat{H}$ be the Hamiltonian operator of the quantum system, $|0\rangle$ be its time-independent ground state and let the boundary conditions of the path integral in (3.3) be chosen such that

$$e^{\frac{i}{\hbar}W[I,J]} = \langle 0|\hat{U}_{I,J}(+\infty,-\infty)|0\rangle = \langle 0|T\exp\left\{-\frac{i}{\hbar}\int_{-\infty}^{+\infty}dt\left[\hat{H}-J(t)\hat{q}-I(t)\hat{p}\right]\right\}|0\rangle\,. \tag{3.7}$$

Then the effective Hamiltonian action defined in (3.4) is related in the following way

$$\Gamma\left[\bar{p},\bar{q}\right] = \underset{|\psi_\pm,t\rangle}{\text{ext}}\left(\int_{-\infty}^{+\infty}dt\,\langle\psi_-,t|i\hbar\partial_t-\hat{H}|\psi_+,t\rangle\right) \tag{3.8}$$

to an extremum with respect to variations of the two states $|\psi_\pm,t\rangle$ preserving the constraints

$$\langle\psi_-,t|\psi_+,t\rangle = 1 \quad , \quad \langle\psi_-,t|\hat{q}|\psi_+,t\rangle = \bar{q}(t) \quad , \quad \langle\psi_-,t|\hat{p}|\psi_+,t\rangle = \bar{p}(t) \tag{3.9}$$

for any t, and the boundary conditions

$$\lim_{t\to\mp\infty}|\psi_\pm,t\rangle = |0\rangle\,. \tag{3.10}$$

A sketch of the proof of this statement is given in Appendix A.2. A special role is played by time-independent $\bar{p}$ and $\bar{q}$, because the previous proposition reduces to $\Gamma[\bar{p},\bar{q}] = -\mathcal{E}(\bar{p},\bar{q})\int dt$ where $\mathcal{E}$ is the usual energy density functional defined by the minimum

$$\mathcal{E}\left(\bar{p},\bar{q}\right) = \min_{|\psi\rangle}\langle\psi|\hat{H}|\psi\rangle \tag{3.11}$$

with respect to variations of the time-independent state $|\psi\rangle$ preserving the time-independent version of the constraints in (3.9).

This clearly provides an energy interpretation for the effective Hamiltonian action. In particular if one evaluates this action on the constant ($\bar{p}$,$\bar{q}$)-values which make it stationary, the resulting number is just minus the "time volume" times the ground state energy. In principle it is possible to compute all the energy levels by means of Γ, but higher levels require more

work. One possible way is through the two point functions. In a Hamiltonian framework the propagator splits in the entries of the matrix:

$$i\langle \mathcal{T} \begin{pmatrix} (p-\bar{p})_{t'}(p-\bar{p})_t & (q-\bar{q})_{t'}(p-\bar{p})_t \\ (p-\bar{p})_{t'}(q-\bar{q})_t & (q-\bar{q})_{t'}(q-\bar{q})_t \end{pmatrix} \rangle = W^{(2)}_{tt'}[I,J] = \begin{pmatrix} \frac{\delta^2 W}{\delta I_{t'}\delta I_t} & \frac{\delta^2 W}{\delta J_{t'}\delta I_t} \\ \frac{\delta^2 W}{\delta I_{t'}\delta J_t} & \frac{\delta^2 W}{\delta J_{t'}\delta J_t} \end{pmatrix} = \begin{pmatrix} \frac{\delta \bar{p}_t}{\delta I_{t'}} & \frac{\delta \bar{p}_t}{\delta J_{t'}} \\ \frac{\delta \bar{q}_t}{\delta I_{t'}} & \frac{\delta \bar{q}_t}{\delta J_{t'}} \end{pmatrix} \tag{3.12}$$

(where $\mathcal{T}$ is the time ordering operator) so that one could try to think about p and q as different "fields" but should also remember about the existence of an unusual mixed propagator connecting p-legs to q-legs or vice versa. Thanks to (3.5) one can write this matrix in terms of Γ as follows

$$W^{(2)}_{tt'}[I,J] = \begin{pmatrix} \frac{\delta \bar{p}}{\delta I} & \frac{\delta \bar{p}}{\delta J} \\ \frac{\delta \bar{q}}{\delta I} & \frac{\delta \bar{q}}{\delta J} \end{pmatrix}_{tt'} = \begin{pmatrix} \frac{\delta I}{\delta \bar{p}} & \frac{\delta I}{\delta \bar{q}} \\ \frac{\delta J}{\delta \bar{p}} & \frac{\delta J}{\delta \bar{q}} \end{pmatrix}^{-1}_{tt'} = -\begin{pmatrix} \frac{\delta^2 \Gamma}{\delta \bar{p}\delta \bar{p}} & \frac{\delta^2 \Gamma}{\delta \bar{q}\delta \bar{p}} \\ \frac{\delta^2 \Gamma}{\delta \bar{p}\delta \bar{q}} & \frac{\delta^2 \Gamma}{\delta \bar{q}\delta \bar{q}} \end{pmatrix}^{-1}_{tt'} = -\left(\Gamma^{(2)}[\bar{p},\bar{q}]\right)^{-1}_{tt'} . \tag{3.13}$$

In order to make the last expression for the two point function more explicit one needs to invert a matrix whose elements are operators. In the particular case in which all block entries of the original matrix are nonsingular, its inverse is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix} . \tag{3.14}$$

In our case the operator $W^{(2)}_k$ is symmetric and one can use the formula in eqn. (3.14) setting $C = B^T$. Let us stress that in order to put the off-diagonal blocks of this inverse in the form of eqn. (3.14) with $C = B^T$ it is only necessary to assume that B be nonsingular, condition which is met by $\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}}$ unless Γ is extremely pathological. Once we know how to compute the two point functions by means of Γ, we could have access to all the energy gaps $\Delta E_n = E_n - E_0$ through the Källen-Lehmann representation of the propagators

$$\begin{aligned} \frac{\delta^2 W}{\delta I(\tau)\delta I(0)} &= i\sum_{n\neq 0} e^{-i\Delta E_n \tau} |\langle 0|\hat{p}|n\rangle|^2 = -\sum_{n\neq 0}\int \frac{dE}{2\pi} e^{-iE\tau} \frac{2\Delta E_n |\langle 0|\hat{p}|n\rangle|^2}{E^2 - \Delta E_n^2 + i\varepsilon} \\ \frac{\delta^2 W}{\delta J(\tau)\delta J(0)} &= i\sum_{n\neq 0} e^{-i\Delta E_n \tau} |\langle 0|\hat{q}|n\rangle|^2 = -\sum_{n\neq 0}\int \frac{dE}{2\pi} e^{-iE\tau} \frac{2\Delta E_n |\langle 0|\hat{q}|n\rangle|^2}{E^2 - \Delta E_n^2 + i\varepsilon} . \end{aligned}$$

Similar expressions hold for mixed derivatives of W. This tells us that, in principle, by studying the pole structure of the Fourier transformed two point functions we could compute all the ΔE_n. As eq. (3.13) shows, this requires the knowledge of the exact $\Gamma^{(2)}$. In most cases this is not available, and only approximations are possible. In certain contexts one popular approxi-

mation scheme for the computation of the effective action is the derivative expansion. The zeroth order of such an expansion in the present Hamiltonian framework can be called the local Hamiltonian approximation (LHA) and consists of the ansatz: $\Gamma = \int dt\ (\bar{p}\partial_t \bar{q} - H_{\text{eff}}(\bar{p},\bar{q}))$ where the effective Hamiltonian H_{eff}, which is an ultralocal function of its arguments (i.e. it does not depend on their derivatives), can be computed by setting the fields $\bar{p}$ and $\bar{q}$ to constant values. For this choice, since the second derivatives of Γ on constant field configurations commute with each other, the inversion rule (3.14) leads to a simple expression

$$\begin{aligned}
\frac{\delta^2 W}{\delta I(\tau)\delta I(0)} &= -\left[\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{p}} - \frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{p}}\left(\frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{q}}\right)^{-1}\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}}\right]^{-1}_{0\tau} \overset{LHA}{=} -\int \frac{dE}{2\pi} e^{-iE\tau}\frac{\frac{\partial^2 H_{\text{eff}}}{\partial\bar{q}\partial\bar{q}}}{E^2 - \det H^{(2)}_{\text{eff}} + i\varepsilon} \\
\frac{\delta^2 W}{\delta J(\tau)\delta J(0)} &= -\left[\frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{q}} - \frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{q}}\left(\frac{\delta^2\Gamma}{\delta\bar{p}\delta\bar{p}}\right)^{-1}\frac{\delta^2\Gamma}{\delta\bar{q}\delta\bar{p}}\right]^{-1}_{0\tau} \overset{LHA}{=} -\int \frac{dE}{2\pi} e^{-iE\tau}\frac{\frac{\partial^2 H_{\text{eff}}}{\partial\bar{p}\partial\bar{p}}}{E^2 - \det H^{(2)}_{\text{eff}} + i\varepsilon}
\end{aligned} \tag{3.15}$$

and similar formulae hold for mixed derivatives of W. Here $\det H^{(2)} = \partial^2_{\bar{q}\bar{q}} H\, \partial^2_{\bar{p}\bar{p}} H - (\partial^2_{\bar{q}\bar{p}} H)^2$ is the determinant of the Hessian matrix of H. Therefore we see that in the LHA, whenever the second derivatives of H_{eff} commute (as in the case they are single numbers and not matrices), only one pole appears in the propagators at the value $(\det H^{(2)}_{\text{eff}})^{1/2}$. Since we are performing a derivative (low energy) expansion, in general this pole is the one closer to $E = 0$, that is to say the first gap ΔE_1, unless the matrix elements $\langle 0|\hat{p}|1\rangle$ and $\langle 0|\hat{q}|1\rangle$ vanish. Therefore we shall use in the LHA the relations

$$E_0 = H_{\text{eff}}|_{\min} \quad , \quad \Delta E_1 = \sqrt{\det H^{(2)}_{\text{eff}}}\,|_{\min}\,. \tag{3.16}$$

So far we have discussed how many properties of a quantum system can be deduced from the effective Hamiltonian action, but how can we compute this action? One way is to use perturbation theory. First of all one needs to define propagators and vertex functions. We already know that in a Hamiltonian framework the propagators of a theory with Hamiltonian action Γ are given by eq. (3.13). The vertex functions generated by Γ are simply given by:

$$\left.\frac{\delta^m}{\delta\bar{p}^m}\frac{\delta^n\Gamma}{\delta\bar{q}^n}\right|_{\bar{q}=\bar{p}=0}, \quad m+n>2 \tag{3.17}$$

and therefore generically comprehend m p-legs and n q-legs. Since perturbation theory in phase space is built on tree level propagators and vertices, one can read off these ingredients from (3.13) and (3.17) by substituting Γ with the bare action S. For instance, to get the one-

loop result one changes variables of integration in (3.6) according to $p = \bar{p} + \hbar^{\frac{1}{2}} p'$, $q = \bar{q} + \hbar^{\frac{1}{2}} q'$, and Taylor-expands both S and Γ around $\hbar = 0$ up to linear terms

$$\begin{aligned} S(p,q) &= S(\bar{p},\bar{q}) + \frac{\hbar}{2}(p',q')S^{(2)}(\bar{p},\bar{q})(p',q')^T + o(\hbar^2) \\ \Gamma[\bar{p},\bar{q}] &= \Gamma_0[\bar{p},\bar{q}] + \hbar\Gamma_1[\bar{p},\bar{q}] + o(\hbar^2)\,. \end{aligned}$$

The change of variable goes along with a change of measure, due to the Jacobian determinant $\mathrm{Det}\hbar$, such that the new measure becomes $\mu[p',q'] = \mathrm{Det}\frac{1}{2\pi}$. The Gaussian path integral over p' and q' combined with such a measure gives $\Gamma_1[\bar{p},\bar{q}] = \frac{i}{2}\log\mathrm{Det}\left(-iS^{(2)}[\bar{p},\bar{q}]\right)$, where S is the bare Hamiltonian action (together with the obvious result $\Gamma_0[\bar{p},\bar{q}] = S[\bar{p},\bar{q}]$). The block determinant can be written in a more explicit form by means of the general formula

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A\ \det(D - CA^{-1}B) = \det D\ \det(A - BD^{-1}C) \tag{3.18}$$

where the first expression is true if $\det A \neq 0$ and the second if $\det D \neq 0$. Therefore, if $\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}$ is non-vanishing

$$\begin{aligned} \Gamma_1[\bar{p},\bar{q}] &= \frac{i}{2}\log\mathrm{Det}\left[-\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}\left(\frac{\delta^2 S}{\delta\bar{q}\delta\bar{q}} - \frac{\delta^2 S}{\delta\bar{p}\delta\bar{q}}\left(\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}\right)^{-1}\frac{\delta^2 S}{\delta\bar{q}\delta\bar{p}}\right)\right] \\ &= \frac{i}{2}\log\mathrm{Det}\left[\left(-{\partial_t}^2 - \det H^{(2)} + \left(\partial_t\frac{\partial^2 H}{\partial\bar{p}\partial\bar{q}}\right) + \left(\partial_t\log\frac{\partial^2 H}{\partial\bar{p}\partial\bar{p}}\right)\left(\partial_t - \frac{\partial^2 H}{\partial\bar{p}\partial\bar{q}}\right)\right)\delta\right] \end{aligned} \tag{3.19}$$

which reduces to the usual one-loop formula for the effective action in the case of a bare Hamiltonian like the one in (3.2). In the formula above we have used the symbol δ for $\delta(t-t')$. If instead $\frac{\delta^2 S}{\delta\bar{p}\delta\bar{p}}$ vanishes while $\frac{\delta^2 S}{\delta\bar{q}\delta\bar{q}}$ is non-vanishing, the result can be obtained from (3.19) by replacing $\delta_{\bar{p}}$ with $\delta_{\bar{q}}$ and vice versa.

In the rest of this chapter we will work on a non-perturbative setting for the computation of the effective Hamiltonian action and we will choose $\hbar$ as our unit of action.

3.2.1 The average effective Hamiltonian action

Sticking to the framework discussed in the previous sections, we regularize the phase space path integral by means of a modification of the bare action and of the functional measure

$$e^{iW_k[I,J]} = \int [dpdq]\,\mu_k[p,q]e^{i\{S[p,q]+\Delta S_k[p,q]+I\cdot p+J\cdot q\}} \tag{3.20}$$

and ask for $\mu_k \exp\{i\Delta S_k\}$ to become μ as $k \to 0$ and to provide a rising delta functional as $k \to \Lambda$. As traditional, to keep the framework as simple as possible, we choose ΔS_k to be quadratic in the fields

$$\Delta S_k[p,q] = \frac{1}{2}(p,q)\cdot R_k \cdot (p,q)^T \tag{3.21}$$

such that we need $R_k \to 0$ and $\mu_k \to \mu$ when $k \to 0$, as well as $R_k \to \infty$ and $\mu_k \to \left(\mathrm{Det}\frac{R_k}{2\pi}\right)^{\frac{1}{2}}$ when $k \to \Lambda$. These constraints can be satisfied by several choices for the symmetric matrix R_k and for the measure μ_k. In this thesis we will consider only two simple cases in which the only non-vanishing entries of R_k are either off-diagonal and built out of an odd differential operator or diagonal and built out of even differential operators. These respectively read

$$R_k(t,t') = \begin{pmatrix} 0 & r_k(-\partial_t{}^2)\partial_t\delta(t-t') \\ -r_k(-\partial_t{}^2)\partial_t\delta(t-t') & 0 \end{pmatrix} \tag{3.22}$$

$$R_k(t,t') = \begin{pmatrix} \mathcal{R}_k^p(-\partial_t{}^2)\delta(t-t') & 0 \\ 0 & \mathcal{R}_k^q(-\partial_t{}^2)\delta(t-t') \end{pmatrix} \tag{3.23}$$

The first choice again can be interpreted as a k-dependent deformation of the symplectic potential $\lambda = pdq$, by means of an operator $(1+r_k)$. As we have already discussed, this interpretation suggests the appropriate k-dependent deformation of the functional measure. Following this line of thought we can guess a convenient choice for the regularized measure also in the second case of a diagonal regulator. The straightforward adaptation of the previous argument is insisting in adding to the fundamental symplectic matrix our regulator matrix, and then taking its determinant. To summarize, the regularized functional measures we will use together with the regulators (3.22) and (3.23) respectively are

$$\mu_k = \left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix} 0 & \left(1+r_k(-\partial_t{}^2)\right)\partial_t\delta(t-t') \\ -\left(1+r_k(-\partial_t{}^2)\right)\partial_t\delta(t-t') & 0 \end{pmatrix}\right]^{\frac{1}{2}} \tag{3.24}$$

$$\mu_k = \left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix} \mathcal{R}_k^p(-\partial_t{}^2)\delta(t-t') & \partial_t\delta(t-t') \\ -\partial_t\delta(t-t') & \mathcal{R}_k^q(-\partial_t{}^2)\delta(t-t') \end{pmatrix}\right]^{\frac{1}{2}} . \tag{3.25}$$

The definition of the average effective Hamiltonian action (AEHA) $\Gamma_k[\bar{p},\bar{q}]$ is

$$\Gamma_k[\bar{p},\bar{q}] + \Delta S_k[\bar{p},\bar{q}] = \underset{I,J}{\mathrm{ext}}\left(W_k[I,J] - I\cdot\bar{p} - J\cdot\bar{q}\right) .$$

Note that the sources minimizing the r.h.s. will in general depend on k. Again it is easy to write

an integro-differential equation for the AEHA:

$$e^{i\Gamma_k[\bar{p},\bar{q}]} = \int [dpdq]\, \mu_k[p,q] e^{i\left\{S[p,q]+\Delta S_k[p-\bar{p},q-\bar{q}]-(p-\bar{p})\frac{\delta\Gamma_k}{\delta\bar{p}}-(q-\bar{q})\frac{\delta\Gamma_k}{\delta\bar{q}}\right\}}. \tag{3.26}$$

When $k \to 0$ eq. (3.26) trivially reduces to eq. (3.6) and the AEHA becomes the full effective Hamiltonian action. It is not hard to check that when $k \to \Lambda$ the r.h.s. of eq. (3.26) reduces to $\exp\{iS[\bar{p},\bar{q}]\}$ and the AEHA coincides with the bare Hamiltonian action. A sketch of the proof can be found in Appendix A.3.

The relation between the average effective Hamiltonian and Lagrangian actions is the same as for the full effective actions:

$$\Gamma_k[\bar{q}] = \underset{\bar{p}}{\text{ext}}\, \Gamma_k[\bar{p},\bar{q}]\,. \tag{3.27}$$

We observe that this is evident in the simplest possible case, i.e. when the bare action is quadratic in the momenta, as in (3.2), since $\frac{\partial^2 H}{\partial p^2}$ and $\frac{\partial^2 H}{\partial p \partial q}$ are constant (the latter is actually zero). Indeed the integration over p in (3.26) can be performed exactly and in such a case one discovers that also the AEHA must be quadratic in the momenta and that for any k the canonical momentum that extremizes it is $\bar{p} = \partial_t \bar{q}$. As a result, plugging this field configuration in (3.26), using the definition (3.27) and integrating out the momenta, one obtains

$$e^{i\Gamma_k[\bar{q}]} = \int [dq]\, \mu_k[q] e^{i\left\{S[q]+\Delta S_k[q-\bar{q}]-(q-\bar{q})\frac{\delta\Gamma_k}{\delta\bar{q}}\right\}} \tag{3.28}$$

where now $\mu_k[q] \equiv \int [dp]\, \mu_k[p,q] e^{-i\frac{p^2}{2}}$ and $\Delta S_k[q]$ arises from the chosen $\Delta S_k[p,q]$. For example, if one adopts the scheme of eqs. (3.22) and (3.24) then

$$\begin{aligned} \mu_k[q] &= \left[\text{Det}\frac{1}{2\pi}\left(1+r_k(-\partial_t{}^2)\right)^2(-\partial_t{}^2)\delta\right]^{\frac{1}{2}} \\ \Delta S_k[q] &= \frac{1}{2}\partial_t q \cdot (r_k^2+2r_k)\partial_t q\,. \end{aligned}$$

As usual, the $k \to \Lambda$ limit of the average effective Lagrangian action coincides with the bare Lagrangian action while the $k \to 0$ limit gives the full quantum effective Lagrangian action.

We are now interested in the cases which depart from such a simple situation.

3.2.2 RG flow equation for the AEHA

In this section we discuss the translation of the functional integro-differential equation (3.26) in a functional differential equation describing a flow parameterized by k.

Denoting by "$\dot{}$" the operation $k\partial_k$, and acting with it on eq. (3.26) one obtains

$$i\dot{\Gamma}_k = \frac{\dot{\mu}_k}{\mu_k} + i\langle \dot{\Delta S}_k[p-\bar{p}, q-\bar{q}]\rangle_k \,.$$

Since ΔS_k has been chosen quadratic in the fields, the expectation value can be rewritten by means of the k-dependent version of formulae (3.12,3.13). Denoting $\tilde{\Gamma}_k\left[\bar{p},\bar{q}\right] \equiv \Gamma_k\left[\bar{p},\bar{q}\right] + \Delta S_k\left[\bar{p},\bar{q}\right]$, these read

$$\begin{aligned} i\langle \mathcal{T}\begin{pmatrix} (p-\bar{p})_{t'}(p-\bar{p})_t & (q-\bar{q})_{t'}(p-\bar{p})_t \\ (p-\bar{p})_{t'}(q-\bar{q})_t & (q-\bar{q})_{t'}(q-\bar{q})_t \end{pmatrix}\rangle_k = W_k^{(2)}{}_{tt'}[I,J] = \begin{pmatrix} \frac{\delta^2 W_k}{\delta I_{t'}\delta I_t} & \frac{\delta^2 W_k}{\delta J_{t'}\delta I_t} \\ \frac{\delta^2 W_k}{\delta I_{t'}\delta J_t} & \frac{\delta^2 W_k}{\delta J_{t'}\delta J_t} \end{pmatrix} = \\ = \begin{pmatrix} \frac{\delta\bar{p}_t}{\delta I_{t'}} & \frac{\delta\bar{p}_t}{\delta J_{t'}} \\ \frac{\delta\bar{q}_t}{\delta I_{t'}} & \frac{\delta\bar{q}_t}{\delta J_{t'}} \end{pmatrix} = \begin{pmatrix} \frac{\delta I}{\delta\bar{p}} & \frac{\delta I}{\delta\bar{q}} \\ \frac{\delta J}{\delta\bar{p}} & \frac{\delta J}{\delta\bar{q}} \end{pmatrix}^{-1}_{tt'} = -\begin{pmatrix} \frac{\delta^2\tilde{\Gamma}_k}{\delta\bar{p}\delta\bar{p}} & \frac{\delta^2\tilde{\Gamma}_k}{\delta\bar{q}\delta\bar{p}} \\ \frac{\delta^2\tilde{\Gamma}_k}{\delta\bar{p}\delta\bar{q}} & \frac{\delta^2\tilde{\Gamma}_k}{\delta\bar{q}\delta\bar{q}} \end{pmatrix}^{-1}_{tt'} = -\left(\tilde{\Gamma}_k^{(2)}[\bar{p},\bar{q}]\right)^{-1}_{tt'} . \end{aligned}$$

Therefore, for any quadratic regulator, the flow equation can be written as

$$i\dot{\Gamma}_k = \frac{\dot{\mu}_k}{\mu_k} - \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\delta\right)^{-1}\dot{R}_k\delta\right] \tag{3.29}$$

where $R_k\delta = \Delta S_k^{(2)}$. Here one has still freedom for the choice of μ_k as a functional of R_k. By using the inversion formula (3.14) one can find a more explicit form for the flow equation. Adopting the regulator (3.22) affecting only the Legendre transform term of the bare action (i.e. the symplectic potential) and the corresponding minimally deformed Liouville functional measure (3.24), eq. 3.29 becomes

$$\begin{aligned} i\dot{\Gamma}_k \;=\; & \mathrm{Tr}\left[\dot{r}_k\left(1+r_k\right)^{-1}\delta\right] \\ & - \mathrm{Tr}\left[\left(\dot{r}_k\partial\delta\right)\left(\left(r_k\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\right) - \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\left(-r_k\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\right)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\right)^{-1}\right] \end{aligned} \tag{3.30}$$

where we denote $(\partial\delta)_{t_1t_2} = \partial_{t_1}\delta(t_1-t_2)$. Instead, the choice of a diagonal regulator (3.23)

and of the corresponding measure (3.25) leads to the flow equation

$$
\begin{aligned}
i\dot{\Gamma}_k \;=\; & \frac{1}{2}\mathrm{Tr}\left[(\dot{\mathcal{R}}_k^p\delta)\left(\mathcal{R}_k^p\delta-(\partial\delta)\,(\mathcal{R}_k^q\delta)^{-1}\,(-\partial\delta)\right)^{-1}\right] \\
& + \frac{1}{2}\mathrm{Tr}\left[(\dot{\mathcal{R}}_k^q\delta)\left(\mathcal{R}_k^q\delta-(-\partial\delta)\,(\mathcal{R}_k^p\delta)^{-1}\,(\partial\delta)\right)^{-1}\right] \\
& - \frac{1}{2}\mathrm{Tr}\left[(\dot{\mathcal{R}}_k^p\delta)\left(\left(\mathcal{R}_k^p\delta+\frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\right)-\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\left(\mathcal{R}_k^q\delta+\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\right)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\right)^{-1}\right] \\
& - \frac{1}{2}\mathrm{Tr}\left[(\dot{\mathcal{R}}_k^q\delta)\left(\left(\mathcal{R}_k^q\delta+\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\right)-\frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\left(\mathcal{R}_k^p\delta+\frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\right)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\right)^{-1}\right] \qquad (3.31)
\end{aligned}
$$

Notice that, thanks to the regularization of the functional measure, these equations correctly reproduces the non-renormalization of H_k in the trivial cases in which the bare Hamiltonian either vanishes or depends on just one field out of p and q. As far as the reality properties of this equation are concerned, there is no difference with the standard Lagrangian formalism in real time, that is to say, the imaginary unit on the l.h.s. is needed in order to ensure reality of Γ_k. This is because in real time the traces on the r.h.s. usually are integrals of functions with poles on the real axis, which thus lead to imaginary values. An appropriate prescription should be given in order to displace these poles off the real axis. As usual in QFT one adopts the prescription which relates the Minkowskian theory to the Euclidean theory by a continuous Wick rotation. The same can be done in QM. The reader can find more details about this translation to imaginary time in appendix A.4.

The previous flow equations are still too general for a first approach to their meaning and application, therefore let us give more specific and simple forms of the first one of them, eq. (3.30). As a first example let's consider the truncation $\Gamma_k = \int dt\ \left(\bar{p}\partial_t\bar{q} - \frac{1}{2}\bar{p}^2 - V_k(\bar{q})\right)$. Introducing the notation $P_k(-\partial_t{}^2) = -\partial_t{}^2(1+r_k)^2$ one finds the RG flow equation

$$
-i\int dt\,\dot{V}_k(\bar{q}) = \frac{1}{2}\mathrm{Tr}\left[\dot{P}_k P_k^{-1}\right] - \frac{1}{2}\mathrm{Tr}\left[\dot{P}_k\left(P_k - V_k^{(2)}(\bar{q})\right)^{-1}\right] \qquad (3.32)
$$

which is what one gets by the effective average Lagrangian action approach in the local potential approximation (LPA). A more general example is the local Hamiltonian approximation (LHA), i.e. the case in which the flow equation for the truncation $\Gamma_k = \int dt\ (\bar{p}\partial_t\bar{q} - H_k(\bar{p},\bar{q}))$ is evaluated on constant $\bar{q}$ and $\bar{p}$ configurations. For this choice, if the second derivatives of Γ_k commute with each other as in the present case where they are 1-by-1 bosonic matrices, the

operators in the trace can be simplified and one obtains

$$
\begin{aligned}
-i\int dt\,\dot{H}_k(\bar{p},\bar{q}) = \; & - \quad \mathrm{Tr}\left[\left(\frac{\dot{r}_k}{1+r_k}\delta\right)\frac{\mathrm{det}H_k^{(2)}(\bar{p},\bar{q})}{-\partial^2(1+r_k)^2\delta-\mathrm{det}H_k^{(2)}(\bar{p},\bar{q})}\right] \\
& + \quad \mathrm{Tr}\left[\frac{(\dot{r}_k\partial\delta)\,\frac{\delta^2 H_k}{\delta\bar{p}\delta\bar{q}}(\bar{p},\bar{q})}{-\partial^2(1+r_k)^2\delta-\mathrm{det}H_k^{(2)}(\bar{p},\bar{q})}\right]
\end{aligned}
\tag{3.33}
$$

where $\mathrm{det}H_k^{(2)} = \partial^2_{\bar{q}\bar{q}}H_k\,\partial^2_{\bar{p}\bar{p}}H_k - (\partial^2_{\bar{q}\bar{p}}H_k)^2$ is the determinant of the Hessian matrix of H_k. Notice that the second trace vanishes whenever it is possible to evaluate it in Fourier space and when the domain in such space is symmetric around the origin. If this is the case we are left with

$$
i\int dt\,\dot{H}_k(\bar{p},\bar{q}) = \frac{1}{2}\mathrm{Tr}\left[\frac{\dot{P}_k}{P_k}\delta\,\frac{\mathrm{det}H_k^{(2)}(\bar{p},\bar{q})}{P_k\delta-\mathrm{det}H_k^{(2)}(\bar{p},\bar{q})}\right]. \tag{3.34}
$$

Here one could adopt any of the regulators $\mathcal{R}_k$ developed in the vast literature about the average effective Lagrangian action [10, 41, 42], and plug it in the last formula by $P_k(-\partial_t{}^2) = -\partial_t{}^2 + \mathcal{R}_k(-\partial_t{}^2)$. One of the simplest choices for the regulator is a constant r_k, that is to say an operator which is multiplicative in both time and frequency representations; in other words a function of k and Λ only. If no UV cutoff is present, this choice is possible only in quantum mechanics, because it does not produce any coarse graining and therefore it does not regularize the functional traces. Assuming $\dot{r}_k > 0$, $\forall k \in (0,\Lambda)$, one can trade k for the dimensionless parameter r_k. Thus, in LHA and if the second derivatives of H_k commute with each other, assuming that the traces can be written as $\int dt \int \frac{dE}{2\pi}$ (after Fourier transform), and that there is no UV cutoff in the theory, then by Wick rotating the trace ($E \to iE$) one gets

$$
\frac{dH_r}{dr} = -\frac{1}{2(1+r)^2}\left(\mathrm{det}H_r^{(2)}\right)^{\frac{1}{2}}. \tag{3.35}
$$

A different choice which makes the computation of the traces even simpler than for a constant r_k is the square root of the Litim regulator [41, 42]. Denoting by $r_k(E^2)E$ the Fourier transform of $r_k(-\partial_t{}^2)i\partial_t$, and with θ the Heaviside step function, after Wick rotation such a regulator reads

$$
r_k(E^2)E = -(k+E)\theta(k+E)\theta(-E) + (k-E)\theta(k-E)\theta(E)\,.
$$

In the LHA and if the second derivatives of H_k commute with each other, this gives the same

result as (3.34) for $P_k(E^2) = k^2\theta(k^2 - E^2) + E^2\theta(E^2 - k^2)$, that is

$$\dot{H}_k = -\frac{k}{\pi}\frac{\det H_k^{(2)}}{k^2 + \det H_k^{(2)}} \,. \tag{3.36}$$

Of course if one considers $H_k(\bar{p}, \bar{q}) = T_k(\bar{p}) + V_k(\bar{q})$ as an initial condition for the flow, whenever both T_k and V_k are polynomials of degree higher than two, the determinant becomes a function of both $\bar{q}$ and $\bar{p}$ so that the flow generates also mixed $\bar{p}$ and $\bar{q}$ dependence in the effective Hamiltonian. Therefore one should consider a larger truncation in order to track such terms. Also a structure of a σ-model kind, quadratic in the momenta, generates a dependence in the momenta which is more than quadratic. We stress that in general the flow will generate also a dependence on time derivatives of q and p variables. This goes beyond the LHA but it is still compatible with the standard Hamiltonian approach as long as one starts the flow at the UV with a derivatives-free bare Hamiltonian.

3.2.3 Exercise: the ground state energy and gap of models that are more than quadratic in the momenta

As an example of the application of the framework discussed in the previous subsections to specific problems, we will present the computation of the first two energy levels of some exactly solvable systems for which no simple Lagrangian description is available, due to to the fact that the functional integral over the conjugate momenta is not Gaussian. This will serve as a check of the soundness of the formalism, but the reader is invited to remember that the very same simple computations explained in the following would work also for much more complicated models. Let us recall that the functional RG has already been successfully applied to the computation of the spectrum of quantum mechanical models in the configuration space formulation [14, 67].

The systems we are going to address have the following classical Hamiltonian:

$$H_n(p, q) = \left(\frac{p^2 + \omega^2 q^2}{2}\right)^n \,. \tag{3.37}$$

They are easy to solve due to the $O(2)$ symmetry which forces the Hamiltonian to depend only on the "action" and not on the "angle" coordinate in phase space. Even without performing a canonical transformation to such coordinates, the energy spectrum can be built by ladder operators. Rescaling the variables $q = q'/\sqrt{\omega}$ and $p = \sqrt{\omega}p'$ as well as the Hamiltonian $H = \omega^n H'$ we can reduce the problem to the one with $\omega = 1$, therefore in the following we will restrict to such a case. The operator algebra of these quantum models is completely

described by

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}} \quad , \quad \hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}} \quad , \quad \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \, . \tag{3.38}$$

The Hamiltonian operator is just the n-th power of $\left(\hat{N} + \frac{1}{2}\right)$ where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator. This is enough to deduce the whole energy spectrum for any positive integer n.

In order to reproduce such a spectrum by means of the RG flow equation, the first step is to specify the initial condition for the integration of the flow. From the discussion of the previous subsections follows that the most suitable initial condition is $\Gamma_{k=\Lambda} = S$, where S is the bare action to be inserted in a path integral, as the input specifying which system is being studied. At this point it is necessary to recall that such a bare action is in one-to-one correspondence with the Hamiltonian of the operator representation: the bare Hamiltonian is just the Weyl symbol of the Hamiltonian operator. Let us remind that an operator $\hat{O}(\hat{p}, \hat{q})$, can always be written as a sum of symmetrized (in $\hat{p}$ and $\hat{q}$) operators

$$\hat{O} = \hat{O}_S + \sum_i \hat{O}_{iS} = \hat{O}_W \tag{3.39}$$

which is what one calls the Weyl-ordered version of $\hat{O}$. Also, its average on coordinate $\hat{q}$ eigenstates with eigenvalues x and y is conveniently given by

$$\langle x|\hat{O}|y\rangle = \int dp \, \langle x|p\rangle \, O_W\left(p, \frac{x+y}{2}\right) \langle p|y\rangle \, . \tag{3.40}$$

The function O_W in the right hand side of eq. (3.40) is called the Weyl symbol of $\hat{O}$, and it can be considered as the classical counterpart of $\hat{O}$. There are many ways to compute this function; one is to Weyl-order $\hat{O}$ and then to replace the operators in $\hat{O}_W$ with c-numbers. Another way is through the relation

$$O_W(p,q) = \int dx \, e^{ipx} \langle q - \frac{x}{2}|\hat{O}(\hat{p},\hat{q})|q + \frac{x}{2}\rangle \tag{3.41}$$

where the bra's and ket's are again eigenstates of the $\hat{q}$ operator. For instance, considering the models in Eq. (3.37), in the $n = 2$ and $n = 3$ cases such symbols read

$$H_{2W}(p,q) = \left(\frac{p^2+q^2}{2}\right)^2 - \frac{1}{4} \quad , \quad H_{3W}(p,q) = \left(\frac{p^2+q^2}{2}\right)^3 - \frac{5}{4}\left(\frac{p^2+q^2}{2}\right) \, . \tag{3.42}$$

Notice that both subtraction terms above, generated by Weyl ordering, are proportional to $\hbar^2$, but in natural units such a dependence disappears.

Inserting these initial conditions in the flow equation for the LHA one can compute the full quantum effective Hamiltonian at $k = 0$. Such a task can be performed by numerically integrating the flow equation. However, if one is interested in simple quantities as the first two energy levels, this might be unnecessary: it could be enough to truncate the LHA to a polynomial in $z \equiv (p^2+q^2)/2$ of finite order. Indeed if the bare Hamiltonian depends on p and q only through z, in the LHA approximation also H_k can be shown to respect this symmetry, for suitable cutoff operators.

We started by studying these polynomial truncated flows as generated by equations (3.35) and (3.36) finding that singularities appear at nonvanishing values of k. This happens because at some k the radius of convergence of the necessary expansion of the r.h.s. in powers of z goes to zero, a fact related to the vanishing of the terms quadratic in the fields in the bare Hamiltonian of the $n = 2$ model. If no expansion is performed, as in the numerical integration of the flow equation for H_k, no singularity is met and the ground state and gap can be estimated by the value of H_k and of $(\det H_k^{(2)})^{1/2}$ at the minimum. However these estimates do not reach a great accuracy either because of spurious dependence on the boundary conditions (which can be controlled by some nonlinear redefinitions of H_k) or because of numerical errors: typically we reached no more than two digit accuracy in the region around the minimum. In order to get stable predictions with a precision better than 1% we turned to a different choice of regulators, curing the problem about the polynomial expansion of the flow equation. Such a choice is that of a diagonal regulator, as in eq. (3.43). We chose this regulator to be constant, i.e. $\mathcal{R}_k^p = \mathcal{R}_k^q = \mathcal{R}$ a multiplicative operator (recall that we are assuming $\omega^2 = 1$ therefore even if $\mathcal{R}_k^p$ and $\mathcal{R}_k^q$ have different dimensions we can set them equal if we assume their ratio to be some power of ω). We also introduced a UV cutoff Λ in order to control the convergence of the flow for $\mathcal{R} \to \infty$. As a result we observed that, for such a constant regulator, Λ can be removed only after the integration of the flow from $\mathcal{R} = \infty$ to $\mathcal{R} = 0$. The resulting flow equation in the LHA is

$$\partial_{\mathcal{R}} H_{\mathcal{R}} = -\frac{1}{\pi} \arctan\left(\frac{\Lambda}{\mathcal{R}}\right) + \frac{2\mathcal{R} + \partial^2_{\bar{p}\bar{p}} H_{\mathcal{R}} + \partial^2_{\bar{q}\bar{q}} H_{\mathcal{R}}}{2\pi \mathcal{D}_{\mathcal{R}}} \arctan\left(\frac{\Lambda}{\mathcal{D}_{\mathcal{R}}}\right) \tag{3.43}$$

where we defined

$$\mathcal{D}_{\mathcal{R}} = \sqrt{\mathcal{R}^2 + \mathcal{R}\left(\partial^2_{\bar{p}\bar{p}} H_{\mathcal{R}} + \partial^2_{\bar{q}\bar{q}} H_{\mathcal{R}}\right) + \det H^{(2)}_{\mathcal{R}}}\,.$$

In this scheme good estimates for the ground state energy E_0 and the energy gap $\Delta E_1 = E_1 - E_0$ can be obtained by simple polynomial truncations. For a bare Hamiltonian which is a polynomial of order n we consider two cases: a truncation with a polynomial of the same order n

and another of order $n+1$. In the latter case we add a suffix $^{+1}$ to the corresponding quantities E_0^{+1} and ΔE_1^{+1}. We give the results obtained by choosing as an initial condition both the Weyl-ordered H_{nW} and the Weyl-uncorrected Hamiltonian H_n:

Bare Hamiltonian	E_0^{exact}	E_0	E_0^{+1}	$\Delta E_1^{\text{exact}}$	ΔE_1	ΔE_1^{+1}
H_{2W}	1/4	0.24936	0.24936	2	1.99871	1.99871
H_2	1/2	0.49989	0.49994	2	1.99867	1.99985
H_{3W}	1/8	0.12492	0.124886	13/4	3.24736	3.24905
H_3	3/4	0.749849	0.74856	9/2	4.4991	4.4939

Tab.1 The ground state energy E_0 and the first energy gap ΔE_1 for the bare Hamiltonians of eqs. (3.37) and (3.42), as computed from the flow eq. (3.43) by means of two polynomial truncations: up to the same order or the bare Hamiltonian and up to the next order ($^{+1}$ superscript).

We note that the quantities E_0 and ΔE_1 depend on the local properties of the effective Hamiltonian at the minimum ($\bar{p}=\bar{q}=0$) and therefore can be extracted with a good approximation adopting simple polynomial truncations. From the table we see that there is no clear pattern on the change of the precision of the results when increasing the order of the truncation. In the worst case we find a relative error of order 10^{-3}. In order to achieve a better accuracy, going to next-to-leading order in the derivative expansion would probably do the job.

We remark that for the first time in the functional RG approach one faces the ordering problem in the choice of the bare Hamiltonian function which corresponds to the initial condition for the flow. This feature generally extends to QFT, therefore one needs to keep it in mind before interpreting the results obtained by choosing an initial condition which is non-separable in p and q.

3.2.4 The average effective Hamiltonian action in fermionic quantum mechanics

Since fermions usually have a first order dynamics, the Hamiltonian formulation of it is identical to the Lagrangian one. Therefore the AEHA formalism in this case is identical to the traditional Lagrangian approach, that was discussed in section 2.2.2 For completeness we will briefly give the real time version of the formulas presented there.

For a system whose Lagrangian variables are n real Grassmann-valued functions of time: $\left\{\theta^i(t)\right\}_{i=1,\dots,n}$, evolving according to the following Lagrangian:

$$L(\theta(t),\partial_t\theta(t)) = \frac{1}{2}\theta^i(t) i\partial_t\theta^j(t)\delta_{ij} - V(\theta^i(t))\,. \tag{3.44}$$

Defining the momenta π_i as the right partial derivatives of L with respect to $\partial_t\theta^i$ we find n second class primary constraints:

$$\chi_i(t) = \pi_i(t) - \frac{i}{2}\delta_{ij}\theta^j(t) = 0 \tag{3.45}$$

which cause the canonical Hamiltonian $H = \pi_i\partial_t\theta^i - L = V(\theta^i)$ to be independent of π_i. Therefore the functional integral over the reduced phase space reads

$$Z = \int [d\theta]\mu[\theta]e^{iS[\theta]} \quad , \quad S[\theta] = \int dt \left[\frac{1}{2}\theta^i\, i\partial_t\theta^j\delta_{ij} - V(\theta^i)\right]. \tag{3.46}$$

Following the same coarse-graining scheme explained in the previous subsections we add the term $\Delta S_k[\theta] = \int dt \left[\frac{1}{2}\theta^i r_k(-\partial_t{}^2)\, i\partial_t\theta^j\delta_{ij}\right]$ to the bare action and we deform the functional measure: $\mu_k = (\mathrm{SDet}\,\frac{\tilde{\sigma}_k}{2\pi})^{1/2} = \mu\ (\mathrm{SDet}(1+r_k)\delta)^{1/2}$, where δ stands for a product of Dirac and Kronecker deltas. Then the modified path integral reads

$$Z_k[J_i] = \int [d\theta]\mu_k[\theta]e^{i\left\{S[\theta]+\Delta S_k[\theta]+J_i\cdot\theta^i\right\}}. \tag{3.47}$$

Starting from it, one defines the AEHA

$$\Delta S_k[\bar{\theta}] + \Gamma_k[\bar{\theta}^i] = \underset{J_i}{\mathrm{ext}}\left(W_k[J^i] - J_i\cdot\bar{\theta}^i\right) \tag{3.48}$$

which satisfy the following integro-differential equation

$$e^{i\Gamma_k[\bar{\theta}^i]} = \int [d\theta]\mu_k[\theta]e^{i\left\{S[\theta]+\Delta S_k[\theta-\bar{\theta}]-\Gamma_k\frac{\overleftarrow{\delta}}{\delta\bar{\theta}^i}\cdot(\theta-\bar{\theta})^i\right\}}. \tag{3.49}$$

and therefore the $k\to\Lambda$ limit of Γ_k is just the bare action. The flow equation for Γ_k reads

$$i\dot{\Gamma}_k = -\frac{1}{2}\mathrm{Tr}\left[\dot{r}_k(1+r_k)^{-1}\delta\right] + \frac{1}{2}\mathrm{Tr}\left[(\dot{r}_k\, i\partial_t\delta)\left(r_k\, i\partial_t\delta + \frac{\overrightarrow{\delta}}{\delta\bar{\theta}}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\bar{\theta}}\right)^{-1}\right]. \tag{3.50}$$

where the trace is over $\{i,j\}$ indices as well and, as in the bosonic case, in the matrix $r_k i\partial\delta$ the derivatives act on the first index.

3.3 The effective Hamiltonian action in quantum field theory

There are at least two possible generalizations of the previous formalism to quantum field theory (QFT).

The simplest can be obtained by embracing the traditional Hamiltonian formulation of field theory, where one associates a canonically conjugate field (momentum) to the time derivative of each Lagrangian coordinate. This choice leads to a non covariant formulation. The translation of all previous formulas to this framework can be obtained by replacing the bare Hamiltonian with the spatial integral of a Hamiltonian density, and promoting the integrals and functional traces to sums over spatial positions as well as time instants. In this way one can obtain a formal definition of the non covariant effective Hamiltonian action and extend all the previous discussions developed in section 2.1.

However, in so doing, willing to construct the corresponding coarse-graining procedure for the flow of the average effective Hamiltonian action, one faces the necessity to regularize the spatial part of these summations, which are otherwise ill-defined. In other words the regulator matrix R_k, appearing in ΔS_k and μ_k, must now contain operators depending on spatial derivatives too. For instance, choosing an off diagonal R_k one could consider

$$\begin{aligned} R_k(x,x') &= \begin{pmatrix} 0 & r_k(-\Box)\partial_0\delta(x-x') \\ -r_k(-\Box)\partial_0\delta(x-x') & 0 \end{pmatrix} \\ \mu_k &= \left[\mathrm{Det}\frac{1}{2\pi}\begin{pmatrix} 0 & (1+r_k(-\Box))\,\partial_0\delta(x-x') \\ -(1+r_k(-\Box))\,\partial_0\delta(x-x') & 0 \end{pmatrix}\right]^{\frac{1}{2}} \end{aligned}$$

but this choice would explicitely break Lorentz symmetry. Instead it would be easy to write more general regulators preserving such a symmetry, even if in an implicit form. In both cases one may study the AEHA defined by the integro-differential equation

$$e^{i\Gamma_k[\bar\pi,\bar\varphi]} = \int [d\pi d\varphi]\,\mu_k[\pi,\varphi] e^{i\left\{S[\pi,\varphi]+\Delta S_k[\pi-\bar\pi,\varphi-\bar\varphi]-(\pi-\bar\pi)\frac{\delta\Gamma_k}{\delta\bar\pi}-(\varphi-\bar\varphi)\frac{\delta\Gamma_k}{\delta\bar\varphi}\right\}}.$$

This road could be useful if one is interested in non-relativistic field theories, but for relativistic systems, since Lorentz invariance is not manifest, in this framework it is hard to distinguish truncations for Γ_k that are Lorentz symmetric from those that are not (one would have to deal with Ward-Takahashi-Slavnov-Taylor identities).

Another possibility is to choose a covariant Hamiltonian formalism, in which one introduces a momentum field for each first order partial derivative of the Lagrangian coordinates, thus preserving manifest Lorentz covariance. In the following we will give the two simplest examples of how this could work: spin zero and spin one half field theories. There are several choices one can do. In this thesis we shall attempt to use a *reduced* approach, which has the advantage of being the minimal extension, which on one side preserves the general results in $0+1$ dimensions (QM) and on the other side leads to the usual QFT results in the case of

quadratic bare Hamiltonians. More general formulations as well as specific applications will be considered elsewhere.

3.3.1 Covariant Hamiltonian scalar field theory

Let us build the covariant Hamiltonian formulation of a classical unconstrained single scalar field in d spacetime dimensions with the standard Lagrangian density

$$\mathcal{L}(\varphi, \partial_\nu \varphi) = -\frac{1}{2} (\partial_\nu \varphi) (\partial^\nu \varphi) - V(\varphi)$$

(in a Minkowski mostly-plus signature). The covariant Hamiltonian density is defined as the extremum

$$\mathcal{H}(\pi^\nu, \varphi) = \underset{\partial_\nu \varphi}{\text{ext}} \left(-\pi^\nu \partial_\nu \varphi - \mathcal{L}(\varphi, \partial_\nu \varphi)\right) = -\frac{\pi_\nu \pi^\nu}{2} + V(\varphi) \tag{3.51}$$

and by demanding the stationarity of the Hamiltonian action

$$S = \int d^d x \, \left[-\pi^\nu \partial_\nu \varphi - \mathcal{H}\right] \tag{3.52}$$

one finds the De Donder-Weyl equations

$$\pi^\nu = \partial^\nu \varphi \quad , \quad \partial_\nu \pi^\nu = V'(\varphi)$$

i.e. a first order system equivalent to $\Box \varphi - V'(\varphi) = 0$. Here the dynamics of π^ν and φ seem to be completely coupled, however this is not the case. In fact the Lorentz vector π^ν can be decomposed into a transverse and a gradient part $\pi^\nu = \pi^\nu_\perp + \pi^\nu_\parallel$, by means of the standard projectors $\Pi^{\mu\nu}_\parallel = \partial^\mu \partial^\nu / \Box$ and $\Pi^{\mu\nu}_\perp = \eta^{\mu\nu} - \Pi^{\mu\nu}_\parallel$. Rewriting the Hamiltonian action density in terms of these reduced degrees of freedom (and assuming that the boundary terms coming from integration by parts do not contribute) one finds $-\pi^\nu_\parallel \partial_\nu \varphi - \mathcal{H}$ with

$$\mathcal{H}(\pi^\nu_\perp, \pi^\nu_\parallel, \varphi) = -\frac{\pi_{\perp\nu} \pi^\nu_\perp}{2} - \frac{\pi^\nu_\parallel \pi_{\parallel\nu}}{2} + V(\varphi)$$

and the corresponding Hamiltonian equations

$$\pi^\nu_\parallel = \partial^\nu \varphi \quad , \quad \partial_\nu \pi^\nu_\parallel = V'(\varphi) \quad , \quad \pi^\nu_\perp = 0 \, .$$

Hence the transverse momenta are classically irrelevant if the Hamiltonian is quadratically depending on them. This translates into the following quantum property: if the bare Hamiltonian is separable in π and φ and quadratically depending on π, the functional integration over trans-

verse momenta factorizes from those on the other two fields.

Now let us address the possibility to extend this formalism to covariant Hamiltonian densities that are more than quadratic in the momenta. The classical decoupling of the transverse momenta, i.e. their factorization in the functional integral, can happen also for non-quadratic Hamiltonians, such as for instance $\mathcal{H} = T(\pi^\mu \pi_\mu) + V(\varphi)$. Insisting in the validity of the classical variational principle for the action (3.52) the classical equations read

$$\partial_\nu \pi^\nu_{\parallel} = \frac{\delta}{\delta \varphi} \int d^d x \mathcal{H} \quad , \quad -\partial_\nu \varphi = \frac{\delta}{\delta \pi^\nu_{\parallel}} \int d^d x \mathcal{H} \quad , \quad \frac{\delta}{\delta \pi^\nu_{\perp}} \int d^d x \mathcal{H} = 0 \, .$$

The interesting question now is whether the third equation is a constraint or it gives a dynamics to the transverse momenta. If $\mathcal{H}$ does not contain derivatives of $\pi^\nu_{\perp}$, and if one can perform some sort of Fourier transform such that $\pi^\nu_{\perp}$ can be considered orthogonal to ∂^ν with respect to the metric in Minkowski spacetime, then the third equation cannot contain derivatives of $\pi^\nu_{\perp}$. Therefore, under these assumptions, one can always solve the third equation by writing $\pi^\nu_{\perp}$ as a local (if $\mathcal{H}$ is local) function of $\pi^\nu_{\parallel}$, φ and their derivatives. By substituting this solution in the first two equations one gets a coupled dynamics for the unconstrained variables $\pi^\nu_{\parallel}$ and φ only. That is, under these assumptions the transverse momenta do not have their own independent dynamics and behave only as redundant variables which can be eliminated without loosing the locality of the action. However even in this case the quantization of the theory containing the $\pi^\nu_{\perp}$ fields is not equivalent to the quantization of the theory in which one got rid of them by means of the classical equations, since in the first case one has a full functional integral over $\pi^\nu_{\perp}$, whose stationary phase approximation gives the second quantum theory. Nevertheless, considering an Hamiltonian action depending on parallel momenta only, although it is not the most general case, is already a consistent and covariant generalization of the standard non-covariant Hamiltonian approach, reproducing the known results for quadratic Hamiltonians. Therefore in this thesis we will restrict ourselves to such a case.

The aim of the rest of this section is to give meaning to the quantization of the classical theory with the bare action (3.52) under the assumption that $\mathcal{H}$ depends on $\pi^\nu_{\parallel}$ only. Since in this case the bare action S does not depend on $\pi^\nu_{\perp}$, we are in presence of a gauge symmetry: by introducing projectors where needed, S can be rewritten in a form which is manifestly invariant under the infinitesimal transformation: $\delta \pi^\nu(x) = \Pi^{\nu\rho}_{\perp} \varepsilon_\rho(x)$, for any infinitesimal vector field ε. In this chapter we will discuss the functional integral quantization of the theory by means of the introduction of the constraint $\Pi^{\nu\rho}_{\perp} \pi_\rho = 0$ in the functional measure (something like a

sharp gauge fixing[1]). Thus, the generating functional of the theory will be

$$Z[I_\nu, J] = e^{iW[I_\nu, J]} = \int \left[d\pi^\nu d\varphi\right] \delta\left[\Pi_\perp^{\nu\rho}\pi_\rho\right] \mu\, e^{i\{S[\pi^\nu,\varphi]+I_\nu\cdot\pi^\nu+J\cdot\varphi\}} . \tag{3.53}$$

Notice that, depending on which regularization and precise definition of the functional integral is chosen, the functional integration over $\left[d\pi^\nu_\parallel\right]$ and the constrained integration $\left[d\pi^\nu\right]\delta\left[\Pi_\perp^{\nu\rho}\pi_\rho\right]$ could differ by a field-independent Jacobian determinant. A skeletonized definition in Fourier space, i.e. the use of a discretization of Fourier space, would make this Jacobian to be equal to one. Whenever such a Jacobian is unity, since the constraint kills all but one of the integrals over the π's, the usual functional measure $\mu = \mathrm{Det}\frac{1}{2\pi}$ provides the normalization needed in order to reproduce the known results for bare Hamiltonian actions quadratic in the momenta. Otherwise μ needs to be different (but still field-independent) in order to balance the Jacobian determinant. Starting from eq. (3.53) the definition of the effective Hamiltonian action is again

$$\Gamma\left[\bar\pi^\nu, \bar\varphi\right] = \underset{I_\nu, J}{\mathrm{ext}} \left(W[I_\nu, J] - I_\nu\cdot\bar\pi^\nu - J\cdot\bar\varphi\right) \tag{3.54}$$

which is equivalent to state that Γ is the solution of the following integro-differential equation with suitable boundary conditions

$$e^{i\Gamma[\bar\pi^\nu,\bar\varphi]} = \int \left[d\pi^\nu d\varphi\right] \delta\left[\Pi_\perp^{\nu\rho}\pi_\rho\right] \mu\, e^{i\left\{S[\pi^\nu,\varphi]-(\pi-\bar\pi)^\nu\frac{\delta\Gamma}{\delta\bar\pi^\nu}-(\varphi-\bar\varphi)\frac{\delta\Gamma}{\delta\bar\varphi}\right\}} . \tag{3.55}$$

In the following we shall try to give a definition of the integrals (3.53) and (3.55) based on an RG flow equation for the average version of the effective action. First of all, one has to introduce k-dependent operators that disappear in the $k \to 0$ limit and that provide a rising delta functional in the $k \to \Lambda$ limit. As before let us denote this regularization as follows

$$Z_k[I_\nu, J] = \int \left[d\pi^\nu d\varphi\right] \delta\left[\Pi_\perp^{\nu\rho}\pi_\rho\right] \mu_k\, e^{i\{S[\pi^\nu,\varphi]+\Delta S_k[\pi^\nu,\varphi]+I_\nu\cdot\pi^\nu+J\cdot\varphi\}} .$$

We will choose a regularization corresponding to a k-dependent deformation of the term whose one-dimensional version is the Legendre transform term, i.e. $-\pi^\mu\partial_\mu\varphi$. In other words, we will

[1]Dirac's classification of constraints and the consequent quantization schemes for gauge theories are based on the non-covariant Hamiltonian formalism and therefore are not straightforwardly applicable to the present case. However classical constrained dynamics has been extensively discussed in the literature about the covariant Hamiltonian formalism(s) [57] and some proposals have been provided about the corresponding path integral quantization of gauge theories [65].

restrict to an off-diagonal R_k, or more explicitly

$$\begin{aligned}\Delta S_k[\pi^\nu,\varphi] &= \int d^dx\,[-\pi^\nu r_k(-\Box)\partial_\nu\varphi] \qquad (3.56)\\ \mu_k &= \mu\left[\mathrm{Det}\begin{pmatrix}0 & -\left(1+r_k(-\Box)\right)\partial_\nu\delta(x-x')\\ \left(1+r_k(-\Box)\right)\partial_\nu\delta(x-x') & 0\end{pmatrix}\right]^{\frac{1}{2}}.\end{aligned}$$

The definition of the AEHA is the same as in quantum mechanics

$$\Gamma_k\left[\bar\pi^\nu,\bar\varphi\right]+\Delta S_k\left[\bar\pi^\nu,\bar\varphi\right]=\underset{I_\nu,J}{\mathrm{ext}}\left(W_k[I_\nu,J]-I_\nu\cdot\bar\pi^\nu-J\cdot\bar\varphi\right) \qquad (3.57)$$

wherefrom the usual integro-differential equation

$$e^{i\Gamma_k[\bar\pi^\nu,\bar\varphi]}=\int\left[d\pi^\nu d\varphi\right]\delta\left[\Pi_\perp^{\nu\rho}\pi_\rho\right]\mu_k\,e^{i\left\{S[\pi^\nu,\varphi]+\Delta S_k[(\pi-\bar\pi)^\nu,\varphi-\bar\varphi]-(\pi-\bar\pi)^\nu\frac{\delta\Gamma_k}{\delta\bar\pi^\nu}-(\varphi-\bar\varphi)\frac{\delta\Gamma_k}{\delta\bar\varphi}\right\}}. \qquad (3.58)$$

By taking the $k\partial_k$ derivative of eq. (3.58) one finds

$$i\dot\Gamma_k=\frac{\dot\mu_k}{\mu_k}-i\int d^dx\langle(\pi-\bar\pi)^\nu\dot r_k\partial_\nu(\varphi-\bar\varphi)\rangle\,. \qquad (3.59)$$

For the second term, we need to write the two point function in terms of derivatives of Γ_k. Since this theory contains one Lagrangian coordinate and one momentum, $\Gamma^{(2)}$ is a two-dimensional square matrix, as in quantum mechanics. However, our momentum is a vector field bringing a Lorentz index, and even if it lies in a one-dimensional subspace, such a subspace varies from point to point in spacetime. Thus, unless we want to choose a frame in the tangent bundle such that at every spacetime point x the vector $\pi^\nu(x)$ has only one and the same non-vanishing component, we are forced to deal with it as a generic Lorentz vector. Since we prefer to write formulas in a generic frame, we will treat $\Gamma^{(2)}$ as a generic $(d+1)$-dimensional square matrix, whose entries can be written as four blocks: a $(1,1)$ tensor (d-by-d square matrix), one contravariant (column) vector, one covariant (row) vector, and one Lorentz scalar. Because the momenta enter the theory naturally with high indices (to be contracted with derivatives), we will treat them as column vectors. Therefore the source I will become a row vector. We will denote by $()^t$ the transposition of these objects, that is the canonical isomorphism defined by the spacetime metric. Thus π^t and I^t will denote row and column vectors respectively. Of course derivatives with respect to contravariant (covariant) vectors will be considered covariant (con-

travariant). Going back to the task of computing the two point functions, since

$$i\langle\mathcal{T}\begin{pmatrix}(\pi-\bar\pi)_x\otimes(\pi-\bar\pi)^t_{x'} & (\pi-\bar\pi)_x(\varphi-\bar\varphi)_{x'}\\ (\varphi-\bar\varphi)_x(\pi-\bar\pi)^t_{x'} & (\varphi-\bar\varphi)_x(\varphi-\bar\varphi)_{x'}\end{pmatrix}\rangle_k = W_k^{(2)}{}_{xx'}[I,J] = \begin{pmatrix}\frac{\delta W_k}{\delta I_x}\otimes\left(\frac{\overleftarrow{\delta}}{\delta I_{x'}}\right)^t & \frac{\delta^2 W_k}{\delta J_{x'}\delta I_x}\\ \left(\frac{\delta^2 W_k}{\delta I_{x'}\delta J_x}\right)^t & \frac{\delta^2 W_k}{\delta J_{x'}\delta J_x}\end{pmatrix}$$

one needs an explicit expression for the vector $\frac{\delta^2 W_k}{\delta J\delta I}$ in terms of Γ_k. This can be found by using

$$I_\mu = r_k\partial_\mu\bar\varphi - \frac{\delta\Gamma_k}{\delta\bar\pi^\mu}\quad,\quad J = -r_k\partial_\nu\bar\pi^\nu - \frac{\delta\Gamma_k}{\delta\bar\varphi}$$

thus getting

$$\begin{aligned}W_k^{(2)}{}_{xx'}[I,J] &= \begin{pmatrix}\bar\pi\otimes\left(\frac{\overleftarrow{\delta}}{\delta I}\right)^t & \frac{\delta\bar\pi}{\delta J}\\ \left(\frac{\delta\bar\varphi}{\delta I}\right)^t & \frac{\delta\bar\varphi}{\delta J}\end{pmatrix}_{xx'} = \begin{pmatrix}I^t\otimes\frac{\overleftarrow{\delta}}{\delta\bar\pi} & \left(\frac{\delta I}{\delta\bar\varphi}\right)^t\\ \frac{\delta J}{\delta\bar\pi} & \frac{\delta J}{\delta\bar\varphi}\end{pmatrix}^{-1}_{xx'}\\ = & -\begin{pmatrix}\left(\frac{\delta\Gamma_k}{\delta\bar\pi}\right)^t\otimes\frac{\overleftarrow{\delta}}{\delta\bar\pi} & \left(-r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\varphi\delta\bar\pi}\right)^t\\ r_k\partial\delta+\frac{\delta^2\Gamma_k}{\delta\bar\pi\delta\bar\varphi} & \frac{\delta^2\Gamma_k}{\delta\bar\varphi\delta\bar\varphi}\end{pmatrix}^{-1}_{xx'} \equiv -\begin{pmatrix}A & B\\ B^T & D\end{pmatrix}^{-1}_{xx'}\end{aligned}$$

where $(r_k\partial\delta)_{x_1x_2} = r_k(-\partial^2_{x_1})\partial_{x_1}\delta(x_1-x_2)$ is a Lorentz covariant (row) vector. This matrix is manifestly symmetric with respect to full transposition T of both Lorentz and spacetime-position indeces. Since the building blocks B and B^T are not square matrices, we cannot use formula (3.14). Anyway, if A and $(D-B^TA^{-1}B)$ are non singular this becomes

$$W_k^{(2)}[I,J] = -\begin{pmatrix}A^{-1}+A^{-1}B(D-B^TA^{-1}B)^{-1}B^TA^{-1} & -A^{-1}B(D-B^TA^{-1}B)^{-1}\\ -(D-B^TA^{-1}B)^{-1}B^TA^{-1} & (D-B^TA^{-1}B)^{-1}\end{pmatrix} \tag{3.60}$$

if instead D and $(A-BD^{-1}B^T)$ are non singular, then we can write

$$W_k^{(2)}[I,J] = -\begin{pmatrix}(A-BD^{-1}B^T)^{-1} & -(A-BD^{-1}B^T)^{-1}BD^{-1}\\ -D^{-1}B^T(A-BD^{-1}B^T)^{-1} & D^{-1}+D^{-1}B^T(A-BD^{-1}B^T)^{-1}BD^{-1}\end{pmatrix}. \tag{3.61}$$

The off-diagonal entries of these matrices can be finally plugged into eq. (3.59). Thus, if for

instance A and $(D - B^T A^{-1} B)$ are non singular the final flow equation is

$$
\begin{aligned}
i\dot{\Gamma}_k \;=\; & \mathrm{Tr}\left[\dot{r}_k \left(1 + r_k\right)^{-1} \delta\right] - \mathrm{Tr}\left[\left(r_k \partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{\pi}\delta\bar{\varphi}}\right)\left(\frac{\delta^2\Gamma_k}{\delta\bar{\pi}\delta\bar{\pi}}\right)^{-1}(\dot{r}_k\partial\delta)\right. \\
& \left.\left[\frac{\delta^2\Gamma_k}{\delta\bar{\varphi}\delta\bar{\varphi}} - \left(r_k\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{\pi}\delta\bar{\varphi}}\right)\left(\frac{\delta^2\Gamma_k}{\delta\bar{\pi}\delta\bar{\pi}}\right)^{-1}\left(r_k\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{\pi}\delta\bar{\varphi}}\right)^T\right]^{-1}\right]. \quad (3.62)
\end{aligned}
$$

Here for sake of notational simplicity we dropped the symbols for tensor products and Lorentz transpositions. By means of eq. (3.61) the reader can write down a similar flow equation for the case in which D and $(A - BD^{-1}B^T)$ are non singular.

As an example let's discuss the LHA for a scalar theory enjoying Z_2-symmetry under simultaneous reflections: $\pi^\nu \to -\pi^\nu$, $\varphi \to -\varphi$. In other words, we are going to insert the approximation

$$
\Gamma_k = \int d^d x \left[-\bar{\pi}^\nu \partial_\nu \bar{\varphi} - \mathcal{H}_k\left(\frac{\bar{\pi}^2}{2}, \frac{\bar{\varphi}^2}{2}\right)\right] \quad (3.63)
$$

where $\bar{\pi}^2 \equiv \bar{\pi}^\nu \bar{\pi}_\nu$, in the previous flow equation. In order to project the r.h.s. of the flow equation inside such an ansatz for Γ_k, one usually evaluates it on constant field configurations. This can be done also in the present case, without contradicting the assumption that the momenta $\bar{\pi}^\nu$ be longitudinal, by choosing the Fourier transform of $\bar{\pi}^\nu$ pointing in the same direction of the Fourier variable and being proportional to a delta function. We will denote by $\mathcal{H}_k^{(i,j)}$ the result of differentiating $\mathcal{H}_k$ i-times w.r.t. $\frac{\bar{\pi}^2}{2}$ and j-times w.r.t. $\frac{\bar{\varphi}^2}{2}$. Let us recall the notation already used in quantum mechanics (see eq. (3.34)) for the regulator in the LHA, i.e. $P_k(-\Box) = (1 + r_k(-\Box))^2(-\Box)$. Let us also introduce for convenience the function

$$
\sigma_d(a) = {}_2F_1\left(\frac{1}{2}, 1; \frac{d}{2}; a\right) \quad (3.64)
$$

and the following threshold functional

$$
l_0^d[a, \beta] = \frac{1}{4} v_d^{-1} k^{-d} \int \frac{d^d p}{(2\pi)^d} \frac{\dot{P}_k(p^2)}{P_k(p^2) + k^2 \beta(p^2)} \sigma_d(a(p^2)) \quad (3.65)
$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(\frac{d}{2})$. Then the flow equation for the dimensionful average effective Hamiltonian density can be written

$$
i\dot{\mathcal{H}}_k = 2 v_d k^d \left(l_0^d[a_{\mathcal{H}}, \beta_{\mathcal{H}}] - l_0^d[a_{\mathcal{H}}, 0]\right) \quad (3.66)
$$

where we further defined the dimensionless quantities

$$a_{\mathcal{H}}(p^2) = \frac{P_k(p^2)}{P_k(p^2) + k^2\beta_{\mathcal{H}}} \frac{\bar{\pi}^2 \mathcal{H}_k^{(2,0)}}{\mathcal{H}_k^{(1,0)} + \bar{\pi}^2 \mathcal{H}_k^{(2,0)}} \tag{3.67}$$

$$\beta_{\mathcal{H}} = \frac{1}{k^2}\left[\bar{\pi}^2\bar{\varphi}^2\left(\mathcal{H}_k^{(1,1)}\right)^2 \frac{\mathcal{H}_k^{(1,0)}}{\mathcal{H}_k^{(1,0)} + \bar{\pi}^2\mathcal{H}_k^{(2,0)}} - \mathcal{H}_k^{(1,0)}\left(\mathcal{H}_k^{(0,1)} + \bar{\varphi}^2\mathcal{H}_k^{(0,2)}\right)\right] \tag{3.68}$$

the second of which is not a function of p^2. First of all let us notice that if we make the ansatz that the theory be quadratic in the momenta at every scale, then the vanishing of $\mathcal{H}_k^{(2,0)}$ entails the vanishing of $a_{\mathcal{H}}$ and we recover the Lagrangian flow in the LPA. If instead $a_{\mathcal{H}}$ is non-vanishing, the presence of a p-dependent denominator in the argument of the function σ_d in general makes the analytic computation of l_0^d quite hard. For this reason it is wise to choose the regulator in such a way to kill the p-dependence of all the denominators. In the LHA this can be accomplished by means of the optimized regulator $r_k(p^2) = \left(k/\sqrt{p^2} - 1\right)\theta(k^2 - p^2)$, i.e. $P_k(p^2) = (k^2 - p^2)\theta(k^2 - p^2)$. For such a choice

$$a_{\mathcal{H}}(p^2) = \frac{1}{1 + \beta_{\mathcal{H}}} \frac{\bar{\pi}^2 \mathcal{H}_k^{(2,0)}}{\mathcal{H}_k^{(1,0)} + \bar{\pi}^2 \mathcal{H}_k^{(2,0)}} \tag{3.69}$$

is p-independent and the threshold function for constant argument becomes

$$l_0^d[a, \beta] = \frac{2}{d}\frac{1}{1+\beta}\sigma_d(a)\,.$$

To sum up, for the optimized regulator the flow equation of the LHA reads (after Wick rotation)

$$\dot{\mathcal{H}} = -\frac{4}{d} v_d k^d \frac{\beta_{\mathcal{H}}}{1 + \beta_{\mathcal{H}}} \sigma_d(a_{\mathcal{H}}) \tag{3.70}$$

with $\beta_{\mathcal{H}}$ and $a_{\mathcal{H}}$ given by (3.68) and (3.69). The function σ_d takes simpler forms for integer d. For instance, in $d = 2$, $d = 3$ and $d = 4$ it respectively reads

$$\sigma_2(a) = (1-a)^{-\frac{1}{2}} \quad , \quad \sigma_3(a) = \frac{\operatorname{arctanh}(\sqrt{a})}{\sqrt{a}} \quad , \quad \sigma_4(a) = \frac{2}{a}\left[1 - (1-a)^{-\frac{1}{2}}\right]\,. \tag{3.71}$$

Equation (3.66) can be taken as a first step towards the nonperturbative study of scalar QFT in the covariant Hamiltonian formalism. In particular, one of the first questions to be addressed is whether such an equation admits non-Gaussian fixed points. In case a positive answer exists, these could provide a possible solution to the triviality problem of scalar QFT in four dimensions (see section 5.1 for a detailed discussion of this problem). In fact, choosing the engi-

neering dimensions of the fields in such a way that the coefficients of the $\bar{\pi}^2$ and Legendre terms be dimensionless, dimensional analysis tells us that the coupling multiplying the operator $(\bar{\pi}^2)^i(\bar{\varphi}^2)^j$ has dimensionality $d_{ij} = (1-i-j)d + 2j$. Therefore in $d = 4$ the only momentum dependent non IR-irrelevant term is $\bar{\pi}^2$, all other terms with positive integers (i, j) being IR-irrelevant. In other words, scalar theories more than quadratic in the momenta are expected to be highly favored in the UV and to be well approximated by quadratic theories in the IR. From this point of view it seems reasonable to look for the UV completion of four dimensional scalar QFT in a general Hamiltonian framework. For instance this could be done according to the paradigm of asymptotic safety described in the introducing chapter of this thesis. On the other hand this very same argument in the case of a simpler scalar QFT in configuration space is often used for a qualitative understanding of the absence of Z_2-symmetric non-Gaussian fixed points in $d = 4$: in this case the only IR-relevant monomial-like operator is the mass term, all other monomials being either marginal or IR-irrelevant. Nevertheless, since in the present formulation the theory contains not only a scalar field but also a longitudinal vector field, we believe that the understanding of this issue requires explicit computations in order to reveal the details of the underlying dynamics.

Another interesting question regarding eq. (3.66) is whether it can teach us to what extent the covariant Hamiltonian framework adopted in this chapter is sound and useful. In particular, it would be interesting to compare, within a fixed approximation such as the LHA, the RG flow of the traditional non-covariant Hamiltonian formulation with that of the covariant one allowing for longitudinal momenta only (the present case) and with the one allowing also for transverse momenta. These and other questions will be left open by the present work.

3.3.2 Spinor field theory

Let us build the covariant Hamiltonian formulation of a classical Lagrangian field theory for a single Dirac field in a number d (allowing Dirac spinors) of spacetime dimensions with the standard Lagrangian density

$$\mathcal{L}(\psi, \partial_\nu \psi) = -\bar{\psi} i \partial\!\!\!/ \psi - V(\bar{\psi}, \psi)$$

(in a Minkowski mostly-plus signature) where $\bar{\psi} = i\psi^\dagger \gamma^0$. Defining the momenta π^ν as the right partial derivatives of $-\mathcal{L}$ with respect to $\partial_\nu \psi$ we find d second class primary constraints:

$$\chi^\nu(x) = \pi^\nu(x) - \bar{\psi}(x)\gamma^\nu = 0 \tag{3.72}$$

whose solution is $\bar{\psi} = \frac{1}{d}\pi^\nu \gamma_\nu$. These constraints boil down the momenta to functions of just one field, hence there is no room here for the other $d-1$ conjugate fields that in the bosonic case could be identified with the transverse momenta. The relevant phase space is the surface $\mathcal{S}$ defined by (3.72), the only independent coordinate on it is ψ and the functional integral is to be taken over all histories $\psi(x)$. The covariant Hamiltonian density is defined as

$$\mathcal{H}(\pi^\nu, \psi) = \underset{\partial_\nu \psi}{\text{ext}} \left(-\pi^\nu \partial_\nu \psi - \mathcal{L}(\psi, \partial_\nu \psi)\right) = V(-\frac{1}{d}\pi^\nu \gamma_\nu, \psi)$$

and on $\mathcal{S}$ it is just $V(\bar{\psi}, \psi)$. Thus the covariant AEHA formalism in this case is equivalent to the usual Lagrangian approach, exactly as was previously described for fermionic QM, one has just to replace time derivatives with $i\partial\!\!\!/$ operators.

3.4 Outlook

In this chapter we have focused on the description of quantum dynamics by means of the quantum effective Hamiltonian action (EHA). We have first reviewed its properties by a discussion in quantum mechanics, taking advantage of the fact that QM and non covariant QFT's are very similar in this respect. We have then discussed how to compute the effective action. For instance we have derived a general one loop formula, which can be useful to compare the results obtained by other approaches, and we have generalized the variational definition provided a long time ago by Jackiw and Kerman [66] for its Lagrangian counterpart. But the main goal of this chapter is to provide an alternative non-perturbative tool to compute the EHA. This is an Hamiltonian generalization of the so-called functional renormalization group, in particular of the formulation by Wetterich based on the average effective (Lagrangian) action [8].

Such a generalization, which is one of the main results of this chapter, is straightforward in QM, even if the one-parameter-dependent family of cutoff operators is wider and in general the formulae are more cumbersome. Starting from the most general flow equation we have derived simpler equations like the one associated to the so called local Hamiltonian approximation (LHA), i.e. the leading order in the derivative expansion. In order to show that the approach is trustworthy, we have studied, as an example, a family of quantum mechanical systems with bare Hamiltonians non quadratic in the momenta, we have computed for two cases the ground state energy and the first energy gap, and we have successfully compared them to the exact results, employing different kind of schemes and approximations. We stress that for the models under consideration we needed to take into account, as expected, the issue of Weyl ordering, which turns out to be at the base of the present flow equation quantization as it is well known to be for the functional integral quantization. This fact calls for some care in defining the concept of

a bare non separable Hamiltonian action.

The application of the formalism developed for QM to the QFT case is straightforward and quickly discussed but, as in all Hamiltonian approaches to QFT, one must pay full generality and manifest unitarity with non-manifest Lorentz covariance. This is unpleasant and complicates the job of performing approximations without breaking such a symmetry. For this reason, in the second part of the chapter, we have discussed the possibility to generalize the EHA formalism to include also covariant Hamiltonian QFT. Functional integral quantizations of such theories have already been addressed in the literature, especially for gauge theories. In the present chapter we have addressed the simplest cases of scalar and spinor degrees of freedom. Actually, for scalar QFT we further restricted our work to the presence of one conjugate momentum only, namely a longitudinal vector field. In this specific case we have provided an RG flow equation representation of the corresponding QFT, and we have worked out its explicit form in the LHA.

Let us close this chapter addressing the issue of the physical motivations for the formalism built in it and of its usefulness. Clearly, the use of this framework is related to Hamiltonian systems non quadratic in momenta, therefore we should comment on the question: where are them or why should we look for them?

Quantum mechanical systems more than quadratic in the momenta may be interesting on the base of first principles (think about the action of the free relativistic particle) or arise as effective descriptions of physical systems. Also, they could appear as intermediate technical tools for the description of more complicated systems. For instance, within the worldline formalism, one-loop computations are reduced to quantum mechanical path integrals with Hamiltonians which sometimes are non-quadratic in the momenta [68]. In these cases one can hope to use this approach as an alternative or a complementary tool to perturbation theory.

Theories more than quadratic in the momenta, when reduced to the Lagrangian formulation, show a nonlinear dependence on the derivatives of the fields. This dependence, if expanded in powers and truncated, typically generates violations of unitarity. Nevertheless before truncation nothing prevents such theories from being unitary. That is, there might be some interesting non trivial extensions of quantum models which are non-quadratic in the momenta and that make perfectly sense from a quantum mechanical point of view.

Why should we look for them? As already commented at the end of the section on scalar QFT, the study of the RG flow on the Hamiltonian theory space might show new possibilities for the UV or IR behavior of systems that at some intermediate scale are well approximated by simple Lagrangian theories. Stated in different words, keeping both phase space variables could make easier the task of parameterizing the quantum dynamics far from that intermediate

simple Lagrangian scale. One reason for such an expectation is the following: we know that the effective actions are in general non-local, and that integrating out non-Gaussian degrees of freedom is responsible for such non-localities, therefore avoiding to integrate out the momenta should be of help in the hard task of reducing as far as possible the importance of non-local interactions. Restated one more time: even by studying the running of approximate local actions on the Hamiltonian theory space one can, just by putting the momenta on-shell, have access to at least part of the running of non-local actions in the Lagrangian theory space. For these reasons also the study of theories whose bare actions are quadratic but that flow to AEHA's more than quadratic in the momenta could benefit from this first order formulation. Examples are the covariant Hamiltonian formulation of Yang-Mills theory and generic nonlinear sigma models, which in our opinion deserve future investigations within the present framework.

The analysis of Hamiltonian flows might open the intriguing possibility of finding systems belonging to new universality classes, by looking for fixed points of the flow in the Hamiltonian formulation. We have started to consider this challenging problem within the "reduced" covariant formulation of scalar QFT presented in this chapter, and we hope to report on this soon. The results of all these studies will in general depend on the kind of Hamiltonian formulation we choose, a fact that enables one to quantitatively compare different quantization prescriptions as well as to look for physical systems described by each of them. Thus, in our opinion, a vast playground lies open, waiting for future investigations.

4

Scalar O(N) models

An Hamiltonian representation of both linear and nonlinear O(N) models is discussed, generalizing the treatment of a single scalar field theory presented in the previous chapter within the Lorentz covariant approach. This representation preserves the manifest covariance under all the linear symmetries of the models, Lorentz included, while invariance under infinitesimal nonlinear transformations is ensured for all the truncations satisfying a set of modified Ward-Takahashi identities. The problem of allowing for global effects in target space is reviewed and then discussed within this representation. The guiding lines of this ongoing research as well as the motivations and the expectations are briefly described.

4.1 The linear and the nonlinear model

O(N) models are field theories possessing the following symmetries: the Lorentz group SO(1,$d-1$) in spacetime, or the rotation group SO(d) in space, translations in spacetime, and a global inner O(N) symmetry in the target manifold. The scalar version of these models is obtained allowing only trivial representations of the Lorentz group on configuration space, i.e. assuming that configuration space is coordinatized by scalar fields. Different scalar O(N) models can be obtained by choosing different representations of the O(N) symmetry. Two choices

are the most studied in the literature. The name "linear O(N) model", or O(N) vector model, is usually reserved to a linear irreducible representation of the full O(N) group. In this case the target manifold can be imagined as a Euclidean vector space. Given a set of global coordinates $\{\gamma^a\}_{a=1,\dots,N}$ on $\mathcal{M}$ we define N scalar fields $\varphi^a(x) = \gamma^a(q(x))$, which we collect for convenience in a vector $\boldsymbol{\varphi}$. The elements of the symmetry group are represented by $N \times N$ orthogonal matrices and the action of the group on configuration space is represented by matrix multiplication between these matrices and the vector $\boldsymbol{\varphi}$. The name "nonlinear O(N) model", usually refers to a representation that is linear and irreducible for the subgroup O(N−1) and nonlinear for the coset O(N)/O(N−1).

What is known about such representations? Given a compact, connected, semisimple Lie group G of dimension d_g and a continuous subgroup H of dimension d_h, the classification of the nonlinear representations of G that become linear when restricted to H was shown in [69] to be equivalent to the classification of the representations of G on a d_g-dimensional manifold $\mathcal{F}$ such that H is the stabilizer group of some reference point $f_\star$ on $\mathcal{F}$. Calling ρ the action of G on $\mathcal{F}$, since $\rho(H)$ is the stabilizer of $f_\star$, it is possible to choose local coordinates for $\mathcal{F}$ in a neighborhood of $f_\star$ such that ρ restricted to H is linear. Furthermore, one can restrict ρ to a specific submanifold of $\mathcal{F}$ and still have a well defined representation. This submanifold is the orbit of $f_\star$ under $\rho(G)$, it has dimension $d_g - d_h$ and we will call it $\mathcal{M}$. One can provide local coordinates for $\mathcal{M}$ in a neighborhood of $f_\star$ introducing a Cartan decomposition of the Lie algebra into $\{t_i\}_{i=1,\dots,d_h}$ generators of H and $\{t_a\}_{a=d_h+1,\dots,d_g}$ generators of the right coset G/H, the two sets being orthogonal with respect to the Cartan inner product, and taking advantage of the exponential map. The local coordinates of $m \in \mathcal{M}$ are called $\varphi^a(m)$ and are defined by $m = \rho\left(e^{i\varphi^a(m)t_a}\right) f_\star$. These coordinates transform nonlinearly under G according to: $\varphi \to \varphi'(\varphi, g)$ where $\rho(g)m = e^{\varphi'^a(m)t_a} f_\star$. In conclusion, the problem of building a nonlinear representation of G that becomes linear when restricted to H leads one to choose a target manifold $\mathcal{M}$ that locally can be built as the orbit of some reference point $f_\star$ under the action of G/H.

If we restrict ourselves to the special groups, in the present case G =SO(N), H =SO(N−1) and $G/H \approx S^{N-1} = \mathcal{M}$. Therefore one could be lead to take as an equivalent definition of scalar nonlinear SO(N) model: a scalar field theory whose target manifold is the N−1 sphere. In this case, nonlinear O(N) models would fall in the wider category of the nonlinear sigma models [70, 71], i.e. field theories whose target is a smooth Riemannian manifold. One reason to question the equivalence of the two definitions is related to the possibility to extend the arguments of [69] beyond the need for a reference point in target manifold and the confinement inside its neighborhood. It often happens in field theory that local expansions in powers of field fluctuations are equivalent to small coupling expansions, therefore the extension be-

yond a local analysis is often considered nonperturbative. In the literature many such analyses have been performed, revealing important global effects, such as instantons, in field theories defined on topologically nontrivial manifolds. Therefore, regarless the possible inequivalence of the two definitions, it is very interesting to allow for global effects in such models.

On the other hand, the emphasis on the presence of a reference point is understood on the basis of the development of such models for the description of the dynamics of Nambu-Goldstone bosons. In fact, spontaneous symmetry breaking is often described as the formation of a nonvanishing vacuum expectation value (vev) for the fields, that is not invariant under the full global symmetry G but only over a subgroup H. Then, such a vev provides a reference point in the target manifold $\mathcal{F}$ of the theory describing the symmetric regime, and applying the construction cited above one can formulate a new field theory for the broken regime, in a new target manifold $\mathcal{M}$ on which new fields, the Nambu-Goldstone bosons, take values. The presence of such a reference point does not clash with the nonperturbative request of a global and geometric definition for the model: since the real configuration space of the theory is a set of histories taking values on the sphere, the fact that the average history be sitting at some point $f_\star$ for any $x \in \mathcal{S}$ does not break the freedom to change coordinates on the sphere.

Therefore one good phenomenological motivation for addressing the study of linear and nonlinear O(N) models altogether is: the same physical system in two different regimes can be described by these two different models. Another very good reason is related to the hypotesis that these two systems belong to the same universality class. This is just a particular case of the general hypothesis that two short-range theories in the same spacetime dimension and with the same symmetries belong to the same universality class. For the case of the O(N) models, many studies performed with many different methods suggest that this is indeed the case [72, 73]. This is why in this chapter we will formulate the two theories in the most similar way, and try to study both of them within the same framework, by means of the fRG.

4.2 fRG and the global definition of the nonlinear model

Our analysis will take into account the different kind of representations of the O(N) symmetry in the two cases, but it will not address the peculiar nontrivial topological properties of the nonlinear model. In fact in the fRG approach it is far from obvious how to properly take into account global effects. The problem about kinks in QM [74, 75, 76] is just a soft version of the challenge provided by topologically nontrivial sectors of quantum field theory excitations. Traditionally, nonlinear sigma models such as the O(N) or the $\mathbb{C}\mathrm{P}^N$ model have been reference models for the study of these aspects, and were considered toy models for the understanding of Yang-Mills theory [77]. Therefore it would be very important to be able to reproduce the

known global effects by fRG computations.

4.2.1 The geometric background field method

All the studies of nonlinear sigma models performed so far within the renormalization flow representation [17, 22, 78, 79, 80, 81, 82] were based on the traditional background field method (BFM) for the construction of the geometric effective action [18, 19, 83], that from here on we will call geometric BFM (gBFM). In this framework, one starts with a reference point $f_\star$ on $\mathcal{M}$ and couples external sources to the derivative w.r.t. $f_\star$ of the geodesic displacement from this point. Since the geodesic displacement is well defined only locally, because of the existence of caustics, the resulting theory will also be restricted to a local chart, regardless of the representation chosen for the quantum dynamics (functional integrals, Schwinger-Dyson equations, RG flow equations or anything else), unless one adds further information as we will explain soon. On the other hand, the main advantage of this method is the fact that any variable appearing in it has a well defined geometrical, though local, meaning, independent of the choice of local coordinates. One negative aspect of this method is the appearance of two external sources, the one coupled to the gradient of the geodesic displacement and the reference point itself. This property translates in the doubling of the theory space in a fRG framework, with a consequent doubling of complexity in the choice of approximation schemes and in the computations. To understand how variations in the choice of the reference point can be undone by variations in the choice of the other source, or equivalently by variations in the field vev, one needs to work out the so called Nielsen identities [19] which, if solved together with the dinamical equations, again reduce the dimension of theory space by a factor 2. Another way to prevent this doubling proposed by B. DeWitt is the identification of the reference point with the vev of the quantum field, but this leads to a definition of the quantum effective action again by means of two entangled equations, thus conserving the double-complexity spell. Apart for these limitations, the gBFM has been fruitfully applied in the fRG framework to nonlinear sigma models, gauge theories [84] and gravity [85]. Let us also mention that attempts to get global information from the geometric gBFM has been started for nonlinear sigma models in [82]

Motivated by the monopolizing success of this method, and by some apparent lack of success in the computation of the critical exponents for the O(N) nonlinear model [81], we would like to develop here and alternative approach for the fRG study of the latter. The goal we put ahead of us is to compare the results with the ones obtained by the gBFM, by ordinary perturbation theory, lattice simulations, the conformal field theory operatorial approach, thermodinamic Bethe ansatz, spin wave theory and all other available methods. This approach is based on the phase space formulation discussed in the previous chapter and so far shares all the negative

qualities and lacks the main advantage of the gBFM. It lacks manifest invariance under coordinate transformations on the target $\mathcal{M}$, since the sources are coupled to the coordinates of the points in $\mathcal{M}$ and of the covectors in $\mathcal{T}^*\mathcal{M}$. It doubles the theory space because it adds sources also for the momenta. In order to ensure the invariance under the nonlinear global symmetry one also needs to deal with further constraints called Ward-Takahashi identities (WTI), which are to be solved together with the dinamical equations. Nevertheless, it could have some other qualities, a possibility which will be discussed in the following sections.

For example, the breaking of general coordinate transformations by means of the sources in principle does not forbid to give a geometrical interpretation to the formulation and to ensure general coordinate transformation invariance in a non-manifest way, by means of some WTI. All one needs to this end is that the theory admits a geometrical interpretation in absence of sources. For instance, in a functional integral representation this requires that both the bare action and the functional measure be invariant under local coordinate transformations. Also, in case the theory is globally well defined in absence of sources, one should be able to reproduce global effects even if the source terms are local. This is again what happens for example in the ordinary (i.e. not gBFM-based) functional integral formulation of nonlinear sigma models, where the functional measure is postulated to contain the appropriate dependence on the metric in target space, while it does not happen in the standard gBFM formulation, where one either integrates over the tangent space at $f_\star$ or on a local patch of $\mathcal{M}$ bounded by the divergence of the van Vleck-Morette determinant. One the other hand, one could try to build a different gBFM by postulating that the van Vleck-Morette determinant should not appear in the functional measure, i.e. by starting from a globally defined functional integral representation and *then* introducing in it the source and the reference point, instead of following the original formulation that *starts* from the introduction of the source and ends in a functional integral.[1] We will try to provide arguments for both these qualities, general coordinate transformation invariance and global definition on $\mathcal{M}$, to be possibly enjoyed by some phase space formulations. We will also propose a phase space functional integral formulation for nonlinear sigma models with a functional measure that, though field independent, enjoys manifest Lorentz invariance, and we will deduce from it a corresponding RG flow equation. Even if the previous qualities turned out not to be present in this approach, yet at least we could say that the Hamiltonian formulation allows for applications to Hamiltonians that are more than quadratic in the momenta.

As a final note about the comparison between the background field and the phase space

[1] In other words, it seems to us that the traditional arguments in favor of the equivalence between the Schwinger variational principle and the Feynman integral formulation, such as for example a functional Fourier transform representation of the solution of the Schwinger-Dyson equations, apply only locally.

methods, let us stress a trivial fact: the two ideas do not exclude each other. In other words, one could apply the gBFM to a phase space formulation, introducing three different external fields: a reference point $f_\star$, a source for the derivative of the geodesic displacement w.r.t. $f_\star$ and an extra source coupling to a new composite field, depending on $f_\star$ and the phase space variables. The requirements on this new composite field are clear: whenever the target metric is trivial (locally or globally) it must reduce to the conjugate momenta, it must be a scalar at the point $\mathcal{M}$ corresponding to the position of the fluctuating Lagrangian variable, it must be a vector at $f_\star$, just as the corresponding source I. Since by construction the conjugate momenta are vectors at the point set by the Lagrangian variable, in order to buid vectors at $f_\star$ out of them we need to transport them there in a covariant way. Therefore the new composite field can be chosen as the contraction of the momenta with the geodesic parallel displacement bivector at the two points mentioned above. Such a construction will not be explored in these pages, being many steps further in complexity and because we don't see strong motivations to develop it.

4.2.2 A geometric definition for phase-space flows?

We intend now to give a general definition of the setting we are going to investigate. For this it is useful to start with simple and widely accepted notions and see how they relate to the new formulation. Let's then go back to QM. Since we would like to lay down a single framework for both linear and nonlinear models, we will assume the target $\mathcal{M}$ to be a generic Riemannian manifold of dimension D. In drawing the general strategy we will try to preserve a geometrical definition for the model, the main reason being that, as we will see when we will explicitely build the nonlinear representation of the O(N) symmetry, invariance under reparameterizations of the target manifold automatically ensure invariance under this global symmetry, because of the arguments recalled above.

Given a coordinate system $\{\gamma^a\}_{a=1,\dots,D}$ in $\mathcal{M}$ we have a corresponding frame $\{\partial_a\}_{a=1,\dots,D}$ in $\mathcal{T}\mathcal{M}$ and the dual frame $\{d^a\}_{a=1,\dots,D}$ in $\mathcal{T}^*\mathcal{M}$. We will denote with g_{ab} the corresponding metric components. Under changes of coordinates $\gamma^a \to \gamma'^a = (\tau \circ \gamma)^a$, where τ is the transition function, the frames change according to $\partial'_a = \frac{\partial \gamma^b}{\partial \gamma'^a}\partial_b$ and $d'^a = \frac{\partial \gamma'^a}{\partial \gamma^b} d^b$, and correspondingly the metric components $g'_{ab} = \frac{\partial \gamma^c}{\partial \gamma'^a}\frac{\partial \gamma^d}{\partial \gamma'^b} g_{cd}$. Let us recall that the Lagrangian variable q is a map from spacetime $\mathcal{S}$ (just time in QM) to $\mathcal{M}$ and that the conjugate momentum p is a map from $\mathcal{S}$ to $\mathcal{T}^*\mathcal{M}$, such that $p(x) \in \mathcal{T}_{q(x)}\mathcal{M}$. Then we define D scalar fields for q by $\varphi^a(x) = \gamma^a(q(x))$ and D scalar fields for p by $p(x) = \pi_a(x) d^a|_{q(x)}$. Under changes of coordinates these fields transform to $\varphi'^a(x) = \gamma'^a(q(x)) = \tau^a(\varphi(x))$ and $\pi'_a(x) = \pi_b(x)\frac{\partial \gamma^b}{\partial \gamma'^a}|_{q(x)}$. Therefore, if the functional integral is an integral over p and q, a geometrical well defined coordinate representation of it does not require any nontrivial measure, because the Jacobian de-

terminant linking $[d\pi_a d\varphi^a]$ to $[d\pi'_a d\varphi'^a]$ is bound to be field independent in any regularization scheme preserving the geometrical interpretation. This is one main advantage of the phase space path integral in QM: the functional measure does not explicitely depend on the target manifold, it does only implicitely through the topological properties and possibly the boundary conditions.

Such a property is enjoyed also by the phase space path integral in the traditional Hamiltonian formulation of QFT. On the other hand such an advantage is overcome by the negative aspect that Lorentz symmetry is not manifest. Therefore in the literature it is much more common to find studies of nonlinear sigma models based on the Lagrangian functional integral, which has a nontrivial functional measure given by the formal expression $\left(\mathrm{Det} g_{ab}(\varphi(x))\right)^{\frac{1}{2}}$. In those regularization schemes for which such an expression is nonvanishing, one has to add the logarithm of it as an extra dynamical sector in the bare action, or to introduce auxiliary fields to translate this sector into a local ghost sector. Our philosophy in dealing with this issue will be trying to have both manifest Lorentz invariance and a trivial functional measure by embedding the Lorentz-non-covariant Hamiltonian formulation into a Lorentz covariant one.

Has we have seen in the previous chapter, in the covariant Hamiltonian formalism one needs many conjugate momenta, for the contraction with a gradient in spacetime. In other words, the covariant Legendre transform term will be $\pi^\nu_a(x)\partial_\nu\varphi^a(x)$. Thus, for a D dimensional target our momenta will contain a total of $D \times d$ real d.o.f., the geometrical interpretation of which is simple. The push forward of the vectors ∂_μ in $\mathcal{TS}$ by q gives the vector $q_*(\partial_\mu) = \partial_\mu\varphi^a(x)\partial_a|_{q(x)}$, such that we have the transformation rule: $\partial_\mu\varphi'^a(x) = \partial_\mu\varphi^b(x)\frac{\partial\gamma'^a}{\partial\gamma^b}|_{q(x)}$. Then we could think about the conjugate momentum as a map from $\mathcal{S}$ to d copies of the cotangent space, in other words as d maps, labelled by an index ν, associating to each point $x \in \mathcal{S}$ a corresponding covector $p^\nu(x) = \pi^\nu_a(x)d^a|_{q(x)}$. As a consequence of this interpretation, for each ν the transformation rule under changes of coordinates will be: $\pi'^\nu_a(x) = \pi^\nu_b(x)\frac{\partial\gamma^b}{\partial\gamma'^a}|_{q(x)}$. The covariant Legendre transform term can be intepreted in two equivalent ways as the contraction between a vector and a covector: either $p^\nu(x) \in \mathcal{T}^*\mathcal{M}|_{q(x)}$ and $q_*(\partial_\nu) \in \mathcal{TM}|_{q(x)}$ or $q^*(p^\nu(x)) \in \mathcal{T}^*\mathcal{S}|_x$ and $\partial_\nu \in \mathcal{TS}|_x$. This reduces to the one explained above for QM if $d \to 1$. So far we have not constrained the conjugate momenta, therefore we have a larger number of p's than q's. This could be expected to create unpleasant features and in fact it does: if we had to build a functional integral for these variables, the corresponding functional measure either would not make any geometrical, i.e. coordinate independent, sense or it would be field dependent. This is because under a change of coordinates $[d\pi^\nu_a] \to [d\pi'^\nu_a]\left(\mathrm{Det}\frac{\partial\gamma'^a}{\partial\gamma^b}|_{q(x)}\right)^{\frac{d}{2}}$ while $[d\varphi^a] \to [d\varphi'^a]\left(\mathrm{Det}\frac{\partial\gamma^b}{\partial\gamma'^a}|_{q(x)}\right)^{\frac{1}{2}}$.

There are at least two ways to solve this problem: the first is reducing the number of momenta and the other is adding more variables transforming in the right way to balance the mo-

menta in excess. Since we want to preserve both Lorentz and general coordinate invariance, the first choice requires a projector preserving both symmetries. Of course the projection studied in the first chapter, on momenta longitudinal with respect to the operator ∂_μ, cannot be consistently required to hold in every coordinates system. A possible solution would be to replace ∂_μ with the covariant derivative operator ∇_μ depending on the metric g_{ab} and on the derivatives $\partial_\nu\varphi^a$. This would entail a differential constraint on π^ν_a depending on the field φ^a, and we are not eager to deal with such a problem, though it could be that this projection turned out to be safe and the computations possible. On the other hand, such a projection would not be justified by the same argument that led us to the one based on ∂_ν in the previous chapter: the covariant Legendre transform term couples both ∇-longitudinal and ∇-transverse momenta to φ^a. Anyway one could consider the possibility to ask the momenta to be ∂-longitudinal only in one coordinate system, everywhere on $\mathcal{M}$, i.e. for any $\varphi^a(x)$ in this frame. This projection would not be covariant w.r.t. reparameterizations of $\mathcal{M}$, but it would not break such a symmetry. Also, in a generic coordinate system the momenta, though not longitudinal, would comprehend only D d.o.f. and therefore the functional measure would be field-independent.

Now let us turn to the second possibility. This choice is inspired by the example of the quantum mechanical case: having two dual set of vectors, transforming in opposite ways under changes of coordinates, should give us a chance to build invariant actions and measures. Thus, let us consider a theory with $2(D \times d)$ d.o.f.: half of them grouped in a momentum field π^ν_a transforming as a contravariant vector under Lorentz transformations and as a covariant vector under changes of coordinates in $\mathcal{M}$, the other half of them belonging to a vector field v^a_ν that is covariant under Lorentz and contravariant under changes of coordinates. What happened to the old field φ^a, taking values on $\mathcal{M}$ and not on $\mathcal{TM}$? Is there a way to extract it from v^a_ν? The usual procedure to extract a scalar from a vector field is to look at the longitudinal component of the vector: $v^a_{\parallel\nu} = \partial_\nu\varphi^a$. In any coordinate system, we are free to split any a-th component of v into longitudinal and transverse parts w.r.t. ∂_ν, but these parts have no well defined geometrical meaning. Therefore, for the definition of $v^a_\parallel$ we have again two choices: either we define it by the operator ∂ in one specific and preferred frame or we replace ∂ with ∇. In the first case we ask it is possible to find a coordinate system such that for any a-th component

$$v^a_\nu(x) = \partial_\nu\varphi^a(x) + v^a_{\perp\nu}(x) \quad , \quad \partial^\nu v^a_{\perp\nu}(x) = 0 \tag{4.1}$$

with φ^a having the same old meaning. In this particular coordinate system we can write:

$$\pi^\nu_a v^a_\nu = \pi^\nu_{\parallel a}\partial_\nu\varphi^a + \pi^\nu_{\perp a} v^a_{\perp\nu} \tag{4.2}$$

such that the extra $D \times (d-1)$ d.o.f. couple to a part of the momentum vector that does not contribute to the covariant Legendre transform term. Notice that, because of the contraction of latin indices, inside $\pi^{\nu}_{\|a}$ there are still $D-1$ d.o.f. not contributing to such term, but this is already true in QM, hence we don't worry about it. In a different coordinate system $v^a_{\|\nu} \neq \partial_\nu \varphi^a$ and the reasoning does not apply any more, but still $v^a_\nu - \partial_\nu \varphi^a$ must have only $D \times (d-1)$ independent components. In case we choose a definition of the longitudinal component based on the covariant derivative, the fact that transverse components do not contribute to the covariant Legendre transform term holds only locally. Thus, our options are essentially the same we met in trying to project on one momentum only. In fact, if the bare action is chosen to be linear in the extra variables $v^a_{\perp\nu}$, i.e. if the bare Hamiltonian does not dependent on them, they can be integrated out thus constraining the conjugate momenta to be longitudinal.

In this chapter we will restrict our analysis of nonlinear models to a piece of the manifold $\mathcal{M}$, therefore we will ignore the issue of a global definition of the longitudinal fields. Thus, the general framework we are going to address is essentially the same that we already developed in the previous chapter, in which phase space is parametrized by D scalar fields φ^a and a vector field π^ν_a longitudinal w.r.t. ∂_ν.

4.2.3 Detailed construction of the nonlinear model

A unitary linear representation of the global O(D) symmetry with generators $(t^c)^a{}_b$ acts on the fields through multiplication by the generic matrix $U = e^{i\varepsilon_c t^c}$

$$\begin{aligned} \varphi'^a(x) &= U^a{}_b \varphi^a(x) = \varphi^a(x) + i\varepsilon_c (t^c)^a{}_b \varphi^b(x) + O(\varepsilon^2) \\ \pi'^\nu_a(x) &= U^{\dagger b}{}_a \pi^\nu_b(x) = \pi^\nu_a(x) - i\varepsilon_c (t^c)^b{}_a \pi^\nu_b(x) + O(\varepsilon^2) \end{aligned}$$

in such a way that the covariant Legendre transform term is invariant. Let us stress that since this transformation is global, we are free to interpret it in two different ways: active and passive. By active transformation we mean here a rotation of the histories $q(x)$ and $p(x)$ leaving the coordinate system unaltered, while by passive we mean a rotation of the coordinates leaving the histores unchanged. This double possibility remains true also for nonlinear representations, as long as the symmetry is global. If no other global symmetry is present (apart for Poincaré), we can think about $\mathcal{M}$ as a D-dimensional vector space and by setting $D = N$ we get the linear O(N) model. A regularization ΔS_k preserving such a symmetry can be easily built and thus the invariance of the theory under the previous two simultaneous transformations holds for any k.

In the case of the O(N) nonlinear model, the manifold $\mathcal{M}$ must be a sphere of dimension $D = N - 1$, the previous symmetry is imposed and preserved by suitable regulators for any

k, regardless of the choice of coordinates on the sphere, but must be augmented by a nonlinear realization of $\mathrm{O(N)/O(N-1)}$. Following the general arguments recalled in the previous section, this realization can be built by choosing a reference point $f_\star$ on $\mathcal{M}$, by considering the action of $\mathrm{O(N)}$ onto a larger manifold and projecting it on $\mathcal{M}$ in such a way that the corresponding linear representation of $\mathrm{O(N-1)}$ be the stabilizer of $f_\star$. The standard way to do this for $\mathrm{O(N)}$ models is to consider a linear representation in N-dimensional Euclidean space, parameterized by Cartesian coordinates $\{X^i\}_{i=1,\dots,N}$. Without loss of generality one can assume $f_\star$ to be the "south pole" with coordinates $X^N = -R < 0$, $\{X^a = 0\}_{a=1,\dots,N-1}$, such that the stabilizer group is given by rotations of $\mathrm{O(N-1)}$ leaving the N-th direction unaltered. The submanifold $\mathcal{M}$ is chosen as the $D = (N-1)$-dimensional sphere of radius R such that on it $(X^N)^2 = R^2 - \sum_{a=1}^{N-1}(X^a)^2$. The $\{X^a\}_{a=1,\dots,D}$ can be considered independent coordinates on the sphere, confined to a neighborhood of $f_\star$. From the simple law $\delta X^i = i\varepsilon_l (t^l)^i{}_j X^j$ one sees that if the indices $i, j < N$ (i.e. if we restrict it to the stabilizer of $f_\star$) the transformation of the independent coordinates $\{X^a\}_{a=1,\dots,D}$ is linear. On the other hand, if we set first $i < N, j = N$ and then $i = N, j < N$ (i.e. if we restrict it to $\mathrm{O(N)/O(N-1)}$) we have nonlinear transformations

$$\delta X^a = i\varepsilon_l (t^l)^a{}_N X^N \equiv \varepsilon'^a X^N \quad , \quad \delta X^N = i\varepsilon_l (t^l)^N{}_a X^a \equiv -\varepsilon'_a X^a$$

where ε' is real and X^N is a function of $\{X^a\}_{a=1,\dots,D}$. Such transformations preserve the identity $(X^N)^2 = R^2 - \sum_{a=1}^{N-1}(X^a)^2$. Choosing any other coordinate system $\{\gamma^a\}_{a=1,\dots,D}$ in a neighborhood of $f_\star$, the representation for it can be obtained from the one for the Cartesian coordinates $\{X^a\}_{a=1,\dots,D}$ by writing X^a on the sphere as functions of γ^a. For example, in stereographic coordinates from the point opposite to $f_\star$ (the "north pole")

$$\gamma^a = X^a \frac{R}{R - X^N}$$

one gets

$$\delta\gamma^a = -\varepsilon^a \frac{R}{2}\left(1 - \frac{\delta_{bc}\gamma^b\gamma^c}{R^2}\right) - \gamma^a \frac{\varepsilon_b \gamma^b}{R}\,.$$

and an identical transformation holds for the fields $\varphi^a(x) = \gamma^a(q(x))$, and for rotations of q at γ^a fixed. Since we can look at rotations of histories as rotations of coordinates, the geometrical interpretation of the conjugate momenta fixes their nonlinear transformation under $\mathrm{O(N)/O(N-1)}$. Its representation acts as a change of frame by a matrix

$$\frac{\delta\gamma'^a}{\delta\gamma^b} = \delta^a{}_b \left(1 - \frac{\varepsilon_c\gamma^c}{R}\right) + \frac{\varepsilon^a\gamma_b - \gamma^a\varepsilon_b}{R}\,.$$

Therefore in conclusion the fields in phase space transform under $\mathrm{O(N)/O(N-1)}$ according to

$$\begin{aligned}\delta\varphi^a(x) &= -\varepsilon^a\frac{R}{2}\left(1-\frac{\boldsymbol{\varphi}^2(x)}{R^2}\right)-\varphi^a(x)\frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\varphi}(x)}{R}\\ \delta\pi^\nu_a(x) &= \frac{1}{R}\left(\varepsilon_a\boldsymbol{\varphi}(x)\cdot\boldsymbol{\pi}^\nu(x)+\pi^\nu_a(x)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varphi}(x)-\varphi_a(x)\boldsymbol{\varepsilon}\cdot\boldsymbol{\pi}^\nu(x)\right)\end{aligned} \tag{4.3}$$

where bold symbols denote vectors collecting the D components of the fields labeled by latin indices, and the dot denotes contraction of these indices by means of the trivial metric δ_{ab} As it should, the Legendre transform term is invariant under these transformations.

As we anticipated in the previous sections, we are going to formulate the dynamics of both the linear and the nonlinear model by means of the same functional integral which looks just like the one in (3.53) with the important corrections that now the fields φ and π^ν bring also a-indices, and that for the nonlinear model one needs to give a careful definition of transverse and longitudinal momenta. Let us try to understand what the two general choices for this definition that we described in the previous subsection actually entail in the present context, and from the particular point of view of an observer sitting on the south pole $f_\star$, associating to each point of the sphere (apart for the north pole, that is out of her scope) the above mentioned stereographic coordinates. The south pole itself is the only point with vanishing coordinates. At every point with coordinates γ^a the metric tensors read

$$g_{ab}(\gamma)=\frac{4}{\left(1+\frac{\boldsymbol{\gamma}^2}{R^2}\right)^2}\delta_{ab}\quad,\quad g^{ab}(\gamma)=\frac{1}{4}\left(1+\frac{\boldsymbol{\gamma}^2}{R^2}\right)^2\delta^{ab}\,. \tag{4.4}$$

such that the curvature of the manifold can be neglected as long as $\forall a$, $(\gamma^a)^2 \ll R^2$. If the observer arbitrarily decides to neglect all quantum fluctuations (i.e. histories) violating this bound then the issue of a careful definition of longitudinal momenta losts of interest, since there is practically no difference between covariant and standard derivatives. If she allows the histories to explore a larger neighborhood of the south pole then the two projections become very different and they entail different dynamics, that coincide only locally. In fact, let us assume for simplicity that the bare Hamiltonian be quadratic in the longitudinal momenta, regardless of their definition. If the observer sticks to the first choice of requiring the momenta to be longitudinal w.r.t. ∂ everywhere (apart for the north pole), then she can safely integrate out such momenta and get the traditional Lagrangian formulation in terms of her particular coordinates. If instead she asks the momenta to be covariantly longitudinal, by integrating them out she will find a different and unusual Lagrangian formulation that looks as the usual

one only for histories close to her. This tells us that if she wants to reproduce the traditional Lagrangian dynamics in the present Hamiltonian formulation she has to stick to the first definition of longitudinal momenta and forget about reparameterization invariance. She will be forced to drop the passive interpretation of (4.3) but she will be able to require this nonlinear symmetry regardless of having fixed a preferred reference frame, at least as long as the rotation of the histories doesn't push them through the north pole. This is also what we are going to do, neglecting for the moment global effects on the dynamics of the model.

4.3 Flow of the linear model

In this section we are going to describe how to compute the flow equation of the AEHA of a linear O(N) model by means of the general formulas derived in the previous chapter. We still assume a d-dimensional Minkowskian spacetime with a trivial metric having a mostly-pluses signature, and we are still interested in the leading order of the derivative expansion, i.e. to the LHA, that is encoded in the following projection

$$\Gamma_k[\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\varphi}}] = \int d^d x \left[-\bar{\boldsymbol{\pi}}^\nu \cdot \partial_\nu \bar{\boldsymbol{\varphi}} - \mathcal{H}\left(\frac{\bar{\boldsymbol{\pi}}^2}{2}, \frac{\bar{\boldsymbol{\varphi}}^2}{2}, \frac{(\bar{\boldsymbol{\pi}} \cdot \bar{\boldsymbol{\varphi}})^2 - \bar{\boldsymbol{\pi}}^2 \bar{\boldsymbol{\varphi}}^2}{2} \right) \right]. \tag{4.5}$$

Recall that bold symbols denote vectors collecting the N components of the fields labeled by latin indices, and that the dot denotes contraction of these indices by means of the trivial metric δ_{ab}. Here we extended this convention by denoting: $\bar{\boldsymbol{\pi}}^2 \equiv \bar{\boldsymbol{\pi}}^\nu \bar{\boldsymbol{\pi}}_\nu$ and $(\bar{\boldsymbol{\pi}} \cdot \bar{\boldsymbol{\varphi}})^2 \equiv (\bar{\boldsymbol{\pi}}^\nu \cdot \bar{\boldsymbol{\varphi}})(\bar{\boldsymbol{\pi}}_\nu \cdot \bar{\boldsymbol{\varphi}})$. With respect to the $N = 1$ case discussed in the previous chapter, here we have a new invariant under Lorentz, O(N) and simultaneous reflections $\bar{\pi}^\nu_a \to -\bar{\pi}^\nu_a$ and $\bar{\varphi}^a \to -\bar{\varphi}^a$, namely $(\bar{\boldsymbol{\pi}} \cdot \bar{\boldsymbol{\varphi}})^2$. Thus, $\mathcal{H}$ has as third argument a new degree of freedom that automatically vanishes in the $N = 1$ case. Remember that we still limit our discussion to the case in which the momenta are longitudinal. In the present case this means that every a-th component specifies a radial vector field in Fourier space. Then, obviously it is not possible to ask that this vector field be homogeneous over spacetime, but it is possible to make this request for its magnitude. The more the magnitude is homogeneous in spacetime, the more localized around the origin is the vector field in Fourier space, still remaining radial. The limiting case of this process is of course mathematically ill-defined, but physically we just require the Fourier transform of the momenta to be longitudinal and strongly peaked at the origin (almost a delta function), such that we can neglect the remaining support. Furthermore, if $\forall a$ the momenta are longitudinal, the direction in Fourier space is independent of a. Therefore in the present framework the LHA can be studied by assuming a factorization $\bar{\pi}^\nu_a(q) = \delta(q) w^\nu \bar{\pi}_a$, with $w^\nu w_\nu = 1$, as well as the usual $\bar{\varphi}^a(q) = \delta(q)\bar{\varphi}^a$. Thus from here on bold symbols will once more denote only O(N)

vectors that are Lorentz scalars. For this choice of field configuration the subset of theory space defined in (4.5) is closed under the RG flow.

We restrict our discussion to an off-diagonal regulator as in eq. (3.56). If we denote the fields in Fourier space by $\Phi(q) \equiv (\bar{\pi}^v_a(q), \bar{\varphi}^a(q))$, in the LHA the fluctuation matrix $\tilde{\Gamma}^{(2)}_k \equiv \Gamma^{(2)}_k + R_k\delta$ is computed as follows

$$\tilde{\Gamma}^{(2)}_k = \frac{\overrightarrow{\delta}}{\delta\Phi(-p_2)}\tilde{\Gamma}_k\frac{\overleftarrow{\delta}}{\delta\Phi(p_1)} \equiv \begin{pmatrix} A & B(p_1) \\ B^T(p_1) & D \end{pmatrix}\delta(p_1 - p_2)$$

where

$$\begin{aligned}
A^{ab}_{\mu\nu} =& -\delta_{\mu\nu}\left[\delta^{ab}\left(\mathcal{H}^{(100)} - \bar{\varphi}^2\mathcal{H}^{(001)}\right) + \bar{\varphi}^a\bar{\varphi}^b\mathcal{H}^{(001)}\right] \\
& - w_\mu w_\nu\left[\bar{\pi}^a\bar{\pi}^b\left(\mathcal{H}^{(200)} - 2\bar{\boldsymbol{\varphi}}^2\mathcal{H}^{(101)} + \bar{\boldsymbol{\varphi}}^4\mathcal{H}^{(002)}\right)\right. \\
& \left. + (\bar{\pi}^a\bar{\varphi}^b + \bar{\varphi}^a\bar{\pi}^b)\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}}\left(\mathcal{H}^{(101)} - \bar{\boldsymbol{\varphi}}^2\mathcal{H}^{(002)}\right) + \bar{\varphi}^a\bar{\varphi}^b(\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})^2\mathcal{H}^{(002)}\right] \\
B^a_{vb}(p_1) =& \ (1 + r_k(p_1^2))ip_{1v}\delta^a_b - w_v\left[\delta^a_b(\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})\mathcal{H}^{(001)} + \bar{\pi}^a\bar{\pi}_b\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}}\left(\mathcal{H}^{(101)} - \bar{\boldsymbol{\varphi}}^2\mathcal{H}^{(002)}\right)\right. \\
& + \bar{\pi}^a\bar{\varphi}_b\left(\mathcal{H}^{(110)} - 2\mathcal{H}^{(001)} - \bar{\boldsymbol{\varphi}}^2\mathcal{H}^{(011)} - \bar{\boldsymbol{\pi}}^2\mathcal{H}^{(101)} + \bar{\boldsymbol{\pi}}^2\bar{\boldsymbol{\varphi}}^2\mathcal{H}^{(002)}\right) \\
& \left. + \bar{\varphi}^a\bar{\pi}_b\left(\mathcal{H}^{(001)} + (\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})^2\mathcal{H}^{(002)}\right) + \bar{\varphi}^a\bar{\varphi}_b(\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})\left(\mathcal{H}^{(011)} - \bar{\boldsymbol{\pi}}^2\mathcal{H}^{(002)}\right)\right] \\
D_{ab} =& -\left[\delta_{ab}\left(\mathcal{H}^{(010)} - \bar{\boldsymbol{\pi}}^2\mathcal{H}^{(001)}\right) + \bar{\pi}_a\bar{\pi}_b\left(\mathcal{H}^{(001)} + (\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})^2\mathcal{H}^{(002)}\right)\right. \\
& + (\bar{\pi}^a\bar{\varphi}^b + \bar{\varphi}^a\bar{\pi}^b)\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}}\left(\mathcal{H}^{(011)} - \bar{\boldsymbol{\pi}}^2\mathcal{H}^{(002)}\right) \\
& \left. + \bar{\varphi}_a\bar{\varphi}_b\left(\mathcal{H}^{(020)} - 2\bar{\boldsymbol{\pi}}^2\mathcal{H}^{(011)} + \bar{\boldsymbol{\pi}}^4\mathcal{H}^{(002)}\right)\right]
\end{aligned}$$

and the entries of B^T can be obtained from the ones of B by exchanging the latin indices and changing sign to the Fourier variable. Then, using eq. (3.60) we can write

$$\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma^{(2)}_k + R_k\delta\right)^{-1}\dot{R}_k\delta\right] = \frac{1}{2}\mathrm{Tr}\left[(D - B^TA^{-1}B)^{-1}\tilde{\partial}_t(D - B^TA^{-1}B)\right]$$

where $\tilde{\partial}_t$ is a differentiation w.r.t. $t \equiv \log(k/k_0)$ acting on r_k only. The matrix inversions and products can be easily performed due to the fact that the fluctuation matrix has a simple structure involving only diagonal tensors or tensor products of vectors. However the algebraic combination of the coefficients of these tensors is complicated and can be dealt with only by some software for symbolic algebraic manipulation. So far we have not managed to get a compact expression for these coefficients therefore in the following we will avoid to attempt any explicit definition of them. However, the implicit definition descends from the fluctuation matrix given above. In order to get a useful expression for the running of the Hamiltonian density of the linear O(N) model, another nontrivial step is needed apart for the algebra mentioned

above: the integration over Fourier space. As was already the case for the simple scalar model discussed in the previous chapter, there are two nontrivial integrations to be performed, one over the angle between the loop Fourier variable p_ν and the direction of the momenta w^ν, and one over the radial variable p^2. The latter can be made trivial by adopting suitable regulators such as the optimized one $r_k(p^2) = \left(k/\sqrt{p^2} - 1\right)\theta(k^2 - p^2)$. The former instead considerably increases the degree of mathematical complexity of the r.h.s., that for a generic $d \in \mathbb{R}$ develops a dependence on hypergeometric functions as in the $N = 1$ case discussed in the previous chapter. To sum up, by working out all the above mentioned computational steps, we obtain a flow equation equation for the linear O(N) model which we depict as

$$\dot{\mathcal{H}}_k\left(\frac{\bar{\boldsymbol{\pi}}^2}{2}, \frac{\bar{\boldsymbol{\varphi}}^2}{2}, \frac{(\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})^2 - \bar{\boldsymbol{\pi}}^2\bar{\boldsymbol{\varphi}}^2}{2}\right) = F\left(\frac{\bar{\boldsymbol{\pi}}^2}{2}, \frac{\bar{\boldsymbol{\varphi}}^2}{2}, \frac{(\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})^2 - \bar{\boldsymbol{\pi}}^2\bar{\boldsymbol{\varphi}}^2}{2}, \mathcal{H}^{(l\,m\,n)}, d, N\right) \tag{4.6}$$

for $l,\ m,\ n \in \{0, 1, 2\}$ with the constraint $0 < (l + m + n) < 3$. As it should, this new expression for the running of $\mathcal{H}$ reduces to eq. (3.66) if one sets $N = 1$ and $(\bar{\boldsymbol{\pi}}\cdot\bar{\boldsymbol{\varphi}})^2 = \bar{\boldsymbol{\pi}}^2\bar{\boldsymbol{\varphi}}^2$.

4.4 Modified Ward-Takahashi identities

Since the representation of the O(N)/O(N−1) symmetry is nonlinear, any ΔS_k quadratic in the field in not invariant under it. Therefore, the RG flow generated by eq. (4.6) cannot manifestly respect this symmetry. This means that if we start at $k = \Lambda$ with a traditional bare Hamiltonian such as

$$\mathcal{H}_\Lambda(x) = -\frac{1}{2}g^{ab}\left(\varphi(x)\right)\bar{\pi}^\nu_a(x)\bar{\pi}_{\nu b}(x)$$

that is manifestly invariant under the transformations (4.3), at $k \neq \Lambda$ the flow equation will produce an average effective Hamiltonian that does not enjoy this symmetry. On the other hand, the flow must contain information about such a symmetry of $\mathcal{H}_\Lambda$ in a nonmanifest way. In other words, if we choose a bare action, i.e. an initial condition for the integration of the flow equation, enjoying this nonlinear symmetry, the corresponding RG flow is constrained by a set of equations encoding this symmetric choice at $k = \Lambda$. We will call these equations modified Ward-Takahashi identities (mWTI). As the standard WTI they can be obtained from the functional integral representation, this time eq. (3.58), by changing integration variables according to an infinitesimal symmetry transformation, in our case give by eq. (4.3), thus getting

$$\int d^dx\left[\langle\delta\pi^\nu_a(x)\rangle\frac{\delta\Gamma_k[\bar{\boldsymbol{\pi}},\bar{\boldsymbol{\varphi}}]}{\delta\bar{\pi}^\nu_a(x)} + \langle\delta\varphi^a(x)\rangle\frac{\delta\Gamma_k[\bar{\boldsymbol{\pi}},\bar{\boldsymbol{\varphi}}]}{\delta\bar{\varphi}^a(x)}\right] = \langle\delta\Delta S_k[\boldsymbol{\pi} - \bar{\boldsymbol{\pi}}, \boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}]\rangle \tag{4.7}$$

We call them modified indentities because of the new contributions coming from the regulator, that make them differ from the standard identities for any $0 < k < \Lambda$. In the case of the transformations (4.3) these identities bring $D = N - 1$ constraints on Γ_k, that are needed to define the theory space of the nonlinear O(N) model. The role of these constraints on the functional RG flow has already been discussed for gauge theories in many works, see [13, 14] and references therein. Before any truncation is performed, the flow equation and the mWTI are compatible, i.e. it is sufficient to solve the mWTI only once at some scale k_0, i.e. to pick an initial point on the restricted theory space, and the integration of the flow equation will map this into solutions of the mWTI at different scales k, i.e. the flow will stay on the restricted theory space. Because of truncations, it is necessary to constrain by hand the flow to lie on such space. This can be achieved by an appropriate choice of the truncation itself, that is, by taking a truncation for the linear model, or even larger, and again projecting this subset of the linear theory space into a subset of the nonlinear one, by asking that the mWTI be satisfied at every scale. This is tantamount to dividing the parameters of the truncation in the larger space into two classes, the dependent and the independent ones, the former being determined as functions of the latter upon solution of the mWTI. Unfortunately, these functions are not unique, because of the nonlinearity of the mWTI, therefore physical arguments are needed in order to identify the meaningful solutions and to drop the spurious ones, generated by the truncation.

Notice that for the transformations (4.3) the mWTI, unlike the flow equation, involves not only two point but also three point functions, appearing on the r.h.s. of eq. (4.7). This fact, together with momentum conservation at the vertices in the LHA, entails that the r.h.s. be a two-loop contribution, involving also the vertex $\tilde{\Gamma}^{(3)}$. This makes the translation of eq. (4.7) into explicit constraints for some specific truncation a bit lenghtier than for the flow equation itself. In fact, in this thesis we are not presenting the final form of the mWTI in the LHA, because its computation is still incomplete. In order to illustrate how this constraint can be treated and how the symmetry requirement is encoded in it, we are going to discuss the extremely simple case of QM on a circle, that is $d = 1$ and $N = 2$. But, before moving on to this little exercise, let us stress another important difference in structure between the flow equation and the mWTI: while the flow equation requires one loop integration regulated by the $(\log k)$-derivative of $\Delta S_k^{(2)}$, the mWTI entails two loop integrations in which no derivative of the regulator appears. Therefore, a choice of ΔS_k as an IR regulator does not allow to regularize the mWTI. To get a Wilsonian setup, completely free of divergences, in general one has to assume that $\Delta S_{k,\Lambda}$ provides both an IR and a UV regularization.

This issue is of course absent in QM, where we are free to choose any regulator providing a

rising delta function. Furtheremore, by restricting to $d = 1$ and $D = N - 1 = 1$ we do not have to take care of Lorentz or O(N−1) indices, that make the general computation long and tedious. Thus, let's work out this explicit example. In this case the transformations (4.3) read:

$$\delta\pi = \varepsilon\pi\frac{\varphi}{R} \quad , \quad \delta\varphi = -\varepsilon\frac{R}{2}\left(1 + \frac{\varphi^2}{R^2}\right) \tag{4.8}$$

such that

$$\begin{aligned}
\langle\delta\pi(p)\rangle &= \frac{\varepsilon}{R}\int_q \langle\pi(p-q)\varphi(q)\rangle \\
\langle\delta\varphi(p)\rangle &= -\varepsilon\frac{R}{2}\left(\delta(p) + \frac{1}{R^2}\int_q \langle\varphi(p-q)\varphi(q)\rangle\right) \\
\langle\delta\Delta S_k[\boldsymbol{\pi}-\bar{\boldsymbol{\pi}}, \boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}]\rangle &= -\frac{\varepsilon}{R}\int_{p_1,p_2} r_k(p_1^2) i p_1 \Big[\bar{\varphi}(-p_1)\langle\pi(p_2)\varphi(p_1-p_2)\rangle \\
&\qquad + \frac{1}{2}\bar{\pi}(-p_1)\langle\varphi(p_2)\varphi(p_1-p_2)\rangle\Big] \\
&\quad + \frac{\varepsilon}{R}\int_{p_1,p_2}\left[r_k(p_2^2) i p_2 + \frac{1}{2} r_k(p_1^2) i p_1\right]\langle\pi(p_1)\varphi(p_2)\varphi(-p_1-p_2)\rangle
\end{aligned}$$

where $\int_p \equiv \int \frac{dp}{2\pi}$. Translating the two- and three-points functions in terms of derivatives of $\tilde{\Gamma}_k$, setting the average fields in Fourier space proportional to a delta function, and performing a Wick rotation inside the loop integrals one gets

$$\begin{aligned}
\int_x \left[\langle\delta\pi_a^\nu(x)\rangle\frac{\delta\Gamma_k[\bar{\boldsymbol{\pi}},\bar{\boldsymbol{\varphi}}]}{\delta\bar{\pi}_a^\nu(x)} + \langle\delta\varphi^a(x)\rangle\frac{\delta\Gamma_k[\bar{\boldsymbol{\pi}},\bar{\boldsymbol{\varphi}}]}{\delta\bar{\varphi}^a(x)}\right] &= \varepsilon\delta(0)\frac{\bar{\varphi}}{R}\Bigg\{\bar{\pi}^2 H_k^{(10)} - \frac{R^2}{2}H_k^{(01)}\left(1+\frac{\bar{\varphi}^2}{R^2}\right) \\
&\quad + \left[\bar{\pi}^2 H_k^{(10)}H_k^{(11)} + \frac{1}{2}\left(H_k^{(10)} + \bar{\pi}^2 H_k^{(20)}\right)\right]\int_p \frac{1}{P_k(p^2) + \det H_k^{(2)}}\Bigg\}
\end{aligned} \tag{4.9}$$

where we slightly modified the notations of the previous chapter as follows: H_k is now a function of $\bar{\pi}^2/2$ and $\bar{\varphi}^2/2$, and as a consequence

$$\det H_k^{(2)} = \left(H_k^{(10)} + \bar{\pi}^2 H_k^{(20)}\right)\left(H_k^{(01)} + \bar{\varphi}^2 H_k^{(02)}\right) - \bar{\pi}^2\bar{\varphi}^2 H_k^{(11)2}\,.$$

As far as the regulator contribution is concerned, we can write it as

$$\langle\delta\Delta S_k[\boldsymbol{\pi}-\bar{\boldsymbol{\pi}}, \boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}]\rangle = \frac{\varepsilon}{R}\delta(0)\left(C_1 I_1 + C_2 I_2 + C_3 I_3\right)$$

where the C_i are polynomial functions of the derivatives of H_k whose precise form is not needed

for the present qualitative discussion, and

$$
\begin{aligned}
I_1 &= \int_{p_1}\int_{p_2} r_k(p_1^2) i p_1 \frac{(1+r_k(p_1^2))ip_1}{P_k(p_1^2)+\det H_k^{(2)}} \frac{1}{P_k(p_2^2)+\det H_k^{(2)}} \frac{1}{P_k((p_1+p_2)^2)+\det H_k^{(2)}} \\
I_2 &= \int_{p_1}\int_{p_2} r_k(p_1^2) i p_1 \frac{1}{P_k(p_1^2)+\det H_k^{(2)}} \frac{(1+r_k(p_2^2))ip_2}{P_k(p_2^2)+\det H_k^{(2)}} \frac{1}{P_k((p_1+p_2)^2)+\det H_k^{(2)}} \\
I_3 &= \int_{p_1}\int_{p_2} r_k(p_1^2) i p_1 \frac{(1+r_k(p_1^2))ip_1}{P_k(p_1^2)+\det H_k^{(2)}} \frac{(1+r_k(p_2^2))ip_2}{P_k(p_2^2)+\det H_k^{(2)}} \frac{(1+r_k((p_1+p_2)^2))i(p_1+p_2)}{P_k((p_1+p_2)^2)+\det H_k^{(2)}}
\end{aligned}
$$

are the two loop integrals. Obviously when $k \to 0$ theses integrals vanish and so does the r.h.s of eq. (4.7), that reduces to the standard WTI. Also, when $k \to \Lambda$ (or equivalently $+\infty$) the presence of a delta function requires $r_k \to +\infty$, therefore also in this limit all the loop integrals, both the ones above and the one in eq. (4.9), must vanish. Thus, in the UV the only nonvanishing contribution to the mWTI is the one on the r.h.s. of the first line of eq. (4.9), that strongly differs from the standard WTI because it does not involve any quantum correction. In fact, this limit of the mWTI just states the invariance of any bare action on the restricted theory space under the present nonlinear O(2) symmetry, since

$$
\delta H\left(\frac{\bar\pi^2}{2},\frac{\bar\varphi^2}{2}\right) = \delta\bar\pi\,\bar\pi H^{(10)} + \delta\bar\varphi\,\bar\varphi H^{(01)} = \varepsilon\frac{\bar\varphi}{R}\left[\bar\pi^2 H_k^{(10)} - \frac{R^2}{2}H_k^{(01)}\left(1+\frac{\bar\varphi^2}{R^2}\right)\right] .
$$

Therefore the mWTI completely defines the non-manifest symmetry of the RG flow, without the need to refer to any other representation such as the functional integral.

4.5 Summary and expectations

In this chapter we discussed the foundations and the line of development of an ongoing research project, whose aim is to build and use a single flow equation representation suitable for the study of both the linear and the nonlinear O(N) scalar model. To this end the usual configuration space formulation is ill-suited because of the field dependent functional measure, and our approach is a particular way to exponentiate this measure by means of extra degrees of freedom, going back to the phase-space functional formula that is the same for both models. This is tantamount to representing the dynamics by a single Hamiltonian flow equation. Such a unified treatment as far as we know has not been studied in a functional RG framework, and we hope that it will provide a new tool for the description of the nonperturbative RG flow of these models, helping in understanding the difficulties met so far in the computation of the critical exponents of the nonlinear model by means of the fRG of the geometric effective action.

We stress that, although the representation relies on the definition of the AEHA, it is applicable also to the cases in which the bare action is quadratic in the momenta. However in this case, while the flow of the linear model reduces to the usual Lagrangian one, the flow of the nonlinear model is described in a radically non-Lagrangian way, because the sector of the momenta becomes nonquadratic after an infinitesimal RG step. From this point of view, for nonlinear models it seems natural to allow for bare actions quadratic in the momenta and to look for the UV completion of these theories in a general Hamiltonian framework.

On the other hand, also the case of linear models could become interesting if one drops the requirement that the Hamiltonian of the theory be exactly quadratic at some scale. For example, while Lagrangian O(N) models are a simple and popular benchmark for the study of the spontaneous symmetry breaking (SSB) mechanism, we do not known if this mechanism could take place in other ways involving a nontrivial dynamics of the momenta. After all, our formulation has a built-in longitudinal vector field whose interactions with the scalar could open new paths towards the generation of a mass. We hope that this possibility will be easy to address by studying truncations of the effective Hamiltonian just as in the Lagrangian case one studies truncations of the effective potential.

The task of computing the flow of the nonlinear model will certainly be more complicated, because of the need to deal with mWTI. Let us stress that in principle the mWTI are not dynamical equations drawing a difference between the representations used for the linear and for the nonlinear model. In the exact formulation, both kinds of dynamics are given only by the flow equation, and the mWTI is just an identity that could or could not be satisfied by the initial condition we choose. Thus, before any truncation, the only thing that makes the difference between linear or nonlinear realizations is which bare action we choose. Therefore the mWTI could be interpreted as a way to distinguish between fair and poor truncations of the exact flow. Of course, simply choosing truncations that are manifestly symmetric under the same symmetry of the bare action is a poor truncation, because of the presence of the regulator, that has the double role of letting the flow equation generate non-symmetric actions and of forcing the mWTI to interpolate between the classical symmetry at $k = \Lambda$ and the standard WTI at $k = 0$.

The first nontrivial test of the reliability of this theoretical framework we plan to perform, is the computation of the ground state energy and gap of a free particle on a D-dimensional sphere in QM. This computation should also teach us if neglecting global effects on target space is a legitimate approximation. In fact, one important fundamental problem still open is how to describe these effects by means of the present construction. But this is an issue that will be left open by the work outlined in this chapter.

5

Gauged chiral Yukawa systems

The UV dynamics of a gauged chiral Yukawa system, serving as a toy model of the electroweak theory or of grand unification theories, is analyzed by means of a functional RG equation, in order to address the triviality and the hierarchy problems beyond the range of applicability of perturbative expansions. Within the chosen truncation of the effective average action, a UV-attractive non-Gaußian fixed point is discovered in the spontaneous symmetry breaking regime, which could provide a UV completion of the theory within the paradigm of asymptotic safety.

5.1 Triviality and hierarchy problems

In the construction of QFT models for high energy physics, great emphasis is traditionally given to the requirement of renormalizability. In the past decades such a requirement enabled physicists to strongly constrain and restrict the set of sensible and interesting models, and served as a guide for the extension of known theories and for the prediction of new physics. The two outstanding examples of the success of this attitude towards QFT are provided by the electroweak model and by QCD. Such models were conceived on the basis of perturbative expansions, i.e. weak coupling expansions, and by construction they were satisfying a pertur-

bative renormalizability requirement in the sense of Weinberg's theorem: they contained only dimensionless and positive-dimension interactions.

However, already before the design of these cornerstones of theoretical physics, it was known that such a requirement is in general not enough to build a self-consistent model, with an *a priori* indefinite range of applicability. This lesson was learnt directly from QED itself: in order to make predictions we need to regularize the theory and to deal with its renormalization flow, and in so doing, sooner or later we have to stop trusting weak coupling expansions. This is the case for QED because of the appearence of a Landau pole [86, 87]. Computing the one loop beta-function of the fine structure constant one finds $\beta_\alpha = 2/(3\pi)\alpha^2$ and this entails that the physical coupling $\alpha(\mu)$ at some scale μ is related to the renormalized coupling α_R (defined by the value of some measurable quantity at some convenient renormalization point, for example by the potential between heavy charges in the Gell-Mann-Low scheme [87]) by the following relationship

$$\alpha(\mu) = \frac{\alpha_R}{1 - \frac{2}{3\pi}\alpha_R \log(\frac{\mu}{m_e})} \,. \tag{5.1}$$

This shows that, for a fixed renormalized coupling and electron mass, the physical coupling increases with increasing μ, and that at some *finite* scale we leave the domain of applicability of perturbative expansions (naively at $\mu_\infty < \infty$, defined by $1 = \frac{2}{3\pi}\alpha_R \log(\frac{\mu_\infty}{m_e})$, the coupling diverges and predictions are no longer possible). Going to higher orders of the perturbative expansion does not change this qualitative picture. Within a Wilsonian interpretation of the renormalization group, this can be restated in a slightly different way. If we regularize the theory with an ultraviolet (UV) cutoff Λ, and assume a corresponding bare coupling α_Λ at this scale, this is related to the physical coupling at the scale μ by

$$\alpha_\Lambda = \frac{\alpha(\mu)}{1 - \frac{2}{3\pi}\alpha(\mu) \log(\frac{\Lambda}{\mu})} \,. \tag{5.2}$$

Demanding that physics be independent of the regularization, i.e. that $\alpha(\mu)$ be independent of Λ, we see how we have to adjust α_Λ as a function of Λ. But this independence cannot be provided inside the domain of validity of the last relationship, because at some finite Λ_∞ the bare α_{Λ_∞} would need to be infinite. The only value of α_Λ allowing for the removal of the UV cutoff is $\alpha_\Lambda = 0$, leading to a free theory. That's why this is also called a "triviality" problem [4]. This issue historically even shook the trust in the usefulness of QFT, and led to the search for other frameworks such as current algebra or Regge theory.

One possible way out of this problem is just to change theory, embedding it in a larger model that does not show Landau poles and that is applicable at every energy. This point of view

is often accompanied by the so-called perturbativity requirement [88, 89], i.e. the condition that acceptable quantum field theories should be consistent and fully predictive without ever needing to go beyond weak coupling expansions. Another possible solution is to consider the theory affected by this problem as an effective theory which is indeed valid only in a restricted range of scales, because at strong coupling new physics shows up or the set of effective degrees of freedom radically changes. The latter case is in fact realized in QED, since at strong coupling the screening of electric charge triggers chiral symmetry breaking (χSB) in a way very similar to what is known to take place in QCD [90, 91, 92]. In other words, it would be impossible to experimentally measure the value of the fine structure constant at scales close to the Landau pole. Yet, this very example shows how nonperturbative phenomena such as χSB might be crucial for the understanding of the Landau poles. One more possible solution is just that the Landau pole be an artifact of weak coupling expansions and that it might be possible to make sense of the theory at any scale without changing the set of fundamental symmetries and of degrees of freedom. In general, in order to explore such a possibility, one needs a framework for describing the dynamics of the system even in strongly coupled regimes.

The triviality problem is a big fundamental issue left open by the standard model (SM), even more so after the discovery of a boson that could be extremely similar to the Higgs scalar of this model. This is because, apart for the U(1) sector, also the Higgs sector is plagued by Landau poles [4, 93, 94, 95, 96, 97, 98], and in this sector the position of these singularities, i.e. the order of magnitude in energy that one could reach without being forced to change the theory (within the big desert hypotesis), is much lower than in the gauge sector [92]. In fact, it could be smaller than the grand unification scale [88, 99, 100, 101, 102]. Since this issue is at the base of several proposals of new physics, we believe it is interesting to understand if nonperturbative effects in the SM can change the picture that one gets from perturbation theory.

Another closely related topic, one of the few unpleasent features of the SM, the understanding of which could receive benefits from nonperturbative investigations, is the hierarchy problem, namely: how does it come that the masses of the particles are scattered over several orders of magnitude, instead of being closer to each other? As other similar questions that have to deal with the concept of "magnitude in diversity", also this one can be reformulated in terms of the choice of some initial condition, in this case the UV starting point for the renormalization flow of the theory. If these initial conditions were chosen in the vicinity of the GFP, we would expect positive-dimension couplings to grow towards the IR and negative-dimension interactions to become negligible. The masses, being of the first kind, would be expected to grow at approximately the same rate and as a consequence only by choosing big differences in their initial conditions one could get big differences in their low energy values. Hence, in order

for this fine-tuning not to be required, one needs a dynamical mechanism able to explain how different rates of change could arise for parameters with the same canonical dimension. Again, one could try to achieve this by staying close to the GFP, i.e. within perturbation theory, and introducing new physics. But another possibility is to allow for the initial conditions of the RG flow to lie well outside the neighborhood of the GFP and to look for other important dynamical regimes, different from the weak coupling one, determining such a hierarchy.

In this chapter we will try to address both the triviality and the hierarchy problems from this nonperturbative point of view, studying the dynamics of a gauged chiral Yukawa system, serving as a toy model of the electroweak model and of grand unification theories (GUT). The method by which we will get such nonperturbative information is still the Euclidean functional RG equation for the Lagrangian AEA, truncated within the general scheme of the derivative expansion at next-to-leading order. We will provide first hints that in this kind of models both the triviality and the hierarchy problems might indeed find a solution without invoking new physics, the reason being encoded in an asymptotic safety (AS) scenario.

How does AS relate to the triviality and hierarchy problems? Of course, if a theory is asymptotically safe no Landau pole can show up in the coupling constants and the theory is consistent and predictive up to arbitrarily high energies. More subtle is the link with the generation of hierarchies. For this, one has to recall that a possible dynamical definition of the canonical dimension of some coupling can be given as the critical exponent associated with the RG flow along the corresponding direction of theory space at the GFP. The critical exponents at some FP are defined by the linearized RG flow close to the FP (see eq. (1.3) and the discussion after it). Therefore they encode the rate of change of the couplings along such flow in a neighborhood of the FP. In case a NGFP exists, couplings that correspond to eigendirections with the same eigenvalue at the GFP (same canonical dimension), could show a very different linearized flow around the NGFP. Hence, similar initial conditions chosen in the vicinity of the NGFP would correspond to different values of these parameters getting out of the NGFP regime. Also, the presence of the NGFP singles out and justifies a preferred set of initial conditions: those close to the FP. Yet again we see that for this mechanism to work some kind of fine-tuning is still necessary, the strength of it being related to the question: *how close* to the NGFP must we choose the initial condition? Since the rate of change of the couplings is determined by the combination of the distance from the FP and by the magnitude of the critical exponents, the smaller the critical exponents, the smaller the differences in the output at the end of the FP regime for different initial conditions, and hence the weaker the required fine-tuning. In conclusion, for AS to provide an answer to the hierarchy problem, one needs small critical exponents and the "right" FP values of the couplings.

As a final motivation for being interested in AS scenarios in high energy physics, let us recall that the existence of a finite dimensional UV critical surface, providing a finite dimensional parameterization of the theory in terms of the so-called relevant couplings, allows one to write all the remaining couplings as functions of this finite set. This opens the intriguing possibility that the number of independent couplings be smaller than the one currently used within perturbation theory, in which case one would be able to predict relationships among some parameters of the SM. Let us mention that SM-alternatives without a fundamental Higgs field have been based on AS scenarios for fermionic theories [103], which, however, are generically plagued by a strong hierarchy problem; see also [104] for an asymptotically free fermionic model where nonlocal interactions lead to an improved hierarchy. The possible relevance of AS scenarios in the SM has also been investigated from the point of view of gauged unitary nonlinear sigma models in [22, 79, 80].

The study of AS scenarios in Yukawa systems was started in [105, 106]. The strategy followed in these works to trigger AS is not based on strong coupling peculiarities but on a conformal threshold behavior of the scalar vacuum expectation value (vev). In the spontaneous symmetry breaking (SSB) regime a whole new sector of the theory develops containing interactions mediated by the condensate. The running of the modes interacting with such a condensate gets frozen since these modes acquire a mass and decouple from the flow. The running of the vev itself can be qualitatively described as follows. If one looks at the dimensionless squared vev $\kappa = v^2/(2k^2)$ its beta-function has the structure

$$\partial_t \kappa \equiv \partial_t \frac{v^2}{2k^2} = -2\kappa + \text{interaction terms}, \qquad \partial_t = k\frac{d}{dk}. \tag{5.3}$$

If the interaction terms are absent, the Gaußian fixed point $\kappa = 0$ is the only conformal point, corresponding to a free massless theory. If the interaction terms are nonvanishing, as is the case if they approach a NGFP by themselves, the sign of these terms decides about a possible conformal behavior. A positive contribution from the interaction terms gives rise to a FP at $\kappa > 0$ which can control the conformal running over many scales. If they are negative, no conformal vev is possible. Since fermions and bosons contribute with opposite signs to the interaction terms, the existence of a fixed point $\kappa_* > 0$ crucially depends on the relative strength between bosonic and fermionic fluctuations. More specifically, the bosons have to win out over the fermions.

A first analysis for simple Yukawa systems [105] containing one scalar field and N_f Dirac fermions revealed that the necessary bosonic dominance actually produces a NGFP, but it occurs only for $N_f \lesssim 0.3$. This result motivated the study of chiral Yukawa systems [106] with N_L left-handed fermion species and one right-handed fermion as well as N_L complex scalars.

On the one hand such a structure was inteded to mimic the coupling between the SM Higgs scalar and the left- and right-handed components of the top quark, also involving Yukawa couplings to the left-handed bottom (for $N_{\rm L} = 2$) and further bottom-like quarks (for $N_{\rm L} > 2$) in the same family. (When the scalar field develops a vev upon symmetry breaking, the top quark acquires a Dirac mass, whereas the bottom-type quarks remain massless in a way similar to neutrinos.) On the other hand this setting was offering a control parameter for boson dominance and for the potential existence of a NGFP: the number $N_{\rm L}$. Indeed, to leading-order of the derivative expansion (local-potential approximation), this left-right asymmetric model was exhibiting the desired NGFP for $1 \leq N_{\rm L} \leq 57$. Moreover, one of the admissible fixed points had only one UV-attractive direction, thus implying that only one physical parameter had to be fixed, e.g., the vev $v = 246$GeV, whereas all other IR quantities such as the Higgs or the top mass would have been a pure prediction of the theory.

Unfortunately, the NGFP was destabilized at higher order in the derivative expansion. In fact, a systematic derivative expansion of the AEA for computing the RG flow of the model is reliable if the momentum dependence of full effective vertices takes only little influence on the flow. A direct means for measuring this influence is the size of the anomalous dimensions η of the fields, since next-to-leading order contributions couple to the leading-order derivative expansion only via terms $\sim \eta$. Monitoring the size of η thus gives us a direct estimate of the reliability of the results. In the aforementioned computation the anomalous dimension of the right-handed fermions was taking unacceptably large values at the NGFP, thus signaling the dramatic unreliability of the results. One reason for the size of $\eta_{\rm R}$ lies in the fact that the massless Nambu-Goldstone bosons (NGB's) and massless bottom-type fermions contribute strongly. This is because they are not damped by massive threshold effects induced by couplings to the condensate, and they contribute with a large multiplicity $\sim N_{\rm L}$. However, as the massless NGB's are not present in the SM due to the gauging of the SU($N_{\rm L}$) symmetry, one could expect that this instability of the NGFP could be much weaker or completely absent in a gauged version of this chiral Yukawa model. This expectation is a fundamental motivation for the present work, in which the gauging is taken into account and its effect on the RG flow of the system is analyzed.

This chapter is organized as follows. In section 5.2, we give a detailed definition of the model under consideration and of the nonperturbative approximations considered in this work. In section 5.3, we give the generic set of equations defining the RG flow of the model. In section 5.4, 5.5 and 5.6 we analyze the flow. Conclusions are presented in Sect. 5.7. Appendix B describe the notations, the analytic manipulations and the approximations leading to the flow equations studied in the present work.

5.2 Chiral Yukawa systems coupled to gauge degrees of freedom

The field content of the model we are going to investigate in the present work is the following: one right-handed fermion ψ_{R}, N_{L} left-handed fermions ψ_{L}^a, N_{L} complex scalars φ^a and a $\mathrm{SU}(N_{\mathrm{L}})$ Yang-Mills vector potential W_ν^a. The matter fields belong to the fundamental representation (indices from the beginning of the latin alphabet) while the gauge potential belong to the adjoint representation (indices from the middle of the latin alphabet) of $\mathrm{SU}(N_{\mathrm{L}})$. Apart for this continuous local symmetry we require also invariance of the theory under global $\mathrm{U}(N_{\mathrm{L}})_{\mathrm{L}}$ transformations of the left-handed fermion and the scalar as well as under global $\mathrm{U}(1)_{\mathrm{R}}$ transformations of the right-handed fermion and the scalar. The projections on the left-/right-handed fermion contributions are carried out via the projection operators

$$P_{\mathrm{L/R}} = \frac{1}{2}(1 \pm \gamma_5)\,. \tag{5.4}$$

The bosons can also be expressed in terms of a real field basis by defining

$$\varphi^a = \frac{1}{\sqrt{2}}(\varphi_1^a + i\varphi_2^a), \quad \varphi^{a\dagger} = \frac{1}{\sqrt{2}}(\varphi_1^a - i\varphi_2^a)\,, \tag{5.5}$$

where $\varphi_1^a, \varphi_2^a \in \mathbb{R}$. The above-mentioned symmetries require the scalar self-interactions to be written in terms of the invariant $\rho := \varphi^{a\dagger}\varphi^a$. We denote by

$$D_\nu^{ab} = \partial_\nu \delta^{ab} - i\bar{g}W_\nu^i (T^i)^{ab} \tag{5.6}$$

the covariant derivatives for the matter fields in the fundamental representation, by f the structure constants of the $\mathfrak{su}(N_{\mathrm{L}})$ algebra $[T^i, T^j] = if^{ijl}T^l$ and by $F_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\mu W_\nu^i + \bar{g}f^{ijl}W_\mu^j W_\nu^l$ the nonabelian field strength (here and in the following a sum over repeated indices is understood, unless differently stated, regardless of the position of indices, since every metric is assumed to be positive definite).

In the spirit of perturbative (power-counting) renormalizabiliy the classical action subject to perturbative quantization would read

$$\begin{aligned} S_{\mathrm{cl}} = \int d^d x \Big[&\frac{1}{4}F_{\mu\nu}^i F^{i\mu\nu} + (D^\mu\varphi)^\dagger (D_\mu\varphi) + \bar{m}^2\rho + \frac{\bar{\lambda}}{2}\rho^2 \\ &+ i(\bar{\psi}_{\mathrm{L}}^a \not{D}^{ab}\psi_{\mathrm{L}}^b + \bar{\psi}_{\mathrm{R}}\not{\partial}\psi_{\mathrm{R}}) + \bar{h}\bar{\psi}_{\mathrm{R}}\varphi^{\dagger a}\psi_{\mathrm{L}}^a - \bar{h}\bar{\psi}_{\mathrm{L}}^a\varphi^a\psi_{\mathrm{R}}\Big] \end{aligned} \tag{5.7}$$

where the classical parameter space would be spanned by the boson mass $\bar{m}$, the scalar self-interaction $\bar{\lambda}$, the Yukawa coupling $\bar{h}$ and the gauge coupling $\bar{g}$. In order to quantize this action

one would need some kind of gauge-fixing. As in this system the Higgs mechanism is known to take place, a particularly useful gauge-fixing choice would be the so-called renormalizable R_α gauge, that is defined in such a way to cancel the derivative interactions between the gauge vector and the NGB's. Separating the scalar field into the vev $v/\sqrt{2}$ and the fluctuations $\Delta\varphi$ around the vev

$$\varphi^a = \frac{v}{\sqrt{2}}\hat{n}^a + \Delta\varphi^a, \quad \Delta\varphi^a = \frac{1}{\sqrt{2}}(\Delta\varphi_1^a + i\Delta\varphi_2^a) \tag{5.8}$$

where $\hat{n}$ is a unit vector ($\hat{n}_a^\dagger \hat{n}^a = 1$) defining the direction for the vev, one could fix the gauge by the condition

$$G^i(W) = \partial_\mu W_\mu^i + \sqrt{2}i\alpha v\bar{g}T^i_{\hat{n}a}\Delta\varphi^a = 0 \tag{5.9}$$

where α is a gauge-fixing parameter interpolating between the unitarity gauge at $\alpha \to \infty$ and the Landau gauge at $\alpha \to 0$. From here on the label $\hat{n}$ in place of a fundamental color index denotes the contraction of that index with the unit vector $\hat{n}$ (or $\hat{n}^\dagger$, depending on the position of the index). In order to affect the NGB's and not the massive scalar, one usually excludes in the gauge-fixing the terms depending on $\Delta\varphi_1^{\hat{n}}$. To implement such a gauge-fixing, one could introduce the following term in the microscopic action

$$S_{\rm gf} \;=\; \frac{1}{2\alpha}\int d^dx\, G^i(W)G^i(W)\,.$$

Accordingly, one would need to include ghost fields c^i and $\bar{c}^i$, with a bare action

$$S_{\rm gh} = -\int d^dx\, \bar{c}^i \mathcal{M}^{ij} c^j$$

where

$$\mathcal{M}^{ij} = -\partial_\mu^2\delta^{ij} - \bar{g}f^{ilj}\partial_\mu W^{l\mu} + \sqrt{2}\alpha v\bar{g}^2 T^i_{\hat{n}a}T^j_{ab}\Delta\varphi^b \tag{5.10}$$

again excluding $a = \hat{n}$ in the sum involving $\Delta\varphi_1^b$.

In conclusion, perturbative quantization of the action (5.7) would go through a bare action of the form $S_\Lambda = S_{\rm cl} + S_{\rm gf} + S_{\rm gh}$. As already explained, in AS scenarios power-counting renormalizability is in general not respected by the microscopic action, therefore we do not want to restrict ourselves to such bare actions. However we *assume* that $\mathrm{SU}(N_{\rm L})$ gauge symmetry be a fundamental symmetry, as opposed to an emergent one. That is, even if we do not know the precise form of the bare action, we expect that it will be separable into two parts, $S_\Lambda[\Phi, W, \bar{W}] = \bar{S}_\Lambda[\Phi, \bar{W}] + S_\Lambda^{\rm gauge}[\Phi, W, \bar{W}]$, the first one enjoying gauge symmetry and the second one being some kind of generalized gauge-fixing, vanishing if one sets $W = \bar{W}$. The fundamental gauge symmetry then ties these two parts together by means of the BRST invari-

ance of the bare action. The same considerations of course apply to the possible FP microscopic action. Since we cannot analyze the infinite possible forms for these actions, we need to make some ansatz for them. For this task, the requirement of perturbative renormalizability can be of some help, since we know that at "low" energies some IR-relevant parameters must be non-vanishing for phenomenological reasons. Thus, we can consider a truncation of theory space containing these power-counting renormalizable terms, plus some other perturbatively non-renormalizable interactions parameterizing our ignorance about the high-energy dynamics.

In this work the choice of the truncation is very simple: we RG-improve the perturbatively renormalizable bare action, allowing for a scale dependence of the coupling constants already appearing in it, furthermore we admit a generic potential for the scalar field, and we also introduce wave function renormalizations for the matter fields. Explicitly

$$\begin{aligned}\Gamma_k = \int d^d x \Big[& U_k(\rho) + Z_{\varphi,k}(D^\mu \varphi)^\dagger (D_\mu \varphi) + i(Z_{\mathrm{L},k}\bar{\psi}^a_{\mathrm{L}} \not{D}^{ab} \psi^b_{\mathrm{L}} + Z_{\mathrm{R},k}\bar{\psi}_{\mathrm{R}} \not{\partial} \psi_{\mathrm{R}}) \\ & + \bar{h}_k \bar{\psi}_{\mathrm{R}} \varphi^{a\dagger} \psi^a_{\mathrm{L}} - \bar{h}_k \bar{\psi}^a_{\mathrm{L}} \varphi^a \psi_{\mathrm{R}} + \frac{Z_{W,k}}{4} F^i_{\mu\nu} F^{i\mu\nu} + \frac{Z_{\varphi,k}}{2\alpha} G^i G^i - \bar{c}^i \mathcal{M}^{ij} c^j \Big] . \end{aligned} \tag{5.11}$$

As far as the gauge-fixing G^i and the corresponding operator $\mathcal{M}$ are concerned, we would like them to enjoy the good properties of both background-gauge and R_α-gauge. Therefore we choose them as in (5.9,5.10), but in these expressions we replace the ordinary differentiation with covariant derivatives $\bar{D}$ w.r.t. the background field $\bar{W}$. However, due to the extreme simplification of the present truncation, and since we are interested only in an approximate one-loop beta function for the gauge coupling, this choice is practically indistinguishable from a non-background-covariant R_α-gauge. Let us stress that we neglect all local non-derivative terms in the spinor and in the ghost sectors different from the ones in (5.11) and that we keep a naive pattern of gauge-breaking BRST-preserving interactions. Furthermore in our study we neglect the difference between the interactions involving the background gauge field and those involving its quantum expectation value. In the particular case $N_{\mathrm{L}} = 2$ a different family of chiral Yukawa couplings exists, involving the Levi-Civita tensor ε_{ab}. In the present work we are not going to take the possibly important effect of these operators into account.

To sum up, the subset of theory space we are considering is parameterized by $Z_{\varphi,k}$, $Z_{\mathrm{L},k}$, $Z_{\mathrm{R},k}$, $Z_{W,k}$, α, $\bar{h}_k$, v and all the parameters contained in U_k different from v itself. It is also useful to introduce a simplifying notation for the masses in the SSB regime, which however are not independent from the parameters listed above. The mass matrix for the gauge bosons is given by

$$\bar{m}^{2\ ij}_W = \frac{1}{2} Z_\varphi \bar{g}^2 v^2 \{T^i, T^j\}_{\hat{n}\hat{n}} \tag{5.12}$$

and since it is diagonalizable, we can (and we will) choose a basis in adjoint color space where

$$\bar{m}_W^{2\,ij} = \bar{m}_{W,i}^2 \delta^{ij} \quad \text{(no sum over i)} \,. \tag{5.13}$$

The scalar mass matrix

$$\bar{m}_\varphi^{2\,ab} = \nu^2 U_k'' \left(\frac{\nu^2}{2} \right) \hat{n}^a \hat{n}^{\dagger b} \,.$$

In a diagonalizing basis becomes $\bar{m}_\varphi^{2\,ab} = \bar{m}_{\varphi,a}^2 \delta^{ab}$ (no sum over a), and of course there is only one nonvanishing eigenvalue for the radial mode. Furthermore the "top mass", i.e. the mass of the $\psi_{\mathrm{L}}^{\hat{n}}$ mode, is given by

$$\bar{m}_\psi = \frac{\bar{h}_k \nu}{\sqrt{2}} \,. \tag{5.14}$$

5.3 Renormalization flow

All the parameters inside the truncation are expected to have a nontrivial dependence on the scale k, as indicated by the subscripts we introduced. A large part of this dependence can be trivially predicted, at least in a neighborhood of the GFP, on the basis of the canonical dimensionality of these parameters, therefore it is useful to redefine the couplings in such a way to wash out this dimensional scaling. In so-doing, we are taking a free theory as a reference, by assigning to it a vanishing RG flow. Since this redefinition corresponds to a rescaling of the fields, we can include in it also the wave function renormalizations, so that in the new Lagrangian the fields will be canonically normalized at every scale. One calls the corresponding parameters "dimensionless" and they are given by

$$h_k^2 = \frac{k^{d-4} \bar{h}_k^2}{Z_\varphi Z_{\mathrm{L}} Z_{\mathrm{R}}}, \qquad g^2 = \frac{\bar{g}^2}{Z_W k^{4-d}},$$
$$u_k(\tilde{\rho}) = k^{-d} U_k(Z_\varphi^{-1} k^{d-2} \tilde{\rho}), \qquad \kappa_k = \frac{Z_\varphi \nu^2}{2k^{d-2}} = \tilde{\rho}_{\min}$$

where the last equation is for the value of $\tilde{\rho}$ that minimizes the effective potential at every scale k. Accordingly, the dimensionless masses are defined as

$$m_{W,i}^2 = \frac{\bar{m}_{W,i}^2}{Z_W k^2}, \quad m_{\varphi,a}^2 = \frac{\bar{m}_{\varphi,a}^2}{Z_\varphi k^2}, \quad m_\psi = \frac{\bar{m}_\psi}{\sqrt{Z_{\mathrm{L}} Z_{\mathrm{R}}} k} \,.$$

The remaining parameter a, that is already dimensionless, will be required to vanish at every scale, thus fixing Landau gauge at every RG step. At a FP also the beta functions of the previous dimensionless couplings must vanish. The contribution of the non-canonical field renor-

malizations to these beta functions is encoded in the following scale-dependent anomalous dimensions

$$\begin{aligned}\eta_\varphi &= -\partial_t \log Z_\varphi\,, \qquad \eta_W = -\partial_t \log Z_W \\ \eta_{\rm L} &= -\partial_t \log Z_{\rm L}\,, \qquad \eta_{\rm R} = -\partial_t \log Z_{\rm R}\,.\end{aligned}$$

Setting the anomalous dimensions to zero defines the leading-order derivative expansion. At next-to-leading order, it is important to distinguish between $Z_{{\rm L},k}$ and $Z_{{\rm R},k}$ as they acquire different loop contributions, see below. Let us stress that, since we are going to compute the running of only one coupling in the pure gauge sector, by projecting the flow on the squared field-strength, $\bar{g}$ can be considered as k-independent and the nontrivial renormalization of g is completely encoded in η_W. For our purposes, we use a linear regulator function R_k that is optimized for the present truncation [41, 42]. The definitions of this regulator and of the threshold functions which appear in the beta functions of the matter sector are given in App. B.1.

5.3.1 Flow equations for the matter couplings

Detailed information about the derivation of the flow equations for the matter sector of this truncation, in arbitrary Euclidean space dimensions d and for any $N_{\rm L}$, is given in App. B.2. Introducing the abbreviation $v_d = 1/(2^{d+1}\pi^{d/2}\Gamma(d/2))$, and calling d_γ the dimension of the representation of the Clifford algebra (i.e. our γ's are $d_\gamma \times d_\gamma$ matrices), without performing further approximations apart for the ones already discussed, we find the set of equations showed in the following. The flow of the potential is described by

$$\begin{aligned}\partial_t u_k &= -d u_k + (d-2+\eta_\varphi)\tilde{\rho} u_k' + 2v_d\Big\{ -2(N_{\rm L}^2-1) l_0^{({\rm G})d}(0) \\ &+ \sum_{i=1}^{N_{\rm L}^2-1}\Big[(d-1) l_{0{\rm T}}^{({\rm GB})d}\left(m_{W,i}^2\right) + l_{0{\rm L}}^{({\rm GB})d}(0)\Big] + (2N_{\rm L}-1) l_0^d(u') + l_0^d(u'+2\tilde{\rho}u'') \\ &- d_\gamma\Big[(N_{\rm L}-1) l_{0{\rm L}}^{({\rm F})d}(0) + l_{0{\rm L}}^{({\rm F})d}\left(\tilde{\rho}h_k^2\right) + l_{0{\rm R}}^{({\rm F})d}\left(\tilde{\rho}h_k^2\right)\Big]\Big\}\end{aligned} \tag{5.15}$$

For the symmetric phase, one usually expands the effective potential around zero field,

$$u_k = \sum_{n=1}^{N_p} \frac{\lambda_{n,k}}{n!}\tilde{\rho}^n = m_k^2\tilde{\rho} + \frac{\lambda_{2,k}}{2!}\tilde{\rho}^2 + \frac{\lambda_{3,k}}{3!}\tilde{\rho}^3 + \cdots \tag{5.16}$$

while for the SSB phase, where the minimum of the effective potential u_k acquires a nonzero value $\kappa_k := \tilde{\rho}_{\min} > 0$, it is more efficient to use a different expansion

$$u_k = \sum_{n=2}^{N_p} \frac{\lambda_{n,k}}{n!}(\tilde{\rho} - \kappa_k)^n = \frac{\lambda_{2,k}}{2!}(\tilde{\rho} - \kappa_k)^2 + \frac{\lambda_{3,k}}{3!}(\tilde{\rho} - \kappa_k)^3 + \cdots . \tag{5.17}$$

Given the flow of u_k (5.15), the flows of m_k^2 or $\lambda_{n,k}$ in both phases can be read off from an expansion of the flow equation and a comparison of coefficients. For the flow of κ_k, we use the fact that the first derivative of u_k vanishes at the minimum, $u_k'(\kappa_k) = 0$. This implies

$$\begin{aligned} 0 = \partial_t u_k'(\kappa_k) &= \partial_t u_k'(\tilde{\rho})|_{\tilde{\rho}=\kappa_k} + (\partial_t \kappa_k) u_k''(\kappa_k) \\ \Rightarrow \partial_t \kappa_k &= -\frac{1}{u_k''(\kappa_k)} \partial_t u_k'(\tilde{\rho})|_{\tilde{\rho}=\kappa_k} . \end{aligned} \tag{5.18}$$

Note that the expansion coefficients $\lambda_{n,k}$ in Eqs. (5.16) and (5.17) are not identical. Since there is little risk that the notation of the different regimes interferes with each other, we refrain from introducing different symbols.

In the SSB regime, the flow of the Yukawa coupling and the scalar anomalous dimension for the NGB can, in principle, be different from that of the radial mode. As the NGB's as such are not present in the standard model, we compute the Yukawa coupling and the scalar anomalous dimension by projecting the flow onto the radial scalar operators in the SSB regime. Note that this strategy is different from that used for critical phenomena in other Yukawa or bosonic systems, where the Nambu-Goldstone modes can dominate criticality. Accordingly, the flow of the Yukawa coupling h_k can be derived and we find the same result already presented in [106], that is

$$\begin{aligned} \partial_t h_k^2 &= (d - 4 + \eta_\varphi + \eta_L + \eta_R) h_k^2 + 4 v_d h_k^4 \Big\{ - l_{1,1}^{(FB)d}(\tilde{\rho} h_k^2, u_k') + l_{1,1}^{(FB)d}(\tilde{\rho} h_k^2, u_k' + 2\tilde{\rho} u_k'') \\ &+ (2\tilde{\rho} u_k'') l_{1,2}^{(FB)d}(\tilde{\rho} h_k^2, u_k') - (6\tilde{\rho} u_k'' + 4\tilde{\rho}^2 u_k''') l_{1,2}^{(FB)d}(\tilde{\rho} h_k^2, u_k' + 2\tilde{\rho} u_k'') \\ &+ (2\tilde{\rho} h_k^2) l_{2,1}^{(FB)d}(\tilde{\rho} h_k^2, u_k') - (2\tilde{\rho} h_k^2) l_{2,1}^{(FB)d}(\tilde{\rho} h_k^2, u_k' + 2\tilde{\rho} u_k'') \Big\} . \end{aligned} \tag{5.19}$$

The vanishing of the gauge contributions to the running of this coupling is a special feature of

Landau gauge, and it is shortly explained in App. B.2. Finally the anomalous dimensions

$$\begin{aligned}\eta_\varphi &= \frac{8v_d}{d}\Big(\tilde\rho(3u_k''+2\tilde\rho u_k''')^2 m_{2,2}^d(u_k'+2\tilde\rho u_k'') + (2N_{\rm L}-1)\tilde\rho u_k''^2 m_{2,2}^d(u_k') \\ &+ d_\gamma h_k^2 m_4^{({\rm F})4}(\tilde\rho h_k^2) - d_\gamma\tilde\rho h_k^4 m_2^{({\rm F})4}(\tilde\rho h_k^2)\Big) - \frac{8v_d(d-1)}{d}\sum_{i=1}^{N_{\rm L}^2-1}\Bigg\{ \\ &2g^2\sum_{a=1}^{N_{\rm L}} T^i_{\hat n a}T^i_{a\hat n}\, l_{1,1}^{({\rm BGB})d}\left(u_k', m_{W,i}^2\right) + \frac{m_{W,i}^4}{\tilde\rho}\left[2a_1^d\left(m_{W,i}^2\right) + m_2^{({\rm GB})d}\left(m_{W,i}^2\right)\right]\Bigg\}\end{aligned} \tag{5.20}$$

$$\eta_{\rm R} = \frac{4v_d}{d}h_k^2\left[m_{1,2}^{({\rm FB})d}(\tilde\rho h_k^2, u_k'+2\tilde\rho u_k'') + m_{1,2}^{({\rm FB})d}(\tilde\rho h_k^2, u_k') + 2(N_{\rm L}-1)m_{1,2}^{({\rm FB})d}(0,u_k')\right] \tag{5.21}$$

$$\begin{aligned}\eta_{\rm L} &= \frac{4v_d}{d}h_k^2\left[m_{1,2}^{({\rm FB})d}(\tilde\rho h_k^2, u_k'+2\tilde\rho u_k'') + m_{1,2}^{({\rm FB})d}(\tilde\rho h_k^2, u_k')\right] + \frac{8v_d(d-1)}{d}g^2\sum_{i=1}^{N_{\rm L}^2-1}\Bigg\{ \\ &\sum_{a=1}^{N_{\rm L}} T^i_{\hat n a}T^i_{a\hat n}\left[m_{1,2}^{({\rm FGB})d}\left(0,m_{W,i}^2\right) - a_3^d\left(0,m_{W,i}^2\right)\right] \\ &+ (T^i_{\hat n\hat n})^2\left[m_{1,2}^{({\rm FGB})d}\left(\tilde\rho h_k^2, m_{W,i}^2\right) - m_{1,2}^{({\rm FGB})d}\left(0,m_{W,i}^2\right) - a_3^d\left(\tilde\rho h_k^2, m_{W,i}^2\right) + a_3^d\left(0,m_{W,i}^2\right)\right]\Bigg\}.\end{aligned} \tag{5.22}$$

If, in the chosen basis in fundamental color algebra, the direction of the vev $\hat n$ has a single non-vanishing component, i.e. if $\hat n^a \propto \delta^{aA}$, the anomalous dimension of the left-handed fermion takes a simpler form

$$\begin{aligned}\eta_{\rm L} &= \frac{4v_d}{d}h_k^2\left[m_{1,2}^{({\rm FB})d}(\tilde\rho h_k^2, u_k'+2\tilde\rho u_k'') + m_{1,2}^{({\rm FB})d}(\tilde\rho h_k^2, u_k')\right] \\ &+ \frac{8v_d(d-1)}{d}g^2\sum_{a=1}^{N_{\rm L}}\sum_{i=1}^{N_{\rm L}^2-1} T^i_{Aa}T^i_{aA}\left[m_{1,2}^{({\rm FGB})d}\left(\delta^{aA}\tilde\rho h_k^2, m_{W,i}^2\right) - a_3^d\left(\delta^{aA}\tilde\rho h_k^2, m_{W,i}^2\right)\right].\end{aligned} \tag{5.23}$$

In all the eqs. (5.19-5.23) the whole r.h.s. is to be evaluated at the value $\tilde\rho = \kappa_k$ that minimizes the potential u_k. The explicit form of these equations for the linear regulator can be found in App. B.2.

5.3.2 Flow equation for the gauge coupling

While for the matter couplings we performed the computation for generic d, d_γ, $N_{\rm L}$ and without further approximations apart for the chosen truncation and gauge, for the gauge coupling

we limit ourselves to an approximate one-loop beta function in $d = 4$, $d_\gamma = 2$ and $N_{\rm L} = 2, 3$. The further approximation applies to the massive threshold behavior of the radial left-handed spinor, and the details of it and of the derivation of the flow equation can be found in App. B.3. This approximation is expected not to affect our qualitative investigations because we are interested only in the regime of weak gauge coupling. To summarize, we can write the gauge one-loop β-function as

$$\begin{aligned} \partial_t g^2 &= g^2 \ \eta_W \\ \eta_W &= \frac{-g^2}{16\pi^2}\Big(\frac{22N_{\rm L}}{3}\,{\rm L}_W(m^2_{W,i}) \,-\, \frac{2N_{\rm L}}{3}{\rm L}_\psi(m^2_\psi) - \frac{N_{\rm L}}{3}{\rm L}_\varphi(m^2_{\varphi,a})\Big) \end{aligned} \tag{5.24}$$

and for SU(2) we have, in the regime of SSB

$$\begin{aligned} {\rm L}_W(m^2_W) &= \frac{1}{44}\left(21 + \frac{21}{1+m^2_W} + 2\right)\,, & m^2_W &= \frac{g^2\kappa_k}{2} \\ {\rm L}_\psi(m^2_\psi) &= \frac{1}{4}\left(1 + \frac{1}{1+m^2_\psi}\right)\,, & m^2_\psi &= h^2_k\kappa_k \\ {\rm L}_\varphi(m^2_\varphi) &= \frac{1}{4}\left(1 + \frac{1}{1+m^2_\varphi}\right)\,, & m^2_\varphi &= 2\lambda_{2,k}\kappa_k \end{aligned} \tag{5.25}$$

where the threshold function ${\rm L}_\psi(m^2_\psi)$ is not exact but accurate enough for our purposes. If we specify a constant pseudo-abelian magnetic background field

$$F^i_{\mu\nu} = \hat{m}^i F_{\mu\nu} \quad , \quad \hat{m}_i\hat{m}^i = 1 \quad , \quad F_{\mu\nu} = B\varepsilon^\perp_{\mu\nu}$$

where the constant antisymmetric tensor ε characterizes the space directions which are affected by the constant magnetic field upon the Lorentz force, the definition of the threshold functions for general $N_{\rm L}$ can be given in terms of v_a, the eigenvalues of $(\hat{m}_i T^i)^{ab}$ for the fundamental representation, and in terms of v_i, the eigenvalues of $(\hat{m}_l\tau^l)^{ij} = i\hat{m}_l f^{ilj}$ for the adjoint representation, as follows (using the linear regulator)

$$\begin{aligned} L_W(m^2_W) &= \frac{3}{22N_{\rm L}}\left[\frac{21}{3}\sum_{i=0}^{N^2_{\rm L}-1}\frac{|v_i|^2}{1+m^2_{W,i}} + \frac{N_{\rm L}}{3}\right] \\ L_\psi(m^2_\psi) &= \frac{2}{N_{\rm L}}\sum_{a=1}^{N_{\rm L}}\frac{1}{1+h^2\kappa\delta^{aA}}|v_a|^2 \\ L_\varphi i(m^2_\varphi) &= \frac{2}{N_{\rm L}}\sum_{a=1}^{N_{\rm L}}\frac{1}{1+m^2_{\varphi,a}}|v_a|^2\,. \end{aligned}$$

Recall that for all the gauge groups only one particular component of $m^2_{\varphi,a}$ is nonvanishing and equal to $2\lambda_2\kappa$. Since we are interested in the search for FP's with a vanishing gauge coupling, the exact threshold behaviour is unnecessary and for $N_{\rm L} > 2$ we can roughly approximate the gauge beta function by

$$\partial_t g^2 = -\frac{g^4}{16\pi^2}\left(\frac{22N_{\rm L}}{3(1+\frac{2\kappa g^2}{N_{\rm L}})} - \frac{2}{3} - \frac{1}{6}\left(\frac{N_{\rm L}^2-1}{N_{\rm L}^2} + \frac{1}{N_{\rm L}^2(1+2\kappa\lambda)}\right)\right) . \tag{5.26}$$

5.3.3 Parameter constraints

Let us finally discuss several constraints on the couplings as, e.g., dictated by physical requirements as well as by our truncation. As our truncation is based on a derivative expansion, satisfactory convergence is expected if the higher derivative operators take little influence on the flow of the leading-order terms. In the present case, the leading-order effective potential receives higher-order contributions only through the anomalous dimensions. Therefore, convergence of the derivative expansion requires

$$\eta_{\rm L}, \eta_{\rm R}, \eta_\varphi \lesssim \mathcal{O}(1).$$

This condition will serve as an important quality criterion for our truncation. The symmetric regime is characterized by a minimum of u_k at vanishing field. A simple consequence is that the mass term needs to be positive. Also, the potential should be bounded from below, which in the polynomial expansion translates into a positive highest nonvanishing coefficient,

$$m_k^2,\ \lambda_{n_{\max},k} > 0.$$

In the SSB regime, the minimum must be positive, $\kappa_k > 0$, the potential should be bounded, and in addition the potential at the minimum must have positive curvature,

$$\kappa_k,\ \lambda_{n_{\max},k},\ \lambda_{2,k} > 0.$$

Osterwalder-Schrader positivity requires

$$h_k^2 > 0.$$

Finally, if one accepts Dyson's suggestion it might be necessary to ask

$$g^2 > 0$$

in order to avoid unitarity violations. Beyond that, there are no constraints on the size of the couplings as in perturbation theory.

5.3.4 Mass parameterization

In the next sections we will present an analysis of the flow equations introduced in the previous subsections, and the results of the search for NGFP's. There, as a first step towards a complete understanding of the RG within our truncation, we will consider a simple polynomial expansion of the effective potential and we will set $\lambda_{3,k}$ and higher terms to zero. The anomalous dimensions of the matter fields can then be expressed exclusively in terms of the couplings $\kappa, \lambda_2, h^2, g^2$. In this case, substituting the η's by these expressions, we obtain the following flow equations in the matter sector

$$\begin{aligned} \partial_t \kappa &= \beta_\kappa(\kappa, \lambda_2, h^2, g^2) \\ \partial_t \lambda_2 &= \beta_\lambda(\kappa, \lambda_2, h^2, g^2) \\ \partial_t h^2 &= \beta_h(\kappa, \lambda_2, h^2, g^2). \end{aligned}$$

It is useful to reparameterize the system in terms of the particle masses that occur in the arguments of the threshold functions, therefore, dropping numerical factors, we introduce the mass parameters

$$\mu_{\mathrm{H}} := \kappa\lambda_2, \quad \mu_{\mathrm{F}} := \kappa h^2, \quad \mu_{\mathrm{G}} := \kappa g^2 .$$

The actual squared masses, m_φ^2, m_ψ^2 and m_W^2, are then proportional to these parameters. The flow equations for the μ's can then be computed from the original set of flow equations for g^2, κ, λ_2 and h^2, yielding

$$\begin{aligned} \partial_t \mu_{\mathrm{H}} &= (\partial_t \kappa)\lambda_2 + \kappa(\partial_t \lambda_2), \\ \partial_t \mu_{\mathrm{F}} &= (\partial_t \kappa) h^2 + \kappa(\partial_t h^2), \\ \partial_t \mu_{\mathrm{G}} &= (\partial_t \kappa) g^2 + \kappa(\partial_t g^2). \end{aligned}$$

Aside these three equations, we need to take into account the flow of g as defined by eqs. (5.24,5.25).

5.4 Limit of asymptotically free gauge coupling

Because of the mechanism based on the conformal threshold behavior of the vev explained in the first section of this chapter, we focus our attention on the SSB regime. This is also supported by the fact that at a first numerical analysis no NGFP has been found in the symmetric regime.

Thus, within the simple mass parameterization of theory space described above, we observe that changing the gauge coupling by hand leads us to the FP depicted in Fig.5.1

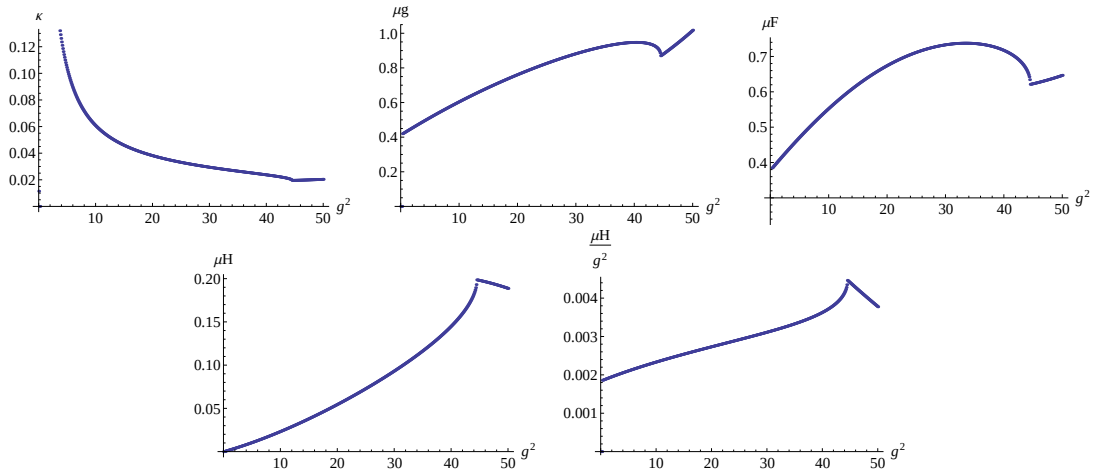

Figure 5.1: Fixed points in the mass parameterization for $N_L = 2$ depending on the gauge coupling.

Thus we suspect the following behavior of the mass terms as $g^2 \to 0$:

$$\mu_H \to 0 \quad (g^2 \to 0), \tag{5.27}$$
$$\frac{\mu_H}{g^2} \to \text{const.} \quad (g^2 \to 0), \tag{5.28}$$
$$\mu_F \to \text{const.} \quad (g^2 \to 0), \tag{5.29}$$
$$\mu_G \to \text{const.} \quad (g^2 \to 0). \tag{5.30}$$

We now want to discuss the limit $g^2 \to 0$ in the flow equations for the mass parameters. As the ratio μ_H/g^2 converges to a constant we set $\mu_H = \chi g^2$ in the flow equations and perform the limit $g^2 \to 0$ afterwards, yielding

$$\partial_t g^2 = 0, \quad \partial_t \mu_H = 0, \quad \partial_t \mu_F = \frac{\mu_F}{\mu_G}(\partial_t \mu_G), \tag{5.31}$$

As the flow equations for g^2 and μ_H vanish identically in this limit they are suitable for a fixed-point solution, however, not to determine the position of this fixed point for the other parameters (μ_G, μ_F), as they don't provide nontrivial algebraic equations. Further, the flow equations for μ_G and μ_F are proportional in this limit and therefore we are left with only one equation for the three remaining parameters, given by the flow of μ_G, which is discussed in the next sections for different values of N_L.

5.4.1 Fixed point for $N_L = 2$

The case $N_L = 2$ is the simplest one, concerning the algebra. Recall that here we miss a possibly important invariant in the truncation, the Yukawa coupling involving the Levi-Civita tensor ε_{ab}. However, the qualitative features of the algebra concerning its fixed-point structure remain the same as we go to higher N_L, see next section. Here, we find

$$\partial_t \mu_G = -2\mu_G + \frac{9\mu_G - 8\mu_F + (10 + 9\mu_F - 2\mu_G)\mu_G\mu_F}{16\pi^2\chi(1+\mu_F)^2(2+\mu_G)^2}. \tag{5.32}$$

To determine a line of fixed points of the mass parametrization in the asymptotically free limit we set up the flow equation for the ratio $\chi = \mu_H/g^2$:

$$\partial_t \chi = \partial_t\left(\frac{\mu_H}{g^2}\right), \tag{5.33}$$

which is supposed to vanish at the fixed point. This yields

$$\chi^* = -\frac{1}{32\pi^2}\left(\frac{2\mu_F(1+3\mu_F)}{(1+\mu_F)^3\mu_G} - \frac{9(2+3\mu_G)}{(2+\mu_G)^3}\right). \tag{5.34}$$

We can plug this result into the flow equation for μ_G to solve for the fixed point giving three solutions for μ_F as a function of μ_G. The flows of μ_F and μ_G are proportional and their ratio remains undetermined for the moment.

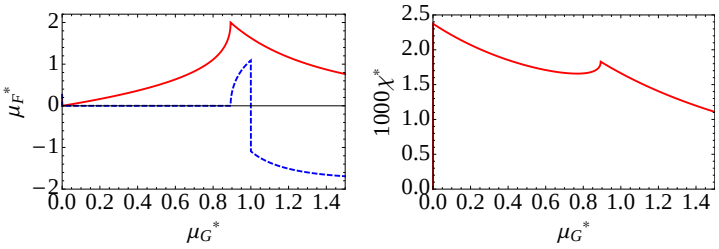

Figure 5.2: Fixed points in the mass parameterization for $N_L = 2$. Left panel: Real part (red solid) and imaginary part (blue dashed).

If we want to single out the FP of Fig.5.1 we have to ask for vanishing beta function of the Yukawa coupling. This beta function is exactly zero in the limit $g^2 \to 0$. Using finite values for g^2 instead and tuning the gauge coupling towards zero we find that $\mu_G^* = 0.413552$ which fits to the plot of Fig.5.1. Thus the FP is given by

$$(\mu_H^*, \mu_F^*, \mu_G^*) = (0, 0.38, 0.41)\,. \tag{5.35}$$

5.4.2 Fixed point for $N_L = 3$

The case $N_L = 3$ (and even higher ones) follows from a straightforward generalization of the previous section.

$$\begin{aligned}\partial_t \mu_G &= -2\mu_G + \frac{1}{16\pi^2\chi(1+\mu_F)^2(2+\mu_G)^2(3+2\mu_G)^2} \\ &\times\Big(24 \quad (6+\mu_F(5+6\mu_F))\mu_G + 2(90+\mu_F(107+90\mu_F))\mu_G^2 \\ &+ \quad ((57+\mu_F(58+57\mu_F))\mu_G^3 - 8\mu_F\mu_G^4 - 72\mu_F\Big). \end{aligned} \tag{5.36}$$

and

$$\begin{aligned}\chi^* &= -\frac{1}{32\pi^2}\Big(\frac{2\mu_F(1+3\mu_F)}{(1+\mu_F)^3\mu_G} + \frac{48}{(2+\mu_G)^3} \\ &\quad -\frac{36}{(2+\mu_G)^2} + \frac{54}{(3+2\mu_G)^3} - \frac{27}{(3+2\mu_G)^2}\Big). \end{aligned} \tag{5.37}$$

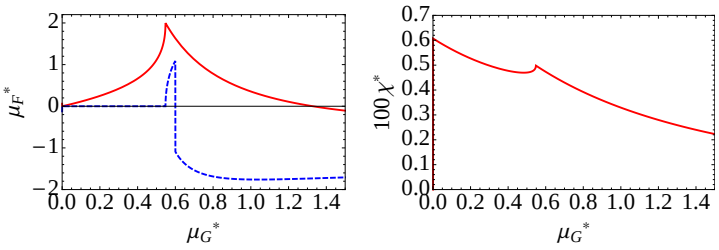

Figure 5.3: Fixed points in the mass parameterization for $N_L = 3$. Left panel: Real part (red solid) and imaginary part (blue dashed).

5.5 Critical Exponents

We define the stability matrix

$$B_i^{\ j} = \frac{\partial \beta_i}{\partial g_j}\Big|_{g^*}, \tag{5.38}$$

in terms of the couplings g_j from the mass parameterization $\{g_*^2, \mu_F^*, \mu_G^*, \chi^*\}$. The negative eigenvalues of the stability matrix are the critical exponents and determine the stability properties of our system. In the following we perform a stability analysis of our system in mass parameterization for $N_L = 2$ and $N_L = 3$.

For $N_L = 2$ the critical exponents depending on μ_g are depicted in Fig. 5.4. Two of the four critical exponents vanish and thus are hardly visible. The other two are distinguishable for $\mu_g < 0.38$. For $\mu_g > 0.38$ they are a complex pair which can be seen on the right plot in Fig. 5.4. Note that the critical exponents corresponding to the fixed point $(g_*^2, \mu_F^*, \mu_g^*, \chi^*) =$

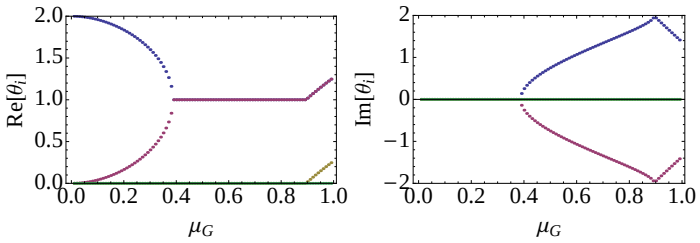

Figure 5.4: Critical exponents for $N_L = 2$.

$(0, 0.38, 0.41, 0.0018)$ are

$$\theta_{1/2} = 1 \pm 0.36i\,, \theta_3 = \theta_4 = 0\,. \tag{5.39}$$

The first two critical exponents are building a complex pair and thus cause a spiraling approach towards the fixed point on the corresponding hypersurface. The normal direction to this spiral is the vector $v_n = (0, 0.00044, -0.00040, 0)$. The last two are marginal directions and we have to investigate them in more details.

For $N_L = 3$ the critical exponents are shown in Fig. 5.5.

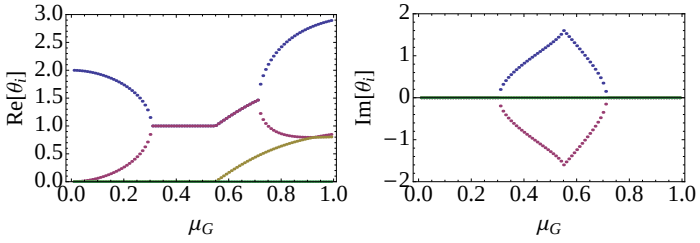

Figure 5.5: Critical exponents for $N_L = 3$.

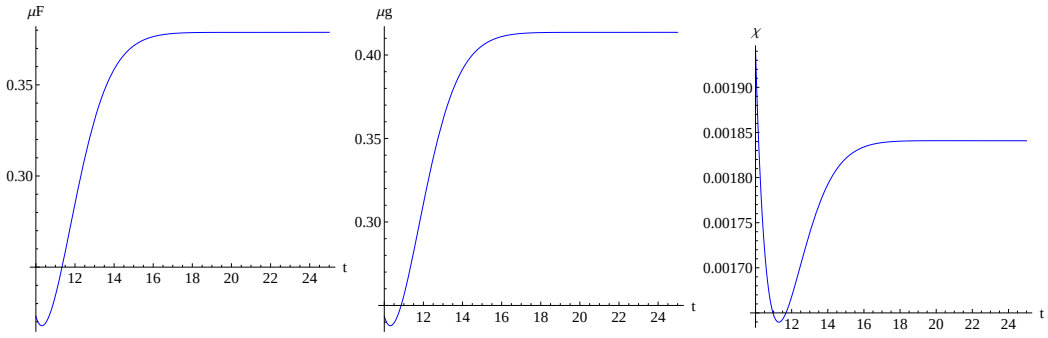

Figure 5.6: Flow of the dimensionless couplings from the UV to IR for $N_{\mathrm{L}} = 2$, with starting values $t_{\mathrm{UV}} = 25$, $g^2_{\mathrm{UV}} = 1/(2\pi)$, $\mu_{\mathrm{G}}^{\mathrm{UV}} = \mu_g^*$, $\mu_{\mathrm{F}}^{\mathrm{UV}} = \mu_{\mathrm{F}}^*$, $\chi^{\mathrm{UV}} = \chi^*$.

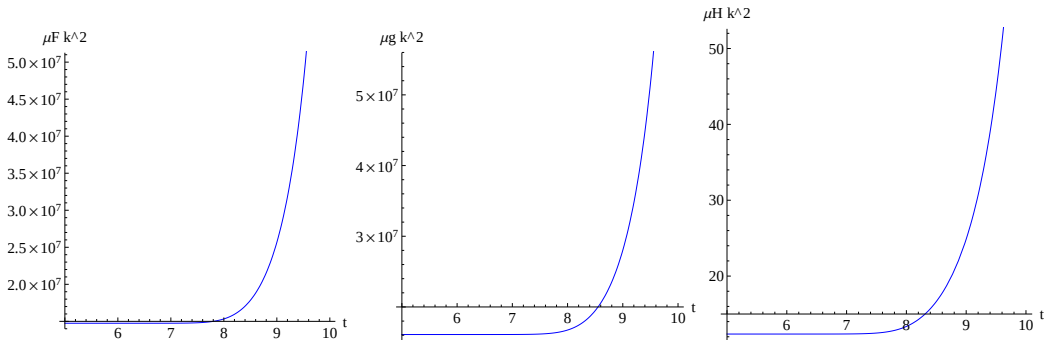

Figure 5.7: Flow of the dimensional masses from the UV to IR for $N_{\mathrm{L}} = 2$ with starting values $t_{\mathrm{UV}} = 10$, $g^2_{\mathrm{UV}} = 1/(2\pi)$, $\mu_g^{\mathrm{UV}} = \mu_g^*$, $\mu_{\mathrm{F}}^{\mathrm{UV}} = \mu_{\mathrm{F}}^*$, $\mu_{\mathrm{H}}^{\mathrm{UV}} = 1.05(\chi^* g^2_{\mathrm{UV}})$.

5.6 Flow from the ultraviolet to the electroweak scale

We observe that the flow to the infrared is divided into three regimes. The first is a fixed point regime: at a large scale (say $k \sim \Lambda$), the system is near the NGFP and would approach to it even further when moving Λ towards the UV. In this regime the flow is fairly described by the critical exponents of the NGFP. The second is a transition regime, where the system in dimensionless couplings runs rapidly away from the FP. This is shown in the plots of Fig. 5.6 at around $t \simeq 12$. In the standard model, that would correspond to scales up to a few orders of magnitude larger than the electroweak scale. The third is a freeze-out regime, where the dimensional masses almost stop running, a behavior that can be immediately spotted in Fig. 5.7 at around $t \simeq 7$. In terms of the dimensionless quantities of Fig. 5.6 this corresponds to an exponential behavior $\sim e^{-2t} = 1/k^2$ and it occurs approximately for $t < 12$. Roughly speaking, this is the analog of the electroweak scale in the standard model.

Apart for the ones just mentioned, following the flow towards the IR we meet more relevant

scales, that however are unphysical and due to the truncation and therefor we do not show them. For example, after the "electroweak freeze-out" near $t \simeq 7$ of Fig. 5.7, there seems to be another rapid transition in the dimensional masses near $t \simeq -10$ in the scalar Higgs mass and also a small response in the fermion mass. We suspect that this is an artifact generated by the impossibility, within our truncation, to distinguish the radial mode self-interactions from those of the "quasi" NG modes. This is an old problem in the deep IR of these types of truncations based on a Cartesian scalar basis, where the NG modes can contribute to the flow of the radial quantities "forever" towards the deep IR, even though in a non-linear basis it is immediately clear that these modes decouple. Even more in the IR, our flow eventually hits the Landau pole of the gauge coupling and gets destabilized. In summary and in practice, we should just stop the flow after the electroweak freeze out in order to stay away from both: the Nambu-Goldstone artifacts at and the gauge coupling blow up.

Interestingly, if we read off the IR masses in the freeze-out regime we observe that for all starting values that lead to reasonable IR flows the ratio of the top quark mass to the Higgs mass and the gauge boson mass to the Higgs mass are limited to a range between 30 and 70. Furthermore the ratio of the top quark mass and the gauge boson mass is equal for all reasonable flows.

5.7 Summary

In this chapter we have presented a computation of the RG flow of a gauged chiral Yukawa system, including N_{L} left- and one right-handend spinors, as well as N_{L} complex scalars, plus a $\mathrm{SU}(N_{\mathrm{L}})$ vector gauge boson, coupled via a chiral Yukawa and minimal gauge terms, plus a scalar self-interaction. At next-to-leading order in the derivative expansion of the AEA we have derived the complete beta functions and anomalous dimensions of this matter sector, for generic dimension d and rank N_{L}. We have supplemented them with a one loop beta function of the gauge coupling including approximate massive threshold effects in the regime of spontaneous symmetry breaking. With this set of flow equations we have started analyzing the latter regime founding evidence for the presence of a non-Gaußian fixed point (NGFP) in $d = 4$ for both $N_{\mathrm{L}} = 2$ and $N_{\mathrm{L}} = 3$, and we believe that this FP should survive also for bigger values of N_{L}. We have also computed the critical exponents finding values of order one, while the anomalous dimensions remain small and thus support the reliability of the derivative expansion. The analysis of the flow for larger truncations of the potential and for general d and N_{L} is still to be completed and unfortunately we are not able to comment further on it in this thesis.

One motivation for this computation was the opportunity to check that a mechanism for triggering the formation of NGFP's in the RG flow of a theory is provided by the conformal

threshold behavior of the scalar vev. While previous investigations were leading to partial results, spoiled by the dominance of unphysical Nambu-Goldstone boson contributions, in the present approach these contributions are killed by the Higgs mechanism and the regime of spontaneous symmetry breaking indeed exhibits at least a NGFP.

The physical consequences of the presence of such a FP could be many and profound, and they provided further motivations for embarking in these computations. If its existence were confirmed by further studies, this NGFP could provide the solution to one fundamental and long-standing problem of the standard model: the triviality of the scalar sector. This would be the case if the NGFP had the right properties to render the theory asymptotically safe. Furthermore, if the corresponding critical exponents were small enough, this FP could allow also for a partial or total smoothing of the hierarchy problem. Of course, a big difference between the present system and the electroweak sector of the standard model is the presence of a $\mathrm{U}(1)$ sector in the latter, that also brings its own triviality problem. However this is absent in grand unification theories, that could also present the same basic dynamical mechanisms underlying the apparent asymptotic safety of our gauged chiral Yukawa system. Also in these theories, the knowledge of the RG flow outside the perturbative regime is of primary importance for establishing the soundness and the physical predictions of the model. For these reasons we plan to properly analyze the flow of our model also for $N_{\mathrm{L}} > 2$.

In order to confirm the asymptotic safety scenario, future studies will have to consider more general truncations of theory space, in order to rule out the possibility that this NGFP be an artifact of the approximations adopted in the present analysis, and in order to confirm that the number of relevant and marginally-relevant directions for this NGFP is finite and sufficiently small to be useful for phenomenology. This would be an abstract or computational approach to the consolidation and validation of the present theoretical picture, while another possibility for future investigations could be taking a more physical point of view and trying to deduce testable phenomenological consequences of asymptotic safety scenarios in the standard model and in grand unification pictures.

In the present fRG approach and by truncations similar to the ones studied in this chapter, one could also address the important problem of the stability of the quantuum vacuum of the Higgs sector, that is likely to require a nonperturbative approach for its proper understanding.

6
Conclusions

In this thesis we have discussed a representation of quantum mechanics and quantum and statistical field theory based on a functional renormalization flow equation for the one-particle-irreducible average effective action, and we have employed it to get information on some specific systems.

In chapter 2 we have derived this representation from a functional integral representation, stressing one necessary condition for these two frameworks to be equivalent: the presence of a rising delta functional in the regularized path integral. To this end we have derived a proper normalization of this flow equation from a regularization in phase space and we have discussed the consequences of this natural normalization on the contribution of free fields to the constant term of the effective potential, finding an automatic disappearance of the need for quartically divergent counter-terms.

In chapter 3 we have extended this representation to get a description of the full dynamics in phase-space in terms of average effective Hamiltonian actions. Such an extension provides a tool for the nonperturbative quantization also of theories whose bare Hamiltonians are not quadratic in the momenta. We have checked its reliability and its power for computing observables such as the ground state energy and gap of these theories, by a simple exercise in quantum mechanics. We have also considered the possibility to apply this phase-space representation to field theory, either in a Lorentz non-covariant Hamiltonian setup or in a particular

Lorentz covariant formulation. For the latter we have derived a first truncated flow that could deserve future numerical studies in order to look for new universality classes and to test this unusual Lorentz covariant quantization. Then we have discussed more possible applications of this Hamiltonian representation.

One of these applications has been started in chapter 4, were we have addressed the study of linear and nonlinear O(N) models by means of this new tool. Regarding the linear model, it could be interesting to investigate if spontaneous symmetry breaking can occur in a new way, intrinsically involving a nontrivial momentum sector. As far as the nonlinear model is concerned, this representation allows to study these systems in a fully Hamiltonian framework even for bare actions that are quadratic in the momenta. Our starting point in this case has been a critical comparison between this representation and the geometric background field method that is usually adopted for the nonlinear systems. We have outlined how modified Ward-Takahashi identities can be used to take into account the nonlinear symmetry along the RG flow. The final goal of this research project is to study the fixed point structure and the universality classes of both systems and to compare the results with all the other available methods. Unfortunately this work is unfinished because the research is still ongoing.

In chapter 5 we have employed the traditional RG flow equation representation in configuration space for the study of a toy model of the electroweak sector of the standard model or of grand unification theories, in order to see if this tool can give new insights in the understanding of the triviality and of the hierarchy problems. To this end we have derived nonperturbative approximate beta functions for the couplings of this gauged chiral Yukawa system. Specifically, we computed the running of the matter sector at next to leading order in the derivative expansion for generic spacetime and unitary gauge group dimensions, while we improved the one-loop running of the gauge coupling by approximate massive threshold effects. The preliminary analysis of the RG flow shows the presence of an interacting fixed point in the spontaneous symmetry breaking regime, for both SU(2) and SU(3) gauge groups, thus confirming that a conformal threshold behaviour of the scalar vacuum expectation value can lead to such fixed points. We also started the analysis of the linearized RG flow around such a FP as well as of the subsequent flow towards the IR. These result provide a first positive hint for the possible asymptotic safety of the model, that would solve the triviality problem and could lead to a partial or total smoothening of the hierarchy problem.

More detailed discussions about the results of the corresponding investigations can be found in the final sections of each chapter.

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A

Appendices to chapter 3

A.1 The effective Hamiltonian action as the generating functional of 1PI vertex functions

In this appendix we are going to prove that the effective Hamiltonian action is the generating functional of the one particle irreducible (1PI) proper vertices, in the sense that the tree level amplitudes computed with vertex functions and propagators extracted from it are equal to the full perturbative series generated by the bare Hamiltonian action. For the ease of the explanation we limit this discussion to the QM case, choosing $\hbar = 1$ as a unit of action. The proof works just as for the usual Lagrangian effective action [107, 108].

1. Write down a path integral based on a Hamiltonian bare action which is $(1/g)$-times the Hamiltonian effective action, with g an external parameter. This rescaling of the action entails a corresponding rescaling of the Liouville form $\lambda_g \equiv \frac{1}{g}\lambda = \frac{1}{g}\bar{p}d\bar{q}$. Thus, in order to define the new path integral we must adopt a functional measure $\mu_g = \sqrt{\mathrm{Det}\sigma_g}$ corresponding to the symplectic structure $\sigma_g = d\lambda_g$:

$$e^{iW_g[I,J]} = \int [dpdq]\, \mu_g[p,q] e^{\frac{i}{g}\{\Gamma[\bar{p},\bar{q}]+I\cdot\bar{p}+J\cdot\bar{q}\}} . \tag{A.1}$$

2. Recognize that the parameter g allows one to distinguish different loop orders in the perturbative evaluation of this path integral. In fact eqs.(3.17,3.13) show that in the perturbation theory generated by $\Gamma_g \equiv \frac{1}{g}\Gamma$ the vertex functions are proportional to $1/g$ while propagators are proportional to g. Thus any graph with i internal lines and v vertices gives a contribution proportional to g^{i-v}. Since the number of loops is $l = i - v + 1$, any loop expansion is an

expansion in powers of g of the kind

$$W_g[I,J] = \sum_{l=0}^{\infty} g^{l-1} W_{g,l}[I,J] \,. \tag{A.2}$$

3. Evaluate the same path integral by a stationary phase method, an approximation that can be made arbitrarily good by tuning g arbitrarily close to zero. Since by definition the exponent at the stationarity point gives the $W[I,J]$ of eqn. (3.4), one gets

$$e^{iW_g[I,J]} \underset{g\to 0}{\sim} \mu_g[p,q] \left(\mathrm{Det} \frac{1}{2\pi g} \Gamma\,[\bar{p},\bar{q}]^{(2)} \right)^{-\frac{1}{2}} e^{\frac{i}{g} W[I,J]} \,. \tag{A.3}$$

4. Expand the logarithm of the last result in powers of g. Because

$$\begin{aligned} \log \mu_g[p,q] &= -\mathrm{Tr} \log g + \log \mu[p,q] \\ \log \left(\mathrm{Det} \frac{1}{2\pi g} \Gamma\,[\bar{p},\bar{q}]^{(2)} \right)^{-\frac{1}{2}} &= \mathrm{Tr} \log g + \log \left(\mathrm{Det}\, \frac{1}{2\pi} \Gamma\,[\bar{p},\bar{q}]^{(2)} \right)^{-\frac{1}{2}} \end{aligned}$$

the combination of eqs. (A.2) and (A.3) gives

$$\sum_{l=0}^{\infty} g^{l-1} W_{g,l}[I,J] \underset{g\to 0}{\sim} \frac{1}{g} W[I,J] - i \log \left\{ \mu[p,q] \left(\mathrm{Det}\, \frac{1}{2\pi} \Gamma\,[\bar{p},\bar{q}]^{(2)} \right)^{-\frac{1}{2}} \right\}$$

that is: $W_{g,0}[I,J] = W[I,J]$.

A.2 The effective Hamiltonian action from a variational formula on the Hilbert space

This appendix is to prove the proposition of section 2 about the possibility to define the effective Hamiltonian action in the operator representation by means of a variational principle. The following arguments are not original, but just the obvious extension of those presented in [66]. In order to compute the extremum (3.8) with the constraints (3.9) one introduces three Lagrange multipliers $w(t), I(t), J(t)$ and looks for the extremum of $\langle \psi_-, t | i\partial_t - \hat{H} + J(t)\hat{q} + I(t)\hat{p} - w(t) | \psi_+, t \rangle$ with respect to the two states $|\psi_\pm, t\rangle$. Setting the two functional derivatives to zero gives

$$\left(i\partial_t - \hat{H} + J(t)\hat{q} + I(t)\hat{p} \right) |\psi_+, t\rangle = w(t) |\psi_+, t\rangle \tag{A.4}$$

$$\left(i\partial_t - \hat{H} + J(t)\hat{q} + I(t)\hat{p} \right) |\psi_-, t\rangle = w^*(t) |\psi_-, t\rangle \,. \tag{A.5}$$

It is possible to define the states

$$|+,t\rangle = \exp\left\{ i\int_{-\infty}^{t} dt'\, w(t') \right\} |\psi_+,t\rangle \quad , \quad |-,t\rangle = \exp\left\{ -i\int_{t}^{+\infty} dt'\, w^*(t') \right\} |\psi_-,t\rangle \tag{A.6}$$

which solve the following Schrödinger equation

$$\left(i\partial_t - \hat{H} + J(t)\hat{q} + I(t)\hat{p} \right) |\pm,t\rangle = 0 \tag{A.7}$$

and satisfy the boundary conditions: $\lim_{t\to\mp\infty} |\pm,t\rangle = |0\rangle$. In other words, $|+,t\rangle = \hat{U}_{I,J}(t,-\infty)|0\rangle$ and $\langle -,t| = \langle 0|\hat{U}_{I,J}(+\infty,t)$, such that

$$e^{iW[I,J]} = \langle 0|\hat{U}_{I,J}(+\infty,-\infty)|0\rangle = \langle -,t|+,t\rangle = e^{\,i\int_{-\infty}^{+\infty} dt'\, w(t')}\,, \tag{A.8}$$

that is: $W[I,J] = \int_{-\infty}^{+\infty} dt'\, w(t')$. On the other hand, by contracting eq. (A.4) with $\langle \psi_-,t|$ and using the previous equation, along with the constraints (3.9), one finds that for the stationarity states the following relation holds

$$\int_{-\infty}^{+\infty} dt\, \langle \psi_-,t|i\hbar\partial_t - \hat{H}|\psi_+,t\rangle = W[I,J] - \int_{-\infty}^{+\infty} dt\, [J(t)\bar{q}(t) + I(t)\bar{p}(t)] \ . \tag{A.9}$$

To prove that the values of I and J on the r.h.s. are the extremal ones it is necessary to take derivatives of this equation with respect to the sources, and remember that on the l.h.s. the extremal value cannot depend on the Lagrange multipliers, nor can the constraint points $\bar{p}$ and $\bar{q}$ on the r.h.s.

A.3 The realization of the rising delta functional when $k \to \Lambda$

In order to analyze the $k \to \Lambda$ limit of eq. (3.26) we first perform a change of variables in the path integral:

$$p' = p - \bar{p} + (r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar{q}} \quad , \quad q' = q - \bar{q} - (r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar{p}}$$

and then define the complex variable: $z = (p' - iq')/\sqrt{2}$. The result of these manipulations is:

$$\begin{aligned} e^{i\Gamma_k[\bar{p},\bar{q}]} = \int [dz]\,\mu_k \exp i\Bigg\{ \frac{1}{2}\int dt\ (z^* r_k\, i\partial_t z - z r_k\, i\partial_t z^*) - \frac{\delta\Gamma_k}{\delta\bar{q}}\cdot(r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar{p}} \\ + S\left[\bar{p} - (r_k\partial_t)^{-1}\frac{\delta\Gamma_k}{\delta\bar{q}} + \sqrt{2}\Re(z), \bar{q} + (r_k\partial_t)^{-1}\frac{\delta H_k}{\delta\bar{p}} - \sqrt{2}\Im(z) \right] \Bigg\} . \end{aligned}$$

Under the assumption that Γ_k stays finite for any $k \in [0,\Lambda]$, when $k \to \Lambda$ every Γ_k-dependent term on the right hand side (r.h.s.) gets killed by the divergence of r_k. On the other hand, since $\mu_k = \mathrm{Det}\left(\frac{1+r_k}{2\pi}\delta\right)$ (excluding the possible zero eigenvalues), the first term in the exponent to-

gether with the regularized functional measure provides a rising delta functional, constraining z, i.e. $(p-\bar{p})$ and $(q-\bar{q})$, to vanish.[1] Thus in this limit the r.h.s. reduces to $\exp\{iS[\bar{p},\bar{q}]\}$ and the AEHA coincides with the bare Hamiltonian action. To show that a rising delta functional is indeed realized we need to prove that the quadratic form $(z^* r_k\, i\partial_t z - z r_k\, i\partial_t z^*)$ is positive definite. This is not obvious since $i\partial_t$ is a real operator on the spaces of functions one is usually interested in, but whose sign is not fixed. However, if the domain of the functional integral is such that all contributions coming from the time boundaries are vanishing, and if the Fourier transform is allowed, then one can write (the reader should interpret the integrals as generic sums over unspecified domains)

$$\begin{aligned}
& \frac{i}{2}\int_t (z(t)^* r_k\, i\partial_t z(t) - z(t) r_k\, i\partial_t z(t)^*) = \int_t p'(t) r_k\, i\partial_t q'(t) \\
= \;& \frac{1}{2}\int_E r_k(E^2) E\,(p'(-E)q'(E) - q'(-E)p'(E)) \\
= \;& \int_E \theta(E) r_k(E^2) E\,(p'(-E)q'(E) - q'(-E)p'(E)) \\
= \;& i\int_E \theta(E) r_k(E^2) E\,(x_-(E)^* x_-(E) - x_+(E)^* x_+(E))
\end{aligned}$$

where we assumed $q(t)$ and $p(t)$ real, such that for their Fourier transforms satisfy $p(-E) = p(E)^*$ and $q(-E) = q(E)^*$, we defined $x_\pm(E) = (p'(E) \pm iq'(E))/\sqrt{2}$, and we denoted by θ the Heaviside step function. The last equation shows that the diagonalization of the quadratic form gives two complex Gaussians which can be independently rotated to real Gaussians with positive definite inverse variances. In reality they might be not positive definite and allow for zero modes, but we will not discuss this possibility in the present work.

A.4 The average effective Hamiltonian action in Euclidean space and Wick rotation

Of course the Hamiltonian formalism without time makes little sense. However it could be nice to forget about the evaluation of integrals with poles once and for all by working in Euclidean space from the very beginning. In this appendix the reader will find the translation, of some of the main formulas of the present work to Euclidean space and a discussion on the possible equivalence of the theories in Minkowski and Euclidean space, i.e. on the feasibility of a Wick rotation to imaginary time.

Let's start with scalar QM. In this case Wick rotation $(t \to -i\tau)$ of eq. (3.3) with action (3.1) is safe and leads to a convergent path integral

$$e^{W[I,J]} = \int [dpdq]\, \mu[p,q] e^{-\{S[p,q]-Jq-Ip\}}$$

[1] Although the quadratic form $(r_k\, i\partial_t \delta)$ in the exponent and the operator in the measure $(1+r_k)\delta$ asymptotically differ for a factor $i\partial_t$, the path integral is properly normalized [**?**] in such a way to be finite for a free system ($\forall k \in [0,\Lambda]$) and to show a k-independent divergence in the $H=0$ case.

with action

$$S[p,q] = \int d\tau \left[-p(\tau) i\partial_\tau q(\tau) + H(p(\tau), q(\tau))\right] . \tag{A.10}$$

The regularization goes as usual

$$e^{W_k[I,J]} = \int [dpdq]\, \mu_k e^{-\{S[p,q]+\Delta S_k[p,q]-Jq-Ip\}}$$

with ΔS_k and μ_k which can still be chosen according to formulas (3.21) to (3.25) if we replace ∂_t with $-i\partial_\tau$ (the minus sign here is due to the global minus factorized in front of the action). The definition of the AEHA is

$$\Gamma_k[\bar{p},\bar{q}] + \Delta S_k[\bar{p},\bar{q}] = \underset{I,J}{\text{ext}} \left(I\cdot\bar{p} + J\cdot\bar{q} - W_k[I,J]\right)$$

which is equivalent to

$$e^{-\Gamma_k[\bar{p},\bar{q}]} = \int [dpdq]\, \mu_k[p,q] e^{-\left\{S[p,q]+\Delta S_k[p-\bar{p},q-\bar{q}]-(p-\bar{p})\frac{\delta\Gamma_k}{\delta\bar{p}}-(q-\bar{q})\frac{\delta\Gamma_k}{\delta\bar{q}}\right\}} . \tag{A.11}$$

From it the flow equation follows

$$\dot{\Gamma}_k = \frac{1}{2}\text{Tr}\left[\left(\Gamma_k^{(2)} + R_k\delta\right)^{-1}\dot{R}_k\delta\right] - \frac{\dot{\mu}_k}{\mu_k} \tag{A.12}$$

where $R_k\delta = \Delta S_k^{(2)}$. We see that this equation formally differs from the Minkowskian one (3.29) by the absence of the imaginary factor i on the l.h.s, by a global minus factor on the r.h.s. and by the fact that inside R_k we find the operator $i\partial_\tau$ instead of ∂_t. Thus, for instance, in the particular case of an off-diagonal regulator the explicit form of the flow equation becomes

$$\begin{aligned}\dot{\Gamma}_k &= \text{Tr}\left[(-\dot{r}_k i\partial\delta)\left(\left(-r_k i\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{p}}\right) - \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{p}}\left(r_k i\partial\delta + \frac{\delta^2\Gamma_k}{\delta\bar{p}\delta\bar{q}}\right)^{-1}\frac{\delta^2\Gamma_k}{\delta\bar{q}\delta\bar{q}}\right)^{-1}\right] \\ &- \text{Tr}\left[\dot{r}_k\left(1+r_k\right)^{-1}\delta\right] .\end{aligned} \tag{A.13}$$

Next let's consider scalar covariant Hamiltonian QFT. Since π^ν is a vector, Wick rotation involves also its zero component, regardless of whether or not we allow for transverse momenta: $x^0 \to -ix^4$ and $\pi^0 \to -i\pi^4$. However, performing such a replacement in the action (3.52) with Hamiltonian (3.51) one finds that

$$iS \to \int d^dx \left[\frac{1}{2}(\pi - \partial\varphi)^2 - \frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right]$$

therefore the integral over π diverges. In other words such a Wick rotation cannot be performed. The main difference from the case of QM, or the reason for such a failure, is the fact that the momenta are assumed to rotate along with time. Despite this problem, one possible reason for studying a Euclidean covariant Hamiltonian formulation is that we know that the

Euclidean non-covariant Hamiltonian theory makes perfectly sense because it is related by a continuos Wick rotation to the corresponding Minkowskian theory. Therefore the Euclidean covariant formulation can be derived from the non-covariant Hamiltonian formulation and studied as a generalization of it. By definition the bare action of such a covariant Hamiltonian Euclidean theory reads

$$S[\pi^\nu, \varphi] = \int d^d x \left[-\pi^\nu i\partial_\nu \varphi + \mathcal{H}(\pi^\nu, \varphi) \right] . \tag{A.14}$$

Its Hamiltonian quantization in a scheme where only longitudinal momenta are present is based on the functional integral

$$Z[I_\nu, J] = \int \left[d\pi^\nu d\varphi \right] \delta\left[\Pi_\perp^{\nu\rho} \pi_\rho \right] \mu \, e^{-\{S[\pi^\nu,\varphi] - I_\nu \cdot \pi^\nu - J\cdot\varphi\}} .$$

Again, to get a functional RG flow equation representation of this integral on introduces a k-dependence in the bare action and in the measure. In the following we choose an off diagonal quadratic regularization, i.e. of the kind (3.56), but with ∂_ν replaced by $i\partial_\nu$. The definition of the AEHA is the same as in Euclidean quantum mechanics

$$\Gamma_k \left[\bar{\pi}^\nu, \bar{\varphi} \right] + \Delta S_k \left[\bar{\pi}^\nu, \bar{\varphi} \right] = \underset{I_\nu, J}{\text{ext}} \left(I_\nu \cdot \bar{\pi}^\nu + J \cdot \bar{\varphi} - W_k[I_\nu, J] \right) \tag{A.15}$$

wherefrom the usual integro-differential equation

$$e^{-\Gamma_k[\bar{\pi}^\nu,\bar{\varphi}]} = \int \left[d\pi^\nu d\varphi \right] \delta\left[\Pi_\perp^{\nu\rho} \pi_\rho \right] \mu_k \, e^{-\left\{ S[\pi^\nu,\varphi] + \Delta S_k[(\pi-\bar{\pi})^\nu, \varphi-\bar{\varphi}] - (\pi-\bar{\pi})^\nu \frac{\delta\Gamma_k}{\delta\bar{\pi}^\nu} - (\varphi-\bar{\varphi}) \frac{\delta\Gamma_k}{\delta\bar{\varphi}} \right\}} . \tag{A.16}$$

Again, the Euclidean flow equation can be obtained from the Minkowskian one by stripping the imaginary i on the l.h.s., by changing the global sign on the r.h.s. and by replacing $r_k \partial\delta$ with $r_k i\partial\delta$.

As far as fermions are concerned, no new behavior under Wick rotation shows up, because of the identification of configuration space with the reduced phase space.

B

Appendices to chapter 5

B.1 Threshold functions

In general one can choose different regulators for the scalar bosons (B), for the gauge bosons (GB), for the ghosts (G) and for the spinor fermions (F). Then the regularized kinetic (or squared kinetic) terms are given by

$$\begin{aligned} P_{\mathrm{B}} &= p^2(1+r_{k\mathrm{B}})\,, \qquad & P_{\mathrm{F}} &= p^2(1+r_{k\mathrm{F}})^2, \\ P_{\mathrm{GB}} &= p^2(1+r_{k\mathrm{GB}})\,, \qquad & P_{\mathrm{G}} &= p^2(1+r_{k\mathrm{G}})\,. \end{aligned}$$

Accordingly, the loop momentum integrals appearing on the r.h.s. of the flow equation can be classified and corresponding threshold functions defined. If one denotes with $\tilde{\partial}_t$ differentiation w.r.t. $t = \log(k/k_0)$ acting on the regulators only, with $\int_p \equiv \int \frac{d^d p}{(2\pi)^d}$ and with

$v_d = 1/(2^{d+1}\pi^{d/2}\Gamma(d/2))$, these functions read

$$
\begin{aligned}
l_n^d(\omega) &= (n+\delta_{n,0})\frac{k^{2n-d}}{4v_d}\int_p \frac{\partial_t P_{\rm B}}{Z_\varphi (P_{\rm B}+\omega k^2)^{n+1}}\,,\\
l_{n,{\rm L/R}}^{({\rm F})d}(\omega) &= (n+\delta_{n,0})\frac{k^{2n-d}}{2v_d}\int_p \frac{\partial_t\left(Z_{\rm L/R}\,p^2\,r_{\rm F}\right)(1+r_{\rm F})}{Z_{\rm L/R}(P_{\rm F}+\omega k^2)^{n+1}}\\
l_{n{\rm T}}^{({\rm GB})d}(\omega) &= (n+\delta_{n,0})\frac{k^{2n-d}}{4v_d}\int_p \frac{\partial_t\left(Z_W\,p^2\,r_{\rm GB}\right)}{Z_W(P_{\rm GB}+\omega k^2)^{n+1}}\\
l_{n{\rm L}}^{({\rm GB})d}(\omega) &= (n+\delta_{n,0})\frac{k^{2n-d}}{4v_d}\int_p \frac{\partial_t\left(Z_\varphi\,p^2\,r_{\rm GB}\right)}{Z_\varphi(P_{\rm GB}+\omega k^2)^{n+1}}\\
l_n^{({\rm G})d}(\omega) &= (n+\delta_{n,0})\frac{k^{2n-d}}{4v_d}\int_p \frac{\partial_t\left(p^2\,r_{\rm G}\right)}{(P_{\rm G}+\omega k^2)^{n+1}}\\
l_{n_1,n_2}^{({\rm FB})d}(\omega_1,\omega_2) &= -\frac{k^{2(n_1+n_2)-d}}{4v_d}\int_p \tilde\partial_t \frac{1}{(P_{\rm F}+\omega_1 k^2)^{n_1}(P_{\rm B}+\omega_2 k^2)^{n_2}}\\
l_{n_1,n_2}^{({\rm BGB})d}(\omega_1,\omega_2) &= -\frac{k^{2(n_1+n_2)-d}}{4v_d}\int_p \tilde\partial_t \frac{1}{(P_{\rm B}+\omega_1 k^2)^{n_1}(P_{\rm GB}+\omega_2 k^2)^{n_2}}\\
m_2^{({\rm F})d}(\omega) &= -\frac{k^{6-d}}{4v_d}\int_p p^2\tilde\partial_t\left(\frac{\partial}{\partial p^2}\frac{1}{P_{\rm F}+\omega k^2}\right)^2\\
m_4^{({\rm F})d}(\omega) &= -\frac{k^{4-d}}{4v_d}\int_p p^4\tilde\partial_t\left(\frac{\partial}{\partial p^2}\frac{1+r_{\rm F}}{P_{\rm F}+\omega k^2}\right)^2\\
m_2^{({\rm GB})d}(\omega) &= -\frac{k^{6-d}}{4v_d}\int_p p^2\tilde\partial_t\left(\frac{\partial}{\partial p^2}\frac{1}{P_{\rm GB}+\omega k^2}\right)^2\\
m_{n_1,n_2}^{d}(\omega_1,\omega_2) &= -\frac{k^{2(n_1+n_2-1)-d}}{4v_d}\int_p p^2\tilde\partial_t\left(\frac{\frac{\partial}{\partial p^2}P_{\rm B}}{(P_{\rm B}+\omega_1 k^2)^{n_1}}\frac{\frac{\partial}{\partial p^2}P_{\rm B}}{(P_{\rm B}+\omega_2 k^2)^{n_2}}\right)\\
m_{n_1,n_2}^{({\rm FB})d}(\omega_1,\omega_2) &= -\frac{k^{2(n_1+n_2-1)-d}}{4v_d}\int_p p^2\tilde\partial_t\left(\frac{1+r_{\rm F}}{(P_{\rm F}+\omega_1 k^2)^{n_1}}\frac{\frac{\partial}{\partial p^2}P_{\rm B}}{(P_{\rm B}+\omega_2 k^2)^{n_2}}\right)\\
m_{n_1,n_2}^{({\rm FGB})d}(\omega_1,\omega_2) &= -\frac{k^{2(n_1+n_2-1)-d}}{4v_d}\int_p p^2\tilde\partial_t\left(\frac{1+r_{\rm F}}{(P_{\rm F}+\omega_1 k^2)^{n_1}}\frac{\frac{\partial}{\partial p^2}P_{\rm GB}}{(P_{\rm GB}+\omega_2 k^2)^{n_2}}\right)\\
a_1^d(\omega) &= -\frac{k^{6-d}}{16v_d}\int_p \frac{1}{p^2}\tilde\partial_t\left(\frac{1}{P_{\rm GB}+\omega k^2}\right)^2\\
a_3^d(\omega_1,\omega_2) &= -\frac{k^{4-d}}{4v_d}\int_p \tilde\partial_t\left(\frac{1+r_{\rm F}}{P_{\rm F}+\omega_1 k^2}\frac{1}{P_{\rm GB}+\omega_2 k^2}\right)\,.
\end{aligned}
$$

We choose the same optimized regulator for the scalar bosons, for the gauge bosons and for the ghosts

$$
y r_{\rm B/GB/G}(y) = (1-y)\theta(1-y), \tag{B.1}
$$

where $y = q^2/k^2$. Instead, for the spinor fermions the function $r_{\mathrm{F}}(y)$ is chosen such that $y(1+r_{\mathrm{B}}) = y(1+r_{\mathrm{F}})^2$. Using this regulator in the Wetterich equation, we can perform all momentum integrations analytically, obtaining

$$
\begin{aligned}
l_n^d(\omega) &= \frac{2(n+\delta_{n,0})}{d}\frac{1-\frac{\eta_\varphi}{d+2}}{(1+\omega)^{n+1}}\,,\\
l_{n,\mathrm{L/R}}^{(\mathrm{F})d}(\omega) &= \frac{2(n+\delta_{n,0})}{d}\frac{1-\frac{\eta_{\mathrm{L/R}}}{d+1}}{(1+\omega)^{n+1}}\,,\\
l_{n\mathrm{T}}^{(\mathrm{GB})d}(\omega) &= \frac{2(n+\delta_{n,0})}{d}\frac{1-\frac{\eta_{\mathrm{F}}}{d+2}}{(1+\omega)^{n+1}}\,,\\
l_{n\mathrm{L}}^{(\mathrm{GB})d}(\omega) &= \frac{2(n+\delta_{n,0})}{d}\frac{1-\frac{\eta_\varphi}{d+2}}{(1+\omega)^{n+1}}\,,\\
l_n^{(\mathrm{G})d}(\omega) &= \frac{2(n+\delta_{n,0})}{d}\frac{1}{(1+\omega)^{n+1}}\,,\\
l_{n_1,n_2}^{(\mathrm{FB})d}(\omega_1,\omega_2) &= \frac{2}{d}\left[n_1\frac{1-\frac{\eta_{\mathrm{L}}+\eta_{\mathrm{R}}}{2(d+1)}}{(1+\omega_1)^{1+n_1}(1+\omega_2)^{n_2}}+n_2\frac{1-\frac{\eta_\varphi}{d+2}}{(1+\omega_1)^{n_1}(1+\omega_2)^{1+n_2}}\right],\\
l_{n_1,n_2}^{(\mathrm{BGB})d}(\omega_1,\omega_2) &= \frac{2}{d}\left[n_1\frac{1-\frac{\eta_\varphi}{d+2}}{(1+\omega_1)^{1+n_1}(1+\omega_2)^{n_2}}+n_2\frac{1-\frac{\eta_{\mathrm{F}}}{d+2}}{(1+\omega_1)^{n_1}(1+\omega_2)^{1+n_2}}\right],\\
m_2^{(\mathrm{F})d}(\omega) &= \frac{1}{(1+\omega)^4}\,,\\
m_4^{(\mathrm{F})d}(\omega) &= \frac{1}{(1+\omega)^4}+\frac{1-\frac{1}{2}(\eta_{\mathrm{L}}+\eta_{\mathrm{R}})}{(d-2)(1+\omega)^3}-\left(\frac{1-\frac{1}{2}(\eta_{\mathrm{L}}+\eta_{\mathrm{R}})}{2d-4}+\frac{1}{4}\right)\frac{1}{(1+\omega)^2}\,,\\
m_2^{(\mathrm{GB})d}(\omega) &= \frac{1}{(1+\omega)^4}\,,\\
m_{n_1,n_2}^d(\omega_1,\omega_2) &= \frac{1}{(1+\omega_1)^{n_1}(1+\omega_2)^{n_2}}\,,\\
m_{n_1,n_2}^{(\mathrm{FB})d}(\omega_1,\omega_2) &= \frac{1-\frac{\eta_\varphi}{d+1}}{(1+\omega_1)^{n_1}(1+\omega_2)^{n_2}}\,,\\
m_{n_1,n_2}^{(\mathrm{FGB})d}(\omega_1,\omega_2) &= \frac{1-\frac{\eta_{\mathrm{F}}}{d+1}}{(1+\omega_1)^{n_1}(1+\omega_2)^{n_2}}\,,\\
a_1^d(\omega) &= \frac{1-\frac{\eta_{\mathrm{F}}}{d}}{d-2}\frac{1}{(1+\omega)^3}\,,\\
a_3^d(\omega_1,\omega_2) &= \frac{2}{d-1}\frac{1-\frac{\eta_{\mathrm{F}}}{d+1}}{(1+\omega_1)(1+\omega_2)^2}+\frac{1}{d-1}\frac{\left(1-\frac{\eta_{\mathrm{L}}}{d}\right)-\omega_1\left(1-\frac{\eta_{\mathrm{R}}}{d}\right)}{(1+\omega_1)^2(1+\omega_2)}\,.
\end{aligned}
$$

B.2 Derivation of the flow equations for the matter sector

We have to evaluate the r.h.s of eq. (1.1), for which we need the $\Gamma_k^{(2)}$ matrix. Let us consider the fields φ_i, ψ_{L}, ψ_{R}, W, c, $\bar{c}$ as column vectors, with a number of components respectively given by N_{L}, $d_\gamma N_{\mathrm{L}}$, d_γ, $d(N_{\mathrm{L}}^2-1)$, (N_{L}^2-1), (N_{L}^2-1). Accordingly let us consider $\bar{\psi}_{\mathrm{L}}$ and $\bar{\psi}_{\mathrm{R}}$ as row vectors. Taking care of the partly Grassmann-valued field components and the Fourier conventions, let us denote by $\Phi^{\mathrm{T}}(q)$ the row vector with components $\varphi_1^{\mathrm{T}}(q)$, $\varphi_2^{\mathrm{T}}(q)$, $\psi_{\mathrm{L}}^{\mathrm{T}}(q)$, $\bar{\psi}_{\mathrm{L}}(-q)$, $\psi_{\mathrm{R}}^{\mathrm{T}}(q)$, $\bar{\psi}_{\mathrm{R}}(-q)$, $W^{\mathrm{T}}(q)$, $c^{\mathrm{T}}(q)$, $\bar{c}^{\mathrm{T}}(q)$, and by $\Phi(p)$ the column vector given by its transposition. Then $\Gamma_k^{(2)}$ is computed as follows

$$\Gamma_k^{(2)} = \frac{\overrightarrow{\delta}}{\delta\Phi^{\mathrm{T}}(-p)}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\Phi(q)}\,.$$

For a proper IR regularization, a regulator which is diagonal in field space is sufficient and convenient,

$$R_k(q,p) = \delta(p-q)\begin{pmatrix} R_{k\mathrm{B}} & 0 & 0 & 0 \\ 0 & R_{k\mathrm{F}} & 0 & 0 \\ 0 & 0 & R_{k\mathrm{GB}} & 0 \\ 0 & 0 & 0 & R_{k\mathrm{G}} \end{pmatrix},$$

with a $2N_{\mathrm{L}}\times 2N_{\mathrm{L}}$ matrix for the scalar bosonic sector

$$R_{k\mathrm{B}} = \begin{pmatrix} Z_{\varphi,k}\delta^{ab}p^2 r_{k\mathrm{B}} & 0 \\ 0 & Z_{\varphi,k}\delta^{ab}p^2 r_{k\mathrm{B}} \end{pmatrix},$$

an $2d_\gamma(N_{\mathrm{L}}+1)\times 2d_\gamma(N_{\mathrm{L}}+1)$ matrix for the spinor sector

$$R_{k\mathrm{F}} = -\begin{pmatrix} 0 & Z_{\mathrm{L},k}\delta^{ab}\not{p}^{\mathrm{T}} & 0 & 0 \\ Z_{\mathrm{L},k}\delta^{ab}\not{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_{\mathrm{R},k}\not{p}^{\mathrm{T}} \\ 0 & 0 & Z_{\mathrm{R},k}\not{p} & 0 \end{pmatrix} r_{k\mathrm{F}}\,,$$

a diagonal $d(N_{\mathrm{L}}^2-1)\times d(N_{\mathrm{L}}^2-1)$ matrix for the gauge vector boson

$$R_{k\mathrm{GB}} = Z_W\left(\Pi_\perp^{\mu\nu} + \frac{Z_\varphi}{aZ_W}\Pi_\parallel^{\mu\nu}\right)\delta^{ij}p^2 r_{k\mathrm{GB}}\,,$$

where the Π's are the usual longitudinal and transverse projectors with respect to p_μ, and a $2(N_{\mathrm{L}}^2-1)\times 2(N_{\mathrm{L}}^2-1)$ matrix for the ghosts

$$R_{k\mathrm{G}} = \begin{pmatrix} 0 & \delta^{ij}p^2 r_{k\mathrm{G}} \\ -\delta^{ij}p^2 r_{k\mathrm{G}} & 0 \end{pmatrix}.$$

See. App. B.1 for the definition of the shape functions r_k.

B.2.1 Flow equation for the potential

The flow of the potential can be computed by setting the field φ^a to a constant value and by killing all the other average fields on the r.h.s. of the flow equation. Then, in Landau gauge the matrix $\Gamma_k^{(2)} + R_k$ can be easily inverted. Multiplying with the derivative of the regulator, and taking the supertrace yields the result. This can be interpreted as an improved one-loop computation for a 0-point function, i.e. a sum over all the one-loop graphs with no external legs. The gauge contribution takes the form of a closed gauge boson propagator, and since it does not involve any vertex, it should not explicitly depend on $\bar{g}$. Indeed we get

$$\begin{aligned}
\partial_t U_k \;=\; & \frac{1}{2}\int_p \partial_t P_{\mathrm{B}} \left[\frac{2N_{\mathrm{L}}-1}{Z_\varphi P_{\mathrm{B}} + U_k'} + \frac{1}{Z_\varphi P_{\mathrm{B}} + U_k' + 2\rho U_k''}\right] \\
& - d_\gamma \int_p \left\{ \left[(N_{\mathrm{L}}-1) + \frac{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}}}{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2}\right] \frac{\partial_t [Z_{\mathrm{L}} r_{k\mathrm{F}}]}{Z_{\mathrm{L}}(1+r_{k\mathrm{F}})} + \frac{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}}}{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2} \frac{\partial_t [Z_{\mathrm{R}} r_{k\mathrm{F}}]}{Z_{\mathrm{R}}(1+r_{k\mathrm{F}})} \right\} \\
& + \frac{1}{2}\sum_{i=1}^{N_{\mathrm{L}}^2-1} \int_p \left[(d-1)\frac{\partial_t (Z_W p^2 r_{k\mathrm{GB}})}{Z_W P_{\mathrm{GB}} + \bar{m}_{W,i}^2} + \frac{\partial_t (Z_\varphi p^2 r_{k\mathrm{GB}})}{Z_\varphi P_{\mathrm{GB}}}\right] - \int_p \frac{(N_{\mathrm{L}}^2-1) p^2 \partial_t r_{k\mathrm{G}}}{P_{\mathrm{G}}}
\end{aligned}$$

that is, in terms of threshold functions

$$\begin{aligned}
\partial_t U_k \;=\; & 2 v_d k^d \Big\{ (2N_{\mathrm{L}}-1) l_0^d \left(\tfrac{U_k'}{Z_\varphi k^2}\right) + l_0^d \left(\tfrac{U_k' + 2\rho U_k''}{Z_\varphi k^2}\right) \\
& - d_\gamma \left[(N_{\mathrm{L}}-1) l_{0\mathrm{L}}^{(\mathrm{F})d}(0) + l_{0\mathrm{L}}^{(\mathrm{F})d}\left(\tfrac{\rho \bar{h}_k^2}{Z_{\mathrm{L}} Z_{\mathrm{R}} k^2}\right) + l_{0\mathrm{R}}^{(\mathrm{F})d}\left(\tfrac{\rho \bar{h}_k^2}{Z_{\mathrm{L}} Z_{\mathrm{R}} k^2}\right)\right] \\
& - 2(N_{\mathrm{L}}^2-1) l_0^{(\mathrm{G})d}(0) + \sum_{i=1}^{N_{\mathrm{L}}^2-1} \left[(d-1) l_{0\mathrm{T}}^{(\mathrm{GB})d}\left(\tfrac{\bar{m}_{W,i}^2}{Z_W k^2}\right) + l_{0\mathrm{L}}^{(\mathrm{GB})d}(0)\right] \Big\}
\end{aligned}$$

where U_k is a function of ρ. Switching over to dimensionless quantities this becomes eq. (5.15), and for our choice of the optimized regulator we eventually get

$$\begin{aligned}
\partial_t u_k \;=\; & -d u_k + (d - 2 + \eta_\varphi)\tilde{\rho} u_k' + \frac{4 v_d}{d} \Bigg\{ \\
& \frac{2N_{\mathrm{L}}-1}{1+u_k'} \left(1 - \frac{\eta_\varphi}{d+2}\right) + \frac{1}{1 + u_k' + 2\tilde{\rho} u_k''} \left(1 - \frac{\eta_\varphi}{d+2}\right) \\
& - d_\gamma \left[\left(1 - \frac{\eta_{\mathrm{L}}}{d+1}\right) \left((N_{\mathrm{L}}-1) + \frac{1}{1+\tilde{\rho} h_k^2}\right) + \left(1 - \frac{\eta_{\mathrm{R}}}{d+1}\right) \frac{1}{1+\tilde{\rho} h_k^2} \right] \\
& + \sum_{i=1}^{N_{\mathrm{L}}^2-1} \left[(d-1) \frac{1 - \frac{\eta_{\mathrm{F}}}{d+2}}{1 + m_{W,i}^2} + \left(1 - \frac{\eta_\varphi}{d+2}\right)\right] - 2\left(N_{\mathrm{L}}^2 - 1\right) \Bigg\}.
\end{aligned}$$

B.2.2 Flow equation for the Yukawa coupling

For the derivation of the flow of the Yukawa coupling, we first separate the bosonic field into the vev and a purely radial deviation from the vev, i.e. according to (5.8) where we set $\Delta\varphi_2 = 0$, since we are mainly interested in the Yukawa coupling between the fermions and the radial mode. The projection of the flow equation on such an operator reads

$$\partial_t \bar{h}_k = -\frac{1}{\sqrt{2}} \frac{\overrightarrow{\delta}}{\delta \bar{\psi}_{\mathrm{L}}^{\hat{n}}(p)} \frac{\overrightarrow{\delta}}{\delta \Delta\varphi_1^{\hat{n}}(p')} \partial_t \Gamma_k \frac{\overleftarrow{\delta}}{\delta \psi_{\mathrm{R}}(q)} \bigg|_0 . \tag{B.2}$$

The vertical line indicates that the equation is evaluated at vanishing momenta $p' = p = q = 0$ and at vanishing fluctuation fields. Next, we can decompose the matrix $(\Gamma_k^{(2)} + R_k)$ into two parts. One part, which we call $(\Gamma_{k,0}^{(2)} + R_k)$, contains only v and is independent of the fluctuations. The remaining part, $\Delta\Gamma_k^{(2)}$, contains all fluctuating fields. Recalling that the r.h.s. of the flow equation can be written in terms of the operator $\tilde{\partial}_t$ acting on the logarithm of $(\Gamma_k^{(2)} + R_k)$, inserting in it this decomposition of $(\Gamma_k^{(2)} + R_k)$ and expanding by means of the Mercator series one can write

$$\partial_t \Gamma_k = \frac{1}{2} \tilde{\partial}_t \mathrm{STr} \log(\Gamma_{k,0}^{(2)} + R_k) + \frac{1}{2} \sum_{s=1}^{\infty} \frac{(-)^{s+1}}{s} \tilde{\partial}_t \mathrm{STr} \left[\left(\frac{\Delta\Gamma_k^{(2)}}{\Gamma_{k,0}^{(2)} + R_k} \right)^s \right] . \tag{B.3}$$

Plugging this expression into equation (B.2), only the term to third power in $\Delta\Gamma_k^{(2)}$ survives the projection. Since we took three derivatives of the Wetterich equation, the diagrammatic interpretation of the result is in terms of one-loop graphs with three external legs: two fermions of opposite chirality and one scalar. The gauge contribution comes from triangular loops with three different propagators: one scalar, one spinor and one gauge vector. It always involves the two-scalars-one-vector vertex. This vertex is proportional to the difference of incoming scalar momenta, while the gauge boson propagator in Landau gauge is transverse. These two facts plus conservation of momentum entail that the direct gauge contribution to the momentum-independent Yukawa coupling under consideration vanishes at one loop. In agreement, performing the matrix calculations and taking the supertrace, we get from the flow equation

$$\begin{aligned} \partial_t \bar{h}_k = -\frac{\bar{h}_k^3}{2} \int \frac{d^d p}{(2\pi)^d} \tilde{\partial}_t \Bigg[& \frac{1}{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2} \left(\frac{2\rho U_k''}{(Z_\varphi P_{\mathrm{B}} + U_k')^2} - \frac{6\rho U_k'' + 4\rho^2 U_k'''}{(Z_\varphi P_{\mathrm{B}} + U_k' + 2\rho U_k'')^2} \right) \\ & + \frac{2\rho \bar{h}_k^2}{(Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2)^2} \left(\frac{1}{Z_\varphi P_{\mathrm{B}} + U_k'} - \frac{1}{Z_\varphi P_{\mathrm{B}} + U_k' + 2\rho U_k''} \right) \\ & - \frac{1}{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2} \left(\frac{1}{Z_\varphi P_{\mathrm{B}} + U_k'} - \frac{1}{Z_\varphi P_{\mathrm{B}} + U_k' + 2\rho U_k''} \right) \Bigg] \end{aligned}$$

where the whole r.h.s. is to be evaluated at the value $\rho = \frac{1}{2}v^2$ that minimizes the potential U_k. In terms of the threshold functions as defined in App. B.1 this reads

$$\begin{aligned}\partial_t \bar{h}_k^2 = \frac{4v_d \bar{h}_k^4}{Z_{\rm L} Z_{\rm R} Z_\varphi k^{4-d}} \Big[& \frac{2\rho U''}{Z_\varphi k^2} l_{12}^{({\rm FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}, \tfrac{U'}{Z_\varphi k^2}\right) - \frac{6\rho U'' + 4\rho U'''}{Z_\varphi k^2} l_{12}^{({\rm FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}, \tfrac{U'+2\rho U''}{Z_\varphi k^2}\right) \\ & + \frac{2\rho\bar{h}_k^2}{k^2} l_{21}^{({\rm FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}, \tfrac{U'}{Z_\varphi k^2}\right) - \frac{2\rho\bar{h}_k^2}{k^2} l_{21}^{({\rm FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}, \tfrac{U'+2\rho U''}{Z_\varphi k^2}\right) \\ & - l_{11}^{({\rm FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}, \tfrac{U'}{Z_\varphi k^2}\right) + l_{11}^{({\rm FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}, \tfrac{U'+2\rho U''}{Z_\varphi k^2}\right) \Big] .\end{aligned}$$

Switching over to dimensionless quantities, we end up with the representation (5.19) given in the main text. In the case of a optimized cutoff it reads

$$\begin{aligned}\partial_t h_k^2 &= (d - 4 + \eta_\varphi + \eta_{\rm L} + \eta_{\rm R}) h_k^2 - 4v_d h_k^4 \Bigg\{ \\ &- \frac{1}{d}\frac{4\tilde{\rho}_k u_k''}{(1+\tilde{\rho}_k h_k^2)(1+u_k')^2}\left[\frac{1}{1+\tilde{\rho}_k h_k^2}\left(1 - \frac{\frac{1}{2}(\eta_{\rm R}+\eta_{\rm L})}{d+1}\right) + \frac{2}{1+u_k'}\left(1 - \frac{\eta_\varphi}{d+2}\right)\right] \\ &+ \frac{1}{d}\frac{12\tilde{\rho}_k u_k'' + 8\tilde{\rho}_k^2 u_k'''}{(1+\tilde{\rho}_k h_k^2)(1+u_k'+2\tilde{\rho}_k u_k'')^2} \\ &\qquad \times \left[\frac{1}{1+\tilde{\rho}_k h_k^2}\left(1 - \frac{\frac{1}{2}(\eta_{\rm R}+\eta_{\rm L})}{d+1}\right) + \frac{2}{1+u_k'+2\tilde{\rho}_k u_k''}\left(1 - \frac{\eta_\varphi}{d+2}\right)\right] \\ &+ \frac{2}{d}\frac{1}{(1+\tilde{\rho}_k h_k^2)(1+u_k')}\left[\frac{1}{1+\tilde{\rho}_k h_k^2}\left(1 - \frac{\frac{1}{2}(\eta_{\rm R}+\eta_{\rm L})}{d+1}\right) + \frac{1}{1+u_k'}\left(1 - \frac{\eta_\varphi}{d+2}\right)\right] \\ &- \frac{2}{d}\frac{1}{(1+\tilde{\rho}_k h_k^2)(1+u_k'+2\tilde{\rho}_k u_k'')} \\ &\qquad \times \left[\frac{1}{1+\tilde{\rho}_k h_k^2}\left(1 - \frac{\frac{1}{2}(\eta_{\rm R}+\eta_{\rm L})}{d+1}\right) + \frac{1}{1+u_k'+2\tilde{\rho}_k u_k''}\left(1 - \frac{\eta_\varphi}{d+2}\right)\right] \\ &- \frac{1}{d}\frac{4\tilde{\rho}_k h_k^2}{(1+\tilde{\rho}_k h_k^2)^2(1+u_k')}\left[\frac{2}{1+\tilde{\rho}_k h_k^2}\left(1 - \frac{\frac{1}{2}(\eta_{\rm R}+\eta_{\rm L})}{d+1}\right) + \frac{1}{1+u_k'}\left(1 - \frac{\eta_\varphi}{d+2}\right)\right] \\ &+ \frac{1}{d}\frac{4\tilde{\rho}_k h_k^2}{(1+\tilde{\rho}_k h_k^2)^2(1+u_k'+2\tilde{\rho}_k u_k'')} \\ &\qquad \times \left[\frac{2}{1+\tilde{\rho}_k h_k^2}\left(1 - \frac{\frac{1}{2}(\eta_{\rm R}+\eta_{\rm L})}{d+1}\right) + \frac{1}{1+u_k'+2\tilde{\rho}_k u_k''}\left(1 - \frac{\eta_\varphi}{d+2}\right)\right] \Bigg\} .\end{aligned}$$

B.2.3 Flow of the scalar anomalous dimension

For the derivation of the flow of $Z_{\varphi,k}$, we decompose the bosonic field as in App. B.2.2. The projection of the Wetterich equation onto the massive scalar kinetic term leads us to

$$\partial_t Z_{\varphi,k} = -\frac{\partial}{\partial(p'^2)} \frac{\delta}{\delta\Delta\varphi_1^{\hat{n}}(p')} \frac{\delta}{\delta\Delta\varphi_1^{\hat{n}}(q')} \partial_t \Gamma_k \Big|_0 .$$

As before the vertical line indicates that the equation is evaluated at vanishing momenta $p' = q' = 0$ and at vanishing fluctuation fields. Expanding again the r.h.s. of the flow equation according to eq. (B.3), this time only the second order term ($s = 2$) contributes. Since we took two derivatives of the flow equation, the diagrammatic interpretation of the result is in terms of one-loop graphs with two external scalar legs. From a one-loop analysis we expect the direct gauge contributions to be of two kinds. One is due to the two-scalars-one-vector vertex and produces a loop containing one scalar and one gauge boson propagator. This is present in both the symmetric and in the spontaneously broken regimes. Another is due to the two-scalars-two-vectors vertex and, if one identifies two external scalar legs with the vev, it produces a loop containing two gauge boson propagators. Therefore this contribution will be present only in the SSB regime. Indeed, performing the matrix calculations and taking the supertrace we find

$$\begin{aligned}
\partial_t Z_\varphi = \frac{1}{d}\int_p \tilde{\partial}_t \Bigg\{ & \Bigg[(3\sqrt{2\rho}U_k'' + 2\sqrt{2\rho^3}U_k''')^2 p^2 Z_\varphi^2 \left(\frac{\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}} + U_k' + 2\rho U_k'')^2} \right)^2 \\
& + (2N_{\mathrm{L}} - 1) 2\rho U_k''^2 p^2 Z_\varphi^2 \left(\frac{\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}} + U_k')^2} \right)^2 \Bigg] \\
& + d_\gamma \left[2\bar{h}_k^2 p^4 Z_{\mathrm{L}} Z_{\mathrm{R}} \left(\frac{\partial}{\partial p^2} \frac{1 + r_{\mathrm{F}}}{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2} \right)^2 - 2\rho \bar{h}_k^4 p^2 \left(\frac{\partial}{\partial p^2} \frac{1}{Z_{\mathrm{L}} Z_{\mathrm{R}} P_{\mathrm{F}} + \rho \bar{h}_k^2} \right)^2 \right] \\
& - 4(d-1)\bar{g}^2 Z_\varphi^2 \sum_{a=1}^{N_{\mathrm{L}}} \sum_{i=1}^{N_{\mathrm{L}}^2 - 1} \frac{T^i_{\hat{n}a} T^i_{a\hat{n}}}{(Z_\varphi P_{\mathrm{B}} + U_k')(Z_W P_{\mathrm{GB}} + \bar{m}_{W,i}^2)} \\
& + \frac{(d-1)}{\rho} \sum_{i=1}^{N_{\mathrm{L}}^2 - 1} \bar{m}_{W,i}^4 \left(\frac{1}{p^2 (Z_W P_{\mathrm{GB}} + \bar{m}_{W,i}^2)^2} + 2p^2 \left(\frac{\partial}{\partial p^2} \frac{1}{Z_W P_{\mathrm{GB}} + \bar{m}_{W,i}^2} \right)^2 \right) \Bigg\}
\end{aligned}$$

and again the whole r.h.s. is to be evaluated at the value $\rho = \frac{1}{2}v^2$ that minimizes the potential U_k. Translating this result in terms of threshold functions

$$\begin{aligned}
\partial_t Z_\varphi &= -\frac{8v_d}{Z_\varphi^2 k^{6-d} d}\left[\left(3\sqrt{\rho}U_k'' + 2\sqrt{\rho^3}U_k'''\right)^2 m_{22}^d\left(\tfrac{U_k'+2\rho U_k''}{Z_\varphi k^2}, \tfrac{U_k'+2\rho U_k''}{Z_\varphi k^2}\right)\right. \\
&\quad \left. +(2N_{\rm L}-1)\rho U_k''^2 m_{22}^d\left(\tfrac{U_k'}{Z_\varphi k^2}, \tfrac{U_k'}{Z_\varphi k^2}\right)\right] \\
&\quad -\frac{4v_d d_\gamma}{d}\left[\frac{2\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^{4-d}} m_4^{({\rm F})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}\right) - \frac{2\rho\bar{h}_k^4}{Z_{\rm L}^2Z_{\rm R}^2k^{6-d}} m_2^{({\rm F})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\rm L}Z_{\rm R}k^2}\right)\right] \\
&\quad +\frac{16v_d(d-1)}{d}\frac{\bar{g}^2 Z_\varphi}{k^{4-d}Z_W}\sum_{a=1}^{N_{\rm L}}\sum_{i=1}^{N_{\rm L}^2-1} T^i_{\hat{n}a}T^i_{a\hat{n}}\, l_{11}^{({\rm BGB})d}\left(\tfrac{U_k'}{Z_\varphi k^2}, \tfrac{\bar{m}_{W,i}^2}{Z_W k^2}\right) \\
&\quad -\frac{8v_d(d-1)}{d}\sum_{i=1}^{N_{\rm L}^2-1}\frac{\bar{m}_{W,i}^4}{Z_W^2 k^{6-d}\rho}\left[2a_1^d\left(\tfrac{\bar{m}_{W,i}^2}{Z_W k^2}\right) + m_2^{({\rm GB})d}\left(\tfrac{\bar{m}_{W,i}^2}{Z_W k^2}\right)\right]
\end{aligned}$$

and then in terms of dimensionless quantities we end up with eq. (5.20). For the optimized cutoff the explicit form is

$$\begin{aligned}
\eta_\varphi &= \frac{4v_d}{d}\frac{18u_k''^2\tilde{\rho}+24u_k''u_k'''\tilde{\rho}^2+8u_k'''^2\tilde{\rho}^3}{(1+u_k'+2\tilde{\rho}u_k'')^4} + \frac{(2N_{\rm L}-1)8v_d}{d}\frac{\tilde{\rho}u_k''^2}{(1+u_k')^4} - \frac{8v_d d_\gamma}{d}\frac{\tilde{\rho}h_k^4}{(1+\tilde{\rho}h_k^2)^4} \\
&+ \frac{8v_d d_\gamma}{d}h_k^2\left(\frac{1}{(1+\tilde{\rho}h_k^2)^4} + \frac{1-\frac{1}{2}(\eta_{\rm L}+\eta_{\rm R})}{(d-2)(1+\tilde{\rho}h_k^2)^3} - \left(\frac{1-\frac{1}{2}(\eta_{\rm L}+\eta_{\rm R})}{2d-4}+\frac{1}{4}\right)\frac{1}{(1+\tilde{\rho}h_k^2)^2}\right) \\
&- \frac{32v_d}{d}g^2(1-\tfrac{1}{d})\sum_{a=1}^{N_{\rm L}}\sum_{i=1}^{N_{\rm L}^2-1} T^i_{\hat{n}a}T^i_{a\hat{n}}\left(\frac{1-\frac{\eta_\varphi}{d+2}}{(1+u_k')^2(1+m_{W,i}^2)} + \frac{1-\frac{\eta_{\rm F}}{d+2}}{(1+u_k')(1+m_{W,i}^2)^2}\right) \\
&+ 8v_d(1-\tfrac{1}{d})\sum_{i=1}^{N_{\rm L}^2-1}\frac{m_{W,i}^4}{\tilde{\rho}}\left(\frac{\frac{2}{d-2}(1-\frac{\eta_{\rm F}}{d})}{(1+m_{W,i}^2)^3} + \frac{1}{(1+m_{W,i}^2)^4}\right).
\end{aligned}$$

B.2.4 Flow of the spinor anomalous dimensions

For the anomalous dimensions of the spinors, the procedure is very similar to the one explained for the scalar. Since one of the left spinors becomes massive in the broken regime, we compute the anomalous dimension of that component (the $\hat{n}-$ th) only. We start with the projection

$$\partial_t Z_{{\rm L/R},k} = -\frac{1}{2v_d d_\gamma}{\rm tr}\gamma^\mu\frac{\partial}{\partial p'^\mu}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}^{\hat{n}}_{\rm L/R}(p')}\partial_t\Gamma_k\frac{\overleftarrow{\delta}}{\delta\psi^{\hat{n}}_{\rm L/R}(q')}\bigg|_0$$

where again the vertical line denotes that the equation is evaluated at vanishing momenta $p' = q' = 0$ and at vanishing fluctuation fields. Expanding again the r.h.s. of the flow equation according to eq. (B.3) only the second order term ($s = 2$) contributes. Obviously the right handed fermion does not receive direct corrections from the gauge boson while the left handed fermion does, since its interactions allow for a gauge propagator to appear in the relevant Feyn-

man graphs already at one loop. The gauge-independent contributions differ from the results of [106] for a factor 2, which was due to a harmless mistake.

For the right-handed spinor the result is

$$\partial_t Z_{\mathrm{R}} = \frac{\bar{h}_k^2}{d}\int\frac{d^dp}{(2\pi)^d}\,p^2\tilde{\partial}_t\left[\frac{Z_{\mathrm{R}}(1+r_{\mathrm{F}})}{Z_{\mathrm{L}}Z_{\mathrm{R}}P_{\mathrm{F}}+\rho\bar{h}_k^2}\left(\frac{Z_\varphi\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}}+U_k'+2\rho U_k'')^2}+\frac{Z_\varphi\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}}+U_k')^2}\right)\right.$$
$$\left.+2(N_{\mathrm{L}}-1)\frac{Z_{\mathrm{R}}(1+r_{\mathrm{F}})}{Z_{\mathrm{L}}Z_{\mathrm{R}}P_{\mathrm{F}}}\frac{Z_\varphi\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}}+U_k')^2}\right].$$

Introducing threshold functions

$$\partial_t Z_{\mathrm{R}} = -\frac{4v_d\bar{h}_k^2}{dZ_\varphi Z_{\mathrm{L}}k^{4-d}}\left[m_{12}^{(\mathrm{FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\mathrm{L}}Z_{\mathrm{R}}k^2},\tfrac{U_k'+2\rho U_k''}{Z_\varphi k^2}\right)+m_{12}^{(\mathrm{FB})d}\left(\tfrac{\rho\bar{h}_k^2}{Z_{\mathrm{L}}Z_{\mathrm{R}}k^2},\tfrac{U_k'}{Z_\varphi k^2}\right)\right.$$
$$\left.+2(N_{\mathrm{L}}-1)m_{12}^{(\mathrm{FB})d}\left(0,\tfrac{U_k'}{Z_\varphi k^2}\right)\right].$$

In terms of dimensionless quantities we recover eq. (5.21), which for the optimized cutoff becomes

$$\eta_{\mathrm{R}} = \frac{4v_d}{d}h_k^2\left[\frac{1-\frac{\eta_\varphi}{d+1}}{(1+\tilde{\rho}h_k^2)(1+u_k'+2\tilde{\rho}u_k'')^2}+\frac{1-\frac{\eta_\varphi}{d+1}}{(1+\tilde{\rho}h_k^2)(1+u_k')^2}+2(N_{\mathrm{L}}-1)\frac{1-\frac{\eta_\varphi}{d+1}}{(1+u_k')^2}\right].$$

For the left-handed fermion the result is

$$\begin{aligned}\partial_t Z_{\mathrm{L}} &= \int\frac{d^dp}{(2\pi)^d}\,\tilde{\partial}_t\left\{\frac{\bar{h}_k^2}{d}\frac{Z_{\mathrm{L}}p^2(1+r_{\mathrm{F}})}{Z_{\mathrm{L}}Z_{\mathrm{R}}P_{\mathrm{F}}+\rho\bar{h}_k^2}\left(\frac{Z_\varphi\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}}+U_k'+2\rho U_k'')^2}+\frac{Z_\varphi\frac{\partial}{\partial p^2}P_{\mathrm{B}}}{(Z_\varphi P_{\mathrm{B}}+U_k')^2}\right)\right.\\
&-\frac{(d-1)}{d}\bar{g}^2Z_{\mathrm{L}}^2\sum_{i=1}^{N_{\mathrm{L}}^2-1}\left[(T^i_{\hat{n}\hat{n}})^2\left(\frac{2}{Z_WP_{\mathrm{GB}}+\bar{m}_{W,i}^2}+2p^2\frac{\partial}{\partial p^2}\frac{1}{Z_WP_{\mathrm{GB}}+\bar{m}_{W,i}^2}\right)\right.\\
&\qquad\left.\times\left(\frac{Z_{\mathrm{R}}(1+r_{\mathrm{F}})}{Z_{\mathrm{L}}Z_{\mathrm{R}}P_{\mathrm{F}}+\rho\bar{h}_k^2}-\frac{Z_{\mathrm{R}}(1+r_{\mathrm{F}})}{Z_{\mathrm{L}}Z_{\mathrm{R}}P_{\mathrm{F}}}\right)\right.\\
&+\left.\left.\sum_{a=1}^{N_{\mathrm{L}}}T^i_{\hat{n}a}T^i_{a\hat{n}}\left(\frac{2}{Z_WP_{\mathrm{GB}}+\bar{m}_{W,i}^2}+2p^2\frac{\partial}{\partial p^2}\frac{1}{Z_WP_{\mathrm{GB}}+\bar{m}_{W,i}^2}\right)\frac{Z_{\mathrm{R}}(1+r_{\mathrm{F}})}{Z_{\mathrm{L}}Z_{\mathrm{R}}P_{\mathrm{F}}}\right]\right\}\end{aligned}$$

in terms of threshold functions

$$\begin{aligned}\partial_t Z_{\rm L} &= -\frac{4v_d h_k^2}{dZ_\varphi Z_{\rm R} k^{4-d}}\left[m_{12}^{({\rm FB})d}\left(\tfrac{\rho h_k^2}{Z_{\rm L}Z_{\rm R}k^2},\tfrac{U_k'+2\rho U_k''}{Z_\varphi k^2}\right)+m_{12}^{({\rm FB})d}\left(\tfrac{\rho h_k^2}{Z_{\rm L}Z_{\rm R}k^2},\tfrac{U_k'}{Z_\varphi k^2}\right)\right]\\
&- \frac{8v_d(d-1)}{d}\frac{\bar g^2 Z_{\rm L}}{Z_W k^{4-d}}\sum_{i=1}^{N_{\rm L}^2-1}\Bigg\{(T^i_{\hat n\hat n})^2\left[m_{12}^{({\rm FGB})d}\left(\tfrac{\rho h_k^2}{Z_{\rm L}Z_{\rm R}k^2},\tfrac{\bar m_{W,i}^2}{Z_W k^2}\right)-m_{12}^{({\rm FGB})d}\left(0,\tfrac{\bar m_{W,i}^2}{Z_W k^2}\right)\right.\\
&\qquad\left.-a_3^d\left(\tfrac{\rho h_k^2}{Z_{\rm L}Z_{\rm R}k^2},\tfrac{\bar m_{W,i}^2}{Z_W k^2}\right)+a_3^d\left(0,\tfrac{\bar m_{W,i}^2}{Z_W k^2}\right)\right]\\
&+\sum_{a=1}^{N_{\rm L}}T^i_{\hat n a}T^i_{a\hat n}\left[m_{12}^{({\rm FGB})d}\left(0,\tfrac{\bar m_{W,i}^2}{Z_W k^2}\right)-a_3^d\left(0,\tfrac{\bar m_{W,i}^2}{Z_W k^2}\right)\right]\Bigg\}\end{aligned}$$

and translated in dimensionless quantities this results in eq. (5.22). Explicitely, for the optimized cutoff this reads

$$\begin{aligned}\eta_{\rm L} &= \frac{4v_d}{d}h_k^2\left(\frac{1-\frac{\eta_\varphi}{d+1}}{(1+\tilde\rho h_k^2)(1+u_k'+2\tilde\rho u_k'')^2}+\frac{1-\frac{\eta_\varphi}{d+1}}{(1+\tilde\rho h_k^2)(1+u_k')^2}\right)\\
&+\frac{8v_d g^2}{d}\sum_{i=1}^{N_{\rm L}^2-1}(T^i_{\hat n\hat n})^2\left[\frac{(d-3)(1-\frac{\eta_{\rm F}}{d+1})}{(1+m_{W,i}^2)^2}\left(\frac{1}{1+\tilde\rho h_k^2}-1\right)\right.\\
&\left.-\frac{1-\frac{\eta_{\rm L}}{d}}{(1+m_{W,i}^2)}\left(\frac{1}{(1+\tilde\rho h_k^2)^2}-1\right)+\frac{\tilde\rho h_k^2\left(1-\frac{\eta_{\rm R}}{d}\right)}{(1+m_{W,i}^2)(1+\tilde\rho h_k^2)^2}\right]\\
&+\frac{8v_d g^2}{d}\sum_{a=1}^{N_{\rm L}}\sum_{i=1}^{N_{\rm L}^2-1}T^i_{\hat n a}T^i_{a\hat n}\left[\frac{(d-3)(1-\frac{\eta_{\rm F}}{d+1})}{(1+m_{W,i}^2)^2}-\frac{1-\frac{\eta_{\rm L}}{d}}{(1+m_{W,i}^2)}\right].\end{aligned}$$

If, in the chosen basis in fundamental color algebra, the direction of the vev $\hat n$ has a single non-vanishing component, i.e. if $\hat n^a\propto\delta^{aA}$, this anomalous dimension takes a simpler form, given in eq. (5.23), and for the optimized cutoff that becomes

$$\begin{aligned}\eta_{\rm L} &= \frac{4v_d}{d}h_k^2\left(\frac{1-\frac{\eta_\varphi}{d+1}}{(1+\tilde\rho h_k^2)(1+u_k'+2\tilde\rho u_k'')^2}+\frac{1-\frac{\eta_\varphi}{d+1}}{(1+\tilde\rho h_k^2)(1+u_k')^2}\right)\\
&+\frac{8v_d g^2}{d}\sum_{a=1}^{N_{\rm L}}\sum_{i=1}^{N_{\rm L}^2-1}T^i_{Aa}T^i_{aA}\left[\frac{(d-3)(1-\frac{\eta_{\rm F}}{d+1})}{(1+m_{W,i}^2)^2(1+\delta^{aA}\tilde\rho h_k^2)}-\frac{1-\frac{\eta_{\rm L}}{d}}{(1+m_{W,i}^2)(1+\delta^{aA}\tilde\rho h_k^2)^2}\right.\\
&\left.+\frac{\delta^{aA}\tilde\rho h_k^2\left(1-\frac{\eta_{\rm R}}{d}\right)}{(1+m_{W,i}^2)(1+\delta^{aA}\tilde\rho h_k^2)^2}\right].\end{aligned}$$

B.3 Flow equation for the gauge coupling

In this appendix we will set $d=4$, $d_\gamma=2$. Since we want to compute the one loop beta function we set all the wave function renormalizations to one ($\mathcal{O}(\partial_t Z)$ terms on the r.h.s. of

the flow equation lead to higher loop corrections of the β_{g^2} function).

B.3.1 Contribution from the gauge modes

For the gauge contribution, the relevant part of the effective Lagrangian is

$$\begin{aligned}\mathcal{L}_k \;&\ni\; \frac{1}{4}(F^i_{ab})^2 + \bar{g}^2 W^i_\mu W^j_\mu \varphi^{\dagger a} T^i_{ab} T^j_{bc} \varphi^c + \mathcal{L}_{k,\mathrm{gf}} + \mathcal{L}_{k,\mathrm{gh}} \\ &=\; \frac{1}{4}(F^i_{ab})^2 + \frac{\bar{g}^2 v^2}{2} \hat{n}^{\dagger a} T^i_{ab} T^j_{bc} \hat{n}^c W^i_\mu W^j_\mu + \cdots .\end{aligned}$$

This defines the mass matrix for the gauge bosons, as given in eq. (5.12). As the generators are real, the mass matrix has real eigenvalues. In order to compute the running coupling, we use the BFM and project on the operator $F^2/4$. For the flow we need $\bar{\Gamma}^{(2)}_k$, the part of the action that is invariant under gauge transformations of the background field, because to the present order of the calculation the terms arising from $\Gamma^{\mathrm{gauge}}_k$ can be ignored. The Hessian for the fluctuating W-boson reads

$$\bar{\Gamma}^{(2)\,ij}_{k\;\mu\nu}\Big|_W = \mathcal{D}_{\mathrm{T}}{}^{lj}_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right) D^{il}_\mu D^{lj}_\nu + \bar{m}^{2\;ij}_W \delta_{\mu\nu}$$

the contributions from ghost fluctuations are

$$\bar{\Gamma}^{(2)\,ij}_k\Big|_{gh} = -D^{il}_\mu D^{lj}_\mu + O(\alpha^2)$$

and as we will later consider Landau gauge ($\alpha \to 0$), we ignore from now on the ghost-Higgs contributions. For a covariantly-constant background field, projectors onto the longitudinal and transverse subspaces (w.r.t. the background field) exist

$$\Pi_\perp + \Pi_\parallel = 1 \quad , \quad \Pi^2_{\perp/\parallel} = \Pi_{\perp/\parallel} \quad , \quad \Pi_\perp \Pi_\parallel = 0$$

such that

$$\begin{aligned}\bar{\Gamma}^{(2)\,ij}_{k\;\mu\nu}\Big|_W = {}& \Pi_{\perp}{}^{il}_{\mu\lambda} \left[\mathcal{D}_{\mathrm{T}}{}^{lj}_{\lambda\nu} + \bar{m}^{2\;ij}_W \delta_{\lambda\nu}\right] \\ &+ \Pi_{\parallel}{}^{il}_{\mu\lambda} \left[\frac{1}{\alpha}\mathcal{D}_{\mathrm{T}}{}^{lj}_{\lambda\nu} + \bar{m}^{2\;ij}_W \delta_{\lambda\nu}\right]\end{aligned}$$

see [16, 109] for the definition of $\mathcal{D}_{\mathrm{T}}$, $\Pi_\perp$, $\Pi_\parallel$. We choose a similar decomposition for the regulator

$$R_k\Big|_W = \Pi_\perp \mathcal{D}_{\mathrm{T}} r_k\left(\frac{\mathcal{D}_{\mathrm{T}}}{k^2}\right) + \Pi_\parallel \frac{1}{\alpha}\mathcal{D}_{\mathrm{T}} r_k\left(\frac{\mathcal{D}_{\mathrm{T}}}{k^2}\right)$$

hence also the functional trace on the r.h.s. of the flow equation decomposes into these two sectors. Using the important property that

$$\mathrm{Tr}\left[\Pi_\parallel f(\mathcal{D}_{\mathrm{T}})\right] = \mathrm{Tr}\left[f(-D^2)\right]$$

we get

$$\mathrm{Tr}\left[\frac{\partial_t R_k}{\Gamma_k^{(2)}+R_k}\right]_W = \mathrm{Tr}\left[\Pi_\perp \frac{\mathcal{D}_\mathrm{T}\partial_t r_k\left(\frac{\mathcal{D}_\mathrm{T}}{k^2}\right)}{\mathcal{D}_\mathrm{T}\left(1+r\left(\frac{\mathcal{D}_\mathrm{T}}{k^2}\right)\right)+\bar{m}_W^2}\right] + \mathrm{Tr}\left[\frac{(-D^2)\partial_t r_k\left(\frac{-D^2}{k^2}\right)}{(-D^2)\left(1+r\left(\frac{-D^2}{k^2}\right)\right)+a\bar{m}_W^2}\right]$$

and writing $\Pi_\perp = 1 - \Pi_\parallel$ in the first term we obtain two unconstrained traces for different differential operators. The ghost contribution gives

$$\mathrm{Tr}\left[\frac{\partial_t R_k}{\Gamma_k^{(2)}+R_k}\right]_{gh} = -2\mathrm{Tr}\left[\frac{(-D^2)\partial_t r_k\left(\frac{-D^2}{k^2}\right)}{(-D^2)\left(1+r\left(\frac{-D^2}{k^2}\right)\right)}\right]$$

such that the total contribution reads $(a \to 0)$

$$\begin{aligned}\mathrm{STr}\left[\frac{\partial_t R_k}{\Gamma_k^{(2)}+R_k}\right]_W &= \mathrm{Tr}\left[\frac{\mathcal{D}_\mathrm{T}\partial_t r_k\left(\frac{\mathcal{D}_\mathrm{T}}{k^2}\right)}{\mathcal{D}_\mathrm{T}\left(1+r\left(\frac{\mathcal{D}_\mathrm{T}}{k^2}\right)\right)+\bar{m}_W^2}\right] - \mathrm{Tr}\left[\frac{(-D^2)\partial_t r_k\left(\frac{-D^2}{k^2}\right)}{(-D^2)\left(1+r\left(\frac{-D^2}{k^2}\right)\right)+\bar{m}_W^2}\right] \\ &- \mathrm{Tr}\left[\frac{(-D^2)\partial_t r_k\left(\frac{-D^2}{k^2}\right)}{(-D^2)\left(1+r\left(\frac{-D^2}{k^2}\right)\right)}\right].\end{aligned}$$

For the ease of the calculation we choose a basis in adjoint color space where the gauge boson mass matrix is diagonal, as in (5.13), and we also specify a constant pseudo-abelian magnetic background field

$$F^i_{\mu\nu} = \hat{m}^i F_{\mu\nu} \quad , \quad \hat{m}_i\hat{m}^i = 1 \quad , \quad F_{\mu\nu} = B\varepsilon^\perp_{\mu\nu}$$

where $\hat{m}$ is the above-mentioned basis in the Cartan of the color algebra, and the constant antisymmetric tensor ε characterizes the space directions which are affected by the constant magnetic field upon the Lorentz force. Then, recalling that the adjoint generators are $(\tau^l)_{ij} = if^{ilj}$ we can call v_i the eigenvalues of $if^{ilj}\hat{m}^l$ such that the covariant derivative

$$D^{ij}_\mu = (\partial_\mu - iv_i)\delta^{ij} \quad \text{(non sum over i)}$$

is also diagonal and so are D^2 and $\mathcal{D}_\mathrm{T}$. Hence $\mathcal{D}_\mathrm{T}$ and $\bar{m}_W^2$, as well as D^2 and $\bar{m}_W^2$ commute. Then equation (B.4) can be brought to the propertime form:

$$\mathrm{STr}\left[\frac{\partial_t R_k}{\Gamma_k^{(2)}+R_k}\right]_W = -\int_0^\infty ds\,\tilde{h}(s,0)\mathrm{Tr}\left[e^{-s\frac{-D^2}{k^2}}\right] + \int_0^\infty ds\,\mathrm{Tr}\left[\tilde{h}\left(s,m_W^2\right)\left(e^{-s\frac{\mathcal{D}_\mathrm{T}}{k^2}} - e^{-s\frac{-D^2}{k^2}}\right)\right]$$

where $\tilde{h}$ is the Laplace transform of the following function

$$h\left(y,m_W^2\right) = \frac{y\partial_t r_k(y)}{y(1+r_k(y))+m_W^2}$$

that is

$$h\left(y, m_W^2\right) = \int_0^\infty ds\, \tilde{h}\left(s, m_W^2\right) e^{-sy}$$

and as before $m_W^2 = \bar{m}_W^2/k^2$. The heat kernel traces are known, see [109]

$$\begin{aligned}
\mathrm{Tr}\left[\tilde{h}\left(s, m_W^2\right) e^{-s\frac{\mathcal{D}_\mathrm{T}}{k^2}}\right] &= \frac{\Omega k^4}{4\pi^2 s^2} \sum_{i=1}^{N_\mathrm{L}^2-1} \tilde{h}\left(s, m_{W,i}^2\right) \\
&\quad \times \left\{ \frac{\frac{sb_i}{k^2}}{\sinh\left(\frac{sb_i}{k^2}\right)} + \frac{sb_i}{k^2} \sinh\left(\frac{sb_i}{k^2}\right) \right\} \\
\mathrm{Tr}\left[\tilde{h}\left(s, m_W^2\right) e^{-s\frac{-D^2}{k^2}}\right] &= \frac{\Omega k^4}{16\pi^2 s^2} \sum_{i=1}^{N_\mathrm{L}^2-1} \tilde{h}\left(s, m_{W,i}^2\right) \frac{\frac{sb_i}{k^2}}{\sinh\left(\frac{sb_i}{k^2}\right)} \\
\mathrm{Tr}\left[\tilde{h}(s,0) e^{-s\frac{-D^2}{k^2}}\right] &= \frac{\Omega k^4}{16\pi^2 s^2} \sum_{i=1}^{N_\mathrm{L}^2-1} \tilde{h}(s,0) \frac{\frac{sb_i}{k^2}}{\sinh\left(\frac{sb_i}{k^2}\right)}
\end{aligned} \tag{B.4}$$

where $b_i = \bar{g}|v_i|B$ and Ω is the spacetime volume. The first trace above is over spacetime and Lorentz and color indeces, the other two only over spacetime and color indeces. For the running gauge coupling we just need the terms of order b_i^2. Since on the l.h.s. of the flow equation we have $\partial_t \Gamma_k = \Omega B^2 \partial_t Z_W/2$, in terms of the renormalized coupling g^2 and the anomalous dimension (such that $\partial_t g^2 = \beta_{g^2} = \eta_W g^2$) we obtain

$$\eta_W\Big|_W = \frac{-g^2}{32\pi^2} \sum_{i=0}^{N_\mathrm{L}^2-1} \left[21 h\left(0, m_{W,i}^2\right) + h(0,0)\right] \frac{|v_i|^2}{3}. \tag{B.5}$$

Using

$$h(y,x) = \frac{y\partial_t r(y)}{y(1+r(y))+x} = -2\frac{y^2 r'(y)}{y(1+r(y))+x}$$

we find that

$$h(y,0) = -2\frac{y r'(y)}{(1+r(y))+x}\,.$$

In the BFM the $y \to 0$ limit is constrained; the only regulators permitted must satisfy $h(y \to 0, 0) = 2$. In the massless limit we thus obtain

$$\eta_W\Big|_W = -\frac{1}{16\pi^2}\frac{22}{3} g^2 \sum_{i=0}^{N_\mathrm{L}^2-1} |v_i|^2 = -\frac{1}{16\pi^2}\frac{22}{3} g^2 N_\mathrm{L}\,.$$

which agrees with standard perturbation theory. Let us work out the massive case using the linear regulator (B.1). In this case $h(y,x) = 2(1+x)^{-1}\theta(1-y)$, the gauge contribution to

the gauge β_{g^2} function thus reeds

$$\beta_{g^2}\Big|_W = \eta_W\Big|_W g^2 = -\frac{g^4}{16\pi^2}\left[\frac{21}{3}\sum_{i=0}^{N_L^2-1}\frac{|v_i|^2}{1+m_{W,i}^2}+\frac{N_L}{3}\right].$$

The first term now depends on the choice of $\hat{n}^a$ the direction of the vev in fundamental color space. This is expected, as for higher gauge groups different breaking patterns and gauge masses can arise. This term also depends in general on $|v_i|^2$, i.e. on $\hat{m}^i$. This is also plausible, as the directions of the vev implicitly also allows for the definition of different couplings: depending on the relative direction of the gauge fluctuation w.r.t. the vev, the fluctuations can couple differently to matter.

For SU(2) things become simpler

$$\bar{m}_W^{2\ ij} = \frac{g^2v^2}{4}\hat{n}^{\dagger a}\sigma^i_{ab}\sigma^j_{bc}\hat{n}^c = \frac{g^2v^2}{4}(\delta^{ij}+i\varepsilon^{ijl}(\hat{n}^\dagger\sigma^l\hat{n})) \tag{B.6}$$

such that $\mathrm{tr}\,\bar{m}_W^{2\ ij} = 3g^2v^2/4$. Let us denote $c^l = (\hat{n}^\dagger\sigma^l\hat{n})$. This is a vector in adjoint color space which is an eigenvector of the mass matrix, with eigenvalue $g^2v^2/4$. One can choose a diagonalizing orthonormal basis $\{e_1, e_2 = c/|c|, e_3\}$ in adjoint color space such that the mass matrix takes the form

$$\bar{m}_W^2 = \frac{g^2v^2}{4}\begin{pmatrix}2&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

Now recall that the $|v_i|$ are the eigenvalues of $(-if^{ijl}\hat{m}^l)$, that for SU(2) simply is $(-i\varepsilon^{ijl}\hat{m}^l)$. Therefore in SU(2) the eigenvalues are $(1,-1,0)$ for any choice of $\hat{m}$. However, depending on the direction of $\hat{m}$ w.r.t. the basis defined above, the v_i could be $\{v_1=1, v_2=-1, v_3=0\}$ or any permutation thereof. The two extreme cases for SU(2) are maximal or minimal decoupling. Maximal decoupling happens if $|v_1| = |v_2| = 1$ and $v_3 = 0$, and in this case

$$\beta_{g^2}\Big|_W = -\frac{g^4}{16\pi^2}\left[\frac{21}{3}\left(\frac{1}{1+\frac{g^2v^2}{2k^2}}+\frac{1}{1+\frac{g^2v^2}{4k^2}}\right)+\frac{2}{3}\right] \tag{B.7}$$

while minimal decoupling happens if $v_3 = 0$ and $|v_2| = |v_3| = 1$, and correspondingly

$$\beta_{g^2}\Big|_W = -\frac{g^4}{16\pi^2}\left[\frac{21}{3}\left(1+\frac{1}{1+\frac{g^2v^2}{4k^2}}\right)+\frac{2}{3}\right]. \tag{B.8}$$

For SU(2) the ambiguity of the β-function arises solely from the ambiguity of defining a coupling in the presence of a vev. In fact, there are more quadratic invariants than the only F^2, such as for example $\hat{n}^{\dagger a}F^i_{\mu\nu}T^i_{ab}T^j_{bc}F^j_{\mu\nu}\hat{n}^c$. For higher groups, even the mass matrix depends on the choice of $\hat{n}^a$.

B.3.2 Contribution from scalar modes

The contribution from scalar fluctuations to the gauge β function arises from the scalar kinetic term. The calculation is very similar to that of the longitudinal modes with two differences: the field is complex and lives in the fundamental representation. Moreover the dimensionless scalar mass matrix in the broken regime reads $m_\varphi^{2\,ab} = (\lambda_2 v^2/2k^2)\hat{n}^a\hat{n}^{\dagger b}$. Let us not attempt to solve the problem in full generality as for the gauge modes, but confine ourselves to a simple choice of backgrounds. Most importantly, we choose the color direction of the pseudo abelian background to satisfy

$$W_\mu^i = \hat{m}^i W_\mu \,, \quad \hat{m}_i\hat{m}^i = 1 \,, \quad [(\hat{m}_i T^i), \hat{n}\otimes\hat{n}^\dagger] = 0 \,. \tag{B.9}$$

It is important to note that this does not constrain the choice of the vev-direction $\hat{n}^a$. This is because we can always choose a basis in fundamental color space such that the projector $P_{\hat{n}} = \hat{n}\otimes\hat{n}^\dagger$ is diagonal. Then the commutation relation (B.9) can be satisfied by choosing $(\hat{m}_i T^i)^{ab}$ to be in the Cartan, i.e. by choosing it to be diagonal in that basis.

Let's consider as an example SU(2). Let $\hat{n} = (0,1)$. Then we choose $\hat{m} = (0,0,1)$ such that

$$(\hat{m}_i T^i)^{ab} = \frac{1}{2}\sigma^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Before we continue with the scalar fluctuations, let us work out the consequences of this choice for the gauge modes of the preceding section. The vector c for this choice becomes $c = (0,0,-1)$ and the mass matrix for the gauge modes, given by (B.6), is

$$\bar{m}_W^{2\;ij} = \frac{\bar{g}^2 v^2}{4}(\delta^{ij} - i\varepsilon^{ij3}) = \frac{\bar{g}^2 v^2}{4}\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.10}$$

The definition of v_i, right above (B.4), combined with the choice $\hat{m} = (0,0,1)$ requires us to compute the eigenvalues of

$$-i\varepsilon^{ij3} = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The simultaneous eigenvectors of this matrix and of $\bar{m}_W^2$ are given by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad , \quad v_3 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

with the corresponding set of eigenvalues: $\{\bar{m}_{W,1}^2 = \frac{\bar{g}^2 v^2}{4}, v_1 = 0\}$, $\{\bar{m}_{W,2}^2 = \frac{\bar{g}^2 v^2}{2}, v_2 = 1\}$, $\{\bar{m}_{W,3}^2 = 0, v_3 = -1\}$. This choice of $\hat{m}$ corresponds to the minimal decoupling case of eq. (B.8). These considerations tell us that the maximal decoupling solution of eq. (B.7) might not be permitted, as it would not correspond to a legitimate choice of $\hat{m}$ such that $\hat{m}_i T^i$ be in the

Cartan (which we had also assumed in the gluonic case in eq. (B.4)). It seems that this choice of $\hat{m}$ satisfying (B.9) corresponds to defining the coupling with respect to the unbroken part of the gauge group.

Let us now return to the scalar fluctuations; eq. (B.9) ensures that the covariant derivative in the fundamental representation satisfies

$$[D_\mu, \hat{n} \otimes \hat{n}^\dagger] = 0$$

for our choice of the background field. Then also $[-D^2, \hat{n} \otimes \hat{n}^\dagger] = 0$ and thus $[-D^2, m_\varphi^2] = 0$ follow, such that $-D^2$ and m_φ^2 can be simultaneously diagonalized. Therefore

$$\mathrm{Tr}\left[\frac{\partial_t R_k}{\Gamma^{(2)} + R_k}\right]_\varphi = \mathrm{Tr}\left[\frac{\frac{-D^2}{k^2}\partial_t r_k\left(\frac{-D^2}{k^2}\right)}{\frac{-D^2}{k^2}\left(1 + r_k\left(\frac{-D^2}{k^2}\right)\right) + m_\varphi^2}\right] .$$

Because of the above considerations, we can rewrite the previous expression in the propertime form

$$\begin{aligned}\mathrm{Tr}\left[\frac{\partial_t R_k}{\Gamma^{(2)} + R_k}\right]_\varphi &= \int_0^\infty ds\, \mathrm{Tr}\left[\tilde{h}(s, m_\varphi^2) e^{-s\frac{-D^2}{k^2}}\right] \\ &= \frac{\Omega}{16\pi^2}\int_0^\infty ds \sum_{a=1}^{N_\mathrm{L}} \tilde{h}(s, m_{\varphi,a}^2)\left(-\frac{1}{6}b_a^2\right)\end{aligned}$$

where we extracted from the third equation of (B.4) the term of order b_i^2, and we denoted with $m_{\varphi,a}^2$ the eigenvalues of the mass matrix (there is only one nonvanishing eigenvalue for the radial mode), and $b_a = \bar{g}|v_a|B$, with v_a being now the eigenvalues of $(\hat{m}_i T^i)^{ab}$, now related to the fundamental representation. In particular, using the standard normalization for the generators of the fundamental representation

$$\sum_{a=1}^{N_\mathrm{L}} |v_a|^2 = \mathrm{tr}\left[\left(\hat{m}_i T^i\right)^2\right] = \hat{m}_i \hat{m}_j \frac{1}{2}\delta^{ij} = \frac{1}{2} .$$

Another difference from the gauge case is that the scalar field is complex and thus there is no factor $1/2$ in front of the trace on the r.h.s. of the flow equation. Hence, analogous to (B.5), the contribution of the scalar to the flow of Z_W reads

$$\eta_W\Big|_\varphi = \frac{g^2}{16\pi^2}\sum_{a=1}^{N_\mathrm{L}} h(0, m_{\varphi,a}^2)\frac{|v_a|^2}{3} . \tag{B.11}$$

In the massless case, since $h(0,0) = 2$, $\eta_W|_\varphi = \frac{g^2}{16\pi^2}\frac{1}{3}$ in agreement with perturbation theory.

In the general massive case and using the linear regulator, we get

$$\eta_W\Big|_\varphi = \frac{g^2}{16\pi^2}\frac{2}{3}\sum_{a=1}^{N_L}\frac{1}{1+m_{\varphi,a}^2}|v_a|^2\,.$$

For all the gauge groups only one particular component of $m_{\varphi,a}^2$ is nonvanishing and equal to $2\lambda_2\kappa$. For SU(2) the v_a are unique and equal to $\{-\frac{1}{2},-\frac{1}{2}\}$. Therefore in this case

$$\eta_W\Big|_\varphi = \frac{g^2}{16\pi^2}\frac{1}{3}\left[\frac{1}{2}+\frac{1}{2}\frac{1}{1+2\lambda_2\kappa_k}\right]\,.$$

B.3.3 Contribution from fermion modes

The relevant part of the effective Lagrangian is

$$\mathcal{L}_k \ni i(\bar\psi_L^a \not{D}^{ab}\psi_L^b + \bar\psi_R\not{\partial}\psi_R) + \bar h_k(\bar\psi_R\varphi^{a\dagger}\psi_L^a - \bar\psi_L^a\varphi^a\psi_R)$$

in which again we set any wave function renormalization to one. For (5.8) we can choose a gauge background field such that D_μ^{ab} and $P_{\hat n}=\hat n\otimes\hat n^\dagger$ as well as $P_{(1-\hat n)}=1-P_{\hat n}$ commute, such that the above parts of $\mathcal{L}_k$ can be written as

$$\begin{aligned}\mathcal{L}_k \ni\ & i(\bar\psi_L^a \not{D}^{ab}P^{bc}_{(1-\hat n)}\psi_L^c) \\ & + i(\bar\psi_L^{\hat n}\not{D}\psi_L^{\hat n} + \bar\psi_R\not{\partial}\psi_R) + \frac{\bar h_k v}{\sqrt{2}}(\bar\psi_R\psi_L^{\hat n} - \bar\psi_L^{\hat n}\psi_R)\end{aligned} \tag{B.12}$$

where the second $\not{D}$ is projected along $\hat n$. The first line corresponds to the massless bottom-type fermions. Their contribution is the standard perturbative contribution weighted by eigenvalues v_a in the orthogonal complement. Let $\hat n$ point into the A-direction: $\hat n^a=\delta^{aA}$. Then the contribution of the massless fermions to the running coupling is

$$\partial_t g^2\Big|_{\psi_{(1-\hat n)}} = \frac{g^4}{16\pi^2}\frac{4}{3}\sum_{a=1,a\neq A}^{N_L}|v_a|^2\,.$$

If the sum ran over all a's we would get $\sum_{a=1}^{N_L}|v_a|^2=1/2$ leading to the correct perturbative result. Combining $\psi_L^{\hat n}$ and ψ_R into a Dirac spinor $\Psi=\begin{pmatrix}\psi_L^{\hat n}\\ \psi_R\end{pmatrix}$, the second line of (B.12) can be written

$$\mathcal{L}_k \ni i\bar\Psi\not{D}_{AL}\Psi + \bar m_\psi\bar\Psi\gamma_5\Psi$$

where $\not{D}_{AL}=\gamma_\mu(\partial^\mu-\bar g v_A W^\mu P_L)$, with the usual definition of the left-projector $P_L=\frac{1}{2}(1-\gamma_5)$, and we introduced the "top-mass" $\bar m_\psi$ as defined in eq. (5.14). Since the regularized fluctuation operator for Ψ satisfies $(\Gamma_k^{(2)}+R_k)^2=\not{D}_{AL}^2(1+r_k)^2+\bar m_\psi^2$ and since $\mathrm{tr}[\gamma_5\not{D}_{AL}]=0$

one gets

$$\mathrm{Tr}\left[\frac{2\,\partial_t R_k}{\Gamma^{(2)}+R_k}\right]_\Psi = \mathrm{Tr}\left[\frac{\not{D}^2_{\mathrm{AL}}\left(1+r_k\left(\frac{\not{D}^2_{\mathrm{AL}}}{k^2}\right)\right)\partial_t r\left(\frac{\not{D}^2_{\mathrm{AL}}}{k^2}\right)}{\not{D}^2_{\mathrm{AL}}\left(1+r_k\left(\frac{\not{D}^2_{\mathrm{AL}}}{k^2}\right)\right)^2+\bar{m}^2_\psi}\right]$$
$$= \int_0^\infty ds\,\tilde{h}(s,\bar{m}^2_\psi)\mathrm{Tr}\left[e^{-s\frac{\not{D}^2_{\mathrm{AL}}}{k^2}}\right] \tag{B.13}$$

Here we need to know the spectrum of

$$\begin{aligned}\not{D}^2_{\mathrm{AL}} &= \gamma_\mu(\partial^\mu - g v_A W^{\mu} P_{\mathrm{L}})\gamma_\nu(\partial^\nu - g v_A W^{\nu} P_{\mathrm{L}}) = \gamma_\mu\gamma_\nu(\partial^\mu - g v_A W^{\mu} P_{\mathrm{R}})(\partial^\nu - g v_A W^{\nu} P_{\mathrm{L}})\\ &= \gamma_\mu\gamma_\nu(D^\mu_{\mathrm{R}}+\partial^\mu_{\mathrm{L}})(D^\nu_{\mathrm{L}}+\partial^\nu_{\mathrm{R}}) = \not{D}^2_{\mathrm{L}} + \gamma_\mu\gamma_\nu(\partial^\mu_{\mathrm{L}}D^\mu_{\mathrm{L}} + D^\mu_{\mathrm{R}}\partial^\nu_{\mathrm{R}})\end{aligned} \tag{B.14}$$

where we denoted $\partial^\mu_{\mathrm{L/R}} = \partial^\mu P_{\mathrm{L/R}}$ and took advantage of: $\not{D}^2_{\mathrm{L}} = \gamma_\mu\gamma_\nu D^\mu_{\mathrm{R}} D^\nu_{\mathrm{L}}$ and $\partial^\mu_{\mathrm{L}}\partial^\nu_{\mathrm{R}} = 0$. Let us take a shortcut at this point. We already know that the contribution of (B.13) to the β-function in the massless limit must be of the form

$$\partial_t g^2\Big|_{\psi_{\hat{n}}} = \frac{g^4}{16\pi^2}\frac{4}{3}|v_A|^2\,.$$

This fixes the $\mathcal{O}(s^0)$-term in $\mathrm{Tr}\left[e^{-s\frac{\not{D}^2_{\mathrm{AL}}}{k^2}}\right]$ to be the same as the $\mathcal{O}(s^0)$-term in $\mathrm{Tr}\left[e^{-s\frac{\not{D}^2_{\mathrm{L}}}{k^2}}\right]$. These heat-kernel traces could differ to higher orders in s, due to the two extra terms in (B.14). These higher-order terms could (unlike as for $\not{D}^2_{\mathrm{L}}$) in principle contain terms of order B^2 and thus contribute to the beta function via functions of the form

$$f_p(m^2_\psi) = \int_0^\infty ds\,\tilde{h}(s,m^2_\psi)s^p = \left[\left(-\frac{\partial}{\partial y}\right)^p h(y,m^2_\psi)\right]_{y=0}$$

where m^2_ψ is the dimensionless top mass squared: $m^2_\psi = h^2_k\kappa_k$. Because of what just said about the massless limit we must have $f_p(0) = 0$. Furthermore f_p has also to show threshold behavior, that is: $f_p(m^2_\psi \to \infty) \to 0$. As the precise dependence of $h(0,m^2_\psi)$ is anyway regulator-dependent, we can ignore the potentially nonvanishing $f_p(m^2_\psi)$ for all qualitative discussions. Therefore, without any further explicit calculation, we approximate the threshold behavior of the massive fermion mode by the same form as for the other modes

$$\partial_t g^2|_\psi = \frac{g^4}{16\pi^2}\frac{4}{3}\sum_{a=1}^{N_{\mathrm{L}}}\frac{1}{1+h^2\kappa\delta^{aA}}|v_a|^2\,.$$

for SU(2) this implies

$$\partial_t g^2|_\psi = \frac{g^4}{16\pi^2}\frac{2}{3}\left(\frac{1}{2}+\frac{1}{2}\frac{1}{1+h^2\kappa}\right)\,.$$

To summarize, we can write the gauge one-loop β-function approximately as given in the main text in eqs. (5.24,5.25).