Classical limit of the Nelson model

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CHAPTER 1

Introduction.

Since the development of quantum mechanics it has been natural to analyze the connection between classical and quantum mechanical descriptions of physical systems. In particular one should expect that in some sense when quantum mechanical effects becomes negligible the system will behave like it is dictated by classical mechanics. One famous relation between classical and quantum theory dates back to early days of quantum mechanics and it is due to Ehrenfest [**Ehr27**]. This result was later developed and put on firm mathematical foundations by Hepp [**Hep74**]. He proved that matrix elements of bounded functions of quantum observables between suitable coherents states (that depend on Planck's constant h) converge to classical values evolving according to the expected classical equations when $h \to 0$. Furthermore he also provides information about the quantum fluctuations of the system in the classical limit: their dynamics is obtained linearizing quantum evolution equation around the classical solution. His results were later generalized by Ginibre and Velo [**GV79, GV80**] to bosonic systems with infinite degrees of freedom and scattering theory. Recently Ginibre, Nironi and Velo applied this method to perform a partially classical limit of the Nelson model [**GNV06**] where only the number of relativistic particles goes to infinity while the number of non-relativistic particles remain fixed. Even more recently some authors used the results of [**GV79**] to obtain estimates on the rate of convergence of transition amplitudes of normal ordered products of creation and annihilation operators in the mean field limit of bosonic systems; it has first been done by Rodnianski and Schlein [**RS09**] and then refined by Chen and Lee [**CL11**] and by Chen, Lee and Schlein [**COS11**].

In this work we will analyze the complete classical limit of the Nelson model with cut off, when both non-relativistic and relativistic particles number goes to infinity. We will prove convergence of quantum observables to the solutions of classical equations, and find the evolution of quantum fluctuations around the classical solution. Furthermore we analyze the convergence of transition amplitudes of normal ordered products of creation and annihilation operators between different types of initial states. Below we describe the main setting of the problem and give a heuristic preview of the results, stripped from most technicalities.

1. Introduction to Fock Space.

Since the theory we want to study is set in a Fock space, we will introduce its basic notions here. The Fock space and second quantization were introduced by Vladimir Fock [**Foc32**] and put on a firm mathematical basis by Cook [**Coo51**]. Let $\mathscr{H}$ be a Hilbert space, and denote $\mathscr{H}_n$ its n-fold tensor product $\mathscr{H}_n = \mathscr{H} \otimes \ldots \otimes \mathscr{H}$. Now on $\mathscr{H}_n$ define S_n the symmetrizing operator. In the case where $\mathscr{H} = L^2(\mathbb{R})$, $S_n\mathscr{H}_n$ is the subspace of $L^2(\mathbb{R}^n)$ of all functions invariant under any permutation of the variables. Set $\mathscr{H}_0 = \mathbb{C}$, we define the symmetric Fock space over $\mathscr{H}$:

$$\mathscr{F}_s(\mathscr{H}) = \bigoplus_{n=0}^{\infty} S_n\mathscr{H}_n \ .$$

The basic operators of the Fock space are the annihilation and creation operators. We will define them only for $\mathscr{H} = L^2(\mathbb{R})$; formally we introduce the following operator-valued distributions, defined by their action on $\mathscr{H}_n$:

$$(a(x)\Phi)_n(x_1,\dots,x_n) = \sqrt{n+1}\,\Phi_{n+1}(x,x_1,\dots,x_n)$$
$$(a^*(x)\Phi)_n(x_1,\dots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\delta(x-x_j)\Phi_{n-1}(x_1,\dots,\hat{x}_j,\dots,x_n)\ ,$$

where $\hat{x}_j$ indicates this variable has been omitted. Then for any $f \in L^2(\mathbb{R})$ the annihilation and creation operators are defined as:

$$a(f) = \int \mathrm{d}x\, f(x)a(x)$$
$$a^*(f) = \int \mathrm{d}x\, f(x)a^*(x)\ .$$

Formally a and a^* satisfy the following commutation relation:

$$[a(x), a^*(x')] = \delta(x-x')$$
$$[a(x), a(x')] = [a^*(x), a^*(x')] = 0\ .$$

For a more rigorous definition of these operators and a list of useful properties the reader can consult the Appendix. Another useful operator is the number particle operator

$$N = \int \mathrm{d}x\, a^*(x)a(x)\ .$$

This operator "counts the number of particles" of a state in the Fock space, in fact acting on a vector $\Phi_n \in \mathscr{H}_n$ we have

$$N\Phi_n = n\Phi_n\ .$$

2. The Quantum Theory.

The system we want to study has been introduced in physics and was called the polaron model; in particular it has been discussed by Gross [**Gro62**]. In mathematical physics it was introduced by Edward Nelson [**Nel64**], and still is referred to as the Nelson model. He used it to describe the mathematical existence of a theory of non-relativistic nucleons interacting with a meson field. However recent developments in condensed matter theory, especially regarding the so-called optical lattices, showed this model could be used also to describe systems of bosons trapped in an electromagnetic field.

We will call $\mathscr{H}$ the Hilbert space of the theory, and it is the tensor product of two symmetric Fock spaces over $L^2(\mathbb{R}^3)$. Define

$$\mathscr{H}_{p,n} = \left\{\Phi_{p,n} :\ \Phi_{p,n}(X_p;K_n) \in L^2(\mathbb{R}^{3p+3n})\right\}\ ,$$

where $X_p = \{x_1,\dots,x_p\}$, $K_n = \{k_1,\dots,k_n\}$ and $\Phi_{p,n}$ is separately symmetric with respect to the first p and the last n variables. So we have that

$$\mathscr{H} = \bigoplus_{p,n=0}^{\infty} \mathscr{H}_{p,n}\ .$$

The vacuum state will be denoted by Ω (namely the unit vector of $\mathscr{H}_{0,0}$). We will use freely the following properties of the tensor product of Hilbert spaces

$$\mathscr{H}_{p,n} = \mathscr{H}_{p,0} \otimes \mathscr{H}_{0,n}\ ,$$

and

$$\mathscr{H} = \Big(\bigoplus_{p=0}^{\infty} \mathscr{H}_{p,0}\Big) \otimes \Big(\bigoplus_{n=0}^{\infty} \mathscr{H}_{0,n}\Big) = \bigoplus_{p=0}^{\infty} \mathscr{H}_p \ ,$$

with

$$\mathscr{H}_p = \mathscr{H}_{p,0} \otimes \bigoplus_{n=0}^{\infty} \mathscr{H}_{0,n} \ .$$

We will call $\psi^{\#}(f)$ the annihilation and creation operators corresponding to the p Fock space, $a^{\#}(f)$ the ones corresponding to the n Fock space (the $^{\#}$ stands for either nothing or *). On $\mathscr{H}_p$ we define the following operators: let $f \in L^{\infty}(\mathbb{R}^{3p}, L^2(\mathbb{R}^3))$, then:

$$(b(f)\Phi)_{p,n}(X_p;K_n) = \sqrt{n+1}\int \mathrm{d}k\, f(X_p;k)\Phi_{p,n+1}(X_p;k,K_n)$$
$$(b^*(f)\Phi)_{p,n}(X_p;K_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(X_p;k_j)\Phi_{p,n-1}(X_p;K_n \setminus k_j) \ .$$

The Hamiltonian operator describing our system, the Nelson model with cut off, is formally the following:

$$H = \frac{1}{2M}\int \mathrm{d}x\, (\nabla\psi)^*(x)\nabla\psi(x) + \int \mathrm{d}k\, \omega(k)a^*(k)a(k)$$
$$+\lambda\int \mathrm{d}x\, \varphi(x)\psi^*(x)\psi(x) \ ; \ \omega(k) = \sqrt{k^2+\mu^2},$$

$M > 0$, $\mu \geq 0$ and $\lambda > 0$. Furthermore:

$$\varphi(x) = \int \frac{\mathrm{d}k}{(2\pi)^{3/2}}\frac{1}{(2\omega)^{1/2}}\chi(k)\Big(a(k)e^{ikx} + a^*(k)e^{-ikx}\Big) \text{ with}$$
$$\chi(k) = \begin{cases} 1 \text{ if } |k| \leq \sigma \\ 0 \text{ if } |k| > \sigma \end{cases} .$$

It is the sum of two bilinear terms, representing the free evolution respectively of a Schrödinger and a Klein Gordon field, and a trilinear Yukawa-type interaction between them. The cut off χ in the interaction guarantees that the theory is well defined even for large momenta of the Klein Gordon field. As we will see the closure of this operator is self-adjoint on a suitable domain. We will call $U(t)$ the evolution generated by H, $U_0(t)$ the evolution generated by the free part H_0:

$$H_0 = \frac{1}{2M}\int \mathrm{d}x\, (\nabla\psi)^*(x)\nabla\psi(x) + \int \mathrm{d}k\, \omega(k)a^*(k)a(k) \ .$$

Let u and $\alpha \in L^2(\mathbb{R}^3)$, then we define the Weyl operators as

$$C(u,\alpha) = \exp\big\{\big(\psi^*(u) - \psi(\bar{u})\big) + \big(a^*(\alpha) - a(\bar{\alpha})\big)\big\} \ .$$

These operators, when acting on the vacuum state Ω of the Fock space, generate the so-called coherent states. Formally the average of $\psi^{\#}(x)$ and $a^{\#}(k)$ on such states is:

$$\langle C(u,\alpha)\Omega, \psi^{\#}(x)C(u,\alpha)\Omega\rangle = u^{\#}(x)$$
$$\langle C(u,\alpha)\Omega, a^{\#}(k)C(u,\alpha)\Omega\rangle = \alpha^{\#}(k) \ .$$

The Weyl operators are treated in greater detail and with mathematical rigour in the Appendix. We write here a formula to be used later in this introduction:

$$C(u,\alpha)^*\psi(x)C(u,\alpha) = \psi(x) + u(x)$$
$$C(u,\alpha)^*a(k)C(u,\alpha) = a(k) + \alpha(k) \ .$$

3. The classical equations.

We use the following convention for the Fourier transform $\hat{f}$ of f:

$$\hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x \, e^{-ikx} f(x) \ ;$$

then the inverse transform $\check{f}$ such that $\hat{\check{f}} = f = \check{\hat{f}}$ is

$$\check{f}(k) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x \, e^{ikx} f(x) \ .$$

In the Heisenberg picture the quantum evolution of the fields $\psi(t)$ and $a(t)$ is given by

$$\begin{cases} i\partial_t \psi(t) = [\psi(t), H] \\ i\partial_t a(t) = [a(t), H] \end{cases}$$

which can be explicitly written as

$$\begin{cases} i\partial_t \psi = -\dfrac{1}{2M}\Delta\psi + \lambda\varphi\psi \\ i\partial_t a = \omega a + \lambda \dfrac{\chi}{\sqrt{2\omega}} (\widehat{\psi^*\psi}) \end{cases} . \tag{3.1}$$

Let s be the initial time, we will write $\psi(s) = \psi_0$, $a(s) = a_0$. $\psi_0^\#$ and $a_0^\#$ are the usual creation and annihilation operators of the Fock space we defined above, however when we consider evolution in the Heisenberg picture we will use this notation to avoid confusion with the time evolved operators $\psi^\#(t)$ and $a^\#(t)$ (since sometimes we omit the explicit dependence on t, as in the equation above).

We will denote by $u(x)$ and $\alpha(x)$ the classical counterparts of ψ and a respectively, in a sense that will be explained in the following section of this introduction. Classical evolution is then dictated by the following system of equations:

$$\begin{cases} i\partial_t u = -\dfrac{1}{2M}\Delta u + (2\pi)^{-3/2}(\check{\chi} * A)u \\ i\partial_t \alpha = \omega\alpha + (2\pi)^{-3/2}(2\omega)^{-1/2}\check{\chi} * |u|^2 \end{cases} , \tag{3.2}$$

where $A = (2\omega)^{-1/2}(\alpha + \bar{\alpha})$ and for all q

$$(\omega^q\alpha)(x) = (2\pi)^{-3/2} \int \mathrm{d}\xi \, e^{i\xi x} (\mu^2 + |\xi|^2)^{q/2} \hat{\alpha}(\xi) \ .$$

To be precise $\alpha(x)$ is the classical correspondent of $\hat{a}$, the Fourier transform of a.

4. The classical limit.

As we said we want to study the behaviour of the system in the classical limit. This limit can be thought as a mean field limit, when the number of non-relativistic particles and excitations of the relativistic field tend to infinity. We expect that in the limit the wave function describing a nonrelativistic particle is coupled with the classical field. In order to see that is the case, we need to choose a suitable series of states to average the quantum fields, such that the number of both particles increases to infinity. We will use the Weyl operators introduced before. If p and

n are positive integers, the sequence of operators $C(p^{1/2}u, n^{1/2}\alpha)$ applied to any fixed state Φ is such that

$$\langle C(p^{1/2}u, n^{1/2}\alpha)\Phi, \psi^{\#}(f)C(p^{1/2}u, n^{1/2}\alpha)\Phi\rangle \sim p^{1/2}$$
$$\langle C(p^{1/2}u, n^{1/2}\alpha)\Phi, a^{\#}(f)C(p^{1/2}u, n^{1/2}\alpha)\Phi\rangle \sim n^{1/2} \ ,$$

that goes to infinity in a suitable way when the number of particles p and n goes to infinity (since the annihilation and creation operators behave like the square root of the number particle operator in Fock spaces). However, in order to obtain a non trivial limiting equation for (3.1) when $p, n \to \infty$ we need to relate λ to p and n, according to $p = n = \lambda^{-2}$. So the mean field limit is also a weak coupling limit. From now on we will use $\lambda \to 0$ as the parameter to perform the classical limit. We want to find the classical counterparts of ψ and a, however their average over any fixed state goes to infinity in the classical limit; on the other hand the averages of $\lambda\psi$ and $\lambda\alpha$ have a finite limit when $\lambda \to 0$, so we expect such operators to have classical limits.

In the following we will explain how to find such limit. Let $\delta^* > 0$, $\Phi \in D((P+N)^{\delta^*})$ a state that does not depend on λ such that $\|\Phi\| = 1$; $u, \alpha, f \in L^2(\mathbb{R}^3)$. We call $(u(t), \alpha(t))$ the solution of (3.2) with initial data (u, α) and define $u_\lambda \equiv u/\lambda$, $\alpha_\lambda \equiv \alpha/\lambda$. Then we know that at time zero

$$\langle C(u_\lambda, \alpha_\lambda)\Phi, \lambda\psi^{\#}(\bar{f}^{\#})C(u_\lambda, \alpha_\lambda)\Phi\rangle = \langle f, u\rangle^{\#} \ .$$

What happens if we introduce evolution in time, as dictated by Nelson's hamiltonian? We have to study

$$\langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^{\#}(\bar{f}^{\#})U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle$$

in the limit $\lambda \to 0$. We use the following equality, valid for all $y, z \in L^2(\mathbb{R}^3)$:

$$\begin{aligned} U^\dagger(t)\psi^{\#}(\bar{f}^{\#})U(t) &= U^\dagger(t)C(y,z)C^\dagger(y,z)\psi^{\#}(\bar{f}^{\#})C(y,z)C^\dagger(y,z)U(t) \\ &= U^\dagger(t)C(y,z)\big(\psi^{\#}(\bar{f}^{\#}) + \langle f, y\rangle^{\#}\big)C^\dagger(y,z)U(t) \ ; \end{aligned}$$

to obtain

$$\begin{aligned} &\langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^{\#}(\bar{f}^{\#})U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle - \langle f, y\rangle^{\#} \\ &= \lambda\langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)C(y_\lambda, z_\lambda)\psi^{\#}(\bar{f}^{\#})C^\dagger(y_\lambda, z_\lambda)U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle \ . \end{aligned}$$

Taking the absolute value and using Schwarz's inequality we obtain:

$$\begin{aligned} &\big|\langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^{\#}(\bar{f}^{\#})U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle - \langle f, y\rangle^{\#}\big| \\ &\qquad \leq \lambda \big\|\psi^{\#}(\bar{f}^{\#})C^\dagger(y_\lambda, z_\lambda)U(t)C(u_\lambda, \alpha_\lambda)\Phi\big\| \ ; \end{aligned}$$

then using standard estimates (proved in appendix) of creation and annihilation operators we have

$$\begin{aligned} &\big|\langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^{\#}(\bar{f}^{\#})U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle - \langle f, y\rangle^{\#}\big| \\ &\qquad \leq \lambda \|f\|_2 \Big\|(P+1)^{1/2}C^\dagger(y_\lambda, z_\lambda)U(t)C(u_\lambda, \alpha_\lambda)\Phi\Big\| \\ &\leq \lambda \|f\|_2 \Big\|(P+N+1)^{1/2}C^\dagger(y_\lambda, z_\lambda)U(t)C(u_\lambda, \alpha_\lambda)\Phi\Big\| \ . \end{aligned}$$

We remark that the left hand side of last inequality is a linear functional of $L^2(\mathbb{R}^3)$ applied to f; and that on right hand side f appears only as $\|f\|_2$. Then by Riesz's Lemma we can deduce that

$$\begin{aligned} &\big\|\langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^{\#}(\cdot)U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle - y^{\#}(\cdot)\big\|_{L^2(\mathbb{R}^3)} \\ &\qquad \leq \lambda \Big\|(P+N+1)^{1/2}C^\dagger(y_\lambda, z_\lambda)U(t)C(u_\lambda, \alpha_\lambda)\Phi\Big\| \ . \end{aligned}$$

If we can bound, in a suitable way in λ, the norm on the right hand side of this inequality we are done. However for general y and z we are able only to give an estimate of the type

$$\left\|(P+N+1)^{1/2}C^\dagger(y_\lambda,z_\lambda)U(t)C(u_\lambda,\alpha_\lambda)\Phi\right\| \sim \lambda^{-6}\left\|(P+N+1)^2\Phi\right\| .$$

That would lead to a divergent quantity in the limit $\lambda\to 0$. Only for a particular choice of (y,z) we are able to obtain a bound convergent in λ. *We have to set* $(y,z)=(u(t),\alpha(t))$ *solution of the classical equations.* In fact define

$$W(t,s)=C^\dagger(u_\lambda(t),\alpha_\lambda(t))U(t-s)C(u_\lambda(s),\alpha_\lambda(s))e^{i\Lambda(t,s)} ,$$

with

$$\Lambda(t,s)=-\frac{1}{\lambda^2}\int_s^t \mathrm{d}t'\int \mathrm{d}x\,(\check{\chi}*A)(t')\bar{u}(t')u(t') ;$$

then we are able to prove estimates of the type

$$\left\|(P+N+1)^\delta W(t,s)\Phi\right\|^2 \leq K_1(t,s)(1+\lambda)e^{\lambda|t-s|+K_2(t,s)} \left\|(P+N+1)^{6\delta+3/2}\Phi\right\|^2 ;$$

where $K_1(t,s)$ and $K_2(t,s)$ are independent of λ and δ is integer. So if we choose $\delta^*\geq 6+3/2$, then $\Phi\in D((P+N)^{6+3/2})$ and we obtain

$$\begin{aligned}&\left\|\langle C(u_\lambda,\alpha_\lambda)\Phi,U^\dagger(t)\lambda\psi^\#(\cdot)U(t)C(u_\lambda,\alpha_\lambda)\Phi\rangle-u^\#(t,\cdot)\right\|_{L^2(\mathbb{R}^3)}\\ &\leq \lambda K_1(t,s)(1+\lambda)^{1/2}e^{(\lambda|t-s|+K_2(t,s))/2}\left\|(P+N+1)^{6+3/2}\Phi\right\| \underset{\lambda\to 0}{\longrightarrow} 0 .\end{aligned}$$

An analogous result can be proved with $a^\#$ and $\alpha^\#(t)$ instead of $\psi^\#$ and $u^\#(t)$. So we have made clear in what sense the quantum operators $\lambda\psi^\#$ and $\lambda a^\#$ converge to the classical solutions u and α: we have the convergence of time evolved transition amplitudes of such operators (between coherent states), in the L^2-norm. Then we can write

$$\langle U(t)C(u_\lambda,\alpha_\lambda)\Phi,\lambda\psi^\#(\cdot)U(t)C(u_\lambda,\alpha_\lambda)\Phi\rangle \overset{L^2(\mathbb{R}^3)}{\underset{\lambda\to 0}{\longrightarrow}} u^\#(t)$$

$$\langle U(t)C(u_\lambda,\alpha_\lambda)\Phi,\lambda a^\#(\cdot)U(t)C(u_\lambda,\alpha_\lambda)\Phi\rangle \overset{L^2(\mathbb{R}^3)}{\underset{\lambda\to 0}{\longrightarrow}} \alpha^\#(t) .$$

Now that we know the classical limit of creation and annihilation operators, it is natural to study the behaviour of the quantum fluctuations around the classical limit. In order to make clear what we mean by fluctuations we analyse in more detail the evolution of quantum operators by Heisenberg equation. First of all observe that if we write H as a function of $\lambda\psi$ and λa we have that

$$H=\lambda^{-2}h(\lambda\psi,\lambda a) ,$$

with

$$h(\psi,a)=\frac{1}{2M}\int \mathrm{d}x\,(\nabla\psi)^*\nabla\psi+\int \mathrm{d}k\,\omega a^*a+\int \mathrm{d}x\,\varphi\psi^*\psi .$$

The time evolution of $\lambda\psi$ and λa is then dictated by Heisenberg equations

$$\begin{cases} i\partial_t\lambda\psi=[\lambda\psi,H]\\ i\partial_t\lambda a=[\lambda a,H]\end{cases} . \tag{4.1}$$

If we call (u,α) the classical solution, we would like to expand h around (u,α) as follows:

$$h(\lambda\psi,\lambda a)=h(u,\alpha)+h_1(\lambda\psi-u,\lambda a-\alpha)+h_2(\lambda\psi-u,\lambda a-\alpha) +h_3(\lambda\psi-u,\lambda a-\alpha) ,$$

where h_1, h_2 and h_3 have total degree 1, 2 and 3 respectively in the variables $\lambda\psi - u$, $\lambda a - \alpha$ and their hermitian conjugates. In particular we have:

$$\begin{aligned} h_1(\psi, a) =& -\frac{1}{2M}\int \mathrm{d}x\, \Delta u \psi^* + \int \mathrm{d}k\, \omega \alpha a^* + \int \mathrm{d}x\, \big(\frac{1}{2}\left|u\right|^2 \varphi \\ &+ (\check{\chi} * A) u \psi^*\big) + \text{h.c.}\ , \\ h_2(\psi, a) =& \frac{1}{2M}\int \mathrm{d}x\, (\nabla\psi)^* \nabla\psi + \int \mathrm{d}k\, \omega a^* a + \Big[\int \mathrm{d}x\, \big(\frac{1}{2}(\check{\chi} * A)\psi^*\psi \\ &+ u\varphi\psi^*\big) + \text{h.c.}\Big]\ , \\ h_3(\psi, a) =& \int \mathrm{d}x\, \varphi \psi^* \psi\ . \end{aligned}$$

Now we define

$$\begin{aligned} h_{k,\psi}(\psi, a) &= [\psi, h_k(\psi, a)] \\ h_{k,a}(\psi, a) &= [a, h_k(\psi, a)] \end{aligned} \quad \text{with } k = 1, 2, 3.$$

Equation (4.1) then could be rewritten as

$$\begin{cases} i\partial_t u + i\partial_t(\lambda\psi - u) = h_{1,\psi} + h_{2,\psi}(\lambda\psi - u, \lambda a - \alpha) \\ \qquad\qquad + h_{3,\psi}(\lambda\psi - u, \lambda a - \alpha) \\ i\partial_t \alpha + i\partial_t(\lambda a - \alpha) = h_{1,a} + h_{2,a}(\lambda\psi - u, \lambda a - \alpha) \\ \qquad\qquad + h_{3,a}(\lambda\psi - u, \lambda a - \alpha) \end{cases} .$$

Since (u, α) is the classical solution, we obtain

$$\begin{cases} i\partial_t(\psi - u_\lambda) = h_{2,\psi}(\psi - u_\lambda, a - \alpha_\lambda) + \lambda h_{3,\psi}(\psi - u_\lambda, a - \alpha_\lambda) \\ i\partial_t(a - \alpha_\lambda) = h_{2,a}(\psi - u_\lambda, a - \alpha_\lambda) + \lambda h_{3,a}(\psi - u_\lambda, a - \alpha_\lambda) \end{cases} , \tag{4.2}$$

where as before

$$\begin{cases} u_\lambda = \frac{1}{\lambda} u \\ \alpha_\lambda = \frac{1}{\lambda}\alpha \end{cases} .$$

We want to study equations (4.2), in particular their limit $\lambda \to 0$; in fact, since (u, α) is the classical solution, these limit equations will describe the quantum fluctuations around the classical solution in the mean field limit. However we have to define new variables with suitable initial conditions, independent of λ: define $\theta(t)$ and $c(t)$ by

$$\begin{cases} \theta(t) = C(u_\lambda(s), \alpha_\lambda(s))^\dagger (\psi(t) - u_\lambda(t)) C(u_\lambda(s), \alpha_\lambda(s)) \\ c(t) = C(u_\lambda(s), \alpha_\lambda(s))^\dagger (a(t) - \alpha_\lambda(t)) C(u_\lambda(s), \alpha_\lambda(s)) \end{cases} ,$$

such that $\theta(s) = \psi_0$ and $c(s) = a_0$. So the initial problem (4.1) is now reduced to finding two families of operators $\theta(t)$ and $c(t)$ satisfying the initial conditions

$$\begin{cases} \theta(s) = \psi_0 \\ c(s) = a_0 \end{cases} ,$$

and equations

$$\begin{cases} i\partial_t \theta = h_{2,\psi}(\theta, c) + \lambda h_{3,\psi}(\theta, c) \\ i\partial_t c = h_{2,a}(\theta, c) + \lambda h_{3,a}(\theta, c) \end{cases} . \tag{4.3}$$

Furthermore we want to take the limit $\lambda \to 0$ of (4.3) in a suitable sense.

It can be seen that formally the solution of (4.3) is given by

$$\begin{cases} \theta(t) = & W^\dagger(t,s)\psi_0 W(t,s) \\ c(t) = & W^\dagger(t,s) a_0 W(t,s) \end{cases} ,$$

where $W(t,s)$ is the unitary two-parameter group introduced before. We said this solution is formal: in fact, as we will discuss in the main body of the work, $W(t,s)$ is not differentiable in t or s on a suitable subset of $\mathscr{H}$; however the operator in the interaction picture $\widetilde{W}(t,s) = U_0^\dagger(t) W(t,s) U_0(s)$ is differentiable on a dense domain (U_0 is the free evolution group).

Consider now the limit equations of (4.3):

$$\begin{cases} i\partial_t \psi_2 = & h_{2,\psi}(\psi_2, a_2) \\ i\partial_t a_2 = & h_{2,a}(\psi_2, a_2) \end{cases} , \tag{4.4}$$

with initial condition as before

$$\begin{cases} \psi_2(s) = \psi_0 \\ a_2(s) = a_0 \end{cases} .$$

The formal solution of such system is

$$\begin{cases} \psi_2(t) = & U_2^\dagger(t,s)\psi_0 U_2(t,s) \\ a_2(t) = & U_2(t,s) a_0 U_2(t,s) \end{cases} ,$$

where $U_2(t,s)$ is a two-parameter unitary group we will define precisely later. Again is $\widetilde{U}_2(t,s) = U_0^\dagger(t) U_2(t,s) U_0(s)$ rather than $U_2(t,s)$ to be differentiable.

A crucial result we will prove is the convergence in the strong topology of $\widetilde{W}(t,s)$ to $\widetilde{U}_2(t,s)$:

$$\operatorname*{s-lim}_{\lambda\to 0} \widetilde{W}(t,s) = \widetilde{U}_2(t,s) .$$

Since U_0, $\widetilde{W}(t,s)$ and $\widetilde{U}_2(t,s)$ are unitary operators this implies also the sought convergence:

$$\operatorname*{s-lim}_{\lambda\to 0} W(t,s) = U_2(t,s) ;$$

and that clarifies in what sense problem (4.3) converges to (4.4) when $\lambda \to 0$. Furthermore this strong convergence implies that, for any family of bounded and suitably regular functions $(R_i(\psi), R_j(a))$ and for any family of times $\{t_i\}$ and $\{t_j\}$, $i = 1, \dots l$, $j = l+1, \dots m$:

$$\begin{split} \operatorname*{s-lim}_{\lambda\to 0} C(u_\lambda, \alpha_\lambda)^\dagger \prod_{i=1}^{l} R_i(\psi(t_i) - u_\lambda(t_i)) \prod_{j=l+1}^{m} R_j(a(t_j) - \alpha_\lambda(t_j)) \\ C(u_\lambda, \alpha_\lambda) = \prod_{i=1}^{l} R_i(\psi_2(t_i)) \prod_{j=l+1}^{m} R_j(a_2(t_j)) . \end{split} \tag{4.5}$$

This convergence can be interpreted in terms of correlation functions on coherent states.

5. Normal ordered products of creation and annihilation operators.

As we just stated in equation (4.5) we could prove a convergence in terms of correlation functions of bounded functions of creation and annihilation operators. Then it is natural to ask if we could say something about unbounded functions of creation and annihilation operators. We focused on the analysis of normal ordered products of creation and annihilation operators at a fixed time t. We studied the average of such products not only between coherent states, but also between fixed particle states.

Let $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$, with norm one. then for any $p, n \in \mathbb{N}$ we define

$$\begin{aligned}
&\Lambda = C(\sqrt{p}\, u_0, \sqrt{n}\, \alpha_0)\Omega \; ; \\
&\Psi = u_0^{\otimes p} \otimes C(\sqrt{n}\, \alpha_0)\Omega \in \mathscr{H}_p \; ; \\
&\Theta = u_0^{\otimes p} \otimes \alpha_0^{\otimes n} \in \mathscr{H}_{p,n} \; .
\end{aligned}$$

By definition Λ is a coherent state; Ψ is a tensor product state of a fixed number p of non-relativistic particles state and a coherent relativistic particles state; Θ is a state with p non-relativistic and n relativistic particles. Each state has norm one. We remark that in the classical limit p and n will go to infinity as λ^{-2}, as discussed before. The quantum evolution of such states is dictated by $U(t)$ so define:

$$\begin{aligned}
&\Lambda(t) = U(t)\Lambda \; , \\
&\Psi(t) = U(t)\Psi \; , \\
&\Theta(t) = U(t)\Theta \; .
\end{aligned}$$

Consider now the following transition amplitudes:

$$\begin{aligned}
\Gamma_{\Lambda(t)}(X_{q+r}; K_{i+j}) =& \langle \Lambda(t), \prod_{a=1}^{q} \lambda\psi^*(x_a) \prod_{b=q+1}^{r} \lambda\psi(x_b) \prod_{c=1}^{i} \lambda a^*(k_c) \\
&\prod_{d=i+1}^{j} \lambda a(k_d) \Lambda(t) \rangle \\
\Gamma_{\Psi(t)}(X_{q+r}; K_{i+j}) =& \langle \Psi(t), \prod_{a=1}^{q} \lambda\psi^*(x_a) \prod_{b=q+1}^{r} \lambda\psi(x_b) \prod_{c=1}^{i} \lambda a^*(k_c) \\
&\prod_{d=i+1}^{j} \lambda a(k_d) \Psi(t) \rangle \\
\Gamma_{\Theta(t)}(X_{q+r}; K_{i+j}) =& \langle \Theta(t), \prod_{a=1}^{q} \lambda\psi^*(x_a) \prod_{b=q+1}^{r} \lambda\psi(x_b) \prod_{c=1}^{i} \lambda a^*(k_c) \\
&\prod_{d=i+1}^{j} \lambda a(k_d) \Theta(t) \rangle \; ;
\end{aligned}$$

we are interested in their behavior when $\lambda \to 0$ (or equivalently $p, n \to \infty$ as λ^{-2}). We will show that $\Gamma_{\Lambda(t)}$, $\Gamma_{\Psi(t)}$ and $\Gamma_{\Theta(t)} \in L^2(\mathbb{R}^{3(q+r+i+j)})$ and prove their convergence when $\lambda \to 0$ in that space.

The idea is to use the estimates of $\left\| (P + N + 1)^\delta W(t, s)\Phi \right\|$ discussed in the previous section, and the convergence result of $\widetilde{W}(t, s)$ towards $\widetilde{U}_2(t, s)$ to obtain a better convergence rate in λ. We can manage to write fixed particle states as particular combinations of coherent states, so we can apply the method to any initial state.

We will prove the following results:

$$\begin{aligned}
&\left\| \Gamma_{\Lambda(t)} - \bar{u}_t^{\otimes q} u_t^{\otimes r} \bar{\alpha}_t^{\otimes i} \alpha_t^{\otimes j} \right\|_2 \leq \lambda^2 \mathcal{C}_\Lambda(t) \; , \\
&\left\| \Gamma_{\Psi(t)} - \delta_{qr} \bar{u}_t^{\otimes q} u_t^{\otimes r} \bar{\alpha}_t^{\otimes i} \alpha_t^{\otimes j} \right\|_2 \leq \delta_{qr} \lambda^2 \mathcal{C}_\Psi(t) \; ,
\end{aligned}$$

where $\mathcal{C}_\Lambda(t)$ and $\mathcal{C}_\Psi(t)$ are functions of time independent of λ, δ_{qr} equals 1 when $q = r$ and is zero otherwise, and (u_t, α_t) is the solution of the classical equations (3.2) with initial data $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$. The norm $\|\cdot\|_2$ is the $L^2(\mathbb{R}^{3(q+r+i+j)})$ one. So we see that not only time ordered products of annihilation and creation operators converge to products of the classical solutions in L^2 when $\lambda \to 0$, but also that they do it with a rate of convergence at least of λ^2 (this also enhances the rate of the convergence stated above of $\lambda\psi^\#$ and $\lambda a^\#$ to the corresponding solutions of the classical problem).

The result concerning Θ states is quite unexpected and deserves a comment. If (u_t, α_t) is the solution of (3.2) with initial data (u_0, α_0), we define $(u_t(\theta), \alpha_t(\theta))$ to be the solution of the same equation but with initial data $(u_0, e^{-i\theta}\alpha_0)$. We remark that the classical solution depends continuously on initial data in the L^2 topology, so $(u_t(\theta), \alpha_t(\theta))$ converges to (u_t, α_t) in L^2 when $\theta \to 0$.

Now we are able to state the result about Θ states:

$$\left\| \Gamma_{\Theta(t)} - \delta_{qr} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{u}_t^{\otimes_q}(\theta) u_t^{\otimes_r}(\theta) \bar{\alpha}_t^{\otimes_i}(\theta) \alpha_t^{\otimes_j}(\theta) \right\|_2 \leq \delta_{qr} \lambda^2 \mathcal{C}_\Theta(t) \; .$$

So the classical limit of product of operators in such case is not purely classical; we mean that the limit is not simply the expected product of classical solutions, but some sort of average of such a product over different initial conditions. This result, as we mentioned, is quite unexpected and could suggest that fixed particle states are not very useful when dealing with a field theory that does not preserve the number of such particles.

6. Future developments.

The previous results concern the classical limit of the Nelson model with cut off. The next question is to study the classical limit of the same model without cut off. In that case the quantum hamiltonian of the theory is ill defined because of ultraviolet divergences. The standard procedure to handle that situation is to introduce a cut off which allows us to subtract to the hamiltonian operator a suitable scalar quantity, depending on the cut off in such a way that the renormalized hamiltonian has a limit when the cut off is removed. This leads to a new hamiltonian with however domain of definition different from that of the free hamiltonian. This is the source of the main difficulties in the treatment of the Nelson model. See [**Gro62, Nel64, Fro74**]; also [**Amm00**] and the references thereof contained for more recent developments.

Preliminary attempts to deal with the classical limit for the full theory without cut off show unexpected difficulties. As an example it appears that the U_2 evolution of fluctuations around the classical solution does not exist. The natural continuation of this thesis is to extend its results to that more singular situation.

CHAPTER 2

The classical Klein-Gordon/Schrödinger system of equations.

We recall the convention for the Fourier transform $\hat{f}$ of f:

$$\hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x\, e^{-ikx} f(x) \; ;$$

and of the inverse transform $\check{f}$:

$$\check{f}(k) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x\, e^{ikx} f(x) \; .$$

Furthermore let $\sigma \in \mathbb{R}^+$, then we define the function $\chi \in L^p(\mathbb{R}^3)$ for all $1 \leq p \leq \infty$ as following:

$$\chi(x) = \begin{cases} 1 \text{ if } |x| \leq \sigma \\ 0 \text{ if } |x| > \sigma \end{cases} .$$

Let $\alpha_0, u_0 \in L^2(\mathbb{R}^3)$, and define $U_{01}(t) \equiv \exp(i\Delta t/2)$, $U_{02}(t) \equiv \exp(-i\omega t)$, with

$$(\omega^\lambda \alpha)(x) = (2\pi)^{-3/2} \int \mathrm{d}\xi\, e^{i\xi x} (\mu^2 + |\xi|^2)^{\lambda/2} \hat{\alpha}(\xi) \; , \; \mu \geq 0.$$

We consider the following system of integral equations:

$$\text{(E)} \qquad \begin{cases} u(t) = U_{01}(t) u_0 - i(2\pi)^{-3/2} \displaystyle\int_0^t \mathrm{d}\tau\, U_{01}(t-\tau) u(\tau) (\check{\chi} * A(\tau)) \\ \alpha(t) = U_{02}(t) \alpha_0 - i \dfrac{(2\pi)^{-3/2}}{\sqrt{2}} \displaystyle\int_0^t \mathrm{d}\tau\, U_{02}(t-\tau) \omega^{-1/2} \check{\chi} * (|u(\tau)|^2) \end{cases}$$

where $A(t) = \omega^{-1/2}(\alpha(t) + \bar{\alpha}(t))$. We want to prove the existence of a unique solution of the system in $\mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$. We follow the method used by Bachelot [**Bac84**] to study more singular potentials.

1. Existence and uniqueness of solution.

Lemma 1.1. *Let $V \in \mathscr{C}^0(\mathbb{R}, L^\infty(\mathbb{R}^3))$. Then, for all $u_0 \in L^2(\mathbb{R}^3)$, $\exists! u \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3))$ solution of*

$$u(t) = U_{01}(t) u_0 - i \int_0^t \mathrm{d}\tau\, U_{01}(t-\tau) V(\tau) u(\tau) \; . \tag{1.1}$$

Furthermore if

$$u_j(t) = U_{01}(t) u_0 - i \int_0^t \mathrm{d}\tau\, U_{01}(t-\tau) V_j(\tau) u_j(\tau) \text{ with } j = 1, 2$$

we have the following estimate:

$$\|u_1(t) - u_2(t)\|_2 \leq \left\| u_2; \mathscr{C}^0([0,t], L^2) \right\| \int_0^t \mathrm{d}\tau\, \|(V_1 - V_2)(\tau)\|_\infty \exp\left(\left| \int_0^t \mathrm{d}\tau\, \|V_1(\tau)\|_\infty \right| \right) . \tag{1.2}$$

Finally if V is real then $\|u(t)\|_2 = \|u_0\|_2$ for all t (the charge is conserved).

PROOF. We start with uniqueness. Let u and u' be solutions of (1.1), then $u_-(\cdot) = (u - u')(\cdot)$ satisfies:

$$u_-(t) = -i \int_0^t \mathrm{d}\tau \, U_{01}(t-\tau) V(\tau) u_-(\tau) \; ,$$

$$\|u_-(t)\|_2 \leq \int_0^t \mathrm{d}\tau \; \|V(\tau)\|_\infty \|u_-(\tau)\|_2 \; ,$$

so applying the Lemma of Gronwall we have $u_- = 0$.

Now we turn to existence. Let $u_0(t) = U_{01}(t)u_0$; then for all $j \geq 1$ we define iteratively:

$$u_j(t) = u_0(t) - i \int_0^t \mathrm{d}\tau \, U_{01}(t-\tau) V(\tau) u_{j-1}(\tau) \; . \tag{1.3}$$

By definition we have that

$$u_n(t) = u_0(t) + \sum_{j=1}^n (-i)^j \int\limits_{t \geq t_1 \cdots \geq t_j \geq 0} \mathrm{d}t_1 \ldots \mathrm{d}t_j \, U_{01}(t-t_1) V(t_1) \\ U_{01}(t_1 - t_2) V(t_2) \ldots U_{01}(t_{j-1} - t_j) V(t_j) u_0(t_j) \; ,$$

so for all $n > m$:

$$(u_n - u_m)(t) = \sum_{j=m+1}^n (-i)^j \int\limits_{t \geq t_1 \cdots \geq t_j \geq 0} \mathrm{d}t_1 \ldots \mathrm{d}t_j \, U_{01}(t-t_1) V(t_1) \\ U_{01}(t_1 - t_2) V(t_2) \ldots U_{01}(t_{j-1} - t_j) V(t_j) u_0(t_j) \; .$$

We then obtain the following estimate:

$$\begin{aligned} \|(u_n - u_m)(t)\|_2 &\leq \sum_{j=m+1}^n \int\limits_{t \geq t_1 \cdots \geq t_j \geq 0} \mathrm{d}t_1 \ldots \mathrm{d}t_j \; \|V(t_1)\|_\infty \ldots \|V(t_j)\|_\infty \\ &\quad \sup_{0 \leq \tau \leq t} \|u_0(\tau)\|_2 \\ &\leq \sum_{j=m+1}^n \frac{1}{j!} \left(\int_0^t \mathrm{d}\tau \; \|V(\tau)\|_\infty \right)^j \sup_{0 \leq \tau \leq t} \|u_0(\tau)\|_2 \; . \end{aligned}$$

So $\exists \lim_{j \to \infty} u_j(t) \equiv u(t)$ in $\mathscr{C}(I, L^2(\mathbb{R}^3))$ for all compact interval I. Taking the limit of both members of (1.3) we see that $u(t)$ satisfies (1.1).

To prove (1.2) we write:

$$\|u_1(t) - u_1(t)\|_2 \leq \int_0^t \mathrm{d}\tau \Big(\|V_1(\tau) - V_2(\tau)\|_\infty \|u_2(\tau)\|_2 + \|V_1(\tau)\|_\infty \\ \|u_1(\tau) - u_2(\tau)\|_2 \Big) \; .$$

Equation (1.2) then follows applying the Lemma of Gronwall.

To prove charge conservation we define $\widetilde{u}(t) \equiv U_{01}(-t)u(t)$. So we have

$$\widetilde{u}(t) = u_0 - i \int_0^t \mathrm{d}\tau \, U_{01}(-\tau) V(\tau) u(\tau) \; ,$$

then since U_{01} is continuous, $\widetilde{u}(t)$ is differentiable in t and

$$i \partial_t \widetilde{u}(t) = U_{01}(-t) V(t) u(t) \; .$$

Now we can write:

$$\begin{aligned}\partial_t\langle u(t),u(t)\rangle &= \partial_t\langle\widetilde{u}(t),\widetilde{u}(t)\rangle = 2\mathrm{Re}\langle\widetilde{u}(t),\partial_t\widetilde{u}(t)\rangle = 2\mathrm{Im}\langle\widetilde{u}(t),i\partial_t\widetilde{u}(t)\rangle \\ &= 2\mathrm{Im}\langle\widetilde{u}(t),U_{01}(-t)V(t)u(t)\rangle = 2\mathrm{Im}\langle u(t),V(t)u(t)\rangle = 0\ ,\end{aligned}$$

since V is real. ■

Now we can prove the existence of a unique solution of the system (E), this is done in the following proposition:

PROPOSITION 1. *Let* $u_0,\alpha_0\in L^2(\mathbb{R}^3)$. *Then* $\exists!(u(\cdot),\alpha(\cdot))$ *in* $\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ *solution of the integral system* (E).

PROOF. First of all we solve for all $j=1,2,\dots$ the systems:

$$(\mathrm{E}_j)\qquad\begin{cases} u_j(t) = u_0(t) - i(2\pi)^{-3/2}\displaystyle\int_0^t \mathrm{d}\tau\, U_{01}(t-\tau)u_j(\tau)(\check{\chi}*A_{j-1}(\tau)) \\ \alpha_j(t) = \alpha_0(t) - i\dfrac{(2\pi)^{-3/2}}{\sqrt{2}}\displaystyle\int_0^t \mathrm{d}\tau\, U_{02}(t-\tau)\omega^{-1/2}\check{\chi}*(|u_{j-1}(\tau)|^2)\end{cases}$$

with $u_0(t)\equiv U_{01}(t)u_0$, $\alpha_0(t)\equiv U_{02}(t)\alpha_0$. Observe that if $\alpha_j\in\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3))$, we have using Sobolev's inequality $A_j\in\mathscr{C}^0(\mathbb{R},L^3(\mathbb{R}^3))$; so $\check{\chi}*A_j\in\mathscr{C}^0(\mathbb{R},L^\infty(\mathbb{R}^3))$ by Young's inequality. Furthermore if $u_j\in\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3))$, $\check{\chi}*(|u_{j-1}(\tau)|^2)\in\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3))$. Using Lemma 1.1 we have a unique solution $u_j\in\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3))$ of the first equation of (E_j), while the second equation defines $\alpha_j\in\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3))$. We want now to prove that $\exists\lim_{j\to\infty}(u_j,\alpha_j)$ in $\mathscr{C}^0(I,L^2(\mathbb{R}^3))$ for a suitable compact I. Let $t\in I$, and define the map S on $\mathscr{C}^0(I,L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ as

$$S\begin{pmatrix}u(t)\\ \alpha(t)\end{pmatrix} = \begin{pmatrix} u_0(t) - i(2\pi)^{-3/2}\displaystyle\int_0^t \mathrm{d}\tau\, U_{01}(t-\tau)u(\tau)(\check{\chi}*A(\tau)) \\ \alpha_0(t) - i\dfrac{(2\pi)^{-3/2}}{\sqrt{2}}\displaystyle\int_0^t \mathrm{d}\tau\, U_{02}(t-\tau)\omega^{-1/2}\check{\chi}*(|u(\tau)|^2)\end{pmatrix}\ ,$$

so we can write (E_j) as

$$\begin{pmatrix}u_j(t)\\ \alpha_j(t)\end{pmatrix} = S\begin{pmatrix}u_{j-1}(t)\\ \alpha_{j-1}(t)\end{pmatrix}\ .$$

So if S a contraction map on $\mathscr{C}^0(I,L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ for a suitable I, the Banach fixed point theorem provides that the limit exists in that space and is the unique solution of (E). In order to do that we have to calculate the norm of

$$S\begin{pmatrix}u_1(t)\\ \alpha_1(t)\end{pmatrix} - S\begin{pmatrix}u_2(t)\\ \alpha_2(t)\end{pmatrix} = \begin{pmatrix}u_1'(t)\\ \alpha_1'(t)\end{pmatrix} - \begin{pmatrix}u_2'(t)\\ \alpha_2'(t)\end{pmatrix}\ ,$$

where $u_1,u_2,\alpha_1,\alpha_2\in\mathscr{C}^0(I,L^2(\mathbb{R}^3))$. Let $I=[0,\epsilon]$; we use estimate (1.2) of Lemma 1.1 and conservation of charge to obtain:

$$\begin{aligned}\sup_{t\in I}\|u_1'(t)-u_2'(t)\|_2 &\leq (2\pi)^{-3/2}\left\|u_2;\mathscr{C}^0(I,L^2)\right\| \\ \int_0^t \mathrm{d}\tau\ \|\check{\chi}*(A_1-A_2)(\tau)\|_\infty &\exp\left((2\pi)^{-3/2}\left|\int_0^t \mathrm{d}\tau\ \|\check{\chi}*A_1(\tau)\|_\infty\right|\right) \\ \leq C_s(2\pi)^{-3/2}\epsilon\,\|\check{\chi}\|_{3/2} &\exp\Big(C_s(2\pi)^{-3/2}\epsilon\,\|\check{\chi}\|_{3/2}\max_{j=1,2}\left\|\alpha_j;\mathscr{C}^0(I,L^2)\right\|\Big) \\ &\max_{j=1,2}\left\|u_j;\mathscr{C}^0(I,L^2)\right\|\left\|\alpha_1-\alpha_2;\mathscr{C}^0(I,L^2)\right\|\ .\end{aligned}$$

We consider now $\alpha_1' - \alpha_2'$:

$$\sup_{t\in I} \|\alpha_1'(t) - \alpha_2'(t)\|_2 \leq \frac{(2\pi)^{-3/2}}{\sqrt{2}} \left\|\omega^{-1/2}\check{\chi}\right\|_2 \int_0^t \mathrm{d}\tau \, \left\|(|u_1|^2 - |u_2|^2)(\tau)\right\|_1$$

$$\leq \frac{(2\pi)^{-3/2}}{\sqrt{2}} \|\check{\chi}\|_2 \int_0^t \mathrm{d}\tau \left(\|u_1 + u_2\|_2 \|u_1 - u_2\|_2\right)(\tau)$$

$$\leq \sqrt{2}(2\pi)^{-3/2}\epsilon \|\check{\chi}\|_2 \max_{j=1,2} \left\|u_j; \mathscr{C}^0(I, L^2)\right\| \left\|u_1 - u_2; \mathscr{C}^0(I, L^2)\right\| \ .$$

So choosing ϵ small enough it follows that S is a strict contraction on $\mathscr{C}^0(I, L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ and so admits a unique fixed point $(u(t), \alpha(t))$ solution of (E) on that space. The charge of $u(t)$ is conserved as proved in Lemma 1.1, so for all real t we have $\|u(t)\|_2 = \|u_0\|_2$. Using this fact we can extend the solution to $\mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$. ■

Remark 1.1.1. If $(u(t), \alpha(t))$ is the solution of (E) in $\mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$, define $(\widetilde{u}(t), \widetilde{\alpha}(t)) \equiv (U_{01}(-t)u(t), U_{02}(-t)\alpha(t))$.

Then $(\widetilde{u}(t), \widetilde{\alpha}(t)) \in \mathscr{C}^1(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$ and we have that:

$$i\partial_t\widetilde{u}(t) = (2\pi)^{-3/2}U_{01}(-t)\left(\check{\chi} * A(t)\right)u(t) \ ,$$

$$i\partial_t\widetilde{\alpha}(t) = \frac{(2\pi)^{-3/2}}{\sqrt{2}}U_{02}(-t)\left(\omega^{-1/2}\check{\chi} * |u(t)|^2\right) \ .$$

2. Continuity of solution with respect to initial conditions.

Lemma 2.1. *Let $(u_1(\cdot), \alpha_1(\cdot))$ and $(u_2(\cdot), \alpha_2(\cdot))$ be the solutions of (E) in $\mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$ corresponding respectively to initial data (u_0, α_{01}) and (u_0, α_{02}) both in $L^2 \otimes L^2$. Then if $\alpha_{01} \to_{L^2} \alpha_{02}$, then $(u_1(\cdot), \alpha_1(\cdot)) \to (u_2(\cdot), \alpha_2(\cdot))$ in $\mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$.*

Proof. Let $0 < t \in [0, T]$ compact. From (E) using equation (1.2) we find the following estimates:

$$\begin{cases} \|u_1(t) - u_2(t)\|_2 \leq f(t) \displaystyle\int_0^t \mathrm{d}\tau \, \|(\alpha_1 - \alpha_2)(\tau)\|_2 \\ \|\alpha_1(t) - \alpha_2(t)\|_2 \leq \|\alpha_{01} - \alpha_{02}\|_2 + g(t) \displaystyle\int_0^t \mathrm{d}\tau \, \|(u_1 - u_2)(\tau)\|_2 \end{cases} ;$$

where both f and g are locally bounded positive functions. Using the first equation, the second one becomes:

$$\|\alpha_1(t) - \alpha_2(t)\|_2 \leq \|\alpha_{01} - \alpha_{02}\|_2 + g(t) \int_0^t \mathrm{d}\tau \, f(\tau) \int_0^\tau \mathrm{d}\tau' \, \|(\alpha_1 - \alpha_2)(\tau')\|_2$$

$$\leq \|\alpha_{01} - \alpha_{02}\|_2 + C(T) \int_0^t \mathrm{d}\tau \, \|(\alpha_1 - \alpha_2)(\tau)\|_2 \ ,$$

so using Gronwall's Lemma we obtain:

$$\|\alpha_1(t) - \alpha_2(t)\|_2 \leq \|\alpha_{01} - \alpha_{02}\|_2 \, e^{TC(T)} \ ,$$

so we have

$$\begin{cases} \left\|u_1 - u_2; \mathscr{C}^0([0, T], L^2)\right\|_2 \leq C_1(T) \|\alpha_{01} - \alpha_{02}\|_2 \\ \left\|\alpha_1 - \alpha_2; \mathscr{C}^0([0, T], L^2)\right\|_2 \leq C_2(T) \|\alpha_{01} - \alpha_{02}\|_2 \end{cases} ,$$

for any compact interval $[0, T]$. ■

CHAPTER 3

The quantum theory.

To describe the quantum theory we will use standard results on the theory of operators in Hilbert spaces, that can be found for example in [**RS72, RS75**].

The space where the quantum theory is defined is the tensor product of two symmetric Fock spaces, representing the nonrelativistic and relativistic fields. As in the introduction, for any $p, n \in \mathbb{N}$ we define

$$\mathscr{H}_{p,n} = \left\{ \Phi_{p,n} : \ \Phi_{p,n}(X_p; K_n) \in L^2(\mathbb{R}^{3p+3n}) \right\} ,$$

where $X_p = \{x_1, \dots, x_p\}$, $K_n = \{k_1, \dots, k_n\}$ and $\Phi_{p,n}$ is separately symmetric with respect to the first p and the last n variables. The Hilbert space $\mathscr{H}$ of the theory is taken to be the direct sum of the $\mathscr{H}_{p,n}$:

$$\mathscr{H} = \bigoplus_{p,n=0}^{\infty} \mathscr{H}_{p,n} .$$

We will use freely the following properties of the tensor product of Hilbert spaces:

$$\mathscr{H}_{p,n} = \mathscr{H}_{p,0} \otimes \mathscr{H}_{0,n} ,$$

and

$$\mathscr{H} = \Bigl(\bigoplus_{p=0}^{\infty} \mathscr{H}_{p,0} \Bigr) \otimes \Bigl(\bigoplus_{n=0}^{\infty} \mathscr{H}_{0,n} \Bigr) = \bigoplus_{p=0}^{\infty} \mathscr{H}_{p} ,$$

with

$$\mathscr{H}_{p} = \mathscr{H}_{p,0} \otimes \bigoplus_{n=0}^{\infty} \mathscr{H}_{0,n} .$$

We call $\mathscr{C}_0(P,N)$ the space of finite particle vectors; $\mathscr{C}_0(P,N)$ is dense in $\mathscr{H}$. We will eventually denote with $\mathscr{B}(\mathscr{H})$ the space of bounded operators of $\mathscr{H}$, and $|||\cdot|||$ its norm. If B is an operator in $L^2(\mathbb{R}^3)$ with domain D, we call $\mathrm{d}\Gamma_p(B)$ the operator on $\bigoplus_p \mathscr{H}_{p,0}$ that acts on $\mathscr{H}_{p,0}$ as $B \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes B \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes B$; a domain of essential self-adjointness for $\mathrm{d}\Gamma_p(B)$ is the domain D_B, the subspace of $\mathscr{C}_0(P)$ of functions $\Phi = \{\Phi_0, \Phi_1, \dots, \Phi_j, \dots\}$ such that for each j either Φ_j is zero or in $\bigotimes_{k=1}^{j} D$. The definition of $\mathrm{d}\Gamma_n(B)$ is perfectly analogous. We denote with X_p the set of variables $\{x_1, \dots, x_p\}$ and accordingly with K_n the set $\{k_1, \dots, k_n\}$; then introduce the formal operator valued distributions, $\psi(x)$, $\psi^*(x)$, $a(k)$ and $a^*(k)$:

$$\begin{aligned}
(\psi(x)\Phi)_{p,n}(X_p; K_n) &= \sqrt{p+1}\,\Phi_{p+1,n}(x, x_1, \dots, x_p; K_n) ,\\
(\psi^*(x)\Phi)_{p,n}(X_p; K_n) &= \frac{1}{\sqrt{p}} \sum_{i=1}^{p} \delta(x - x_i) \Phi_{p-1,n}(X_p \setminus x_i; K_n) ,\\
(a(k)\Phi)_{p,n}(X_p; K_n) &= \sqrt{n+1}\,\Phi_{p,n+1}(X_p; k, k_1, \dots, k_n) ,\\
(a^*(k)\Phi)_{p,n}(X_p; K_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(k - k_j) \Phi_{p,n-1}(X_p; K_n \setminus k_j) .
\end{aligned}$$

It is easy to see that

$$\begin{cases} [\psi(x), \psi^*(x')] = \delta(x - x') \\ [\psi(x), \psi(x')] = [\psi^*(x), \psi^*(x')] = 0 \\ [a(k), a^*(k')] = \delta(k - k') \\ [a(k), a(k')] = [a^*(k), a^*(k')] = 0 \end{cases} .$$

Integrating the distributions with functions we obtain the annihilation and creation operators. Let $f \in L^2(\mathbb{R}^3)$ and define:

$$(\psi(f)\Phi)_{p,n}(X_p; K_n) = \sqrt{p+1} \int \mathrm{d}x\, f(x)\Phi_{p+1,n}(x, x_1, \dots, x_p; K_n) \; ,$$

$$(\psi^*(f)\Phi)_{p,n}(X_p; K_n) = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} f(x_i)\Phi_{p-1,n}(X_p \setminus x_i; K_n) \; ,$$

$$(a(f)\Phi)_{p,n}(X_p; K_n) = \sqrt{n+1} \int \mathrm{d}k\, f(k)\Phi_{p,n+1}(X_p; k, k_1, \dots, k_n) \; ,$$

$$(a^*(f)\Phi)_{p,n}(X_p; K_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(k_j)\Phi_{p,n-1}(X_p; K_n \setminus k_j) \; .$$

In Appendix A we provide a detailed mathematical description of creation and annihilation operators in Fock space. If X_p, K_n are sets of variables, then

$$\psi^\#(X_p) = \prod_{i=1}^{p} \psi^\#(x_i) \; , \; a^\#(K_n) = \prod_{j=1}^{n} a^\#(k_j) \; .$$

On $\mathscr{H}_p$ we also define slightly different relativistic annihilation and creation operators; consider now $f \in L^\infty(\mathbb{R}^{3p}, L^2(\mathbb{R}^3))$ and define

$$(b(f)\Phi)_{p,n}(X_p; K_n) = \sqrt{n+1} \int \mathrm{d}k\, f(X_p, k)\Phi_{p,n+1}(X_p; k, K_n) \; ,$$

$$(b^*(f)\Phi)_{p,n}(X_p; K_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(X_p, k_j)\Phi_{p,n-1}(X_p; K_n \setminus k_j) \; .$$

We also define the particle number operators of $\mathscr{H}$, together with their domains of self-adjointness:

$$P = \mathrm{d}\Gamma_p(1) \otimes 1 \; , \; D(P) = \left\{ \Phi \in \mathscr{H} \;\Big|\; \sum_{p,n=0}^{\infty} p^2 \left\| \Phi \right\|_{p,n}^2 < \infty \right\} ,$$

$$N = 1 \otimes \mathrm{d}\Gamma_n(1) \; , \; D(N) = \left\{ \Phi \in \mathscr{H} \;\Big|\; \sum_{p,n=0}^{\infty} n^2 \left\| \Phi \right\|_{p,n}^2 < \infty \right\} ,$$

$$Q = P + N \; , \; D(Q) = \left\{ \Phi \in \mathscr{H} \;\Big|\; \sum_{p,n=0}^{\infty} (p+n)^2 \left\| \Phi \right\|_{p,n}^2 < \infty \right\} .$$

1. Nelson's Hamiltonian (self-adjointness).

Now we can define the free Hamiltonian function of the system:

Definition (Free Hamiltonian). $H_0 = \overline{(H_{01} + H_{02})}$, with domain of self-adjointness $D(H_0)$ and

$$H_{01} = \frac{1}{2M}\int \mathrm{d}x\, (\nabla\psi)^*(x)\nabla\psi(x) = \mathrm{d}\Gamma_p\Big(-\frac{\Delta}{2M}\Big)\otimes 1\ ,\ M > 0\ ;$$
$$H_{02} = \int \mathrm{d}k\, \omega(k)a^*(k)a(k) = 1\otimes \mathrm{d}\Gamma_n\big(\omega(\cdot)\big)\ ,\ \omega(k) = \sqrt{k^2+\mu^2}\ ,$$
$$\mu \geq 0\ .$$

To introduce the interaction we define the operator valued function $\varphi(x) = \varphi^-(x) + (\varphi^-(x))^\dagger$, with

$$\varphi^-(x) = \int \mathrm{d}k\, \frac{\chi_\sigma(k)}{(2\pi)^{3/2}(2\omega(k))^{1/2}}a(k)e^{ikx}\ ;$$

and the operators:

$$H_I^- = \lambda\int \mathrm{d}x\, \varphi^-(x)\psi^*(x)\psi(x)\ ,\ H_I^+ = (H_I^-)^\dagger\ ,$$
$$H_I\big|_p = \lambda\int \mathrm{d}x\, \varphi(x)\psi^*(x)\psi(x) = b(\bar{f}) + b^*(f)\ ,\ \lambda > 0\ ;$$
$$f = \sum_{j=1}^p f_j\ ,\ f_j = \lambda f_0 e^{-ik\cdot x_j}\ ,\ f_0 = (2\pi)^{-3/2}(2\omega)^{-1/2}\chi_\sigma\ .$$

We remark that for all $\sigma\in\mathbb{R}$, $f_0\in L^2(\mathbb{R}^3)$ with $\omega^\delta f_0\in L^2(\mathbb{R}^3)$ for all $\delta\geq -1/2$, even when $\mu = 0$.

Definition (Interaction Hamiltonian). $H_I = H_I^- + H_I^+$, defined on $D(P^2+N)$.

Definition (Nelson's Hamiltonian). $H = \overline{(H_0 + H_I)}$.

We remark that for all $\Phi\in D(H_0)\cap D(P^2+N)$ we have that $H\Phi = (H_0+H_I)\Phi$. Both H_0 and, as we will see, H are self-adjoint operators on $\mathscr{H}$, so we associate with each one a unitary evolution operator using the Theorem of Stone; in particular we define:

Definition (Evolution operators (Stone's Theorem)). $U_0(t) = \exp\{-itH_0\}$ and $U(t) = \exp\{-itH\}$.

Prior to prove self-adjointness of H we formulate a useful lemma:

Lemma 1.1. *Let $f\in L^\infty(\mathbb{R}^{3p}, L^2(\mathbb{R}^3))$ such that also $\omega^{-1/2}(k)f(X_p,k)$ is in $L^\infty(\mathbb{R}^{3p}, L^2(\mathbb{R}^3))$. Then, for all $\Phi\in D(H_{02}^{1/2})\cap\mathscr{H}_p$, intended as the domain on which the RHS is finite, the following estimates hold:*

$$\|b(f)\Phi\|^2 \leq \Big\|\omega^{-1/2}f\Big\|_*^2\Big\|H_{02}^{1/2}\Phi\Big\|^2\ ;$$
$$\|b^*(f)\Phi\|^2 \leq \Big\|\omega^{-1/2}f\Big\|_*^2\Big\|H_{02}^{1/2}\Phi\Big\|^2 + \|f\|_*^2\|\Phi\|^2\ ;$$

where $\|\cdot\|_*$ is the $L^\infty(\mathbb{R}^{3p}, L^2(\mathbb{R}^3))$-norm.

Let now $f\in L^\infty(\mathbb{R}^{3p}, L^2(\mathbb{R}^3))$, and $\Phi\in D(N^{1/2})\cap\mathscr{H}_p$, then:

$$\|b(f)\Phi\| \leq \|f\|_*\Big\|N^{1/2}\Phi\Big\|\ ;$$
$$\|b^*(f)\Phi\| \leq \|f\|_*\Big\|(N+1)^{1/2}\Phi\Big\|\ .$$

Proof. These estimates are easily provable by a direct calculation on $\mathscr{H}_{p,n}$, using Schwarz's inequality and the symmetry of $\Phi_{p,n}$. See also Lemma 2.1 of [**GNV06**] as a reference. ■

Corollary. *For all* $\Phi \in D(P^2+N)\cap D(PN^{1/2})$ *we have that:*

$$\|H_I\Phi\| \le 2\lambda \|f_0\|_2 \left\|P(N+1)^{1/2}\Phi\right\| \le \lambda \|f_0\|_2 \left\|(P^2+N+1)\Phi\right\| \ .$$

Proof. We use the second couple of inequalities of the above lemma on $\mathscr{H}_p$, then sum over all p. ■

Let Φ_p, $H_0\big|_p$ and $H\big|_p$ be the projections of $\Phi \in \mathscr{H}$, H_0 and H respectively on $\mathscr{H}_p$. Then we can formulate the following proposition:

Proposition 2 (Self-adjointness of H).

i. $H\big|_p$ *is self-adjoint on* $\mathscr{H}_p$ *with domain* $D(H_0\big|_p)$.

ii. H *is self-adjoint on* $\mathscr{H}$ *with domain* $D(H)$ *defined as following:*

$$D(H) = \left\{\Phi \in \mathscr{H} : \sum_{p=0}^{\infty} \left\|H\big|_p \Phi_p\right\|^2 < \infty,\ \Phi_p \in D(H_0\big|_p)\right\} .$$

iii. On $\mathscr{H}$*, we have the following inclusions:*

$$D(H_0) \supseteq D(H)\cap D(P^2+N) \ ,$$
$$D(H) \supseteq D(H_0)\cap D(P^2+N) \ .$$

Proof. *i.* In order to prove the self-adjointness of $H\big|_p$ we will show that $H_I\big|_p$ is a Kato perturbation of $H_0\big|_p$. Using Lemma 1.1 for all $\Phi_p \in D(H_0\big|_p)$ we obtain

$$\|H_I\Phi_p\|^2 \le 4\lambda^2 \left\|\omega^{-1/2} f_0\right\|_2^2 \left\|P H_{02}^{1/2}\Phi_p\right\|^2 + 2\lambda^2 \|f_0\|_2^2 \|P\Phi_p\|^2 \ .$$

But since $AB \le (1/2\rho^2)A^2 + (\rho^2/2)B^2$ for all $A, B \ge 0$ with $\rho > 0$ it follows that, for all $\epsilon > 0$:

$$\|H_I\Phi_p\|^2 \le \epsilon^2 \|H_{02}\Phi_p\|^2 + \frac{4\lambda^4}{\epsilon^2}\left\|\omega^{-1/2} f_0\right\|_2^4 \left\|P^2\Phi_p\right\|^2 + 2\lambda^2 \|f_0\|_2^2 \|P\Phi_p\|^2 \ ;$$

so choosing $\epsilon < 1$ we prove that $H_I\big|_p$ is a Kato perturbation since P is a bounded operator on $\mathscr{H}_p$.

ii. Since $H\big|_p$ is self-adjoint on $\mathscr{H}_p$ we can define a self-adjoint operator H on $\mathscr{H}$ as the direct sum

$$H = \bigoplus_{p=0}^{\infty} H\big|_p$$

with the maximal domain such that $H\big|_p\Phi_p$ is defined for all p and the norm $\|H\Phi\|$ is finite.

iii. To prove the first relation we proceed as following: from the fact that $H_0 = H - H_I$ we can write

$$\|H_0\Phi\| \le \|H\Phi\| + \|H_I\Phi\| \ .$$

Then using the Corollary of Lemma 1.1 we obtain for all $\Phi \in D(H)\cap D(P^2+N)$

$$\|H_0\Phi\| \le \|H\Phi\| + K\left\|(P^2+N+1)\Phi\right\| \ ,$$

with K a positive constant. So whenever both the norms of $H\Phi$ and $(P^2+N)\Phi$ are finite, also the norm of $H_0\Phi$ is finite and that proves the assertion. The second relation is proved in analogous fashion, writing $H = H_0 + H_I$ to obtain

$$\|H\Phi\| \le \|H_0\Phi\| + K\left\|(P^2+N+1)\Phi\right\| \ ,$$

for all $\Phi \in D(H_0)\cap D(P^2+N)$. ■

PROPOSITION 3 (Self-adjointness of H (direct proof)). *$H_0 + H_I$ is essentially self-adjoint on $D(H_0) \cap \mathscr{C}_0(P, N) \equiv D$.*

PROOF. Only in this proof P and N will be two positive integers and not the particle number operators. We define the orthogonal projector $\mathbb{Q}_{P,N}$ as follows:

$$(\mathbb{Q}_{P,N}\Phi)_{p,n} = \begin{cases} \Phi_{p,n} \text{ if } \begin{cases} p \leq P \\ n \leq N \end{cases} \\ \quad 0 \text{ otherwise.} \end{cases}$$

Using Lemma 2.1 we will prove in the next section we have

$$\begin{aligned} b(\bar{f})\mathbb{Q}_{P,N} &= \mathbb{Q}_{P,N-1} b(\bar{f})\mathbb{Q}_{P,N} \\ b^*(f)\mathbb{Q}_{P,N} &= \mathbb{Q}_{P,N+1} b^*(f)\mathbb{Q}_{P,N} \ . \end{aligned}$$

$H_0 + H_I$ is trivially symmetric. It will be then sufficient to show that $(z - (H_0 + H_I))D$ is dense in $\mathscr{H}$ for all $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$. Let $\Psi \in \mathscr{H}$ such that

$$\langle \Psi, (z - (H_0 + H_I))\Phi \rangle = 0 \tag{1.1}$$

for all $\Phi \in D$. We will prove $\Psi = 0$. From (1.1) it follows

$$\langle \Psi, H_0 \Phi \rangle = z\langle \Psi, \Phi \rangle - \langle \Psi, b(\bar{f})\Phi \rangle - \langle \Psi, b^*(f)\Phi \rangle \ . \tag{1.2}$$

Choose Φ to have the only component different from zero to be $\Phi_{p,n} \in \mathscr{H}_{p,n}$. A direct calculation let us rewrite (1.2) as

$$\begin{aligned} \langle \Psi_{p,n}, H_0\big|_{p,n} \Phi_{p,n} \rangle - z\langle \Psi_{p,n}, \Phi_{p,n} \rangle = n^{1/2} \int \mathrm{d}X_p \mathrm{d}K_n \, \bar{f}(X_p, k_1) \\ \overline{\Psi_{p,n-1}(X_p; K_{n-1})} \Phi_{p,n}(X_p; k_1 \cup K_{n-1}) \\ +(n+1)^{1/2} \int \mathrm{d}X_p \mathrm{d}K_n \mathrm{d}k \, f(X_p, k) \overline{\Psi_{p,n+1}(X_p; k \cup K_n)} \Phi_{p,n}(X_p; K_n) \ . \end{aligned} \tag{1.3}$$

From (1.3) it follows

$$\begin{aligned} \left| \langle \Psi_{p,n}, H_0\big|_{p,n} \Phi_{p,n} \rangle \right| \leq \|\Phi_{p,n}\|_{\mathscr{H}_{p,n}} \Big[|z| + p\lambda \, \|f_0\|_2 \Big(n^{1/2} \, \|\Psi_{p,n-1}\|_{\mathscr{H}_{p,n-1}} \\ +(n+1)^{1/2} \, \|\Psi_{p,n+1}\|_{\mathscr{H}_{p,n+1}} \Big) \Big] \ . \end{aligned}$$

Since $H_0\big|_{p,n}$ is self-adjoint $\Psi_{p,n} \in D(H_0\big|_{p,n})$. So for all P and N, $\mathbb{Q}_{P,N}\Psi \in D$. We remark that

$$\begin{aligned} &\langle \Psi, H_0 \mathbb{Q}_{P,N}\Psi \rangle = \langle \mathbb{Q}_{P,N}\Psi, H_0 \mathbb{Q}_{P,N}\Psi \rangle \\ &\langle \Psi, \mathbb{Q}_{P,N}\Psi \rangle = \langle \mathbb{Q}_{P,N}\Psi, \mathbb{Q}_{P,N}\Psi \rangle \\ &\langle \Psi, b(\bar{f})\mathbb{Q}_{P,N}\Psi \rangle + \langle \Psi, b^*(f)\mathbb{Q}_{P,N}\Psi \rangle = \langle \mathbb{Q}_{P,N}\Psi, b(\bar{f})\mathbb{Q}_{P,N}\Psi \rangle \\ &\qquad + \langle \mathbb{Q}_{P,N}\Psi, b^*(f)\mathbb{Q}_{P,N}\Psi \rangle + \langle (1 - \mathbb{Q}_{P,N})\Psi, b(\bar{f})\mathbb{Q}_{P,N}\Psi \rangle \\ &\qquad\qquad + \langle (1 - \mathbb{Q}_{P,N})\Psi, b^*(f)\mathbb{Q}_{P,N}\Psi \rangle \ , \end{aligned}$$

so we obtain

$$\begin{aligned} \mathrm{Im} z \, \|\mathbb{Q}_{P,N}\Psi\|^2 = \mathrm{Im} \Big[\langle (1 - \mathbb{Q}_{P,N})\Psi, b(\bar{f})\mathbb{Q}_{P,N}\Psi \rangle \\ + \langle (1 - \mathbb{Q}_{P,N})\Psi, b^*(f)\mathbb{Q}_{P,N}\Psi \rangle \Big] \ . \end{aligned} \tag{1.4}$$

By $\mathbb{Q}_{P,N-1}(1 - \mathbb{Q}_{P,N}) = 0$ we have

$$\langle (1 - \mathbb{Q}_{P,N})\Psi, b(\bar{f})\mathbb{Q}_{P,N}\Psi \rangle = \langle \mathbb{Q}_{P,N-1}(1 - \mathbb{Q}_{P,N})\Psi, b(\bar{f})\mathbb{Q}_{P,N}\Psi \rangle = 0 \ ;$$

also

$$\begin{aligned}\langle(1-\mathbb{Q}_{P,N})\Psi, b^*(f)\mathbb{Q}_{P,N}\Psi\rangle &= \langle(\mathbb{Q}_{P,N+1}-\mathbb{Q}_{P,N})\Psi, b^*(f)\mathbb{Q}_{P,N}\Psi\rangle \\ = \sum_{p\le P}\langle b(\bar f)\Psi_{p,N+1}, \mathbb{Q}_{P,N}\Psi\rangle &= \sum_{p\le P}\langle b(\bar f)\Psi_{p,N+1}, \Psi_{p,N}\rangle \\ &= (N+1)^{1/2}\sum_{p\le P}\int \mathrm{d}X_p \mathrm{d}K_N \mathrm{d}k\, \overline{f(X_p,k)} \\ &\quad \overline{\Psi_{p,N+1}(X_p;k\cup K_N)}\Psi_{p,N}(X_p;K_N)\ .\end{aligned}$$

Equation (1.4) then becomes:

$$\begin{aligned}\mathrm{Im} z \sum_{\substack{p\le P\\ n\le N}} \|\Psi_{p,n}\|^2_{\mathscr{H}_{p,n}} = (N+1)^{1/2}\sum_{p\le P}\int \mathrm{d}X_p \mathrm{d}K_N \mathrm{d}k\, \overline{f(X_p,k)} \\ \overline{\Psi_{p,N+1}(X_p;k\cup K_N)}\Psi_{p,N}(X_p;K_N)\ ,\end{aligned}$$

so we obtain

$$\begin{aligned}\frac{|\mathrm{Im} z|}{(N+1)^{1/2}}\sum_{\substack{p\le P\\ n\le N}} \|\Psi_{p,n}\|^2_{\mathscr{H}_{p,n}} &\le \lambda \|f_0\|_2 \sum_{p\le P} p \|\Psi_{p,N+1}\|_{\mathscr{H}_{p,N+1}} \|\Psi_{p,N}\|_{\mathscr{H}_{p,N}} \\ &\le \frac{\lambda}{2} P \|f_0\|_2 \left(\sum_{p\le P}\|\Psi_{p,N+1}\|^2_{\mathscr{H}_{p,N+1}} + \sum_{p\le P}\|\Psi_{p,N}\|^2_{\mathscr{H}_{p,N}}\right)\ .\end{aligned} \tag{1.5}$$

We fix P. If we have

$$\sum_{\substack{p\le P\\ 0\le n<\infty}} \|\Psi_{p,n}\|^2_{\mathscr{H}_{p,n}} \equiv S_P^2 < \infty\ ,$$

then $\exists N(P)$ such that $\forall N \ge N(P)$

$$\frac{1}{2}S_P^2 \le \sum_{\substack{p\le P\\ 0\le n\le N}} \|\Psi_{p,n}\|^2_{\mathscr{H}_{p,n}} \le S_P^2\ .$$

From (1.5) we would have for all $N \ge N(P)$

$$\frac{|\mathrm{Im} z|}{(N+1)^{1/2}} S_P^2 \le \lambda P \|f_0\|_2 \left(\sum_{p\le P}\|\Psi_{p,N+1}\|^2_{\mathscr{H}_{p,N+1}} + \sum_{p\le P}\|\Psi_{p,N}\|^2_{\mathscr{H}_{p,N}}\right)\ ,$$

hence

$$\begin{aligned}|\mathrm{Im} z|\, S_P^2 \sum_{N(P)\le N\le N'} (N+1)^{-1/2} &\le 2\lambda P \|f_0\|_2 \sum_{\substack{p\le P\\ N(P)\le N\le N'+1}} \|\Psi_{p,N}\|^2_{\mathscr{H}_{p,N}} \\ &\le 2\lambda P \|f_0\|_2\, S_P^2\ ,\end{aligned}$$

and that is absurd, since

$$\sum_{N(P)\le N<\infty} (N+1)^{-1/2}$$

is divergent, unless $\Psi = 0$. ∎

2. Invariance of domains

In order to prove invariance under evolution of some useful domains we formulate the following lemma:

LEMMA 2.1 (Ordered products of creation and annihilation operators).
Let $F(\lambda)$ be the spectral family of the operator $P+N$, $f(P,N)$ any F-measurable operator-valued function, with domain $D(f)$; consider now the operator

$$B = \int \mathrm{d}X_q \mathrm{d}Y_r \mathrm{d}K_i \mathrm{d}M_j \, g(X_q, Y_r, K_i, M_j) \psi^*(X_q)\psi(Y_r) a^*(K_i) a(M_j) \ ,$$

defined on $D(B)$, with $q, r, i, j \in \mathbb{N}$ and $q + r + i + j = \delta$. Then:

i. The following equality holds:

$$f(P,N)B\Psi = Bf(P+q-r, N+i-j)\Psi$$

for suitable Ψ.

ii. For all $g \in L^2(\mathbb{R}^{3\delta})$ and $\Phi \in D(Q^\delta)$ the following estimate holds:

$$\|B\Phi\| \leq \|g\|_{L^2(\mathbb{R}^{3\delta})} \left\| \frac{\sqrt{P!(P+q-r)!N!(N+i-j)!}}{(P-r)!(N-j)!} \theta(P-r)\theta(N-j)\Phi \right\|$$

where $\theta(\xi) = 1$ if $\xi \geq 0$ and zero otherwise, with $\xi \in \mathbb{Z}$.

PROOF. This Lemma is an extension of Lemmas 1.2 and 2.4 of Appendix A, and the proof is perfectly analogous. ■

PROPOSITION 4 (Invariance of $D((P^2+N))$ and $D(H_0)$ under evolution). *$U_0(\cdot)$ transforms $D(H_0)$ and $D(f(P,N))$, where f is any F-measurable function (see Lemma 2.1), into themselves, $U(\cdot)$ transforms $D(H)$ and $D((P^2+N))$ into themselves; in particular we have:*

i. $U_0(t)\Phi \in D(H_0)$ for all $t \in \mathbb{R}$, $\Phi \in D(H_0)$, and

$$\|H_0 U_0(t)\Phi\| = \|H_0\Phi\| \ ;$$

ii. $U_0(t)\Phi \in D(f(P,N))$ for all $t \in \mathbb{R}$, $\Phi \in D(f(P,N))$, and

$$\|f(P,N)U_0(t)\Phi\| = \|f(P,N)\Phi\| \ ;$$

iii. $U(t)\Phi \in D(H)$ for all $t \in \mathbb{R}$, $\Phi \in D(H)$, and

$$\|HU(t)\Phi\| = \|H\Phi\| \ .$$

iv. $U(t)\Phi \in D((P^2+N)^\delta)$ for all $t \in \mathbb{R}$, $\Phi \in D((P^2+N)^\delta)$, $\delta \in \mathbb{R}$; and

$$\left\|(P^2+N+1)^\delta U(t)\Phi\right\| \leq \exp(|\delta|\, \mu_\delta \lambda \left\|f_0\right\|_2 |t|) \left\|(P^2+N+1)^\delta \Phi\right\| \ ,$$

with $\mu_\delta = \max(3, 1+2^{|\delta|})$.

PROOF.

i. and *iii.* Both statements are an easy application of Stone's Theorem.

ii. The proof is straightforward since P and N commute with H_0 and $U_0(t)$ is unitary for all $t \in \mathbb{R}$.

iv. Let $\Phi \in D(H_0\big|_p)$, $0 < h(N)$ a bounded operator on $\mathscr{H}_p$ such that $\operatorname{Ran} h(N) \subset D(N^{1/2})$. Define the differentiable quantity

$$M(t) \equiv \frac{1}{2} \left\|h(N)U(t)\Phi\right\|^2 \ .$$

With a bit of manipulation and since H_0 commutes with N we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t) &= \operatorname{Im}\langle h(N)U(t)\Phi, a(f)\Big(h(N-1)h(N)^{-1} - 1\Big)h(N)U(t)\Phi\rangle \\ &\quad + \operatorname{Im}\langle a(f)\Big(h(N)h(N-1)^{-1} - 1\Big)h(N)U(t)\Phi, h(N)U(t)\Phi\rangle.\end{aligned}$$

So we can bound the derivative of $M(t)$ by

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}M(t)\right| \leq 2p\lambda \left\|f_0\right\|_2 \Bigg[|||\sqrt{N}\big(h(N-1)h(N)^{-1}-1\big)||| + |||\sqrt{N}\big(h(N)h(N-1)^{-1}-1\big)||| \Bigg] M(t) \; .$$

Let $h \in \mathscr{C}^1$, $h(\cdot)$ and $|h'(\cdot)|$ non-increasing; then

$$|h(N-1)-h(N)| \leq |h'(N-1)| \; ,$$

and

$$K \equiv \Big[\dots\Big] \leq \sup_{n=0,1,\dots} \sqrt{n}\,|h'(n-1)|\,h^{-1}(n) + \sup_{n=0,1,\dots} \sqrt{n}\,|h'(n-1)|\,h^{-1}(n-1) \; .$$

We are interested in the case $h(n) = (n+j+1)^{-\delta}$, with $\delta \geq 1/2$ (so $\operatorname{Ran} h(N) \subset D(N^{1/2})$) and $j \geq 1$. h satisfies the hypothesis above and $h'(n) = -\delta(n+j+1)^{-\delta-1}$. So we have that

$$|h'(n-1)|\,h^{-1}(n) = \delta(n+j)^{-1}\Big(1+\frac{1}{n+j}\Big)^{\delta} \leq \delta 2^{\delta}(n+j)^{-1} \; ,$$
$$|h'(n-1)|\,h^{-1}(n-1) = \delta(n+j)^{-1} \; .$$

The function $g(x) = \sqrt{x}/(x+j)$, with $x \geq 0$ has a maximum when $x = j$, so

$$g(x) \leq g(j) \leq \frac{1}{2}j^{-1/2} \; ,$$
$$K \leq \frac{1}{2}\delta(1+2^{\delta})j^{-1/2} \; .$$

We have then the following differential inequality for $M(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) \leq pj^{-1/2}\lambda \left\|f_0\right\|_2 \delta(1+2^{\delta})M(t) \; ,$$

so the Gronwall Lemma implies

$$M(t) \leq e^{pj^{-1/2}\delta(1+2^{\delta})\lambda\|f_0\|_2 t} M(0) \; .$$

Set now $j = p^2$, with $p \geq 1$:

$$\left\|(N+p^2+1)^{-\delta}U(t)\Phi\right\| \leq e^{\delta(1+2^{\delta})\lambda\|f_0\|_2 t} \left\|(N+p^2+1)^{-\delta}\Phi\right\| \; ; \tag{2.1}$$

for all $\delta \geq 1/2$ and $\Phi \in D(H_0\big|_p)$. Interpolating between $\delta = 0$ and $\delta = 1$ we extend the result to $0 \leq \delta \leq 1$:

$$\left\|(N+p^2+1)^{-\delta}U(t)\Phi\right\| \leq e^{3\delta\lambda\|f_0\|_2 t} \left\|(N+p^2+1)^{-\delta}\Phi\right\| \; . \tag{2.2}$$

These results extend immediately to all $\Phi \in \mathscr{H}_p$. Now let $A = (N+p^2+1)^{\delta}$, so we write equations (2.1) and (2.2) (depending on the value of δ) in compact notation as

$$\left\|A^{-1}U(t)A\Phi\right\| \leq a(t) \left\|\Phi\right\| \; ,$$

for all $\Phi \in D(A)$. Let $\Psi \in \mathscr{H}_p$ and $\Phi \in D(A)$; then

$$\left|\langle\Psi, A^{-1}U(t)A\Phi\rangle\right| = \left|\langle U(-t)A^{-1}\Psi, A\Phi\rangle\right| \leq a(t) \left\|\Psi\right\| \left\|\Phi\right\| \; ,$$

so $U(-t)A^{-1}\Psi \in D(A^\dagger) = D(A)$ since A is self-adjoint, so

$$\begin{aligned}
&\langle AU(-t)A^{-1}\Psi, \Phi\rangle = \langle U(-t)A^{-1}\Psi, A\Phi\rangle \ ,\\
&\left\|AU(-t)A^{-1}\Psi\right\| \leq a(t)\,\|\Psi\| \ ,\\
&a(t) = e^{\delta\mu_\delta\lambda\|f_0\|_2 t} \ ,\\
&\mu_\delta = \max(3, 1+2^\delta) \ .
\end{aligned}$$

The result on $\mathscr{H}$ follows by taking the direct sum of all p. ■

3. Weyl operators.

Weyl operators are described in great detail in Appendix A, however we will give here their definition, and state the properties we will use the most as a proposition, to help readability. Proofs can be recovered in the Appendix.

Let u and α in $L^2(\mathbb{R}^3)$, then the following operators are skew self-adjoint:

$$\begin{aligned}
&\overline{(\psi^*(u) - \psi(\bar{u}))} = -\left(\overline{(\psi^*(u) - \psi(\bar{u}))}\right)^\dagger\\
&\overline{(a^*(\alpha) - a(\bar{\alpha}))} = -\left(\overline{(a^*(\alpha) - a(\bar{\alpha}))}\right)^\dagger \ .
\end{aligned}$$

Definition (Weyl operators). For all $u, \alpha \in L^2(\mathbb{R}^3)$ we define the following unitary operators:

$$\begin{aligned}
&C_p(u) = \exp\Big[\overline{(\psi^*(u) - \psi(\bar{u}))}\Big] \text{ defined on } \bigoplus_{p=0}^{\infty}\mathscr{H}_{p,0} \ ;\\
&C_n(\alpha) = \exp\Big[\overline{(a^*(\alpha) - a(\bar{\alpha})}\Big] \text{ defined on } \bigoplus_{n=0}^{\infty}\mathscr{H}_{0,n} \ ;\\
&C(u,\alpha) = C_p(u)\otimes C_n(\alpha) \ .
\end{aligned}$$

Proposition 5 (Properties of Weyl Operators).

i. *$C(u,\alpha)$ is unitary and strongly continuous as a function of u or α in $L^2(\mathbb{R}^3)$. Furthermore, for any $\Phi \in D(\psi(\bar{\gamma}))$ and $\Psi \in D(a(\bar{\gamma}))$, with $\gamma \in L^2(\mathbb{R}^3)$, $C(u,\alpha)\Phi \in D(\psi(\bar{\gamma}))$, $C(u,\alpha)\Psi \in D(a(\bar{\gamma}))$ and the following identities hold:*

$$\begin{aligned}
&C(u,\alpha)^\dagger\psi(\bar{\gamma})C(u,\alpha)\Phi = \psi(\bar{\gamma})\Phi + \langle\gamma,u\rangle_2\Phi \ ;\\
&C(u,\alpha)^\dagger a(\bar{\gamma})C(u,\alpha)\Psi = a(\bar{\gamma})\Psi + \langle\gamma,\alpha\rangle_2\Psi \ ;\\
&C(u,\alpha)^\dagger\psi^*(\gamma)C(u,\alpha)\Phi = \psi^*(\gamma)\Phi + \langle u,\gamma\rangle_2\Phi \ ;\\
&C(u,\alpha)^\dagger a^*(\gamma)C(u,\alpha)\Psi = a^*(\gamma)\Psi + \langle\alpha,\gamma\rangle_2\Psi \ .
\end{aligned}$$

ii. *Let $u,\alpha : t \to u(t),\alpha(t) \in \mathscr{C}^1(\mathbb{R}, L^2)$. Then $C(u(t),\alpha(t))$ is strongly differentiable in t from $D(P^{1/2})\cap D(N^{1/2})$ to $\mathscr{H}$. The derivative is given by*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}C(u(t),\alpha(t)) = C(u(t),\alpha(t))\big[&\psi^*(\dot{u}) - \psi(\dot{\bar{u}}) + i\mathrm{Im}\langle u,\dot{u}\rangle\\
&+a^*(\dot{\alpha}) - a(\dot{\bar{\alpha}}) + i\mathrm{Im}\langle\alpha,\dot{\alpha}\rangle\big]\\
\big[&\psi^*(\dot{u}) - \psi(\dot{\bar{u}}) - i\mathrm{Im}\langle u,\dot{u}\rangle + a^*(\dot{\alpha})\\
&-a(\dot{\bar{\alpha}}) - i\mathrm{Im}\langle\alpha,\dot{\alpha}\rangle\big]C(u(t),\alpha(t)) \ .
\end{aligned}$$

where $\dot{u}$, $\dot{\alpha}$ are the time derivatives respectively of u and α.

iii. Let $u, \alpha \in L^2(\mathbb{R}^3)$*. Then for all* $\delta \in \mathbb{R}$*, we have the following invariances:*

$$C(u,\alpha)\Phi \in D(N^\delta) \ \forall \Phi \in D(N^\delta) \ ,$$
$$C(u,\alpha)\Phi \in D(P^\delta) \ \forall \Phi \in D(P^\delta) \ ,$$
$$C(u,\alpha)\Phi \in D(Q^\delta) \ \forall \Phi \in D(Q^\delta) \ .$$

iv. We recall the definition of $U_{01}(t) \equiv \exp(i\Delta t/2)$ *and* $U_{02}(t) \equiv \exp(-i\omega t)$ *given in the previous chapter. They are unitary operators on* $L^2(\mathbb{R}^3)$*. Now define* $\tilde{u}(t) = U_{01}^\dagger(t)u(t)$*,* $\tilde{\alpha}(t) = U_{02}^\dagger(t)\alpha(t)$ *for all* $u, \alpha \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3))$*. Then the following equality holds* $\forall \Phi \in \mathscr{H}$ *and* $t \in \mathbb{R}$*:*

$$U_0^\dagger(t)C(u(t),\alpha(t))U_0(t) = C(\tilde{u}(t),\tilde{\alpha}(t)) \ .$$

CHAPTER 4

The quantum fluctuations.

We will now study the time-dependent evolution operator U_2 that describes the quantum fluctuations of the system. We start with some definitions:

Definitions ($Q(B)$, $\mathscr{H}^\delta$, $\mathscr{B}(\delta';\delta)$). Let $B \geq 0$ a self-adjoint operator, we define $Q(B) \subseteq \mathscr{H}$ the form domain of B, *i.e.* $Q(B) = D(B^{1/2})$. $Q(B)$ is a Hilbert space with norm $\left\|(B+1)^{1/2}\Phi\right\|$. We denote $Q^*(B)$ the completion of $\mathscr{H}$ in the norm $\left\|(B+1)^{-1/2}\Phi\right\|$. Finally we define the Hilbert spaces $\mathscr{H}^\delta$, $\delta \in \mathbb{R}$: $\mathscr{H}^\delta = Q((P+N)^\delta)$ for $\delta \geq 0$, and $\mathscr{H}^\delta = Q^*((P+N)^{|\delta|})$ for $\delta < 0$; $\mathscr{H}^\delta$ is a Hilbert space in the norm

$$\|\Phi\|_\delta = \left\|(P+N+1)^{\delta/2}\Phi\right\| \ .$$

We will denote $\mathscr{B}(\delta';\delta)$ the space of bounded operators from $\mathscr{H}^{\delta'}$ to $\mathscr{H}^\delta$.

1. The operator $\widetilde{V}(t)$.

Definition ($V(t)$). We define the operator $V(t) \equiv V_{--}(t)+V_{-+}(t)+V_{+-}(t)+V_{++}(t)+V_0(t)$, defined on $D(V(t))$, where ($-$ is related to annihilation, and $+$ to creation):

$$V_{\#\#}(t) = \int \mathrm{d}x\mathrm{d}k \, v_{\#\#}(t,x,k)\psi^{\#}(x)a^{\#}(k) \ ,$$

$$V_0(t) = \int \mathrm{d}x \, (\check{\chi} * A(t))(x)\psi^*(x)\psi(x) = \mathrm{d}\Gamma_p((\check{\chi} * A(t))(\cdot)) \otimes 1 \ ,$$

$v_{\#\#} \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3 \otimes \mathbb{R}^3))$ and $A \in \mathscr{C}^0(\mathbb{R}, L^3(\mathbb{R}^3))$ (so $\check{\chi} * A \in \mathscr{C}^0(\mathbb{R}, L^\infty(\mathbb{R}^3))$). Let $u \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3))$, then we can write explicitly the specific $v_{\#\#}$s of the system:

$$v_{\#-} = f_0(k)e^{ik\cdot x}u^{\#}(t,x) \ ;$$

$$v_{\#+} = f_0(k)e^{-ik\cdot x}u^{\#}(t,x) \ .$$

Lemma 1.1 (Self-adjointness of $V(t)$). *For all $t \in \mathbb{R}$, $V(t)$ is essentially self-adjoint on any core of Q.*

Proof. Using estimates in Lemma 2.1 we can apply Lemma 4.2 of Appendix A. ∎

We would like to define the evolution operator of the quantum fluctuations as the evolution group generated by $H_2 = H_0 + V(t)$; however this could be done with mathematical rigour only passing to the so-called interaction representation.

Definition ($\widetilde{V}(t)$ (interaction representation)).

$$\widetilde{V}(t) = U_0^\dagger(t)V(t)U_0(t) \ .$$

Obviously we have $D(\widetilde{V}(t)) = U_0^\dagger(t)[D(V(t))]$ for all $t \in \mathbb{R}$.

Remark. $\widetilde{V}(t)$ is essentially self-adjoint on $D(Q)$ for all $t \in \mathbb{R}$. This is due to the fact that $U_0(t)[D(Q)] \subseteq D(Q)$ and $\widetilde{V}(t) = U_0^\dagger(t)V(t)U_0(t)$.

LEMMA 1.2. *For all* $\Phi \in D(\widetilde{V}(t))$ *we have* $\widetilde{V}(t) = \widetilde{V}_{--}(t) + \widetilde{V}_{-+}(t) + \widetilde{V}_{+-}(t) + \widetilde{V}_{++}(t) + \widetilde{V}_0(t)$*, with:*

$$\begin{aligned}
&\widetilde{V}_{\#\#}(t) = \int \mathrm{d}x\mathrm{d}k\, \widetilde{v}_{\#\#}(t,x,k)\psi^{\#}(x)a^{\#}(k)\ ,\\
&\widetilde{v}_{--}(x,k) = U_{01}(t)U_{02}(t)v_{--}(x,k)\ ,\\
&\widetilde{v}_{-+}(x,k) = U_{01}(t)U_{02}^{\dagger}(t)v_{-+}(x,k)\ ,\\
&\widetilde{v}_{+-}(x,k) = U_{01}^{\dagger}(t)U_{02}(t)v_{+-}(x,k)\ ,\\
&\widetilde{v}_{++}(x,k) = U_{01}^{\dagger}(t)U_{02}^{\dagger}(t)v_{++}(x,k)\ ,\\
&\widetilde{V}_0 = \mathrm{d}\Gamma_p\Big(U_{01}^{\dagger}(t)(\check{\chi} * A(t))(\cdot)U_{01}(t)\Big)\otimes 1\ .
\end{aligned}$$

Furthermore $\left\|\widetilde{v}_{\#\#}(t)\right\|_2 = \left\|v_{\#\#}(t)\right\|_2$ *for all* $t \in \mathbb{R}$.

PROOF. If we set $E_0(t) = U_{01}(t)U_{02}(t)$, from the definition of $\widetilde{V}_{--}(t)$ we obtain for all $\Phi \in D(\widetilde{V}(t))$ (the explicit dependence on variables X_p and K_n is omitted):

$$\Big(\widetilde{V}_{--}(t)\Phi\Big)_{p,n} = \sqrt{(p+1)(n+1)}\langle\overline{v_{--}}(\cdot,\cdot,t), E_0(t)\Phi_{p+1,n+1}(\cdot,\cdot)\rangle_2\ .$$

Since $E_0(t)$ is unitary on $L^2(\mathbb{R}^3\otimes\mathbb{R}^3)$, we obtain the sought result from the relation

$$\langle\overline{v_{--}}(\cdot,\cdot,t), E_0(t)\Phi_{p+1,n+1}(\cdot,\cdot)\rangle_2 = \langle E_0^{\dagger}(t)\overline{v_{--}}(\cdot,\cdot,t), \Phi_{p+1,n+1}(\cdot,\cdot)\rangle_2.$$

We proceed in the same manner for $\widetilde{V}_{-+}(t)$. From these results we obtain the ones on $\widetilde{V}_{++}(t)$ and $\widetilde{V}_{+-}(t)$ using the definition of adjoint operator. Finally we can check directly that

$$\begin{aligned}
(\exp(itH_0)V_0(t)\exp(-itH_0)\Phi)_{p,n}(X_p;K_n) = \sum_{j=1}^{p}&\exp\{-it\Delta_j/2M\}\\
&(\check{\chi} * A(t))(x_j)\exp\{it\Delta_j/2M\}\Phi_{p,n}(X_p;K_n)\ .
\end{aligned}$$

■

LEMMA 1.3. *Let* $\widetilde{V}(t)$ *be defined as above, with* $u \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3))$ *and* $(\check{\chi}*A) \in \mathscr{C}^0(\mathbb{R}, L^{\infty}(\mathbb{R}^3))$*. Then* $\forall\delta \in \mathbb{R}$*,* $\widetilde{V}(t)$ *belongs to* $\mathscr{B}(\delta+2;\delta)$*; furthermore is norm continuous as a function of* t*. We have in fact the following estimates:*

$$\begin{aligned}
&\left\|\widetilde{V}_{--}\Phi\right\|_{\delta}^2 \leq \frac{1}{2}C_{\delta}\left\|v_{--}(t)\right\|_2^2\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle\ ;\\
&\left\|\widetilde{V}_{-+}\Phi\right\|_{\delta}^2 \leq \left\|v_{-+}(t)\right\|_2^2\Big(\frac{1}{2}\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle + \langle\Phi,(Q+1)^{\delta+1}\Phi\rangle\Big)\ ;\\
&\left\|\widetilde{V}_{+-}\Phi\right\|_{\delta}^2 \leq \left\|v_{+-}(t)\right\|_2^2\Big(\frac{1}{2}\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle + \langle\Phi,(Q+1)^{\delta+1}\Phi\rangle\Big)\ ;\\
&\left\|\widetilde{V}_{++}\Phi\right\|_{\delta}^2 \leq C_{-\delta}\left\|v_{++}(t)\right\|_2^2\Big(\frac{1}{2}\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle + 2\langle\Phi,(Q+1)^{\delta+1}\Phi\rangle\\
&\qquad\qquad + \langle\Phi,(Q+1)^{\delta}\Phi\rangle\Big)\ ;\\
&\left\|\widetilde{V}_0\Phi\right\|_{\delta}^2 \leq \left\|(\check{\chi} * A(t))\right\|_{\infty}^2\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle\ ,
\end{aligned}$$

where $C_{\delta} = 1$ *if* $\delta \geq 0$*,* $C_{\delta} = 3^{|\delta|}$ *otherwise.*

PROOF. Throughout the proof let $\Phi \in \mathscr{H}^{\delta+2}$. We start proving the boundedness of $\widetilde{V}_{--}$; for the sake of simplicity we will perform calculations in the norm $|||\cdot|||_{\delta}$ defined as:

$$|||\cdot|||_{\delta} = \left\|(Q+3)^{\delta/2}\cdot\right\|\ ;$$

such norm is clearly equivalent to $\|\cdot\|_\delta$ for all $\delta \in \mathbb{R}$. Using Lemma 2.1 we obtain:

$$\begin{aligned}\left\|(Q+3)^{\delta/2}\widetilde{V}_{--}\Phi\right\|^2 &= \left\|\widetilde{V}_{--}(Q+1)^{\delta/2}\Phi\right\|^2 \\ &\le \|\widetilde{v}_{--}\|_2^2 \left\|P^{1/2}N^{1/2}(Q+1)^{\delta/2}\Phi\right\|^2 \\ &\le \frac{1}{2}\|v_{--}(t)\|_2^2 \langle\Phi,(Q+1)^{\delta+2}\Phi\rangle \ ;\end{aligned}$$

the sought result following from the inequality $(Q+1)^\delta \le C_\delta (Q+3)^\delta$ with

$$C_\delta = \begin{cases} 3^{|\delta|} \text{ if } \delta < 0 \ , \\ \ 1 \text{ if } \delta \ge 0 \ ; \end{cases}$$

Consider now $\widetilde{V}_{-+}$; again using Lemma 2.1 we have:

$$\begin{aligned}\left\|(Q+1)^{\delta/2}\widetilde{V}_{-+}\Phi\right\|^2 &= \left\|\widetilde{V}_{-+}(Q+1)^{\delta/2}\Phi\right\|^2 \\ \le \|\widetilde{v}_{-+}\|_2^2 &\left(\left\|P^{1/2}N^{1/2}(Q+1)^{\delta/2}\Phi\right\|^2 + \left\|P^{1/2}(Q+1)^{\delta/2}\Phi\right\|^2\right) \\ &\le \|v_{-+}(t)\|_2^2 \left(\frac{1}{2}\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle + \langle\Phi,(Q+1)^{\delta+1}\Phi\rangle\right) .\end{aligned}$$

In the same fashion we obtain:

$$\begin{aligned}\left\|(Q+1)^{\delta/2}\widetilde{V}_{+-}\Phi\right\|^2 \le \|v_{+-}(t)\|_2^2 &\left(\frac{1}{2}\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle\right. \\ &\left.+\langle\Phi,(Q+1)^{\delta+1}\Phi\rangle\right) .\end{aligned}$$

Again using Lemma 2.1 we have for $\widetilde{V}_{++}(t)$:

$$\begin{aligned}\left\|(Q+1)^{\delta/2}\widetilde{V}_{++}\Phi\right\|^2 \le \|v_{++}(t)\|_2^2 &\Big(\langle\Phi,PN(Q+3)^\delta\Phi\rangle \\ &+\langle\Phi,P(Q+3)^\delta\Phi\rangle + \langle\Phi,N(Q+3)^\delta\Phi\rangle \\ &+\langle\Phi,(Q+3)^\delta\Phi\rangle\Big) \ ;\end{aligned}$$

so, since $(Q+3)^\delta \le C_{-\delta}(Q+1)^\delta$, we can write

$$\begin{aligned}\left\|\widetilde{V}_{++}\Phi\right\|_\delta^2 \le C_{-\delta}\|v_{++}(t)\|_2^2 &\Big(\frac{1}{2}\langle\Phi,(Q+1)^{\delta+2}\Phi\rangle \\ &+\langle\Phi,(Q+1)^{\delta+1}\Phi\rangle + \langle\Phi,(Q+1)^{\delta+1}\Phi\rangle \\ &+\langle\Phi,(Q+1)^{\delta}\Phi\rangle\Big) .\end{aligned}$$

Finally the estimate for $\widetilde{V}_0(t)$ is trivial, since it commutes with Q:

$$\begin{aligned}\left\|\widetilde{V}_0\Phi\right\|_\delta = \left\|\widetilde{V}_0(Q+1)^{\delta/2}\Phi\right\| &\le \|\check{\chi} * A(t)\|_\infty \left\|P(Q+1)^{\delta/2}\Phi\right\| \\ &\le \|\check{\chi} * A(t)\|_\infty \|\Phi\|_{\delta+2} .\end{aligned}$$

■

2. The evolution with cut off $\widetilde{U}_{2;\mu}(t,s)$.

To construct the evolution operator $\widetilde{U}_2(t,s)$ generated by $\widetilde{V}(t)$, we will use the Dyson series. However in order to do that we have to introduce a cut off in the total number of particles: let $\sigma_1 \in \mathscr{C}^1(\mathbb{R}^+)$, positive and decreasing, $\sigma_1(s) = 1$ if $s \le 1$, $\sigma_1(s) = 0$ if $s \ge 2$; define σ_μ the operator $\sigma_1(Q/\mu)$ in $\mathscr{H}$. Then we set $\widetilde{V}_\mu(t) = \sigma_\mu \widetilde{V}(t)\sigma_\mu$, for all $\mu \ge 1$.

LEMMA 2.1. *Let $\widetilde{V}_\mu(t)$ be defined as above, then:*

i. $\widetilde{V}_\mu(t)$ *satisfies Lemma 1.3, with uniform bound in* μ. *Furthermore* $\widetilde{V}_\mu(t)$ *is in* $\mathscr{B}(\delta;\delta)$ *for all* $\delta \in \mathbb{R}$ *and is norm continuous as a function of* t.

ii. For all δ *in* $\mathbb{R}$, $\widetilde{V}_\mu(t) \to \widetilde{V}(t)$ *when* μ *goes to infinity, in norm on* $\mathscr{B}(\delta+2+\varepsilon;\delta)$, $\varepsilon > 0$, *and strongly in* $\mathscr{B}(\delta+2;\delta)$, *uniformly in* t *on bounded intervals.*

Proof. To prove i. observe that σ_μ belongs to $\mathscr{B}(\delta;\delta')$ for all δ and δ' and $\|\sigma_\mu \Phi\|_{\delta'}^2 \le C(\mu)\,\|\Phi\|_\delta$, with

$$C(\mu) = \sup_{p+n\le 2\mu}\left[\sigma_1^2\Big(\frac{p+n}{\mu}\Big)(p+n+1)^{\delta'-\delta}\right] .$$

Obviously if $\delta' \le \delta$, $C(\mu) \le 1$ for all $\mu \ge 1$, and we have a uniform bound in μ. Point i. follows from that using Lemma 1.3.

Concerning point ii. the strong convergence of $\widetilde{V}_\mu(t)$ to $\widetilde{V}(t)$ in $\mathscr{B}(\delta+2;\delta)$ follows from the obvious strong convergence on $\mathscr{C}_0(P,N)$, since $\widetilde{V}_\mu(t)$ is bounded in $\mathscr{B}(\delta+2;\delta)$ uniformly in μ. Norm convergence on $\mathscr{B}(\delta+2+\varepsilon;\delta)$ follows from the fact that $(1-\sigma_\mu)(Q+1)^{-\varepsilon}$ goes to zero in norm as an operator in $\mathscr{H}$; as a matter of fact we have

$$\begin{aligned}\left\|(1-\sigma_\mu)(Q+1)^{-\varepsilon}\Phi\right\|^2 &= \sum_{p,n}\frac{(1-\sigma_1(\frac{p+n}{\mu}))^2}{(p+n+1)^{2\varepsilon}}\,\|\Phi\|_{p,n}^2\\ &\le \sum_{p+n\ge\mu}(p+n+1)^{-2\varepsilon}\,\|\Phi\|_{p,n}^2 \le (\mu+1)^{-2\varepsilon}\,\|\Phi\|^2\end{aligned}$$

i.e.

$$\left\|(1-\sigma_\mu)(Q+1)^{-\varepsilon}\right\| \le (\mu+1)^{-2\varepsilon} \to 0 ,$$

when μ goes to infinity. ■

Definition ($\widetilde{U}_2(t,s)$). The unitary group $\widetilde{U}_{2;\mu}(t,s)$ is defined by means of a Dyson series:

$$\widetilde{U}_{2;\mu}(t,s) = \sum_{m=0}^{\infty}(-i)^m\int_s^t \mathrm{d}t_1\int_s^{t_1}\mathrm{d}t_2\cdots\int_s^{t_{m-1}}\mathrm{d}t_m\,\widetilde{V}_\mu(t_1)\ldots\widetilde{V}_\mu(t_m) .$$

Using previous Lemma we see that the series converge in norm on $\mathscr{B}(\delta;\delta)$ and $\widetilde{U}_{2;\mu}(t,s)$ is continuous and differentiable in norm with respect to t on $\mathscr{B}(\delta;\delta)$ for all real δ. We list below some useful properties of the family $\widetilde{U}_{2;\mu}(t,s)$, whose proof is immediate since $\widetilde{V}_\mu \in \mathscr{B}(\delta;\delta)$ for all $\delta\in\mathbb{R}$:

Lemma 2.2.

i. $\widetilde{U}_{2;\mu}(s,s) = 1$, $\widetilde{U}_{2;\mu}(t,r)\widetilde{U}_{2;\mu}(r,s) = \widetilde{U}_{2;\mu}(t,s)$ *for all* $r,s,t\in\mathbb{R}$.

ii. $\widetilde{U}^\dagger_{2;\mu}(t,s) = \widetilde{U}_{2;\mu}(s,t)$, *and* $\widetilde{U}_{2;\mu}(t,s)$ *are unitary in* $\mathscr{H}$.

iii. $\widetilde{U}_{2;\mu}(t,s)$ *is norm differentiable on* $\mathscr{B}(\delta;\delta)$ *for all real* δ, *and*

$$i\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{U}_{2;\mu}(t,s) = \widetilde{V}_\mu(t)\widetilde{U}_{2;\mu}(t,s) ;$$

$$i\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{U}_{2;\mu}(t,s) = -\widetilde{U}_{2;\mu}(t,s)\widetilde{V}_\mu(s) .$$

The operators $\widetilde{U}_{2;\mu}(t,s)$ also satisfy the following crucial boundedness property:

Lemma 2.3. *Let* $u \in \mathscr{C}^0(\mathbb{R},L^2)$ *and* $\check{\chi} * A \in \mathscr{C}^0(\mathbb{R},L^\infty)$. *Then the operator* $\widetilde{U}_{2;\mu}(t,s)$ *is bounded on* $\mathscr{H}^\delta$ *uniformly in* μ *for all real* δ. *More precisely:*

$$\left\|\widetilde{U}_{2;\mu}(t,s)\right\|_{\mathscr{B}(\delta;\delta)} \le \exp\left\{\frac{|\delta|}{2}\left(\ln 3 + \sqrt{2}\rho_\delta\left|\int_s^t \mathrm{d}\tau\,\|v_{--}(\tau)\|_2\right|\right)\right\} , \tag{2.1}$$

with $\rho_\delta = \max(4, 3^{|\delta|/2}+1)$.

PROOF. Let $\Phi \in \mathscr{H}$, $0 < h(Q)$ a bounded operator on $\mathscr{H}$ such that $\mathrm{Ran}\, h(Q) \subset D(Q)$. Define the differentiable quantity

$$M(t) \equiv \frac{1}{2} \left\| h(Q)\widetilde{U}_{2;\mu}(t,s)\Phi \right\|^2 .$$

With a bit of manipulation we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t) = \mathrm{Im}\langle h(Q)\widetilde{U}_{2;\mu}(t,s)\Phi, \widetilde{V}_{--;\mu}(t)\Big(h(Q-2)h(Q)^{-1} \\ -1\Big)h(Q)\widetilde{U}_{2;\mu}(t,s)\Phi\rangle + \mathrm{Im}\langle \widetilde{V}_{--;\mu}(t)\Big(h(Q)h(Q-2)^{-1} \\ -1\Big)h(Q)\widetilde{U}_{2;\mu}(t,s)\Phi, h(Q)\widetilde{U}_{2;\mu}(t,s)\Phi\rangle.\end{aligned}$$

So we can bound the derivative of $M(t)$ by

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}M(t)\right| \le \frac{1}{\sqrt{2}} \|v_{--}\|_2 \Big[|||(Q+1)\big(h(Q-2)h(Q)^{-1}-1\big)||| \\ + |||(Q+1)\big(h(Q)h(Q-2)^{-1}-1\big)|||\Big] M(t) .\end{aligned}$$

Let $h \in \mathscr{C}^1$, $h(\cdot)$ and $|h'(\cdot)|$ non-increasing; then

$$|h(Q-2) - h(Q)| \le 2\,|h'(Q-2)| ,$$

and

$$\begin{aligned}K \equiv \Big[\dots\Big] \le 2 \sup_{p,n=0,1,\dots} (p+n+1)\,|h'(p+n-2)|\,h^{-1}(p+n) \\ +2 \sup_{p,n=0,1,\dots} (p+n+1)\,|h'(p+n-2)|\,h^{-1}(p+n-2) .\end{aligned}$$

We are interested in the case $h(p+n) = (p+n+3)^{-\delta}$, with $\delta \ge 1$ (in order to fulfill the condition $\mathrm{Ran}\, h(Q) \subset D(Q)$). h satisfies the hypothesis above and $h'(p+n) = -\delta(p+n+3)^{-\delta-1}$. So we have that

$$K \le 2 \sup_{p,n=0,1,\dots} \Big(\delta\big(\frac{p+n+3}{p+n+1}\big)^{\delta} + \delta\Big) = 2\delta(3^\delta+1)$$

We have then the following differential inequality for $M(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) \le \sqrt{2}\,\|v_{--}\|_2\,\delta(3^\delta+1)M(t) ,$$

so the Gronwall's Lemma implies

$$\begin{aligned}\left\|(P+N+3)^{-\delta}\widetilde{U}_{2;\mu}(t,s)\Phi\right\| \le e^{\sqrt{2}\delta(3^\delta+1)\left|\int_s^t \mathrm{d}\tau\, \|v_{--}(\tau)\|_2\right|} \\ \left\|(P+N+3)^{-\delta}\Phi\right\|\end{aligned}$$

for all $\delta \ge 1$. We then obtain by interpolation the result for $0 \le \delta \le 1$:

$$\begin{aligned}\left\|(P+N+3)^{-\delta}\widetilde{U}_{2;\mu}(t,s)\Phi\right\| \le e^{4\sqrt{2}\delta\left|\int_s^t \mathrm{d}\tau\, \|v_{--}(\tau)\|_2\right|} \\ \left\|(P+N+3)^{-\delta}\Phi\right\| .\end{aligned}$$

Finally by duality we extend the result to all $\delta \in \mathbb{R}$. ∎

3. The unitary evolution $\widetilde{U}_2(t,s)$.

We are ready to define the fluctuations evolution operator in interaction representation $\widetilde{U}_2(t,s)$. We will do that in the following Proposition that also describes its key properties.

PROPOSITION 6 (Quantum fluctuations evolution operator). *Let $\widetilde{V}(t)$ defined as above, with $u \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3))$, $\check{\chi} * A \in \mathscr{C}^0(\mathbb{R}, L^\infty(\mathbb{R}^3))$. Then exists a family of operators $\widetilde{U}_2(t,s)$ satisfying the following properties:*

i. for all $\delta \in \mathbb{R}$, $\widetilde{U}_2(t,s)$ is bounded and strongly continuous with respect to t and s on $\mathscr{H}^\delta$ and satisfies

$$\left\|\widetilde{U}_2(t,s)\right\|_{\mathscr{B}(\delta;\delta)} \leq \exp\left\{\frac{|\delta|}{2}\left(\ln 3 + \sqrt{2}\rho_\delta \left|\int_s^t \mathrm{d}\tau \, \|v_{--}(\tau)\|_2\right|\right)\right\} , \tag{3.1}$$

with $\rho_\delta = \max(4, 3^{|\delta|/2} + 1)$.

ii. $\widetilde{U}_2(t,s)$ is unitary in $\mathscr{H}$.

iii. $\widetilde{U}_2(s,s) = 1$, $\widetilde{U}_2(t,r)\widetilde{U}_2(r,s) = \widetilde{U}_2(t,s)$ for all r, s and t in $\mathbb{R}$.

iv. For all $\delta \in \mathbb{R}$, $\widetilde{U}_2(t,s)$ is strongly differentiable from $\mathscr{H}^{\delta+2}$ to $\mathscr{H}^\delta$; in particular is strongly differentiable from $D(Q)$ to $\mathscr{H}$. Furthermore:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{U}_2(t,s) = \widetilde{V}(t)\widetilde{U}_2(t,s) \; ;$$

$$i\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{U}_2(t,s) = -\widetilde{U}_2(t,s)\widetilde{V}(s) \; .$$

v. For all $\Psi \in D(Q)$ and $\Phi \in \mathscr{H}$

$$i\partial_t \langle \Psi, \widetilde{U}_2(t,s)\Phi\rangle = \langle \widetilde{V}(t)\Psi, \widetilde{U}_2(t,s)\Phi\rangle \; .$$

vi. Let $U_2(t,s) = U_0(t)\widetilde{U}_2(t,s)U_0^{-1}(s)$; for all $\Psi \in D(Q) \cap D(H_0)$, $\Phi \in \mathscr{H}$

$$i\partial_t \Big\langle \Psi, U_2(t,s)\Phi\Big\rangle = \Big\langle \Big(H_0 + V(t)\Big)\Psi, U_2(t,s)\Phi\Big\rangle \; .$$

PROOF. i. For all couples of positive integers μ and ν, write

$$\widetilde{U}_{2;\mu}(t,s) - \widetilde{U}_{2;\nu}(t,s) = -i\int_s^t \mathrm{d}\tau \, \widetilde{U}_{2;\nu}(t,\tau)(\widetilde{V}_\mu(\tau) - \widetilde{V}_\nu(\tau))\widetilde{U}_{2;\mu}(\tau,s),$$

as a Riemann integral in norm on $\mathscr{B}(\delta;\delta)$ for all δ. Then, using point i. of Lemma 2.1 and equation (2.1) we obtain

$$\left\|\widetilde{U}_{2;\mu}(t,s) - \widetilde{U}_{2;\nu}(t,s)\right\|_{\mathscr{B}(\delta+2+\varepsilon;\delta)} \leq |t-s| \, e^{\gamma\left|\int_s^t \mathrm{d}\tau \, \|v_{--}(\tau)\|_2\right|} \cdot \sup_{\tau\in[s,t]} \left\|\widetilde{V}_\mu(\tau) - \widetilde{V}_\nu(\tau)\right\|_{\mathscr{B}(\delta+2+\varepsilon;\delta)} ,$$

where γ depends on δ and ε. Utilizing then point ii of Lemma 2.1, we see that for all $\delta \in \mathbb{R}$, $\widetilde{U}_{2;\mu}(t,s)$ converges in norm on $\mathscr{B}(\delta+2+\varepsilon;\delta)$ when $\mu \to \infty$ uniformly in t and s on every compact interval. The resulting limit $\widetilde{U}_2(t,s)$ is continuous in the norm of $\mathscr{B}(\delta+2+\varepsilon;\delta)$ with respect to t and s. The norm convergence just proved and the estimate (2.1), uniform in μ, imply the strong convergence of $\widetilde{U}_{2;\mu}(t,s)$ to $\widetilde{U}_2(t,s)$ on $\mathscr{B}(\delta;\delta)$ uniformly in t and s on every compact interval. Consequently $\widetilde{U}_2(t,s)$ satisfies the estimate (3.1) and is strongly continuous in t and s.

ii. The result follows from the unitarity of $\widetilde{U}_{2;\mu}(t,s)$ on $\mathscr{H}$ and from the strong convergence of $\widetilde{U}_{2;\mu}(t,s)$ and its adjoint $\widetilde{U}_{2;\mu}(s,t)$.

iii. The result is an immediate consequence of the corresponding properties of $\widetilde{U}_{2;\mu}(t,s)$.

iv. We prove the result regarding the derivative with respect to t, the other being analogous. Write $\widetilde{U}_{2;\mu}(t,s)\Phi$, with $\Phi \in \mathscr{H}^{\delta+2}$, as a strong Riemann integral on $\mathscr{H}^{\delta}$:

$$\widetilde{U}_{2;\mu}(t,s)\Phi = \Phi - i\int_s^t \mathrm{d}\tau\, \widetilde{V}_\mu(\tau)\widetilde{U}_{2;\mu}(\tau,s)\Phi \; .$$

Using point ii. of Lemma 2.1 and the strong convergence proved above we can go to the limit $\mu \to \infty$ in previous equation. The result then following from Lemma 1.3 and from point i. of this Lemma.

v. To prove this point remember that for all f and g continuous functions from $\mathbb{R}$ to $\mathbb{C}$ the following identity holds:

$$i\frac{\mathrm{d}}{\mathrm{d}t}f = g \Leftrightarrow f(t) - f(s) = -i\int_s^t \mathrm{d}t'\, g(t') \; . \tag{3.2}$$

Consider now both Ψ and Θ in $D(Q)$, then using previous point:

$$i\partial_t\langle\Psi, \widetilde{U}_2(t,s)\Theta\rangle = \langle\widetilde{V}(t)\Psi, \widetilde{U}_2(t,s)\Theta\rangle \; ,$$

i.e. using (3.2)

$$\langle\Psi, \widetilde{U}_2(t,s)\Theta\rangle - \langle\Psi,\Theta\rangle = -i\int_s^t \mathrm{d}t'\, \langle\widetilde{V}(t')\Psi, \widetilde{U}_2(t',s)\Theta\rangle \; . \tag{3.3}$$

Consider now $\{\Phi_j\} \in D(Q)$ such that $\mathscr{H} - \lim_j \Phi_j = \Phi \in \mathscr{H}$, that is allowed since $D(Q)$ is dense in $\mathscr{H}$. For all Φ_j equation (3.3) holds, furthermore both $\widetilde{V}(t)\Psi$ and $\widetilde{U}_2(t,s)\Phi_j$ are uniformly bounded in t, so we use the dominated convergence theorem to go to the limit $j \to \infty$, then again equation (3.2) to obtain the desired result.

vi. With the aid of previous point, we calculate explicitly, for $\Psi \in D(Q)\cap D(H_0)$, $\Phi \in \mathscr{H}$ the derivative:

$$\begin{aligned} i\partial_t\Big\langle\Psi, U_2(t,s)\Phi\Big\rangle &= \lim_{h\to 0} i\Big\{\Big\langle\frac{U_0^{-1}(t+h) - U_0^{-1}(t)}{h}\Psi, \widetilde{U}_2(t+h,s) \\ &\quad\cdot U_0^{-1}(s)\Phi\Big\rangle - \Big\langle U_0^{-1}(t)\Psi, \frac{\widetilde{U}_2(t+h,s) - \widetilde{U}_2(t,s)}{h} \\ &\quad\cdot U_0^{-1}(s)\Phi\Big\rangle\Big\} = \Big\langle H_0U_0^{-1}(t)\Psi, \widetilde{U}_2(t,s)U_0^{-1}(s)\Phi\Big\rangle \\ &\quad+ \Big\langle\widetilde{V}(t)U_0^{-1}(t)\Psi, \widetilde{U}_2(t,s)U_0^{-1}(s)\Phi\Big\rangle \; , \end{aligned}$$

where the second term of the right hand side of the equality makes sense because $D(Q)\cap D(H_0)$ is invariant under the action of $U_0^{-1}(t)$ since Q and H_0 commute. The result follows immediately.

∎

We want to emphasize that, even if $U_2(t,s)$ defined above is formally generated by $H_0 + V(t)$, *i.e.* formally satisfies the equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}U_2(t,s) = \big(H_0 + V(t)\big)U_2(t,s) \; ,$$

we can only assert that U_2 is weakly differentiable in the sense make explicit in point vi. of the previous Proposition. We are not able to determine any strong differentiability property for U_2, and we need to use the interaction representation in order to take strong derivatives. However we have the following uniqueness result regarding U_2:

LEMMA 3.1 (Uniqueness of $U_2(t,s)$). *Let $s \in \mathbb{R}$, $\Phi(\cdot) \in \mathscr{C}_W(\mathbb{R}, \mathscr{H})$ with $\Phi(s) \equiv \Phi$, such that*

$$i\partial_t \big|\big\langle \Psi, \Phi(t) \big\rangle\big| = \big|\big\langle \big(H_0 + V(t)\big)\Psi, \Phi(t)\big\rangle\big| \ ,$$

for all $\Psi \in D(Q) \cap D(H_0)$ and $\Phi \in \mathscr{H}$. Then $\Phi(t) = U_2(t,s)\Phi$.

PROOF. Define $\widetilde{\Phi}(t) \equiv U_0^{-1}(t)\Phi(t)$. Let $\Psi \in D(Q) \cap D(H_0)$ and $\Phi \in \mathscr{H}$, then we obtain, using the previous Proposition:

$$\langle \Psi, \widetilde{\Phi}(t)\rangle - \langle \Psi, U_0^{-1}(s)\Phi\rangle = -i \int_s^t \mathrm{d}t' \, \langle \widetilde{V}(t')\Psi, \widetilde{\Phi}(t')\rangle \ . \tag{3.4}$$

Now consider $\Psi \in D(Q)$ and a sequence $\Psi_j \in D(Q) \cap D(H_0)$ such that $\Psi_j \to \Psi$ in $D(Q)$, then equation (3.4) holds for all $\Psi \in D(P+N)$, using dominated convergence theorem. Then we obtain

$$i\partial_t \langle \widetilde{U}_2(t,s)\Psi, \widetilde{\Phi}(t)\rangle = 0 \ ,$$

i.e. $\langle \Psi, \widetilde{U}_2^{-1}(t,s)\widetilde{\Phi}(t)\rangle = \langle \Psi, U_0^{-1}(s)\Phi\rangle$ for all $t \in \mathbb{R}$, and that proves our assertion. ■

CHAPTER 5

The convergence of $\widetilde{W}(t,s)$ to $\widetilde{U}_2(t,s)$.

1. $W(t,s)$ and $\widetilde{W}(t,s)$.

Definition ($W(t,s)$). We define the unitary evolution of the quantum system between coherent states as

$$W(t,s) = C^\dagger(u_\lambda(t),\alpha_\lambda(t))U(t-s)C(u_\lambda(s),\alpha_\lambda(s))e^{i\Lambda(t,s)} ,$$

where $\Lambda(t,s)$ is a phase function, and $(u(\cdot),\alpha(\cdot))$ is the $\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ unique solution of the classical system of equations (E) corresponding to initial data $(u(s),\alpha(s))\in L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)$.

Remark. Here we just made an abuse of notation. If we want to be precise, we take $(u(\cdot),\check{\alpha}(\cdot))$ to be the $\mathscr{C}^0(\mathbb{R},L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ unique solution of the classical system of equations (E) corresponding to initial data $(u(s),\check{\alpha}(s))\in L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)$; then $(u(\cdot),\alpha(\cdot))$ is the Fourier transform (in α) of the solution of (E).

However throughout the rest of the paper we will continue to call $(u(\cdot),\alpha(\cdot))$ the solution of (E).

Definition ($\widetilde{W}(t,s)$). In the interaction picture, we will write $\widetilde{W}(t,s) = U_0^\dagger(t)W(t,s)U_0(s)$, so using the last point of Proposition 5 we can write it as following:

$$\widetilde{W}(t,s) = C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))U_0^\dagger(t)U(t-s)U_0(s)C(\widetilde{u}_\lambda(s),\widetilde{\alpha}_\lambda(s))e^{i\Lambda(t,s)} ;$$

observe that by Remark 1.1.1 $(\widetilde{u}(\cdot),\widetilde{\alpha}(\cdot))\in\mathscr{C}^1(\mathbb{R},L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$, and

$$\begin{aligned} i\partial_t\widetilde{u}(t) &= (2\pi)^{-3/2}U_{01}(-t)\Big(\check{\chi}*A(t)\Big)u(t) \\ i\partial_t\widetilde{\alpha}(t) &= \frac{(2\pi)^{-3/2}}{\sqrt{2}}U_{02}(-t)\Big(\omega^{-1/2}\chi\widehat{(\bar{u}u)}(t)\Big) . \end{aligned}$$

We remark again that the solution $\alpha(t)$ we are considering is the Fourier transform of the one we considered in Chapter 2, and here $\omega(k)=\sqrt{k^2+\mu^2}$.

Definitions ($Z(t)$, $\mathscr{D}$, $\mathscr{D}^\delta$).

$$Z(t) = C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))U_0^\dagger(t)U(t)e^{i\Lambda(t,0)} ;$$

so we can write $\widetilde{W}(t,s)=Z(t)Z^\dagger(s)$. Define also the domains

$$\begin{aligned} \mathscr{D} &= \{\Psi\in D(Q)|C(\widetilde{u}_\lambda(s),\widetilde{\alpha}_\lambda(s))\Psi\in D(H_0)\} , \\ \mathscr{D}^\delta &= \{\Psi\in\mathscr{H}^\delta|C(\widetilde{u}_\lambda(s),\widetilde{\alpha}_\lambda(s))\Psi\in D(H_0)\} . \end{aligned}$$

Now, using Propositions 4 and 5 we can formulate the following remark:

Remark 1.0.1. $\widetilde{W}(t,s)$ is unitary on $\mathscr{H}$ and such that $\widetilde{W}^\dagger(t,s)=\widetilde{W}(s,t)$.

2. Differentiability of $\widetilde{W}(t,s)$.

PROPOSITION 7 (Differentiability of $\widetilde{W}$). *$\widetilde{W}(t,s)$ is strongly differentiable in t from $\mathscr{D}$ to $\mathscr{H}$; $\widetilde{W}^\dagger(t,s)$ is strongly differentiable in t from $D(P^2+N)$ to $\mathscr{H}$. More precisely if*

$$\Lambda(t,s) = -\frac{1}{\lambda^2}\int_s^t \mathrm{d}t' \int \mathrm{d}x\, (\check{\chi} * A(t'))\bar{u}(t')u(t') \, ,$$

then for all $\Psi \in \mathscr{D}$, $\Theta \in D(P^2+N)$

$$i\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{W}(t,s)\Psi = \Bigl(U_0^\dagger(t)H_I U_0(t) + \widetilde{V}(t)\Bigr)\widetilde{W}(t,s)\Psi \; ; \tag{2.1}$$

$$i\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{W}^\dagger(t,s)\Theta = -\widetilde{W}^\dagger(t,s)\Bigl(U_0^\dagger(t)H_I U_0(t) + \widetilde{V}(t)\Bigr)\Theta \, . \tag{2.2}$$

PROOF. In order to prove the proposition, we formulate the following lemmas:

LEMMA 2.1. *$Z(t)$ is strongly differentiable from $D(H_0)\cap D(P^2+N)$ to $\mathscr{H}$. In particular for all $\Psi \in D(H_0)\cap D(P^2+N)$ we have*

$$i\partial_t Z(t)\Psi = \Bigl(U_0^\dagger(t)H_I U_0(t) + \widetilde{V}(t)\Bigr)Z(t)\Psi \, .$$

PROOF. We write

$$\begin{aligned}
\frac{i}{h}\Bigl(Z(t+h)-Z(h)\Bigr)\Psi =& \frac{i}{h}\Bigl(C^\dagger(\widetilde{u}_\lambda(t+h),\widetilde{\alpha}_\lambda(t+h)) - C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))\Bigr)\\
& U_0^\dagger(t)U(t)e^{i\Lambda(t,0)}\Psi\\
& + C^\dagger(\widetilde{u}_\lambda(t+h),\widetilde{\alpha}_\lambda(t+h))\frac{i}{h}\Bigl(U_0^\dagger(t+h)-U_0^\dagger(t)\Bigr)\\
& U(t)e^{i\Lambda(t,0)}\Psi\\
& + C^\dagger(\widetilde{u}_\lambda(t+h),\widetilde{\alpha}_\lambda(t+h))U_0^\dagger(t+h)\\
& \frac{i}{h}\Bigl(U(t+h)-U(t)\Bigr)e^{i\Lambda(t,0)}\Psi\\
& + C^\dagger(\widetilde{u}_\lambda(t+h),\widetilde{\alpha}_\lambda(t+h))U_0^\dagger(t+h)U(t+h)\\
& \frac{i}{h}\Bigl(e^{i\Lambda(t+h,0)} - e^{i\Lambda(t,0)}\Bigr)\Psi\\
\equiv& \Bigl(J_1+J_2+J_3+J_4\Bigr)\Psi \, .
\end{aligned}$$

$J_1\Psi$ converges strongly to:

$$\begin{aligned}
J_1\Psi \underset{h\to 0}{\longrightarrow} C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))\Bigl(&-\psi^*(i\dot{\widetilde{u}}_\lambda) + \psi(i\bar{\dot{\widetilde{u}}}_\lambda) - \mathrm{Im}\langle\widetilde{u}_\lambda,\dot{\widetilde{u}}_\lambda\rangle\\
&-a^*(i\dot{\widetilde{\alpha}}_\lambda) + a(i\bar{\dot{\widetilde{\alpha}}}_\lambda) - \mathrm{Im}\langle\widetilde{\alpha}_\lambda,\dot{\widetilde{\alpha}}_\lambda\rangle\Bigr)U_0^\dagger(t)U(t)e^{i\Lambda(t,0)}\Psi \, ,
\end{aligned}$$

since $C(u,\alpha)$ is strongly differentiable on $\mathscr{H}^1 \supset D(P^2+N)$ and $D(P^2+N)$ is left unvaried by U_0 and U. Consider now $J_2\Psi$, it converges to:

$$J_2\Psi \underset{h\to 0}{\longrightarrow} C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))U_0^\dagger(t)H_0U(t)e^{i\Lambda(t,0)}\Psi \, ,$$

since $C^\dagger$ is continuous and $U(t)\Psi \in D(H_0)\cap D(P^2+N)$ if $\Psi \in D(H_0)\cap D(P^2+N)$. With J_3 and J_4 we proceed in an analogous fashion. We have the following limits:

$$J_3\Psi \underset{h\to 0}{\longrightarrow} C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))U_0^\dagger(t)HU(t)e^{i\Lambda(t,0)}\Psi \, ,$$

$$J_4\Psi \underset{h\to 0}{\longrightarrow} C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))U_0^\dagger(t)\Bigl(-\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t,0)\Bigr)U(t)e^{i\Lambda(t,0)}\Psi \, .$$

So we can write, for all $\Psi \in D(H_0) \cap D(P^2+N)$:

$$\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}t}Z(t)\Psi =& C^\dagger(\widetilde{u}_\lambda(t),\widetilde{\alpha}_\lambda(t))U_0^\dagger(t)\Big\{U_0(t)\Big(-\psi^*(i\dot{\widetilde{u}}_\lambda)+\psi(i\dot{\bar{\widetilde{u}}}_\lambda)\\
&-\mathrm{Im}\langle\widetilde{u}_\lambda,\dot{\widetilde{u}}_\lambda\rangle - a^*(i\dot{\widetilde{\alpha}}_\lambda)+a(i\dot{\bar{\widetilde{\alpha}}}_\lambda)-\mathrm{Im}\langle\widetilde{\alpha}_\lambda,\dot{\widetilde{\alpha}}_\lambda\rangle\Big)U_0^\dagger(t)\\
&+H_I-\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t,0)\Big\}U(t)e^{i\Lambda(t,0)}\Psi\\
=&U_0^\dagger(t)C^\dagger(u_\lambda(t),\alpha_\lambda(t))\Big\{-\frac{1}{\lambda}\Big(\psi^*((\check{\chi}*A)u)+\psi((\check{\chi}*A)\bar{u})\Big)\\
&-\frac{1}{\lambda^2}\mathrm{Im}\langle u,(\check{\chi}*A)u\rangle-\frac{1}{\lambda}\Big(a^*((2\omega)^{-1/2}\chi\widehat{(\bar{u}u)})\\
&+a((2\omega)^{-1/2}\chi\widehat{(\bar{u}u)})\Big)-\frac{1}{\lambda^2}\mathrm{Im}\langle\alpha,(2\omega)^{-1/2}\chi\widehat{(\bar{u}u)}\rangle\\
&+C(u_\lambda(t),\alpha_\lambda(t))H_IC^\dagger(u_\lambda(t),\alpha_\lambda(t))\\
&-\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t,0)\Big\}U_0(t)Z(t)\Psi\ .
\end{aligned}$$

The result then follows immediately using Lemma 8.2 of Appendix A. ■

LEMMA 2.2. *$Z^\dagger(t)$ is strongly differentiable from $D(P^2+N)$ to $\mathscr{H}$, and for all $\Theta \in D(P^2+N)$ we have:*

$$i\frac{\mathrm{d}}{\mathrm{d}t}Z^\dagger(t)\Theta = -Z^\dagger(t)\Big(U_0^\dagger(t)H_IU_0(t)+\widetilde{V}(t)\Big)\Theta\ .$$

PROOF. Let $B \equiv U_0^\dagger(t)H_IU_0(t)+\widetilde{V}(t)$. From previous Lemma and the results of Chapters 3 and 4 we know that:

i. For all $\Psi \in D(H_0)\cap D(P^2+N)$ we have $i\partial_t Z(t)\Psi = BZ(t)\Psi$;
ii. $\|B\Theta\| \le C\,\big\|(P^2+N)\Theta\big\|$ for all $\Theta \in D(P^2+N)$, C being a constant and

$$\langle\Theta_1,B\Theta_2\rangle = \langle B\Theta_1,\Theta_2\rangle \text{ for all } \Theta_j \in D(P^2+N);$$

iii. $Z(t)[D(H_0)\cap D(P^2+N)] \subseteq D(P^2+N)$, and both $Z(t)$ and $Z^\dagger(t)$ strongly continuous in t.

So we can write for all $\Theta \in D(P^2+N)$:

$$i\partial_t\langle Z^\dagger(t)\Theta,\Psi\rangle = \langle Z^\dagger(t)B\Theta,\Psi\rangle\ ,$$

so integrating both members we find

$$i\Big(\langle Z^\dagger(t)\Theta,\Psi\rangle - \langle Z^\dagger(0)\Theta,\Psi\rangle\Big) = \int_0^t \mathrm{d}\tau\,\langle Z^\dagger(\tau)B\Theta,\Psi\rangle\ ,$$

but since $Z^\dagger(\tau)B\Theta$ is continuous in τ for all $\Theta \in D(P^2+N)$ we infer

$$i\frac{\mathrm{d}}{\mathrm{d}t}Z^\dagger(t)\Theta = -Z^\dagger(t)B\Theta\ .$$

■

The proof of the Proposition follows recalling that $\widetilde{W}(t,s) = Z(t)Z^\dagger(s)$. ■

REMARK 2.2.1. $\widetilde{W}(t,s)$ maps $\mathscr{D}^\delta$ into $\mathscr{H}^{\delta/2}$.

PROOF. By Proposition 5 Weyl operators map $\mathscr{H}^\delta$ into itself. By Proposition 4 U_0 maps $\mathscr{H}^\delta$ into itself, and U maps $\mathscr{H}^\delta \subset D((P^2+N)^{\delta/4})$ into $D(P^2+N)^{\delta/4} \subset \mathscr{H}^{\delta/2}$. ■

3. Strong convergence of $\widetilde{W}(t,s)$ to $\widetilde{U}_2(t,s)$.

THEOREM 1 (Strong limit of $\widetilde{W}(t,s)$). *Let $\widetilde{W}(t,s)$ and $\widetilde{U}_2(t,s)$ defined as above. Then the following strong limit exists*

$$\mathrm{s}-\lim_{\lambda\to 0}\widetilde{W}(t,s)=\widetilde{U}_2(t,s)\ ,$$

uniformly in t,s on compact intervals.

PROOF. We will prove the existence of the limit on $\mathscr{D}^{\delta^*}$ with $\delta^*\geq 4$, dense in $\mathscr{H}$. $\widetilde{W}$ is strongly differentiable on such domain and $\widetilde{W}[\mathscr{D}^{\delta^*}]\subseteq\mathscr{H}^{\delta^*/2}$ (Proposition 2.1 and Remark 2.2.1); while $\widetilde{U}_2$ is strongly differentiable on $\mathscr{H}^{\delta^*/2}$, when $\delta^*\geq 4$. Then we can write the following inequalities for all $\Phi\in\mathscr{D}^{\delta^*}$, every term being well-defined and the integrals making sense as strong Riemann integrals on $\mathscr{H}$:

$$\begin{aligned}
\left\|\Big(\widetilde{W}(t,s)-\widetilde{U}_2(t,s)\Big)\Phi\right\|^2 &= 2\,\mathrm{Re}\left\langle\Phi,\Big(1-\widetilde{U}_2^\dagger(t,s)\widetilde{W}(t,s)\Big)\Phi\right\rangle\\
&= -2\,\mathrm{Re}\left\langle\Phi,\int_s^t \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{U}_2^\dagger(\tau,s)\widetilde{W}(\tau,s)\Phi\right\rangle\\
&= 2\,\mathrm{Im}\int_s^t\mathrm{d}\tau\left\langle H_I U_0(\tau)\widetilde{U}_2(\tau,s)\Phi, U_0(\tau)\widetilde{W}(\tau,s)\Phi\right\rangle\\
&\leq 2\,\|\Phi\|\left|\int_s^t\mathrm{d}\tau\,\left\|H_I U_0(\tau)\widetilde{U}_2(\tau,s)\Phi\right\|\right|\\
&\leq 2\lambda\,\|f_0\|_2\,\|\Phi\|\left|\int_s^t\mathrm{d}\tau\,\left\|\widetilde{U}_2(\tau,s)\Phi\right\|_{\mathscr{H}^4}\right|\\
\leq 2\lambda\,\|f_0\|_2\left|\int_s^t\mathrm{d}\tau\,\exp\left\{2\left(\ln 3+10\sqrt{2}\left|\int_s^\tau\mathrm{d}\tau'\,\|v_{--}(\tau')\|_2\right|\right)\right\}\right|&\,\|\Phi\|\,\|\Phi\|_{\delta^*}
\end{aligned}$$

that tends to zero when $\lambda\to 0$, uniformly in t and s on compact intervals. ■

COROLLARY. *Let $W(t,s)$ and $U_2(t,s)$ defined as above. Then also the following strong limit exists*

$$\mathrm{s}-\lim_{\lambda\to 0}W(t,s)=U_2(t,s)\ ,$$

uniformly in t,s on compact intervals.

CHAPTER 6

A crucial bound of $\|W(t,s)\Phi\|_\delta$.

1. The cut off evolution $\widetilde{W}_\mu(t,s)$.

From now on we will use the notation $\widetilde{H}_I(t) = U_0^\dagger(t) H_I U_0(t)$. We also define the orthogonal projectors $\mathbb{P}_{\leq\mu}$ and $\mathbb{N}_{\leq\mu}$ as following:

$$(\mathbb{P}_{\leq\mu}\Phi)_{p,n} = \begin{cases} \Phi_{p,n} \text{ if } p \leq \mu \\ 0 \text{ if } p > \mu \end{cases}$$

$$(\mathbb{N}_{\leq\mu}\Phi)_{p,n} = \begin{cases} \Phi_{p,n} \text{ if } n \leq \mu \\ 0 \text{ if } n > \mu \end{cases}$$

Definition ($\widetilde{W}_\mu(t,s)$). We define $\mathbb{R}_\mu = \mathbb{P}_{\leq\mu}\mathbb{N}_{\leq\mu}$, so $X_\mu(t) \equiv \mathbb{R}_\mu\Big(\widetilde{H}_I(t) + \widetilde{V}(t)\Big)\mathbb{R}_\mu$ is bounded in $\mathscr{H}$. Then by means of a Dyson series we obtain:

$$\widetilde{W}_\mu(t,s) = \sum_{m=0}^{\infty} (-i)^m \int_s^t \mathrm{d}t_1 \int_s^{t_1} \mathrm{d}t_2 \cdots \int_s^{t_{m-1}} \mathrm{d}t_m X_\mu(t_1)\cdots X_\mu(t_m) \ .$$

The family $\widetilde{W}_\mu(t,s)$ satisfies the following Lemma, since it is defined by a Dyson series:

Lemma 1.1.

i. $\widetilde{W}_\mu(s,s) = 1$, $\widetilde{W}_\mu(t,r)\widetilde{W}_\mu(r,s) = \widetilde{W}_\mu(t,s)$ *for all* $r,s,t \in \mathbb{R}$.

ii. $\widetilde{W}_\mu^\dagger(t,s) = \widetilde{W}_\mu(s,t)$, *and* $\widetilde{W}_\mu(t,s)$ *are unitary in* $\mathscr{H}$.

iii. $\widetilde{W}_\mu(t,s)$ *is strongly differentiable on* $\mathscr{H}$ *and*

$$i\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{W}_\mu(t,s) = X_\mu(t)\widetilde{W}_\mu(t,s) \ ;$$

$$i\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{W}_\mu(t,s) = -\widetilde{W}_\mu(t,s)X_\mu(s) \ .$$

Furthermore we can prove that $\widetilde{W}_\mu(t,s)$ maps $\mathscr{H}^\delta$ into itself:

Lemma 1.2. *Let* $\Phi \in \mathscr{H}^{2\delta}$, $\delta \in \mathbb{R}$, $\mu \geq 1$. *Then*

$$\Big\|\widetilde{W}_\mu(t,s)\Phi\Big\|_{2\delta} \leq \exp\Bigg\{\sqrt{\mu}\lambda\,|\delta|\,\nu_\delta\,\|f_0\|_2\,|t-s| + |\delta|\,\Bigg(\ln 3 + \rho_\delta\left|\int_s^t \mathrm{d}\tau\ \|v_{--}(\tau)\|_2\right|\Bigg)\Bigg\}\,\|\Phi\|_{2\delta} \ ,$$

with $\nu_\delta = \max(5/2, 2^{|\delta|} + 1/2)$, $\rho_\delta = \max(4, 3^{|\delta|} + 1)$.

Proof. Define, for all $\Phi \in \mathscr{H}$,

$$M(t,s) = \frac{1}{2}\left\|(Q+3)^{-\delta}\widetilde{W}_\mu(t,s)\Phi\right\|^2 \ ,$$

with $\delta \geq 1$, differentiable in t and s. Set $(Q+3)^{-\delta} \equiv h(Q)$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t,s) =& \mathrm{Im}\langle h(Q)\widetilde{W}_\mu(t,s)\Phi, \mathbb{R}_\mu \widetilde{H}_I^-(t)\mathbb{R}_\mu \\ & \Big(h(Q-1)h(Q)^{-1}-1\Big)h(Q)\widetilde{W}_\mu(t,s)\Phi\rangle \\ & +\mathrm{Im}\langle \mathbb{R}_\mu \widetilde{H}_I^-(t)\mathbb{R}_\mu\Big(h(Q)h(Q-1)^{-1}-1\Big) \\ & h(Q)\widetilde{W}_\mu(t,s)\Phi, h(Q)\widetilde{W}_\mu(t,s)\Phi\rangle \\ & +\mathrm{Im}\langle h(Q)\widetilde{W}_\mu(t,s)\Phi, \widetilde{V}_{--;\mu}(t)\Big(h(Q-2)h(Q)^{-1} \\ & -1\Big)h(Q)\widetilde{W}_\mu(t,s)\Phi\rangle \\ & +\mathrm{Im}\langle \widetilde{V}_{--;\mu}(t)\Big(h(Q)h(Q-2)^{-1} \\ & -1\Big)h(Q)\widetilde{W}_\mu(t,s)\Phi, h(Q)\widetilde{W}_\mu(t,s)\Phi\rangle\end{aligned}$$

The last two terms of the right hand side of the equality are bounded in Lemma 2.3. So we obtain:

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}M(t,s)\right| \leq\ & 2\lambda\|f_0\|_2\Big[|||\mathbb{R}_\mu P\sqrt{N}\Big(h(Q)h(Q-1)^{-1}-1\Big)||| \\ & +|||\mathbb{R}_\mu P\sqrt{N}\Big(h(Q-1)h(Q)^{-1}-1\Big)|||\Big]M(t,s) \\ & +\sqrt{2}\|v_{--}\|_2\,\delta(3^\delta+1)M(t,s)\ .\end{aligned}$$

We have then

$$\begin{aligned}K \equiv \Big[\dots\Big] \leq \delta\mu\sqrt{\mu}\Big(\frac{(2\mu+3)^\delta}{(2\mu+2)^\delta}+\frac{(2\mu+3)^{\delta-1}}{(2\mu+3)^\delta}\Big) &= \delta\frac{\mu\sqrt{\mu}}{\mu+1}(2^{\delta-2}+2^{-1}) \\ &\leq \delta\sqrt{\mu}\Big(2^\delta+\frac{1}{2}\Big)\ .\end{aligned}$$

Applying now Gronwall's Lemma we have

$$\begin{aligned}\Big\|(Q+3)^{-\delta}\widetilde{W}_\mu(t,s)\Phi\Big\| \leq \exp\Big\{&\sqrt{\mu}\delta(2^\delta+\frac{1}{2})\lambda\|f_0\|_2(t-s) \\ &+\frac{1}{\sqrt{2}}\delta(3^\delta+1)\left|\int_s^t \mathrm{d}\tau\ \|v_{--}(\tau)\|_2\right|\Big\}\left\|(Q+3)^{-\delta}\Phi\right\|^2\ ,\end{aligned}$$

for all $\delta \geq 1$. Interpolating between $\delta = 0$ and $\delta = 1$ we obtain for all $\delta \geq 0$:

$$\begin{aligned}\Big\|(Q+3)^{-\delta}\widetilde{W}_\mu(t,s)\Phi\Big\| \leq \exp\Big\{&\sqrt{\mu}\delta\nu_\delta\lambda\|f_0\|_2(t-s) \\ &+\frac{1}{\sqrt{2}}\delta\rho_\delta\left|\int_s^t \mathrm{d}\tau\ \|v_{--}(\tau)\|_2\right|\Big\}\left\|(Q+3)^{-\delta}\Phi\right\|^2\ ,\end{aligned}$$

with $\nu_\delta = \max(5/2, 2^{|\delta|}+1/2)$, $\rho_\delta = \max(4, 3^{|\delta|}+1)$. By duality we extend the result to all $\delta \in \mathbb{R}$. ■

2. A preliminary (not so good) bound of $\left\|\widetilde{W}(t,s)\Phi\right\|_\delta$.

We prove here a Lemma about $\exp\{i\varphi(f)\}$, as defined in Appendix A.

LEMMA 2.1. *Let $b \geq 1/2$. Then for all $m = 1, 2, \dots$ and $\Psi \in D(N^m)$ we have*

$$\|(N+b)^m \exp\{i\varphi(f)\}\Psi\| \leq 6^{m/2} \left\| \prod_{j=0}^{m-1} (N + b + \|f\|_2^2 + j)\Psi \right\|$$
$$\leq 6^{m/2}(1+2(m-1))^m (1 + 2\|f\|_2^2)^m \|(N+b)^m\Psi\| \ .$$

PROOF. Recalling that $N = \mathrm{d}\Gamma(1)$, we can use Lemma 5.7 of Appendix A to write:

$$\exp\{-i\varphi(f)\}(N+b)\exp\{i\varphi(f)\} = N + b + \|f\|_2^2 + a(\overline{if}) + a^*(if) \ ;$$

and such equality holds on $D(N)$. So if $m = 1$, $\Psi \in D(N^m)$:

$$\|(N+b)\exp\{i\varphi(f)\}\Psi\|^2 = \left\|(N+b+\|f\|_2^2)\Psi + a(\overline{if})\Psi + a^*(if)\Psi\right\|^2$$
$$\leq 3\Big(\left\|(N+b+\|f\|_2^2)\Psi\right\|^2 + \left\|a(\overline{if})\Psi\right\|^2 + \|a^*(if)\Psi\|^2\Big)$$
$$\leq 3\langle\Psi, \Big((N+b+\|f\|_2^2)^2 + 2\|f\|_2^2 N + \|f\|_2^2\Big)\Psi\rangle \ .$$

Now if $b \geq 1/2$ we have $2\|f\|_2^2 N + \|f\|_2^2 \leq (N+b+\|f\|_2^2)^2$.

Suppose the result is verified for m, and verify it for $m+1$. Let

$$h_m(N) = \prod_{j=0}^{m-1} (N + b + \|f\|_2^2 + j) \ .$$

Then

$$\left\|(N+b)^{m+1}\exp\{i\varphi(f)\}\Psi\right\|^2 \leq \left\|h_m(N)(N+b+\|f\|_2^2 + a + a^*)\Psi\right\|^2$$
$$= \left\|\Big((N+b+\|f\|_2^2)h_m(N) + a h_m(N-1) + a^* h_m(N+1)\Big)\Psi\right\|^2$$
$$\leq 3\langle\Psi, h_m(N+1)^2\Big((N+b+\|f\|_2^2)^2 + 2N\|f\|_2^2 + \|f\|_2^2\Big)\Psi\rangle \ .$$

■

LEMMA 2.2. *$C(u,\alpha)$ maps $\mathscr{H}^{2\delta}$ into itself for any positive δ. In particular, let $u, \alpha \in L^2$, $\delta \geq 0$, $\Phi \in \mathscr{H}^{2\delta}$; then*

$$\|C(u,\alpha)\Phi\|_{2\delta} \leq K_\delta(u,\alpha)\|\Phi\|_{2\delta} \ , \tag{2.1}$$

with

$$K_\delta(u,\alpha) = 6^{\delta/2}(1+2(d_- - 1))^{\frac{d_-(d_+-\delta)}{d_+-d_-}}(1+2(d_+-1))^{\frac{d_+(\delta-\delta_-)}{d_+-d_-}}$$
$$\big(1 + 2\|u\|_2^2 + 2\|\alpha\|_2^2\big)^\delta \ ,$$

where

$$d_- = \max_{m\in\mathbb{N}}\{m \leq \delta\}$$
$$d_+ = \min_{m\in\mathbb{N}}\{m \geq \delta\} \ .$$

PROOF. The result is a direct consequence of Lemma 2.1 when δ is an integer. By interpolation we extend it to all real δ: let $\delta \in \mathbb{R}$, and define the integers d_- and d_+ as above; then interpolating between d_- and d_+ we obtain

$$\|C(u,\alpha)\Phi\|_{2\delta} \leq K_\delta(u,\alpha)\|\Phi\|_{2\delta} \ ,$$

with

$$K_\delta(u,\alpha) = 6^{\frac{d_-(d_+-\delta)}{2(d_+-d_-)}}(1+2(d_--1))^{\frac{d_-(d_+-\delta)}{d_+-d_-}}\left(1+2\|u\|_2^2+2\|\alpha\|_2^2\right)^{\frac{d_-(d_+-\delta)}{d_+-d_-}}$$
$$6^{\frac{d_+(\delta-d_-)}{2(d_+-d_-)}}(1+2(d_+-1))^{\frac{d_+(\delta-d_-)}{d_+-d_-}}\left(1+2\|u\|_2^2+2\|\alpha\|_2^2\right)^{\frac{d_+(\delta-d_-)}{d_+-d_-}}$$
$$= 6^{\delta/2}(1+2(d_--1))^{\frac{d_-(d_+-\delta)}{d_+-d_-}}(1+2(d_+-1))^{\frac{d_+(\delta-d_-)}{d_+-d_-}}$$
$$\left(1+2\|u\|_2^2+2\|\alpha\|_2^2\right)^{\delta} .$$

■

REMARK 2.2.1. Let $\Phi \in \mathscr{H}^{4\delta}$, with positive integer δ and $\lambda \leq 1$; then

$$\left\|\widetilde{W}(t,s)\Phi\right\|_{2\delta} \leq K_\delta(t,s)\lambda^{-6\delta}\exp\Big\{|\delta|\,\mu_\delta\lambda\,\|f_0\|_2\,|t-s|\Big\}\,\|\Phi\|_{4\delta} ,$$

with $\mu_\delta = \max(3, 1+2^{|\delta|})$ and

$$K_\delta(t,s) = K_\delta(u(t),\alpha(t))K_{2\delta}(u(s),\alpha(s)) .$$

PROOF. We can pass $(Q+1)^\delta$ to the right of U_0 since it commutes with H_0, and to the right of C using Lemma 2.2. So we have

$$\left\|\widetilde{W}(t,s)\Phi\right\|_{2\delta} \leq K_\delta(t)\lambda^{-2\delta}\,\|U(t-s)C(u_\lambda(s),\alpha_\lambda(s))\Phi\|_{2\delta} \;,$$

where

$$K_\delta(t) = K_\delta(u(t),\alpha(t)) ,$$

since

$$K_\delta(u_\lambda(t),\alpha_\lambda(t)) \leq \lambda^{-2\delta}K_\delta(u(t),\alpha(t))$$

when $\lambda \leq 1$. Using the fact that $P^2 + N \geq Q$ we can write:

$$\left\|\widetilde{W}(t,s)\Phi\right\|_{2\delta} \leq K_\delta(t)\lambda^{-2\delta}\left\|(P^2+N+1)^\delta U(t-s)C(u_\lambda(s),\alpha_\lambda(s))U_0(s)\Phi\right\| \;,$$

and use Proposition 4 to obtain

$$\left\|\widetilde{W}(t,s)\Phi\right\|_{2\delta} \leq K_\delta(t)\lambda^{-2\delta}\exp\Big\{|\delta|\,\mu_\delta\lambda\,\|f_0\|_2\,|t-s|\Big\}$$
$$\|C(u_\lambda(s),\alpha_\lambda(s))U_0(s)\Phi\|_{4\delta} \;;$$

since $P^2 + N \leq Q^2$. Now using again Lemma 2.2 we obtain the sought result with

$$K_\delta(t,s) = K_\delta(t)K_{2\delta}(s) .$$

■

3. The good bound of $\left\|\widetilde{W}(t,s)\Phi\right\|_\delta$.

The bound we just proved in Remark 2.2.1 is divergent when $\lambda \to 0$, as $\lambda^{-6\delta}$. So it is not suitable to be applied in the classical limit. However using it and the one regarding the cut off operator $\widetilde{W}_\mu$ proved in Lemma 1.2 we can obtain a bound of $\widetilde{W}$ that behaves well when $\lambda \to 0$.

PROPOSITION 8. *For all positive δ exists a $\delta^* > \delta$ such that $\widetilde{W}(t,s)$ maps $\mathscr{H}^{\delta^*}$ into $\mathscr{H}^\delta$.*

In particular let $\Phi \in \mathscr{H}^{\delta^*}$. Then for all $\lambda \leq 1$, $\delta^* = \max(4, 6\delta+3)$:

$$\left\|\widetilde{W}(t,s)\Phi\right\|_\delta^2 \leq \Big(\mathcal{K}_1(t,s) + \lambda\mathcal{K}_2(t,s)\Big)e^{\lambda\mathcal{C}_1|t-s|+\mathcal{K}_3(t,s)}\,\|\Phi\|_{\delta^*}^2 \;,$$

where $\mathcal{C}_1$ is a positive constant depending on δ; $\mathcal{K}_j(t,s)$, $j = 1,2,3$, positive functions depending also on δ.

Proof. Let $\Phi \in \mathscr{H}^{\delta^*}$, with $\delta^* \geq 4$. Due to the properties of $\widetilde{W}(t,s)$ and $\widetilde{W}_\mu(t,s)$ all the steps of the following proof are well defined, and the integrals make sense as strong Riemann integrals on $\mathscr{H}$. We evaluate separately each term of the right hand side of the identity

$$\langle \widetilde{W}(t,s)\Phi, (Q+1)^\delta \widetilde{W}(t,s)\Phi\rangle = \langle \widetilde{W}_\mu(t,s)\Phi, (Q+1)^\delta \widetilde{W}_\mu(t,s)\Phi\rangle + \langle \widetilde{W}(t,s)\Phi, (Q+1)^\delta \Big(\widetilde{W}(t,s) - \widetilde{W}_\mu(t,s)\Big)\Phi\rangle + \langle \Big(\widetilde{W}(t,s) - \widetilde{W}_\mu(t,s)\Big)\Phi, (Q+1)^\delta \widetilde{W}_\mu(t,s)\Phi\rangle\ .$$

The estimate for the first one is provided by Lemma 1.2. Consider now the second term:

$$\begin{aligned}\Big|\langle \widetilde{W}(t,s)\Phi, (Q+1)^\delta \Big(\widetilde{W}(t,s) - \widetilde{W}_\mu(t,s)\Big)\Phi\rangle\Big| &= \Big|\langle \widetilde{W}^\dagger(t,s) (Q+1)^\delta \widetilde{W}(t,s)\Phi, \int_s^t \mathrm{d}\tau\, \widetilde{W}^\dagger(\tau,s)\Big(\mathbb{R}_\mu\big(\widetilde{H}_I(\tau)+ \widetilde{V}(\tau)\big)\mathbb{R}_\mu - \widetilde{H}_I(\tau) - \widetilde{V}(\tau)\Big)\widetilde{W}_\mu(\tau,s)\Phi\rangle\Big| \\ &\leq K_\delta(t,s)\lambda^{-6\delta}\exp\Big\{\delta\mu_\delta\lambda\left\|f_0\right\|_2 |t-s|\Big\}\Big|\int_s^t \mathrm{d}\tau\ \left\|\Phi\right\|_{4\delta} \\ &\quad \Big\|\Big(\mathbb{R}_\mu\big(\widetilde{H}_I(\tau)+\widetilde{V}(\tau)\big)\mathbb{R}_\mu - \widetilde{H}_I(\tau) - \widetilde{V}(\tau)\Big)\widetilde{W}_\mu(\tau,s)\Phi\Big\|\Big|;\end{aligned}$$

where in the inequality we used Remark 2.2.1. To evaluate the last norm we use the commutation properties defined in Lemma 2.2 of Part I to move $\mathbb{R}_\mu$ to the left, the usual estimates of H_I and $\widetilde{V}$ and then the fact that for every j we have

$$(1-\mathbb{R}_\mu) \leq \frac{(Q+1)^{2j}}{\sqrt{\mu}^{4j}} \leq \frac{(Q+1)^{2j}}{\sqrt{\mu-1}^{4j}}$$

to obtain:

$$\begin{aligned}\Big|\langle \widetilde{W}(t,s)\Phi, (Q+1)^\delta \Big(\widetilde{W}(t,s) - \widetilde{W}_\mu(t,s)\Big)\Phi\rangle\Big| &\leq K_\delta(t,s) \exp\Big\{\delta\mu_\delta\lambda\left\|f_0\right\|_2 |t-s|\Big\}\Big|\int_s^t \mathrm{d}\tau\, \Big(\left\|f_0\right\|_2 \Big(\frac{1}{\lambda\sqrt{\mu-1}}\Big)^{6\delta-1} \\ &\quad +\big(8\left\|v_{--}(\tau)\right\|_2 + 2\left\|\check{\chi} * A(\tau)\right\|_\infty\big)\Big(\frac{1}{\lambda\sqrt{\mu-1}}\Big)^{6\delta}\Big)\Big| \\ &\quad \left\|\Phi\right\|_{4\delta}\left\|\widetilde{W}_\mu(\tau,s)\Phi\right\|_{6\delta+3}\ .\end{aligned}$$

Now we use Lemma 1.2 to write

$$\begin{aligned}\Big|\langle \widetilde{W}(t,s)\Phi, (Q+1)^\delta \Big(\widetilde{W}(t,s) - \widetilde{W}_\mu(t,s)\Big)\Phi\rangle\Big| &\leq K_\delta(t,s) \exp\Big\{\delta\mu_\delta\lambda\left\|f_0\right\|_2 |t-s|\Big\}\Big|\int_s^t \mathrm{d}\tau\, \Big(\left\|f_0\right\|_2 \Big(\frac{1}{\lambda\sqrt{\mu-1}}\Big)^{6\delta-1} \\ &\quad +\big(8\left\|v_{--}(\tau)\right\|_2 + 2\left\|\check{\chi} * A(\tau)\right\|_\infty\big)\Big(\frac{1}{\lambda\sqrt{\mu-1}}\Big)^{6\delta}\Big) \\ &\quad \exp\Big\{\sqrt{\mu}\lambda(3\delta+3/2)\nu_{3\delta+3/2}\left\|f_0\right\|_2|\tau-s| \\ &\quad +(3\delta+3/2)\Big(\ln 3 + \rho_{3\delta+3/2}\Big|\int_s^\tau \mathrm{d}\tau'\ \left\|v_{--}(\tau')\right\|_2\Big|\Big)\Big\}\Big| \\ &\quad \left\|\Phi\right\|_{4\delta}\left\|\Phi\right\|_{6\delta+3}\ .\end{aligned}$$

The last term is easier to estimate, we have to use again Lemma 1.2 and the standard estimates for H_I and $\widetilde{V}$, and obtain:

$$\begin{aligned}\Big|\langle\Big(\widetilde{W}(t,s)-\widetilde{W}_\mu(t,s)\Big)\Phi,(Q+1)^\delta\widetilde{W}_\mu(t,s)\Phi\rangle\Big| \\ \leq \exp\Big\{\sqrt{\mu}\lambda\delta\nu_\delta\left\|f_0\right\|_2|t-s| \\ +\delta\Big(\ln 3+\rho_\delta\Big|\int_s^t \mathrm{d}\tau\ \left\|v_{--}(\tau)\right\|_2\Big|\Big)\Big\} \\ \Big|\int_s^t \mathrm{d}\tau\ \Big(\lambda\left\|f_0\right\|_2+4\left\|v_{--}(\tau)\right\|_2+\left\|\check{\chi}*A(\tau)\right\|_\infty\Big) \\ \exp\Big\{\sqrt{\mu}\lambda 2\nu_2\left\|f_0\right\|_2|\tau-s|+2\Big(\ln 3+\rho_2\Big|\int_s^\tau \mathrm{d}\tau'\ \left\|v_{--}(\tau')\right\|_2\Big|\Big)\Big\}\Big| \\ \left\|\Phi\right\|_{2\delta}\left\|\Phi\right\|_{\mathscr{H}^4}\ ;\end{aligned}$$

where we denoted the norm of $\mathscr{H}^4$ with $\|\cdot\|_{\mathscr{H}^4}$ to avoid confusion with the L^4-norm. Now we can choose $\mu=1+1/\lambda^2$, and finally obtain the sought result:

$$\begin{aligned}\Big\|\widetilde{W}(t,s)\Phi\Big\|_\delta^2 \leq K_\delta(t,s)\exp\Big\{\delta\mu_\delta\lambda\left\|f_0\right\|_2|t-s|\Big\}\Big|\int_s^t \mathrm{d}\tau\ \Big(\left\|f_0\right\|_2+8\left\|v_{--}(\tau)\right\|_2 \\ +2\left\|\check{\chi}*A(\tau)\right\|_\infty\Big)\exp\Big\{\sqrt{1+\lambda^2}(3\delta+3/2)\nu_{3\delta+3/2}\left\|f_0\right\|_2|\tau-s| \\ +(3\delta+3/2)\Big(\ln 3+\rho_{3\delta+3/2}\Big|\int_s^\tau \mathrm{d}\tau'\ \left\|v_{--}(\tau')\right\|_2\Big|\Big)\Big\}\Big| \\ \left\|\Phi\right\|_{4\delta}\left\|\Phi\right\|_{6\delta+3} \\ +\exp\Big\{\sqrt{1+\lambda^2}\delta\nu_\delta\left\|f_0\right\|_2|t-s|+\delta\Big(\ln 3+\rho_\delta\Big|\int_s^t \mathrm{d}\tau\ \left\|v_{--}(\tau)\right\|_2\Big|\Big)\Big\} \\ \Big|\int_s^t \mathrm{d}\tau\ \Big(\lambda\left\|f_0\right\|_2+4\left\|v_{--}(\tau)\right\|_2+\left\|\check{\chi}*A(\tau)\right\|_\infty\Big) \\ \exp\Big\{\sqrt{1+\lambda^2}2\nu_2\left\|f_0\right\|_2|\tau-s|+2\Big(\ln 3+\rho_2\Big|\int_s^\tau \mathrm{d}\tau'\ \left\|v_{--}(\tau')\right\|_2\Big|\Big)\Big\}\Big| \\ \left\|\Phi\right\|_{2\delta}\left\|\Phi\right\|_{\mathscr{H}^4} \\ +\exp\Big\{\sqrt{1+\lambda^2}\delta\nu_{\delta/2}\left\|f_0\right\|_2|t-s|+\delta\Big(\ln 3+\rho_{\delta/2}\Big|\int_s^t \mathrm{d}\tau\ \left\|v_{--}(\tau)\right\|_2\Big|\Big)\Big\} \\ \left\|\Phi\right\|_\delta^2\ ,\end{aligned}$$

with $\delta^*=\max(4,6\delta+3)$. ■

In order to obtain the good estimate in λ, we have to restrict to a subspace of $\mathscr{H}$ smaller than expected. We could *a priori* expect, since $D((P^2+N)^\delta)$ and not $\mathscr{H}^\delta$ is invariant for $\widetilde{W}$, to bound the δ-norm of $\widetilde{W}\Phi$ with the 2δ-norm of Φ; however this leads to the (divergent) bound of previous section. In fact we need an even smaller subspace if we want the estimate to remain finite in the limit $\lambda\to 0$.

COROLLARY. *Also $W(t,s)$ maps $\mathscr{H}^{\delta^*}$ into $\mathscr{H}^\delta$. The same estimate as for $\widetilde{W}(t,s)$ holds:*

$$\left\|W(t,s)\Phi\right\|_\delta^2 \leq \Big(\mathcal{K}_1(t,s)+\lambda\mathcal{K}_2(t,s)\Big)e^{\lambda\mathcal{C}_1|t-s|+\mathcal{K}_3(t,s)}\left\|\Phi\right\|_{\delta^*}^2\ .$$

PROOF. This corollary is a direct consequence of the fact that U_0 commutes with every function of P and N. ■

CHAPTER 7

The classical limit of creation and annihilation operators.

1. The evolution of quantum fields of fluctuations.

The $\widetilde{U}_2$-evolution does not preserve the number of particles, however the evolution of quantum fields applied to the vacuum remains a state with only one particle. Using this fact we will be able to improve the convergence of creation and annihilation operators in the classical limit, in a sense we will explain below.

DEFINITION (The projector on the one particle subspace of $\mathscr{H}$). We define $\mathbb{P}_0\mathbb{N}_1 + \mathbb{P}_1\mathbb{N}_0$ to be the orthogonal projector onto $\mathscr{H}_{0,1} \oplus \mathscr{H}_{1,0}$

PROPOSITION 9 ($\widetilde{U}_2$-evolution of quantum fields). *Let $\mathbf{g} = \{g_i\}_{i=1}^4$ be four $L^2(\mathbb{R}^3)$ functions, and consider the field $\varphi(\mathbf{g}) = \psi^*(g_1) + \psi(g_2) + a^*(g_3) + a(g_4)$. Then*

$$\widetilde{U}_2^\dagger(t,s)\varphi(\mathbf{g})\widetilde{U}_2(t,s)\Omega = (\mathbb{P}_0\mathbb{N}_1 + \mathbb{P}_1\mathbb{N}_0)\widetilde{U}_2^\dagger(t,s)\varphi(\mathbf{g})\widetilde{U}_2(t,s)\Omega \; .$$

So $\widetilde{U}_2^\dagger(t,s)\varphi(\mathbf{g})\widetilde{U}_2(t,s)\Omega$ belongs to the subspace of $\mathscr{H}$ with only one particle.

PROOF. Let $\Theta \in (\mathscr{H}_{0,1} \oplus \mathscr{H}_{1,0})^\perp$, and define:

$$\begin{aligned} X(t) = &\sup_{g_1\in L^2} \frac{1}{\|g_1\|_2} \left|\langle \Theta, \widetilde{U}_2^\dagger(t,s)\psi^*(g_1)\widetilde{U}_2(t,s)\Omega\rangle\right| \\ &+ \sup_{g_2\in L^2} \frac{1}{\|g_2\|_2} \left|\langle \Theta, \widetilde{U}_2^\dagger(t,s)\psi(g_2)\widetilde{U}_2(t,s)\Omega\rangle\right| \\ &+ \sup_{g_3\in L^2} \frac{1}{\|g_3\|_2} \left|\langle \Theta, \widetilde{U}_2^\dagger(t,s)a^*(g_3)\widetilde{U}_2(t,s)\Omega\rangle\right| \\ &+ \sup_{g_4\in L^2} \frac{1}{\|g_4\|_2} \left|\langle \Theta, \widetilde{U}_2^\dagger(t,s)a(g_4)\widetilde{U}_2(t,s)\Omega\rangle\right| \; . \end{aligned}$$

First of all observe that $X(s) = 0$. So if we prove that exists a positive finite constant C such that

$$X(t) \leq C \int_s^t \mathrm{d}\tau\, X(\tau) \; , \tag{1.1}$$

we can use Gronwall's Lemma to assert $X(t) = 0$ for all $t \geq s$. But if $X(t) = 0$ then

$$\left|\langle \Theta, \widetilde{U}_2^\dagger(t,s)\varphi(\mathbf{g})\widetilde{U}_2(t,s)\Omega\rangle\right| \leq \sup_{i\in\{1,2,3,4\}} \|g_i\|_2\, X(t) = 0 \; ;$$

for all $\Theta \in (\mathscr{H}_{0,1} \oplus \mathscr{H}_{1,0})^\perp$, so $\widetilde{U}_2^\dagger(t,s)\varphi(\mathbf{g})\widetilde{U}_2(t,s)\Omega \in \mathscr{H}_{0,1} \oplus \mathscr{H}_{1,0}$. We need to show (1.1); we will prove such relation only for the first term of $X(t)$, since for the others is very similar; define then

$$X_1(t) = \sup_{g_1\in L^2} \frac{1}{\|g_1\|_2} \left|\langle \Theta, \widetilde{U}_2^\dagger(t,s)\psi^*(g_1)\widetilde{U}_2(t,s)\Omega\rangle\right| \; .$$

Using the differentiability properties of $\widetilde{U}_2$, we have

$$i\partial_t\langle\Theta,\widetilde{U}_2^\dagger(t,s)\psi^*(g_1)\widetilde{U}_2(t,s)\Omega\rangle = \langle\Theta,\widetilde{U}_2^\dagger(t,s)[\psi^*(g_1),\widetilde{V}(t)]\widetilde{U}_2(t,s)\Omega\rangle \ .$$

Performing the commutation, integrating and taking the absolute value we obtain

$$\begin{aligned}\left|\langle\Theta,\widetilde{U}_2^\dagger(t,s)\psi^*(g_1)\widetilde{U}_2(t,s)\Omega\rangle\right| \leq \int_s^t \mathrm{d}\tau \Big|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)\Big(a(g_{1-})\\ +a^*(g_{1+})+\psi^*(g_{10})\Big)\widetilde{U}_2(\tau,s)\Omega\rangle\Big| \ ,\end{aligned}$$

with

$$\begin{aligned}g_{1-}(t,\cdot) &= \int \mathrm{d}x\, g_1(x)\widetilde{v}_{--}(t,x,\cdot) \ ,\\ g_{1+}(t,\cdot) &= \int \mathrm{d}x\, g_1(x)\widetilde{v}_{-+}(t,x,\cdot) \ ,\\ g_{10}(t,\cdot) &= U_{01}^\dagger(t)(\check{\chi}*A(t))(\cdot)U_{01}(t)g_1(\cdot) \ .\end{aligned}$$

Multiply now both members by $\|g_1\|_2^{-1}$, and calculate the supremum in g_1:

$$\begin{aligned}X_1(t) \leq &\int_s^t \mathrm{d}\tau \sup_{g_1\in L^2}\frac{1}{\|g_1\|_2}\left|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)a(g_{1-})\widetilde{U}_2(\tau,s)\Omega\rangle\right|\\ +&\int_s^t \mathrm{d}\tau \sup_{g_1\in L^2}\frac{1}{\|g_1\|_2}\left|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)a^*(g_{1+})\widetilde{U}_2(\tau,s)\Omega\rangle\right|\\ +&\int_s^t \mathrm{d}\tau \sup_{g_1\in L^2}\frac{1}{\|g_1\|_2}\left|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)\psi^*(g_{10})\widetilde{U}_2(\tau,s)\Omega\rangle\right| \ .\end{aligned}$$

Now we change the supremum function obtaining:

$$\begin{aligned}X_1(t) \leq &\int_s^t \mathrm{d}\tau \sup_{g_{1-}\in L^2}\frac{\|g_{1-}\|_2}{\|g_1\|_2}\frac{1}{\|g_{1-}\|_2}\left|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)a(g_{1-})\widetilde{U}_2(\tau,s)\Omega\rangle\right|\\ +&\int_s^t \mathrm{d}\tau \sup_{g_{1+}\in L^2}\frac{\|g_{1+}\|_2}{\|g_1\|_2}\frac{1}{\|g_{1+}\|_2}\left|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)a^*(g_{1+})\widetilde{U}_2(\tau,s)\Omega\rangle\right|\\ +&\int_s^t \mathrm{d}\tau \sup_{g_{10}\in L^2}\frac{\|g_{10}\|_2}{\|g_1\|_2}\frac{1}{\|g_{10}\|_2}\left|\langle\Theta,\widetilde{U}_2^\dagger(\tau,s)\psi^*(g_{10})\widetilde{U}_2(\tau,s)\Omega\rangle\right| \ .\end{aligned}$$

Using the following estimates

$$\begin{aligned}\|g_{1-}(t)\|_2 &\leq \|f_0\|_2\,\|u(t)\|_2\,\|g_1\|_2 \ ,\\ \|g_{1+}(t)\|_2 &\leq \|f_0\|_2\,\|u(t)\|_2\,\|g_1\|_2 \ ,\\ \|g_{10}(t)\|_2 &\leq \|\check{\chi}*A(t)\|_\infty\,\|g_1\|_2 \ .\end{aligned}$$

we obtain the sought result

$$\begin{aligned}X_1(t) &\leq C\int_s^t \mathrm{d}\tau\, X(\tau) \ ,\\ C &= \sup_{\tau\in[s,t]}\Big(2\,\|f_0\|_2\,\|u(\tau)\|_2 + \|\check{\chi}*A(\tau)\|_\infty\Big) \ .\end{aligned}$$

■

2. The general case: convergence as λ.

Using only the bound of Proposition 8 we can obtain a first result of convergence of the creation and annihilation operators towards the classical solutions when $\lambda\to 0$. Obviously since $\psi^\#$ and $a^\#$ are operators on $\mathscr{H}$, while $u^\#$ and $\alpha^\#$ are functions in $L^2(\mathbb{R}^3)$, we should expect that the average of the former between suitable states will converge to the latter in some sense. Consider a set of λ-dependent states constructed by means of Weyl operators, defined as follows.

DEFINITION. Let $\Phi \in \mathscr{H}^\delta$, with $\delta \geq 1$, such that $\|\Phi\| = 1$; $(u, \alpha) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. Then

$$\{C(u_\lambda, \alpha_\lambda)\Phi \equiv C(u/\lambda, \alpha/\lambda)\Phi \ , \ 0 < \lambda \leq 1\}$$

is a λ-dependent set of vectors in $\mathscr{H}^\delta$.

From this definition we immediately see that if we choose $\delta \geq 2$ and $f \in L^2(\mathbb{R}^3)$, we can study the well-defined averages

$$\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi} = \langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^\#(\bar{f}^\#)U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle$$
$$\langle \lambda a^\#(\bar{f}^\#)(t)\rangle_{C\Phi} = \langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda a^\#(\bar{f}^\#)U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle \ .$$

The reason we consider $\lambda\psi^\#$ and $\lambda a^\#$ instead of $\psi^\#$ and $a^\#$ as the quantum operators to have classical limit is discussed in Section 4 of Chapter 1; however it is evident that at time zero the averages above would diverge, when $\lambda \to 0$, if we substitute $\lambda\psi^\#$ and $\lambda a^\#$ with $\psi^\#$ and $a^\#$.

LEMMA 2.1. *Let $\Phi \in \mathscr{H}^\delta$, $\delta \geq 9$, $f \in L^2(\mathbb{R}^3)$. Furthermore let $(u(\cdot), \alpha(\cdot)) \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$ be the solution of* (E) *with initial conditions $(u, \alpha) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. Then*

$$\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi} \to \langle f^\#, u^\#(t)\rangle_2$$
$$\langle \lambda a^\#(\bar{f}^\#)(t)\rangle_{C\Phi} \to \langle f^\#, \alpha^\#(t)\rangle_2$$

when $\lambda \to 0$. More precisely we have the following bounds:

$$\left|\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi} - \langle f^\#, u^\#(t)\rangle_2\right| \leq \lambda \|f\|_2 \Big(\mathcal{K}_1(t,0) + \lambda\mathcal{K}_2(t,0)\Big)^{1/2} e^{(\lambda\mathcal{C}_1|t| + \mathcal{K}_3(t,0))/2} \|\Phi\|_{\mathscr{H}^9}$$
$$\left|\langle \lambda a^\#(\bar{f}^\#)(t)\rangle_{C\Phi} - \langle f^\#, \alpha^\#(t)\rangle_2\right| \leq \lambda \|f\|_2 \Big(\mathcal{K}_1(t,0) + \lambda\mathcal{K}_2(t,0)\Big)^{1/2} e^{(\lambda\mathcal{C}_1|t| + \mathcal{K}_3(t,0))/2} \|\Phi\|_{\mathscr{H}^9} \ ,$$

where the constants are defined in Proposition 8.

PROOF. We prove the result for $\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi}$, the other case being perfectly analogous. Using Proposition 5 we can write

$$\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi} = \langle W(t,0)\Phi, \lambda\psi^\#(\bar{f}^\#)W(t,0)\Phi\rangle + \langle f^\#, u^\#(t)\rangle_2 \ .$$

Then we have that

$$\left|\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi} - \langle f^\#, u^\#(t)\rangle_2\right| \leq \lambda \left\|\psi^\#(\bar{f}^\#)W(t,0)\Phi\right\| \leq \lambda \|f\|_2 \|W(t,0)\Phi\|_{\mathscr{H}^1} \ ,$$

using the Corollary of Proposition 8 we obtain the result. ■

DEFINITION. We define formally the following complex-valued functions of $\mathbb{R}^3$:

$$\langle \lambda\psi^\#(t,\cdot)\rangle_{C\Phi} = \langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda\psi^\#(\cdot)U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle$$
$$\langle \lambda a^\#(t,\cdot)\rangle_{C\Phi} = \langle C(u_\lambda, \alpha_\lambda)\Phi, U^\dagger(t)\lambda a^\#(\cdot)U(t)C(u_\lambda, \alpha_\lambda)\Phi\rangle \ .$$

By this definition we have formally the following relations:

$$\langle \lambda\psi^\#(\bar{f}^\#)(t)\rangle_{C\Phi} = \int \mathrm{d}x\, \bar{f}^\#(x)\langle \lambda\psi^\#(t,x)\rangle_{C\Phi}$$
$$\langle \lambda a^\#(\bar{f}^\#)(t)\rangle_{C\Phi} = \int \mathrm{d}k\, \bar{f}^\#(k)\langle \lambda a^\#(t,k)\rangle_{C\Phi} \ .$$

LEMMA 2.2. *$\langle\lambda\psi^{\#}(\bar{f}^{\#})(t)\rangle_{C\Phi}$ and $\langle\lambda a^{\#}(\bar{f}^{\#})(t)\rangle_{C\Phi}$ (as functionals of $f^{\#}$) are bounded antilinear functionals of $L^2(\mathbb{R}^3)$.*

PROOF. Linearity follows from the linearity of $\psi^{\#}(\cdot)$ and $a^{\#}(\cdot)$ on suitable domains as proved in Appendix A. Boundedness follows from usual estimates: let $\Phi \in \mathscr{H}^{\delta}$, with $\delta \geq 2$, then exists a constant C independent of f such that:

$$\left|\langle\lambda\psi^{\#}(\bar{f}^{\#})(t)\rangle_{C\Phi}\right| \leq \lambda \left\|f\right\|_2 \left\|U(t)C(u_\lambda,\alpha_\lambda)\Phi\right\|_{\mathscr{H}^1} \leq C\left\|f\right\|_2 \ .$$

The case of $\langle\lambda a^{\#}(\bar{f}^{\#})(t)\rangle_{C\Phi}$ being analogous. ■

COROLLARY. *$\langle\lambda\psi^{\#}(t)\rangle_{C\Phi}$ and $\langle\lambda a^{\#}(t)\rangle_{C\Phi}$ are in $L^2(\mathbb{R}^3)$.*

PROOF. Consider $\langle\lambda\psi^{\#}(t)\rangle_{C\Phi}$, the other case is identical. We use Riesz's Lemma and the result above to infer that exists a function $g \in L^2(\mathbb{R}^3)$ such that

$$\langle f^{\#}, g\rangle_2 = \langle\lambda\psi^{\#}(\bar{f}^{\#})(t)\rangle_{C\Phi} \ .$$

Then accordingly to the formal definition of $\langle\lambda\psi^{\#}(t)\rangle_{C\Phi}$ we set $g \equiv \langle\lambda\psi^{\#}(t)\rangle_{C\Phi}$. ■

PROPOSITION 10. *Let $\Phi \in \mathscr{H}^{\delta}$, $\delta \geq 9$; $(u(\cdot),\alpha(\cdot)) \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ be the solution of* (E) *with initial conditions $(u,\alpha) \in L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)$. Then $\langle\lambda\psi^{\#}(t)\rangle_{C\Phi} \rightarrow_{L^2(\mathbb{R}^3)} u^{\#}(t)$, $\langle\lambda a^{\#}(t)\rangle_{C\Phi} \rightarrow_{L^2(\mathbb{R}^3)} \alpha^{\#}(t)$ when $\lambda \to 0$. More precisely we have the following bounds:*

$$\left\|\langle\lambda\psi^{\#}(t)\rangle_{C\Phi} - u^{\#}(t)\right\|_2^2 \leq \lambda\Big(\mathcal{K}_1(t,0) + \lambda\mathcal{K}_2(t,0)\Big)e^{\lambda\mathcal{C}_1|t| + \mathcal{K}_3(t,0)}\left\|\Phi\right\|_{\mathscr{H}^9}$$
$$\left\|\langle\lambda a^{\#}(t)\rangle_{C\Phi} - \alpha^{\#}(t)\right\|_2^2 \leq \lambda\Big(\mathcal{K}_1(t,0) + \lambda\mathcal{K}_2(t,0)\Big)e^{\lambda\mathcal{C}_1|t| + \mathcal{K}_3(t,0)}\left\|\Phi\right\|_{\mathscr{H}^9}$$

where the constants are defined in Proposition 8.

PROOF. A straightforward consequence of Lemma 2.1 and the Corollary of Lemma 2.2. ■

3. Coherent states: convergence as λ^2.

Using the evolution of quantum fluctuations, in particular the result proved in this chapter, we can improve the bound obtained in the proposition above. However the result applies only to a particular set of states called coherent, namely the states we obtain applying Weyl operators to the vacuum.

DEFINITION 3.1 (Classical limit coherent states Λ). Let $(u,\alpha) \in L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)$, Ω the vacuum state of $\mathscr{H}$ (see Section 2 of Chapter 1). Then

$$\{\Lambda \equiv C(u_\lambda,\alpha_\lambda)\Omega \ , \ 0 < \lambda \leq 1\}$$

is a set of λ-dependent states in $\mathscr{H}^{\delta}$ for all positive δ.

We will proceed as in the previous section. So we will define the following quantities:

DEFINITION. Let $f \in L^2(\mathbb{R}^3)$, then we define

$$\langle\lambda\psi^{\#}(\bar{f}^{\#})(t)\rangle_{\Lambda} = \langle\Lambda, U^{\dagger}(t)\lambda\psi^{\#}(\bar{f}^{\#})U(t)\Lambda\rangle$$
$$\langle\lambda a^{\#}(\bar{f}^{\#})(t)\rangle_{\Lambda} = \langle\Lambda, U^{\dagger}(t)\lambda a^{\#}(\bar{f}^{\#})U(t)\Lambda\rangle \ .$$

Using the results of previous section we can also define the following functions of $L^2(\mathbb{R}^3)$:

$$\langle\lambda\psi^{\#}(t,\cdot)\rangle_{\Lambda} = \langle\Lambda, U^{\dagger}(t)\lambda\psi^{\#}(\cdot)U(t)\Lambda\rangle$$
$$\langle\lambda a^{\#}(t,\cdot)\rangle_{\Lambda} = \langle\Lambda, U^{\dagger}(t)\lambda a^{\#}(\cdot)U(t)\Lambda\rangle \ .$$

PROPOSITION 11. *Let* $(u(\cdot),\alpha(\cdot)) \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3))$ *be the solution of* (E) *with initial conditions* $(u,\alpha)\in L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)$. *Then we have two positive constants* K_1 *and* K_2 *such that:*

$$\begin{aligned}\left\|\langle\lambda\psi^{\#}(t)\rangle_\Lambda - u^{\#}(t)\right\|_2 &\leq \lambda^2 K_1\,|t|\,e^{K_2|t|}\\ \left\|\langle\lambda a^{\#}(t)\rangle_\Lambda - \alpha^{\#}(t)\right\|_2 &\leq \lambda^2 K_1\,|t|\,e^{K_2|t|}\ .\end{aligned}$$

PROOF. To prove the proposition we proceed as in the previous section, in particular it will be sufficient to prove the following lemma:

LEMMA 3.1. *Let* $f\in L^2(\mathbb{R}^3)$, *then we have the following bounds:*

$$\begin{aligned}\left|\langle\lambda\psi^{\#}(\bar f^{\#})(t)\rangle_\Lambda - \langle f^{\#}, u^{\#}(t)\rangle_2\right| &\leq \lambda^2\,\|f\|_2\,K_1\,|t|\,e^{K_2|t|}\\ \left|\langle\lambda a^{\#}(\bar f^{\#})(t)\rangle_\Lambda - \langle f^{\#}, \alpha^{\#}(t)\rangle_2\right| &\leq \lambda^2\,\|f\|_2\,K_1\,|t|\,e^{K_2|t|}\end{aligned}$$

with C *and* K *positive constants.*

PROOF. As usual we prove the result for $\langle\lambda\psi^{\#}(\bar f^{\#})(t)\rangle_\Lambda$, the other case being perfectly analogous. Using Proposition 5 we can write

$$\begin{aligned}\left|\langle\lambda\psi^{\#}(\bar f^{\#})(t)\rangle_\Lambda - \langle f^{\#}, u^{\#}(t)\rangle_2\right| &= \lambda\left|\langle W(t,0)\Omega, \psi^{\#}(\bar f^{\#})W(t,0)\Omega\rangle\right|\\ &= \lambda\left|\langle \widetilde W(t,0)\Omega, U_0^{\dagger}(t)\psi^{\#}(\bar f^{\#})U_0(t)\widetilde W(t,0)\Omega\rangle\right|\ .\end{aligned}$$

Using Lemma 6.1 of Appendix A we then obtain:

$$\left|\langle\lambda\psi^{\#}(\bar f^{\#})(t)\rangle_\Lambda - \langle f^{\#}, u^{\#}(t)\rangle_2\right| = \lambda\left|\langle \widetilde W(t,0)\Omega, \psi^{\#}(\bar{\tilde f}^{\#})\widetilde W(t,0)\Omega\rangle\right|\ .$$

Now using the equality

$$\begin{aligned}\langle \widetilde W(t,0)\Omega, \psi^{\#}(\bar{\tilde f}^{\#})\widetilde W(t,0)\Omega\rangle &= \langle \widetilde U_2(t,0)\Omega, \psi^{\#}(\bar{\tilde f}^{\#})\widetilde U_2(t,0)\Omega\rangle\\ &+\langle(\widetilde W(t,0)-\widetilde U_2(t,0))\Omega, \psi^{\#}(\bar{\tilde f}^{\#})\widetilde U_2(t,0)\Omega\rangle\\ &+\langle \widetilde W(t,0)\Omega, \psi^{\#}(\bar{\tilde f}^{\#})(\widetilde W(t,0)-\widetilde U_2(t,0))\Omega\rangle\end{aligned}$$

we obtain

$$\begin{aligned}\left|\langle\lambda\psi^{\#}(\bar f^{\#})(t)\rangle_\Lambda - \langle f^{\#}, u^{\#}(t)\rangle_2\right| &\leq \lambda\Big(\left|\langle \widetilde U_2(t,0)\Omega, \psi^{\#}(\bar{\tilde f}^{\#})\widetilde U_2(t,0)\Omega\rangle\right|\\ &+\left|\langle(\widetilde W(t,0)-\widetilde U_2(t,0))\Omega, \psi^{\#}(\bar{\tilde f}^{\#})\widetilde U_2(t,0)\Omega\rangle\right|\\ &+\left|\langle \widetilde W(t,0)\Omega, \psi^{\#}(\bar{\tilde f}^{\#})(\widetilde W(t,0)-\widetilde U_2(t,0))\Omega\rangle\right|\Big)\\ &\equiv \lambda\Big(X_1+X_2+X_3\Big)\ .\end{aligned}$$

By Proposition 9 we have that $X_1=0$. Then we bound X_2 as follows:

$$\begin{aligned}X_2 &= \left|\langle\int_0^t \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde W^{\dagger}(\tau,0)\widetilde U_2(\tau,0)\Omega, \widetilde W^{\dagger}(t,0)\psi^{\#}(\bar{\tilde f}^{\#})\widetilde U_2(t,0)\Omega\rangle\right|\\ &\leq \left|\int_0^t \mathrm{d}\tau\,\left\|\widetilde H_I(\tau)\widetilde U_2(\tau,0)\Omega\right\|\,\left\|\psi^{\#}(\bar{\tilde f})\widetilde U_2(t,0)\Omega\right\|\right|\\ &\leq \lambda\,\|f\|_2\,\|f_0\|_2\left|\int_0^t \mathrm{d}\tau\,\left\|\widetilde U_2(\tau,0)\Omega\right\|_{\mathscr{H}^4}\left\|\widetilde U_2(t,0)\Omega\right\|_{\mathscr{H}^1}\right|\ .\end{aligned}$$

Using Proposition 6 we obtain

$$\begin{aligned}X_2 \leq \lambda\,\|f\|_2\,\|f_0\|_2\exp&\left\{\frac12\left(\ln 3+4\sqrt2\left|\int_0^t \mathrm{d}\tau\,\|v_{--}(\tau)\|_2\right|\right)\right\}\\ &\left|\int_0^t \mathrm{d}\tau\,\exp\left\{2\left(\ln 3+10\sqrt2\left|\int_0^\tau \mathrm{d}\tau'\,\|v_{--}(\tau')\|_2\right|\right)\right\}\right|\ .\end{aligned}$$

To bound X_3 we use a similar method:

$$X_3 = \left| \langle \widetilde{W}^\dagger(t,0) \big(\psi^\#(\bar{\tilde{f}}^\#)\big)^\dagger \widetilde{W}(t,0)\Omega, \int_0^t \mathrm{d}\tau\, \widetilde{W}^\dagger(\tau,0) \widetilde{H}_I(\tau) \widetilde{U}_2(\tau,0)\Omega\rangle \right|$$

$$\leq \lambda \|f\|_2 \|f_0\|_2 \left\| \widetilde{W}(t,0)\Omega \right\|_{\mathscr{H}^1} \left| \int_0^t \mathrm{d}\tau \, \left\| \widetilde{U}_2(\tau,0)\Omega \right\|_{\mathscr{H}^4} \right| ,$$

then using Propositions 6 and 8 we obtain

$$X_3 \leq \lambda \|f\|_2 \|f_0\|_2 \Big(\mathcal{K}_1(t,0) + \lambda \mathcal{K}_2(t,0) \Big)^{1/2} e^{(\lambda \mathcal{C}_1 |t| + \mathcal{K}_3(t,0))/2}$$
$$\left| \int_0^t \mathrm{d}\tau \, \exp\left\{ 2 \left(\ln 3 + 10\sqrt{2} \left| \int_0^\tau \mathrm{d}\tau' \, \|v_{--}(\tau')\|_2 \right| \right) \right\} \right| .$$

■

■

4. The classical limit of creation and annihilation operators.

In this section we will sum up in a theorem the results we proved above.

THEOREM 2. *Let $\Phi \in \mathscr{H}^\delta$, with $\delta \geq 9$, $(u, \alpha) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. We call $(u(t,x), \alpha(t,k)) \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$ the solution of* (E) *with initial conditions $(u, \alpha) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. Then the following statements are valid:*

i. The following limits exist in $L^2(\mathbb{R}^3)$ when $\lambda \to 0$:

$$\langle \lambda\psi(t,\cdot)\rangle_{C\Phi} \underset{\lambda\to 0}{\overset{L^2(\mathbb{R}^3)}{\longrightarrow}} u(t,\cdot)$$
$$\langle \lambda\psi^*(t,\cdot)\rangle_{C\Phi} \underset{\lambda\to 0}{\overset{L^2(\mathbb{R}^3)}{\longrightarrow}} \bar{u}(t,\cdot)$$
$$\langle \lambda a(t,\cdot)\rangle_{C\Phi} \underset{\lambda\to 0}{\overset{L^2(\mathbb{R}^3)}{\longrightarrow}} \alpha(t,\cdot)$$
$$\langle \lambda a^*(t,\cdot)\rangle_{C\Phi} \underset{\lambda\to 0}{\overset{L^2(\mathbb{R}^3)}{\longrightarrow}} \bar{\alpha}(t,\cdot) .$$

ii. There are two positive constants K_1 and K_2 such that

$$\|\langle \lambda\psi(t,\cdot)\rangle_{C\Phi} - u(t,\cdot)\|_2 \leq \lambda K_1 |t| e^{K_2|t|} \|\Phi\|_\delta^2$$
$$\|\langle \lambda\psi^*(t,\cdot)\rangle_{C\Phi} - \bar{u}(t,\cdot)\|_2 \leq \lambda K_1 |t| e^{K_2|t|} \|\Phi\|_\delta^2$$
$$\|\langle \lambda a(t,\cdot)\rangle_{C\Phi} - \alpha(t,\cdot)\|_2 \leq \lambda K_1 |t| e^{K_2|t|} \|\Phi\|_\delta^2$$
$$\|\langle \lambda a^*(t,\cdot)\rangle_{C\Phi} - \bar{\alpha}(t,\cdot)\|_2 \leq \lambda K_1 |t| e^{K_2|t|} \|\Phi\|_\delta^2 .$$

iii. If $\Phi = \Omega$, the vacuum state of $\mathscr{H}$, then

$$\|\langle \lambda\psi(t,\cdot)\rangle_{C\Omega} - u(t,\cdot)\|_2 \leq \lambda^2 K_1 |t| e^{K_2|t|}$$
$$\|\langle \lambda\psi^*(t,\cdot)\rangle_{C\Omega} - \bar{u}(t,\cdot)\|_2 \leq \lambda^2 K_1 |t| e^{K_2|t|}$$
$$\|\langle \lambda a(t,\cdot)\rangle_{C\Omega} - \alpha(t,\cdot)\|_2 \leq \lambda^2 K_1 |t| e^{K_2|t|}$$
$$\|\langle \lambda a^*(t,\cdot)\rangle_{C\Omega} - \bar{\alpha}(t,\cdot)\|_2 \leq \lambda^2 K_1 |t| e^{K_2|t|} .$$

With this theorem we clarify in what sense $(u(t), \alpha(t))$ are the classical counterparts of $(\lambda\psi(t), \lambda a(t))$. We remark that if we did not have used, in the definition of $W(t,s)$, the solution of classical equations (E) then we could not have found a λ-convergent bound of $\|W(t,s)\Phi\|_\delta$ and calculate the limit $\lambda \to 0$. Furthermore we see that the analysis of quantum fluctuations enables us to improve the result of convergence we would have obtained using only the properties of quantum evolution $W(t,s)$.

CHAPTER 8

Normal ordered products of operators.

The method we used in the previous chapter can be used to calculate the classical limit of averages of normal ordered products of creation and annihilation operators. The limit of such averages can be calculated not only between coherent states, but also between fixed particle states. We will state a theorem with the general result, however it will be proved in appendix; here we will discuss in detail two simpler cases.

1. The set of states to calculate the classical limit.

We will consider three different kind of states to calculate transition amplitudes. We call Ω_p the vacuum state of $\bigoplus_p \mathscr{H}_{p,0}$, Ω_n the one of $\bigoplus_n \mathscr{H}_{0,n}$.

Definition (Λ, Ψ, Θ). Let $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$ such that $\|u_0\|_2 = \|\alpha_0\|_2 = 1$. Then we define, for any $\tilde{p}, \tilde{n} \in \mathbb{N}$ the following set of states:

$$\{\Lambda = C(\sqrt{\tilde{p}}\, u_0, \sqrt{\tilde{n}}\, \alpha_0)\Omega \ , \ 1 \leq \tilde{p}, \tilde{n} < \infty\}$$
$$\{\Psi = u_0^{\otimes \tilde{p}} \otimes C_n(\sqrt{\tilde{n}}\, \alpha_0)\Omega_n \in \mathscr{H}_{\tilde{p}} \ , \ 1 \leq \tilde{p}, \tilde{n} < \infty\}$$
$$\{\Theta = u_0^{\otimes \tilde{p}} \otimes \alpha_0^{\otimes \tilde{n}} \in \mathscr{H}_{\tilde{p},\tilde{n}} \ , \ 1 \leq \tilde{p}, \tilde{n} < \infty\} \ .$$

Λ is the coherent state we already defined, Ψ is a tensor product of a state with fixed number of particles and a coherent one, finally Θ is a state with fixed number of particles. We want to evolve such states with the quantum evolution $U(\cdot)$, so we introduce the notation:

$$\Lambda(t) = U(t)\Lambda \ ,$$
$$\Psi(t) = U(t)\Psi \ ,$$
$$\Theta(t) = U(t)\Theta \ .$$

From now on, we fix $\tilde{p}$ and $\tilde{n}$ as following: $\tilde{p} = \tilde{n} = \lambda^{-2}$, so the Weyl operators above become $C(u_0/\lambda, \alpha_0/\lambda)$ and $C(\alpha_0/\lambda)$. In the case of fixed particle states the limit $\lambda \to 0$ corresponds to the limit when the number of particles becomes infinite. The reason we introduced here the parameters $\tilde{p}$ and $\tilde{n}$ is that it is more natural to write a fixed particle state having $\tilde{p}$ or $\tilde{n}$ components than λ^{-2}.

Definition ($\alpha_0(\theta)$, $(u_\theta(t), \alpha_\theta(t))$). Let $\alpha_0 \in L^2(\mathbb{R}^3)$, we define

$$\alpha_0(\theta) \equiv \exp(-i\theta)\alpha_0 \ .$$

Furthermore we call $(u_\theta(t), \alpha_\theta(t)) \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3))$ the solution of (E) with initial condition $(u_0, \alpha_0(\theta)) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ (while $(u(t), \alpha(t))$ is the solution associated to (u_0, α_0)).

We will formulate here three lemmas that relate coherent states to states with fixed number of particles. We adapted these results to be suitable for our set of vectors Ψ and Θ. The basic formula used to prove the first and third of such lemmas in literature is:

$$\exp\big((a^*(f) - a(\bar{f}))^{\overline{\ \ }}\big) = e^{-\|f\|_2^2/2} \exp(a^*(f)) \exp(-a(\bar{f})) \ .$$

However this equality makes sense only applied to vectors that permits us to write such exponentials as power series. The properties of exponentials as series and their associated vectors are studied in Appendix D. The second result is proved using sharp estimates of Laguerre polynomials recently obtained in a work by Krasikov [**Kra05**].

By definition of creation operators we can write the following identities:

$$\Psi = \frac{\psi^*(u_0)^{\tilde{p}}}{\sqrt{\tilde{p}!}} \otimes C_n(\sqrt{\tilde{n}}\alpha_0)\Omega = \frac{\psi^*(u_0)^{\lambda^{-2}}}{\sqrt{(\lambda^{-2})!}} \otimes C_n(\alpha_0/\lambda)\Omega \ ,$$

$$\Theta = \frac{\psi^*(u_0)^{\tilde{p}}}{\sqrt{\tilde{p}!}} \frac{a^*(\alpha_0)^{\tilde{n}}}{\sqrt{\tilde{n}!}}\Omega = \frac{\psi^*(u_0)^{\lambda^{-2}}}{\sqrt{(\lambda^{-2})!}} \frac{a^*(\alpha_0)^{\lambda^{-2}}}{\sqrt{(\lambda^{-2})!}}\Omega \ .$$

Definition (d_x function).

$$d_x \equiv \frac{\sqrt{x!}}{e^{-x/2}x^{x/2}} \sim x^{1/4} \ .$$

Lemma 1.1. *Let $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$, such that $\|u_0\|_2 = \|\alpha_0\|_2 = 1$. Then we obtain the following identities:*

$$\Psi = d_{\tilde{p}}\, \mathbb{P}_{\tilde{p}}\, C(\sqrt{\tilde{p}}u_0, \sqrt{\tilde{n}}\alpha_0)\Omega = d_{\lambda^{-2}}\mathbb{P}_{\lambda^{-2}}C(u_0/\lambda, \alpha_0/\lambda)\Omega$$

$$\Theta = d_{\tilde{p}}\, d_{\tilde{n}}\, \mathbb{P}_{\tilde{p}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{i\tilde{n}\theta} C(\sqrt{\tilde{p}}u_0, \sqrt{\tilde{n}}\alpha_0(\theta))\Omega$$

$$= d^2_{\lambda^{-2}}\mathbb{P}_{\lambda^{-2}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{i\lambda^{-2}\theta} C(u_0/\lambda, \alpha_0(\theta)/\lambda)\Omega \ ,$$

where $\mathbb{P}_p$ is the orthogonal projector on $\mathscr{H}_p$.

Proof. The first relation is a rewriting of equation (4.2) of [**COS11**], the second utilizes both the one just cited and Lemma 4.1 of [**RS09**]. ■

Lemma 1.2. *Let $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$ such that $\|u_0\|_2 = \|\alpha_0\|_2 = 1$. Then there are two constants K_Ψ and K_Θ independent of λ and θ such that:*

$$\left\|(P+1)^{-1/2}C^\dagger(u_0/\lambda, \alpha_0/\lambda)\Psi\right\| \leq K_\Psi d^{-1}_{\lambda^{-2}} \ ,$$

$$\left\|(P+1)^{-1/2}(N+1)^{-1/2}C^\dagger(u_0/\lambda, \alpha_0(\theta)/\lambda)\Theta\right\| \leq K_\Theta d^{-2}_{\lambda^{-2}} \ .$$

Proof. This result is stated in Lemma 7.1 of [**COS11**] and proved in [**CL11**]. ■

Lemma 1.3. *Consider Ψ, Θ defined as above, then for all $\theta \in \mathbb{R}$*

$$(\mathbb{P}_0\mathbb{N}_1 + \mathbb{P}_1\mathbb{N}_0)C^\dagger(u_0/\lambda, \alpha_0/\lambda)\Psi = 0$$

$$(\mathbb{P}_0\mathbb{N}_1 + \mathbb{P}_1\mathbb{N}_0)C^\dagger(u_0/\lambda, \alpha_0(\theta)/\lambda)\Theta = 0 \ .$$

Proof. The part about $\mathbb{P}_1$ of the Ψ case is proved in[**COS11**], page 13, as well as the result for Θ. Obviously also $\mathbb{N}_1C^\dagger(u_0/\lambda, \alpha_0/\lambda)\Psi = 0$. ■

2. Definition of the transition amplitudes.

Definition (Transition amplitudes). Let $q, r, i, j \in \mathbb{N}$, $\delta = q + r + i + j$, $g \in L^2(\mathbb{R}^{3(q+r)}) \otimes L^2(\mathbb{R}^{3(i+j)}) \equiv L^2(\mathbb{R}^{3\delta})$ and

$$B = \int \mathrm{d}X_q\mathrm{d}Y_r\mathrm{d}K_i\mathrm{d}M_j\, \bar{g}(X_q, Y_r, K_i, M_j)\psi^*(X_q)\psi(Y_r)a^*(K_i)a(M_j) \ .$$

Then we define the following transition amplitudes:

$$\begin{aligned}
\langle B\rangle_\Lambda(t) &\equiv \lambda^\delta\langle\Lambda(t), B\Lambda(t)\rangle = \lambda^\delta\langle\Lambda, U^\dagger(t)BU(t)\Lambda\rangle \\
\langle B\rangle_\Psi(t) &\equiv \lambda^\delta\langle\Psi(t), B\Psi(t)\rangle = \lambda^\delta\langle\Psi, U^\dagger(t)BU(t)\Psi\rangle \\
\langle B\rangle_\Theta(t) &\equiv \lambda^\delta\langle\Theta(t), B\Theta(t)\rangle = \lambda^\delta\langle\Theta, U^\dagger(t)BU(t)\Theta\rangle \ .
\end{aligned}$$

Furthermore we define the following complex-valued functions of $\mathbb{R}^{3\delta}$:

$$\begin{aligned}
\langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Lambda(t) &\equiv \lambda^\delta\langle\Lambda(t), \psi^*(X_q)\psi(Y_r)a^*(K_i)a(M_j)\Lambda(t)\rangle \\
\langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Psi(t) &\equiv \lambda^\delta\langle\Psi(t), \psi^*(X_q)\psi(Y_r)a^*(K_i)a(M_j)\Psi(t)\rangle \\
\langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Theta(t) &\equiv \lambda^\delta\langle\Theta(t), \psi^*(X_q)\psi(Y_r)a^*(K_i)a(M_j)\Theta(t)\rangle \ .
\end{aligned}$$

We remark that λ^δ is necessary if we want these transition amplitudes to remain finite even at $t = 0$ in the limit $\lambda \to 0$ ($\lambda\psi$ and λa rather than ψ and a are the quantum operators that have classical limit). Formally we have that

$$\begin{aligned}
\langle B\rangle_\Lambda(t) &= \langle g, \langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Lambda(t)\rangle_{L^2(\mathbb{R}^{3\delta})} \\
\langle B\rangle_\Psi(t) &= \langle g, \langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Psi(t)\rangle_{L^2(\mathbb{R}^{3\delta})} \\
\langle B\rangle_\Theta(t) &= \langle g, \langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Theta(t)\rangle_{L^2(\mathbb{R}^{3\delta})} \ .
\end{aligned}$$

Now we will prove such relation is not only formal, because $\langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle$ is in $L^2(\mathbb{R}^{3\delta})$ for all states.

LEMMA 2.1. *Λ, Ψ and Θ are in $\mathscr{H}^\delta$ for all positive δ.*

PROOF. The proof is trivial for fixed particle states. We know that $C(u,\alpha)$ map $\mathscr{H}^\delta$ into itself (Proposition 5) and Ω is obviously in $\mathscr{H}^\delta$ for all positive δ. Using the results of Appendix A we also know that $C(\alpha)$ maps $D(N^\delta)$ into itself and Ω_n is obviously in $D(N^\delta)$ for all δ. ■

LEMMA 2.2. *For all positive δ and real t there is a finite constant C such that for all $g \in L^2(\mathbb{R}^{3\delta})$*

$$\begin{aligned}
\langle B\rangle_\Lambda(t) &\leq C\left\|g; L^2(\mathbb{R}^{3\delta})\right\| \\
\langle B\rangle_\Psi(t) &\leq C\left\|g; L^2(\mathbb{R}^{3\delta})\right\| \\
\langle B\rangle_\Theta(t) &\leq C\left\|g; L^2(\mathbb{R}^{3\delta})\right\| \ .
\end{aligned}$$

PROOF. We proof the result for Λ, the other cases being identical. Using Lemma 2.1 of Chapter 3 we obtain:

$$\left|\langle B\rangle_\Lambda(t)\right| \leq \lambda^\delta\left\|g; L^2(\mathbb{R}^{3\delta})\right\|\left\|\Lambda\right\|\left\|\frac{\sqrt{P!(P+q-r)!N!(N+i-j)!}}{(P-r)!(N-j)!}\right. \\ \left.\theta(P-q)\theta(N-i)U(t)\Lambda\right\| .$$

Using now the fact that $Q \leq P^2 + N$, we have that

$$\left|\langle B\rangle_\Lambda(t)\right| \leq \lambda^\delta\left\|g; L^2(\mathbb{R}^{3\delta})\right\|\left\|\Lambda\right\|\left\|(P^2+N+q+i)^{\delta/2}U(t)\Lambda\right\| \ .$$

Proposition 4 then leads to

$$\left|\langle B\rangle_\Lambda(t)\right| \leq C\left\|g; L^2(\mathbb{R}^{3\delta})\right\|\left\|\Lambda\right\|\left\|\Lambda\right\|_{2\delta} \ .$$

Now $\left\|\Lambda\right\|_{2\delta}$ is finite by Lemma 2.1, so we obtain the sought result. ■

PROPOSITION 12. $\langle\psi^*(q)\psi(r)a^*(i)a(j)\rangle_X(t) \in L^2(\mathbb{R}^{3\delta})$, *for $X \in \{\Lambda, \Psi, \Theta\}$.*

PROOF. Again we prove the result only for Λ. $\langle B\rangle_\Lambda(t)$ defines a linear functional on $L^2(\mathbb{R}^{3\delta})$, as a direct consequence of Lemma 2.2, since the bound is independent of g. The applying Riesz's Lemma we know that there is a function $h \in L^2(\mathbb{R}^{3\delta})$ such that $\langle B\rangle_\Lambda(t) = \langle g, h\rangle_{L^2(\mathbb{R}^{3\delta})}$. Then using the formal result stated above we infer $h \equiv \langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Lambda(t)$. ■

3. The general result.

We state here the theorem summing up the results we found about the classical limit of the transition amplitudes defined above. However we will give its proof, that is rather intricate, in Appendix E. We also formulate a lemma that will be useful in the proof of the theorem as well as in the next section.

DEFINITION. Let $w_1, w_2, w_3, w_4 \in \mathscr{C}^0(\mathbb{R}, L^2(\mathbb{R}^3))$, then we define

$$w_1^{\otimes_q} w_2^{\otimes_r} w_3^{\otimes_i} w_4^{\otimes_j}(t) \equiv \prod_{a=1}^{q} w_1(t, x_a) \prod_{b=1}^{r} w_2(t, y_b) \prod_{c=1}^{i} w_3(t, k_c) \prod_{d=1}^{j} w_4(t, m_d).$$

THEOREM 3. *Let $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$ such that $\|u_0\|_2 = \|\alpha_0\|_2 = 1$; Λ, Φ, Θ, $(u(t), \alpha(t))$, $(u_\theta(t), \alpha_\theta(t))$ defined as above. Then the following statements are valid for all $q, r, i, j \in \mathbb{N}$, $\delta = q + r + i + j$:*

i. The following limits exist in $L^2(\mathbb{R}^{3\delta})$ when $\lambda \to 0$:

$$\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Lambda(t) \overset{L^2(\mathbb{R}^{3\delta})}{\underset{\lambda\to 0}{\longrightarrow}} \bar{u}^{\otimes_q} u^{\otimes_r} \bar{\alpha}^{\otimes_i} \alpha^{\otimes_j}(t)$$

$$\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Psi(t) \overset{L^2(\mathbb{R}^{3\delta})}{\underset{\lambda\to 0}{\longrightarrow}} \delta_{qr}\bar{u}^{\otimes_q} u^{\otimes_r} \bar{\alpha}^{\otimes_i} \alpha^{\otimes_j}(t)$$

$$\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Theta(t) \overset{L^2(\mathbb{R}^{3\delta})}{\underset{\lambda\to 0}{\longrightarrow}} \delta_{qr}\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{u}_\theta^{\otimes_q} u_\theta^{\otimes_r} \bar{\alpha}_\theta^{\otimes_i} \alpha_\theta^{\otimes_j}(t)\ ,$$

δ_{qr} being the function equal to 1 when $q = r$, 0 otherwise.

ii. For all $X \in \{\Lambda, \Psi, \Theta\}$ there are two positive constants $K_1(X)$ and $K_2(X)$ that depend on p, q, i, j such that

$$\|\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Lambda(t) - \bar{u}^{\otimes_q} u^{\otimes_r} \bar{\alpha}^{\otimes_i} \alpha^{\otimes_j}(t)\|_{L^2(\mathbb{R}^{3\delta})} \leq \lambda^2 K_1(\Lambda)\,|t|\,e^{K_2(\Lambda)|t|}$$

$$\|\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Psi(t) - \delta_{qr}\bar{u}^{\otimes_q} u^{\otimes_r} \bar{\alpha}^{\otimes_i} \alpha^{\otimes_j}(t)\|_{L^2(\mathbb{R}^{3\delta})} \leq \delta_{qr}\lambda^2 K_1(\Psi)\,|t|\,e^{K_2(\Psi)|t|}$$

$$\|\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_\Theta(t) - \delta_{qr}\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{u}_\theta^{\otimes_q} u_\theta^{\otimes_r} \bar{\alpha}_\theta^{\otimes_i} \alpha_\theta^{\otimes_j}(t)\|_{L^2(\mathbb{R}^{3\delta})} \leq \delta_{qr}\lambda^2 K_1(\Theta)\,|t|\,e^{K_2(\Theta)|t|}\ .$$

LEMMA 3.1. *Let B as above, then for all $g \in L^2(\mathbb{R}^{3\delta})$ and $\Phi \in \mathscr{H}^\delta$ the following identity holds:*

$$C^\dagger(u,\alpha)BC(u,\alpha)\Phi = \int \mathrm{d}X_q \mathrm{d}Y_r \mathrm{d}K_i \mathrm{d}M_j\, \bar{g}(X_q, Y_r, K_i, M_j) \prod_{a=1}^{q}(\psi^*(x_a) + \bar{u}(x_a)) \prod_{b=1}^{r}(\psi(y_b) + u(y_b)) \prod_{c=1}^{i}(a^*(k_c) + \bar{\alpha}(k_c)) \prod_{d=1}^{j}(a(m_d) + \alpha(m_d))\Phi\ .$$

PROOF. Let $\delta_1 = q + r$, $\delta_2 = i + j$. We call $\mathrm{Sep}(L^2(\mathbb{R}^{3\delta}))$ the set of functions that are product of functions in $L^2(\mathbb{R}^3)$. So if $g \in \mathrm{Sep}(L^2(\mathbb{R}^{3\delta}))$ we can write:

$$g(X_q, Y_r, K_i, M_j) = \prod_{a=1}^{q} g_a(x_a) \prod_{b=1}^{r} g_{q+b}(y_b) \prod_{c=1}^{i} g_{\delta_1+c}(k_c) \prod_{d=1}^{j} g_{\delta_1+i+d}(m_d) .$$

For all $g \in \mathrm{Sep}(L^2(\mathbb{R}^{3\delta}))$ the result holds. As a matter of fact in that case

$$\begin{aligned} C^\dagger(u,\alpha)BC(u,\alpha)\Phi = C^\dagger(u,\alpha)\psi^*(\bar{g}_1)C(u,\alpha)\dots C^\dagger(u,\alpha)\psi(\bar{g}_{q+1})C(u,\alpha) \\ \dots C^\dagger(u,\alpha)a^*(\bar{g}_{\delta_1+1})C(u,\alpha)\dots C^\dagger(u,\alpha)a(\bar{g}_{\delta_1+i+1})C(u,\alpha)\dots\Phi \\ = (\psi^*(\bar{g}_1) + \langle u, \bar{g}_1\rangle_2)\dots(\psi(\bar{g}_{q+1}) + \langle g_{q+1}, u\rangle_2) \\ \dots(a^*(\bar{g}_{\delta_1+1}) + \langle \alpha, \bar{g}_{\delta_1+1}\rangle_2)\dots(a(\bar{g}_{\delta_1+i+1}) + \langle g_{\delta_1+i+1}, \alpha\rangle_2)\dots\Phi \end{aligned}$$

because we can apply Proposition 5 to every term since $\psi^\#$ and $a^\#$ map $\mathscr{H}^\delta$ into $\mathscr{H}^{\delta-1}$. The result then holds also for finite linear combinations of functions in $\mathrm{Sep}(L^2(\mathbb{R}^{3\delta}))$. Then since $L^2(\mathbb{R}^{3\delta})$ admits a basis of functions in $\mathrm{Sep}(L^2(\mathbb{R}^{3\delta}))$, we can extend the result to all $g \in L^2(\mathbb{R}^{3\delta})$ with a density argument. ■

4. Two examples: $\langle\psi^*(x_1)\psi(x_2)\rangle_\Psi(t)$ and $\langle a^*(k_1)a(k_2)\rangle_\Theta(t)$.

We want to explain the procedure that leads to the results of Theorem 3. In order to do that we will study two simple examples, respectively concerning Ψ and Θ vectors. We do not discuss Λ vectors here since we did it in Chapter 7.

DEFINITION ($\langle\psi^*(x_1)\psi(x_2)\rangle_\Psi(t)$ and $\langle a^*(k_1)a(k_2)\rangle_\Theta(t)$). We recall that

$$\begin{aligned} \langle\psi^*(x_1)\psi(x_2)\rangle_\Psi(t) &= \lambda^2\langle\Psi, U^\dagger(t)\psi^*(x_1)\psi(x_2)U(t)\Psi\rangle \\ \langle a^*(k_1)a(k_2)\rangle_\Theta(t) &= \lambda^2\langle\Theta, U^\dagger(t)a^*(k_1)a(k_2)U(t)\Theta\rangle . \end{aligned}$$

We define for all $g \in L^2(\mathbb{R}^6)$:

$$\begin{aligned} \langle g\rangle_\Psi(t) &= \lambda^2\langle\Psi, U^\dagger(t)\int \mathrm{d}x_1\mathrm{d}x_2\, \bar{g}(x_1,x_2)\psi^*(x_1)\psi(x_2)U(t)\Psi\rangle \\ \langle g\rangle_\Theta(t) &= \lambda^2\langle\Theta, U^\dagger(t)\int \mathrm{d}k_1\mathrm{d}k_2\, \bar{g}(k_1,k_2)a^*(k_1)a(k_2)U(t)\Theta\rangle . \end{aligned}$$

LEMMA 4.1. *There are two positive constants K_1 and K_2 such that*

$$\left|\langle g\rangle_\Psi(t) - \langle g(x_1,x_2), \bar{u}(t,x_1)u(t,x_2)\rangle_{L^2(\mathbb{R}^6)}\right| \le \lambda^2 \left\|g; L^2(\mathbb{R}^6)\right\| K_1\,|t|\,e^{K_2|t|}$$

PROOF. Using Lemma 1.1 and setting $m = \lambda^{-2}$ we can write:

$$\langle g\rangle_\Psi(t) = m^{-1}d_m\langle\Psi, C(\sqrt{m}\,u_0, \sqrt{m}\,\alpha_0)W^\dagger(t,0)XW(t,0)\Omega\rangle ,$$

where

$$\begin{aligned} X \equiv C^\dagger(\sqrt{m}\,u(t), \sqrt{m}\,\alpha(t))\int \mathrm{d}x_1\mathrm{d}x_2\, \bar{g}(x_1,x_2)\psi^*(x_1)\psi(x_2) \\ C(\sqrt{m}\,u(t), \sqrt{m}\,\alpha(t)) \end{aligned}$$

By Lemma 3.1 we obtain:

$$\begin{aligned} X = \int \mathrm{d}x_1\mathrm{d}x_2\, \bar{g}(x_1,x_2)\psi^*(x_1)\psi(x_2) + \sqrt{m}\int \mathrm{d}x_1\mathrm{d}x_2\, \bar{g}(x_1,x_2)\big(\bar{u}(t,x_1) \\ \psi(x_2) + \psi^*(x_1)u(t,x_2)\big) + m\int \mathrm{d}x_1\mathrm{d}x_2\, \bar{g}(x_1,x_2)\bar{u}(t,x_1)u(t,x_2) \\ \equiv X_1 + X_2 + X_3 . \end{aligned}$$

First of all consider X_3:

$$m^{-1}d_m\langle\Psi, C(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)W^\dagger(t,0)X_3W(t,0)\Omega\rangle$$
$$= \int \mathrm{d}x_1\mathrm{d}x_2\,\bar g(x_1,x_2)\bar u(t,x_1)u(t,x_2)\langle\Psi, d_mC(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)\Omega\rangle\ .$$

So since $\Psi=\mathbb{P}_m\Psi$ we obtain:

$$m^{-1}d_m\langle\Psi, C(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)W^\dagger(t,0)X_3W(t,0)\Omega\rangle = \langle g,\bar u(t)u(t)\rangle_{L^2(\mathbb{R}^6)}.$$

Then we can write:

$$\big|\langle g\rangle_\Psi(t)-\langle g,\bar u(t)u(t)\rangle_{L^2(\mathbb{R}^6)}\big| \le m^{-1}d_m\big|\langle\Psi, C(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)W^\dagger(t,0)$$
$$(X_1+X_2)W(t,0)\Omega\rangle\big| \le Y_1+Y_2\ ,$$

and we rewrite Y_1 and Y_2 to obtain:

$$Y_1 = m^{-1}\Big|\Big\langle d_m(P+1)^{-1/2}C^\dagger(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)\Psi, \sqrt{(P+1)}W^\dagger(t,0)$$
$$X_1W(t,0)\Omega\Big\rangle\Big|\ ;$$
$$Y_2 = m^{-1/2}d_m\Big|\langle C^\dagger(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)\Psi, \widetilde{W}^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle\Big|\ ;$$

where $\varphi(\tilde{\mathbf{g}})$ is the field defined in Proposition 9, with

$$\tilde g_1(x) = U_{01}^\dagger(t)\int \mathrm{d}x'\,\bar g(x,x')u(t,x')\ ,$$
$$\tilde g_2(x) = U_{01}(t)\int \mathrm{d}x'\,\bar g(x',x)\bar u(t,x')\ ,$$
$$\tilde g_3=\tilde g_4=0\ .$$

We remark that by Schwarz's inequality we have $\|\tilde g_j\|_2 \le \big\|g;L^2(\mathbb{R}^6)\big\|\,\|u(t)\|_2$ for $j=1,2$.

Consider Y_1: then using Schwarz's inequality and Lemma 1.2 we can write

$$Y_1 \le m^{-1}K_\Psi\left\|W^\dagger(t,0)X_1W(t,0)\Omega\right\|_{\mathscr{H}^1}\ .$$

To bound the last norm we use two times the Corollary of Proposition 8, so we obtain ($m=\lambda^{-2}$)

$$Y_1 \le \lambda^2K_\Psi\left\|g;L^2(\mathbb{R}^6)\right\|\Big(\mathcal{K}_1(\delta=1,0,t)+\lambda\mathcal{K}_2(\delta=1,0,t)\Big)$$
$$e^{\lambda\mathcal{C}_1(\delta=1)|t|+\mathcal{K}_3(\delta=1,0,t)}\Big(\mathcal{K}_1(\delta=19/2,t,0)+\lambda\mathcal{K}_2(\delta=19/2,t,0)\Big)$$
$$e^{\lambda\mathcal{C}_1(\delta=19/2)|t|+\mathcal{K}_3(\delta=19/2,t,0)}\ ,$$

where the constants are the ones introduced in Proposition 8 and we made explicit the dependence on δ.

Consider now Y_2: define $\Phi\equiv C^\dagger(\sqrt{m}\,u_0,\sqrt{m}\,\alpha_0)\Psi$. Then we can write

$$\langle\Phi,\widetilde{W}^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle = \langle\Phi,\widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{U}_2(t,0)\Omega\rangle$$
$$+\langle\Phi,(\widetilde{W}^\dagger(t,0)-\widetilde{U}_2^\dagger(t,0))\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle$$
$$+\langle\Phi,\widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})(\widetilde{W}(t,0)-\widetilde{U}_2(t,0))\Omega\rangle \equiv Z_1+Z_2+Z_3\ .$$

$Z_1=0$ using Proposition 9 and Lemma 1.3. We study then Z_2, using Schwarz's inequality and Lemma 1.2 we obtain:

$$|Z_2| \le d_m^{-1}K_\Psi\left\|(\widetilde{W}^\dagger(t,0)-\widetilde{U}_2^\dagger(t,0))\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\right\|_{\mathscr{H}^1} \le d_m^{-1}K_\Psi$$
$$\left|\int_0^t \mathrm{d}\tau\,\left\|\widetilde{W}^\dagger(\tau,0)\widetilde{H}_I\widetilde{U}_2(\tau,0)\widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\right\|_{\mathscr{H}^1}\right|\ .$$

Then we use: Proposition 8; Lemma 2.1 of Chapter 3 to move $(P+N)^\delta$ to the right of H_I; the usual estimate of H_I (that gives the factor $m^{-1/2}$); Proposition 6 two times; again Lemma 2.1 of Chapter 3 to move $(P+N)^\delta$ to the right of $\varphi(\tilde{\mathbf{g}})$; the usual estimate for $\varphi(\tilde{\mathbf{g}})$ (that gives $\|g; L^2(\mathbb{R}^6)\|$); Proposition 8 to finally obtain

$$|Z_2| \leq m^{-1/2} d_m^{-1} \|g; L^2(\mathbb{R}^6)\| K_\Psi C\, |t|\, e^{K|t|} ,$$

where C and K are positive constants.

To bound $|Z_3|$ we proceed in the same way, and we obtain

$$|Z_3| \leq m^{-1/2} d_m^{-1} \|g; L^2(\mathbb{R}^6)\| K_\Psi C'\, |t|\, e^{K'|t|} .$$

with again C' and K' positive constants. ■

LEMMA 4.2. *There are two positive constants K_3 and K_4 such that*

$$\left| \langle g\rangle_\Theta(t) - \langle g(k_1,k_2), \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{\alpha}_\theta(t,k_1)\alpha_\theta(t,k_2)\rangle_{L^2(\mathbb{R}^6)} \right| \leq \lambda^2 \|g; L^2(\mathbb{R}^6)\| \, K_3\, |t|\, e^{K_4|t|}$$

PROOF. Using Lemma 1.1 and setting $m = \lambda^{-2}$ we can write:

$$\langle g\rangle_\Theta(t) = m^{-1} d_m^2 \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{im\theta} \langle \Theta, C(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta)) W_\theta^\dagger(t,0) X \, W_\theta(t,0)\Omega\rangle ,$$

where

$$X \equiv C^\dagger(\sqrt{m}\, u_\theta(t), \sqrt{m}\, \alpha_\theta(t)) \int \mathrm{d}k_1 \mathrm{d}k_2\, \bar{g}(k_1,k_2) a^*(k_1) a(k_2) \, C(\sqrt{m}\, u_\theta(t), \sqrt{m}\, \alpha_\theta(t))$$

and $W_\theta(t,s)$ is the operator W with classical solution $(u_\theta(t), \alpha_\theta(t))$. Using Lemma 3.1 we obtain:

$$\begin{aligned} X = &\int \mathrm{d}k_1 \mathrm{d}k_2\, \bar{g}(k_1,k_2) a^*(k_1) a(k_2) + \sqrt{m} \int \mathrm{d}k_1 \mathrm{d}k_2\, \bar{g}(k_1,k_2) \big(\bar{\alpha}_\theta(t,k_1) \\ & a(k_2) + a^*(k_1)\alpha_\theta(t,k_2)\big) + m \int \mathrm{d}k_1 \mathrm{d}k_2\, \bar{g}(k_1,k_2) \bar{\alpha}_\theta(t,k_1)\alpha_\theta(t,k_2) \\ & \equiv X_1 + X_2 + X_3 . \end{aligned}$$

First of all consider X_3:

$$\begin{aligned} m^{-1} d_m^2 &\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{im\theta} \langle \Theta, C(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta)) W_\theta^\dagger(t,0) X_3 W_\theta(t,0)\Omega\rangle \\ &= \int \mathrm{d}k_1 \mathrm{d}k_2\, \bar{g}(k_1,k_2) \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{\alpha}_\theta(t,k_1)\alpha_\theta(t,k_2) \\ & \qquad \langle \Theta, d_m^2 e^{im\theta} C(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta))\Omega\rangle . \end{aligned}$$

In the transition amplitude on the right hand side we proceed as following:

$$\begin{aligned} \langle \Theta, d_m^2 e^{im\theta} C(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta))\Omega\rangle &= \langle \frac{(\psi^*(\alpha_0))^m}{\sqrt{m!}} \Omega_p, \, d_m \mathbb{P}_m C_p(\sqrt{m}\, u_0)\Omega_p\rangle \\ & \quad \langle \frac{(a^*(\alpha_0))^m}{\sqrt{m!}} \Omega_n, d_m e^{im\theta} \mathbb{N}_m C_n(\sqrt{m}\, \alpha_0(\theta))\Omega_n\rangle \\ &= \langle \frac{(a^*(\alpha_0(\theta)))^m}{\sqrt{m!}} \Omega_n, d_m \mathbb{N}_m C_n(\sqrt{m}\, \alpha_0(\theta))\Omega_n\rangle = 1 . \end{aligned}$$

Here we used two times the relation in equation (4.2) of [**COS11**] (the same holds for ψ and Ω_p):

$$d_m \mathbb{N}_m C_n(\sqrt{m}\alpha)\Omega_n = \frac{(a^*(\alpha))^m}{\sqrt{m!}}\Omega_n \; .$$

So we finally obtain for X_3:

$$m^{-1} d_m^2 \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{im\theta} \langle \Theta, C(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta)) W_\theta^\dagger(t,0) X_3 W_\theta(t,0)\Omega\rangle$$
$$= \int \mathrm{d}k_1 \mathrm{d}k_2\, \bar{g}(k_1,k_2) \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{\alpha}_\theta(t,k_1)\alpha_\theta(t,k_2) \; .$$

Then we can write:

$$\Bigl| \langle g\rangle_\Theta(t) - \langle g, \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{\alpha}_\theta(t)\alpha_\theta(t)\rangle_{L^2(\mathbb{R}^6)} \Bigr| \leq m^{-1} d_m^2 \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}$$
$$\Bigl| \langle \Theta, C(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta)) W_\theta^\dagger(t,0)(X_1 + X_2) W_\theta(t,0)\Omega\rangle \Bigr| \leq Y_1 + Y_2 \; ,$$

and we rewrite Y_1 and Y_2 to obtain:

$$Y_1 = m^{-1} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \Bigl| \Bigl\langle d_m^2 (P+1)^{-1/2}(N+1)^{-1/2} C^\dagger(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta))\Theta,$$
$$\sqrt{(P+1)(N+1)} W_\theta^\dagger(t,0) X_1 W_\theta(t,0)\Omega \Bigr\rangle \Bigr| \; ;$$
$$Y_2 = m^{-1/2} d_m^2 \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \Bigl| \langle C^\dagger(\sqrt{m}\, u_0, \sqrt{m}\, \alpha_0(\theta))\Theta, \widetilde{W}_\theta^\dagger(t,0)\varphi(\tilde{\mathbf{g}})$$
$$\widetilde{W}_\theta(t,0)\Omega\rangle \Bigr| \; ;$$

where $\varphi(\tilde{\mathbf{g}})$ is the field defined in Proposition 9, with

$$\tilde{g}_1 = \tilde{g}_2 = 0 \; ,$$
$$\tilde{g}_3(k) = U_{02}^\dagger(t) \int \mathrm{d}k'\, \bar{g}(k,k')\alpha_\theta(t,k') \; ,$$
$$\tilde{g}_4(k) = U_{02}(t) \int \mathrm{d}k'\, \bar{g}(k',k)\bar{\alpha}_\theta(t,k') \; .$$

We remark that by Schwarz's inequality we have $\|\tilde{g}_j\|_2 \leq \bigl\| g; L^2(\mathbb{R}^6) \bigr\| \, \|\alpha_\theta(t)\|_2$ for $j = 3,4$.

Consider Y_1: then using Schwarz's inequality and Lemma 1.2 we can write

$$Y_1 \leq m^{-1} K_\Theta \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \Bigl\| W_\theta^\dagger(t,0) X_1 W_\theta(t,0)\Omega \Bigr\|_{\mathscr{H}^2} \; .$$

To bound the last norm we use two times the Corollary of Proposition 8, so we obtain ($m = \lambda^{-2}$)

$$Y_1 \leq \lambda^2 K_\Theta \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bigl\| g; L^2(\mathbb{R}^6) \bigr\| \Bigl(\mathcal{K}_1(\delta = 2, 0, t) + \lambda \mathcal{K}_2(\delta = 2, 0, t) \Bigr)$$
$$e^{\lambda \mathcal{C}_1(\delta=2)|t| + \mathcal{K}_3(\delta=2,0,t)} \Bigl(\mathcal{K}_1(\delta = 31/2, t, 0) + \lambda \mathcal{K}_2(\delta = 31/2, t, 0) \Bigr)$$
$$e^{\lambda \mathcal{C}_1(\delta=31/2)|t| + \mathcal{K}_3(\delta=31/2,t,0)} \; ,$$

where the constants are the ones introduced in Proposition 8 and we made explicit the dependence on δ. The dependence in θ of the constants is in the form of norms $\|\check{\chi} * A_\theta(t)\|_\infty$ and $\|\alpha_\theta(t)\|_2$, and by Lemma 2.1 of Chapter 2 we know the solution of the classical equation is continuous in $L^2(\mathbb{R}^3)$ with respect to a change of initial α-data. So we can integrate in θ on a finite interval, and the global constants are finite.

Consider now Y_2: define $\Phi \equiv C^\dagger(\sqrt{m}\,u_0, \sqrt{m}\,\alpha_0(\theta))\Theta$ and $\widetilde{U}_{2,\theta}$ the operator with $\alpha_\theta(t)$ replacing $\alpha(t)$. Then we can write

$$\langle\Phi, \widetilde{W}^\dagger_\theta(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}_\theta(t,0)\Omega\rangle = \langle\Phi, \widetilde{U}^\dagger_{2,\theta}(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{U}_{2,\theta}(t,0)\Omega\rangle$$
$$+\langle\Phi, (\widetilde{W}^\dagger_\theta(t,0) - \widetilde{U}^\dagger_{2,\theta}(t,0))\varphi(\tilde{\mathbf{g}})\widetilde{W}_\theta(t,0)\Omega\rangle$$
$$+\langle\Phi, \widetilde{U}^\dagger_{2,\theta}(t,0)\varphi(\tilde{\mathbf{g}})(\widetilde{W}_\theta(t,0) - \widetilde{U}_{2,\theta}(t,0))\Omega\rangle \equiv Z_1 + Z_2 + Z_3 \; .$$

$Z_1 = 0$ using Proposition 9 and Lemma 1.3. We study then Z_2, using Schwarz's inequality and Lemma 1.2 we obtain:

$$|Z_2| \le d_m^{-2} K_\Theta \left\| (\widetilde{W}^\dagger_\theta(t,0) - \widetilde{U}^\dagger_{2,\theta}(t,0))\varphi(\tilde{\mathbf{g}})\widetilde{W}_\theta(t,0)\Omega \right\|_{\mathscr{H}^2} \le d_m^{-2} K_\Theta$$
$$\left| \int_0^t \mathrm{d}\tau \left\| \widetilde{W}^\dagger_\theta(\tau,0)\widetilde{H}_I\widetilde{U}_{2,\theta}(\tau,0)\widetilde{U}^\dagger_{2,\theta}(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}_\theta(t,0)\Omega \right\|_{\mathscr{H}^2} \right| \; .$$

Then we use: Proposition 8; Lemma 2.1 of Chapter 3 to move $(P+N)^\delta$ to the right of H_I; the usual estimate of H_I (that gives the factor $m^{-1/2}$); Proposition 6 two times; again Lemma 2.1 of Chapter 3 to move $(P+N)^\delta$ to the right of $\varphi(\tilde{\mathbf{g}})$; the usual estimate for $\varphi(\tilde{\mathbf{g}})$ (that gives $\left\|g; L^2(\mathbb{R}^6)\right\|$); Proposition 8 to finally obtain

$$|Z_2| \le m^{-1/2} d_m^{-2} \left\|g; L^2(\mathbb{R}^6)\right\| K_\Theta C(\theta)\, |t|\, e^{K(\theta)|t|} \; ,$$

where $C(\theta)$ and $K(\theta)$ are positive and continuous in θ since the latter appears only in $\|\check{\chi} * A_\theta(t')\|_\infty$ and $\|\alpha_\theta(t')\|_2$, for some $t' \in [0,t]$.

To bound $|Z_3|$ we proceed in the same way, and we obtain

$$|Z_3| \le m^{-1/2} d_m^{-2} \left\|g; L^2(\mathbb{R}^6)\right\| K_\Theta C'(\theta)\, |t|\, e^{K'(\theta)|t|} \; .$$

with again $C'(\theta)$ and $K'(\theta)$ positive and continuous in θ. ∎

Proposition 13. *Let $u_0, \alpha_0 \in L^2(\mathbb{R}^3)$ such that $\|u_0\|_2 = \|\alpha_0\|_2 = 1$; Φ, Θ, $(u(t), \alpha(t))$, $(u_\theta(t), \alpha_\theta(t))$ defined as above. Then there are four positive constants K_j, $j = 1,2,3,4$, such that*

$$|\langle\psi^*(x_1)\psi(x_2)\rangle_\Psi(t) - \bar{u}(t,x_1)u(t,x_2)\|_{L^2(\mathbb{R}^6)} \le \lambda^2 K_1\, |t|\, e^{K_2|t|}$$
$$\|\langle a^*(k_1)a(k_2)\rangle_\Theta(t) - \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \bar{\alpha}_\theta(t,k_1)\alpha_\theta(t,k_2)\|_{L^2(\mathbb{R}^6)} \le \lambda^2 K_3\, |t|\, e^{K_4|t|} \; .$$

Proof. The result is a direct consequence of the bounds proved in Lemmas 4.1 and 4.2. ∎

APPENDIX A

Mathematical aspects of second quantization.

1. Annihilation and creation operators.

Let $\mathscr{H}$ be a (symmetric) Fock space over $L^2(\mathbb{R}^d)$, such that

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \ ,$$

with $\mathscr{H}_0 = \mathbb{C}$, and

$$\mathscr{H}_n = \{\Phi_n : \Phi_n(x_1, \dots, x_n) \in L^2(\mathbb{R}^{nd}), \text{ totally symmetric}\} \ .$$

So a vector $\Phi \in \mathscr{H}$ will be of the form $\Phi = (\Phi_0, \Phi_1, \dots, \Phi_n, \dots)$, with $\Phi_j \in \mathscr{H}_j$ for each $j = 0, 1, \dots$; denote with Ω the Fock vacuum, *i.e.* $\Omega = (1, 0, \dots, 0, \dots)$ and with $\mathscr{C}_0(N)$ the subset of $\mathscr{H}$ of vectors with finite number of particles: that is the set of vectors $(\Phi_0, \Phi_1, \dots)$ for which $\Phi_n = 0$ for all but finitely many n. The Fock space is a Hilbert space equipped with the norm

$$\|\Phi\| = \Big(\sum_{n=0}^{\infty} \|\Phi_n\|^2_{\mathscr{H}_n} \Big)^{1/2} \ .$$

We will now define annihilation and creation operators $a(f)$ and $a^*(f)$: let $f \in L^2(\mathbb{R}^d)$ and define X_n the set of variables $\{x_1, \dots, x_n\}$, then:

$$\begin{aligned}(a(f)\Phi)_n(X_n) &= \sqrt{n+1} \int \mathrm{d}x_0 \, f(x_0) \Phi_{n+1}(x_0, X_n) \\ &\equiv \sqrt{n+1} \langle \bar{f}, \Phi_{n+1} \rangle_{\mathscr{H}_1}(X_n) \ ,\end{aligned}$$

with domain

$$D(a(f)) = \left\{ \Phi : \sum_{n \geq 0} (n+1) \int \mathrm{d}X_n \left| \int \mathrm{d}x_0 \, f(x_0) \Phi_{n+1}(x_0, X_n) \right|^2 < \infty \right\}$$

and

$$(a^*(f)\Phi)_n(X_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \Phi_{n-1}(X_n \setminus x_j) \equiv \sqrt{n} S_n (f \otimes \Phi_{n-1})(X_n)$$

where S_n is the symmetrizing operator on $\mathscr{H}_n$, with domain

$$D(a^*(f)) = \left\{ \Phi : \sum_{n \geq 1} \frac{1}{n} \int \mathrm{d}X_n \left| \sum_{j=1}^{n} f(x_j) \Phi_{n-1}(X_n \setminus x_j) \right|^2 < \infty \right\} .$$

Define P_j the orthogonal projector on $\mathscr{H}$:

$$P_j(\Phi_0, \Phi_1, \dots, \Phi_n, \dots) \equiv (\Phi_0, \Phi_1, \dots, \Phi_j, 0, \dots, 0, \dots) \ .$$

Finally let

$$a_0^{\#}(f) \equiv a^{\#}(f) \Big|_{\mathscr{C}_0(N)} \ ,$$

where $^{\#}$ stands for either nothing or *.

Lemma 1.1. $\overline{a_0^{\#}(f)} = a^{\#}(f)$.

Proof. a) $a^{\#}(f)$ is closed.

In order to prove that let $\Phi^{(j)} \in D(a(f))$ such that

$$\Phi^{(j)} \longrightarrow \Phi \,,$$
$$a(f)\Phi^{(j)} \longrightarrow \Theta \,;$$

we have that $(a(f)\Phi^{(j)})_n \to \Theta_n$ in $\mathscr{H}_n$, so by definition

$$\sqrt{n+1}\int \mathrm{d}x_0\, f(x_0)\Phi_{n+1}^{(j)}(x_0, X_n) \underset{j\to\infty}{\longrightarrow} \Theta_n \text{ in } L^2(X_n).$$

Now, since $\Phi_n^{(j)} \to \Phi_n$ in $L^2(X_n)$ for all n we have that in $L^2(X_n)$

$$\int \mathrm{d}x_0\, f(x_0)\Phi_{n+1}^{(j)}(x_0, X_n) \to \int \mathrm{d}x_0\, f(x_0)\Phi_{n+1}(x_0, X_n)\,.$$

In order to prove this last convergence we estimate:

$$\int \mathrm{d}X_n \left|\int \mathrm{d}x_0\, f(x_0)\big(\Phi_{n+1}^{(j)}(x_0, X_n) - \Phi_{n+1}(x_0, X_n)\big)\right|^2 \le \|f\|_2^2 \left\|\Phi_{n+1}^{(j)} - \Phi_{n+1}; L^2(X_{n+1})\right\| \to 0\,.$$

Now

$$\sqrt{n+1}\int \mathrm{d}x_0\, f(x_0)\Phi_{n+1}(x_0, X_n) = \Theta_n\,;$$

and since $\sum \|\Theta_n\|^2_{\mathscr{H}_n} < \infty$ we have that $\Phi \in D(a(f))$ and $a(f)\Phi = \Theta$.

Consider now $\Phi^{(j)} \in D(a^*(f))$ such that

$$\Phi^{(j)} \longrightarrow \Phi \,,$$
$$a^*(f)\Phi^{(j)} \longrightarrow \Theta \,.$$

So

$$(a^*(f)\Phi^{(j)})_n = \sqrt{n}S_n(f\otimes\Phi_{n-1}^{(j)}) \longrightarrow \Theta_n \in \mathscr{H}_n\,.$$

Since $\Phi_n^{(j)} \to \Phi_n$ in $L^2(X_n)$ for all n,

$$S_n(f\otimes\Phi_{n-1}^{(j)}) \longrightarrow S_n(f\otimes\Phi_{n-1}) \in L^2(X_n)\,,$$

then $\sqrt{n}S_n(f\otimes\Phi_{n-1}) = \Theta$. Since $\sum \|\Theta_n\|^2_{\mathscr{H}_n} < \infty$ we have that $\Phi \in D(a^*(f))$ and $a^*(f)\Phi = \Theta$.

b) $\overline{a_0^{\#}(f)} \supset a^{\#}(f)$.

Let $\Phi \in D(a(f))$, $a(f)\Phi = \Theta$. Define $\Phi^{(j)} \equiv P_j\Phi$; obviously $\Phi^{(j)} \to \Phi$ when $j \to \infty$. Then

$$a(f)P_j\Phi = a_0(f)P_j\Phi = P_{j-1}a_0(f)P_j\Phi\,.$$

So we have that

$$\big(a(f)(\Phi - P_j\Phi)\big)_n = \begin{cases} 0 & \text{if } n+1 \le j \\ \sqrt{n+1}\displaystyle\int \mathrm{d}x_0\, f(x_0)\Phi_{n+1}(x_0, X_n) & \text{if } n+1 > j \end{cases}$$

and

$$\|a(f)\Phi - a_0(f)P_j\Phi\|^2 = \sum_{n\ge j}\|\Theta_n\|^2_{\mathscr{H}_n} \underset{j\to\infty}{\longrightarrow} 0\,.$$

The case of creation operator is similar: let $\Phi \in D(a^*(f))$, $a^*(f)\Phi = \Theta$. Again $\Phi^{(j)} \equiv P_j\Phi$ with $\Phi^{(j)} \to \Phi$ when $j \to \infty$. Then we have that

$$a^*(f)P_j\Phi = a_0^*(f)P_j\Phi$$

and

$$\big(a^*(f)(\Phi - P_j\Phi)\big)_n(X_n) = \begin{cases} 0 & \text{if } n-1 \le j \\ (a^*(f)\Phi)_n & \text{if } n \ge j+2 \end{cases} .$$

Again

$$\|a^*(f)\Phi - a_0^*(f)P_j\Phi\|^2 = \sum_{n\ge j} \|\Theta_n\|^2_{\mathscr{H}_n} \underset{j\to\infty}{\longrightarrow} 0 .$$

■

Let A and B be two operators on $\mathscr{H}$ with domain $D(A)$ and $D(B)$, we will call the sum $A+B$ the operator defined on $D(A)\cap D(B)$ as

$$(A+B)\Phi = A\Phi + B\Phi, \forall \Phi \in D(A)\cap D(B).$$

Corollary. *Let $f, g \in L^2(\mathbb{R}^d)$, then*

$$\overline{(a_0^\#(f) + a_0^\#(g))} = \overline{(a^\#(f) + a^\#(g))} = a^\#(f+g) .$$

We also remark that $a^\#_{(0)}(\lambda f) = \lambda a^\#_{(0)}(f)$ for all $\lambda \in \mathbb{C}$.

Proof. Obviously we have $D(a^\#(f)) \cap D(a^\#(g)) \subset D(a^\#(f+g))$, and

$$a^\#(f)\Phi + a^\#(g)\Phi = (a^\#(f) + a^\#(g))\Phi = a^\#(f+g)\Phi$$

for all $\Phi \in D(a^\#(f)) \cap D(a^\#(g)) \supset \mathscr{C}_0(N)$. So we have that

$$a_0^\#(f+g) = (a_0^\#(f) + a_0^\#(g)) \subset (a^\#(f) + a^\#(g)) \subset a^\#(f+g) .$$

If we now consider closures we obtain

$$\overline{a_0^\#(f+g)} = \overline{(a_0^\#(f) + a_0^\#(g))} \subset \overline{(a^\#(f) + a^\#(g))} \subset a^\#(f+g) .$$

■

Now let $h : \mathbb{N} \longrightarrow \mathbb{C}$. We define the operator $h(N)$ on $\mathscr{H}$ by

$$\begin{cases} (h(N)\Phi)_n = h(n)\Phi_n \\ D(h(N)) = \{\Phi : \sum_{n=0}^{\infty} |h(n)|^2 \|\Phi_n\|^2_{\mathscr{H}_n} < \infty\} \end{cases}$$

By direct inspection we see that:

1) $h(N)\mathscr{C}_0(N) \subset \mathscr{C}_0(N)$
2) $a_0(f)h(N) = h(N+1)a_0(f)$
   $a_0^*(f)h(N) = h(N-1)a_0^*(f)$
3) $h(N)$ is a closed operator.

Lemma 1.2. *Let $h(N)$ as above. Then:*

i. If $\Phi \in D(a(f)) \cap D(h(N))$ and $a(f)\Phi \in D(h(N+1))$, then $h(N)\Phi \in D(a(f))$ and

$$a(f)h(N)\Phi = h(N+1)a(f)\Phi .$$

ii. If $\Phi \in D(a^*(f)) \cap D(h(N))$ and $a^*(f)\Phi \in D(h(N-1))$, then $h(N)\Phi \in D(a^*(f))$ and

$$a^*(f)h(N)\Phi = h(N-1)a^*(f)\Phi .$$

Remark. If $h(N)$ is a bounded operator, the hypotheses on the domains of $h(N)$ and $h(N\pm 1)$ are superfluous.

PROOF. i. Let $\Phi \in \mathscr{H}$. Then $P_j\Phi \longrightarrow \Phi$.
From $\Phi \in D(a(f))$ it follows that $a_0(f)P_j\Phi = P_{j-1}a(f)\Phi \longrightarrow a(f)\Phi$.
From $\Phi \in D(h(N))$ it follows that $h(N)P_j\Phi \longrightarrow h(N)\Phi$.
From $a(f)\Phi \in D(h(N+1))$ it follows that $h(N+1)P_{j-1}a(f)\Phi \longrightarrow h(N+1)a(f)\Phi$.
Since $h(N+1)P_{j-1}a(f)\Phi = h(N+1)a_0(f)P_j\Phi = a_0(f)h(N)P_j\Phi$ also the right hand side of previous equality converges. Now the proof is completed using the fact that $a(f)$ is closed.

ii. The proof is analogous to the point above.

■

For every operator A, we will denote its adjoint by $A^\dagger$. From the definition of $a(\bar{f})$ and $a^*(f)$ it is quite clear that they are, in some sense, one the adjoint of the other. In fact, that is the case, as proved in the following lemma:

LEMMA 1.3. *Let $f \in L^2(\mathbb{R}^d)$, then:*

$$a_0(\bar{f})^\dagger = a(\bar{f})^\dagger = a^*(f)$$
$$a_0^*(f)^\dagger = a^*(f)^\dagger = a(\bar{f}) \ .$$

PROOF. 1) Let $\Theta, \Phi \in \mathscr{C}_0(N)$, then $\langle \Theta, a(\bar{f})\Phi\rangle = \langle a_0^*(f)\Theta, \Phi\rangle$.
The proof is by means of a direct calculation on $\mathscr{H}_n$.

2) Let $\Phi \in \mathscr{C}_0(N)$ and $\Theta \in D(a^*(f))$, then $\langle \Theta, a_0(\bar{f})\Phi\rangle = \langle a^*(f)\Theta, \Phi\rangle$, *i.e.*

$$a_0(\bar{f})^\dagger \supset a^*(f) \ .$$

Let j so that $P_j\Phi = \Phi$. Then

$$\begin{aligned}\langle \Theta, a_0(\bar{f})\Phi\rangle &= \langle \Theta, a_0(\bar{f})P_j\Phi\rangle = \langle \Theta, P_{j-1}a_0(\bar{f})\Phi\rangle = \langle P_{j-1}\Theta, a_0(\bar{f})\Phi\rangle \\ &= \langle a_0^*(f)P_{j-1}\Theta, \Phi\rangle = \langle P_j a^*(f)\Theta, \Phi\rangle = \langle a^*(f)\Theta, P_j\Phi\rangle = \langle a^*(f)\Theta, \Phi\rangle\end{aligned}$$

3) Let $\Phi \in D(a(\bar{f}))$ and $\Theta \in \mathscr{C}_0(N)$, then $\langle \Theta, a(\bar{f})\Phi\rangle = \langle a_0^*(f)\Theta, \Phi\rangle$, *i.e.*

$$a_0^*(f)^\dagger \supset a(\bar{f}) \ .$$

The proof is analogous to the one above (point 2).

4) $a_0(\bar{f})^\dagger \subset a^*(f)$.
Let Θ, Θ^* such that

$$\langle \Theta, a_0(\bar{f})\Phi\rangle = \langle \Theta^*, \Phi\rangle \ ,$$

for all $\Phi \in \mathscr{C}_0(N)$. Choose now $\Phi = (0, \dots, 0, \Phi_n, 0, \dots)$, so

$$a_0(\bar{f})\Phi = (0, \dots, 0, \sqrt{n}\int \mathrm{d}x_1\, \bar{f}(x_1)\Phi_n(x_1, X_{n-1}), 0, \dots) \ .$$

The equation above then becomes

$$\sqrt{n}\langle \Theta_{n-1}(X_{n-1}f(x_1)), \Phi_n(x_1, X_{n-1})\rangle = \langle \Theta_n^*, \Psi_n\rangle \ ,$$

so $\Theta_n^* = \sqrt{n}S_n f \otimes \Theta_{n-1}$ that implies $\Theta \in D(a^*(f))$ and $\Theta^* = a^*(f)\Theta$.

5) $a_0^*(f)^\dagger \subset a(\bar{f})$.
The proof is analogous to point 4.

We see that points 2 and 4 imply $a_0(\bar{f})^\dagger = a^*(f)$; 3 and 5 imply that $a_0^*(f)^\dagger = a(\bar{f})$. We have concluded the proof since by Lemma 1.1 and the fact that $(\overline{A})^\dagger = A^\dagger$ we can write $a_0(\bar{f})^\dagger = (\overline{a_0(\bar{f})})^\dagger = a(\bar{f})^\dagger$ (and the same for $a_0^*(f)^\dagger$). ■

2. Domains of $a^{\#}(f)$.

Now we will define another useful category of operators on the Fock space. Let X be any self-adjoint operator on $L^2(\mathbb{R}^d)$ with domain of essential self-adjointness D. Let $D_X = \{\Phi \in \mathscr{C}_0(N) : \Phi_n \in \bigotimes_{k=1}^n D \text{ for each } n\}$. Then we define the second quantization of X, that we will call $\mathrm{d}\Gamma(X)$, on $D_X \cap \mathscr{H}_n$ as

$$\mathrm{d}\Gamma(X)\Big|_{D_X\cap\mathscr{H}_n} = X \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes X \otimes \cdots \otimes 1 + \ldots + 1 \otimes 1 \otimes \cdots \otimes X \; .$$

Using properties of tensor products of operators in Hilbert spaces it can be proved that $\mathrm{d}\Gamma(X)$ is essentially self adjoint on D_X. A first useful example of an operator constructed in that way is

$$N = \mathrm{d}\Gamma(1) \; .$$

It is called number operator since its eigenstates are vectors with the only non-zero component in some $\mathscr{H}_n$: let $\Phi = (0, \ldots, 0, \Phi_n, 0, \ldots)$, then $N\Phi = n\Phi$. Since $D_1 = \mathscr{C}_0(N)$, we know that N is essentially self-adjoint on $\mathscr{C}_0(N)$. Furthermore we have the following couple of lemmas, whose proof is trivial:

Lemma 2.1. *Let*

$$D(N) = \Big\{\Phi : \sum_{n=0}^{\infty} n^2 \left\|\Phi_n\right\|^2_{\mathscr{H}_n} < \infty\Big\} \; ,$$

then N is self-adjoint on $D(N)$.

Lemma 2.2. *For all $\delta \in \mathbb{R}^+$, let*

$$D(N^\delta) = \Big\{\Phi : \sum_{n=0}^{\infty} n^{2\delta} \left\|\Phi_n\right\|^2_{\mathscr{H}_n} < \infty\Big\} \; ,$$

then N^δ is self-adjoint on $D(N^\delta)$.

We remark that the operator $h(N)$ defined above could also be defined using the spectral theorem as a function of N, and this justifies the notation used. The number operator N is also related to $a^{\#}(f)$: for every $f \in L^2(\mathbb{R}^d)$ we have that $D(a^{\#}(f)) \supset D(N^{1/2})$. This is proved using the following estimates:

Lemma 2.3. *Let $f \in L^2(\mathbb{R}^d)$, then for all $\Phi \in D(N^{1/2})$ the following inequalities hold:*

$$\left\|a(f)\Phi\right\| \leq \left\|f\right\|_2 \left\|N^{1/2}\Phi\right\|$$
$$\left\|a^*(f)\Phi\right\| \leq \left\|f\right\|_2 \left\|(N+1)^{1/2}\Phi\right\| \; .$$

Proof. Using the definition of $(a(f)\Phi)_n$ and Schwarz's inequality we obtain that

$$\left\|(a(f)\Phi)_n\right\|^2_{\mathscr{H}_n} \leq (n+1) \left\|f\right\|_2^2 \left\|\Phi_{n+1}\right\|^2_{\mathscr{H}_{n+1}} \; ,$$

so summing over all n we obtain the sought inequality.

Using the definition of $(a^*(f)\Phi)_n$ and triangle inequality we can write

$$\left\|(a^*(f)\Phi)_n\right\|^2_{\mathscr{H}_n} = n \left\|S_n(f \otimes \Phi_{n-1})\right\|^2_{\mathscr{H}_n} \leq n \left\|f\right\|_2^2 \left\|\Phi_{n-1}\right\|^2_{\mathscr{H}_{n-1}} \; ;$$

the sought inequality again is obtained summing over all n. ∎

More in general we can prove directly that if $\Phi \in D(N^{\lambda/2})$ with $\lambda \geq 1$, then $a^{\#}(f)\Phi \in D(N^{(\lambda-1)/2})$ for all $f \in L^2(\mathbb{R}^d)$. Then

$$A_{n,m} = \prod_{l=1}^{n} a^*(f_l) \prod_{k=1}^{m} a(\bar{g}_k)$$

is applicable to every $\Phi \in D(N^{\lambda/2})$ if $\lambda \geq n+m$. Defining for any integer a the function $\theta(N-a)$ as

$$(\theta(N-a)\Phi)_n = \begin{cases} \Phi_n \text{ if } n-a \geq 0 \\ 0 \text{ if } n-a < 0 \end{cases},$$

we have the following estimate:

LEMMA 2.4. *Let $f_l \in L^2(\mathbb{R}^d)$ for all $l = 1,\ldots,n$, $g_k \in L^2(\mathbb{R}^d)$ for all $k = 1,\ldots,m$ and*

$$\mu_{n,m} = \prod_{l=1}^{n} \|f_l\|_2 \prod_{k=1}^{m} \|g_k\|_2 \ .$$

Then for all $\Phi \in D(N^{\lambda/2})$:

$$\|A_{n,m}\Phi\| \leq \mu_{n,m} \left\| \Big(\frac{N!}{(N-m)!}\Big)^{1/2} \Big(\frac{(N-m+n)!}{(N-m)!}\Big)^{1/2} \theta(N-m)\Phi \right\| \ .$$

PROOF. Let $\Phi \in D(N^{\lambda/2})$, $\Psi \in D(N^{n/2})$, furthermore define $F(Z_n) \equiv \prod_{l=1}^{n} f(z_l)$, $G(Y_m) \equiv \prod_{l=1}^{m} g(y_l)$. So we can write

$$\langle \Psi, A_{n,m}\Phi \rangle = \langle \prod_{l=1}^{n} a(\bar{f}_l)\Psi, \prod_{k=1}^{m} a(\bar{g}_k)\Phi \rangle = \sum_{j\geq 0} \int \mathrm{d}Z_n \mathrm{d}Y_m \, F(Z_n)\bar{G}(Y_m)$$
$$\frac{[(j+n)!(j+m)!]^{1/2}}{j!} \int \mathrm{d}X_j \, \overline{\Psi_{n+j}}(Z_n, X_j)\Phi_{m+j}(Y_m, X_j) \ ,$$

so

$$|\langle \Psi, A_{n,m}\Phi \rangle| \leq \mu_{n,m} \sum_{j\geq 0} \frac{[(j+n)!(j+m)!]^{1/2}}{j!} \|\Psi_{n+j}\|_{\mathscr{H}_{n+j}} \|\Phi_{m+j}\|_{\mathscr{H}_{m+j}}$$
$$\leq \mu_{n,m} \|\theta(N-n)\Psi\| \left\| \frac{(N!(N-m+n)!)^{1/2}}{(N-m)!} \theta(N-m)\Phi \right\| \ .$$

The inequality above can be extended to all $\Psi \in \mathscr{H}$. So we have completed the proof since $\Phi \in D(N^{\lambda/2})$ and

$$\frac{N!}{(N-m)!} \leq N^m \ , \ \frac{(N-m+n)!}{(N-m)!} \leq (N-m+n)^n \leq (N+n)^n \ .$$

■

3. Canonical Commutation Relations (CCR).

a and a^* satisfy well-known commutation relations that are called canonical. Precisely, we would like to give a rigorous meaning to the following statements:

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0$$
$$[a(\bar{f}), a^*(g)] = \langle f, g \rangle_{L^2(\mathbb{R}^d)} \ ,$$

where $[A, B] = AB - BA$. If $\Phi \in \mathscr{C}_0(N)$, a direct calculation leads to CCR:

(CCR1) $(a(f)a(g) - a(g)a(f))\Phi = 0$

(CCR2) $(a^*(f)a^*(g) - a^*(g)a^*(f))\Phi = 0$

(CCR3) $(a(\bar{f})a^*(g) - a^*(g)a(\bar{f}))\Phi = \langle f, g \rangle_{L^2(\mathbb{R}^d)}\Phi$.

In bilinear form we write for all $\Theta, \Phi \in \mathscr{C}_0(N)$:

(CCR1′)

$$\langle a^*(\bar{f})\Theta, a(g)\Phi\rangle - \langle a^*(\bar{g})\Theta, a(f)\Phi\rangle = 0$$

(CCR2′)

$$\langle a(\bar{f})\Theta, a^*(g)\Phi\rangle - \langle a(\bar{g})\Theta, a^*(f)\Phi\rangle = 0$$

(CCR3′)

$$\langle a^*(f)\Theta, a^*(g)\Phi\rangle - \langle a(\bar{g})\Theta, a(\bar{f})\Phi\rangle = \langle f, g\rangle_{L^2(\mathbb{R}^d)}\langle \Theta, \Phi\rangle \ .$$

Now if we consider (CCR3′) with $f = g$ and $\Theta = \Phi \in \mathscr{C}_0(N)$ we get:

$$\|a^*(f)\Theta\|^2 = \left\|a(\bar{f})\Theta\right\|^2 + \|f\|^2_{L^2(\mathbb{R}^d)} \|\Theta\|^2 \ .$$

Then let $\Phi \in D(a(\bar{f}))$. Rewrite equation above with $\Theta = \Phi_j \equiv P_j\Phi$:

$$\|a^*(f)\Phi_j\|^2 = \left\|a(\bar{f})\Phi_j\right\|^2 + \|f\|^2_{L^2(\mathbb{R}^d)} \|\Phi_j\|^2 \ . \tag{3.1}$$

$a(\bar{f})$ is closed, so

$$\begin{cases} \Phi_j \longrightarrow \Phi \\ a(\bar{f})\Phi_j \longrightarrow a(\bar{f})\Phi \end{cases},$$

then (3.1) implies $a^*(f)\Phi_j$ is convergent. Now also $a^*(f)$ is closed, so $\Phi \in D(a^*(f))$ and $a^*(f)\Phi_j \longrightarrow a^*(f)\Phi$. This argument leads to

$$D(a(\bar{f})) \subseteq D(a^*(f)) \ .$$

Repeating the argument with $a(\bar{f})$ and $a^*(f)$ exchanged we obtain

$$D(a^*(f)) \subseteq D(a(\bar{f})) \ .$$

Remark. For all $f \in L^2(\mathbb{R}^d)$, $D(a^*(f)) = D(a(\bar{f}))$.

Using the above remark on domains we have that:

- (CCR1′) holds for $\Theta, \Phi \in D(a(f)) \cap D(a(g))$,
- (CCR2′) holds for $\Theta, \Phi \in D(a(\bar{f})) \cap D(a(\bar{g}))$,
- (CCR3′) holds for $\Theta, \Phi \in D(a(\bar{f})) \cap D(a(\bar{g}))$.

Furthermore if $\Phi \in D(N)$ from (CCR1′), (CCR2′) and (CCR3′) we can write, using a limiting procedure,

$$\langle \Theta, (a(f)a(g) - a(g)a(f))\Phi\rangle = 0$$
$$\langle \Theta, (a^*(f)a^*(g) - a^*(g)a^*(f)\Phi\rangle = 0$$
$$\langle \Theta, (a(\bar{f})a^*(g) - a^*(g)a(\bar{f}))\Phi\rangle = \langle f, g\rangle_{L^2(\mathbb{R}^d)}\langle \Theta, \Phi\rangle$$

for all $\Theta \in \mathscr{H}$. So (CCR1), (CCR2) and (CCR3) hold for all $\Phi \in D(N)$.

4. The field operator (self-adjointness).

We want to study the field operator of the Fock space.

Lemma 4.1 (direct proof). *For all* $f \in L^2(\mathbb{R}^d)$, $(a(\bar{f}) + a^*(f))^{-} = (a_0(\bar{f}) + a_0^*(f))^{-}$ *is self-adjoint.* [$a(\bar{f}) + a^*(f)$ is defined on $D(a(\bar{f})) = D(a^*(f))$]

Proof. $(a_0(\bar{f}) + a_0^*(f))^{-}$ is essentially self-adjoint on $\mathscr{C}_0(N)$ if and only if $\mathrm{Ran}((a_0(\bar{f}) + a_0^*(f))^{-} - \lambda)$ is dense in $\mathscr{H}$, for all λ such that $\mathrm{Im}\lambda \neq 0$. That is equivalent to say that $\mathrm{Ran}((a_0(\bar{f}) + a_0^*(f))^{-} - \lambda)^{\perp} = \{0\}$. Now, let $\Theta \in \mathrm{Ran}((a_0(\bar{f}) + a_0^*(f))^{-} - \lambda)^{\perp}$, then for all $\Phi \in \mathscr{C}_0(N)$:

$$\langle \Theta, ((a_0(\bar{f}) + a_0^*(f))^{-} - \lambda)\Phi\rangle = 0 \ . \tag{4.1}$$

We remark that on $\mathscr{C}_0(N)$ we have $\overline{(a_0(\bar{f})+a_0^*(f))} = a_0(\bar{f})+a_0^*(f)$. Choose now

$$\Phi = \sum_{j=0}^{n} Q_j\Theta \ ,$$

where Q_j is the orthogonal projector on $\mathscr{H}_j$, such that

$$Q_j(\Theta_0, \Theta_1, \dots,) = (0, \dots, 0, \Theta_j, 0, \dots) \ .$$

Then equation (4.1) becomes:

$$\begin{aligned} \lambda \sum_{j=0}^{n} \|Q_j\Theta\|^2 &= \langle \Theta, a_0(\bar{f}) \sum_{j=1}^{n} Q_j\Theta\rangle + \langle \Theta, a_0^*(f) \sum_{j=0}^{n} Q_j\Theta\rangle \\ &= \sum_{j=1}^{n} \langle Q_{j-1}\Theta, a_0(\bar{f})Q_j\Theta\rangle + \sum_{j=0}^{n} \langle Q_{j+1}\Theta, a_0^*(f)Q_j\Theta\rangle \ . \end{aligned}$$

Now if we consider the imaginary part we obtain:

$$\begin{aligned} 2i\mathrm{Im}\lambda \sum_{j=0}^{n} \|Q_j\Theta\|^2 &= \sum_{j=1}^{n} \langle Q_{j-1}\Theta, a_0(\bar{f})Q_j\Theta\rangle - \sum_{j=0}^{n} \langle a_0^*(f)Q_j\Theta, Q_{j+1}\Theta\rangle \\ &\quad + \sum_{j=0}^{n} \langle Q_{j+1}\Theta, a_0^*(f)Q_j\Theta\rangle - \sum_{j=1}^{n} \langle a_0(\bar{f})Q_j\Theta, Q_{j-1}\Theta\rangle \\ &= -\langle Q_n\Theta, a_0(\bar{f})Q_{n+1}\Theta\rangle + \langle a_0(\bar{f})Q_{n+1}\Theta, Q_n\Theta\rangle \end{aligned}$$

so

$$\begin{aligned} |\mathrm{Im}\lambda|\, S_n \equiv |\mathrm{Im}\lambda| \sum_{j=0}^{n} \|Q_j\Theta\|^2 &\le \|Q_n\Theta\|\, \|a_0(f)Q_{n+1}\Theta\| \\ &\le \|Q_n\Theta\|\, (n+1)^{1/2}\, \|f\|_2\, \|Q_{n+1}\Theta\| \\ &\le \frac{1}{2}(n+1)^{1/2}\, \|f\|_2 \left(\|Q_n\Theta\|^2 + \|Q_{n+1}\Theta\|^2\right) \ . \end{aligned}$$

However since

$$S \equiv \sum_{j=0}^{\infty} \|Q_j\Theta\| < \infty \ ,$$

a $\bar{n}$ exists such that for all $n \ge \bar{n}$,

$$\sum_{j=0}^{n} \|Q_j\Theta\|^2 \ge \frac{S}{2} \ .$$

So for all $n \ge \bar{n}$

$$\frac{S\,|\mathrm{Im}\lambda|}{(n+1)^{1/2}} \le \|f\|_2 \left(\|Q_n\Theta\|^2 + \|Q_{n+1}\Theta\|^2\right) \ ,$$

that implies Θ cannot have finite norm, unless $\Theta = 0$.

So $(a_0(\bar{f})+a_0^*(f))^\dagger = \overline{(a_0(\bar{f})+a_0^*(f))}$. On the other hand $(a(\bar{f})+a^*(f)$ is symmetric):

$$\begin{aligned} (\overline{(a_0(\bar{f})+a_0^*(f))})^\dagger \supseteq (\overline{(a(\bar{f})+a^*(f))})^\dagger \supseteq \overline{(a(\bar{f})+a^*(f))} \\ \supseteq \overline{(a_0(\bar{f})+a_0^*(f))} \end{aligned}$$

so $\overline{(a(\bar{f})+a^*(f))} = (a(\bar{f})+a^*(f))^\dagger$. ∎

A general method to prove self-adjointness of $\overline{(a(\bar{f})+a^*(f))}$ is given by the following lemma:

LEMMA 4.2 (Self-adjointness criterion). *Let N be a self-adjoint operator such that $N \geq 1$, and A_0 symmetric with domain $D_0 \subset D(N)$, core of N. Furthermore let*

(4.2) $\|A_0\Theta\| \leq a\,\|N\Theta\| \;\; \forall\Theta \in D_0$

(4.3) $|\langle A_0\Theta, N\Theta\rangle - \langle N\Theta, A_0\Theta\rangle| \leq b\langle\Theta, N\Theta\rangle \;\forall\Theta \in D_0$.

Then:

1) $A = \overline{A_0}$ has domain $D(A) \supset D(N)$ and

(4.2′) $\|A\Theta\| \leq a\,\|N\Theta\| \;\; \forall\Theta \in D(N)$

(4.3′) $|\langle A\Theta, N\Theta\rangle - \langle N\Theta, A\Theta\rangle| \leq b\langle\Theta, N\Theta\rangle \;\forall\Theta \in D(N)$.

*2) A_0 is essentially self-adjoint (*i.e. *$\overline{A_0} = A = A^\dagger$), and for all $D_1 \subset D(N)$ core of N, if we call $A_1 \equiv A\big|_{D_1}$ we have $\overline{A_1} = A$.*

PROOF. 1): Let $\Theta \in D(N)$ and $\{\Theta_j\} \in D_0$ such that $\Theta_j \longrightarrow \Theta$ and $N\Theta_j \longrightarrow N\Theta$. From (4.2) it follows that the suite $A_0\Theta_j$ converges, then $\Theta \in D(\overline{A_0})$ and $A_0\Theta_j \longrightarrow \overline{A_0}\,\Theta$. Obviously from (4.2) and (4.3) follow (4.2′) and (4.3′).

2): To prove A is self-adjoint it will be sufficient to show that for at least one $\rho \in \mathbb{R} \setminus \{0\}$ the equation

(4.4) $A^\dagger\Phi = i\rho\Phi$ with $\Phi \in D(A^\dagger)$

does not have a solution different from 0. Let $\Phi \in D(A^\dagger)$ and $\Theta = N^{-1}\Phi \in D(N)$. Then:

$$\begin{aligned}\left|2i\mathrm{Im}\langle\Theta, A^\dagger\Phi\rangle\right| &= \left|\langle\Theta, A^\dagger\Phi\rangle - \langle A^\dagger\Phi, \Theta\rangle\right| = |\langle A\Theta, \Phi\rangle - \langle\Phi, A\Theta\rangle| \\ &= |\langle A\Theta, N\Theta\rangle - \langle N\Theta, A\Theta\rangle| \leq b\langle\Theta, N\Theta\rangle\ .\end{aligned}$$

Now if Φ satisfies (4.4), the inequality above leads to

$$2\,|\rho|\,\langle\Theta, N\Theta\rangle \leq b\langle\Theta, N\Theta\rangle\ ,$$

so $2\,|\rho| \leq b$. Therefore for all $|\rho| > b/2$ equation (4.4) does not have solution $\Phi \neq 0$, and thus $A^\dagger = A$. Let now $D_1 \subset D(N)$ is a core of N, (D_1, A_1) satisfy the same assumptions of (D_0, A_0). Then $\overline{A_1}$ is self-adjoint, and

$$\overline{A_0} \supset A_1\ ,\ \overline{A_1} \supset A_0 \Rightarrow \overline{A_0} = \overline{A_1} = A\ .$$

■

DEFINITION. $\varphi(f) \equiv \overline{(a(\bar{f}) + a^*(f))} = \overline{(a_0(\bar{f}) + a_0^*(f))}$.

We remark that if we want to use $a(\bar{f})+a^*(f)$ instead of $\varphi(f)$ we have to apply it to a vector $\Phi \in D(a(f))$:

$$\varphi(f)\Phi = a(\bar{f})\Phi + a^*(f)\Phi\ .$$

In particular the equality above is true $\forall\Phi \in D(N^{1/2})$. Observe also that $\varphi(\lambda f) = \lambda\varphi(f)$ for all $\lambda \in \mathbb{R}$.

5. Invariance of domains under the action of $\exp(i\varphi(f)t)$.

Now we are able to formulate some important results about the invariance of useful domains under the action of $e^{i\varphi(f)}$.

- Stone's theorem provides us with a fundamental information: $\forall\Phi \in D(\varphi(f))$,

  $\Theta(t) \equiv \exp(i\varphi(f)t)\Phi \in \mathscr{C}^1(\mathbb{R}, \mathscr{H})$,

  $\Theta(t) \in D(\varphi(f))$ and

  $-i\dfrac{\mathrm{d}}{\mathrm{d}t}\Theta(t) = \varphi(f)\Theta(t)$.

- Let $h(N)$ be a bounded function such that $\operatorname{Ran} h(N) \subset D(N^{1/2})$ (we remark that $D(N^{1/2}) \subset D(\varphi(f))$ and on $D(N^{1/2})$ $\varphi(f) = a(\bar{f}) + a^*(f)$). Define $M(t) = \|h(N)\Theta(t)\|$. Then we have:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}M(t)^2 = -\mathrm{Im}\langle h(N)\Theta, h(N)\varphi(f)\Theta\rangle &= -\mathrm{Im}\langle \varphi(f)h(N)^2\Theta, \Theta\rangle \\ &= -\mathrm{Im}\langle (a(\bar{f}) + a^*(f))h(N)^2\Theta, \Theta\rangle \\ &= -\mathrm{Im}\langle \Big(h(N+1)a(\bar{f}) + h(N-1)a^*(f) \\ &\quad -h(N)(a(\bar{f}) + a^*(f))\Big)h(N)\Theta, \Theta\rangle \\ &= \mathrm{Im}\langle h(N)\Theta, \Big(a(\bar{f})(h(N) - h(N-1)) \\ &\quad +a^*(f)(h(N) - h(N+1))\Big)\Theta\rangle\ .\end{aligned}$$

So we can estimate:

(∗)
$$\begin{aligned}\left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}M(t)^2\right| \le M(t)\,\|f\|_2 \Big\{&\Big\|N^{1/2}(h(N) - h(N-1))\Theta\Big\| \\ &+ \Big\|(N+1)^{1/2}(h(N) - h(N+1))\Theta\Big\|\Big\}\ ,\end{aligned}$$

that could be continued to

(∗∗)
$$\begin{aligned}\left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}M(t)^2\right| \le M(t)^2\,\|f\|_2 \Big\{&\sup_{n\ge 1}\Big|\sqrt{n}\Big(h(n-1)h(n)^{-1} - 1\Big)\Big| \\ &+\sup_{n\ge 1}\Big|\sqrt{n}\Big(1 - h(n)h(n-1)^{-1}\Big)\Big|\Big\}\ .\end{aligned}$$

In order to pass from (∗) to (∗∗) we have to put some restrictions on the choice of $h(N)$.

- We make a suitable choice of $h(N)$: $h(N) = (N+j+1)^{-\delta}$, $j \ge 1$. In order to have $\operatorname{Ran} h(N) \subset D(N^{1/2})$ we have to restrict ourselves to $\delta \ge 1/2$. The general result for all δ will be recovered at the end by interpolation of the results for $\delta = 1/2$ (or better $\delta = 1$) and $\delta = 0$ (unitarity of $\exp(i\varphi(f)t)$), or by introduction and subsequent elimination of a cut off in (∗). For the moment we will continue with $\delta \ge 1/2$. However the following calculations are valid for all $\delta > 0$.

$$(h(n-1)h(n)^{-1} - 1) = (1 + (n+j)^{-1})^{\delta} - 1 \le \frac{\delta}{n+j}\begin{cases} 1 & \text{if } \delta \le 1 \\ \Big(\dfrac{j+2}{j+1}\Big)^{\delta-1} & \text{if } \delta \ge 1\end{cases}$$

$$(1 - h(n)h(n-1)^{-1}) = 1 - (1 - (n+j+1)^{-1})^{\delta} \le \frac{\delta}{n+j+1}\begin{cases} 1 & \text{if } \delta \ge 1 \\ \Big(\dfrac{j+2}{j+1}\Big)^{1-\delta} & \text{if } \delta \le 1\end{cases}$$

We remark that

$$\sup_{n\ge 1} n^{1/2}(n+j)^{-1} \le \frac{1}{2}j^{-1/2}\ .$$

The term between braces in the right hand side of (∗∗) is then bounded by:

$$\Big\{\dots\dots\Big\} \le \delta\Big(\sup_{n\ge 1}\frac{n^{1/2}}{n+j}\Big(\frac{j+2}{j+1}\Big)^{\delta-1} + \sup_{n\ge 1}\frac{n^{1/2}}{n+j+1}\Big) \le 2\delta$$

$$\sup_{n\ge 1}\frac{n^{1/2}}{n+j}\Big(\frac{j+2}{j+1}\Big)^{\delta-1} \le \delta j^{-1/2}\Big(\frac{j+2}{j+1}\Big)^{\delta-1} \quad \text{if } \delta \ge 1.$$

$$\Big\{\ldots\ldots\Big\} \le \delta\Big(\sup_{n\ge1}\frac{n^{1/2}}{n+j}+\sup_{n\ge1}\frac{n^{1/2}}{n+j+1}\Big(\frac{j+2}{j+1}\Big)^{1-\delta}\Big)$$
$$\le \delta j^{-1/2}\Big(\frac{j+2}{j+1}\Big)^{1-\delta} \text{ if } \delta\le 1.$$

Summing up we can write for all $\delta\ge0$:

$$\Big\{\ldots\ldots\Big\} \le \delta j^{-1/2}\Big(\frac{j+2}{j+1}\Big)^{|\delta-1|} \le \delta j^{-1/2}\Big(\frac{3}{2}\Big)^{|\delta-1|} .$$

So recalling $(*)$ and $(**)$ we obtain with $h(N)=(N+j+1)^{-\delta}$, $j\ge1$, $\delta\ge 1/2$:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\|(N+j+1)^{-\delta}\Theta(t)\right\|\right| \le \delta j^{-1/2}\Big(\frac{3}{2}\Big)^{|\delta-1|}\|f\|_2$$
$$\left\|(N+j+1)^{-\delta}\Theta(t)\right\| \ ;$$

then we can write

$$\left\|(N+j+1)^{-\delta}\exp(i\varphi(f)t)\Phi\right\| \le \exp\Big\{\delta j^{-1/2}\Big(\frac{3}{2}\Big)^{|\delta-1|}\|f\|_2\,|t|\Big\}$$
$$\left\|(N+j+1)^{-\delta}\Phi\right\| \ ,$$

for all $\Phi\in D(\varphi(f))$ and then $\forall\Phi\in\mathscr{H}$.

To obtain the result for all $\delta>0$ it would be sufficient to interpolate between $\delta=1$ and $\delta=0$:

$$\left\|(N+j+1)^{-\delta}\exp(i\varphi(f)t)\Phi\right\| \le \exp(\mu\,|t|)\left\|(N+j+1)^{-\delta}\Phi\right\| \tag{5.1}$$

with

$$\mu=\delta j^{-1/2}\,\|f\|_2\left(\frac{3}{2}\right)^{\max\{0,(\delta-1)\}}$$

for all $0\le\delta$, $\Phi\in\mathscr{H}$.

- We would like to obtain the differential inequality of the previous point with δ positive but not restricted to $\delta\ge1/2$. In order to do that we introduce

$$h(N)=e^{-\varepsilon N}h_1(N) \text{ with } h_1(N)=(N+j+1)^{-\delta}\ ,$$

where δ is positive, condition $\operatorname{Ran} h(N)\subset D(N^{1/2})$ being satisfied. Inequality $(*)$ then becomes:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\|e^{-\varepsilon N}h_1(N)\Theta(t)\right\|\right| \le \|f\|_2\Big(\Big\|N^{1/2}e^{-\varepsilon(N-1)}\Big((1+(N+j)^{-1})^{\delta}$$
$$-e^{-\varepsilon}\Big)h_1(N)\Theta(t)\Big\|+\Big\|(N+1)^{1/2}e^{-\varepsilon(N+1)}\Big(e^{\varepsilon}$$
$$-(1-(N+j+2)^{-1})^{\delta}\Big)h_1(N)\Theta(t)\Big\|\Big)$$
$$\le\Big(\Big(\frac{2}{e}\Big)^{1/2}\varepsilon^{1/2}+\sup_{n\ge1}n^{1/2}\Big(\big(1+(n+j)^{-1}\big)^{\delta}-1\Big)$$
$$+\sup_{n\ge1}n^{1/2}\Big(1-\big(1-(n+j+1)^{-1}\big)^{\delta}\Big)\Big)\,\|f\|_2\,\|h_1(N)\Theta(t)\|$$

since $\sup_{n\geq 1} n^{1/2}e^{-\varepsilon n} \leq (2\varepsilon e)^{-1/2}$. Integrating we obtain:

$$\begin{aligned}\left\|e^{-\varepsilon N}h_1(N)\Theta(t)\right\| \leq \left\|e^{-\varepsilon N}h_1(N)\Phi\right\| + \Bigg\{ \left(\frac{2}{e}\right)^{1/2}\varepsilon^{1/2} \\ +\sup_{n\geq 1} n^{1/2}\Big(\big(1+(n+j)^{-1}\big)^{\delta}-1\Big) \\ +\sup_{n\geq 1} n^{1/2}\Big(1-\big(1-(n+j+1)^{-1}\big)^{\delta}\Big)\Bigg\} \\ \|f\|_2 \int_0^t \mathrm{d}t' \; \|h_1(N)\Theta(t')\| \; .\end{aligned}$$

Taking the limit $\varepsilon \longrightarrow 0$ on both sides of the inequality, we obtain a linear inequality identical to the one obtained in the case $\delta \geq 1/2$ but in integral form. The rest of the calculation can be completed using previous estimates. We remark that the inequality obtained by interpolation when $0 < \delta < 1$ is better than the one obtained here by direct calculation.

- Let $A = (N+j+1)^{\delta}$, then inequality (5.1) can be rewritten as

  $$\left\|A^{-1}e^{i\varphi(f)t}A\Psi\right\| \leq a(t)\,\|\Psi\| \;\; \forall\Psi \in D(A)\;.$$

  Now if $\Phi \in \mathscr{H}$:

  $$\left|\langle\Phi, A^{-1}e^{i\varphi(f)t}A\Psi\rangle\right| = \left|\langle e^{-i\varphi(f)t}A^{-1}\Phi, A\Psi\rangle\right| \leq a(t)\,\|\Phi\|\,\|\Psi\| \;,$$

  so $e^{-i\varphi(f)t}A^{-1}\Phi \in D(A^{\dagger}) = D(A)$ since $A = A^{\dagger}$ and

  $$\left\|Ae^{-i\varphi(f)t}A^{-1}\Phi\right\| \leq a(t)\,\|\Phi\| \;.$$

So we just proved the following lemma:

LEMMA 5.1. *Let $\delta \geq 0$, then for all $j \geq 1$ we have*

1) $\left\|(N+j+1)^{-\delta}\exp(i\varphi(f)t)\Phi\right\| \leq \exp(\mu_\delta\,|t|)\left\|(N+j+1)^{-\delta}\Phi\right\|$ *for all $\Phi \in \mathscr{H}$, where*

$$\mu_\delta = \delta j^{-1/2}\,\|f\|_2\left(\frac{3}{2}\right)^{\max\{0,(\delta-1)\}} \;.$$

2) $\forall\Phi \in D(N^{\delta})$, $\exp(i\varphi(f)t)\Phi \in D(N^{\delta})$ *and*

$$\left\|(N+j+1)^{\delta}\exp(i\varphi(f)t)\Phi\right\| \leq \exp(\mu_\delta\,|t|)\left\|(N+j+1)^{\delta}\Phi\right\| \;.$$

This lemma has a lot of useful applications, for example we can prove regularity results like the one following:

LEMMA 5.2. *Let B an operator such that for some $\delta \geq 0$ satisfies:*

(1) $D(N^{\delta}) \subset D(B)$,

(2) $\|B\Phi\| \leq b\left\|(N+2)^{\delta}\Phi\right\|$ *for all $\Phi \in D(N^{\delta})$.*

Define then $\Theta(t) = \exp(i\varphi(f)t)\Theta_0$. Then the function $t \longrightarrow B\Theta(t)$ is continuous for all $\Theta_0 \in D(N^{\delta})$ and differentiable for all $\Theta_0 \in D(N^{\delta+1/2})$, the derivative being equal to the formal one.

PROOF. Using Lemma 5.1 and

$$\Theta(t) - \Theta(t_0) = \Big(\exp(i\varphi(f)(t-t_0)) - 1\Big)\exp(i\varphi(f)t_0)\Theta_0$$

it will be sufficient to prove continuity and differentiability at $t = 0$.

Continuity. Let $\Theta_0 \in D(N^{2\delta})$. We estimate

$$\begin{aligned}\|B(\Theta(t)-\Theta_0)\|^2 &\le b^2 \left\|(N+2)^\delta(\Theta(t)-\Theta_0)\right\|^2 \\ &\le b^2 \left\|(N+2)^{2\delta}(\Theta(t)-\Theta_0)\right\| \|\Theta(t)-\Theta_0\|\end{aligned}$$

and since

$$\left\|(N+2)^{2\delta}(\Theta(t)-\Theta_0)\right\| \le (1+\exp(\mu_\delta\,|t|))\left\|(N+2)^{2\delta}\Theta_0\right\| ,$$

we obtain $\|B(\Theta(t)-\Theta_0)\| \longrightarrow 0$ when $t \to 0$. Since Graph$(N^{2\delta})$ is dense in Graph(N^δ), a 2ε-argument let us extend the result to $\Theta_0 \in D(N^\delta)$. ∎

Differentiability. We expect that

$$\frac{\mathrm{d}}{\mathrm{d}t}B\Theta(t) = iB\varphi(f)\Theta(t) .$$

Let $\Theta_0 \in D(N^{2\delta+1/2})$. If we define $\chi(t) = (\Theta(t)-\Theta_0)/t$ we get

$$(*)\qquad \begin{aligned}\|B(\chi(t)-i\varphi(f)\Theta_0)\|^2 \le b^2 \Big(&\left\|(N+2)^{2\delta}\chi(t)\right\| \\ &+ \left\|(N+2)^{2\delta}\varphi(f)\Theta_0\right\|\Big)\|\chi(t)-i\varphi(f)\Theta_0\| .\end{aligned}$$

Using formula

$$\chi(t) = \frac{1}{t}\int_0^t \mathrm{d}t'\,(i\varphi(f))\exp(i\varphi(f)t')\Theta_0 ,$$

we can bound

$$\begin{aligned}\left\|(N+2)^{2\delta}\chi(t)\right\| &\le \|f\|_2 \sup_{0\le t'\le t}\left\|(N+3)^{2\delta+1/2}\exp(i\varphi(f)t')\Theta_0\right\| \\ &\le \|f\|_2\exp(\mu_{2\delta+1/2}\,|t|)\left\|(N+3)^{2\delta+1/2}\Theta_0\right\| .\end{aligned}$$

So the braket $(\dots)$ on the right hand side of $(*)$ has the estimate:

$$\Big(\dots\Big) \le \|f\|_2\Big(1+\exp(\mu_{2\delta+1/2}\,|t|)\Big)\left\|(N+3)^{2\delta+1/2}\Theta_0\right\| .$$

Then we can conclude that $\|B(\chi(t)-i\varphi(f)\Theta_0)\| \longrightarrow 0$ when $t \to 0$. The extension of the domain of Θ_0 to $D(N^{\delta+1/2})$ is performed with an argument identical to the one used in continuity. ∎

∎

Lemma 5.3. *Let $\Phi \in D(a(\bar g)) = D(a^*(g))$, then $\exp(i\varphi(f)t)\Phi \in D(a(\bar g))$ and*

$$\exp(-i\varphi(f)t)a(\bar g)\exp(i\varphi(f)t)\Phi = a(\bar g)\Phi + it\langle g,f\rangle_{L^2}\Phi \tag{5.2}$$

$$\exp(-i\varphi(f)t)a^*(g)\exp(i\varphi(f)t)\Phi = a^*(g)\Phi - it\langle f,g\rangle_{L^2}\Phi \tag{5.3}$$

Proof. We restrict to (5.2), the proof of (5.3) being identical. Let $\Theta \in D(N^{1/2})$ and $\Phi \in D(N)$. Using Lemma 5.2 with $B = a(\bar g)$, $\delta = 1/2$ we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\exp(i\varphi(f)t)\Theta, a(\bar g)\exp(i\varphi(f)t)\Phi\rangle &= \langle i\varphi(f)\exp(i\varphi(f)t)\Theta, a(\bar g)\exp(i\varphi(f)t)\Phi\rangle \\ &\quad+\langle\exp(i\varphi(f)t)\Theta, a(\bar g)i\varphi(f)\exp(i\varphi(f)t)\Phi\rangle\end{aligned}$$

On $D(N^{1/2})$ we have $\varphi(f) = a(\bar f) + a^*(f)$, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\exp(i\varphi(f)t)\Theta, a(\bar g)\exp(i\varphi(f)t)\Phi\rangle &= i\langle\exp(i\varphi(f)t)\Theta,[a(\bar g),a(\bar f) \\ &\quad+a^*(f)]\exp(i\varphi(f)t)\Phi\rangle = i\langle g,f\rangle_{L^2}\langle\Theta,\Phi\rangle .\end{aligned}$$

Integrating in t we obtain (5.2) when $\Phi \in D(N)$. To extend it to the general case we do as following: given $\Phi \in D(a(\bar g))$, let $\Phi_j \in D(N)$ such that $\Phi_j \longrightarrow \Phi$

and $a(\bar{g})\Phi_j \longrightarrow a(\bar{g})\Phi$. From (5.2) with Φ_j it follows that $a(\bar{g})\exp(i\varphi(f)t)\Phi_j$ converges. Then, since $a(\bar{g})$ is closed, $\exp(i\varphi(f)t)\Phi \in D(a(\bar{g}))$ and (5.2) holds for all $\Phi \in D(a(\bar{g}))$. ■

We are now able to prove Weyl's formula.

LEMMA 5.4 (Weyl's formula.). *We have:*

$$\exp(i\varphi(f))\exp(i\varphi(g)) = \exp(i\varphi(f+g))\exp(-i\mathrm{Im}\langle f,g\rangle_{L^2})\ .$$

PROOF. We call $U(tf) \equiv \exp(i\varphi(f)t) = \exp(i\varphi(ft))$. For all $\Phi \in D(N^{1/2})$ we set $\Theta(t) \equiv U(tf)U(tg)U(-t(f+g))\Phi$. Formally we calculate the derivative of Θ using Leibniz's rule:

$$\begin{aligned}-i\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t) &= \Big(U(tf)\varphi(f)U(tg)U(-t(f+g)) \\ &\quad +U(tf)U(tg)\varphi(g)U(-t(f+g)) \\ &\quad -U(tf)U(tg)\varphi(f+g)U(-t(f+g))\Big)\Phi \\ &= U(tf)U(tg)\Big(\varphi(g)+\varphi(f)-\varphi(f+g) \\ &\quad +it\Big(\langle f,g\rangle_{L^2}-\langle g,f\rangle_{L^2}\Big)\Big)U(-t(f+g))\Phi\ ,\end{aligned}$$

last equality is obtained using Lemma 5.3. We remark that we deal at any point of this chain of equalities with vectors in $D(N^{1/2})$. Since on such vectors $\varphi(f+g) = \varphi(f)+\varphi(g)$, we obtain:

$$-i\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t) = i2it\mathrm{Im}\langle f,g\rangle_{L^2}\Theta\ ,$$

so

$$\Theta(t) = \exp(-it^2\mathrm{Im}\langle f,g\rangle_{L^2})$$

and therefore when $t = 1$ the Weyl's formula. The effective calculation consists in justifying the formal derivative above:

$$\begin{aligned}h^{-1}\Big(\Theta(t+h)-\Theta(t)\Big) &= h^{-1}\Big(U((t+h)f)U((t+h)g)U(-(t+h)(f+g)) \\ &\qquad -U(tf)U(tg)U(-t(f+g))\Big)\Phi \\ &= U((t+h)f)U((t+h)g)\frac{U(-(t+h)(f+g)-U(-t(f+g)))}{h}\Phi \\ &\quad +U((t+h)f)\frac{U((t+h)g)-U(tg)}{h}U(-t(f+g))\Phi \\ &\quad +\frac{U((t+h)f)-U(tf)}{h}U(tg)U(-t(f+g))\Phi\end{aligned}$$

Using Lemma 5.2, the limit of each term on the right hand side of the equality above exists when $h \to 0$, so we obtain the formal result. ■

To transform with $\exp(i\varphi(f))$ bilinear operators more general than N, the following lemmas will be useful. We will use the following notation: let u an operator from $D \subset L^2$ in L^2, then if $\Psi_n(X_n) = \prod_{k=1}^{n} w_k(x_k)$, for all $j = 1,\dots,n$ we define $u_j\Psi_n(X_n) = w_1(x_1)\cdots(uw_j)(x_j)\cdots w_n(x_n)$. By linearity, and eventually continuity, we extend the definition of $u_j\Psi_n$ to all $\Psi_n \in \mathscr{H}_n$.

LEMMA 5.5. *Let u be a bounded operator in L^2, $u^\dagger$ its adjoint, $\{e_j\}$ an orthonormal basis of L^2. Then, $\forall \Phi \in D(N^{1/2})$, we have:*

$$\left\|(\mathrm{d}\Gamma(u^\dagger u))^{1/2}\Phi\right\|^2 = \langle \Phi, \mathrm{d}\Gamma(u^\dagger u)\Phi\rangle = \sum_j \left\|a(\overline{u^\dagger e_j})\Phi\right\|^2$$

$$\leq \left\|u; L^2 \to L^2\right\| \left\|N^{1/2}\Phi\right\|^2 \; .$$

PROOF.

$$\left\|(\mathrm{d}\Gamma(u^\dagger u))^{1/2}\Phi\right\|^2 = \sum_{n\geq 1}\langle \Phi_n, (u_1^\dagger u_1 + \ldots + u_n^\dagger u_n)\Phi_n\rangle = \sum_{n\geq 1} n\langle \Phi_n, u_1^\dagger u_1 \Phi_n\rangle$$

$$= \sum_{n\geq 0}(n+1)\left\|u_1\Phi_{n+1}\right\|^2_{\mathscr{H}_{n+1}} = \sum_{n\geq 0}(n+1)\int \mathrm{d}X_n \sum_j \left|\langle e_j, u_1\Phi_{n+1}\rangle_1(X_n)\right|^2$$

$$= \sum_{n\geq 0}(n+1)\int \mathrm{d}X_n \sum_j \left|\langle u^\dagger e_j, \Phi_{n+1}\rangle_1(X_n)\right|^2$$

$$= \sum_{n\geq 0}\int \mathrm{d}X_n \sum_j \left|(a(\overline{u^\dagger e_j})\Phi)_n(X_n)\right|^2 = \sum_j\sum_n \left\|(a(\overline{u^\dagger e_j})\Phi)_n\right\|^2_{\mathscr{H}_n} \; .$$

We could exchange $\sum_n \int \sum_j$ because they all have positive terms. ∎

LEMMA 5.6. *Let v be a bounded operator in L^2, $\{e_j\}$ an orthonormal basis of L^2, $\Theta \in \mathscr{H}$, $\Phi \in D(N^{1/2})$. Then*

$$\sum_j \langle \Theta, a(\overline{ve_j})\Phi\rangle \overline{\langle e_j, g\rangle}_{L^2} = \langle \Theta, a(\overline{vg})\Phi\rangle = \sum_n \langle \Theta_n, \langle g, v_1^\dagger \Phi_{n+1}\rangle_1\rangle_n (n+1)^{1/2}$$

and we have the following bound:

$$\left|\sum_n \langle \Theta_n, \langle g, v_1^\dagger \Phi_{n+1}\rangle_{\mathscr{H}_1}\rangle_{\mathscr{H}_n}(n+1)^{1/2}\right| \leq \|g\|_2 \|\Theta\| \left\|(\mathrm{d}\Gamma(vv^\dagger))^{1/2}\Phi\right\| \; .$$

PROOF. We remark that

$$\langle \Theta, a(\overline{ve_j})\Phi\rangle = \sum_n \langle \Theta_n, \langle ve_j, \Phi_{n+1}\rangle_1\rangle_{\mathscr{H}_n}(n+1)^{1/2} \; .$$

We bound the following double sum in absolute value:

$$\sum_{j,n}\left|\langle \Theta_n, \langle ve_j, \Phi_{n+1}\rangle_{\mathscr{H}_1}\rangle_{\mathscr{H}_n}(n+1)^{1/2}\langle e_j, g\rangle_{L^2}\right| \leq \sum_n (n+1)^{1/2}$$

$$\langle |\Theta_n|, \sum_j |\langle e_j, v^\dagger \Phi_{n+1}\rangle_{\mathscr{H}_1}|\,|\langle e_j, g\rangle_{L^2}|\rangle_{\mathscr{H}_n}$$

$$\leq \sum_n (n+1)^{1/2}\langle |\Theta_n|, \left\|v_1^\dagger \Phi_{n+1}\right\|_{\mathscr{H}_1}\rangle_{\mathscr{H}_n}\|g\|_2$$

$$\leq \|g\|_2 \|\Theta\| \left\|(\mathrm{d}\Gamma(vv^\dagger))^{1/2}\Phi\right\|$$

$$\leq \|g\|_2 \left\|v; L^2 \to L^2\right\| \|\Theta\| \left\|N^{1/2}\Phi\right\| \; .$$

We now continue with the calculation of the series:

$$\begin{aligned}
&\sum_j \langle \Theta_n, \langle v e_j, \Phi_{n+1}\rangle_{\mathscr{H}_1}\rangle_{\mathscr{H}_n} \overline{\langle e_j, g\rangle}_{L^2}(n+1)^{1/2} \\
&= \sum_j \langle \Theta_n, \langle e_j, v_1^\dagger \Phi_{n+1}\rangle_{\mathscr{H}_1}\rangle_{\mathscr{H}_n} \overline{\langle e_j, g\rangle}_{L^2}(n+1)^{1/2} \\
&= \langle \Theta_n, \langle g, v_1^\dagger \Phi_{n+1}\rangle_{\mathscr{H}_1}\rangle_{\mathscr{H}_n}(n+1)^{1/2} \\
&= \langle \Theta_n, \langle v g, \Phi_{n+1}\rangle_{\mathscr{H}_1}\rangle_{\mathscr{H}_n}(n+1)^{1/2} \\
&= \langle \Theta_n, (a(\overline{vg})\Phi)_n\rangle_{\mathscr{H}_n}
\end{aligned}$$

Then summing over n and exchanging $\sum_j$ and $\sum_n$ we complete the proof. ∎

LEMMA 5.7. *Let u be a bounded operator in L^2. Then $\forall \Phi \in D(N^{1/2})$ we have:*

$$\begin{aligned}
\left\| (\mathrm{d}\Gamma(u^\dagger u))^{1/2} \exp(i\varphi(f))\Phi \right\|^2 &= \left\| (\mathrm{d}\Gamma(u^\dagger u))^{1/2}\Phi \right\|^2 + \|uf\|_2^2 \|\Phi\|^2 \\
&\quad + \langle \Phi, \big(a(\overline{u^\dagger u i f}) + a^*(u^\dagger u i f)\big)\Phi\rangle \\
&\le 2\Big(\left\| (\mathrm{d}\Gamma(u^\dagger u))^{1/2}\Phi \right\|^2 + \|uf\|_2^2 \|\Phi\|^2 \Big)
\end{aligned}$$

In particular since $\mathrm{d}\Gamma(u^\dagger u)$ and $\big(a(\overline{u^\dagger u i f}) + a^*(u^\dagger u i f)\big)$ are symmetric we can write

$$\begin{aligned}
\exp(-i\varphi(f))\mathrm{d}\Gamma(u^\dagger u)\exp(i\varphi(f))\Phi &= \mathrm{d}\Gamma(u^\dagger u)\Phi + \|uf\|_2^2 \Phi \\
&\quad + a(\overline{u^\dagger u i f})\Phi + a^*(u^\dagger u i f)\Phi \ ;
\end{aligned}$$

for all $\Phi \in D(N)$.

PROOF. From Lemma 5.5 we obtain

$$\left\| (\mathrm{d}\Gamma(u^\dagger u))^{1/2} \exp(i\varphi(f))\Phi \right\|^2 = \sum_j \left\| a(\overline{u^\dagger} e_j)\exp(i\varphi(f))\Phi \right\|^2 .$$

We can continue using Lemma 5.3 and obtain

$$\begin{aligned}
\left\| (\mathrm{d}\Gamma(u^\dagger u))^{1/2} \exp(i\varphi(f))\Phi \right\|^2 &= \sum_j \left\| a(\overline{u^\dagger} e_j)\Phi + i\langle e_j, uf\rangle_{L^2}\Phi \right\|^2 \\
&= \left\| (\mathrm{d}\Gamma(u^\dagger u))^{1/2}\Phi \right\|^2 + \|uf\|_2^2 \|\Phi\|^2 \\
&\quad + 2\mathrm{Im} \sum_j \langle \Phi, a(\overline{u^\dagger} e_j)\Phi\rangle \overline{\langle e_j, uf\rangle}_{L^2} .
\end{aligned}$$

Lemma 5.6 leads to the first equality in the statement. The following inequality is obtained from the $\sum_j$ above or using Schwarz's inequality and Lemma 5.5 or recalling the calculation done in the proof of Lemma 5.6. ∎

LEMMA 5.8. *Let u = multiplication by ω^λ ($\omega = \omega(k) = |k|$, $\lambda \ge 0$). Let $f \in L^2(\mathbb{R}^d)$ with $\omega^\lambda f \in L^2(\mathbb{R}^d)$. If $\Phi \in D((\mathrm{d}\Gamma(\omega^{2\lambda}))^{1/2})$, then $\exp(i\varphi(f))\Phi \in D((\mathrm{d}\Gamma(\omega^{2\lambda}))^{1/2})$ and*

$$\left\| (\mathrm{d}\Gamma(\omega^{2\lambda}))^{1/2} \exp(i\varphi(f))\Phi \right\|^2 \le 2\Big(\left\| (\mathrm{d}\Gamma(\omega^{2\lambda}))^{1/2}\Phi \right\|^2 + \left\| \omega^\lambda f \right\|_2^2 \|\Phi\|^2 \Big) .$$

PROOF. We apply Lemma 5.7 with $u = u_\sigma = \omega^\lambda \chi_{\omega\leq\sigma}$, so u is a multiplication by ω^λ with cut off. For all $M \geq 1$ we have

$$\begin{aligned}\sum_{n=1}^{M}\langle(\exp(i\varphi(f))\Phi)_n, \Big(\sum_{j=1}^{n}\omega_j^{2\lambda}\chi_{\omega_j\leq\sigma}\Big)(\exp(i\varphi(f))\Phi)_n\rangle_{\mathscr{H}_n} \\ = \sum_{n=1}^{M}\Big\|(\mathrm{d}\Gamma(\omega^{2\lambda}\chi_{\omega\leq\sigma}))^{1/2}\exp(i\varphi(f))\Phi\Big\|^2 \\ \leq 2\Big(\Big\|(\mathrm{d}\Gamma(\omega^{2\lambda}\chi_{\omega\leq\sigma}))^{1/2}\Phi\Big\|^2 + \|\omega^\lambda\chi_{\omega\leq\sigma}f\|_2^2\,\|\Phi\|^2\Big) \\ \leq 2\Big(\Big\|(\mathrm{d}\Gamma(\omega^{2\lambda}))^{1/2}\Phi\Big\|^2 + \|\omega^\lambda f\|_2^2\,\|\Phi\|^2\Big)\,.\end{aligned}$$

The result follows taking the limit $\sigma\to\infty$ in the left hand side of the inequality (monotone convergence theorem) and then taking the limit $M\to\infty$. ■

In the work we need regularity results of the exponential of the field as we change the function that appears inside the field operator. To be more precise we set

$$U(f) \equiv \exp(i\varphi(f))$$

and study its regularity as a function of $f \in L^2(\mathbb{R}^d)$.

LEMMA 5.9. *1) $U(f)$ is strongly continuous as a function of f.*
2) Let $f\in\mathscr{C}^1(I, L^2)$. Then, $\forall\Phi\in D(N^{1/2})$, $U(f(\cdot))\Phi$ is differentiable and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U(f(t))\Phi &= \Big(i\varphi(\dot f(t)) + i\mathrm{Im}\langle\dot f(t), f(t)\rangle\Big)U(f(t))\Phi \\ &\quad U(f(t))\Big(i\varphi(\dot f(t)) - i\mathrm{Im}\langle\dot f(t), f(t)\rangle\Big)\Phi\end{aligned}$$

PROOF. The following basic formula is obtained from Weyl's formula (Lemma 5.4):

$$(*)\qquad\begin{aligned}U(f) - U(g) &= (U(f)U(-g) - 1)U(g) = (U(f-g)e^{i\mathrm{Im}\langle f,g\rangle_{L^2}} - 1)U(g) \\ &= U(f-g)(e^{i\mathrm{Im}\langle f,g\rangle_{L^2}} - 1)U(g) + (U(f-g)-1)U(g) \\ &= U(g)U(f-g)(e^{-i\mathrm{Im}\langle f,g\rangle_{L^2}} - 1) + U(g)(U(f-g)-1)\end{aligned}$$

Another important formula is the following: $\forall\Phi\in D(N^{1/2})$ we have

$$(**)\qquad (U(f)-1)\Phi = \int_0^1 \mathrm{d}s\, U(sf)i\varphi(f)\Phi\,.$$

1) Use $(**)$ in $(*)$ with $\Phi\in D(N^{1/2})$ to obtain:

$$\begin{aligned}\|(U(f)-U(g))\Phi\| &\leq |\mathrm{Im}\langle f,g\rangle_{L^2}|\,\|\Phi\| + \|\varphi(f-g)\Phi\| \\ &\leq |\mathrm{Im}\langle f,g\rangle_{L^2}|\,\|\Phi\| + 2\,\|f-g\|_2\,\Big\|(N+1)^{1/2}\Phi\Big\|\end{aligned}$$

and this goes to zero if $f\to_{L^2} g$. The elimination of condition $\Phi\in D(N^{1/2})$ is done by standard methods.

2) We set, for brevity, $F\equiv F(h,t)\equiv f(t+h)-f(t)$. We write

$$\begin{aligned}\frac{U(f(t+h)) - U(f(t))}{h} &= U(f(t))U(F)\Big(\frac{e^{-i\mathrm{Im}\langle f(t+h),f(t)\rangle_{L^2}} - 1}{h}\Big) \\ &\quad + U(f(t))\Big(\frac{U(F)-1}{h}\Big)\,.\end{aligned}$$

Now we have that

$$\lim_{h\to 0}\frac{e^{-i\mathrm{Im}\langle f(t+h),f(t)\rangle_{L^2}}-1}{h}=-i\mathrm{Im}\langle\dot f(t),f(t)\rangle$$

and, $\forall\Phi\in D(N^{1/2})$,

$$\frac{U(F(h,t))-1}{h}\Phi=\int_0^1 \mathrm{d}s\, U(sF(h,t))i\varphi\Big(\frac{F(h,t)}{h}\Big)\Phi\underset{h\to 0}{\longrightarrow} i\varphi(\dot f)\Phi\ .$$

Then we obtain the second formula of the derivative. To obtain the first we follow the same procedure.

■

6. Interaction picture.

We will study now how the annihilation and creation operators, and then also $\exp(i\varphi(f))$, modify under the action of a class of unitary operators. Consider an operator v from $D\subset L^2$ in L^2, self adjoint. Define $u_0(t)\equiv\exp(-itv)$, $U_0(t)\equiv\exp(-it\mathrm{d}\Gamma(v))$. We remark that, under suitable domain conditions, $\mathrm{d}\Gamma(v)$ commutes with N^δ for all δ, so if $\Phi\in D(N^\delta)$, then $U_0(t)\Phi\in D(N^\delta)$.

Lemma 6.1. *Let $\Phi\in D(N^{1/2})$, $f\in L^2(\mathbb{R}^d)$. Define $\tilde f\equiv u_0^\dagger(t)f$, then*

$$U_0^\dagger(t)a(\bar f)U_0(t)\Phi=a(\bar{\tilde f})\Phi$$
$$U_0^\dagger(t)a^*(f)U_0(t)\Phi=a^*(\tilde f)\Phi$$

Proof. The proof is done by means of a direct calculation on $\mathscr{H}_n$.

$$\begin{aligned}(U_0^\dagger(t)a(\bar f)U_0(t)\Phi)_n(X_n)&=(n+1)^{1/2}\exp(it\sum_{j=1}^n v_j)\int\mathrm{d}x\,\bar f(x)\\&\qquad\exp(-it\sum_{j=1}^n v_j)\exp(-itv_x)\Phi_{n+1}(x,X_n)\\&=(n+1)^{1/2}\int\mathrm{d}x\,\bar f(x)\exp(-itv_x)\Phi_{n+1}(x,X_n)\\&=(n+1)^{1/2}\langle f,u_0(t)\Phi_{n+1}\rangle_{\mathscr{H}_1}(X_n)\\&=(n+1)^{1/2}\langle u_0^\dagger(t)f,\Phi_{n+1}\rangle_{\mathscr{H}_1}(X_n)\ .\end{aligned}$$

Now that we proved the one about $a(\bar f)$, the other is obtained by taking the adjoint: let $\Theta,\Phi\in D(N^{1/2})$, then

$$\begin{aligned}\langle U_0^\dagger(t)a^*(f)U_0(t)\Theta,\Phi\rangle=\langle\Theta,U_0^\dagger(t)a(\bar f)U_0(t)\Phi\rangle&=\langle\Theta,a(\bar{\tilde f})\Phi\rangle\\&=\langle a^*(\tilde f)\Theta,\Phi\rangle\ .\end{aligned}$$

The equality between the first and the last term then can be extended to all $\Phi\in\mathscr{H}$, and that completes the proof. ■

Lemma 6.2. *Let $f\in\mathscr{C}^0(I,L^2)$. Then*

$$U_0^\dagger(t)\exp(i\lambda\varphi(f(t)))U_0(t)=\exp(i\lambda\varphi(\tilde f(t)))\ .$$

Proof. The proof is an application of Stone's theorem. Let $\Phi\in D(N^{1/2})$, and define $W(\lambda)\equiv U_0^\dagger(t)\exp(i\lambda\varphi(f(t)))U_0(t)$. Then $W(\lambda)$ is differentiable on $D(N^{1/2})$, and $W(\lambda)\Phi\in D(N^{1/2})$, furthermore using Lemma 6.1 (on $D(N^{1/2})$ we have $\varphi(f)=a(\bar f)+a^*(f)$) we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}W(\lambda)\Phi=i\Big(a(\bar{\tilde f}(t))+a^*(\tilde f(t))\Big)W(\lambda)\Phi\ .$$

Then by Stone's theorem $W(\lambda) = \exp\Big(i\lambda\Big[a(\bar{\tilde{f}}(t)) + a^*(\tilde{f}(t))\Big]^{-}\Big)$. ■

7. From $\exp(i\varphi(f))$ to $C(\alpha)$.

In order to be in agreement with the literature on the classical limit of field theories we have to change the notation we used above. Let $f = -i\alpha$. Then we have

$$\varphi(f) \equiv (a(\bar{f}) + a^*(f))^{-} = i(a(\bar{\alpha}) - a^*(\alpha))^{-} ,$$

$$U(f) \equiv \exp(i\varphi(f)) = \exp\big((a^*(\alpha) - a(\bar{\alpha}))^{-}\big) .$$

Formulas of Lemma 5.3 become: $\forall \Phi \in D(a(\bar{g}))$

$$C(\alpha)^\dagger a(\bar{g}) C(\alpha)\Phi = a(\bar{g})\Phi + \langle g, \alpha\rangle_{L^2}\Phi$$
$$C(\alpha)^\dagger a^*(g) C(\alpha)\Phi = a^*(g)\Phi + \langle \alpha, g\rangle_{L^2}\Phi .$$

The derivative of Lemma 5.9 becomes: $\forall \Phi \in D(N^{1/2})$

$$\frac{\mathrm{d}}{\mathrm{d}t} C(\alpha)\Phi = \Big(a^*(\dot{\alpha}) - a(\dot{\bar{\alpha}}) - i\mathrm{Im}\langle \alpha, \dot{\alpha}\rangle\Big) C(\alpha)\Phi$$
$$C(\alpha)\Big(a^*(\dot{\alpha}) - a(\dot{\bar{\alpha}}) + i\mathrm{Im}\langle \alpha, \dot{\alpha}\rangle\Big)\Phi .$$

8. The case of $\mathscr{H} = \mathscr{F}_s(L^2(\mathbb{R}^3)) \otimes \mathscr{F}_s(L^2(\mathbb{R}^3))$.

To apply the results of this appendix to the system we studied we have to extend them to the case of a tensor product of Fock spaces. Let $\mathscr{H} = \mathscr{F}_s(L^2(\mathbb{R}^3)) \otimes \mathscr{F}_s(L^2(\mathbb{R}^3))$, then we will have two types of annihilation and creation operators, namely $\psi^\#(f)$ and $a^\#(f)$ that satisfy all lemmas above, and two number operators P and N. If we want to be precise we have this abuse of notation:

$$\psi^\#(f) \equiv \psi^\#(f) \otimes 1 ,\ a^\#(f) \equiv 1 \otimes a^\#(f)$$
$$P \equiv P \otimes 1 ,\ N \equiv 1 \otimes N .$$

We have also Weyl operators $C(u) \equiv C(u) \otimes 1$, $C(\alpha) \equiv 1 \otimes C(\alpha)$, $C(u,\alpha) = C(u)C(\alpha)$. The results above can be applied suitably for $C(u), C(\alpha), C(u,\alpha)$.

Let $B \geq 0$ a self-adjoint operator, we define $Q(B) \subseteq \mathscr{H}$ the form domain of B, *i.e.* $Q(B) = D(B^{1/2})$. $Q(B)$ is a Hilbert space with norm $\big\|(B+1)^{1/2}\Phi\big\|$.

We denote $Q^*(B)$ the completion of $\mathscr{H}$ in the norm $\big\|(B+1)^{-1/2}\Phi\big\|$. Finally we define the Hilbert spaces $\mathscr{H}^\delta$, $\delta \in \mathbb{R}$: $\mathscr{H}^\delta = Q((P+N)^\delta)$ for $\delta \geq 0$, and $\mathscr{H}^\delta = Q^*((P+N)^{|\delta|})$ for $\delta < 0$; $\mathscr{H}^\delta$ is a Hilbert space in the norm

$$\|\Phi\|_\delta = \Big\|(P+N+1)^{\delta/2}\Phi\Big\| .$$

For any $p, n \in \mathbb{N}$ we also define

$$\mathscr{H}_{p,n} = \big\{\Phi_{p,n} :\ \Phi_{p,n}(X_p; K_n) \in L^2(\mathbb{R}^{3p+3n})\big\} ,$$

where $X_p = \{x_1, \ldots, x_p\}$, $K_n = \{k_1, \ldots, k_n\}$ and $\Phi_{p,n}$ is separately symmetric with respect to the first p and the last n variables. $\mathscr{H}$ is then the direct sum of the $\mathscr{H}_{p,n}$:

$$\mathscr{H} = \bigoplus_{p,n=0}^{\infty} \mathscr{H}_{p,n} .$$

We will use the following properties of the tensor product of Hilbert spaces

$$\mathscr{H}_{p,n} = \mathscr{H}_{p,0} \otimes \mathscr{H}_{0,n} ,$$

and

$$\mathscr{H} = \Big(\bigoplus_{p=0}^{\infty} \mathscr{H}_{p,0}\Big) \otimes \Big(\bigoplus_{n=0}^{\infty} \mathscr{H}_{0,n}\Big) = \bigoplus_{p=0}^{\infty} \mathscr{H}_p \ ,$$

with

$$\mathscr{H}_p = \mathscr{H}_{p,0} \otimes \bigoplus_{n=0}^{\infty} \mathscr{H}_{0,n} \ .$$

We want to extend some results on the invariance of domains under the action of $C(u,\alpha)$ we proved above to $\mathscr{H}^\delta$. When the proof is almost identical to the one already done above, we will omit it. In particular we modify Lemma 5.1 to read as:

LEMMA 8.1. *$C(u,\alpha)$ maps $\mathscr{H}^{2\delta}$ into itself for all $\delta \in \mathbb{R}$. In particular we have for all $\Phi \in \mathscr{H}^{2\delta}$, $j \geq 1$*

$$\left\|(P+N+j+1)^\delta C(u,\alpha)\Phi\right\| \leq \exp(\rho_\delta) \left\|(P+N+j+1)^\delta \Phi\right\| \ ,$$

with $\rho_\delta \sim \delta(\|u\|_2 + \|\alpha\|_2)$.

We did not calculate the constant ρ_δ explicitly since we are interested only in the invariance of the domains.

Finally we want to study the behaviour of a particular class of trilinear operators of $\mathscr{H}$, when they are transformed by $C(u,\alpha)$. Consider the following operator on $L^2(\mathbb{R}^3) \otimes D(N^{1/2})$: let $f(\cdot) \in L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3))$, define

$$\varphi(x) \equiv a(\bar{f}(x)) + a^*(f(x)) \ .$$

Then we consider the operator $\mathrm{d}\Gamma_p(\varphi)$, defined on $D(P) \otimes D(N^{1/2})$. On $\mathscr{H}_{p,0} \otimes D(N^{1/2})$ it acts as

$$\mathrm{d}\Gamma_p(\varphi)\Big|_{\mathscr{H}_{p,0}\otimes D(N^{1/2})} = \sum_{j=1}^{p} \varphi(x_j) \ .$$

Furthermore define another operator on $L^2(\mathbb{R}^3) \otimes \bigoplus_{n\geq 0} \mathscr{H}_{0,n}$: let $\alpha \in L^2(\mathbb{R}^3)$, $f(\cdot) \in L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3))$, then

$$\langle f, \alpha\rangle(x) \equiv 2\mathrm{Re}\langle f(x), \alpha\rangle_{L^2(\mathbb{R}^3)} \ .$$

Using Lemma 5.3 on $\mathscr{H}_{p,0} \otimes D(N^{1/2})$, we have that

$$C^*(\alpha)\mathrm{d}\Gamma_p(\varphi)C(\alpha)\Big|_{\mathscr{H}_{p,0}\otimes D(N^{1/2})} = \Big(\mathrm{d}\Gamma_p(\varphi) + \mathrm{d}\Gamma_p(\langle f,\alpha\rangle)\Big)\Big|_{\mathscr{H}_{p,0}\otimes D(N^{1/2})} \ .$$

Finally applying the last formula of Lemma 5.7 on $D(P^2+N)$ to

$$C^*(u)\Big(\mathrm{d}\Gamma_p(\varphi) + \mathrm{d}\Gamma_p(\langle f,\alpha\rangle)\Big)C(u)$$

we prove the following lemma (we write it in the language of second quantization for clarity):

LEMMA 8.2. *Let $\Phi \in D(P^2+N)$, $f \in L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3))$ then:*

$$\begin{aligned} C^*(u,\alpha) \int \mathrm{d}x\mathrm{d}k\, \Big(\bar{f}(x,k)a(k) + f(x,k)a^*(k)\Big)\psi^*(x)\psi(x)C(u,\alpha)\Phi \\ = \int \mathrm{d}x\mathrm{d}k\, \Big(\bar{f}(x,k)(a(k)+\alpha(k)) + f(x,k)(a^*(k)+\bar{\alpha}(k))\Big) \\ (\psi^*(x)+\bar{u}(x))(\psi(x)+u(x))\Phi \ . \end{aligned}$$

APPENDIX B

Direct sum of operators on Hilbert spaces.

1. Direct sum of domains.

- Let $\mathscr{H}_n$ be a Hilbert space for all n, then we define

  $$\mathscr{H} = \bigoplus_{n=1}^{\infty} \mathscr{H}_n \ .$$

  P_n is the orthogonal projector on $\mathscr{H}_n$ and

  $$P_{\leq n} = \{\text{projector on } \bigoplus_{j=1}^{n} \mathscr{H}_j\} = \sum_{j=1}^{n} P_j \ .$$

- If $D_n \subset \mathscr{H}_n$, then

  $$\bigoplus_{n=1}^{\infty} D_n = \{x \in \mathscr{H},\ P_n x \in D_n \text{ for all } n\}$$

  $$\bigoplus_{\text{fin}} D_n = \{x \in \mathscr{H},\ P_n x = 0\ \forall n \geq n_0(x) \text{ and } P_n x \in D_n \text{ when} \neq 0\}.$$

- It is clear that

  $$\Big(\bigoplus_{\text{fin}} D_n\Big)^{-} \supset \bigoplus_{n=1}^{\infty} D_n$$

  for every $x \in \bigoplus_{n=1}^{\infty} D_n$ can be approximated as well as we want by $\sum_{n=1}^{N} P_n x$. Then

  $$\Big(\bigoplus_{\text{fin}} D_n\Big)^{-} = \Big(\bigoplus_{n=1}^{\infty} D_n\Big)^{-} .$$

- We remark that if

  $$S \equiv \bigoplus_{n=1}^{\infty} D_n = \bigoplus_{n=1}^{\infty} D'_n \ ,$$

  taking the P_n-projection of a generic $x \in S$ we obtain $D_n = D'_n$ for all n.

Lemma 1.1.

$$\Big(\bigoplus_{\text{fin}} D_n\Big)^{-} = \Big(\bigoplus_{n=1}^{\infty} D_n\Big)^{-} = \bigoplus_{n=1}^{\infty} D_n^{-} .$$

Proof. Let $x \in \bigoplus D_n^{-}$. Then $P_n x \in D_n^{-}$. For all ε_n, $\exists y_n \in D_n$ such that $\|P_n x - y_n\|_n < \varepsilon_n$. Since $\|y_n\|_n^2 \leq 2\,\|P_n x\|_n^2 + 2\,\|P_n x - y_n\|_n^2$, $\sum_{n=1}^{\infty} \|y_n\|_n^2 < \infty$, so $y = (y_1, \dots, y_n, \dots) \in \bigoplus_n D_n$. It follows that $\|x - y\|^2 \leq \sum_n \varepsilon_n^2$ that can become suitably small. Then

$$\Big(\bigoplus_{n=1}^{\infty} D_n\Big)^{-} \supset \bigoplus_{n=1}^{\infty} D_n^{-} .$$

Let now $x \in \left(\bigoplus D_n\right)^{-}$. For all ε, $\exists y$ with $P_n y \in D_n$ such that $\|x-y\| < \varepsilon$. It follows that $\|P_n x - P_n y\| < \varepsilon$. Hence $P_n x \in D_n^{-}$, and $x \in \bigoplus_n D_n^{-}$. Then

$$\left(\bigoplus_{n=1}^{\infty} D_n\right)^{-} \subset \bigoplus_{n=1}^{\infty} D_n^{-} .$$

■

Corollary 1.1. *If $D_n^{-} = \mathscr{H}_n$ then*

$$\left(\bigoplus_{\text{fin}} D_n\right)^{-} = \left(\bigoplus_{n=1}^{\infty} D_n\right)^{-} = \mathscr{H} .$$

2. Direct sum of operators.

- Let A_n be linear operator on $\mathscr{H}_n$ defined on $D(A_n)$. Define

  $$D(A) = \{x \in \mathscr{H} : \; P_n x \in D(A_n), \; \sum_{n=1}^{\infty} \|A_n P_n x\|_n^2 < \infty\}$$

  $$Ax = \sum_{n=1}^{\infty} A_n P_n x = \lim_{N\to\infty} \sum_{n=1}^{N} A_n P_n x \; , \; \forall x \in D(A) .$$

  We will use the notation $A = \bigoplus A_n$. Define also

  $$A_{\text{fin}} \equiv A\Big|_{\bigoplus_{\text{fin}} D(A_n)} .$$

- Observe that, by definition, $\{x, y\} \in \mathscr{G}(A) \subset \mathscr{H} \oplus \mathscr{H}$ if and only if $x \in \bigoplus_n D(A_n)$, $y \in \bigoplus_n \operatorname{Ran}(A_n)$ and $P_n y = A_n P_n x$ for all n. Since $\mathscr{H} \oplus \mathscr{H} = \bigoplus_n (\mathscr{H}_n \oplus \mathscr{H}_n)$ we can write

  $$\mathscr{G}(A) = \bigoplus_n \mathscr{G}(A_n)$$

  and we have

  $$\mathscr{G}(A_{\text{fin}}) = \bigoplus_{\text{fin}} \mathscr{G}(A_n) .$$

  We remark that $\ker(A) = \bigoplus_n \ker(A_n)$ and $\operatorname{Ran}(A) = \bigoplus_n \operatorname{Ran}(A_n)$.

Lemma 2.1.

$$\mathscr{G}(A_{\text{fin}})^{-} = \left(\bigoplus_{\text{fin}} \mathscr{G}(A_n)\right)^{-} = \mathscr{G}(A)^{-} = \left(\bigoplus_{n=1}^{\infty} \mathscr{G}(A_n)\right)^{-} = \bigoplus_{n=1}^{\infty} \mathscr{G}(A_n)^{-} .$$

Proof. A direct consequence of Lemma 1.1 ■

- If $D(A_n)^{-} = \mathscr{H}_n$ by Corollary 1.1 $\left(\bigoplus_{\text{fin}} D(A_n)\right)^{-} = \mathscr{H}$ and since $D(A) \supset \bigoplus_{\text{fin}} D(A_n)$ it follows that $D(A)^{-} = \mathscr{H}$.

Corollary 2.1. *If all A_n are closable, then A is closable and*

$$\mathscr{G}(A^{-}) = \bigoplus_{n=1}^{\infty} \mathscr{G}(A_n^{-}) \; , \; A^{-} = \bigoplus_{n=1}^{\infty} A_n^{-} \; , \; A_{\text{fin}}^{-} = A^{-} .$$

PROOF. If A_n is closable then $\mathscr{G}(A_n)^{-} = \mathscr{G}(\overline{A_n})$. Hence by Lemma 2.1 $\mathscr{G}(A)^{-} = \bigoplus_n \mathscr{G}(\overline{A_n})$, then

$$\mathscr{G}(A)^{-} = \mathscr{G}(\bigoplus_n \overline{A_n}) \ , \ \overline{A} = \bigoplus_n \overline{A_n} \ .$$

■

LEMMA 2.2. *Let A_n be densely defined for all n. Since $A = \bigoplus_n A_n$, then $A^\dagger = \bigoplus_n A_n^\dagger$.*

PROOF. Let τ be the operator on $\mathscr{H} \oplus \mathscr{H}$ such that $\tau\{x, y\} = \{-y, x\}$. If τ_n is the operator τ defined on $\mathscr{H}_n \oplus \mathscr{H}_n$ and M_n a linear subspace $\subset \mathscr{H}_n \oplus \mathscr{H}_n$, setting $\mathscr{H} \oplus \mathscr{H} = \bigoplus_n (\mathscr{H}_n \oplus \mathscr{H}_n)$ we obtain

$$\tau\Big(\bigoplus_n M_n\Big) = \bigoplus_n \tau_n M_n \ .$$

Also, by direct inspection, it is obvious that

$$\Big(\bigoplus_n M_n\Big)^{\perp} = \bigoplus_n M_n^{\perp} \ .$$

Because $\mathscr{G}(B^\dagger) = (\tau\mathscr{G}(B))^{\perp} = \tau(\mathscr{G}(B))^{\perp}$, taking the orthogonal complement of

$$\mathscr{G}(A) = \bigoplus_n \mathscr{G}(A_n)$$

we obtain

$$\mathscr{G}(A)^{\perp} = \bigoplus_n \mathscr{G}(A_n)^{\perp} \ ;$$

then

$$\mathscr{G}(A^\dagger) = \tau(\mathscr{G}(A))^{\perp} = \bigoplus_n \tau_n(\mathscr{G}(A_n))^{\perp} = \bigoplus_n \mathscr{G}(A_n^\dagger) \ .$$

■

LEMMA 2.3. *Let $A = \bigoplus_n A_n$ with $D(A_n) \subset \mathscr{H}_n$, $D(A) \subset \mathscr{H} = \bigoplus_n \mathscr{H}_n$. Then A is continuous if and only if A_n is continuous for all n and $\sup \|A_n\|_n < \infty$. If that is the case then $\|A\| = \sup_n \|A_n\|_n$.*

PROOF. We recall that

$$\|x\|^2 = \sum_{n=1}^{\infty} \|P_n x\|_n^2$$

$$\|Ax\|^2 = \sum_{n=1}^{\infty} \|A_n P_n x\|_n^2 \ .$$

Let A continuous. From $\sum_n \|A_n P_n x\|_n^2 \le \|A\|^2 \sum_n \|P_n x\|_2^2$ setting $x = P_m x$ we have $\|A_m\| \le \|A\|$ and $\sup_n \|A_n\|_n \le \|A\|$. On the other hand:

$$\sum_n \|A_n P_n x\|_n^2 \le \sup_m \|A_m\|_m^2 \sum_n \|P_n x\|_n^2 \ ,$$

then $\|A\|^2 \le \sup_n \|A_n\|_n^2$, so $\|A\| = \sup_n \|A_n\|_n$.

Let now A_n continuous for all n and $\sup_n \|A_n\|_n < \infty$. Then $\|Ax\|^2 \le \sup_n \|A_n\|_n^2 \|x\|^2$ hence A is continuous and we can use the reasoning above. ■

APPENDIX C

Scale of spaces and interpolation.

The spaces $\mathscr{H}^\delta$ defined here are different from the ones defined in Chapter 4. The latter are the form domains of (powers of) the operator under consideration, the former are the domains of definition of the same operator. As a matter of fact to translate the results obtained here to the spaces of Chapter 4 you have to call $\mathscr{H}^{2\delta}$ every space that here is $\mathscr{H}^\delta$.

1. Scale of spaces.

- Let $\mathscr{H}$ be a Hilbert space with scalar product $\langle \cdot\,, \cdot\rangle$ and norm $\|\cdot\|$. Then let N be a self-adjoint operator in $\mathscr{H}$, $N \geq 1$; $\delta \in \mathbb{R}$.

  If $\delta > 0$, we define $\mathscr{H}^\delta = D(N^\delta)$ the pre-Hilbert space with respect to the scalar product

  $$\langle x, y\rangle_\delta = \langle N^\delta x, N^\delta y\rangle \ .$$

  $\mathscr{H}_\delta$ is complete since N is closed and $N \geq 1$, so it is a Hilbert space. If $\delta = 0$, we set $\mathscr{H}^0 = \mathscr{H}$.

  If $\delta < 0$, let $\mathscr{H}^\delta$ be the completion of $\mathscr{H}$ with respect to the norm

  $$\|x\|_\delta = \left\|N^\delta x\right\| \ .$$

  $\mathscr{H}^\delta$ is a Hilbert space by construction.
- Let now $\delta_1 \leq \delta_2$. We introduce the following family of identity operators $i_{\delta_1,\delta_2} x = x$ from a subspace of $\mathscr{H}^{\delta_2}$ in $\mathscr{H}^{\delta_1}$. Their domains and ranges are:

  $$D(i_{\delta_1,\delta_2}) = \begin{cases} D(N^{\delta_2}) = \mathscr{H}^{\delta_2} \text{ if } \delta_2 \geq 0 \\ \mathscr{H} \subset \mathscr{H}^{\delta_2} \text{ if } \delta_2 < 0 \end{cases}$$

  $$\operatorname{Ran}(i_{\delta_1,\delta_2}) = \begin{cases} D(N^{\delta_2}) \subset D(N^{\delta_1}) = \mathscr{H}^{\delta_1} \text{ if } \delta_1 \geq 0 \\ D(N^{\delta_2}) \subset \mathscr{H} \subset \mathscr{H}^{\delta_1} \text{ if } \delta_1 < 0 \end{cases} .$$

  With this notation we have

(∗) $\qquad \|i_{\delta_1,\delta_2} x\|_{\delta_1} \leq \|x\|_{\delta_2} \ \ \forall x \in D(i_{\delta_1,\delta_2})$

(∗∗) $\qquad i_{\delta_1,\delta_2} i_{\delta_2,\delta_3} = i_{\delta_1,\delta_3}$ if $\delta_1 \leq \delta_2 \leq \delta_3$.

  Furthermore

  $$\langle x, y\rangle_\delta = \langle N^\delta i_{0,\delta} x, N^\delta i_{0,\delta} y\rangle_0 \text{ for } \delta \geq 0,\ \forall x, y \in \mathscr{H}^\delta$$

  $$\langle i_{\delta,0} x, i_{\delta,0} y\rangle_\delta = \langle N^\delta x, N^\delta y\rangle_0 \text{ for } \delta < 0,\ \forall x, y \in \mathscr{H}^0 \ .$$

  Consider now i_{δ_1,δ_2} with $\delta_1 < \delta_2 < 0$. Then $D(i_{\delta_1,\delta_2}) = \mathscr{H}$ with $\mathscr{H}$ dense in $\mathscr{H}^{\delta_2}$. Equation (∗) implies that i_{δ_1,δ_2} can be extended in a unique way to a continuous application i'_{δ_1,δ_2} with $D(i'_{\delta_1,\delta_2}) = \mathscr{H}^{\delta_2}$ and values in $\mathscr{H}^{\delta_1}$. i'_{δ_1,δ_2} verifies

(∗′) $\qquad \left\|i'_{\delta_1,\delta_2} x\right\|_{\delta_1} \leq \|x\|_{\delta_2} \ \ \forall x \in \mathscr{H}^{\delta_2}$.

We can now prove i'_{δ_1,δ_2} is injective. Let $i'_{\delta_1,\delta_2}x = 0$ with $x \in \mathscr{H}^{\delta_2}$. Then exists $\{x_n\}_{n\in\mathbb{N}} \in \mathscr{H} \subset \mathscr{H}^{\delta_2}$ such that $x_n \to x$ when $n \to \infty$ in $\mathscr{H}^{\delta_2}$ and $i'_{\delta_1,\delta_2}x_n \to 0$ in $\mathscr{H}^{\delta_1}$. So $\left\|N^{\delta_1}x\right\|_0 \to 0$, $\left\|N^{\delta_2}x_n\right\|_0 \to \|x\|_{\delta_2}$ and $\left\|N^{\delta_2}x_n - N^{\delta_1}x_m\right\|_0 \to 0$ when $n, m \to \infty$. If we define $y_n = N^{\delta_1}x_n$ we can rewrite: $\|y_n\|_0 \to 0$, $\left\|N^{\delta_2-\delta_1}y_n\right\| \to \|x\|_{\delta_2}$ and

$$\left\|N^{\delta_2-\delta_1}y_n - N^{\delta_2-\delta_1}y_m\right\|_0 \to 0 \ .$$

Since $N^{\delta_2-\delta_1}$ is a closed operator, $\mathscr{H}-\lim N^{\delta_2-\delta_1}y_n = 0$ so $x = 0$.

Furthermore equation $(**)$ can be extended to i', since from $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x$ in $\mathscr{H}^0$ it follows that $i_{\delta',\delta}x_n$ converges in $\mathscr{H}^{\delta'}$ for all $\delta' < \delta$:

$$(**') \qquad i'_{\delta_1,\delta_2}i'_{\delta_2,\delta_3} = i'_{\delta_1,\delta_3} \ .$$

- Let $\delta > 0$. There is a natural isometry between $\mathscr{H}^{-\delta}$ and $(\mathscr{H}^{\delta})^*$, where the latter is the vector space of antilinear continuous functionals on $\mathscr{H}^{\delta}$. To a fixed $y \in \mathscr{H}^0$, the application

$$\mathscr{H}^{\delta} \ni x \longrightarrow \langle i_{0,\delta}x, y\rangle_0$$

defines an element of $(\mathscr{H}^{\delta})^*$ by

$$\langle i_{0,\delta}x, y\rangle_0 = \langle x, i^*_{0,\delta}y\rangle_{\mathscr{H}^{\delta},(\mathscr{H}^{\delta})^*}$$

where $i^*_{0,\delta}$ is linear and continuous from $\mathscr{H}$ to $(\mathscr{H}^{\delta})^*$. Furthermore since $N^{\delta}D(N^{\delta}) = \mathscr{H}$:

$$\left\|i^*_{0,\delta}y\right\|_{(\mathscr{H}^{\delta})^*} = \sup_{x\in\mathscr{H}^{\delta}} \frac{\left|\langle x, i^*_{0,\delta}y\rangle_{\mathscr{H}^{\delta},(\mathscr{H}^{\delta})^*}\right|}{\|x\|_{\delta}} = \sup_{x\in\mathscr{H}^{\delta}} \frac{\left|\langle N^{\delta}i_{0,\delta}x, N^{-\delta}y\rangle_0\right|}{\|N^{\delta}i_{o,\delta}x\|_0} = \left\|N^{-\delta}y\right\|_0 \ .$$

Then it follows that $i^*_{0,\delta}$ is injective from $\mathscr{H}$ in $(\mathscr{H}^{\delta})^*$ and to $\mathrm{Ran}(i^*_{0,\delta})$ can be given a structure of pre-Hilbert space with scalar product

$$\langle i^*_{0,\delta}x, i^*_{0,\delta}y\rangle_{(\mathscr{H}^{\delta})^*} = \langle N^{-\delta}x, N^{-\delta}y\rangle_0 \ .$$

$\mathrm{Ran}(i^*_{0,\delta})$ is dense in $(\mathscr{H}^{\delta})^*$, since $\langle x, i^*_{0,\delta}y\rangle_{\mathscr{H}^{\delta},(\mathscr{H}^{\delta})^*} = 0$ for all $y \in \mathscr{H}^0$ implies $x = 0$. Because

$$\langle i_{-\delta,0}x, i_{-\delta,0}y\rangle_{-\delta} = \langle N^{-\delta}x, N^{-\delta}y\rangle_0 \ \forall x, y \in \mathscr{H} \ ,$$

$i^*_{0,\delta}i^{-1}_{-\delta,0}$ preserves the scalar product between $\mathrm{Ran}(i_{-\delta,0})$ (dense in $\mathscr{H}^{-\delta}$) and $\mathrm{Ran}(i^*_{0,\delta})$ (dense in $(\mathscr{H}^{\delta})^*$). This isometry is extended in a unique way to an isometry between $\mathscr{H}^{-\delta}$ and $(\mathscr{H}^{\delta})^*$, so they can be identified.
- All isometries i, i', i^* are treated as the identity operator (being coherent between one another).

2. Operators on the scale of spaces. Interpolation.

- We will call $D_0 = \mathscr{C}_0(N)$ the set of vectors Φ such that $\exists a = a(\Phi)$ implying $\mathrm{d}\,\|P_\lambda\Phi\| = 0$ for $\lambda > a(\Phi)$, where P_λ is the spectral family of projectors of N.
- Let $\delta > 0$. T_0 a linear operator defined on D_0 with $T_0D_0 \subset D(N^{\delta})$ and $\exists c_\delta \geq 0$ such that

$$(\dagger) \qquad \|T_0\Phi\|_{\delta} \leq c_\delta \|\Phi\|_{\delta} \ \ \forall \Phi \in D_0 \ .$$

The biggest admissible domain of definition for T_0 is $D = D(N^{\delta})$ and in that case we suppose $T_0D(N^{\delta}) \subset D(N^{\delta})$ and $(\dagger)$ for all $\Phi \in D$.

Let now $\delta \leq 0$. T_0 linear operator defined on D_0 and $\exists c_\delta$ such that (†) holds. In this case the biggest admissible domain of definition for T_0 is $D = \mathscr{H}$, and that (†) holds for all $\Phi \in D$.

It is evident that (†) is equivalent to

(†′) $$\left\| N^\delta T_0 N^{-\delta}\Phi \right\| \leq c_\delta \left\|\Phi\right\| \ \forall \Phi \in D_0 \ .$$

- Under the preceding assumptions it is clear that, for all $\delta \in \mathbb{R}$, T_0 has a unique continuous extension T_δ to all of $\mathscr{H}^\delta$ that satisfies

  $$\left\|T_\delta \Phi\right\|_\delta \leq c_\delta \left\|\Phi\right\|_\delta \ \forall \Phi \in \mathscr{H}^\delta \ .$$

  If $\delta < 0$, T_δ is defined on a set bigger than $\mathscr{H}$.

  If $\delta > 0$, T_δ is defined on $D(N^\delta) \subset \mathscr{H}$ with values in $D(N^\delta)$, and so we can write

  $$\left\|N^\delta T_\delta \Phi\right\| \leq c_\delta \left\|\Phi\right\| \ \forall \Phi \in D(N^\delta) \ .$$
- If (†) holds for δ_1 and δ_2, $\delta_1 < \delta_2$ then $T_{\delta_2} \subset T_{\delta_1}$. To prove this assertion, let $\Phi \in \mathscr{H}^{\delta_2}$ and $\{\Phi_j\} \in D_0$ such that $\mathscr{H}^{\delta_2} - \Phi_j \to \Phi$. Since the topology of $\mathscr{H}^{\delta_2}$ is stronger than that of $\mathscr{H}^{\delta_1}$, we have $\mathscr{H}^{\delta_1} - \Phi_j \to \Phi$. Then from $\mathscr{H}^{\delta_2} - T_0\Phi_j \to T_{\delta_2}\Phi$ and $\mathscr{H}^{\delta_1} - T_0\Phi_j \to T_{\delta_1}\Phi$ it follows that $T_{\delta_2}\Phi = T_{\delta_1}\Phi$.

Proposition 14 (Interpolation). *If T_0 satisfies (†) for δ_1 and δ_2 with $\delta_1 < \delta_2$, then for all δ such that $\delta_1 \leq \delta \leq \delta_2$ and for all $\Phi \in D_0$ we have that*

$$\left\|N^\delta T_0 \Phi\right\| \leq c_\delta \left\|N^\delta \Phi\right\|$$

with

$$c_\delta = c_{\delta_1}^{\frac{\delta_2-\delta}{\delta_2-\delta_1}} c_{\delta_2}^{\frac{\delta-\delta_1}{\delta_2-\delta_1}} \ .$$

Proof. We will use Hadamard three-lines theorem:

Lemma (Hadamard three-lines theorem). *Let $f(z) = f(x+iy)$ a function with values in a Banach space, bounded and continuous in the closed strip $\delta_1 \leq x \leq \delta_2$, $-\infty < y < +\infty$, analytic on its interior. Suppose that*

$$\left\|f(\delta_1 + iy)\right\| \leq c_{\delta_1} \ , \ \left\|f(\delta_2 + iy)\right\| \leq c_{\delta_2} \ .$$

Then, $\forall z$ in the closed strip defined above

$$\left\|f(z)\right\| \leq c_{\delta_1}^{\frac{\delta_2-\delta}{\delta_2-\delta_1}} c_{\delta_2}^{\frac{\delta-\delta_1}{\delta_2-\delta_1}} \ .$$

Here f is defined as following:

(††) $$f(z) = N^z T_0 N^{-z}\Phi \text{ with } \Phi \in D_0 \ .$$

Observe that $(z = x + iy)$

$$\left\|N^z\Phi\right\|^2 = \int \left|\lambda^z\right|^2 \mathrm{d}\left\|P_\lambda \Phi\right\|^2 = \int \lambda^{2x} \mathrm{d}\left\|P_\lambda\Phi\right\|^2 \ ,$$

for all $\Phi \in D(N^x)$, hence N^{iy} is a unitary operator. We will show that $f(z)$, as defined by (††), satisfies the hypotheses of Hadamard's theorem.

a) $f(z)$ is well defined and bounded on the closed strip.

Write

(† † †) $$f(z) = N^{z-\delta_2}(N^{\delta_2} T_0 N^{-\delta_2}) N^{-z+\delta_2}\Phi \ .$$

$N^{z-\delta_2}\Phi \in D_0$ because $\Phi \in D_0$ so

$$\left\|N^{-z+\delta_2}\Phi\right\|^2 = \int_1^{\mu(\Phi)} \lambda^{2(\delta_2 - x)} \mathrm{d}\left\|P_\lambda \Phi\right\|^2 \ , \forall x, y \in \mathbb{R}$$

where

$$\mu(\Phi) = \sup\{\lambda : \lambda \in \mathrm{Supp}\, \|P(\cdot)\Phi\|^2\} \ .$$

Hence for all $\delta_1 \le x \le \delta_2$

$$\left\|N^{\delta_2} T_0 N^{-\delta_2} N^{-z+\delta_2}\Phi\right\|^2 \le c_{\delta_2} \int_1^{\mu(\Phi)} \lambda^{2(\delta_2-\delta_1)} \mathrm{d}\, \|P_\lambda \Phi\|^2 \ .$$

On the other hand $N^{z-\delta_2}$ is a bounded operator when z is in the closed strip:

$$\left\|N^{z-\delta_2}\Psi\right\|^2 = \int_1^\infty \lambda^{2(x-\delta_2)} \mathrm{d}\, \|P_\lambda \Psi\|^2 \le \|\Psi\|^2 \ .$$

From the inequalities above and († † †) it follows that f is bounded on the closed strip. Furthermore for $j = 1, 2$

$$f(\delta_j + iy) = N^{\delta_j + iy} T_0 N^{-\delta_j - iy}\Phi$$

hence

$$\|f(\delta_j + iy)\| = \left\|N^{\delta_j} T_0 N^{-\delta_j} N^{-iy}\Phi\right\| \le c_{\delta_j} \left\|N^{-iy}\Phi\right\| = c_{\delta_j} \|\Phi\| \ .$$

b) $f(z)$ is continuous on the closed strip.

We will use $f(z)$ in the form written in († † †). $N^{-z+\delta_2}\Phi$ is continuous in $z \in \mathbb{C}$:

$$\left\|(N^{-z+\delta_2} - N^{-z'+\delta_2})\Phi\right\|^2 = \int_1^{\mu(\Phi)} \lambda^{2\delta_2} \left|\lambda^{-z} - \lambda^{-z'}\right|^2 \mathrm{d}\, \|P_\lambda \Phi\|^2$$

that goes to zero when $z' \to z$ by Lebesgue's theorem. On the other hand

$$\left\|(N^{z-\delta_2} - N^{z'-\delta_2})\Psi\right\|^2 = \int_1^\infty \lambda^{2(\mathrm{Re}z-\delta_2)} \left|1 - \lambda^{z'-z}\right|^2 \mathrm{d}\, \|P_\lambda \Psi\|^2$$

for all $\Psi \in \mathscr{H}$, that goes to zero when $z' \to z$ in the closed strip by Lebesgue's theorem. Hence from († † †) it follows the continuity of $f(z)$.

c) $f(z)$ is analytic on the open strip.

We recall that strong analyticity is equivalent to weak analyticity. Observe that

$$\langle \Psi, N^{-z+\delta_2}\Phi\rangle = \int_1^{\mu(\Phi)} \lambda^{-z+\delta_2} \mathrm{d}\langle \Psi, P(\lambda)\Phi\rangle, \ \forall \Phi \in D_0, \ \forall \Psi \in \mathscr{H}$$

is analytic for $z \in \mathbb{C}$. Analogously

$$\langle \Psi, N^{z-\delta_2}\Theta\rangle = \int_1^\infty \lambda^{z-\delta_2} \mathrm{d}\langle \Psi, P(\lambda)\Theta\rangle, \ \forall \Psi, \Theta \in \mathscr{H}$$

is analytic for $\mathrm{Re}z < \delta_2$. Naming $z - \delta_2 = \tilde{z}$ and $N^{-\delta_2} T_0 N^{\delta_2} = \widetilde{T}_0$, we write using († † †)

$$\begin{aligned} \frac{1}{h}(f(z+h) - f(z)) &= \frac{1}{h}(N^{\tilde{z}+h}\widetilde{T}_0 N^{-\tilde{z}-h}\Phi - N^{\tilde{z}}\widetilde{T}_0 N^{-\tilde{z}}\Phi) \\ &= \frac{1}{h}(N^{\tilde{z}+h} - N^{\tilde{z}})\widetilde{T}_0 N^{-\tilde{z}}\Phi + N^{\tilde{z}+h}\widetilde{T}_0 \frac{1}{h}(N^{-\tilde{z}-h} - N^{-\tilde{z}})\Phi \end{aligned}$$

and both terms of the right hand side converge, when $h \to 0$, in $\mathscr{H}$ as a consequence of the above analyticities.

■

APPENDIX D

Exponentials of operators written as power series.

1. Preliminaries.

Definition ($f(\delta, N)$).

$$f(\delta, N) = \Big[\frac{(N+\delta)!}{N!}\Big]^{1/2} ,$$

$N, \delta \in \mathbb{N}$ and $0! = 1$.

Lemma 1.1. *$\forall \varepsilon > 0$, we have that*

$$\sup_{N\geq 0} f(\delta, N) e^{-\varepsilon N} \leq e^{\varepsilon}(\delta!)^{1/2}(1-e^{-2\varepsilon})^{-\delta/2} .$$

Proof.

$$\sup_{N\geq 0} f(\delta, N) e^{-\varepsilon N} = \Big\{\sup_{N\geq 0} u_N\Big\}^{1/2}$$

with $u_N = (N+1)(N+2)\cdots(N+\delta)e^{-2\varepsilon N}$. The fraction

$$\frac{u_N}{u_{N-1}} = \frac{N+\delta}{N} e^{-2\varepsilon} = (1+\delta/N)e^{-2\varepsilon}$$

with $N \geq 1$ is decreasing in N.

Suppose that exists $N_0 \geq 1$ such that

$$\frac{u_{N_0}}{u_{N_0-1}} \geq 1 > \frac{u_{N_0+1}}{u_{N_0}} .$$

When $1 \leq N \leq N_0$ we have

$$\frac{u_N}{u_{N-1}} \geq \frac{u_{N_0}}{u_{N_0-1}} \geq 1$$

and when $N_0 < N$ we have

$$1 > \frac{u_{N_0+1}}{u_{N_0}} > \frac{u_{N+1}}{u_N} .$$

$$u_{N_0} > u_{N_0-1} > \cdots > u_0$$
$$u_{N_0} > u_{N_0+1} > \cdots > u_0 ;$$

so $\sup_N u_N = u_{N_0}$.

If for all $1 \leq N$, $u_N/u_{N-1} < 1$ then $\sup_N u_N = u_0$. Let $\gamma = e^{2\varepsilon} - 1$, so that

$$\frac{u_N}{u_{N-1}} = \frac{1+\delta/N}{1+\gamma} .$$

If $\delta/\gamma \geq 1$ then $\exists N_0 \geq 1$ such that $1 \leq N_0 \leq \delta/\gamma < N_0 + 1$. If $\delta/\gamma < 1$ we are in the situation $u_N < \ldots < u_0$ for all N. In every case we have

$$\sup_{N\geq 0} u_N = u_{N_0} \text{ with } N_0 = [\delta/\gamma] \text{ ([ ] stands for the integer part).}$$

So we can bound

$$u_{N_0} = ([\delta/\gamma]+1)\cdots([\delta/\gamma]+\delta)e^{-2\varepsilon[\delta/\gamma]} = \delta! \begin{pmatrix} +\delta \\ \delta \end{pmatrix} e^{-2\varepsilon[\delta/\gamma]}$$

$$\le \delta! \frac{(1+x)^{[\delta/\gamma]+\delta}}{x^\delta} e^{-2\varepsilon[\delta/\gamma]} \ \forall x \ge 0 \ .$$

$$u_{N_0} \le \delta! e^{-2\varepsilon[\frac{\delta}{\gamma}]} \inf_{x>0} \Big(\frac{(1+x)^{1+[\frac{\delta}{\gamma}]/\delta}}{x} \Big)^\delta \le \delta! e^{2\varepsilon} e^{-2\varepsilon \frac{\delta}{\gamma}} \inf_{x>0} \Big(\frac{(1+x)^{1+1/\gamma}}{x} \Big)^\delta$$

$$= \delta! e^{2\varepsilon} e^{-2\varepsilon\frac{\delta}{\gamma}} \Big(\frac{(1+\gamma)^{1+1/\gamma}}{\gamma} \Big)^\delta = \delta! e^{2\varepsilon}(1-e^{-2\varepsilon})^{-\delta} \ .$$

■

2. Exponentials of creation and annihilation operators as series.

If A is an operator on a symmetric Fock space over $L^2(\mathbb{R}^3)$. We define formally

$$\exp A = \sum_{m=0}^{\infty} \frac{A^m}{m!} \ .$$

Rigorously, for all $\Phi \in \bigcup_{m=0}^{\infty} D(A^m)$, so that

$$\exists \lim_{l\to\infty} \sum_{m=0}^{l} A^m \Phi/m! \ ,$$

we define

$$(\exp A)\Phi = \sum_{m=0}^{\infty} \frac{A^m}{m!}\Phi \ .$$

The set defined above is the natural domain of definition for $\exp A$, however it is useful to work with the subset

$$\Big\{ \Phi : \sum_{m=0}^{\infty} \frac{\|A^m\Phi\|}{m!} < \infty \Big\} \equiv D(\exp A) \ .$$

Consider now the case $A = a^*(f_1) + a(f_2)$, $f_1, f_2 \in L^2(\mathbb{R}^3)$.

LEMMA 2.1. *$\forall \varepsilon > 0$, $D(\exp \varepsilon N) \subset D(\exp A)$ and for all $\Phi \in D(\exp \varepsilon N)$*

$$\|\exp(a^*(f_1)+a(f_2))\Phi\| \le e^{\varepsilon} \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} (\|f_1\|_2 + \|f_2\|_2)^m (1-e^{-2\varepsilon})^{-m/2} \|(\exp \varepsilon N)\Phi\| \ .$$

PROOF. We estimate

$$\|(a^*(f_1)+a(f_2))^m\Phi\| \le \prod_{j=0}^{m-1} \big\| f(j)(a^*(f_1)+a(f_2))f(j+1)^{-1} \big\| \, \|f(m)\Phi\| \ ,$$

where $f(j) = f(j,N)$. However

$$f(j,N)a^*(f_1)f(j+1,N)^{-1} = a^*(f_1)f(j,N+1)f(j+1,N)^{-1}$$

$$= a^*(f_1)(N+1)^{-1/2}$$

$$f(j,N)a(f_2)f(j+1,N)^{-1} = a(f_2)f(j,N-1)f(j+1,N)^{-1}$$

$$= a(f_2)\Big[\frac{N}{(j+N)(j+N+1)}\Big]^{1/2} \ ,$$

and

$$\left\| a^*(f_1)(N+1)^{-1/2} \right\| \leq \|f_1\|_2$$
$$\left\| a(f_2)\Big[\frac{N}{(j+N)(j+N+1)}\Big]^{1/2} \right\| \leq \|f_2\|_2 \ .$$

So we have that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left\| (a^*(f_1) + a(f_2))^m \Phi \right\| \leq \sum_{m=0}^{\infty} \frac{1}{m!} (\|f_1\|_2 + \|f_2\|_2)^m \left\| f(m,N) e^{-\varepsilon N} \right\| \left\| e^{\varepsilon N} \Phi \right\|$$

hence the result using Lemma 1.1. ■

It is clear that if $A = a^*(f_1) + a(f_2)$ and $\Phi \in D(\exp \varepsilon N)$, then exists

$$\frac{\mathrm{d}}{\mathrm{d}x} (\exp Ax)\Phi = A(\exp Ax)\Phi = (\exp Ax)A\Phi \ .$$

The calculation is easily done using the series. This leads to the following lemma:

Lemma 2.2. *Let $A = a^*(f_1) + a(f_2)$, $B = a^*(g_1) + a(g_2)$, $f_j, g_j \in L^2(\mathbb{R}^3)$ if $j = 1, 2$, then for all $\varepsilon > 0$, $\Phi \in D(\exp \varepsilon N)$:*

$$\exp(A+B)\Phi = (\exp A)(\exp B)\exp(-[A,B]/2)\Phi \ .$$

Proof. First of all we remark that $[A, B]$ is a number, then commutes with A and B.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\Big\{ \exp(-[A,B]x^2/2)\exp(-(A+B)x)\exp(Ax)\exp(Bx)\Phi - \Phi \Big\} \\ &= \exp(-[A,B]x^2/2)\exp(-(A+B)x)\Big\{ -B - [A,B]x \Big\} \exp(Ax)\exp(Bx)\Phi \\ &\quad + \exp(-[A,B]x^2/2)\exp(-(A+B)x)\exp(Ax)B\exp(-Ax) \\ &\quad \exp(Ax)\exp(Bx)\Phi = 0 \end{aligned}$$

and we used Lemma 5.3 of Appendix A on the second term of the right hand side of the above equality. The result follows immediately. ■

Remark. If $f_2 = -\bar{f}_1$, iA is essentially self-adjoint. Then we can define the unitary operator $\exp(\overline{A})$ by Stone's theorem. For all $\Phi \in D(\exp A)$, $(\exp(\overline{A}))\Phi = (\exp A)\Phi$, where the latter is the exponential defined above using the series.

Corollary. *If $f_2 = 0$, $g_1 = 0$, $g_2 = -\bar{f}_1$, then for all $\varepsilon > 0$, $\Phi \in D(\exp \varepsilon N)$:*

$$\exp\big(\overline{(a^*(f_1) - a(\bar{f}_1))}\big)\Phi = e^{-\|f_1\|_2^2/2} \exp(a^*(f_1))\exp(-a(\bar{f}_1))\Phi \ .$$

APPENDIX E

Proof of Theorem 3.

We will prove the theorem for vectors Λ and Ψ. For vectors Θ the proof is very similar to the case Ψ, with a little bit of care to deal with θ dependence and integration. However since θ appears only in $\|\alpha_\theta(t)\|_2$ for some finite t, and the solution of (E) is continuous in $L^2(\mathbb{R}^3)$ with respect to a change of initial data in $L^2(\mathbb{R}^3)$, then $\|\alpha_\theta(t)\|_2$ is continuous in θ, and integration on a finite θ-interval is well-defined.

1. General remarks.

We recall the definition of operator B and transition amplitudes $\langle B\rangle_X(t)$, with $X \in \{\Lambda, \Psi\}$.

Definition. Let $q, r, i, j \in \mathbb{N}$, $\delta = q+r+i+j$, $g \in L^2(\mathbb{R}^{3(q+r)}) \otimes L^2(\mathbb{R}^{3(i+j)}) \equiv L^2(\mathbb{R}^{3\delta})$. Then

$$B = \int \mathrm{d}X_q \mathrm{d}Y_r \mathrm{d}K_i \mathrm{d}M_j \, \bar{g}(X_q, Y_r, K_i, M_j) \psi^*(X_q) \psi(Y_r) a^*(K_i) a(M_j) \; .$$

Also we define the following transition amplitudes:

$$\langle B\rangle_\Lambda(t) \equiv \lambda^\delta \langle \Lambda(t), B\Lambda(t)\rangle = \lambda^\delta \langle \Lambda, U^\dagger(t) B U(t) \Lambda\rangle$$
$$\langle B\rangle_\Psi(t) \equiv \lambda^\delta \langle \Psi(t), B\Psi(t)\rangle = \lambda^\delta \langle \Psi, U^\dagger(t) B U(t) \Psi\rangle \; .$$

Since, as we proved in Proposition 12, $\langle \psi^*(q)\psi(r)a^*(i)a(j)\rangle_X(t) \in L^2(\mathbb{R}^{3\delta})$, for $X \in \{\Lambda, \Psi\}$, it would be sufficient to bound $\left|\langle B\rangle_X(t) - \langle g, \bar{u}^{\otimes_q} u^{\otimes_r} \bar{\alpha}^{\otimes_i} \alpha^{\otimes_j}\rangle_{L^2(\mathbb{R}^{3\delta})}\right|$, then the results of the theorem follows immediately applying Riesz's Lemma.

Definition ($B^{(d)}$). Let B be defined as above, $0 \leq d \leq \delta$. We establish the following correspondence:

$$\psi(x) \longleftrightarrow \frac{1}{\lambda} u_t(x) \; ,$$
$$\psi^*(x) \longleftrightarrow \frac{1}{\lambda} \bar{u}_t(x) \; ,$$
$$a(k) \longleftrightarrow \frac{1}{\lambda} \alpha_t(k) \; ,$$
$$a^*(k) \longleftrightarrow \frac{1}{\lambda} \bar{\alpha}_t(k) \; .$$

We will call $B^{(d)}$ the operator obtained substituting in any possible way d creation or annihilation operators of B with functions, following the correspondence above.

$B^{(d)}$ is the sum of $\binom{\delta}{d}$ operators of type B, but with $\delta - d$ creation or annihilation operators. So we can formulate the following Lemma:

Lemma 1.1. *Let $B^{(d)}$ as above. Then for any $0 \leq d \leq \delta$ exists a function $C_d(t)$, depending on $\|u(t)\|_2$ and $\|\alpha(t)\|_2$, such that for all $\Phi \in \mathscr{H}^\delta$ we have the following* *inequality:*

$$\begin{aligned}\left\|B^{(d)}\Phi\right\| &\leq \lambda^{-d}C_d(t)\left\|g;L^2(\mathbb{R}^{3\delta})\right\|\left\|(P+N+q+i)^{\delta/2}\Phi\right\| \\ &\leq \lambda^{-d}(q+i)^{\delta/2}C_d(t)\left\|g;L^2(\mathbb{R}^{3\delta})\right\|\left\|\Phi\right\|_\delta \ .\end{aligned}$$

Proof. The proof utilizes Lemma 2.1 of Chapter 3, and the fact that for any r and j such that $q+r+i+j\leq\delta$

$$\frac{\sqrt{P!(P+q-r)!N!(N+i-j)!}}{(P-r)!(N-j)!}\theta(P-r)\theta(N-j)\leq(P+N+q+i)^{\delta/2} \ . \tag{1.1}$$

Let's see it in detail for $B^{(1)}$, the other cases being similar. We have that

$$\begin{aligned}B^{(1)} = \frac{1}{\lambda}\Big(\int \mathrm{d}X_q\mathrm{d}Y_r\mathrm{d}K_i\mathrm{d}M_j\ \bar{u}(t,x_1)g(X_q,Y_r,K_i,M_j)\psi^*(X_q\setminus x_1) \\ \psi(Y_r)a^*(K_i)a(M_j)+\dots\Big) \ .\end{aligned}$$

Consider the first term, written explicitly. Using Lemma 2.1 of Chapter 3 and equation (1.1) we obtain, for all $\Phi\in\mathscr{H}^\delta$

$$\left\|B^{(1)}\Phi\right\| \leq \frac{1}{\lambda}\left\|\int \mathrm{d}x_1\ \bar{u}(t,x_1)g(x_1,\cdot);L^2(\mathbb{R}^{3(\delta-1)})\right\|\left\|(P+N+q+i)^{\delta/2}\Phi\right\| \\ +\dots \ ;$$

where the L^2-norm is intended on the $3\delta-3$ variables of g excluded x_1. Using now Schwarz's inequality and the fact that $(P+N+q+i)^{\delta/2}\leq(q+i)^{\delta/2}(P+N+1)^{\delta/2}$ we obtain

$$\left\|B^{(1)}\Phi\right\| \leq \frac{1}{\lambda}(q+i)^{\delta/2}\left\|u(t)\right\|_2\left\|g;L^2(\mathbb{R}^{3\delta})\right\|\left\|\Phi\right\|_\delta+\dots \ .$$

So $C_d(t)$ would be in general the sum of products of the L^2-norms of $u(t)$ and $\alpha(t)$, and

$$C_1(t) = (q+r)\left\|u(t)\right\|_2+(i+j)\left\|\alpha(t)\right\|_2 \ .$$

■

Lemma 1.2. *For all $\Phi\in\mathscr{H}^\delta$ the following identity holds:*

$$B'\Phi\equiv C^\dagger(u(t)/\lambda,\alpha(t)/\lambda)BC(u(t)/\lambda,\alpha(t)/\lambda)\Phi=\sum_{d=0}^{\delta}B^{(d)}\Phi \ .$$

Proof. The result is a restatement of Lemma 3.1 of Chapter 8. ■

Lemma 1.3. *Let $\varphi(\mathbf{g})$ defined as in Proposition 9. Then for all $\Phi\in\mathscr{H}^\delta$ the following equality holds:*

$$B^{(\delta-1)}\Phi = \lambda^{-\delta+1}\varphi(\mathbf{g})\Phi \ ,$$

with

$$\begin{aligned}
g_1(x) =& \sum_{\alpha=1}^{q} \int \mathrm{d}(X_q \setminus x_\alpha)\mathrm{d}Y_r\mathrm{d}K_i\mathrm{d}M_j\, g(\ldots, x_{\alpha-1}, x, x_{\alpha+1}, \ldots, Y_r, K_i, M_j) \\
& \bar{u}_t^{\otimes_{q-1}}(X_q \setminus x_\alpha)u_t^{\otimes_r}(Y_r)\bar{\alpha}_t^{\otimes_i}(K_i)\alpha_t^{\otimes_j}(M_j)\ , \\
g_2(x) =& \sum_{\alpha=1}^{r} \int \mathrm{d}X_q\mathrm{d}(Y_r \setminus y_\alpha)\mathrm{d}K_i\mathrm{d}M_j\, g(X_q, \ldots, y_{\alpha-1}, y, y_{\alpha+1}, \ldots, K_i, M_j) \\
& \bar{u}_t^{\otimes_q}(X_q)u_t^{\otimes_{r-1}}(Y_r \setminus y_\alpha)\bar{\alpha}_t^{\otimes_i}(K_i)\alpha_t^{\otimes_j}(K_j)\ , \\
g_3(k) =& \sum_{\alpha=1}^{i} \int \mathrm{d}X_q\mathrm{d}X_r\mathrm{d}(K_i \setminus k_\alpha)\mathrm{d}M_j\, g(X_q, Y_r, \ldots, k_{\alpha-1}, k, k_{\alpha+1}, \ldots, M_j) \\
& \bar{u}_t^{\otimes_q}(X_q)u_t^{\otimes_r}(Y_r)\bar{\alpha}_t^{\otimes_{i-1}}(K_i \setminus k_\alpha)\alpha_t^{\otimes_j}(M_j)\ , \\
g_4(k) =& \sum_{\alpha=1}^{j} \int \mathrm{d}X_q\mathrm{d}Y_r\mathrm{d}K_i\mathrm{d}(M_j \setminus m_\alpha)\, g(X_q, Y_r, K_i, \ldots, m_{\alpha-1}, k, m_{\alpha+1}, \ldots) \\
& \bar{u}_t^{\otimes_q}(X_q)u_t^{\otimes_r}(Y_r)\bar{\alpha}_t^{\otimes_i}(K_i)\alpha_t^{\otimes_{j-1}}(M_j \setminus m_\alpha)\ .
\end{aligned}$$

Proof. By definition of $B^{(\delta-1)}$. We restricted the result to $\mathscr{H}^\delta$ since for our purposes $B^{(\delta-1)}$ will be applied only to such vectors. ■

2. The proof for Λ vectors.

Proposition 15. *Two constants $K_j(\Lambda)$ with $j = 1, 2$ exist such that for all $g \in L^2(\mathbb{R}^{3\delta})$*

$$\left|\langle B\rangle_\Lambda(t) - \langle g, \bar{u}^{\otimes_q}u^{\otimes_r}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})}\right| \leq \lambda^2 \left\|g; L^2(\mathbb{R}^{3\delta})\right\| K_1(\Lambda)\, |t|\, e^{K_2(\Lambda)|t|}\ .$$

Proof. Write explicitly $\langle B\rangle_\Lambda(t) = \lambda^\delta\langle\Lambda(t), B\Lambda(t)\rangle$:

$$\langle B\rangle_\Lambda(t) = \lambda^\delta\langle\Lambda, U^*(t)BU(t)C(u_0/\lambda, \alpha_0/\lambda)\Omega\rangle\ ;$$

that could be reformulated to show a dependence on $W(t, s)$:

$$\begin{aligned}
\langle B\rangle_\Lambda(t) &= \lambda^\delta\langle\Lambda, C(u_0/\lambda, \alpha_0/\lambda)W^\dagger(t, 0)B'W(t, 0)\Omega\rangle \\
&= \lambda^\delta\langle\Omega, W^\dagger(t, 0)B'W(t, 0)\Omega\rangle\ ,
\end{aligned}$$

where $B' = C^*(u_t/\lambda, \alpha_t/\lambda)BC(u_t/\lambda, \alpha_t/\lambda)$. Then by Lemma 1.2 we obtain

$$\langle B\rangle_\Lambda(t) - \langle g, \bar{u}^{\otimes_q}u^{\otimes_r}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})} = \sum_{d=0}^{\delta-1}\lambda^\delta\langle\Omega, W^\dagger(t, 0)B^{(d)}W(t, 0)\Omega\rangle.$$

Using Lemma 1.1 for $0 \leq d \leq \delta - 2$ we obtain

$$\begin{aligned}
\left|\langle B\rangle_\Lambda(t) - \langle g, \bar{u}^{\otimes_q}u^{\otimes_r}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})}\right| \leq& \sum_{d=0}^{\delta-2}\lambda^{\delta-d}(q+i)^{\delta/2}C_d(t) \\
& \left\|g; L^2(\mathbb{R}^{3\delta})\right\| \left\|W(t, 0)\Omega\right\|_\delta + \lambda^\delta \left|\langle\Omega, W^\dagger(t, 0)B^{(\delta-1)}W(t, 0)\Omega\rangle\right|\ .
\end{aligned}$$

We are interested in the region where $\lambda < 1$, so $\lambda^a \leq \lambda^2$ for any $a \geq 2$, and we know that $\|\Omega\|_\delta = 1$ for any δ. So we can apply the Corollary of Proposition 8 and the considerations above to write

$$\begin{aligned}
\left|\langle B\rangle_\Lambda(t) - \langle g, \bar{u}^{\otimes_q}u^{\otimes_r}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})}\right| \leq& \lambda^2 \|g\|_2 K_1 |t|\, e^{K_2|t|} \\
& + \lambda^\delta \left|\langle\Omega, W^\dagger(t, 0)B^{(\delta-1)}W(t, 0)\Omega\rangle\right|\ ,
\end{aligned}$$

with

$$K_1 \, |t| \, e^{K_2|t|} \geq \sum_{d=0}^{\delta-2} (q+i)^{\delta/2} C_d(t) \cdot \sup_{\lambda<1} \left[\Big(\mathcal{K}_1(t,0) + \lambda \mathcal{K}_2(t,0) \Big) e^{\lambda \mathcal{C}_1 |t| + \mathcal{K}_3(t,0)} \right]^{1/2} ;$$

the calligraphic functions and constants are the ones defined in Proposition 8.

We have to use a different approach to estimate the last term of the inequality above, namely

$$X \equiv \lambda^{\delta} \left| \langle \Omega, W^{\dagger}(t,0) B^{(\delta-1)} W(t,0)\Omega \rangle \right| \; ;$$

because the procedure just described would lead to a bound by λ instead of λ^2. Using Lemma 1.3 we obtain

$$X = \lambda \left| \langle \Omega, W^{\dagger}(t,0) \varphi(\mathbf{g}) W(t,0)\Omega \rangle \right| \; .$$

Now we pass to the interaction representation since we will need to differentiate, so

$$X = \lambda \left| \langle \Omega, \widetilde{W}^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0)\Omega \rangle \right| \; ,$$

with

$$\begin{aligned}
\tilde{g}_1(x) &= U_{01}^{\dagger}(t) g_1(x) \; , \\
\tilde{g}_2(x) &= U_{01}(t) g_2(x) \; , \\
\tilde{g}_3(k) &= U_{02}^{\dagger}(t) g_3(k) \; , \\
\tilde{g}_4(k) &= U_{02}(t) g_4(x) \; .
\end{aligned}$$

Then, using the following identity

$$\begin{aligned}
\langle \Omega, \widetilde{W}^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0)\Omega \rangle = \langle \Omega, \widetilde{U}_2^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) \widetilde{U}_2(t,0)\Omega \rangle \\
+ \langle \Omega, (\widetilde{W}^{\dagger}(t,0) - \widetilde{U}_2^{\dagger}(t,0)) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0)\Omega \rangle \\
+ \langle \Omega, \widetilde{U}_2^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) (\widetilde{W}(t,0) - \widetilde{U}_2(t,0))\Omega \rangle \; ,
\end{aligned}$$

and Proposition 9 we obtain

$$\begin{aligned}
X \leq & \lambda \Big(\left| \langle \Omega, (\widetilde{W}^{\dagger}(t,0) - \widetilde{U}_2^{\dagger}(t,0)) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0)\Omega \rangle \right| \\
& + \left| \langle \Omega, \widetilde{U}_2^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) (\widetilde{W}(t,0) - \widetilde{U}_2(t,0))\Omega \rangle \right| \Big) \equiv \lambda (X_1 + X_2) \; .
\end{aligned}$$

We define $\|\mathbf{g}\|_2 = \|g_1\|_2 + \|g_2\|_2 + \|g_3\|_2 + \|g_4\|_2$. To bound X_1 we proceed as follows, every term being well defined due to the properties of $\widetilde{W}(t,s)$ and $\widetilde{U}_2(t,s)$, and the integrals making sense as strong Riemann integrals on $\mathscr{H}$:

$$\begin{aligned}
X_1 &= \left| \langle \Big(1 - \widetilde{W}^{\dagger}(t,0) \widetilde{U}_2(t,0) \Big) \Omega, \widetilde{W}^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0)\Omega \rangle \right| \\
&\leq \left| \int_0^t \mathrm{d}\tau \, \left\| \widetilde{W}^{\dagger}(\tau,0) U_0^{\dagger}(\tau) H_I U_0(\tau) \widetilde{U}_2(\tau,0)\Omega \right\| \right| \left\| \widetilde{W}^{\dagger}(t,0) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0)\Omega \right\| \\
&\leq \lambda \, \|f_0\|_2 \, \|\tilde{\mathbf{g}}\|_2 \left\| \widetilde{W}(t,0)\Omega \right\|_{\mathscr{H}^1} \left| \int_0^t \mathrm{d}\tau \, \left\| \widetilde{U}_2(\tau,0)\Omega \right\|_{\mathscr{H}^4} \right| ;
\end{aligned}$$

where we used the standard estimates for H_I and $\varphi(\tilde{\mathbf{g}})$. We remark that $\|\tilde{\mathbf{g}}\|_2 = \|\mathbf{g}\|_2 \leq C_{\delta-1}(t) \left\| g; L^2(\mathbb{R}^{3\delta}) \right\|$, with $C_{\delta-1}(t)$ defined in Lemma 1.1. Now using Proposition 6, Proposition 8 and the fact that $\|\Omega\|_{\delta} = 1$ for any real δ, we obtain

$$X_1 \leq \lambda \, \|g\|_2 \, K_1' \, |t| \, e^{K_2'|t|} \; ,$$

with

$$K_1' \, |t| \, e^{K_2'|t|} \geq \|f_0\|_2 \, C_{\delta-1}(t) \sup_{\lambda<1} \Big[\Big(\mathcal{K}_1(\delta=1,t,0) + \lambda \mathcal{K}_2(\delta=1,t,0) \Big) e^{\lambda \mathcal{C}_1(\delta=1)|t| + \mathcal{K}_3(\delta=1,t,0)} \Big]^{1/2} \Big| \int_0^t \mathrm{d}\tau \, \exp\Big\{ 2\Big(\ln 3 + 10\sqrt{2} \Big| \int_0^\tau \mathrm{d}\tau' \, \|v_{--}(\tau')\|_2 \Big| \Big) \Big\} \Big| \, .$$

To bound X_2 we proceed in an analogous fashion:

$$\begin{aligned} X_2 &= \Big| \langle \widetilde{W}^\dagger(t,0) \varphi^\dagger(\bar{\tilde{\mathbf{g}}}) \widetilde{U}_2(t,0)\Omega, \Big(1 - \widetilde{W}^\dagger(t,0)\widetilde{U}_2(t,0)\Big)\Omega\rangle \Big| \\ &\leq \Big\| \varphi^\dagger(\bar{\tilde{\mathbf{g}}}) \widetilde{U}_2(t,0)\Omega \Big\| \, \Big| \int_0^t \mathrm{d}\tau \, \Big\| H_I U_0(\tau) \widetilde{U}_2(\tau,0)\Omega \Big\| \Big| \\ &\leq \lambda K_1'' \, |t| \, e^{K_2''|t|} \, \|g\|_2 \ , \end{aligned}$$

with

$$K_1'' \, |t| \, e^{K_2''|t|} \geq \|f_0\|_2 \, C_{\delta-1}(t) \exp\Big\{ \frac{1}{2}\Big(\ln 3 + 4\sqrt{2} \Big| \int_0^t \mathrm{d}\tau \, \|v_{--}(\tau)\|_2 \Big| \Big) \Big\} \Big| \int_0^t \mathrm{d}\tau \, \exp\Big\{ 2\Big(\ln 3 + 10\sqrt{2} \Big| \int_0^\tau \mathrm{d}\tau' \, \|v_{--}(\tau')\|_2 \Big| \Big) \Big\} \Big| \, .$$

■

3. The proof for Ψ vectors.

To improve readability we make the following definitions:

DEFINITIONS ($K_W(\delta,t,s)$, $K_U(\delta,t,s)$).

$$K_W(\delta,t,s) = \Big(\mathcal{K}_1(\delta,t,s) + \lambda \mathcal{K}_2(\delta,t,s) \Big) e^{\lambda \mathcal{C}_1(\delta)|t-s| + \mathcal{K}_3(\delta,t,s)}$$

where the functions and constants on the right hand side are defined in Proposition 8, with δ-dependence made explicit.

$$K_U(\delta,t,s) = \exp\Big\{ \frac{|\delta|}{2} \Big(\ln 3 + \sqrt{2}\rho_\delta \Big| \int_s^t \mathrm{d}\tau \, \|v_{--}(\tau)\|_2 \Big| \Big) \Big\} \ ,$$

with $\rho_\delta = \max(4, 3^{|\delta|/2} + 1)$.

PROPOSITION 16. *Two constants $K_j(\Psi)$ with $j = 1,2$ exist such that for all $g \in L^2(\mathbb{R}^{3\delta})$*

$$\Big| \langle B \rangle_\Psi(t) - \delta_{qr} \langle g, \bar{u}^{\otimes_q} u^{\otimes_r} \bar{\alpha}^{\otimes_i} \alpha^{\otimes_j}(t) \rangle_{L^2(\mathbb{R}^{3\delta})} \Big| \leq \delta_{qr} \lambda^2 \Big\| g; L^2(\mathbb{R}^{3\delta}) \Big\| \, K_1(\Psi) \, |t| \, e^{K_2(\Psi)|t|} \, .$$

PROOF. This proof is quite similar to the one of Proposition 15 above, so we will emphasize mostly the differences between the two. Write:

$$\langle B \rangle_\Psi(t) = \lambda^\delta \langle \Psi, U^\dagger(t) B U(t) \frac{\psi^*(u_0)^{\lambda^{-2}}}{\sqrt{\lambda^{-2}!}} C(\alpha_0/\lambda)\Omega \rangle \, .$$

Observe that when $q \neq r$ we have $\langle B \rangle_\Psi(t) = 0$ since P commutes with H and B doesn't preserve the number of non-relativistic particles. So we will set $q = r$ for the rest of the proof. Using Lemma 1.2 and Lemma 1.1 of Chapter 8 we can write:

$$\begin{aligned} \langle B \rangle_\Psi(t) &= \lambda^\delta d_{\lambda^{-2}} \langle \Psi, C(u_0/\lambda, \alpha_0/\lambda) W^\dagger(t,0) B' W(t,0)\Omega \rangle \\ &= \lambda^\delta d_{\lambda^{-2}} \langle \frac{1}{\sqrt{P+1}} C^\dagger(u_0/\lambda, \alpha_0/\lambda)\Psi, \sqrt{P+1} W^\dagger(t,0) B' W(t,0)\Omega \rangle \, . \end{aligned}$$

So we can write:

$$\langle B\rangle_\Psi(t)-\langle g,\bar{u}^{\otimes_q}u^{\otimes_q}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})}=\sum_{d=0}^{\delta-1}\lambda^{\delta-d}d_{\lambda^{-2}}$$
$$\langle\frac{1}{\sqrt{P+1}}C^\dagger(u_0/\lambda,\alpha_0/\lambda)\Psi,\sqrt{P+1}W^\dagger(t,0)B^{(d)}W(t,0)\Omega\rangle.$$

Using Lemma 1.2 of Chapter 8 for $0\leq d\leq\delta-2$ we obtain

$$\begin{aligned}&\left|\langle B\rangle_\Psi(t)-\langle g,\bar{u}^{\otimes_q}u^{\otimes_q}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})}\right|\\ &\leq\sum_{d=0}^{\delta-2}\lambda^{\delta-d}K_\Psi\left\|\sqrt{P+1}W^\dagger(t,0)B^{(d)}W(t,0)\Omega\right\|\\ &+\lambda^\delta d_{\lambda^{-2}}\left|\langle C^\dagger(u_0/\lambda,\alpha_0/\lambda)\Psi,W^\dagger(t,0)B^{(\delta-1)}W(t,0)\Omega\rangle\right|\ .\end{aligned}$$

We are interested in the region where $\lambda<1$, so $\lambda^a\leq\lambda^2$ for any $a\geq 2$, and we know that $\|\Omega\|_\delta=1$ for any δ. We can apply two times the Corollary of Proposition 8, Lemma 2.1 of Chapter 3 and Lemma 1.1:

$$\begin{aligned}\left\|\sqrt{P+1}W^\dagger(t,0)B^{(d)}W(t,0)\Omega\right\|&\leq K_W(1,t,0)\\ \left\|B^{(d)}(Q+1+q)^3W(t,0)\Omega\right\|&\leq(q+i)^{\delta/2}q^{1/2}C_d(t)K_W(1,t,0)\\ \left\|g;L^2(\mathbb{R}^{3\delta})\right\|\|W(t,0)\Omega\|_{\delta+6}&\leq(q+i)^{\delta/2}q^{1/2}C_d(t)K_W(1,t,0)\\ &\quad K_W(\delta+6,t,0)\left\|g;L^2(\mathbb{R}^{3\delta})\right\|\ .\end{aligned}$$

So we can write:

$$\begin{aligned}\left|\langle B\rangle_\Psi(t)-\langle g,\bar{u}^{\otimes_q}u^{\otimes_q}\bar{\alpha}^{\otimes_i}\alpha^{\otimes_j}(t)\rangle_{L^2(\mathbb{R}^{3\delta})}\right|&\leq\lambda^2K_1|t|e^{K_2|t|}\left\|g;L^2(\mathbb{R}^{3\delta})\right\|\\ &+\lambda^\delta d_{\lambda^{-2}}\left|\langle C^\dagger(u_0/\lambda,\alpha_0/\lambda)\Psi,W^\dagger(t,0)B^{(\delta-1)}W(t,0)\Omega\rangle\right|\ ,\end{aligned}$$

with

$$K_1|t|e^{K_2|t|}\geq\sum_{d=0}^{\delta-2}(q+i)^{\delta/2}q^{1/2}C_d(t)K_W(1,t,0)K_W(\delta+6,t,0)\ .$$

We have to use a different approach to estimate the last term of the inequality above, namely

$$X\equiv\lambda^\delta d_{\lambda^{-2}}\left|\langle C^\dagger(u_0/\lambda,\alpha_0/\lambda)\Psi,W^\dagger(t,0)B^{(\delta-1)}W(t,0)\Omega\rangle\right|\ .$$

By Lemma 1.3:

$$X=\lambda d_{\lambda^{-2}}\left|\langle C^\dagger(u_0/\lambda,\alpha_0/\lambda)\Psi,W^\dagger(t,0)\varphi(\mathbf{g})W(t,0)\Omega\rangle\right|\ .$$

Now we pass to the interaction representation since we will need to differentiate, obtaining

$$X=\lambda d_{\lambda^{-2}}\left|\langle C^\dagger(u_0/\lambda,\alpha_0/\lambda)\Psi,\widetilde{W}^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle\right|\ ,$$

with

$$\begin{aligned}\tilde{g}_1(x)=&U_{01}^\dagger(t)g_1(x)\ ,\\ \tilde{g}_2(x)=&U_{01}(t)g_2(x)\ ,\\ \tilde{g}_3(k)=&U_{02}^\dagger(t)g_3(k)\ ,\\ \tilde{g}_4(k)=&U_{02}(t)g_4(x)\ ;\end{aligned}$$

since $U_0(0) = 1$. We will use the following identity:

$$\begin{aligned}\langle \Phi, \widetilde{W}^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle &= \langle \Phi, \widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{U}_2(t,0)\Omega\rangle \\ &+\langle \Phi, (\widetilde{W}^\dagger(t,0) - \widetilde{U}_2^\dagger(t,0))\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle \\ &+\langle \Phi, \widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})(\widetilde{W}(t,0) - \widetilde{U}_2(t,0))\Omega\rangle \ ;\end{aligned}$$

with $\Phi = C^\dagger(u_0/\lambda, \alpha_0/\lambda)\Psi$. Using Lemma 1.3 and Proposition 9 we obtain

$$\begin{aligned}X \leq &\lambda d_{\lambda^{-2}}\Big(\Big|\langle \Phi, (\widetilde{W}^\dagger(t,0) - \widetilde{U}_2^\dagger(t,0))\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\rangle\Big| \\ &+ \Big|\langle \Phi, \widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})(\widetilde{W}(t,0) - \widetilde{U}_2(t,0))\Omega\rangle\Big|\Big) \equiv \lambda d_{\lambda^{-2}}(X_1 + X_2) \ .\end{aligned}$$

We define $\|\mathbf{g}\|_2 = \|g_1\|_2 + \|g_2\|_2 + \|g_3\|_2 + \|g_4\|_2$. To bound X_1 we proceed as follows, every term being well defined due to the properties of $\widetilde{W}(t,s)$ and $\widetilde{U}_2(t,s)$, and the integrals making sense as strong Riemann integrals on $\mathscr{H}$:

$$\begin{aligned}X_1 \leq &\left|\int_0^t \mathrm{d}\tau \left|\Big\langle \frac{1}{\sqrt{P+1}}\Phi, (P+1)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\widetilde{W}^\dagger(\tau,0)\widetilde{U}_2(\tau,0)\Big)\widetilde{U}_2^\dagger(t,0)\right.\right. \\ &\left.\left.\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\Big\rangle\right|\right| \\ &\leq K_\Psi d_{\lambda^{-2}}^{-1}\left|\int_0^t \mathrm{d}\tau \left\|\widetilde{W}^\dagger(\tau,0)U_0^*(\tau)H_I U_0(\tau)\widetilde{U}_2(\tau,0)\right.\right. \\ &\left.\left.\widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\right\|_{\mathscr{H}^1}\right| \ .\end{aligned}$$

We remark that $\|\tilde{\mathbf{g}}\|_2 = \|\mathbf{g}\|_2 \leq C_{\delta-1}(t)\left\|g; L^2(\mathbb{R}^{3\delta})\right\|$, with $C_{\delta-1}(t)$ defined in Lemma 1.1. Now using Proposition 8 three times, the fact that U_0 commutes with P and N, the usual estimates for H_I and $\varphi(\tilde{\mathbf{g}})$, Lemma 2.1 of Chapter 3, Proposition 6 and the fact that $\|\Omega\|_\delta = 1$ for any real δ, we obtain

$$X_1 \leq \lambda d_{\lambda^{-2}}^{-1} K_1' \, |t| \, e^{K_2'|t|} \left\|g; L^2(\mathbb{R}^{3\delta})\right\| \ ,$$

with

$$\begin{aligned}K_1' \, |t| \, e^{K_2'|t|} \geq &K_\Psi 2^{11} \, \|f\|_0 \, C_{\delta-1}(t)K_U(13,t,0)K_W(14,t,0) \\ &\left|\int_0^t \mathrm{d}\tau \, K_W(1,\tau,0)K_U(13,\tau,0)\right| \ ;\end{aligned}$$

For the sake of completeness we will write explicitly the most relevant steps:

$$\begin{aligned}X_1 \leq &K_\Psi \lambda d_{\lambda^{-2}}^{-1} 2^{9/2} \, \|f\|_0 \left|\int_0^t \mathrm{d}\tau \, K_W(1,\tau,0)\right\|(P+N+1)^{13/2}\widetilde{U}_2(\tau,0) \\ &\left.\widetilde{U}_2^\dagger(t,0)\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\right\|\Big| \\ \leq &K_\Psi \lambda d_{\lambda^{-2}}^{-1} 2^{9/2} \, \|f\|_0 \left|\int_0^t \mathrm{d}\tau \, K_W(1,\tau,0)K_U(13,\tau,0)K_U(13,t,0)\right. \\ &\left.\left\|(Q+1)^{13/2}\varphi(\tilde{\mathbf{g}})\widetilde{W}(t,0)\Omega\right\|\right|\end{aligned}$$

$$\leq K_\Psi \lambda d_{\lambda^{-2}}^{-1} 2^{11} \|f\|_0 C_{\delta-1}(t) \left\|g; L^2(\mathbb{R}^{3\delta})\right\| \left| \int_0^t \mathrm{d}\tau \, K_W(1,\tau,0) \right.$$
$$\left. K_U(13,\tau,0) K_U(13,t,0) \left\| (Q+1)^7 \widetilde{W}(t,0)\Omega \right\| \right|$$
$$\leq K_\Psi \lambda d_{\lambda^{-2}}^{-1} 2^{11} \|f\|_0 C_{\delta-1}(t) \left\|g; L^2(\mathbb{R}^{3\delta})\right\| K_U(13,t,0) K_W(14,t,0)$$
$$\left| \int_0^t \mathrm{d}\tau \, K_W(1,\tau,0) K_U(13,\tau,0) \right| .$$

To bound X_2 we proceed in an analogous fashion:

$$X_2 \leq \left| \int_0^t \mathrm{d}\tau \left| \left\langle \frac{1}{\sqrt{P+1}} \Phi, (P+1)^{1/2} \widetilde{U}_2^*(t,0) \varphi(\tilde{\mathbf{g}}) \widetilde{W}(t,0) \right. \right. \right.$$
$$\left. \left. \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\widetilde{W}^*(\tau,0) \widetilde{U}_2(\tau,0) \right) \Omega \right\rangle \right| \right| .$$

A calculation perfectly analogous to the one performed above for X_1 leads to:

$$X_2 \leq \lambda d_{\lambda^{-2}}^{-1} K_1'' \, |t| \, e^{K_2''|t|} \left\|g; L^2(\mathbb{R}^{3\delta})\right\| \ ,$$

with

$$K_1'' \, |t| \, e^{K_2''|t|} \geq K_\Psi 2^{47} \|f_0\|_2 K_U(1,t,0) K_W(2,t,0) \left| \int_0^t \mathrm{d}\tau \, K_W(15,\tau,0) \right.$$
$$\left. K_U(97,\tau,0) \right| .$$

■

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