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THE GEOMETRY OF THE MODULI SPACE OF POLYGONS IN THE EUCLIDEAN SPACE.

Ph.D. Dissertation in Mathematics

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"Io sono qui,
sono venuto a suonare,
sono venuto ad amare
e di nascosto a danzare."
P.Conte.

To $\mathcal{R}$.,
dancer,
but only on the sly.

Contents

Introduction

The study of the geometry of moduli spaces of polygons in Euclidean space has had, since the 1990's, a remarkable importance in symplectic geometry, especially thanks to the works of Kapovich–Millson and Hausmann–Knutson; many others have also contributed to this lively field.

These moduli spaces have a very rich structure, which arises from two possible descriptions of them as symplectic quotients. Let $\mathcal{S}_r = \prod_{j=1}^{n} S^2_{r_j}$ be the product of n spheres of radii $r_1, \ldots, r_n$ respectively; $\mathcal{S}_r$ is a symplectic manifold and a Hamiltonian $SO(3)$-space with associated moment map

$$\begin{array}{rcl} \mu : \mathcal{S}_r & \rightarrow & \mathrm{Lie}(SO(3))^* \simeq \mathbb{R}^3 \\ \vec{e} = (e_1, \ldots, e_n) & \mapsto & e_1 + \ldots + e_n. \end{array}$$

For a (suitably chosen) length vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$ (the positive quadrant) the symplectic quotient $\mathcal{S}_r /\!\!/ SO(3)$ at the 0-level set is a smooth manifold, and it is defined to be the moduli space M_r (Kapovich–Millson [KM]). Note that the condition $\mu(\vec{e}) = 0$ is the closing condition for a polygon with edge vectors $e_1, \ldots, e_n$ starting at an arbitrary basepoint. Thus M_r can be identified with the set of polygons in $\mathbb{R}^3$, with n sides of lengths $r_1, \ldots r_n$, modulo rigid motions.

M_r can also be described as the symplectic reduction for the natural action of the torus U_1^n, of diagonal matrices in the unitary group U_n, on the complex Grassmannian of 2-planes $Gr_2(\mathbb{C}^n)$ (Hausmann–Knutson [HK97]); the moment map $\mu_{U_1^n} : Gr_{2,n} \rightarrow \mathbb{R}^n$ associated to this Hamiltonian action maps the plane $\langle a, b \rangle$ generated by the vectors $a, b \in \mathbb{C}^n$ into $\mu_{U_1^n}(\langle a, b \rangle) = (|a_1|^2 + |b_1|^2, \ldots, |a_n|^2 + |b_n|^2)$. Then M_r is the topological quotient $\mu_{U_1^n}^{-1}(r)/U_1^n$.

In this thesis we present original results which describes the cobordism class and the volume of M_r. Then we use the volume formula to calculate the cohomology ring of M_r applying the Duistermaat–Heckman theorem. This calculation also use a original description of how the diffeotype of M_r changes as r change region of regular values.

The idea to calculate the complex cobordism class of M_r in terms of data localized at the fixed point set $M_r^{S^1}$, when S^1 acts by bending along a diagonal, is due to L. Migliorini and A. Reznikov.

The first result (Theorem 4.4) we present is an explicit characterization of the oriented S^1-cobordism class of M_r which depends uniquely on the lengths vector r: Let $r \in \mathbb{R}^n_+$ such that M_r is smooth, then M_r is the disjoint union of a finite number of oriented $(n-3)$-dimensional complex projective spaces, i.e.

$$M_r \simeq \coprod_{\substack{I \in \mathcal{I} \\ \ell = \sharp I}} (-1)^{n-\ell} \mathbb{P}^{n-3}(\mathbb{C})$$

where $\mathcal{I}$ is the set of $I \subset \{1, \ldots, n-2\}$ which satisfy a system of inequalities that depends only on the r_i's (thus also the orientation of the projective spaces $\mathbb{P}^{n-3}(\mathbb{C})$ only depends on the r_i's). In particular if n is even M_r is cobordant to 0.

The main ingredients to prove this result are the bending action (Kapovich-Millson [KM]) and results presented by Ginzburg, Guillemin and Karshon ([GGK96, GGK02]) that, under suitable hypothesis, link the S^1-cobordism class of a even-dimensional manifold M to data associated to the connected components of the fixed points set $(M)^{S^1}$.

The second main result of the paper is a formula (Theorem 3.3.1) that describes the volume of M_r as a piecewise polynomial function in the r_i's (in accordance with the Duistermaat–Heckman theorem): Let $r \in \mathbb{R}^n_+$ such that M_r is smooth, then

$$vol(M_r) = -\frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{I \in \mathcal{I} \\ |I| = k}} (R_I^+ - R_I^-)^{n-3}$$

where $R_I^- = \sum_{i\in I} r_i$, $R_I^+ = \sum_{i\notin I} r_i$ and $\mathcal{I} = \{I \subset \{1,\ldots,n\}/R_I^+ - R_I^- > 0\}$.

The main tools to prove this result are localization theorems in equivariant cohomology (Martin [Ma], Jeffrey–Kirwan [JK], Guillemin–Kalkman [GK96]) together with an equivariant integration formula for symplectic quotients by non-abelian groups (Martin [Ma2]).

An interesting application of the volume formula for M_r is the calculation of the cohomology ring $H^*(M_r)$ of M_r. $H^*(M_r)$ has been already determined by Hausmann and Knutson ([HK98]), but the technique that we present involves a thorough analysis of how the diffeotype of M_r changes as r crosses a wall in $\mu_{U_1^n}(Gr_{2,n})$ (which we believe has an independent interest) and perhaps gives a geometrically more direct comprehension of the cohomology ring $H^*(M_r)$.

The convex polytope $\Xi := \mu_{U_1^n}(Gr_{2,n})$ is itself the union of convex polytopes Δ_i (which are the regions of regular values of $\mu_{U_1^n}$), separated by $n-1$-dimensional subspaces, called walls, which are the set of points fixed by some subgroup $H \simeq S^1$ of U_1^n. A wall in Ξ has equations $\sum_{i\in I_p} r_i = \sum_{j\in I_q}$ for some partions I_p and I_q of $\{1,\ldots,n\}$. The index subsets I_p and I_q (together with a suitably choosen orientation) determines the type (p,q) of the wall. In chapter 4 we prove the following theorem (Theorem 4.2.4):

As the length vector r crosses a wall of type (p,q) in Ξ, the diffeotype of the moduli space of polygons M_r change by blowing up the $(q-2)$-dimensional submanifold $M_{I_p}(r)$ isomorphic to $\mathbb{CP}^{q-2}$ and blowing down the (projectivized normal bundle) of a submanifold $M_{I_q}(r)$ isomorphic to $\mathbb{CP}^{p-2}$.

$M_{I_p}(r)$ and $M_{I_q}(r)$ are moduli space of polygons themselves (of lower dimensions). They are resolutions of the singularity corresponding to the lined polygon P in M_{r^c}.

The last original contribution presented in this thesis is the following theorem (Theorem 5.2.4)

$$H^*(M_r, \mathbb{Q}) \simeq \mathbb{Q}[x_1,\ldots,x_n]/ann(volM_r)$$

where a polynomial $Q(x_1, \dots, x_n) \in ann(volM_r)$ *if and only if* $Q\Big(\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}\Big) vol(M_r) = 0.$

This result gives a way to compute the cohomology ring of smooth moduli space of polygons M_r in terms of cohomological classes which arise naturally from the symplectic quotient structure of M_r.

The main steps to prove Theorem 5.2.4 is to show that the cohomology ring $H^*(M_r, \mathbb{Q})$ of the moduli space of polygons M_r, when M_r is a smooth manifold, is generated by the first Chern classes $c_1, \dots, c_n$ of the n complex line bundles associated to the fibration $\mu^{-1}(r_1, \dots, r_n) \to M_r$. This has been done using Gysin and Mayer–Vietoris sequences togheter with the decomposition theorem.

Then the result follows applying the Duistermaat–Heckman Theorem, pecisely by showing that the Duistermaat–Heckman polynomial encodes all the necessary information on the generators and relators of $H^*(M_r)$, as it is the case for the cohomology ring of a flag manifold (Guillemin–Sternberg [GS95]).

The moduli space M_r is introduced in chapter 1, which is a quick overview of the results presented in literature which will be used in the proofs of the original theorems in the following chapters.

In chapter 2 we describe the theorems by Ginzburg, Guillemin and Karshon that are the main tool to prove Theorem 2.0.2 and give the proof of it in detail. We shall see some examples in 2.3.

In chapter 3 we first recall some very basic facts in equivariant cohomology and describe some of the results that S. Martin proved in his PhD thesis. Then we use these to prove the volume formula (Theorem 3.3.1) and we give some examples.

In chapter 4 we recall some results from [GS89] and calculate how the diffotype of M_r changes as r crosses a wall. In particular in section 4.2.1 we describe the wall-crossing phenomena in terms of moduli spaces.

In chapter 5, section 5.1, we first describe how to apply the Duistermaat–

Heckman Theorem to describe the cohomology ring $H^*(M_r)$— following Guillemin–Sternberg [GS89]. Then, in section 5.2, we use the wall-crossing analysis to prove that the Chern classes described above actually generates $H^*(M_r)$, and so we prove Theorem 5.2.4

Chapter 1

The Moduli Space M_r of Polygons in the Euclidean Space

In this chapter we introduce the moduli space of polygons M_r both as the symplectic quotient–at the 0-level set– for the diagonal action of $SO(3)$ on the product of n spheres and as the symplectic quotient–at the r-level set– for the standard action of the torus U_1^n of diagonal matrices in the unitary group U_n. Moment maps and symplectic reductions are nowdays classic construction is symplectic geometry. And the subject there is a wide literature, we refer in particular to D. McDuff–D. Salamon [McDS], A. Cannas da Silva [Ca, Ca01] and M. Audin [Au].

In section 1.2 we describe in detail the bending action, as it has been introduced by Kapovich–Millson.

1.1 The moduli space M_r

A $n-$gon P in the Euclidean space $\mathbb{E}^3$ is determined by its n vertices $v_1, \ldots, v_n$ joined by the oriented edges $e_j = v_{j+1} - v_j$ ($e_n = v_1 - v_n$). A polygon is said to be degenerate if it lies on a line. Let $\mathcal{P}_n$ be the space of all $n-$gons in $\mathbb{E}^3$. Two polygons $P = (e_1, \ldots, e_n)$ and $Q = (\varepsilon_1, \ldots, \varepsilon_n)$ are identified if there exists an orientation preserving isometry g of $\mathbb{E}^3$ such that $g(e_i) = \varepsilon_i, 1 \leq i \leq n$. Let $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, the moduli space M_r is defined to be the space of $n-$gons with fixed side lengths $r_1, \dots, r_n$ respectively modulo isometries as above.

Because $\mathcal{P}_n$ is the space of polygons of n sides in the Euclidean space without fixing the lengths, we can observe that the group $\mathbb{R}_+$ acts on $\mathcal{P}_n$ by scaling and this induces an isomorphism $M_r \cong M_{\lambda r}$ for each λ in $\mathbb{R}_+$. Moreover, also the group S_n of permutations on n elements acts on $\mathcal{P}_n$ by permuting the order of the edges, and this induce an isomorphism between M_r and $M_{\sigma(r)}$ for each $\sigma \in S_n$.

Let S^2_t be the sphere in $\mathbb{R}^3$ of radius t and center the origin. Let $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, the product $\mathcal{S}_r = \prod_{j=1}^n S^2_{r_j}$ of n copies of spheres is a smooth manifold which can be endowed of a symplectic structure: if $p_j : \mathcal{S}_r \to S^2_{r_j}$ is the projection on the j-th factor and ω_j is the volume form on the sphere $S^2_{r_j}$, then the 2-form $\omega = \sum_{j=1}^n \frac{1}{r_j} p_j^* \omega_j$ on $\mathcal{S}_r$ is closed and non-degenerate and $(\mathcal{S}_r, \omega)$ is a symplectic manifold. The group $SO(3)$ acts diagonally on $\mathcal{S}_r$ or, equivalently, identifying the sphere $S^2_{r_j}$ with a $SO(3)$-coadjoint orbit, on each sphere the $SO(3)$-action is the coadjoint one.The choice of an invariant inner product on the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ induce an identification $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ between the dual of $\mathfrak{so}(3)$ and $\mathbb{R}^3$. So, on each single sphere $S^2_{r_j}$, the moment map associated to the coadjoint action is the inclusion of $S^2_{r_j}$ in $\mathbb{R}^3$. It follows that the diagonal action of $SO(3)$ on $\mathcal{S}_r$ is still Hamiltonian and, by linearity, has moment map

$$\begin{aligned} \mu : \mathcal{S}_r &\to \mathbb{R}^3 \\ \vec{e} = (e_1, \dots, e_n) &\mapsto e_1 + \cdots + e_n. \end{aligned}$$

The level set $\mu^{-1}(0) := \tilde{M}_r = \{\vec{e} = (e_1, \dots, e_n) \in \mathcal{S}_r : \sum_{i=1}^n e_i = 0\}$ is a submanifold of $\mathcal{S}_r$ because 0 is a regular value for μ.

Intuitively, if we think at the e_j's as edges of a "broken line" L in $\mathbb{R}^3$ then the condition $\sum_{i=1}^n e_i = 0$ is the closing condition for L and L is actually a polygon in $\mathbb{R}^3$, so $\tilde{M}_r = \mathcal{P}_n$. The existence of an isometry g such that $g\vec{e} = \vec{\varepsilon}$ is equivalent to the existence of $A \in SO(3)$ such that $A\vec{e} = \vec{\varepsilon}$. Thus the topological quotient $\tilde{M}_r/SO(3)$ is the moduli space M_r of $n-$gons of fixed side lengths described above and M_r is realized as the symplectic quotient $\mathcal{S}_r/\!\!/SO(3)$. We also point out that the existence of degenerate polygons in M_r translates the existence of a partition $I_1 = \{i_1, \ldots, i_s\}$ and $I_2 = \{i_{s+1}, \ldots, i_n\}$ of $\{1, \ldots, n\}$ such that $r_{i_1} + \ldots + r_{i_s} - r_{i_{s+1}} - \cdots r_{i_n} = 0$. For details and formal proofs of these arguments see [KM].

Proposition 1.1.1. *M_r is a smooth manifold if and only if the vector of lengths r does not admit degenerate polygons.*

If $r \in \mathbb{R}^n_+$ is such that in M_r there exists polygons on a line, then M_r has singularities, which have been studied by Kapovich and Millson in [KM]. Precisely, they proved that M_r is a complex analytic space with isolated singularities corresponding to the degenerate n-gons in M_r, and these singularities are "equivalent" to homogeneous quadratic cones.

Remark 1. Observe that for $\vec{e} \in \tilde{M}_r$ and and $u, v \in T_{\vec{e}}\tilde{M}_r$ the formulas

$$\langle u, v\rangle = \sum_{j=1}^n \tfrac{1}{r_j}\langle u_j, v_j\rangle_S$$
$$\omega(u, v) = \sum_{j=1}^n \langle \tfrac{e_j}{r_j^2}, u_j \wedge v_j\rangle_S$$
$$J(u) = (\tfrac{e_1}{r_1} \wedge u_1, \ldots, \tfrac{e_n}{r_n} \wedge u_n)$$

(where $\langle\, ,\rangle_S$ is the standard scalar product in $\mathbb{R}^3$) are $SO(3)$-invariant, and determine to an inner product $\langle\, ,\rangle$,, a symplectic form ω, and a complex structure J on M_r

1.2 The Bending Action

In this section we describe bending flows introduced by Kapovich and Millson in [KM]. The geometrical idea underlying the construction is the following: let P be a $n-$gon and μ_k its k-th diagonal, i.e. $\mu_k = e_1 + \cdots + e_{k+1}$. Consider the surface S bounded by P; S is the union of the triangles $\Delta_1, \ldots, \Delta_n$ where Δ_j has edges $\mu_{j-1}, e_{j+1}, \mu_j$. Each (nonzero) diagonal breaks S in two pieces, S' and S'', S' being the union of $\Delta_1, \ldots, \Delta_k$ and S'' the union of the remaining ones. The bending action along the k-th diagonal is the S^1-action which bends S' along μ_k and let S'' fixed.

More formally, let $\vec{e} = (e_1, \dots, e_n) \in \tilde{M}_r$ and as before let μ_k be its k-th diagonal, the function $f_k(\vec{e}) = \frac{1}{2}\|\mu_k\|^2$ is $SO(3)-$invariant, and thus will be identified with the function it induces on the quotient space M_r. from now on the construction will depend only formally on the representative of the classes, and $SO(3)$-invariance should be kept in mind. The bending flow around the k-th diagonal is the Hamiltonian flow of the Hamiltonian vector field H_{f_k}

$$H_{f_k}(e_1, \dots, e_n) = (\mu_k \wedge e_1, \dots, \mu_k \wedge e_{k+1}, 0, \dots, 0)$$

associated to the function f_k.

Remark 2. For all k and l the functions f_k and f_l Poisson-commute, i.e. $\{f_k, f_l\} = 0$.

The Hamiltonian flow φ_k^t associated to f_k is the solution of the differential system

$$\begin{cases} \frac{de_i}{dt} = \mu_k \wedge e_i, & 1 \leq i \leq k+1 \\ \frac{de_i}{dt} = 0, & k+2 \leq i \leq n. \end{cases}$$

In [KM] Kapovich and Millson proved that if P is the polygon in M_r of edges $e_1, \dots, e_n$, then $\varphi_k^t(P)$ is the polygon of edges $e_1(t), \dots, e_n(t)$ where

$$\begin{cases} e_i(t) = \exp(t\,\mathrm{ad}_{\mu_k})e_i & 1 \leq i \leq k+1 \\ e_i(t) = e_i, & k+2 \leq i \leq n. \end{cases}$$

From now on we will denote by β_k the S^1-action just described of bending along the k-diagonal.

Remark 3. Here we are using the identification of the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$, thus, for each $u, v \in \mathbb{R}^3$, $ad_u v = u \wedge v$ and $exp(ad_u) \in SO(3)$ is the sum of power series $\sum_{n=0}^{\infty} \frac{(ad_u)^n}{n!}$; in other words $exp(ad_u)$ is the rotation in the plane orthogonal to u of an angle of $\|u\|$ radians.

Let $\ell_k : M_r \to \mathbb{R}$ be the function that associates to each polygon $P = \vec{e}$ the length of its k-th diagonal, i.e. $\ell_k(P) = \|e_i + \ldots + e_{k+1}\|$, then the curve $\varphi_k^t(P)$ is periodic of period $2\pi/\ell_k(P)$ if $\ell_k(P) \neq 0$, otherwise P is a fixed point for φ_k^t and the flow $\varphi_k^t(P)$ has infinite period. It is possible to normalize the flow so that the bending action bends polygons with constant velocity up to excluding the polygons P such that $\ell_k(P) = 0$. Let M'_r be the open subset of M_r consisting of those polygons such that no diagonal μ_i has zero length; the choice of a system of $n-3$ non intersecting diagonals in M'_r allows one to define an action β of a $(n-3)$-dimensional T^{n-3} torus on M'_r by applying progressively the bending actions $\beta_1, \ldots, \beta_{n-3}$; β will be called (toric) bending action.

Restricting to the dense open subset $M_r^0 \subset M'_r$ of polygons such that the i-th diagonal μ_i is not collinear to e_{i+1}, Kapovich and Millson showed in [KM] that this system is completely integrable and introduced on M_r^0 action-angle coordinates. Precisely, the action coordinates are the lengths ℓ_i of the diagonals and the angle coordinates are $\theta_i = \pi - \hat{\theta}_i$, where $\hat{\theta}_i$ is the dihedral angle between Δ_i and Δ_{i+1}. (Note that under the hypothesis that no μ_i is collinear to e_{i+1} none of the Δ_i is degenerate, thus all the θ_i are well defined).

Thus the moment map for the bending action β is

$$\begin{aligned} \mu_{T^{n-3}} : M_r &\longrightarrow (\mathfrak{t}^{n-3})^* \simeq \mathbb{R}^{n-3} \\ \vec{e} &\longmapsto (\ell_1(\vec{e}), \ldots, \ell_{n-3}(\vec{e})). \end{aligned}$$

Remark 4. If $n = 4, 5, 6$ then M_r is toric for generic r's (i.e. for r's such that no degenerate polygons are possible), see [KM].

1.3 Polygon spaces and Grassmannians

In this section we will give an alternative description of the moduli space M_r of polygons as the symplectic reduction of the Grassmannian of 2-planes in $\mathbb{C}^n$ by the action of the maximal torus U_1^n of diagonal matrices in U_n. This point of view has been introduced by Hausmann and Knutson in [HK97] and has been used by them (also) to give a nice description of the bending action as the residual torus action coming from the Gel'fand–Cetlin system on $Gr_{2,n}$. This approach made it possible the study of wall-crossing problems (see chapter 4.2) and the description of the cohomology ring $H^*(M_r)$ applying the Duistermaat–Heckman Theorem (see chapter 5).

Let the group of unitary matrices U_2 act by right multiplication on the manifold $\mathcal{M}_{2\times n}(\mathbb{C})$ of $n \times 2$ complex matrices, this action is Hamiltonian with associated moment map

$$\begin{array}{rcl} \mu_{U_2} : \mathcal{M}_{2\times n}(\mathbb{C}) & \to & \mathfrak{u}(2)^* \\ A & \mapsto & iA^*A \end{array}$$

where A^* is the conjungate transpose of A. The Stiefel manifold of orthonormal 2 frames in $\mathbb{C}^n$, defined as follows

$$St_{2,n} = \left\{ \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \in \mathcal{M}_{n\times 2}(\mathbb{C}) : \sum_{i=1}^n |a_i|^2 = 1, \sum_{i=1}^n |b_i|^2 = 1, \sum_{i=1}^n a_i \bar{b_i} = 0 \right\}$$

can be realized as the preimage $(\mu_{U_2})^{-1}(iI)$ of the matrix $iI = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. Let $Gr_{2,n}$ be the Grassmannian of 2-planes in $\mathbb{C}^n$. The application

$$p : St_{2,n} \to Gr_{2,n}$$

that maps an element $(a, b) \in St_{2,n}$ into the plane generated by a and b is actually the projection of $St_{2,n}$ on the orbit space $St_{2,n}/U_2$, and this realize the Grassmannian $Gr_{2,n}$ as the symplectic quotient $\mathcal{M}_{n\times 2}(|C) /\!\!/ U_2$.

The unitary group U_n acts by left-multiplication on the manifold $\mathcal{M}_{n\times 2}(\mathbb{C})$ (and by restriction on $St_{2,n}$), the projection p defined above is U_n-equivariant and so the U_n-action descends to an action on the quotient $Gr_{2,n}$.

We recall here some results proved by Hausmann and Knutson in [HK97], section 4, to which we refer for proofs and further details.

The action of the maximal torus U_1^n on $Gr_{2,n}$ is Hamiltonian with associated moment map $\mu_{U_1^n} : Gr_{2,n} \to \mathbb{R}^n$ such that, if $\Pi = \langle a, b \rangle$ is the plane generated by $a, b \in \mathbb{C}^n$, then

$$\mu_{U_1^n}(\Pi) = (|a_1|^2 + |b_1|^2, \ldots, |a_n|^2 + |b_n|^2).$$

Then the image of the moment map $\mu_{U_1^n}(Gr_{2,n})$ is the hypersimplex Ξ

$$\mu_{U_1^n}(Gr_{2,n}) = \Xi = \left\{ (r_1, \ldots, r_n) \in \mathbb{R}^n | 0 \le r_i \le 1, \quad \sum_{i=1}^n r_i = 2 \right\}$$

and the set of critical values of $\mu_{U_1^n}$ consists of those points $(r_1,\ldots,r_n)\in\Xi$ satisfying one of the following conditions

a) one of the r_i's vanishes;

b) one of the r_i's is equal to 1;

c) there exists $\varepsilon_i=\pm1$ such that $\sum_{i=1}^n \varepsilon_i r_i = 0$ with at least two ε_i's for each sign.

Remark 5. Note that points satisfying a) and b) constitute the boundary of Ξ, while points satisfying condition c) are the inner walls of Ξ.

Remark 6. By this construction of M_r from the set of orthonormal 2-frames, the choice of the fixed perimeter $\mathfrak{p}=\sum_{i=1}^n r_i = 2$ arises naturally, and actually agrees with the common choice in literature. Still, we already pointed out that $M_r \simeq M_{\lambda r}$ $\forall\lambda\in\mathbb{R}_+$, thus sometimes (especially in the examples), we will choose $\mathfrak{p}\neq 2$, i.e. we will not re-normalize the r_i. In this case the image $\Xi_{\mathfrak{p}}$, (which still will be called Ξ when this will create no confusion), is $\Xi_{\mathfrak{p}} = \{(r_1,\ldots,r_n)\in\mathbb{R}^n | 0\le r_i\le \mathfrak{p}/2, \quad \sum_{i=1}^n r_i = \mathfrak{p}\}$.

From the identification of the bending flows with the residual torus action coming from the Gel'fand–Cetlin system ([HK97] theorem 5.2), Hausmann and Knutson prove that the action coordinates $\ell_1,\ldots,\ell_{n-3}$ satisfy the system

$$\begin{cases} r_{i+2}\le \ell_i+\ell_{i+1}\\ \ell_i\le r_{i+2}+\ell_{i+2}\\ \ell_{i+1}\le r_{i+2}+\ell_i\end{cases}\tag{1.1}$$

where, following our notation, ℓ_i is the length of the i-th diagonal, i.e. $\ell_i(\vec{e}) = |e_1+\ldots+e_{n-3}|$.

In the case $n=5$ the choice of the two (proper) diagonals from the first vertex, i.e. $\mu_1 = e_1+e_2$ and $\mu_2 = e_1+e_2+e_3 = -(e_4+e_5)$, allows us to define a toric bending action. We can nicely describe the image of the moment map $\mu_{T^2}: M_r\to\mathbb{R}^2$ associated to this bending action. Let us rename the action coordinates: $\ell_1 =: x$ and $\ell_2 =: y$; moreover note that $\ell_0 = r_1$, $\ell_3 = r_5$ and so the system 1.1 makes sense for $i=0,1,2$.

For $i = 0$

$$(1.1) \iff \begin{cases} r_2 \le r_1 + x \\ r_1 \le r_2 + x \\ x \le r_2 + r_1 \end{cases} \iff |r_1 - r_2| \le x \le r_1 + r_2; \qquad (1.2)$$

and similarly for $i = 2$,

$$(1.1) \iff \begin{cases} r_4 \le y + r_5 \\ y \le r_4 + r_5 \\ r_5 \le r_4 + y \end{cases} \iff |r_4 - r_5| \le x \le r_4 + r_5. \qquad (1.3)$$

Finally, for $i = 1$ we get the system

$$\begin{cases} x + y \ge r_3 \\ x \le r_3 + y \\ y \le r_3 + x. \end{cases} \qquad (1.4)$$

So, if I is the rectangle

$$I = \Big[|r_1 - r_2|, r_1 + r_2\Big] \times \Big[|r_4 - r_5|, r_4 + r_5\Big]$$

and Υ is the region

$$\Upsilon = \{(x, y) \in \mathbb{R}^2 : y \ge -x + r_3;\ y \ge x - r_3;\ y \le x + r_3\},$$

then

$$\mu_{T^2}(M_r) = I \cap \Upsilon. \qquad (1.5)$$

For some examples we refer to section 2.3.

Note that that the polytope $\mu_{T^2}(M_r)$ encodes all the informations on the region $r \in \Delta \subset \Xi$ of regular values for the moment map $\mu_{U_1^n}$. To see this let us introduce some notation: let A be the line of equation $y = |r_4 - r_5|$, $B:\ y = r_4 + r_5$, $C:\ x = |r_1 - r_2|$ and $D:\ x = r_1 + r_2$. Moreover let 1 be the line $y = x + r_3$, 2 the line $y = -x + r_3$ and 3 the line $y = x - r_3$. By $A1$ we will denote the intersection point of the lines A and 1, and similarly for the other intersection points. The positions of these intersection points

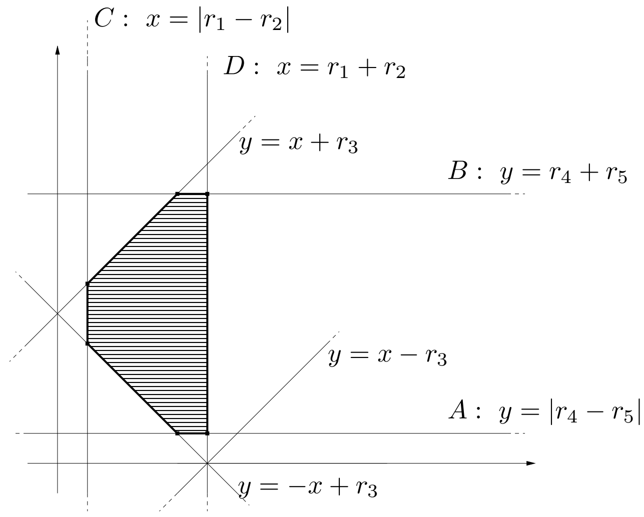

Figure 1.1: $\mu_{T^2}(M_r)$

determines the region Δ in which r lies; so, if we assume $r_1 < r_2$ and $r_4 < r_5$, in the example as in figure 1.1 we get the order of the intersection points on A is

$$AC < A2 < AD < A3$$

which, read on the x-coordinates, is the following system of inequalities:

$$\begin{cases} -r_1 + r_2 < r_3 + r_4 - r_5 \\ r_3 + r_4 - r_5 < r_1 + r_2 \\ r_1 + r_2 < r_3 - r_4 + r_5. \end{cases} \tag{1.6}$$

Similar inequalities follow from the order of the intersection points on the lines B, C and D. Recalling the description of the walls given above it is clear that these inequalities determine the region Δ. This remark will be useful in section 2.3.

Proposition 1.3.1. *For regular values $r \in \Xi$ of the moment map $\mu_{U_1^n}$*

$$M_r \simeq \mu_{U_1^n}^{-1}(r)/U_1^n = Gr_{2,n} /\!\!/ U_1^n(r)$$

the space of polygons of fixed side lengths r is the symplectic reduction relative to the U_1^n-action on the Grassmaniann $Gr_{2,n}$ at the level set r.

Proof. Let p be the standard projection of $St_{2,n}$ on the orbit space $St_{2,n}/U_2$. An element $(a,b) \in St_{2,n}$ such that $p(a,b) \in \mu^{-1}_{U_1^n}(r)$ satisfies $|a_\ell|^2 + |b_\ell|^2 = r_i$ for each i. Thus the couple of complex numbers (a_i, b_i) can be thought as an element of the 3-dimensional sphere $S^3_{\sqrt{r_\ell}}$ and so

$$p^{-1}(\mu^{-1}_{U_1^n}(r)) \hookrightarrow \prod_{\ell=1}^{n} S^3_{\sqrt{r_\ell}}.$$

Recall that the 3-sphere can be identified with the group of unit quaternions $U_1(\mathbb{H})$; where we denote by $\mathbb{H} := \mathbb{C} \oplus i\mathbb{C}$ be the field of quaternions and the space $I\mathbb{H}$ of pure imaginary quaternions is equipped with the standard basis i, j, k.

The Hopf map $H : \mathbb{H} \to I\mathbb{H}$ maps q into $\bar{q}iq$; (we are using the notation of Hausmann–Knutson, [HK97], not the original one introduced by Hopf).

H maps the 3-sphere of radius $\sqrt{r_\ell}$ in H onto the 2-sphere of radius r_ℓ in $I\mathbb{H}$, precisely, for $q = a_\ell + ib_\ell$, $a_\ell, b_\ell \in \mathbb{C}$, then

$$H(a_\ell, b_\ell) = i\,[(|a_\ell|^2 - |b_\ell|^2) + 2\bar{a}_\ell b_\ell j].$$

From now on we will also identify $H(a_\ell, b_\ell)$ with the element $(|a_\ell|^2 - |b_\ell|^2, 2\mathrm{Re}(\bar{a}_\ell b_\ell), 2\mathrm{Im}(\bar{a}_\ell b_\ell)) \in \mathbb{R}^3$.

So the application

$$\begin{array}{rcl} H^n : \prod_\ell S^3_{\sqrt{r_l}} & \to & \prod_\ell S^2_{r_\ell} \\ (a,b) & \mapsto & (H(a_1,b_1), \dots, H(a_n,b_n)) \end{array}$$

maps the submanifold $p^{-1}(\mu^{-1}_{U_1^n}(r))$ onto $\tilde{M}_r = \{(e_1, \dots, e_n) \in \prod_\ell S^2_{r_\ell} : \sum_i e_i = 0\}$.

Thus the orbit space of the residual $U_2/U_1 \simeq SO(3)$ action is the moduli space of polygons as introduced in section 1.1. □

Chapter 2

The Cobordism Class of M_r

The aim of this chapter is to give a explicit description of the oriented cobordism class of the moduli space of polygons M_r. In the first section we will recall some notions on G-cobordism and state S^1-cobordism results for symplectic manifolds due to V. Ginzburg, V. Guillemin and Y. Karshon (see [GGK96] and [GGK02]) and to S. Martin ([Ma]). In section 2.2 we use the S^1-action on M_r of bending along a chosen diagonal to prove the following result.

Definition 2.1. For each index set $I \subset \{1, \ldots, n-2\}$ let $\varepsilon_i = 1$ if $i \in I$ and $\varepsilon_i = -1$ if $i \in I^c = \{1, \ldots, n-2\} \setminus I$. An index set I is said to be *admissible* if and only if $\sum_{i=1}^{n-2} \varepsilon_i r_i > 0$ and the following inequalities hold:

$$\begin{cases} \sum \varepsilon_i r_i + r_{n-1} - r_n > 0 \\ \sum \varepsilon_i r_i - r_{n-1} + r_n > 0 \\ -\sum \varepsilon_i r_i + r_{n-1} + r_n > 0. \end{cases} \tag{2.1}$$

We will denote with $\mathcal{I}$ the set of all admissible I.

Theorem 2.0.2. *For $r \in \mathbb{R}^n_+$ such that M_r is a smooth manifold*

$$M_r \simeq \coprod_{\substack{I \in \mathcal{I} \\ \ell = |I|}} (-1)^{n-\ell} \mathbb{CP}^{n-3}.$$

In particular $M_r \simeq 0$ if n is even.

2.1 S^1-cobordism

Let M be a smooth oriented manifold, as usual we will denote with $-M$ the same manifold with opposite orientation and with $\amalg$ the disjoint union (or topological sum) of smooth manifolds.

Definition 2.2. Let M and M' be two smooth oriented n-dimensional manifolds. M and M' are said to be *oriented cobordant* if there exists a smooth compact oriented manifold with boundary X such that δX with its induced orientation is diffeomorphic (under a orientation preserving diffoemorphism) to $M \amalg -M'$. Let G be a compact Lie group and a and a' be actions of G on M and M' respectively, both preserving the orientations on M and M'. M and M' are said to be *equivariantly oriented cobordant* if there exists X as above and an (orientation preserving) action α of G on X such that $\delta X \simeq M \amalg -M'$ and $\alpha_{|_M} = a$ and $\alpha_{|_{M'}} = a'$.

The following theorem was shown by V. Ginzburg, V. Guillemin and Y. Karshon in [GGK96] (see also [GGK02]). Similar results were also proved by S. Martin in [Ma].

Theorem 2.1.1. (V. Ginzburg, V. Guillemin, Y. Karshon)

Let M be a oriented $2d$-dimensional manifold on which the group S^1 acts. Suppose that this action is quasi-free and has finitely many fixed points. Then M is cobordant a disjoint union of N copies of $\pm\mathbb{CP}^d$, where N is the number of fixed points.

Proof. The circle S^1 acts on $M \times \mathbb{C}$ by the product of the action on M and the standard action on $\mathbb{C}$. The fixed points of this action are $q_k = (p_k, 0)$, $p_k \in M^{S^1}$. Denote by U_k an S^1-invariant open ball around q_k (with respect to some invariant Riemann metric). Let W be the subset of $M \times \mathbb{C}$ obtained by excising the U_k's and the set $|z| > 1$. Since S^1 acts freely on W, the quotient W/S^1 is a compact manifold with boundary and

$$\delta(W/S^1) = M \cup \coprod_k (\delta U_k)/S^1. \tag{2.2}$$

The linear isotropy action of S^1 on the tangent space $T_{q_k}(M \times \mathbb{C})$ is free except at the origin, hence there is an $\mathbb{R}$-linear identification

$$T_{q_k}(M \times \mathbb{C}) \simeq \mathbb{C}^{d+1} \tag{2.3}$$

which converts this action multiplication by $e^{iw_j\theta_j}$ on each copy of $\mathbb{C}$, the w's being the weights of the isotropy action. Via 2.3 one can identify U_k with the set $\|z\| < \varepsilon$ and hence identify $\delta U_k/S^1$ with $\mathbb{P}^d(\mathbb{C})$. Note that the isomorphism 2.3 may not respect the orientation, if it does the $\mathbb{P}^d(\mathbb{C})$ comes with standard orientation it inherits from $\mathbb{C}^{n+1}$, if it does not $\mathbb{P}^d(\mathbb{C})$ comes with the opposite one. So from 2.2 we get that M is cobordant to the disjoint union of N copies of $\pm\mathbb{P}^d(\mathbb{C})$; this cobordism is equivariant because the S^1 action on the first component of $M \times \mathbb{C}$ commutes with the diagonal action and thus descends to W/S^1. □

Both the assumptions on the action are extremely strong, much can be said with weaker assumptions. If we do not ask the S^1 action to be quasi-free (but still to have finitely many fixed points) than the space W/S^1 has in general singularities, but it is still possible to prove a result on equivariant orbifold cobordism between M and the disjoint union of twisted projective spaces ([GGK96], [GGK02]). On the other hand, if we assume the action to be quasi-free but we allow the fixed points set not to be finite, still it is possible to describe nicely the equivariant cobordism class of M.

Theorem 2.1.2. (V. Ginzburg, V. Guillemin, Y. Karshon)

Let M be a oriented $2d$-dimensional manifold endowed of a quasi-free S^1 action. Let $X_k, k = 1, \ldots, N$, be the connected components of the fixed points set M^{S^1}. Then

$$M \sim \coprod_{k=1}^{N} B_k,$$

where B_k is a fibration over X_k with fiber $\mathbb{CP}^{m_k}$, $m_k = codim_{\mathbb{C}} X_k$.

It is also possible to describe the equivariant orbifold cobordism class of M when the S^1 action is not quasi-free and M^{S^1} is not finite. In this more general case a result similar to 2.1.2 holds, but the fibrations over the connected components of M^{S^1} have now fibers which are twisted projective spaces.

2.2 Proof of Theorem 2.0.2

In light of the results presented in the previous section we investigate the set of fixed points for a bending action. Let β be the action of S^1 on M_r by bending along the $n-3$ diagonal $\mu_{(n-3)} = e_1 + e_2 + \cdots + e_{n-2}$, i.e.

$$\begin{array}{lrcl} \beta: & S^1 \times M_r & \rightarrow & M_r \\ & (t, [(e_1, \ldots e_n)]) & \mapsto & [(exp(tad_{\mu_{(n-3)}})e_1, \ldots, exp(tad_{\mu_{(n-3)}})e_{n-2}, e_{n-1}, e_n)]. \end{array}$$

The action β is quasi-free, in fact the stabilizers of points are connected (they are S^1 for fixed points, $\{0\}$ otherwise).

A point $P \in M_r$ is fixed by β if it is of one of the following two types:

(I) $[P] = [\vec{e}], e_1, \ldots, e_{n-2}$ are collinear as in figure 2.1

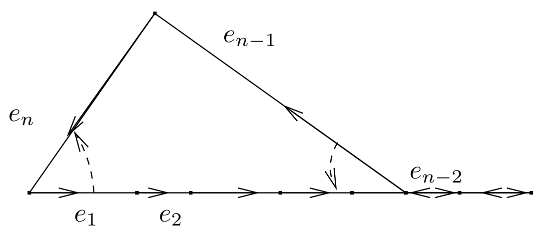

Figure 2.1: Fixed point of type I

In this case the action β fixes not just $[P]$ but also each representative.

(II) $[P] = [\vec{e}], e_{n-1}, e_n$ are collinear as in figure 2.2.

In this case the action β changes the the representative $\vec{e}$ but not the $SO(3)$ class.

The fixed points set $M_r^{S^1}$ is then the (disjoint) union of the sets $(M_r^{S^1})_{isol}$ of fixed points of type I and $(M_r^{S^1})_{subm}$ of fixed points of type II.

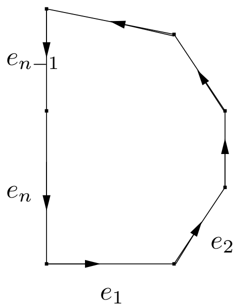

Figure 2.2: Fixed point of type II

Remark 7. It is now possible to give a geometric interpretation of the notion of admissibility for an index set I introduced in 2.1.

Let us consider a fixed point of type I; because P is planar it is not restrictive to assume it lies in the plane (x, y). Moreover, let us assume that the coordinate axis x is oriented as the $(n-3)$-diagonal $\mu_{(n-3)} := \mu$, then the triangle in figure 2.1 has side lengths r_n, r_{n-1}, and $\sum \varepsilon_i r_i$ where $\varepsilon_1 = 1$ if $e_i = \frac{r_i}{\|\mu\|}\mu$, $\varepsilon_i = -\frac{r_i}{\|\mu\|}\mu$. Then the index set I counts the number of "forward tracks", or, more formally, if $\ell = |I|$, then

$$\ell = \sharp\{e_j / e_j \cdot \mu > 0\}$$

and the inequalities in system 2.1 are just the "triangle inequalities". So I is admissible if and only if the triangle of side lengths r_n, r_{n-1}, and $\sum \varepsilon_i r_i$ (as in figure 2.1) closes.

If $[P]$ is a fixed point of type II then $[P] \in X_k$, X_k submanifold of fixed points. In particular X_k is the space of polygons of $n-1$ sides $M_{\bar{r}}$, with $\bar{r} = (r_1, \ldots, r_{n-2}, \pm r_{n-1} \pm r_n) \in \mathbb{R}_+^{n-1}$.(The sign $\pm$ are determined according to the orientation of the edges e_{n-1} and e_n.) Then $codim_{\mathbb{C}} X_k = 1$, so, for 2.1.2, X_k contributes to the cobordism of M_r with the total space B_k of a fibration on X_k with fiber $\mathbb{CP}^1$. This imply that $B_k \sim 0$ because it is the boundary of an associated fibration $\tilde{B}_k$ on X_k with fiber the disk D $(\delta D = S^2 \simeq \mathbb{CP}^1$.

Fixed points of type I are instead isolated and so from theorem 2.1.1

contribute to the cobordism class of M_r with a copy of $\mathbb{CP}^{n-3}$. The orientation of this projective space comes from the generator of the bending action and may not agree with the orientation that $\mathbb{CP}^{n-3}$ inherits from the symplectic structure of M_r. In fact for each $[P] \in (M_r^{S^1})_{isol}$ the symplectic form ω on M_r defines a complex structure J on $T_{[P]}M_r$ by $\omega_{[P]}(u,v) = g(u, Jv)$, where g is a Riemannian metric on $\mathbb{R}^{3n}$. The bending action defines too a complex structure on $T_{[P]}M_r$: differentiating β in $(\theta, [P])$ and valuating it in $1 \in \mathbb{R} \simeq Lie(S^1)$ we obtain an endomorphism of $T_{[p]}M_r$ and this define also a S^1-action (the linear isotropy action) on $T_{[p]}M_r$:

$$\begin{array}{rccl} d_{[P]}\beta : & S^1 & \rightarrow & End(T_{[p]}M_r) \\ & \theta & \mapsto & d_{(\theta,[P])}\beta(1) \end{array}$$

under which $T_{[p]}M_r$ decompose in the direct sum

$$T_{[p]}M_r = \bigoplus_{w \in \mathbb{Z}} V_w$$

so that on each V_w the S^1-action is "multiplication by $e^{iw\theta}$"; the w's are the isotropy weights and, because the action is semi-free (for S^1-actions quasi-free and semi free are equivalent), they are 0 or ± 1. The differential of $d_{[P]}\beta$

$$A = \frac{d}{d\theta}\big(d_{[P]}\beta\big)_{|=0}(1) : T_{[p]}M_r \rightarrow T_{[p]}M_r$$

is the generator of the bending action (note that on each V_w, A is the multiplication by iw).

To determine the cobordism class of M_r we will calculate the orientation that A induce on the projective spaces $\mathbb{CP}^{n-3}$. The proof will go as follows: first we will calculate

$$\hat{A} = \frac{d}{d\theta}\big(d_{\vec{e}}\hat{\beta}\big)_{|=0}(1) : T_{\vec{e}}\tilde{M}_r \rightarrow T_{\vec{e}}\tilde{M}_r$$

where $\hat{\beta}$ is the bending action on the level set $\tilde{M}_r = \{\vec{e} \in \prod_{j=1}^n S^2(r_j)/e_1 + \ldots + e_n = 0\}$, i.e.

$$\begin{array}{rrcl} \hat{\beta} : & S^1 \times \tilde{M}_r & \rightarrow & \tilde{M}_r \\ & (t,(e_1,\ldots e_n)) & \mapsto & (exp(tad_{\mu_{(n-3)}})e_1,\ldots,exp(tad_{\mu_{(n-3)}})e_{n-2},e_{n-1},e_n). \end{array}$$

Then identifying $T_{[P]}M_r$ with the orthogonal $T_P^{\perp}(SO(3)\cdot P)$ to tangent space to the $SO(3)$ orbit through P in $\tilde{M}_r$ we will project $\hat{A}$ on $T_{[P]}M_r$ and write explicitly A. Finally we will verify that A is a complex structure and compare it with J by checking when a J-positive basis of T_PM_r is also A-positive.

Remark 8. Observe that $\hat{A}$ is well defined because if $P=[\vec{e}]$ is a fixed point of type I then $\vec{e}$ is a fixed point for $\hat{\beta}$ (i.e. β fixes each representative of the class, not just the class).

2.2.1 Determine the complex structure A

Determine $\hat{A}: T_P\tilde{M}_r \to T_P\tilde{M}_r$

The action $\hat{\beta}$ described above still bends the first $(n-2)$ sides of a polygon along its $(n-3)$-diagonal. An element of $T_P\tilde{M}_r$ is of the form $\frac{d}{d\varepsilon}(P+\varepsilon Q)|_{\varepsilon=0}$, $P+\varepsilon Q=(e_1+\varepsilon v_1,\dots,e_n+\varepsilon v_n)$. Let μ be the $(n-3)$-diagonal of the polygon P, i.e. $\mu=e_1+\dots+e_{n-2}$, and ν be the $(n-3)$-diagonal of $P+\varepsilon Q$, i.e. $\nu=\sum_{i=1}^{n-2}e_i+\varepsilon\sum_{i=1}^{n-2}v_i:=\mu+\varepsilon\xi$. Now on, when $\vec{v}$ is understood, we will write ξ for $\xi(\vec{v})=\sum_{i=1}^{n-2}v_i$.

Let R_ε be the rotation that take ν on the x-axis and let b_θ the rotation of angle θ around the x-axis. The bending action $\hat{\beta}$ can be described in terms of R_ε and b_θ, precisely:

$$\hat{\beta}(P+\varepsilon Q)=(\dots,R_\varepsilon^{-1}b_\theta R_\varepsilon(e_j+\varepsilon v_j),\dots,e_{n-1}+\varepsilon v_{n-1},e_n+\varepsilon v_n).$$

So

$$\begin{array}{rccc}\hat{A}: & T_P\tilde{M}_r & \to & T_P\tilde{M}_r\\ & v & \mapsto & \hat{A}(v)\end{array}$$

with

$$\hat{A}(v)=\frac{d}{d\theta}_{|\theta=0}\frac{d}{d\varepsilon}_{|\varepsilon=0}(\dots,R_\varepsilon^{-1}b_\theta R_\varepsilon(e_j+\varepsilon v_j),\dots,e_{n-1}+\varepsilon v_{n-1},e_n+\varepsilon v_n).$$

Remark 9. We will use the notation $\underline{j}\wedge\underline{k}$ for the matrix $\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$ of the rotation around the x-axis. In general, for u_1,u_2 in $\mathbb{R}^3$, $u_1\wedge u_2$ is the rotation which takes u_1 on u_2, i.e.

$$(u_1 \wedge u_2)(v) = \langle u_1, v\rangle u_2 - \langle u_2, v\rangle u_1 \quad \forall v \in \mathbb{R}^3.$$

Proposition 2.2.1.

$$\frac{d}{d\theta}_{|\theta=0}\Big(\frac{d}{d\varepsilon}_{|\varepsilon=0} R_\varepsilon^{-1} b_\theta R_\varepsilon (e_j + \varepsilon v_j)\Big) = -\frac{\langle \mu, e_j\rangle}{\|\mu\|^2}\underline{j} \wedge \underline{k}(\xi) + \underline{j} \wedge \underline{k}(v_j).$$

Proof. Using the notation as in remark 9,the rotation R_ε is $exp(-\Theta\frac{\mu\wedge\varepsilon\xi}{\|\mu\|\|\varepsilon\xi\|})$, where the angle of rotation is $\Theta = \frac{\|\varepsilon\xi\|}{\|\mu\|}$, and b_θ is $exp(\theta\underline{j} \wedge \underline{k})$. The first order Taylor expansions of $R_\varepsilon^{-1}, R_\varepsilon$ and b_θ are

$$R_\varepsilon^{-1} = id + \varepsilon\frac{\mu \wedge \xi}{\|\mu\|^2} + o(\varepsilon),\ R_\varepsilon = id - \varepsilon\frac{\mu \wedge \xi}{\|\mu\|^2} + o(\varepsilon),\ b_\theta = id + \theta\underline{j} \wedge \underline{k} + o(\theta).$$

So

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} R_\varepsilon^{-1} b_\theta R_\varepsilon (e_j + \varepsilon v_j) = \Big[\frac{\mu \wedge \xi}{\|\mu\|^2} b_\theta (e_j + \varepsilon v_j) - b_\theta \frac{\mu \wedge \xi}{\|\mu\|^2}(e_j + \varepsilon v_j) + b_\theta v_j\Big]_{|\varepsilon=0} =$$

$$\frac{\mu \wedge \xi}{\|\mu\|^2} b_\theta e_j - b_\theta \frac{\mu \wedge \xi}{\|\mu\|^2} e_j + b_\theta v_j.$$

Now

$$\frac{d}{d\theta}_{|\theta=0}\frac{d}{d\varepsilon}_{|\varepsilon=0} R_\varepsilon^{-1} b_\theta R_\varepsilon (e_j + \varepsilon v_j) = \frac{d}{d\theta}_{|\theta=0}\Big(\frac{\mu \wedge \xi}{\|\mu\|^2} b_\theta e_j - b_\theta \frac{\mu \wedge \xi}{\|\mu\|^2} e_j + b_\theta v_j\Big) =$$

$$\frac{\mu \wedge \xi}{\|\mu\|^2}\underline{j} \wedge \underline{k}(e_j) - \underline{j} \wedge \underline{k}\frac{\mu \wedge \xi}{\|\mu\|^2}(e_j) + \underline{j} \wedge \underline{k}(v_j) =$$

$$\frac{\mu \wedge \xi}{\|\mu\|^2}(\underbrace{\langle \underline{j}, e_j\rangle}_{=0}\underline{k} - \underbrace{\langle \underline{k}, e_j\rangle}_{=0}\underline{j}) - \frac{\underline{j} \wedge \underline{k}}{\|\mu\|^2}(\langle \mu, e_j\rangle\xi - \underbrace{\langle \xi, e_j\rangle}_{=0}\mu) + \underline{j} \wedge \underline{k}(v_j) =$$

$$-\frac{\langle \mu, e_j\rangle}{\|\mu\|^2}\underline{j} \wedge \underline{k}(\xi) + \underline{j} \wedge \underline{k}(v_j).$$

∎

So

$$\begin{array}{rcll} \hat{A}: & T_P\tilde{M}_r & \to & T_P\tilde{M}_r \\ & v & \mapsto & (\hat{A}_1(v),\dots,\hat{A}_k(v),0,0)=\hat{A}(v), \end{array}$$

where

$$\hat{A}_j(v) = -\frac{\langle \mu, e_j\rangle}{\|\mu\|^2}\underline{j}\wedge\underline{k}(\xi)+\underline{j}\wedge\underline{k}(v_j). \tag{2.4}$$

When the vector v will be clear we will write $\hat{A}$ for $\hat{A}(v)$, and $\hat{A}_j$ for $\hat{A}_j(v)$.

Passage to the quotient $M_r = \tilde{M}_r \big/ SO(3)$

The action of $SO(3)$ on $\tilde{M}_r$ allows us to decompose the tangent space in P at $\tilde{M}_r$ as follows:

$$T_P\tilde{M}_r = T_P(SO(3)\cdot P)\oplus T_P^{\perp}(SO(3)\cdot P).$$

It is possible to identify $T_P^{\perp}(SO(3)\cdot P)$ with $T_{[P]}M_r$; to project $\hat{A}$ on $T_{[P]}M_r \simeq T_P^{\perp}(SO(3)\cdot P)$ we determine an orthogonal basis $\delta^1,\delta^2,\delta^3$ of $T_p(SO(3)\cdot P)$ and the projection A is given by:

$$A(v) = \hat{A}(v) - \frac{\langle\hat{A}(v),\delta^1\rangle}{\|\delta^1\|^2}\delta^1 - \frac{\langle\hat{A}(v),\delta^2\rangle}{\|\delta^2\|^2}\delta^2 - \frac{\langle\hat{A}(v),\delta^3\rangle}{\|\delta^3\|^2}\delta^3. \tag{2.5}$$

The generators of the $SO(3)$ action are the rotations around the axes. So $\hat{\delta}^1 = (e_1\wedge\underline{i},\dots,e_n\wedge\underline{i})$, $\hat{\delta}^2 = (e_1\wedge\underline{j},\dots,e_n\wedge\underline{j})$, $\hat{\delta}^3 = (e_1\wedge\underline{k},\dots,e_n\wedge\underline{k})$ define a basis of $T_p(SO(3)\cdot P)$. This base in general is not orthogonal with respect to the metric associated to the symplectic structure and we will orthonormalize it using the Gram-Schmidt formula. So, in order to write explicitly the basis $\hat{\delta}^1,\hat{\delta}^2$ and $\hat{\delta}^3$ of the $SO(3)$-orbit trough P in $\tilde{M}_r$ let us fix a representative $\vec{e}$ in $[P]$. As before (remark 7) let us assume that $\vec{e}$ lies in the (x,y)-plane and let the first $e_1,\dots,e_{n-2}$ edges lie on the x-axis. In $[P]$ there are two polygons with these properties, one such that $\mu=\|\mu\|\underline{i}$ and the other such that $\mu=-\|\mu\|\underline{i}$. Let $\vec{e}=P$ be the polygon such that $\mu=\|\mu\|\underline{i}$ (this is equivalent to require $\sum\varepsilon_i r_i>0$ as in definition 2.1). These assumptions are not restrictive. Let us also assume that the first ℓ edges are oriented as the x-axis, i.e.

$$e_i = (r_i,0,0),\quad \forall i=1,\dots,\ell,$$

and that the following $(n-2-\ell)$ edges are conversely oriented, i.e.

$$e_i = (-r_i, 0, 0) \quad \forall i = \ell+1, \ldots, n-2.$$

This assumption is instead restrictive, we are in fact choosing a particular class $[P]$; this assumption is useful in order to keep the notation more compact. We will say some more words about what happen if we consider another class in remark 14.

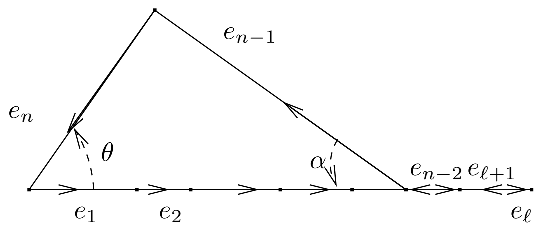

Figure 2.3: toy model for $[P]$ fixed point of type I

Under these assumptions the polygon P is as in figure 2.3 the edges e_n and e_{n-1} are $e_n = (-r_n cos\theta, -r_n sin\theta, 0)$, $e_{n-1} = (-r_{n-1}cos\alpha, r_{n-1}sin\alpha, 0)$. We can express $cos\alpha$ and $sin\alpha$ in function of θ, r_{n-1} and r_n as follows:

$$sin\alpha = \frac{r_n}{r_{n-1}} sin\theta, \quad cos\alpha = \frac{\|\mu\| - r_n cos\theta}{r_{n-1}}.$$

It is now easy to determine the vectors $\hat{\delta}^1$, $\hat{\delta}^2$ and $\hat{\delta}^3$.
$e_i \wedge \underline{i} = 0 \quad \forall i = 1, \ldots, n-2.$
$e_{n-1} \wedge \underline{i} = (-r_{n-1}cos\alpha, r_{n-1}sin\alpha, o) \wedge (1,0,0) = -r_{n-1}sin\alpha\, \underline{k}.$
$e_n \wedge \underline{i} = (-r_n cos\theta, -r_n sin\theta, 0) \wedge (1,0,0) = r_n sin\theta\, \underline{k}.$
So

$$\hat{\delta}^1 = (0, \ldots, 0, -r_{n-1}sin\alpha\, \underline{k}, r_n sin\theta\, \underline{k}).$$

$e_i \wedge \underline{j} = r_i\, \underline{k} \quad \forall i = 1, \ldots, \ell$
$e_i \wedge \underline{j} = -r_i\, \underline{k} \quad \forall i = 1, \ldots, \ell$
$e_{n-1} \wedge \underline{j} = (-r_{n-1}cos\alpha, r_{n-1}sin\alpha, o) \wedge (0,1,0) = -r_{n-1}cos\alpha\, \underline{k}.$

$e_n \wedge \underline{j} = (-r_n cos\theta, -r_n sin\theta, 0) \wedge (0,1,0) = -r_n cos\theta\, \underline{k}$.
So

$$\hat{\delta}^2 = (r_1\, \underline{k}, \dots, r_\ell\, \underline{k}, -r_{\ell+1}\, \underline{k}, \dots, -r_{n-2}\, \underline{k}, -r_{n-1} cos\alpha\, \underline{k}, -r_n cos\theta\, \underline{k}).$$

$e_i \wedge \underline{k} = -r_i\, \underline{j} \quad \forall i = 1, \dots, \ell$
$e_i \wedge \underline{k} = r_i\, \underline{j} \quad \forall i = 1, \dots, \ell$
$e_{n-1} \wedge \underline{k} = (-r_{n-1} cos\alpha, +r_{n-1} sin\alpha, o) \wedge (0,0,1) = r_{n-1} sin\alpha\, \underline{i} + r_{n-1} cos\alpha\, \underline{j}$.
$e_n \wedge \underline{k} = (-r_n cos\theta, -r_n sin\theta, 0) \wedge (0,0,1) = -r_n sin\theta\, \underline{i} + r_n cos\theta\, \underline{j}$.
So

$$\hat{\delta}^3 = (-r_1\underline{k}, \dots, -r_\ell\underline{k}, r_{\ell+1}\underline{k}, \dots, r_{n-2}\underline{k}, r_{n-1} sin\alpha\underline{i} + r_{n-1} cos\alpha\underline{j}, -r_n sin\theta\underline{i} + r_n cos\theta\underline{j}).$$

The orthogonal basis of $T_p(SO(3)\cdot P)$ We now apply Gram-Schmidt to built a orthogonal basis $\{\delta^1, \delta^2, \delta^3\}$ from $\{\hat{\delta}^1, \hat{\delta}^2, \hat{\delta}^3\}$.

Remark 10. The scalar product on $T_P\tilde{M}_r$ is $\langle u, v\rangle = \sum_{i=1}^n \frac{1}{r_i}\langle u_i, v_i\rangle_S$ where $\langle\cdot,\cdot\rangle_S$ is the standard scalar product in $\mathbb{R}^3$. Of course $\|\delta^i\|^2$ will be the inner product $\langle \delta^i, \delta^i\rangle$.

$\delta^1 := \hat{\delta^1}$, and $\delta^2 := \hat{\delta^2} - \frac{\langle\hat{\delta}^2, \delta^1\rangle}{\langle\delta^1, \delta^1\rangle}\delta^1$. Now,

$$\langle\hat{\delta^1}, \hat{\delta^2}\rangle = r_{n-1} cos\alpha sin\alpha - r_n cos\theta sin\theta =$$

$$r_n sin\theta \frac{\|\mu\| - r_n cos\theta}{r_{n-1}} - r_n cos\theta sin\theta =$$

$$\frac{\|\mu\| r_n sin\theta - r_n^2 sen\theta cos\theta - r_{n-1} r_n cos\theta sin\theta}{r_{n-1}} =$$

$$\frac{r_n sin\theta(\|\mu\| - cos\theta(r_{n-1} + r_n))}{r_{n-1}}.$$

and

$$\langle\hat{\delta^1}, \hat{\delta^1}\rangle = r_{n-1} sin^2\alpha + r_n sin^2\theta = \frac{r_n^2}{r_{n-1}} sin^2\theta + r_n sin^2\theta =$$

$$\frac{r_n}{r_{n-1}} sin^2\theta(r_{n-1} + r_n).$$

So

$$\frac{\langle \hat{\delta^1}, \hat{\delta^2}\rangle}{\langle \hat{\delta^1}, \hat{\delta^1}\rangle} = \frac{\|\mu\| - cos\theta(r_{n-1}+r_n)}{sin\theta(r_{n-1}+r_n)}.$$

Thus

$$\delta^2 = \Big(r_1\underline{k}, \dots, r_\ell\underline{k}, -r_{\ell+1}\underline{k}, \dots, -r_{n-2}\underline{k},$$

$$(-(\|\mu\|r_n sin\theta) + \frac{\|\mu\| - cos\theta(r_{n-1}+r_n)}{sin\theta(r_{n-1}+r_n)} r_n sin\theta)\underline{k},$$

$$(-r_n cos\theta - \frac{\|\mu\| - cos\theta(r_{n-1}+r_n)}{sin\theta(r_{n-1}+r_n)} r_n sin\theta)\underline{k}\Big) =$$

$$\Big(r_1\underline{k}, \dots, r_\ell\underline{k}, -r_{\ell+1}\underline{k}, \dots, -r_{n-2}\underline{k}, -\frac{r_{n-1}\|\mu\|}{r_{n-1}+r_n}\underline{k}, -\frac{r_n\|\mu\|}{r_{n-1}+r_n}\underline{k}\Big).$$

$\delta^3 := \hat{\delta^3} - \frac{\langle \hat{\delta^3}, \delta^1\rangle}{\langle \hat{\delta^1}, \hat{\delta^1}\rangle}\hat{\delta^1} - \frac{\langle \hat{\delta^3}, \delta^2\rangle}{\langle \delta^2, \delta^2\rangle}\delta^2.$

We can observe that $\langle \hat{\delta^3}, \delta^1\rangle = 0$ and also $\langle \hat{\delta^3}, \delta^2\rangle = 0$.

To summarize, an orthogonal basis of $T_p(SO(3)\cdot P)$ is given by:

$$\delta^1 = (0, \dots, 0, -r_n sin\theta\underline{k}, r_n sin\theta\underline{k}),$$

$$\delta^2 = (r_1\underline{k}, \dots, r_\ell\underline{k}, -r_{\ell+1}\underline{k}, \dots, -r_{n-2}\underline{k}, -\frac{r_{n-1}\|\mu\|}{r_{n-1}+r_n}\underline{k}, -\frac{r_n\|\mu\|}{r_{n-1}+r_n}\underline{k}),$$

$\delta^3 = (-r_1\underline{k}, \dots, -r_\ell\underline{k}, r_{\ell+1}\underline{k}, \dots, r_{n-2}\underline{k}, r_n sin\theta\underline{i} + (\|\mu\| - r_n cos\theta)\underline{j},$
$-r_n sin\theta\underline{i} + r_n cos\theta\underline{j}).$

Calculate $A(v)$

Recall that $A(v) = \hat{A}(v) - \frac{\langle \hat{A}(v), \delta^1\rangle}{\|\delta^1\|^2}\delta^1 - \frac{\langle \hat{A}(v), \delta^2\rangle}{\|\delta^2\|^2}\delta^2 - \frac{\langle \hat{A}(v), \delta^3\rangle}{\|\delta^3\|^2}\delta^3$, where

$$\hat{A}(v)_j = -\frac{\langle \mu, e_j\rangle}{\|\mu\|^2}\underline{j}\wedge\underline{k}(\xi) + \underline{j}\wedge\underline{k}(v_j) =$$

$$\Big(-\frac{\langle \mu, e_j\rangle}{\|\mu\|^2}\langle \underline{j}, \xi\rangle + \langle \underline{j}, v_j\rangle\Big)\underline{k} + \Big(\frac{\langle \mu, e_j\rangle}{\|\mu\|^2}\langle \underline{k}, \xi\rangle - \langle \underline{k}, v_j\rangle\Big)\underline{j} \quad \forall j = 1, \dots, n-2,$$

and $\hat{A}(v)_j = 0$ if $j = n-1, n$. Let ε_j denote the direction of e_j, i.e.

$$\varepsilon_j = \begin{cases} 1, & j = 1, \dots, \ell \\ -1 & j = \ell+1, \dots, n-2; \end{cases}$$

It is straightforward to verify that $\langle \hat{A}(v), \delta^1 \rangle = 0$;

$$\langle \hat{A}(v), \delta^2 \rangle = \sum_{j=1}^{k} \varepsilon_j \Big(- \frac{\langle \mu, e_j \rangle}{\|\mu\|^2} \langle \underline{j}, \xi \rangle + \langle \underline{j}, v_j \rangle \Big) =$$

$$\sum_{j=1}^{l} \Big(- \frac{r_j}{\|\mu\|} \langle \underline{j}, \xi \rangle + \langle \underline{j}, v_j \rangle \Big) - \sum_{j=l+1}^{k} \Big(\frac{r_j}{\|\mu\|} \langle \underline{j}, \xi \rangle + \langle \underline{j}, v_j \rangle \Big);$$

similarly

$$\langle \hat{A}(v), \delta^3 \rangle = \sum_{j=1}^{k} (-\varepsilon_j) \Big(- \frac{\langle \mu, e_j \rangle}{\|\mu\|^2} \langle \underline{k}, \xi \rangle + \langle \underline{k}, v_j \rangle \Big) =$$

$$- \sum_{j=1}^{l} \Big(- \frac{r_j}{\|\mu\|} \langle \underline{k}, \xi \rangle + \langle \underline{k}, v_j \rangle \Big) + \sum_{j=l+1}^{k} \Big(\frac{r_j}{\|\mu\|} \langle \underline{k}, \xi \rangle + \langle \underline{k}, v_j \rangle \Big)$$

So, for each $v \in T_{[P]} M_r$ the components of $A(v)$ are:

$$A(v)_j = \Big(- \frac{r_j}{\|\mu\|} \langle \underline{j}, \xi \rangle + \langle \underline{j}, v_j \rangle - \frac{\langle \hat{A}(v), \delta^2 \rangle}{\|\delta^2\|^2} r_j \Big) \underline{k} +$$

$$\Big(\frac{r_j}{\|\mu\|} \langle \underline{k}, \xi \rangle - \langle \underline{k}, v_j \rangle + \frac{\langle \hat{A}(v), \delta^3 \rangle}{\|\delta^3\|^2} r_j \Big) \underline{j},$$

for $j = 1, \cdots, \ell$;

$$A(v)_j = (\frac{r_j}{\|\mu\|} \langle \underline{j}, \xi \rangle + \langle \underline{j}, v_j \rangle + \frac{\langle \hat{A}(v), \delta^2 \rangle}{\|\delta^2\|^2} r_j) \underline{k} +$$

$$\Big(- \frac{r_j}{\|\mu\|} \langle \underline{k}, \xi \rangle - \langle \underline{k}, v_j \rangle - \frac{\langle \hat{A}(v), \delta^3 \rangle}{\|\delta^3\|^2} r_j \Big) \underline{j},$$

for $j = \ell + 1, \cdots, n-2$;

$$A(v)_{n-1} = \frac{\langle \hat{A}(v), \delta^2 \rangle}{\|\delta^2\|^2} \frac{r_{n-1}\|\mu\|}{r_{n-1} + r_n} \underline{k} - \frac{\langle \hat{A}(v), \delta^3 \rangle}{\|\delta^3\|^2} (r_n sin\theta \underline{i} + (\|\mu\| - r_n cos\theta)) \underline{j};$$

$$A(v)_n = \frac{\langle \hat{A}(v), \delta^2 \rangle}{\|\delta^2\|^2} \frac{r_n\|\mu\|}{r_{n-1} + r_n} \underline{k} - \frac{\langle \hat{A}(v), \delta^3 \rangle}{\|\delta^3\|^2} (-r_n sin\theta \underline{i} + r_n cos\theta) \underline{j};$$

2.2.2 Compare the complex structures A and J

Determine a basis of $T_{[P]}M_r$

We have already seen that $T_{[P]}M_r$ can be identified with $T_P^{\perp}(SO(3)\cdot P)$, trough this identification it is possible to write the equations of $T_{[P]}M_r$ as a subspace of $\mathbb{R}^{3n}$, which are:

i) $\sum_{i=1}^n v_i = 0$,

ii) $e_i \cdot v_i = 0 \quad \forall i = 1, \ldots, n$,

iii) $\sum_{i=1}^n \frac{1}{r_i}(e_i \wedge v_i) = 0$.

Let

$$u_i = (0, \ldots, 0, \underbrace{\underline{j}}_{i}, \underbrace{-\underline{j}}_{i+1}, 0, \ldots, 0), i = 1, \ldots, \ell - 1$$
$$\hat{u}_i = (0, \ldots, 0, \underbrace{\underline{j}}_{i}, \underbrace{-\underline{j}}_{i+1}, 0, \ldots, 0), i = \ell + 1, \ldots, n - 3$$
$$v_i = (0, \ldots, 0, \underbrace{\underline{k}}_{i}, \underbrace{-\underline{k}}_{i+1}, 0, \ldots, 0), i = 1, \ldots, \ell - 1$$
$$\hat{v}_i = (0, \ldots, 0, \underbrace{\underline{k}}_{i}, \underbrace{-\underline{k}}_{i+1}, 0, \ldots, 0), i = \ell, \ldots, n - 3.$$

The vectors $u_i, \hat{u}_i, v_i, \hat{v}_i$ verifies the conditions $i)$, $ii)$, $iii)$, so they are in $T_{[P]}M_r$.

Remark 11. A vector of the form $(0, \ldots, 0, \underbrace{\underline{j}}_{\ell}, \underbrace{-\underline{j}}_{\ell+1}, 0, \ldots, 0)$ would not satisfy condition iii).

In the case $\ell = n - 2$ the vectors above are $2(n-3)$ and are linearly independent, so they form a basis of $T_{[P]}M_r$.
If instead $\ell \neq n - 2$ then the vectors above are $2(n-4)$ and it is necessary to complete them to a basis. To do this we look for a vector of the form

$$w = (\lambda\underline{k}, \ldots, \lambda\underline{k}, \gamma\underline{k}, \ldots, \gamma\underline{k}, \lambda_{n-1}\underline{k}, \lambda_n\underline{k}),$$

with $\lambda, \gamma, \lambda_{n-1}, \lambda_n \in \mathbb{R}$, and we impose that w satisfy the conditions i), ii), iii). Condition $iii)$ is straightforward verified by w. Condition $i)$ holds if and only if

$$\ell\lambda + (n-\ell-2)\gamma + \lambda_{n-1} + \lambda_n = 0. \tag{2.6}$$

Remembering that

$$\frac{1}{r_i} e_i \wedge w_i = -\lambda \underline{j} \qquad \forall i = 1, \ldots, \ell;$$

$$\frac{1}{r_i} e_i \wedge w_i = \gamma \underline{j} \qquad \forall i = \ell+1, \ldots, n-2;$$

$$\frac{1}{r_{n-1}} e_{n-1} \wedge w_{n-1} = \Big(-\frac{\|\mu\| - r_n cos\theta}{r_{n-1}}, \frac{r_n}{r_{n-1}} sin\theta, 0 \Big) \wedge (0, 0, \lambda_{n-1})$$
$$= \Big(\lambda_{n-1} \frac{r_n}{r_{n-1}} sin\theta, \lambda_{n-1} \frac{\|\mu\| - r_n cos\theta}{r_{n-1}}, 0 \Big);$$

$$\frac{1}{r_n} e_n \wedge w_n = (-cos\theta, -sin\theta, 0) \wedge (0, 0, \lambda_n) = (-\lambda_n sin\theta, \lambda_n cos\theta, 0);$$

we obtain that condition $iii)$ holds if and only if

$$-\ell\lambda + (n-\ell-2)\gamma + \lambda_{n-1} \frac{\|\mu\| - r_n cos\theta}{r_{n-1}} + \lambda_n cos\theta = 0 \tag{2.7}$$

and

$$\lambda_{n-1} \frac{r_n}{r_{n-1}} sen\theta - \lambda_n sin\theta = 0. \tag{2.8}$$

So to determine w we want to solve the system (2.6), (2.7), (2.8).

$$(4.6) \iff \lambda_n = \frac{r_n}{r_{n-1}} \lambda_{n-1}.$$

Using this in (2.7), we get $-\ell\lambda + (n-\ell-2)\gamma + \frac{\|\mu\|}{r_{n-1}} \lambda_{n-1} = 0$, which is

$$\lambda_{n-1} = \frac{r_{n-1}}{\|\mu\|} (\ell\lambda - (n-\ell-2)\gamma) \tag{2.9}$$

and using (2.9) in (2.8):

$$\lambda_n = \frac{r_n}{\|\mu\|} (\ell\lambda - (n-\ell-2)\gamma). \tag{2.10}$$

So (2.9) and (2.10) in (2.6) give

$$\lambda = -\frac{n-\ell-2}{\ell} \, \gamma \, \frac{\|\mu\| - r_{n-1} - r_n}{\|\mu\| + r_{n-1} + r_n}.$$

Choosing $\gamma = \frac{1}{2(n-\ell-2)}(\|\mu\| + r_{n-1} + r_n)$ and $\lambda = -\frac{1}{2\ell}(\|\mu\| - r_{n-1} - r_n)$, it follows that $\ell\lambda - (n-\ell-2)\gamma = -\|\mu\|$. So $\lambda_{n-1} = -r_{n-1}$ and $\lambda_n = -r_n$.

The vector $w = (\lambda\underline{k}, \dots, \lambda\underline{k}, \gamma\underline{k}, \dots, \gamma\underline{k}, -r_{n-1}\underline{k}, -r_n\underline{k})$, λ, γ as above, is linearly independent with the vectors $u_i, \hat{u}_i, v_i, \hat{v}_i$. Remember that J is the complex structure associated to the symplectic form, so $-J(w)$ is linearly independent with $u_i, \hat{u}_i, v_i, \hat{v}_i, w$ and complete to a basis of $T_{[P]}M_r$. Remembering that $Jw = (\frac{e_1}{r_1} \wedge w_1, \dots, \frac{e_n}{r_n} \wedge w_n)$, it follows that

$$\frac{e_i}{r_i} \wedge w_i = (1,0,0) \wedge (0,0,\lambda) = -\lambda\underline{j} \qquad \forall i = 1, \dots, \ell;$$

$$\frac{e_i}{r_i} \wedge w_i = (-1,0,0) \wedge (0,0,\gamma) = \gamma\underline{j} \qquad \forall i = \ell+1, \dots, n-2;$$

$$\frac{e_{n-1}}{r_{n-1}} \wedge w_{n-1} = \frac{1}{r_{n-1}}(-\|\mu\| + r_n cos\theta, r_n sin\theta, 0) \wedge (0,0,-r_{n-1}) =$$

$$-r_n sin\theta\underline{i} - (\|\mu\| - r_n cos\theta)\underline{j};$$

$$\frac{e_n}{r_n} \wedge w_n = \frac{1}{r_n}(-r_n cos\theta, -r_n sin\theta, 0) \wedge (0,0,-r_n)$$

$$r_n sin\theta\underline{i} - r_n cos\theta\underline{j}.$$

Thus

$$-Jw = (\lambda\underline{j}, \dots, \lambda\underline{j}, -\gamma\underline{j}, -\gamma\underline{j}, r_n sin\theta\underline{i} + (\|\mu\| - r_n cos\theta)\underline{j}, -r_n sin\theta\underline{i} + r_n cos\theta\underline{j}).$$

So $\mathcal{B}_1 = \{u_1, v_1, \dots, u_{\ell-1}, v_{\ell-1}, \hat{u}_{\ell+1}, -\hat{v}_{\ell+1}, \dots, \hat{u}_{n-3}, -\hat{v}_{n-3}, Jw, w\}$ is a basis of $T_{[P]}M_r$ and it is positive.

Remark 12. This is the standard convention, in fact

$$Ju_i = (\dots, \frac{r_i}{r_i}\underline{i} \wedge \underline{j}, \frac{r_i}{r_i}\underline{i} \wedge (-\underline{j}), 0, \dots, 0) = (0, \dots, 0, \underline{k}, -\underline{k}, 0, \dots, 0) = v_i;$$

$$J\hat{u}_i = (\dots, -\frac{r_i}{r_i}\underline{i} \wedge \underline{j}, -\frac{r_i}{r_i}\underline{i} \wedge (-\underline{j}), 0, \dots, 0) = (0, \dots, 0, -\underline{k}, \underline{k}, 0, \dots, 0) = -\hat{v}_i;$$

and $Jv_i = -u_i$, $J(-\hat{v}_i) = -\hat{u}_i$ and $J(-Jw) = w$.

A is a complex structure

In this section we will verify that A is a complex structure. To check that $A^2 = -Id$ we write the matrix of A with respect to the basis $\mathcal{B}_1$ (with a little abuse of notation, we will call this matrix A).

First of all we can note that $\xi(u_i) = \xi(\hat{u}_i) = \xi(v_i) = \xi(\hat{v}_i) = 0$ (remember that $\xi(v) = \sum_{i=1}^{n-2} v_i$ for all $v \in \mathbb{R}^{3n}$). So

$$\langle \hat{A}(u_i), \delta^2 \rangle = \langle \underline{j}, \underline{j} \rangle + \langle \underline{j}, -\underline{j} \rangle = 0$$

and similarly $\langle \hat{A}(\hat{u}_i), \delta^2 \rangle = \langle \hat{A}(v_i), \delta^3 \rangle = \langle \hat{A}(\hat{v}_i), \delta^3 \rangle = 0$. Moreover it is trivial to see that $\langle \hat{A}(u_i), \delta^3 \rangle = \langle \hat{A}(\hat{u}_i), \delta^3 \rangle = \langle \hat{A}(v_i), \delta^2 \rangle = \langle \hat{A}(\hat{v}_i), \delta^2 \rangle = 0$. Now it is easy to verify that

$$Au_i = (0, \ldots, 0, \underline{k}, -\underline{k}, 0, \ldots, 0) = v_i, \quad \forall i = 1, \ldots \ell - 1;$$

$$Av_i = (0, \ldots, 0, -\underline{j}, \underline{j}, 0, \ldots, 0) = -u_i, \quad \forall i = 1, \ldots \ell - 1;$$

$$A\hat{u}_i = (0, \ldots, 0, \underline{k}, -\underline{k}, 0, \ldots, 0) = \hat{v}_i, \quad \forall i = \ell, \ldots n - 3;$$

$$A\hat{v}_i = (0, \ldots, 0, -\underline{j}, \underline{j}, 0, \ldots, 0) = -\hat{u}_i, \quad \forall i = \ell, \ldots n - 3;$$

Also

$$A(-Jw) = b_1 v_1 + \ldots + b_{k-2} \hat{v}_{k-2} + bw$$

and

$$A(w) = a_1 u_1 + \ldots + a_{n-3} \hat{u}_{n-3} + a(-Jw)$$

$a_i, b_i, a, b \in \mathbb{R}$, then the matrix A is:

$$A = \left(\begin{array}{cccc|cccc|cc}
0 & 1 & & & & & & & 0 & a_1 \\
-1 & 0 & & & & & & & b_1 & 0 \\
& & \ddots & & & & & & \vdots & \vdots \\
& & & 0 \;\; 1 & & & & & 0 & a_{\ell-1} \\
& & & -1 \;\; 0 & & & & & b_{\ell-1} & 0 \\
\hline
& & & & 0 & -1 & & & 0 & a_\ell \\
& & & & 1 & 0 & & & b_\ell & 0 \\
& & & & & & \ddots & & \vdots & \vdots \\
& & & & & & & 0 \;\; -1 & 0 & a_{n-3} \\
& & & & & & & 1 \;\; 0 & b_{n-3} & 0 \\
\hline
& & & & & & & & 0 & a \\
& & & & & & & & b & 0
\end{array}\right).$$

So $A^2 = -Id \iff ab = -1$.

Determine a and b. First of all we can notice that the last two components of $A(-Jw)$ and of $A(w)$ are enough to determine a and b because the vectors $u_i, \hat{u}_i, v_i, \hat{v}_i$ have no influence on the final components.
Observing that $\langle \hat{A}(-Jw), \delta^3 \rangle = 0$ (because $-Jw$ has no not 0 components along $\underline{k}$), it follows that:

$$A(-Jw) = \Big(\dots, \frac{\langle \hat{A}(-Jw), \delta^2 \rangle}{\|\delta^2\|^2} \frac{r_{n-1}\|\mu\|}{r_{n-1}+r_n}\underline{k}, \frac{\langle \hat{A}(-Jw), \delta^2 \rangle}{\|\delta^2\|^2} \frac{r_n\|\mu\|}{r_{n-1}+r_n}\underline{k}\Big).$$

Now, recalling that $w = (\lambda\underline{k}, \dots, \lambda\underline{k}, \gamma\underline{k}, \dots, \gamma\underline{k}, -r_{n-1}\underline{k}, -r_n\underline{k})$ we get

$$b = -\frac{\langle \hat{A}(-Jw), \delta^2 \rangle}{\|\delta^2\|^2} \frac{\|\mu\|}{r_{n-1}+r_n}. \tag{2.11}$$

Similarly it is possible to observe that $\langle \hat{A}(w), \delta^2 \rangle = 0$, thus

$$A(w) = \Big(\dots, -\frac{\langle \hat{A}(w), \delta^3 \rangle}{\|\delta^3\|^2}\big(r_n sin\theta\underline{i} + (\|\mu\| - r_n cos\theta)\underline{j}\big),$$

$$-\frac{\langle \hat{A}(w), \delta^3 \rangle}{\|\delta^3\|^2}\big(-r_n sin\theta\underline{i} + r_n cos\theta\underline{j}\big)\Big).$$

Comparing $A(w)$ with the last two components of $-Jw$ we get:

$$a = -\frac{\langle \hat{A}(w), \delta^3 \rangle}{\|\delta^3\|^2}. \tag{2.12}$$

Remember that $\xi(-Jw) = \sum_{i=1}^{n-2}(-Jw)_i = \ell\lambda - (n-\ell-2)\gamma = -\|\mu\|\underline{j}$. Then

$$\langle \hat{A}(-Jw), \delta^2 \rangle = \sum_{j=1}^{\ell}\Big(\frac{r_j}{\|\mu\|}\|\mu\| + \lambda\Big) - \sum_{j=\ell+1}^{n-2}\Big(-\frac{r_j}{\|\mu\|}\|\mu\| - \gamma\Big) =$$

$$\sum_{j=1}^{n-2} r_j + \ell\lambda + (n-\ell-2)\gamma = \sum_{j=1}^{n-2} r_j + r_{n-1} + r_n = 2.$$

$$\|\delta^2\|^2 = \sum_{j=1}^{n-2} r_j + \frac{\|\mu\|^2(r_{n-1}+r_n)}{(r_{n-1}+r_n)^2} = \frac{(r_{n-1}+r_n)\sum_{j=1}^{n-2} r_j + \|\mu\|^2}{r_{n-1}+r_n}.$$

So

$$b = -\frac{2\|\mu\|}{(r_{n-1}+r_n)\sum_{j=1}^{n-2} r_j + \|\mu\|^2}.$$

Similarly, $\xi(w) = \ell\lambda + (n-\ell-2)\gamma = -\frac{1}{2}(\|\mu\| - r_{n-1} - r_n) + \frac{1}{2}(\|\mu\| + r_{n-1} + r_n) = r_{n-1} + r_n$.

$$\langle \hat{A}(w), \delta^3 \rangle = \sum_{j=1}^{\ell}\Big(-\frac{r_j}{\|\mu\|}(r_{n-1}+r_n) + \lambda\Big) + \sum_{j=\ell+1}^{n-2}\Big(-\frac{r_j}{\|\mu\|}(r_{n-1}+r_n) - \gamma\Big) =$$

$$-\frac{r_{n-1}+r_n}{\|\mu\|}\sum_{j=1}^{n-2} r_j + \ell\lambda - (n-\ell-2)\gamma = -\frac{(r_{n-1}+r_n)\sum_{j=1}^{n-2} r_j + \|\mu\|^2}{\|\mu\|}.$$

$\|\delta^3\|^2 = \sum_{j=1}^{n-2} r_j + r_{n-1} + r_n = 2.$

So

$$a = \frac{(r_{n-1}+r_n)\sum_{j=1}^{n-2} r_j + \|\mu\|^2}{2\|\mu\|}. \tag{2.13}$$

It is now straightforward to verify that $ab = -1$, and so $A^2 = -Id$.

Conclusions

$\mathcal{B}_2 = \{\ldots, u_i, Au_i, \ldots, \hat{u}_i, A\hat{u}_i, \ldots, -Aw, w\}$ is a A-positive basis of $T_{[P]}M_r$. Then $\mathcal{B}_2$ is also J-positive if and only if the determinant of the matrix of the change of base $M_{\mathcal{B}_2\mathcal{B}_1} = M$ is positive; in this case the orientation induced by A is positive (or concord with the one induced by J).
Let $-Aw = \alpha_1 u_1 + \ldots + \alpha_n \hat{u}_n + \alpha(-Jw)$, from the description of A given in the previous section the matrix of the change of coordinates is

$$M = \left(\begin{array}{ccccc|ccccc|cc}
1 & & & & & & & & & & \alpha_1 & 0 \\
 & 1 & & & & & & & & & 0 & 0 \\
 & & 1 & & & & & & & & \alpha_2 & 0 \\
 & & & \ddots & & & & & & & \vdots & \vdots \\
 & & & & 1 & & & & & & 0 & 0 \\
\hline
 & & & & & 1 & & & & & \alpha_l & 0 \\
 & & & & & & -1 & & & & 0 & 0 \\
 & & & & & & & \ddots & & & \vdots & \vdots \\
 & & & & & & & & 1 & & \alpha_{n-4} & 0 \\
 & & & & & & & & & -1 & 0 & 0 \\
\hline
 & & & & & & & & & & \alpha & 0 \\
 & & & & & & & & & & 0 & 1
\end{array}\right)$$

So $det M = (-1)^{n-3-\ell}\alpha$.

Now, $\alpha = -a$, from 2.13 it follows that

$$\alpha = -\frac{(r_{n-1} + r_n)\sum_{j=1}^{n-2} r_j + \|\mu\|^2}{2\|\mu\|} < 0.$$

So $sgn(det(M)) = (-1)^{n-\ell}$ and $[P]$ contributes to the cobordism class of M_r with $(-1)^{n-\ell}\mathbb{P}^{n-3}(\mathbb{C})$.

Remark 13. We already observed that if $\ell = n-2$ then the vectors $u_i, v_i, \hat{u}_i, \hat{v}_i$ is a basis of $T_{[P]}M_r$. In this case it is straightforward to see that the orientation induced by A and J agree, i.e. $det(M) = 1$. So the result $sgn(det(M)) = (-1)^{n-\ell}$ holds for each $\ell = 1, \ldots, n-2$.

Remark 14. We assumed in 2.2.1 that the first ℓ edges are oriented as the x-axis and the following $n-\ell-2$ are conversely oriented. We already pointed out that this assumption is equivalent to choosing a particular class $[P]$. Let us consider another fixed point $[Q] = [\vec{e}]$ of type I. Because the first $n-2$ edges are on the x-axis and $\mu = e_1 + \ldots + e_{n-2} = \|\mu\|\underline{i}$, then there exist two subsets I and I^c of $\{1, \ldots, n-2\}$ such that $I \cap I^c = \emptyset$, $I \cup I^c = \{1, \ldots, n-2\}$, and such that

$$e_i = (r_i, 0, 0) \quad \forall i \in I$$

$$e_i = (-r_i, 0, 0) \quad \forall i \in I^c.$$

Let ℓ be the cardinality of I. If $I = \{1, \ldots \ell\}$ then this is the case that we studied in detail. Otherwise, the proof extends word by word just changing $\{1, \ldots, \ell\}$ with I and $\{\ell+1, \ldots, n-2\}$ with I^c. So a generic point $[Q]$ contributes to the cobordism class of M_r with $(-1)^{n-\ell}\mathbb{P}^{n-3}(\mathbb{C})$ where ℓ is the number of forward tracks, i.e. $\ell = \sharp\{e_j \mid e_j \cdot \mu > 0\}$.

Remark 15. If $n = 2m$ then the odd dimensional projective space $\mathbb{CP}^{n-3}$ is the total space of a sphere bundle over the quaternion projective space $\mathbb{HP}^{m-2}$, and hence is the boundary of an associated disk bundle. So, if n is even $M_r \sim 0$.

2.3 Some examples

The case $n = 5$

We will now calculate explicitly the cobordism class of the moduli space M_r when $n = 5$ for some choices of r such that $\tilde{M}_r$ does not contain degenerate polygons, or equivalently such that M_r is a smooth manifold. For each length vector r we will analyse which index sets I are admissible (see definition 2.1). We point out that if I does not satisfy the closing conditions (system 2.1), also its complement $I^c := \{1, \ldots, 5\} \setminus I$ does not. Moreover if I is admissible then I^c can't be admissible too, in fact just one between $\sum_{i \in I} \varepsilon_i r_i > 0$ and $\sum_{i \in I^c} \varepsilon_i r_i > 0 = -\sum_{i \in I} \varepsilon_i r_i > 0$ is true. In this section we will denote an element of $(M_r^{S^1})_{isol}$ just by giving the signs of the vectors e_1, e_2, e_3, so for example $++-$ say us

$$e_1 = (r_1, 0, 0), e_2 = (r_2, 0, 0), e_3 = (-r_3, 0, 0),$$

and the remaining edges e_4, e_5 are determined up to rotations. So the class (uniquely) determined in M_r by $++-$ will be denoted by P_{++-}.

In the examples the vector of lengths is not normalized (i.e. $\sum_i r_i \neq 2$), this will keep the notation cleaner and is not restrictive because $M_r \simeq M_{\lambda r}$ for all $\lambda \in \mathbb{R}^+$.

Each of the following examples is obtained by its previous one by crossing a inner wall in Ξ, or equivalently (because M_r is toric for $n = 5$) by chopping off a vertex in the moment polytope $\mu_{T^2} M_r$. We will go back on this remark at the end of this section, but this should be kept in mind as looking at the moment polytopes.

1. **r=(1,1.5,4,1,2)**
   M_r is a manifold, and the only configuration that is admissible for this choice of r is:

$--+ \Rightarrow$ the associated index set I is $\{3\}$, so $\ell = 1$; on $T_{P_{--+}} M_r$ $A = (-1)^{n-\ell} J$, so the $\mathbb{CP}^2$ produced with the surgery around P_{--+} comes with sign $(-1)^{5-1} = 1$, i.e. it comes with the standard orientation.

Other configurations are not admissible, in fact:

$+++ \Rightarrow$ closing conditions 2.1 not satisfied;

$++- \Rightarrow$ associated to $I = \{1, 2\}$ for which $\sum_{i \in I} \varepsilon_i r_i < 0$ (I is the complement of $\{3\}$);

$+-+$ $\Rightarrow$ closing conditions not satisfied (and so for $-+-$);

$-++$ $\Rightarrow$ closing conditions not satisfied (and so for $+--$).

Thus
$$M_r \simeq \mathbb{CP}^2.$$
In this case the image $\mu_{T^2}(M_r)$ (as described by equation (1.5)) is as in figure 2.4.

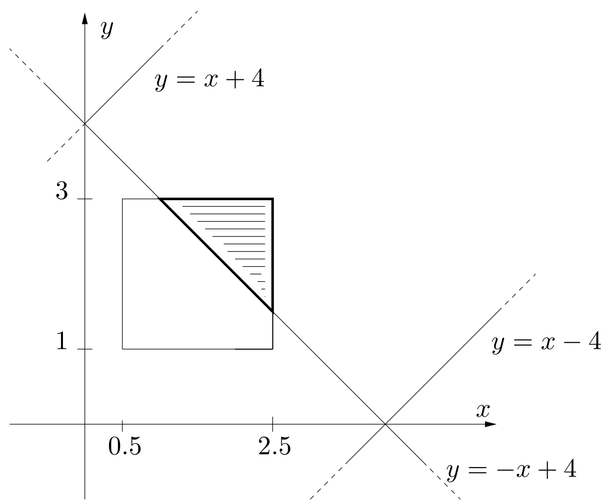

Figure 2.4: $\mu_{T^2}(M_r)$, $M_r \simeq \mathbb{CP}^2$.

2. **r=(0.5,2,4,1,2)**

M_r is a manifold, and the configurations that this length vector admits are:

$-++$ $\Rightarrow l = 2 \Rightarrow$ on $T_{P_{-++}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

$--+$ $\Rightarrow l = 3; \Rightarrow$ on $T_{P_{+++}}M_r$, $A = J$ and $\mathbb{CP}^2$ comes with the standard orientation.

Thus

$$M_r \simeq \mathbb{CP}^2 \amalg -\mathbb{CP}^2 \simeq 0.$$

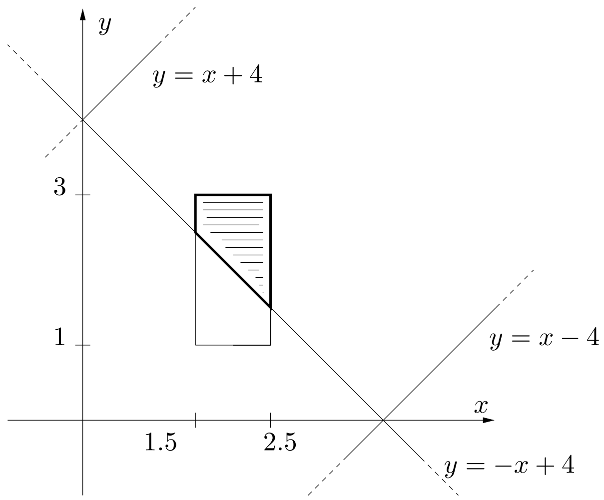

Figure 2.5: $\mu(M_r)$, $M_r \simeq 0$.

For this choice of r the image $\mu_{T^2}(M_r)$ is as in figure 2.5

3. **r=(2,0.5,4,0.5,2.5)**

   M_r is a manifold, and the only admissible configuration is:

$-++$ $\Rightarrow$ associated $I = \{2,3\}$, so $\ell = 2 \Rightarrow$ on $T_{P_{-++}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

The other configurations are not admissible, in fact:

$+++$ $\Rightarrow$ closing condition not satisfied;

$++-$ $\Rightarrow$ closing condition not satisfied (and so for $--+$);

$--+$ $\Rightarrow$ closing condition not satisfied (and so for $++-$);

$+-+$ $\Rightarrow$ closing condition not satisfied (and so for $-+-$);

$+--$ $\Rightarrow$ associated $I = \{1\}$, $\sum_{i \in I} \varepsilon_i r_i < 0$ (I is the complement of $\{2,3\}$).

Thus

$$M_r \simeq -\mathbb{CP}^2.$$

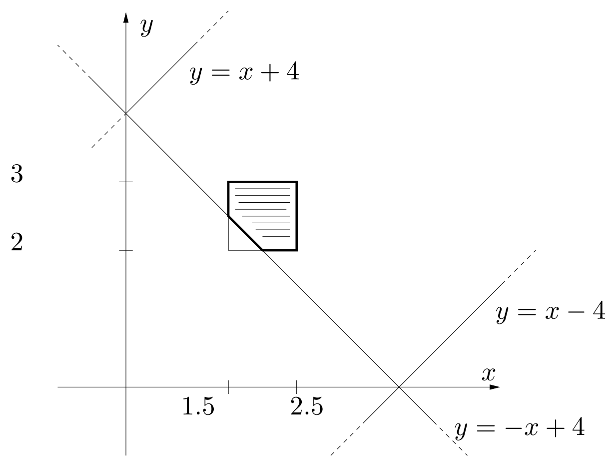

Figure 2.6: $\mu_{T^2}(M_r)$, $M_r \simeq -\mathbb{CP}^2$.

The image $\mu_{T^2}(M_r)$ of M_r is then the 5-sides polytope in figure 2.6.

4. **r=(2,3.5,4,1,2)**

   M_r is a manifold, and the admissible configurations are:

$++-$ $\Rightarrow$ $\ell = 2 \Rightarrow$ on $T_{P_{++-}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

$+-+$ $\Rightarrow$ $\ell = 2 \Rightarrow$ on $T_{P_{+-+}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

It is possible to check that no other configuration are admissible, thus

$$M_r \simeq -\mathbb{CP}^2(\mathbb{C}) \amalg -\mathbb{CP}^2(\mathbb{C}) \simeq -2\mathbb{CP}^2(\mathbb{C}).$$

As before, it is immediate to draw the polytope $\mu_{T^2}(M_r)$, see figure 2.7.

5. **r=(2,3.5,4,3.5,2.5)**

   M_r is a manifold, and the configurations that this vector of lengths admits are:

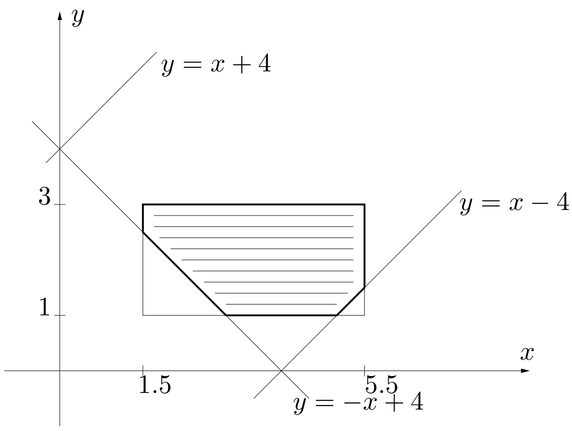

Figure 2.7: $\mu_{T^2}(M_r)$, $M_r \simeq -2\mathbb{CP}^2$.

$++-$ $\Rightarrow$ $\ell = 2$ $\Rightarrow$ on $T_{P_{++-}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

$+-+$ $\Rightarrow$ $\ell = 2$ $\Rightarrow$ on $T_{P_{+-+}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

$-++$ $\Rightarrow$ $\ell = 2$ $\Rightarrow$ on $T_{P_{-++}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

Thus

$$M_r \simeq -\mathbb{CP}^2 \amalg -\mathbb{CP}^2 \amalg -\mathbb{CP}^2 \simeq -3\mathbb{CP}^2.$$

For this choice of the length vector r the image $\mu_{T^2}(M_r)$ is as in figure 2.8.

6. **r=(5,1,4,5,1)**

M_r is a manifold; for this choice of r the set $(M_r^{S^1})_{isol}$ is empty, in fact none of the configuration $+++$, $++-$, $+-+$, $-++$ satisfy the closing condition, thus

$$M_r \simeq 0.$$

and $\mu_{T^2}(M_r)$ is as in figure 2.9.

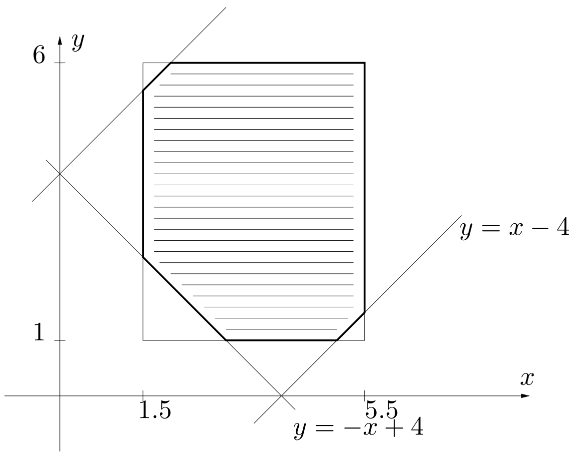

Figure 2.8: $\mu_{T^2}(M_r),\ M_r \simeq -3\mathbb{CP}^2$.

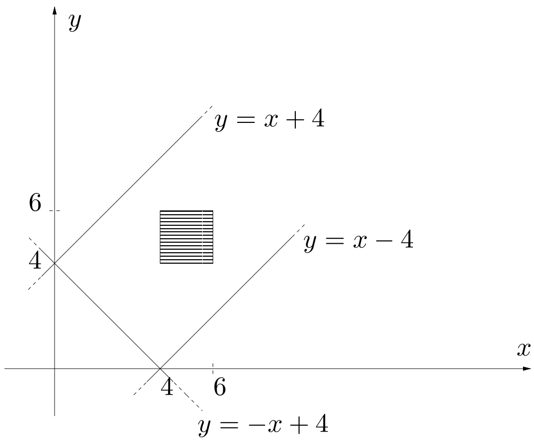

Figure 2.9: $\mu_{T^2}(M_r),\ M_r \simeq 0$.

7. **r=(1,1.5,3.5,3,3.5)**

   M_r is a manifold, and the configurations that this vector of lengths admits are:

$+++ \Rightarrow \ell = 3; \Rightarrow$ on $T_{P_{+++}}M_r$, $A = J$ and $\mathbb{CP}^2$ comes with the standard orientation.

$+-+ \Rightarrow \ell = 2 \Rightarrow$ on $T_{P_{+-+}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

$-++ \Rightarrow \ell = 2 \Rightarrow$ on $T_{P_{-++}}M_r$, $A = -J$ and $\mathbb{CP}^2$ comes with the opposite orientation to the standard one.

$--+ \Rightarrow \ell = 1; \Rightarrow$ on $T_{P_{+++}}M_r$, $A = J$ and $\mathbb{CP}^2$ comes with the standard orientation.

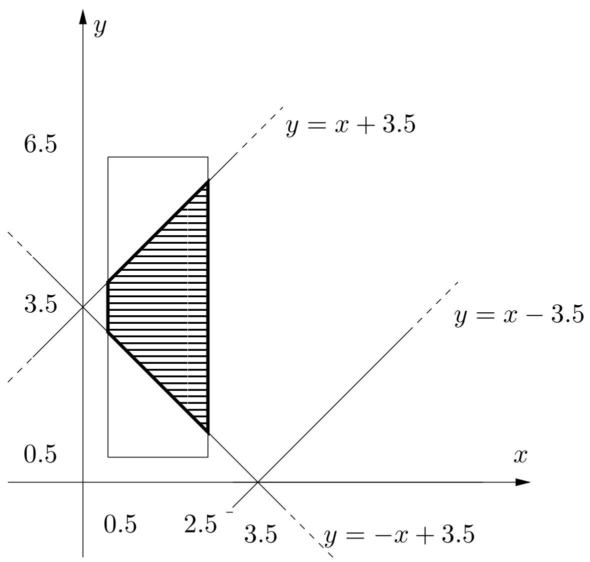

Figure 2.10: $\mu_{T^2}(M_r)$, $M_r \simeq 0$.

Thus

$$M_r \simeq \mathbb{CP}^2 \amalg \mathbb{CP}^2 \amalg -\mathbb{CP}^2 \amalg -\mathbb{CP}^2 \simeq 0$$

and the moment image $\mu_{T^2}(M_r)$ is as in figure 2.10

Note that the examples above are built by "chopping off a vertex" at each step. This has a formal description: "chopping a vertex" corresponds to a wall crossing in Ξ, for example the passage from r's such that $\mu_{T^2}(M_r)$ is as in figure 2.4 to r's such that the moment image $\mu_{T^2}(M_r)$ is as in figure 2.5 corresponds to the crossing of the wall $r_1 + r_3 = r_2 + r_4 + r_5$, (which is the coincidence condition of the intersection points BC and $B2$, see section 1.3).

This is an expected phenomenon, in fact in the 4-dimensional case ($n = 5$) crossing a wall has the effect to blow up a fixed point (or blow down, depending on the wall-crossing direction). In chapter 4 we will give a detailed description of this, together with a complete analysis of wall problems in higher dimension (using arguments presented by Guillemin and Sternberg in [GS89]).

By the notion of admissibility for an index subset I, it follows that for $n = 5$ these are all the possible cobordism types of M_r. Moreover for r's in the same region of regular values $\Delta \subset \Xi$, the moment polytope $\mu_{T^2}(M_r)$ has the same "shape", and its number of edges is an invariant of cobordism.

Remark 16. The manifolds M_r as in examples 2 and 6-7 have the same cobordism type ($M_r \simeq 0$), but different diffeotype, and thus different symplectomorphism type. The moment polytope $\mu_{T^2}(M_r)$ contains all the informations needed to recover the (T^2-equivariant) symplectomorphism type (see Delzant [De], Lerman-Tolman [LT]). For M_r's such that the moment polytope is as in example 6,and more generally when the opposite edges of the polytope $\mu_{T^2}(M_r)$ are parallel, it is well-known that the manifold M_r is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, see, for example, [ACL].

Let us now analyze the cases such that the moment polytope has shape as in figures 2.4 and 2.10. Karshon [Ka] finds explicitly the (S^1-equivariant) symplectomorphism types for these examples, and, in particular, establishes when they are the same. A possible way to see it is the following: because $\mu_{T^2}(M_r)$ is the intersection of the regions I and Υ, its edges are either horizontal, vertical or have slope ± 1, moreover there is always a pair of opposite edges which are parallel. If the normals to the other opposites edges (the non-parallel ones) generate the lattice $\mathbb{Z}^2$ then M_r is diffeomorphic to $\mathbb{CP}^2$ blown up at a point; otherwise, if they generate a sublattice of $\mathbb{Z}^2$ of index two, it is diffeomorphic to $S^2 \times S^2 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$.

Chapter 3

The Volume of M_r

The main result of this chapter is a volume formula (Theorem 3.3.1) that describes the volume of the moduli space M_r as a piece-wise polynomial function in the r_i's. We saw in the previous chapter that the bending action along a system of $n-3$ non-intersecting diagonals allows one to define a system of action-angle coordinates on a open dense subset of M_r. Because of the complete integrability of the system it is possible to calculate the volume of M_r from the convex polytope image of the manifold via the moment map; anyway the impossibility to give a unique formula for generic n and, even fixing the dimension n, for the possible different quotients suggests to approach the problem in a different way. The study of the integration of equivariant cohomology classes developed by S. Martin ([Ma] and [Ma2]) and separately by V. Guillemin and J. Kalkman ([GK96]) is the key to prove theorem 3.3.1. An interesting application of this result will be the calculation of the cohomology ring $H^*(M_r)$, see chapter 5.

In the first section some basic facts and definitions of equivariant cohomology are recalled, and in the second there is a quick overview on the results that S. Martin proved in his thesis. In section 3.3 the volume formula comes, then some examples are given.

3.1 Equivariant cohomology

In this section we will give some basic definitions and results in equivariant cohomology that will be helpful in the next sections, and in particular in the proof of the volume theorem 3.3.1 which is the main result of this chapter. About equivariant cohomology there is a rich literature, in particular we suggest the beautiful survey papers [AB] and [Du], and also the book [Ki].

Let G be a compact Lie group acting on a smooth manifold in a Hamiltonian way, with associate moment map $\mu : M \to \mathfrak{g}^*$. The equivariant cohomology of M is defined to be the ordinary cohomology of $M_G := EG \times_G M$, where EG is the total space of the universal bundle $EG \to BG$, BG being the classifying space of the group G.

If the action is locally free, then the equivariant cohomology of M is canonically isomorphic to the de Rham cohomology of the quotient space M/G, and the isomorphism is given by pullback $\pi^* : H^*(M/G) \to H^*_G(M)$ induced in cohomology by the projection $\pi : M \to M/G$ of M on the topological quotient M/G.

Let ξ be a regular value for the moment map μ, so that the level set $\mu^{-1}(\xi)$ is a smooth compact submanifold of M, of codimension equal to the dimension of the Lie algebra $\mathfrak{g}$. Moreover, under the assumptions above $\mu^{-1}(\xi)$ is G-invariant and and G acts locally freely on $\mu^{-1}(\xi)$, thus the orbit space $\mu^{-1}(\xi)/G := M/\!\!/G(\xi)$ is an orbifold.

Call π_{xi} the projection $\pi_\xi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G := M/\!\!/G(\xi)$ that associate to $x \in \mu^{-1}(\xi)$ its G-orbit $G \cdot x$; and let i_ξ denote the inclusion $\mu^{-1}(\xi) \hookrightarrow M$ of the ξ-level set in the manifold M.

Then, using the gradient flow of the function $x \mapsto \|\mu(x)\|^2$ on M, F. Kirwan [Ki] proved that the map

$$i_\xi^* : H^*_G(M) \to H^*(\mu^{-1}(\xi))$$

is surjective. Because the G-action on the level set $\mu^{-1}(\xi)$ is free then π_ξ^* : $H^*(M/\!\!/G(\xi)) \to H^*G(\mu^{-1}(\xi))$ is an isomorphism. These two combines to give the following homomorphism

$$k_\xi := (\pi_\xi^*)^{-1} \circ i_\xi : H_G^*(M) \to H^*(M /\!\!/ G(\xi))$$

which is known as *Kirwan map*, and is one of the fundamental tools in equivariant cohomology.

The surjectivity of the Kirwan map rises the hope that a good deal of informations on $H^*(M /\!\!/ G(\xi))$ of a reduced space can be computed from the equivariant cohomology $H_G^*(M)$ of M, and not from its ordinary cohomology $H^*(M)$, which is often much simpler than the one of the reduced space. The extra informations encoded by the equivariant cohomology turns out to be related with the orbit structure of the G-action, and in this sense equivariant cohomology is the natural setting for results, known as "localization theorems", which enable many computation to be reduced to the fixed point set of the G-action. In the next section we will see one of these results, due to S. Martin, which will enable us to prove the volume formula 3.3.1.

Remark 17. As pointed out in [AB], the functorial nature of the construction that to M associate M_G enables one to define equivariant correspondents of the concepts of ordinary cohomology in a "natural" way. In particular, if V is a vector bundle over M, then any action of G on V lifting the action on M can be used to define a vector bundle $V_G = EG \times_G V$ over M_G that extends the bundle $V \to M$. Thus, for example, the first Chern class of V_G, $c_1(V_G)$, naturally lies in $H^*(M_G) =: H_G^*(M)$ and is called the equivariant first Chern class of V, denoted by $c_1^G(V)$. All other equivariant characteristic classes are defined in a similar way.

3.2 S. Martin's Results

Let X be a symplectic manifold endowed of a Hamiltonian action of a torus T with associated moment map $\mu : X \to \mathfrak{t}^*$..

Definition 3.1. Let p_0 and p_1 be two regular value of the moment map $\mu *$. A *transverse path* Z is a one-dimensional submanifold $Z \subset \mathfrak{t}^*$ with boundary $\{p_0, p_1\}$ such that Z is transverse to μ. A *wall* in $\mathfrak{t}^*$ is defined to be a connected component of $\mu(X^H)$ where X^H is the fixed points set for some oriented subgroup $H \simeq S^1$ of T.

We point out that by transversality theory the preimage $\mu^{-1}(Z)$ of a transverse path Z is a submanifold of X.

Definition 3.2. Associated to each transverse path Z there is a finite set $\mathcal{D}(Z)$ which we refer to as the *wall crossing data for* Z. $\mathcal{D}(Z)$ is defined to be the set of pairs (H, q) such that $\mu(X^H)$ is a wall crossed by Z at the point q; the orientation of H is defined by the wall-crossing direction: let us orient Z from p_0 to p_1, each positive tangent vector field in T_qZ, thought as an element of $\mathfrak{t}^*$, defines a functional on $\mathfrak{t}$ which restricts to a nonzero functional on $\mathfrak{h}:=$ Lie(H). The orientation of H is defined to be the positive one with respect to this functional.

Theorem 3.2.1. Localization Theorem
Let p_0 and p_1 be regular values of the moment map μ joined by a transverse path Z having a single wall crossing at q and let $H \simeq S^1$ be the (oriented) subgroup associated to the wall. There exists a map

$$\lambda_H : H^*_T(X) \to H^*_{T/H}(X^H),$$

called localization map such that, for any $a \in H^*_T(X)$,

$$\int_{X/\!\!/T(p_0)} k_0(a) - \int_{X/\!\!/T(p_1)} k_1(a) = \int_{X^H/\!\!/T(q)} k_q(\lambda_H(a_{|_{X^H}}))$$

where the maps $k_i : H^*_T(X) \to H^*(X/\!\!/T(p_i))$ are the Kirwan maps, $X^H/\!\!/T(q)$ is the symplectic quotient of $\mu^{-1}_{|_{X^H}}(q) \cap X^H$ by the quotient subgroup T/H and $k_q : H^*_{T/H}(X^H) \to H^*(X^H/\!\!/T(q))$ is the associated Kirwan map.

Remark 18. The quotient $X^H/\!\!/T(q)$ looks of a wild geometrical nature because q is not a regular value for the moment map μ (it lies on a wall), but it is instead well defined. H acts trivially on its fixed points set X^H, thus $\mu(X^H)$ lies in an affine translate $\mathcal{T}$ of $Lie(T/H)^*$. Then q is a regular value for the map $\mu_{|_{X^H}} : X^H \to \mathcal{T}$, and so $\mu^{-1}_{|_{X^H}}(q) \cap X^H$ is a compact closed submanifold of X^H. $X^H /\!\!/ T(q)$ is defined to be the quotient of $\mu^{-1}_{|_{X^H}}(q) \cap X^H$ by the group T/H and is a symplectic orbifold.

It is possible to describe the localization map λ in terms of equivariant characteristic classes. To this aim, we introduce briefly the definition of (ordinary and equivariant) weighted Chern and Segre classes. Let $V \to Y$ be a complex vector bundle and $\mathbb{P}(V) \to Y$ be its projectivitation (see[BT]). Suppose that the bundle V is endowed of a S^1-action linear on the fibers and such that the set of fixed points equals the zero section. Let $S(V)$ be the unit sphere bundle in V (with respect to some S^1-invariant metric). If S^1 acts with weight 1 on the fibers, i.e. on each fiber the action is the standard multiplication by $e^{i\theta}$, then there is an induced isomorphism $S(V)/S^1 \simeq \mathbb{P}(V)$. For generic weights of the S^1-action it is possible to define a cohomology class (the weighted Chern class $c^w(V)$ of the pair (V, S^1)) that restricts to the total Chern class $c(V)$ when the weights are all 1. (For definition and properties of $c(V)$ we refer to [BT]).

Note that under the S^1-action V splits into isotypic subbundles $V \simeq \oplus_{i\in\mathbb{Z}} V_i$. The weighted Chern class $c^w(V)$ of V is defined to be the product

$$c^w(V) = \prod_i c^w(V_i),$$

where $c^w(V_i) = i^r + i^{r-1}c_1(V_i) + \ldots + c_r(V_i)$, $c_j(V_i)$ being the j-th Chern class of V_i and r the rank of V_i. Observe that $c^w(V)$ is invertible because none of the V_i is acted on with weight zero (this is equivalent to assume that the zero section equals the set of points fixed by the action). The weighted Segre class $s^w(V)$ is defined to be its inverse, i.e. $s^w(V)c^w(V) = 1$.

Now, applying the arguments described in remark 17, we can define equivariant weighted Chern and Segre classes: let G be a Lie group acting on V, and suppose that the actions of S^1 and G commute. The G-equivariant weighted Chern class $(c^w)^G(V)$ is defined to be the ordinary weighted Chern class $c^w(V_G)$ of $V_G = EG \times V$. Similarly, the G-equivariant weighted Segre class $(s^w)^G(V)$ is by definition $s^w(V_G)$.

Now let us go back to the localization theorem' setting, and recall that $H \simeq S^1$ is the subgroup of T associated to the wall we are examining. Let $T' \subset T$ be a complement to H, i.e. $T = T' \times H$, this defines an isomorphism $H_T^*(X^H) \cong H_{T'}^*(X^H) \otimes H_H^*(X^H)$. Note that $H_H^*(X^H) \cong H^*(BH)$ (it is enough to remember that $H_H^*(X^H)$ is defined to be the ordinary cohomology ring $H^*(EH \times_H X^H)$ and note H acts trivially on its fixed points set X^H), so

$$H_T^*(X^H) \cong H_{T'}^*(X^H) \otimes H^*(BH).$$

It follows that the restriction to X^H of any class $a \in H_T^*(X)$ decomposes $a_{|_{X^H}} = \sum_{i \geq 0} a_i \otimes u^i$ where u is the positive generator of $H^*(BH)$ and the a_i are elements in $H_{T'}^*(X^H)$.

Proposition 3.2.2. *With the notation above*

$$\lambda_H(a) = k(X_i^H) \sum_{i \geq 0} a_i \smile s_{i-\rho+1}^w$$

where $k(X_i^H)$ is the greatest common divisor of the weights of the H-action on the fibers of $\nu X_i^H \to X_i^H$, s_j^w denotes the j-th T'-equivariant Segre class of $(\nu X^H, H)$ and ρ is the function (constant on the connected components of X^H) such that $2\rho = \operatorname{rank}(\nu X^H)$.

Next result (see [Ma2]) relates the integration over the symplectic quotient $X/\!\!/G$ of a G-manifold X with the integration over the associated quotient $X/\!\!/T$ by a maximal subtorus $T \subset G$.

Let G be a connected compact Lie group which acts on the smooth manifold X in a Hamiltonian way (with associated moment map μ_G) and let T be a maximal subtorus in G; the restriction of the action of G defines a Hamiltonian action of T on X (with associated moment map μ_T). There is a natural restriction map $r_T^G : H_G^*(X) \to H_T^*(X)$ between the equivariant (respect to G and T) cohomology rings. To fix the notation, $\mathbb{C}_{(w)}^m$ denotes the complex space $\mathbb{C}^m$ endowed of the S^1-action with weight w and $\underline{\mathbb{C}}_{(w)}^m := X \times \mathbb{C}_w$ is the total space of an equivariant line bundle over X.

Theorem 3.2.3. Equivariant integration formula.
For all $a \in H^*_G(X)$,

$$\int_{X /\!\!/ G} k_G(a) = \frac{1}{|W|} \int_{X /\!\!/ T} k_T\Big(r^G_T(a) \smile \prod_{\alpha \in \Delta} c^T_1(\underline{\mathbb{C}}_\alpha) \Big).$$

where $|W|$ *is the order of the Weyl group of* G *and* Δ *is the set of roots of* G.

3.3 The Volume Theorem

Theorem 3.3.1. *For* $r \in \mathbb{R}^n_+$ *such that* M_r *is a smooth manifold,*

$$vol(M_r) = -\frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{I \in \mathcal{I} \\ |I| = k}} (R^+_I - R^-_I)^{n-3},$$

where

$$R^-_I = \sum_{i \in I} r_i, \quad R^+_I = \sum_{i \notin I} r_i$$

and

$$\mathcal{I} = \{ I \subset \{1, \ldots, n\} : R^+_I - R^-_I > 0 \}.$$

Proof. The first step in the proof is to apply theorem 3.2.3 and write the volume of M_r as

$$vol(M_r) = \frac{1}{2} \int_{\mathcal{S}_r /\!\!/ S^1} k_{S^1}(r^{SO(3)}_{S^1}(a) \smile c^{S^1}_1(\underline{\mathbb{C}}_{(1)}) \smile c^{S^1}_1(\underline{\mathbb{C}}_{(-1)}))$$

where $a \in H^*_{SO(3)}(\mathcal{S}_r)$ is such that $k_{SO(3)}(a)$ is the volume form on $\mathcal{S}_r /\!\!/ SO(3) = M_r$ and S^1 is a (arbitrarily chosen) maximal subtorus of $SO(3)$. (We have already entered in the formula that the Weyl group of $SO(3)$ is $\mathbb{Z}/2\mathbb{Z}$ and that the set of roots of $SO(3)$ is $\{\pm 1\}$.)

The second step is to apply the localization theorem 3.2.1 to localize the calculation of the integral above to data associated to the fixed points set of the S^1-action.

Remember that the symplectic structure on $\mathcal{S}_r$ is defined by the 2-form $\omega = \sum_{j=1}^n \frac{1}{r_j} p_j^* \omega_j$, where $p_j : \mathcal{S}_r \to S^2_{r_j}$ is the natural projection on the j-th factor and ω_j is the volume form on the sphere $S^2_{r_j}$. It is a calculation to check that, if α is the volume form on the unit sphere and ω_{FS} is the Fubini–Study form on $\mathbb{CP}^1 \simeq S^2$, then $\omega_j = r_j\alpha = 2r_j\omega_{FS}$.

On each sphere consider the line bundle $\mathcal{O}(2r_j) \to S^2_j$, the tensor product of the pullbacks $p_j^*\mathcal{O}(2r_j)$ of the line bundles $\mathcal{O}(2r_j)$ defines on $\mathcal{S}_r$ the line bundle $\mathcal{L} := \mathcal{O}(2r_1) \boxtimes \ldots \boxtimes \mathcal{O}(2r_n)$ (known in literature as the prequantum line bundle (of $\mathcal{S}_r$)). Observe that ω_{FS} "is" the first Chern class of $\mathcal{O}(1)$, precisely $[\frac{\omega_{FS}}{2\pi}] = c_1(\mathcal{O}(1))$, it follows by the definition of the symplectic form ω on $\mathcal{S}_r$ that

$$\left[\frac{\omega}{2\pi}\right] = c_1(\mathcal{O}(2r_1) \boxtimes \ldots \boxtimes \mathcal{O}(2r_n)) = c_1(\mathcal{L}).$$

The construction above is well defined just for integral $r_1, \ldots, r_n$, so let us restrict to the case $r \in \mathbb{Z}^n_+$ and prove the stated result for the volume of M_r. Then, for each $\lambda \in \mathbb{R}^+$, we get the volume of $M_{\lambda r}$ by rescaling, i.e. $vol(M_{\lambda r}) = (\lambda)^{n-3}(vol M_r)$, thus the formula holds also for rational r_i. Finally, by density, the result extends to $r \in \mathbb{R}^n_+$.

Let a be the $(n-3)$-th power of the first equivariant Chern class $c_1^{SO(3)}(\mathcal{L})$ of the prequantum line bundle $\mathcal{L}$ (normalized by a factor $\frac{(2\pi)^{n-3}}{(n-3)!}$), then its image $k(a)$ trough the Kirwan map $k : H^*_{SO(3)}(\mathcal{S}_r) \to H^*(\mathcal{S}_r /\!\!/ SO(3))$ is the volume form on M_r :

$$vol(M_r) = \frac{(2\pi)^{n-3}}{(n-3)!} \int_{M_r} k(c_1^{SO(3)}(\mathcal{L})^{n-3}).$$

We now apply the integration formula 3.2.3: the restriction $r^{SO(3)}_{S^1}$ maps $c_1^{SO(3)}(\mathcal{L})^{n-3}$ in $c_1^{S^1}(\mathcal{L})^{n-3}$, thus

$$vol(M_r) = \frac{1}{2}\frac{(2\pi)^{n-3}}{(n-3)!} \int_{\mathcal{S}_r /\!\!/ S^1} k_{S^1}(c_1^{S^1}(\mathcal{L})^{n-3} \smile c_1^{S^1}(\underline{\mathbb{C}}_{(1)}) \smile c_1^{S^1}(\underline{\mathbb{C}}_{(-1)}))$$

and the first step is done.

In order to apply the localization theorem 3.2.1 we make an explicit choice of a maximal subtorus $S^1 \subset SO(3)$: let S^1 be the subgroup that acts on each sphere by rotation along the z-axis, i.e.

$$\begin{array}{rcl} S^1 \times S^2_{r_j} & \to & S^2_{r_j} \\ (\theta, e_j) & \mapsto & A_\theta e_j \end{array}$$

where $A_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\,\theta & sin\,\theta \\ 0 & -sin\,\theta & cos\,\theta \end{pmatrix}$ and $e_j = \begin{pmatrix} e_{1,j} \\ e_{2,j} \\ e_{3,j} \end{pmatrix}$.
This action is Hamiltonian with moment map the height function

$$\begin{array}{rcl} \mu : S^2_{r_j} & \longrightarrow & \mathfrak{s}^1 \simeq \mathbb{R} \\ e_j & \mapsto & ht(e_j) := e_{3,j}. \end{array}$$

Note that the fixed points of this action are the north pole N_j and the south pole S_j and the image $\mu(S^2_{r_j})$ is the segment $[\mu(S_j), \mu(N_j)] = [-r_j, r_j]$ (in agreement with the convexity theorem.)

These observations extends easily to the product manifold $\mathcal{S}_r$: a point in $\mathcal{S}_r$ is given by the n-tuple $(e_1, \ldots, e_n)$, and the action of the maximal torus S^1 is

$$\begin{array}{rcl} S^1 \times \mathcal{S}_r & \longrightarrow & \mathcal{S}_r \\ (\theta, (e_1, \ldots e_n)) & \mapsto & (A_\theta e_1, \ldots, A_\theta e_n). \end{array}$$

This action is clearly still Hamiltonian and, by linearity, has moment map the sum of the heights, i.e. if $e_j = (x_j, y_j, z_j)$ then $\mu(e_1, \ldots, e_n) = \sum_j z_j$.

A point $(e_1, \ldots, e_n)$ is fixed by this action if and only if $e_j \in \{N_j, S_j\}$ for each $j \in \{1, \ldots, n\}$, and these points are isolated. For these points we introduce a more handy notation: let I be any subset of $\{1, \ldots, n\}$, we define f_I to be the point $(e_1, \ldots, e_n) \in \mathcal{S}_r$ such that e_j is a south pole if $j \in I$, a north pole otherwise. Thus all the fixed points are a f_I for some index set I and

$$\mu(f_I) = \sum_{i \notin I} r_i - \sum_{i \in I} r_i := R_I^+ - R_I^-.$$

Remark 19. Note that $R_I^+ - R_I^- \neq 0$ for all I; in fact $R_I^+ - R_I^- = 0$ would mean that polygons on a line are possible, which is against the assumption that M_r is a manifold. This implies that 0 is a regular value of the moment map μ, in fact $d_x\mu$ is identically 0 if and only if $x = f_I$: for each tangent vector $v = (v_1, \ldots, v_n) \in T_x\mathcal{S}$, $d_x\mu(v) = \sum_j \zeta_j$ where ζ_j is the third component of v_j. So

$$d_x\mu \equiv 0 \iff \zeta_j = 0 \quad \forall j \iff x = f_I.$$

For the Atiyah and Guillemin-Sternberg convexity theorem, the image $\mu(\mathcal{S}_r)$ is the convex hull of the points $\mu(f_I)$, i.e.

$$\mu(\mathcal{S}_r) = \Big[-\sum_{i=1}^n r_i, \sum_{i=1}^n r_i \Big].$$

The idea is now to apply theorem 3.2.1 to calculate the volume of $\mathcal{S}_r /\!\!/ S^1$. Choose $p_0 = 0$ and $p_1 > \sum_{i=1}^n r_i$, so that $\mu^{-1}(p_1)$ is empty, this imply that the integral over $\mathcal{S}_r /\!\!/ S^1(p_1)$ is zero and

$$\int_{\mathcal{S}_r /\!\!/ S^1} k(\tilde{a}) = \sum \int_{X^H /\!\!/ T(q)} k_q(\lambda_H(\tilde{a}_{|_{X^H}}))$$

where the sum is made over the walls $\mu(X^{H_i})$ that the path $Z = [0, p_1] \subset \mathbb{R}$ crosses at q_i.

Moreover note that the walls in $\mu(\mathcal{S}_r)$ are just the points $\mu(f_I)$, and the path Z crosses the walls $\mu(f_I)$ only for those I such that $R_I^+ - R_I^- > 0$. We call these I admissible and define $\mathcal{I}$ to be the set of all the admissible I. We can also point out that the quotient spaces $X^H /\!\!/ T(q)$ are just points, thus

$$\int_{\mathcal{S}_r /\!\!/ S^1} k(\tilde{a}) = \sum_{I \in \mathcal{I}} k_I(\lambda_I(\tilde{a}_{|_{f_I}})).$$

Now we will study the normal bundle νf_I in order to work out the details necessary to use the equivariant description 3.2.2 of λ_{f_I}.

The f_I's are points thus for each I the normal bundle νf_I is the direct sum of copies of $T_{N_j} S^2_{r_j}$ and $T_{S_j} S^2_{r_j}$. So, if k is the number of south poles in f_I, i.e. $k = |I|$, then

$$\nu f_I \simeq \mathbb{C}^{n-k}_{(1)} \oplus \mathbb{C}^k_{(-1)}.$$

The equivariant Segre classes that appear in 3.2.2 formally lie in $H^*_{T/H}(f_I)$, where $H \simeq S^1$ is the subgroup of T associated to the wall $\mu(f_I)$; in our case T is S^1 itself, then $s^w(\nu f_I)$ lies in the de Rham cohomology ring $H^*(f_I)$. The bundle νf_I has rank one, and the j-th Chern classes $c_j(\mathbb{C}_{(\pm 1)})$ are zero for each i and j (because, for each I, νf_I is a line bundle over a point), then

$$c^w(\nu f_I) = \prod_i i = (-1)^{n-k}$$

where k is the number of south poles. Then

$$s_j^w(\nu f_I)\begin{cases}(-1)^{n-k} & j=0, k=|I|\\ 0 & \text{otherwise}\end{cases}$$

Moreover, $k(f_I)=1$ for each I because the the weights are all ± 1.

We have now all the ingredients to apply the equivariant formula 3.2.2 and calculate $\lambda_I(\tilde{a}_{|f_I})$, with $\tilde{a}=c_1^{S^1}(\mathcal{L})^{n-3}\smile c_1^{S^1}(\underline{\mathbb{C}}_{(1)})\smile c_1^{S^1}(\underline{\mathbb{C}}_{(-1)})$.

From the construction of the line bundle $\mathcal{L}$ we made above, it follows that $\mathcal{L}_{|f_I}=\mathbb{C}_{(R_I^+-R_I^-)}$ where again I is the index set that detects the south poles. Thus

$$c_1^{S^1}(\mathcal{L})_{|f_I}=(R_I^+-R_I^-)u,$$

where u is the positive generator of the equivariant cohomology of a point $H^*_{S^1}(f_I)$. Similarly,

$$c_1^{S^1}\left(\underline{\mathbb{C}}_{(1)}\right)_{|f_I}=u, \qquad c_1^{S^1}\left(\underline{\mathbb{C}}_{(-1)}\right)_{|f_I}=-u.$$

So

$$\tilde{a}_{|f_I}=-(R_I^+-R_I^-)^{n-3}u^{n-1}$$

and

$$\lambda_I(\tilde{a}_{|f_I})=-(-1)^k(R_I^+-R_I^-)^{n-3}u^{n-1}.$$

To finish the proof we should now apply the Kirwan map $k_q: H^*_{T/H}(X^H)\to H^*(X^H/\!\!/T(q))$ as in 3.2.1. Note that in our case T is S^1 itself and the fixed points sets X^H are the f_I's, so $k_q: H^*(f_I)\to H^*(f_I)$ is the identity map. Thus, summing on all the admissible I, the result follows.

□

3.4 Examples

Let $r=(2,1,8,2,4)$. We proved in 2.3 that M_r is cobordant $\mathbb{CP}^2$.
$I=\emptyset\Rightarrow R_I^+-R_I^-=17\Rightarrow\emptyset$ is admissible;
$I=\{1\}\Rightarrow R_I^+-R_I^-=13\Rightarrow\{1\}$ is admissible;

$I=\{2\} \Rightarrow R_I^+ - R_I^- = 15 \Rightarrow \{2\}$ is admissible;
$I=\{3\} \Rightarrow R_I^+ - R_I^- = 1 \Rightarrow \{3\}$ is admissible;
$I=\{4\} \Rightarrow R_I^+ - R_I^- = 13 \Rightarrow \{4\}$ is admissible;
$I=\{5\} \Rightarrow R_I^+ - R_I^- = 9 \Rightarrow \{5\}$ is admissible;
$I=\{1,2\} \Rightarrow R_I^+ - R_I^- = 11 \Rightarrow \{1,2\}$ is admissible;
$I=\{1,3\} \Rightarrow R_I^+ - R_I^- = -3 \Rightarrow \{2,4,5\}$ is admissible;
$I=\{1,4\} \Rightarrow R_I^+ - R_I^- = 9 \Rightarrow \{1,4\}$ is admissible;
$I=\{1,5\} \Rightarrow R_I^+ - R_I^- = 5 \Rightarrow \{1,5\}$ is admissible;
$I=\{2,3\} \Rightarrow R_I^+ - R_I^- = -1 \Rightarrow \{1,4,5\}$ is admissible;
$I=\{2,4\} \Rightarrow R_I^+ - R_I^- = 11 \Rightarrow \{2,4\}$ is admissible;
$I=\{2,5\} \Rightarrow R_I^+ - R_I^- = 7 \Rightarrow \{2,5\}$ is admissible;
$I=\{3,4\} \Rightarrow R_I^+ - R_I^- = -3 \Rightarrow \{1,2,5\}$ is admissible;
$I=\{3,5\} \Rightarrow R_I^+ - R_I^- = -7 \Rightarrow \{1,2,4\}$ is admissible;
$I=\{4,5\} \Rightarrow R_I^+ - R_I^- = 5 \Rightarrow \{4,5\}$ is admissible.

Thus, summing on all the admissible I's we get:

$$vol(M_r) = -\pi^2\Big(\underbrace{17^2}_{|I|=0} - \underbrace{(13^2+15^2+1+13^2+9^2)}_{|I|=1} + \underbrace{(11^2+9^2+5^2+11^2+7^2+5^2)}_{|I|=2} -$$

$$-\underbrace{(3^2+1+3^2+7^2)}_{|I|=3}\Big) = -\pi^2(-2) = 2\pi^2.$$

More generally, let Δ_0 be the region in $\Xi \in \mathbb{R}^5$ of regular values for the moment map $\mu_{U_1^6}$ delimited by the following walls:

$$r_i < \sum_{j\neq i} r_i; \qquad \forall i,$$

$$\sum_{i\in\{3,j\}} r_i > \sum_{i\in\{3,j\}^c} r_i; \quad \forall j=1,2,4,5; \quad \{3,j\}^c := \{1,\dots,5\}\setminus\{3,j\}$$

$$\sum_{i\in\{j,k,\ell\}} r_i > \sum_{i\in\{j,k,\ell\}^c} r_i; \quad \forall j,k,\ell=1,2,3,4; \quad \{j,k,\ell\}^c := \{1,\dots,5\}\setminus\{j,k,\ell\}.$$

So for r's in Δ_0 the set of admissible I's is

$$\mathcal{I} = \big\{\{t\}, \{3, j\}, \{j, k, \ell\}/t = 1, \ldots, 5; \quad j, k, \ell = 1, 2, 4, 5\big\}.$$

Then, applying theorem 3.3.1, it is a calculation to check that the volume for the associated symplectic quotient M_r is

$$\mathrm{vol} M_r = 2\pi^2(r_1 + r_2 - r_3 + r_4 + r_5)^2$$

and, because the perimeter $\sum_{i=1}^n r_i = \mathfrak{p}$ is fixed, in particular $\mathfrak{p} = 2$ on Ξ, then we get

$$\mathrm{vol} M_r = 2\pi^2(2 - 2r_3)^2.$$

Remark 20. Note that $r = (2, 1, 8, 2, 4) \notin \Xi$, in fact $\sum_{i=1}^n r_i = 17$. Normalizing it $\bar{r} = \frac{2}{17}(2, 1, 8, 2, 4)$, and the volume of the associated symplectic quotient $M_{\bar{r}}$ can be deduced by the volume of M_r by rescaling, i.e.

$$\mathrm{vol} M_{\bar{r}} = \Big(\frac{2}{17}\Big)^2 \mathrm{vol} M_r = \Big(\frac{2}{17}\Big)^2 2\pi^2$$

in accordance with the formula above.

Let us now calculate the volume of M_r for $r = (1, 4, 8, 2, 4)$, (which is the second example we examined in section 2.3), and in general for r's in the region of regular values Δ_1 such that $r = (1, 4, 8, 2, 4) \in \Delta_1$. Of course it was possible to calculate the set $\mathcal{I}$ of admissible I for $r \in \Delta_1$ as done in the previous example. Anyway recall that Δ_1 can be reached from Δ_0 by crossing the wall $r_1 + r_3 = r_2 + r_4 + r_5$. It is immediate to check that for $r^0 \in \Delta_0$ then $r_1^0 + r_3^0 > r_2^0 + r_4^0 + r_5^0$ (and in fact $I = \{2, 4, 5\}$ is admissible) and $r_1^1 + r_3^1 < r_2^1 + r_4^1 + r_5^1$ for $r^1 \in \Delta_1$. So for $r^1 \in \Delta_1$, $\{2, 4, 5\}$ is no longer admissible, while its complement is.

So Δ_1 is the region delimited by

$$r_i < \sum_{j \neq i} r_i; \qquad \forall i,$$

$$\sum_{i \in \{3,j\}} r_i > \sum_{i \in \{3,j\}^c} r_i; \quad \forall j = 2, 4, 5; \quad \{3, j\}^c := \{1, \ldots, 5\} \setminus \{3, j\}$$

$$r_1 + r_3 < r_2 + r_4 + r_5; \quad r_1 + r_2 + r_5 > r_3 + r_4; \quad r_1 + r_2 + r_4 < r_3 + r_5.$$

So for $r \in \Delta_1$ the admissible I are

$$\mathcal{I} = \{\{t\}; \{3, j\}; \{1, 2, 5\}; \{1, 2, 4\}; \{2, 4, 5\}/t = 1, \ldots, 5; j = 2, 4, 5\}.$$

Again, applying theorem 3.3.1, it is a (long, but not very instructive) calculation to verify that

$$\frac{\mathrm{vol}(M_r)}{4\pi^2} = 2r_1(\mathfrak{p} - r_1 - 2r_3)$$

where as usual $\mathfrak{p}$ is the fixed perimeter.

Remark 21. Note that the quantity $R_I^+ - R_I^-$ associated to an admissible I is actually the Euclidean distance of r from the wall $\sum_{i \in I^c} r_i > \sum_{i \in I} r_i$. So as $r \to r^c$, r^c being the wall crossing point, $R_I^+ - R_I^-$ decreases (and it is zero on the wall-crossing point).

Chapter 4

Crossing the Walls

In this chapter we will mainly focus our attention on the description of M_r as the symplectic quotient of the Grassmannians $Gr_{2,n}$ of 2-planes in $\mathbb{C}^n$ by the action of the maximal torus U_1^n (as in section 1.3) and we explicitly describe how the diffeotype of the manifold M_r changes as r crosses a wall in $\Xi = \mu_{U_1^n}(Gr_{2,n})$.

The image $\Xi = \mu_{U_1^n}(Gr_{2,n})$ via the moment map associated to the U_1^n-action is, by the convexity theorem ([At],[GS82]), a convex polytope; the regions of regular values in ξ, which will be denoted by Δ_i, are separated by walls, i.e. by the images $\mu_{U_1^n}(Gr_{2,n}^H)$ of the sets of points fixed by the subgroups $H \simeq S^1$ of U_1^n. In section 4.1 we will describe how the wall-crossing direction determines the orientation of the subgroup H associated to the wall; in section 4.2 we show that crossing a wall can be interpreted in terms of blow up and down of submanifolds, and characterize these submanifolds in terms of moduli space of lower dimension.

4.1 Associate the orientation to the wall-crossing direction

From now on, when this will keep the notation more handy, we will denote by X the the complex Grassmannian $Gr_{2,n}$ and by T the maximal torus U_1^n

of diagonal matrices in the unitary subgroup U_n.

Let r^0 and r^1 be regular values of the moment map $\mu_T : X \to \Xi \subset \mathbb{R}^n$ lying in different regions, Δ_0 and Δ_1 respectively, of regular values. From remark 5 we know that the wall W between Δ_0 and Δ_1 has equation

$$\sum_{j=1}^{p} r_{i_j} = \sum_{k=1}^{q} r_{i_k}, \qquad p + q = n \tag{4.1}$$

for some $I_p = \{i_1, \dots i_p\}$ and $I_q = \{i_1, \dots, i_q\}$ partition of $\{1, \dots, n\}$. It is not restrictive to assume that $r^0 \in \Delta_0$ satisfies

$$\sum_{i \in I_p} r_i^0 > \sum_{i \in I_q} r_i^0 \tag{4.2}$$

and $r^1 \in \Delta_1$ satisfies

$$\sum_{i \in I_p} r_i^1 < \sum_{i \in I_q} r_i^1. \tag{4.3}$$

Definition 4.1. A wall of equation (4.1) together with the data of a wall crossing direction from Δ_0 to Δ_1 as in (4.2) and (4.3) is said a *wall of type* (p, q).

Note that the change of the wall crossing direction changes the type of the wall, i.e. a wall of equation (4.1) crossed from Δ_1 to Δ_0 is a wall of type (q, p).

For simplicity, let us first analyze in detail the case $I_p = \{1, \dots, p\}$ and $I_q = \{p + 1, \dots, n\}$ when the wall W has equation

$$r_1 + \dots + r_p = r_{p+1} + \dots + r_n.$$

The directions normal to this wall are $\pm v_0 = \pm(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q})$ and the subgroup H of T associated to the wall (i.e. such that W is a connected component of $\mu_T(X^H)$) is $H = \langle \pm v_0 \rangle$, in other words

$$H = \{\mathrm{diag}(\underbrace{e^{\pm i\theta}, \dots, e^{\pm i\theta}}_{p}, \underbrace{e^{\mp i\theta}, \dots, e^{\mp i\theta}}_{q})/\theta \in S^1\}.$$

It is clear that the choice of the sign $\pm$ determines the orientation of the subgroup H; we want to orient it accordingly with the wall crossing direction. (We borrow the following notation from Martin [Ma]). Assume we want to cross the wall W from Δ_0 to Δ_1 and let Z be a transverse path from r^0 to r^1 (note that this defines an orientation on Z) which crosses the wall W at $q = Z \cap \mu_T(X^H)$. Then a positive tangent vector in T_qZ, thought as an element of $\mathfrak{t}^*$, defines a linear functional on $\mathfrak{t}$ which restricts to a non-zero functional on the Lie algebra $\mathfrak{h}$ of H. We orient H to be positive respect to this functional.

In our case, let Z be the segment

$$Z(t) = (1-t)r^0 + t\,r^1, \quad t \in [0,1],$$

and assume we cross the wall at the time t_q, i.e. $q = Z(t_q)$. The tangent vector $\frac{d}{dt}Z(t)_{|t=t_q} = -r^0 + r^1$ in T_qZ defines on $\mathfrak{h}$ the functional that associates to $v \in \mathfrak{h}$ the inner product $\langle -r^0 + r^1, v\rangle$.

One between v_0 and $-v_0$ satisfy the condition that the above inner product is positive, and that determine on H the orientation positive with respect to the wall-crossing direction. Precisely

$$H = \langle -v_o \rangle = \langle (\underbrace{-1,\dots,-1}_{p},\underbrace{1,\dots,1}_{q})\rangle.$$

In fact, from (4.2) and (4.3) follows

$$\langle -r^0 + r^1, -v_0\rangle = \sum_{i=1}^{p} r_i^0 - \sum_{i=1}^{p} r_i^1 - \sum_{i=p+1}^{n} r_i^0 + \sum_{i=p+1}^{n} r_i^1 > 0.$$

In the general case of a wall W of equations $\sum_{i\in I_p} r_i = \sum_{i\in I_q} r_i$ (as in (4.1)) similar arguments hold. Precisely, let $\varepsilon_i = 1$ if $i \in I_p$ and $\varepsilon_i = -1$ if $i \in I_q$, then the normals to the wall are $\pm v_0 = (\varepsilon_1, \dots, \varepsilon_n)$.

The orientation of H positive with respect to the wall crossing from Δ_0 to Δ_1 (as defined in (4.2) and in (4.3)) is the one determined by $-v_0$.

4.2 Crossing a wall of type (p, q)

In [GS89] Guillemin and Sternberg give a thorough analysis of wall crossing problems relative to (quasi-free) S^1-actions. They also point out that their construction can be made H'-equivariant, when H' is a compact group commuting with the S^1-action. In particular, this is the situation when studying how things change as we cross a inner wall in the moment image by the action of a torus - as it is our case, where U_1^n decomposes in the product of the subgroup $H \simeq S^1$ associated to the wall and (one of) its complements.

In the more general setting of the action of a torus, Guillemin and Sternberg show that, as one passes through a inner wall of the momentum polytope, the diffeotype of the associated reduced space changes a by blowing up followed by a blowing down.

We want to point out some of the arguments from Guillemin and Sternberg (see [GS89] §11) that will be useful in our proof: recall that each linear S^1-action on $\mathbb{C}^n$ (which is the local model) is diagonalizable with eigenvalues $e^{ik\theta}$; under the assumption that the action is quasi-free the weights are $k = \pm 1$ (the case $k = 0$ will not appear in the case of the moduli space of polygons, and, even in the general case, the coordinates acted on with weight 0 corresponds to trivial actions which can be factored out). Moreover, 0 is the only fixed point and the Hessian in 0 of the associated moment map has signature $(2\chi^-, 2\chi^+)$, where χ^+ is the number of positive weights and χ^- is the number of negative ones. Guillemin and Sternberg shows that crossing a wall associated to this S^1-action change the diffeotype of the associated reduced manifold by blowing up a submanifold of complex dimension $\chi^+ - 1$ followed by the blowing down of a submanifold of complex dimension $\chi^- - 1$.

Remark 22. Note that the wall-crossing direction is clear because S^1 is intended with its standard orientation. If we wish to cross the wall the other way round, i.e. if we choose on S^1 the "clockwise" orientation, this changes the roles of χ^+ and χ^- and -as we would expect- this wall crossing has the effect to blow up a submanifolds of dimension $\chi^- - 1$ and then blow down a submanifold of dimension $\chi^+ - 1$.

Remark 23. On the nature of these two submanifolds, and how they are related one to the other, we will be more explicit later on, giving all the details for the specific case of moduli space of polygons. We can though anticipate that these two submanifold are actually two possible resolution of the (conic) singularities that correspond to lined polygons in the singular quotient M_{r^c}.

In this section we will work out all the details in the case of the moduli space M_r and we will use its nice structure to give a characterization of the blowed up and down manifolds in terms of (littler) moduli spaces.

Remember that in section 1.3 we described the Grassmannian as the symplectic quotient of the manifold $M_{n\times 2}(\mathbb{C})$ of matrices $n \times 2$ by the action of the unitary group U_2, when the level set is the Stiefel manifold of orthonormal 2-frames in $\mathbb{C}^n$. The torus U_1^n of diagonal matrices in the unitary group U_n acts (by multiplication on the left) on $M_{n\times 2}(\mathbb{C})$; this action commutes with the U_2-action and thus descends to a (Hamiltonian) action on the Grassmannian $Gr_{2,n}$. The symplectic quotient relative to this action at the level set r is the moduli space M_r of polygons with fixed side length r. This is summarized by the following diagram, (where the vertical arrows are the symplectic quotients):

$$\begin{array}{ccccc}
St_{2,n} & \hookrightarrow & M_{n\times 2}(\mathbb{C}) & & \\
 & \searrow & \downarrow{\scriptstyle U_2} & & \\
\mu_{U_1^n}^{-1}(r) & \hookrightarrow & Gr_{2,n} & \xrightarrow{\mu_{U_1^n}} & \Xi \subset \mathbb{R}_+^n \\
 & \searrow & \downarrow{\scriptstyle U_1^n} & & \\
 & & M_r & &
\end{array}$$

Let H be the oriented subgroup associated to the wall W and H' be a complement of H, i.e. $U_1^n = H \times H'$; then both H and H' acts on $Gr_{2,n}$ (by restriction of the U_1^n action), and these actions are still Hamiltonian with moment maps $\mu_H := p_H \circ \mu_{U_1^n}$ and $\mu_{H'} := p_{H'} \circ \mu_{U_1^n}$, where p_H and $p_{H'}$ are the natural Lie algebra's projections of $\mathfrak{t}^*$ onto $\mathfrak{h}^*$ and $(\mathfrak{h}')^*$ respectively. So we want to determine the signature $(2\chi^+, 2\chi^-)$ of the Hessian of μ_H at fixed points "modulo H'" (this can be formally described in terms of reduction in stages, first quotienting by H', then applying the analysis due to Guillemin and Sternberg to the residual H-action).

Now on, when this will create no confusion, the moment map $\mu_{U_1^n}$ relative to the action of the torus U_1^n will be denoted just by μ. Let X^H be the set of fixed points by the H-action on the Grassmannian and r^c be the point of wall crossing. Moreover let $\nu := \nu X^H|_{X^H \cap \mu^{-1}(r^c)}$ be the restriction to $\mu^{-1}(r^c)$ of the normal bundle to the fixed points set X^H; then the normal bundle

$$\nu \to X^H \cap \mu^{-1}(r^c)$$

splits under the H-action into the direct sum $\nu^+ \oplus \nu^-$ of two subbundles such that H acts on ν^+ with positive weights (the action is quasi-free, so the positive weights are all $+1$), and on ν^- with negative weights. So χ^+ and χ^- are just the weights of the H action on $\nu X^H|_{X^H \cap \mu^{-1}(r^c)}$.

As in section 4.1 we will first analyse the wall crossing of a wall W of equations

$$\sum_{i=1}^{p} r_i = \sum_{i=p+1}^{q} r_i, \quad p + q = n.$$

Moreover suppose again that we are crossing the wall W from Δ_0 to Δ_1, as defined in 4.2 and 4.3, so that W is a wall of type (p, q) and the subgroup $H \simeq S^1$ associated to W is

$$H = \langle (\underbrace{-1, \ldots, -1}_{p}, \underbrace{1, \ldots, 1}_{q}) \rangle,$$

i.e. $H = \{\mathrm{diag}(\underbrace{e^{-i\theta}, \ldots, e^{-i\theta}}_{p}, \underbrace{e^{i\theta}, \ldots, e^{i\theta}}_{q})\}$. H acts on the space of matrices $M_{n\times 2}$ by multiplication: $\forall (a, b) \in M_{n\times 2}$

$$\mathrm{diag}(e^{-i\theta}, \ldots, e^{i\theta}) \cdot (a, b) = \begin{pmatrix} e^{-i\theta} a_1 & e^{-i\theta} b_1 \\ \vdots & \vdots \\ e^{i\theta} a_n & e^{i\theta} b_n \end{pmatrix}$$

and, for each fixed point P, the weights of the H-action on $T_P M_{n\times 2}$ are

$$\begin{pmatrix} -1 & -1 \\ \vdots & \vdots \\ -1 & -1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{pmatrix},$$

i.e. under the H-action the tangent space $T_p M_{n\times 2}$ get decomposed in the direct sum of four spaces, which we will call A_+, A_-, B_+, B_-; A_+ corresponds to the directions of the a_i coordinates acted on with weight $+1$; A_- corresponds to the a_i's acted on with weight -1 and B_+ and B_- are similarly defined with respect to the b_i's. Note that

$$\dim_{\mathbb{C}} A_+ = q,\ \dim_{\mathbb{C}} B_+ = q,\ \dim_{\mathbb{C}} A_- = p,\ \dim_{\mathbb{C}} B_- = p.$$

The set X^H of points fixed by the H-action is

$$X^H := \left\{ [(a, b)] \in St_{2,n}/U_2 : \begin{cases} a_i = 0 & \forall i \geq p+1 \\ b_i = 0 & \forall i \leq p \end{cases} \right\}$$

$\mu^{-1}(r^c)$ intersects X^H in a submanifold of real dimension $n-1$, in fact, recalling that $\mu(a, b) = (|a_1|^2 + |b_1|^2, \ldots, |a_n|^2 + |b_n|^2)$ (see §1.3 and [HK97]), the equations $|a_j|^2 = r_j^c$ and $|b_j|^2 = r_j^c$ do not determine the phases of a_j and b_j. This submanifold is a U_1^n-orbit in $\mu^{-1}(r^c) \cap X^H$, for example it is the

U_1^n-orbit of $P \in \mu^{-1}(r^c) \cap X^H$ defined as follow:

$$P := \begin{pmatrix} \sqrt{r_1^c} & 0 \\ \vdots & \vdots \\ \sqrt{r_p^c} & 0 \\ 0 & \sqrt{r_{p+1}^c} \\ \vdots & \vdots \\ 0 & \sqrt{r_n^c} \end{pmatrix}.$$

This, using the quaternionic Hopf map introduced in section 1.3, identifies explicitly the quotient $(\mu^{-1}(r^c) \cap X^H)/U_1^n$ with the degenerate (i.e. lined) polygon $[P] \in M_r$, $[P] = (r_1\, \underline{i}, \ldots, r_p\, \underline{i}, -r_{p+1}\, \underline{i}, \ldots, -r_n\, \underline{i})$, where $\underline{i} = (1, 0, 0)$.

By definition

$$\nu X^H = TX_{|_{X^H}} \big/ TX^H$$

and

$$\nu X^H{}_{|(\mu^{-1}(r^c)\cap X^H)} = TX_{|(\mu^{-1}(r^c)\cap X^H)} \big/ TX^H_{|(\mu^{-1}(r^c)\cap X^H)}.$$

By U_1^n-equivariance, the weights of the H-action on $\nu X^H{}_{|_Q}$ are constant for each $Q \in \mu^{-1}(r^c) \cap X^H$, and precisely they are the weights of the H-action on $\nu X^H{}_{|(\mu^{-1}(r^c)\cap X^H)}$.

Thus we will calculate the weights of the H-action on $\nu X^H{}_{|_P}$. In this situation we can identify the orthogonal to $T_P X^H$ with the restriction of the normal bundle to X^H, i.e.

$$\nu X^H{}_{|_P} \simeq T_P^{\perp} X^H.$$

Moreover, from the description of $Gr_{2,n}$ as the orbit space for the U_2-action on $St_{2,n}$ it follows

$$T_P Gr_{2,n} \simeq T_P^{\perp}(U_2 \cdot P).$$

(With a little abuse of notation we use the same symbol P for the element in $St_{2,n}$ and its class in $Gr_{2,n}$).

So to calculate the weights of the H-action in $\nu X^H{}_{|_P}$ we will use the following identifications:

$$\begin{array}{rcccccccc} T_P St_{2,n} & = & T_P(U_2\cdot P) & \oplus & T_P^{\perp}(U_2\cdot P) & & & & \\ & & & & \wr\| & & & & \\ & & & & T_P Gr_{2,n} & = & T_P X^H & \oplus & T_P^{\perp} X^H \\ & & & & & & & & \wr\| \\ & & & & & & & & \nu X^H{}_{|P}. \end{array}$$

Now, let

$$\tilde{X}^H = \left\{ \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \in St_{2,n} : \begin{cases} a_i = 0 & \forall i \geq p+1 \\ b_i = 0 & \forall i \leq p \end{cases} \right\} \subseteq St_{2,n},$$

then because the U_2-action and the U_1^n-action (and thus also H-action) commutes, the following identification holds

$$\nu X^H{}_{|P} \simeq T_P^{\perp}(\tilde{X}^H \cap (U_2\cdot P)) \subseteq T_P St_{2,n}.$$

We now determine the equations for $T_P^{\perp}(\tilde{X}^H)$, $T_P^{\perp}(U_2\cdot P)$ and $T_P St_{2,n}$; let $(a,b)\in M_{n\times 2}$

$$(a,b)\in T_P\tilde{X}^H \iff \begin{cases} a_{p+1} = \ldots = a_n = 0 \\ b_1 = \ldots b_p = 0 \end{cases}$$

thus

$$(a,b)\in T_P^{\perp}\tilde{X}^H \iff \begin{cases} a_1 = \ldots = a_p = 0 \\ b_{p+1} = \ldots = b_n = 0 \end{cases} \tag{4.4}$$

Moreover

$$(a,b)\in St_{2,n} \iff \begin{cases} |a| = \sum_{i=1}^n \langle a_i, a_i\rangle = 1 \\ |b| = \sum_{i=1}^n \langle b_i, b_i\rangle = 1 \\ \langle a,b\rangle = \sum_{i=1}^n \langle a_i, b_i\rangle = 0; \end{cases}$$

so, recalling that $\langle u,v\rangle$ is the standard Hermitian product, i.e. $\langle u,v\rangle = \frac{1}{2}u\bar{v} + \bar{u}v$, and differentiating the relations above at P we get:

$$(a,b)\in T_PSt_{2,n} \iff \begin{cases} \sum_{i=1}^n a_i\bar{\alpha}_i+\bar{a}_i\alpha_i=0 \\ \sum_{i=1}^n b_i\bar{\beta}_i+\bar{b}_i\beta_i=0 \\ \sum_{i=1}^n a_i\bar{\beta}_i+\bar{\alpha}_i b_i=0 \end{cases} \tag{4.5}$$

Note that $(4.5)_1$ and $(4.5)_2$ are actually real equations while $(4.5)_3$ is a complex equation.

Choosed a basis for $T_P(U_2\cdot P)$, it is possible to compute the equations for $T_P^{\perp}(U_2\cdot P)$, which are:

Lemma 4.2.1.

$$(a,b)\in T_P^{\perp}(U_2\cdot P) \iff \begin{cases} \sum_{i=1}^n a_i\bar{\alpha}_i-\bar{a}_i\alpha_i=0 \\ \sum_{i=1}^n b_i\bar{\beta}_i-\bar{b}_i\beta_i=0 \\ \sum_{i=1}^n a_i\bar{\beta}_i-\bar{\alpha}_i b_i=0 \end{cases} \tag{4.6}$$

Proof. Let $\begin{pmatrix} ue^{i\theta} & v \\ -\bar{v} & \bar{u}e^{i\theta} \end{pmatrix}$ be a generic element in U_2, $(u,v\in\mathbb{C},\ \theta\in S^1)$. Then, derivating, a generic element on the Lie algebra $\mathfrak{u}_2$ is $\begin{pmatrix} \nu+i\theta & \eta \\ -\bar{\eta} & \bar{\nu}+i\theta \end{pmatrix}$ with ν pure complex, i.e. $\nu=i\phi$, $\phi\in\mathbb{R}$, and $\eta\in\mathbb{C}$. Thus the infinitesimal action on $P=(\alpha,\beta)$ is

$$\begin{pmatrix} i(\theta+\phi) & \eta \\ -\bar{\eta} & i(\theta-\phi) \end{pmatrix}(\alpha,\beta)=\begin{pmatrix} i(\theta+\phi)\alpha_1 & -\bar{\eta}\alpha_1 \\ \vdots & \vdots \\ i(\theta+\phi)\alpha_p & -\bar{\eta}\alpha_p \\ \eta\beta_{p+1} & i(\theta-\phi)\beta_{p+1} \\ \vdots & \vdots \\ \eta\beta_n & i(\theta-\phi)\beta_n \end{pmatrix}. \tag{4.7}$$

The following choices

$$\begin{cases}\phi=\theta\\ \beta=0\end{cases};\quad \begin{cases}\phi=-\theta\\ \beta=0\end{cases};\quad \begin{cases}\phi=0\\ \theta=0\end{cases}$$

determine a basis of the Lie algebra $\mathfrak{u}_2$. Thus a basis of $T_P(U_2\cdot P)$ is determined by entering the choices above in 4.7. An element $(a,b)\in T_PSt_{2,n}$ is in

$T_P^{\perp}(U_2 \cdot P)$ if it is orthogonal to each element of a basis of $T_P(U_2 \cdot P)$. This, for the basis we chose, gives the following system:

$$\begin{cases} \sum_{j=1}^{p} \langle i\alpha_j, a_j \rangle = 0 \\ \sum_{j=p+1}^{n} \langle j\beta_j, b_j \rangle = 0 \\ \sum_{j=1}^{p} \langle -\bar{\eta}\alpha_j, b_j \rangle + \sum_{j=p+1}^{n} \langle \eta\beta_j, a_j \rangle = 0 \end{cases}$$

from which, recalling that $\langle u, v \rangle = \frac{1}{2} u\bar{v} + \bar{u}v$ is the standard hermitian product, the result follow.

□

Again $(4.6)_1$ and $(4.6)_2$ are pure imaginary equations, and sums up with $(4.5)_1$ and $(4.5)_2$ to give two complex equations, one in the a_i's and one in the b_i's [1].

Thus

$$(a, b) \in T_P^{\perp}(\tilde{X}^H \cap (U_2 \cdot P)) \subseteq T_P St_{2,n} \iff \begin{cases} \sum a_i \bar{\alpha}_i = 0 \\ \sum b_i \bar{\beta}_i = 0 \\ \sum a_i \bar{\beta}_i = 0 \\ \sum b_i \bar{\alpha}_i = 0 \\ a_1 = \ldots = a_p = 0 \\ b_{p+1} = \ldots = b_n = 0 \end{cases} \tag{4.8}$$

$(4.8)_1$ and $(4.8)_2$ becomes trivial when assuming $(a, b) \in T_P^{\perp} \tilde{X}^H$ (i.e. when requiring $(4.8)_5$ and $(4.8)_6$).

$(4.8)_3$ solves in A_+, and $(4.8)_4$ solves in B_-.

The p conditions $(4.8)_5$ solve in A_- (and actually they kill it all), and the q conditions $(4.8)_6$ solve in B_+.

Geometrically, this means:

$$\begin{aligned} \dim_{\mathbb{C}}(A_+ \cap \nu X^H{}_{|P}) &= q - 1, \\ \dim_{\mathbb{C}}(A_- \cap \nu X^H{}_{|P}) &= 0, \\ \dim_{\mathbb{C}}(B_- \cap \nu X^H{}_{|P}) &= p - 1, \\ \dim_{\mathbb{C}}(B_+ \cap \nu X^H{}_{|P}) &= 0. \end{aligned}$$

[1] The fact that 4.5 and 4.6 combines to give four complex equations reflects that we are actually considering the GIT quotient by the complexification $GL_2\mathbb{C}$ of the real group U_2.

and thus the weights of the H-action on $\nu X^H{}_{|P}$ are

$$q-1\left\{\left(\begin{array}{cc} & -1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \\ & -1 \end{array}\right)\right\}p-1$$

So $\chi^- = p-1, \chi^+ = q-1$.

By [GS89], this mean that when r crosses the wall $\sum_{i=1}^{p} r_i = \sum_{i=p+1}^{n} r_i$ from $r^0 \in \Delta_0$ to $r^1\Delta_1$ the associated symplectic quotient M_r changes by blowing up a submanifold of M_{r^0} diffeomorphic to $\mathbb{CP}^{p-2}$ to a $\mathbb{CP}^{q-2}$-bundle followed by blowing down this divisor, viewed as a $\mathbb{CP}^{p-2}$-bundle over a submanifold of M_{r^1} diffeomorphic to $\mathbb{CP}^{q-2}$.

These submanifolds will be characterized in terms of moduli spaces of lower dimension in section 4.2.1.

The general case: let W be the wall of equation $\sum_{i\in I_p} r_i = \sum_{i\in I_q} r_i$ for generic disjoint index subsets $I_p, I_q \subset \{1,\dots,n\}$ of cardinality p and q respectively, $p+q=n$. By 4.1, the associated subgroup H to this wall is

$$H = \langle(\varepsilon_1,\dots,\varepsilon_n)\rangle$$

where

$$\varepsilon_i = \begin{cases} -1 & \text{if} \quad i\in I_p \\ 1 & \text{if} \quad i \in I_q \end{cases}$$

and so the weights of the H-action on $T_P M_{n\times 2}$, P fixed point, are

$$\left(\begin{array}{cc} \varepsilon_1 & \varepsilon_1 \\ \vdots & \vdots \\ \varepsilon_n & \varepsilon_n \end{array}\right).$$

As before, this means that under the H-action the tangent space $T_P M_{n\times 2}$ get decomposed into the direct sum of subspaces A_+, A_-, B_+, B_- such that H acts on A_+ and B_+ with weight 1, and $\dim_{\mathbb{C}} A_+ = \dim_{\mathbb{C}} B_+ = q$; H acts on A_- and B_- with weight -1, and $\dim_{\mathbb{C}} A_- = \dim_{\mathbb{C}} B_- = p$.

The set of fixed points X^H is

$$X^H := \left\{[(a,b)] \in St_{2,n}/U_2 : \begin{cases} a_i = 0 & \forall i \in I_q \\ b_i = 0 & \forall i \in I_p \end{cases}\right\}.$$

Again, $X^H \cap \mu^{-1}(r^c)$ is a submanifold of real dimension $n-1$ and it is the U_1^n-orbit of P,

$$P_1 := (\delta^p \alpha, \delta^q \beta)$$

where $\delta^p\alpha$ is the column vector $(\delta_1^p\alpha_1, \ldots, \delta_n^p\alpha_n)$ and $\delta^q\beta$ is the column vector $(\delta_1^q\beta_1, \ldots, \delta_n^q\beta_n)$, with

$$\delta_i^p = \begin{cases} 1 & \text{if} \quad i \in I_p \\ 0 & \text{if} \quad i \in I_q. \end{cases} \qquad \delta_i^q = \begin{cases} 0 & \text{if} \quad i \in I_p \\ 1 & \text{if} \quad i \in I_q. \end{cases}$$

Now, arguments similar to the ones used before hold, and we can lift both X^H to $\tilde{X^H}$ inside the Stiefel manifold $St_{2,n}$. This let us calculate the equations for $T_P^{\perp}\tilde{X}^H \subset T_P St_{2,n}$ and in the same way; so we get the following conditions:

$$(a,b) \in T_P^{\perp}\tilde{X}^H \iff \begin{cases} a_i = 0 & \forall i \in I_p \\ b_i = 0 & \forall i \in I_q; \end{cases}$$

$$(a,b) \in T_P St_{2,n} \iff \begin{cases} \sum_{i=1}^n a_i\delta_i^p\bar{\alpha}_i + \bar{a}_i\delta_i^p\alpha_i = 0 \\ \sum_{i=1}^n b_i\delta_i^q\bar{\beta}_i + \bar{b}_i\delta_i^q\beta_i = 0 \\ \sum_{i=1}^n a_i\delta_i^q\bar{\beta}_i + b_i\delta_i^p\bar{\alpha}_i = 0; \end{cases}$$

$$(a,b) \in T_P^{\perp}(U_2 \cdot P) \iff \begin{cases} \sum_{i=1}^n a_i\delta_i^p\bar{\alpha}_i - \bar{a}_i\delta_i^p\alpha_i = 0 \\ \sum_{i=1}^n b_i\delta_i^q\bar{\beta}_i - \bar{b}_i\delta_i^q\beta_i = 0 \\ \sum_{i=1}^n a_i\delta_i^q\bar{\beta}_i - b_i\delta_i^p\bar{\alpha}_i = 0. \end{cases}$$

These sums up to the following system

$$(a,b) \in T_P^{\perp}(\tilde{X}^H \cap (U_2 \cdot P)) \subseteq T_P St_{2,n} \iff \begin{cases} \sum a_i\delta_i^p\bar{\alpha}_i = 0 \\ \sum b_i\delta_i^q\bar{\beta}_i = 0 \\ \sum a_i\delta_i^q\bar{\beta}_i = 0 \\ \sum b_i\delta_i^p\bar{\alpha}_i = 0 \\ a_i = 0 \,\forall\, i \in I_p \\ b_i = 0 \,\forall\, i \in I_q. \end{cases} \tag{4.9}$$

The systems $(4.9)_5$ and $(4.9)_6$ solve in A_- and B_+ respectively. When $(4.9)_5$ and $(4.9)_6$ hold, the equations $(4.9)_1$ and $(4.9)_2$ are trivially verified. $(4.9)_3$ solves in A_+ and $(4.9)_4$ solves in B_-, thus the weights of the H-action on $\nu X_1^H{}_{|P_1}$ are $(p-1, q-1)$:

$$q-1\left\{\begin{pmatrix} & -1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \\ & -1 \end{pmatrix}\right\} p-1$$

This calculation proves the following:

Proposition 4.2.2. *Let W be a wall of type (p, q) and let H be the circle subgroup generated by the normal direction to the wall, positive with respect to the wall crossing direction. Then along $X_H \cap \mu^{-1}(r^c)$, the weights of the H-action are -1, with real multiplicity $2(p-1)$, and $+1$, with real multiplicity $2(q-1)$.*

Therefore, according to Guillemin and Sternberg [GS89], this means that M_{r^0} and M_{r^1} are related by a birational map which is the composite of a blow-up and a blow-down.

Explicitly, first we blow-up a copy C_0 of $\mathbb{CP}^{p-2}$ in M_{r^0}. This blow-up is a new manifold $\tilde{M}$ in which C_0 is replaced by its projectivized normal bundle, which is a $\mathbb{CP}^{q-2}$-bundle over $C_0 \cong \mathbb{CP}^{p-2}$ called the exceptional divisor $\mathcal{E}$ in $\tilde{M}$ (note that $\mathcal{E}$ has complex dimension $n-4$, hence has complex codimension one in $\tilde{M}$). There is a map $p_0 \colon \tilde{M} \to M_{r^0}$ which is a bijection everywhere except over C_0.

Second, $\mathcal{E}$ is also a $\mathbb{CP}^{p-2}$-bundle over $\mathbb{CP}^{q-2}$ and we can blow-down $\tilde{M}$ by replacing this bundle with its base. We thus obtain a map $p_1 \colon \tilde{M} \to M_{r^1}$ which is the blow-up of a submanifold $C_1 \cong \mathbb{C}P^{q-2}$.

This is what the top half of the figure 4.2 describes.

Remark 24. Note that the wall-crossing analysis done so far applies to the outer walls $r_i = 1$ (which are just the walls with $\sum_{j\neq i} r_j - r_i = 0$). (Note that r is always a regular value if $\mu^{-1}(r)$ is empty!) In this case, the wall crossing has type $(n-1,1)$ or $(1,n-1)$, and so the weights are $(n-2,0)$ or $(0,n-2)$. Hence wall crossing replaces the empty set by a copy of $\mathbb{CP}^{n-3}$ if r crosses the wall from outside Ξ to inside, or vice versa it replaces a copy of $\mathbb{CP}^{n-3}$ with the empty set if r crosses the wall W from inside Ξ to outside .

4.2.1 Crossing the walls in terms of moduli spaces of polygons

In this section we will give a characterization of the blown up and blown down submanifolds $\mathbb{CP}^{q-2}$ and $\mathbb{CP}^{p-2}$ in terms of moduli spaces of polygons. This characterization arises very naturally by the geometry of M_r, and can be constructed by looking carefully at the birational map between M_{r^0} and M_{r^1} described by [GS89].

We already pointed out that the moduli space M_r is a symplectic manifolds long as the lengths are chosen so that there are no polygons on a line. Equivalently, this means that for every partition $I, I^c \subseteq \{1,\ldots,n\}$

$$\varepsilon_I(r) = \sum_{i\in I} r_i - \sum_{i\in I^c} r_i$$

must be nonzero. Moreover, let r^c is a critical value of the momentum map $\mu_{U_1^n}$ lying on a wall of type (p,q). Then, by the description of the walls due to Kapovich–Millson (see section 1.3), there exists an index subset $I_p \subset \{1,\ldots,n\}$, of cardinality $|I| = p$, such that

$$\varepsilon_{I_p}(r^c) = 0.$$

As before, let r^0 and r^1 be regular values lying in different regions Δ_0 and Δ_1, precisely:

$$\sum_{i\in I_p} r_i^0 > \sum_{i\in I_q} r_i^0; \qquad \sum_{i\in I_p} r_i^1 < \sum_{i\in I_q} r_i^1.$$

When r moves along a path from r^0 to r^1 then $\varepsilon_{I_p}(r) \xrightarrow[r\to r^c]{} 0$, $\varepsilon_{I_p}(r^0) > 0$ and $\varepsilon_{I_p}(r^1) < 0$.

Let

$$M_{I_q}(r) = M\Big(r_{i_1},\dots,r_{i_p},\sum_{j\in I_q} r_j,\Big)$$

be the moduli space of $p+1$-gons of fixed side length $\Big(r_{i_1},\dots,r_{i_p},\sum_{j\in I_q} r_j,\Big)$, and similarly let

$$M_{I_p}(r) = M\Big(\sum_{i\in I_p} r_i, r_{j_1},\dots,r_{j_q}\Big)$$

be the moduli space of $q+1$-gons of fixed side length $\Big(\sum_{i\in I_p} r_i, r_{j_1},\dots,r_{j_q}\Big)$.

When $\varepsilon_{I_p}(r) > 0$, $M_{I_q}(r)$ has complex dimension $p-2$ and $M_{I_p}(r)$ is empty. When $\varepsilon_{I_p}(r) < 0$, $M_{I_p}(r)$ has complex dimension $q-2$ and $M_{I_q}(r)$ is empty. When $\varepsilon_{I_p}(r) = 0$, $M_{I_p}(r) = M_{I_q}(r)$ and is a singular point (a polygon on a line).

$M_{I_p}(r)$ and $M_{I_q}(r)$ can be identified with (eventually empty) submanifolds of M_r as follow. If $I_p = \{1,\dots,p\}$ and $I_q = \{p+1,\dots,p+q\}$, then $M_{I_p}(r)$ can be identified with the submanifold of M_r such that the first p edges are all oriented in the same direction (and hence collinear), i.e. $M_{I_p}(r)$ is the submanifold of polygons as in figure 4.1.

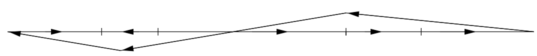

Figure 4.1: Polygon in $M_{I_p}(r)$.

Similarly, $M_{I_q}(r)$ is the submanifold in M_r of polygons such that the last q edges are collinear. If $r = r^c \in W$ then $M_{I_p}(r) = M_{I_q}(r) = P$ where P is the singular point corresponding to the lined polygon in the singular quotient M_{r^c}.

Lemma 4.2.3. *When not empty,* $M_{I_p}(r) \simeq \mathbb{CP}^{q-2}$ *and* $M_{I_q}(r) \simeq \mathbb{CP}^{p-2}$.

Proof. Let us first analyze $M_{I_p}(r)$. If $\sum_{i\in I_p} r_i - \sum_{j\in I_q} r_j > 0$ then $M_{I_p}(r)$ is empty and we have nothing to prove.

Assume $\sum_{i\in I_p} r_i - \sum_{j\in I_q} r_j < 0$, and let $\rho := \Big(\sum_{i\in I_p} r_i, r_{j_1},\dots,r_{j_q}\Big) \in \mathbb{R}_+^{q+1}$, so that $M_{I_p}(r) = M_\rho$. ρ lies in an external region of $\Xi \subset \mathbb{R}_+^{q+1}$, delimited by the outer wall W_1 of type $(1, q)$ of equation

$$\rho_1 = \rho_2 + \ldots + \rho_{q+1}.$$

Δ_0 is the region of "regular values" ρ^0 such that $\rho_1 > \sum_{i=2}^{q+1} \rho_i^0$. This imply that $\mu_{U_n^{q+1}}(\rho^0)$ is empty for each $\rho^0 \in \Delta_0$.

By proposition 4.2.2, or, more precisely, by remark 24, as ρ crosses the wall W_1 from Δ_0 to Δ_1 the diffeotype of M_ρ changes by replacing the empty set with a copy of $\mathbb{CP}^{q-2}$. so $M_\rho = M_{I_p}(r) \simeq \mathbb{CP}^{q-2}$ as we wanted to prove.

Similar arguments holds for $M_{I_q}(r)$.

□

For generic I_p and I_q this argument shows that the moduli space $M_{I_q}(r)$ can be identified with a submanifold of $M(r_{i_1}, \ldots, r_{i_n}) = M_{\sigma(r)}$ for some $\sigma \in S_n$ permutation on n elements. Moreover $M(r_{i_1}, \ldots, r_{i_n}) \simeq M_r$ as observed in section 1, and we can think $M_{I_q}(r)$ as a submanifold of M_r isomorphic to $\mathbb{CP}^{p-2}$ if $r \in \Delta_0$, empty if $r \in \Delta_1$.

For the same argument, $M_{I_p}(r)$ can be thought as a submanifold of M_r isomorphic to $\mathbb{CP}^{q-2}$ if $r \in \Delta_1$, empty if $r \in \Delta_0$.

Note that as $r \longrightarrow r^c$, $r \in \Delta_0$, the width $\varepsilon_{I_p}(r)$ of polytopes in $M_{I_q}(r) \subset M_r$ goes linearly to zero, and it is zero for $r = r^c$. So the $(p-2)$-dimensional submanifold $M_{I_q}(r)$ collapses to a point as r crosses the wall W. Similarly, as r leaves from the wall W to the interior of Δ_1, the width $\varepsilon_{I_q}(r)$ of the polytopes in $M_{I_p}(r)$ increases, and $M_{I_p}(r)$ is the $(q-2)$-dimensional submanifold that is born as crossing the wall W.

In figure 4.2, the map π_0 collapses $M_{I_q}(r)$ to the point P and the map π_1^{-1} resolves the singularity in P by "blowing it up" to give $M_{I_p}(r)$.

At the light of this analysis, proposition 4.2.2 implies:

Theorem 4.2.4. *As the length vector r crosses a wall of type (p, q) in Ξ, the diffeotype of the moduli space of polygons M_r change by blowing up the $(q-2)$-dimensional submanifold $M_{I_p} \simeq \mathbb{CP}^{q-2}$ and blowing down the (projectivized normal bundle) of $M_{I_q} \simeq \mathbb{CP}^{p-2}$.*

$M_{I_p}(r)$ and $M_{I_q}(r)$ are resolutions of the singularity corresponding to the lined polygon P in M_{r^c}, and both are dominated by the blow up of M_{r^c} at the singular point.

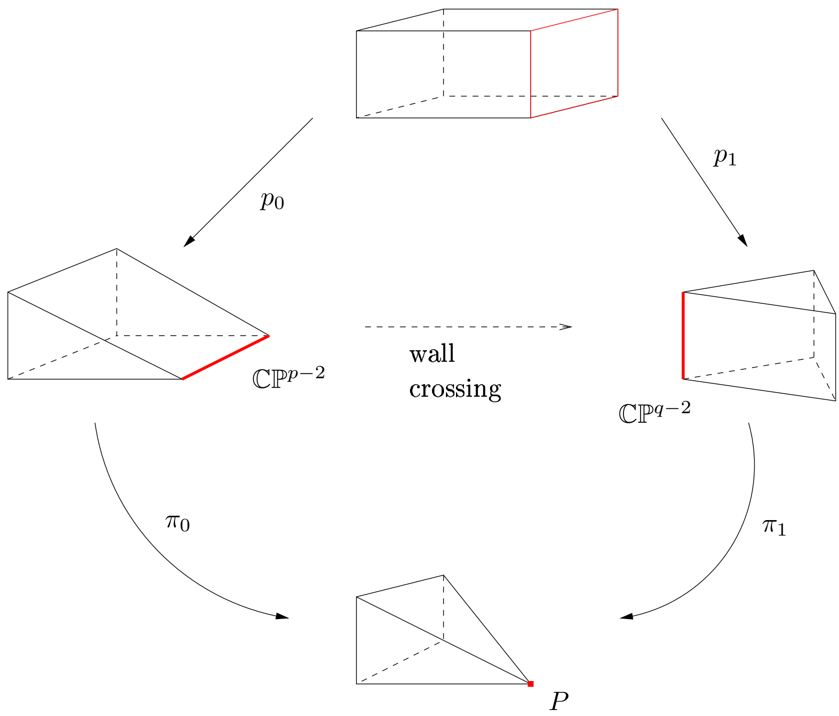

Figure 4.2: Crossing a wall of type (p, q)

Chapter 5

The Cohomology Ring of M_r

In this section we will study how the cohomology ring of M_r changes as r cross a wall in the moment polytope $\Xi = \mu_{U_1^n}(Gr_{2,n})$, and we will apply the Duistermaat–Heckman theorem together with the volume formula 3.3.1 to describe explicitly the cohomology ring $H^*(M_r)$.

The study of the cohomology ring structure of a reduced space $M/\!\!/G$ (even in the good case of a compact connect Lie group G acting on a compact manifold M) has been since the 1980's one of the leading and most interesting topics in symplectic topology. Many beautiful results has been achieved, and between them we like to cite the works of L. Jeffrey and F. Kirwan (in particular [JK]) and of J.Kalkman [Ka]. The problem is not closed though, in fact in practice to give an explicit description of the cohomology ring $H^*(M/\!\!/G, \mathbb{Q})$ from the formulas mentioned above still some (non trivial) work need to be done. This was already pointed out by Guillemin and Sternberg in [GS95], who observed that in "nice" situations (essentially when the Chern class of the fibration $\mu^{-1}(\xi) \to M/\!\!/G$ generates the cohomology ring), then a good deal of information on its cohomology ring can be deduced from the Duistermaat–Heckman theorem, if the polynomial that describe the volume of a symplectic reduction is known. This is the point of view we will take in our analysis.

5.1 The cohomology ring of reduced spaces

In this section we summarize the main ideas and theorems in [GS95] using the notation of moduli spaces. These arguments are valid in more general settings, and actually they have been applied in [GS95] to flag manifolds and toric manifolds associated with a simplicial fan. For proofs and more details we refer to Guillemin and Sternberg, [GS95]. As before, let $T = U_1^n$ be the maximal subtorus of diagonal matrices in the unitary group U_n acting on the compact manifold $X = Gr_{2,n}(\mathbb{C})$ with associated moment map μ. Moreover, let r and r^0 be regular values of μ lying in the same region of regular values, and denote by (M_r, ω_r) and (M_{r^0}, ω_{r^0}) the associated symplectic quotients. Using this notation we now state the Duistermaat-Heckman theorem ([DH]), which relates the cohomology classes $[\omega_r]$ and $[\omega_{r^0}]$ of the symplectic reduced forms ω_r and ω_{r^0}.

Theorem 5.1.1. (J.J. Duistermaat, G.J. Heckman) *As differentiable manifolds $M_r = M_{r^0}$, and*

$$[\omega_r] = [\omega_{r^0}] + \sum_{i=1}^{n} (r_i - r_i^0) c_i$$

where $c = (c_1, \ldots, c_n)$ is the Chern class of the fibration $\mu^{-1}(r) \to M_r$

By definition of symplectic volume, we have:

$$vol(M_r) = \int_{M_r} exp([\omega_r]) = \int_{M_{r^0}} exp([\omega_{r^0}] + \sum_{i=1}^{n} (r_i - r_i^0) c_i).$$

$Vol(M_r)$ is a polynomial (on each region of regular values) of degree $n-3$ and

$$\frac{\partial^\alpha}{\partial r^\alpha} vol(M_r)_{|_{r^0}} = \frac{1}{k!} \int_{M_{r^0}} [\omega_{r^0}]^k \cdots c_1^{\alpha_1} \cdots c_n^{\alpha_n}$$

for α multindex, $|\alpha| = \alpha_1 + \ldots \alpha_n = n - 3 - k$, with $0 \leq k \leq n-3$.

In particular, if $|\alpha| = n-3$ then

$$\frac{\partial^\alpha}{\partial r^\alpha} vol(M_r)_{|_{r^0}} = \int_{M_{r^0}} c_1^{\alpha_1} \cdots c_n^{\alpha_n}. \tag{5.1}$$

Thus the leading coefficients of the polynomial $vol(M_r)$ determine the cohomology pairing 5.1

As Guillemin and Sternberg point out, if the c_i generates the cohomology ring $H^*(M_r, \mathbb{Q})$ then it is possible to read from 5.1 the multiplicative relations

$$\sum_{|\beta|+|\gamma|=n-3} a_\beta c^{\beta+\gamma} \tag{5.2}$$

between the generators c_i and, by Poincaré duality[1],

$$\sum a_\beta c^\beta, \quad 0 \leq |\beta| \leq n-3. \tag{5.3}$$

Writing the identities above as

$$Q\Big(\frac{\partial}{\partial r}\Big)\Big(\frac{\partial^\gamma}{\partial r^\gamma} vol(M_r)\Big) = 0$$

where Q is the polynomial $Q(x) = a_\beta x^\beta$, then

Theorem 5.1.2. *If the $c_1, \dots, c_n$ generate the cohomology ring $H^*(M_r, \mathbb{Q})$, then $H^*(M_r, \mathbb{Q})$ is isomorphic to the abstract ring*

$$\mathbb{Q}[x_1, \dots, x_n]/ann(vol)$$

where $Q(x_1, \dots, x_n) \in ann(vol)$ if and only if $Q\Big(\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}\Big) vol(M_r) = 0$.

It is now clear that it is a central problem to determine when the c_i generates the cohomology ring $H^*(M_r)$. When M_r is toric it is well known that this is the case (see for example [Fu]). We already observed that for $n = 4, 5, 6$ the toric action of bending along a system of $(n-3)$ non intersecting diagonals realize M_r as a toric manifold (for r's such that M_r is smooth). This is not the case for higher dimensions; in general let Δ be the set of regular values of μ_T in the convex polytope $\Xi = \mu_T(X)$. The connected components $\Delta_1, \dots, \Delta_\ell$ of Δ are themselves complex polytopes, and by Duistermaat-Heckman theorem the diffeotype of the reduced space M_r (thus also its cohomology ring) depends only on the region Δ_i which contains r.

[1] Note that M_r is compact, so its cohomology ring $H^*(M_r, \mathbb{Q})$ is finite dimensional and satisfy Poincaré duality

Theorem 5.1.3. *Suppose that the closure of Δ_i contains a vertex of Ξ, then its associated reduced space is a toric manifold.*

This result enlighten the importance of the wall crossing analysis we did in chapter 4. In fact, we know that for r^0 in a "external" region Δ_0 (i.e. such that Δ_0 contains a vertex of Ξ), the cohomology ring $H^*(M_{r^0})$ is generated by the c_i. To prove that this is true for each regular value r we will show that crossing a wall has the effect to kill some relations, and so (very roughly speaking) some of the generators that were "hidden" appear.

In 5.2 we will study how the cohomology ring $H^*(M_r)$ changes as r crosses a inner wall, before that let us calculate the cohomology ring for some examples in the case $n = 5$, where we already know that $H^*(M_r)$ is generated by the c_i.

5.1.1 Some examples

In 3.4 we calculated that for $r \in \Delta_0$, Δ_0 being the region of regular values such that $r^0 = (1, \frac{1}{2}, 4, 1, 2) \in \Delta_0$,

$$vol(M_r) = 2\pi^2(\mathfrak{p} - 2r_3)^2$$

$\mathfrak{p}$ being the (fixed) perimeter of the polygons in M_r, i.e. $\mathfrak{p} = \sum_{i=1}^n r_i$. From the cobordism result it also follows that, for $r \in \Delta_0$, M_r is cobordant $\mathbb{CP}^2$.

Because M_r is toric, the generators of the cohomology ring $H^*(M_r)$ are $c_1, \ldots, c_5$, and using the description 5.1 of the multiplicative relations between them we get:

$$\begin{array}{rcl} \frac{\partial^2}{\partial r_i \partial r_j} \frac{vol(M_r)}{4\pi^2} = 0 \quad \text{if } i,j \neq 3 & \Rightarrow & c_i c_j = 0 \quad \forall i,j \neq 3 \\ \frac{\partial^2}{\partial^2 r_3} \frac{vol(M_r)}{4\pi^2} = \frac{\partial}{\partial r_3}(-2(\mathfrak{p} - 2r_3)) = 4 & \Rightarrow & c_3^2 = 4 \end{array}$$

So all the c_i for $i = 1, 2, 4, 5$ are "hidden" by the relations above, and we can conclude

$$H^*(M_r) = \frac{\mathbb{Q}[c_3]}{\{c_3^3 = 0\}}.$$

Let us now analyze the second example we saw about both the cobordism and volume formulas: let $r^1 = (\frac{1}{2}, 2, 4, 1, 2)$, and let $\mathfrak{p}$ be the fixed perimeter and Δ_1 be the region of regular values determined by r^1.

From the volume formula it follows that for $r \in \Delta_1$

$$\frac{volM_r}{4\pi^2} = 2r_1(\mathfrak{p} - r_1 - 2r_3).$$

The only second partial derivatives which are not zero are $\frac{\partial^2}{\partial r_3 \partial r_1}$ and $\frac{\partial^2}{\partial^2 r_1}$, so $c_i c_j = 0$ if $i, j \notin \{1, 3\}$ and

$$\left.\begin{array}{l} \frac{\partial^2}{\partial r_3 \partial r_1} \frac{vol(M_r)}{4\pi^2} = 4 \\ \frac{\partial^2}{\partial^2 r_1} \frac{vol(M_r)}{4\pi^2} = -4 \end{array}\right\} \Rightarrow c_1^2 + c_1 c_3 = c_1(c_1 + c_3) = 0$$

which gives us the multiplicative relation between the generators $c_1, c_1 + c_3$; so, up to rescaling,

$$H^*(M_r) = \frac{\mathbb{Q}[c_1, c_1 + c_3]}{\{c_1^2 = -1, (c_1 + c_3)^2 = 1, c_1(c_1 + c_3) = 0\}}.$$

5.2 Wall crossing and Cohomology

By theorem 4.2.4, when r crosses a wall of type (p, q) the diffeotype of the reduced manifold M_r changes by replacing a copy of $\mathbb{CP}^{p-2}$ in M_r by a $\mathbb{CP}^{q-2}$ by means of a blow-up followed by a blow-down.

In this section we will study how the cohomology ring $H^*(M_r)$ changes as r crosses a wall; the main tools to prove our result will be the Mayer–Vietoris sequence and the Gysin sequence, for which we refer to [BT], together with the decomposition theorem as presented in [BBD] and [CM05].

Suppose r crosses a wall of type (p, q), let M be the moduli space of polygons before the wall crossing and M' be the moduli space of polygons after the wall crossing.

Remark 25. Note that, because the diffeotype of the moduli space M_r depends only on the region of regular values Δ_i which contains r, in the study of the cohomology ring structure there is no lost of informations in forgetting the length vector r and keeping track just of the regular values' region.

Moreover, the type of the wall already determine the wall crossing direction from Δ_0 (see (4.2)) to Δ_1 (see (4.3)), thus $M \simeq M_{r^0}$ for all $r^0 \in \Delta_0$ and $M' \simeq M_{r^1}$ for all $r^1 \in \Delta_1$.

Let us fix some notation:

$$\begin{aligned} V = N_\varepsilon \mathbb{CP}^{p-2} &= \text{tubular neightborhood of } \mathbb{CP}^{q-2} \subset M \\ V' = N_\varepsilon \mathbb{CP}^{q-2} &= \text{tubular neightborhood of } \mathbb{CP}^{p-2} \subset M' \\ U &= M \setminus \mathbb{CP}^{p-2} \\ U' &= M' \setminus \mathbb{CP}^{q-2}. \end{aligned}$$

By the wall-crossing theorem 4.2.4, $U = U'$ and $U \cap V = U' \cap V' := S_\varepsilon$. The Mayer–Vietoris sequences for the manifolds M and M' are:

$$\ldots \to H^{k-1}(S_\varepsilon) \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(S_\varepsilon) \to \ldots$$

$$\ldots \to H^{k-1}(S_\varepsilon) \to H^k(M') \to H^k(U') \oplus H^k(V') \to H^k(S_\varepsilon) \to \ldots$$

Because $H^k(U) = H^k(U')$ the change in the cohomology ring structure is enclose in how $H^k(V')$ and $H^k(V)$ map into $H^k(S_\varepsilon)$. These map will be brought into the light in the proof the next proposition.

Proposition 5.2.1.

$$H^*(S_\varepsilon) = H^*(\mathbb{CP}^{\min(p,q)-2}) \otimes H^*(S^{2\max(p,q)-3})$$

Proof. By construction, $N_\varepsilon \mathbb{CP}^{p-2}$ is the total space of a fibration in disks over $\mathbb{CP}^{p-2}$, and S_ε is the total space of the associated fibration in spheres:

$$\begin{array}{ccc} N_\varepsilon \mathbb{CP}^{p-2} & & S_\varepsilon \\ \Big\downarrow {\scriptstyle D^{2q-2}} & \Rightarrow & \Big\downarrow {\scriptstyle S^{2q-3}} \\ \mathbb{CP}^{p-2} & & \mathbb{CP}^{p-2} \end{array}$$

The fibration in spheres $\pi : S_\varepsilon \xrightarrow{S^{2q-3}} \mathbb{CP}^{p-2}$ induce the following Gysin sequence

$$\longrightarrow H^k(\mathbb{CP}^{p-2}) \xrightarrow{\pi^*} H^k(S_\varepsilon) \xrightarrow{\pi_*} H^{k-(2q-3)}(\mathbb{CP}^{p-2}) \xrightarrow{\wedge e} H^{k+1}(\mathbb{CP}^{p-2}) \longrightarrow$$

where π^* is the map induced in cohomology by the projection map π, π_* is the integration along the fibers and $\wedge e$ is the wedge product with the Euler class.

Recall that

$$H^k(\mathbb{CP}^{p-2}) = \begin{cases} \mathbb{Q} & \text{if } k = 0, 2, \ldots, 2(p-2) \\ 0 & \text{otherwise,} \end{cases}$$

and suppose that $q \geq p$, then the first bit of the Gysin map is

$$\mathbb{Q} \xrightarrow{\pi^*} H^0(S_\varepsilon) \xrightarrow{\pi_*} 0 \xrightarrow{\wedge e} \mathbb{Q} \xrightarrow{\pi^*} H^2(S_\varepsilon) \xrightarrow{\pi_*} 0 \xrightarrow{\wedge e}$$

$$\xrightarrow{\wedge e} 0 \xrightarrow{\pi^*} H^3(S_\varepsilon) \xrightarrow{\pi_*} 0 \longrightarrow \cdots$$

till $k = 2p - 2$ (in fact for all $0 \leq k \leq 2q - 3$, $H^{k-(2q-3)}(\mathbb{CP}^2) \simeq 0$ for dimensional reasons, i.e. $k - 2q - 3 \leq 0$). So

$$H^k(S_\varepsilon) \simeq H^k(\mathbb{CP}^{p-2}) \quad \forall 0 \leq k \leq 2(p-2).$$

At $k = 2q - 3$ the Gysin sequence goes as follow:

$$0 \xrightarrow{\pi^*} H^{2q-3}(S_\varepsilon) \xrightarrow{\pi_*} \mathbb{Q} \xrightarrow{\wedge e} 0 \xrightarrow{\pi^*} H^{2q-3}(S_\varepsilon) \xrightarrow{\pi_*} 0 \xrightarrow{\wedge e}$$

$$\xrightarrow{\wedge e} 0 \xrightarrow{\pi^*} H^{2q-1}(S_\varepsilon) \xrightarrow{\pi_*} \mathbb{Q} \xrightarrow{\wedge e} 0 \longrightarrow \cdots$$

(to check this second bit of the Gysin sequence the only thing to keep in mind is that $H^k(\mathbb{CP}^{p-2}) \simeq 0 \quad \forall k \geq 2q-3$, in fact $k \geq 2q-3 \geq 2p-3 > 2(p-2)$). Observing that $k - 2q + 3 = 2(p-2) \iff k = 2(p+q) - 7$,

$$H^k(S_\varepsilon) \simeq H^{k-(2q-3)}(\mathbb{CP}^{p-2}) \quad \forall 2q - 3 \leq k \leq 2n - 7.$$

$$H^k(S_\varepsilon) \simeq 0 \quad \forall\, k : \, 2(p-2) < k < 2q - 3, k \geq 2n - 6.$$

So, under the assumption $q \geq p$ we proved

$$H^*(S_\varepsilon) = H^*(\mathbb{CP}^{p-2}) \otimes H^*(S^{2q-3});$$

it is easy to check that if we assume $p \geq q$ then p and q exchange role, and the result follow. □

Because V retracts on $\mathbb{CP}^{p-2}$, $H^*(V) = H^*(\mathbb{CP}^{p-2})$; similarly $H^*(V') = H^*(\mathbb{CP}^{q-2})$, and we have all the ingredients to write the Mayer–Vietoris sequences for M and M' :

$$H^0(M) \to H^0(U) \oplus \mathbb{Q} \to \mathbb{Q} \to H^1(M) \to H^1(U) \oplus 0 \to 0 \to$$

$$\to H^2(M) \to H^2(U) \oplus \mathbb{Q} \to \mathbb{Q} \to$$

and

$$H^0(M') \to H^0(U') \oplus \mathbb{Q} \to \mathbb{Q} \to H^1(M') \to H^1(U') \oplus 0 \to 0 \to$$

$$\to H^2(M') \to H^2(U') \oplus \mathbb{Q} \to \mathbb{Q} \to$$

so, till the degree $2(p-2)$, the two sequences above are the same, thus

$$H^k(M) = H^k(M') \quad 0 \leq k \leq 2(p-2).$$

At $2(p-2)+1$ the Mayer–Vietoris sequences of the manifolds of M and M' are:

$$\to H^{2p-3}(M) \to H^{2p-3}(U) \oplus 0 \to 0 \to H^{2p-2}(M) \to H^{2p-2}(U) \oplus 0 \to 0 \to$$

$$\to H^{2p-3}(M') \to H^{2p-3}(U') \oplus 0 \to 0 \to H^{2p-2}(M') \to H^{2p-2}(U') \oplus \mathbb{Q} \to 0 \to$$

and, till the degree $2q-3$ the two sequences differ by the fact that $H^k(V') = \mathbb{Q}$ for k even, $2p-3 \leq k \leq 2q-3$, and $H^k(V) \simeq 0$. Thus

$$\dim(H^k(M')) = \dim(H^k(M)) + 1 \quad \text{if } k \text{ even, } 2p-3 \leq k \leq 2q-3$$

$$H^k(M') = H^k(M) = 0 \quad \text{for } k \text{ odd.}$$

At $2q-3$ the Mayer–Vietoris sequences for M and M' are

$$\to H^{2q-3}(M) \to H^{2q-3}(U) \oplus 0 \to 0 \to H^{2q-2}(M) \to H^{2q-2}(U) \oplus 0 \to 0 \to$$

$$\to H^{2q-3}(M') \to H^{2q-3}(U') \oplus 0 \to 0 \to H^{2q-2}(M') \to H^{2q-2}(U') \oplus 0 \to 0 \to$$

and so again (just as for $0 \leq k \leq 2(p-2)$) are

$$H^k(M') \simeq H^k(M) \quad k \geq 2q-3.$$

If $p \geq q$ the same arguments holds, just exchanging the roles of p and q, so:

$$\begin{array}{ll}
H^k(M') = H^k(M) = 0 & \text{if } k \text{ is odd;} \\
H^k(M') = H^k(M) & \begin{cases} 0 \leq k \leq 2(\min(p,q) - 2), \\ k \geq 2\max(p,q) - 3; \end{cases} \\
\dim(H^k(M')) = \dim(H^k(M)) + 1 & \begin{cases} k \,\text{even}, \\ 2\min(p,q) - 3 \leq k \leq 2\max(p,q) - 3. \end{cases}
\end{array}$$

This calculation, done using the Mayer–Vietoris sequences of the manifolds M and M', tells us in which degree the cohomology groups of the symplectic quotient M_r change as r crosses a wall of type (p, q).

Even though it is quite natural- by the construction- to expect that the new born cohomological classes are polynomial in the class of the blown up manifold $\mathbb{CP}^{q-2}$, this calculation does not give us such precise informations. We use the decomposition theorem due to Beilinson–Bernstein–Deligne [BBD] to identify precisely the new born classes that increase the dimension of the cohomology groups of "middle" degrees.

Let $f : X \to Y$ be a map of algebraic manifolds (i.e. manifolds which are the set of common zeros of a finite number of polynomials). For each $k > 0$, let

$$Y_k = \{y \in Y : \dim(f^{-1}(y)) \geq k\}. \tag{5.4}$$

Definition 5.1.

$$f \;\text{ is } \textit{small} \iff \dim Y_k + 2k < \dim X; \tag{5.5}$$

$$f \;\text{ is } \textit{semi-small} \iff \dim Y_k + 2k \leq \dim X. \tag{5.6}$$

Let $r^c \in W$ be a point of wall-crossing between Δ_0 and Δ_1, and let M_{r^c} the singular symplectic quotient associated to r^c. Let f be the inverse of the resolution in M of the singularity in M_{r^c}, and let f' be the inverse of its resolution in M', i.e.

$$\begin{array}{ccc}
M & & M' \\
& \searrow^{f} \quad {}^{f'}\swarrow & \\
& M_{r^c} &
\end{array}$$

and $\exists\ y_s \in M_{r^c}$ such that $(f^{-1}(y_s)) = \mathbb{CP}^{p-2}$ and $((f')^{-1}(y_s)) = \mathbb{CP}^{q-2}$.

Proposition 5.2.2. *At least one between f and f' is small.*

Proof. Assume $q > p$, then

- f is small, in fact $Y_k = \{y_s\}\ \forall 1 \leq k \leq 2(p-2)$ and the "biggest" of the inequalities 5.5 (i.e. corresponding to $k = 2(p-2)$) is verified, in fact it is

$$4(p-2) < 2(n-3) \iff 4p - 8 < 2p + 2q - 6 \iff p - 1 < q.$$

- f' is not semi-small (thus even not small); in fact again $Y_k = \{y \in Y : \dim((f^{-1})^{-1}(y)) \geq k\} = \{y_s\}$ and the "biggest" of the inequalities 5.6 is

$$4(q-2) \leq 2(n-3) \iff q - 1 \leq p,$$

  which is false.

If $p > q$, then f' is small and f is not semi-small. Note that if $p = q$ then both f and f' are small. □

Suppose that $f : M \to M_{r^c}$ is small, then

$$H^*(M) = IH^*(M_{r^c}),$$

where $IH^*(M_{r^c})$ is the intersection cohomology of the singular manifold M_{r^c}, see the survey paper by M. de Cataldo and L. Migliorini [CM].

We state the decomposition theorem just for the special situation of f and f' resolution of the singularity corresponding to the lined polygon in M_{r^c}. For the statement in full generality, proofs and more details we refer to the original paper by Beilinson–Bernstein–Deligne [BBD], and to [CM05] by de Cataldo–Migliorini, where an alternative proof is given.

In our setting, the decomposition theorem says that $H^*(M')$ is isomorphic to the intersection cohomology $IH^*(M_{r^c})$ of M_{r^c} plus polynomials in the cohomological classes of submanifolds $\mathcal{C}_i$ of M. In the moduli space situation, these submanifolds are just the preimages of the points $y_i \in Y_k$ as defined in (5.4).

If we assume $q \geq p$ (which is equivalent to assume f small), then $Y_k = \{y_S\}$ for $0 \leq k \leq 2(q-2)$. Thus $\mathcal{C} := f^{-1}(y_S)$ is the only submanifold in M' such that its class was born in the wall-crossing. By theorem 4.2.4,

$$\mathcal{C} = M\Big(\sum_{i \in I_p} r_i, r_{j_1}, \ldots, r_{j_q}\Big) \simeq \mathbb{CP}^{q-2}.$$

Similar arguments hold if $p \geq q$.

So, applying the decomposition theorem and the fact that one between f and f' is small, the following holds:

Theorem 5.2.3.

- *If* $q \geq p$,

$$H^*(M') = H^*(M) \oplus \bigoplus_{\alpha=0}^{q-p} \mathbb{Q}\Big([\mathbb{CP}^{q-2}] \wedge c_1^{\alpha}(\mathcal{N}')\Big)$$

where $\mathcal{N}'$ *is the normal bundle to* $\mathbb{CP}^{q-2} \subseteq M'$ *and* $[\mathbb{CP}^{q-2}] \in H^{2p-2}(M')$ *is the class of* $M\Big(\sum_{i \in I_p} r_i, r_{j_1}, \ldots, r_{j_q}\Big) \subset M'$.

- *If* $p \geq q$

$$H^*(M) = H^*(M') \oplus \bigoplus_{\alpha=0}^{p-q} \mathbb{Q}\Big([\mathbb{CP}^{p-2}] \wedge c_1^{\alpha}(\mathcal{N})\Big)$$

where $\mathcal{N}$ *is the normal bundle to* $\mathbb{CP}^{p-2} \subseteq M$ *and* $[\mathbb{CP}^{p-2}] \in H^{2q-2}(M)$ *is the class of* $M\Big(r_{i_1}, \ldots, r_{i_p}, \sum_{j \in I_q} r_j\Big) \subset M$.

At the light of this result, to prove that $H^*(M_r)$ is generated by the Chern classes c_i we need to express the classes $[\mathbb{CP}^{q-2}]$, $[\mathbb{CP}^{p-2}]$ and their wedge products $[\mathbb{CP}^{q-2}] \wedge c_1^{\alpha}(\mathcal{N}')$ and $[\mathbb{CP}^{p-2}] \wedge c_1^{\alpha}(\mathcal{N})$ as combinations of the c_i.

As before, let us assume $q \geq p$. Then, by Poincaré duality, $[\mathbb{CP}^{q-2}] \in H^{2p-2}(M')$ and thus we want to show that, for some constants A_α,

$$[\mathbb{CP}^{q-2}] = \sum_{\sum \alpha_i = 2p-2} A_\alpha c_1^{\alpha_1} \cdots c_n^{\alpha_n}.$$

To this aim, let us write explicitly who are the Chern classes c_i that appear in theorem 5.1.1.

Let r be a regular value in Δ_1 such that the reduced manifold M_r has the same diffeotype as M'. If $p : St_{2,n} \to Gr_{2,n}$ is the projection of $St_{2,n}$ on the orbit space $St_{2,n}/U_2$, denote by $p^{-1}(\mu^{-1}(r))$ the preimage on $St_{2,n}$ of the r-level set in $Gr_{2,n}$. Then $p^{-1}(\mu^{-1}(r))$ is the set of $(a,b) \in St_{2,n}$ such that each row has norm r_i, i.e.

$$p^{-1}(\mu^{-1}(r)) = \{(a,b) \in St_{2,n} : |a_i|^2 + |b_i|^2 = r_i \quad \forall i = 1, \dots n\}.$$

Because $a, b \in \mathbb{C}$, this naturally defines an inclusion map

$$p^{-1}(\mu^{-1}(r)) \hookrightarrow \prod_j S^3_{\sqrt{r_j}}.$$

Recall now that the quaternionic Hopf map

$$H(a_i, b_i) = i[(|a_i|^2 - |b_i|^2) - 2a_i b_i j]$$

as defined in section 1.3 gives us a way to associate to the i-th row of (a,b) a vector of length r_i in $\mathbb{R}^3$.

Thus the map

$$\begin{array}{rcl} H^n : \prod_j S^3_{\sqrt{r_j}} & \to & \prod_j S^2_{r_j} \\ (a,b) & \mapsto & (H(a_1,b_1), \dots, H(a_n,b_n)) \end{array}$$

maps $p^{-1}(\mu^{-1}(r))$ into the zero level set $\tilde{M}_r$ of the moment map relative to the $SO(3)$-action on the product of spheres as seen in section 1,

$$H^n(p^{-1}(\mu^{-1}(r))) = \{(e_1, \dots, e_n) \in \prod_j S^2_{r_j} : \sum_i x_i = 0\} = \tilde{M}_r.$$

In fact, thinking at the description of M_r as the quotient by the right action of U_2 and by the left action of U_1^n on $M_{n\times 2}$, it follows that to reach M_r from $H^n(p^{-1}(\mu^{-1}(r)))$ we still have to quotient by the residual U_2/U_1-action.

Thus the following diagram enclose (somehow) the beautiness of the rich geometric structure of M_r :

$$
\begin{array}{ccc}
St_{2,n} \supseteq p^{-1}(\mu^{-1}(r)) & \hookrightarrow & \prod_j S^3_{\sqrt{r_j}} \\
\downarrow p & & \downarrow H^n \\
Gr_{2,n} \supseteq \mu{-}1(r) & & H^n(p^{-1}(\mu^{-1}(r))) \subseteq \prod_j S^2_{r_j} \\
\downarrow U_1^n & \swarrow & \\
M_r & &
\end{array}
$$

It is now clear that the classes c_i relative to the fibration $\mu^{-1}(r) \to M_r$ are actually the classes c_i relative to the fibration $\prod_j S^3_{\sqrt{r_j}} \to \prod_j S^2_{r_j}$ (in fact the fibration $\prod_j S^2_{r_j} \to M_r$ is trivial), and these are well known to be the pullbacks of the volume forms on each sphere, i.e.

$$c_j = p_j^* \omega_j \tag{5.7}$$

where $p_j : \prod_j S^2_{r_j} \to S^2_{r_j}$ is the canonical projection on the j-th factor and ω_j is the volume formula $S^2_{r_j}$.

In theorem 4.2.4 we characterized the blown up and down manifolds $\mathbb{CP}^{q-2}$ and $\mathbb{CP}^{p-2}$ in terms of moduli spaces. Precisely, as crossing a wall of type (p, q) the blown up manifold is $M(\sum_{i \in I_p} r_i, r_{j_1} \dots, r_{j_q}) \simeq \mathbb{CP}^{q-2}$.

Let $\rho = (\sum_{i \in I_p} r_i, r_{j_1} \dots, r_{j_q})$, then the moduli space M_ρ verifies itself the construction above, precisely there exists a fibration

$$\prod_j S^3_{\sqrt{\rho_j}} \to S^2_{\rho_j}$$

which, identifying each $S^2_{\rho_j}$ with $\mathbb{CP}^1$, looks as follow

$$(\mathcal{O}(r_{i_1}) \oplus \dots \oplus \mathcal{O}(r_{i_p})) \boxtimes \mathcal{O}(r_{j_1}) \boxtimes \dots \boxtimes \mathcal{O}(r_{j_q}) \to \prod_{q+1} \mathbb{CP}^1$$

where, if p_{j_0} is the projection from $\prod_{j=0}^{q} S^2_{\rho_j}$ onto $S^2_{\sum_{i\in I_p} r_i}$ and if p_{j_i} is the projections on the sphere $S^2_{r_{j_i}}$ for each $i = 1, \ldots, q$, then

$$(\mathcal{O}(r_{i_1}) \oplus \ldots \oplus \mathcal{O}(r_{i_p})) \boxtimes \mathcal{O}(r_{j_1}) \boxtimes \ldots \boxtimes \mathcal{O}(r_{j_q}) =$$

$$= p^*_{j_0}(\mathcal{O}(r_{i_1}) \oplus \ldots \oplus \mathcal{O}(r_{i_p})) \oplus p^*_{j_1}\mathcal{O}(r_{j_1}) \oplus \ldots \oplus \mathcal{O}(r_{j_q}).$$

We saw in section 4.2 that not just $M_\rho \simeq \mathbb{CP}^{q-2}$, but moreover ρ lies in an "external" region of regular values for the moment map $\mu_{U_1^{q+1}}$, and thus M_ρ is toric. So the cohomology ring of M_ρ is generated by the components of the first Chern class of the fibration

$$\mu_{U_1^{q+1}}(\rho) \to M_\rho.$$

These Chern classes are

$$c_i = c_1(\mathcal{O}(r_i)) \quad \forall i \in I_q$$

and

$$c_1 = c_1(\mathcal{O}(r_{i_1}) \oplus \ldots \oplus \mathcal{O}(r_{i_p})) = \prod_{i\in I_p} c_1(\mathcal{O}(r_i)) = \prod_{i \in I_p} c_i \qquad (5.8)$$

where the first Chern classes on the right hand side of 5.8 are the Chern classes in 5.7.

So, if $r \in \mathbb{R}^n_+$ is such that $M(\sum_{i\in I_p} r_i, r_{j_1}, \ldots, r_{j_q})$ is an empty submanifold of M_r, (as it is the case before the wall-crossing) then the calculation tells us that

$$c_{i_1} \cdots c_{i_p} = 0.$$

When r crosses the wall $\sum_{i\in I_p} r_i = \sum_{j \in I_q} r_j$, then $M(\sum_{i\in I_p} r_{j_1}, \ldots, r_{j_q})$ is a non empty submanifold of M_r (and in fact it is a submanifold of dimension $(q-2)$ isomorphic to $\mathbb{CP}^{q-2}$). So, crossing the wall, the multiplicative relation $c_{i_1} \cdots c_{i_p}$ stops to be zero and in fact

$$[\mathbb{CP}^{q-2}] = \Big[M\Big(\sum_{i\in I_p} r_{j_1}, r_{j_1} \ldots, r_{j_q}\Big)\Big] = c_{i_1} \cdots c_{i_p} \in H^{2p-2}(M')$$

as we wanted to prove.

Analogously we can prove that the other Chern classes are can be expressed as a product of c_i's, and thus, by theorem 5.1.2, we proved the following:

Theorem 5.2.4. *The cohomology ring $H^*(M_r, \mathbb{Q})$ of the moduli space of polygons M_r, when M_r is a smooth manifold, is generated by the first Chern classes $c_1, \ldots, c_n$ of the n complex line bundles associated to the fibration $\mu^{-1}(r_1, \ldots, r_n) \to M_r$. So*

$$H^*(M_r, \mathbb{Q}) \simeq \mathbb{Q}[x_1, \ldots, x_n]/ann(volM_r)$$

where a polynomial $Q(x_1, \ldots, x_n) \in ann(volM_r)$ if and only if $Q\Big(\frac{\partial}{\partial r_1}, \ldots, \frac{\partial}{\partial r_n}\Big) vol(M_r) = 0$.

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