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Local Risk-Minimization for Defaultable Markets

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Introduction

Over the last thirty years, mathematical finance and financial engineering have been rapidly expanding fields of science. The main reason is the success of sophisticated quantitative methodologies in helping professional manage financial risk. Hence it may be reasonable that newly developed credit derivatives industry will also benefit from the use of advanced mathematics. What does it justify the considerable growth and development of this kind of industry?

The answer is given by the need to handle credit risk, which is one of the fundamental factors of financial risk. Indeed, a great interest has grown in the development of advanced mathematical models for finance and at the same time we can note a tremendous acceleration in research efforts aimed to a better understanding, modelling and hedging this kind of risk.

But what does *credit risk* mean exactly?

A **default risk** is the possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract. If this happens, we say that the party defaults, or that a default event occurs.

More generally, by **credit risk** we mean the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads and default events (bankruptcy, insolvency, missed payments).

It is important to make a clear distinction between the *reference (credit) risk* and the *counterparty (credit) risk.* The first term refers to the situation where

both parties involved in a contract are supposed to be default-free, but the underlying assets are defaultable. *Credit derivatives* are recently developed financial instruments that allow to trade and transfer the reference credit risk, either completely or partially, between the counterparties.
Let us now consider the counterparty risk. This kind of risk emerges in a clear way in such contracts as *defaultable claims*. These derivatives are contingent agreements that are traded over-the-counter between default-prone parties. Each side of contract is exposed to the counterparty risk of the other party but we should stress that the underlying assets are assumed to be *insensitive* to credit risk (for an extensive survey of this subject see [13]).
A classical example of defaultable claim is a European defaultable option, that is an option contract in which the payoff at maturity depends on whether a default event, associated with the option's writer, has occurred before maturity or not (see for instance Chapter 3 which deals with the case of a defaultable put).
The main objective of this thesis is right the study of the problem of pricing and hedging defaultable claims, in particular by using the local risk-minimization, one of the main competing quadratic hedging approaches. The thesis is divided into six parts, consisting of Chapters 1-5 and a final Appendix.

Chapter 1 is completely devoted to a review of the main results of the theory of the so-called *quadratic criteria*: the local risk-minimization and the mean-variance hedging. For an exhaustive survey of relevant results we refer to [22], [25] and [35], while a numerical comparison study can be found in [26].
The local risk-minimization approach was first introduced by Föllmer and Sondermann in [23] when the risky asset is represented by a martingale. Successively it was extended to the general semimartingale case by Schweizer in [32] and [33] and by Föllmer and Schweizer in [22].
The main feature of the local risk-minimization approach is the fact that one has to work with strategies which are not self-financing. Given a contingent

claim H, according to this method, we look for a hedging strategy that perfectly replicates H, but renouncing to the self-financing constraint. Under this assumption, the strategy needs an instantaneous adjustment represented by the *cost process*. It is clear that a "good" strategy should have a minimal cost. The locally risk-minimizing strategy is characterized by two properties:

- the cost process C is a *martingale* (so the strategy is at least "mean-self-financing");
- the cost process C is *strongly orthogonal* to the martingale part of the underlying asset.

A locally risk-minimizing strategy exists if and only if the contingent claim H admits the so-called "Föllmer-Scweizer decomposition", that can be seen as generalization of the Galtchouk-Kunita-Watanabe decomposition from martingale theory. In particular, if the discounted risky asset price X is continuous, the Föllmer-Scweizer decomposition can be obtained as Galtchouk-Kunita-Watanabe decomposition computed under the so-called *minimal martingale measure*.

The mean-variance hedging method insists on the self-financing constraint and looks for the best approximation of a contingent claim by the terminal value of a self-financing portfolio. The use of a quadratic criterion to measure the quality of this approximation has been proposed for the first time by Bouleau and Lamberton in [14], in the case of assets represented by martingales which are also functions of a Markov process. We can obtain the mean-variance optimal strategy by projecting the discounted value of a contingent claim H on a suitable space of stochastic integrals, which represents the attainable claims. The dual problem is to find the so-called *variance optimal measure*. It can be proved (see [16] and [31]) that if the density of this martingale measure is known, the variance-optimal portfolio and its initial value are completely characterized. The *mean-variance hedging* has been extensively studied in the context of defaultable markets by [7], [8], [9] and [10]. In Chapter 3 we extend some of their results to the case of *stochastic* drift

μ and volatility σ in the dynamics (2.5) of the risky asset price, and *random* recovery rate. Empirical analysis of recovery rates shows that they may depend on several factors, among which default delays (see for example [15]).

In Chapter 2, we describe our general framework into details, emphasizing in particular the presence of defaultable claims in the market. We consider a simple market model with two non-defaultable primary assets (the money market account B and the discounted risky asset X) and a (discounted) defaultable claim H. Then we discuss our choice to investigate defaultable markets by means of quadratic hedging criteria and in particular the choice of the local risk-minimization. Finally, the last section presents an outline of the thesis.

In Chapter 3 we start the study of defaultable markets by means of local risk-minimization. According to [1], we apply the local risk-minimization approach to a defaultable put option with random *recovery at maturuty* and we compare it with intensity-based evaluation formulas and the mean-variance hedging. We solve analytically the problem of finding respectively the hedging strategy and the associated portfolio for the three methods in the case where the default time and the underlying Brownian motion are supposed to be independent.
The following two chapters are devoted to the application of the local risk-minimization in the general case. First we study defaultable claims with random recovery scheme *at maturity*, then *at default time*.

In Chapter 4 we extend the previous results and consider a more general case: according to [2] we apply the local risk-minimization approach to a generic defaultable claim with recovery scheme at *maturity* in a more general setting where the dynamics of the discounted risky asset X may be influenced by the occurring of a default event and also the default time τ itself may depend on the assets prices behavior.

In Chapter 5 we study the problem of pricing and hedging a defaultable claim with random recovery scheme *at default time*, i.e. a random recovery payment is received by the owner of the contract in case of default at time of default. Here according to [3], we provide the pseudo-locally risk-minimizing strategy in the case when the agent information takes into account the possibility of a default event. We conclude by discussing the problem of finding a pseudo-locally risk-minimizing strategy in the case when the agent obtains her information only by observing the asset prices on the non-defaultable market before the default happens.

In the Appendix, we summarize for the reader's convenience the definition and the main properties of the *predictable projection*, an important subject of Probability Theory that we have used in Chapter 4.

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Alessandra Cretarola

Chapter 1

Quadratic Hedging Methods in Incomplete Markets

1.1 Introduction

In this chapter we provide a review of the main results of the theory of local risk-minimization and mean-variance hedging. These are "quadratic" hedging methods used for valuation and hedging of derivatives in incomplete markets. For an extensive survey of both approaches, we refer to [22], [35] and [25]. A numerical comparison can be found in [26].

If we deal with non-attainable contingent claims, it is by definition impossible to find a hedging strategy allowing a perfect replication which is at the same time self-financing. From a financial point of view, this means that such a claim will have an *intrinsic risk*.

The main feature of the local risk-minimization approach is the fact that one has to work with strategies which are not self-financing and the purpose becomes to minimize the riskiness in a suitable way. If we consider a not attainable contingent claim H, a defaultable claim for instance, according to this method we look for a hedging strategy with minimal cost that perfectly replicates H.

The mean-variance hedging approach insists on the self-financing constraint and looks for the best approximation of a contingent claim by the terminal value of a self-financing portfolio. The use of a quadratic criterion to measure the quality of this approximation has been proposed for the first time by Bouleau and Lamberton in [14], in the case of assets represented by martingales which are also functions of a Markov process.

1.2 Setting

This section lays out the general background for the two approaches in an uniform framework.
We start with a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ and a fixed time horizon $T \in (0, \infty)$. We consider a simple model of financial market in continuous time with two non-defaultable primary assets available for trade a risky asset and the money market account described by the processes S and B respectively, and a contingent claim whose discounted value H is given by a random variable on $(\Omega, \mathcal{G}, \mathbb{Q})$.

- We assume that the processes S and B are adapted to a filtration $(\mathcal{G}_t)_{0\leq t\leq T}$ satisfying the usual hypotheses of completeness and right-continuity. Adaptedness ensures that the prices at time t are $\mathcal{G}_t$-measurable. In particular the money market account is given by $B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right)$, where r_t is a $\mathcal{G}_t$-predictable process and used as discounting factor

- Furthermore we assume that r and the dynamics of S are such that the discounted price process $X_t := \dfrac{S_t}{B_t}$ belongs to $L^2(\mathbb{Q})$, $\forall t \in [0, T]$. In addition, we assume that there exists an equivalent martingale measure $\mathbb{Q}^*$ with square-integrable density for the discounted price process X. Hence we can exclude arbitrage opportunities in the market. Mathematically, this implies that X is a semimartingale under the basic measure $\mathbb{Q}$.

- Finally we suppose that the discounted payoff H at time T is described by a $\mathcal{G}_T$-measurable square-integrable random variable. Hence $H \in L^2(\mathcal{G}_T, \mathbb{Q})$.

It should be clear that completeness now means that any contingent claim H can be represented as a stochastic integral with respect to X. The integrand provides the hedging strategy which is self-financing and which creates the discounted payoff at the maturity T of the contract without any risk.
Generally, given a contingent claim H with expiration date T, there are at least two things a trader may want to do: *pricing* by assigning a value to H at times $t < T$ and *hedging* by covering himself against potential losses arising from a sale of H, in particular by means of dynamic trading strategies based on X. Since under the previous assumptions X is a $\mathbb{Q}$-semimartingale, we can use stochastic integrals with respect to X and introduce the set $L(X)$ of all $\mathcal{G}$-predictable X-integrable processes.

Definition 1.2.1. *An* admissible *strategy is any pair $\varphi = (\xi, \eta)$, where $\xi \in L(X)$ and η is a real-valued $\mathcal{G}$-adapted process such that the discounted value process $V_t(\varphi) := \xi_t X_t + \eta_t$, $0 \le t \le T$, is right-continuous.*

In an *incomplete* market a general claim is not necessarily a stochastic integral with respect to X. For instance, in the case of defaultable claims, the presence of default adds an ulterior source of randomness that makes the market incomplete. Hence it is interesting to introduce the main *quadratic* hedging approaches used to price and hedge derivatives in incomplete financial markets.

1.3 Local risk-minimization

Problem: in the financial market outlined in Section 1.2, we look for an *admissible strategy with minimal cost* which replicates a given contingent claim H.

If H is not attainable we cannot work with self-financing strategies and so the purpose is to reduce the risk. The local risk-minimization criterion

for measuring the riskiness of a strategy was first introduced by Föllmer and Sondermann in [23] when the risky asset is represented by a martingale. Successively it was extended to the general semimartingale case by Schweizer in [32] and [33] and by Föllmer and Schweizer in [22].
First we briefly discuss the simple special case where X is a $\mathbb{Q}$-martingale. Consequently we motivate and investigate the general case. We address the first problem in the following section, the second in Section 1.3.2.

1.3.1 The martingale case

For the case where X is a $\mathbb{Q}$-martingale, this method has been defined and developed by Föllmer and Sondermann under the name of *risk-minimization*. In the market model outlined in Section 1.2 we introduce $L^2(X)$, the space of all $\mathcal{G}$-predictable processes ξ such that

$$\|\xi\|_{L^2(X)} := \left(E\left[\int_0^T \xi_s^2 \mathrm{d}[X]_s\right]\right)^{\frac{1}{2}} < \infty.$$

Definition 1.3.1. *An* RM-strategy *is an admissible strategy* $\varphi = (\xi, \eta)$ *with* $\xi \in L^2(X)$ *and such that the discounted value process* $V_t(\varphi) = \xi_t X_t + \eta_t$, $0 \le t \le T$ *is square-integrable.*

Definition 1.3.2. *For any RM-strategy* φ*, the* cost process *is defined by*

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \xi_s \mathrm{d}X_s, \quad 0 \le t \le T. \tag{1.1}$$

$C_t(\varphi)$ *describes the total costs incurred by* φ *over the interval* $[0, T]$. *The* risk process *of* φ *is defined by*

$$R_t(\varphi) := E\left[(C_T(\varphi) - C_t(\varphi))^2 \middle| \mathcal{G}_t\right], \quad 0 \le t \le T. \tag{1.2}$$

Definition 1.3.3. *An RM-strategy* φ *is called* risk-minimizing *if for any RM-strategy* $\tilde{\varphi}$ *such that* $V_T(\tilde{\varphi}) = V_T(\varphi)$ $\mathbb{Q}$*-a.s., we have*

$$R_t(\varphi) \le R_t(\tilde{\varphi}) \quad \mathbb{Q} - a.s. \quad \textit{for every } t \in [0, T].$$

The following results provide a characterization of a risk-minimization strategy.

Lemma 1.3.4. *An RM-strategy φ is risk-minimizing if and only if*

$$R_t(\varphi) \leq R_t(\tilde{\varphi}) \quad \mathbb{Q} - a.s.$$

for every $t \in [0, T]$ and for every RM-strategy $\tilde{\varphi}$ which is an admissible continuation of φ from t on in the sense that $V_t(\tilde{\varphi}) = V_t(\varphi)$ $\mathbb{Q}$-a.s., $\tilde{\xi}_s = \xi_s$, for $s \leq t$ and $\tilde{\eta}_s = \eta_s$ for $s < t$.

Proof. See Lemma 2.1 of [34] for the proof. □

Definition 1.3.5. *An RM-strategy φ is called* mean-self-financing *if its cost process $C(\varphi)$ is a $\mathbb{Q}$-martingale.*

Lemma 1.3.6. *If φ is a risk-minimizing strategy, then it is also mean-self-financing.*

Proof. See Lemma 2.3 of [35]. □

If X is a $\mathbb{Q}$-martingale, the risk-minimization problem is always solvable by applying the Galtchouk-Kunita-Watanabe decomposition. Since the set $I^2(X) = \{\int \xi \mathrm{d}X | \xi \in L^2(X)\}$ is a stable subspace of $\mathcal{M}_0^2(\mathbb{Q})$, i.e. the space of square-integrable $\mathbb{Q}$-martingales null at 0 (see Lemma 2.1 of [35]), any $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ can be uniquely written as

$$H = E[H] + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H \quad \mathbb{Q} - \text{a.s.} \tag{1.3}$$

for some $\xi^H \in L^2(X)$ and some $L^H \in \mathcal{M}_0^2(\mathbb{Q})$ strongly orthogonal to $I^2(X)$. The next result was obtained by Föllmer and Sondermann in [23] for the one-dimensional case under the assumption that X is a square-integrable $\mathbb{Q}$-martingale. Schweizer has proved this result for a general local $\mathbb{Q}$-martingale X.

Theorem 1.3.7. *If X is a $\mathbb{Q}$-martingale, then every contingent claim $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ admits a unique risk-minimizing strategy φ^* such that $V_T(\varphi^*) = H$. In terms of decomposition* (1.3)*, the risk-minimizing strategy φ^* is explicitly given by*

$$\begin{aligned} \xi^* &= \xi^H, \\ V_t(\varphi^*) &= E\left[H \middle| \mathcal{G}_t\right], \quad 0 \leq t \leq T, \\ C(\varphi^*) &= E\left[H\right] + L^H. \end{aligned}$$

Proof. See Theorem 2.4 of [35] for the proof. □

1.3.2 The semimartingale case

The generalization to the semimartingale case is due to Schweizer (see [32] and [33]), who called the resulting concept *local risk-minimization*. When X is a semimartingale under $\mathbb{Q}$, a contingent claim H admits in general no risk-minimizing strategy φ with $V_T(\varphi) = H$ $\mathbb{Q}$-a.s. The proof is based on an explicit counterexample in discrete times and can be found in [32].

We analyze here only the continuous-time framework. The basic idea of this approach is to control hedging errors at each instant by minimizing the conditional variances of instantaneous cost increments sequentially over time. This involves (local) variances and so we require more specific assumptions on the discounted price process X.

- We remark that in our model X belongs to the space $\mathcal{S}^2(\mathbb{Q})$ of semimartingales so that it can be decomposed as follows:

  $$X_t = X_0 + M_t^X + A_t^X, \quad t \in [0, T],$$

  where M^X is a square-integrable (local) $\mathbb{Q}$-martingale null at 0 and A^X is a predictable process of finite variation null at 0.

- We say that the so-called **Structure Condition (SC)** is satisfied in our model if the *mean-variance tradeoff process*

  $$\widehat{K}_t(\omega) := \int_0^t \alpha_s^2(\omega) \mathrm{d}\langle M^X \rangle_s \qquad (1.4)$$

is almost surely finite $\forall t \in [0,T]$, where α is a $\mathcal{G}$-predictable process. Since there exists an equivalent martingale measure for X by hypothesis, it is automatically satisfied if X is continuous (see [35]).

We denote by Θ_s the space of $\mathcal{G}$-predictable processes ξ on Ω such that

$$E\left[\int_0^T \xi_s^2 \mathrm{d}[M^X]_s\right] + E\left[\left(\int_0^T \left|\xi_s \mathrm{d}A_s^X\right|\right)^2\right] < \infty. \tag{1.5}$$

Definition 1.3.8. *An* L^2-strategy *is an admissible strategy* $\varphi = (\xi, \eta)$ *such that* $\xi \in \Theta_s$ *and the discounted value process* $V(\varphi)$ *is square-integrable, i.e.* $V_t(\varphi) \in L^2(\mathbb{Q})$ *for each* $t \in [0,T]$.

Definition 1.3.9. *An* L^2*-strategy* φ *is called* mean-self-financing *if its cost process* $C(\varphi)$ *is a* $\mathbb{Q}$*-martingale.*

Remark 1.3.10. *We should stress that we consider strategies which are in general not self-financing. It is clear that an admissible strategy is self-financing if and only if the cost process* C *is constant and the risk process* R *is identically zero. Hence the cost process represents the instantaneous adjustment needed by the self-financing part of the portfolio in order to perfectly replicate the contingent claim* H *at time* T *of maturity.*

A *small perturbation* is an L^2-strategy $\Delta = (\delta, \epsilon)$ such that δ is bounded, the variation of $\int \delta(\mu - r)X\mathrm{d}t$ is bounded (uniformly in t and ω) and $\delta_T = \epsilon_T = 0$. Given an L^2-strategy φ a small perturbation Δ and a partition $\pi \in [0,T]$, set

$$r^\pi(\varphi, \Delta) := \sum_{t_i, t_{i+1} \in \pi} \frac{R_{t_i}\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) - R_{t_i}(\varphi)}{E[\langle(\sigma X) \cdot W\rangle_{t_{i+1}} - \langle(\sigma X) \cdot W\rangle_{t_i} | \mathcal{G}_{t_i}]} \mathbb{I}_{(t_i, t_{i+1}]}.$$

The next definition formalizes the intuitive idea that changing an optimal strategy over a small time interval increases the risk, at least asymptotically.

Definition 1.3.11. *We say that* φ *is* locally risk-minimizing *if*

$$\liminf_{n \to \infty} r^{\pi_n}(\varphi, \Delta) \geq 0 \qquad (\mathbb{Q} \otimes \langle M^X \rangle) - a.e. \ on \ \Omega \times [0,T],$$

for every small perturbation Δ and every increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions going to zero.

In particular, how to characterize a locally risk-minimizing strategy is shown in the next result valid for the one-dimensional case.

Theorem 1.3.12. *Suppose that X satisfies (SC), $\langle M^X\rangle$ is $\mathbb{Q}$-a.s. strictly increasing, A^X is $\mathbb{Q}$-a.s. continuous and $E\left[\hat{K}_T\right] < \infty$. Let $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ be a contingent claim and φ an L^2-strategy with $V_T(\varphi) = H$ $\mathbb{Q}$-a.s. Then φ is locally risk-minimizing if and only if φ is mean-self-financing and the martingale $C(\varphi)$ is strongly orthogonal to M^X.*

Proof. See Proposition 2.3 of [33] for the proof. □

Theorem 1.3.12 motivates the following:

Definition 1.3.13. *Let $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ be a contingent claim. An L^2-strategy φ with $V_T(\varphi) = H$ $\mathbb{Q}$-a.s. is called* pseudo-locally risk-minimizing *for H if φ is mean-self-financing and the martingale $C(\varphi)$ is strongly orthogonal to M^X.*

Definition 1.3.13 is given for the general multi-dimensional case. If we consider a one-dimensional model and X is sufficiently well-behaved, then pseudo-locally and locally risk-minimizing strategies coincide. But in general, pseudo-locally risk-minimizing strategies are easier to find and to characterize, as shown in the next result.
Let $\mathcal{M}_0^2(\mathbb{Q})$ be the space of all the square-integrable $\mathbb{Q}$-martingale null at 0.

Proposition 1.3.14. *A contingent claim $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ admits a pseudo-locally risk-minimizing strategy φ (in short* plrm-strategy*) if and only if H can be written as*

$$H = H_0 + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H \quad \mathbb{Q} - \text{a.s.} \tag{1.6}$$

with $H_0 \in \mathbb{R}$, $\xi^H \in \Theta_S$, $L^H \in \mathcal{M}_0^2(\mathbb{Q})$ strongly $\mathbb{Q}$-orthogonal to M^X. The plrm-strategy is given by

$$\xi_t = \xi_t^H, \quad 0 \leq t \leq T$$

with minimal cost

$$C_t(\varphi) = H_0 + L_t^H, \quad 0 \le t \le T.$$

If (1.6) holds, the optimal portfolio value is

$$V_t(\varphi) = C_t(\varphi) + \int_0^t \xi_s \mathrm{d}X_s = H_0 + \int_0^t \xi_s^H \mathrm{d}X_s + L_t^H,$$

and

$$\zeta_t = \zeta_t^H = V_t(\varphi) - \xi_t^H X_t.$$

Proof. It follows from the definition of pseudo-optimality and Proposition 2.3 of [22]. □

Decomposition (1.6) is well known in literature as the *Föllmer-Schweizer decomposition* (in short *FS decomposition*). In the martingale case it coincides with the Galtchouk-Kunita-Watanabe decomposition. We see now how one can obtain the FS decomposition by choosing a convenient martingale measure for X following [22].

Definition 1.3.15 (The Minimal Martingale Measure). *A martingale measure $\widehat{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ with square-integrable density is called **minimal** if $\widehat{\mathbb{Q}} \equiv \mathbb{Q}$ on $\mathcal{G}_0$ and if* any square-integrable $\mathbb{Q}$-local martingale which is strongly orthogonal to M^X under $\mathbb{Q}$ remains a local martingale under $\widehat{\mathbb{Q}}$.

The minimal measure is the equivalent martingale measure that modifies the martingale structure as little as possible.

Theorem 1.3.16. *Suppose X is continuous and hence satisfies (SC). Suppose that the strictly positive local $\mathbb{Q}$-martingale*

$$\hat{Z}_t = E\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\bigg|\mathcal{G}_t\right] = \mathcal{E}\left(-\int \alpha \mathrm{d}M^X\right)_t$$

is a square-integrable martingale and define the process $\widehat{V}^H$ as follows

$$\widehat{V}_t^H := \widehat{E}[H|\mathcal{G}_t], \qquad 0 \le t \le T,$$

where $\widehat{E}[\,\cdot\,|\mathcal{G}_t]$ *denotes the conditional expectation under* $\widehat{\mathbb{Q}}$. *Let*

$$\widehat{V}_T^H = \widehat{E}[H|\mathcal{G}_T] = \widehat{V}_0^H + \int_0^T \widehat{\xi}_s^H \mathrm{d}X_s + \widehat{L}_T^H \tag{1.7}$$

be the GKW decomposition of $\widehat{V}_t^H$ *with respect to* X *under* $\widehat{\mathbb{Q}}$. *If either* H *admits a FS decomposition or* $\hat{\xi}^H \in \Theta_s$ *and* $\hat{L}^H \in \mathcal{M}_0^2(\mathbb{Q})$, *then (1.7) for* $t = T$ *gives the FS decomposition of* H *and* $\hat{\xi}^H$ *gives a plrm-strategy for* H. *A sufficient condition to guarantee that* $\widehat{Z} \in \mathcal{M}_0^2(\mathbb{Q})$ *and the existence of a FS decomposition for* H *is that the* mean-variance tradeoff process $\widehat{K}_t$ *is uniformly bounded.*

Proof. For the proof, see Theorem 3.5 of [35]. □

Theorem 1.3.16 shows that for X continuous, finding a pseudo-locally risk-minimizing strategy for a given contingent claim $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ essentially leads us to find the Galtchouk-Kunita-Watanabe decomposition of H under the minimal martingale measure $\widehat{\mathbb{Q}}$.

1.4 Mean-variance hedging

This sections presents the second of the two main quadratic hedging approaches: mean-variance hedging. While local risk-minimization insists on the replication requirement $V_T = H$ $\mathbb{Q}$-a.s., mean-variance hedging is concerned on the self-financing constraint.

In this method, hedging performance is defined as the L^2-norm of the difference, at maturity date T, between the discounted payoff H and the hedging portfolio V_T:

$$\left\| H - V_0 - \int_0^T \xi_s \mathrm{d}X_s \right\|_{L^2(\mathbb{Q})}^2 .$$

Given an admissible self-financing hedging strategy $\varphi = (\xi, \eta)$ according to Definition 1.2.1, the discounted value process $V(\varphi)$ is given by

$$V_t(\varphi) = V_0 + \int_0^t \xi_s \mathrm{d}X_s.$$

Then η is completely determined by the pair (V_0, ξ):

$$\eta_t = V_0 + \int_0^t \xi_s \mathrm{d}X_s - \xi_t X_t, \quad 0 \leq t \leq T.$$

The difference $H - V_0 - \int_0^T \xi_s \mathrm{d}X_s$ is then the net loss at time T from paying out the claim H after having traded according to (V_0, ξ) and mean-variance hedging simply minimizes the expected net squared loss. Hence we can formulate the mean-variance problem as follows:

Problem: finding an admissible hedging strategy (V_0, ξ) which solves the following minimization problem:

$$\min_{(V_0,\xi)} E\left[\left(H - V_0 - \int_0^T \xi_s \mathrm{d}X_s\right)^2\right],$$

where ξ belongs to

$$\Theta = \left\{\xi \in L(X) : \int_0^t \xi_s \mathrm{d}X_s \in L^2(\mathcal{G}_T, \mathbb{Q})\right\},$$

where we recall that $L(X)$ denotes the set of all $\mathcal{G}$-predictable X-integrable processes. If such strategy exists, it is called *Mean-Variance Optimal Strategy* (in short mvo-strategy) and denoted by $(\tilde{V}_0, \tilde{\xi})$. V_0 is called *approximation price*.

To give another interpretation, we note that $H - V_0 - \int_0^T \xi_s \mathrm{d}X_s$ is the cost on $(0, T]$ of an admissible strategy φ with $V_T(\varphi) = H$, initial capital V_0 and stock component ξ. Hence we minimize the risk at time 0 only instead of the entire risk process as in the previous section. Since R_0 depends only on V_0 and ξ, it is not necessary to minimize over the entire pair $\varphi = (\xi, \eta)$.

Dual Problem: finding an equivalent martingale measure $\tilde{\mathbb{Q}}$ such that its density is square-integrable and its norm:

$$\left\|\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right\|^2 = E\left[\left(\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right)^2\right]$$

is minimal over the set of all the equivalent probability measures $\mathcal{P}_e^2(X)$ for X. By [16] this probability measure exists if X is continuous and $\mathcal{P}_e^2(X) \neq \emptyset$

and it is called *Variance-Optimal Measure* since:

$$\left\|\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right\|^2 = 1 + Var\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right].$$

Remark 1.4.1. *From a mathematical point of view, mean-variance hedging leads us to project the random variable H on the linear space generated by constants and stochastic integrals with respect to X. In the case where X is a local $\mathbb{Q}$-martingale, the problem is solved by the Galtchouk-Kunita-Watanabe decomposition. Moreover the mvo-strategy coincides with the plrm-strategy in the martingale case, but it is not necessarily true in the semimartingale case.*

The main result is given by the following Theorem:

Theorem 1.4.2. *Suppose Θ is closed and let X be a continuous process such that $\mathcal{P}^2_e(X) \neq \emptyset$. Let $H \in L^2(\mathcal{G}_T, \mathbb{Q})$ be a contingent claim and write the Galtchouk-Kunita-Watanabe decomposition of H under $\tilde{\mathbb{Q}}$ with respect to X as*

$$H = \tilde{E}[H] + \int_0^T \tilde{\xi}^H_u \mathrm{d}X_u + \tilde{L}_T = \tilde{V}_T, \tag{1.8}$$

with

$$\tilde{V}_t := \tilde{E}[H|\mathcal{G}_t] = \tilde{E}[H] + \int_0^t \tilde{\xi}^H_u \mathrm{d}X_u + \tilde{L}_t, \quad 0 \le t \le T, \tag{1.9}$$

where $\tilde{E}[\,\cdot\,|\mathcal{G}_t]$ denotes the conditional expectation under $\tilde{\mathbb{Q}}$. Then the mean-variance optimal Θ-strategy for H exists and it is given by

$$\tilde{V}_0 = \tilde{E}[H]$$

and

$$\begin{aligned}\tilde{\theta}_t &= \tilde{\xi}^H_t - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - \tilde{E}[H] - \int_0^t \tilde{\theta}_u \mathrm{d}X_u\right)\\ &= \tilde{\xi}^H_t - \tilde{\zeta}_t\left(\frac{\tilde{V}_0 - \tilde{E}[H]}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u}\mathrm{d}\tilde{L}_u\right), \qquad 0 \le t \le T,\end{aligned}$$

where

$$\tilde{Z}_t = \tilde{E}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\middle|\mathcal{G}_t\right] = \tilde{Z}_0 + \int_0^t \tilde{\zeta}_u \mathrm{d}X_u, \qquad 0 \le t \le T \tag{1.10}$$

Proof. The proof can be found in [31]. □

It is clear that the solution of the mean-variance hedging problem depends on $\tilde{\mathbb{Q}}$, $\tilde{Z}$ and $\tilde{\zeta}$.

It should be clear that both approaches aim at minimizing squared hedging costs. The only difference is that mean-variance hedging does this over a long term whereas local risk-minimization approach applies the quadratic criterion "on each infinitesimal interval".

Chapter 2

Quadratic Hedging Methods for Defaultable Markets

2.1 Introduction

In this chapter we motivate our choice to study defaultable markets by means of quadratic hedging criteria and in particular by applying the local risk-minimization.

First we provide a careful description of the general setting of our model, in particular emphasizing the presence of the possibility of a default event in the financial market. Then we explain why the market extended with the defaultable claim is incomplete and our idea to apply the local risk-minimization approach and its role in literature. Finally Section 2.4 lays out the outline of the thesis.

2.2 General setting

This section describes the general framework of our model and in particular it emphasizes the presence of defaultable claims that make the market incomplete.

We start with a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ and a fixed time horizon $T \in$

$(0, \infty)$. We consider a simple model of financial market in continuous time with two non-defaultable primary assets available for trade, a risky asset and the money market account, and with *defaultable claims*, i.e. contingent agreements that are traded over-the-counter between default-prone parties. Each side of contract is exposed to the *counterparty risk* of the other party but the underlying assets are assumed to be *insensitive* to credit risk.
The random time of default is represented by a stopping time $\tau : \Omega \to [0, T] \cup \{+\infty\}$, defined on the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, satisfying: $\mathbb{Q}(\tau = 0) = 0$ and $\mathbb{Q}(\tau > t) > 0$ for any $t \in [0, T]$. For a given default time τ, we introduce the associated *default process* $H_t = \mathbb{I}_{\{\tau \leq t\}}$, for $t \in [0, T]$ and denote by $(\mathcal{H}_t)_{0 \leq t \leq T}$ the filtration generated by the process H, i.e. $\mathcal{H}_t = \sigma(H_u : u \leq t)$ for any $t \in [0, T]$.
Let W_t be a standard Brownian motion on the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ and $(\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of W_t. The reference filtration is then $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, for any $t \in [0, T]$, i.e. the information at time t is captured by the σ-field $\mathcal{G}_t$. In addition we assume that τ is a $\mathcal{G}_t$-*totally inaccessible* stopping time (see [13]).It should be emphasized that the default time τ is a stopping time with respect to the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ and not with respect to the Brownian filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, otherwise it would be necessarily a predictable stopping time. Moreover we postulate that the Brownian motion W *remains a (continuous) martingale* (and then a Brownian motion) *with respect to the enlarged filtration* $(\mathcal{G}_t)_{0 \leq t \leq T}$. In the sequel we refer to this assumption as the hypothesis (H). We remark that all the filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity.

- We introduce the $\mathcal{F}$-*hazard process* of τ under $\mathbb{Q}$:

$$\Gamma_t = -\ln(1 - F_t), \quad \forall t \in [0, T],$$

where

$$F_t = \mathbb{Q}(\tau \leq t | \mathcal{F}_t) \tag{2.1}$$

is the conditional distribution function of the default time τ. In particular $F_t < 1$ for $t \in [0, T]$. Let, in addition, the process F be absolutely continuous with respect to the Lebesgue measure, so that

$$F_t = \int_0^t f_s \mathrm{d}s, \quad \forall t \in [0,T],$$

for some $\mathcal{F}$-progressively measurable process f. Then the $\mathcal{F}$-hazard process Γ of τ admits the following representation:

$$\Gamma_t = \int_0^t \lambda_s \mathrm{d}s, \quad t \in [0,T], \tag{2.2}$$

where λ_t is a non-negative, $\mathcal{F}_t$-adapted process given by

$$\lambda_t = \frac{f_t}{1-F_t}, \quad \forall t \in [0,T]. \tag{2.3}$$

The process λ is called $\mathcal{F}$-*intensity* or *hazard rate*. By Proposition 5.1.3 of [13] we obtain that the compensated process $\hat{M}$ given by

$$\hat{M}_t := H_t - \int_0^{t\wedge\tau} \lambda_u \mathrm{d}u = H_t - \int_0^t \tilde{\lambda}_u \mathrm{d}u, \quad \forall t \in [0,T] \tag{2.4}$$

follows a martingale with respect to the filtration $(\mathcal{G}_t)_{0\le t\le T}$. Notice that for the sake of brevity we have denoted $\tilde{\lambda}_t := \mathbb{I}_{\{\tau \ge t\}}\lambda_t$. We note that since Γ_t is a continuous increasing process, by Lemma 5.1.6 of [13] the stopped process $W_{t\wedge\tau}$ follows a $\mathcal{G}_t$-martingale.

- We denote the money market account by $B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right)$, where r_t is a $\mathcal{G}_t$-predictable process, and represent the risky asset price by a continuous stochastic process S_t on $(\Omega, \mathcal{G}, \mathbb{Q})$, whose dynamics is given by the following equation:

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \\ S_0 = s_0, \quad s_0 \in \mathbb{R}^+ \end{cases} \tag{2.5}$$

where $\sigma_t > 0$ a.s. for every $t \in [0,T]$ and μ_t, σ_t, r_t are $\mathcal{G}_t$-adapted processes such that the discounted price process $X_t := \dfrac{S_t}{B_t}$ belongs to $L^2(\mathbb{Q})$, $\forall t \in [0,T]$. Furthermore we assume that the dynamics of S_t is such that it admits an equivalent martingale measure $\mathbb{Q}^*$ for X_t and this implies that X is a semimartingale under the basic measure $\mathbb{Q}$. We denote by

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t} \tag{2.6}$$

the *market price of risk* and we also assume that μ, σ and r are such that the density $\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}} := \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T$ is square-integrable. Hence we can exclude arbitrage opportunities in the market.

In addition we make the following assumptions, in order to apply the local risk-minimization and the mean-variance hedging.

- We remark that in our model the discounted risky asset price $X = \frac{S}{B}$ belongs to the space $\mathcal{S}^2(\mathbb{Q})$ of semimartingales so that it can be decomposed as follows:

$$X_t = X_0 + \int_0^t (\mu_s - r_s) X_s \mathrm{d}s + \int_0^t \sigma_s X_s \mathrm{d}W_s, \qquad t \in [0, T],$$

  where $\int_0^t \sigma_s X_s \mathrm{d}W_s$ is a square-integrable (local) $\mathbb{Q}$-martingale null at 0 and $\int_0^t (\mu_s - r_s) X_s \mathrm{d}s$ is a predictable process of finite variation null at 0. Moreover, in our case we recall that X is a continuous process.

- In our model we have that the so-called **Structure Condition (SC)** is satisfied, i.e. the *mean-variance tradeoff*

$$\widehat{K}_t(\omega) := \int_0^t \theta_s^2(\omega) \mathrm{d}s \tag{2.7}$$

  is almost surely finite, where θ is the market price of risk defined in (2.6), since X is continuous and $\mathcal{P}_e^2(X) \neq \emptyset$ by hypothesis (see [35]). *In particular, from now on we assume that $\widehat{K}_t$ is uniformly bounded in t and ω, i.e. there exists K such that*

$$\widehat{K}_t(\omega) \leq K, \qquad \forall t \in [0, T], \text{ a.s.} \tag{2.8}$$

Remark 2.2.1. *This assumption guarantees the existence of the minimal martingale measure for X (see Definition 1.3.15). It is possible to choose* *different hypotheses. However assumption* (2.8) *is the simplest condition that can be assumed. For a complete survey and a discussion of the others, we refer to [35].*

In this context Θ_s denotes the space of all $\mathcal{G}$-predictable processes ξ on Ω such that

$$E\left[\int_0^T (\xi_s \sigma_s X_s)^2 \mathrm{d}s\right] + E\left[\left(\int_0^T |\xi_s(\mu_s - r_s)X_s| \mathrm{d}s\right)^2\right] < \infty. \tag{2.9}$$

As mentioned above, in this market model we can find defaultable claims, which are represented by a quintuple $(\bar{X}, \tilde{X}, Z, A, \tau)$, where:

- the *promised contingent claim* $\bar{X}$ represents the payoff received by the owner of the claim at time T, if there was no default prior to or at time T. In particular we assume it is represented by a $\mathcal{G}_T$-measurable random variable $\bar{X} \in L^2(\mathbb{Q})$;

- the *recovery claim* $\tilde{X}$ represents the recovery payoff at time T, if default occurs prior to or at the maturity date T. It is supposed to be a $\mathcal{G}_T$-measurable random variable $\tilde{X} \in L^2(\mathbb{Q})$;

- the *recovery process* Z represents the recovery payoff at the time of default, if default occurs prior to or at the maturity date T. We postulate that the process Z is predictable with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$;

- the process A represents the *promised dividends*, that is the stream of cash flows received by the owner of the claim prior to default. It is given by a finite variation process which is supposed to be predictable with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$.

We restrict our attention to the case of $A \equiv 0$. Hence the discounted value of a defaultable claim H can be represented as follows:

$$H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{\tilde{X}}{B_T}\mathbb{I}_{\{\tau \le T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau \le T\}}. \tag{2.10}$$

In particular we obtain that $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$.

2.3 Quadratic Hedging Methods for Defaultable Claims

In this section, we explain why we have decided to investigate defaultable markets by means of quadratic hedging criteria and in particular the choice of the local risk-minimization.
We recall that we consider a financial market model with two non-defaultable primary assets, the risky asset S and the money market account B. The presence of a possible default event adds a further source of randomness in the market. Hence the market model extended with the defaultable claim is incomplete since it is impossible to hedge against the occurrence of a default by using a portfolio consisting only of the (non-defaultable) primary assets. Moreover, even if we assume to trade with $\mathcal{G}_t$-adapted strategies, the process $\hat{M}_t$ does not represent the value of any tradable asset. Then it makes sense to apply some of the methods used for pricing and hedging derivatives in incomplete markets. In particular we focus here on quadratic hedging approaches, i.e. local risk-minimization and mean-variance hedging whose theory and main results have been provided in the previous chapter. The mean-variance hedging method has been already extensively studied in the context of defaultable markets by [7], [8], [9] and [10]. For instance in [8], they provide an explicit formula for the optimal trading strategy which solves the mean-variance hedging problem, in the case of a defaultable claim represented by a $\mathcal{G}_T$-measurable square-integrable random variable.[1]Moreover they compare the results obtained using strategies adapted to the Brownian filtration, to the ones obtained using strategies based on the enlarged filtration, which encompasses also the observation of the default time.
In the next chapter we extend some of their results to the case of *stochastic* drift μ and volatility σ in the dynamics (2.5) of the risky asset price, and

[1] $\mathcal{G}_t$ denotes the enlarged filtration $\mathcal{F}_t \vee \mathcal{H}_t$ generated by the Brownian motion and the natural filtration of the jump process H. This is a usual setting in the literature concerning defaultable markets (see for example [13] and related works)

random recovery rate.
We should stress that in our model we have introduced the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ in order to distinguish between the different sources of randomness that an agent faces on the market:

1. the variation in value of the non-defaultable assets is represented as depending on the fluctuation of the driving Brownian motion W;

2. the loss arising from the trading of a defaultable claim, if the counterpart fails to fulfill her/his contractual commitments, is modelled through the default time τ and its associated filtration (*default risk*).

Even if we admit a reciprocal influence between the occurring of the default and the asset prices (we will consider this situation into details in Chapter 4), two different kinds of risk affect the market. Mathematically this is reflected by the fact that the martingale structure is generated by W and H.
The main contribution of this thesis is to collect and discuss extensively our results (see [1], [2], [3]), where, to the best of our knowledge, we have applied for the first time in literature the local risk-minimization method to the pricing and hedging of defaultable claims.

2.4 Outline

The thesis is organized as follows. First we are going to apply the local risk-minimization approach to the case of a defaultable put, where we also make a comparison with the intensity-based evaluation formulas and the mean-variance hedging. We solve analytically the problem of finding respectively the hedging strategy and the associated portfolio for the three methods in the case of a defaultable put option with random *recovery at maturity.*
Then we study the general case by considering two different possible recovery schemes for a generic defaultable claim.

- We apply the local risk-minimization approach to a defaultable claim with *recovery scheme at maturity* in a more general setting where the dynamics of the risky asset X may be influenced by the occurring of a default event and also the default time τ itself may depend on the assets prices behavior. We are able to provide the Föllmer-Schweizer decomposition and compute explicitly the pseudo-locally risk-minimizing strategy in two examples.

- Finally, we study the local risk-minimization approach for defaultable claims with random *recovery scheme at default time*, i.e. a random recovery payment is received by the owner of the contract in case of default *at time of default.* Even in this case we are able to provide the Föllmer-Schweizer decomposition and in particular we apply the results to the case of a Corporate bond. Moreover we discuss the problem of finding a pseudo-locally risk-minimizing strategy if we suppose the agent obtains her information only by observing the non-defaultable assets.

Chapter 3

Local Risk-Minimization for a Defaultable Put

3.1 Introduction

In this chapter we start the study of defaultable markets by means of local risk-minimization. As a first step, we apply the local risk-minimization approach to a certain defaultable claim and we compare it with intensity-based evaluation formulas and mean-variance hedging, only in the case where the default time and the underlying Brownian motion are supposed to be independent. More precisely, under this assumption we solve analytically the problem of finding respectively the hedging strategy and the associated portfolio for the three methods in the special case of a defaultable put with *random recovery* at maturity.

In the market model outlined in Section 2.2, by following the approach of [8], [11] and [13], we first consider the so-called "intensity-based approach", where a defaultable claim is priced by using the risk-neutral valuation formula as the market would be complete. However we recall that the market model extended with the defaultable claim is incomplete since it is impossible to hedge against the occurrence of a default by using a portfolio consisting only of the (non-defaultable) primary assets. Hence this method can only

provide pricing formulas for the discounted defaultable payoff H, since it is impossible to find a replicating portfolio for H consisting only of the risky asset and the bond. Then it makes sense to apply the quadratic hedging methods introduced in Chapter 1, used for pricing and hedging derivatives in incomplete markets. Local risk-minimization and mean-variance hedging provide arbitrage-free valuations and in the case of a complete market reproduce the usual arbitrage-free prices and riskless hedging strategies. Hence they can be considered as a consistent extension from the complete to the incomplete market case.
The main goal of this chapter is to apply the local risk-minimization method to the pricing and hedging of a certain defaultable claim and provide a comparison with other two hedging methods. According to [1], we investigate the particular case of a defaultable put option with random recovery rate and solve explicitly the problem of finding a pseudo-local risk-minimizing strategy and the portfolio with minimal cost. As mentioned previously, the mean-variance hedging method has been already extensively studied in the context of defaultable markets by [7], [8], [9] and [10]. Here we extend some of their results to the case of *stochastic* drift μ and volatility σ in the dynamics (2.5) of the risky asset price, and *random* recovery rate. Empirical analysis of recovery rates shows that they may depend on several factors, among which default delays (see for example [15]). For the sake of simplicity here we assume that the recovery rate depends only on the random time of default.

3.2 Setting

Since the default time and the underlying Brownian motion are supposed to be independent and we consider here only the case of a defaultable put, we need additional assumptions:

- the risky asset price S and the risk-free bond B are both defined on the probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$, endowed with the Brownian filtration

$(\mathcal{F}_t)_{0\le t\le T}$;

- the default time τ is represented by a totally inaccessible stopping time on the probability space $(\hat{\Omega}, \mathcal{H}, \nu)$, endowed with the filtration $(\mathcal{H}_t)_{0\le t\le T}$.

Hence we consider the following product probability space

$$(\Omega, \mathcal{G}, \mathbb{Q}) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F} \otimes \mathcal{H}, \mathbb{P} \otimes \nu)$$

endowed with the filtration

$$\mathcal{G}_t = \mathcal{H}_t \otimes \mathcal{F}_t, \quad \forall t \in [0, T].$$

Since $\mathcal{H}_t$ is independent of $\mathcal{F}_t$ for every $t \in [0, T]$, the cumulative distribution function of τ is given by:

$$F_t = \mathbb{Q}(\tau \le t) = \nu(\tau \le t) \tag{3.1}$$

and the intensity λ is a non-negative, integrable function. Furthermore:

- the short-term interest rate r is a *deterministic* function, $\mu = \mu(\tilde{\omega})$, $\sigma = \sigma(\tilde{\omega})$ are $\mathcal{F}$-adapted processes.

- μ is adapted to the filtration $\mathcal{F}^S$ generated by S. We remark that if σ has a right-continuous version, then it is $\mathcal{F}^S$-adapted (see [22]) since

$$\int_0^t \sigma_s^2 S_s^2 \mathrm{d}s = \lim_{\sup_i |t_{i+1}-t_i| \to 0} \sum_i^n |S_{t_{i+1}} - S_{t_i}|^2,$$

  where $0 = t_0 \le t_1 \le \cdots t_n = t$ is a partition of $[0, t]$. Hence we obtain that $\mathcal{F}_t^S = \mathcal{F}_t$ for any $t \in [0, T]$ and from now on we assume $\mathcal{F}_t$ as the *reference filtration* on $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$.

- μ, σ and r are such that there exists a unique equivalent martingale measure for the discounted price process X whose density $\dfrac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} := \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T$ is square-integrable. Hence the non-defaultable market is *complete*.

Definition 3.2.1. *The buyer of a* defaultable put *has to pay a premium to the seller who undertakes the default risk linked to the underlying asset. If a* credit event *occurs before the maturity date T of the option, the seller has to pay to the put's owner an amount* (default payment)*, which can be fixed or variable.*

If we restrict our attention to the simple case of

$$Z \equiv 0,$$

the defaultable put is given by a triplet $(\bar{X}, \tilde{X}, \tau)$, where

1. the promised claim is given by the payoff of a standard put option with *strike price* and *exercise date* T:

$$\bar{X} = (K - S_T)^+; \tag{3.2}$$

2. the recovery payoff at time T is given by

$$\tilde{X} = \delta(K - S_T)^+, \tag{3.3}$$

   where $\delta = \delta(\omega)$ is supposed to be a ***random*** recovery rate.

In particular we assume that $\delta(\omega) = \delta(\tilde{\omega}, \hat{\omega}) = \delta(\hat{\omega})$ is represented by a $\mathcal{H}_T$-measurable random variable in $L^2(\hat{\Omega}, \mathcal{H}_T, \nu)$, i.e.

$$\delta(\omega) = h(\tau(\omega) \wedge T) \tag{3.4}$$

for some square-integrable Borel function $h : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $0 \leq h \leq 1$. Here we differ from the approach of [13], since we assume that $\tilde{X}$ is $\mathcal{G}_T$-measurable and not necessarily $\mathcal{F}_T$-measurable. This is due to the fact that in our model we allow the recovery rate δ to depend on the default time τ. This represents a generalization of the models presented in [8] and [13].

Example 3.2.2. *We remark that here we restrict our attention to the case when the recovery rate depends only on the random time of default. For example $\delta(\omega)$ can be of the form:*

$$\delta(\omega) = \delta_1 \mathbb{I}_{\{\tau \leq T_0\}} + \delta_2 \mathbb{I}_{\{T \geq \tau > T_0\}},$$

when $\delta_1, \delta_2 \in \mathbb{R}_0^+$ *and* $0 < T_0 < T$*. In this example we are considering a case when we obtain a portion of the underlying option according to the fact that the default occurs before or after a certain date. The recovery claim is always handled out at time* T *of maturity.*

In this case the discounted value of the defaultable put can be represented as follows:

$$
\begin{aligned}
H &= \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{\tilde{X}}{B_T}\mathbb{I}_{\{\tau \leq T\}} \\
&= \frac{(K - S_T)^+}{B_T}\left(\mathbb{I}_{\{\tau > T\}} + \delta(\omega)\mathbb{I}_{\{\tau \leq T\}}\right) \\
&= \frac{(K - S_T)^+}{B_T}\left(1 + (\delta(\omega) - 1)\mathbb{I}_{\{\tau \leq T\}}\right), \qquad (3.5)
\end{aligned}
$$

where δ is given in (3.4). Our aim is now to apply the local risk-minimization in this framework and compare the results with the ones obtained through the intensity-based approach and mean-variance hedging.

3.3 Reduced-form model

In this section we present the main results that can be obtained through the intensity-based approach to the valuation of defaultable claims and then we apply them to the case of a defaultable put. We follow here the approach of [8], [11] and [13].

We remark that under the assumption of Section 3.2 the non-defaultable market is complete since there exists a unique equivalent martingale measure $\mathbb{P}^*$ for the discounted price process $X_t = \dfrac{S_t}{B_t}$. See [28] for further details. We put

$$\mathbb{Q}^* = \mathbb{P}^* \otimes \nu$$

in the sequel. Note that by hypothesis (H), $\mathbb{Q}^*$ is still a martingale measure for X_t with respect to the filtration $\mathcal{G}_t$.

By using no-arbitrage arguments, in Section 8.1.1 of [13] they show that a valuation formula for a defaultable claim can be obtained by the usual risk-neutral valuation formula as follows.

Let H be the defaultable claim given in (2.10). We restrict our attention to the case of $\tilde{X} = 0$ since the more general case can be handled with similar techniques. The following result provides an alternative representation for the price process of a defaultable claim whose discounted value is given by

$$H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{t<\tau\leq T\}}. \tag{3.6}$$

Lemma 3.3.1. *The price process V of a defaultable claim H given in* (3.6) *admits the following representation:*

$$V_t = B_t E^* \left[\int_t^T \frac{Z_u}{B_u}\tilde{\lambda}_u \mathrm{d}u + \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} \middle| \mathcal{G}_t \right]. \tag{3.7}$$

Proof. See Proposition 8.3.1 of [13] for the proof. □

The next result plays a key role in the martingale approach to valuation of defaultable claims.

Theorem 3.3.2. *Let Z and $\bar{X}$ be an $\mathcal{F}$-predictable process and an $\mathcal{F}_T$-measurable random variable respectively. Consider the process*

$$U_t = \tilde{B}_t E^* \left[\int_t^T \frac{Z_u}{\tilde{B}_u}\lambda_u \mathrm{d}u + \frac{\bar{X}}{\tilde{B}_T} \middle| \mathcal{G}_t \right] \tag{3.8}$$

where

$$\tilde{B}_t = \exp\left(\int_0^t (r(u) + \lambda_u)\mathrm{d}u \right)$$

($R_t = r(t) + \lambda_t$ denotes the **default-risk-adjusted interest rate***). Then*

$$\begin{aligned}
\mathbb{I}_{\{t<\tau\}}U_t &= B_t E^* \left[\frac{(Z_\tau + \Delta U_\tau)}{B_\tau}\mathbb{I}_{\{t<\tau\leq T\}} + \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} \middle| \mathcal{G}_t \right] \\
&= B_t \left(E^* \left[H \middle| \mathcal{G}_t \right] + E^* \left[\frac{\Delta U_\tau}{B_\tau}\mathbb{I}_{\{t<\tau\leq T\}} \middle| \mathcal{G}_t \right] \right).
\end{aligned}$$

Proof. See Proposition 8.3.2 of [13] for the proof. □

The following Corollary appears to be useful in the study of the case of Brownian filtration.

Corollary 3.3.3. *Let the processes V and U be defined by* (3.7) *and* (3.8), *respectively. Then*

$$V_t = \mathbb{I}_{\{t<\tau\}} \left(U_t - B_t E^* \left[\frac{\Delta U_\tau}{B_\tau} \mathbb{I}_{\{t<\tau\le T\}} \middle| \mathcal{G}_t \right] \right).$$

If $\Delta U_\tau = 0$, then

$$V_t = \mathbb{I}_{\{t<\tau\}} U_t \quad \textit{for every } t \in [0,T]$$

and

$$V_t = \mathbb{I}_{\{t<\tau\}} \tilde{B}_t E^* \left[\int_t^T \frac{Z_u}{\tilde{B}_u} \lambda_u \mathrm{d}u + \frac{\bar{X}}{\tilde{B}_T} \middle| \mathcal{G}_t \right].$$

Remark 3.3.4. *The continuity condition $\Delta U_\tau = 0$ seems to be rather difficult to verify in a general set-up. It can be established, however, if certain additional restrictions are imposed on underlying filtrations $(\mathcal{F}_t)_{0\le t\le T}$ and $(\mathcal{G}_t)_{0\le t\le T}$. For instance, when the filtration $(\mathcal{F}_t)_{0\le t\le T}$ is generated by a Brownian motion under $\mathbb{Q}^*$, the continuity of U is trivial.*

Example 3.3.5. *We compute now the price process of a defaultable put whose recovery process Z is given by a constant d. We assume in addition that the intensity λ, drift μ and volatility σ are constant.Hence the discounted value H can be represented as follows:*

$$H = e^{-rT}(K - S_T)^+ \mathbb{I}_{\{\tau>T\}} + d e^{-r\tau} \mathbb{I}_{\{t<\tau\le T\}}. \tag{3.9}$$

By applying Theorem 3.3.2 and Corollary 3.3.3, the price process V at time t of a defaultable put defined in (3.9) *is given by:*

$$\begin{aligned}
V_t &= \mathbb{I}_{\{t<\tau\}} U_t \\
&= \mathbb{I}_{\{t<\tau\}} e^{(r+\lambda)t} E^* \left[(K - S_T)^+ e^{-(r+\lambda)T} + \int_t^T d e^{-(r+\lambda)s} \lambda \mathrm{d}s \middle| \mathcal{G}_t \right] \\
&= \mathbb{I}_{\{t<\tau\}} e^{(r+\lambda)t} E^* \left[(K - S_T)^+ e^{-(r+\lambda)T} - \frac{\lambda d}{r+\lambda} e^{-(r+\lambda)t} \left(e^{-(r+\lambda)(T-t)} - 1 \right) \middle| \mathcal{G}_t \right] \\
&= \mathbb{I}_{\{t<\tau\}} \left(e^{-(r+\lambda)(T-t)} E^* \left[(K - S_T)^+ \middle| \mathcal{G}_t \right] + \frac{\lambda d}{r+\lambda} \left(1 - e^{-(r+\lambda)(T-t)} \right) \right) \\
&= \mathbb{I}_{\{t<\tau\}} \left(e^{-(r+\lambda)(T-t)} P_t + \frac{\lambda d}{r+\lambda} \left(1 - e^{-(r+\lambda)(T-t)} \right) \right),
\end{aligned}$$

where P represents the well-known price of a standard put option:

$$P_t = B_t E^* \left[\left. \frac{(K - S_T)^+}{B_T} \right| \mathcal{G}_t \right] = K e^{-r(T-t)} \mathcal{N}(x_1) - S_t \mathcal{N}(x_2),$$

with

$$x_1 = \frac{\lg\left(\frac{K}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$x_2 = \frac{\lg\left(\frac{K}{S_t}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Let us turn on the defaultable put H defined in (3.5). Under the probability measure $\mathbb{Q}^*$, the discounted price process of the defaultable put at time t is given by:

$$\begin{aligned}
\frac{V_t}{B_t} &= E^* \left[\left. \frac{\bar{X}}{B_T} \mathbb{I}_{\{\tau > T\}} + \frac{\tilde{X}}{B_T} \mathbb{I}_{\{\tau \le T\}} \right| \mathcal{G}_t \right] \\
&= B_t E^* \left[\left. \frac{(K - S_T)^+}{B_T} \left(1 + (\delta(\omega) - 1)\mathbb{I}_{\{\tau \le T\}}\right) \right| \mathcal{G}_t \right] \\
&= \underbrace{B_t E^* \left[\left. \frac{(K - S_T)^+}{B_T} \right| \mathcal{G}_t \right]}_{a)} \underbrace{E^* \left[(1 + (\delta(\omega) - 1) H_T) \middle| \mathcal{G}_t \right]}_{b)},
\end{aligned}$$

where the last equality follows from the fact that S_T and H_T are independent. We compute separately the terms a) and b).

a) This term represents the well-known price P_t of a standard put option:

$$\begin{aligned}
P_t = B_t E^* \left[\left. \frac{(K - S_T)^+}{B_T} \right| \mathcal{G}_t \right] &= E^* \left[\left. e^{-\int_t^T r(s)\mathrm{d}s} (K - S_T)^+ \right| \mathcal{G}_t \right] \\
&= E^* \left[\left. e^{-\int_t^T r(s)\mathrm{d}s} (K - S_T)^+ \right| \mathcal{F}_t \right] \\
&= K e^{-\int_t^T r(s)\mathrm{d}s} E^* \left[\mathbb{I}_A \middle| \mathcal{F}_t \right] - S_t E^{\mathbb{Q}^{*,X}} \left[\mathbb{I}_A \middle| \mathcal{F}_t \right],
\end{aligned} \tag{3.10}$$

where by [24] we have

$$\frac{\mathrm{d}\mathbb{Q}^{*,X}}{\mathrm{d}\mathbb{Q}^*} = \frac{X_T}{X_0}.$$

b) It remains to compute the second term:

$$E^* \left[1 + (\delta(\omega) - 1) H_T \middle| \mathcal{G}_t \right] = 1 + \underbrace{E^* \left[\delta(\omega) H_T \middle| \mathcal{G}_t \right]}_{\mathbf{c)}} - E^* \left[H_T \middle| \mathcal{G}_t \right].$$

Then, we have to examine the conditional expectation $E^* \left[H_T \middle| \mathcal{G}_t \right]$. First we note that

$$E^* \left[H_T \middle| \mathcal{G}_t \right] = E^* \left[H_T \middle| \mathcal{H}_t \right].$$

Lemma 3.3.6. *The process M given by the formula*

$$M_t = \frac{1 - H_t}{1 - F_t}, \quad \forall t \in \mathbb{R}^+, \tag{3.11}$$

where F_t is given by (3.1)*, follows a martingale with respect to the filtration $(\mathcal{H}_t)_{0 \le t \le T}$. Moreover, for any $t < s$, the following equality holds:*

$$E^* \left[1 - H_s \middle| \mathcal{H}_t \right] = (1 - H_t) \frac{1 - F_s}{1 - F_t}. \tag{3.12}$$

Proof. We refer to Corollary 4.1.2 of [13]. □

Note that the cumulative distribution function of τ is the same both under $\mathbb{Q}^*$ and $\mathbb{Q}$ since $\mathbb{Q}^*(\tau \le t) = \nu(\tau \le t) = \mathbb{Q}(\tau \le t)$. We apply (3.12) to get

$$\begin{aligned} E^* \left[H_T \middle| \mathcal{H}_t \right] &= 1 - \underbrace{\left(\frac{1 - H_t}{1 - F_t} \right)}_{M_t} (1 - F_T) \\ &= 1 - (1 - F_T) M_t. \end{aligned} \tag{3.13}$$

To complete the computations, we evaluate the conditional expectation **c)**.

c) In view of the Corollary 4.1.3 and the Corollary 5.1.1 of [13], using (3.4) we have:

$$\begin{aligned} E \left[\delta(\omega) H_T \middle| \mathcal{G}_t \right] &= E \left[h(\tau \wedge T) H_T \middle| \mathcal{G}_t \right] \\ &= h(\tau \wedge T) H_t + (1 - H_t) e^{\int_0^t \lambda_u \mathrm{d}u} E^* \left[\mathbb{I}_{\{\tau > t\}} h(\tau \wedge T) H_T \right] \\ &= h(\tau \wedge T) H_t + (1 - H_t) e^{\int_0^t \lambda_u \mathrm{d}u} E^* \left[\mathbb{I}_{\{t < \tau < T\}} h(\tau \wedge T) \right] \\ &= h(\tau \wedge T) H_t + (1 - H_t) \int_t^T h(s) \lambda_s e^{-\int_t^T \lambda_u \mathrm{d}u} \mathrm{d}s. \end{aligned}$$

Finally, gathering the results, we obtain the following Proposition.

Proposition 3.3.7. *In the market model outlined in Sections 2.2 and 3.2, we obtain that the discounted value at time t of the replicating portfolio according to the intensity-based approach is:*

$$\frac{V_t}{B_t} = E^* \left[\frac{\bar{X}}{B_T} \mathbb{I}_{\{\tau > T\}} + \frac{\tilde{X}}{B_T} \mathbb{I}_{\{\tau \leq T\}} \middle| \mathcal{G}_t \right]$$

$$= P_t \left[H_t h(\tau \wedge T) + (1 - H_t) \left(\int_t^T h(s) \lambda_s e^{-\int_t^s \lambda_u \mathrm{d}u} \mathrm{d}s \right) + (1 - F_T) M_t \right], \tag{3.14}$$

where P_t is the hedging portfolio value for a standard put option given in (3.10).

Example 3.3.8. *In this simple example we compute explicitly the replicating portfolio of a defaultable put whose recovery claim $\tilde{X}$ is given by $\delta(t)(K - S_T)^+$, for $t \in [0, T]$, where δ is a* deterministic *function. In addition, we suppose that the intensity λ is constant. This implies that $F_t = F$, for every $t \in [0, T]$, i.e. the conditional distribution function of τ is constant. Hence, by Proposition 3.3.7, we obtain that the discounted value at time t of the replicating portfolio is given by:*

$$\frac{V_t}{B_t} = P_t \left(\delta(t) + \mathbb{I}_{\{\tau > t\}} \left(1 - \delta(t) e^{-\lambda(T-t)} \right) \right), \quad 0 \leq t \leq T.$$

If the intensity λ is supposed to be a deterministic *function, we have:*

$$\frac{V_t}{B_t} = P_t \left(\delta(t) + \mathbb{I}_{\{\tau > t\}} \left(\frac{1 - F(T)}{1 - F(t)} - \delta(t) e^{-\int_t^T \lambda(s) \mathrm{d}s} \right) \right), \quad 0 \leq t \leq T.$$

Remark 3.3.9. *Since in our market there are non-defaultable primary assets, finding a self-financing portfolio that replicates our put option perfectly is not possible (see [8] for further details). Hence, we have restricted our attention to the pricing problem, according to [13].*

3.4 Local risk-minimization

In Section 3.3 we have computed in Proposition 3.3.7 the discounted portfolio value that replicates our defaultable option. The main idea of the intensity-based approach is to assume that the market is complete. However, due to the possibility of default, one cannot perfectly hedge a defaultable claim in this framework, since only non-defaultable assets are present in our market model and $\hat{M}$ does not represent the value of any tradable asset. Now we are going to apply the local risk-minimization to particular case of a defaultable put defined in (3.5). We wish to find a portfolio "with minimal cost" that perfectly replicates H according to the local risk-minimizing criterion. We remark that we focus on the case of trading strategies *adapted to the full filtration* $\mathcal{G}_t$ (see [8]).

Lemma 3.4.1. *The minimal martingale measure for X_t with respect to $\mathcal{G}_t$ exists and coincides with $\mathbb{Q}^*$.*

Proof. Since W and $\hat{M}$ defined in (2.4) have the predictable representation property for the space of square-integrable local martingale on the product probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F} \otimes \mathcal{H}, \mathcal{F}_t \otimes \mathcal{H}_t, \mathbb{P} \otimes \nu)$, the result follows by Definition 1.3.15. See also [5] and [27]. In fact by Definition 1.3.15, the minimal martingale measure is the unique equivalent martingale measure for X with square-integrable density such that any square-integrable $\mathbb{Q}$-local martingale strongly orthogonal to $\int \sigma X \mathrm{d}W$ remains a $\mathbb{Q}$-local martingale.
Consider a square-integrable local martingale L under $\mathbb{Q}$ strongly orthogonal to $\int \sigma X \mathrm{d}W$. We note that the Brownian motion W and $\hat{M}$ defined in (2.4) have the predictable representation property for the space of square-integrable local martingale on the product probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F} \otimes \mathcal{H}, \mathcal{F}_t \otimes \mathcal{H}_t, \mathbb{P} \otimes \nu)$. Hence

$$L_t = L_0 + \int_0^t \varphi_s^W \mathrm{d}W_s + \int_0^t \varphi_s^{\hat{M}} \mathrm{d}\hat{M}_s, \quad \forall t \in [0, T].$$

Since L is strongly orthogonal with respect to the martingale part of X, we have $\varphi_t^W \equiv 0, \ \forall t \in [0, T]$. Hence

$$L_t = L_0 + \int_0^t \varphi_s^{\hat{M}} \mathrm{d}\hat{M}_s.$$

If $Z_t = E\left[\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}} \middle| \mathcal{G}_t\right]$ is the density process associated to this change of measure, then we obtain

$$Z_t = \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_t$$

where θ is the market price of risk, because this change of law does not affect ν. Then LZ is a local martingale since $(W, \hat{M})$ are strongly orthogonal. Since the density of $\mathbb{P}^*$ is supposed to be square-integrable, then

$$\widehat{\mathbb{Q}} = \mathbb{Q}^* = \mathbb{P}^* \otimes \nu$$

is the minimal measure for X. □

Proposition 3.4.2. *Let $\hat{M}$ be the compensated process defined in (2.4) and X the discounted price process. The pair $(X, \hat{M})$ has the predictable representation property on $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}^*)$, i.e. for every $H \in L^1(\Omega, \mathcal{G}_T, \mathbb{Q}^*)$, there exists a pair of $\mathcal{G}$-predictable processes $(\tilde{\Phi}, \tilde{\Psi})$ such that*

$$H = c + \int_0^T \tilde{\Phi}_s \mathrm{d}X_s + \int_0^T \tilde{\Psi}_s \mathrm{d}\hat{M}_s \tag{3.15}$$

and

$$\int_0^T \tilde{\Phi}_s^2 \mathrm{d}\langle X\rangle_s + \int_0^T \tilde{\Psi}_s^2 \mathrm{d}[\hat{M}]_s < \infty \quad a.s.$$

Proof. Since there exists a unique equivalent martingale measure $\mathbb{P}^*$ for the continuous asset process X_t on $(\tilde{\Omega}, \mathcal{F}, \mathcal{F}_t)$, then by Theorem 40 of Chapter IV of [29] we have that X_t has the predictable representation property for the local martingales on $(\tilde{\Omega}, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^*)$.
By Proposition 4.1 of [4] the compensated default process $\hat{M}$ has the predictable representation property for the local martingales on $(\hat{\Omega}, \mathcal{H}, \mathcal{H}_t, \nu)$.
Hence, since X and $\hat{M}$ are strongly orthogonal, by Proposition A.2 of [5] and by using a limiting argument we obtain that $(X, \hat{M})$ has the predictable representation property on the product probability space

$$(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}^*) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F} \otimes \mathcal{H}, \mathcal{F}_t \otimes \mathcal{H}_t, \mathbb{P}^* \otimes \nu).$$

□

We remark that the market is incomplete even if we trade with $\mathcal{G}_t$-adapted strategies since $\hat{M}$ does not represent the value of any tradable asset.
We can apply Proposition 3.4.2 to obtain a *plrm*-strategy for $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$.

Proposition 3.4.3. *Let $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ be the value of a defaultable claim. Then a plrm-strategy for H exists and it is given by*

$$\Phi_t = \tilde{\Phi}_t$$

with minimal cost

$$C_t = c + \int_0^t \tilde{\Psi}_s \mathrm{d}\hat{M}_s,$$

where $\tilde{\Phi}_t, \tilde{\Psi}_t$ are the same as in Proposition 3.4.2.

Proof. Let $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$. We note that since $\dfrac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} \in L^2(\mathbb{Q})$, we have that $L^2(\Omega, \mathcal{G}_T, \mathbb{Q}) \subset L^1(\Omega, \mathcal{G}_T, \widehat{\mathbb{Q}})$. Then $H \in L^1(\widehat{\mathbb{Q}})$ and we can apply Proposition 3.4.2 to obtain decomposition (3.15) for H given by

$$H = c + \int_0^T \tilde{\Phi}_s \mathrm{d}X_s + \int_0^T \tilde{\Psi}_s \mathrm{d}\hat{M}_s. \tag{3.16}$$

The martingale $\hat{M}$ is strongly orthogonal to the martingale part of X, hence (3.16) gives the Galtchouk-Kunita-Watanabe decomposition of H under $\widehat{\mathbb{Q}}$. Since by hypothesis $\dfrac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \dfrac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}} \in L^2(\mathbb{Q})$ and X is continuous, then by Theorem 3.5 of [22] the associated density process

$$Z_t = \widehat{E}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\middle|\,\mathcal{G}_t\right] = \widehat{E}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\middle|\,\mathcal{F}_t\right]$$

is a square-integrable martingale. Moreover since hypothesis (2.8) is in force, we can apply Theorem 1.3.16 and conclude that (3.15) is the FS decomposition of H. □

Remark 3.4.4. *It is possible to choose different hypotheses that guarantee that decomposition (3.15) gives the FS decomposition. We recall that assumption (2.8) is the simplest condition that can be assumed.*

Under the equivalent martingale probability measure $\widehat{\mathbb{Q}}$, the discounted price process $\hat{V}_t$ of the defaultable put at time t, is given by:

$$\begin{aligned}
\hat{V}_t &= \widehat{E}\left[H|\, \mathcal{G}_t\right] \\
&= \widehat{E}\left[\frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{\tilde{X}}{B_T}\mathbb{I}_{\{\tau\le T\}}\middle|\, \mathcal{G}_t\right] \\
&= \widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\, \mathcal{G}_t\right]\cdot \widehat{E}\left[1+(\delta(\omega)-1)H_T|\, \mathcal{G}_t\right] \\
&= \underbrace{\widehat{E}\left[\frac{(K-S_T)^+}{B_T}\middle|\, \mathcal{G}_t\right]}_{\mathbf{a)}}\cdot \underbrace{\widehat{E}\left[(1+(\delta(\omega)-1)H_T)|\, \mathcal{G}_t\right]}_{\mathbf{b)}}. \qquad (3.17)
\end{aligned}$$

We need only to find the Föllmer-Schweizer decomposition of $\hat{V}_t$ as illustrated in (1.6).

a) By Section 5 of [6] and using the "*change of numéraire*" technique of [24], we have

$$\begin{aligned}
\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\, \mathcal{G}_t\right] &= \widehat{E}\left[\frac{(K-S_T)^+}{B_T}\middle|\, \mathcal{G}_t\right] \\
&= \widehat{E}\left[\frac{(K-S_T)}{B_T}\mathbb{I}_{\underbrace{\{K\ge S_T\}}_{A}}\middle|\, \mathcal{G}_t\right] \\
&= K\widehat{E}\left[\frac{1}{B_T}\mathbb{I}_A\middle|\, \mathcal{G}_t\right] - \widehat{E}\left[\frac{S_T}{B_T}\mathbb{I}_A\middle|\, \mathcal{G}_t\right] \\
&= \frac{K}{B_T}\widehat{E}\left[\mathbb{I}_A|\, \mathcal{G}_t\right] - \widehat{E}\left[X_T\mathbb{I}_A|\, \mathcal{G}_t\right] \\
&= \frac{K}{B_T}\widehat{E}\left[\mathbb{I}_A|\, \mathcal{G}_t\right] - X_t\widehat{E}\left[\mathbb{I}_A|\, \mathcal{G}_t\right],
\end{aligned}$$

where

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^X}{\mathrm{d}\widehat{\mathbb{Q}}} = \frac{X_T}{X_0}$$

is well-defined since $X_T \in L^2(\mathbb{Q})$ by hypothesis and hence $X_T \in L^1(\widehat{\mathbb{Q}})$. In addition by (3.15) we obtain that $\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\, \mathcal{G}_t\right]$ admits the decomposition

$$\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\, \mathcal{G}_t\right] = c + \int_0^t \xi_s \mathrm{d}X_s. \qquad (3.18)$$

Since

$$E^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{G}_t\right] = E^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{F}_t\right]$$

because $\mathbb{I}_A$ is independent of τ, by [24] we have that

$$\xi_t = E^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{F}_t\right]. \tag{3.19}$$

b) It remains to calculate the term $\widehat{E}\left[1 + (\delta(\omega) - 1)H_T|\,\mathcal{G}_t\right]$. First we note that

$$\begin{aligned}
\widehat{E}\left[1 + (\delta(\omega) - 1)H_T|\,\mathcal{G}_t\right] &= 1 + \widehat{E}\left[\delta(\omega)H_T|\,\mathcal{G}_t\right] - \widehat{E}\left[H_T|\,\mathcal{G}_t\right] \\
&= 1 + \widehat{E}\left[\delta(\omega)H_T|\,\mathcal{G}_t\right] - (1 - (1 - F_T)M_t) \\
&= \widehat{E}\left[\delta(\omega)H_T|\,\mathcal{G}_t\right] + (1 - F_T)M_t,
\end{aligned}$$

by (3.13). Since $\delta(\omega)H_T = f(\tau)$ for some integrable Borel function $f : \mathbb{R}^+ \to [0, 1]$, by Proposition 4.3.1 of [13], we have

$$\widehat{E}\left[1 + (\delta(\omega) - 1)H_T|\,\mathcal{G}_t\right] = c_h + \int_0^t \hat{f}(s)\mathrm{d}\hat{M}_s + (1 - F_T)M_t,$$

where $c_h = E^{\widehat{\mathbb{Q}}}[f(\tau)]$ and the function $\hat{f} : \mathbb{R}^+ \to \mathbb{R}$ is given by the formula

$$\hat{f}(t) = f(t) - e^{\Gamma_t}E^{\widehat{\mathbb{Q}}}[\mathbb{I}_{\{\tau>t\}}f(\tau)]. \tag{3.20}$$

Note that

$$f(x) = h(x \wedge T)\mathbb{I}_{\{x<T\}},$$

where h is introduced in (3.4). We only need to find the relationship between M_t and $\hat{M}_t$.

Lemma 3.4.5. *Let M and $\hat{M}$ be defined by (2.4) and (3.11) respectively. The following equality holds:*

$$\mathrm{d}M_t = -\frac{1}{1 - F_t}\mathrm{d}\hat{M}_t, \quad 0 \leq t \leq T. \tag{3.21}$$

Proof. To obtain (3.21), it suffices to apply Itô's formula. For further details see Section 6.3 of [13]. $\square$

Finally, gathering the results we obtain

$$\begin{aligned}
\hat{V}_t &= \widehat{E}\left[H|\,\mathcal{G}_t\right] \\
&= \left(\underbrace{c + \int_0^t \xi_s \mathrm{d}X_s}_{\Phi_t}\right) \cdot \left(E^{\widehat{\mathbb{Q}}}[f(\tau)] + \int_0^t \hat{f}(s)\mathrm{d}\hat{M}_s + (1-F_T)M_t\right) \\
&= \Phi_t \cdot \left(\underbrace{E^{\widehat{\mathbb{Q}}}[f(\tau)] + \int_0^t \left(\hat{f}(s) - \frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s}_{\Psi_t}\right).
\end{aligned}$$

Since

$$\mathrm{d}[\Phi,\Psi]_t = \xi_t \left(\hat{f}(t) - \frac{1-F_T}{1-F_t}\right)\mathrm{d}[X,\hat{M}]_t = 0,$$

applying Itô's formula we get

$$\begin{aligned}
\mathrm{d}\hat{V}_t &= \Phi_t \mathrm{d}\Psi_t + \Psi_{t-}\mathrm{d}\Phi_t + \mathrm{d}[\Phi,\Psi]_t \\
&= \left(c + \int_0^t \xi_s \mathrm{d}X_s\right)\left(\hat{f}(t) - \frac{1-F_T}{1-F_t}\right)\mathrm{d}\hat{M}_t \\
&\qquad + \left(E^{\widehat{\mathbb{Q}}}[f(\tau)] + \int_0^t \left(\hat{f}(s) - \frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s\right)\xi_t \mathrm{d}X_t.
\end{aligned} \tag{3.22}$$

Hence we can conclude that:

Proposition 3.4.6. *In the market model outlined in Sections 2.2 and 3.2, under hypothesis* (2.8) *the local risk-minimizing portfolio for H defined in* (3.5) *is given by*

$$\hat{V}_t = c_1 + \int_0^t \Phi^1_s \mathrm{d}X_s + \hat{L}_t, \tag{3.23}$$

where the plrm strategy is

$$\Phi^1_t = \left(E^{\widehat{\mathbb{Q}}}[f(\tau)] + \int_0^t \left(\hat{f}(s) - \frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s\right)\xi_t \tag{3.24}$$

and the minimal cost is

$$\hat{L}_t = \int_0^t \left(c + \int_0^s \xi_u \mathrm{d}X_u\right)\left(\hat{f}(s) - \frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s, \tag{3.25}$$

where ξ_t is given by (3.19), *$\hat{f}(s)$ by* (3.20) *and F_t by* (3.1).

Proof. Proposition 3.4.3 guarantees that (3.22) provides the FS decomposition for H, i.e. that Φ_t^1 and $\hat{L}_t$ satisfy the required integrability conditions. □

3.5 Mean-variance hedging

Finally to conclude this chapter, we consider the mean-variance hedging approach. This method has been already applied to defaultable markets in [7], [8], [9] and [10]. Here we extend their results to the case of general coefficients in the dynamics of X and random recovery rate and compute explicitly the mean-variance strategy in the particular case of a defaultable put option. Again we focus on the case of $\mathcal{G}$-adapted hedging strategies.

We can interpret the presence on the market of a default possibility as a particular case of "*incomplete information*". Hence the results of [5] and [4], where the variance-optimal measure is characterized as the solution of an equation between Doléans exponentials, can also be applied in this context to compute $\tilde{\mathbb{Q}}$. In particular by Theorem 2.16 and Section 3 (α) of [5], it follows that the variance-optimal measure coincides with the minimal one. In this case

$$\tilde{\mathbb{Q}} = \widehat{\mathbb{Q}} = \mathbb{Q}^*. \tag{3.26}$$

First of all we check that the space Θ is closed.

By Proposition 4.2 of [5], we have that Θ is closed if and only if for every stopping time η, with $0 \leq \eta \leq T$, the following condition holds for some constant M

$$\bar{E}\left[\exp\left(\int_\eta^T \theta_s^2 \mathrm{d}s\right)\Big|\mathcal{G}_\eta\right] \leq M, \tag{3.27}$$

where $\dfrac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} := \mathcal{E}\left(-\int 2\theta \mathrm{d}W\right)_T$. Note that since we are assuming that $\widehat{\mathbb{Q}}$ exists and it is square-integrable, then $\bar{\mathbb{Q}}$ also exists and $\exp\left(\int_0^T \theta_t^2 \mathrm{d}t\right)$ is $\bar{\mathbb{Q}}$-integrable ([5], Section 3(α)). Here we obtain that condition (3.27) is a verified for every $\mathcal{G}$-stopping time η such that $0 \leq \eta \leq T$ as a consequence of our assumption (2.8). Then we need to check that condition (3.27) holds for every $\mathcal{G}$-stopping time η such that $0 \leq \eta \leq T$. Let $A \in \mathcal{G}_\eta$, then we have that

$$\begin{aligned}\int_A \bar{E}\left[\exp\left(\int_\eta^T \theta_s^2 \mathrm{d}s\right)\Big|\mathcal{G}_\eta\right]\mathrm{d}\bar{\mathbb{Q}} &= \int_A e^{\int_\eta^T \theta_s^2 \mathrm{d}s}\mathrm{d}\bar{\mathbb{Q}} \\ &= \int_A e^{\widehat{K}_T-\widehat{K}_\eta}\mathrm{d}\bar{\mathbb{Q}} \\ &\leq K\cdot\bar{\mathbb{Q}}(A),\end{aligned}$$

where the last inequality is a consequence of (2.8).
Hence $\bar{E}\left[\exp\left(\int_\eta^T \theta_s^2 \mathrm{d}s\right)\Big|\mathcal{G}_\eta\right]$ is uniformly bounded and we conclude that Θ is closed. Then we can use Theorem 1.4.2 to obtain the mean-variance optimal Θ-strategy for H. The process $\tilde{V}_t$ at time t, is given by:

$$\begin{aligned}\tilde{V}_t &= \tilde{E}\left[H|\,\mathcal{G}_t\right] \\ &= \tilde{E}\left[\frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}}+\frac{\tilde{X}}{B_T}\mathbb{I}_{\{\tau\leq T\}}\Big|\,\mathcal{G}_t\right] \\ &= \tilde{E}\left[\frac{(K-S_T)^+}{B_T}\left(1+(\delta(\omega)-1)\mathbb{I}_{\{\tau\leq T\}}\right)\Big|\,\mathcal{G}_t\right].\end{aligned}$$

By Section 3 (α) in [5], we also obtain that

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \mathcal{E}\left(-\int\beta\mathrm{d}X\right)_T\frac{1}{E\left[(-\beta\mathrm{d}X)\right]},$$

where $\beta_t = \dfrac{\theta_t-h_t}{\sigma_t X_t}$ and h_t solves the equation

$$\mathcal{E}\left(\int h\mathrm{d}\bar{W}\right)_T = \frac{\exp(\int_0^T \theta_t^2\mathrm{d}t)}{\bar{E}\left[\exp\left(\int_0^T \theta_t^2\mathrm{d}t\right)\right]}$$

with $\bar{W}_t := W_t + 2\int_0^t \theta_s\mathrm{d}s$ and $\dfrac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \mathcal{E}\left(-\int 2\theta\mathrm{d}W\right)_T$. Hence we have that

$$\tilde{Z}_t = \tilde{E}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\Big|\mathcal{F}_t\right] = \frac{\mathcal{E}\left(-\int\beta\mathrm{d}X\right)_t}{E\left[\mathcal{E}\left(-\int\beta\mathrm{d}X\right)_T\right]} \tag{3.28}$$

and $\mathrm{d}\tilde{Z}_t = \beta_t\tilde{Z}_t\mathrm{d}X_t$. Consequently we can compute decomposition (1.10) and obtain

$$\tilde{\zeta}_t = \tilde{Z}_t\beta_t. \tag{3.29}$$

Since $\tilde{\mathbb{Q}} = \widehat{\mathbb{Q}} = \mathbb{Q}^*$, we can use (3.23), (3.24) and (3.25), to obtain the mean-variance optimal Θ-strategy $(\tilde{V}_0, \tilde{\theta})$ for H.

Proposition 3.5.1. *In the market model outlined in Sections 2.2 and 3.2, under hypothesis* (2.8) *the mean-variance hedging strategy for H defined in* (3.5) *is given by:*

- ***Approximation Price***

$$\tilde{V}_0 = \tilde{E}[H] = \tilde{E}\left[\frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{\tilde{X}}{B_T}\mathbb{I}_{\{\tau \leq T\}}\right].$$

  We note that the optimal price for the mean-variance hedging criterion coincides with the optimal price for the locally risk-minimizing criterion.

- ***Mean-Variance Optimal Strategy***

$$\tilde{\theta}_t = \Phi_t^1 - \tilde{\zeta}_t \int_0^{t-} \frac{1}{\tilde{Z}_u} \mathrm{d}\tilde{L}_u, \tag{3.30}$$

  where Φ^1, $\tilde{Z}$ and $\tilde{\zeta}$ are given by (3.24), (3.28) and (3.29) respectively and

$$\mathrm{d}\tilde{L}_t = \mathrm{d}\hat{L}_t = \left(c + \int_0^t \xi_s \mathrm{d}X_s\right)\left(\hat{f}(t) - \frac{1 - F_T}{1 - F_t}\right)\mathrm{d}\hat{M}_t. \tag{3.31}$$

Chapter 4

Local Risk-Minimization for Defaultable Claims with Recovery Scheme at Maturity

4.1 Introduction

In the previous chapter, according to [1] we have investigated the local risk-minimization method but only in the case of a defaultable put and under the assumption that the default time and the underlying Brownian motion were independent. Here according to [2], we extend these results and consider a more general case: we apply the local risk-minimization approach to a generic defaultable claim with recovery scheme at *maturity* in a more general setting, where the dynamics of the discounted risky asset X may be influenced by the occurring of a default event and also the default time τ itself may depend on the assets prices behavior. The main goal of this chapter is to provide the Föllmer-Schweizer decomposition of a generic defaultable claim with random recovery rate in this general setting.

In particular we focus on two cases where we compute explicitly the pseudo-locally risk-minimizing strategy and the optimal cost. First we consider the situation where the default time τ depends on the behavior of the risky asset price, but not vice versa. In the second case we assume that drift μ and volatility σ of the underlying discounted asset are affected by τ and we show how our result fits in the approach of [22] of local risk-minimization for financial markets affected by *incomplete information*.

4.2 Local risk-minimization for defaultable claims

All the hypotheses outlined in Section 2.2 are supposed to hold in this framework. In particular we should emphasize that here the default time occurring and the risky asset behavior can influence each other. Mathematically this means that $(\Omega, \mathcal{G}, \mathbb{Q})$ is not necessarily a product probability space and the reference filtration is then

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \text{ for any } t \in [0, T].$$

Under the hypotheses of Section 2.2, we study now the local risk-minimization approach for a defaultable claim H with random recovery scheme at maturity. In this case the discounted value of H can be represented as follows:

$$\begin{aligned} H &= \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{\tilde{X}}{B_T}\mathbb{I}_{\{\tau \le T\}} \\ &= \frac{\bar{X}}{B_T}\left(\mathbb{I}_{\{\tau > T\}} + \delta(\omega)\mathbb{I}_{\{\tau \le T\}}\right) \\ &= \frac{\bar{X}}{B_T}\left(1 + (\delta(\omega) - 1)\mathbb{I}_{\{\tau \le T\}}\right), \end{aligned} \tag{4.1}$$

where δ is given in (3.4) and the promised contingent claim $\bar{X}$ is given by a $\mathcal{G}_T$-measurable random variable. The next result guarantees the existence of a pseudo-locally risk-minimizing strategy for H.

Proposition 4.2.1. *Assume that the hazard process Γ is continuous. Then for any $\mathcal{G}_t$-martingale N_t under $\mathbb{Q}$ we have*

$$N_t = N_0 + \int_0^t \xi_u^N \mathrm{d}W_u + \int_{]0,t]} \zeta_u^N \mathrm{d}\hat{M}_u = N_0 + M_t^N + L_t^N,$$

where ξ^N *and* ζ^N *are* $\mathcal{G}$*-predictable processes such that for every* $t \in [0,T]$

$$\int_0^t (\xi_u^N)^2 \mathrm{d}u + \int_0^t (\zeta_u^N)^2 \lambda_u \mathrm{d}u < +\infty.$$

The continuous $\mathcal{G}_t$*-martingale* M_t^N *and the purely discontinuous* $\mathcal{G}_t$*-martingale* L_t^N *are mutually orthogonal.*

Proof. See Corollary 5.2.4 of [13] for the proof. □

Under assumption (2.8) we know that the minimal martingale measure $\widehat{\mathbb{Q}}$ exists and it is unique. In particular we have

Proposition 4.2.2. *Assume hypothesis* (2.8) *is in force. Consider the probability measure* $\bar{\mathbb{Q}}$ *with Radon-Nykodym density*

$$\frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T,$$

where θ *is defined in* (2.6)*. Then* $\bar{\mathbb{Q}} = \widehat{\mathbb{Q}}$*.*

Proof. It follows by hypothesis (2.8), Definition 1.3.15 and Proposition 4.2.1. □

By Proposition 4.2.2 we have that $\hat{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$ is a $\mathcal{G}_t$-Brownian motion under $\widehat{\mathbb{Q}}$ and the results of Proposition 4.2.1 can be reformulated in terms of $(\hat{W}, \hat{M})$. In fact $\hat{M}_t = H_t - \int_0^t \tilde{\lambda}_s \mathrm{d}s$ is also a $\widehat{\mathbb{Q}}$-martingale since the orthogonal martingale structure is not affected by the change of measure from $\mathbb{Q}$ to $\widehat{\mathbb{Q}}$. Hence we obtain that, since the hazard process Γ_t is continuous by hypothesis (2.2), every $\mathcal{G}_t$-martingale $\hat{N}_t$ under $\widehat{\mathbb{Q}}$ is of the form

$$\hat{N}_t = \hat{N}_0 + \int_0^t \xi_u^{\hat{N}} \mathrm{d}\hat{W}_u + \int_{]0,t]} \zeta_u^{\hat{N}} \mathrm{d}\hat{M}_u. \tag{4.2}$$

We now find the plrm-strategy for H by computing the decomposition (4.2) for $\widehat{E}\left[H \middle| \mathcal{G}_t\right]$ under $\widehat{\mathbb{Q}}$. Theorem 1.3.16 and our hypothesis (2.8) guarantee that this is indeed the FS-decomposition for H.

Under the equivalent martingale probability measure $\widehat{\mathbb{Q}}$, the discounted optimal portfolio value $\hat{V}_t$ of the defaultable claim H at time t, is given by:

$$\begin{aligned}\hat{V}_t &= \widehat{E}\left[H \middle| \mathcal{G}_t\right]\\ &= \widehat{E}\left[\frac{\bar{X}}{B_T}\left(1+(h(\tau\wedge T)-1)H_T\right)\middle| \mathcal{G}_t\right]\\ &= \underbrace{\widehat{E}\left[\frac{\bar{X}}{B_T}\middle| \mathcal{G}_t\right]}_{a)} + \underbrace{\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle| \mathcal{G}_t\right]}_{b)}.\end{aligned} \tag{4.3}$$

a) Since $\bar{X}\in L^1(\mathcal{G}_T,\widehat{\mathbb{Q}})$, by (4.2) we have

$$\widehat{E}\left[\frac{\bar{X}}{B_T}\middle| \mathcal{G}_t\right] = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^t \bar{\xi}_s \mathrm{d}\hat{W}_s + \int_0^t \bar{\eta}_s \mathrm{d}\hat{M}_s, \tag{4.4}$$

where $\bar{\xi}_t$ and $\bar{\eta}_t$ are $\mathcal{G}_t$-predictable process for every $t\in[0,T]$ and $\hat{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$ is a Brownian motion under $\widehat{\mathbb{Q}}$.
b) It remains to compute the term $\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle| \mathcal{G}_t\right]$. First by Corollary 5.1.2 of [13] we can obtain the following decomposition

$$\begin{aligned}\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle| \mathcal{G}_t\right] &= H_t\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\mathcal{F}_t\vee\mathcal{H}_T\right] +\\ &+(1-H_t)\widehat{E}\left[(1-H_t)e^{\int_0^t\lambda_s\mathrm{d}s}\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle| \mathcal{F}_t\right] =\\ &= H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right] + (1-H_t)e^{\int_0^t\lambda_s\mathrm{d}s}\cdot\\ &\quad \underbrace{\cdot E^*\left[\mathbb{I}_{\{t<\tau\le T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle| \mathcal{F}_t\right]}_{c)}\end{aligned} \tag{4.5}$$

We focus now on the conditional expectation c). We introduce here the σ-algebra

$$\mathcal{F}_{\tau-} = \sigma\left(A\cap\{\tau>t\},\ A\in\mathcal{F}_t,\ 0\le t\le T\right)$$

of the *events strictly prior* to τ. We set

$$N := \widehat{E}\left[(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle| \mathcal{F}_{\tau-}\right] \tag{4.6}$$

and note that

$$N = \widehat{E}\left[(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T}\middle| \mathcal{F}_{\tau-}\right] = (h(\tau \wedge T) - 1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle| \mathcal{G}_{\tau-}\right]$$

since $\mathcal{F}_{\tau-} = \mathcal{G}_{\tau-}$ and the $\mathcal{G}$-stopping time τ is $\mathcal{G}_{\tau-}$-measurable by Theorem 5.6 on page 118 of [18].

Lemma 4.2.3. *Let N be defined in* (4.6)*. Then*

$$\widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T}\middle| \mathcal{F}_t\right] = \widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}N\middle| \mathcal{F}_t\right], \quad \forall t \in [0,T].$$

Proof. Consider an arbitrary event $A \in \mathcal{F}_t$. By using the definition of the conditional expectation, we have

$$\begin{aligned}
\int_A \mathbb{I}_{\{t<\tau\leq T\}}(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T}\mathrm{d}\mathbb{Q} &= \int_{A\cap\{\tau>t\}} \mathbb{I}_{\{\tau\leq T\}}(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T}\mathrm{d}\mathbb{Q} \\
&= \int_{A\cap\{\tau>t\}} \widehat{E}\left[\mathbb{I}_{\{\tau\leq T\}}(h(\tau \wedge T) - 1)\frac{\bar{X}}{B_T}\middle| \mathcal{F}_{\tau-}\right]\mathrm{d}\mathbb{Q} \\
&= \int_{A\cap\{\tau>t\}} \mathbb{I}_{\{\tau\leq T\}}N\mathrm{d}\mathbb{Q} \\
&= \int_A \mathbb{I}_{\{t<\tau\leq T\}}N\mathrm{d}\mathbb{Q},
\end{aligned}$$

since the event $\{\tau \leq T\}$ is in $\mathcal{F}_{\tau-}$ and $\mathcal{F}_{\tau-} = \mathcal{G}_{\tau-}$ (see Lemma 5.1.3 of [13]). $\square$

By Theorem 67 page 125 in [18] and since $\mathcal{F}_{\tau-} = \mathcal{F}_\tau$ by Chapter XX of [17], page 148, we know that there exists an $\mathcal{F}_t$-predictable[1] process $\bar{Z}_t$ such that

$$\bar{Z}_\tau = N. \tag{4.7}$$

[1]By Theorem 67 on page 125 in [18], there exists a $\mathcal{G}_t$-predictable process Z^G such that $Z^G_\tau = N$. Since $\mathcal{F}_{\tau-} = \mathcal{F}_\tau$ by Chapter XX of [17], page 148, we have that N is also $\mathcal{F}_\tau$-measurable. But for every $A \in \mathcal{F}_\tau$ the process $A_t = \mathbb{I}_A\mathbb{I}_{\{\tau\leq t\}}$ is càdlàg, $\mathcal{F}_t$-adapted and $A_\tau = \mathbb{I}_A$. Hence $\mathcal{F}_\tau \subseteq \sigma(Y_\tau, Y$ càdlàg $\mathcal{F}_t -$ adapted processes$)$ and there exists a $\mathcal{F}_t$-predictable process $\bar{Z}_t$ such that $\bar{Z}_\tau = N$.

Hence we obtain

$$\begin{aligned}\widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\bigg|\,\mathcal{F}_t\right] &= \widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}N\big|\,\mathcal{F}_t\right]\\ &= \widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}\bar{Z}_\tau\big|\,\mathcal{F}_t\right]\\ &= \widehat{E}\left[\int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\bigg|\,\mathcal{F}_t\right], \qquad (4.8)\end{aligned}$$

where the last equality holds in view of Proposition 5.1.1 (ii) of [13] and the $\mathcal{F}$-predictability of $\bar{Z}$ (see page 148 of [13]). Hence we can rewrite (4.5) as follows:

$$\begin{aligned}\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\bigg|\,\mathcal{G}_t\right] &= H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\bigg|\,\mathcal{F}_t\vee\mathcal{H}_T\right]+\\ &\quad+(1-H_t)e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[\int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\bigg|\,\mathcal{F}_t\right]. \qquad (4.9)\end{aligned}$$

We put

$$D_t := e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[\int_t^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\bigg|\,\mathcal{F}_t\right] \qquad (4.10)$$

and we introduce the $\mathcal{F}_t$-martingale m_t by setting

$$m_t = \widehat{E}\left[\int_0^T \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\bigg|\,\mathcal{F}_t\right]. \qquad (4.11)$$

Following the proof of Proposition 5.2.1 of [13], we write D_t in terms of the $\mathcal{F}_t$-martingale m_t

$$D_t = e^{\int_0^t \lambda_s \mathrm{d}s}m_t - e^{\int_0^t \lambda_s \mathrm{d}s}\int_0^t \bar{Z}_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s.$$

By applying the Itô integration by parts formula, we obtain

$$\begin{aligned}D_t = m_0 &+ \int_{]0,t]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_0^t m_s e^{\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s - \int_0^t \bar{Z}_s\lambda_s \mathrm{d}s\\ &- \int_0^t e^{\int_0^s \lambda_u \mathrm{d}u}\int_0^s \bar{Z}_v e^{-\int_0^v \lambda_u \mathrm{d}u}\lambda_v \mathrm{d}v\lambda_s \mathrm{d}s\end{aligned}$$

which implies that

$$D_t = m_0 + \int_{]0,t]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_0^t (D_s - \bar{Z}_s)\lambda_s \mathrm{d}s,$$

Furthermore, since D_t is a continuous process, we have

$$(1 - H_t)D_t = m_0 + \int_{]0,t\wedge\tau]} \mathrm{d}D_s - \mathbb{I}_{\{\tau\le t\}}D_\tau.$$

Hence

$$\begin{aligned}(1 - H_t)D_t &= m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_0^{t\wedge\tau}(D_s - \bar{Z}_s)\lambda_s \mathrm{d}s - \mathbb{I}_{\{\tau\le t\}}D_\tau \\ &= m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \left(\mathbb{I}_{\{\tau\le t\}}D_\tau - \int_0^{t\wedge\tau} D_s\lambda_s\mathrm{d}s\right) - \int_0^{t\wedge\tau}\bar{Z}_s\lambda_s\mathrm{d}s \\ &= m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_0^t D_s\mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau}\bar{Z}_s\lambda_s\mathrm{d}s.\end{aligned}$$

Consequently we can rewrite (4.9) as follows:

$$\begin{aligned}\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\,\mathcal{G}_t\right] &= H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right] + \\ & m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_0^t D_s\mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau}\bar{Z}_s\lambda_s\mathrm{d}s. \qquad (4.12)\end{aligned}$$

A useful result is given by the following Lemma stated in [2].

Lemma 4.2.4. *Let Z_t be the $\mathcal{F}_t$-predictable process given by* (4.7)*. Then the following equality holds:*

$$H_tZ_\tau = H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right], \quad \forall t\in[0,T]. \qquad (4.13)$$

Proof. It is clear that

$$H_tZ_\tau = \widehat{E}\left[H_t(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{F}_{\tau-}\right].$$

Hence we need only to show that

$$\widehat{E}\left[H_t(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{F}_{\tau-}\right] = H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right]. \quad (4.14)$$

By using the definition of conditional expectation and the fact that conditioning with respect to $\mathcal{G}_t$ can be replaced by conditioning with respect to

$\mathcal{F}_t \vee \mathcal{H}_T$ on the event $\{\tau \leq t\}$ (see Lemma 5.1.5 of [13]), given an arbitrary event A in $\mathcal{F}_s$, with $0 < s \leq t$, for any $t \in [0, T]$, we have

$$\begin{aligned}\int_{A\cap\{\tau>s\}} H_t(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\mathrm{d}\mathbb{Q} &= \int_{A\cap\{s<\tau\leq t\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\mathrm{d}\mathbb{Q}\\ &= \int_{A\cap\{s<\tau\leq t\}}\widehat{E}\left[(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{G}_t\right]\mathrm{d}\mathbb{Q}\\ &= \int_{A\cap\{\tau>s\}} H_t(h(\tau\wedge t)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\,\mathcal{G}_t\right]\mathrm{d}\mathbb{Q}\\ &= \int_{A\cap\{\tau>s\}} H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right]\mathrm{d}\mathbb{Q},\end{aligned}$$

since

$$H_t(h(\tau\wedge T)-1) = H_t(h(\tau\wedge t)-1), \quad \forall t\in[0,T].$$

Then the statement is proved since (4.14) is verified on the generators. □

Finally gathering the results, we obtain by using (4.13)

$$\begin{gathered}\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\,\mathcal{G}_t\right] =\\ H_t Z_\tau + m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_0^t D_s\mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau}\bar{Z}_s\lambda_s\mathrm{d}s =\\ m_0 + \int_0^t \bar{Z}_s\mathrm{d}\hat{M}_s + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_0^t D_s\mathrm{d}\hat{M}_s =\\ m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_0^t(\bar{Z}_s - D_s)\mathrm{d}\hat{M}_s =\\ m_0 + \int_0^t (1-H_s)e^{\int_0^s \lambda_u \mathrm{d}u}\xi_s^m\mathrm{d}\hat{W}_s + \int_0^t(\bar{Z}_s - D_s)\mathrm{d}\hat{M}_s, \qquad (4.15)\end{gathered}$$

where we have used the fact that the continuous $\mathcal{F}_t$-martingale m_t admits the following integral representation with respect to the Brownian motion $\hat{W}_t$

$$m_t = m_0 + \int_0^t \xi_s^m \mathrm{d}\hat{W}_s, \qquad (4.16)$$

for some $\mathcal{F}_t$-predictable process ξ^m, such that $\forall t$, $\int_0^t(\xi_s^m)^2\mathrm{d}s < +\infty$.

Proposition 4.2.5. *In the market model outlined in Section 2.2 and under the assumption of Section 4.2, the FS decomposition for H defined in* (2.10) *is given by*

$$\begin{aligned} \hat{V}_t &= \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_0^t \left(\bar{\xi}_s + \mathbb{I}_{\{\tau\geq s\}}\xi_s^m e^{\int_0^s \lambda_u \mathrm{d}u}\right)\mathrm{d}\hat{W}_s + \\ &\quad + \int_0^t (\bar{Z}_s - D_s + \bar{\eta}_s)\mathrm{d}\hat{M}_s \qquad (4.17)\\ &= \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_0^t \frac{1}{\sigma_s X_s}\left(\bar{\xi}_s + \mathbb{I}_{\{\tau\geq s\}}\xi_s^m e^{\int_0^s \lambda_u \mathrm{d}u}\right)\mathrm{d}X_s + \\ &\quad + \int_0^t (\bar{Z}_s - D_s + \bar{\eta}_s)\mathrm{d}\hat{M}_s, \qquad (4.18) \end{aligned}$$

where the processes m, $\bar{Z}$, D, $\bar{\xi}$, $\bar{\eta}$, ξ^m *and* $\hat{M}$ *are defined in* (4.11), (4.6), (4.10), (4.4), (4.16) *and* (2.4). *In particular we have that the* plrm-strategy *is given by*

$$\xi_t^H = \frac{1}{\sigma_t X_t}\left(\bar{\xi}_t + \mathbb{I}_{\{\tau\geq t\}}\xi_t^m e^{\int_0^t \lambda_s \mathrm{d}s}\right) \qquad (4.19)$$

and the minimal cost *is*

$$C_t^H = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_0^t (\bar{Z}_s - D_s + \bar{\eta}_s)\mathrm{d}\hat{M}_s. \qquad (4.20)$$

Proof. It follows by hypothesis (2.8) and Theorem 1.3.16. □

Proposition 4.2.5 extends the main result of [1], where decomposition (5.23) was already proved in the case when the trajectories of X_t are $\mathcal{F}_t$-adapted and $\mathcal{F}_t$ and $\mathcal{H}_t$ are independent for every $t \in [0, T]$.
In general if $\dfrac{\bar{X}}{B_T}$ is $\mathcal{F}_T$-measurable, we have $\bar{\eta}_t = 0$ in decomposition (4.4) and

$$\bar{Z}_t = (h(t\wedge T) - 1)\left(\widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^t \bar{\xi}_s \mathrm{d}\hat{W}_s\right)$$

in equation (4.8). In fact by (4.4) and Theorem 67 page 125 in [18], we get

$$\begin{aligned}
\widehat{E}\left[\left.\frac{\bar{X}}{B_T}\right|\mathcal{F}_{\tau-}\right] &= \widehat{E}\left[\left.\frac{\bar{X}}{B_T}\right|\mathcal{G}_{\tau-}\right] \\
&= \widehat{E}\left[\left.\widehat{E}\left[\left.\frac{\bar{X}}{B_T}\right|\mathcal{G}_{\tau}\right]\right|\mathcal{G}_{\tau-}\right] \\
&= \widehat{E}\left[\left.\widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^{\tau}\bar{\xi}_s \mathrm{d}\hat{W}_s\right|\mathcal{G}_{\tau-}\right] \\
&= \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^{\tau}\bar{\xi}_s \mathrm{d}\hat{W}_s. \qquad (4.21)
\end{aligned}$$

Note that here we are using implicitly hypothesis (H) under $\widehat{\mathbb{Q}}$.

Remark 4.2.6. *The introduction of the process $\bar{Z}$ in* (4.7) *may appear artificial. However it is necessary to find decomposition* (4.8). *We have already seen that $\bar{Z}_t$ can be explicitly calculated if $\frac{\bar{X}}{B_T}$ is $\mathcal{F}_T$-measurable. This is already a quite general case since we do not require the trajectories of X_t to be $\mathcal{F}_t$-adapted or the independence of τ from $\mathcal{F}_t$.*
Another example is the following. We suppose that under $\widehat{\mathbb{Q}}$, the discounted asset price X_t is of the form

$$X_t = x_0 e^{\sigma(\tau)W_t - \frac{1}{2}\sigma(\tau)^2 t}, \quad x_0 > 0,$$

where σ is a sufficiently integrable positive Borel function, and $\frac{\bar{X}}{B_T} = X_T^2$. In this case $\frac{\bar{X}}{B_T}$ is (strictly) $\mathcal{G}_T$-measurable. We obtain

$$\begin{aligned}
\widehat{E}\left[\left.\frac{\bar{X}}{B_T}\right|\mathcal{F}_{\tau-}\right] &= \widehat{E}\left[\left.x_0^2 e^{2\sigma(\tau)W_T - \frac{1}{2}\sigma(\tau)^2 T}\right|\mathcal{G}_{\tau-}\right] \\
&= x_0^2 e^{-\sigma(\tau)^2 T}\widehat{E}\left[\left.e^{2\sigma(\tau)W_T}\right|\mathcal{G}_{\tau-}\right] \\
&= x_0^2 e^{\sigma(\tau)^2 T}\widehat{E}\left[\left.e^{2\sigma(\tau)W_T - 2\sigma(\tau)^2 T}\right|\mathcal{G}_{\tau-}\right] \\
&= x_0^2 e^{\sigma(\tau)^2 T} e^{2\sigma(\tau)W_\tau - 2\sigma(\tau)^2 \tau}
\end{aligned}$$

and

$$\bar{Z}_t = x_0^2 (h(t \wedge T) - 1) e^{\sigma(t)^2 T} e^{2\sigma(t)W_t - 2\sigma(t)^2 t}.$$

We remark that $\bar{Z}_t$ is not uniquely defined. However in the case that several possible $\mathcal{F}_t$-predictable process $\bar{Z}_t$ exist satisfying equation (4.7)*, they all provide the same conditional expectation* (4.8)*. We refer also to [13], page 148, for a further discussion of this issue.*

We compute decomposition (4.18) in two particular cases.

4.3 Example 1: τ dependent on X

We consider first the case where the default process may depend on the evolution of the asset price, but the dynamics of the money market account and of the stock are *not* influenced by the presence of the default in the market. We represent this fact by assuming that the interest rate, the drift and volatility in (2.5) are $\mathcal{F}_t$-adapted processes.
Since the promised contingent claim $\bar{X}$ is written on the underlying non-defaultable assets S_t and B_t, in this setting $\bar{X}$ is $\mathcal{F}_T$-measurable and we have

$$\widehat{E}\left[\left.\frac{\bar{X}}{B_T}\right|\mathcal{G}_t\right] = \widehat{E}\left[\left.\frac{\bar{X}}{B_T}\right|\mathcal{F}_t\right],$$

as a consequence of our hypothesis (H) under $\widehat{\mathbb{Q}}$. Hence we get $\bar{\eta} = 0$ in (4.4).
We show now how to hedge a Corporate bond with a Treasury bond by using the local risk-minimizing approach, i.e. we compute the plrm-strategy for a defaultable claim H whose promised contingent claim $\bar{X}$ is equal to 1, i.e. $\bar{X} = p(T,T) = 1$, where the process $p(t,T)$ represents the price of a Treasury bond that expires at time T. For the sake of simplicity we put

$$B_t \equiv 1, \ \forall t \in [0,T].$$

Hence the discounted value of H can be represented as follows:

$$H = 1 + (h(\tau \wedge T) - 1)H_T. \tag{4.22}$$

In addition we assume the following hypotheses:

- λ_t is an affine process, in particular it satisfies the following equation under $\widehat{\mathbb{Q}}$:

$$\begin{cases} \mathrm{d}\lambda_t &= (b+\beta\lambda_t)\mathrm{d}t + \alpha\sqrt{\lambda_t}\mathrm{d}\hat{W}_t \\ \lambda_0 &= 0, \end{cases} \tag{4.23}$$

where $b, \alpha \in \mathbb{R}^+$ and β is arbitrary. It is the Cox-Ingersoll-Ross model and we know it has a unique strong solution $\lambda \geq 0$ for every $\lambda_0 \geq 0$. You can see [20] for further details.

- The Borel function $h : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$h(x) = \alpha_0 \mathbb{I}_{\{x \leq T_0\}} + \alpha_1 \mathbb{I}_{\{x > T_0\}}, \tag{4.24}$$

where $\alpha_0, \alpha_1 \in \mathbb{R}^+$ with $0 \leq \alpha_0 < \alpha_1$ and T_0 is a fixed date before the maturity T.

Under the equivalent martingale probability measure $\widehat{\mathbb{Q}}$, the discounted optimal portfolio value $\hat{V}_t$ of the defaultable claim H given in (5.33) at time t, is given by:

$$\begin{aligned} \hat{V}_t &= \widehat{E}\left[H \middle| \mathcal{G}_t\right] \\ &= 1 + \widehat{E}\left[(h(\tau \wedge T) - 1)H_T \middle| \mathcal{G}_t\right] \\ &= 1 + m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s + \int_0^t (h(s) - 1 - D_s)\mathrm{d}\hat{M}_s, \end{aligned} \tag{4.25}$$

where h is given in (4.24) and m, D and $\hat{M}$ are the processes introduced in (4.11), (4.10) and (2.4) respectively (see also Corollary 5.2.2 of [13]). We focus now on the $\mathcal{F}_t$-martingale m_t, that means we compute the conditional expectation $\widehat{E}\left[\int_0^T (h(s)-1)e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right]$.

$$\begin{aligned} m_t &= \widehat{E}\left[\int_0^T e^{-\int_0^s \lambda_u \mathrm{d}u}\left((\alpha_0 - \alpha_1)\mathbb{I}_{\{s\leq T_0\}} + (\alpha_1 - 1)\mathbb{I}_{\{s \leq T\}}\right)\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= (\alpha_0 - \alpha_1)\widehat{E}\left[\int_0^{T_0} e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] + (\alpha_1 - 1)\widehat{E}\left[\int_0^T e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= (\alpha_1 - \alpha_0)\underbrace{\widehat{E}\left[e^{-\int_0^{T_0}\lambda_s \mathrm{d}s} \middle| \mathcal{F}_t\right]}_{a)} + (1-\alpha_1)\underbrace{\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s} \middle| \mathcal{F}_t\right]}_{b)} + \alpha_0 - 1. \end{aligned}$$

b) Since λ is an affine process whose dynamics is given in (5.35) , we have

$$\begin{aligned}\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right] &= e^{-\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[e^{-\int_t^T \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right]\\ &= e^{-\int_0^t \lambda_s \mathrm{d}s}\cdot e^{-A(t,T)-B(t,T)\lambda_t},\end{aligned}$$

where the functions $A(t,T)$, $B(t,T)$ satisfy the following equations:

$$\partial_t B(t,T) = \frac{\alpha^2}{2}B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0 \tag{4.26}$$

$$\partial_t A(t,T) = -bB(t,T), \quad A(T,T) = 0, \tag{4.27}$$

that admit explicit solutions (see for instance [21]). It is clear that the $\mathcal{F}_t$-martingale $\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right]$ admits the integral representation with respect to the underlying Brownian motion $\hat{W}_t$, then it must be of the form

$$\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right] = \widehat{E}\left[^{-\int_0^T \lambda_s \mathrm{d}s}\right] + \int_0^t \varphi_s \mathrm{d}\hat{W}_s,$$

for a suitable φ. The Itô formula yields

$$\begin{aligned}&\mathrm{d}\left(e^{-\int_0^t \lambda_s \mathrm{d}s}\cdot e^{-A(t,T)-B(t,T)\lambda_t}\right) =\\ &= e^{-\int_0^t \lambda_s \mathrm{d}s}\underbrace{\mathrm{d}\left(e^{-A(t,T)-B(t,T)\lambda_t}\right)}_{c)} - e^{-A(t,T)-B(t,T)\lambda_t}e^{-\int_0^t \lambda_s \mathrm{d}s}\lambda_t \mathrm{d}t.\end{aligned} \tag{4.28}$$

We focus now on c).applying Itô formula we get

$$\begin{aligned}\mathrm{d}\left(e^{-A(t,T)-B(t,T)\lambda_t}\right) &= e^{-A(t,T)}\mathrm{d}\left(e^{-B(t,T)\lambda_t}\right) + e^{-B(t,T)\lambda_t}\mathrm{d}\left(e^{-A(t,T)}\right)\\ &= e^{-A(t,T)-B(t,T)\lambda_t}\Bigg[\Bigg(-\frac{\partial}{\partial_t}B(t,T)\lambda_t - bB(t,T) - \beta B(t,T)\lambda_t +\\ &\quad + \frac{1}{2}\alpha^2 B^2(t,T)\lambda_t - \frac{\partial}{\partial_t}A(t,T)\Bigg)\mathrm{d}t - \alpha B(t,T)\sqrt{\lambda_t}\mathrm{d}\hat{W}_t\Bigg].\end{aligned} \tag{4.29}$$

By plugging (4.29) into (4.28) and by (4.26) and (4.27), we obtain

$$\mathrm{d}\left(e^{-\int_0^t \lambda_s \mathrm{d}s}\cdot e^{-A(t,T)-B(t,T)\lambda_t}\right) = -e^{-\int_0^t \lambda_s \mathrm{d}s - A(t,T)-B(t,T)\lambda_t}\left(\alpha B(t,T)\sqrt{\lambda_t}\mathrm{d}\hat{W}_t\right).$$

Hence

$$\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right] = e^{-A(0,T)} - \int_0^t \alpha e^{-\int_0^s \lambda_u \mathrm{d}u - A(s,T)-B(s,T)\lambda_s}B(s,T)\sqrt{\lambda_s}\mathrm{d}\hat{W}_s \tag{4.30}$$

Similarly we can compute a) and we get

$$\widehat{E}\left[e^{-\int_0^{T_0}\lambda_s \mathrm{d}s}\middle| \mathcal{F}_t\right] = e^{-A(0,T_0)}+$$
$$-\int_0^t \alpha \mathbb{I}_{\{s\le T_0\}} e^{-\int_0^s \lambda_u \mathrm{d}u - A(s,T_0)-B(s,T_0)\lambda_s} B(s,T_0)\sqrt{\lambda_s}\mathrm{d}\hat{W}_s. \tag{4.31}$$

Finally gathering the results, we obtain

$$\begin{aligned} m_t = &\alpha_0 - 1 + (\alpha_1-\alpha_0)e^{-A(0,T_0)} + (1-\alpha_1)e^{-A(0,T)}+ \\ &-\int_0^t \alpha e^{-\int_0^s \lambda_u \mathrm{d}u}\Bigg((\alpha_1-\alpha_0)\mathbb{I}_{\{s\le T_0\}}e^{-A(s,T_0)-B(s,T_0)\lambda_s}B(s,T_0)+ \\ &+(1-\alpha_1)e^{-A(s,T)-B(s,T)\lambda_s}B(s,T)\Bigg)\sqrt{\lambda_s}\mathrm{d}\hat{W}_s. \end{aligned}$$

Consequently D_t is given by

$$\begin{aligned} D_t &= e^{\int_0^t \lambda_s \mathrm{d}s} m_t - e^{\int_0^t \lambda_s \mathrm{d}s}\int_0^t (h(s)-1)e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \\ &= e^{\int_0^t \lambda_s \mathrm{d}s} m_t + (\alpha_0-\alpha_1)(1-e^{-\int_t^{T_0}\lambda_s \mathrm{d}s})\mathbb{I}_{\{t\le T_0\}} \\ &\quad + [(\alpha_0-\alpha_1)e^{-\int_t^{T_0}\lambda_s \mathrm{d}s} - (\alpha_0-1)e^{\int_0^t \lambda_s \mathrm{d}s} + \alpha_1 - 1] \end{aligned} \tag{4.32}$$

Finally we can write explicitly decomposition (4.25) that provides the FS decomposition for H:

$$\begin{aligned} \hat{V}_t = &\alpha_0 + (\alpha_1-\alpha_0)e^{-A(0,T_0)} + (1-\alpha_1)e^{-A(0,T)}+ \\ &-\int_{t\wedge\tau}\frac{\alpha}{\sigma_s X_s}\Bigg((\alpha_1-\alpha_0)\mathbb{I}_{\{s\le T_0\}}e^{-A(s,T_0)-B(s,T_0)\lambda_s}B(s,T_0)+ \\ &(1-\alpha_1)e^{-A(s,T)-B(s,T)\lambda_s}B(s,T)\Bigg)\sqrt{\lambda_s}\mathrm{d}X_s + \int_0^t (h(s)-1-D_s)\mathrm{d}\hat{M}_s, \end{aligned} \tag{4.33}$$

where A, B, h, D and M are given in (4.27), (4.26), (4.24), (4.32) and (2.4) respectively.

4.4 Example 2: X dependent on τ

We study now the case when the default time may influence the dynamics of the asset price but not vice versa. We suppose then that the default time

$\tau = \tau(\eta)$ and the underlying Brownian motion $W = W(\tilde{\omega})$ are independent and defined on the product space $\Omega = \tilde{\Omega} \times E$, endowed with the product filtration $\mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{H}_t$, $\forall t \in [0,T]$ and the product probability $\mathbb{Q} = \mathbb{Q}^W \otimes \nu$, where $\mathbb{Q}^W$ is the Wiener measure and ν is the law of $H_t = \mathbb{I}_{\{\tau \leq t\}}$. Note that now with respect to the previous setting we have $\omega = (\tilde{\omega}, \eta)$. In particular following [6], we assume that the dynamics of S_t are of the form

$$\mathrm{d}S_t = S_t \left[\mu_t(\eta)\mathrm{d}t + \sigma_t(\eta)\mathrm{d}W_t \right], \tag{4.34}$$

and that the hypotheses outlined in Section 2.2 still hold. Note that here we are focusing on the case where drift and volatility *depend only* on η, seen as an *exterior source of randomness.*

Consider now the larger filtration $\tilde{\mathcal{G}}_t := \mathcal{F}_t \otimes \mathcal{H}_T$, obtained by adding to $\mathcal{G}_t$ the full information about η since the initial instant $t = 0$: it follows that $\mathcal{G}_t \subset \tilde{\mathcal{G}}_t$, $0 \leq t < T$. We suppose that W_t *is a Brownian motion with respect to* $\tilde{\mathcal{G}}_t$.

Proposition 4.4.1. *Under the hypotheses outlined above the process* ξ^H_t *given in* (4.19) *coincides with the predictable projection*[2] *of the* $\tilde{\mathcal{G}}_t$*-predictable process* $\tilde{\xi}^H_t$ *such that* $\int_0^T (\tilde{\xi}^H_s)^2 \mathrm{d}s < \infty$ *a.s. and*

$$\frac{\bar{X}}{B_T} = \widehat{E}\left[\frac{\bar{X}}{B_T} \middle| \tilde{\mathcal{G}}_0 \right] + \int_0^T \tilde{\xi}^H_s \mathrm{d}\hat{W}_s.$$

Proof. Since $\mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$, we may prove the Proposition in the case when the $\mathcal{G}_T$-measurable random variable $\dfrac{\bar{X}}{B_T}$ is of the form $\dfrac{\bar{X}}{B_T} = (1 - H_T)F$, for some $\mathcal{F}_T$-measurable random variable F. We compute first decomposition (4.4) for $\dfrac{\bar{X}}{B_T}$. We note that

$$\frac{\bar{X}}{B_T} = (1 - H_T)F = (1 - H_T)e^{\int_0^T \lambda_u \mathrm{d}u} \bar{F} = L_T \bar{F},$$

where the process $L_t = (1 - H_t)e^{\int_0^t \lambda_u \mathrm{d}u}$ is a $\mathcal{G}_t$-martingale (see Lemma 5.1.7 of [13] for further details) and $\bar{F} = e^{-\int_0^t \lambda_u \mathrm{d}u} F$ is an $\mathcal{F}_T$-measurable, integrable

[2] For an extensive discussion of this subject we refer to Appendix A

random variable. First by the martingale representation property of the Brownian filtration, we have

$$\bar{F} = \widehat{E}\left[\bar{F}\right] + \int_0^T \xi_u \mathrm{d}\hat{W}_u,$$

where ξ_t is a $\mathcal{F}_t$-predictable process. Then

$$\frac{\bar{X}}{B_T} = L_T\left(\widehat{E}\left[\bar{F}\right] + \int_0^T \xi_u \mathrm{d}\hat{W}_u\right) = L_T \widehat{E}\left[\bar{F}\right] + \int_0^T L_T \xi_t \mathrm{d}\hat{W}_t, \tag{4.35}$$

i.e. $\dfrac{\bar{X}}{B_T}$ is attainable with respect to the larger filtration $\tilde{\mathcal{G}}_t$. If we put $G_t := \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right]$, we have

$$\frac{\bar{X}}{B_T} = L_T \bar{F} = L_T \widehat{E}\left[F|\mathcal{F}_T\right] = L_T G_T.$$

By Proposition 5.1.3 of [13] we have $L_t = \mathcal{E}(-M)_t$, where $\hat{M}_t = H_t - \int_0^{t\wedge\tau} \lambda_u \mathrm{d}u$. Hence $[L, G]_t = 0$, for every $t \in [0, T]$ and the Itô integration by parts formula yields

$$\begin{aligned}
\frac{\bar{X}}{B_T} &= L_0 G_0 + \int_0^T L_{t-} \mathrm{d}G_t + \int_0^T G_t \mathrm{d}L_t + [L, G]_T \\
&= \widehat{E}\left[\bar{F}\right] + \int_0^T L_{t-}\xi_t \mathrm{d}\hat{W}_t + \int_0^T \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right] \mathrm{d}L_t \\
&= \widehat{E}\left[\bar{F}\right] + \int_0^T L_{t-}\xi_t \mathrm{d}\hat{W}_t - \int_0^T \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right] L_t \mathrm{d}\hat{M}_t,
\end{aligned} \tag{4.36}$$

since $G_t = \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right]$ is continuous. On the other hand by (4.2), we get

$$\frac{\bar{X}}{B_T} = L_T \bar{F} = \widehat{E}\left[L_T \bar{F}\right] + \int_0^T \bar{\xi}_t \mathrm{d}\hat{W}_t + \int_0^T \bar{\eta}_t \mathrm{d}\hat{M}_t, \tag{4.37}$$

and the uniqueness of the decomposition implies that

$$\bar{\xi}_t = L_{t-}\xi_t = (L_T \xi_{\cdot})^p_t,$$

i.e. $\bar{\xi}$ coincides with the predictable projection of the process $L_T\xi_t$.
Analogously we compute the decomposition of $\widehat{E}\left[\dfrac{\bar{X}}{B_T}(h(\tau\wedge T) - 1)H_T\middle|\tilde{\mathcal{G}}_t\right]$,

that is given by

$$\begin{aligned}
&\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\tilde{\mathcal{G}}_t\right]\\
&\qquad=(h(\tau\wedge T)-1)H_T\left(\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\tilde{\mathcal{G}}_0\right]+\int_0^T L_T\xi_u\mathrm{d}\hat{W}_u\right)\\
&\qquad=(h(\tau\wedge T)-1)H_T\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\tilde{\mathcal{G}}_0\right]+\int_0^T \underbrace{L_s\xi_u(h(\tau\wedge T)-1)H_T}_{\tilde{\Psi}_u}\mathrm{d}\hat{W}_u.
\end{aligned}$$

With a similar argument as before we can conclude that the integrand

$$\Psi_t=(1-H_t)e^{\int_0^t\lambda_s\mathrm{d}s}\xi_t^m$$

appearing in decomposition (4.15) of $\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\,\mathcal{G}_t\right]$ is the predictable projection of $\tilde{\Psi}_t$. □

In particular we note that we obtain again the results of Theorem 4.6 and Theorem 4.16 of [22]. Hence (4.36) is the FS decomposition in the case of *incomplete information.* Namely if the trader would have access to the larger filtration $\tilde{\mathcal{G}}_t$ which contains at any time the information on past and future behavior of the default time, the market would be complete because the volatility and drift are deterministic with respect to $\tilde{\mathcal{G}}_t$.

Example 4.4.2. *We apply these results to find the plrm-strategy for a defaultable claim H whose promised contingent claim $\bar{X}$ is given by the standard payoff of a call option, i.e. $\bar{X}=(S_T-K)^+$, where $K\in\mathbb{R}_+$ represents the exercise price. Hence the discounted value of H can be represented as follows:*

$$H=\frac{(S_T-K)^+}{B_T}(1+(h(\tau\wedge T)-1)H_T) \tag{4.38}$$

and with respect to $\tilde{\mathcal{G}}_t$, the discounted replicating portfolio $\tilde{V}_t$ for H is given *by:*

$$
\begin{aligned}
\tilde{V}_t &= \widehat{E}[H|\tilde{\mathcal{G}}_t] \\
&= \widehat{E}\left[\frac{(S_T-K)^+}{B_T}(1+(h(\tau\wedge T)-1)H_T)\middle|\tilde{\mathcal{G}}_t\right] \\
&= (1+(h(\tau\wedge T)-1)H_T)\widehat{E}\left[\frac{(S_T-K)^+}{B_T}\middle|\tilde{\mathcal{G}}_t\right] \\
&= (1+(h(\tau\wedge T)-1)H_T)\left(X_t\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_t]-\frac{K}{B_T}\widehat{E}[\mathbb{I}_A|\tilde{\mathcal{G}}_t]\right) \\
&= (1+(h(\tau\wedge T)-1)H_T)\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_t]X_t \\
&\qquad -(1+(h(\tau\wedge T)-1)H_T)\frac{K}{B_T}\widehat{E}[\mathbb{I}_A|\tilde{\mathcal{G}}_t], \qquad (4.39)
\end{aligned}
$$

where A denotes the event $\{S_T\geq K\}$ and by [6] we have that the minimal martingale measure under the numéraire X_t satisfies

$$
\left.\frac{\mathrm{d}\widehat{\mathbb{Q}}^X}{\mathrm{d}\widehat{\mathbb{Q}}}\right|_{\tilde{\mathcal{G}}_t}=\frac{X_T}{X_0}
$$

since X_t is a square-integrable $\tilde{\mathcal{G}}_t$-martingale under $\widehat{\mathbb{Q}}$. By standard delta-hedging arguments the process $\tilde{\xi}^H_t=(1+(h(\tau\wedge T)-1)H_T)\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_t]$ represents the component invested in the discounted risky asset X_t of the replicating portfolio with respect to the filtration $\tilde{\mathcal{G}}_t$.
By Proposition 4.4.1 we only need to compute the predictable projection ξ^H of the process $\tilde{\xi}^H$.
By Theorem VI.43 of [19], we need to check that for every predictable $\mathcal{G}_t$-stopping time $\hat{\tau}$

$$
\xi_{\hat{\tau}}\mathbb{I}_{\{\hat{\tau}<\infty\}}=\widehat{E}\left[(h(\tau\wedge T)-1)H_T\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_{\hat{\tau}}]\mathbb{I}_{\{\hat{\tau}<\infty\}}\middle|\mathcal{G}_{\hat{\tau}-}\right],
$$

i.e.

$$
\begin{aligned}
\xi_{\hat{\tau}} \mathbb{I}_{\{\hat{\tau}<\infty\}} &= \widehat{E}\left[(h(\tau \wedge T) - 1) H_T \frac{\widehat{E}[X_T \mathbb{I}_A | \tilde{\mathcal{G}}_{\hat{\tau}}]}{\widehat{E}[X_T | \tilde{\mathcal{G}}_{\hat{\tau}}]} \mathbb{I}_{\{\hat{\tau}<\infty\}} \middle| \mathcal{G}_{\hat{\tau}-} \right] \\
&= \widehat{E}\left[(h(\tau \wedge T) - 1) H_T \frac{1}{X_{\hat{\tau}}} \widehat{E}[X_T \mathbb{I}_A | \tilde{\mathcal{G}}_{\hat{\tau}}] \mathbb{I}_{\{\hat{\tau}<\infty\}} \middle| \mathcal{G}_{\hat{\tau}-} \right] \\
&= \widehat{E}\left[\widehat{E}\left[(h(\tau \wedge T) - 1) H_T \frac{X_T}{X_{\hat{\tau}}} \mathbb{I}_A \middle| \tilde{\mathcal{G}}_{\hat{\tau}} \right] \mathbb{I}_{\{\hat{\tau}<\infty\}} \middle| \mathcal{G}_{\hat{\tau}-} \right] \\
&= \widehat{E}\left[(h(\tau \wedge T) - 1) H_T \frac{X_T}{X_{\hat{\tau}}} \mathbb{I}_A \mathbb{I}_{\{\hat{\tau}<\infty\}} \middle| \mathcal{G}_{\hat{\tau}-} \right] \\
&= \widehat{E}^X \left[(h(\tau \wedge T) - 1) H_T \mathbb{I}_A \mathbb{I}_{\{\hat{\tau}<\infty\}} \middle| \mathcal{G}_{\hat{\tau}-} \right].
\end{aligned}
$$

If we suppose that the process $\widehat{E}^X[(h(\tau \wedge T) - 1) H_T \mathbb{I}_A | \mathcal{G}_{t-}]$ *has a left-continuous version, then it coincides with the* $\mathcal{G}_t$*-predictable projection under the probability* $\widehat{\mathbb{Q}}^X$*. Hence the plrm-strategy for* H*, whose promised contingent claim* $\bar{X}$ *is given by the standard payoff of a call option, is given by*

$$
\xi_t^H = \widehat{E}^X \left[\mathbb{I}_A \left(1 + \left(h(\tau \wedge T) - 1\right) H_T\right) \middle| \mathcal{G}_{t-} \right]. \tag{4.40}
$$

Chapter 5

Local Risk-Minimization for Defaultable Claims with Recovery Scheme at Default Time

5.1 Introduction

In this chapter we study the local risk-minimization approach for defaultable claims with a random *recovery scheme at default time*, i.e. a random recovery payment is received by the owner of the contract in case of default *at time of default.*

In Chapter 3 we have applied for the first time the local risk-minimization approach to defaultable markets, in particular to price and hedge a defaultable put only under the assumption that the default time occurring and the risky asset behavior are independent. In Chapter 4 we have extended these results to the case of a general defaultable claim with random recovery *at maturity*, in a more general setting, assuming a mutual dependence of the risky asset behavior and the default time.

Here according to [3], we consider a general defaultable claim with random *recovery at default time*, represented by a predictable stochastic process. Our goal is to provide the pseudo-locally risk-minimizing strategy in the case when the agent information takes into account the possibility of a default event. Moreover in Section 5.2.2 we discuss the problem of finding a pseudo-locally risk-minimizing strategy if we suppose the agent obtains her information only by observing the non-defaultable assets.

5.2 Local risk-minimization for defaultable claims

All the hypotheses outlined in Section 2.2 are supposed to hold in this framework. In particular we assume that the short-term interest rate r is a $\mathcal{F}$-predictable process and that the promised contingent claim $\bar{X}$ is represented by a $\mathcal{F}_T$-measurable random variable. Under the hypotheses of Section 2.2, we investigate now the local risk-minimization approach for a defaultable claim H with random recovery scheme at default time and zero-recovery at maturity. Hence the discounted value of H can be represented as follows:

$$H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{Z}{B_\tau}\mathbb{I}_{\{\tau \leq T\}} \tag{5.1}$$

where the recovery process Z is given by a $\mathcal{F}$-predictable process. In particular we obtain that $H \in L^2(\mathcal{G}_T, \mathbb{Q})$. In this setting we study the problem of a trader wishing to price and hedge a defaultable claim H which pays a positive random recovery in case of default *at default time* τ. We recall that our market model is incomplete even if we assume to trade with $\mathcal{G}_t$-adapted strategies because $\hat{M}_t$ does not represent the value of any tradable asset. According to [3], we are able to provide a pseudo-locally risk-minimizing strategy for such defaultable claim. Since in practice hedging a credit derivative after default time is usually of minor interest and in our model we have only a single default time, we follow the approach of [12] and assume that *hedging stops after default*. Hence we need to reformulate the local risk-minimization approach, that can be applied to contingent claims that ensure one payment at a fixed date. Here we have a defaultable claim which guarantees a payment at a fixed date, but in this case it can be maturity or default time, if a default event occurs before the expiration date of the contract. Hence we look for a hedging strategy φ for H given in (5.1) with minimal cost C and such that the discounted value process satisfies

$$V_{\tau\wedge T}(\varphi) = H.$$

First we look for $\mathcal{G}_t$-strategies, i.e. we admit that the agent information takes into account the possibility of a default event. Then we wish to investigate the problem of finding a pseudo-locally risk-minimizing strategy in the case when the agent obtains her information only by observing the asset prices on the non-defaultable market before the default happens. This is equivalent to look for a pseudo-locally risk-minimizing strategy in the class of $\mathcal{F}_t$-strategies, i.e. adapted to the smaller filtration generated by the Brownian motion. We discuss this issue in Section 5.2.2.

5.2.1 Local risk-minimization with $\mathcal{G}_t$-strategies

By following [22] and [35] we introduce the $\mathcal{G}$-pseudo-locally risk-minimizing strategy for defaultable claims with recovery scheme at default time. We denote by $\Theta_s^{\mathcal{G}}$ the space of $\mathcal{G}$-predictable processes ξ on Ω such that

$$E\left[\int_0^T (\xi_s\sigma_s X_s)^2 \mathrm{d}s\right] + E\left[\left(\int_0^T |\xi_s(\mu_s - r_s)X_s|\mathrm{d}s\right)^2\right] < \infty. \tag{5.2}$$

Definition 5.2.1. *Let* $H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau\leq T\}} \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ *be the discounted value of a defaultable claim. A pair* $\varphi^{\mathcal{G}} = (\xi, \eta)$ *of stochastic processes is said a* $\mathcal{G}$-pseudo-locally risk-minimizing strategy *(in short* $\mathcal{G}$-plrm-strategy*) if*

1. $\xi_t \in \Theta_s^{\mathcal{G}}$;
2. η_t *is* $\mathcal{G}_t$*-adapted;*
3. *The discounted value process* $V_t(\varphi^{\mathcal{G}}) = \xi_t X_t + \eta_t$ *is such that*

$$V_t(\varphi^{\mathcal{G}}) = \int_0^t \xi_s \mathrm{d}X_s + C_t(\varphi^{\mathcal{G}}), \qquad t \in [\![0, \tau\wedge T]\!] \tag{5.3}$$

where C_t is the cost process and it is a square-integrable $\mathcal{G}_t$-martingale strongly orthogonal to the martingale part of X_t, and $V_{\tau\wedge T}(\varphi^{\mathcal{G}}) = H$, i.e.

$$V_T(\varphi^{\mathcal{G}}) = \frac{\bar{X}}{B_T} \text{ if } \tau > T, \qquad V_\tau(\varphi^{\mathcal{G}}) = \frac{Z_\tau}{B_\tau} \text{ if } \tau \leq T. \tag{5.4}$$

In the next result we can see how to characterize a $\mathcal{G}$-plrm strategy for the defaultable claim H given in (5.1).
We recall that $\mathcal{M}_0^2(\mathbb{Q})$ is the space of all $\mathbb{Q}$-square-integrable martingales with zero initial value.

Proposition 5.2.2. *A defaultable claim $H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau\leq T\}}$ belonging to $L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ admits a $\mathcal{G}$-plrm-strategy $\varphi^{\mathcal{G}} = (\xi, \eta)$ if and only if H can be written as*

$$H = H_0 + \int_0^{\tau\wedge T} \xi_s^H \mathrm{d}X_s + L_{\tau\wedge T}^H \quad \mathbb{Q} - a.s. \tag{5.5}$$

where $H_0 \in \mathbb{R}$, $\xi^H \in \Theta_s^{\mathcal{G}}$ and $L^H \in \mathcal{M}_0^2(\mathbb{Q})$ is strongly orthogonal to the martingale part of X. The $\mathcal{G}$-plrm-strategy $\varphi^{\mathcal{G}}$ is given by

$$\xi_t = \xi_t^H, \quad t \in [\![0, \tau\wedge T]\!]$$

with minimal cost

$$C_t(\varphi^{\mathcal{G}}) = H_0 + L_t^H, \quad t \in [\![0, \tau\wedge T]\!].$$

If (5.5) *holds, the optimal portfolio value is*

$$V_t(\varphi^{\mathcal{G}}) = C_t(\varphi^{\mathcal{G}}) + \int_0^t \xi_s \mathrm{d}X_s = H_0 + \int_0^t \xi_s \mathrm{d}X_s + L_t^H, \quad t \in [\![0, \tau\wedge T]\!]$$

and

$$\eta_t = \eta_t^H = V_t(\varphi^{\mathcal{G}}) - \xi_t^H X_t, \quad t \in [\![0, \tau\wedge T]\!].$$

Decomposition (5.5) is the (stopped) Föllmer-Schweizer decomposition (in short *FS decomposition*) of H. Again we need to find the minimal martingale measure (see Definition 1.3.15) in this framework. Under assumption (2.8) we know that the minimal martingale measure $\widehat{\mathbb{Q}}$ exists and it is unique by Proposition 4.2.2. In addition, we recall that by Proposition 4.2.2 the pair

$(\hat{W}, \hat{M})$, where $\hat{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$, for every $t \in [0,T]$, has the predictable representation property under $\widehat{\mathbb{Q}}$. Hence every $\mathcal{G}_t$-martingale N_t under $\widehat{\mathbb{Q}}$ can be written as

$$N_t = N_0 + \int_0^t \xi_s^N \mathrm{d}\hat{W}_s + \int_{]0,t]} \zeta_s^N \mathrm{d}\hat{M}_s, \tag{5.6}$$

for every $t \in [0,T]$. Again, how to use $\widehat{\mathbb{Q}}$ to characterize the $\mathcal{G}$-plrm strategy is shown in Theorem 1.3.16. The next result guarantees the existence of the pseudo-locally risk-minimizing strategy for H.

Proposition 5.2.3. *Let $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ be the defaultable claim given in (5.1) and define the $\mathcal{G}_t$-martingale $G_t^H = \widehat{E}\left[H | \mathcal{G}_t\right]$, $t \in [0,T]$. Then there exists a pair of $\mathcal{G}$-predictable processes $(\tilde{\xi}, \tilde{\zeta})$ satisfying*

$$\int_0^t \tilde{\xi}_s^2 \mathrm{d}s + \int_0^t \tilde{\zeta}_s^2 \mathrm{d}[\hat{M}]_s < \infty \quad t \in [0,T] \ a.s.$$

such that

$$G_t^H = G_0^H + \int_0^t \tilde{\xi}_s \mathrm{d}\hat{W}_s + \int_{]0,t]} \tilde{\zeta}_s \mathrm{d}\hat{M}_s, \quad \forall t \in [\![0, \tau \wedge T]\!] \tag{5.7}$$

under $\widehat{\mathbb{Q}}$, where $\hat{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$, for every $t \in [0,T]$.

Proof. We can rewrite H as follows:

$$H = \underbrace{\frac{\bar{X}}{B_T}(1 - H_T)}_{a)} + \underbrace{\frac{Z_\tau}{B_\tau} H_T}_{b)}.$$

a) We note that

$$(1 - H_T)\frac{\bar{X}}{B_T} = (1 - H_T) e^{\int_0^T \lambda_s \mathrm{d}s} F = L_T F,$$

where we have put

$$F = e^{-\int_0^T \lambda_s \mathrm{d}s} \frac{\bar{X}}{B_T}, \tag{5.8}$$

and the process $L_t = (1 - H_t) e^{\int_0^t \lambda_s \mathrm{d}s}$ is a $\mathcal{G}_t$-martingale (see Lemma 5.1.7 of [13] for further details) such that $\int_0^t \left((1 - H_s) e^{\int_0^s \lambda_u \mathrm{d}u}\right)^2 \mathrm{d}s < \infty$, $\forall t \in$

$[0, T]$. By decomposition (46) that can be found in the proof of Proposition 6.1 of [2], we obtain the following representation of $(1 - H_T)\frac{\bar{X}}{B_T}$:

$$
\begin{aligned}
&(1 - H_T)\frac{\bar{X}}{B_T} \\
&= \widehat{E}[F] + \int_0^T \mathbb{I}_{\{\tau > t\}} e^{\int_0^t \lambda_u \mathrm{d}u} \xi_t \mathrm{d}\hat{W}_t - \int_{]0,T]} \mathbb{I}_{\{\tau \geq t\}} e^{\int_0^t \lambda_u \mathrm{d}u} \widehat{E}\left[F \middle| \mathcal{F}_t\right] \mathrm{d}\hat{M}_t \\
&= \widehat{E}[F] + \int_0^{\tau \wedge T} e^{\int_0^t \lambda_u \mathrm{d}u} \xi_t \mathrm{d}\hat{W}_t - \int_{]0,\tau \wedge T]} e^{\int_0^t \lambda_u \mathrm{d}u} \widehat{E}\left[F \middle| \mathcal{F}_t\right] \mathrm{d}\hat{M}_t, \qquad (5.9)
\end{aligned}
$$

where ξ_t is the $\mathcal{F}_t$-predictable process such that $\int_0^t (\xi_s)^2 \mathrm{d}s < \infty$ for every $t \in [0, T]$, that appears in the following integral representation of the $\mathcal{F}_t$-martingale $\tilde{E}\left[F \middle| \mathcal{F}_t\right]$ with respect to the Brownian motion $\hat{W}_t$ given by

$$
\widehat{E}\left[F \middle| \mathcal{F}_t\right] = \widehat{E}[F] + \int_0^t \xi_s \mathrm{d}\hat{W}_s. \qquad (5.10)
$$

b) It remains to decompose the term $\frac{Z_\tau}{B_\tau} H_T$. By following Section 4 of [2], we have:

$$
\begin{aligned}
&\widehat{E}\left[\frac{Z_\tau}{B_\tau} H_T \middle| \mathcal{G}_t\right] \\
&= H_t \widehat{E}\left[\frac{Z_\tau}{B_\tau} H_T \middle| \mathcal{F}_t \vee \mathcal{H}_T\right] + (1 - H_t)\widehat{E}\left[(1 - H_t) e^{\int_0^t \lambda_s \mathrm{d}s} \frac{Z_\tau}{B_\tau} H_T \middle| \mathcal{F}_t\right] \\
&= H_t \widehat{E}\left[\frac{Z_\tau}{B_\tau} \middle| \mathcal{F}_t \vee \mathcal{H}_T\right] + (1 - H_t) e^{\int_0^t \lambda_s \mathrm{d}s} \cdot \underbrace{\widehat{E}\left[\mathbb{I}_{\{t < \tau \leq T\}} \frac{Z_\tau}{B_\tau} \middle| \mathcal{F}_t\right]}_{c)}. \qquad (5.11)
\end{aligned}
$$

We focus now on the conditional expectation c). Since $\frac{Z}{B}$ is a $\mathcal{F}$-predictable process, in view of Proposition 5.1.1 of [13] the following equality holds

$$
\widehat{E}\left[\mathbb{I}_{\{t < \tau \leq T\}} \frac{Z_\tau}{B_\tau} \middle| \mathcal{F}_t\right] = \widehat{E}\left[\int_t^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right]. \qquad (5.12)
$$

Hence we can rewrite (5.11) as follows:

$$
\begin{aligned}
&\widehat{E}\left[\frac{Z_\tau}{B_\tau} H_T \middle| \mathcal{G}_t\right] \\
&= H_t \widehat{E}\left[\frac{Z_\tau}{B_\tau} \middle| \mathcal{F}_t \vee \mathcal{H}_T\right] + (1 - H_t) e^{\int_0^t \lambda_s \mathrm{d}s} \widehat{E}\left[\int_t^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right]. \\
&\qquad (5.13)
\end{aligned}
$$

We consider the process D introduced in (4.10) that is given in this case by

$$D_t = e^{\int_0^t \lambda_s \mathrm{d}s} \tilde{E} \left[\int_t^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right] \tag{5.14}$$

and the $\mathcal{F}_t$-martingale m_t introduced in (4.11):

$$m_t = \widehat{E} \left[\int_0^T \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right]. \tag{5.15}$$

Following the same procedure applied in the previous chapter, we write D_t in terms of the $\mathcal{F}_t$-martingale m_t and by applying the Itô integration by parts formula, we obtain

$$D_t = m_0 + \int_{]0,t]} e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s + \int_0^t \left(D_s - \frac{Z_s}{B_s} \right) \lambda_s \mathrm{d}s.$$

Furthermore, since D_t is a càdlàg process, we have

$$(1 - H_t) D_t = m_0 + \int_{]0,t\wedge\tau]} \mathrm{d}D_s - \mathbb{I}_{\{\tau \le t\}} D_\tau.$$

Hence

$$(1 - H_t) D_t = m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s - \int_0^t D_s \mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau} \frac{Z_s}{B_s} \lambda_s \mathrm{d}s.$$

Consequently we can rewrite (5.13) as follows:

$$\begin{aligned} \widehat{E} \left[\frac{Z_\tau}{B_\tau} H_T \middle| \mathcal{G}_t \right] &= H_t \widehat{E} \left[\frac{Z_\tau}{B_\tau} \middle| \mathcal{F}_t \vee \mathcal{H}_T \right] \\ &\quad + m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s - \int_0^t D_s \mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau} \frac{Z_s}{B_s} \lambda_s \mathrm{d}s. \end{aligned} \tag{5.16}$$

To express the right-hand side of (5.16) as a stochastic integral with respect to m and $\hat{M}$, we need the following Lemma.

Lemma 5.2.4.

$$H_t \widehat{E} \left[\frac{Z_\tau}{B_\tau} \middle| \mathcal{F}_t \vee \mathcal{H}_T \right] = \int_0^t \frac{Z_s}{B_s} \mathrm{d}H_s = H_t \frac{Z_\tau}{B_\tau}, \quad \forall t \in [0, T]. \tag{5.17}$$

Proof. We recall the σ-algebra

$$\mathcal{F}_{\tau-} = \sigma\left(A \cap \{\tau > t\},\ A \in \mathcal{F}_t,\ 0 \le t \le T\right)$$

of the *events strictly prior* to τ. We note that since $\mathcal{F}_{\tau-} = \mathcal{G}_{\tau-}$ by Lemma 5.1.1 of [13] and the recovery process Z refers to a recovery payment in the interval $[0, T]$ only, then the following holds:

$$\widehat{E}\left[\frac{Z_\tau}{B_\tau}\bigg|\,\mathcal{F}_{\tau-}\right] = \widehat{E}\left[\frac{Z_\tau}{B_\tau}\bigg|\,\mathcal{G}_{\tau-}\right] = \widehat{E}\left[\frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau<\infty\}}\bigg|\,\mathcal{G}_{\tau-}\right]. \tag{5.18}$$

Moreover $\dfrac{Z_t}{B_t}$ is in particular a $\mathcal{G}_t$-predictable process and τ is a $\mathcal{G}_t$-stopping time. Therefore we can apply Theorem 88C page 141 of [18] and obtain

$$\widehat{E}\left[\frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau<\infty\}}\bigg|\,\mathcal{G}_{\tau-}\right] = \frac{Z_\tau}{B_\tau}. \tag{5.19}$$

Lemma 4.4 of [2] guarantees that

$$H_t\widehat{E}\left[\frac{Z_\tau}{B_\tau}\bigg|\,\mathcal{F}_{\tau-}\right] = H_t\widehat{E}\left[\frac{Z_\tau}{B_\tau}\bigg|\,\mathcal{F}_t \vee \mathcal{H}_T\right], \quad \forall t \in [0, T].$$

Hence, the equality (5.17) follows. □

Finally gathering the results, we obtain by Lemma 5.2.4

$$\begin{aligned}
&\widehat{E}\left[\frac{Z_\tau}{B_\tau}H_T\bigg|\,\mathcal{G}_t\right]\\
&= H_t\frac{Z_\tau}{B_\tau} + m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_0^t D_s\mathrm{d}\hat{M}_s - \int_0^{t\wedge\tau}\frac{Z_s}{B_s}\lambda_s\mathrm{d}s\\
&= m_0 + \int_0^t \frac{Z_s}{B_s}\mathrm{d}\hat{M}_s + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_0^t D_s\mathrm{d}\hat{M}_s\\
&= m_0 + \int_{]0,t\wedge\tau]} e^{\int_0^s \lambda_u \mathrm{d}u}\xi_s^m\mathrm{d}\hat{W}_s + \int_0^t\left(\frac{Z_s}{B_s} - D_s\right)\mathrm{d}\hat{M}_s,
\end{aligned} \tag{5.20}$$

where we have used the fact that the continuous $\mathcal{F}_t$-martingale m_t admits the following integral representation with respect to the Brownian motion $\hat{W}_t$

$$m_t = m_0 + \int_0^t \xi_s^m\mathrm{d}\hat{W}_s, \tag{5.21}$$

for some $\mathcal{F}_t$-predictable process ξ^m, such that $\widehat{E}\left[\int_0^t (\xi_s^m)^2 \mathrm{d}s\right] < +\infty$, $\forall t \in [0,T]$. Moreover, since all the integrability conditions are satisfied we have

$$\frac{Z_\tau}{B_\tau} H_T = m_0 + \int_{]0,\tau\wedge T]} e^{\int_0^s \lambda_u \mathrm{d}u} \xi_s^m \mathrm{d}\hat{W}_s + \int_0^T \left(\frac{Z_s}{B_s} - D_s\right) \mathrm{d}\hat{M}_s.$$

We conclude that the asserted formula holds, with the following processes:

$$\tilde{\xi}_t = e^{\int_0^t \lambda_s \mathrm{d}s} \mathbb{I}_{\{\tau \geq t\}} (\xi_t + \xi_t^m) \quad \text{and} \quad \tilde{\zeta}_t = e^{\int_0^t \lambda_s \mathrm{d}s} \widehat{E}\left[F \mid \mathcal{F}_t\right] + \frac{Z_t}{B_t} - D_t. \tag{5.22}$$

for every $t \in [\![0, \tau \wedge T]\!]$. □

We use now Proposition 5.2.3 to find the $\mathcal{G}$-plrm-strategy for H by computing the Galtchouk-Kunita-Watanabe decomposition of H under $\widehat{\mathbb{Q}}$. Theorem 1.3.16 and hypothesis (2.8) guarantee that this is indeed the FS-decomposition for H.

Proposition 5.2.5. *In the market model outlined in Section 2.2 and under the assumptions of Section 5.2, the FS decomposition for H defined in* (5.1) *is given by*

$$\begin{aligned} \hat{V}_t = \widehat{E}\left[F\right] + m_0 + \int_0^t \mathbb{I}_{\{\tau \geq s\}} \left(\frac{e^{\int_0^s \lambda_u \mathrm{d}u}(\xi_s + \xi_s^m)}{\sigma_s X_s}\right) \mathrm{d}X_s \\ + \int_0^t \left(e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}\left[F \mid \mathcal{F}_s\right] + \frac{Z_s}{B_s} - D_s\right) \mathrm{d}\hat{M}_s, \end{aligned} \tag{5.23}$$

where m, ξ, ξ^m, F and D are given in respectively in (5.15), (5.9), (5.21), (5.8) *and* (5.14). *In particular we have that the $\mathcal{G}$-plrm-strategy $\varphi^{\mathcal{G}}$ is given by*

$$\xi_t^H = \mathbb{I}_{\{\tau \geq t\}} \frac{e^{\int_0^t \lambda_s \mathrm{d}s}(\xi_t + \xi_t^m)}{\sigma_t X_t}, \qquad \forall t \in [\![0, \tau \wedge T]\!] \tag{5.24}$$

and the minimal cost *is*

$$C_t^H = \widehat{E}\left[F\right] + m_0 + \int_0^t \left(\mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}\left[F \mid \mathcal{F}_s\right] + \frac{Z_s}{B_s} - D_s\right) \mathrm{d}\hat{M}_s, \tag{5.25}$$

$\forall t \in [\![0, \tau \wedge T]\!]$.

Proof. Since $\sigma_t > 0$ for every $t \in [0,T]$, by Proposition 5.2.3 we can rewrite decomposition (5.9) in terms of X_t:

$$
\begin{aligned}
&(1-H_T)\frac{\bar{X}}{B_T} \\
&\quad = \widehat{E}\left[F\right] + \int_0^T \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \xi_s \mathrm{d}\hat{W}_s - \int_{]0,T]} \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}\left[F \middle| \mathcal{F}_t\right] \mathrm{d}\hat{M}_s \\
&\quad = \widehat{E}\left[F\right] + \int_0^T \mathbb{I}_{\{\tau \geq s\}} \left(\frac{e^{\int_0^s \lambda_u \mathrm{d}u} \xi_s}{\sigma_s X_s} \right) \mathrm{d}X_t - \int_{]0,T]} \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}\left[F \middle| \mathcal{F}_t\right] \mathrm{d}\hat{M}_s.
\end{aligned}
$$

Analogously we have

$$
\begin{aligned}
\frac{Z_\tau}{B_\tau} H_T &= m_0 + \int_{]0,T]} \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \xi_s^m \mathrm{d}\hat{W}_s + \int_0^T \left(\frac{Z_s}{B_s} - D_s \right) \mathrm{d}\hat{M}_s \\
&= m_0 + \int_{]0,T]} \mathbb{I}_{\{\tau \geq s\}} \left(\frac{e^{\int_0^s \lambda_u \mathrm{d}u} \xi_s^m}{\sigma_s X_s} \right) \mathrm{d}X_s + \int_0^T \left(\frac{Z_s}{B_s} - D_s \right) \mathrm{d}\hat{M}_s.
\end{aligned}
$$

Then hypothesis (2.8) and Theorem 1.3.16 guarantee that (5.23) gives the FS decomposition of H. □

5.2.2 Local risk-minimization with $\mathcal{F}_t$-strategies

We remark that we have assumed that replication refers to the behavior of the discounted value process on the random interval $[\![0, \tau \wedge T]\!]$ only. The following Lemma is essential to introduce the problem of local risk-minimization with $\mathcal{F}_t$-strategies.

Lemma 5.2.6. *Let $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, $t \in [0,T]$ and F be the process defined in* (2.1)*. Then for any $\mathcal{G}_t$-predictable process $\tilde{\phi}_t$ there exists a $\mathcal{F}_t$-predictable process ϕ_t such that*

$$
\mathbb{I}_{\{\tau \geq t\}} \phi_t = \mathbb{I}_{\{\tau \geq t\}} \tilde{\phi}_t, \qquad t \in [0,T]. \tag{5.26}
$$

If in addition, the inequality $F_t = \mathbb{Q}(\tau \leq t | \mathcal{F}_t) < 1$ holds for every $t \in [0,T]$, then the process ϕ_t satisfying (5.26) *is unique.*

Proof. See [17], page 186, for the proof. □

By Lemma 5.2.6 we obtain that there exists a $\mathcal{F}_t$-predictable process $\tilde{X}_t$ such that

$$\tilde{X}_t \mathbb{I}_{\{\tau \geq t\}} = X_t \mathbb{I}_{\{\tau \geq t\}}, \qquad t \in [0, T].$$

Following [10] and [12] we refer to $\tilde{X}_t$ as the *pre-default value* of X_t. In practice, the agent observes the pre-default (discounted) value $\tilde{X}$ and hedges by using $\tilde{X}$ until the default happens. Hence it is sufficient to consider the prices of primary non-defaultable assets stopped at $\tau \wedge T$ in order to hedge defaultable claims of the form $(\bar{X}, Z, \tau)$, following the approach of [10] and [12]. In addition, Lemma 5.2.6 allows us to assume in the dynamics of X that the processes μ and σ are $\mathcal{F}$-predictable. This also justifies that in Section 5.2 we are already supposing the promised contingent claim $\bar{X}$ to be $\mathcal{F}_T$-measurable. If it wouldn't be the case, by Lemma 5.2.6 we can always replace $\bar{X}$ by its pre-default value, since $\bar{X}$ appears multiplied by $(1 - H_T)$ in the definition (5.1) of the defaultable claim H.
We denote by $\Theta_s^{\mathcal{F}}$ the space of $\mathcal{F}$-predictable processes ξ on Ω such that

$$E\left[\int_0^T (\xi_s \sigma_s X_s)^2 \mathrm{d}s\right] + E\left[\left(\int_0^T |\xi_s(\mu_s - r_s) X_s| \mathrm{d}s\right)^2\right] < \infty. \tag{5.27}$$

We observe that there not exist $\mathcal{F}_t$-pseudo-locally risk-minimizing strategies. In fact, finding a $\mathcal{F}_t$-pseudo-locally risk-minimizing strategy $\phi^{\mathcal{F}} = (\xi, \eta)$ is equivalent to find a pair a processes (ξ, C) such that:

- $\xi_t \in \Theta_s^{\mathcal{F}}$;

- the cost process C_t is a $\mathcal{F}_t$-martingale strongly orthogonal to the martingale part of X_t,

with

$$V_t(\phi^{\mathcal{F}}) = \int_0^t \xi_s \mathrm{d}X_s + C_t(\phi^{\mathcal{F}}), \qquad t \in [\![0, \tau \wedge T]\!]$$

and $V_{\tau \wedge T}(\phi^{\mathcal{F}}) = H$. Clearly, since H is a $\mathcal{G}_T$-measurable random variable, it may be not replicable by the $\mathcal{F}_T$-measurable random variable V_T. In fact we cannot hedge against the occurring of a default by using only the information contained in the pre-default asset prices. This is one of the differences with respect to the mean-variance hedging, where the optimal $\mathcal{F}_t$-strategy is given by the replicating strategy for $E\left[H|\,\mathcal{F}_t\right]$, (if it exists). See [7] for further details.

However, one can think that the agent invests in the risky asset according to the information provided by the asset behavior before default and adjusts the portfolio value (by adding or spending money, i.e. modifying the cost), depending on the occurrence or not of the default. Then it may be reasonable to give the following definition.

Definition 5.2.7. *Let* $H = \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau\le T\}} \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$. *A pair* $\varphi^{\mathcal{F}} = (\xi, C)$ *of stochastic processes is said a* $\mathcal{F}$-pseudo-locally risk-minimizing strategy *(in short* $\mathcal{F}$-plrm-strategy*) if*

1. $\xi_t \in \Theta_s^{\mathcal{F}}$;

2. C_t *is* $\mathcal{G}_t$*-martingale strongly orthogonal to the martingale part of* X_t*;*

3. *The discounted value process* $V_t(\varphi^{\mathcal{F}}) = \xi_t X_t + \eta_t$ *is such that*

$$V_t(\varphi^{\mathcal{F}}) = \int_0^t \xi_s \mathrm{d}X_s^\tau + C_t(\varphi^{\mathcal{F}}), \tag{5.28}$$

where $V_{\tau\wedge T}(\varphi^{\mathcal{F}}) = H$*, i.e.*

$$V_T(\varphi^{\mathcal{F}}) = \frac{\bar{X}}{B_\tau} \text{ if } \tau > T, \qquad V_\tau(\varphi^{\mathcal{F}}) = \frac{Z_\tau}{B_\tau} \text{ if } \tau \le T. \tag{5.29}$$

Clearly the component η *invested in the money market account, is given by*

$$\eta_t = V_t(\varphi^{\mathcal{F}}) - \xi_t\tilde{X}_t = C_t(\varphi^{\mathcal{F}}), \qquad t \in [\![0, \tau\wedge T]\!].$$

The key result to find a $\mathcal{F}$-plrm strategy for H is given by the following Lemma.

Lemma 5.2.8. *Given a* $\mathcal{G}$*-predictable process* ϕ *such that* $\widehat{E}\left[\int_0^T \phi_s^2 \mathrm{d}\langle X\rangle_s\right] < \infty$*, let* $\tilde{\phi}$ *be the* $\mathcal{F}$*-predictable process such that* $\mathbb{I}_{\{\tau\ge t\}}\phi_t = \mathbb{I}_{\{\tau\ge t\}}\tilde{\phi}_t$*. Then for every* $t \le T$

$$\int_0^t \tilde{\phi}_s \mathrm{d}X_s^\tau = \int_0^t \phi_s \mathrm{d}X_s^\tau, \qquad \forall t \in [0, T].$$

Proof. Since X is a continuous martingale under $\widehat{\mathbb{Q}}$ and ϕ is square-integrable with respect to X, we have that for $t \le T$

$$\int_0^t \phi_s \mathrm{d}X_s^\tau = \int_0^\tau \phi_s \mathrm{d}X_s =$$
$$\int_0^t \mathbb{I}_{\{s \le \tau\}} \phi_s \mathrm{d}X_s = \int_0^t \mathbb{I}_{\{s \le \tau\}} \tilde{\phi}_s \mathrm{d}X_s = \int_0^t \tilde{\phi}_s \mathrm{d}X_s^\tau.$$

We only need to check that the integral $\int_0^t \tilde{\phi}_s \mathrm{d}X_s^\tau$ exists and it is well-defined if the integral $\int_0^t \phi_s \mathrm{d}X_s^\tau$ exists and it is well-defined. This is clear since if $\widehat{E}\left[\int_0^T \phi_s^2 \mathrm{d}\langle X^\tau \rangle_s\right] < \infty$, we have

$$\begin{aligned} \infty > \widehat{E}\left[\int_0^T \phi_s^2 \mathrm{d}\langle X^\tau \rangle_s\right] &= \widehat{E}\left[\left(\int_0^T \phi_s \mathrm{d}X_s^\tau\right)^2\right] \\ &= \widehat{E}\left[\left(\int_0^T \phi_s \mathbb{I}_{\{\tau \ge s\}} \mathrm{d}X_s\right)^2\right] \\ &= \widehat{E}\left[\int_0^T \tilde{\phi}_s^2 \mathrm{d}\langle X^\tau \rangle_s\right], \end{aligned}$$

since $\mathbb{I}_{\{\tau \ge t\}} \phi_t = \mathbb{I}_{\{\tau \ge t\}} \tilde{\phi}_t$ by hypothesis. □

Proposition 5.2.9. *In the market model outlined in Section 2.2, under the assumptions of Section 5.2, the FS decomposition for H defined in* (5.1) *is given by*

$$\begin{aligned} \hat{V}_t = \widehat{E}\left[F\right] + m_0 + \int_0^t \hat{\xi}_s(\xi_s + \xi_s^m) \mathrm{d}X_s^\tau \\ + \int_0^t \left(e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}\left[F \mid \mathcal{F}_s\right] + \frac{Z_s}{B_s} - D_s\right) \mathrm{d}M_s, \end{aligned} \tag{5.30}$$

where m, ξ, ξ^m, F and D are introduced respectively in (5.15), (5.9), (5.21), (5.8) *and* (5.14) *and the process $\mathcal{F}$-predictable process $\hat{\xi}$ is given by*

$$\hat{\xi}_t = \frac{e^{\int_0^t \lambda_s \mathrm{d}s}}{\tilde{\sigma}_t \tilde{X}_t} \qquad \forall t \in [\![0, \tau \wedge T]\!],$$

where $\tilde{\sigma}$ and $\tilde{X}$ are the pre-default values of σ and X respectively. In particular we have that the pre-$\mathcal{F}$-plrm-strategy is given by

$$\xi_t^H = \hat{\xi}_t(\xi_t + \xi_t^m), \qquad \forall t \in [\![0, \tau \wedge T]\!] \tag{5.31}$$

and the minimal cost *is*

$$C_t^H = \widehat{E}\left[F\right] + m_0 + \int_0^t \left(e^{\int_0^s \lambda_u \mathrm{d}u} \widehat{E}\left[F \middle| \mathcal{F}_s\right] + \frac{Z_s}{B_s} - D_s \right) \mathrm{d}\hat{M}_s, \tag{5.32}$$

$\forall t \in [\![0, \tau \wedge T]\!]$.

Proof. It follows by Proposition 5.2.5 and Lemma 5.2.8. □

5.3 Example

In this example, we wish to find the $\mathcal{G}$-plrm strategy for a Corporate bond that we hedge by using a Treasury bond. This example is similar to one computed in the previous chapter, but now we suppose to have a recovery at default and we work under a different model for r_t. To simplify the computations, we assume that hypothesis (2.8) is satisfied and we *work out the example directly under* $\widehat{\mathbb{Q}}$.
We fix $T > 0$ and assume that the discounted price process X_t is $\mathcal{F}_t$-adapted. Here we assume that the process X represents the discounted price of a Treasury bond that expires at time T with the following representation

$$X_t = \widehat{E}\left[e^{-\int_0^T r_s \mathrm{d}s} \middle| \mathcal{F}_t \right], \tag{5.33}$$

and that the discounted recovery process $\dfrac{Z}{B}$ is given by

$$\frac{Z_t}{B_t} = \delta X_t, \quad t \in [0, T],$$

where δ is a constant belonging to the interval $]0, 1[$. As we said, we put $\bar{X} = 1$ and the discounted value of H can be represented as follows:

$$H = \frac{1}{B_T}(1 - H_T) + \delta X_\tau H_T. \tag{5.34}$$

We make also the following hypotheses:

- r is an affine process, in particular it satisfies the following equation under $\widehat{\mathbb{Q}}$:

$$\begin{cases} \mathrm{d}r_t &= (b + \beta r_t)\mathrm{d}t + \alpha\sqrt{r_t}\mathrm{d}\hat{W}_t \\ r_0 &= 0, \end{cases} \tag{5.35}$$

where $b, \alpha \in \mathbb{R}^+$ and β is arbitrary. This is the Cox-Ingersoll-Ross model and we know it has a unique strong solution $r \geq 0$ for every $r_0 \geq 0$. See [20] for further details.

- The $\mathcal{F}$-intensity λ is supposed to be a positive *deterministic* function.

Under the equivalent martingale probability measure $\widehat{\mathbb{Q}}$, the discounted optimal portfolio value $\hat{V}_t$ of the defaultable claim H given in (5.34) at time t, is given by:

$$\begin{aligned}\hat{V}_t = \widehat{E}\left[H\middle|\,\mathcal{G}_t\right] &= \widehat{E}\left[\frac{1}{B_T}(1-H_T)+\delta X_\tau H_T\middle|\,\mathcal{G}_t\right]\\ &= e^{-\int_0^T \lambda(s)\mathrm{d}s}\widehat{E}\left[\frac{1}{B_T}\right]+m_0+\int_0^t \mathbb{I}_{\{\tau\geq s\}}\left(\frac{e^{\int_0^s \lambda(u)\mathrm{d}u}(\xi_s+\xi_s^m)}{\sigma_s X_s}\right)\mathrm{d}X_s\\ &\quad+\int_0^t\left(e^{-\int_s^T \lambda(u)\mathrm{d}u}\widehat{E}\left[\frac{1}{B_T}\middle|\,\mathcal{F}_s\right]+\delta X_s - D_s\right)\mathrm{d}\hat{M}_s, \qquad (5.36)\end{aligned}$$

where m, ξ, ξ^m, and D are given in respectively in (5.15), (5.9), (5.21) and (5.14).

We compute now the terms appearing in decomposition (5.36). First, we note that in this case the $\mathcal{F}_T$-random variable F introduced in (5.8) is given by

$$F = e^{-\int_0^T \lambda(u)\mathrm{d}u}\frac{1}{B_T}.$$

Hence

$$\xi_t = e^{-\int_0^T \lambda(u)\mathrm{d}u}\xi_t^{\bar{X}},$$

where $\xi_t^{\bar{X}}$ is the $\mathcal{F}_t$-predictable process appearing in the integral representation of $\frac{1}{B_T}$ with respect to the Brownian motion $\hat{W}_t$:

$$\frac{1}{B_T} = \widehat{E}\left[\frac{1}{B_T}\right]+\int_0^T \xi_t^{\bar{X}}\mathrm{d}\hat{W}_t. \qquad (5.37)$$

By following Section 4.4 of the previous chapter, since r is an affine process whose dynamics is given in (5.35), we have

$$\begin{aligned}\widehat{E}\left[\frac{1}{B_T}\middle|\,\mathcal{F}_t\right] &= e^{-\int_0^t r_s\mathrm{d}s}e^{-A(t,T)-B(t,T)r_t}\\ &= e^{-A(0,T)}-\int_0^t e^{-A(s,T)-B(s,T)r_s}\frac{B(s,T)}{B_s}\sqrt{r_s}\mathrm{d}\hat{W}_s, \qquad (5.38)\end{aligned}$$

where the functions $A(t,T)$, $B(t,T)$ satisfy the following equations:

$$\partial_t B(t,T) = \frac{\alpha^2}{2} B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0 \tag{5.39}$$

$$\partial_t A(t,T) = -b B(t,T), \quad A(T,T) = 0, \tag{5.40}$$

that admit explicit solutions (see for instance [21]). Hence we can rewrite decomposition (5.36) as follows:

$$\begin{aligned}
&\hat{V}_t \\
&= e^{-\int_0^T \lambda(u)\mathrm{d}u - A(0,T)} + m_0 \\
&+ \int_0^t \mathbb{I}_{\{\tau \geq s\}} \frac{1}{\sigma_s X_s} \left(-e^{-\int_s^T \lambda(u)\mathrm{d}u - A(s,T) - B(s,T) r_s} \frac{B(s,T)}{B_s} \sqrt{r_s} + e^{\int_0^s \lambda(u)\mathrm{d}u} \xi_s^m \right) \mathrm{d}X_s \\
&+ \int_0^t \left[X_s \left(e^{-\int_s^T \lambda(u)\mathrm{d}u} + \delta \right) - D_s \right] \mathrm{d}\hat{M}_s.
\end{aligned} \tag{5.41}$$

We focus now on the process D. By applying the Fubini-Tonelli Theorem, we have

$$\begin{aligned}
D_t &= e^{\int_0^t \lambda(s)\mathrm{d}s} \tilde{E} \left[\int_t^T \delta X_s e^{-\int_0^s \lambda(u)\mathrm{d}u} \lambda(s)\mathrm{d}s \middle| \mathcal{F}_t \right] \\
&= \int_t^T e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda(s) \delta \tilde{E} \left[X_s \middle| \mathcal{F}_t \right] \mathrm{d}s \\
&= \int_t^T e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda(s) \delta X_t \mathrm{d}s \\
&= \delta X_t \int_t^T e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda(s)\mathrm{d}s \\
&= \delta X_t \left(1 - e^{-\int_t^T \lambda(s)\mathrm{d}s} \right),
\end{aligned}$$

since λ is a deterministic function. We can modify the integral with respect to M in decomposition (5.41), as follows:

$$\begin{aligned}
&\hat{V}_t \\
&= e^{-\int_0^T \lambda(u)\mathrm{d}u - A(0,T)} + m_0 \\
&+ \int_0^t \mathbb{I}_{\{\tau \geq s\}} \frac{1}{\sigma_s X_s} \left(-e^{-\int_s^T \lambda(u)\mathrm{d}u - A(s,T) - B(s,T) r_s} \frac{B(s,T)}{B_s} \sqrt{r_s} + e^{\int_0^s \lambda(u)\mathrm{d}u} \xi_s^m \right) \mathrm{d}X_s \\
&+ (\delta + 1) \int_0^t X_s e^{-\int_s^T \lambda(u)\mathrm{d}u} \mathrm{d}\hat{M}_s.
\end{aligned} \tag{5.42}$$

It only remains to compute the $\mathcal{F}_t$-martingale m_t introduced in (5.15) and in particular its integral representation with respect to the Brownian motion $\hat{W}_t$. Since

$$D_t = e^{\int_0^t \lambda(s)\mathrm{d}s} m_t - e^{\int_0^t \lambda(s)\mathrm{d}s} \int_0^t \delta X_s e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}s,$$

we can rewrite m_t in terms of D_t:

$$\begin{aligned}
& m_t \\
&= e^{-\int_0^t \lambda(s)\mathrm{d}s} D_t + \delta \int_0^t X_s e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}s \\
&= \delta \left[X_t \left(e^{-\int_0^t \lambda(s)\mathrm{d}s} - e^{-\int_0^T \lambda(s)\mathrm{d}s} \right) + \int_0^t X_s e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}s \right] \\
&= \delta \Bigg[\Bigg(e^{-A(0,T)} - \int_0^t \underbrace{e^{-A(s,T)-B(s,T)r_s} \frac{B(s,T)}{B_s} \sqrt{r_s}}_{\varphi_s} \mathrm{d}\hat{W}_s \Bigg) \left(e^{-\int_0^t \lambda(s)\mathrm{d}s} - e^{-\int_0^T \lambda(s)\mathrm{d}s} \right) \\
&\qquad + \int_0^t \left(e^{-A(0,T)} - \int_0^s \varphi_u \mathrm{d}\hat{W}_u \right) e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}s \Bigg] \\
&= \delta \Bigg[e^{-A(0,T)} \left(1 - e^{-\int_0^T \lambda(s)\mathrm{d}s} \right) - \left(e^{-\int_0^t \lambda(s)\mathrm{d}s} - e^{-\int_0^T \lambda(s)\mathrm{d}s} \right) \int_0^t \varphi_s \mathrm{d}\hat{W}_s \\
&\qquad - \int_0^t \int_0^s \varphi_u e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s) \mathrm{d}\hat{W}_u \mathrm{d}s \Bigg],
\end{aligned}$$

where φ_t is a $\mathcal{F}_t$-predictable process such that

$$\widehat{E} \left[\int_0^T (e^{-A(s,T)-B(s,T)r_s} \frac{B(s,T)}{B_s} \sqrt{r_s})^2 \mathrm{d}s \right] < +\infty.$$

Moreover we note that

$$\widehat{E} \left[\int_0^T \int_0^T \varphi_u^2 e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}u\mathrm{d}s \right] = \int_0^T \int_0^T \widehat{E}\left[\varphi_u^2\right] e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}u\mathrm{d}s < \infty,$$

since $\widehat{E}\left[\varphi_T\right]$ is bounded because X takes values in $(0,1)$ (see (5.38)). Since all the integrability conditions are satisfied, by applying the Fubini's Theorem for stochastic integrals, we have

$$\begin{aligned}
-\int_0^t \int_0^t e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s) \varphi_u \mathbb{I}_{\{u \le s\}} \mathrm{d}\hat{W}_u \mathrm{d}s &= \int_0^t \left(-\int_u^t e^{-\int_0^s \lambda(v)\mathrm{d}v} \lambda(s)\mathrm{d}s \right) \varphi_u \mathrm{d}\hat{W}_u \\
&= \int_0^t \left(e^{-\int_0^t \lambda(v)\mathrm{d}v} - e^{-\int_0^u \lambda(v)\mathrm{d}v} \right) \varphi_u \mathrm{d}\hat{W}_u.
\end{aligned}$$

In particular

$$m_0 = \delta e^{-A(0,T)} \left(1 - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right)$$

and

$$m_t = \delta \left[e^{-A(0,T)} \left(1 - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right) + \int_0^t \left(e^{-\int_0^T \lambda(v)\mathrm{d}v} - e^{-\int_0^u \lambda(v)\mathrm{d}v}\right) \varphi_u \mathrm{d}\hat{W}_u \right] \tag{5.43}$$

Finally, gathering the results we obtain

$$\begin{aligned} &\hat{V}_t \\ &= e^{-A(0,T)} \left[e^{-\int_0^T \lambda(u)\mathrm{d}u} + \delta \left(1 - e^{-\int_0^T \lambda(u)\mathrm{d}u}\right)\right] \\ &\quad - \int_0^t \mathbb{I}_{\{\tau \geq s\}} \frac{1}{\sigma_s X_s} \varphi_s \mathrm{d}X_s + (\delta + 1) \int_0^t X_s e^{-\int_s^T \lambda(u)\mathrm{d}u} \mathrm{d}\hat{M}_s. \end{aligned} \tag{5.44}$$

where the function $A(t,T)$ and $B(t,T)$ are provided by (5.40) and (5.39) respectively. In particular the $\mathcal{G}$-plrm-strategy is given by

$$\xi_t^H = -\frac{1}{\sigma_t X_t} \varphi_t \tag{5.45}$$

and the minimal cost is

$$\begin{aligned} C_t^H &= e^{-\int_0^T \lambda(u)\mathrm{d}u - A(0,T)} + \delta e^{-A(0,T)} \left(1 - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right) \\ &\quad + (\delta + 1) \int_0^t X_s e^{-\int_s^T \lambda(u)\mathrm{d}u} \mathrm{d}\hat{M}_s \end{aligned} \tag{5.46}$$

for every $t \in [\![0, \tau \wedge T]\!]$.

Remark 5.3.1. *The pair* (ξ^H, C^H) *also provides a* $\mathcal{F}$*-plrm strategy for* H*. In fact the process* ξ^H *belongs to* $\Theta_s^{\mathcal{F}}$*, since we have assumed that the drift and volatility in* (2.5) *are* $\mathcal{F}$*-predictable processes.*

Appendix A

The predictable projection

We recall the definition of predictable projection of a measurable process endowed with some suitable integrability properties and the main properties. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Theorem A.0.2 (Predictable Projection)**.** *Let X be a measurable process either positive or bounded. There exists a predictable process Y such that*

$$E\left[X_\tau \mathbb{I}_{\{\tau<\infty\}} | \mathcal{F}_{\tau-}\right] = Y_\tau \mathbb{I}_{\{\tau<\infty\}} \quad a.s. \tag{A.1}$$

for every predictable stopping time τ.

The process Y is called the *predictable projection* of X and it is denoted by X^p.

Proof. See [19] or [30] for the proof. □

The predictable projection has the following fundamental properties:

1. In the discrete case, where the space Ω is endowed with a filtration $(\mathcal{F}_n)_{n\geq 0}$, the predictable projection of a process $(X_n)_{n\geq 0}$ is the process

   $$Y_n = E\left[X_n | \mathcal{F}_{n-1}\right], \ (n \geq 0),$$

   with the convention $\mathcal{F}_{-1} = \mathcal{F}_0$, if $\mathcal{F}_{-1}$ is not otherwise specified.

2. To prove (A.1) it is sufficient to prove that, without conditioning,

$$E\left[X_\tau \mathbb{I}_{\{\tau<\infty\}}\right] = E\left[Y_\tau \mathbb{I}_{\{\tau<\infty\}}\right], \text{ for every predictable stopping time } \tau. \tag{A.2}$$

3. If X is measurable and H is a bounded predictable process, then

$$(HX)^p = HX^p,$$

i.e. the predictable projection of an integrable and predictable process is the process itself.

4. It is possible to give a definition of predictable projection also for measurable processes which are neither positive or bounded. We say that the predictable projection of a measurable process X exists if the predictable projection of the positive measurable process $|X|$ is indistinguishable[1] from a finite process and then we set

$$Y = X^p = (X^+)^p - (X^-)^p.$$

To check that $|X|^p$ is indistinguishable from a finite process, it is sufficient to verify that

$$[X_\tau \mathbb{I}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}] < \infty, \text{ a.s.}$$

for every predictable stopping time τ, i.e. the generalized conditional expectation[2] $E[X_\tau \mathbb{I}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}]$ exists, and then

$$Y_\tau \mathbb{I}_{\{\tau<\infty\}} = E\left[X_\tau \mathbb{I}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}\right] \text{ a.s.}$$

[1] Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X and Y are *indistinguishable* if for almost all $\omega \in \Omega$

$$X_t(\omega) = Y_t(\omega) \text{ for all } t.$$

[2] Given an arbitrary random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$, we say that X has *generalized conditional expectation* if $E[X^+|\mathcal{F}_t]$ and $E[X^-|\mathcal{F}_t]$ are finite a.s. $\forall t \geq 0$, and then we set

$$E[X|\mathcal{F}_t] = E[X^+|\mathcal{F}_t] - [X^-|\mathcal{F}_t].$$

The definition of Y is formally the same as (A.1) and characterizes Y uniquely.

5. If H is an integrable random variable and $(H_t)_{t\geq 0}$ denotes a càdlàg version of the martingale $E[H|\mathcal{F}_t]$, the predictable projection of the process $X_t(\omega) = H(\omega)$, which is constant through time, is the process $(H_{t-})_{t\geq 0}$. More generally, if H is a local martingale and τ a predictable stopping time, the conditional expectation $E[H_\tau|\mathcal{F}_{\tau-}]$ exists and takes the value $H_{\tau-}$. According to (4), this means that H has a predictable projection, which is the process $(H_{t-})_{t\geq 0}$.

Definition A.0.3. *An* increasing *process is any process $(A_t)_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, whose paths are positive, increasing, finite, right-continuous on* $[0,\infty[$.

The differences of increasing processes are called *processes of finite variation.* The following Theorem which involves increasing processes, provides the characteristic properties of predictable projections.

Theorem A.0.4. *Let X be a positive measurable process and Y its predictable projection. Let A be an increasing predictable process. Then*

$$E\left[\int_{[0,\infty[} X_s \mathrm{d}A_s\right] = E\left[\int_{[0,\infty[} Y_s \mathrm{d}A_s\right]. \tag{A.3}$$

Proof. See Theorem 57, page 122 of [19] for the proof. □

We note that if we take $A_t = \mathbb{I}_{\{\tau\leq t\}}$, where τ is a predictable stopping time, formula (A.3) reduces to $E\left[X_\tau \mathbb{I}_{\{\tau<\infty\}}\right] = E\left[Y_\tau \mathbb{I}_{\{\tau<\infty\}}\right]$ and this property is equivalent to the definition of the predictable projection.
Moreover we note the analogous formula on an interval $[\tau,\infty[$

$$E\left[\int_{[T,\infty[} X_s \mathrm{d}A_s \middle| \mathcal{F}_{\tau-}\right] = E\left[\int_{[T,\infty[} Y_s \mathrm{d}A_s \middle| \mathcal{F}_{\tau-}\right],$$

where in particular τ and A are predictable.

We conclude with an intuitive interpretation of this projection. The σ-field $\mathcal{F}_t$ describes the entire information available at time t. If we consider a measurable process H which is not adapted, it is not possible to capture X behavior but we can estimate it. Theorem A.0.2 says that we can estimate the whole path of X and the computation of the evaluation at time t should not depend on what the process is doing at that time, but only on its behavior strictly before t.

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