Sequential compound options and investments valuation

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Preface

The analysis of R&D ventures and start-up companies is one of the most difficult investment problem; this is due to the property that much of the value of new ventures is associated with future cash flows that are contingent on intermediate decisions. The real options perspective on these investments has therefore acquired importance since traditional DCF-based approaches seem unsuitable to explain the dynamics in the value of R&D investment strategies. As a result, the recent body of research on the use of option pricing to R&D investments leads to a considerable literature which captures many different features of these investments.

The motivation for this thesis is to:

−provide a survey of the models on investment optimal characteristics (with respect to R&D investments in particular) and sequential investments valuation, commonplace in the real option literature;

−provide a few comprehensive models to value new ventures which take into account different R&D features;

−provide real examples of multiple compound real options exercise strategies at each stage until the research and development is completed;

−explain, through intuitive explanations, the motivation for modelling R&D investment with a jump-diffusion process;

−provide a model which relies on simple mathematics to price options with a jump-diffusion process.

−illustrate the mathematical concepts by numerical implementations.

This thesis is structured as follows:

−The first part of the thesis (chapter 1) discusses several articles in the literature related to the option valuation analysis of R&D ventures and start-up companies. We focus both on the classical reading which deal with optimal investment characteristics and the articles which study R&D investments as complex options.

−The second part (chapter 2) deals with the valuation of new ventures possessing flexibility in the form of multiple real options. For this purpose the chapter develops a multicompound options approach to value sequential investments, as R&D projects, when firms make the intermediate investment decisions to continue, expand, contract, suspend or abandon the project.

−The third part (chapter 3) deals with the valuation of R&D investments with a jump-diffusion process. This process better describes the evolution of project value and can be applied to a wide array of real-world investment context. This chapter develops a multicompound options approach to value sequential investment opportunities where the underlying asset value is subject to market and technical uncertainty. These features are modelled by assuming that the underlying project value follows a jump-diffusion process. By assuming as in Merton (1976) that the technical uncertainty is completely diversifiable and that the jump distribution is lognormal, closed-form solutions for simple multicompound and for multicompound exchange options are obtained.

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Part I

Sequential compound options and investments valuation

Chapter 1

An overview

1.1 Introduction

An increasing number of academics and corporate practitioners have been interested in the real options theory in order to accommodate operating flexibility and other strategic considerations. This new view of investment that treats opportunities as corporate real options has already enriched modern corporate finance. Thus, the real options approach is reaching advanced textbook status and is rapidly gaining reputation and influence. Such an approach better suits reality by taking into account project optimal characteristics such as multiple real option, withdrawal, sequential investment, crisis management etc. In that sense real option theory leads to a decision criterion that adapts to each particular project assessment.

The real option literature has made the argument that traditional discounted cash flow models do not capture the value of the options embedded in many corporate actions and that these options need to be not only considered explicitly and valued, but also that the value of these options can be substantial. In fact, many investments that would not be justifiable otherwise will be value enhancing, if the options embedded in them are considered. A most famous result of the real options literature is the invalidation of the standard net present value rule; it consists in investing if and only if the sum of the project discounted benefits is higher than the sum of its discounted costs. Such a criterion does have several weakness. Among many others, the following facts are often mentioned:

−The NPV method does not take into account potential uncertainty of future cash flows;

−It uses an explicit calculation for the cost of risk;

−It focuses on present time: the investment decision can only be taken now or never.

But, reality is often more complex and flexible including, for instance, optimal components for the project: a firm may have the opportunity, but not the obligation, to undertake the project not only at a precise and given time, but during a whole period of time. In that sense, these characteristics may be related to that of an American call option, the underlaying asset being, for example, the cash flows generated by the project. Sometimes, the underlying asset may be a package consisting of the project plus the value of other embedded corporate real options, to later expand production scale, to abandon the project for its salvage value, etc. The techniques derived from option pricing can help quantify management's ability to adapt its future plans to capitalize on favorable investment opportunities or to respond to undesirable development in a dynamic environment by cutting losses.

The use of a method based on option theory, such as the real option theory would improve the optimality of the investment decision. Several articles appear as benchmark in this field. The seminal articles of Brennan and Schwartz (1985) , McDonald and Siegel (1986) , Pindyck (1991) and Trigeorgis (1996) are often quoted as they present the fundamentals of this method, using particularly dynamic programming and arbitrage techniques.

The analysis of R&D ventures and start-up companies is one of the most difficult investment problem:

−A simple NPV calculation used to value these projects would suggest that they should not to be undertaken;

−New ventures take time-to-build. The development of a drug, for example, can take ten or more years to complete;

−Investments have to be made without reaping any of the possible benefits of the investment during construction;

−New ventures are subject to several, qualitatively different sources of risk: there is substantial uncertainty about the sales, costs and cash flows that they will generate;

−There is a significant probability of having to put an end to the effort for technical or economic reasons. Decision to continue with R&D are made conditioning on the resolution of systematic as well as unsystematic uncertainty.

−Learning by doing also plays an important role in determining the patterns of returns earned over the life of the project.

The approach taken in the real option literature is to treat the R&D project as complex options on the variables underlying the value of the project, which are the expected costs to completion and the estimated cash flows after completion. Uncertainty is introduced in the analysis by allowing these variables to follow stochastic processes through time. Although the real options theory suggested the use of more suitable technique to value R&D projects with these features, such investments are hard to value even with the real options approach. The main reason for this is that there are multiple sources of uncertainty in R&D investment projects and that they interact in complicated way. In practice, the bulk of the literature on the R&D valuation using real option theory have dealt with the development of numerical simulation algorithms solving optimal stopping time problems.

In the following section we give a brief overview of the literature on the investment valuation based on option theory; we start from the most popular articles in the real option literature and we continue with more recent and complex models on this topic.

We first focus on the classical readings which take into account different important features of R&D ventures and start-up companies: time-to build, abandonment, crises, etc. We also review the articles in the real option literature which studied the R&D process as a contingent claim on the value of an underlying asset.

In section 3, we overview the models of a higher complexity which depart from the traditional Black-Scholes model: compound option, exchange option, etc. The so-called exotic option allows for richer specifications (more complicated payoff function) than more traditional call and put option and overcomes the lack of flexibility of standard vanilla options; this is closer in spirit to the real option approach.

In section 4 we give a brief introduction to modelling financial derivatives with jump-diffusion process. Jump processes are introduced in the real option literature to study the impact of market crises on investment decisions. The research-oriented industries, as pharmaceutical, depend heavily on the impact of market crises.

Section 5 is devoted to game option. To analyze investment decision in industries with competitive pressure a game-theoretic analysis of options exercise strategies is essential.

Section 6 concludes.

1.2 Literature review

In recent years more and more attention has been given to stochastic models of financial markets which start from the traditional Black-Scholes/Merton model. Since Black and Scholes (1973) provide a framework to value non-dividend-paying European options, lots of practitioners and academics have dealt with the option pricing methodology. In many cases, the options are not on financially traded assets, such as stocks or commodities, but are real options, such as those on projects or other embedded corporate real options. In this section and throughout this survey, our main concern will be with investment decisions that have very important characteristics of R&D ventures and start up companies.

The basic model of irreversible investment (McDonald and Siegel, 1986) demonstrated a close analogy between a firm's option to invest and a financial call option. McDonald and Siegel studied the optimal timing of investment in an irreversible project where the benefits from the project and the investment cost follow continuous-time stochastic processes. The optimal time to invest and an explicit formula for the value of the option to invest are derived. The rule "invest if benefits exceed costs" does not properly account for the option value of waiting. Simulations show that this option value can be significant, and that for surprisingly reasonable parameter values it may be optimal to wait until benefits are twice the investment cost. Similarly, Paddock, Siegel and Smith (1988) examined the option to defer in valuing offshore petroleum leases, and Tourinho (1979) in valuing reserves of natural resources. Ingersoll and Ross (1992) reconsider the decision to wait in light of the beneficial impact of a potential future interest rate decline on project value.

As future prices and costs fluctuate, the operating profit of a project in place may turn negative; however, most firms have some escape routes available. In Dixit (1989) a firm's entry and exit decisions when the output price follows a random walk are examined. An idle firm and an active firm are viewed as assets that are call options on each other. The solution is a pair of trigger prices for entry and exit. The entry trigger exceeds the variable cost plus the interest on the entry cost, and the exit trigger is less than the variable cost minus the interest on the exit cost. These gap produces 'hysteresis'. Numerical solutions are obtained for several parameter values; hysteresis is found to be significant even with small sunk cost. Myers and Majd (1990) analyzed the option to permanently abandon a project for its salvage value.

They use the analogy with financial options and value the option to abandon as an American put option on the value of the underlying project where the strike price equals the salvage value. The option to abandon a project in exchange for its salvage (or best alternative use) value is also studied in McDonald and Siegel (1986). American put option is analyzed in Brennan and Schwartz (1977), Geske and Johnson (1984), and Barone-Adesi and Whaley (1987). Instead of abandoning permanently a project management might exercise the option to temporarily shut down whenever the output price would not be sufficient to cover the variable costs of production. Several articles suppose that a loss-making project may be temporarily suspended, and its operation resumed later if and when it becomes profitable again; among these we mention the seminal articles of McDonald and Siegel (1985) and Brennan and Schwartz (1985).

Starting from today's capacity (after a call option to invest has already been exercised) a firm usually has further call option to invest in extending today's capacity. Similarly, it often has a put option to reduce today's capacity. The put option may sometimes give them an actual cash inflow from disinvestment (e.g. proceeds from sale of redundant land or buildings, or from scrapped machinery). In other case exit may require a costly payment, which is justified if it allows them to terminate existing operating losses. Options to expand or contract capacity are further examples of the strategic dimension of R&D ventures and start up companies. Several articles examine a firm's decisions to expand or contract capacity in a more general context. Among these we mention the works of Trigeorgis and Mason (1987), Carr (1988), Pindyck (1988), Trigeorgis (1993), Abel, Dixit, Eberly and Pindyck (1996) and Dixit and Pindyck (1998).

Carr (1988) obtained a closed form solution to a compound exchange option integrating work on compound option pricing by Geske (1979) with work on exchange option pricing by Margrabe (1978). As a result, the general valuation formula may be used to value real options, as for example options to expand or contract capacity and option to switch inputs or outputs in production. Exercise of this instrument involves delivering one asset in return for an exchange option. The option received upon delivery may then be used to make another exchange at a later date. In this case the sequential expansion/contraction decision can be viewed to be similar to the exercise of a call/put compound exchange option. In Trigeorgis (1993,1996), option to expand is similar to a call option to acquire an additional part ($x\%$) by incurring a follow-on cost I_E as exercise price. The investment opportunity with the option to expand can be viewed as the base-scale project plus a call option on future investment, i.e., $V + \max(xV - I_E, 0)$. For example, management may adopt a more expensive technology to expand production if and when it becomes desirable. Similarly, management may reduce the scale of the operations, by $c\%$, thereby saving part of the planned investment outlays I_C. This flexibility to mitigate loss is analogous to a put option on part $c\%$ of the base-scale project with exercise price equal to the potential cost savings, giving $\max(I_C - cV, 0)$. The option to contract , like the option to expand, may be particularly valuable in choosing among technologies. Abel, Dixit, Eberly and Pindyck (1996) show how opportunities for future expansion or contraction can be valued as options, how this valuation relates to the q-theory of investment, and how these options affect the incentive to invest. Abel, Dixit, Eberly and Pindyck showed that a firm that makes an investment that is partially or totally reversible acquires a put option. This option has value if future uncertainty involves a sufficiently large downside with a positive probability that the firm will want to exercise the option. Recognition of this put option will make the firm more willing to invest than it would be under a naive NPV calculation that assumes that the project continues for its physical time life and omits the possibility of future disinvestment. Likewise, a firm that can expand by making an investment now or in the future (at a cost) is exercising a call option, namely it is acting now when it might have waited. This option has value if future uncertainty has a sufficiently large downside that waiting would have been preferable. Therefore recognition of this call option will make the firm more willing to invest than it would be under a naive NPV calculation that assumes that the project must be started now or never, and ignores the possibility of a future optimal start-up decision. For many real world investments, both of these options exist to some degree. Firms typically have at least some ability to expand their capacity at a time of their choosing, and sometimes can partially reverse their decisions by selling off capital and recovering part of their investment. The net effect of these two options will in general be ambiguous, depending on the degrees of reversibility and expansibility, and the extent and nature of the uncertainty. In Pindyck (1988), uncertainty over future market conditions affects investment decisions through the option that firms hold, operating options, which determine the value of capital in place, and options to add more capital, which, when investment is irreversible, determine the opportunity cost of investing. By treating capital as homogeneous and focusing on incremental investment decisions, Pindyck has tried to clarify the ways in which uncertainty and irreversibility affect the values of these options, and thereby affect the firm's optimal capacity and its market value. Dixit and Pindyck (1998) develop continuous-time models of capacity choice when demand fluctuates stochastically, and the firm has limited opportunities to expand or contract. Specifically, they consider costs of investing or disinvesting that vary with time, or with the amount of capacity already installed. The firm's limited opportunities to expand or contract create call and put options on incremental units of capital; they show how the values of these options affect the firm's investment decisions.

Most articles described above consider a single initial investment decision. In many situations, however, investment decisions are made sequentially and in particular order. Majd and Pindyck (1987) study investments with time-to-build. Many investment projects have the following characteristics: (1) spending decisions and cash outlays occur sequentially over time, (2) there is a maximum rate at which outlays and construction can proceed, that is it takes 'time-to-build', and (3) the project yields no cash return until it is actually completed. Furthermore, the pattern of investment outlays is usually flexible and can be adjusted as new information arrives. For such projects traditional discounted cash flow criteria, which treat the spending pattern as fixed, are inadequate as a guide for project evaluation. Majd and Pindyck develops an explicit model of investment projects with these characteristics, and uses option pricing methods to derive optimal decision rules for investment outlays over the entire construction program. Numerical solutions are used to demonstrate how time-to-build, opportunity cost, and uncertainty interact in affecting the investment decision. They show that with moderate levels of uncertainty over the future value of the completed project, a simple NPV rule could lead to gross over-investment. Also, They show how the contingent nature of the investment program magnifies the depressive effect of increased uncertainty on investment spending. Similarly, Bal-Ilan and Strange (1996) study the effect of investment lags in the simplest possible model of an uncertain, irreversible investment. Their paper makes two specific contributions. First, they present an analytic solution to the investment problem with lags. Second, they show that conventional results on the effect of price uncertainty on investment are weakened or reversed when there are lags. In particular, it is possible that an increase in uncertainty hastens the decision to invest. Thus, investment lags offset uncertainty and tend to reduce inertia, contrary to conventional wisdom. Finally, Grenadier (2000,2002) studied time-to-build options using option-game approach in continuous time.

Seminal works in the real option literature allowed a firm to hold and operate a large number of projects, to add new projects, and perhaps to retire old ones, etc. Real-life projects are often more complex in that they involve a collection of multiple real options, whose value may interact. Multiple real options are studied by Brennan and Schwartz (1985), Trigeorgis (1993) and Kulatilaka (1995).

Brennan and Schwartz (1985) determine the combined value of the options to shut-down and restart a mine, and to abandon it for salvage. They recognize that partial irreversibility resulting from the costs of switching the mine's operating state may create hysteresis or inertia effect, making it optimal in the long term to remain in the same operating state even if short-term cash-flow considerations seem to favour early switching. Although hysteresis is a form of interaction between early and later decisions, Brennan and Schwartz do not explicitly address the interactions among individual option values. In Trigeorgis (1993), managerial flexibility is regarded as a set of real option, for example the option to defer, abandon, contract, or expand investment, or switch investment to an alternative use. The real options literature has tended to focus on individual options, one type of operating option at time; however, managerial flexibility embedded in investment projects typically takes the form of a collection of real options. Trigeorgis demonstrates that interactions among real options present in combination generally make their individual values non-additive. Although many reader may intuit that certain options do in fact interact, the nature of such interactions and the conditions under which they may be small or large, as well as positive or negative, may not be trial. In particular, Trigeorgis illustrates through a generic project the size and type of interactions among the options to defer, abandon, contract, expand and switch use. The combined value of operating options can have a large impact on the value of the project. However, the incremental value of an additional option often tends to be lower the greater the number of other options already present. Neglecting a particular option while including others may not necessarily cause significant valuation errors. However, valuing each option individually and summing these separate option values can substantially overstate the value of a project. Configuration of real options that can exhibit precisely the opposite behavior are also identified. Kulatilaka (1995) examines the impact of interactions among such options on their optimal exercise schedules. The recent recognition of the interdependencies of real options should make possible a smoother transition from a theoretical stage to an application phase.

The focus of all these models is on optimal investment characteristics in a more general context rather than with respect to R&D ventures. The next section reviews a number of articles which have studied R&D process as a contingent claim on the value of the cash flows on completion of the R&D.

1.2.1 R&D as real options

Several papers in the economics and finance literature study dynamic R&D policies and R&D valuation using real option approach. Myers and Howe (1997) present a life cycle model of investments in pharmaceutical R&D programs. Uncertainty is explicitly accounted in the model, which is solved using Monte Carlo simulation.

Childs and Triantis (1999) develop and numerically implement a model of dynamic R&D that highlights the interactions across projects. They solve for and interpret optimal policies for a firm with multiple R&D projects, which can run in parallel or sequentially, and calculate the values of the real options such problems present. Childs and Triantis analyze in detail the intensity and timing of optimal investment policies.

Schwartz and Moon (2000) have studied R&D investment projects in the pharmaceutical industry using a real options framework. In this articles, they numerically solve a continuous-time model to value R&D projects allowing for three types of uncertainty. There is technical uncertainty associated with the success of the R&D process itself. There is an exogenous chance for obsolescence, during and after the development process and there is uncertainty about the value of the project on completion of the R&D. Schwartz and Moon solve for optimal investment policies, provide comparative statics regarding the option component of the project's value, and compare the project's value to the NPV rule.

Schwartz (2003) develops and implements a simulation approach to value patents and patents-protected R&D projects based on the real option approach. It takes into account uncertainty in the cost to completion of the project, uncertainty in the cash flows to be generated from the project, and the possibility of catastrophic events that could put an end to the effort before it is completed. This paper differs from Schwartz and Moon (2000) both in the formulation of the problem and in the solution procedure. In Schwartz and Moon (2000), once investment in R&D is completed, the owner of the project receives the value of the approved drug in the form of the single cash flow. In that framework calendar time does not enter into the solution of the problem. In Schwartz (2003), upon the approval the owner starts receiving cash flows with timing depending on the duration of the R&D investment. If a patent is obtained before the completion of the R&D investment, the duration of the cash flows will depend critically on the duration of the investment; that is, it will be path dependent.

Berk, Green and Naik (2004) also develop a dynamic model of multi-stage investment project that captures many features of R&D ventures and start-up companies. Their model assumes different sources of risk and allow to study their interaction in determining the value and risk premium of the venture. Technical uncertainty is modelled differently from Schwartz and Moon (2000) In Schwartz and Moon, the expected cost to completion of the project is an exogenous stochastic process with drift and diffusion coefficients that depend on its current value and on the current level of investment. Given this process, optimal investment policies are obtained. Berk, Green and Naik take as exogenous the technology for randomly advancing through stages of the project and then derives optimal investment policies. These will lead to an endogenous process for expected cost to completion.

Errais and Sadowsky (2005) introduce a general discrete time dynamic framework to value pilot investments that reduce idiosyncratic uncertainty with respect to the final cost of a project. In this model, the pilot phase requires N stages of investment for completion that they value as a compound perpetual Bermudan option. They work in an incomplete market setting where market uncertainty is spanned by tradable assets and technical uncertainty is idiosyncratic to the firm. The value of the option to invest as well as the optimal exercise policy are solved by an approximate dynamic programming algorithm that relies on the independence of the state variables increments.

The next section will give a brief introduction to the exotic option literature. Some academics and corporate practitioners provide applications of these models to value firm's investments and financial arrangements commonplace in the real world.

1.2.2 Exotic Options

Before we start to describe exotic options and their utility for real options purpose, it is necessary for us to summarize some basic concept of plain vanilla options. Financial literature classifies standard options, into two groups: call options and put options. A call (put) option gives its holder the right to buy (sell) the corresponding underlying asset at a specified strike price. If the strike price of a call option is lower (higher) than the spot price of the underlying asset, it is called an in-the-money call option (out-of-the-money call option). If the strike price of a call option is equal to the spot price of the underlying asset, it is called an at-the-money call option. Furthermore, vanilla options share a few common characteristics: one underlying asset; the effective starting time is present; only the price of the underlying asset at the option's maturity affects the payoff of the option; whether an option is a call or a put is known when sold; the payoff is always the difference between the underlying asset price and the strike price, and so on.

Plain vanilla options have many limitations resulting from their lack of flexibility. Exotic options[1] differ from standard options to this respect. Each type of exotic options, in fact, overcomes one particular limitation of plain vanilla options; in this way, exotic options can somehow serve a special purpose which standard options cannot do conveniently. We will look at this literature in some detail considering that the exotic options models may be relevant to the valuation of real investment opportunities. Exotic options, in fact, allow for richer specifications, more complicated payoff functions, than more traditional plain vanilla options which can be applied to a wide array of real-world investment context. For instance, since the approach taken in the real option literature is to treat the R&D project as complex options, the exotic options literature provides the insight for the analysis of growth options[2] through compound options pricing. Furthermore, real options analysis combines the two elements of compoundness and the option to exchange in the analysis of sequential investments opportunities which involves the option to switch between alternative technologies, and so on. In general, both exotic options and real options instruments are traded between companies, banks and other financial intermediaries and not quoted on an exchange.

Compound options are options written on other standard options. As there are two kinds of vanilla options, calls and puts, there are four kinds of compound options: a call option written on a call option, a call option written on a put option, a put option written on a call option, and a put option written on a put option. As a result a compound option has two expiration dates and two strike prices. Compound options are often used to hedge difficult investments which are contingent on other conditions. The buyer of a compound option normally pays an initial up-front premium for an option which he/she may need later on. The buyer will have to pay an additional premium only if this option is needed. If the buyer finds that this

[1] Exotic options literature is well summarized in Zhang (1997).

[2] Trigeorgis (1996) highlights that R&D projects are essentially real growth options because the value of these early projects derives not so much from their expected cash flows as from the follow-on opportunities they may create.

option is not necessary, he/she can simply give up the right. Geske (1979) derives a closed-form formula for a European call on European call, or compound option, and shows that the standard Black and Scholes framework is a special case such a formula. Rubinstain (1991) generalizes this result to all four possible combinations: call on a call, put on a call, call on a put and put on a put, and includes techniques for American options. Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a jump-diffusion process, applying these results to value extendible options, American call options on stocks that pay discrete dividends and American options on assets that pay continuous proportional dividends. Agliardi and Agliardi (2005) study multicompound options in the case of time-dependent volatility and interest rate. This assumption seems more suitable due to the sequential nature of many early projects. Multicompound options are merely N-fold options of options. Basically the procedure consists of solving N-nested Black-Scholes partial differential equations: at the first step the underlying option is priced according to the Black-Scholes method; then, compound options are priced as options on the securities whose values have already been found in the earlier steps. Roll (1977), Whaley (1981), Geske and Johnson (1894) and Selby and Hodges (1987) also study compound options.

Option to exchange one risky asset for another is simultaneously a call option on asset one and a put on asset two. Margrabe (1978) gives a closed-form solution for variuous exchange options and shows that the Black and Scholes call option formula is a special case of the simple exchange option formula. The exchange option formula can be used to value options, including real options, when both the strike price and the underlying asset are uncertain. McDonald and Siegel (1985) derive the analogous formula for American perpetual exchange option[3]. Stulz (1982) examines similar European options on the minimum or maximum of two risky asset. Options written on the better or worse performing (the maximum or minimum) of two or more underlying assets are often called rainbow options. Rainbow options are useful in many financial applications such as pricing foreign currency debts, compensation plans, and risk-sharing contracts.

As an exchange option a basket option is written on a basket of assets rather than one single asset. Basket options are also called portfolio options. The popular basket options are those written on baskets of currencies. As correlations among various components in a basket largely determine the

[3] See section 1.2

characteristic of the basket, basket options are correlation options. They can be used by portfolio managers to hedge their positions on the basis of their whole portfolio performance, instead of individual assets within the portfolio. Or they can be used to speculate based on the same information about their portfolios. Grannis (1992) explained how options written on baskets of currencies work. Dembo and Patel (1992) studied synthetic basket options of stocks. Gentle (1993) and Huynh (1994) also priced basket options.

Barrier options are probably the oldest of all exotic options. A barrier option is a derivative product that either becomes worthless, must be exercised, or comes into existence if the underlying asset price reaches a certain level during a certain period of time. Snyder (1969) discussed "down-and-out" options. For example, a down-and-out call has similar features to a vanilla call option, except that it becomes nullified when the asset price falls below a knock-out level. Because the holder of the option loses some of the right, the price of such an option is lower than a vanilla call option. However, if the asset price is always higher than the knock-out level, then the two options are actually the same. Therefore such a call option is more attractive than a vanilla call option for people who expect the price to rise. A knock-in option is a contract that comes into existence if the asset price crosses a barrier. For example, a "down-and-in" call with a lower barrier B_l expires worthless unless the asset price reaches the lower barrier from above prior to or at expiry. If it crosses the lower barrier from above at some time before expiry, then the option becomes a vanilla option. For closed-form expressions for prices of various barrier options and numerical methods, we refer to Reiner and Rubinstein (1991), Kunitono and Ikeda (1992), Carr (1995), Cheuk and Vorst (1996) Ritchken (1995) and Roberts and Shortland (1997). Heynen and Kat (1994) examine partial barrier options, that is, barrier options in which the underlying price is monitored for barrier hits only during a specified period during option's life. So called double barrier options are treated in Geman and Yor (1996), Hui (1996) and Pelsser (2000b).

A lookback option is an option whose payoff is determined not only by the settlement price but also by the maximum or minimum prices of the underlying asset within the option's lifetime. There are two kind of lookback options: floating-strike and fixed-strike lookback options. Floating-strike lookback options are true "no regret" options because their payoffs are the maximum. Specifically, the payoff of a floating-strike lookback call option is the difference between the settlement price and the minimum price of the underlying asset during the option's lifetime, and the payoff of a floating-strike lookback put option is the difference between the maximum price and the settlement price of the underlying asset during the option's lifetime. Thus, the payoffs of these call and put options are the greatest that could be possibly achieved.

The payoff of a fixed-strike lookback call (put) option is the difference between the maximum price of the underlying asset and the fixed strike price (the difference between the fixed strike and the minimum price) during the life of the option. Lookback options can somehow capture investors' fantasy of buying low and selling high, to minimize regret, as Goldman, Sosin, and Gatto (1979) argued. However, the no-arbitrage principle guarantees that these options are expensive to buy. The high premiums of lookback options prevent them from being widely used. So-called partial lookback options were examined by Conze and Viswanathan (1991).

Forward-start options are options with up-front premium payments, yet they start in specified future time with strike prices equal to the starting underlying asset prices. Thus, forward-start options can be considered as simple spread options in which the spreads are the differences between the prices of the same underlying asset at two different time points compared to standard simple spread options over the differences of two underlying assets. Forward-start options normally exist in the interest-rate markets where investors can use them to bet on interest-rate fluctuations. Forward-start options are studied by Rubinstein (1991).

Zhang (1993, 1994) introduced Asian options. Asian options are options with payoffs determined by some averages of the underlying asset prices during a specified period of time before the option expiration, they are also called average-price or average-rate options. Asian options can be used by a company to reduce its risk in purchasing raw materials; they can also reduce the risk in selling foreign currency through buying a put whose payoff depends on the difference between the exercise price and the average exchange rate. If the exchange rate drops, the company can get some compensation from the option for the loss in selling foreign currency. Asian options are also studied by Rubinstein (1991) and Geman and Yor (1993).

As in the case of standard American option, the owner of a Russian option has the right to choose the exercise time, τ. However, the Russian option pays the owner either S_τ, or the maximum stock price achieved up to the exercise date, whichever is larger, discounted at $e^{-r\tau}$. An analysis of an optimal stopping problem associated with the valuation of the Russian option was done in Shepp and Shiryaev (1993, 1994) and Kramkov and Shiryaev (1994).

The following section contains an application of the compound option models to venture capitals.

1.3 An example

Traditional tools based on DCF methods fail to capture the value of new ventures because of their dependence on future events that are uncertain at the time of the initial decision. The primary value of new ventures lies in the physical options it creates. These options refer to a future market opportunity resulting from a contingent claim on new product patents, knowledge, and the competitive position being created. The investment in R&D, for instance, generates value primarily by creating options for future products development. Some academics have suggested that R&D investments are essentially real growth options[4] since investment decisions are made sequentially and in a particular order. Staging investment involves firms either with some degree of flexibility in proceeding with the investment or when there is a maximum rate at which outlays or construction can proceed:

−Investing in new oil production capacity take time-to-build. First, reserves of oil must be obtained. Second, development wells and pipelines must be built so that the oil can be produced from these reserves.

−An investment in a new drug by a pharmaceutical company begins with research that (with some probability) leads to a new compound, and continues with extensive testing until the authority approval is obtained, and concludes with the construction of a production facility and marketing of the product.

−Investing in an initial-scale project in a software development requires to the venture capitalist to continue making investments in up-dating its technology and marketing its product just to keep up.

The real option literature suggests that sequential high-risk projects such as pharmaceuticals and new technologies development can be seen as compound options where each stage can be viewed as an option on the value of the subsequent stage. A compound option is simply an option on an

[4] See Trigeorgis (1996) for a discussion.

option; the exercise payoff of the compound option involves the value of the underlying option. Take the example of a European style call on a call. On the first expiration date T_1, the firm has the right to invest (to buy a new call) by paying the strike price I_1. The new call has expiration date T_2 (to complete the project) and strike price I_2. The procedure consists of following two steps: first, the underlying option is priced according to the Black-Scholes method; second, the compound option is priced as an option on the opportunity to invest whose value has already been found in the first step.

The following sections contain applications to new ventures in which we take flexibility into account at each stage of the product development, in practice, R&D managers have the flexibility to defer, contract or expand expenditures, or alternatively to abandon the R&D after funding ceases.

1.3.1 Value of expansion opportunities

Let us consider the investment decision by a venture capital fund that is evaluating the project of a single start-up company providing software tools in the computer industry. We interpret this type of corporate finance transaction as a sequential compound option.[5]

Suppose the inverse demand function for the product, giving price in terms of quantity Q is $P = YD(Q)$, where Y is a stochastic shift variable[6]. We will assume throughout that the agent is risk neutral and can borrow and lend freely at a constant interest rate, $r > 0$. Moreover, the investment project, once completed, produces one unit of output per year at zero operating costs. Hence, we assume P follows a stochastic differential equation of the form:

$$dP = \alpha P dt + \sigma P dz,$$

where dz is the increment of the standard Wiener process; σ is the instantaneous standard deviation of the spot price at time t and α is the trend rate in the price[7]. The profit flow is P in perpetuity, and its expected value

[5] See Agliardi and Agliardi (2005) for a more detailed development of the multicompoud option formula.

[6] This assumption is standard in the real options literature; see for example Dixit and Pindyck (1994).

[7] Further, it is possible to include both a time-varying variance and time-varying interest rate. This assumption seems more suitable due to the sequential nature of start-up projects ; see Agliardi and Agliardi (2003) and Amin (1993) for a discussion of this point.

grows at the trend rate α. We will assume that the price uncertainty is spanned by capital market, so that contingent claim methods can be used. Let μ denote the risk-adjusted discount rate that applies to P, where $\mu > \alpha$. We will let δ denote the difference between μ and α, that is, $\delta = \mu - \alpha$; thus we are assuming $\delta > 0$. The parameter δ has a standard role in this model, so we refer the interested reader to Dixit and Pindyck (1994) for further details[8].

If future revenues are discounted at μ, then the expected present value, V, of the project when the current price is P is just given by $V = \frac{P}{\delta}$. In this case V, being a constant multiple of P, also follows a geometric Brownian motion with the same parameters α and σ.

The second stage investment

We find the value of the option to complete the investment in the second stage of the project, $F\left(V\left(P\right), t\right)$, by constructing a risk-free portfolio, determining its expected rate of return, and equating the expected rate of return to the risk-free rate of interest. Let I_2 be the amount of investment required for completion of the second-stage. Since the option to complete the project expires at time T, the value of the option depends on the current time t. Thus, the risk-less portfolio will consist of one option to invest F and a short position of $n = \frac{\partial F}{\partial V}$ units of the project; as usual its value is $\Phi = F - \frac{\partial F}{\partial V}V$, and and the instantaneous change in this value is $d\Phi = dF - \frac{\partial F}{\partial V}dV$. The short position in this portfolio will require paying out $\delta V \frac{\partial F}{\partial V}dt$. Thus, by using Ito's Lemma we get the partial differential equation that the option to invest must satisfy:

$$\frac{\partial F}{\partial t} = rF - \left(r - \delta\right) V \frac{\partial F}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2}, \tag{1.1}$$

which must satisfy the boundary condition:

$$F_T = \max\left[V_T - I_2, 0\right]. \tag{1.2}$$

If at time T, the value of the project is greater than I_2, the option will be exercised, otherwise it is unworthy. As is well-known (see Merton,1973) the solution to (1) subject to the boundary condition (2) is:

[8] For example, δ can be interpreted as the shortfall in the expected rate of return from holding the option to complete rather than the completed project.

$$F(V,t) = Ve^{-\delta(T-t)}N_1(d_1) - I_2 e^{-r(T-t)}N_1(d_2),$$

with:

$$d_1 = \frac{\ln\left(\frac{V}{I_2}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\frac{V}{I_2}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

and $N(.)$is the cumulative normal distribution function.

The first stage investment

Given $F(V,t)$, we can now back up to the first stage of the investment, and find the value of the first-stage project, that can be represented functionally as $c = f(F,t) = f(F(V,t),t)$, with exercise price I_1, and expiration date t^*, $t^* \leq T$. Since the option to complete the project is a function of the value of the firm and time, this call option can be regarded as an European call on an European call. By going through the usual step, we can determine that c will satisfy the following partial differential equation:

$$\frac{\partial c}{\partial t} = rc - (r-\delta)V\frac{\partial c}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 c}{\partial V^2}, \tag{1.3}$$

which must satisfy the boundary condition at $t = t^*$:

$$c_{t^*} = \max\left[F_{t^*} - I_1, 0\right]. \tag{1.4}$$

where F_{t^*} is the value of the asset underlying the underlying option after time t^*. Let $\bar{V}$ denote the value of the firm which solves the integral equation $F_\tau - I_1 = 0$, where $\tau = T - t^*$. For values of the firm less then $\bar{V}$ the option to get started the project will remain unexercised, while if the value of the firm is greater than $\bar{V}$, the option will be exercised.

Let us define now:

$$h_1 = \frac{\ln\left(\frac{V}{\bar{V}}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(t^* - t)}{\sigma\sqrt{t^* - t}},$$

and:

$$h_2 = \frac{\ln\left(\frac{V}{\bar{V}}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(t^* - t)}{\sigma\sqrt{t^* - t}}.$$

In order to solve (3) and (4), we make the following substitutions:

$$c(V,t) = e^{-r(t^*-t)}\tilde{c}(u,z), \tag{1.5}$$

where:

$$u = -\ln\left(\frac{V}{\bar{V}}\right) - \left[r - \delta - \frac{\sigma^2}{2}\right](t^* - t), \tag{1.6}$$

and:

$$z = \frac{1}{2}\sigma^2(t^* - t). \tag{1.7}$$

In term of the new independent variables the fundamental equation for c becomes:

$$\frac{\partial \tilde{c}}{\partial z} = \frac{\partial^2 \tilde{c}}{\partial u^2}, \quad -\infty < u < +\infty, \quad z \geq 0. \tag{1.8}$$

The partial differential equation (8) subject to the initial value condition $\tilde{c}(u,0)$, has a unique solution which we use to write c as follows:

$$c(V,t) = e^{-r(t^*-t)} \int_{-\infty}^{+\infty} \tilde{c}(\xi,0) \frac{1}{2\sqrt{\pi z}} e^{-(u-\xi)^2/4z} d\xi.$$

Substituting the solution for $F(V,t)$ into this expression and changing the variable u with h_2, gives the following identity:

$$c(V,t) =$$

$$e^{-r(t^*-t)} \left\{ \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(h_2 + \xi/\sqrt{2z}\right)^2/2} V e^{-\delta(T-t^*)} N_1(d_1(t^*))\, d\xi + \right.$$

$$-\int_{-\infty}^{0}\frac{1}{2\sqrt{\pi z}}e^{-\left(h_2+\xi/\sqrt{2z}\right)^2/2}I_2e^{-r(T-t^*)}N_1\left(d_2\left(t^*\right)\right)d\xi+$$

$$\left.-\int_{-\infty}^{0}\frac{1}{2\sqrt{\pi z}}e^{-\left(h_2+\xi/\sqrt{2z}\right)^2/2}I_1d\xi\right\}. \tag{1.9}$$

The third term in (8) can be written in the form:

$$-I_1e^{-r(t^*-t)}N_1\left(h_2\left(t\right)\right).$$

In order to solve the remaining integrals, let us set $x=h_1+\xi/\sqrt{2z}$ in the integral of the first term in (6) and $x=h_2+\xi/\sqrt{2z}$ in the second term in (8); moreover, let us set:

$$\rho\left(t\right)=\sqrt{\frac{t^*-t}{T-t}},$$

as in Geske (1979). The first term can be written in the form:

$$Ve^{-\delta(T-t)}\int_{-\infty}^{h_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}N_1\left(\frac{d_1\left(t\right)-\rho x}{\sqrt{1-\rho^2}}\right)dx,$$

and the second term in the form:

$$-e^{-r(T-t)}I_2\int_{-\infty}^{h_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}N_1\left(\frac{d_2\left(t\right)-\rho x}{\sqrt{1-\rho^2}}\right)dx.$$

The solutions to these integrals can be found by using the fact that:

$$\int_{-\infty}^{h}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}N\left(\frac{d-\rho x}{\sqrt{1-\rho^2}}\right)dx,$$

can be written in the form:

$$\int_{-\infty}^{h}\int_{-\infty}^{d}\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\left(\frac{\left(x^2+y^2-2xy\rho\right)}{\left(2-2\rho^2\right)}\right)}dxdy,$$

which is the bivariate cumulative normal distribution $N\left(h,d;\rho\right)$ with ρ as the correlation coefficient. Thus, we obtain:

$$c\left(V,t\right)= \tag{1.10}$$

$$Ve^{-\delta(T-t)}N_2\left(h_1\left(t\right),d_1\left(t\right);\rho\left(t\right)\right)-I_2e^{-r(T-t)}N_2\left(h_2\left(t\right),d_2\left(t\right);\rho\left(t\right)\right)+$$
$$-I_1e^{-r(t^*-t)}N_1\left(h_2\left(t\right)\right),$$

where the $h_i s$, the $d_i s$ and the ρ are as defined previously.

The formula (7) captures the value of the investment opportunity in a computer software. Venture capitalists need to compute $c\left(V,t\right)$ in order to evaluate whether initially invest in such start-up firm.

1.3.2 Value with abandonment option

Many start-up companies rely upon venture capitalists to begin operations. Typically, after the initial injection of funds, addition funding is provided as the firm reaches certain performance targets. The payment of the first funding round is comparable to an initial option premium. Further payments are contingent claims: the right but not the obligation to continue financially supporting the project. If at any point, the venture capitalist ceases to pay, the project is assumed to end. Therefore, the venture capitalist can be thought of injecting funds that not only keep the project alive but also retain the right to pay the remaining payments in the future.

Computer software and Internet start-ups need two or more rounds of financing to be implemented[9]. However, venture capitalist has an alternative option to permanent abandonment the project if the operating project becomes negative; in practice, after the initial injection of funds, it gets a project in place and an option to abandon to save the follow-on expenditures. This possibility has limited applicability in most real investment projects because of the high cost of abandonment. In such cases, it would

[9] We refer the reader to Briginshaw (2002) for Internet valuation.

not make sense to abandon, unless the cash flows on the project are even more negative. To keep the analysis simple, we will ignore the fact that the abandonment may create costs and will consider the second stage investment being analogous to the exercise of financial put option on the business value with strike price equals the salvage value from abandonment[10].

Let $F_1(V,t;\varsigma_1)$ denote the value of a European call/put option with exercise price I_1 and expiration date T_1[11]. The binary option operator $\varsigma_1 = \pm 1$, when the option is a call/put. Let us now define inductively a call/put option, with value $F_2(F_1(V,t;\varsigma_1),t;\varsigma_2)$, on the call/put option whose value is F_1, with exercise price I_2 and expiration date T_2, where we assume $T_2 \leq T_1$, and $\varsigma_2 = \pm 1$, when the compound option is a call/put.

The value of the investment opportunity $F_1(V,t;\varsigma_1)$ is:

$$F_1(V,t;\varsigma_1) = \varsigma_1 V e^{-\delta(T_1-t)} N_1(\varsigma_1 a_1(t)) - \varsigma_1 I_1 e^{-r(T_1-t)} N_1(\varsigma_1 b_1(t)), \quad (1.11)$$

$$b_1(t) = \frac{\ln\left(\frac{V}{I_1}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \quad (1.12)$$

and:

$$a_1(t) = b_1(t) + \sigma\sqrt{T_1 - t}. \quad (1.13)$$

In this case, $\varsigma_1 = -1$ and $F_1(V,t)$ is the value of the option to permanent abandonment.

We can now back up to the first stage of the investment and find the value of the installment option. Let us rewrite the partial differential equation that the value of the project must satisfy as:

$$\frac{\partial F_k}{\partial t} = rF_k - (r-\delta)V\frac{\partial F_k}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F_k}{\partial V^2}, \ t \leq T_k, \ k = 1,2; \ T_1 \geq T_2. \quad (1.14)$$

A formula for the value of $F_2(V,t;\varsigma_1,\varsigma_2)$ of a European compound option can be derived by solving the partial differential equation (14) subject to the boundary condition at $t = T_2$:

[10]See Myers and Majd (1990) for abandonment option.
[11]See also Zhang (1998 pp. 607-617) for more details.

$$F_2\left(F_1\left(V, T_2; \varsigma_1\right), T_2; \varsigma_1, \varsigma_2\right) =$$

$$\max\left(\varsigma_2 F_1\left(V, T_2; \varsigma_1\right) - \varsigma_2 I_2, 0\right), \tag{1.15}$$

where $F_1\left(V, T_2; \varsigma_1\right)$ stands for the price of the underlying option. Let V_2^* denote the value of V such that $F_1\left(V, T_2; \varsigma_1\right) - I_2 = 0$. Then for V greater than V_2^* the compound option will be exercised, while for values less than V_2^* it will remain unexercised. Let us define now:

$$b_2\left(t\right) = \frac{\ln\left(\frac{V}{V_2^*}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)\left(T_2 - t\right)}{\sigma\sqrt{T_2 - t}}, \tag{1.16}$$

and:

$$a_2\left(t\right) = b_2\left(t\right) + \sigma\sqrt{T_2 - t}; \tag{1.17}$$

moreover, we set:

$$\rho_{12}\left(t\right) = \sqrt{\frac{T_2 - t}{T_1 - t}}. \tag{1.18}$$

Following the steps $(5) - (9)$, we find the value of the compound option $F_2\left(V, t; \varsigma_1, \varsigma_2\right)$:

$$F_2\left(V, t; \varsigma_1, \varsigma_2\right) = \tag{1.19}$$

$$\varsigma_2\varsigma_1 V e^{-\delta(T_1 - t)} N_2\left(\varsigma_2\varsigma_1 a_2\left(t\right), \varsigma_1 a_1\left(t\right); \varsigma_2\rho_{12}\left(t\right)\right) +$$

$$-\varsigma_2\varsigma_1 I_1 e^{-r(T_1 - t)} N_2\left(\varsigma_2\varsigma_1 b_2\left(t\right), \varsigma_1 b_1\left(t\right); \varsigma_2\rho_{12}\left(t\right)\right)$$

$$-\varsigma_2 I_2 e^{-r(T_2 - t)} N_1\left(\varsigma_2 b_2\left(t\right)\right),$$

where the $a_i s$, the $b_i s$ and the correlation coefficient, ρ_{12}, are as defined previously. In this case, $\varsigma_1 = -1$, $\varsigma_2 = +1$ and $F_2\left(V, t\right)$ is the value of the project with the opportunity to abandon the investment.

1.3.3 Value with temporary suspension

Venture capital is a typical multi-stage investment. In practice, the full-scale business can be viewed as an initial injection of funds plus additional funding, only when the research or management goal of earlier stage is achieved. Due to this feature, the real options literature emphasizes that start-up ventures are analogous to the exercising of multicompound options, as the progress towards completion usually requires a sequence of successful investments, each of which opens the possibility to undertake the next phase of expenditures.

Instead of abandoning, venture capitalist may choose to keep its project alive by maintaining its initial installment and renouncing to future investments (i.e. advertising and upgrading expenditures). For our purpose, we will consider that the project may be temporarily and costlessly suspended and that this opportunity can be seen as a sequence of three or more operating call/put options; that is, when a venture capitalist exercises its option to get started the project, it gets a project in place and a chain of interrelated options, to temporarily and costlessly shut-down or to continue funding. If it exercises the option to suspend, it gets the option to invest again or to continue suspending, and so on.

Let examine the opportunity to temporarily shut-down in more detail through a three-stage model, generalizing the well-known Geske's expression to the case of a call on a put on a call. As before, we can work backwards to determine the value of the operating option in each stage of the project.

Let $F(V,t;\varsigma_1)$ denote the value of a European option with exercise price I_1 and expiration date T_1. An analytic expression for $F(V,t;\varsigma_1)$ is given by (11):

$$F_1(V,t;\varsigma_1) = \varsigma_1 V e^{-\delta(T_1-t)} N_1(\varsigma_1 a_1(t)) - \varsigma_1 I_1 e^{-r(T_1-t)} N_1(\varsigma_1 b_1(t)),$$

where $a_1(t)$ equals (13) , $b_1(t)$ equals (12) and $N(.)$ is the cumulative normal distribution function. In this case, $\varsigma_1 = +1$ and $F_1(V,t)$ is the value of the option to restart the project.

Let us find now the value of the put option, F_2, to temporarily shut-down in the second stage, for its salvage value I_2 and expiration date T_2, where $T_2 \leq T_1$. In the section 3.2 the value of the European compound option, F_2 is found to be:

$$F_2(V,t;\varsigma_1,\varsigma_2) = \tag{1.20}$$

$$\varsigma_2\varsigma_1 V e^{-\delta(T_1-t)} N_2\left(\varsigma_2\varsigma_1 a_2(t), \varsigma_1 a_1(t); \varsigma_2\rho_{12}(t)\right) +$$
$$-\varsigma_2\varsigma_1 I_1 e^{-r(T_1-t)} N_2\left(\varsigma_2\varsigma_1 b_2(t), \varsigma_1 b_1(t); \varsigma_2\rho_{12}(t)\right)$$
$$-\varsigma_2 I_2 e^{-r(T_2-t)} N_1\left(\varsigma_2 b_2(t)\right).$$

In this case $\varsigma_1 = +1$, $\varsigma_2 = -1$, and (20) is a put on a call. Once we know the value of the option to costlessly shut-down and restart a project, we can now back up to the first stage of the investment and find the value of the compound option F_3 to get-started the funding; this option has exercise price, I_3, and expiration date T_3, $T_3 \leq T_2 \leq T_1$. Because F_3 is a function of V and t, the usual riskless hedging argument yields the partial differential equations (14) where in this case $k = 3$; the boundary condition at $t = T_3$ is:

$$F_3\left(F_2\left(V, T_3; \varsigma_1, \varsigma_2\right), T_3; \varsigma_1, \varsigma_2, \varsigma_3\right) =$$
$$\max\left(\varsigma_3 F_2\left(V, T_3; \varsigma_1, \varsigma_2\right) - \varsigma_3 I_3, 0\right), \tag{1.21}$$

Let us set V_3^* the value of V such that $F_2\left(V, T_3; \varsigma_1, \varsigma_2\right) - I_3 = 0$. Then for V greater than V_3^* the $3^{rd}-$ compound option will be exercised, while for values less than V_3^* it will remain unexercised. Let us define now:

$$b_3(t) = \frac{\ln\left(\frac{V}{V_3^*}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(T_3 - t)}{\sigma\sqrt{T_3 - t}}, \tag{1.22}$$

and:

$$a_3(t) = b_3(t) + \sigma\sqrt{T_3 - t}. \tag{1.23}$$

Finally, let $\Xi(t)$ denote the 3-dimension symmetric correlation matrix:

$$\Xi(t) = \begin{bmatrix} 1 & \varsigma_2\rho_{12} & \varsigma_3\varsigma_2\rho_{13} \\ \varsigma_2\rho_{12} & 1 & \varsigma_3\rho_{23} \\ \varsigma_3\varsigma_2\rho_{13} & \varsigma_3\rho_{23} & 1 \end{bmatrix}, \tag{1.24}$$

with entries $\rho_{ij}(t) = \sqrt{\frac{T_j - t}{T_i - t}}$, for $1 \leq i < j \leq 3$. The solution of the Black-Scholes partial differential equation (14) is written in the form:

$$F_3(V,t) = e^{-r(T_3-t)} \int_{-\infty}^{+\infty} \tilde{c}(\xi, 0) \frac{1}{2\sqrt{\pi z}} e^{-(u-\xi)^2/4z} d\xi.$$

Substituting the expression for F_2 into this solution and changing the variable u with $\varsigma_3\varsigma_2\varsigma_1 b_3$, yields the following identity:

$$\begin{aligned}
&F_3(V,t;\varsigma_1,\varsigma_2,\varsigma_3) = \\
&e^{-r(T_3-t)} \Bigg\{ \varsigma_3\varsigma_2\varsigma_1 V e^{-\delta(T_1-T_3)} \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_3\varsigma_2\varsigma_1 b_3 + \xi/\sqrt{2z}\right)^2/2} \times \\
&N_2\left(\varsigma_2\varsigma_1 a_2(T_3), \varsigma_1 a_1(T_3); \varsigma_2\rho_{12}(T_3)\right) d\xi + \\
&-\varsigma_3\varsigma_2\varsigma_1 I_1 e^{-r(T_1-T_3)} \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_3\varsigma_2\varsigma_1 b_3 + \xi/\sqrt{2z}\right)^2/2} \\
&N_2\left(\varsigma_2\varsigma_1 b_2(T_3), \varsigma_1 b_1(T_3); \varsigma_2\rho_{12}(T_3)\right) d\xi + \\
&-\varsigma_3\varsigma_2 I_2 e^{-r(T_2-T_3)} \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_3\varsigma_2\varsigma_1 b_3 + \xi/\sqrt{2z}\right)^2/2} N_1\left(\varsigma_2\varsigma_1 b_2(T_3)\right) \\
&-\varsigma_3 I_3 \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_3\varsigma_2\varsigma_1 b_3 + \xi/\sqrt{2z}\right)^2/2} d\xi \Bigg\}.
\end{aligned} \tag{1.25}$$

The last term in the expression of $F_3(V,t;\varsigma_1,\varsigma_2,\varsigma_3)$ can be written in the form:

$$-\varsigma_3 I_3 e^{-r(T_3-t)} N_1\left(\varsigma_3\varsigma_2\varsigma_1 b_3(t)\right).$$

In order to write the third term in a form that resembles the one in Geske (1979), let us change to variables $x = \varsigma_3\varsigma_2\varsigma_1 b_3(t) + \xi/\sqrt{2z}$:

$$-\varsigma_3\varsigma_2 I_2 e^{-r(T_2-t)} \int_{-\infty}^{\varsigma_3\varsigma_2\varsigma_1 b_3(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} N_1\left(\frac{\varsigma_2\varsigma_1 b_2(t) - x\varsigma_3\rho_{23}}{\sqrt{1-\varsigma_3\rho_{23}^2}}\right) dx,$$

thus, we obtain:

$$-\varsigma_3\varsigma_2 I_2 e^{-r(T_2-t)} N_2\left(\varsigma_3\varsigma_2\varsigma_1 b_3\left(t\right), \varsigma_2\varsigma_1 b_2\left(t\right); \varsigma_3\rho_{23}\left(t\right)\right).$$

Let us now set the other terms in (25) in the required form. Changing to variables $x = \varsigma_3\varsigma_2\varsigma_1 a_3\left(t\right) + \xi/\sqrt{2z}$ in the integral of the first term above and $x = \varsigma_3\varsigma_2\varsigma_1 b_3\left(t\right) + \xi/\sqrt{2z}$ in the second term; finally, we replace $\rho_{12}\left(T_3\right)$ with a function of t, according to the following rule:

$$\rho_{12}\left(T_3\right) = \frac{\left(\rho_{12}\left(t\right) - \rho_{13}\left(t\right)\rho_{23}\left(t\right)\right)}{\sqrt{\left(1-\rho_{13}^2\left(t\right)\right)\left(1-\rho_{23}^2\left(t\right)\right)}}. \tag{1.26}$$

The second term can be written in the form:

$$-I_1 e^{-r(T_1-t)} \int_{-\infty}^{\varsigma_3\varsigma_2\varsigma_1 b_3(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times$$

$$N_2\left(\frac{\varsigma_2\varsigma_1 b_2(t) - x\varsigma_3\rho_{23}(t)}{\sqrt{\left(1-\rho_{23}^2(t)\right)}}, \frac{\varsigma_1 b_1(t) - x\varsigma_3\varsigma_2\rho_{13}(t)}{\sqrt{\left(1-\rho_{13}^2(t)\right)}}; \varsigma_2 \frac{\left(\rho_{12}(t) - \rho_{13}(t)\rho_{23}(t)\right)}{\sqrt{\left(1-\rho_{13}^2(t)\right)\left(1-\rho_{23}^2(t)\right)}}\right);$$

and first term:

$$\varsigma_3\varsigma_2\varsigma_1 V e^{-\delta(T_1-t)} \int_{-\infty}^{\varsigma_3\varsigma_2\varsigma_1 a_3(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times \tag{1.27}$$

$$N_2\left(\frac{\varsigma_2\varsigma_1 a_2(t) - x\varsigma_3\rho_{23}(t)}{\sqrt{\left(1-\rho_{23}^2(t)\right)}}, \frac{\varsigma_1 a_1(t) - x\varsigma_3\varsigma_2\rho_{13}(t)}{\sqrt{\left(1-\rho_{13}^2(t)\right)}}; \varsigma_2 \frac{\left(\rho_{12}(t) - \rho_{13}(t)\rho_{23}(t)\right)}{\sqrt{\left(1-\rho_{13}^2(t)\right)\left(1-\rho_{23}^2(t)\right)}}\right).$$

Let $N_3\left(\varsigma_3\varsigma_2\varsigma_1 b_3, \varsigma_2\varsigma_1 b_2, \varsigma_1 b_1; \Xi\left(t\right)\right)$ denote the $3-$ dimension multinormal cumulative distribution function, with upper limits of integration $\varsigma_1 b_1$, $\varsigma_2\varsigma_1 b_2$ and $\varsigma_3\varsigma_2\varsigma_1 b_3$ and correlation matrix $\Xi\left(t\right)$ defined in (24).

Now we remind that:

$$\int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} N_2\left(\frac{h-\rho_{hk}x}{\sqrt{1-\rho_{hk}^2}}, \frac{d-\rho_{dk}x}{\sqrt{1-\rho_{dk}^2}}; \frac{(\rho_{dh}-\rho_{dk}\rho_{hk})}{\sqrt{\left(1-\rho_{dk}^2\right)\left(1-\rho_{hk}^2\right)}}\right) dx,$$

can be written in the form:

$$N_3(k,h,d;\Sigma) = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{|\Sigma|}} \int_{-\infty}^{k}\int_{-\infty}^{h}\int_{-\infty}^{d} e^{-\frac{1}{2}\left(X'\Sigma^{-1}X\right)} dX,$$

$$\Sigma(t) = \begin{bmatrix} 1 & \rho_{dh} & \rho_{dk} \\ \rho_{dh} & 1 & \rho_{hk} \\ \rho_{dk} & \rho_{hk} & 1 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $N_3(k,h,d;\rho_{hk},\rho_{dk},\rho_{dh})$ is the trivariate cumulative normal distribution.

Finally, we succeed in writing (27) and (28) in terms of the trivariate cumulative function according to the following generalized rule:

$$\int_{-\infty}^{\varsigma_3\varsigma_2\varsigma_1 b_3} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$N_2\left(\frac{\varsigma_2\varsigma_1 b_2 - x\varsigma_3\rho_{23}}{\sqrt{\left(1-\rho_{23}^2\right)}}, \frac{\varsigma_1 b_1 - x\varsigma_3\varsigma_2\rho_{13}}{\sqrt{\left(1-\rho_{13}^2\right)}}; \varsigma_2\frac{(\rho_{12}-\rho_{13}\rho_{23})}{\sqrt{\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right)}}\right) dx =$$

$$= N_3\left(\varsigma_3\varsigma_2\varsigma_1 b_3, \varsigma_2\varsigma_1 b_2, \varsigma_1 b_1; \Xi(t)\right).$$

The value of the compound option, $F_3(V,t;\varsigma_1,\varsigma_2,\varsigma_3)$, is:

$$F_3(V,t;\varsigma_1,\varsigma_2,\varsigma_3) =$$

$$\varsigma_3\varsigma_2\varsigma_1 V e^{-\delta(T_1-t)} N_3\left(\varsigma_3\varsigma_2\varsigma_1 a_3(t), \varsigma_2\varsigma_1 a_2(t), \varsigma_1 a_1(t); \Xi(t)\right) +$$

$$-\varsigma_3\varsigma_2\varsigma_1 I_1 e^{-r(T_1-t)} N_3\left(\varsigma_3\varsigma_2\varsigma_1 b_3(t), \varsigma_2\varsigma_1 b_2(t), \varsigma_1 b_1(t); \Xi(t)\right) +$$

$$
\begin{aligned}
-\varsigma_3\varsigma_2 I_2 e^{-r(T_2-t)} N_2\left(\varsigma_3\varsigma_2\varsigma_1 b_3\left(t\right), \varsigma_2\varsigma_1 b_2\left(t\right); \varsigma_3\rho_{23}\left(t\right)\right) + \\
-\varsigma_3 I_3 e^{-r(T_3-t)} N_1\left(\varsigma_3\varsigma_2\varsigma_1 b_3\left(t\right)\right). \qquad (1.28)
\end{aligned}
$$

Unlike most compound options in the financial market, it is perfectly possible for the firm to suspend investment at a certain time T_2, and resume it later; so, in the case, $\varsigma_1 = +1$, $\varsigma_2 = -1$, $\varsigma_3 = +1$ and $F_3\left(V, t\right)$ is a call on a put on a call.

1.4 Real option modelling with jump processes

1.4.1 Introduction

This section reviews some models with jumps[12] and jump-diffusion processes in particular. These processes have become increasingly popular for modelling market fluctuations, both for option pricing and for real option valuation.

Empirical evidence confirms the systematic mispricing of the Black-Scholes call option pricing model. A number of explanations for the systematic price bias have been suggested[13]. Among these is the presence of jumps in price. Diffusion models cannot properly capture sudden, discontinuous moves in price. This well-known fact leads to the argument that using continuous or discontinuous models has important consequences for the representation of the risk.

Merton (1976) have suggested that incorporating jumps in option valuation models may explain some of the large empirical biases exhibited by the Black-Scholes model. According to the Merton specification, the arrival of normal information leads to price changes which can be modeled

[12] Financial models with jumps fall into two categories. In the first category, called jump-diffusion models, the "normal" evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. Here the jumps represent rare events, as crashes and large drawdowns. The second category consists of models with infinite number of jumps in every interval, which are called *infinite activity* models. In these models Brownian component is not needed since the dynamics of jumps is already rich enough that such models give a more realistic description of the price process at various time scales. The interested reader can see Cont and Tankov (2004) for the necessary tools for understanding these models and the concepts behind them.

For our purpose we deal with the reasons which motivate the use of jump processes for modelling real options and R&D investments in particular.

[13] See for example Geske and Roll (1984).

as a lognormal diffusion, while the arrival of abnormal information gives rise to lognormally distributed jumps in the security return, which can be modelled as a Poisson process. If the underlying security follows a mixed jump-diffusion process, then the resultant equilibrium option price will systematically differ from the Black-Scholes equilibrium option price.

Real option studies are usually written in a continuous time framework for the underlying dynamics. However, the existence of crises and shocks on investment market generates discontinuities. The impact of these crises on the decision process is then an important feature to consider. The assumption of jump-diffusion process better describe the evolution of asset value due to the risky nature of many early investments. Of course new ventures are subject to several, qualitatively different sources of risk. There is the uncertainty associated with the market factors outside the control of the firm, that causes marginal changes in the asset value. This is related to the demand for the product and production costs and is modeled by a standard geometric Brownian motion. There is the exogenous risk associated with the actions of a competitor, and finally, there is the technical uncertainty which is idiosyncratic to the firm. The technical risk which represents the discontinuous arrival of new information has more than a marginal effect on the asset value. This component is modelled by a jump process reflecting the non-marginal impact of information.

Let us consider the following examples:

−The policy process is particular relevant for the firm engaged in R&D and other new ventures. Governments can not only deploy measures to reduce the uncertainty facing potential investors, they can also create uncertainty through the prospect of policy changes. It is commonly believed that expectations of shifts of policy can have powerful effects on decisions to invest in these early projects. However, policy uncertainty is not likely to be well captured by a Brownian motion process; it is more likely to be a Poisson jump.

−R&D in pharmaceuticals and biotechnologies frequently involves upward jumps or downward jumps, for example drugs can turn into mega-selling blockbuster products or suffer clinical trial failures and withdrawal from the markets. Hence, real R&D investment appraisal should rely on a model focusing on these aspects, rather on standard Brownian motion.

−The opportunities for a firm to continuously expand its technology represents a critical component of the software providing industry's investment decisions. The firms' ability to later expand capacity is clearly more valuable for more volatile business with higher returns on project, such as computer software or biotechnology, than it is for traditional business, as real estate or automobile production. Nevertheless, when the new software product comes together with technological innovations, there is also considerable uncertainty with respect to the actions of a competitor or changes in environment before or soon after technological improvements. For example, a software product may fail because of technological advances in hardware.

The current valuation of investments based on option methodology assumes a continuous cash-flow generation process which is inadequate when these types of risk jointly determine the value of a new venture.

1.4.2 Merton's approach

Merton (1976) extended the Black-Scholes model to include situations when the underlying asset returns are discontinuous. As in many economic models, the discontinuity is modeled with a Poisson process. The Poisson distributed event is the arrival of an important piece of information about the underlying instrument. The arrivals of information are assumed to be independently and identically distributed. The probability of an event during a time interval of length h (h can be as small as possible) can be written as

$$Prob.\,[\text{the event does not occur in the time interval } (t, t+h)] = 1 - \lambda h + O(h),$$

$$Prob.\,[\text{the event occurs in the time interval } (t, t+h)] = \lambda h + O(h),$$

$$Prob.\,[\text{the event occurs more than once in the time interval } (t, t+h)] = O(h),$$

where $O(h)$ represents a function of h which goes to zero faster than h.

With the above description of the Poisson distribution, Merton (1976) assumed the following stochastic process for the underlying asset:

$$\frac{dS}{S} = (\mu - \lambda k)\,dt + \sigma dZ + (Y-1)\,dq, \tag{1.29}$$

where μ is the instantaneous expected return on the underlying asset; σ is the instantaneous standard deviation of the return, conditional on no arrivals of important new information[14]; dz is the standard Brownian motion; dq is the independent Poisson process with rate λt (the mean number of jumps per unit time); $(Y-1)$ is the proportional change in the stock price if the Poisson event occurs and $k \equiv E[Y-1]$, where E is the expectation operator over the random variable Y; dq and dz are assumed to be independent [so that $E(dzdq) = 0$].

Merton's approach proposes to ignore risk premia for jumps. He assumes that nonsystematic risk (jump component) is completely diversifiable, that is, the firm will not demand any additional return over the risk free rate for being exposed to this source of risk. This fact will allow us to specify a unique equivalent risk-neutral measure by setting the market price of risk of q to zero. A contingent claim F on the stock price must satisfy:

$$\frac{\partial F}{\partial \tau} = rF - (r - \lambda k) S \frac{\partial F}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - \lambda E\{F(SY,t) - F(S,t)\},$$

where τ is the time time to expiration and r is the risk-less interest rate. With the Poisson distribution assumption of information arrivals and the underlying asset return distribution process (19) Merton obtained a pricing expression for the European call option with strike price K, as follows:

$$F(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \left\{E_n\left[W\left(SX_n e^{-\lambda k\tau}, \tau, K, \sigma^2, r\right)\right]\right\}, \quad (1.30)$$

where X_n has the same distribution as the product of n independently and identically distributed variables Y, and $X_0 = 1$; E_n represents the expectation operator over the distribution of X_n and W is the standard Black-Scholes formula for a European call option.

Merton noted a special case of no little interest when the random variable Y has a log-normal distribution. Let δ^2 denote the variance of $\log Y$, and let $\gamma \equiv \log(1+k)$. In that case, the Black-Scholes model for normally distributed stock prices with a constant variance might be adjusted, where the adjusted interest rate is:

[14] Further, it is possible to include both a time-varying variance and time-varying interest rate; see Agliardi and Agliardi (2003) and Amin (1993) for a discussion of this point.

$$r_n = r - \lambda k + \frac{n\gamma}{\tau}.$$

Merton's solution is written as the adjusted Poisson distribution function times the adjusted Black-Scholes option price $W(S)$:

$$F(S,\tau) = \sum_{n=0}^{\infty} \frac{e^{-\acute{\lambda}\tau}\left(\acute{\lambda}\tau\right)^n}{n!}\left\{W\left[S,\tau;K,\sigma_n,r_n\right]\right\}, \tag{1.31}$$

where:

$$\sigma_n = \sqrt{\sigma^2 + \delta^2\left(\frac{n}{\tau}\right)},$$

and:

$$\acute{\lambda} = \lambda\left(1+k\right).$$

Merton also showed an adjusted 'delta' for the mixed diffusion-jump process model, that is the number of shares that should be bought per each call option sold that would create a riskless hedge, which is the change in the option price per change in the stock price. He argued that there is a delta which eliminates all systematic risk (assuming the jumps are not systematic), which is the first derivative of equation (21) :

$$N^* = \sum_{n=0}^{\infty} \frac{e^{-\acute{\lambda}\tau}\left(\acute{\lambda}\tau\right)^n}{n!}\left\{\Phi\left[d\left(n\right)\right]\right\},$$

where:

$$d(n) = \frac{\log\left(\frac{S}{E}\right) + \left(r_n + \frac{\sigma^2}{2}\right)\tau + \frac{n\delta^2}{2}}{\sqrt{\sigma^2\tau + n\delta^2}},$$

and $\Phi(.)$ is the cumulative distribution function of a standard normal distribution.

Of course, when $\lambda = 0$, (21) reduces to the Black-Scholes formula.

1.4.3 Further reading

Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a jump-diffusion process, applying these results to value extendible options, American call options on stocks that pay discrete dividends and American options on assets that pay continuous proportional dividends.

As before, Gukhal considers a frictionless continuous time economy where information arrives both continuously and discontinuously. This is modeled by a mixed diffusion-jump process (19). Consider a compound call option written on the European call $C_E(K,T)$ with expiration date T_1 and strike price K_1, where $T_1 < T$. Let $CC[C_E(K,T), K_1, T_1]$ denote this compound option. This compound option is exercised at time T_1 when the value of the underlying asset, $C_E(S_1, K, T_1, T)$, exceeds the strike price K_1. When $C_E(S_1, K, T_1, T) < K_1$, it is not optimal to exercise the compound option and hence expires worthless. The asset price at which one is indifferent between exercising and not exercising is specified by the following relation:

$$C_E(S_1^*, K, T_1, T) = K_1.$$

When it is optimal to exercise the compound at time T_1, the option holder pays K_1 and receives the European call $C_E(K, T_1, T)$. This European call can in turn be exercised at time T when S_T exceed K and expires worthless otherwise. Hence, the cashflows to the compound option are an outflow of K_1 at time T_1 when $S_1 > S_1^*$, a net cashflow at time T of $S_T - K$ when $S_1 > S_1^*$ and $S_T > K$, and none in the other states.

The value of the compound option is the expected present value of these cashflows and is given by:

$$\begin{gathered} CC[C_E(K,T), K_1, 0, T_1] = \\ E_0\left[e^{-rT}(S_T - K)\, 1_{\{S_T > K\}} 1_{\{S_1 > S_1^*\}}\right] + E_0\left[e^{-rT_1}(-K_1)\, 1_{\{S_1 > S_1^*\}}\right] = \\ E_0\left[e^{-rT} C_E(S_1, K, T_1, T)\, 1_{\{S_1 > S_1^*\}}\right] - E_0\left[e^{-rT_1} K_1 1_{\{S_1 > S_1^*\}}\right], \end{gathered} \tag{1.32}$$

where $C_E(S_1, K, T_1, T)$ is given in Merton (1976).

To examine option pricing when the asset price dynamics include the possibility of non-local changes, Gukhal conditions the expectations on the number of jumps in the intervals $[0, T_1]$, and $(0, T_1]$, denoted by n_1 and n_2, respectively. The first expectation in (22) can then be written as:

$$E_0\left[e^{-rT}C_E\left(S_1, K, T_1, T\right)1_{\left\{S_1>S_1^*\right\}}\right] =$$

$$E_0\left\{\sum_{n_1=0}^{\infty}\left[e^{-rT_1}E_{T_1}\left(\sum_{n_2=0}^{\infty}\left[e^{-r(T-T_1)}\left(S_T-K\right)1_{\{S_T>K\}} \mid n_2\right]prob\left(n_2\right)\right)\times\right.\right.$$
$$\left.\left. 1_{\left\{S_1>S_1^*\right\}} \mid n_1\right]prob\left(n1\right)\right\}.$$

The evaluation of this expectation requires the joint density of two Poisson weighted sums of correlated normals. Thus, it is useful to work with the logarithmic return, $x_t = \ln\left(\frac{S_t}{S_0}\right)$, rather than the price.

With Y log-normally distributed, Gukhal obtained a pricing expression for a European compound call option:

$$\sum_{n_1=0}^{\infty}\frac{e^{-\lambda T_1(\lambda T_1)^{n_1}}}{n_1!}K_1e^{-rT_1}N\left[a_2\right]+$$
$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{e^{-\lambda T_1(\lambda T_1)^{n_1}}}{n_1!}\frac{e^{-\lambda v(\lambda v)^{n_2}}}{n_2!}\left\{S_0N_2\left[a_1, b_1, \rho_{1T}\right]+\right.$$
$$\left.-Ke^{-rT}N_2\left[a_2, b_2, \rho_{1T}\right]\right\},$$

where:

$$a_1 = \frac{\ln\left(S_0/S_1^*\right)+\left(\mu_{JD1}+\sigma_{JD1}^2/2\right)T_1}{\sigma_{JD1}\sqrt{T_1}},$$

$$a_2 = a_1 - \sigma_{JD1}\sqrt{T_1},$$

$$b_1 = \frac{\ln\left(S_0/K\right)+\left(\mu_{JD1}+\sigma_{JD}^2/2\right)T}{\sigma_{JD}\sqrt{T}},$$

$$b_2 = b_1 - \sigma_{JD}\sqrt{T},$$

and:

$$\rho_{1T} = \frac{cov\left(x_{T_1}, x_T\right)}{\sqrt{var\left(x_{T_1}\right) var\left(x_T\right)}},$$

where ρ_{1T} is the correlation coefficient between x_{T_1} and x_T.

Ball and Torous (1985) note that the Black-Scholes call option pricing model exhibits systematic empirical biases and that Merton call option pricing model, which explicitly admits jumps in the underlying security return process, may potentially eliminate these biases. They provide statistical evidence consistent with the existence of lognormally distributed jumps in a majority of the daily returns of a sample of NYSE listed common stocks. However, they find no operationally significant differences between the Black-Scholes and Merton model prices of the call options written on the sampled common stocks.

Amin (1993) develops a simple, discrete time model to value options when the underlying process follows a jump-diffusion process. Multivariate jumps are superimposed on the binomial model of Cox-Ross-Rubinstein (1979) to obtain a model with a limiting jump diffusion process. This model incorporates the early exercise feature of American options as well as arbitrary jump distributions. It yields an efficient computational procedure that can be implemented in practice.

Cont and Tankov (2004) provide an overview of theoretical, numerical and empirical research on the use of jump processes in financial modelling. The goals of the book are to explain the motivation for using Lévy processes[15] in financial modelling and to provide real examples of uses of jump processes in option pricing and risk management.

[15] A Lévy process is a process with stationary and independent increments which is based on a more general distribution than the normal distribution. In order to represent skewness and excess kurtosis, the distribution in a Lévy process has to be infinitely divisible. For every such infinitely divisible distribution, there is a stochastic process $X = \{X_t, t \geq 0\}$ called Lévy process, which starts at zero and has stationary and independent increments such that the distribution of an increment over $[s, s+t]$, s, $t \geq 0.$, i.e $X_{t+s} - X_s$ has $(\phi(u))^t$ as its characteristic function.

Generally a Lévy process may consist of three independent components, namely a linear drift, a Brownian diffusion and a pure jump. The latter is characterized by the density of jumps, which is called the Lévy density. Let $f(x)$ denote this density. The Lévy density has the same mathematical requirements as a probability density except that it does not need to be integrable and must be zero at the origin. In relation a Lévy density is expressed

Finally, Cox and Ross (1976), Naik and Lee (1990) and Bates (1996) deal with jump processes in modelling financial derivatives.

More recently, some papers in the finance literature study Lévy processes in real options pricing. Barrieu and Bellamy (2005) analyze the impact of market crises on investment decision via real option theory. The investment project, modelled by its profits/costs ratio, is characterized by a mixed diffusion process, whose jumps represent the consequences of crises on the investment field. This paper is dedicated to the analysis of the exercising time properties in an unstable framework. The modelling of the underlying dynamics involves a mixed-diffusion, made up of Brownian motion and Poisson process. The jumps are negative as to represent troubles and difficulties occurring in the underlying market.

This paper focus on a single investor evolving in a universe, defined as a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. He has to decide whether he will undertake a given investment project and, if so, when it is optimal to invest. An infinite time horizon for the investment is considered. The investment opportunity at time $t = 0$ is given by:

$$C_0 = \sup_{\tau \in \Upsilon} \mathbb{E}\left[\exp(-\mu\tau)(S_\tau - 1)^+\right],$$

where $\mathbb{E}$ is the expectation with respect to the prior probability measure $\mathbb{P}$, Υ is the set of the $(\mathcal{F}_t)-$stopping times, $(S_t, t \geq 0)$ is the process of

as $\nu(dx) = f(x)\,dx$.

A Lévy process can be completely specified by its moment generating function $E\left[e^{uX_t}\right] = e^{\kappa(u)t}$. κ is the Lévy exponent. The equivalent description is in terms of the characteristic exponent, ζ, definable from the representation $E\left[e^{i\xi X(t)}\right] = e^{-\zeta(\xi)t}$, where $i = \sqrt{-i}$ and hence $\kappa(u) = \zeta(-iu)$. For example, if the jump component is a compound Poisson process, then the Lévy exponent is

$$\kappa(u) = u\gamma^2 + \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{+\infty} (e^{yu} - 1) f(dy),$$

and $f(dy)$ satisfies

$$\int_{\mathbb{R}\{0\}} \min\{1, |y|\} f(dy) < +\infty.$$

The cumulant characteristic function $\zeta(u)$ is often called the characteristic exponent, which satisfies the following $L\acute{e}vy - Khintchine\ formula$,

$$\zeta(u) = iu\gamma - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{+\infty} \left(e^{yui} - 1 - yui1_{\{|y|<1\}}\right) \nu(dy),$$

where the triplet of the Lévy characteristics is given by $\left(\gamma, \sigma^2, \nu\right)$.

the profits/costs ratio and 1 is the strike price. The profits/costs ratio is characterized by the following dynamics

$$\begin{cases} dS_t = S_{t^-}\left[\alpha dt + \sigma dW_t + \varphi dM_t\right] \\ S_0 = s_0 \end{cases};$$

where $(W_t, t \geq 0)$ is a standard $(\mathbb{P}, (\mathcal{F}_t))$ −Brownian motion and $(M_t, t \geq 0)$ is the compensated martingale associated with a $(\mathbb{P}, (\mathcal{F}_t))$ −Poisson process N. The Poisson process is assumed to have a constant intensity λ and the considered filtration is defined by $\mathcal{F}_t = \sigma\left(W_s, M_s, 0 \leq s \leq t\right)$.

Equivalently, the process $(S_t, t \geq 0)$ may be written in the form $S_t = s_0 \exp\left(X_t\right)$ where $(X_t, t \geq 0)$ is a Lévy process with the Lévy exponent Ψ

$$\mathbb{E}\left(\exp\left(\xi X_t\right)\right) = \exp\left(t\Psi\left(\xi\right)\right),$$

with

$$\Psi\left(\xi\right) = \xi^2\frac{\sigma^2}{2} + \xi\left(\alpha - \lambda\varphi - \frac{\sigma^2}{2}\right) - \lambda\left(1 - (1+\varphi)^{\xi}\right);$$

Hence

$$\begin{aligned} \mathbb{E}\left(\exp\left(iX_1\right)\right) &= \exp\left(i\xi\left(\alpha - \lambda\varphi - \frac{\sigma^2}{2}\right) - \xi^2\frac{\sigma^2}{2} + \lambda\left(e^{i\xi\ln(1+\varphi)} - 1\right)\right) \\ &= \exp\left(-\Phi\left(\xi\right)\right). \end{aligned}$$

Therefore, the Lévy measure associated with the characteristic exponent Φ is expressed in terms of the Dirac measure δ as: $\nu\left(dx\right) = \lambda\delta_{\ln(1+\varphi)}\left(dx\right)$. Furthermore, Barrieu and Bellamy assume: (1) $0 < s_0 < 1$, that is, delaying the project realization is only relevant in the case the profits/costs ratio is less than one; (2) $\sigma > 0$ and (3) $0 > \varphi > -1$. The last assumption states that the jump size is negative. This hypothesis together with the identity: $S_t = s_0\left(1+\varphi\right)^{N_t} \times e^{(\alpha-\lambda\varphi)t} \times e^{\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)}$ ensure that the process S remains strictly positive. Finally, they impose the integrability condition: $\mu > \sup\left(\alpha, 0\right)$.

There exists an optimal frontier L_φ^* such that:

$$\sup_{\tau\in\Upsilon}\mathbb{E}\left[\exp\left(-\mu\tau\right)\left(S_\tau - 1\right)^+\right] = \mathbb{E}\left[\exp\left(-\mu\tau_{L_\varphi^*}\right)\left(S_{\tau_{L_\varphi^*}} - 1\right)^+\right];$$

where τ_L is the first hitting time of the boundary L by the process S, defined as $\tau_L = \inf\{t \geq 0; S_t \geq L\}$. After having analyzed the implications of different modelling choices, Barrieu and Bellamy study the real option associated with the investment project. Mordecki (2002) also studies optimal stopping and perpetual options for Lévy processes.

The focus of this chapter up to this point is on optimal investment characteristics with respect to single firm's investment decisions. However, the impact of competitive pressure is an important feature to take into account when dealing with R&D investments.

1.5 Real option and game theory

1.5.1 Introduction

A feature that the vast majority of the real option articles have in common is the lack of strategic interaction across option holders. In this context, investment strategies are formulated in isolation, without regard to the potential impact of other firm's exercise strategies. The standard real options framework, (Dixit and Pindick,1994), has been based on the assumption that an investment opportunity in a monopoly is exclusive so, since increasing uncertainty increases the value of the option, the optimal timing choice leads to delay investment. However, the complexity of reality suggests that strategic relationships between the economic agents, may play an important role. Investigating investment decisions when first movers can preempt and enjoy an advantage, leads to more general models involving strategic dimensions and game theory; in this context, results differ from those viewed in monopoly and lead to Stackelberg leader-follower framework, where the commitment of an irreversible investment and the threat of a first mover may cancel the strategic effect resulting from single-agent framework. Under imperfect competition the investment timing strategy of an incumbent firm is coupled to the timing tactics of the rivals and leads to hasten, rather than delay, the cost of investment. All those considerations make the optimal timing choice a very complex problem.

Strategic consideration, particularly with regard to R&D activities are recently being treated in the real options literature. The timing and value of many early investments depend critically on competitive interaction. For example, the payoff from R&D projects may be substantial if the firm has monopolistic access to the technology research, but can become significantly modified if competitors have similar access. That is the case of two or more firms engaged in a patent race in the pharmaceutical industry; the only non-negative cash flow is received by the winning firm at the end of the race. In a few instances when related options are held by a small number of firms with an advantage to the first mover, each firm's ability to delay is undermined by the fear of preemption. Thus, many models of competitive strategy using real options have focused on the trade-off between flexibility and commitment in capacity investments.

A large economics literature studies strategic exercise games applying these to a wide array of real-world investment contexts, where the payoffs from a firm's investments are fundamentally affected by the investment strategies of its competitors. Several articles appear as benchmark in this field. The seminal articles of Smit and Trigeorgis (1997, 2006), Baldursson (1998), Grenadier (1996, 2002), Lambrecht and Perraudin (2003) and Miltersen and Schwartz (2004) illustrate the use of real option and game theory principles to analyze prototypical investment opportunities involving important competitive/strategic decisions under uncertainty.

1.5.2 Grenadier's model

Grenadier (2002) develops an equilibrium framework for strategic options exercise games. The author provides a tractable approach for deriving equilibrium investment strategies in continuous time Cournot-Nash framework. In this model each firm from the n-firms oligopoly holds a sequence of investment opportunities that are like call options over a production project of capacity addiction. The first assumption is that all firms are equal, with technology to produce a single, homogeneous good. The output is infinitively divisible and the unity price of the product is $P(t)$. At time t, firm i produces $q_i(t)$ units of output at price $P(t) = D[X(t), Q(t)]$, where D is the inverse demand function, $X(t)$ is an exogenous shock process to demand and $Q(t)$ is the industry supply process. The model assumes throughout that investors are risk neutral and can borrow and lend freely at a constant safe interest rate $r > 0$. Firm i has no variable costs of production. At any instant the profit flow for firm i is given by $\pi_i[X(t), q_i(t), Q_{-i}(t)] \equiv q_i(t) \cdot D[X(t), q_i(t) + Q_{-i}(t)]$ given the current state of the industry. Firms choose output as their strategy variable. At any point in time, each firm can invest in additional capacity to increase its output by an infinitesimal increment $dq_i \equiv \frac{dQ}{n}$. The cost of increasing output is K per unit of output. The shock $X(t)$, follows a time-homogeneous diffusion process of the form $dX = \mu(X)\,dt + \sigma(X)\,dz$, where dz is a Wiener process[16]. Let $V^i[X, q_i, Q_{-i}; q_i(t), Q_{-i}(t)]$ denote the value of firm i. Given the initial values of the state variable $[x, q_i, Q_{-i}]$ and strategies for all firms, q_i, $i = 1, .., n$, V^i can be written as the discounted expectation of future cash flows:

$$V^i[X, q_i, Q_{-i}; q_i(t), Q_{-i}(t)] =$$
$$E_{x,q}\left\{\int_0^\infty e^{-rt}\pi_i[X(t), q_i(t), Q_{-i}(t)]\,dt - \int_0^\infty e^{-rt}K\,dq_i(t)\right\},$$

where the expectation operator is conditional on the initial values $[X, q_i, Q_{-i}]$. The strategies $q^*(t)$ constitute a Counot-Nash equilibrium if

$$V^i\left[X, q_i, Q_{-i}; q_i^*(t), Q_{-i}^*(t)\right] = \sup_{\{q_i(t):t>0\}} V^i\left[X, q_i, Q_{-i}; q_i(t), Q_{-i}^*(t)\right], \ \forall i.$$

Firms choose quantities $q_i^*(t)$ $(i = 1, .., n)$ maximizing their payoffs and considering the competitors best response Q_{-i}^*. Grenadier focuses on the case of symmetric Nash equilibrium $q_i^*(t) = q_j^*(t)$ for all i, j. Therefore, the optimal output for n-firms oligopoly symmetric Nash equilibrium is $q_i^*(t) = \frac{Q^*(t)}{n}$.

Each firm faces a dynamic programming problem of determining its optimal investment strategy, taking into account its competitors' investment strategies. The derivation of firm's value can be obtained by applying the option pricing approach. In more specific terms, each firm holds a sequence of options on the marginal flow of profits, fully recognizing that the exercise of investment options by its competitor will impact its own payoff from exercise. Let us begin by considering firm $i's$ optimal investment strategies, where firm i takes all competitor's strategies as given. Thus, while firm i controls the evolution of the process $q_i(t)$, it recognizes that the evolution of the process $Q_{-i}(t)$, which is beyond firm i' s control, helps determine the payoff from the exercise of its investment options. Given these properties, the optimal exercise policies of all of these options will take the form of trigger strategies: the option to add capacity is exercised by firm i when the demand shock $X(t)$ reaches a threshold level that is a function of the current state of the industry.

Grenadier summarizes the equilibrium investment strategies in proposition 1, with a differential equation and three boundary conditions. The first

[16] If $\alpha(X) = \alpha \cdot X$ and $\sigma(X) = \sigma \cdot X$, we get the known geometric Brownian motion.

and second boundary conditions are the value-matching and smooth-pasting conditions, which are very known in continuous-time real options framework. However, the third condition is the strategic one, requiring that each firm i is maximizing its value V^i given the competitor's strategies. The third condition is a value-matching at the competitors' threshold $\bar{X}^{-i}\left(q_i, Q_{-i}\right)$, which is equal to $\bar{X}^i\left(q_i, Q_{-i}\right)$, due to the symmetric equilibrium. The third condition is also like a fixed-point search over the best response maps. However, this condition will not be necessary with the Grenadier's Proposition 2, extending the myopic optimality concept to oligopolies. Proposition 2 tells that the myopic firm threshold is equal to the firm's strategic threshold. Proposition 3 establishes the symmetric Nash equilibrium: each firm will exercise its investment option whenever $X(t)$ rises to the trigger $X^*(Q)$. Let the value of the myopic firm's marginal output be denoted by $m^i(X, q_i, Q_{-i})$ with $m^i(X, q_i, Q_{-i}) = \frac{\partial M^i}{\partial qi}(X, q_i, Q_{-i})$, where $M^i(X, q_i, Q_{-i})$ is the value of the myopic firm. Finally, given the symmetry $X_i^*(q_i, Q_{-i}) = X^*(Q)$ because $q_i = \frac{Q}{n}$ and $Q_{-i} = \frac{(n-1)}{n}Q$.

If $m(X, Q)$ denote the value of a myopic firm's marginal investment, the following differential equation and two boundary conditions determine both $X^*(Q)$ and $m(X, Q)$:

$$\frac{1}{2}\sigma(X)^2 m_{XX} + \mu(X) m_X - r \cdot m + D(X, Q) + \frac{Q}{n} D_Q(X, Q) = 0,$$

subject to a value-matching condition at $X^*(Q)$:

$$m[X^*(Q), Q] = K,$$

and a smooth-pasting condition:

$$\frac{\partial m}{\partial X}[X^*(Q), Q] = 0.$$

Grenadier's section 3 investigates the impact of an increasing competition on the value of the option to invest. The author demonstrates that the presence of competition drastically erodes the value of the option to wait. While for reasonable parameter values a monopolist may not invest until the NPV is double the cost of investment, with competition the traditional NPV rule becomes approximately correct, even for industries with only a few competitors.

1.5.3 Further reading

Several articles integrated real options and game-theoretic framework in a more general context. Baldursson (1998) studied an oligopoly where firms facing a stochastic inverse demand curve use capacity as strategic variable. Capacity may be adjusted continuously over time with linear cost. The analysis uses the technique of a fictitious social planner and the theory of irreversible investment under uncertainty. Examples indicate that qualitatively the price process will be similar in oligopoly and competitive equilibrium. When firms are nonidentical, e.g. in initial size, and even if they are alike in other respects, substantial time may pass until they are all the same size. Much of that time, one firm may dominate the market.

Mason and Weeds (2001) examine irreversible investment in a project with uncertain return, when there is an advantage to being the first to invest and externalities to investing when others also do so. Preemption decreases and may even eliminate the option values created by irreversibility and uncertainty. Externalities introduce inefficiencies in investment decisions. Preemption and externalities combined can hasten, rather than delay, investment, contrary to the usual outcome.

Lambrecht and Perraudin (2003) introduces incomplete information and preemption into an equilibrium model of firms facing real investment decisions. The optimal investment strategy may lie anywhere between the zero-NPV trigger level and the optimal strategy of a monopolist, depending on the distribution of competitors' costs and the implied fear of preemption. The model implies that the equity returns of firms which hold real options and are subject to preemption will contain jumps and positive skewness.

Lambrecht (2004) analyzes the timing of mergers motivated by economies of scale. He shows that firms have an incentive to merge in periods of economic expansion. Relaxing the assumption that firms are price takers, he finds that market power strengthens the firms' incentive to merge and speeds up merger activity. Finally, comparing mergers with hostile takeovers Lambrecht shows that the way merger synergies are divided not only influences the acquirer's and acquiree's returns from merging, but also the timing of the restructuring.

Ziegler (2004) shows how to combine game theory and option pricing in order to analyze dynamic multi-person decision problems in continuous time and under uncertainty. The basic intuition of the method is to separate the problem of the valuation of payoffs from the analysis of strategic interactions. Whereas the former is to be handled using option pricing, the latter can be addressed by game theory. The text shows how both instruments can be combined and how game theory can be applied to complex problems of corporate finance and financial intermediation. Besides providing theoretical foundations and serving as a guide to stochastic game theory modelling in continuous time, the text contains numerous applications to the theory of corporate and financial intermediation, such as the design of debt contracts, capital structure choice, the structure of banking deposit contracts, and the incentive effects of deposit insurance. By combining arbitrage-free valuation techniques with strategic analysis, the game theory analysis of options actually provides the link between markets and organizations.

A new stream in this literature combines real options with game theory to analyze the competitive interactions of R&D ventures. Smit and Trigeorgis (1997) integrated real options with industrial organization framework to model competitive advantage in strategic R&D investments by analyzing two stage games where the option value of R&D depends on endogenous competitive reactions. The models illustrate the trade-off between the flexibility value and the strategic commitment value of R&D using numerical example and provide different investment strategies based on Fudenberg and Tirole (1984) framework. Smit and Trigeorgis (2006), therefore, analyze the impact of uncertainty, incomplete information and learning on R&D value with endogenous market structure and quantities in the product market. Contrary to standard option valuation, the value of an R&D investment opportunity may no longer increase monotonically with option parameters because strategic preemption may cause value discontinuities. They analyze how the sign and magnitude of this effect on R&D value depends on the existence of learning effects in production, technical R&D uncertainty and the degree of incomplete information, and cooperation via joint research ventures. Kulatilaka and Perrotti (1998) analyze strategic growth opportunities involving by adopting a costs reducing technology under uncertainty and imperfect competition. The model shows that contrary to the result found in the real options literature, the effect of uncertainty on the value of the strategic growth options is ambiguous. Lambrecht (1999) derives the optimal investment thresholds for two investors who compete to obtain a patent that gives its holder an option to invest in a following project, which consists of the commercialisation of the invention. Huisman (2000) studies a dynamic duopoly in which firms compete in the adoption of new technologies. The model assumes that both firms have the possibility to adopt a current technology or wait a better technology that arrives at an unknown point of time in the future. Mason and Weeds (2001) examine the irreversible adoption of technology whose returns are uncertain, when there is the first-mover advantage and a network advantage to adopting when others also do so. Abel and Eberly (2004) develop a model in which the opportunity for a firm to upgrade its technology to the frontier (at a cost) leads to growth options in the value of the firm; that is, a firm's value is the sum of value generated by its current technology plus the value of the option to upgrade. Miltersen and Schwartz (2004) develop a model to analyze patent-protected R&D investment projects when there is multiple sources of uncertainty in R&D stages and imperfect competition in the development and marketing of the resulting product. Garlappi (2004) analyzes the impact of competition on the risk premia of R&D ventures engaged in a multi-stage patent race with technical and market uncertainty. R&D competition causes substantial rent dissipation, excess R&D spending and higher and more volatile risk premia. On the other side, competition dramatically reduces the completion time and the failure rate of research within the industry.

More recently, a number of articles studies new derivatives securities called game options, or Israeli options. These are contracts which enable both their buyer and seller to stop them at any time and then the buyer can exercise the right to buy (call option) or to sell (put option) a specified security for certain agreed price. If the contract is terminated by the seller he must pay certain penalty to the buyer. More precisely, the contract may be specified in terms of two stochastic processes $(L_t)_{t\in[0,T]}$, $(U_t)_{t\in[0,T]}$ with

$$L_t \leq U_t \text{ for } t \in [0,T) \text{ and } L_T = U_T.$$

If A terminates the contract at time t before it is exercised by B, he has to pay B the amount U_t. If B exercises the option before it is terminated by A, he is paid L_t. An example is a put option of game type with constant penalty $\delta > 0$. If S^1 denotes the price process of the underlying and K the strike price, then $L_t = \left(K - S_t^1\right)^+$ and $U_t = \left(K - S_t^1\right)^+ + \delta 1_{\{t<T\}}$.

Kifer (2000) introduced the Israeli option and examined the pricing of such contracts using game theory, determining the value of a Dynkin game, under a slightly modified general Black Scholes non-arbitrage framework. Kifer shows that the valuation can be reduced to evaluating a stochastic saddle point problem associated with a Dynkin game. Game options can be sold cheaper than usual American options and their introduction could diversify financial markets.

Kallsen & Kühn (2003) expanded the above approach for incomplete markets by use of indifference arguments, again by determining the value of a Dynkin game and adjusting the measure.

1.6 Final remark

This thesis deals with the modeling of R&D projects and start up companies by a multicompound options approach and develops two frameworks of multi-stage investment projects that captures many features of new ventures.

In the first part we focus on the valuation of new ventures possessing flexibility in the form of multiple real options: to continue, expand, contract, suspend or abandon the project at each step. To this purpose, we obtain both a formula for multicompound call-put option written compound call-put option, and a formula for multicompound call-put option to switch investment to an alternative use.

Recently, a number of papers have studied R&D process as a contingent claim on the value of an underlying asset, which is interpreted as the present value of the cash flows on completion of the R&D stages. There are two paper that are closest to this one. The first one is Agliardi and Agliardi (2005) which obtained the pricing formula for a multicompound call option useful to sequential investments valuation. Although there are no important differences both in the formulation of the problem and in the solution procedure between Agliardi and Agliardi (2005) and chapter 2, we include situations in which the project can be abandoned, suspended, expanded or reduced in order to capitalize on favorable future opportunities.

The second paper that is closely related to chapter 2 is Trigeorgis (1993) which provides the valuation results for a generic project, when particular real options are valued in the presence of others and illustrates option configurations where interactions can be small or large, as well as negative or positive. This model differs from chapter 2 both in the technique and in the function: Trigeorgis, in fact, presents a numerical valuation approach for a generic project's multiple real options.

The second part of this thesis proposes a model of R&D investments which relies on the multicompound options valuation and allows for different sources of uncertainty. In the process, the venture is subject to two types of risk: (1) uncertainty associated the potential future cash flows the project will produce if completed; this is represented by a standard diffusion process, punctuated by (2) jumps at random intervals. This risk pertains to the successful completion of the venture itself and is represented by a Poisson process. There are three papers that are closest to chapter 3.

The first one is Gukhal (2003) which derives an analytical valuation formula for compound options when the underlying asset follows a jump-diffusion process. This is applied to value extendible options, American call options on stocks that pay discrete dividends and American options on assets that pay continuous proportional dividends. Our model starts from the result obtained in Gukhal (2003) and analyzes sequential investment opportunities: R&D ventures and start-up companies; these are analogous to the exercise of an European-style multicompound option because each stage in the investment process is viewed as an option on the value of a subsequent stage. The formula is shown to generalize previous works in real option pricing.

Jump-diffusion process is introduced in chapter 2 to model crises and shocks in the investment market. The value of the multicompound option, in our model, relies on knowing exactly the number of Poisson jumps, n, occurring in each stage during the life of the option. Clearly, its actual value is just the weighted sum of the prices of the option where each weight equals the joint probability that N Poisson random variables will take on the value n in each stage.

The second paper that is closely related to chapter 3 Errais and Sadowsky (2005). This article differs from chapter 3 both in the technique and the treatment of uncertainty. They introduce a general discrete time dynamic framework to value pilot investments; the pilot phase requires N stages of investment for completion that they value as a compound perpetual Bermudan option. In this model, both tradable market uncertainty and idiosyncratic technical uncertainty affect the value of the project's commercial phase, with the diffusion coefficients that drive idiosyncratic shocks being a function of the amount invested in the product's development stage. The setting in the two models are also different. Errais and Sadowsky work in an incomplete market setting; instead, we adopt the Merton's approach and we assume that technical uncertainty is completely diversifiable. This fact will allow us to specify a unique equivalent risk-neutral measure by setting the market price of risk of q[17] to zero.

The last paper that is closely related to chapter 3 is Berk, Green and Naik (2004) which also develops a dynamic model of multi-stage investment project that captures many features of R&D ventures and start-up companies. The valuation method in this paper differs from ours, however. They

[17] The technical uncertainty driven by Poisson process.

also take as exogenous the process describing the cash flows the project will generate, and value these cash flows and the R&D project simultaneously. By deriving the value of the underlying cash flows rather than specifying it exogenously, they are able to focus on the relative systematic risk of R&D and the underlying cash flows and explain the dynamics of the risk premium of the R&D. While their formulation is more fundamental in this respect, it makes it harder to employ in practice.

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Part II

The valuation of new ventures

Chapter 2

2.1 Introduction

[1]Traditional tools based on DCF methods fail to capture the value of new ventures because of their dependence on future events that are uncertain at the time of the initial decision. The primary value of new ventures lies in the physical options it creates. These options refer to a future market opportunity resulting from a contingent claim on new product patents, knowledge, and the competitive position being created. The investment in R&D, for instance, generates value primarily by creating options for future product development. As several researchers have noted R&D investments are essentially real growth options since investment decisions are made sequentially and in a particular order. Staging investment involves firms either with some degree of flexibility in proceeding with the investment or when there is a maximum rate at which outlays or construction can proceed. Very often, firms engage R&D projects to get started a multi-stage process that may eventually reach a commercial phase of launching the new product. For example, the drug development process might follow various stages. The likelihood of passing each stage will depend on the nature of the diseases. A potentially more useful valuation method is real option analysis, which takes flexibility into account at each stage of the development process: R&D management, in fact, has the flexibility to defer, contract or expand expenditures, or alternatively to abandon the R&D project after funding ceases.

Computer software are very similar to R&D projects because of the large investments they require. In particular, investing in a software development

[1]This chapter builds on the working paper DSE n. 554 (2006), University of Bologna and was presented at a recent DSE seminar.

requires to the venture capitalist to continue making investments in updating its technology and marketing its product just to keep up. Computer software is a profitable and dynamic industry, creating market openings for competitors and potential entrants. This feature makes the risk of the business much greater than risk faced by start-ups in other industries, and then makes the valuation problem more easy if we use the real options methodology.

Using real option approach, venture capitals such as pharmaceuticals and new technologies development would be seen as compound options, as the progress towards completion usually requires a sequence of successful investments, each of which opens the possibility to undertake the next operating phase. Each stage can be viewed as an option on the value of subsequent stage and valued as compound option. A compound option is simply an option on an option. The exercise payoff of a compound option involves the value of another option. A compound option then has two expiration dates and two strike price. Take the example of a European style call on a call. On the first expiration date T_1, the holder has the right to buy a new call by paying the strike price I_1. The new call has expiration date T_2 and strike price I_2.

Compound options have been extensively used in corporate finance to hedge difficult investments which are contingent on other conditions. Geske (1979) suggested that when a company has common stock and coupon bonds outstanding, the firm's stock can be viewed as a call option on a call option. Rubinstein (1992) generalized this result to all four possible combinations: call on a call, put on a call, call on a put and put on a put. Carr (1988) obtained a closed form solution to a compound exchange option integrating work on compound option pricing by Geske (1979) with work on exchange option pricing by Margrabe (1978). As a result, the general valuation formula may be used to value real options, as for example options to expand capacity and option to switch inputs or outputs in production. Exercise of this instrument involves delivering one asset in return for an exchange option. The option received upon delivery may then be used to make another exchange at a later date. Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a jump-diffusion process, applying these results to value extendible options, American call options on stocks that pay discrete dividends and American options on assets that pay continuous proportional dividends. Agliardi and Agliardi (2005) obtained a closed-form solution for the price of a multicompound call option, i.e. N-fold options of options, in the case of time-varying variance and time-varying interest rate. In this paper, an explicit pricing formula is proved that generalizes the well-known Geske's expression for compound options, solving N nested Black-Scholes differential equations and using some properties of multivariate normal integrals. Roll (1977), Whaley (1981), Geske and Johnson (1894) and Selby and Hodges (1987) also study compound options.

The objective of this paper is to study the multicompound option approach to value sequential investment opportunities taking into account multiple real options: as the option to permanently abandon, temporarily shutdown, expand and contract the project. Unlike most compound options in the financial market, it is perfectly possible for the firm to suspend investment at a certain fixed time and resume it later, so the first contribution of this paper is to derive a closed-form solution for the price of a multicompound call and put option; this result generalizes the solution of Agliardi and Agliardi (2005) . The second contribution is to derive an analytical formula for the price of a multicompound exchange call and put option that integrates the first result with the work on compound exchange options by Carr (1988). This formula is useful to value sequential expansion and contraction option as the opportunity to upgrade a software or the option to switch to a lower technology.

The structure of our paper is as follows. Section 2 reviews the articles in the literature related to ours. This is followed by a description of the economic model in section 3. Section 4 derives an analytical valuation formula for the price of a multicompound call/put option. Section 5 derives the valuation formula for the price of a multicompound exchange call/put option. In section 6 we illustrate our model by providing numerical results for a two stages investment project which relies on compound options pricing. We focus on the situations in which firms may take the intermediate decision to suspend the project continuously. Section 7 concludes the paper.

2.2 Literature Review

A number of existing research contribution has previously analyzed various aspects of optimal sequential investment behavior for firm facing multistage projects. In the real options setting, the value of these investments is the value of the follow-on opportunities they may create. In this sense, firms undertake these projects not so much for their own returns, but rather to get started a pilot that may eventually reach an operating stage. Take the example of developing a new drug in the pharmaceutical industry. Investing in a new drug by pharmaceutical company is a multistage process, beginning with research that leads with some probability to a new compound; such a project continues with testing and concludes with the construction of a production facility and the marketing of the product. Thus, in order to draw the analogy with the valuation and exercising of financial option, an R&D venture by pharmaceutical company can be compared to multicompound option involving sequential decisions to exercise the options to invest only when the R&D outcomes are successful. Although the preceding analysis suggested the use of more suitable technique when we attempt to value new ventures, such investments are hard to value even with the real options approach. The main reason for this is that there are multiple sources of uncertainty in R&D investments and that they interact in complicated way. In practice, the bulk of the literature have dealt with the development of numerical simulation methods based on optimal stopping time problems.

Majd and Pindyck (1987) study investments with time-to-build when investment projects have the following characteristics: (1) spending decisions and cash outlays occur sequentially over time, (2) there is a maximum rate at which outlays and construction can proceed, that is it takes 'time-to-build', and (3) the project yields no cash return until it is actually completed. Furthermore, the pattern of investment outlays is usually flexible and can be adjusted as new information arrives. Majd and Pindyck develops an explicit model of investment projects with these characteristics, and uses option pricing methods to derive optimal decision rules for investment outlays over the entire construction program. Numerical solutions are used to demonstrate how time-to-build, opportunity cost, and uncertainty interact in affecting the investment decision. Similarly, Bar-Ilan and Strange (1996) study the effect of investment lags in a model of an uncertain, irreversible investment. Finally, Grenadier (1996,2000a) studied time-to-build options using option-game approach in continuous time.

The real option literature have studied the R&D process as a contingent claim on the value of the underlying cash flows on completion of the R&D project. Childs and Triantis (1999) develop and numerically implement a model of dynamic R&D investment that highlights the interactions across projects. Schwartz and Moon (2000) have also studied R&D investment projects in the pharmaceutical industry using a real options framework. In this articles, they numerically solve a continuous-time model to value R&D projects allowing for three types of uncertainty: technical uncertainty associated with the success of the R&D process itself, an exogenous chance for obsolescence and uncertainty about the value of the project on completion of the R&D stages. Schwartz (2003) develops and implements a simulation approach to value patents-protected R&D projects based on the real option approach. It takes into account uncertainty in the cost to completion of the project, uncertainty in the cash flows to be generated from the project, and the possibility of catastrophic events that could put an end to the effort before it is completed. Berk, Green and Naik (2004) develop a dynamic model of multi-stage investment project that captures many features of R&D ventures and start-up companies. Their model assumes different sources of risk and allow to study their interaction in determining the value and risk premium of the venture. Closed-form solutions for important cases are obtained. Errais and Sadowsky (2005) introduce a general discrete time dynamic framework to value pilot investments that reduce idiosyncratic uncertainty with respect to the final cost of a project. In this model, the pilot phase requires N stages of investment for completion that they value as a compound perpetual Bermudan option.

More recently, a number of articles consider strategic interaction features across R&D options holders. Garlappi (2004) analyzes the impact of competition on the risk premium of R&D ventures engaged in a multi-stage patent race with technical and market uncertainty. R&D competition causes substantial rent dissipation, excess R&D spending and higher and more volatile risk premium. On the other side, competition dramatically reduces the completion time and the failure rate of research within the industry. Finally, Miltersen and Schwartz (2004) develop a model to analyze patent-protected R&D investment projects when there is multiple sources of uncertainty in R&D stages and imperfect competition in the development and marketing of the resulting product.

2.2.1 Flexibility of Multiple Compound Real Options

Most work in real options has focused on valuing individual options. However, many real investments often involve a collection of various options, which need to be valued together because their combined value may differ from the sum of their separate values. As it is well known, the operating flexibility and strategic value aspect of intangible investment projects cannot be properly captured by traditional tools, because of their discretionary nature. As new information arrives and uncertainty about market conditions and future cash flows is resolved, firms may have valuable flexibility to alter its initial operating strategy in order to capitalize on favorable future opportunities. For example, management may be able to permanently abandon or temporarily shut-down and restart; to expand or contract capacity.

Many start-up companies rely upon venture capitalists to begin operations. Typically, after the initial injection of funds, addition funding is provided as the firm reaches certain performance targets. The payment of the first funding round is comparable to an initial option premium. Further payments are contingent claims: the right but not the obligation to continue financially supporting the project. If at any point, the venture capitalist ceases to pay, the project is assumed to end. Therefore, the venture capitalist can be thought of injecting funds that not only keep the project alive but also retain the right to pay the remaining payments in the future.

Computer software and Internet start-ups need two or more rounds of financing to be implemented. However, venture capitalist has an alternative option to permanent abandonment the project if the operating project becomes negative; in practice, after the initial injection of funds, it gets a project in place and an option to abandon to save the follow-on expenditures. This possibility has limited applicability in most real investment projects because of the high cost of abandonment. In such cases, it would not make sense to abandon, unless the cash flows on the project are even more negative. To keep the analysis simple, we will ignore the fact that the abandonment may create costs and will consider the following stage investment being analogous to the exercise of financial put option on the business value with strike price equals the salvage value from abandonment[2].

Instead of abandoning, venture capitalist may choose to keep its project alive by maintaining its initial installment and renouncing to future investments (i.e. advertising and upgrading expenditures). For our purpose, we will consider that the project may be temporarily and costlessly suspended and that this opportunity can be seen as a sequence of three or more operating call/put options; that is, when a venture capitalist exercises its option to get started the project, it gets a project in place and a chain of interrelated options, to temporarily and costlessly shut-down or to continue funding. If it exercises the option to suspend, it gets the option to invest again or to continue suspending, and so on.

Starting from today's capacity (after a call option to invest has already been exercised) a firm usually have further call options to invest in extending today's capacity. Similarly, it often has a put option to reduce today's capacity. The put option may sometimes give them an actual cash inflow from disinvestment. In the real options literature the opportunities to expand or contract capacity, to switch to a new technology,etc., are usually valued as a compound exchange option.

[2]See Myers and Majd (1990) for abandonment option.

Multiple real options are mostly studied by Brennan and Schwartz (1985) and Trigeorgis (1993). Real-life projects are often more complex in that they involve a collection of multiple real options, whose value may interact. Brennan and Schwartz (1985) determine the combined value of the options to shut-down and restart a mine, and to abandon it for salvage. Trigeorgis (1993) focuses explicitly on the nature of real option interactions, pointing out that the presence of subsequent options can increase the value of the effective underlying asset for earlier options, while the exercise of prior real options may alter the underlying asset itself, and hence the value of subsequent options on it. Thus, the combined value of a collection of real options may differ from the sum of separate option values. In Pindyck (1988), uncertainty over future market conditions affects investment decisions through the option that firms hold, operating options, which determine the value of capital in place, and options to add more capital, which, when investment is irreversible, determine the opportunity cost of investing. In Dixit (1989) a firm's entry and exit decisions when the output price follows a random walk are examined. An idle firm and an active firm are viewed as assets that are call options on each other. The solution is a pair of trigger prices for entry and exit. Kulatilaka (1995) examines the impact of interactions among such options on their optimal exercise schedules. Abel, Dixit, Eberly and Pindyck (1996) show how opportunities for future expansion or contraction can be valued as options. They showed that a firm that makes an investment that is partially or totally reversible acquires a put option. This option has value if future uncertainty involves a sufficiently large downside with a positive probability that the firm will want to exercise the option. Likewise, a firm that can expand by making an investment now or in the future (at a cost) is exercising a call option, namely it is acting now when it might have waited. This option has value if future uncertainty has a sufficiently large downside that waiting would have been preferable.

This paper deals with the modeling sequential investments by a multicompound options approach and develops two frameworks of multi-stage projects that captures many features of new ventures. Particularly, we focus on the valuation of new ventures possessing flexibility in the form of multiple real options: to continue, expand, contract, suspend or abandon the project at each step. There are two paper that are closest to this one. The first one is Agliardi and Agliardi (2005) which obtained the pricing formula for a multicompound call option useful to sequential investments valuation. Although there are no important differences both in the formulation of the problem and in the solution procedure between Agliardi and Agliardi (2005) and our model, we add flexibility in each step by allowing the multicompound call/put option to be written on further compound call/put options. The second paper that is closely related to chapter 2 is Trigeorgis (1993) which illustrates through a generic project the size and type of interactions among the option to defer, abandon, contract, expand and switch use. This model differs from ours both in the technique and in the function. Indeed, Trigeorgis presents a numerical valuation approach for a generic project's multiple real options.

2.3 Model and Assumptions

Let us consider the investment decision by a venture capital fund that is evaluating the project of start-up company providing software tools in the Internet industry. We assume that the commercial phase of the project can not be launched before a pilot phase consisting on N-stages of investment is completed. Let I be the amount of investment required for completion of any pilot stage. Furthermore, to make the analysis easier we will assume that the project is patent-protected.[3]

Suppose the inverse demand function for the software, giving price in terms of quantity Q is $P = YD(Q)$, where Y is a stochastic shift variable. The risk free rate in our setting will be denoted by $r(t)$. Moreover, the investment project, once completed, produces one unit of output per year at zero operating costs. We assume the price for the software, P, follows a stochastic differential equation of the form:

$$dP = \alpha(t)\,Pdt + \sigma(t)\,Pdz,$$

where dz is the increment of the standard Wiener process; $\sigma(t)$ is the instantaneous standard deviation of the spot price at time t and $\alpha(t)$ is the trend rate in the price. The assumption of time-dependent volatility and interest rate seems more suitable due to the sequential nature of start-up projects (see Agliardi and Agliardi, 2003). Let V the expected present value of the project when the current price is P, in this case V, being a constant multiple of P, also follows a geometric Brownian motion with the same parameters $\alpha(t)$ and $\sigma(t)$.

[3] This assumption, widely used in the real options literature, will allow us to avoid competitive interactions across R&D options holders.

2.3.1 Value of the Option to Continuously Shut - Down

Unlike most compound options in the financial market, it is perfectly possible for the firm to suspend investment on the pilot at a certain time T_k, $k = 1, .., N$, if, for instance, market conditions are not favorable, and resume investment at a later point in time.

Let $F_1(V, t; \varsigma_1)$ denote the value of a European call/put option with exercise price I_1 and expiration date T_1. Let us now define inductively a sequence of call/put options, with value F_k, on the call/put option whose value is F_{k-1}, with exercise price I_k and expiration date T_k, $k = 1, .., N$, where we assume $T_1 \geq T_2 \geq ... \geq T_N$.

Because all the calls and puts are function of the value of the firm V and the time t, the following partial differential equation holds for F_k:

$$\frac{\partial F_k}{\partial t} = r(t) F_k - r(t) V \frac{\partial F_k}{\partial V} - \frac{1}{2}\sigma^2(t) V^2 \frac{\partial^2 F_k}{\partial V^2}, \quad t \leq T_k, k = 1, .., N,$$

$T_1 \geq T_2 \geq ... \geq T_N$.The boundary condition is:

$$\begin{aligned} & F_k\left(F_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right), T_k; \varsigma_1, .., \varsigma_k\right) = \\ & \max\left(\varsigma_k F_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right) - \varsigma_k I_k, 0\right), \end{aligned}$$

where $F_{k-1}(V, T_k; \varsigma_1, .., \varsigma_{k-1})$ stands for the price of the underlying compound option and the binary option operator $\varsigma_k = \pm 1$, $k = 1, .., N$ when the k^{th}-compound option is a call/put and the operator $\varsigma_{k-1} = \pm 1$, when the $(k-1)^{th}$-underlying compound option is a call/put. Naturally, if $k = 1$ the well-known pricing formulae for simple options are obtained.

In order to solve the partial differential equations above subject to their boundary conditions we need to use the following notation: let V_k^* denote the value of V such that $F_{k-1}(V, T_k; \varsigma_1, .., \varsigma_{k-1}) - I_k = 0$ if $k > 1$, and $V_1^* = I_1$. Let us define now:

$$b_k(t) = \frac{\ln\left(\frac{V}{V_k^*}\right) + \int_t^{T_k} r(\tau) - \frac{\sigma^2(\tau)}{2} d\tau}{\left(\int_t^{T_k} \sigma^2(\tau)\, d\tau\right)^{\frac{1}{2}}}, \tag{2.1}$$

and:

$$a_k(t) = b_k(t) + \left(\int\limits_t^{T_k} \sigma^2(\tau)\, d\tau \right)^{\frac{1}{2}}; \tag{2.2}$$

moreover, we set:

$$\rho_{ij}(t) = \left(\frac{\int\limits_t^{T_j} \sigma^2(\tau)\, d\tau}{\int\limits_t^{T_i} \sigma^2(\tau)\, d\tau} \right)^{\frac{1}{2}}, \text{ for } 1 \leq i < j \leq k, \quad t \leq T_k. \tag{2.3}$$

For any k, $1 \leq k \leq N$, let $\Sigma_k^{(N)}(t)$ denote the k-dimension symmetric correlation matrix with typical element $\rho_{ij}(t) = \rho_{N-k+i,N-k+j}(t)$ if $i < j$.

Since we want to derive a valuation formula for the price of N-fold multicompound call/put option, that is for $F_N(V, t; \varsigma_1, .., \varsigma_N)$, $0 \leq t \leq T_N$, let V_N^* denote the value of V such that $F_{N-1}(V, T_N; \varsigma_1, .., \varsigma_{N-1}) - I_N = 0$. Then, for V greater than V_N^* the N^{th}-compound call option will be exercised, otherwise the project will be temporarily suspended.

As usual, it is desirable to transform the partial differential equation for F_N into the diffusion equation. First, we adopt the following change of variables:

$$F_N(V, t) = e^{-\int\limits_t^{T_N} r(\tau) d\tau} \tilde{F}_N(u, z),$$

where:

$$u = -\ln\left(\frac{V}{V_N^*}\right) - \int\limits_t^{T_N} r(\tau) - \frac{\sigma^2(\tau)}{2} d\tau,$$

and:

$$z = \frac{1}{2} \int\limits_t^{T_N} \sigma^2(\tau)\, d\tau.$$

In term of the new independent variables the fundamental equation for F_N becomes:

$$\frac{\partial \tilde{F}_N}{\partial z} = \frac{\partial^2 \tilde{F}_N}{\partial u^2}, \quad -\infty < u < +\infty, \quad z \geq 0.$$

The partial differential equation above subject to the initial value condition $\tilde{F}_N(u,0)$, has a unique solution which we use to write F_N as follows:

$$F_N(V,t) = e^{-\int\limits_t^{T_N} r(\tau)d\tau} \int\limits_{-\infty}^{+\infty} \tilde{F}_N(\xi,0) \frac{1}{2\sqrt{\pi z}} e^{-(u-\xi)^2/4z} d\xi.$$

Substituting the solution for F_{N-1} into this expression and changing the variable u with $\varsigma_N..\varsigma_1 b_N(t)$, gives the following identity:

$$\begin{aligned}
F_N(V,t;\varsigma_1,..,\varsigma_N) = {} & \varsigma_N..\varsigma_1 V e^{-\int\limits_t^{T_N} r(\tau)d\tau} \int\limits_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b_N + \xi/\sqrt{2z}\right)^2/2} \times \\
& N_{N-1}\left(\varsigma_{N-1}..\varsigma_1 a_{N-1}(T_N), .., \varsigma_1 a_1(T_N); \Xi_{N-1}^{(N-1)}(T_N)\right) d\xi + \\
& -\sum_{j=1}^{N-1} \varsigma_N..\varsigma_j I_j e^{-\int\limits_t^{T_j} r(\tau)d\tau} \int\limits_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b_N + \xi/\sqrt{2z}\right)^2/2} \times \\
& N_{N-j}\left(\varsigma_{N-1}..\varsigma_1 b_{N-1}(T_N), .., \varsigma_j b_j(T_N); \Xi_{N-j}^{(N-1)}(T_N)\right) d\xi + \\
& -\varsigma_N I_N e^{-\int\limits_t^{T_N} r(\tau)d\tau} \int\limits_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b_N + \xi/\sqrt{2z}\right)^2/2} d\xi; \qquad (2.4)
\end{aligned}$$

where $N_k(\varsigma_k..\varsigma_1 b_k, .., \varsigma_1 b_1; \Xi_k)$ denotes the k-dimension multinormal cumulative distribution function with upper limits of integration $\varsigma_1 b_1, .., \varsigma_k..\varsigma_1 b_k$ and $\Xi_k^{(N-1)}(T_N)$ denotes the k-dimension modified symmetric correlation matrix:

$$\Xi_k^{(N-1)}(T_N) = \begin{bmatrix} 1 & \varsigma_2\rho_{12} & \cdots & \varsigma_{N-1}..\varsigma_2\rho_{1,N-1} \\ \varsigma_2\rho_{12} & 1 & & \vdots \\ \vdots & & \ddots & \varsigma_{N-1}\rho_{N-2,N-1} \\ \varsigma_{N-1}..\varsigma_2\rho_{1,N-1} & \cdots & \varsigma_{N-1}\rho_{N-2,N-1} & 1 \end{bmatrix}, \tag{2.5}$$

with the entries $\rho_{ij}(T_N) = \rho_{N-1-k+i,N-1-k+j}(T_N)$, $i < j$, defined as above.

The third term can be easily written in the form:

$$-\varsigma_N I_N e^{-\int_t^{T_N} r(\tau)d\tau} N_1(\varsigma_N..\varsigma_1 b_N(t)).$$

In order to solve the remaining integrals, let us set $x = \varsigma_N..\varsigma_1 a_N(t) + \xi/\sqrt{2z}$ in the integral of the first term above and $x = \varsigma_N..\varsigma_1 b_N(t) + \xi/\sqrt{2z}$ in the second; further, we replace any element $\rho_{ij}(T_N)$ in the matrix $\Xi_k^{(N-1)}(T_N)$ with a function of t, according to the following rule:

$$\rho_{ij}(T_N) = \frac{\left(\rho_{ij}(t) - \rho_{iN}(t)\rho_{jN}(t)\right)}{\sqrt{\left(1-\rho_{iN}^2(t)\right)\left(1-\rho_{jN}^2(t)\right)}}, \tag{2.6}$$

for $1 \le i < j \le N, t \le T_N$. The first term can be written in the form:

$$\varsigma_N..\varsigma_1 V \int_{-\infty}^{\varsigma_N..\varsigma_1 a_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times$$

$$N_{N-1}(\varsigma_{N-1}..\varsigma_1 a_{N-1}(t) - x\varsigma_N\rho_{N-1,N}(t)/\sqrt{\left(1-\rho_{N-1,N}^2(t)\right)}, ..$$

$$.., \varsigma_1 a_1(t) - x\varsigma_N..\varsigma_2\rho_{1,N}(t)/\sqrt{\left(1-\rho_{1,N}^2(t)\right)}; \tilde{\Xi}_{N-1}^{(N-1)}(t))dx,$$

and the second term:

$$-\sum_{j=1}^{N-1} \varsigma_N..\varsigma_j I_j e^{-\int_t^{T_j} r(\tau)d\tau} \int_{-\infty}^{\varsigma_N..\varsigma_1 b_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times$$

$$N_{N-j}(\varsigma_{N-1}..\varsigma_1 b_{N-1}(t) - x\varsigma_N \rho_{N-1,N}(t) / \sqrt{\left(1-\rho_{N-1,N}^2(t)\right)}, ..$$
$$.., \varsigma_j b_j(t) - x\varsigma_N..\varsigma_{j+1}\rho_{j,N}(t) / \sqrt{\left(1-\rho_{j,N}^2(t)\right)}; \tilde{\Xi}_{N-j}^{(N-1)}(t))dx;$$

Lemma 1 (generalized) *Let $\Xi_k^{(N)}(t)$ denote the k-dimension correlation matrix with entries $\varsigma_j \rho_{ij}(t)$ for $i<j$ and let $\tilde{\Xi}_k^{(N-1)}$ be the matrix obtained from $\Xi_k^{(N-1)}$ replacing any element ρ_{ij} with*

$$\left(\rho_{ij} - \rho_{iN}\rho_{jN}\right) / \sqrt{(1-\rho_{iN}^2)(1-\rho_{jN}^2)}$$

for $1 \le i < j \le N$. Moreover, let $\varsigma_k = \pm 1$, $k = 1, .., N$, if the k^{th}-compound option is a call/put and $\varsigma_{k-1} = \pm 1$ if the $(k-1)^{th}$-underlying compound option is a call/put. Then, the following identity holds:

$$\int_{-\infty}^{\varsigma_N..\varsigma_1 b_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times$$

$$N_k\left(\frac{\varsigma_{N-1}..\varsigma_1 b_{N-1} - x\varsigma_N\rho_{N-1,N}}{\sqrt{\left(1-\rho_{N-1,N}^2\right)}}, .., \frac{\varsigma_{N-k}b_{N-k} - x\varsigma_N..\varsigma_{N-k+1}\rho_{N-k,N}}{\sqrt{\left(1-\rho_{N-k,N}^2\right)}}; \tilde{\Xi}_k^{(N-1)}\right) dx =$$

$$= N_{k+1}\left(\varsigma_N..\varsigma_1 b_N, .., \varsigma_{N-k} b_{N-k}; \Xi_{k+1}^N\right).$$

Proof. by induction. ■

Applying this argument to the first and the second terms, we have the following result for the value of a multicompound call/put option.

Proposition 2 *The value of the multicompound call/put option F_N with maturity T_N and strike price I_N written on a compound call/put option F_{N-1} with maturity T_{N-1} and strike price I_{N-1} is given by:*

$$F_N(V, t; \varsigma_1, .., \varsigma_N) = \varsigma_N..\varsigma_1 V N_N\left(\varsigma_N..\varsigma_1 a_N(t), .., \varsigma_1 a_1(t); \Xi_N^{(N)}(t)\right) +$$

$$-\sum_{j=1}^{N} \varsigma_N..\varsigma_j I_j e^{-\int_t^{T_j} r(\tau)d\tau} N_{N+1-j}\left(\varsigma_N..\varsigma_1 b_N(t), .., \varsigma_j b_j(t); \Xi_{N+1-j}^{(N)}(t)\right),$$

$$0 \leq t \leq T_N,$$

where the a_is, the b_is and the ρ_{ij}s are as defined previously.

Remark 3 *It can be proved that*

$$\partial_V F_k(V, t; \varsigma_1, .., \varsigma_k) = N_k(\varsigma_k..\varsigma_1 a_k(t), ..., \varsigma_1 a_1(t); \Xi_k^{(k)}(t));$$

thus uniqueness of V_k^* is guaranteed for every k, $1 \leq k \leq N$.

Note that in the particular case when $\varsigma_k = +1$, $k = 1, .., N$ the formula above reduces to Agliardi and Agliardi (2005). This proposition is the main result of the paper and forms the basis for the valuation of sequential investment opportunities, as for example R&D ventures, computer software updating,.. including the possibility of multiple real options.

2.4 An extension

When future returns are uncertain, these features yield two options. First, venture capitalists sometimes engage the pilot either to make further investments or to enter other markets in the future. On the other hand, when a firm gets started a project or installs a new technology that it may later abandon, it acquires a put option. Once again, the opportunities for future expansion or contraction are examples of the strategic dimension of the Internet start-up venture. The firms' ability to later contract or expand capacity is clearly more valuable for more volatile business with higher returns on project, such as computer software or biotechnology, than it is for traditional business, as real estate or automobile production. Next, we recast the main assumptions allowing the firm to face the opportunity to continuously switch to more or less updated technologies. The sequential technological expansion/contraction decision can be viewed to be similar to the exercise of a multicompound exchange option.

As before, we assume the inverse demand function for the software, giving price in terms of quantity Q is $P = YD(Q)$, where Y is a stochastic shift variable. Once again, the variable costs of production are assumed to be zero. Let V_i, $i = 1, .., N$, the price of i–th underlying asset which we could interpret as the value of the operating project with current technology. As before, we assume that the underlying risky assets pay no dividends, and that they follow standard diffusion processes as:

$$dV_i = \alpha_i(t) V_i dt + \sigma_i(t) V_i dz_i, \quad i = 1, .., N,$$

where dz_i, $i = 1, .., N$, are Wiener processes. They are correlated:

$$E[dz_i, dz_j] = \rho_{ij} dt, \quad i, j = 1, .., N, \quad i \neq j,$$

with $\rho_{ij} = \rho_{ji}$, $\rho_{ii} = 1$, $i, j = 1, .., N$, and ρ_{ij} is the correlation coefficient between V_i and V_j. Let $F(V, t)$ the function $F(V_1, V_2, .., V_N; t)$, according to the generalized Ito lemma we can determine that F will satisfy the following partial differential equation:

$$\frac{\partial F}{\partial t} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} \sigma_i(t) \sigma_j(t) V_i V_j \frac{\partial^2 F}{\partial V_i \partial V_j} + r(t) \sum_{i=1}^{N} V_i \frac{\partial F}{\partial V_i} - r(t) F = 0,$$

$0 \leq V_i$, $i = 1, .., N$, $0 \leq t \leq T$.

2.4.1 The mathematical problem and solution

Let $F(V_1, V_2, t; \varsigma_1)$ denote the value of a European call/put option to exchange asset one for asset two which can be exercised at T_1. This option is simultaneously a call/put option on asset two with exercise price V_1 and a put/call option on asset one with exercise price V_2. Taking V_1 as numeraire, the option to exchange asset one for asset two is a call/put option on asset two with exercise price equal to unity and interest rate equal to zero. The option sells for:

$$F(V_1, V_2, t; \varsigma_1) / V_1 = W_1(V, t; \varsigma_1),$$

where $\varsigma_1 = \pm 1$ if is a call/put and $V = V_2/V_1$. An analytic expression for $W_1(V,t)$ was found in Margrabe (1978).

Let us now define inductively a sequence of call/put option, with value W_k, on the call/put option whose value is W_{k-1}, with exercise price q_k[4] and expiration date T_k, $k = 1,..,N$, $T_1 \geq T_2 \geq ... \geq T_N$. Because all the calls and puts are function of the value of the firm V and the time t, the following partial differential equation holds for W_k:

$$\frac{\partial W_k}{\partial t} + \frac{1}{2}\sigma^2(t)\, V^2 \frac{\partial^2 W_k}{\partial V^2} = 0, \quad t \leq T_k, \quad k = 1,..,N,$$

$T_1 \geq T_2 \geq ... \geq T_N$. The boundary conditions can be written in the form:

$$\begin{gathered} W_k\left(W_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right), T_k; \varsigma_1, .., \varsigma_k\right) = \\ \max\left(\varsigma_k W_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right) - \varsigma_k q_k, 0\right), \end{gathered}$$

where $\varsigma_k = \pm 1$, $k = 1,..,N$, if the k^{th}-compound option is a call/put and $\varsigma_{k-1} = \pm 1$, if the $(k-1)^{th}$-underlying compound option is a call/put. Naturally, if $k = 1$ the well-known pricing formula for simple exchange option is obtained.

In order to solve the partial differential equations above subject to their boundary conditions we need to use the following notation: let V_k^* denote the value of V such that $W_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right) - q_k = 0$ if $k > 1$, and $V_1^* = q_1$. Let us define now:

$$b_k'(t) = \frac{\ln\left(\frac{V}{V_k^*}\right) - \int_t^{T_k} \frac{\sigma^2(\tau)}{2} d\tau}{\left(\int_t^{T_k} \sigma^2(\tau)\, d\tau\right)^{\frac{1}{2}}}, \tag{2.7}$$

and:

$$a_k'(t) = b_k'(t) + \left(\int_t^{T_k} \sigma^2(\tau)\, d\tau\right)^{\frac{1}{2}}; \tag{2.8}$$

[4] As in Carr (1988) the exchange ratio q is taken to be constant or, at most, a deterministic function of time.

moreover, we set $\rho_{ij}(t)$ as in (3). For any k, $1 \leq k \leq N$, let $\Sigma_k^{(N)}(t)$ denote the k-dimension symmetric correlation matrix with typical element $\rho_{ij}(t) = \rho_{N-k+i,N-k+j}(t)$ if $i < j$.

Since we want to derive a valuation formula for the price of N-fold multicompound call/put option to exchange asset one for asset two, that is for $W_N(V, t; \varsigma_1, .., \varsigma_N)$, $0 \leq t \leq T_N$, let V_N^* denote the value of V such that $W_{N-1}(V, T_N; \varsigma_1, .., \varsigma_{N-1}) - q_N = 0$. Then, for V greater than V_N^* the N^{th}-compound call option will be exercised, that is, the firm will update to a superior, new technology, otherwise it will contract it.

As usual, it is desirable to transform the partial differential equation for W_N into the diffusion equation. First, we adopt the following change of variables:

$$W_N(V, t) = \tilde{W}_N(u, z),$$

where:

$$u = -\ln\left(\frac{V}{V_N^*}\right) - \int_t^{T_N} \frac{\sigma^2(\tau)}{2} d\tau,$$

and:

$$z = \frac{1}{2} \int_t^{T_N} \sigma^2(\tau) d\tau.$$

In term of the new independent variables the fundamental equation for F_N becomes:

$$\frac{\partial \tilde{W}_N}{\partial z} = \frac{\partial^2 \tilde{W}_N}{\partial u^2}, \quad -\infty < u < +\infty, \quad z \geq 0.$$

The partial differential equation above subject to the initial value condition $\tilde{W}_N(u, 0)$, has a unique solution which we use to write W_N as follows:

$$W_N(V, t) = \int_{-\infty}^{+\infty} \tilde{W}_N(\xi, 0) \frac{1}{2\sqrt{\pi z}} e^{-(u-\xi)^2/4z} d\xi.$$

Substituting the solution for W_{N-1} into this expression and changing the variable u with $\varsigma_N..\varsigma_1 b'_N(t)$, gives the following identity:

$$W_N(V,t;\varsigma_1,..,\varsigma_N) = \varsigma_N..\varsigma_1 V \int\limits_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b'_N + \xi/\sqrt{2z}\right)^2/2} \times$$

$$N_{N-1}\left(\varsigma_{N-1}..\varsigma_1 a'_{N-1}(T_N),..,\varsigma_1 a'_1(T_N);\Xi^{(N-1)}_{N-1}(T_N)\right) d\xi +$$

$$-\sum_{j=1}^{N-1} \varsigma_N..\varsigma_j q_j \int\limits_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b'_N + \xi/\sqrt{2z}\right)^2/2} \times$$

$$N_{N-j}\left(\varsigma_{N-1}..\varsigma_1 b'_{N-1}(T_N),..,\varsigma_j b'_j(T_N);\Xi^{(N-1)}_{N-j}(T_N)\right) d\xi +$$

$$-\varsigma_N q_N \int\limits_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b'_N + \xi/\sqrt{2z}\right)^2/2} d\xi; \tag{2.9}$$

where $N_k(\varsigma_k..\varsigma_1 b_k,..,\varsigma_1 b_1;\Xi_k)$ denotes the k-dimension multinormal cumulative distribution function with upper limits of integration $\varsigma_1 b'_1,..,\varsigma_k..\varsigma_1 b'_k$ and $\Xi^{(N-1)}_k(T_N)$ denotes the k-dimension modified symmetric correlation matrix with typical element $\rho_{ij}(T_N) = \rho_{N-1-k+i,N-1-k+j}(T_N)$ for $i<j$, as defined in (5). The third term can be easily written in the form:

$$-\varsigma_N q_N N_1(\varsigma_N..\varsigma_1 b'_N(t)).$$

In order to solve the remaining integrals, let us set $x = \varsigma_N..\varsigma_1 a'_N(t) + \xi/\sqrt{2z}$ in the integral of the first term above and $x = \varsigma_N..\varsigma_1 b'_N(t) + \xi/\sqrt{2z}$ in the second; further, we replace any element $\rho_{ij}(T_N)$ in the matrix $\Xi^{(N-1)}_k(T_N)$ with a function of t, according to (6).

The first term can be written in the form:

$$\varsigma_N..\varsigma_1 V \int\limits_{-\infty}^{\varsigma_N..\varsigma_1 a'_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$N_{N-1}(\varsigma_{N-1}..\varsigma_1 a'_{N-1}(t) - x\varsigma_N \rho_{N-1,N}(t) / \sqrt{\left(1-\rho^2_{N-1,N}(t)\right)}, ..$$

$$.., \varsigma_1 a'_1(t) - x\varsigma_N..\varsigma_2 \rho_{1,N}(t) / \sqrt{\left(1-\rho^2_{1,N}(t)\right)}; \tilde{\Xi}^{(N-1)}_{N-1}(t))dx,$$

and the second term:

$$-\sum_{j=1}^{N-1} \varsigma_N..\varsigma_j q_j \int_{-\infty}^{\varsigma_N..\varsigma_1 b'_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$N_{N-j}(\varsigma_{N-1}..\varsigma_1 b'_{N-1}(t) - x\varsigma_N \rho_{N-1,N}(t) / \sqrt{\left(1-\rho^2_{N-1,N}(t)\right)}, ..$$

$$.., \varsigma_j b'_j(t) - x\varsigma_N..\varsigma_{j+1} \rho_{j,N}(t) / \sqrt{\left(1-\rho^2_{j,N}(t)\right)}; \tilde{\Xi}^{(N-1)}_{N-j}(t))dx.$$

Again, in light of the identity obtained before, we finally obtain the following result for the value of a sequential exchange option.

Proposition 4 *Let W_N the value of the multicompound call/put option with maturity T_N and strike price q_N written on a compound call/put option W_{N-1} with maturity T_{N-1} and strike price q_{N-1} , whose value is given by:*

$$W_N(V,t;\varsigma_1,..,\varsigma_N) = \varsigma_N..\varsigma_1 V N_N\left(\varsigma_N..\varsigma_1 a'_N(t), .., \varsigma_1 a'_1(t); \Xi^{(N)}_N(t)\right) +$$

$$-\sum_{j=1}^{N} \varsigma_N..\varsigma_j q_j N_{N+1-j}\left(\varsigma_N..\varsigma_1 b'_N(t), .., \varsigma_j b'_j; \Xi^{(N)}_{N+1-j}(t)\right),$$

$$0 \le t \le T_N.$$

The value of a multicompound call/put option to switch use is:

$$F_N(V_1,V_2,t;\varsigma_1,..,\varsigma_N) = \varsigma_N..\varsigma_1 V_2 N_N\left(\varsigma_N..\varsigma_1 a'_N(t), .., \varsigma_1 a'_1(t); \Xi^{(N)}_N(t)\right) +$$

$$-V_1\sum_{j=1}^{N} \varsigma_N..\varsigma_j q_j N_{N+1-j}\left(\varsigma_N..\varsigma_1 b'_N(t), .., \varsigma_j b'_j; \Xi^{(N)}_{N+1-j}(t)\right),$$

$$0 \le t \le T_N.$$

where the a_is, the b_is and the ρ_{ij}s are as defined previously.

Note that in the particular case when $\varsigma_k = +1$, $k = 1, .., N$ the formula above reduces to Carr (1988).

2.5 Implementation of the approach

In this section, we illustrate our model by providing numerical results for a two stage investment project which relies on compound options pricing. The compound option has two strike prices and two expiration dates. This is a simplification of reality since multi-stage projects, as R&D projects and start up companies require N phases to completion of the investment[5]. For our purpose we consider a simple sequential investment with multiple real options: a firm can make only the intermediate decision to suspend the project at the first exercise date; conversely it will continue investing. This decision is followed by a subsequent decision either to permanently abandonment or to continue, which concludes the process. When the project is successfully completed, it will generate a stream of stochastic cash flows that is given by a standard diffusion process.

A typical new product introduction takes time to build. For instance, the development of a new drug by a pharmaceutical company takes, on average, 10 years to complete. That is, the research and development phase, which takes T years, would give management the right to start the project and continue making investment outlays to find a new molecule that lead to a new drug. The R&D process, in fact, can be seen as N rounds of investments in which brand-new molecules are continuously investigated and tested at a cost.

In the case under consideration we deal with a generic sequential investment project: the construction of the project requires two investment outlays at specific dates during the development phase. For instance, a first outlay of I_1, followed by a subsequent outlay, I_2, which complete the project. The firm does not earn any cash flows from the project until it is complete; indeed, the project generates cash flows during the operating stage that follows the last investment outlay, I_2. Due to our simplification we assume that a firm can make only the intermediate investment outlay I_1, to launch a pilot at T_1; this fact gives the firm an option to complete the next stage of investment. After a new product is ready to be introduced on a market, within the period T, management may decide to invest I_2 for marketing it.

[5] See the discussion of chapter 1.

For our purpose the project possesses flexibility in the form of multiple real options: the firm can decide either to temporarily stop the project at T_1 or to continue investing; next, management can decide either to abandon it at T, or to continue with the investment construction.

These possibilities of stopping midstream make this investment analogous to the exercise of compound option strategies:

(1) Management suspends the project at the first exercise date, T_1, which permits to save the exercise price I_1, and continues the construction at the second exercise date, T, by paying the exercise price I_2; this decision is analogous to the exercise a put on a call;

(2) Management continues the construction of the project at T_1, by paying I_1, and decides to abandon it at T, which permits to save I_2; this decision is analogous to the exercise of a call on a put;

(3) Management suspends the project at the first exercise date, T_1, and decides to abandon it at T. This decision is analogous to the exercise of a put on a put;

(4) Management continues the construction of the project at T_1, and continues the construction at T; this decision is analogous to the exercise of a call on a call.

For the base case we consider the following values of inputs:

(a) The initial project value is $V_0 = 100$, which is simply the present value of expected cash flows from immediately undertaking the project, including the initial cost to get it started and not including the subsequent investment outlays I_1 and I_2, or embedded real options;

(b) Compound option values can be determined by discounting project's future payoffs at the risk-less interest rate, $r = 0.1$[6];

(c) The dividend yield is $\delta = 0.05$[7];

(d) The cash flow volatility parameter is assumed to be $\sigma = 0.2$[8]. Con-

[6] In general, any contingent claim on an asset (traded or not) can be priced in a world with systematic risk by replacing the actual growth rate, α, with a certainty-equivalent rate, $\hat{\alpha} \equiv \alpha - \textit{risk premium}$; this is equivalent to a risk-neutral valuation where the actual drift, α, would be replaced by the risk-neutral equivalent drift $\hat{\alpha} = r - \delta$, where δ is the shortfall in the expected rate of return from holding the option to complete rather than the completed project. See Trigeorgis (1996) for an overview.

[7] Note that the dividend yield is assumed to be zero in section 2.3.

[8] For comparison purpose we assume the same parameter values used in Shaw (1998). Different value of the volatility parameter can be used depending on the project under consideration. For example, in Schwartz (2004) volatility parameter value for R&D projects is obtained as the average implied volatility for traded call option of nine pharmaceutical companies. This is around 0.35.

trarily to the assumption in section 2.3, the risk-free rate of interest r and the volatility σ are constant;

(e) If at any time market conditions deteriorate, construction can be temporarily and costlessly stopped at $T_1 = 0.25$ (or permanently abandoned), and restarted later. Conversely firms will continue investing until $T = 1$. At that date management can both salvage a portion of the investment, I_2, by abandoning the project or launch the product into the market;

(f) The strike on the intermediate date is given by $I_1 = 10$; finally, the cost for launching and marketing is $I_2 = 100$.

2.5.1 Numerical results

Our model is implemented computing the formula derived in section 2.3.1 with Mathematica. This procedure make use of the standard Black-Scholes formula for the critical call and put prices computation at the first exercise date and also uses the bivariate normal distribution for treating two correlated random processes. Let us denote the compound options values by F_2.

Table I displays the values of F_2 corresponding to the exercise of a put on a call.

Table I

put on call			
	$I_1 = 10;$	$I_1 = 15;$	$I_1 = 20;$
$T_1 = 0.25;$	2.4314	5.8681	10.1208
$T_1 = 0.5;$	3.38011	6.64592	10.4943
$T_1 = 0.75;$	4.15786	7.33643	10.9436

Base case parameters are: $V = 100$, $I_1 = 10$, $I_2 = 100, \sigma = 0.2, r = 0.1, \delta = 0.05, T_1 = 0.25, T = 1$.

Numerical evidence demonstrates that increasing value of the first investment outlay, I_1, will increase the value of the compound option F_2. For example, if $T_1 = 0.25$, then an increase in I_1 from 10 to 20 will increase the value of the compound option from 2.4314 to 10.1208. That is, increasing the exercise price of a put to temporarily shut down permits the firm to save 20 instead of 10. Finally, an increase in the intermediate exercise date, T_1, will increase the value of a put on a call. In fact, if $I_1 = 10$, an increase in

T_1 from 0.25 to 0.75 will increase the value of F_2 from 2.4314 to 4.15786. This is a standard option pricing result.

It is interesting to note that the value of options in the presence of others may differ from its value in isolation[9]. For instance, for the base case parameters values:

−the price of the put at T_1, without considering subsequent real options is equal to 0: abandonment strategy is worthless. Instead of abandoning, firm may choose to keep its project alive and renounce to investment I_1: this strategy is worth and its value is shown in Table I. This is because the exercise of an individual put, to abandon early, eliminates the whole portfolio of gross project value plus the value of any future options.

−the value of the call at T, not including the possibility to suspend the construction midstream, is 9.9409. This result does not consider that the exercise of a prior real option may alter the underlying asset value, and hence, the value of subsequent options on it. In this case, in fact, the exercise of the option to temporarily shut-down would increase the value of the project, increasing the value of the call option on it.

Finally, with respect to the strategy (1) , an option-based approach which focus on valuing the project as a sum of two individual options (a put and a call) use to undervalue it.

Table II displays the compound option values, F_2, corresponding to the exercise of a call on a put.

Table II

call on put			
	$I_1 = 10;$	$I_1 = 15;$	$I_1 = 20;$
$T_1 = 0.25;$	0.364158	0.0558511	0.00571632
$T_1 = 0.5;$	0.969139	0.347942	0.104005
$T_1 = 0.75;$	1.59221	0.783484	0.340136

Base case parameters are: $V = 100$, $I_1 = 10$, $I_2 = 100, \sigma = 0.2, r = 0.1, \delta = 0.05, T_1 = 0.25, T = 1.$

Numerical evidence demonstrates that increasing value of the first investment outlay, I_1, will decrease the value the compound option F_2. For example, if $T_1 = 0.25$, then an increase in I_1 from 10 to 20 will decrease

[9] Trigeorgis (1993) provides the valuation results for a generic project, when particular real options are valued in the presence of others and illustrates option configurations where interactions can be small or large, as well as negative or positive.

the value of the compound option from 0.364158 to 0.00571632. Instead, an increase in the intermediate exercise date, T_1, will increase the value of a call on a put. In fact, if $I_1 = 10$, an increase in T_1 from 0.25 to 0.75 will increase the value of F_2 from 0.364158 to 1.59221.

Note that for the base case parameters values:

−the price of the call at the intermediate date T_1, without considering subsequent real options is 89.0047. Clearly, this result relies on the option price in isolation and comes from the discounted cash flows of the project if it would be completed at T_1. Of course, the value of the prior call is lower when followed by a subsequent option because in this case its value depends on the value of another option. This is shown in table II.

−the value of the put to permanently abandon the project at T, not including the prior real option, is 5.3017; of course, this price comes solely from the salvage of I_2. It can be proved that the exercise of the call option to continue investing, would increase the value of the project, causing the value of the subsequent put option to decrease.

Finally, with respect to the strategy (2) , the value of the project viewed as sum of separate options is far from the true value when interdependences[10] (compound options) are considered.

Table III displays the compound option values, F_2, corresponding to the exercise of a put on a put.

Table III

put on put			
	$I_1 = 10;$	$I_1 = 15;$	$I_1 = 20;$
$T_1 = 0.25;$	4.68042	9.25438	14.0865
$T_1 = 0.5;$	5.05548	9.2016	13.725
$T_1 = 0.75;$	5.45405	9.30039	13.5121

Base case parameters are: $V = 100$, $I_1 = 10$, $I_2 = 100, \sigma = 0.2, r = 0.1, \delta = 0.05, T_1 = 0.25, T = 1.$

Numerical evidence demonstrates that increasing values of the first investment outlay, I_1, will increase the value the compound option F_2. For example, if $T_1 = 0.25$, then an increase in I_1 from 10 to 20 will increase the value of the compound option from 4.68042 to 14.0865. This is due to the investment outlay salvage. Instead, an increase in the intermediate exercise

[10] See Trigeorgis (1993) for further details.

date, T_1, both increases and decreases the value of a put on a put. In fact, if $I_1 = 10$, an increase in T_1 from 0.25 to 0.75 will increase the value of the compound option from 4.68042 to 5.45405. Instead, if $I_1 = 20$, an increase in T_1 from 0.25 to 0.75 will decrease F_2 from 14.0865 to 13.5121.

As before, the value of the compound option to temporarily suspend the project is higher than the value of the put to abandon it; of course, increasing flexibility causes both the value of the project to increase and the value of the subsequent put option to decrease.

Finally, by comparison, it is interesting to note that the value of the option to temporarily shut-down is higher when it is followed by another put (to permanently abandon the project). For example, with the base case parameter values, the price of a put on a put is 4.68042; instead, the price of a put on a call is 2.4314.

Table IV displays the compound option values, F_2, corresponding to the exercise of a call on a call.

Table IV

call on call			
	$I_1 = 10$;	$I_1 = 15$;	$I_1 = 20$;
$T_1 = 0.25$;	2.44416	0.998576	0.368981
$T_1 = 0.5$;	3.62277	2.12127	1.20231
$T_1 = 0.75$;	4.62502	3.14854	2.10064

Base case parameters are: $V = 100$, $I_1 = 10$, $I_2 = 100$, $\sigma = 0.2$, $r = 0.1$, $\delta = 0.05$, $T_1 = 0.25$, $T = 1$.

Numerical evidence demonstrates that increasing values of the first investment outlay, I_1, will decrease the value of the compound option F_2. Instead, postponing the intermediate exercise date, T_1, causes the value of the option to rise, as is traditional in option pricing. For example, if $T_1 = 0.25$, then an increase in I_1 from 10 to 20 will decrease the value of the compound option from 2.44416 to 0.368981. Finally, if $I_1 = 10$, an increase in T_1 from 0.25 to 0.75 will increase the value of the compound option from 2.44416 to 4.62502.

We recall that the value of the compound option to continue the project at T_1 is lower than the value of a call in isolation; the reason is because the real options under consideration are written on different assets: (1) the value of a subsequent call option with strike price I_2 and maturity T, and (2) the discounted cash flows of the gross project if it would be completed at

T_1, respectively. Thus, strategy (4) leads to a project valuation which is far from the value obtained by pricing two option separately and then adding their results.

Finally, note that the possibility to launch a pilot at T_1, causes both the value of the project and the value of the subsequent call option to increase.

2.6 Final remarks

New ventures have the property that much of the value of the investment is associated with future cash flows that are contingent on intermediate decisions. This fact makes the valuation of venture capitals and start-up companies one of the most difficult investment problem. The real options approach has therefore acquired importance since traditional DCF-based approaches seem unsuitable to explain the dynamics in the value of these investment projects. As a result, the recent body of research on the use of option pricing leads to a considerable literature which captures many different features of these strategic investments.

This paper deals with sequential investment opportunities, as R&D investments, pilot projects, etc., and provides a valuation method when these ventures possess flexibility in the form of multiple real options. To do it, the paper develops a real options approach where the completion of the project requires $N-$rounds of investment that are analogous to the exercise prices of a multicompound call/put option.

The main conclusions of the paper are discussed below.

First, we obtain a closed-form solution for multicompound option which is simultaneously a call and a put. This result is useful in situations when firms make the intermediate investment decisions to continue, suspend or abandon the project. In doing so, we generalize the setting in Agliardi and Agliardi (2005)

Second, we obtain a closed-form solution for multicompound exchange options which is simultaneously a call and a put. In doing so, we extend the work of Carr (1988) to real options problem when firms make the intermediate investment decisions to continue expand (upgrade/switch technology) or contract the scale of the project.

Third, we show through numerical implementation, that the value of an option in the presence of others may differ from its value in isolation. First, recall that the value of a prior real option would be altered if followed by a subsequent real option because it would be written on the value of another option. Second, the effective underlying asset for the latter option would be altered conditional on prior exercise of an earlier option.

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Part III

Valuing R&D investments with a jump-diffusion process

Chapter 3

3.1 Introduction

[1] As several researchers have noted R&D ventures are essentially real growth options. The value of these early projects derives not so much from their expected cash flows as from the follow-on opportunities they may create. For example, basic R&D option is the option that gives the firm the right to make positive NPV investments if and when the R&D project is successfully completed. Although traditional tools fail to capture the value of these investments, because of their dependence on future events that are uncertain at the time of the initial decision, firms engage the pilot to get started a multi-stage process that may eventually reach a commercial phase of launching the new product. Take the example of developing a new drug. Investing in R&D in the pharmaceutical industry, begins with research that leads with some probability to a new compound; such a project continues with testing and concludes with the construction of a production facility and the marketing of the product. Because many early investments can be seen as chains of interrelated projects, the earlier of which is prerequisite for those to follow, they can be evaluated as multicompound options which involve sequential decisions to exercise the options to invest only when the R&D outcomes are successful. Compound options have been extensively used in corporate finance to evaluate investment opportunities. For example, Geske (1979) suggested that when a company has common stock and coupon bonds outstanding, the firm's stock can be viewed as a call option on a call option. Carr (1988) analyzed sequential compound options, which involve options to acquire subsequent options to exchange an

[1] This chapter builds on the working paper DSE n. 569 (2006), University of Bologna and was presented at a recent DSE seminar.

underlying risky asset for another risky asset. Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a jump-diffusion process. Agliardi and Agliardi (2005) study multicompound call options in the case of time-dependent volatility and interest rate. This assumption seems more suitable due to the sequential nature of many early projects. Multicompound options are merely N-fold options of options. Basically the procedure consists of solving N-nested Black-Scholes partial differential equations: at the first step the underlying option is priced according to the Black-Scholes method; then, compound options are priced as options on the securities whose values have already been found in the earlier steps. Roll (1977), Whaley (1981), Geske and Johnson (1894) and Selby and Hodges (1987) also study compound options.

The objective of this paper is to study the multicompound options approach to value sequential investment opportunities, as research and development and similar investment projects, when the underlying asset follows a jump-diffusion process. Empirical evidence confirms the systematic mispricing of the Black-Scholes call option pricing model[2]. A number of explanations for the systematic price bias have been suggested[3]. Among these is the presence of jumps in price. Diffusion models cannot properly capture sudden, discontinuous moves in price. This well-known fact leads to the argument that using continuous or discontinuous models has important consequences for the representation of the risk. Merton (1976) have suggested that incorporating jumps in option valuation models may explain some of the large empirical biases exhibited by the Black-Scholes model. According to Merton, the arrival of normal information leads to price changes which can be modelled as a lognormal diffusion, while the arrival of abnormal information gives rise to lognormally distributed jumps in the security return, which can be modelled as a Poisson process. If the underlying project value follows a mixed jump-diffusion process the price of the multicompound option will systematically differ from the multicompound option price. Jump models for option pricing are also Cox and Ross (1976), Ball and Torous (1985), Naik and Lee (1990), Amin (1993) and Bates (1996).

More recently, some papers in the finance literature study Lévy processes in real options pricing. For example, Barrieu and Bellamy (2005) analyze the impact of market crises on investment decision via real option theory. The investment project, modelled by its profits/costs ratio, is characterized

[2]Ball and Torous (1995) provide statistical evidence consistent with the existence of lognormally distributed jumps in a majority of the daily returns of a sample of NYSE listed common stocks.

[3]See for example Geske and Roll (1984).

by a mixed diffusion process, whose jumps represent the consequences of crises on the investment field. This paper is dedicated to the analysis of the exercising time properties in an unstable framework. The modelling of the underlying dynamics involves a mixed-diffusion, made up of Brownian motion and Poisson process. The jumps are negative as to represent troubles and difficulties occurring in the underlying market. Finally, Cont and Tankov (2004) provide an overview of theoretical, numerical and empirical research on the use of jump processes in financial modelling.

Real option studies are usually written in a continuous framework for the underlying dynamics. However, the existence of crises and shocks on investment markets generates discontinuities. The impact of these crises on the decision process is then an important feature to consider. The assumption of jump-diffusion process better describe the evolution of asset value due to the risky nature of many early investments. Of course new ventures are subject to several, qualitatively different sources of risk. There is the uncertainty associated with the market factors outside the control of the firm, that causes marginal changes in the asset value. This is related to the demand for the product and production costs and is modelled by a standard geometric Brownian motion. There is the exogenous risk associated with the actions of a competitor, and finally, there is the technical uncertainty which is idiosyncratic to the firm. The technical risk which represents the discontinuous arrival of new information has more than a marginal effect on the asset value. This component is modelled by a jump process reflecting the non-marginal impact of information. Let us consider the following cases. The policy process is particular relevant for the firm engaged in R&D and other new ventures. Governments can not only deploy measures to reduce the uncertainty facing potential investors, they can also create uncertainty through the prospect of policy change. It is commonly believed that expectations of shifts of policy can have powerful effects on decisions to invest in these early projects. However, policy uncertainty is not likely to be well captured by a Brownian motion process; it is more likely to be a Poisson jump. R&D in pharmaceuticals and biotechnologies frequently involves upward jumps or downward jumps, for example drugs can turn into mega-selling blockbuster products or alternatively, suffer clinical trial failures and withdrawal from the markets. Hence R&D investments valuation should rely on a model focusing on these aspects, rather on standard Brownian motion. The current valuation of investments based on option methodology assumes a continuous cash-flow generation process which is inadequate when these types of risk jointly determine the value of a new venture. However, in many cases, closed-form solutions for valuing options with jump processes are not available. The main contribution of this paper is to derive a pricing formula for multicompound options when the jump distribution is lognormal; in doing so, we integrate work on multicompound options[4] by Agliardi and Agliardi (2005) with that on compound options[5] by Gukhal (2003).

The paper is organized as follows. Section 2 reviews the literature on real options and its application to the valuation of R&D ventures and start-up companies. This is followed by a description of the economic model in Section 3. Section 4 derives a closed-form solution for multicompound options in which the equation for the underlying process is replaced by a more general mixed diffusion-jump process. An extension to pricing sequential expansion options is presented in section 5. In section 6 we illustrate our model by providing numerical results for two different type of compound options. First, consider Geske's (1979) compound option formula in which the underlying asset follows a standard geometric Brownian motion; next, we consider the compound option formula where the underlying asset price follows a mixed jump-diffusion process. For the last case we assume that the proportional jump size has a lognormal distribution. Section 7 concludes the paper.

3.2 Literature review

[4] In this paper the multicompound option c_N is expressed in the form:

$$\begin{aligned} c_N(S,t) &= SN_N(h_N(t)+\sqrt{\int_t^{T_N}\sigma^2(\tau)d\tau}..., h_1(t)+\sqrt{\int_t^{T_1}\sigma^2(\tau)d\tau}; \Xi_N^{(N)}(t)) + \\ & -\sum_{j=1}^{N} X_j e^{-\int_t^{T_j} r(\tau)d\tau} N_{N+1-j}(h_N(t),...,h_j(t); \Xi_{N+1-j}^{(N)}(t)), \end{aligned}$$

where

$$h_k(t) = \left(\ln\frac{S}{S_k^*} + \int_t^{T_k}(r(\tau)-\frac{\sigma^2(\tau)}{2})d\tau\right) / \left(\int_t^{T_k}\sigma^2(\tau)d\tau\right)^{\frac{1}{2}};$$

and $\Xi_k^{(N)}(t)$ denotes a k-dimension correlation matrix with typical element

$$\rho_{ij}(t) = \left(\int_t^{T_j}\sigma^2(\tau)d\tau / \int_t^{T_i}\sigma^2(\tau)d\tau\right)^{\frac{1}{2}};$$

for $1 \le i \le j \le k, t \le T_k$.

[5] See section 1.4.3.

A number of existing research contribution has previously analyzed various aspects of optimal sequential investment behavior for firm facing multi-stage projects. Staging investment involves firms either with some degree of flexibility in proceeding with investment or when there is a maximum rate at which outlays or construction can proceed, that is, it takes time-to-build. The real option literature have studied the R&D process as a contingent claim on the value of the underlying cash flows on completion of the R&D project. Majd and Pindyck (1987) develop a continuous investment model with time-to-build. They solve an investment problem in which the project requires a fixed total investment to complete, with a maximum instantaneous rate of investment. Pindyck (1993) also takes into account market and technical uncertainty. Myers and Howe (1997) present a life cycle model of investments in pharmaceutical R&D programs; the problem is solved using Monte Carlo simulation. Childs and Triantis (1999) develop and numerically implement a model of dynamic R&D investment that highlights the interactions across projects. Schwartz and Moon (2000) have also studied R&D investment projects in the pharmaceutical industry using a real options framework. In this articles, they numerically solve a continuous-time model to value R&D projects allowing for three types of uncertainty: technical uncertainty associated with the success of the R&D process itself, an exogenous chance for obsolescence and uncertainty about the value of the project on completion of the R&D stages. Schwartz (2003) develops and implements a simulation approach to value patents-protected R&D projects based on the real option approach. It takes into account uncertainty in the cost to completion of the project, uncertainty in the cash flows to be generated from the project, and the possibility of catastrophic events that could put an end to the effort before it is completed. Errais and Sadowsky (2005) introduce a general discrete time dynamic framework to value pilot investments that reduce idiosyncratic uncertainty with respect to the final cost of a project. In this model, the pilot phase requires N stages of investment for completion that they value as a compound perpetual Bermudan option. Although the preceding articles suggested the use of more suitable technique when we attempt to value intangible project that are linked to the future opportunities they create, such investments are hard to value, even with the real options approach. The main reason for this is that there are multiple sources of uncertainty in R&D investment projects and that they interact in complicated way. In practice, the bulk of the literature on the R&D valuation have dealt with the development of numerical simulation methods based on optimal stopping time problems. Berk, Green and Naik (2004) develop a dynamic model of multi-stage investment project that captures many features of R&D ventures and start-up companies. Their model assumes different sources of risk and allow to study their interaction in determining the value and risk premium of the venture. Closed-form solutions for important cases are obtained. More recently, a number of articles consider strategic interaction features across R&D options holders. Garlappi (2004) analyzes the impact of competition on the risk premium of R&D ventures engaged in a multi-stage patent race with technical and market uncertainty. R&D competition causes substantial rent dissipation, excess R&D spending and higher and more volatile risk premium. On the other side, competition dramatically reduces the completion time and the failure rate of research within the industry. Miltersen and Schwartz (2004) develop a model to analyze patent-protected R&D investment projects when there is multiple sources of uncertainty in R&D stages and imperfect competition in the development and marketing of the resulting product. Grenadier (2002) adds a time-to-build features in a model of option exercise games.

Our study differs from those mentioned above in several crucial respects. We provide a model which relies on simple mathematics to price options with jump-diffusion process. We emphasize that sequential investments opportunities, as for example R&D projects, can be valued in a continuous-time framework based on the Black-Scholes model.

3.3 Model and assumptions

Let us consider the investment decision by a venture capital fund that is evaluating a single R&D project. We assume that the commercial phase of the project cannot be launched before a pilot phase consisting on $N-$ stages of investment is completed. The risk free rate in our setting will be denoted by r. Let I be the amount of investment required for completion of any R&D stage. When the R&D is successfully completed, the project will generate a stream of stochastic cash flows, which we model as a mixed diffusion-jump process:

$$dV_t = (\alpha - \lambda k)\, V_t dt + \sigma V_t dz_t + (Y-1)\, V_t dq_t, \tag{3.1}$$

where α is the instantaneous expected return on the underlying asset; σ is the instantaneous standard deviation of the return, conditional on no arrivals of important new information[6]; dz is the standard Brownian motion; dq is the independent Poisson process with rate λt; $(Y-1)$ is the proportional change in the asset value due to a jump and $k \equiv E[Y-1]$; dq and dz are assumed to be independent.

The total uncertainty in the underlying project is posited to be the composition of two type of risk: the systematic risk and the technical risk. The former is generally related to economic fundamentals that causes marginal changes in the asset value. This is associated with demand for the product and production costs and is modelled by a standard geometric Brownian motion. The technical risk which represents the discontinuous arrival of new information has more than a marginal effect on the asset value. This component is modelled by a jump process reflecting the non-marginal impact of information. Usually, such information is specific to the firm: for example, a new drug may be rendered unnecessary by a superior treatment option, the entry by a new competitor who take out a patent for a drug that is targeted to cure the same disease, the possibility of political and technical unpredictable information that will cause V to jump. Assume that the logarithmic jump amplitude, $\ln(Y)$, is normally distributed with mean (μ_J) and variance $\left(\sigma_J^2\right)$; then, the version of Ito's lemma for a diffusion-jump stochastic process is:

$$dx_t = \left(\alpha - \lambda k - \frac{1}{2}\sigma^2\right) dt + \sigma dz_t + \ln(Y)\, dq_t.$$

As in Merton (1976) we assume that technical uncertainty is completely diversifiable, that is, the firm will not demand any additional return over the risk free rate for being exposed to this source of risk. This fact will allow us to specify a unique equivalent risk-neutral measure by setting the market price of risk of q to zero. Although, it may be too strong an assumption for industries where firms may place an important premium on idiosyncratic risk, this assumption seems unlikely to change results significantly (see Errais and Sadowsky, (2005) for further details). In the particular case when the expected change in the asset price is zero, given that the Poisson event occurs $(i.e.,\ k=0)$[7], by following standard arguments in the financial mathematics literature we can construct the risk-neutral pricing measure under which we

[6]Further, it is possible to include both a time-varying variance and time-varying interest rate; see Agliardi and Agliardi (2003) and Amin (1993) for a discussion of this point.

[7]See Merton (1976, pp. 135-136) for a discussion of this point.

will work for the remaining of the paper[8]. The process for the underlying asset value under $\mathbb{Q}$ is given by:

$$dV_t = rV_t dt + \sigma V_t d\tilde{z}_t + (Y-1)\, V_t dq_t. \tag{3.2}$$

In the real options setting, investment opportunities may be viewed as options; thus, the pricing formula for multicompound option can be applied to evaluate the N-stages pilot we described earlier. In more specific terms, let $F(V,t)$ denote the value of a European call option with exercise price I_1 and expiration date T_1. Let us now define inductively a sequence of call options, with value F_k, on the call option whose value is F_{k-1}, with exercise price I_k and expiration date T_k, $k = 1,..,N$, where we assume $T_1 \geq T_2 \geq ... \geq T_N$. Because all the calls are function of the value of the firm V and the time t, the following partial integro-differential equation holds for F_k:

$$\frac{\partial F_k}{\partial t} = rF_k - rV\frac{\partial F_k}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F_k}{\partial V^2} - \lambda E\left\{F_k(VY,t) - F_k(V,t)\right\},$$

$t \leq T_k$, $k = 1,..,N$, $T_1 \geq T_2 \geq .. \geq T_N$. The boundary condition is:

$$F_k\left(F_{k-1}(V,T_k), T_k\right) = \max\left(F_{k-1}(V,T_k) - I_k, 0\right),$$

where $F_{k-1}(V,T_k)$ stands for the price of the underlying compound option. Naturally, if $k = 1$ the well-known pricing formula for simple option is obtained:

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda T_1}(\lambda T_1)^n}{n!}\left(VN_1(a_1) - I_1 e^{-rT_1} N_1(b_1)\right),$$

with:

$$a_1 = \frac{\ln\left(\frac{V}{I_1}\right) + \left(r + \frac{\nu^2}{2}\right)T_1}{\nu\sqrt{T_1}}, \qquad b_1 = a_1 - \nu\sqrt{T_1},$$

where $t = 0$ and $\nu^2 = \sigma^2 + \frac{n\sigma_J^2}{T_1}$, conditional on the number of jumps n.

[8] We refer the reader to Musiela and Rutkowsky (1998) for additional details.

3.4 Derivation of the valuation formula

We want to determine the value of the investment opportunity $F_k(V,t)$ in each stage T_k, $k = 1, .., N$, of the pilot conditioning on the discontinuous arrival of new information. To simplify notation, we assume that t equals zero. Let V_k^* denote the value of V such that $F_{k-1}(V, T_k) - I_k = 0$ if $k > 1$ and $V_1^* = I_1$. Moreover, let us set $s_k = \sum_{i=k}^{N} n_i$ the total number of jumps in the interval $[0, T_k]$, $k = 1, .., N$, $T_1 \geq T_2 \geq .. \geq T_N$.

Let us define now:

$$b_k = \frac{\ln\left(\frac{V}{V_k^*}\right) + \left(r - \frac{\nu^2}{2}\right) T_k}{\nu\sqrt{T_k}}, \tag{3.3}$$

and:

$$a_k = b_k + \nu\sqrt{T_k}, \tag{3.4}$$

where $\nu^2 T_k = \sigma^2 T_k + s_k \sigma_J^2$, $k = 1, .., N$. Moreover, let:

$$\rho_{ij} = \sqrt{\frac{T_j}{T_i}}, \quad \text{for } 1 \leq i < j \leq N, \tag{3.5}$$

the correlation between the logarithmic returns x_{T_j} and x_{T_i} conditioning on the number of jumps s_j and $S_i = s_i - s_j$. For any k, $1 \leq k \leq N$, let $\Xi_k^{(N)}$ denote a k-dimension symmetric correlation matrix with typical element $\rho_{ij} = \rho_{N-k+i, N-k+j}$. Let $N_k(b_k, ..., b_1; \Xi_k)$ denote the k-dimension multinormal cumulative distribution function, with upper limits of integration $b_1, ..., b_k$ and correlation matrix Ξ_k. Finally, let $\sum_{n_k=0}^{\infty} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda}\lambda^{n_1}}{n_1!}$ denote the joint probability function of k independent Poisson processes with rate λ. We mean the discontinuous arrivals of new information are assumed to be independent of each other[9]. Our aim is to derive a valuation formula for the N-fold multicompound option. Let V_N^* denote the value of V such that $F_{N-1}(V, T_N) - I_N = 0$. Then, for V greater than V_N^* the N^{th}- compound call option will be exercised, while for values less than V_N^* it will remain unexercised.

[9] See Kocherlakota and Kocherlakota (1992) for a more detailed development of the bivariate Poisson distribution.

The value of the multicompound option is the expected present value of the resulting cash flows on the completed project :

$$E_0^{\mathbb{Q}}\left[e^{-rT_1}\left(V-I_1\right)1_{\varepsilon_1}..1_{\varepsilon_N}\right]+\sum_{j=2}^{N}E_0^{\mathbb{Q}}\left[e^{-rT_j}\left(-I_j\right)1_{\varepsilon_j}..1_{\varepsilon_N}\right], \tag{3.6}$$

where $\varepsilon_k=\{V_k\geq V_k^*\}$, $k=1,..,N$. The first term in (3.6) can be written in the form:

$$E_0^{\mathbb{Q}}\left\{e^{-rT_N}E_{T_N}^{\mathbb{Q}}\left[..\left\{e^{-r\tau_1}\left(V-I_1\right)1_{\varepsilon_1}\right\}..\right]1_{\varepsilon_N}\right\}, \tag{3.7}$$

$\tau_k=T_k-T_{k+1}$. To examine option pricing when the asset price dynamics include the possibility of non-local changes, we condition the expectation to the number of jumps between any points in time:

$$E_0^{\mathbb{Q}}\left\{\sum_{n_N=0}^{\infty}\left[..\left\{\sum_{n_1=0}^{\infty}\left[e^{-r\tau_1}\left(V-I_1\right)1_{\varepsilon_1}\mid n_1\right]prob\left(n_1\right)\right\}..1_{\varepsilon_N}\mid n_N\right]prob\left(n_N\right)\right\}. \tag{3.8}$$

The evaluation of the expectation requires the calculation of the joint probability function of N independent Poisson processes with rate λt :

$$\sum_{n_N=0}^{\infty}\frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!}..\sum_{n_1=0}^{\infty}\frac{e^{-\lambda\tau_1}\left(\lambda\tau_1\right)^{n_1}}{n_1!}\times$$

$$E_0^{\mathbb{Q}}\left\{e^{-rT_N}E_{T_N}^{\mathbb{Q}}\left[..\left\{e^{-r\tau_1}\left(V-I_1\right)1_{\varepsilon_1}\right\}..\right]1_{\varepsilon_N}\mid n_1,..,n_N\right\}. \tag{3.9}$$

To evaluate the first expectation we will work with the logarithmic return x_{T_k}, rather than V. Conditioning on the number of jumps, s_k, $\ln x_{T_k}\sim N\left[\eta,\nu^2T_k\right]$ where $\eta T_k=\left(r-\frac{\nu^2}{2}\right)T_k$ and $\nu^2T_k=\sigma^2T_k+s_k\sigma_J^2$. The price of the multicompound option at time 0 equals:

$$\sum_{n_n=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} (\lambda \tau_1)^{n_1}}{n_1!} \times$$

$$e^{-rT_N} \left\{ \int_{-\infty}^{b_N} N'(y) \left(g(y) N_{N-1} \left(\hat{a}_{N-1}, .., \hat{a}_1; \hat{\Xi}_{N-1}^{(N-1)} \right) dy + \right) \right.$$

$$\left. - \int_{-\infty}^{b_N} N'(y) \left(I_1 e^{-r(T_1 - T_N)} N_{N-1} \left(\hat{b}_{N-1}, .., \hat{b}_1; \hat{\Xi}_{N-1}^{(N-1)} \right) \right) dy \right\}, \tag{3.10}$$

where $\hat{a}_k = a_k (g(y), V_k^*, T_k, T_N)$, $\hat{b}_k = b_k (g(y), V_k^*, T_k, T_N)$ for $k = 1, .., N-1$, and the entries of $\hat{\Xi}_k^{(N-1)}$ are $\beta_{N-1-k+i, N-1-k+j}$, where we define $\beta_{ij} = \frac{-\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}$, for $i < j$. Note that the critical values V_k^* above which the k^{th}-multicompound option will be exercised, are determined recursively and their existence and uniqueness are guaranteed in view of the expression of F_{k-1} (see Remark 3).

The function $g : \mathbb{R} \to \mathbb{R}$ is given by the formula:

$$g(y) = V_0 \exp \left[\left(r - \frac{\nu^2}{2} \right) T_N + \nu \sqrt{T_N} \cdot y \right], \tag{3.11}$$

where y has a standard Gaussian probability law under $\mathbb{Q}$. Straightforward calculations yield:

$$\hat{a}_k = \frac{\ln \left(\frac{V_0}{V_k^*} \right) + \left(r + \frac{\nu^2}{2} \right) (T_k - T_N) + \left(r - \frac{\nu^2}{2} \right) T_N}{\nu \sqrt{T_k - T_N}} + y \sqrt{\frac{T_N}{T_k - T_N}}, \tag{3.12}$$

$$\hat{b}_k = \frac{\ln \left(\frac{V}{V_k^*} \right) + \left(r - \frac{\nu^2}{2} \right) T_k}{\nu \sqrt{T_k - T_N}} + y \sqrt{\frac{T_N}{T_k - T_N}}, \text{ for } k = 1, .., N-1. \tag{3.13}$$

The second integral in (3.10) can be expressed in terms of the $N-$ dimension multinormal cumulative distribution function by applying the following

Lemma 5 *Let* $1 \leq k < N$*, and let* $\hat{\Xi}_k^{(N-1)}$ *be the matrix obtained from* $\Xi_k^{(N-1)}$ *replacing any element* β_{ij} *with* $\frac{-\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}$*, by setting*

$$\alpha_k = \frac{\ln\left(\frac{V}{V_k^*}\right) + \left(r - \frac{\nu^2}{2}\right) T_k}{\nu\sqrt{T_k - T_N}},$$

where α *and* β *are real numbers, the following identity holds:*

$$\int_{-\infty}^{b_N} N'(y)\, N_k \left(\alpha_{N-1} + y\beta_{N-1,N}, .., \alpha_{N-k,N} + y\beta_{N-k,N}; \hat{\Xi}_k^{(N-1)}\right) dy =$$

$$N_{k+1}(b_N, ..., b_{N-k}; \Xi_{k+1}^{(N)}).$$

Proof. *by induction after solving the following equation* $\frac{b_k}{\sqrt{1-\rho_{k,N}^2}} = \alpha_k$ *and* $\frac{-\rho_{k,N}}{\sqrt{1-\rho_{k,N}^2}} = \beta_{k,N}$*,* $k = 1, .., N-1$*, for* b_k *and* $\rho_{k,N}$*.* ■

Finally, we succeed in writing the first integral in (3.10) in terms of the cumulative function of the multivariate normal distribution using Lemma 1, after making the following substitution $x = y - \sigma\sqrt{T_N}$; thus we get:

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} (\lambda \tau_1)^{n_1}}{n_1!} \times$$

$$\left[V_0 N_N \left(a_N, .., a_1; \Xi_N^{(N)}\right) - I_1 e^{-rT_1} N_N \left(b_N, .., b_1; \Xi_N^{(N)}\right)\right].$$

The second expectation in (3.6) can be evaluate to give:

$$-\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} .. \sum_{n_j=0}^{\infty} \frac{e^{-\lambda \tau_j} (\lambda \tau_j)^{n_j}}{n_j!} \times$$

$$\left\{ \sum_{j=2}^{N} I_j e^{-rT_j} N_{N+1-j} \left(b_N, .., b_j; \Xi_{N+1-j}^{(N)}\right)\right\}, \quad j = 2, .., N.$$

Hence, we have the following result for the value of a multicompound call option:

Proposition 6 *The value of the multicompound call option F_N with maturity T_N and strike price I_N written on a compound call option F_{N-1} with maturity T_{N-1} and strike price I_{N-1} is given by:*

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1}\left(\lambda \tau_1\right)^{n_1}}{n_1!}\left[V_0 N_N\left(a_N, .., a_1; \Xi_N^{(N)}\right)\right]+$$

$$-\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_j=0}^{\infty} \frac{e^{-\lambda \tau_j}\left(\lambda \tau_j\right)^{n_j}}{n_j!} \times$$

$$\left[\sum_{j=1}^{N} I_j e^{-rT_j} N_{N+1-j}\left(b_N, .., b_j; \Xi_{N+1-j}^{(N)}\right)\right], \quad j = 1, .., N;$$

where the a_is, the b_is and the ρ_{ij}s are as defined previously.

Remark 7 *It can be proved that $\partial_V F_k = N_k(a_k, ..., a_1; \Xi_k^{(k)})$. Thus uniqueness of V_k^* is guaranteed for every k, $1 \leq k \leq N$.*

In the particular case when $\lambda = 0$, the formula reduces to Agliardi and Agliardi (2005) . Note that the value of the compound option in the square brackets is conditional on knowing exactly the number of Poisson jumps occurring during the life of the option. Clearly, the actual value of the multicompound option, F_N, is just the weighted sum of the prices of the option where each weight equals the joint probability that N Poisson random variables with characteristic parameters λt, will take on the value n.

This proposition is the main result of the paper and forms the basis for the valuation of sequential investment opportunities, as for example R&D ventures, including the possibility of jumps in the underlying asset value.

3.5 An extension

In Carr [9] sequential exchange opportunities are valued using the techniques of modern option-pricing theory. The vehicle for analysis is the concept of compound exchange option. Accordingly, the real option literature has suggested that sequential expansion opportunities can be viewed as compound exchange options. Trigeorgis (1996) highlighted that many new business ventures, as R&D and start-up projects, can be seen as the base-scale projects plus an option to make additional investments. For example, the opportunities for a firm to continuously expand its technology represents a critical component of the software providing industry's investment decisions. The firms' ability to later expand capacity is clearly more valuable for more volatile business with higher returns on project, such as computer software or biotechnology, than it is for traditional business, as real estate or automobile production. Nevertheless, the value of these early investments is generally subject to considerable uncertainty, because of their dependence on future events that are uncertain at the time the base-scale takes place. Market factors outside the control of the firm change continuously and have considerable effect on the value of these investment opportunities. Moreover, when the new software product comes together with technological innovations, there is also considerable uncertainty with respect to the actions of a competitor or changes in environment before or soon after technological improvements. For example, a software product may fail because of technological advances in hardware.

In this section we attempt to evaluate sequential technology adoptions as in Carr (1988). As before, we could relax the assumption of a pure diffusion process for the underlying asset value, to illustrate the case where new technology competitors arrive randomly according to an exogenous Poisson distribution. A pricing-formula for multicompound exchange option with jump-diffusion process is obtained.

3.5.1 The mathematical problem and solution

Since this problem and its solution are extensions of the multicompound call option formula, I will use the same notation and assumptions as much as possible. We consider the valuation of a European sequential exchange option $F_k(V_1, V_2, t)$ which can be exercised at T_k, where $T_1 \geq T_2 \geq ... \geq T_N$. Assume that the prices of both assets follow the same stochastic differential equation (1). Let φ_{12} denote the correlation coefficient between the Wiener processes dz_1 and dz_2; dq_i and dz_i are assumed to be independent as well dq_i and dq_j, $i, j = 1, 2$. As suggested by Margrabe (1978), the valuation problem can be reduced to that of a one-asset option if we treat V_1 as numeraire. Accordingly, we define a new random variable $V = \frac{V_2}{V_1}$, which is again lognormal. The option sells for $F_k(V_1, V_2, t)/V_1 = W_k(V, t)$. The risk-free rate in this market is zero. The functional governing the multicompound exchange option's value $W_k(W_{k-1}(V, T_k), T_k)$ is known at expiration to be $\max(W_{k-1}(V, T_k) - q_k, 0)$ where q_k is the exchange ratio of the option[10]. This problem is analogous to that of section 4 if we treat q_k as the exercise price of the option. Our aim is to derive a valuation formula for the N-fold multicompound exchange option, that is for $W_N(V,t)$, $0 \le t \le T_N$. Let V_N^* denote the value of V such that $W_{N-1}(V, T_N) - q_N = 0$ and $V_1^* = q_1$. To simplify notation we will assume again $t = 0$. Let us define now:

$$b_k' = \frac{\ln\left(\frac{V}{V_k^*}\right) - \frac{\nu^2}{2} T_k}{\nu\sqrt{T_k}}, \tag{3.14}$$

and:

$$a_k' = b_k' + \nu\sqrt{T_k}, \tag{3.15}$$

where $\nu^2 = \nu_1^2 - 2\varphi_{12}\nu_1\nu_2 + \nu_2^2$; Finally, we set ρ_{ij} as in (3.5).

The current value of the multicompound exchange option W_N follows by:

$$E_0^{\mathbb{Q}}\left[(V - q_1) 1_{\varepsilon_1}..1_{\varepsilon_N}\right] + \sum_{j=2}^{N} E_0^{\mathbb{Q}}\left[(-q_j) 1_{\varepsilon_j}..1_{\varepsilon_N}\right]. \tag{3.16}$$

The derivation of the pricing formula is standard. We assume that the random variable Y has the same log-normal distribution as we described before. In this case the logarithmic return x_{T_k} will have a normal distribution with mean equals $\eta T_k = \left(r - \frac{\nu^2}{2}\right) T_k$ and variance equals $\nu^2 T_k = \sigma^2 T_k + s_k \sigma_J^2$. The evaluation of the first expectation in (3.16) requires the calculation of the joint probability function of N independent Poisson processes with rate λt. Solving as in $(3.7) - (3.9)$, we obtain:

$$\sum_{n_n=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} (\lambda \tau_1)^{n_1}}{n_1!} \times$$

[10] As in Carr (1988) the exchange ratio q is taken to be constant or, at most, a deterministic function of time.

$$\left\{ \int_{-\infty}^{b_N} N'(y) \left(\tilde{g}\,(y)\, N_{N-1} \left(\tilde{a}_{N-1}, .., \tilde{a}_1; \hat{\Xi}_{N-1}^{(N-1)} \right) \right) dy + \right.$$

$$\left. - \int_{-\infty}^{b_N} N'(y) \left(q_1 N_{N-1} \left(\tilde{b}_{N-1}, .., \tilde{b}_1; \hat{\Xi}_{N-1}^{(N-1)} \right) \right) dy \right\}, \tag{3.17}$$

where $\tilde{g}\,(y)$ equals (3.11), $\tilde{a}_k$ equals (3.12) and $\tilde{b}_k$ equals (3.13) if $r = 0$. The calculation of the second integral in (3.16) is straightforward. Finally, in light of Lemma 1, we obtain the following:

Proposition 8 *The value of the sequential exchange option F_N with maturity T_N and strike price q_N written on a exchange option F_{N-1} with maturity T_{N-1} and strike price q_{N-1} is given by:*

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} (\lambda \tau_1)^{n_1}}{n_1!} \left[V_{02} N_N \left(a'_N, .., a'_1; \Xi_N^{(N)} \right) \right] +$$

$$- \sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} .. \sum_{n_j=0}^{\infty} \frac{e^{-\lambda \tau_j} (\lambda \tau_j)^{n_j}}{n_j!} \times$$

$$\left[V_{01} \sum_{j=1}^{N} q_j N_{N+1-j} \left(b'_N, .., b'_j; \Xi_{N+1-j}^{(N)} \right) \right], \quad j = 1, .., N;$$

where the a'_is, the b'_is and the ρ_{ij}s are as defined previously.

Of course, when $\lambda = 0$, the formula reduces Carr (2005).

3.6 Numerical Results

In this section, we illustrate our model by providing numerical results for two different type of compound options. First, consider Geske's (1979) compound option formula in which the underlying asset dynamics is modelled by a standard geometric Brownian motion. Next, we consider the compound option formula derived in section 4, where the underlying asset price follows a mixed jump-diffusion process. We assume for the latter process that the proportional jump size has a lognormal distribution. For comparison purposes we consider a simple compound option with two strike prices and two expiration dates.

Our model is implemented as follows. Consider a European compound call option written on a European call option. The corresponding values computed using Geske's closed-form solution are reported in Table I under the heading diffusion[11]. This model makes use of the standard Black-Scholes formula for the critical prices computation at the first exercise date, T_1[12], and also uses the bivariate normal distribution for treating two correlated random processes. Conversely, the implementation of the compound option formula with jump-diffusion process makes use of the Merton formula for determining the critical prices at the first exercise date and also uses the bivariate discrete distribution, in addition to the bivariate normal distribution, for treating two uncorrelated Poisson processes. In the computation of this formula we further truncate the sums neglecting higher order terms of the Poisson distribution, where we require that the error committed in this way is lower than an arbitrary taken value. The results of the model using the jump-diffusion compound option formula are reported in Table I under the heading jump-diffusion. Further, we can indicate these two types of compound options by $F_2\left(\sigma^2\right)$ and $F_2\left(\sigma^2, \lambda, \sigma_J^2\right)$, respectively.

As in Shaw (1998) we assume that the compound option has expiration date $T_1 = 0.25$ and strike prices $X_1 = \{5,\ 7.5,\ 10\}$; the call option has expiration date $T_2 = 1$ and strike price $X_2 = 100$. The risk-free rate of interest is fixed at $r = 0.10$, and the dividend yield[13] is $\delta = 0.05$, in annual terms. The annual variance of the diffusion component is $\sigma^2 = 0.04$, and the variance of the asset price return due to each jump occurrence is $\sigma_J^2 = 0.04$. The initial asset value is $V_0 = 100$.

Table I displays the difference between the compound option $F_2\,(0.04)$ and the compound option $F_2\left(0.04, \lambda, \sigma_J^2\right)$ corresponding to different values of the mean number of abnormal information arrivals.

Notice that for moderately large value of λ, significant differences prevail between the Geske value of the compound option $F_2\,(0.04)$, and the

[11] See Shaw (1998) for the corresponding computations.

[12] Contrarily to the assumption in section 3, we assume that the first exercise date is T_1 and the second exercise date is T_2, with $T_1 < T_2$.

[13] See Dixit and Pindyck (1994) and Trigeorgis (1996) for a detailed discussion . Here, δ is the shortfall in the expected rate of return from holding the option to complete rather than the completed project. Note that the dividend yield is assumed to be zero in section 3.

value of the jump-diffusion compound call option $F_2\left(0.04, \lambda, \sigma_J^2\right)$. Further, the actual value of the compound call option $F_2\left(0.04, \lambda, \sigma_J^2\right)$ gets monotonically close to the value computed by using Geske's formula, $F_2(0.04)$, for λ sufficiently small.

TABLE I

VALUE OF THE SEQUENTIAL INVESTMENT OPPORTUNITIES $F_2\left(0.04, \lambda, \sigma_J^2\right)$ AND $F_2(0.04)$.

$V = 100,\ X_2 = 100,\ \sigma = 0.2,\ r = 0.1,\ \delta = 0.05,\ T_1 = 0.25,\ T_2 = 1;$

Diffusion

	$X_1 = 5;$	$X_1 = 7.5;$	$X_1 = 10;$
	5.30918	3.6633	2.44416

$V = 100,\ X_2 = 100,\ \sigma = 0.2,\ \sigma_J = 0.2,\ r = 0.1,\ \delta = 0.05,\ T_1 = 0.25,\ T_2 = 1;$

Jump − Diffusion

λ	$X_1 = 5;$	$X_1 = 7.5;$	$X_1 = 10;$
1.0	7.90894	5.98219	4.41552
1.0×10^{-1}	5.58408	3.89883	2.6387
1.0×10^{-2}	5.33678	3.68685	2.46356
1.0×10^{-3}	5.31194	3.66565	2.4461
1.0×10^{-4}	5.30946	3.66353	2.44435
1.0×10^{-5}	5.30921	3.66332	2.44418
1.0×10^{-6}	5.30918	3.6633	2.44416
1.0×10^{-7}	5.30918	3.6633	2.44416

We graphically summarize this result in Figure 1 where the compound option values, $F_2\left(\sigma^2, \lambda, \sigma_J^2\right)$, are represented as a function of the mean number of abnormal information arrivals, λ.

Table II displays the compound option values $F_2\left(0.04, \lambda, \sigma_J^2\right)$ corresponding to different values of the parameter λ, and increasing values of the standard deviation, σ_J. In particular, numerical evidence demonstrates that increasing values of σ_J will increase the value the compound option $F_2\left(0.04, \lambda, \sigma_J^2\right)$. For example, if $\lambda = 1.0 \times 10^{-1}$ and $X_1 = 5$, then an increase in σ_J from 0.2 to 0.3 will increase the value of the compound option from 5.58408 to 5.83585. Finally, the actual value of the compound option $F_2\left(0.04, \lambda, \sigma_J^2\right)$ gets monotonically close to $F_2(0.04)$ for λ sufficiently small and σ_J sufficiently large.

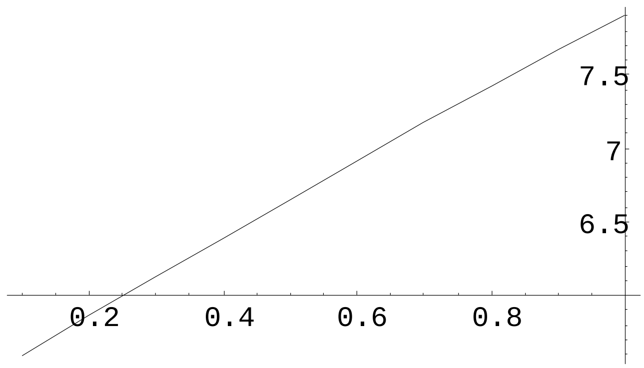

Figure 3.1: Compound option value, $F_2\left(\sigma^2, \lambda, \sigma_J^2\right)$, as a function of the mean number of abnormal information arrivals, λ, per unit time. Parameter values: $V_0 = 100$, $X_2 = 100$, $X_1 = 5$, $\sigma = 0.2$, $\sigma_J = 0.2$, $r = 0.1$, $\delta = 0.05$, $T_1 = 0.25$, $T_2 = 1$.

TABLE II

VALUE OF THE SEQUENTIAL INVESTMENT OPPORTUNITY $F_2\left(0.04, \lambda, \sigma_J^2\right)$ FOR DIFFERENT VALUES OF σ_J^2.

$V = 100$, $X_1 = 5$, $X_2 = 100$, $\sigma = 0.2$, $r = 0.1$, $\delta = 0.05$, $T_1 = 0.25$, $T_2 = 1$

		$X_1 = 5$;	$X_1 = 7.5$;	$X_1 = 10$;
λ	σ_J			
1.0	0.2	7.90894	5.98219	4.41552
1.0×10^{-1}	0.3	5.83585	4.11133	2.8121
1.0×10^{-2}	0.4	5.3896	3.73093	2.49933
1.0×10^{-3}	0.5	5.32009	3.67244	2.4516
1.0×10^{-4}	0.6	5.31056	3.66445	2.4451
1.0×10^{-5}	0.7	5.30935	3.66332	2.44418
1.0×10^{-6}	0.8	5.3092	3.66331	2.44417
1.0×10^{-7}	0.9	5.30918	3.6633	2.44416

Notice that this result is in part similar to numerical evidence in Ball and Tourus (1985) where the Merton value of the call option gets arbitrarily close to the Black-Scholes call option value for λ sufficiently small and σ_J^2 sufficiently large, although not monotonically.

We have shown the effect of changing the parameters λ and σ_J, while keeping all the other parameters constants. The comparative static experiment in Table III is different from that in Table I and II. Table III demonstrates that the value of the sequential investment option is sensitive to changes in the parameters r, σ, δ, T_1, T_2 and V. Responses of the jump-diffusion compound option price to changes in the values of the parameters are essentially maintained with respect to ones in Geske (1979).

For the base case in table below, we set $V = 100$, $X_1 = 5$, $X_2 = 100$, $\sigma = 0.2$, $\sigma_J = 0.2$, $\lambda = 1$, $r = 0.1$, $\delta = 0.05$, $T_1 = 0.25$ and $T_2 = 1$. Table II displays the compound option price F_2 corresponding to these values of the parameters: 7.90894.

(1) Increases in the interest rate r rise the value of the investment opportunity; the sensitivity analysis shows that if r increases to 0.15 the value of the option will increase to 10.0383.

TABLE III

VALUE OF THE SEQUENTIAL INVESTMENT OPPORTUNITY F_2 FOR A WIDE RANGE OF THE PARAMETERS

	$T_1 = .25$	$T_1 = .375$	$T_1 = .5$	$T_2 = 1.5$	$T_2 = 2$	$T_2 = 3$
r						
0.1	7.90894	8.23688	8.59035	10.7385	13.1214	16.8632
0.15	10.0383	10.3217	10.6422	13.8643	17.1101	22.2905
0.2	12.2563	12.5083	12.7996	17.067	21.1544	27.6681
σ						
0.225	8.60684	8.93657	9.29443	11.5371	13.9925	17.826
0.25	9.32774	9.65873	10.0206	12.3681	14.9021	18.8338
0.3	10.8258	11.1582	11.5273	14.1066	16.8101	20.9511
δ						
0.075	6.64528	7.01039	7.3868	8.75235	10.4469	12.886
0.1	5.53617	5.93507	6.32969	7.02418	8.14707	9.56381
0.15	3.74455	4.19154	4.60508	4.28441	4.56537	4.60318
V						
90	3.26294	3.71819	4.13237	5.46279	7.58267	11.1439
110	14.3177	14.5112	14.7556	17.2002	19.5669	23.2261

Note : Entries are calculated using equation for F_2 in the text. Base case parameters are $V = 100$, $X_1 = 5$, $X_2 = 100$, $\sigma = 0.2$, $\sigma_J = 0.2$, $\lambda = 1$, $r = 0.1$, $\delta = 0.05$, $T_1 = 0.25$, $T_2 = 1$.

(2) Increases in σ rise the value of the option; for example an increase of the standard deviation from 0.2 to 0.225 will increase the value of the sequential investment from 7.90894 to 8.60684.

(3) Postponing the exercise dates of the investments causes the value of the option to rise, as is traditional in option pricing; indeed if $r = 0.1$ and the maturity date of the first stage investment, T_1, is delayed from 0.25 to 0.375 of a year, the value of the option will increase from 7.90894 to 8.23688. Moreover, if the maturity date of the second stage investment, T_2, is postponed from 1 to 2 years, the value of the option will increase from 7.90894 to 13.1214.

(4) The value of the option decreases as δ increases. In Table III, F_2 reduces from 7.90894 to 6.64528 as δ goes from 0.05 to 0.075 (when $T_1 = 0.25$).

(5) The option value is also sensitive to change in V from 100 to 110. This sensitivity results in an increase of the option price from 7.90894 to 14.3177.

3.7 Final remarks

R&D and similar investment projects have the property that much of the value of the investment is associated with future cash flows that are contingent on intermediate decisions. Due to this property the analysis of R&D ventures and start-up companies is one of the most difficult investment problem. Starting from the well known difficulty of traditional DCF methods to capture the value of new ventures, the real options literature provides advanced models which focus on different R&D characteristics.

This paper deals with compoundness of R&D investment project and analyses the impact of different sources of uncertainty on its valuation. To do it, the paper develops a real options approach where the R&D process requires $N-$rounds of investment that are analogous to the exercise prices of a multicompound option. Furthermore, the paper assumes that the underlying asset is subject both to market and technical uncertainty: the former is generally related to economic fundamentals and always driving the value of a project, while the latter is idiosyncratic to the firm and associated with the success of the venture itself. These sources of risk are modelled through the assumption that the underlying asset follows a jump-diffusion process.

The paper is organized as follows:

First, we obtain a closed-form solution for multicompound option which allows for different sources of uncertainty. In the process, the R&D venture is subject to two types of risk: (1) uncertainty associated the potential future cash flows the project will produce if completed; this is represented by a standard diffusion process, punctuated by (2) jumps at random intervals. By assuming that the jump risk is completely diversifiable and that the distribution of jump size is lognormal, a European-style multicompond option which extends the work of Agliardi and Agliardi (2005) can be priced according to the risk-neutral valuation method.

Second, we obtain a closed-form solution for multicompound exchange options with a jump-diffusion process which extends the work of Carr (1988) to real options problem when crises and shocks create discontinuities on the investments process.

Third, we show through numerical implementation, that the resultant equilibrium option price will systematically differ from that obtained in Geske (1979) ; particularly, comparative statics results confirm that increasing value of the parameters in the jump component will increase the value of the compound option and leads to an R&D investment appraisal which does not use to undervalue it.

A final remark needs to be mentioned. The existing research contributions in the real options field has previously analyzed various aspects of optimal sequential investment behavior for firm facing multi-stage projects. The focus of these articles is on optimal investment characteristics with respect to single firm's investment decisions. However, the impact of competitive pressure is an important feature to take into account when dealing with R&D investments. Hence, a multi-stage investments appraisal should rely on a game-theoretic analysis of R&D projects exercise strategies. We leave it for further research.

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