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SPECTRAL ANALYSIS OF A PARAMETER-DEPENDENT STURM-LIOUVILLE PROBLEM

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Introduction

This work arises from the study by A. Parmeggiani and M. Wakayama of the differential operator Q, introduced in [17], and of its spectral properties. Q is defined, as an unbounded operator acting on $L^2(\mathbb{R}, \mathbb{C}^2) = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, by

$$Q(x, D_x) = A\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) + J\left(x\partial_x + \frac{1}{2}\right) =$$

$$= \begin{bmatrix} \alpha\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) & -\left(x\partial_x + \frac{1}{2}\right) \\ x\partial_x + \frac{1}{2} & \beta\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) \end{bmatrix}, \qquad x \in \mathbb{R}, \tag{1}$$

with $D_x = -i\partial_x$, $\partial_x = \frac{d}{dx}$ and α, β real and positive parameters such that $\alpha\beta > 1$, and with

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In particular the analysis of the spectral zeta function of Q (see [7]; see also [19]) is rather interesting, since this function is a "deformation" of the Riemann zeta function, which we will denote by $\zeta(s)$. More precisely, let

$$\zeta_Q(s) = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}$$

be the spectral zeta function associated with Q, where

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to +\infty$$

is the sequence of the eigenvalues of Q, written taking into account their multeplicity. Then we have that $\zeta_Q(s)$ can be extended meromorphically to the whole complex plane $\mathbb{C}$. Moreover this extension has one simple pole in $s = 1$ and it vanishes on all non-positive, even, integers (the so called "trivial zeros"), just as $\zeta(s)$ does; these are almost surprising properties.

The deep correspondence between $\zeta_Q(s)$ and $\zeta(s)$ appears also if we notice that $\zeta(s)$ is connected with the spectral zeta of the harmonic oscillator

$$H = -\frac{\partial_x^2}{2} + \frac{x^2}{2}$$

through the relation $\zeta_H(s) = (2^s - 1)\zeta(s)$. In fact, if we put $\alpha = \beta$ in (1) then Q is unitarily equivalent exactly to the scalar harmonic oscillator $\sqrt{\alpha^2 - 1}\ H\ I_{2\times 2}$, which has the eigenvalues given by $(n + \frac{1}{2})\sqrt{\alpha^2 - 1}$, $(n = 0, 1, 2, \dots)$, all with multeplicity 2, and furthermore

$$\zeta_Q\big|_{\alpha=\beta>1}(s) = 2\frac{(2^s - 1)}{(\alpha^2 - 1)^{s/2}}\zeta(s), \quad s \in \mathbb{C}.$$

Whence $\zeta_Q(s)$ is a remarkable deformation (depending on α, β) of $\zeta(s)$. From here the natural problem of studying another possible deformation of $\zeta(s)$ arises, that is to say the spectral ζ-function of the harmonic oscillator, defined on the interval $[-L, L] \subset \mathbb{R}$ with zero Dirichlet conditions, when $L \to +\infty$. The eigenvalue problem of the harmonic oscillator defined on an interval of the real line and with Dirichlet conditions on the boundary has been studied by several authors (see, e.g. [4], [21] and [23]), but it presents relevant difficulties in computations. For this reason we will study here the spectrum of a slightly simplified operator, that is

$$P_L : D(P_L) \longrightarrow L^2(-\pi L, \pi L), \quad (P_L f)(x) = -\frac{1}{2}f''(x) + V_L(x)f(x),$$

with

$$V_L(x) = \frac{L^2}{2}\sin^2\left(\frac{x}{2L}\right)$$

and

$$D(P_L) = H_0^1(-\pi L, \pi L) \cap H^2(-\pi L, \pi L) \subset L^2(-\pi L, \pi L).$$

The study of P_L is related to the aforementioned problems, since V_L tends, in the sense of tempered distributions, to the harmonic potential.

The spectral zeta is more regular than the spectrum, for varying L, because it is defined by means of a trace: $\zeta_{P_L}(s) = \operatorname{Tr} P_L^{-s}$, for sufficiently large s. However, the aim of this work is to study in the first place the eigenvalues, in order to control them as much as possible explicitly. In particular we will study the behaviour of the eigenvalues of P_L when $L \to +\infty$.

Notice that the eigenvalue equation of P_L

$$P_L f = \mu f, \qquad 0 \neq f \in D(P_L),\ \mu \in \mathbb{C},$$

is similar to the Mathieu equation (see, e.g. [13]) and therefore it presents similar difficulties.

The exposition of the aforementioned topics is organized as follows.

In the first chapter we set the spectral problem for P_L on the interval $(-\pi L, \pi L)$. Then we normalize the problem, by removing from $(-\pi L, \pi L)$ the dependence on the parameter L, so that the extremes of the interval are fixed. With this procedure we reduce ourselves to the study of the *semiclassical* problem

$$P(h)f = \lambda f,$$

with

$$P(h) : H_0^1(-\pi, \pi) \cap H^2(-\pi, \pi) \longrightarrow L^2(-\pi, \pi),$$

$$P(h)f(x) = -f''(x) + \frac{1}{h^2}\sin^2\left(\frac{x}{2}\right)f(x) = \lambda f$$

and with $\lambda = \dfrac{2\mu}{h}$, and $h = \dfrac{1}{L^2}$. Then, for the sake of completeness, we recall the selfadjointness of $P(h)$, the discreteness of its spectrum and we state some basic properties of the eigenvalues.

In the second chapter we recall some classical results on Perturbation Theory (see [9]) which we apply to the problem, setting h^{-2} as the perturbative parameter. In this way we obtain, for h fixed, power series expansions, in h^{-2}, of the eigenvalues and of the associated eigenfunctions. Although our attention is to the case $h \to 0^+$, perturbation theory allows us to have information in the intermediate range, where h can be small but cannot tend to 0. Moreover this study provides an orthonormal basis of $L^2(I)$ formed entirely by eigenfunctions of $P(+\infty) = -d^2/dx^2$ in $H_0^1(-\pi,\pi) \cap H^2(-\pi,\pi)$, which turns out to be the natural Fourier basis for the expansion of the eigenfunctions of $P(h)$.

In the third chapter we expand the eigenfunctions with respect to the orthonormal basis of $L^2(-\pi,\pi)$, obtained in the case $h = +\infty$ of the previous chapter, getting a three-term recurrence relation for the Fourier coefficients. From here we obtain an equation, which involves a particular continued fraction derived in a natural way from the recurrence relation; this condition characterizes the eigenvalues of $P(h)$ (as zeros of the "determinant" of an infinite size tridiagonal matrix).

Afterwards we associate, to each eigenvalue, two sequences converging, one from above and the other from below, to the same eigenvalue. We obtain some of these results following the ideas used in [13] for studying the eigenvalues of the

Mathieu equation; we use in particular the theory of polynomials "with interlaced zeros" (which are essentially orthogonal polynomials).

Moreover we give estimates for large eigenvalues (large depending on h^{-1}), that is we study the clustering of the spectrum for high energies.

In the last chapter we study the asymptotics, as $h \to 0^+$, of the lowest eigenvalue, ϖ, of P_L as a function of h. In particular we prove the existence of the limit $\lim_{h\to 0^+} \varpi(h)$. This result could be obtained as a consequence of a theorem by Helffer and Sjöstrand, but it is proved here by following a different approach, using the continued fractions.

To conclude, this analysis is intended to give the spectral results on which we will base our (future) study of the spectral zeta function of the operator P_L and of its relations to the Riemann zeta function. Furthermore the theory of polynomials "with interlaced zeros", as used in the third chapter of this thesis, will be used in the continuation of this work to analyse in detail the (difficult) sets Σ_0 and Σ_∞ which form the spectrum of the system Q (see [15], [16], [17], [14] and [19]).

Contents

Chapter 1

Basic features of the spectrum of the operator P

1.1 P is selfadjoint

In this section the selfadjointness of the operator P will be proved by decomposing P into simpler operators and by using classical theorems.

Let $L > 0$ and let P_L be the unbounded operator defined as follows:

$$P_L : D(P_L) \longrightarrow L^2(-\pi L, \pi L), \quad (P_L f)(x) = -\frac{1}{2} f''(x) + V_L(x) f(x),$$

with

$$V_L(x) = \frac{L^2}{2} \sin^2\left(\frac{x}{2L}\right)$$

and

$$D(P_L) = H_0^1(-\pi L, \pi L) \cap H^2(-\pi L, \pi L) \subset L^2(-\pi L, \pi L).$$

We will study the solutions of the eigenvalue problem related to P_L:

$$P_L(f) = \mu f, \qquad f \in D(P_L),\ \mu \in \mathbb{C}. \tag{1.1}$$

In particular we will analyse the eigenvalues' behaviour in the limit $L \to +\infty$. We normalize the problem in order to remove the parameter L from the interval $(-\pi L, \pi L)$.

Proposition 1.1.1. *Let $u \in H_0^1(-\pi L, \pi L) \cap H^2(-\pi L, \pi L)$ be a solution of the following equation:*

$$-u''(x) + L^2 \sin^2\left(\frac{x}{2L}\right) u(x) = 2\mu u(x). \tag{1.2}$$

Set $\psi(t) = \sqrt{L}\ u(Lt)$. We have that $\psi \in H_0^1(-\pi, \pi) \cap H^2(-\pi, \pi)$ and that ψ is a solution of the equation

$$-\psi''(t) + \frac{\sin^2\left(\frac{t}{2}\right)}{h^2}\ \psi(t) = \lambda\ \psi(t), \tag{1.3}$$

with $h = \dfrac{1}{L^2}$ and $\lambda = \dfrac{2\mu}{h}$.

Moreover if $\psi \in H_0^1(-\pi, \pi) \cap H^2(-\pi, \pi)$ satisfies equation (1.3) then, if we define $u(x) = \frac{1}{\sqrt{L}}\psi\left(\frac{x}{L}\right)$, it follows that $u \in H_0^1(-\pi L, \pi L) \cap H^2(-\pi L, \pi L)$ and u is a solution of (1.2).

Proof. Set

$$U_L : L^2(-\pi L, \pi L) \longrightarrow L^2(-\pi, \pi), \quad f(t) \longmapsto U_L(f)(t) = \sqrt{L} f(Lt).$$

Note that U_L is an isometry between $L^2(-\pi L, \pi L)$ and $L^2(-\pi, \pi)$. Indeed

$$\|U_L(f)\|^2_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} L|f(Lt)|^2 dt.$$

Changing variable in the integral we obtain:

$$\|U_L(f)\|^2_{L^2(-\pi,\pi)} = L\int_{-\pi L}^{\pi L} \frac{1}{L}\ |f(x)|^2 dx = \|f\|^2_{L^2(-\pi L,\pi L)},$$

as was to be proved. Hence $f \in H^1_0(-\pi L, \pi L) \cap H^2(-\pi L, \pi L)$ if and only if $U_L(f) \in H^1_0(-\pi, \pi) \cap H^2(-\pi, \pi)$.

Now let $u \in H^1_0(-\pi L, \pi L) \cap H^2(-\pi L, \pi L)$ be a solution of the equation

$$-u''(x) + L^2 \sin^2\left(\frac{x}{2L}\right) u(x) = 2\mu u(x). \tag{1.4}$$

Set $\psi = U_L(u)$, that is $\psi(t) = \sqrt{L} u(Lt)$ for $t \in [-\pi, \pi]$. Then, posing $x = Lt$, it follows that

$$\frac{d\psi}{dt} = \sqrt{L}\,\frac{du}{dx}\,\frac{dx}{dt} = L^{3/2}\,\frac{du}{dx},$$
$$\frac{d^2\psi}{dt^2} = L^{5/2}\,\frac{d^2u}{dx^2}.$$

From this we have

$$\begin{cases} u(x) = \dfrac{\psi(t)}{\sqrt{L}} \\ \dfrac{du}{dx}(x) = \dfrac{\psi'(t)}{L^{3/2}} \\ \dfrac{d^2u}{dx^2}(x) = \dfrac{\psi''(t)}{L^{5/2}}. \end{cases} \tag{1.5}$$

Recalling that $x = Lt$ and substituting (1.5) in (1.4) we obtain

$$-\frac{1}{\sqrt{L}\,L^2}\,\psi''(t) + L^2\,\frac{\sin^2\left(\frac{t}{2}\right)}{\sqrt{L}}\,\psi(t) = \frac{2\mu}{\sqrt{L}}\,\psi(t),$$

that is

$$-\frac{\psi''(t)}{L^2} + L^2\,\sin^2\left(\frac{t}{2}\right)\,\psi(t) = 2\mu\psi(t). \tag{1.6}$$

Define $\frac{1}{L^2} = h$ and $\lambda = \frac{2\mu}{h}$. Replacing these values in (1.6) gives

$$-\psi''(t) + \frac{\sin^2\left(\frac{t}{2}\right)}{h^2}\,\psi(t) = \lambda\,\psi(t).$$

In a similar way it can be proved that if $\psi \in H^1_0(-\pi, \pi) \cap H^2(-\pi, \pi)$ is a solution of equation (1.3) then, by defining $u(x) = \frac{1}{\sqrt{L}}\psi\left(\frac{t}{L}\right)$, it follows that u belongs to $H^1_0(-\pi L, \pi L) \cap H^2(-\pi L, \pi L)$ and it is a solution of (1.2). □

This Proposition provides a different formulation of problem (1.1), which is more suitable for our purposes. For instance in order to analyse the eigenvalues of the operator P_L we will use in the next chapter perturbation theory, setting $\frac{1}{h^2}$ as the perturbative parameter, when h is near any fixed $h_0 > 0$.

To fix notation for future reference we state again the problem, recalling Proposition 1.1.1.

Let $P\left(\frac{1}{h^2}\right) := P$ be the operator defined by

$$P : D(P) \longrightarrow L^2(-\pi, \pi),$$

with

$$(Pf)(x) = -f''(x) + V(x)f(x), \quad V(x) = \frac{1}{h^2} \sin^2\left(\frac{x}{2}\right)$$

and

$$D(P) = H_0^1(-\pi, \pi) \cap H^2(-\pi, \pi) \subset L^2(-\pi, \pi).$$

From now on we will deal with the eigenvalue problem:

$$P(f) = \lambda f, \qquad f \in D(P),\ \lambda \in \mathbb{C}. \tag{1.7}$$

In the first place it will be shown that P is selfadjoint. Eventually, in the next section we will prove that P has discrete spectrum.

For shortness we will call I an interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$.

To the purpose of showing the selfadjointness of P we will use the integration by parts formula, which holds for all functions in the domain of P. To get this formula we first state the following

Lemma 1.1.2. *Let be $f \in H^2(I)$. Then f' has a continuous representative with finite limits on boundary.*

For the proof of this Lemma see [6] p. 297. The following Remark recalls that all functions in the domain of P vanish on the boundary of I.

Remark 1.1.3. *Notice that if $f \in H_0^1(I)$ then $f(a) = f(b) = 0$.*

The integration by parts formula follows from Lemma 1.1.2 and Remark 1.1.3.

Proposition 1.1.4. *For every $f \in H^2(I)$ and $g \in H_0^1(I)$ we have:*

$$\int_a^b f''(x)g(x)dx = [f'(x)g(x)]_a^b - \int_a^b f'(x)g'(x)dx = -\int_a^b f'(x)g'(x)dx. \quad (1.8)$$

Using (1.8) we can prove the following

Proposition 1.1.5. *The operator*

$$T : D(T) \longrightarrow L^2(I),\ f \longmapsto -f'',$$

with $D(T) = H_0^1(I) \cap H^2(I)$, is selfadjoint.

Hereafter we pose, in the definition of the interval I, $a = -\pi$ and $b = \pi$, thus getting $I = [-\pi, \pi]$.

Now we recall a selfadjointness criterion for the sum of two operators (see [9] p. 287) which yields, along with Proposition 1.1.5, the selfadjointness of P.

Theorem 1.1.6. *Let T be a selfadjoint operator. If A is a bounded symmetric operator such that $D(A) \supset D(T)$, then $T + A$ is selfadjoint.*

Finally we prove the selfadjointness of P.

Proposition 1.1.7. *Let P the operator defined by*

$$P : D(P) \longrightarrow L^2(I), \qquad f \longmapsto P(f),$$

with

$$(Pf)(x) = -f'' + V(x)f(x), \qquad V(x) = \frac{1}{h^2}\sin^2\left(\frac{x}{2}\right).$$

P is selfadjoint.

Proof. We can write $P = T + A$, with

$$T : D(P) \longrightarrow L^2(I), \qquad f \longmapsto -f''$$

and

$$A : L^2(I) \longrightarrow L^2(I), \qquad f \longmapsto Vf.$$

Note that A is bounded, indeed

$$\|A(f)\|_{L^2(I)} \le \frac{1}{h^2}\|f\|_{L^2(I)}, \qquad \forall\, f \in L^2(I).$$

Moreover, from Proposition 1.1.5, T is selfadjoint; from Theorem 1.1.6 the assertion follows. □

1.2 P has discrete spectrum

This section is intended to analyse the spectrum of P and its properties. In particular we will prove that the spectrum of P is an unbounded sequence of real numbers $\lambda_0 < \lambda_1 < \ldots$ such that $\lambda_0 > \frac{1}{4\pi^2}$.

In order to obtain the discreteness of the spectrum of P we use the following embedding theorem to show that the resolvent operator of P is compact (see [6] p. 355):

Theorem 1.2.1 (Rellich). *Let Ω be a bounded open set; the canonical injection*

$$\jmath : H_0^1(\Omega) \longrightarrow L^2(\Omega), \quad u \longmapsto \jmath(u) = u$$

is a compact operator.

Notice that from Theorem 1.2.1 and Proposition 1.1.7 we get that the spectrum of P is contained in $\mathbb{R}$.

Now we state a classical existence result for the solutions of the Cauchy problems (see [1], p. 88). We will use this theorem to obtain the existence of the resolvent operator $(P-\lambda)^{-1}$ for particular values of λ.

Theorem 1.2.2. *Let p, q, g be continuous real-valued functions on the open interval (a,b). Let $x_0 \in (a,b)$. Then the Cauchy problem*

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x), \;\; x \in (a,b), \\ y(x_0) = y_0, \;\; y'(x_0) = y_0', \end{cases}$$

$y_0, \; y_0' \in \mathbb{R}$, *posseses a unique solution $y \in C^2(a,b)$.*

Using Theorem 1.2.2 we get a lower bound for the spectrum of P. More precisely we have the following

Proposition 1.2.3. *The spectrum of P is contained in the set* $\left[\frac{1}{4\pi^2}, +\infty\right)$.

Proof. Let $\lambda \in \mathbb{R}$ be such that $\lambda < \dfrac{1}{4\pi^2}$. Define the operator

$$L_\lambda : D(P) \longrightarrow L^2(I)$$

by

$$L_\lambda u = -u'' + (V-\lambda)\,u, \quad V(x) = \frac{1}{h^2}\, \sin^2\left(\frac{x}{2}\right),$$

for every u in $D(P)$. We want to show that λ belongs to the resolvent set of P, namely that $(P-\lambda)$ is injective and surjective and $(P-\lambda)^{-1}$ is bounded.

To prove the surjectivity of $(P-\lambda)$ we solve the boundary value problem

$$\begin{cases} L_\lambda u = f \\ u(\pi) = u(-\pi) = 0, \end{cases}$$

with $f \in L^2(I)$. Consider the following Cauchy problems:

$$\begin{cases} L_\lambda u = 0 \\ u(-\pi) = 0 \\ u'(-\pi) = 1 \end{cases}, \qquad \begin{cases} L_\lambda u = 0 \\ u(\pi) = 0 \\ u'(\pi) = 1. \end{cases} \tag{1.9}$$

Notice that the functions in the differential equations are continuous on $\mathbb{R}$ and thus on every open interval containing I. Whence, from Theorem 1.2.2, we have the existence and uniqueness of the solution on the close interval $[-\pi, \pi]$. Let u_1 and u_2 be solutions of the first and of the second problem respectively. The Wronksian of u_1 and u_2 is constant on I. Indeed, if

$$W(x) = \det \begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} = u_1(x)u_2'(x) - u_1'(x)u_2(x),$$

then we have

$$W'(x) = \det \begin{pmatrix} u_1(x) & u_2(x) \\ u_1''(x) & u_2''(x) \end{pmatrix} =$$

$$= (V(t) - \lambda) \det \begin{pmatrix} u_1(x) & u_2(x) \\ u_1(x) & u_2(x) \end{pmatrix} = 0.$$

Thus we can write $W(x) = W(0)$, for every $x \in I$.

We will reason by contradiction. Suppose that $W(0) = 0$. Then

$$0 = W(-\pi) = u_1(-\pi)u_2'(-\pi) - u_1'(-\pi)u_2(-\pi) = -u_2(-\pi).$$

Therefore $u_2(-\pi) = 0$. In a similar way from $W(\pi) = 0$ we have that $u_1(\pi) = 0$. Moreover, by the initial value conditions of (1.9), $u_2(\pi) = 0$ and $u_1(-\pi) = 0$. Thus $u_1,\ u_2 \in D(P)$. Now we can write:

$$|u_1(x)|^2 = \left| \int_{-\pi}^{x} u_1'(t)dt \right|^2 \leq \left[\int_{-\pi}^{\pi} |u_1'(t)|dt \right]^2 \leq \left[\sqrt{2\pi} \left(\int_{-\pi}^{\pi} |u_1'(t)|^2 dt \right)^{\frac{1}{2}} \right]^2$$

by Hölder's inequality. Whence we have

$$|u_1(x)|^2 \leq 2\pi \|u_1'\|^2_{L^2(I)}.$$

Integrating both sides of this inequality we obtain Poincaré's inequality

$$\|u_1\|^2_{L^2(I)} \leq 4\pi^2 \|u_1'\|^2_{L^2(I)}. \tag{1.10}$$

Moreover we have

$$\|u_1'\|^2_{L^2(I)} = (u_1', u_1') = -(u_1'', u_1) = -\left((V(x) - \lambda)u_1, u_1\right) =$$

$$= -\left\| \frac{1}{h} \sin\left(\frac{x}{2}\right) u_1 \right\|^2_{L^2(I)} + \lambda \|u_1\|^2_{L^2(I)}.$$

Therefore, recalling (1.10), we have

$$\frac{\|u_1\|^2_{L^2(I)}}{4\pi^2} \leq -\left\| \frac{1}{h} \sin\left(\frac{x}{2}\right) u_1 \right\|^2_{L^2(I)} + \lambda \|u_1\|^2_{L^2(I)},$$

that is

$$\left(\lambda - \frac{1}{4\pi^2}\right) \|u_1\|^2_{L^2(I)} \geq \left\| \frac{1}{h} \sin\left(\frac{x}{2}\right) u_1 \right\|^2_{L^2(I)} \geq 0.$$

From this inequality, since in the beginning we have fixed $\lambda < \dfrac{1}{4\pi^2}$, it follows that $u_1 = 0$ on the whole interval $[-\pi, \pi]$, but this is impossible because $u_1'(-\pi) = 1$. Thus if $\lambda < \dfrac{1}{4\pi^2}$ we have $W(0) \neq 0$.

Define the Green function as follows

$$G(x,y) = \begin{cases} \frac{1}{W(0)} u_2(x) u_1(y), & -\pi \leq y \leq x, \\ \frac{1}{W(0)} u_1(x) u_2(y), & x < y \leq \pi. \end{cases}$$

Set, for $f \in L^2(I)$,

$$u(x) = -\int_{-\pi}^{\pi} G(x,y) f(y) dy =$$

$$= -\int_{-\pi}^{x} \frac{1}{W(0)} u_2(x) u_1(y) f(y) dy - \int_{x}^{\pi} \frac{1}{W(0)} u_1(x) u_2(y) f(y) dy.$$

Notice that by definition u is continuous and $u(\pi) = 0 = u(-\pi)$. Differentiating u gives

$$u'(x) = -\frac{1}{W(0)} u_2(x)u_1(x)f(x) - \frac{u_2'(x)}{W(0)} \int_{-\pi}^{x} u_1(y)f(y)dy + \frac{1}{W(0)} u_2(x)u_1(x)f(x) +$$

$$-\frac{u_1'(x)}{W(0)} \int_{x}^{\pi} u_2(y)f(y)dy =$$

$$= -\frac{u_2'(x)}{W(0)} \int_{-\pi}^{x} u_1(y)f(y)dy - \frac{u_1'(x)}{W(0)} \int_{x}^{\pi} u_2(y)f(y)dy \qquad (1.11)$$

and, using (1.9),

$$u''(x) = -\frac{u_2''(x)}{W(0)} \int_{-\pi}^{x} u_1(y)f(y)dy - \frac{u_2'(x)}{W(0)} u_1(x)f(x) - \frac{u_1''(x)}{W(0)} \int_{x}^{\pi} u_2(y)f(y)dy +$$

$$+\frac{u_1'(x)}{W(0)} u_2(x)f(x) = \left(\frac{1}{h^2}\sin^2\left(\frac{x}{2}\right) - \lambda\right) u(x) - f(x), \qquad (1.12)$$

From (1.11) we obtain the continuity of u' and from (1.12) it follows that $u'' \in L^2(I)$. Therefore $u \in H_0^1(I) \cap H^2(I) = D(P)$ and, from (1.12), u is a solution of the boundary value problem. Thus the surjectivity of $(P - \lambda)$ is proved.

To see that $(P-\lambda)$ is injective it suffices to show that from $(P-\lambda)(f_1-f_2) = 0$ follows $(f_1 - f_2, g) = 0$ for every $g \in L^2(I)$ (where $f_1,\ f_2 \in D(P)$). If $g \in L^2(I)$, because of the surjectivity of $(P-\lambda)$ there exists $q \in D(P)$ such that $(P-\lambda)q = g$. We have:

$$(f_1 - f_2, g) = (f_1 - f_2, (P - \lambda)q) = ((P - \lambda)(f_1 - f_2), q) = 0,$$

because P is selfadjoint.

The boundedness of $(P - \lambda)^{-1}$ follows from the fact that the function $G(x, y)$ is bounded on $I \times I$ (which holds because u_1 and u_2 are continuous on $[-\pi, \pi]$ and therefore bounded). □

Recall that the operator P is selfadjoint from Proposition 1.1.7 and it has compact resolvent because of Theorem 1.2.1. Hence we get the following

Proposition 1.2.4. *P has a discrete spectrum.*

Furthermore, from Proposition 1.2.3, 0 belongs to the resolvent set of P. Therefore from Hilbert-Schmidt Theorem the eigenfunctions of P form a complete orthonormal basis for $L^2(I)$.

Corollary 1.2.5. *The space $L^2(I)$ admits a complete orthonormal basis of eigenfunctions for P.*

In order to get information on the eigenvalues' multiplicity we recall now a classic result about Sturm-Liouville problems (see [5], p. 337):

Theorem 1.2.6. *Let $p = p(t) > 0$, $q = q(t)$ be real valued functions, continuous for $a \leq t \leq b$, with α, $\beta \in \mathbb{R}$. Then there exists an unbounded sequence of real numbers $\lambda_0 < \lambda_1 < \dots$ such that*

1) the equations

$$(p(t)u')' + [q(t) + \lambda]u = 0, \tag{1.13}$$

$$u(a)\cos\alpha - p(a)u'(a)\sin\alpha = 0, \quad u(b)\cos\beta - p(b)u'(b)\sin\beta = 0 \tag{1.14}$$

have a nontrivial solution if and only if $\lambda = \lambda_n$ for some n;

2) if $\lambda = \lambda_n$ and if $u = u_n(t)$ is a nontrivial solution of (1.13), (1.14), then u_n is unique up to a multiplicative constant and u_n has exactly n zeros for $a < t < b$ for $n = 0, 1, \dots$.

Now we apply this result to the eigenvalue problem for P.

Remark 1.2.7. *The Dirichlet problem for P fulfills the hypotheses of Theorem 1.2.6. Hence the eigenvalues of P are all simple.*

To summarize, P has discrete spectrum, which is an unbounded sequence of real numbers $\lambda_0 < \lambda_1 < \ldots$, such that $\lambda_0 \geq \frac{1}{4\pi^2}$ (from Proposition 1.2.3), with simple eigenvalues.

Chapter 2

Perturbative analysis of the eigenfunctions of P

2.1 Topics in Perturbation Theory

In this section we recall some results in Perturbation Theory, which will be used to study the spectrum of the operator P, setting $\frac{1}{h^2}$ as the perturbative, small, parameter. (For a reference see [9].) These results will be used, afterwards, to get power series expansions of eigenvalues and eigenfunctions of $P = P\left(\frac{1}{h^2}\right)$ in the parameter $\frac{1}{h^2}$. Besides, this study provides a particular orthonormal basis of $L^2(I)$, obtained as the set of all eigenfunctions of the operator $P\left(\frac{1}{h^2}\right)$, in which we set $h = +\infty$. This basis is formed entirely by functions of $D(P) = H_0^1(I) \cap H^2(I)$ and, as we are looking for eigenfunctions of P it will be used in the next chapter for the Fourier expansion of the eigenfunctions of P.

Definition 2.1.1. *A family of operators $T(\xi) \in C(X, Y)$, defined for $\xi \in D_0$,* *where* D_0 *is a domain of the complex plane* $\mathbb{C}$, *is said to be* ***holomorphic of type (A)*** *if*

a) $D(T(\xi)) = D$ *is independent of* ξ.

b) $T(\xi)u$ *is holomorphic for* $\xi \in D_0$ *for every* $u \in D$.

In this case $T(\xi)u$ *has a Taylor expansion at every* $u \in D_0$. *For example if* $\xi = 0$ *belongs to* D_0 *we can write*

$$T(\xi)u = T^{(0)}u + \xi T^{(1)}u + \xi^2 T^{(2)}u + \dots, \qquad u \in D \tag{2.1}$$

which converges in a disk $|\xi| < r \in \mathbb{R}$ *indipendent of u;* $T^{(n)}$ *are linear operators from* X *to* Y *with domain* D.

For a reference see [9], p. 375.

We state here a result which we will use for showing that the h-dependent family of operators $P = P(\frac{1}{h^2})$, forms an holomorphic family of type (A) in $\frac{1}{h^2}$ (see [9], p. 377). We will see afterwards that selfadjoint holomorphic families of type (A) admit particular power series expansions for their eigenvalues and eigenfunctions.

Theorem 2.1.2. *Let* $T^{(0)}$ *be a closable operator from* X *to* Y, *with* $D(T^{(0)}) = D$. *Let* $T^{(n)}$, $n = 1, 2, \dots$, *be operators from* X *to* Y *with domains containing* D, *and let there be constants* $a,\ b,\ c \geq 0$ *such that*

$$\left\| T^{(n)}u \right\| \leq c^{n-1}(a\|u\| + b\|T^{(0)}u\|), \qquad u \in D,\ n = 1, 2, \dots . \tag{2.2}$$

Then the series (2.1) defines an operator $T(\xi)$ *with domain* D *for* $|\xi| < \frac{1}{c}$. *If* $|\xi| < (b+c)^{-1}$ *then* $T(\xi)$ *is closable and the closures for such* ξ *form a holomorphic family of type* (A).

Remark 2.1.3. *The operator $P = P\left(\frac{1}{h^2}\right)$ defines, for varying h, an holomorphic family of type (A), with infinite convergence ray.*

Proof. Notice that $P = T^{(0)} + T^{(1)}$, with

$$T^{(0)} : D(P) \longrightarrow L^2(I), \qquad f \longmapsto -f'',$$

$$T^{(1)} : D(P) \longrightarrow L^2(I), \qquad f \longmapsto \frac{1}{h^2}\sin^2\left(\frac{x}{2}\right) f.$$

We have

$$\|T^{(1)}u\| \leq \frac{1}{h^2}\|u\|.$$

Hence, following the notation fixed in Teorem 2.1.2, we can choose $a = \frac{1}{h^2}$, $b = c = 0$, if

$$\left\|T^{(1)}u\right\| \leq a\|u\| + b\|T^{(0)}u\|, \quad u \in D.$$

Then, from Theorem 2.1.2, the assertion follows. □

Note that, by definition, the parameter $\frac{1}{h^2}$ can not vanish. From now on we will call $P(0)$ the operator

$$P(0) : D(P) = D_0 = H_0^1(I) \cap H^2(I) \longrightarrow L^2(I), \qquad f \longmapsto -f''. \tag{2.3}$$

We will show that the family of operators $P = P(\frac{1}{h^2})$ is a selfadjoint holomorphic family; from this it will follow that there exist power series expansions, in terms of $\frac{1}{h^2}$, of eigenvalues and eigenfunctions of P.

In the first place we recall the definition of selfadjoint holomorphic family (see [9], p. 385).

Definition 2.1.4. *Following the notation of Definition 2.1.1, in which we pose $X = Y = H$, where H is an Hilbert space, let $T(\xi)$ be an holomorphic family.*

Moreover let $T(\xi)$ be densely defined for every ξ and let $T(\xi)^* = T(\bar{\xi})$. Then we say that $T(\xi)$ is a ***selfadjoint holomorphic family.***

The conditions of this definition are satisfied by $P(\frac{1}{h^2})$, i.e. we have the following

Remark 2.1.5. *$P(\frac{1}{h^2})$ is a selfadjoint holomorphic family.*

The holomorphic families of type (A) have a particular series expansion for eigenfunctions (see [9], p. 392):

Theorem 2.1.6. *Let $T(\xi)$ a selfadjoint holomorphic family of type (A), defined in a neighborhood of an interval I_0 of the real axis. Furthermore, let $T(\xi)$ have compact resolvent for every ξ. Then all eigenvalues of $T(\xi)$ can be represented by functions which are holomorphic on I_0. More precisely, there is a sequence of scalar-valued functions $\mu_n(\xi)$ and a sequence of vector-valued functions $\varphi_n(\xi)$, all holomorphic on I_0, such that for every $\xi \in I_0$, the $\mu_n(\xi)$ represent all the repeated eigenvalues of $T(\xi)$ and the $\varphi_n(\xi)$ form a complete orthonormal family of the associated eigenvectors of $T(\xi)$.*

This Theorem implies that the eigenvalues of $T(\xi)$ converge to those of $T(0)$, when $\xi \to 0$; in other words we have the following

Remark 2.1.7. *Theorem 2.1.6 implies, in particular, that for $\xi \to 0$ the eigenfunctions $\varphi_n(\xi)$ converge, in norm $L^2(I)$, to the eigenfunctions of $T(0)$ and also that the eigenvalues $\mu_n(\xi)$ converge to those of $T(0)$.*

From Theorem 2.1.6 and Proposition 2.1.3, upon recalling that $P = P(\frac{1}{h^2})$ is selfadjoint with compact resolvent, we can expand all eigenfunctions and eigenvalues of P in power series of $\frac{1}{h^2}$. Furthermore, from Remark 2.1.7 follows that eigenvalues and eigenfunctions of $P(\frac{1}{h^2})$ converge, as $h \to +\infty$, respectively to eigenvalues and eigenfunctions of $P(0)$ (see 2.3).

Proposition 2.1.8. *Let $\psi \in D(P)$ be an eigenfunction for P, associated to the eigenvalue λ. Then we can expand ψ and λ in power series of $\frac{1}{h^2}$, that is*

$$\psi(t) = \sum_{m=0}^{+\infty} \left(\frac{1}{h^2}\right)^m \psi_m(t), \tag{2.4}$$

$$\lambda = \sum_{m=0}^{+\infty} \left(\frac{1}{h^2}\right)^m \lambda_m. \tag{2.5}$$

2.2 Recursive formulas for the eigenfunctions' and eigenvalues' coefficients

Recalling Proposition 2.1.8 and in particular the relations (2.4), (2.5), we will prove in this section a formula which helps in computing the coefficients $\psi_m(t)$ and λ_m. To this aim it is useful to have the following

Proposition 2.2.1. *Let $\psi \in D(P)$ be an eigenfunction of P; by definition it satisfies*

$$-\psi''(t) + \frac{1}{h^2}\sin^2\left(\frac{t}{2}\right)\psi(t) = \lambda\psi(t), \qquad \forall\ t \in [-\pi, \pi], \tag{2.6}$$

with power series expansion given by (2.4). Let $\lambda \in \mathbb{R}$ be the eigenvalue associated to ψ, with power series expansion given by (2.5). Then for the coefficients

$\lambda_m, \psi_m(t)$ we have the following relations:

$$\begin{cases} -\psi_0''(t) - \lambda_0\psi_0(t) = 0 \\ \\ -\psi_1''(t) - \lambda_0\psi_1(t) + \dfrac{1}{2}\psi_0(t) - \dfrac{1}{2}\cos(t)\psi_0(t) - \lambda_1\psi_0(t) = 0 \\ \\ -\psi_m''(t) - \lambda_0\psi_m(t) + \dfrac{1}{2}\psi_{m-1}(t) - \dfrac{1}{2}\cos(t)\psi_{m-1}(t) + \\ \qquad\qquad -\lambda_1\psi_{m-1}(t) - \displaystyle\sum_{j=2}^{m}\lambda_j\psi_{m-j} = 0, \quad \forall\ m \geq 2, \end{cases} \tag{2.7}$$

for every t in $[-\pi, \pi]$.

Proof. Replacing the expansions (2.4) and (2.5) in the terms of the eigenvalue equation for P, (2.6), we have

$$\lambda\psi = \left[\sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m\right]\left[\sum_{j=0}^{+\infty}\left(\frac{1}{h^2}\right)^j \lambda_j\right] = \sum_{m=0}^{+\infty}\left(\sum_{j=0}^{m}\lambda_j\psi_{m-j}\right)\left(\frac{1}{h^2}\right)^m =$$

$$\lambda_0\psi_0 + \sum_{m=1}^{+\infty}\left(\sum_{j=0}^{m}\lambda_j\psi_{m-j}\right)\left(\frac{1}{h^2}\right)^m.$$

Upon setting $i = m-1,\ m = i+1$, we obtain

$$\lambda\psi = \lambda_0\psi_0 + \sum_{i=0}^{+\infty}\left(\sum_{j=0}^{i+1}\lambda_j\psi_{i+1-j}\right)\left(\frac{1}{h^2}\right)^{i+1}. \tag{2.8}$$

Substituting (2.4) in $-\psi''$ gives

$$-\psi'' = -\sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m'' = -\psi_0'' - \sum_{m=1}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m'',$$

whence

$$-\psi'' = -\psi_0'' - \sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^{m+1} \psi_{m+1}''. \tag{2.9}$$

By using (2.4), (2.8) and (2.9) in (2.6) we have

$$-\psi_0''(t) - \sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^{m+1}\psi_{m+1}''(t) + \left(\frac{1}{2h^2} - \frac{1}{2h^2}\cos(t)\right)\sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m(t) +$$

$$-\lambda_0\psi_0(t) - \sum_{m=0}^{+\infty}\left(\sum_{j=0}^{m+1}\lambda_j\psi_{m+1-j}(t)\right)\left(\frac{1}{h^2}\right)^{m+1} = 0,$$

$$-\psi_0''(t) - \lambda_0\psi_0(t) +$$

$$+\sum_{m=0}^{+\infty}\left[-\psi_{m+1}''(t) + \left(\frac{1-\cos(t)}{2}\right)\psi_m(t) - \sum_{j=0}^{m+1}\lambda_j\psi_{m+1-j}(t)\right]\left(\frac{1}{h^2}\right)^{m+1} = 0.$$

This implies that all coefficients of powers of $\frac{1}{h^2}$ must vanish. □

Remark 2.2.2. *From Remark 2.1.7, λ_0 represents an eigenvalue of $P(0)$ and ψ_0 the associated eigenfunction. Therefore we have either $\lambda_0 = n^2$ and $\psi_0(t) = \frac{1}{\sqrt{\pi}}\sin(nt)$, with $n \in \mathbb{N}\backslash\{0\}$, or $\lambda_0 = \frac{(2n+1)^2}{4}$ and $\psi_0(t) = \frac{1}{\sqrt{\pi}}\cos\left(\frac{2n+1}{2}\ t\right)$, $n \in \mathbb{N}$.*

In order to have uniqueness, in (2.4), of the coefficients $\psi_m(t)$ we will impose on the eigenfunctions the following normalization condition

$$\|\psi\|^2 = 1. \tag{2.10}$$

Remark 2.2.3. *Using the notation fixed in Proposition 2.1.8 we have that the normalization condition $\|\psi\|^2 = 1$ holds if and only if*

$$\begin{cases} \|\psi_0\|^2 = 1 \\ \sum_{r=0}^{k}(\psi_{k-r}, \psi_r) = 0, \qquad k \geq 1. \end{cases} \tag{2.11}$$

Proof. Indeed

$$\|\psi\|^2 = (\psi, \psi) = \left(\sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m, \sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m\right) =$$

$$= \sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \sum_{r=0}^{m}(\psi_{m-r}, \psi_r) = 1;$$

as this must hold for generic h we equal to 0 the coefficients of $\left(\frac{1}{h^2}\right)^m$, with $m \in \mathbb{N}\backslash\{0\}$ and we equal to 1 the term $(\psi_0,\ \psi_0)$. □

From the normalization condition (2.10) it follows the uniqueness of the (normalized) eigenfunction associated to a given eigenvalue.

Remark 2.2.4. *Using the notation of Proposition 2.1.8, the eigenvalue* λ *admits a unique eigenfunction which satisfies the normalization condition (2.10).*

Let λ be an eigenvalue of $P = P\left(\frac{1}{h^2}\right)$. From Proposition 2.1.8 λ has the series expansion (2.5)

$$\lambda = \sum_{m=0}^{+\infty} \left(\frac{1}{h^2}\right)^m \lambda_m.$$

From Remark 2.2.2 we have either $\lambda_0 = n^2,\ n \in \mathbb{N}\backslash\{0\}$, or $\lambda_0 = \frac{(2n+1)^2}{4},\ n \in \mathbb{N}$. Hereafter we will assume that λ_0 is a fixed value, chosen in the set of eigenvalues of $P(0)$.

We now make some remarks on the functions ψ_m of formula (2.4). We know that the ψ_m belong to $L^2(I)$. Thus we may expand them with respect to the basis $\{\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\right),\ \frac{1}{\sqrt{\pi}}\sin(kx)\}_{k\in\mathbb{N}}$. Depending on the parity of the eigenfunction ψ, the ψ_m are either even or odd functions of t. We study separately the two cases.

When ψ is an even eigenfunction we can expand $\psi_m(t)$ with respect to

$$\left\{\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\right)\right\}_{k\in\mathbb{N}},$$

so that, in particular,

$$\psi_m(t) = \sum_{k=0}^{+\infty} \psi_{m,2k+1} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2k+1}{2}\,t\right). \tag{2.12}$$

The following results are intended to provide recursive formulas for $\psi_{m,2k+1}$ and for the coefficients λ_m of the expansion (2.5). After that, analogous results on odd eigenfunctions will be given.

In the first place it is useful to make the following

Remark 2.2.5. *Let be* $a, m \in \mathbb{N}$. *We have:*

$$\int_{-\pi}^{\pi} \cos(t)\cos\left(\frac{2a+1}{2}t\right)\cos\left(\frac{2m+1}{2}t\right)dt =$$

$$= \int_{-\pi}^{\pi} \frac{1}{4}[\cos((m+a+2)t)+\cos((m+a)t)+\cos((m-a+1)t)+\cos((m-a-1)t)]dt$$

Proof. Indeed,

$$\int_{-\pi}^{\pi} \cos(t)\cos\left(\frac{2a+1}{2}t\right)\cos\left(\frac{2m+1}{2}t\right)dt =$$

$$= \int_{-\pi}^{\pi} \cos(t)\frac{1}{2}\left[\cos\left(\frac{2a+1+2m+1}{2}t\right)+\cos\left(\frac{2m+1-2a-1}{2}t\right)\right]dt =$$

$$= \int_{-\pi}^{\pi} \cos(t)\frac{1}{2}\left[\cos((a+m+1)t)+\cos((m-a)t)\right]dt$$

and from this the formula we wanted to prove follows. □

In the first place we set the value λ_0 as

$$\lambda_0 := \frac{(2n+1)^2}{4}, \quad n \neq 0. \tag{2.13}$$

We will treat the case $n = 0$ separately. Anyway, as we are analysing a fixed eigenvalue of P, we assume hereafter that n in (2.13) is fixed, but generic.

Theorem 2.2.6. *Let ψ be an even, normalized (see condition (2.11)), eigenfunction of P associated to the eigenvalue λ given by (2.5), with $\lambda_0 = \frac{(2n+1)^2}{4}$, $n \neq 0$. Let (see (2.4))*

$$\psi(t) = \sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m \psi_m(t)$$

be its series expansion. Recall the expansion (2.12)

$$\psi_m(t) = \sum_{k=0}^{+\infty} \psi_{m,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\, t\right).$$

Then the coefficients $\psi_{m,2k+1}$ and λ_m fulfill the following equations:

$$\begin{cases} \lambda_0 = \dfrac{(2n+1)^2}{4} \\ \lambda_1 = \dfrac{1}{2} \\ \lambda_2 = -\dfrac{1}{4}\left[\psi_{1,2n+3} + \psi_{1,2n-1}\right] \\ \lambda_m = -\dfrac{1}{4}\left[\psi_{m-1,2n+3} + \psi_{m-1,2n-1}\right] - \displaystyle\sum_{j=2}^{m-2} \lambda_j \psi_{m-j,2n+1}, \qquad \forall\ m \geq 2. \end{cases} \tag{2.14}$$

$$\begin{cases} \psi_{m,1} = -\dfrac{1}{4n(n+1)}\left[\psi_{m-1,1} + \psi_{m-1,3} + 4\displaystyle\sum_{j=2}^{m-1} \lambda_j \psi_{m-j,1}\right] \\ \psi_{m,2k+1} = \dfrac{1}{4(k-n)(k+n+1)}\left[\psi_{m-1,2(k+1)+1} + \psi_{m-1,2(k-1)+1} + \right. \\ \qquad\qquad \left. +4\displaystyle\sum_{j=2}^{m-1} \lambda_j \psi_{m-j,2k+1}\right], \quad k \neq n,\ k = 1,2,3,\ldots \end{cases} ; \tag{2.15}$$

(Here we use the convention that $\sum_{n=j}^{k} = 0$ when $j > k$.)

Proof. From Proposition 2.2.1 we have

$$\lambda_0 = \frac{(2n+1)^2}{4}, \qquad -\psi_0'' = \lambda_0 \psi_0.$$

Moreover

$$-\psi_1''(t) - \lambda_0 \psi_1(t) + \left(\frac{1-\cos(t)}{2}\right)\psi_0(t) - \lambda_1 \psi_0(t) = 0, \quad \forall\ t \in [-\pi, \pi]. \tag{2.16}$$

By taking on both sides the scalar product with ψ_0 we get

$$-(\psi_1'', \psi_0) - \lambda_0(\psi_1, \psi_0) + \left(\left(\frac{1-\cos(t)}{2}\right)\psi_0, \psi_0\right) - \lambda_1(\psi_0, \psi_0) = 0,$$

that is

$$-(\psi_1, \psi_0'') - \lambda_0(\psi_1, \psi_0) + \left(\left(\frac{1-\cos(t)}{2}\right)\psi_0, \psi_0\right) - \lambda_1(\psi_0, \psi_0) = 0.$$

Then

$$(\psi_1, -\psi_0'' - \lambda_0\psi_0) + \left(\left(\frac{1-\cos(t)}{2}\right)\psi_0, \psi_0\right) - \lambda_1(\psi_0, \psi_0) =$$
$$= \left(\left(\frac{1-\cos(t)}{2}\right)\psi_0, \psi_0\right) - \lambda_1(\psi_0, \psi_0) = 0.$$

From the normalization conditions (2.11) we obtain

$$\lambda_1 = \frac{1}{2} - \frac{1}{2}(\cos(t)\psi_0, \psi_0). \tag{2.17}$$

We have

$$(\cos(t)\psi_0, \psi_0) = \int_{-\pi}^{\pi} \frac{1}{\pi}\cos(t)\cos^2\left(\frac{2n+1}{2}t\right)dt =$$
$$= \int_{-\pi}^{\pi} \frac{1}{\pi}\cos(t)\frac{1}{2}\left[1+\cos((2n+1)t)\right]dt = \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(t)\cos((2n+1)t)dt = 0,$$

therefore, from (2.17), $\lambda_1 = \frac{1}{2}$ for all $n \in \mathbb{N}\backslash\{0\}$.

Now let

$$\psi_1(t) = \sum_{k=0}^{+\infty} \psi_{1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}t\right)$$

be the expansion of ψ_1. Replacing this expansion, and the values of λ_1, λ_0, in equation (2.16) we get

$$\sum_{k=0}^{+\infty}\left(\frac{(2k+1)^2}{4} - \frac{(2n+1)^2}{4}\right)\psi_{1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}t\right) + \frac{1}{2}\psi_0 +$$
$$-\frac{\cos(t)}{2}\psi_0 - \frac{1}{2}\psi_0 = 0,$$

that is

$$\sum_{k=0}^{+\infty}\left(\frac{(2k+1)^2}{4} - \frac{(2n+1)^2}{4}\right)\psi_{1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}t\right) +$$
$$-\frac{1}{4\sqrt{\pi}}\left[\cos\left(\left(\frac{2n+1}{2}+1\right)t\right) + \cos\left(\left(\frac{2n+1}{2}-1\right)t\right)\right] = 0,$$

and finally

$$\sum_{k=0}^{+\infty}\left(\frac{(2k+1)^2}{4} - \frac{(2n+1)^2}{4}\right)\psi_{1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}t\right) +$$

$$-\frac{1}{4\sqrt{\pi}}\left[\cos\left(\left(\frac{2(n+1)+1}{2}\right)t\right)+\cos\left(\left(\frac{2(n-1)+1}{2}\right)t\right)\right]=0,$$

for all t in $[-\pi,\pi]$, $n\neq 0$. It follows that the coefficients of $\cos\left(\left(\frac{2j+1}{2}\right)t\right)$ must vanish for all $j\in\mathbb{N}$. Notice that the coefficient of $\cos\left(\frac{2n+1}{2}\ t\right)$ is 0 for a generic $n\in\mathbb{N}\backslash\{0\}$. Moreover, from the normalization conditions (2.11) we have $(\psi_1,\psi_0)=0$ and then $\psi_{1,2n+1}=0$. Imposing that the coefficients of $\cos\left(\left(\frac{2k+1}{2}\right)t\right)$ be zero for $k\neq n$, gives $\psi_{1,2k+1}=0$ for all $k\neq n+1,\ n-1$ (recall that n is fixed). For $k=n+1$ we

obtain

$$\frac{(2(n+1)+1)^2-(2n+1)^2}{4}\psi_{1,2n+3}-\frac{1}{4}=0,$$

that is

$$((2n+3)^2-4n^2-4n-1)\psi_{1,2n+3}=1.$$

Hence

$$\psi_{1,2n+3}=\frac{1}{4n^2+9+12n-4n^2-4n-1}=\frac{1}{8n+8}.$$

If $k=n-1$ we have

$$\frac{(2n-1)^2-(2n+1)^2}{4}\psi_{1,2n-1}=\frac{1}{4},$$

that is

$$\psi_{1,2n-1}=\frac{1}{1-4n-4n-1}=\frac{1}{-8n}.$$

Therefore we obtain

$$\psi_1(t)=\frac{1}{-8n}\cos\left(\frac{2n-1}{2}\ t\right)+\frac{1}{8n+8}\cos\left(\frac{2n+3}{2}\ t\right).$$

Now we seek a general formula for the remaining coefficients. For $m\geq 2$ we get, from Proposition 2.2.1,

$$-\psi''_m(t)-\lambda_0\psi_m(t)+\frac{1}{2}\psi_{m-1}(t)-\frac{1}{2}\cos(t)\psi_{m-1}(t)-\lambda_1\psi_{m-1}(t)-\sum_{j=2}^{m}\lambda_j\psi_{m-j}=0.$$

Recalling that $\lambda_1 = \frac{1}{2}$ we have

$$-\psi_m''(t) - \lambda_0\psi_m(t) - \frac{1}{2}\cos(t)\psi_{m-1}(t) - \lambda_m\psi_0 - \sum_{j=2}^{m-1}\lambda_j\psi_{m-j} = 0, \tag{2.18}$$

with the convention that when $m = 2$ the last sum vanishes. Note that

$$(-\psi_m'', \psi_0) - (\lambda_0\psi_m, \psi_0) = (\psi_m, -\psi_0'') - (\psi_m, \lambda_0\psi_0) =$$

$$= (\psi_m, \lambda_0\psi_0 - \lambda_0\psi_0) = 0. \tag{2.19}$$

By taking on both sides of (2.18) the scalar product with ψ_0, from (2.19), and the normalization conditions (2.11) we have

$$\lambda_m = -\frac{1}{2}\left(\cos(t)\psi_{m-1}, \psi_0\right) - \sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j}, \psi_0\right). \tag{2.20}$$

Let ψ_{m-1} be given by

$$\sum_{k=0}^{+\infty}\psi_{m-1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\,t\right).$$

We compute the term $(\cos(t)\psi_{m-1}, \psi_0)$ of (2.20) by substituting the expansion of ψ_{m-1} :

$$(\cos(t)\psi_{m-1}, \psi_0) = \sum_{k=0}^{+\infty}\left(\cos(t)\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\,t\right), \psi_0\right)\psi_{m-1,2k+1} =$$

$$= \sum_{k=0}^{+\infty}\left(\cos(t)\cos\left(\frac{2k+1}{2}\,t\right), \cos\left(\frac{2n+1}{2}\,t\right)\right)\frac{\psi_{m-1,2k+1}}{\pi}. \tag{2.21}$$

From Remark 2.2.5 and since $n \neq 0$, by hypothesis, we have:

$$\left(\cos(t)\cos\left(\frac{2k+1}{2}\,t\right), \cos\left(\frac{2n+1}{2}\,t\right)\right) =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\cos((n+k+2)t) + \cos((n+k)t) + \cos((n-k+1)t) + \cos((n-k-1)t)dt =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\cos((n-k+1)t) + \cos((n-k-1)t)dt. \tag{2.22}$$

The term

$$\left(\cos(t)\cos\left(\frac{2k+1}{2}\ t\right), \cos\left(\frac{2n+1}{2}\ t\right)\right)$$

is different from 0 if and only if $k = n-1,\ n+1$. In both cases we have

$$\left(\cos(t)\cos\left(\frac{2k+1}{2}\ t\right), \cos\left(\frac{2n+1}{2}\ t\right)\right) = \frac{\pi}{2}.$$

By substituting in (2.21) we get

$$(\cos(t)\psi_{m-1}, \psi_0) = \frac{\pi}{2}\frac{\psi_{m-1,2n+3}}{\pi} + \frac{\pi}{2}\frac{\psi_{m-1,2n-1}}{\pi} =$$

$$= \frac{\psi_{m-1,2n+3}}{2} + \frac{\psi_{m-1,2n-1}}{2}.$$

Substituting the results obtained up to now in (2.20) gives

$$\lambda_m = -\frac{1}{4}\psi_{m-1,2n+3} - \frac{1}{4}\psi_{m-1,2n-1} - \sum_{j=2}^{m-1}\lambda_j(\psi_{m-j}, \psi_0), \tag{2.23}$$

thus completing the proof of (2.14). Now we consider once again the equation (2.18) to obtain the functions ψ_m. We compute each term separately. We have:

$$-\frac{\cos(t)}{2}\psi_{m-1}(t) = \sum_{k=0}^{+\infty} -\frac{\cos(t)}{2}\psi_{m-1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\ t\right) =$$

$$= \sum_{k=0}^{+\infty} -\frac{1}{4\sqrt{\pi}}\psi_{m-1,2k+1}\left[\cos\left(\frac{2(k-1)+1}{2}\ t\right) + \cos\left(\frac{2(k+1)+1}{2}\ t\right)\right].$$

Then

$$-\frac{\cos(t)}{2}\psi_{m-1}(t) = \sum_{k=0}^{+\infty} -\frac{1}{4\sqrt{\pi}}\psi_{m-1,2k+1}\cos\left(\frac{2(k-1)+1}{2}\ t\right) +$$

$$+\sum_{k=0}^{+\infty} -\frac{1}{4\sqrt{\pi}}\psi_{m-1,2k+1}\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.24}$$

For the first term in the last equality we get:

$$\sum_{k=0}^{+\infty} -\frac{1}{4\sqrt{\pi}}\psi_{m-1,2k+1}\cos\left(\frac{2(k-1)+1}{2}\ t\right) = -\frac{1}{4\sqrt{\pi}}\psi_{m-1,1}\cos\left(-\frac{1}{2}\ t\right) +$$

$$-\frac{1}{4\sqrt{\pi}}\psi_{m-1,3}\cos\left(\frac{1}{2}\ t\right)+\sum_{k=2}^{+\infty}-\frac{1}{4\sqrt{\pi}}\psi_{m-1,2k+1}\cos\left(\frac{2(k-1)+1}{2}\ t\right).$$

By changing index in the last sum we obtain

$$\sum_{k=0}^{+\infty}-\frac{1}{4\sqrt{\pi}}\psi_{m-1,2k+1}\cos\left(\frac{2(k-1)+1}{2}\ t\right)=$$

$$=-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,1}+\psi_{m-1,3}\right]\cos\left(\frac{1}{2}\ t\right)+$$

$$\sum_{k=0}^{+\infty}-\frac{1}{4\sqrt{\pi}}\psi_{m-1,2(k+2)+1}\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.25}$$

Substituting (2.25) in (2.24) gives:

$$-\frac{\cos(t)}{2}\psi_{m-1}(t)=-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,1}+\psi_{m-1,\ 3}\right]\cos\left(\frac{1}{2}\ t\right)+$$

$$+\sum_{k=0}^{+\infty}-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2(k+2)+1}+\psi_{m-1,2k+1}\right]\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.26}$$

We now compute the term $-\psi_m''-\lambda_0\psi_m$. Since $\lambda_0=\frac{(2n+1)^2}{4}$ we have

$$-\psi_m''-\lambda_0\psi_m=\sum_{k=0}^{+\infty}\left(\frac{(2k+1)^2-(2n+1)^2}{4}\right)\psi_{m,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\ t\right)=$$

$$=\left(\frac{1-(2n+1)^2}{4}\right)\psi_{m,1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{t}{2}\right)+$$

$$+\sum_{k=1}^{+\infty}\left(\frac{(2k+1)^2-(2n+1)^2}{4}\right)\psi_{m,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\ t\right).$$

By renaming the index in the last sum we get

$$-\psi_m''-\lambda_0\psi_m=\left(\frac{1-(2n+1)^2}{4}\right)\psi_{m,1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{t}{2}\right)+$$

$$+\sum_{k=0}^{+\infty}\left(\frac{(2(k+1)+1)^2-(2n+1)^2}{4}\right)\psi_{m,2(k+1)+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.27}$$

Substituting the (2.23), (2.27), (2.26) in (2.18) gives

$$\left(\frac{1-(2n+1)^2}{4}\right)\psi_{m,1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{1}{2}\ t\right)+$$

$$+\sum_{k=0}^{+\infty}\left(\frac{(2(k+1)+1)^2-(2n+1)^2}{4}\right)\psi_{m,2(k+1)+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2(k+1)+1}{2}\ t\right)+$$

$$-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,1}+\psi_{m-1,3}\right]\cos\left(\frac{1}{2}\ t\right)+$$

$$+\sum_{k=0}^{+\infty}-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2(k+2)+1}+\psi_{m-1,2k+1}\right]\cos\left(\frac{2(k+1)+1}{2}\ t\right)+$$

$$+\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2n+3}+\psi_{m-1,2n-1}\right]\cos\left(\frac{2n+1}{2}\ t\right)+$$

$$+\sum_{j=2}^{m-1}\lambda_j(\psi_{m-j},\psi_0)\psi_0-\sum_{j=2}^{m-1}\lambda_j\psi_{m-j}=0.$$

By collecting the common factors' coefficients we get

$$\frac{1}{\sqrt{\pi}}\left[\frac{1-(2n+1)^2}{4}\psi_{m,1}-\frac{1}{4}\psi_{m-1,1}-\frac{1}{4}\psi_{m-1,3}\right]\cos\left(\frac{t}{2}\right)+$$

$$+\frac{1}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\left[\frac{(2(k+1)+1)^2-(2n+1)^2}{4}\psi_{m,2(k+1)+1}-\frac{1}{4}\psi_{m-1,2(k+2)+1}+\right.$$

$$\left.-\frac{1}{4}\psi_{m-1,2k+1}\right]\cos\left(\frac{2(k+1)+1}{2}\ t\right)+$$

$$+\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2n+3}+\psi_{m-1,2n-1}\right]\cos\left(\frac{2n+1}{2}\ t\right)+$$

$$+\sum_{j=2}^{m-1}\lambda_j(\psi_{m-j},\psi_0)\psi_0-\sum_{j=2}^{m-1}\lambda_j\psi_{m-j}=0. \tag{2.28}$$

Since $\{\cos\left(\frac{2n+1}{2}\ x\right)\}_{n\in\mathbb{N}}$ is an orthogonal basis for the **even** functions in $L^2(I)$ the coefficients of these functions in (2.28) must vanish. We obtain:

$$\begin{cases}\dfrac{1-(2n+1)^2}{4}\psi_{m,1}-\dfrac{1}{4}\psi_{m-1,1}-\dfrac{1}{4}\psi_{m-1,3}-\displaystyle\sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j},\cos\left(\frac{1}{2}\ t\right)\right)=0\\ \dfrac{(2(k+1)+1)^2-(2n+1)^2}{4}\psi_{m,2(k+1)+1}-\dfrac{1}{4}\psi_{m-1,2(k+2)+1}-\dfrac{1}{4}\psi_{m-1,2k+1}+\\ -\displaystyle\sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j},\cos\left(\frac{2(k+1)+1}{2}\ t\right)\right)=0,\qquad k\neq n-1;k=0,1,2,\ldots\ .\end{cases} \tag{2.29}$$

Note that the coefficient of $\cos\left(\frac{2n+1}{2}\, t\right)$ vanishes for all $n \in \mathbb{N}\backslash\{0\}$. We will check this statement for all n because, although n is fixed, it can assume a generic value in $\mathbb{N}\backslash\{0\}$, as λ is a generic eigenvalue of P. For $n = 1$ we have

$$\frac{9-9}{4}\psi_{m,5} - \frac{1}{4}\psi_{m-1,5} - \frac{1}{4}\psi_{m-1,1} + \frac{1}{4}\left[\psi_{m-1,5} + \psi_{m-1,1}\right] +$$
$$-\sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j}, \cos\left(\frac{3}{2}\, t\right)\right) + \sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j}, \cos\left(\frac{3}{2}\, t\right)\right) = 0;$$

and for $n \geq 2$ we get

$$\frac{(2n+1)^2-(2n+1)^2}{4}\psi_{m,2n+1} - \frac{1}{4}\psi_{m-1,2(n+1)+1} - \frac{1}{4}\psi_{m-1,2(n-1)+1} +$$
$$+\frac{1}{4}\left[\psi_{m-1,2n+3} + \psi_{m-1,2n-1}\right] - \sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j}, \cos\left(\frac{2n+1}{2}\, t\right)\right) +$$
$$-\sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j}, \cos\left(\frac{2n+1}{2}\, t\right)\right) = 0.$$

From (2.29), changing index in the last sum, it follows that

$$\begin{cases} \psi_{m,1} = \dfrac{1}{1-4n^2-4n-1}\left[\psi_{m-1,1} + \psi_{m-1,3} + 4\displaystyle\sum_{j=2}^{m-1}\lambda_j\psi_{m-j,1}\right] \\ \psi_{m,2k+1} = \dfrac{1}{4k^2+4k+1-4n^2-4n-1}\left[\psi_{m-1,2(k+1)+1} + \psi_{m-1,2(k-1)+1} + \right. \\ \qquad\qquad \left. +4\displaystyle\sum_{j=2}^{m-1}\lambda_j\psi_{m-j,2k+1}\right] = 0, \qquad k \neq n,\ k = 1,2,3,\dots\ , \end{cases} \tag{2.30}$$

from this the (2.15) follows. □

Now we consider the case $n = 0$, i.e. $\lambda_0 = \frac{1}{4}$.

Theorem 2.2.7. *Let ψ be an even eigenfunction of P associated to the eigenvalue λ given by (2.5), with $\lambda_0 = \frac{1}{4}$. Let ψ satisfies the normalization conditions (2.11), and let (see (2.4))*

$$\psi(t) = \sum_{m=0}^{+\infty}\left(\frac{1}{h^2}\right)^m\ \psi_m(t)$$

be its series expansion. Moreover we recall expansion (2.12)

$$\psi_m(t) = \sum_{k=0}^{+\infty} \psi_{m,2k+1} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2k+1}{2}\ t\right).$$

Then the coefficients $\psi_{m,2k+1}$ *and* λ_m *fulfill the following equations:*

$$\begin{cases} \lambda_0 = \dfrac{1}{4} \\ \lambda_1 = \dfrac{1}{4} \\ \lambda_2 = -\dfrac{1}{4}\psi_{1,3} \\ \lambda_m = -\dfrac{1}{4}\psi_{m-1,3} - \displaystyle\sum_{j=2}^{m-2} \lambda_j \psi_{m-j,1}, \qquad \forall\ m \geq 2, \end{cases} \tag{2.31}$$

$$\psi_1(t) = \frac{1}{8} \cos\left(\frac{3}{2}t\right), \tag{2.32}$$

$$\begin{aligned} \psi_{m,2k+1} = \frac{1}{4k(k+1)} \Bigg[& \psi_{m-1,2(k+1)+1} + \psi_{m-1,\ 2(k-1)+1} - \psi_{m-1,2k+1} + \\ & +4 \sum_{j=2}^{m-1} \lambda_j \psi_{m-j,2k+1} \Bigg], \qquad k \neq n,\ k = 1, 2, 3, \dots\ . \end{aligned} \tag{2.33}$$

(Here we use the convention that $\sum_{n=j}^{k} = 0$ *when* $j > k$*.)*

Proof. By hypothesis we have $\lambda_0 = \frac{1}{4}$. Then, from (2.7) in Proposition 2.2.1, we get $\psi_0(t) = \cos\left(\frac{1}{2}t\right)$. Moreover we have

$$-\psi_1''(t) - \lambda_0\psi_1(t) + \left(\frac{1-\cos(t)}{2}\right)\psi_0(t) - \lambda_1\psi_0(t) = 0. \tag{2.34}$$

By taking the scalar product with ψ_0 on both sides we obtain

$$-(\psi_1'', \psi_0) - \lambda_0(\psi_1, \psi_0) + \left(\left(\frac{1-\cos(t)}{2}\right)\psi_0, \psi_0\right) - \lambda_1(\psi_0, \psi_0) =$$

$$= \left(\left(\frac{1-\cos(t)}{2}\right)\psi_0, \psi_0\right) - \lambda_1(\psi_0, \psi_0) = 0.$$

From the normalization conditions (2.11) we get

$$\lambda_1 = \frac{1}{2} - \frac{1}{2}(\cos(t)\psi_0, \psi_0). \tag{2.35}$$

We have

$$(\cos(t)\psi_0, \psi_0) = \int_{-\pi}^{\pi} \frac{1}{\pi} \cos(t) \cos^2\left(\frac{1}{2}\, t\right) dt = \int_{-\pi}^{\pi} \frac{1}{\pi} \cos(t) \frac{1}{2} \left[1 + \cos(t)\right] dt =$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos^2(t) dt = \int_{-\pi}^{\pi} \frac{1}{4\pi}(1 + \cos(2t)) dt = \frac{1}{2},$$

whence, from (2.35),

$$\lambda_1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Now let

$$\psi_1(t) = \sum_{k=0}^{+\infty} \psi_{1,2k+1} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2k+1}{2} t\right)$$

be the expansion of ψ_1. By substituting this expansion and the values of $\lambda_1,\ \lambda_0$ in equation (2.34) we obtain

$$\sum_{k=0}^{+\infty} \left(\frac{(2k+1)^2 - 1}{4}\right) \psi_{1,\ 2k+1} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2k+1}{2} t\right) + \frac{1}{2}\psi_0 - \frac{\cos(t)}{2}\psi_0 - \frac{1}{4}\psi_0 = 0,$$

that is

$$\sum_{k=0}^{+\infty} \left(\frac{4k^2 + 1 + 4k - 1}{4}\right) \psi_{1,\ 2k+1} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2k+1}{2} t\right) +$$

$$+ \frac{1}{4\sqrt{\pi}} \cos\left(\frac{t}{2}\right) - \frac{1}{2\sqrt{\pi}} \cos(t) \cos\left(\frac{t}{2}\right) = 0.$$

In other words

$$\sum_{k=0}^{+\infty} (k^2 + k) \psi_{1,2k+1} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2k+1}{2} t\right) +$$

$$+ \frac{1}{4\sqrt{\pi}} \cos\left(\frac{t}{2}\right) - \frac{1}{4\sqrt{\pi}} \cos\left(\frac{3}{2}\, t\right) - \frac{1}{4\sqrt{\pi}} \cos\left(\frac{t}{2}\right) = 0.$$

From here we get

$$\sum_{k=0}^{+\infty}(k^2+k)\psi_{1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\ t\right)-\frac{1}{4\sqrt{\pi}}\cos\left(\frac{3}{2}\ t\right)=0, \qquad \forall\ t\in[-\pi,\pi]. \tag{2.36}$$

From the normalization conditions (2.11) we have $(\psi_1,\psi_0)=0$, whence $\psi_{1,1}=(\psi_1,\psi_0)=0$. Note that the coefficient of $\cos\left(\frac{t}{2}\right)$ in (2.36) vanishes, whence the equation does not provide any condition on $\psi_{1,1}$. From (2.36) the coefficients of $\cos\left(\left(\frac{2j+1}{2}\right)t\right)$ must vanish for all $j\in\mathbb{N}$. We have $\psi_{1,2k+1}=0$ for every k different from 1. For $k=1$ we get the condition

$$2\psi_{1,3}-\frac{1}{4}=0,$$

that is

$$\psi_{1,3}=\frac{1}{8}.$$

Therefore we have obtained (2.32). Now we seek a general formula for the remaining coefficients. When $m\geq 2$ we have, from Proposition 2.2.1,

$$-\psi_m''(t)-\lambda_0\psi_m(t)+\frac{1}{2}\psi_{m-1}(t)-\frac{1}{2}\cos(t)\psi_{m-1}(t)-\lambda_1\psi_{m-1}(t)-\sum_{j=2}^{m}\lambda_j\psi_{m-j}=0.$$

On recalling that $\lambda_1=\frac{1}{4}$ we get

$$-\psi_m''(t)-\lambda_0\psi_m(t)+\frac{1}{4}\psi_{m-1}-\frac{1}{2}\cos(t)\psi_{m-1}(t)-\lambda_m\psi_0-\sum_{j=2}^{m-1}\lambda_j\psi_{m-j}=0, \tag{2.37}$$

with the convention that for $m=2$ the last sum vanishes. Notice that

$$(-\psi_m'',\psi_0)-(\lambda_0\psi_m,\psi_0)=(\psi_m,-\psi_0'')-(\psi_m,\lambda_0\psi_0)=$$

$$=(\psi_m,\lambda_0\psi_0-\lambda_0\psi_0)=0 \tag{2.38}$$

By taking on both sides of (2.37) the scalar product with ψ_0, from (2.38), and the normalization conditions (2.11) we have

$$\lambda_m = \frac{1}{4}(\psi_{m-1},\psi_0) - \frac{1}{2}\left(\cos(t)\psi_{m-1},\psi_0\right) - \sum_{j=2}^{m-1}\lambda_j\left(\psi_{m-j},\psi_0\right). \tag{2.39}$$

Let ψ_{m-1} be given by

$$\sum_{k=0}^{+\infty}\psi_{m-1,2k+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\,t\right).$$

We compute the term $(\cos(t)\psi_{m-1},\psi_0)$ in (2.39) by substituting the expansion of ψ_{m-1} :

$$(\cos(t)\psi_{m-1},\psi_0) = \sum_{k=0}^{+\infty}\left(\cos(t)\frac{1}{\sqrt{\pi}}\cos\left(\frac{2k+1}{2}\,t\right),\psi_0\right)\psi_{m-1,2k+1} =$$

$$= \sum_{k=0}^{+\infty}\left(\cos(t)\cos\left(\frac{2k+1}{2}\,t\right),\cos\left(\frac{1}{2}\,t\right)\right)\frac{\psi_{m-1,2k+1}}{\pi}. \tag{2.40}$$

From Remark 2.2.5 we get:

$$\left(\cos(t)\cos\left(\frac{2k+1}{2}\,t\right),\cos\left(\frac{1}{2}\,t\right)\right) =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\cos((k+2)t)+\cos(kt)+\cos((-k+1)t)+\cos((-k-1)t)dt =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\cos(kt)+\cos((-k+1)t)+\cos((-k-1)t)dt. \tag{2.41}$$

From (2.41) the term

$$\left(\cos(t)\cos\left(\frac{2k+1}{2}\,t\right),\cos\left(\frac{1}{2}\,t\right)\right)$$

is different from 0 if and only if $k = 0, 1$. In both cases we have

$$\left(\cos(t)\cos\left(\frac{2k+1}{2}\,t\right),\cos\left(\frac{1}{2}\,t\right)\right) = \frac{\pi}{2}.$$

Substituting in (2.40) gives

$$(\cos(t)\psi_{m-1},\psi_0) = \frac{\pi}{2}\frac{\psi_{m-1,1}}{\pi} + \frac{\pi}{2}\frac{\psi_{m-1,3}}{\pi} = \frac{\psi_{m-1,1}}{2} + \frac{\psi_{m-1,3}}{2}.$$

By substituting the relations obtained up to now in (2.39) we have

$$\lambda_m = \frac{1}{4}\psi_{m-1,1} - \frac{1}{4}\psi_{m-1,1} - \frac{1}{4}\psi_{m-1,3} - \sum_{j=2}^{m-1} \lambda_j(\psi_{m-j}, \psi_0) =$$

$$= -\frac{1}{4}\psi_{m-1,3} - \sum_{j=2}^{m-1} \lambda_j(\psi_{m-j}, \psi_0). \tag{2.42}$$

This completes the proof of (2.31). We now consider once again equation (2.37) to obtain the functions ψ_m. We compute each term separately. The formula (2.26) obtained in Theorem 2.2.6 holds even in this case, so we recall it:

$$-\frac{\cos(t)}{2}\psi_{m-1}(t) = -\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,1} + \psi_{m-1,\ 3}\right]\cos\left(\frac{1}{2}\ t\right) +$$

$$+\sum_{k=0}^{+\infty} -\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2(k+2)+1} + \psi_{m-1,2k+1}\right]\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.43}$$

We compute the term $-\psi_m'' - \lambda_0\psi_m$ using (2.27), which holds true also in this case. Since $\lambda_0 = \frac{1}{4}$, $n = 0$ we have

$$-\psi_m'' - \lambda_0\psi_m = \sum_{k=0}^{+\infty}\left(\frac{(2(k+1)+1)^2-1}{4}\right)\psi_{m,2(k+1)+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.44}$$

Furthermore we get

$$\frac{1}{4}\psi_{m-1} = \frac{1}{4\sqrt{\pi}}\psi_{m-1,1}\cos\left(\frac{t}{2}\right) + \frac{1}{4\sqrt{\pi}}\sum_{k=1}^{+\infty}\psi_{m-1,2k+1}\cos\left(\frac{2k+1}{2}\ t\right) =$$

$$= \frac{1}{4\sqrt{\pi}}\psi_{m-1,1}\cos\left(\frac{t}{2}\right) + \frac{1}{4\sqrt{\pi}}\sum_{k=0}^{+\infty}\psi_{m-1,2(k+1)+1}\cos\left(\frac{2(k+1)+1}{2}\ t\right). \tag{2.45}$$

By substituting (2.42), (2.44), (2.43) and (2.45) in (2.37) we obtain

$$\sum_{k=0}^{+\infty}\left(\frac{(2(k+1)+1)^2-1}{4}\right)\psi_{m,2(k+1)+1}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2(k+1)+1}{2}\ t\right) +$$

$$+\frac{1}{4\sqrt{\pi}}\psi_{m-1,1}\cos\left(\frac{t}{2}\right) + \frac{1}{4\sqrt{\pi}}\sum_{k=0}^{+\infty}\psi_{m-1,2(k+1)+1}\cos\left(\frac{2(k+1)+1}{2}\ t\right) +$$

$$-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,1}+\psi_{m-1,\ 3}\right]\cos\left(\frac{1}{2}\ t\right)+$$

$$+\sum_{k=0}^{+\infty}-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2(k+2)+1}+\psi_{m-1,2k+1}\right]\cos\left(\frac{2(k+1)+1}{2}\ t\right)+$$

$$+\frac{1}{4\sqrt{\pi}}\psi_{m-1,3}\cos\left(\frac{1}{2}\ t\right)+\sum_{j=2}^{m-1}\lambda_j(\psi_{m-j},\psi_0)\psi_0-\sum_{j=2}^{m-1}\lambda_j\psi_{m-j}=0.$$

Collecting the common factors' coefficients gives:

$$\sum_{k=0}^{+\infty}\left[\left(\frac{(2(k+1)+1)^2-1}{4\sqrt{\pi}}\right)\psi_{m,2(k+1)+1}+\frac{1}{4\sqrt{\pi}}\psi_{m-1,2(k+1)+1}+\right.$$

$$\left.-\frac{1}{4\sqrt{\pi}}\left[\psi_{m-1,2(k+2)+1}+\psi_{m-1,2k+1}\right]\right]\cos\left(\frac{2(k+1)+1}{2}\ t\right)+$$

$$-\sum_{j=2}^{m-1}\lambda_j\psi_{m-j}=0, \tag{2.46}$$

because the coefficient of $\cos\left(\frac{1}{2}\right)$ vanishes. Since $\{\cos\left(\frac{2n+1}{2}\ x\right)\}_{n\in\mathbb{N}}$ is an orthogonal basis for the **even** functions in $L^2(I)$ the coefficients of these functions in (2.46) must vanish. We get:

$$\left(\frac{(2(k+1)+1)^2-1}{4}\right)\psi_{m,2(k+1)+1}+\frac{1}{4}\psi_{m-1,\ 2(k+1)+1}+$$

$$-\frac{1}{4}\left[\psi_{m-1,2(k+2)+1}+\psi_{m-1,\ 2k+1}\right]-\sum_{j=2}^{m-1}\lambda_j\psi_{m-j,2(k+1)+1}=0, \qquad k=0,1,2,\dots \tag{2.47}$$

From (2.47), changing index, it follows that

$$\left(\frac{(2k+1)^2-1}{4}\right)\psi_{m,2k+1}=\frac{1}{4}\left[-\psi_{m-1,\ 2k+1}+\psi_{m-1,2(k+1)+1}+\psi_{m-1,2(k-1)+1}\right]+$$

$$+\sum_{j=2}^{m-1}\lambda_j\psi_{m-j,2k+1}=0, \qquad k=1,2,\dots\ , \tag{2.48}$$

and thus (2.33). □

Now we state an analogous result involving odd eigenfunctions of P. As before we state the following theorem for a generic eigenvalue λ of P, with odd associated eigenfunction. So we consider, hereafter $\lambda_0 = n^2$, where n is fixed but generic.

Theorem 2.2.8. *Let φ be an odd eigenfunction of P, associated to the eigenvalue λ, with series expansion given by the equation:*

$$\varphi(t) = \sum_{m=0}^{+\infty} \left(\frac{1}{h^2}\right)^m \varphi_m(t). \tag{2.49}$$

Let λ be given by (2.5), and set $\lambda_0 = n^2,\ n \in \mathbb{N}\backslash\{0\}$. We suppose that φ satisfies the normalization conditions (2.11):

$$\begin{cases} \|\varphi_0\| = 1, \\ \sum_{r=0}^{k} (\varphi_{k-r}, \varphi_r) = 0, \qquad \forall\ k \geq 1. \end{cases}$$

Moreover we set, as $\varphi_m(t)$ are odd functions of $L^2(I)$,

$$\varphi_m(t) = \sum_{k=0}^{+\infty} \varphi_{m,k} \frac{1}{\sqrt{\pi}} \sin(kt).$$

Then, for coefficients $\varphi_{m,k}$ and λ_m, hold the following equations:

$$\begin{cases} \lambda_0 = n^2 \\ \lambda_1 = \dfrac{1}{2} \\ \lambda_2 = -\dfrac{1}{4}\varphi_{1,3}, \qquad n = 2, \\ \lambda_m = -\dfrac{1}{4}\varphi_{m-1,3} - \displaystyle\sum_{j=2}^{m-2} \lambda_j \varphi_{m-j,2}, \qquad n = 2,\ \forall\ m \geq 2 \\ \lambda_2 = -\dfrac{1}{4}\left[\varphi_{1,n-1} + \varphi_{1,n+1}\right], \qquad n \neq 2 \\ \lambda_m = -\dfrac{1}{4}\left[\varphi_{m-1,n-1} + \varphi_{m-1,n+1}\right] - \displaystyle\sum_{j=2}^{m-2} \lambda_j \varphi_{m-j,n}, \qquad n \neq 2,\ \forall\ m \geq 2. \end{cases} \tag{2.50}$$

$$\begin{cases} \varphi_{m,1} = \dfrac{1}{4(1-n^2)} \left[\varphi_{m-1,2} + 4 \displaystyle\sum_{j=2}^{m-1} \lambda_j \varphi_{m-j,1} \right], \ n \neq 1 \\ \varphi_{m,k} = \dfrac{1}{4(k^2-n^2)} \left[\varphi_{m-1,k-1} + \varphi_{m-1,k+1} + 4 \displaystyle\sum_{j=2}^{m-1} \lambda_j \varphi_{m-j,k} \right], \quad ; \\ k \neq n, \ k = 2, 3, 4, \dots \end{cases} \tag{2.51}$$

(Here we use the convention that $\sum_{n=j}^{k} = 0$ when $j > k$.)

Proof. From Proposition 2.2.1 we have

$$\lambda_0 = n^2, \qquad -\varphi_0'' = \lambda_0 \varphi_0.$$

Moreover,

$$-\varphi_1'' - \lambda_0 \varphi_1 + \left(\frac{1-\cos(t)}{2} \right) \varphi_0 - \lambda_1 \varphi_0 = 0. \tag{2.52}$$

By taking on both sides the scalar product with φ_0 we obtain

$$-(\varphi_1'', \varphi_0) - \lambda_0(\varphi_1, \varphi_0) + \left(\left(\frac{1-\cos(t)}{2} \right) \varphi_0, \varphi_0 \right) - \lambda_1(\varphi_0, \varphi_0) = 0,$$

that is

$$-(\varphi_1, \varphi_0'') - \lambda_0(\varphi_1, \varphi_0) + \left(\left(\frac{1-\cos(t)}{2} \right) \varphi_0, \varphi_0 \right) - \lambda_1(\varphi_0, \varphi_0) = 0,$$

in other words

$$(\varphi_1, -\varphi_0'' - \lambda_0 \varphi_0) + \left(\left(\frac{1-\cos(t)}{2} \right) \varphi_0, \varphi_0 \right) - \lambda_1(\varphi_0, \varphi_0) =$$
$$= \left(\left(\frac{1-\cos(t)}{2} \right) \varphi_0, \varphi_0 \right) - \lambda_1(\varphi_0, \varphi_0) = 0.$$

From the normalization conditions (2.11) we get

$$\lambda_1 = \frac{1}{2} - \frac{1}{2}(\cos(t)\varphi_0, \varphi_0). \tag{2.53}$$

We have

$$-(\cos(t)\varphi_0, \varphi_0) = -\int_{-\pi}^{\pi} \cos(t) \frac{\sin^2(nt)}{\pi} dt =$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\left[\frac{1-\cos(2nt)}{2}\right]dt = \int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(t)\cos(2nt) = 0,$$

hence, from (2.53), $\lambda_1 = \frac{1}{2}$ for all $n \in \mathbb{N}\backslash\{0\}$.

Now let

$$\varphi_1(t) = \sum_{k=1}^{+\infty}\varphi_{1,k}\frac{1}{\sqrt{\pi}}\sin(kt)$$

be the expansion of φ_1. Substituting this expansion and the value of λ_1 in equation (2.52) gives

$$\sum_{k=1}^{+\infty}(k^2-n^2)\varphi_{1,k}\frac{\sin(kt)}{\sqrt{\pi}} + \frac{\sin(nt)}{2\sqrt{\pi}} - \frac{\cos(t)}{2\sqrt{\pi}}\sin(nt) - \frac{\sin(nt)}{2\sqrt{\pi}} = 0,$$

that is

$$\sum_{k=1}^{+\infty}(k^2-n^2)\varphi_{1,k}\sin(kt) - \frac{1}{4}\left[\sin((n+1)t)+\sin((n-1)t)\right] = 0, \qquad \forall\, t \in [-\pi,\pi].$$

It follows that the coefficients of $\sin(jt)$ must vanish for all $j \in \mathbb{N}$. We have $\varphi_{1,k} = 0$ for all $k \neq n+1,\ n-1$, for $\varphi_{1,n}$ vanishes because of the normalization condition. For $k = n+1$ we get

$$((n+1)^2-n^2)\varphi_{1,n+1} - \frac{1}{4} = 0,$$

that is

$$\varphi_{1,n+1} = \frac{1}{4(2n+1)}.$$

If $n = 1$ the term $\sin((n-1)t)$ vanishes, and thus the equation does not provide any further condition on the coefficients $\varphi_{1,k}$. If $n \neq 1$ we have the condition

$$\varphi_{1,n-1}(n^2+1-2n-n^2) - \frac{1}{4} = 0,$$

that is

$$\varphi_{1,n-1} = \frac{1}{4(1-2n)}.$$

Thus, for $\varphi_1(t)$ we have

$$\varphi_1(t) = \frac{1}{4(1+2n)}\sin((n+1)t) + \frac{1}{4(1-2n)}\sin((n-1)t), \qquad n \geq 1.$$

Now we find a general formula for the remaining coefficients. If $m \geq 2$, from Proposition 2.2.1, we have

$$-\varphi_m''(t) - \lambda_0\varphi_m(t) + \frac{1}{2}\varphi_{m-1}(t) - \frac{1}{2}\cos(t)\varphi_{m-1}(t) - \lambda_1\varphi_{m-1}(t) - \sum_{j=2}^{m}\lambda_j\varphi_{m-j}(t) = 0,$$

that is, on recalling that $\lambda_1 = \frac{1}{2}$,

$$-\varphi_m''(t) - \lambda_0\varphi_m(t) - \frac{1}{2}\cos(t)\varphi_{m-1}(t) - \lambda_m\varphi_0(t) - \sum_{j=2}^{m-1}\lambda_j\varphi_{m-j}(t) = 0, \tag{2.54}$$

with the convention that, for $m = 2$ the last sum vanishes. Note that

$$(-\varphi_m'', \varphi_0) - (\lambda_0\varphi_m, \varphi_0) = (\varphi_m, -\varphi_0'') - (\varphi_m, \lambda_0\varphi_0) = (\varphi_m, \lambda_0\varphi_0 - \lambda_0\varphi_0) = 0. \tag{2.55}$$

By taking the scalar product with φ_0 on both sides of (2.54), from (2.55), and the normalization condition, we have

$$\lambda_m = -\frac{1}{2}\left(\cos(t)\varphi_{m-1}, \varphi_0\right) - \sum_{j=2}^{m-1}\lambda_j\left(\varphi_{m-j}, \varphi_0\right). \tag{2.56}$$

Let φ_{m-1} be given by

$$\sum_{k=1}^{+\infty}\varphi_{m-1,k}\frac{1}{\sqrt{\pi}}\sin(kt).$$

We compute the term $(\cos(t)\varphi_{m-1}, \varphi_0)$ of (2.56) by substituting the expansion of φ_{m-1} :

$$(\cos(t)\varphi_{m-1}, \varphi_0) = \sum_{k=1}^{+\infty}\frac{1}{\sqrt{\pi}}\left(\cos(t)\sin(kt), \varphi_0\right)\varphi_{m-1,k}. \tag{2.57}$$

At first we compute $\frac{1}{\sqrt{\pi}}\left(\cos(t)\sin(kt), \varphi_0\right)$. We get

$$\frac{1}{\sqrt{\pi}}\left(\cos(t)\sin(kt), \varphi_0\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(kt)\sin(nt)dt =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \frac{1}{2} \left[\cos[(k-n)t] - \cos[(k+n)t] \right] dt =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{4} \left[\cos[(k-n+1)t] - \cos[(k-n-1)t] - \cos[(k+n+1)t] + \right.$$

$$\left. - \cos[(k+n-1)t] \right] dt.$$

From here we have

$$\frac{1}{\sqrt{\pi}} (\cos(t)\sin(kt), \varphi_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{4} \left[\cos[(k-n+1)t] - \cos[(k-n-1)t] \right] dt, \tag{2.58}$$

where $k,\ n \geq 1$. The term

$$\frac{1}{\sqrt{\pi}} (\cos(t)\sin(kt), \varphi_0)$$

is different from 0 if and only if $k = n-1$, for $n \geq 2$, and $k = n+1$ (recall that n is fixed. In either case we obtain

$$(\cos(t)\sin(kt), \varphi_0) = \frac{1}{2}.$$

By substituting in (2.57) we get

$$(\cos(t)\varphi_{m-1}, \varphi_0) = \frac{1}{2} \left[\varphi_{m-1,n-1} + \varphi_{m-1,n+1} \right],$$

with the convention that if $n = 1$ then the term $\varphi_{m-1,n-1}$ vanishes. Substituting the obtained results in (2.56) gives

$$\lambda_m = -\frac{1}{4}\varphi_{m-1,3} - \sum_{j=2}^{m-1} \lambda_j (\varphi_{m-j}, \varphi_0), \qquad n = 2$$

and

$$\lambda_m = -\frac{1}{4}\varphi_{m-1,n-1} - \frac{1}{4}\varphi_{m-1,n+1} - \sum_{j=2}^{m-1} \lambda_j (\varphi_{m-j}, \varphi_0), \qquad n \geq 2. \tag{2.59}$$

This completes the proof of (2.50).

Now we consider once again the equation (2.54) to obtain the functions φ_m. We compute each term separately. We have:

$$-\frac{\cos(t)}{2}\varphi_{m-1}(t) = \sum_{k=1}^{+\infty} -\frac{\cos(t)}{2}\varphi_{m-1,k}\frac{1}{\sqrt{\pi}}\sin(kt) =$$

$$= \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\left[\sin((k+1)t) + \sin((k-1)t)\right].$$

Thus

$$-\frac{\cos(t)}{2}\varphi_{m-1}(t) =$$

$$= \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\sin((k+1)t) + \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\sin((k-1)t). \tag{2.60}$$

For the second term of (2.60) we get:

$$\sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\sin((k-1)t) =$$

$$= -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,2}\sin(t) + \sum_{k=3}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\sin((k-1)t).$$

Changing index in the last sum we have

$$\sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\sin((k-1)t) =$$

$$= -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,2}\sin(t) + \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k+2}\sin((k+1)t). \tag{2.61}$$

By substituting (2.61) in (2.60) and changing index we obtain:

$$-\frac{\cos(t)}{2}\varphi_{m-1}(t) = \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k}\sin((k+1)t) - \frac{1}{4\sqrt{\pi}}\varphi_{m-1,2}\sin(t) +$$

$$+ \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,k+2}\sin((k+1)t) =$$

$$= -\frac{1}{4\sqrt{\pi}}\varphi_{m-1,2}\sin(t) + \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\left[\varphi_{m-1,k} + \varphi_{m-1,k+2}\right]\sin((k+1)t). \tag{2.62}$$

We compute the term $-\varphi_m'' - \lambda_0\varphi_m$. Since $\lambda_0 = n^2$ we have

$$-\varphi_m'' - \lambda_0\varphi_m = \sum_{k=1}^{+\infty}(k^2 - n^2)\varphi_{m,k}\frac{1}{\sqrt{\pi}}\sin(kt) =$$

$$= (1-n^2)\varphi_{m,1}\frac{1}{\sqrt{\pi}}\sin(t) + \sum_{k=2}^{+\infty}(k^2 - n^2)\varphi_{m,k}\frac{1}{\sqrt{\pi}}\sin(kt) =$$

$$= (1-n^2)\varphi_{m,1}\frac{1}{\sqrt{\pi}}\sin(t) + \sum_{k=1}^{+\infty}((k+1)^2 - n^2)\varphi_{m,k+1}\frac{1}{\sqrt{\pi}}\sin((k+1)t). \quad (2.63)$$

Substituting (2.59), (2.63) and (2.62) in (2.54) gives

$$(1-n^2)\varphi_{m,1}\frac{1}{\sqrt{\pi}}\sin(t) + \sum_{k=1}^{+\infty}((k+1)^2 - n^2)\varphi_{m,k+1}\frac{1}{\sqrt{\pi}}\sin((k+1)t)+$$

$$-\frac{1}{4\sqrt{\pi}}\varphi_{m-1,2}\sin(t) + \sum_{k=1}^{+\infty} -\frac{1}{4\sqrt{\pi}}\left[\varphi_{m-1,k} + \varphi_{m-1,k+2}\right]\sin((k+1)t)+$$

$$+\frac{1}{4\sqrt{\pi}}\varphi_{m-1,n-1}\sin(nt) + \frac{1}{4}\varphi_{m-1,n+1}\frac{1}{\sqrt{\pi}}\sin(nt)+$$

$$+\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\varphi_0)\frac{1}{\sqrt{\pi}}\sin(nt) - \sum_{j=2}^{m-1}\lambda_j\varphi_{m-j} = 0.$$

Collecting the common factors' coefficients gives:

$$\left[(1-n^2)\varphi_{m,1} - \frac{1}{4}\varphi_{m-1,2}\right]\frac{1}{\sqrt{\pi}}\sin(t)+$$

$$+\sum_{k=1}^{+\infty}\left[((k+1)^2 - n^2)\varphi_{m,k+1} - \frac{1}{4}\left[\varphi_{m-1,k} + \varphi_{m-1,k+2}\right]\right]\frac{1}{\sqrt{\pi}}\sin((k+1)t)+$$

$$+\left[\frac{1}{4}\varphi_{m-1,n-1} + \frac{1}{4}\varphi_{m-1,n+1} + \sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\varphi_0)\right]\frac{1}{\sqrt{\pi}}\sin(nt) - \sum_{j=2}^{m-1}\lambda_j\varphi_{m-j} = 0. \quad (2.64)$$

Since $\{\sin(jx)\}_{j\in\mathbb{N}}$ is an orthogonal basis for the **odd** functions in $L^2(I)$ the coefficients of these functions in (2.64) must vanish. Note that the coefficient of $\sin(nt)$ in (2.64) vanishes for all $n \in \mathbb{N}\backslash\{0\}$. Indeed, for $n = 1$, recalling that in this case we pose, by convention $\varphi_{m,n-1} = 0$ for every $m \in \mathbb{N}$, we have:

$$(1-1)\frac{1}{\sqrt{\pi}}\varphi_{m,1} - \frac{1}{4\sqrt{\pi}}\varphi_{m-1,2} + \frac{1}{4\sqrt{\pi}}\varphi_{m-1,1-1} + \frac{1}{4\sqrt{\pi}}\varphi_{m-1,2}+$$

$$+\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\frac{1}{\sqrt{\pi}}\sin(t))-\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\frac{1}{\sqrt{\pi}}\sin(t))=0.$$

For $n\geq 2$ we get:

$$(n^2-n^2)\varphi_{m,n}-\frac{1}{4}\left[\varphi_{m-1,n-1}+\varphi_{m-1,n+1}\right]+\frac{1}{4}\varphi_{m-1,n-1}+$$

$$+\frac{1}{4}\varphi_{m-1,n+1}+\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\varphi_0)-\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\varphi_0)=0.$$

The coefficients of $\sin(kx)$ in (2.64), for $k+1\neq n$ must vanish. We obtain:

$$\begin{cases}(1-n^2)\varphi_{m,1}-\frac{1}{4}\varphi_{m-1,2}-\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\sin(t))=0, \quad n\neq 1\\ ((k+1)^2-n^2)\varphi_{m,k+1}-\frac{\varphi_{m-1,k}}{4}-\frac{\varphi_{m-1,k+2}}{4}-\sum_{j=2}^{m-1}\lambda_j(\varphi_{m-j},\sin((k+1)t))=0,\\ \qquad k+1\neq n,\ k\geq 1;\end{cases} \tag{2.65}$$

that is

$$\begin{cases}(1-n^2)\varphi_{m,1}-\frac{1}{4}\varphi_{m-1,2}-\sum_{j=2}^{m-1}\lambda_j\varphi_{m-j,1}=0, \quad n\neq 1\\ (k^2-n^2)\varphi_{m,k}-\frac{1}{4}\left[\varphi_{m-1,k-1}+\varphi_{m-1,k+1}\right]-\sum_{j=2}^{m-1}\lambda_j\varphi_{m-j,k}=0,\\ \qquad k\neq n,\ k\geq 2.\end{cases} \tag{2.66}$$

Hence

$$\begin{cases}(1-n^2)\varphi_{m,1}=\frac{1}{4}\varphi_{m-1,2}+\sum_{j=2}^{m-1}\lambda_j\varphi_{m-j,1}, \quad n\neq 1\\ (k^2-n^2)\varphi_{m,k}=\frac{1}{4}\left[\varphi_{m-1,k-1}+\varphi_{m-1,k+1}\right]+\sum_{j=2}^{m-1}\lambda_j\varphi_{m-j,k},\\ \qquad k\neq n,\ k\geq 2;\end{cases} \tag{2.67}$$

from which (2.51) follows. □

Chapter 3

Analysis of eigenvalues and continued fractions

3.1 Necessary conditions for eigenfunctions

We analyse the structure of eigenfunctions by using the Fourier series expansion. In particular we want to substitute the Fourier expansion of a generic eigenfunction of P in the eigenvalue equation (1.7) and then differentiate term by term, getting in this way conditions on the Fourier coefficients of eigenfunctions (as distributions). Notice that, since we are studying a Sturm-Liouville problem, the choice of the Fourier basis in using this procedure is fundamental. In fact, if we chose for instance the classic Fourier basis for $L^2(I)$, i.e. $\{1, \cos(nx), \sin(nx); n \in \mathbb{N}\backslash\{0\}\}$, we would not be able to find all eigenvalues of P (the trouble arises because the eigenfunctions vanish on the boundary of I but the $\cos(nx)$ do not).

Notice that the eigenfunctions of the problem belong to $D(P)$, thus a proper basis to be used for their expansion is formed interely by functions in $D(P)$. We will use, to this purpose, the basis of the eigenfunctions of the operator $P(0)$ (see (2.3)), which is suggested by the analysis of the problem made in the previous chapter.

Recall that the eigenfunctions of the operator

$$P(0) : H_0^1(I) \cap H^2(I) \longrightarrow L^2(I), \qquad f \longmapsto -f'',$$

form a complete orthonormal basis of the space $L^2(I)$.

Indeed the operator $P(0)$ is selfadjoint from Proposition 1.1.5, with compact resolvent; from Hilbert-Schmidt's theorem we have that the eigenfunctions of $P(0)$ are a complete orthonormal basis of $L^2(I)$.

The normalized eigenfunctions of the operator $P(0)$ are

$$\frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\, x\right), \quad n \in \mathbb{N}$$

and

$$\frac{1}{\sqrt{\pi}} \sin(nx), \quad n \in \mathbb{N}\backslash\{0\}.$$

We will expand the eigenfunctions of P with respect to the basis of $L^2(I)$ formed by the eigenfunctions of $P(0)$. By substituting this expansion in the eigenvalue equation for P we will get a recurrence relation for the Fourier coefficients of the eigenfunctions. Afterwards we will analyse this recurrence relation using the continued fraction theory. This study will provide necessary and sufficient conditions for the eigenvalues of P.

In the first place we make some basic remarks about the eigenfunctions of P.

Remark 3.1.1. *All eigenfunctions of P are real-valued.*

Proof. P is a self-adjoint operator with real coefficients. □

Remark 3.1.2. *Let ψ be an eigenfunction of P. Then ψ is either even or odd.*

Proof. Using the notation fixed in Proposition 1.1.7 this result follows because the operators A and T preserve the parity of functions, then also P does. □

As the eigenfunctions of $P(0)$ form a complete basis of $L^2(I)$, we can expand every function of this space with respect to this basis. In particular for odd and even functions we can state the following

Remark 3.1.3. *If $v \in D(P)$ is an even function then it admits the following Fourier series expansion:*

$$v(x) = \sum_{n=0}^{+\infty} v_n \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2} x\right), \tag{3.1}$$

with $v_n = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2} x\right) v(x)dx$, for all $n \in \mathbb{N}$.

Remark 3.1.4. *If $u \in D(P)$ is an odd function then it admits the following Fourier series expansion:*

$$u(x) = \sum_{n=1}^{+\infty} u_n \frac{1}{\sqrt{\pi}} \sin(nx), \tag{3.2}$$

with $u_n = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin(nx)u(x)dx$, for all $n \in \mathbb{N}\backslash\{0\}$.

We want to find out necessary conditions on the Fourier coefficients of eigenfunctions of P. By Remark 3.1.2 we can treat separately the odd and the even eigenfunctions and Remarks 3.1.3 and 3.1.4 provide particular Fourier expansions for even and odd eigenfunctions of P.

In the next propositions we assume that all the equalities are intended in the distribution sense, and thus we will use the theorem of differentiation term by term. Then we will find conditions for the convergence of these Fourier coefficients which will justify the use of that theorem.

Proposition 3.1.5. *Let $v \in D(P)$ be an even function with Fourier expansion given by (3.1). We assume that v is an eigenfunction for P associated to the eigenvalue λ, i.e. such that*

$$Pv = -v'' + \frac{1}{h^2}\sin^2\left(\frac{x}{2}\right) v = \lambda v, \qquad on \ \ [-\pi, \pi], \qquad v(\pm\pi) = 0.$$

Then the coefficients v_n of the Fourier expansion of v fulfill the following conditions:

$$v_1 = \left(h^2 + 1 - 4\lambda h^2\right) v_0; \tag{3.3}$$

$$v_{n+1} = \left((2n+1)^2 h^2 + 2 - 4\lambda h^2\right) v_n - v_{n-1}, \qquad n \in \mathbb{N}\backslash\{0\}. \tag{3.4}$$

To prove this proposition we will use the formulas of the following

Lemma 3.1.6. *Let v be an eigenfunction of P, even, associated to the eigenvalue λ. Then we have:*

$$-v''(x) = \frac{v_0}{4\sqrt{\pi}} \cos\left(\frac{1}{2}\ x\right) + \sum_{n=1}^{+\infty} \frac{(2n+1)^2 v_n}{4\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right), \tag{3.5}$$

$$\frac{1}{2h^2} v(x) = \frac{v_0}{2h^2\sqrt{\pi}} \cos\left(\frac{1}{2}\ x\right) + \sum_{n=1}^{+\infty} \frac{v_n}{2h^2\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right), \tag{3.6}$$

$$-\frac{1}{2h^2}\cos(x)v(x) = -\frac{(v_0+v_1)}{4h^2\sqrt{\pi}} \cos\left(\frac{1}{2}x\right) - \sum_{n=1}^{+\infty} \frac{(v_{n-1}+v_{n+1})}{4h^2\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right), \tag{3.7}$$

$$\lambda v(x) = \frac{\lambda v_0}{\sqrt{\pi}} \cos\left(\frac{1}{2}\ x\right) + \sum_{n=1}^{+\infty} \frac{\lambda v_n}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right). \tag{3.8}$$

Proof. Relation (3.5) follows from the equalities

$$-v''(x) = -\left[\sum_{n=0}^{+\infty} \frac{v_n}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right)\right]'' =$$

$$= -\left[\sum_{n=0}^{+\infty} \frac{v_n(2n+1)}{2\sqrt{\pi}} \left(-\sin\left(\frac{2n+1}{2} x\right)\right)\right]' =$$

$$= \sum_{n=0}^{+\infty} \frac{v_n(2n+1)^2}{4\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right) =$$

$$= \frac{v_0}{4\sqrt{\pi}} \cos\left(\frac{1}{2}\ x\right) + \sum_{n=1}^{+\infty} \frac{v_n(2n+1)^2}{4\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right).$$

Relation (3.6) is straightforward on recalling the Fourier expansion of v. In fact

$$\frac{1}{2h^2} v(x) = \sum_{n=0}^{+\infty} \frac{1}{2h^2} \left(\frac{v_n}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right)\right) =$$

$$= \frac{v_0}{2h^2\sqrt{\pi}} \cos\left(\frac{1}{2}\ x\right) + \sum_{n=1}^{+\infty} \frac{v_n}{2h^2\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right).$$

To obtain (3.7) notice that

$$-\frac{1}{2h^2} \cos(x) v(x) = -\frac{1}{2h^2} \cos(x) \left[\sum_{n=0}^{+\infty} \frac{v_n}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\ x\right)\right] =$$

$$= -\sum_{n=0}^{+\infty} \frac{v_n}{2h^2\sqrt{\pi}} \cos(x) \cos\left(\frac{2n+1}{2}\ x\right) =$$

$$= -\sum_{n=0}^{+\infty} \frac{v_n}{4h^2\sqrt{\pi}} \left[\cos\left(\frac{2(n-1)+1}{2}\ x\right) + \cos\left(\frac{2(n+1)+1}{2}\ x\right)\right] =$$

Upon setting $k = n+1$ we get

$$-\frac{1}{2h^2} \cos(x) v(x) = -\sum_{n=0}^{+\infty} \frac{v_n}{4h^2\sqrt{\pi}} \cos\left(\frac{2(n-1)+1}{2}\ x\right) +$$

$$-\sum_{k=1}^{+\infty} \frac{v_{k-1}}{4h^2\sqrt{\pi}} \cos\left(\frac{(2k+1)}{2} x\right) =$$

$$= -\frac{v_0}{4h^2\sqrt{\pi}} \cos\left(-\frac{1}{2}\ x\right) - \frac{v_1}{4h^2\sqrt{\pi}} \cos\left(\frac{1}{2}\ x\right) +$$

$$-\sum_{n=2}^{+\infty}\frac{v_n}{4h^2\sqrt{\pi}}\cos\left(\frac{2(n-1)+1}{2}\;x\right)-\sum_{k=1}^{+\infty}\frac{v_{k-1}}{4h^2\sqrt{\pi}}\cos\left(\frac{(2k+1)}{2}x\right).$$

Substituting $k=n-1$ in the first sum gives

$$-\frac{1}{2h^2}\cos(x)v(x)=-\frac{(v_0+v_1)}{4h^2\sqrt{\pi}}\cos\left(\frac{1}{2}x\right)+$$

$$-\sum_{k=1}^{+\infty}\frac{v_{k+1}}{4h^2\sqrt{\pi}}\cos\left(\frac{(2k+1)}{2}x\right)-\sum_{k=1}^{+\infty}\frac{v_{k-1}}{4h^2\sqrt{\pi}}\cos\left(\frac{(2k+1)}{2}x\right)$$

whence (3.7) follows.

Equation (3.8) is easily obtained from the expansion of v. □

Proof of Proposition 3.1.5. By hypothesis v is an eigenfunction for P associated to λ, namely $Pv=\lambda v$, that is

$$-v''+\frac{1}{h^2}\sin^2\left(\frac{x}{2}\right)v=\lambda v.$$

It follows that

$$-v''+\frac{1}{2h^2}\left(1-\cos(x)\right)v=\lambda v. \tag{3.9}$$

By substituting (3.5), (3.6), (3.7) and (3.8) in (3.9) we get

$$\frac{v_0}{4\sqrt{\pi}}\cos\left(\frac{1}{2}\;x\right)+\sum_{n=1}^{+\infty}\frac{(2n+1)^2v_n}{4\sqrt{\pi}}\cos\left(\frac{2n+1}{2}\;x\right)+$$

$$+\frac{v_0}{2h^2\sqrt{\pi}}\cos\left(\frac{1}{2}\;x\right)+\sum_{n=1}^{+\infty}\frac{v_n}{2h^2\sqrt{\pi}}\cos\left(\frac{2n+1}{2}\;x\right)+$$

$$-\frac{(v_0+v_1)}{4h^2\sqrt{\pi}}\cos\left(\frac{1}{2}\;x\right)-\sum_{n=1}^{+\infty}\frac{(v_{n-1}+v_{n+1})}{4h^2\sqrt{\pi}}\cos\left(\frac{2n+1}{2}\;x\right)=$$

$$=\frac{\lambda v_0}{\sqrt{\pi}}\cos\left(\frac{1}{2}\;x\right)+\sum_{n=1}^{+\infty}\frac{\lambda v_n}{\sqrt{\pi}}\cos\left(\frac{2n+1}{2}\;x\right),$$

that is

$$\left(\frac{v_0}{4}+\frac{v_0}{2h^2}-\frac{(v_1+v_0)}{4h^2}-\lambda v_0\right)\cos\left(\frac{1}{2}\;x\right)+$$

$$+\sum_{n=1}^{+\infty}\left(\frac{(2n+1)^2 v_n}{4}+\frac{v_n}{2h^2}-\frac{(v_{n-1}+v_{n+1})}{4h^2}-\lambda v_n\right)\cos\left(\frac{2n+1}{2}\,x\right)=0. \quad (3.10)$$

Since $\{\cos\left(\frac{2n+1}{2}\,x\right)\}_n$ is an orthogonal basis for the even functions of $L^2(I)$, all the coefficients in (3.10) vanish:

$$\frac{v_0}{4}+\frac{v_0}{2h^2}-\frac{(v_1+v_0)}{4h^2}-\lambda v_0=0,$$

$$\frac{(2n+1)^2 v_n}{4}+\frac{v_n}{2h^2}-\frac{(v_{n-1}+v_{n+1})}{4h^2}-\lambda v_n=0, \quad n\geq 1.$$

From this (3.3) and (3.4) follow. □

In a similar way we get necessary conditions on coefficients of odd eigenfunctions.

Proposition 3.1.7. *Let $u\in D(P)$ be an odd function with Fourier series expansion given by (3.2). Suppose that u is an eigenfunction for P associated to the eigenvalue λ, namely such that*

$$Pu=-u''+\frac{1}{h^2}\sin^2\left(\frac{x}{2}\right)u=\lambda u, \qquad on\ \ [-\pi,\pi], \qquad u(\pm\pi)=0.$$

Then the coefficients u_n of the Fourier expansion of u fulfill the following conditions:

$$u_2=\left(4h^2+2-4\lambda h^2\right)u_1; \quad (3.11)$$

$$u_{n+1}=\left(4n^2h^2+2-4\lambda h^2\right)u_n-u_{n-1}, \qquad n\in\mathbb{N},\ n\geq 2. \quad (3.12)$$

To prove this proposition we will use the formulas of the following

Lemma 3.1.8. *Let u be an odd eigenfunction of P, associated to λ. Then we have:*

$$-u''(x)=\frac{u_1}{\sqrt{\pi}}\sin(x)+\sum_{n=2}^{+\infty}\frac{n^2 u_n}{\sqrt{\pi}}\sin(nx), \quad (3.13)$$

$$\frac{1}{2h^2}u(x) = \frac{u_1}{2h^2\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{u_n}{2h^2\sqrt{\pi}}\sin(nx), \tag{3.14}$$

$$-\frac{1}{2h^2}\cos(x)u(x) = -\frac{u_2}{4h^2\sqrt{\pi}}\sin(x) - \sum_{n=2}^{+\infty}\frac{(u_{n-1}+u_{n+1})}{4h^2\sqrt{\pi}}\sin(nx), \tag{3.15}$$

$$\lambda u(x) = \frac{\lambda u_1}{\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{\lambda u_n}{\sqrt{\pi}}\sin(nx). \tag{3.16}$$

Proof. Relation (3.13) follows from the equalities

$$-u''(x) = -\left(\sum_{n=1}^{+\infty}\frac{u_n}{\sqrt{\pi}}\sin(nx)\right)'' = -\left(\sum_{n=1}^{+\infty}\frac{u_n n}{\sqrt{\pi}}\cos(nx)\right)' =$$

$$= \sum_{n=1}^{+\infty}\frac{u_n n^2}{\sqrt{\pi}}\sin(nx) = \frac{u_1}{\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{u_n n^2}{\sqrt{\pi}}\sin(nx).$$

Equation (3.14) is straightforward on recalling the Fourier expansion of u. In fact

$$\frac{1}{2h^2}u(x) = \frac{1}{2h^2}\sum_{n=1}^{+\infty}\frac{u_n}{\sqrt{\pi}}\sin(nx) = \frac{u_1}{2h^2\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{u_n}{2h^2\sqrt{\pi}}\sin(nx).$$

To get (3.15) we notice that

$$-\frac{1}{2h^2}\cos(x)u(x) = -\frac{1}{2h^2}\sum_{n=1}^{+\infty}\frac{u_n}{\sqrt{\pi}}\cos(x)\sin(nx) =$$

$$= -\sum_{n=1}^{+\infty}\frac{u_n}{4h^2\sqrt{\pi}}\left[\sin(x(n+1)) + \sin(x(n-1))\right] =$$

$$= -\sum_{n=1}^{+\infty}\frac{u_n}{4h^2\sqrt{\pi}}\sin(x(n+1)) - \sum_{n=1}^{+\infty}\frac{u_n}{4h^2\sqrt{\pi}}\sin(x(n-1)).$$

Changing index we have

$$-\frac{1}{2h^2}\cos(x)u(x) =$$

$$= -\sum_{k=2}^{+\infty}\frac{u_{k-1}}{4h^2\sqrt{\pi}}\sin(kx) - \frac{u_2}{4h^2\sqrt{\pi}}\sin(x) - \sum_{n=3}^{+\infty}\frac{u_n}{4h^2\sqrt{\pi}}\sin(x(n-1)) =$$

$$= -\sum_{k=2}^{+\infty}\frac{u_{k-1}}{4h^2\sqrt{\pi}}\sin(kx) - \frac{u_2}{4h^2\sqrt{\pi}}\sin(x) - \sum_{k=2}^{+\infty}\frac{u_{k+1}}{4h^2\sqrt{\pi}}\sin(kx)$$

whence (3.15) follows.

Relation (3.16) is easily obtained from the expansion of u. □

Proof of Proposition 3.1.7. By reasoning as in the proof of Proposition 3.1.5 we get that u satisfies the equation

$$-u'' + \frac{1}{2h^2}(1-\cos(x))\, u = \lambda u. \tag{3.17}$$

Substituting (3.13), (3.14), (3.15) and (3.16) in (3.17) gives

$$\frac{u_1}{\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{n^2 u_n}{\sqrt{\pi}}\sin(nx) + \frac{u_1}{2h^2\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{u_n}{2h^2\sqrt{\pi}}\sin(nx)+$$

$$-\frac{u_2}{4h^2\sqrt{\pi}}\sin(x) - \sum_{n=2}^{+\infty}\frac{(u_{n-1}+u_{n+1})}{4h^2\sqrt{\pi}}\sin(nx) = \frac{\lambda u_1}{\sqrt{\pi}}\sin(x) + \sum_{n=2}^{+\infty}\frac{\lambda u_n}{\sqrt{\pi}}\sin(nx),$$

that is

$$\left(u_1 + \frac{u_1}{2h^2} - \frac{u_2}{4h^2} - \lambda u_1\right)\sin(x)+$$
$$+\sum_{n=2}^{+\infty}\left(n^2 u_n + \frac{u_n}{2h^2} - \frac{(u_{n-1}+u_{n+1})}{4h^2} - \lambda u_n\right)\sin(nx) = 0. \tag{3.18}$$

Since $\{\sin(nx)\}_{n\in\mathbb{N}}$ is an orthogonal basis for the odd functions in $L^2(I)$, all the coefficients in (3.18) vanish:

$$\frac{u_2}{4h^2} = u_1\left(\frac{1}{2h^2} - \lambda + 1\right),$$

$$\frac{u_{n+1}}{4h^2} = u_n\left(n^2 + \frac{1}{2h^2} - \lambda\right) - \frac{u_{n-1}}{4h^2}, \qquad n \geq 2.$$

From this (3.11) and (3.12) follow. □

We next show that the sequences of coefficients of eigenfunctions, $\{v_n\}_n$, $\{u_n\}_n$, fulfill recurrence relations of the form

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}. \tag{3.19}$$

Studying the properties of this type of relation will give information on Fourier coefficients of the eigenfunctions and eventually on the eigenvalues of P.

Remark 3.1.9. *Using the notation fixed in Proposition 3.1.7 and assuming the same hypotheses, let λ be an eigenvalue of P and let $\{a_n\}_{n\geq -1}$ be the sequence defined by*

$$\begin{cases} a_{-1} = 0 \\ a_n = u_{n+1}, \qquad \forall\, n \in \mathbb{N}. \end{cases} \tag{3.20}$$

Then, setting

$$\gamma_n = \gamma_n(\lambda) := 4(n+1)^2h^2 + 2 - 4\lambda h^2, \qquad \forall\, n \in \mathbb{N}, \tag{3.21}$$

the sequence $\{a_n\}_{n\geq -1}$ satisfies the recurrence relation

$$a_{-1} = 0, \qquad a_{n+1} = \gamma_n a_n - a_{n-1}, \qquad \forall\, n \in \mathbb{N}. \tag{3.22}$$

Using Remark 3.1.9 we can write the Fourier series expansion for odd eigenfunctions in a slightly different form. In particular we have the following

Remark 3.1.10. *Let u be an odd eigenfunction of P. Using the notation fixed in Remark 3.1.9 and assuming the same hypotheses, we have*

$$u = \sum_{n=0}^{+\infty} a_n \frac{1}{\sqrt{\pi}} \sin((n+1)x). \tag{3.23}$$

The following remark grants that also the coefficients of the even eigenfunctions verify a recurrence relation of the form (3.19).

Remark 3.1.11. *Using the notation of Proposition 3.1.5 let λ be an eigenvalue of P. We set by definition $v_{-1} = 0$ and*

$$\begin{cases} \delta_0 = h^2 + 1 - 4\lambda h^2 \\ \delta_n = (2n+1)^2h^2 + 2 - 4\lambda h^2, \qquad \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases} \tag{3.24}$$

Then the sequence $\{v_n\}_{n\geq -1}$ satisfies the following recurrence relation

$$v_{-1} = 0, \qquad v_{n+1} = \delta_n v_n - v_{n-1}, \qquad \forall\, n \in \mathbb{N}. \tag{3.25}$$

3.2 Formulas for solution of the recurrence relations

The aim of this section is to provide formulas for solutions of the equation (3.19), for particular values of $\{\vartheta_n\}_n$. In the first place we state some remarks without any conditions on the sequence $\{\vartheta_n\}_n$, analysing both cases $\{\vartheta_n\}_n := \{\delta_n\}_n$, $\{\vartheta_n\}_n := \{\gamma_n\}_n$ afterwards. These results will be useful in determining eigenfunctions of P (recall that, when $\{\vartheta_n\}_n := \{\delta_n\}_n$ or $\{\vartheta_n\}_n := \{\gamma_n\}_n$, the g_n in (3.19) represent the Fourier coefficients of the eigenfunctions, as stated in Remarks 3.1.9, 3.1.11).

Lemma 3.2.1. *Let $\{g_n\}_{n\geq -1}$ be a sequence different from the 0-sequence, that is such that there exists $n_0 \in \mathbb{N}$ with $g_{n_0} \neq 0$. Assume that $\{g_n\}_{n\geq -1}$ satisfies the recurrence equation*

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}. \tag{3.26}$$

Then the sequence $\{g_n\}_{n\geq -1}$ is not definitely 0 (i.e. there is no n_0 such that $g_n = 0$ for every $n \geq n_0$), in particular $g_n = 0$ implies $g_{n+1} \neq 0$ and $g_{n-1} \neq 0$.

Proof. Reasoning by contradiction, let $m \in \mathbb{N}$ such that $g_m \neq 0$ and such that $g_n = 0$ for all $n > m$. From (3.26), with $n = m + 1$, we get

$$g_{m+2} = \vartheta_{m+1} g_{m+1} - g_m,$$

whence, recalling that $g_{m+2} = g_{m+1} = 0$, we have $g_m = 0$, but this is impossible by hypothesis.

In a similar way it can be proved that if $g_n = 0$ then $g_{n+1} \neq 0$. Indeed, were it not so, we would have, from (3.26), that $g_{n+2} = g_{n+3} = \cdots = 0$, but this is impossible because we proved that $\{g_n\}_{n\geq -1}$ is not definitely the 0-sequence. Also $g_{n-1} \neq 0$, indeed, were it not so, we would have

$$g_{n+1} = \vartheta_n g_n - g_{n-1} = 0,$$

which is absurd from what we have just proved. □

Note that Lemma 3.2.1 can be applied to the recurrence relations (3.22), (3.25) obtained for the coefficients of the eigenfunctions of P, $\{a_n\}_n$, $\{v_n\}_n$ (recall (3.23), (3.1)). Indeed these coefficients can never be all equal to 0, because they are the Fourier coefficients of an eigenfunction. Hence Lemma 3.2.1 states that these sequences cannot have two successive terms that are both 0.

Corollary 3.2.2. *In the hypothesis of Lemma 3.2.1 if $g_{-1} = 0$ then $g_0 \neq 0$.*

Proof. By contradiction, if $g_0 = 0$ then from (3.26) we would have $g_n = 0$ for all $n \in \mathbb{N}$, contradicting the hypothesis. □

The following remarks, remaining true for generic recurrence equations, are particulary useful for studying the sequences $\{v_n\}, \{a_n\}$. For this reason now we fix the notation with the following

Definition 3.2.3. *In the sequel the equation*

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N} \tag{3.27}$$

will denote either equation (3.22) or equation (3.25), where we will have, respectively, either

$$(g_n, \vartheta_n) := (a_n, \gamma_n), \qquad \forall\, n \in \mathbb{N},$$

or

$$(g_n, \vartheta_n) := (v_n, \delta_n), \qquad \forall\, n \in \mathbb{N},$$

recalling formula (3.21) for γ_n *and formula (3.24) for* δ_n. *It is not required that* λ, *appearing in (3.21) and (3.24), is an eigenvalue of* P. *Thus, with these assumptions,* ϑ_n *is always a function of the parameter* λ.

It is worth to remark that *when* λ *is an eigenvalue of* P the sequence $\{g_n\}_n$ of Definition 3.2.3 coincides with the sequence of Fourier coefficients of the eigenfunction associated with λ (see Remarks 3.1.9 and 3.1.11).

The following remarks define, through $\{g_n\}_n$, another sequence, $\{w_n\}_n$, which fulfills a "normal form" of the recurrence relation. From this relation we can find a formula to determine $\{w_n\}_n$, and consequently $\{g_n\}_n$.

Lemma 3.2.4. *Let* $\{\vartheta_n\}_{n\geq 0}$ *be a sequence such that* $\vartheta_n \neq 0$ *for all* $n \in \mathbb{N}$. *Let* $\{g_n\}_{n\geq -1}$ *be a sequence such that* $g_{-1} = 0$.

Then $\{g_n\}_{n\geq -1}$ *is a solution of (3.27):*

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}$$

if and only if $\{w_n\}_{n\geq -1}$ *is a solution of*

$$w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n \in \mathbb{N}, \tag{3.28}$$

with

$$\begin{cases} w_{-1} = 0 \\ w_0 = g_0 \\ w_n = \dfrac{g_n}{\vartheta_0 \ldots \vartheta_{n-1}}, \quad n \in \mathbb{N}\backslash\{0\}, \end{cases} \tag{3.29}$$

and

$$\begin{cases} \alpha_{-1} = 1 \\ \alpha_n = \dfrac{1}{\vartheta_n \vartheta_{n+1}}, \;\; n \in \mathbb{N}. \end{cases} \tag{3.30}$$

Proof. Suppose that

$$g_{n+1} = g_n \vartheta_n - g_{n-1}, \qquad n \in \mathbb{N}.$$

Then, upon dividing by $\vartheta_0 \dots \vartheta_n$ we get

$$\frac{g_{n+1}}{\vartheta_0 \dots \vartheta_{n-2}\vartheta_{n-1}\vartheta_n} = \frac{g_n \vartheta_n}{\vartheta_0 \dots \vartheta_{n-2}\vartheta_{n-1}\vartheta_n} - \frac{g_{n-1}}{\vartheta_0 \dots \vartheta_{n-2}\vartheta_{n-1}\vartheta_n}. \tag{3.31}$$

From (3.29) and setting $\vartheta_{-1} = 1$ in (3.31) we obtain

$$w_{n+1} = w_n - \frac{w_{n-1}}{\vartheta_{n-1}\vartheta_n}, \qquad n \in \mathbb{N},$$

which yields, by (3.30), recalling that $w_{-1} = 0$, formula (3.28).

To prove the converse it suffices to invert the procedure. □

In other words, Lemma 3.2.4 states that we can relate the solutions of equations (3.27) and (3.28) if the coefficients ϑ_n are all different from 0. In this hypotesis we can obtain $\{g_n\}_n$ from the values of $\{w_n\}_n$. We will see that this is true also if $\vartheta_n(\lambda) = 0$ for some $n \in \mathbb{N}$.

It is now convenient to assume that all sequences we will consider from now on take values in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 3.2.5. *Given the sequence $\{\alpha_N\}_{N\in\mathbb{N}}$ in $\widehat{\mathbb{C}}$ we denote by*

$$\{[\alpha_0, \dots, \alpha_j]\}_{j\in\mathbb{N}}$$

the sequence defined by recurrence as

$$\begin{cases} [\alpha_0] = 1 - \alpha_0 \\ [\alpha_0, \dots, \alpha_n] = 1 - \dfrac{\alpha_n}{[\alpha_0, \dots, \alpha_{n-1}]}, \qquad \forall\; n \in \mathbb{N}\backslash\{0\}, \end{cases}$$

where we pose, by convention, $\frac{1}{0} = \infty$ *and* $\frac{1}{\infty} = 0$.

The following Proposition provides a formula that gives the terms of $\{w_n\}_n$ depending on the coefficients in (3.28) (for the detailed proof see [15], p. 570).

Proposition 3.2.6. *Let* $\{w_n\}_{n\geq -1}$ *and* $\{\alpha_n\}_{n\geq -1}$ *be two sequences such that* $w_{-1} = 0$ *and* $\alpha_{-1} = 1$. *We assume that* $\{w_n\}_n$ *fulfills the recurrence equation*

$$w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n \in \mathbb{N}.$$

Moreover, put $z_n = [\alpha_0, \ldots, \alpha_n]$ *for every* $n \in \mathbb{N}$ *and let* $\{Z_n\}_{n\geq 0}$ *be the sequence defined by*

$$\begin{cases} Z_0 = Z_1 = 1 \\ Z_N = \prod_{j=0}^{N-2} z_j^*, \ N \geq 2, \end{cases} \tag{3.32}$$

with

$$z_j^* = \begin{cases} z_j & \textit{if} \quad z_j \neq 0, \infty \\ -\alpha_{j+1} & \textit{if} \quad z_j = 0 \\ 1 & \textit{if} \quad z_j = \infty. \end{cases} \tag{3.33}$$

Then we have

$$w_N = \begin{cases} Z_N w_0 & \textit{if} \quad z_{N-2} \neq 0 \\ 0 & \textit{if} \quad z_{N-2} = 0, \end{cases} \tag{3.34}$$

for all $N \in \mathbb{N}$.

proof (sketch). We consider $\{w_n\}_n$ as a sequence of determinants of proper tridiagonal matrix (depending on coefficients α_n). By triangularizing these matrix we obtain essentially z_n as diagonal elements. □

With the following definition we fix the notation we will use hereafter, for notational simplicity.

Definition 3.2.7. *Following the notation of Proposition 3.2.6, we will write (3.34) simply as*

$$w_N = z_0^* z_1^* \dots z_{N-2}^* w_0, \qquad N \in \mathbb{N}, \tag{3.35}$$

using, as before, relation (3.33) for the coefficients z_j^**, but with the convention of setting, when* $j = N-2$*,* $z_{N-2}^* = 0$ *if* $z_{N-2} = 0$*.*

We next obtain, from Proposition 3.2.6, a formula for the coefficients $\{g_n\}_n$. To recall and summarize the notation fixed up to now we give the following

Definition 3.2.8. *Using the notation fixed in Definition 3.2.3, we denote by* $\{\alpha_n\}_{n\geq -1}$ *the sequence defined by (3.30):*

$$\begin{cases} \alpha_{-1} = 1 \\ \alpha_n = \dfrac{1}{\vartheta_n \vartheta_{n+1}}, \qquad n \in \mathbb{N} \end{cases}$$

and we denote with $\{w_n\}_{n\geq -1}$ *the sequence defined by (3.29):*

$$\begin{cases} w_{-1} = 0 \\ w_0 = g_0 \\ w_n = \dfrac{g_n}{\vartheta_0 \dots \vartheta_{n-1}}, \qquad n \in \mathbb{N}\backslash\{0\}. \end{cases}$$

From Proposition 3.2.6 we get a formula for coefficients g_n. In particular we have the following

Lemma 3.2.9. *Following the notation fixed in Proposition 3.2.6 and in Definitions 3.2.3, 3.2.8, if* $\vartheta_n \neq 0$*, for all* $n \in \mathbb{N}$*, the solution* $\{g_n\}_{n\geq -1}$ *of equation (3.26):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}$$

satisfies

$$\begin{cases} g_1 = \vartheta_0 g_0 \\ \\ g_n = \vartheta_0 \dots \vartheta_{n-1} z_0^* \dots z_{n-2}^* \ g_0, \qquad \forall \ n \geq 2, \end{cases} \tag{3.36}$$

with the convention, given in Definition 3.2.7, that if $z_{n-2} = 0$ *then* $g_n = 0$.

Proof. Equation (3.26) fulfills the hypothesis of Lemma 3.2.4 and therefore can be related to the equation (3.28):

$$w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n \in \mathbb{N}.$$

From Proposition 3.2.6, recalling Definition 3.2.7, we prove the assertion just by substituting (3.29) in (3.35):

$$w_n = z_0^* z_1^* \dots z_{n-2}^* w_0, \qquad n \in \mathbb{N}.$$

This concludes the proof. □

Using Lemma 3.2.9 we will be able to study the behaviour of the Fourier coefficients, g_n, of the eigenfunctions as $n \to +\infty$, thus obtaining a necessary and sufficient condition for the eigenvalues. Before doing this, we give results analogous to Lemma 3.2.9 also in the case there exists $n_0 \in \mathbb{N}$ such that $\vartheta_{n_0} = 0$. Since ϑ_n depends on λ we will have $\vartheta_{n_0}(\lambda) = 0$ for particular values of λ. Recalling Definition 3.2.3, (3.21) and (3.24) we have:

Remark 3.2.10. *Let the sequence* $\{\vartheta_n\}_n = \{\vartheta_n(\lambda)\}_n$ *be defined either by*

$$\vartheta_n := \gamma_n = 4(n+1)^2 h^2 + 2 - 4\lambda h^2, \qquad \forall \ n \in \mathbb{N},$$

or by

$$\vartheta_n := \delta_n = \begin{cases} h^2 + 1 - 4\lambda h^2, & \text{if } \ n = 0 \\ \\ (2n+1)^2 h^2 + 2 - 4\lambda h^2, & \forall \ n \in \mathbb{N} \backslash \{0\}. \end{cases}$$

Thus ϑ_n depends linearly on λ and if there exists λ such that $\vartheta_{n_0}(\lambda) = 0$, for some n_0, then $\vartheta_n(\lambda) \neq 0$ for all $n \neq n_0$.

In the hypothesis of Remark 3.2.10 we will obtain for g_n a formula similar to (3.36). In particular, for the first terms of the sequence we get the following

Lemma 3.2.11. *Let $\{g_n\}_{n\geq -1}$ be a solution of (3.27):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}.$$

Suppose there exists $n_0 \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ such that $\vartheta_{n_0}(\lambda) = 0$. Then, using the notation as in Lemma 3.2.4, we have

$$w_{-1} = 0, \qquad w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n = 0, 1, \ldots, n_0. \tag{3.37}$$

Proof. From Remark 3.2.10 if $n \leq n_0 - 1$ we have $\vartheta_n \neq 0$. To prove (3.37) it suffices to follow the procedure used in Lemma 3.2.4. □

From the proof of Proposition 3.2.6 (see [15], p. 570) it follows that formula (3.34) can be used as well for a finite number of terms of the sequence. In particular we get the following

Remark 3.2.12. *In the hypothesis of Lemma 3.2.11, from Proposition 3.2.6 it follows that*

$$w_N = z_0^* \ldots z_{N-2}^* w_0, \qquad N = 0, 1, \ldots, n_0, \tag{3.38}$$

with the convention, fixed in Definition 3.2.7, that if $z_{N-2} = 0$ then $w_N = 0$.

From here we get at once a formula to compute g_n, with $n = 0, 1, \ldots, n_0$.

Lemma 3.2.13. *In the hypothesis of Lemma 3.2.11 we have*

$$g_n = \vartheta_0 \dots \vartheta_{n-1} z_0^* \dots z_{n-2}^*\, g_0, \qquad n = 0, 1, \dots, n_0, \tag{3.39}$$

with the convention, fixed in Definition 3.2.7, that if $z_{n-2} = 0$ *then* $g_n = 0$.

Proof. From Remark 3.2.12 we get

$$w_n = z_0^* \dots z_{n-2}^* w_0, \qquad n = 0, 1, \dots, n_0.$$

Thus, from (3.29), we have

$$g_n = \vartheta_0 \dots \vartheta_{n-1} z_0^* \dots z_{n-2}^* g_0, \qquad n = 0, 1, \dots, n_0,$$

that is, by recalling the convention fixed in Definition 3.2.7, relation (3.39). □

Now we want to prove that (3.27) can be related to (3.28) even in case there exists $n_0 \in \mathbb{N}$ such that $\vartheta_{n_0} = 0$. In particular it will be shown that the sequence satisfies, from a certain index onward, the hypothesis of Lemma 3.2.4. We will treat separately the cases $n_0 = 0$ and $n_0 \in \mathbb{N}\backslash\{0\}$.

Proposition 3.2.14. *Let* $\{g_n\}_{n \geq -1}$ *be a solution of (3.27):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}$$

and let $\vartheta_0 = 0$. *Then, upon setting*

$$\begin{cases} d_{-1} = 0 \\ d_0 = -g_0 \\ d_k = g_{k+2}, \ \forall\, k \in \mathbb{N}\backslash\{0\} \end{cases} \tag{3.40}$$

and

$$\eta_k = \vartheta_{n+2}, \qquad \forall\, n \in \mathbb{N}, \tag{3.41}$$

we get

$$d_{n+1} = \eta_n d_n - d_{n-1}, \qquad n \in \mathbb{N}. \tag{3.42}$$

Proof. Since $\vartheta_0 = 0$ from (3.27) we have

$$g_1 = \vartheta_0 g_0 = 0$$

$$g_2 = \vartheta_1 g_1 - g_0 = -g_0$$

$$g_3 = \vartheta_2 g_2 - g_1 = \vartheta_2 g_2.$$

Whence from (3.40) and (3.41) it follows (3.42) when $n = 0$.

Substituting (3.40) and (3.41) in (3.27) gives (3.42) also when $n \geq 1$. □

When $n_0 \neq 0$ we have the following

Proposition 3.2.15. *Let $\{g_n\}_{n\geq -1}$ be a solution of (3.27):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}.$$

Assume there exists $n_0 \in \mathbb{N}\backslash\{0\}$ such that $\vartheta_{n_0} = 0$. Then

a) If $g_{n_0} \neq 0$ and $g_{n_0-1} \neq 0$ we have

$$f_{n+1} = \mu_n f_n - f_{n-1}, \qquad n \in \mathbb{N}, \tag{3.43}$$

where

$$\begin{cases} f_{-1} = 0 \\ f_0 = g_{n_0} \\ f_1 = -g_{n_0-1} \\ f_k = g_{n_0+k}, \qquad \forall\ k \geq 2 \end{cases}$$

and

$$\begin{cases} \mu_0 = -\dfrac{g_{n_0-1}}{g_{n_0}} \\ \mu_k = \vartheta_{n_0+k}, \qquad \forall\ k \in \mathbb{N}\backslash\{0\}. \end{cases}$$

b) If $g_{n_0} \neq 0$ and $g_{n_0-1} = 0$ we have

$$p_{n+1} = \nu_n p_n - p_{n-1}, \qquad n \in \mathbb{N}, \tag{3.44}$$

where

$$\begin{cases} p_{-1} = 0 \\ p_k = g_{n_0+2+k}, \qquad \forall\ k \in \mathbb{N}. \end{cases}$$

and

$$\nu_k = \vartheta_{n_0+2+k}, \qquad \forall\ k \in \mathbb{N}.$$

c) If $g_{n_0} = 0$ we have

$$q_{n+1} = \rho_n q_n - q_{n-1}, \qquad n \in \mathbb{N}, \tag{3.45}$$

where

$$\begin{cases} q_{-1} = 0 \\ q_0 = -g_{n_0-1} \\ q_k = g_{n_0+1+k}, \qquad \forall\ k \in \mathbb{N}\backslash\{0\}. \end{cases}$$

and

$$\rho_k = \vartheta_{n_0+1+k}, \qquad \forall\ k \in \mathbb{N}.$$

Moreover we have

$$\begin{cases} \mu_k \neq 0 \\ \nu_k \neq 0 \quad , \quad \forall\ k \in \mathbb{N}. \\ \rho_k \neq 0 \end{cases} \tag{3.46}$$

Proof. Notice that if $\vartheta_{n_0} = 0$ then, from Remark 3.2.10, $\vartheta_k \neq 0$ for all $k \neq n_0$.

a) By hypotesis $\{g_n\}_n$ is a solution of (3.27), whence

$$g_{n_0+1} = \vartheta_{n_0} g_{n_0} - g_{n_0-1},$$

that is

$$g_{n_0+1} = -g_{n_0-1} \tag{3.47}$$

because $\vartheta_{n_0} = 0$.

By substituting

$$\begin{cases} f_n = g_{n_0+n}, \\ \mu_n = \vartheta_{n_0+n}, \end{cases} \quad \forall\ n \in \mathbb{N},$$

in (3.27) we easily obtain (3.43) for $n \geq 1$. If $n = 0$, (3.43) becomes

$$f_1 = \mu_0 f_0 - f_{-1}.$$

We will check that, by hypothesis, this equation is satisfied.

As $f_{-1} = 0$, $\mu_0 = -\dfrac{g_{n_0-1}}{g_{n_0}}$ and $f_0 = g_{n_0}$, substituting these values gives

$$f_1 = -\frac{g_{n_0-1}}{g_{n_0}} g_{n_0},$$

that is

$$f_1 = -g_{n_0-1},$$

which is satisfied by hypothesis.

b) By hypotesis we have $g_{n_0-1} = \vartheta_{n_0} = 0$, hence we obtain, from (3.27),

$$\begin{cases} g_{n_0+1} = \vartheta_{n_0} g_{n_0} - g_{n_0-1} = 0 \\ g_{n_0+2} = \vartheta_{n_0+1} g_{n_0+1} - g_{n_0} = -g_{n_0} \\ g_{n_0+3} = \vartheta_{n_0+2} g_{n_0+2} - g_{n_0+1} = \vartheta_{n_0+2} g_{n_0+2}. \end{cases}$$

By substituting

$$\begin{cases} p_{-1} = 0 \\ p_0 = g_{n_0+2} \\ p_1 = g_{n_0+3} \\ \nu_0 = \vartheta_{n_0+2} \end{cases}$$

we get

$$p_1 = \nu_0 p_0 - p_{-1},$$

which is (3.44) for $n = 0$.

Equation (3.44) for $n \geq 1$ is easily obtained by using in (3.27)

$$\begin{cases} g_{n_0+2+k} = p_k \\ \vartheta_{n_0+2+k} = \nu_k \end{cases}, \ k \in \mathbb{N}\backslash\{0\}.$$

c) The proof is similar to the previous one.

Finally (3.46) is a straightforward consequence of Remark 3.2.10, recalling the definitions of $\{\mu_n\}_n, \ \{\nu_n\}_n, \ \{\rho_n\}_n$. □

Similarly to the case $\vartheta_n \neq 0$ we can show the following

Lemma 3.2.16. *In the hypothesis of Proposition 3.2.15 we have*

a) If $g_{n_0} \neq 0$ and $g_{n_0-1} \neq 0$ then

$$\begin{cases} f_1 = \mu_0 f_0 \\ f_n = \mu_0 \dots \mu_{n-1} z_0^* \dots z_{n-2}^* f_0, \qquad \text{if } z_{n-2} \neq 0, \ n \geq 2, \\ f_n = 0, \qquad \text{if } z_{n-2} = 0, \ n \geq 2, \end{cases} \tag{3.48}$$

where

$$\begin{cases} \alpha_{-1} = 1 \\ \alpha_n = \dfrac{1}{\mu_n \mu_{n+1}}, \end{cases} \tag{3.49}$$

and

$$z_j = [\alpha_0, \dots, \alpha_n], \qquad z_j^* = \begin{cases} z_j, & \text{if } z_j \neq 0, \infty \\ -\alpha_{j+1}, & \text{if } z_j = 0 \\ 1 & \text{if } z_j = \infty. \end{cases} \tag{3.50}$$

b) If $g_{n_0} \neq 0$ *and* $g_{n_0-1} = 0$ *then*

$$\begin{cases} p_1 = \nu_0 p_0 \\ p_n = \nu_0 \dots \nu_{n-1} z_0^* \dots z_{n-2}^* p_0, \qquad \textit{if } z_{n-2} \neq 0,\ n \geq 2, \\ p_n = 0, \qquad \textit{if } z_{n-2} = 0,\ n \geq 2, \end{cases} \tag{3.51}$$

with

$$\begin{cases} \alpha_{-1} = 1 \\ \alpha_n = \dfrac{1}{\nu_n \nu_{n+1}}, \end{cases} \tag{3.52}$$

and

$$z_j = [\alpha_0, \dots, \alpha_n], \qquad z_j^* = \begin{cases} z_j, & \textit{if } z_j \neq 0, \infty \\ -\alpha_{j+1}, & \textit{if } z_j = 0 \\ 1 & \textit{if } z_j = \infty. \end{cases} \tag{3.53}$$

c) If $g_{n_0} = 0$ *then*

$$\begin{cases} q_1 = \rho_0 q_0 \\ q_n = \rho_0 \dots \rho_{n-1} z_0^* \dots z_{n-2}^* q_0, \qquad \textit{if } z_{n-2} \neq 0,\ n \geq 2, \\ q_n = 0, \qquad \textit{if } z_{n-2} = 0,\ n \geq 2, \end{cases} \tag{3.54}$$

with

$$\begin{cases} \alpha_{-1} = 1 \\ \alpha_n = \dfrac{1}{\rho_n \rho_{n+1}}, \end{cases} \tag{3.55}$$

and

$$z_j = [\alpha_0, \dots, \alpha_n], \qquad z_j^* = \begin{cases} z_j, & \textit{if } z_j \neq 0, \infty \\ -\alpha_{j+1}, & \textit{if } z_j = 0 \\ 1 & \textit{if } z_j = \infty. \end{cases} \tag{3.56}$$

Proof. The recurrence relation (3.43) fulfill the hypothesis of Lemma 3.2.9. If we set $g_n = f_n$ and $\vartheta_n = \mu_n$ in Lemma 3.2.9, from (3.36) we obtain (3.48).

In a similar way one can prove (3.51) and (3.54). □

3.3 Continued fractions and necessary and sufficient conditions for the eigenvalues

We will now use the theory of continued fractions in order to study the convergence of coefficients of the eigenfunctions of P. To this purpose we recall several definitions and a classical result on 1-periodic continued fractions (see [11] pp. 7, 8, 9, 59, 103, 150). We recall that the sequences used in these arguments take value in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 3.3.1. *A **continued fraction** is an ordered pair*

$$((\{a_n\}_n, \{b_n\}_n), \{f_n\}_n),$$

where the sequences $\{a_n\}_n$, $\{b_n\}_n \subseteq \mathbb{C}$ *and* $\{f_n\}_n \subseteq \widehat{\mathbb{C}}$ *is given by*

$$f_n = S_n(0), \qquad n = 0, 1, 2, \ldots,$$

where

$$S_0(w) = s_0(w), \qquad S_n(w) = S_{n-1}(s_n(w)), \qquad n = 1, 2, 3, \ldots,$$

$$s_0(w) = b_0 + w, \qquad s_n(w) = \frac{a_n}{b_n + w}, \qquad n = 1, 2, 3, \ldots.$$

We will call $\{f_n\}_n$ the sequence of approximants of the continued fraction.

Definition 3.3.2. *Using the notation of Definition 3.3.1 we define the **n-th approximant** of the continued fraction as*

$$f_n = S_n(0) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots + \cfrac{a_n}{b_n}}}.$$

Moreover, setting $f_n = \frac{A_n}{B_n}$, *we call* A_n *and* B_n *the **n-th canonical numerator** and **denominator**, respectively.*

We introduce a concept of convergence for continued fractions.

Definition 3.3.3. *We say that the continued fraction* $((\{a_n\}_n, \{b_n\}_n), \{f_n\}_n)$ *is convergent to* $f \in \widehat{\mathbb{C}}$ *if*

$$\lim_{n\to+\infty} f_n = f.$$

In this case we write

$$f = b_0 + K_{n=1}^{+\infty}(a_n/b_n).$$

We state some properties on the sequences $\{A_n\}_n$ and $\{B_n\}_n$ of Definition 3.3.2, which will be useful in the sequel.

Remark 3.3.4. *Let* $f = b_0 + K_{n=1}^{+\infty}(a_n/b_n)$ *be a continued fraction and let* $\{A_n\}_n$ *and* $\{B_n\}_n$ *be the sequences of canonical numerators and denominators, respectively.*

If we set $A_{-1} = 1,\ \ A_0 = b_0,\ \ B_{-1} = 0,\ \ B_0 = 1$, *then we have:*

$$\begin{cases} A_{n+1} = b_{n+1}A_n + a_nA_{n-1}, \\ B_{n+1} = b_{n+1}B_n + a_nB_{n-1}, \end{cases} \qquad n \in \mathbb{N}. \tag{3.57}$$

Moreover we have

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}\prod_{k=1}^{n} a_k, \qquad n > 1. \tag{3.58}$$

The following definitions will be applied to study the recurrence relations of Fourier coefficients of eigenfunctions of P, found in the previous section. In particular, as we will see in detail, by showing that $\{z_n\}_n$ is a tail sequence for the continued fraction $K_{n=1}^{+\infty}(-\alpha_n/-1)$, we will obtain an equation, involving this continued fraction, which is a necessary and sufficient condition for λ to be an eigenvalue of P.

Definition 3.3.5. *We say that a sequence* $\{t_n\}_{n\in\mathbb{N}} \subseteq \widehat{\mathbb{C}}$ *is a* ***tail sequence*** *for the continued fraction* $b_0 + K_{n=1}^{+\infty}(a_n/b_n)$ *if*

$$t_{n-1} = \frac{a_n}{b_n + t_n} = s_n(t_n), \qquad n = 1, 2, 3, \ldots.$$

Definition 3.3.6. *A continued fraction* $K_{n=1}^{+\infty}(a_n/b_n)$ *is said to be* ***limit 1-periodic*** *if there exist the limits*

$$\lim_{n\to+\infty} a_n = a^*, \qquad \lim_{n\to+\infty} b_n = b^*,$$

with $a^*, b^* \in \widehat{\mathbb{C}}$.

We can associate to each term of a tail sequence a Möbius transformation, in a natural way, so obtaining a sequence of Möbius transformations. Studying the limit transformation of this sequence will give us particular properties of the continued fraction. An important result is obtained if this limit Möbius transformation is loxodromic.

Definition 3.3.7. *Let*

$$t : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \quad w \longmapsto t(w) = \frac{aw+b}{cw+d},$$

with $ad - bc \neq 0$, *be a Möbius transformation. Let* x *and* y *be two fixed points for* t, *that is* $\lim_{n\to+\infty} t^n(x) = x$ *and* $\lim_{n\to+\infty} t^n(y) = y$. *Then* t *is said to be* ***loxodromic*** *if* $x \neq y$ *and*

$$\begin{cases} |cx+d| > |cy+d|, \ \text{if } c \neq 0, \\ |a| \neq |d|, \ \text{if } c = 0. \end{cases}$$

Definition 3.3.8. *A limit 1-periodic continued fraction* $K_{n=1}^{+\infty}(a_n/b_n)$ *is said to be* ***of loxodromic type*** *if*

$$\lim_{n\to+\infty} a_n = a^* \in \mathbb{C}, \qquad \lim_{n\to+\infty} b_n = b^* \in \mathbb{C}$$

and if the following implications hold:

a) *if $a^* \neq 0$ then $T(w) := \frac{a^*}{b^*+w}$ is loxodromic as a Möbius transformations;*

b) *if $a^* = 0$ then $b^* \neq 0$. In this last case T is a singular transformation, with $T(w) = 0$ for all $w \neq b^*$. We say that $x = 0$ is the* ***attractive fixed point*** *of T and $y = -b^*$ is the* ***repulsive fixed point*** *of T.*

We state a very important property of tail sequences of limit 1-periodic continued fractions of loxodromic type.

Theorem 3.3.9. *Let $K_{n=1}^{+\infty}(a_n/b_n)$ be a limit 1-periodic continued fraction of loxodromic type, where T has attractive fixed point x and repulsive fixed point y. Then $K_{n=1}^{+\infty}(a_n/b_n)$ converges to a value $f \in \widehat{\mathbb{C}}$. Moreover, for every tail sequence $\{z_n\}_n$, we have*

$$\lim_{n\to+\infty} z_n = \begin{cases} x & \text{if } z_0 = f \\ y & \text{if } z_0 \neq f. \end{cases} \tag{3.59}$$

For the proof of this theorem see [11], p. 151.

These results will now be used to analyse the convergence of the coefficients of eigenfunctions of P and, moreover, this will provide the necessary and sufficient condition on eigenvalues of P.

Lemma 3.3.10. *Using the notation of Proposition 3.2.6 the sequence $\{z_n\}_n$ is a tail sequence for the continued fraction $K_{n=1}^{+\infty}(-\alpha_n/-1)$.*

Proof. By definition we have

$$z_n = 1 - \frac{\alpha_n}{z_{n-1}}, \qquad \forall\, n \in \mathbb{N}\backslash\{0\},$$

whence

$$-(z_n - 1) = \frac{\alpha_n}{z_{n-1}},$$

that is

$$z_{n-1} = \frac{-\alpha_n}{z_n - 1}, \qquad \forall\, n \in \mathbb{N}\backslash\{0\},$$

which, recalling Definition 3.3.5, proves the claim. □

Thus, since $K_{n=1}^{+\infty}(-\alpha_n/-1)$ is limit 1-periodic of loxodromic type, we can use Theorem 3.3.9 to have information on $\lim_{n\to+\infty} z_n$.

Proposition 3.3.11. *Using the notation fixed in Definition 3.2.8, let be* $z_n = [\alpha_0, \ldots, \alpha_n]$, $n \in \mathbb{N}$, *then*

$$\lim_{n\to+\infty} z_n = \begin{cases} 0 & \textit{if} \quad z_0 = f = K_{n=1}^{+\infty}(-\alpha_n/-1) \\ 1 & \textit{if} \quad z_0 \neq f. \end{cases}$$

Proof. By Definition 3.2.3 and recalling (3.30) we have that $K_{n=1}^{+\infty}(-\alpha_n/-1)$ is limit 1-periodic of loxodromic type. In fact, for every fixed λ we have

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{1}{\gamma_n\gamma_{n+1}} = \lim_{n\to+\infty} \frac{1}{\delta_n\delta_{n+1}} = 0.$$

Besides, following the notation of Definition 3.3.7, we have, in this case $b^* = -1 \neq 0$. Moreover, by Lemma 3.3.10, z_n is a tail sequence for $K_{n=1}^{+\infty}(-\alpha_n/-1)$. From Theorem 3.3.9 we obtain the assertion. □

Now we want to prove that all values of λ such that $\lim_{n\to+\infty} z_n = 0$, and only those values, are related, through the recurrence relations, to the Fourier coefficients of the eigenfunctions associated with λ. In the first place we suppose that $\vartheta_n = \vartheta_n(\lambda) \neq 0$ for every $n \in \mathbb{N}$, analysing the case $\vartheta_m = 0$ for some m afterwards.

Theorem 3.3.12. *Assume that $\vartheta_n = \vartheta_n(\lambda) \neq 0$ for all $n \in \mathbb{N}$. Then*

a) if λ is such that $z_0 = K_{n=1}^{+\infty}(-\alpha_n/-1)$ then λ is an eigenvalue of P and the coefficients in the recurrence relation (3.27) are the Fourier coefficients of an eigenfunction associated to λ;

b) if λ is such that $z_0 \neq K_{n=1}^{+\infty}(-\alpha_n/-1)$ then the coefficients in (3.27) do not converge and the function series associated with them does not represent an eigenfunction of P.

Proof. a) Suppose λ is such that $z_0 = K_{n=1}^{+\infty}(-\alpha_n/-1)$. From Proposition (3.3.11) we have that $\lim_{n\to+\infty} z_n = 0$. In these hypotheses we will prove that if $\{g_n\}_{n\geq-1}$ is a solution of

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \ n \in \mathbb{N}$$

then $g_n \to 0$, as $n \to +\infty$, faster than any negative power of n. From this, recalling Definition 3.2.3, we obtain that the series given by

$$v := \sum_{n=0}^{+\infty} v_n \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\, x\right) \tag{3.60}$$

or by

$$u := \sum_{n=1}^{+\infty} u_n \frac{1}{\sqrt{\pi}} \sin(nx), \tag{3.61}$$

converge uniformly to the eigenfunctions u and v. We show now the convergence of the coefficients. Recalling Lemma 3.2.9 and Proposition 3.2.6 we have:

$$g_n = \vartheta_0 \dots \vartheta_{n-1} z_0^* \dots z_{n-2}^*\, g_0, \qquad n \geq 2,$$

with

$$z_j^* = \begin{cases} z_j & \text{if} \quad z_j \neq 0, \infty, \\ -\alpha_{j+1} & \text{if} \quad z_j = 0, \\ 1 & \text{if} \quad z_j = \infty, \end{cases} \qquad \text{if } j \neq n-2, \tag{3.62}$$

and with

$$z_{n-2}^* = \begin{cases} z_{n-2} & \text{if} \quad z_{n-2} \neq 0,\ \infty, \\ 0 & \text{if} \quad z_{n-2} = 0, \\ 1 & \text{if} \quad z_{n-2} = \infty. \end{cases} \tag{3.63}$$

As already noticed $\lim_{n\to+\infty} z_n = 0$, that is

$$\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N} \text{ such that } |z_n| \leq \varepsilon,\ \forall\, n \geq n_0. \tag{3.64}$$

Fix $\varepsilon < \frac{1}{2}$. We have, for every $n \geq n_0$,

$$|1 - z_{n+1}| \geq 1 - \varepsilon.$$

Since $\{z_n\}_n$ is a tail sequence for $K_{n=1}^{+\infty}(-\alpha_n/-1)$ we have

$$|z_n| = \frac{\left|\frac{1}{\vartheta_{n+1}\vartheta_{n+2}}\right|}{|1 - z_{n+1}|} \leq \frac{\left|\frac{1}{\vartheta_{n+1}\vartheta_{n+2}}\right|}{1 - \varepsilon}. \tag{3.65}$$

Moreover, for all $n \geq n_0$ we get $z_n \neq 0, \infty$. Indeed, were z_{n_0} to vanish for some n_0 we would have $z_{n_0+1} = 1 - \frac{\alpha_{n_0+1}}{z_{n_0}} = \infty$, which is impossible because of (3.64), recalling that $\varepsilon \leq \frac{1}{2}$. Relation (3.64) implies also that $z_n \neq \infty$ for every $n \in \mathbb{N}$.

Hence, recalling (3.62) and (3.63), if $n \geq n_0$ we have $z_n = z_n^*$ and therefore

$$|g_N| = |g_0| \left|\vartheta_0 \ldots \vartheta_{n_0+1} z_0^* \ldots z_{n_0}^* \vartheta_{n_0+2} \ldots \vartheta_{N-1} z_{n_0+1} \ldots z_{N-2}\right|, \quad N \geq 2. \tag{3.66}$$

By (3.36) we have

$$|g_{n_0+2}| = |g_0| \left|\vartheta_0 \ldots \vartheta_{n_0+1} z_0^* \ldots z_{n_0}^*\right|. \tag{3.67}$$

From (3.65) it follows that

$$|g_N| \leq |g_{n_0+2}| \ |\vartheta_{n_0+2} \dots \vartheta_{N-1}| \ \left| \frac{1}{(1-\varepsilon)} \frac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}} \cdots \frac{1}{(1-\varepsilon)} \frac{1}{\vartheta_{N-1}\vartheta_N} \right|,$$

which is, simplifying,

$$|g_N| \leq |g_{n_0+2}| \ \frac{1}{(1-\varepsilon)^{N-n_0-2} \ |\vartheta_{n_0+3} \dots \vartheta_N|}. \tag{3.68}$$

Recall that, by Definition 3.2.3, $\{\vartheta_n\}_n$ denotes either $\{\gamma_n\}_n$, or $\{\delta_n\}_n$, defined respectively by (3.21) or (3.24). Suppose, to fix ideas, that

$$\{\vartheta_n\}_n := \{\delta_n\}_n \tag{3.69}$$

(for $\{\gamma_n\}_n$ the proof is similar.) We will show that the right-hand side of (3.68) tends to zero as $N \to +\infty$. Notice that, by hypothesis, we have $\vartheta_n = \delta_n = \delta_n(\lambda) \neq 0$ for all $n \in \mathbb{N}$, so that (3.68) makes sense. We write δ_n in the form

$$\delta_n = (2n+1)^2 h^2 \left(1 + \frac{\frac{1}{h^2}\,(2 - 4\lambda h^2)}{(2n+1)^2}\right) = (2n+1)^2 h^2 \left(1 + \frac{\frac{2}{h^2} - 4\lambda}{(2n+1)^2}\right) =$$
$$= (2n+1)^2 h^2 \left(1 - \frac{4\lambda - \frac{2}{h^2}}{(2n+1)^2}\right), \quad n = n_0+3, \dots, N. \tag{3.70}$$

By substituting (3.70) in (3.68) we get

$$|g_N| = \frac{|g_{n_0+2}|}{((1-\varepsilon)h^2)^{N-n_0-2} \ [(2n_0+7) \ \dots \ (2N+1)]^2 \prod_{k=n_0+3}^{N} \left|1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2}\right|} \leq$$
$$\leq \frac{|g_{n_0+2}|}{((1-\varepsilon)h^2)^{N-n_0-2} \ [2(n_0+3)2(n_0+4) \ \dots \ 2N]^2 \prod_{k=n_0+3}^{N} \left|1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2}\right|} =$$
$$= \frac{|g_{n_0+2}|}{((1-\varepsilon)h^2 4)^{N-n_0-2} \ [(n_0+3) \ \dots \ N]^2 \prod_{k=n_0+3}^{N} \left|1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2}\right|}. \tag{3.71}$$

Consider the term $\prod_{k=n_0+3}^{N} \left|1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2}\right|$. In the first place we can assume that for all $k \in \mathbb{N}$

$$1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2} \neq 0.$$

In fact, if for some $m \in \mathbb{N}$

$$1 - \frac{4\lambda - \frac{2}{h^2}}{(2m+1)^2} = 0,$$

then δ_m would vanish, contradicting the fact that $\vartheta_n \neq 0$ for all n (recall (3.69)). Furthermore we can suppose that n_0 is such that for every $k \in \mathbb{N}$, with $k \geq n_0+3$, we have

$$\left| \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2} \right| < 1.$$

Thus

$$\prod_{k=n_0+3}^{N} \left| 1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2} \right| \geq \prod_{k=n_0+3}^{N} \left| 1 - \left| \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2} \right| \right| =$$

$$= \prod_{k=n_0+3}^{N} \left(1 - \frac{\left| 4\lambda - \frac{2}{h^2} \right|}{(2k+1)^2} \right).$$

Whence, recalling (3.71), we get

$$|g_N| \leq \frac{|g_{n_0+2}|}{((1-\varepsilon)h^2 4)^{N-n_0-2} \ [(n_0+3) \ \ldots \ N]^2 \prod_{k=n_0+3}^{N} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right)}. \quad (3.72)$$

As $\frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} < 1$ for every $k \geq n_0+3$ we have that $\prod_{k=n_0+3}^{N} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) > 0$ for all N, so we may write

$$\prod_{k=n_0+3}^{N} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) = \exp\left\{ \log\left[\prod_{k=n_0+3}^{N} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) \right] \right\} =$$

$$= \exp\left[\sum_{k=n_0+3}^{N} \log\left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) \right]. \quad (3.73)$$

Since the series

$$\sum_{k=n_0+3}^{+\infty} \log\left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right)$$

converges, taking the limit in (3.73) gives

$$\prod_{k=n_0+3}^{+\infty} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) = \exp\left[\sum_{k=n_0+3}^{+\infty} \log\left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) \right] = a \in \mathbb{R}_+. \quad (3.74)$$

Set

$$\prod_{k=n_0+3}^{N}\left(1-\frac{\left|4\lambda-\frac{2}{h^2}\right|}{(2k+1)^2}\right)=D_N.$$

We multiply and divide (3.72) by

$$\left[(n_0+2)!\right]^2 2\pi N^{2N+1}e^{-2N},$$

and obtain

$$|g_N|\le\frac{|g_{n_0+2}|\left[(n_0+2)!\right]^2 2\pi N^{2N+1}e^{-2N}}{\left((1-\varepsilon)h^2 4\right)^{N-n_0-2}\ (N!)^2 D_N 2\pi N^{2N+1}e^{-2N}}.$$

Upon setting

$$C_N=\frac{2\pi N^{2N+1}e^{-2N}}{(N!)^2}$$

we have, from Stirling's formula (see e.g. [10], p.423),

$$\lim_{N\to+\infty}C_N=1. \tag{3.75}$$

Therefore we have

$$|g_N|\le\frac{|g_{n_0+2}|\left[(n_0+2)!\right]^2 C_N}{\left((1-\varepsilon)h^2 4\right)^{N-n_0-2}\ D_N 2\pi N^{2N+1}e^{-2N}}=$$

$$=\frac{|g_{n_0+2}|\left[(n_0+2)!\right]^2 C_N}{\left((1-\varepsilon)h^2 4\frac{N^2}{e^2}\right)^{N-n_0-2}\ D_N 2\pi N}\left(\frac{e^2}{N^2}\right)^{n_0+2}. \tag{3.76}$$

From (3.74) it follows $\lim_{N\to+\infty}D_N\neq 0$. Then the right-hand side of (3.76), for ε fixed and for $N\to+\infty$, approaches to zero faster than every negative power of N. From this we get that the series given either by (3.60) or by (3.61) converges uniformly on $[-\pi,\pi]$, with all its derivatives and therefore it represents a function of the space $D(P)$ and an eigenfunction associated with λ. Moreover, from what just stated, the eigenfunction obtained in this way is C^∞ on the interval $[-\pi,\pi]$.

b) Conversely, let λ be such that $z_0 \neq K_{n=1}^{+\infty}(-\alpha_n/-1)$ and let $\vartheta_n \neq 0$ for every $n \in \mathbb{N}$. Then from Proposition 3.3.11 we have that $\lim_{n\to+\infty} z_n = 1$. Let $\{g_n\}_n$ be a solution of

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}.$$

We will show that $|g_n| \to +\infty$ as $n \to +\infty$. This implies that the series given by either the expansion (3.60) or the expansion (3.61) does not converge to a function of $D(P)$. Since $\lim_{n\to+\infty} z_n = 1$ we have that, for a fixed $\varepsilon > 0$, exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $|z_n - 1| < \varepsilon$. Let be $\varepsilon < \frac{1}{2}$.

We have that $1-\varepsilon \leq z_n \leq 1+\varepsilon$ for every $n \geq n_0$ and in particular $z_n \neq 0, \infty$, and whence that for all $n \geq n_0$ holds the equality $z_n = z_n^*$. From Lemma 3.2.9 we have

$$|g_N| = |g_0| \left|\vartheta_0 \dots \vartheta_{N-1} z_0^* \dots z_{n_0}^*\right| \left|z_{n_0+1} \dots z_{N-2}\right|,$$

where, recalling that $|g_{n_0+2}| = |g_0| \left|z_0^* \dots z_{n_0}^* \vartheta_0 \dots \vartheta_{n_0+1}\right|$, it follows that

$$|g_N| = |g_{n_0+2}| \left|\vartheta_{n_0+2} \dots \vartheta_{N-1}\right| \left|z_{n_0+1} \dots z_{N-2}\right| \geq$$

$$\geq |g_{n_0+2}| \left|\vartheta_{n_0+2} \dots \vartheta_{N-1}\right| (1-\varepsilon)^{N-2-n_0}. \tag{3.77}$$

As in the proof of a) we show the divergence of g_n only for $\{\vartheta_n\}_n := \{\delta_n\}_n$ (the proof for $\{\gamma_n\}_n$ is similar). We use here, as in a), formula (3.70).

Substituting (3.70) in (3.77) we have

$$|g_N| \geq |g_{n_0+2}| \left[(1-\varepsilon)h^2\right]^{N-2-n_0} \left[(2n_0+5)\dots(2N-1)\right]^2 \prod_{k=n_0+2}^{N-1} \left|1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2}\right| \geq$$

$$\geq |g_{n_0+2}| \left[(1-\varepsilon)h^2\right]^{N-2-n_0} \left[2(n_0+2)\dots 2(N-1)\right]^2 \prod_{k=n_0+2}^{N-1} \left|1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2}\right| =$$

$$= |g_{n_0+2}| \left[(1-\varepsilon)4h^2\right]^{N-2-n_0} [(n_0+2)\dots(N-1)]^2 \prod_{k=n_0+2}^{N-1} \left| 1 - \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2} \right| . \tag{3.78}$$

Suppose that n_0 is such that for every $k \geq n_0 + 2$ we have

$$\left| \frac{4\lambda - \frac{2}{h^2}}{(2k+1)^2} \right| < 1.$$

Thus, from (3.78), as in the proof of a), we obtain

$$|g_N| \geq |g_{n_0+2}| \left[(1-\varepsilon)4h^2\right]^{N-2-n_0} [(n_0+2)\dots(N-1)]^2 \prod_{k=n_0+2}^{N-1} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) \tag{3.79}$$

We prove that the right-hand side of (3.78) goes to infinity as $N \to +\infty$. In a way similar to that of case a) we obtain

$$\lim_{N\to+\infty} \prod_{k=n_0+2}^{N-1} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right) = a \in \mathbb{R}.$$

Now set

$$C_N = \prod_{k=n_0+2}^{N-1} \left(1 - \frac{\left|4\lambda - \frac{2}{h^2}\right|}{(2k+1)^2} \right)$$

and

$$B_N = \frac{[(N-1)!]^2}{2\pi(N-1)(N-1)^{2(N-1)}\, e^{-2(N-1)}}.$$

Note that, by Stirling's formula, $\lim_{N\to+\infty} B_N = 1$. Multiplying and dividing by

$$2\pi(N-1)(N-1)^{2(N-1)}\, e^{-2(N-1)}\, [(n_0+1)!]^2$$

the right-hand side of (3.79) we have

$$|g_N| \geq \frac{|g_{n_0+2}| B_N}{[(n_0+1)!]^2} \left[(1-\varepsilon)4h^2\right]^{N-2-n_0} 2\pi(N-1)(N-1)^{2(N-1)}\, e^{-2(N-1)} C_N,$$

that is

$$|g_N| \geq \frac{|g_{n_0+2}| B_N}{[(n_0+1)!]^2} \left[(1-\varepsilon)4h^2 \left(\frac{N-1}{e} \right)^2 \right]^{N-2-n_0} 2\pi(N-1) \left(\frac{N-1}{e} \right)^{2n_0+2} C_N.$$

Taking the limit as $N \to +\infty$ gives $\lim_{N\to+\infty} |g_N| = +\infty$. Therefore the series (3.60) and (3.61) in this case do not converge to functions of $L^2(I)$ and then they cannot represent any function in $D(P)$. □

The following theorem states an analogous characterization of the eigenvalues of P, even in case λ is such that $\vartheta_{n_0}(\lambda) = 0$, for a certain $n_0 \in \mathbb{N}$.

Theorem 3.3.13. *Let λ be such that $\vartheta_{n_0}(\lambda) = 0$ for a certain $n_0 \in \mathbb{N}$.*

a) Using the notation of Proposition 3.2.14 if $n_0 = 0$, necessary and sufficient condition for λ to be an eigenvalue of P is that

$$1 - \frac{1}{\eta_0\eta_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\eta_n\eta_{n+1}}}{-1}\right).$$

b) Using the notation of Proposition 3.2.15 if $n_0 \neq 0$, $g_{n_0} \neq 0$, $g_{n_0-1} \neq 0$, necessary and sufficient condition for λ to be an eigenvalue of P is that

$$1 - \frac{1}{\mu_0\mu_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\mu_n\mu_{n+1}}}{-1}\right).$$

c) Using the notation of Proposition 3.2.15 if $n_0 \neq 0$, $g_{n_0} \neq 0$, $g_{n_0-1} = 0$, necessary and sufficient condition for λ to be an eigenvalue of P is that

$$1 - \frac{1}{\nu_0\nu_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\nu_n\nu_{n+1}}}{-1}\right).$$

d) Using the notation of Proposition 3.2.15 if $n_0 \neq 0$, $g_{n_0} = 0$, necessary and sufficient condition for λ to be an eigenvalue of P is that

$$1 - \frac{1}{\rho_0\rho_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\rho_n\rho_{n+1}}}{-1}\right).$$

Proof. Lemma 3.2.16 states that, in this case, it is possible to obtain the coefficients g_n, from a certain index onward, using formulas (3.48), (3.51) and (3.54). A procedure similar to the proof of Theorem 3.3.12 proves the assertion. □

From Theorems 3.3.12, 3.3.13 we get further information on eigenfunctions' Fourier coefficients. For instance we can prove that for every $N \in \mathbb{N}$ there exists $n_0 > N$ such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$, where g_n represent, as usual, Fourier coefficients of eigenfunctions. This will be proved for a general solution $\{g_n\}_n$ of the recurrence relation (3.27) of Definition 3.2.3, even if g_n does not represent an eigenfunction's Fourier coefficient (i.e. if λ is not an eigenvalue of P).

Remark 3.3.14. *Let $\lambda \in \mathbb{R}$ and let $\{g_n\}_{n\geq -1}$ be the solution, different from the 0-sequence, of the recurrence relation*

$$g_{-1} = 0, \quad g_{n+1} = \vartheta_n(\lambda) g_n - g_{n-1}, \qquad \forall\, n \in \mathbb{N}, \tag{3.80}$$

where we use the notation fixed in Definition 3.2.3. Then, for every $N \in \mathbb{N}$ there exists $n_0 > N$ such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$.

Proof. By Theorems 3.3.12, 3.3.13, we have either $\lim_{n\to+\infty} |g_n| = 0$ or $\lim_{n\to+\infty} |g_n|$ $= +\infty$. In the second case, that is when λ is not an eigenvalue of P, the assertion follows immediatly.

Let λ be an eigenvalue of P and assume that $g_n = 0$ for infinite values of n (otherwise the assertion follows immediatly).

In the first place we prove that there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$ we have $(g_n, g_{n+2}) \neq (0, 0)$. Set $n_1 \in \mathbb{N}$ such that $\vartheta_n(\lambda) \neq 0$ for all $n > n_1$ (the existence of such an n_1 follows from Definition 3.2.3). By contradiction let

$g_n = g_{n+2} = 0$ for $n > n_1$. Then, by (3.80), we have

$$0 = g_{n+2} = \vartheta_{n+1}(\lambda) g_{n+1} - g_n = \vartheta_{n+1}(\lambda) g_{n+1}.$$

As $\vartheta_{n+1}(\lambda) \neq 0$ this implies that $g_{n+1} = 0$. Since $g_n = 0$ this is a contradiction, by Lemma 3.2.1.

Up to now we have shown that, for all $n > n_1$, $g_n = 0$ implies that both g_{n+1} and g_{n+2} are different from 0. We reason again by contradiction to conclude the proof. Suppose there exists $N \in \mathbb{N}$ such that, for all $n > N$ if $g_n, g_{n+1} \neq 0$ then $g_{n+2} = 0$. Fix $n_0 > \max\{N, n_1\}$, such that $g_{n_0} = 0$ (recall that we are in the hypothesis that $g_n = 0$ for infinite values of n). Then $g_{n_0+1}, g_{n_0+2} \neq 0$. Thus $g_{n_0+3} = 0$ and this implies $g_{n_0+4}, g_{n_0+5} \neq 0$ and so on. Substituting these values in (3.80) gives

$$\begin{aligned}
&g_{n_0} = 0 \\
&g_{n_0+1} = -g_{n_0-1} \neq 0 \\
&g_{n_0+2} = \vartheta_{n_0+1} g_{n_0+1} = -\vartheta_{n_0+1} g_{n_0-1} \neq 0 \\
&g_{n_0+3} = 0 \\
&g_{n_0+4} = -g_{n_0+2} = \vartheta_{n_0+1} g_{n_0-1} \neq 0 \\
&g_{n_0+5} = \vartheta_{n_0+4} g_{n_0+4} = \vartheta_{n_0+4} \vartheta_{n_0+1} g_{n_0-1} \neq 0 \\
&g_{n_0+6} = 0
\end{aligned}$$

By induction, since $|\vartheta_n(\lambda)| \to +\infty$, we see that

$$\lim_{j \to +\infty} |g_{n_0+m_j}| = +\infty, \quad \text{when } \; m_j \in \mathbb{N}, \quad m_j \equiv 1 \bmod 3,$$

and $|g_{n_0+m_k}| = 0$ for all $m_k \in \mathbb{N}$ such that $m_j \equiv 0 \bmod 3$. Thus the sequence $\{|g_n|\}_n$ has not limit, but this is a condradiction because, as λ is an eigenvalue of P, by Theorems 3.3.12, 3.3.13 the sequence $\{|g_n|\}_n$ must converge to 0. □

Looking at the proof of Theorem 3.3.12 (see (3.66), (3.67)) we recall that we have, for a sufficiently large n_0

$$g_N = g_{n_0+2}\vartheta_{n_0+2}\dots\vartheta_{N-1}z_{n_0+1}\dots z_{N-2}, \qquad N > n_0+2, \tag{3.81}$$

where $\{z_n\}_n$ is recursively defined by

$$\begin{cases} z_0 = 1 - \dfrac{1}{\vartheta_0\vartheta_1} \\ z_n = 1 - \dfrac{\dfrac{1}{\vartheta_n\vartheta_{n+1}}}{z_{n-1}}. \end{cases} \tag{3.82}$$

Besides we have, if λ is an eigenvalue of P, that

$$z_0 = 1 - \frac{1}{\vartheta_0\vartheta_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_n\vartheta_{n+1}}}{-1}\right). \tag{3.83}$$

In other words z_0 can be written as a continued fraction. From (3.82) and (3.83) we find out that we can write all z_n in (3.82) as continued fractions which are the tails of the continued fraction in (3.83). To prove this we recall the following statement about tail sequences (see [11], p. 60).

Remark 3.3.15. *Let $\{t_n\}_n$, $\{\tilde{t}_n\}_n$ be two tail sequences for $b_0 + K\left(a_n/b_n\right)$, with $t_k = \tilde{t}_k$ for one index k. Then $t_n = \tilde{t}_n$ for all $n \in \mathbb{N}$.*

Proposition 3.3.16. *Let λ be an eigenvalue of P. Using the notation of Theorem 3.3.12 we have that*

$$z_m = \cfrac{\dfrac{1}{\vartheta_m\vartheta_{m+1}}}{1 - \cfrac{\dfrac{1}{\vartheta_{m+1}\vartheta_{m+2}}}{1-\ddots}}, \qquad m \in \mathbb{N}. \tag{3.84}$$

Proof. By Lemma 3.3.10 $\{z_n\}_n$ is a tail sequence for $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_n\vartheta_{n+1}}}{-1}\right)$. The right-hand side of (3.84), for $m = 1, 2, ...$, is obviously a tail sequence for the same continued fraction (see Definition 3.3.5). The assertion follows from Remark 3.3.15 and Theorem 3.3.12, as the two tail sequences have the first term in common. □

Using this proposition we will give estimates on coefficients z_n, appearing in (3.81). This will be done by recalling the following theorem about continued fractions (for the proof see [11], p. 35).

Theorem 3.3.17 (Worpitzky). *Let be* $\{a_n\}_n \subseteq \mathbb{C}$. *If*

$$|a_n| \leq \frac{1}{4}, \ \forall\ n \in \mathbb{N}\backslash\{0\}$$

then $K_{n=1}^{+\infty}(a_n/1)$ *converges. Moreover all approximants* f_n *verify* $|f_n| < \frac{1}{2}$ *and we have*

$$|f| = \left|K_{n=1}^{+\infty}(a_n/1)\right| \leq \frac{1}{2}.$$

Applying this theorem to (3.84) gives the following

Corollary 3.3.18. *Let* λ *be a real number. There exists* $n_0 \in \mathbb{N}$ *such that* $\left|\frac{1}{\vartheta_n(\lambda)\vartheta_{n+1}(\lambda)}\right| < \frac{1}{4}$ *for all* $n > n_0$, *so that we have*

$$\left|K_{j=n}^{+\infty}\left(\frac{\frac{1}{\vartheta_j(\lambda)\vartheta_{j+1}(\lambda)}}{-1}\right)\right| = \left|\frac{\frac{1}{\vartheta_n(\lambda)\vartheta_{n+1}(\lambda)}}{1-\frac{\frac{1}{\vartheta_{n+1}(\lambda)\vartheta_{n+2}(\lambda)}}{1-\ddots}}\right| \leq \frac{1}{2}, \qquad n > n_0,$$

where we use the notation of Definition 3.2.3.

Proof. We can always find n_0 such that $\left|\frac{1}{\vartheta_n(\lambda)\vartheta_{n+1}(\lambda)}\right| < \frac{1}{4}$ for all $n > n_0$ since $\vartheta_n = \vartheta_n(\lambda) \to +\infty$ as $n \to +\infty$ (recall Definition 3.2.3 and (3.21), (3.24)). Thus, by Theorem 3.3.17, the assertion follows. □

We consider once again equation (3.81), shown in the proof of Theorem 3.3.12. Notice that the recurrence relation (3.80) gives an unique expression for g_{n_0+2} in both cases $\vartheta_m(\lambda) \neq 0$ for all n and $\vartheta_m(\lambda) = 0$ for some m. Moreover we can compute also coefficients z_n, appearing in (3.81), with a procedure independent to whether or not $\vartheta_{m_0}(\lambda)$ vanishes for some m_0. This will be done by computing z_{n_0}, for n_0 large enough, independently to z_0, z_1,..., z_{n_0-1}. Following these ideas we shall find out the following general form of the Fourier coefficients, g_n, of the eigenfunctions of P :

$$g_{n_0+1+m} = \vartheta_{n_0+1} \dots \vartheta_{n_0+m} z_{n_0} \dots z_{n_0+m-1} g_{n_0+1}, \qquad \forall\, m > 0, \tag{3.85}$$

for a sufficiently large n_0, where we have

$$z_{n_0+m} = \cfrac{\cfrac{1}{\vartheta_{n_0+m+1}\vartheta_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+m+2}\vartheta_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\, m \in \mathbb{N}. \tag{3.86}$$

As already stressed, relations (3.85) and (3.86) are fulfilled in both cases $\vartheta_n(\lambda) = 0$ or $\vartheta_n(\lambda) \neq 0$. Therefore, from these equations, we get a general necessary and sufficient condition for the eigenvalues of P that unifies the notation of the two cases considered in Theorems 3.3.12, 3.3.13.

Proposition 3.3.19. *Using the notation of Remark 3.3.14 let $\lambda \in \mathbb{R}$. Let n_0 be such that $|\vartheta_n(\lambda)| > 2$ for all $n \geq n_0$ and such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$. Then λ is an eigenvalue of P if and only if*

$$1 - \cfrac{1}{\cfrac{g_{n_0+1}}{g_{n_0}}\vartheta_{n_0+1}} = \cfrac{\cfrac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}}. \tag{3.87}$$

Furthermore, if λ is an eigenvalue of P, we have

$$g_{n_0+1+m} = \vartheta_{n_0+1} \dots \vartheta_{n_0+m} z_{n_0} \dots z_{n_0+m-1} g_{n_0+1}, \qquad \forall\, m > 0, \tag{3.88}$$

with

$$z_{n_0+m} = \cfrac{\cfrac{1}{\vartheta_{n_0+m+1}\vartheta_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+m+2}\vartheta_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\, m \in \mathbb{N}. \tag{3.89}$$

Notice that $z_{n_0+m} \neq 0$ for all $m \in \mathbb{N}$ and thus $g_{n_0+1+m} \neq 0$ for all $m > 0$.

Proof. The existence of n_0 such that $|\vartheta_n(\lambda)| > 2$ for all $n \geq n_0$ and such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$ is a consequence of Remark 3.3.14 and of Definition 3.2.3. We write the recurrence relation as

$$g_{n_0+1} = \left(\frac{g_{n_0+1}}{g_{n_0}} \right) g_{n_0}, \tag{3.90}$$

$$g_{n_0+m+2} = \vartheta_{n_0+m+1} g_{n_0+m+1} - g_{n_0+m}, \qquad m \in \mathbb{N}. \tag{3.91}$$

Thus we apply Lemma 3.2.4, Proposition 3.2.6 and Proposition 3.3.11 to this recurrence relation as in the proof of Theorem 3.3.12; notice that the analogous of the sequence $\{z_n\}_n$ in this case is

$$\begin{cases} \tilde{z}_0 = 1 - \cfrac{1}{\cfrac{g_{n_0+1}}{g_{n_0}} \vartheta_{n_0+1}} \\ \tilde{z}_n = 1 - \cfrac{\cfrac{1}{\vartheta_n \vartheta_{n+1}}}{\tilde{z}_{n-1}} \end{cases} \tag{3.92}$$

So we have that (3.87) is a necessary and sufficient condition for λ to be an eigenvalue of P and furthermore we have

$$g_{n_0+1+m} = \vartheta_{n_0+1} \dots \vartheta_{n_0+m} \tilde{z}_0^* \dots \tilde{z}_{m-1}^* g_{m_0+1}, \tag{3.93}$$

where $\tilde{z}_j^*$ are defined in analogy with z_j^* in (3.62). We reason as in the proof of Proposition 3.3.16. From (3.87) and Remark 3.3.15, as $\{\tilde{z}_n\}_n$ is a tail sequence for $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_{n_0+n+1}\vartheta_{n_0+n+2}}}{-1}\right)$ we get

$$\tilde{z}_n = \cfrac{\frac{1}{\vartheta_{n_0+n+1}\vartheta_{n_0+n+2}}}{1 - \cfrac{\frac{1}{\vartheta_{n_0+n+2}\vartheta_{n_0+n+3}}}{1 - \ddots}}, \qquad \forall\, n \in \mathbb{N}. \tag{3.94}$$

Notice that

$$\tilde{z}_0 = 1 - \frac{1}{\frac{g_{n_0+1}}{g_{n_0}}\vartheta_{n_0+1}} \neq 0.$$

In fact

$$1 - \frac{1}{\frac{g_{n_0+1}}{g_{n_0}}\vartheta_{n_0+1}} = 0$$

implies

$$\vartheta_{n_0+1} g_{n_0+1} - g_{n_0} = 0,$$

but this is not the case from (3.91), as $g_{n_0+2} \neq 0$. Moreover, as $|\vartheta_n(\lambda)| > 2$ for every $n \geq n_0$, we have, from (3.94) and Corollary 3.3.18, $|\tilde{z}_n| \leq \frac{1}{2}$. From here and by (3.92) we have $\tilde{z}_n \neq 0, \infty$ for all $n \in \mathbb{N}$. Thus $\tilde{z}_n^* = \tilde{z}_n$ for all n, so (3.93) and (3.94) imply (3.88) and (3.89). □

Theorems 3.3.12, 3.3.13, provide necessary and sufficient conditions for λ to be an eigenvalue of P. We now plan to write explicitly the condition of Theorem 3.3.12 in the particular cases of sequences $\{a_n\}_n$, $\{v_n\}_n$ (recall, for the definition of these sequences, Remarks 3.1.9 and 3.1.11). But before doing this, we state a lemma, which will allow us to write the conditions of Theorem 3.3.12 in a simpler form. We recall the definition of equivalent continued fractions (see [11], p. 72). This will be used to prove the lemma.

Definition 3.3.20. *We say that two continued fractions are **equivalent** if they have the same sequence of approximants (see Definition 3.3.2).*

Lemma 3.3.21. *Let $\{\vartheta_n\}_n$ be a sequence such that $\vartheta_n \neq 0$ for all $n \in \mathbb{N}$. Then we have*

$$\vartheta_0 \; K_{n=0}^{+\infty} \left(\frac{-\frac{1}{\vartheta_n \vartheta_{n+1}}}{-1} \right) = K_{n=1}^{+\infty} \left(\frac{-1}{-\vartheta_n} \right). \tag{3.95}$$

Proof. The continued fractions in the left-hand side and in the right-hand side of (3.95) are equivalent (see Definitions 3.3.20 and 3.3.2) so they converge to the same limit. $\square$

The necessary and sufficient condition for the eigenvalues, associated with odd eigenfunctions, are given by the following

Remark 3.3.22. *Let $\lambda \in \mathbb{R}$ be such that $\gamma_n(\lambda) \neq 0$ for all $n \in \mathbb{N}$. Necessary and sufficient condition for λ to be an eigenvalue for P with eigenfunction given by*

$$u(x) = \sum_{n=0}^{+\infty} a_n \sin((n+1)x)$$

is that the following condition holds:

$$1 - \frac{1}{\gamma_0 \gamma_1} = \cfrac{\frac{1}{\gamma_1 \gamma_2}}{1 - \cfrac{\frac{1}{\gamma_2 \gamma_3}}{1 - \ddots}}$$

that is, recalling Lemma 3.3.21,

$$\gamma_1 - \frac{1}{\gamma_0} = \cfrac{1}{\gamma_2 - \cfrac{1}{\gamma_3 - \ddots}}.$$

By substituting the values of γ_n *we get:*

$$1-\frac{1}{(4h^2+2-4\lambda h^2)\,(16h^2+2-4\lambda h^2)}=\cfrac{\cfrac{1}{(16h^2+2-4\lambda h^2)\,(36h^2+2-4\lambda h^2)}}{1-\cfrac{\cfrac{1}{(36h^2+2-4\lambda h^2)\,(64h^2+2-4\lambda h^2)}}{1-\ddots}}$$

or, equivalently,

$$16h^2+2-4\lambda h^2-\frac{1}{4h^2+2-4\lambda h^2}=\cfrac{1}{36h^2+2-4\lambda h^2-\cfrac{1}{64h^2+2-4\lambda h^2-\ddots}}.$$

The necessary and sufficient condition for the eigenvalues, associated with even eigenfunctions, are given by the following

Remark 3.3.23. *Let* $\lambda \in \mathbb{R}$ *be such that* $\delta_n(\lambda) \neq 0$ *for all* $n \in \mathbb{N}$. *Necessary and sufficient condition for* λ *to be an eigenvalue for* P *with eigenfunction given by*

$$v(x)=\sum_{n=0}^{+\infty} v_n \cos\left(\frac{2n+1}{2}\,x\right)$$

is that the following condition holds:

$$1-\frac{1}{\delta_0\delta_1}=\cfrac{\cfrac{1}{\delta_1\delta_2}}{1-\cfrac{\cfrac{1}{\delta_2\delta_3}}{1-\ddots}},$$

that is, recalling Lemma 3.3.21,

$$\delta_1-\frac{1}{\delta_0}=\cfrac{1}{\delta_2-\cfrac{1}{\delta_3-\ddots}}.$$

Substituting the values of δ_n *gives:*

$$1-\frac{1}{(h^2+1-4\lambda h^2)\,(9h^2+2-4\lambda h^2)}=\cfrac{\cfrac{1}{(9h^2+2-4\lambda h^2)\,(25h^2+2-4\lambda h^2)}}{1-\cfrac{\cfrac{1}{(25h^2+2-4\lambda h^2)\,(49h^2+2-4\lambda h^2)}}{1-\ddots}},$$

or, equivalently

$$9h^2 + 2 - 4\lambda h^2 - \frac{1}{h^2 + 1 - 4\lambda h^2} = \cfrac{1}{25h^2 + 2 - 4\lambda h^2 - \cfrac{1}{49h^2 + 2 - 4\lambda h^2 - \ddots}}.$$

We write the characterization of eigenvalues of P in case ϑ_{n_0} vanishes, for a certain $n_0 \in \mathbb{N}$. Recall that, by Definition 3.2.3 we have either $\{\vartheta_n\}_n := \{\gamma_n\}_n$ or $\{\vartheta_n\}_n := \{\delta_n\}_n$, with

$$\gamma_n := 4(n+1)^2 h^2 + 2 - 4\lambda h^2, \qquad \forall\, n \in \mathbb{N}, \tag{3.96}$$

or

$$\begin{cases} \delta_0 = h^2 + 1 - 4\lambda h^2 \\ \\ \delta_n = (2n+1)^2 h^2 + 2 - 4\lambda h^2, \qquad \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases} \tag{3.97}$$

We treat the case $\vartheta_0 = 0$.

Remark 3.3.24. *If $\vartheta_0 = 0$ then, by Definition 3.2.3 we have*

$$\lambda = 1 + \frac{1}{2h^2}, \qquad \textit{when} \;\; \vartheta_n := \gamma_n;$$

$$\lambda = \frac{1}{4} + \frac{1}{4h^2}, \qquad \textit{when} \;\; \vartheta_n := \delta_n.$$

Remark 3.3.25. *Let $\lambda \in \mathbb{R}$ such that $\vartheta_0 = 0$. Necessary and sufficient condition for λ to be an eigenvalue for P is*

$$1 - \frac{1}{\vartheta_2\vartheta_3} = \cfrac{\cfrac{1}{\vartheta_3\vartheta_4}}{1 - \cfrac{\cfrac{1}{\vartheta_4\vartheta_5}}{1 - \ddots}} \tag{3.98}$$

Proof. It is an immediate consequence of Proposition 3.2.14 and of Theorem 3.3.13. □

If λ is such that ϑ_{n_0} vanishes, with $n_0 \in \mathbb{N}\backslash\{0\}$, then we have the following

Remark 3.3.26. *If $\vartheta_{n_0} = 0$ with $n_0 \in \mathbb{N}\backslash\{0\}$, then, by Definition 3.2.3 we have*

$$\lambda = (n_0+1)^2 + \frac{1}{2h^2}, \qquad \text{when} \quad \vartheta_n := \gamma_n;$$

$$\lambda = \frac{(2n_0+1)^2}{4} + \frac{1}{4h^2}, \qquad \text{when} \quad \vartheta_n := \delta_n.$$

The necessary and sufficient conditions for λ to be an eigenvalue for P are given by Proposition 3.2.15 and by Theorem 3.3.13. In particular we have the following

Lemma 3.3.27. *Let $\lambda \in \mathbb{R}$ be such that $\vartheta_{n_0}(\lambda) = 0$ with $n_0 \in \mathbb{N}\backslash\{0\}$. Using the notation of Proposition 3.2.6 necessary and sufficient condition for λ to be an eigenvalue of P is that the following conditions hold:*

a) if $g_{n_0} \neq 0$ and $g_{n_0-1} \neq 0$ then

$$\vartheta_{n_0+1} + \vartheta_{n_0-1} - \frac{1}{\vartheta_{n_0-2} z^*_{n_0-3}} = \cfrac{\frac{1}{\vartheta_{n_0+2}}}{1 - \cfrac{\frac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}};$$

b) if $g_{n_0} \neq 0, \; g_{n_0-1} = 0$ then

$$z^*_{n_0-3} = 0$$

and

$$1 - \frac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}} = \cfrac{\frac{1}{\vartheta_{n_0+3}\vartheta_{n_0+4}}}{1 - \cfrac{\frac{1}{\vartheta_{n_0+4}\vartheta_{n_0+5}}}{1 - \ddots}}; \tag{3.99}$$

c) if $g_{n_0} = 0$ then

$$\frac{1}{\vartheta_{n_0-1}\vartheta_{n_0-2}} = z^*_{n_0-3} \tag{3.100}$$

and

$$1 - \cfrac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}} = \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+3}\vartheta_{n_0+4}}}{1 - \ddots}}. \tag{3.101}$$

Proof. By Theorem 3.3.13, b), and by Proposition 3.2.15 a necessary and sufficient condition for λ to be an eigenvalue of P is

$$1 - \cfrac{1}{-\cfrac{g_{n_0-1}}{g_{n_0}}\vartheta_{n_0+1}} = \cfrac{\cfrac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}},$$

that is

$$1 + \frac{g_{n_0}}{g_{n_0-1}\vartheta_{n_0+1}} = \cfrac{\cfrac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}}. \tag{3.102}$$

We compute the term $\frac{g_{n_0}}{g_{n_0-1}}$. From the relation

$$g_{n_0} = \vartheta_{n_0-1}g_{n_0-1} - g_{n_0-2}$$

it follows that

$$\frac{g_{n_0}}{g_{n_0-1}} = \vartheta_{n_0-1} - \frac{g_{n_0-2}}{g_{n_0-1}}. \tag{3.103}$$

From Lemma 3.2.13 we have

$$g_{n_0-1} = \vartheta_0 \dots \vartheta_{n_0-2} z_0^* \dots z_{n_0-3}^* g_0 \tag{3.104}$$

and

$$g_{n_0-2} = \vartheta_0 \dots \vartheta_{n_0-3} z_0^* \dots z_{n_0-4}^* g_0. \tag{3.105}$$

Notice that $z_{n_0-3}^* \neq 0$, for otherwise we would have (by (3.104)) $g_{n_0-1} = 0$, contradicting the hypothesis stated in a). Replacing (3.104) and (3.105) in (3.103) we get

$$\frac{g_{n_0}}{g_{n_0-1}} = \vartheta_{n_0-1} - \frac{1}{\vartheta_{n_0-2} z^*_{n_0-3}}. \tag{3.106}$$

Substituting (3.106) in (3.102) and multiplying both sides by ϑ_{n_0+1} we have a).

b) From Theorem 3.3.13, c), and from Proposition 3.2.15 a necessary and sufficient condition for λ to be an eigenvalue of P is

$$1 - \frac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}} = \cfrac{\cfrac{1}{\vartheta_{n_0+3}\vartheta_{n_0+4}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+4}\vartheta_{n_0+5}}}{1 - \ddots}},$$

that is the relation (3.99). We have $g_{n_0-1} = 0$ by hypothesis and

$$g_{n_0-1} = \vartheta_0 \dots \vartheta_{n_0-2} z^*_0 \dots z^*_{n_0-3} g_0$$

from Lemma 3.2.13. This implies that $z_{n_0-3} = z^*_{n_0-3} = 0$.

c) If $g_{n_0} = 0$ from

$$0 = g_{n_0} = \vartheta_{n_0-1} g_{n_0-1} - g_{n_0-2}$$

follows that

$$\frac{g_{n_0-2}}{g_{n_0-1}} = \vartheta_{n_0-1} \neq 0. \tag{3.107}$$

Indeed $g_{n_0-1} \neq 0$ from Lemma 3.2.1, because $g_{n_0} = 0$, and $\vartheta_{n_0-1} \neq 0$ from Remark 3.2.10, since $\vartheta_{n_0} = 0$. From Lemma 3.2.13 it follows that (3.104) and (3.105) hold also in this case, so that substiting them in (3.107) gives

$$\frac{1}{\vartheta_{n_0-2} z^*_{n_0-3}} = \vartheta_{n_0-1},$$

from which (3.100) follows. Note that $z_{n_0-3} \neq 0$, by (3.104), because $g_{n_0-1} \neq 0$.

Relation (3.101) follows from Theorem 3.3.13, d), and from Proposition 3.2.15.

□

We recall the definition of z_n^* :

$$z_j^* = \begin{cases} z_j & \text{if} \quad z_j \neq 0, \infty \\ -\alpha_{j+1} & \text{if} \quad z_j = 0 \\ 1 & \text{if} \quad z_j = \infty \end{cases} \tag{3.108}$$

and the definition of z_n :

$$\begin{cases} z_0 = 1 - \alpha_0 \\ z_n = 1 - \dfrac{\alpha_n}{z_{n-1}}, \quad \forall\, n \in \mathbb{N}\backslash\{0\}, \end{cases}$$

with $\alpha_{-1} = 1$, $\alpha_n = \dfrac{1}{\vartheta_n \vartheta_{n+1}}$, $n \in \mathbb{N}$.

In the hypothesis of Lemma 3.3.27 we write in all cases what the conditions for λ to be an eigenvalue are.

Proposition 3.3.28. *Let $\lambda \in \mathbb{R}$ be such that $\vartheta_{n_0}(\lambda) = 0$ with $n_0 \in \mathbb{N}$ and let be $g_{n_0} \neq 0$. Necessary and sufficient condition for λ to be an eigenvalue of P is that the following conditions hold:*

1) if $g_{n_0-1} \neq 0$ then

$$z_{n_0-3} \neq 0, \ \infty$$

and

$$\vartheta_{n_0+1} + \vartheta_{n_0-1} - \cfrac{\frac{1}{\vartheta_{n_0-2}}}{1 - \cfrac{\frac{1}{\vartheta_{n_0-3}\vartheta_{n_0-2}}}{1 - \cfrac{\ddots}{1 - \frac{1}{\vartheta_0 \vartheta_1}}}} = \cfrac{\frac{1}{\vartheta_{n_0+2}}}{1 - \cfrac{\frac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}}; \tag{3.109}$$

2) if $g_{n_0-1} \neq 0,\ z_{n_0-3} = \infty$ *then*

$$1 - \cfrac{\cfrac{1}{\vartheta_{n_0-4}\vartheta_{n_0-3}}}{1 - \cfrac{\ddots}{1 - \cfrac{1}{\vartheta_0\vartheta_1}}} = 0 \tag{3.110}$$

and

$$\vartheta_{n_0+1} + \vartheta_{n_0-1} - \frac{1}{\vartheta_{n_0-2}} = \cfrac{\cfrac{1}{\vartheta_{n_0+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}}; \tag{3.111}$$

3) if $g_{n_0-1} = 0$ *then*

$$z_{n_0-3} = 1 - \cfrac{\cfrac{1}{\vartheta_{n_0-3}\vartheta_{n_0-2}}}{1 - \cfrac{\ddots}{1 - \cfrac{1}{\vartheta_0\vartheta_1}}} = 0$$

and

$$1 - \frac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}} = \cfrac{\cfrac{1}{\vartheta_{n_0+3}\vartheta_{n_0+4}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+4}\vartheta_{n_0+5}}}{1 - \ddots}}. \tag{3.112}$$

Proof. By Lemma 3.3.27 if $g_{n_0},\ g_{n_0-1} \neq 0$ then λ is an eigenvalue of P if and only if

$$\vartheta_{n_0+1} + \vartheta_{n_0-1} - \frac{1}{\vartheta_{n_0-2} z^*_{n_0-3}} = \cfrac{\cfrac{1}{\vartheta_{n_0+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}}, \tag{3.113}$$

hence, from (3.108), if $z_{n_0-3} \neq 0,\ \infty$ we have

$$z^*_{n_0-3} = z_{n_0-3} \tag{3.114}$$

Substituting (3.114) in (3.113) gives relation (3.109), which proves 1).

2) If $z_{n_0-3} = \infty$ then, since by hypothesis $\alpha_{n_0-3} \neq 0$, we have $z_{n_0-4} = 0$, in other words (3.110) holds. In this case, from (3.108), $z^*_{n_0-3} = 1$. Substituting this last relation in (3.113) we obtain (3.111).

3) If $g_{n_0-1} = 0$ then $z^*_{n_0-3} = 0$ (recall Lemma 3.2.13) therefore

$$z^*_{n_0-3} = 1 - \cfrac{\cfrac{1}{\vartheta_{n_0-3}\vartheta_{n_0-2}}}{1 - \cfrac{\ddots}{1 - \cfrac{1}{\vartheta_0\vartheta_1}}} = 0.$$

Relation (3.112) has already been proved in Lemma 3.3.27. $\square$

We now study the case in which $g_{n_0} = 0$.

Proposition 3.3.29. *Let $\lambda \in \mathbb{R}$ be such that $\vartheta_{n_0}(\lambda) = 0$ with $n_0 \in \mathbb{N}$ and let $g_{n_0} = 0$. Necessary and sufficient condition for λ to be an eigenvalue for P is that the following conditions hold:*

1) if $z_{n_0-3} \neq 0, \ \infty$ we have

$$1 - \cfrac{\cfrac{1}{\vartheta_{n_0-2}\vartheta_{n_0-1}}}{1 - \cfrac{\ddots}{1 - \cfrac{1}{\vartheta_0\vartheta_1}}} = 0$$

and

$$1 - \frac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}} = \cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+3}\vartheta_{n_0+4}}}{1 - \ddots}}; \qquad (3.115)$$

2) if $z_{n_0-3}=\infty$ *then*

$$1-\cfrac{\cfrac{1}{\vartheta_{n_0-4}\vartheta_{n_0-3}}}{1-\cfrac{\ddots}{1-\cfrac{1}{\vartheta_0\vartheta_1}}}=0 \tag{3.116}$$

$$\frac{1}{\vartheta_{n_0-1}\vartheta_{n_0-2}}=1 \tag{3.117}$$

$$1-\frac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}}=\cfrac{\cfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1-\cfrac{\cfrac{1}{\vartheta_{n_0+3}\vartheta_{n_0+4}}}{1-\ddots}}. \tag{3.118}$$

Proof. The assertion follows immediatly from c) of Lemma 3.3.27, substituting in (3.100) and (3.101) the possible values of z_j^*, given by (3.108). Note that in this case it can not be $z_{n_0-3}=0$, for otherwise we would have $g_{n_0-1}=0$, which is impossible from Lemma 3.2.1. □

3.4 Upper and lower bounds for eigenvalues

In this section we will provide for each eigenvalue two sequences; one converging to the eigenvalue from above and the other converging to the eigenvalue from below. The following results can be found in [13] and we just give the statements tailored to our particular situation.

We write again the recurrence relations fulfilled by the coefficients of eigenfunctions $\{v_n\}_n$ and $\{a_n\}_n$, recalling that, by the notation fixed in Proposition 1.1.1 we have $\lambda=\dfrac{2\mu}{h}$ (where μ represents, by (1.1), an eigenvalue of P_L). We have

$$\begin{cases} v_{n+1} = \delta_n v_n - v_{n-1}, & \forall\, n \in \mathbb{N} \\ a_{n+1} = \gamma_n a_n - a_{n-1}, & \forall\, n \in \mathbb{N}, \end{cases} \tag{3.119}$$

with

$$\begin{cases} \delta_0 = h^2 + 1 - 8\mu h \\ \delta_n = (2n+1)^2 h^2 + 2 - 8\mu h, & \forall\, n \in \mathbb{N}\backslash\{0\} \end{cases} \tag{3.120}$$

and

$$\gamma_n = 4(n+1)^2 h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}. \tag{3.121}$$

Following the notation fixed in Definition 3.2.3 we will consider $\{g_n\}_{n\geq -1} = \{g_n(\mu)\}_{n\geq -1}$ as a particular sequence of polynomials in μ. We will see that the eigenvalues of P_L are the limits of zeros of these polynomials.

To start this analysis it is useful to give the following definition.

Definition 3.4.1. *Let $\{\Pi_n\}_n$ be a sequence of polynomials with real coefficients. Denote by $r_{n,1} \leq r_{n,2} \leq \cdots \leq r_{n,k}$ the real zeros (in case they exist) of Π_n and put, by definition, $r_{n,0} = -\infty$ and $r_{n,k+1} = +\infty$.*

We shall say that $\{\Pi_n\}_{n\geq 0}$ is a ***sequence of polynomials with interlaced zeros*** *if*

(i) Π_0 is not the zero polynomial, it has degree $d \geq 0$ and all its zeros are real with multeplicity 1.

(ii) Π_1 has degree $d+1$, all its zeros are real-valued with multeplicity 1 and each zero of Π_1 is located between two consecutive zeros of Π_0, i.e.

$$r_{0,i-1} < r_{1,i} < r_{0,i}, \quad i = 1, 2, \ldots, d+1.$$

(iii) There exists a sequence $\{\beta_n\}_n$ *of polynomials of degree* 1 *such that*

$$\Pi_{n+1} = \beta_n \Pi_n + \Pi_{n-1}, \qquad n = 1, 2, \dots \ . \tag{3.122}$$

(iv) $\lim_{n\to+\infty} \Pi_n(\mu) := \Pi_n(+\infty)$ *and* $\lim_{n\to+\infty} \Pi_{n+2}(\mu) := \Pi_{n+2}(+\infty)$ *have opposite signs for all* $n \in \mathbb{N}$.

If Π_0 *has degree* 0 *we say that* $\{\Pi_n\}_n$ *is a sequence of polynomials with interlaced zeros if* $\{\Pi_n\}_n$ *fulfills (i), (iii), (iv).*

Now we change the sequences of coefficients of eigenfunctions $\{v_n\}_n,\ \{a_n\}_n$ so that they satisfy Definition 3.4.1.

Lemma 3.4.2. *Let* $\{b_n\}_n,\ \{c_n\}_n$ *be such that*

$$\begin{cases} b_{2n} = (-1)^n a_{2n}, & n \in \mathbb{N}, \\ b_{2n+1} = (-1)^n a_{2n+1}, & n \in \mathbb{N} \end{cases} \tag{3.123}$$

and

$$\begin{cases} c_{2n} = (-1)^n v_{2n}, & n \in \mathbb{N}, \\ c_{2n+1} = (-1)^n v_{2n+1}, & n \in \mathbb{N}. \end{cases} \tag{3.124}$$

Moreover let $\{\chi_n\}_n,\ \{\psi_n\}_n$ *be such that*

$$\begin{cases} \chi_n = (-1)^n \gamma_n, & n \in \mathbb{N}, \\ \psi_n = (-1)^n \delta_n, & n \in \mathbb{N}. \end{cases} \tag{3.125}$$

Then $\{b_n\}_n,\ \{c_n\}_n$ *satisfy the following recurrence relations:*

$$\begin{cases} b_{n+1} = \chi_n b_n + b_{n-1}, & n \in \mathbb{N}, \\ c_{n+1} = \psi_n c_n + c_{n-1}, & n \in \mathbb{N}. \end{cases} \tag{3.126}$$

In particular $\{b_n\}_n,\ \{c_n\}_n$ *are sequences of polynomials in* μ *with interlaced zeros.*

Proof. It follows immediatly from relations (3.123), (3.124) and (3.125), recalling (3.119), (3.120) and (3.121). □

We next state an important property of sequences of polynomials with interlaced zeros.

Theorem 3.4.3. *If* $\{\Pi_n\}_n$ *is a sequence of polynomials with interlaced zeros, then* Π_n *has all real and distinct zeros, for every* $n \in \mathbb{N}$. *Moreover, for all* $n \geq 1$, *each zero of* Π_{n-1} *is located between two consecutive zeros of* Π_n; *in other words*

$$r_{n-1,i-1} < r_{n,i} < r_{n-1,i}, \qquad i = 1, 2, \ldots, d+n, \qquad \forall\, n \geq 1.$$

The following result defines the polynomials Θ_n and Ψ_n and gives a property for their zeros; we will see that particular sequences defined using these zeros converge to eigenvalues of P_L.

Theorem 3.4.4. *Let* $\{\Pi_n\}_n$ *be a sequence of polynomials with interlaced zeros and define* $\Theta_n = \Pi_n - \Pi_{n-1}$ *and* $\Psi_n = \Pi_n + \Pi_{n-1}$. *Then all the zeros of* Θ_n, Ψ_n *are real and distinct. Furthermore:*

(i) if $\Pi_n(+\infty)$ *and* $\Pi_{n-1}(+\infty)$ *have different signs then the zeros of* Θ_n, *which we denote by* $\rho_{n,1} < \rho_{n,2} < \cdots < \rho_{n,d+n}$, *are such that*

$$r_{n-1,i-1} < \rho_{n,i} < r_{n,i}$$

and the zeros, $\rho'_{n,1} < \rho'_{n,2} < \cdots < \rho'_{n,d+n}$, *of* Ψ_n *are such that*

$$r_{n,i} < \rho'_{n,i} < r_{n-1,i};$$

(ii) if $\Pi_n(+\infty)$ *and* $\Pi_{n-1}(+\infty)$ *have the same sign then we have*

$$r_{n-1,i-1} < \rho'_{n,i} < r_{n,i}$$

and

$$r_{n,i} < \rho_{n,i} < r_{n-1,i}.$$

Next we define the sequence $\{\rho^-_{n,i}\}_n$, which approximates an eigenvalue of P_L from below.

Definition 3.4.5. *Let $\rho_{n,i}$ and $\rho'_{n,i}$ be defined as in Theorem 3.4.4. We set*

$$\rho^-_{n,i} = \min\{\rho_{n,i},\ \rho'_{n,i}\}.$$

From Definition 3.4.5 and from Theorem 3.4.4 it follows that, for every $n \in \mathbb{N}$ and for every $i = 1, 2, \ldots, d + n$, we have

$$\rho^-_{n,i} \in (r_{n-1,i-1},\ r_{n,i})\,.$$

We will use the following definition to prove the monotonicity of $\{\rho^-_{n,i}\}$, for $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

Definition 3.4.6. *Using the notation of Definition 3.4.1 and writing β_n as $\beta_n(\mu) = \xi_n(\mu - B_n)$, for $\xi_n \in \mathbb{R}$, we say that a sequence of polynomials with interlaced zeros is **admissible** if there exist $\xi > 0$ and $N_0 \in \mathbb{N}$ such that $|\xi_n| \geq \xi$ for all $n \in \mathbb{N}$ and $B_{n+1} - B_n > \frac{2}{\xi}$ for all $n \geq N_0$.*

We show that the sequences $\{\chi_n\}_n,\ \{\psi_n\}_n$ introduced in Lemma 3.4.2 satisfy Definition 3.4.6.

Lemma 3.4.7. *The sequences $\{\chi_n\}_n,\ \{\psi_n\}_n$, defined by (3.125) are admissible.*

Proof. Recalling relations (3.125) and (3.120) we have

$$\psi_n(\mu) = (-1)^n((2n+1)^2h^2 + 2 - 8\mu h) = (-1)^{n+1}8h\left(-\frac{(2n+1)^2h}{8} - \frac{1}{4h} + \mu\right),$$

with $n \in \mathbb{N}\backslash\{0\}$. Using the notation fixed in Definition 3.4.6, we set $\xi = 8h$, $\xi_n = (-1)^{n+1}8h$. In order to verify the definition it suffices to prove that there exists $N_0 \in \mathbb{N}$ such that

$$\frac{(2n+3)^2h}{8} - \frac{(2n+1)^2h}{8} \geq \frac{1}{4h} \tag{3.127}$$

for all $n \geq N_0$. From (3.127) we get $N_0 \geq \frac{1}{4h^2} - 1$. In a similar way, recalling (3.125) and (3.121), we obtain that $\{\chi_n\}_n$ fulfills the admissibility hypothesis, for $\xi = 8h$, $\xi_n = (-1)^{n+1}8h$ and $N_0 \geq \frac{1}{4h^2} - \frac{3}{2}$. $\square$

The following theorem ensures the monotonicity of $\{\rho^-_{n,i}\}$, for sufficiently large n, in the case the sequence of polynomials is admissible.

Theorem 3.4.8. *Let $\{\Pi_n\}_n$ be an admissible sequence of polynomials with interlaced zeros and let $r_{n,i}$ be the zeros of Π_n (using the notation fixed in Definition 3.4.1). Fix $i \in \mathbb{N}$. By Definition 3.4.6 there exists $n_i \in \mathbb{N}$ such that $|\beta_{n+1}(\mu)| > 2$ for all $\mu < r_{n,i}$ and for all $n \geq n_i$. Then, for $n \geq n_i$ we have*

$$r_{n+1,i} \in \left[\rho^-_{n,i},\ r_{n,i}\right)$$

and

$$\rho^-_{n+1,i} > \rho^-_{n,i}.$$

As a consequence for every i the sequence $\{r_{n,i}\}_n$ converges and $\rho^-_{n,i}$, with $n \geq n_i$, are lower bounds for $\lim_{n\to+\infty} r_{n,i}$.

The following result provides an estimate on the size of the absolute value of admissible sequences of polynomials with interlaced zeros, for certain values of the variable. This will give us information about eigenvalues of P_L, since the sequence of Fourier coefficients of eigenfunctions converges to zero (see Theorems 3.3.12, 3.3.13), and by Lemma 3.4.2 the same sequence is an admissible sequence of polynomials, with interlaced zeros.

Lemma 3.4.9. *Let $\{\Pi_n\}_n$ be an admissible sequence of polynomials with interlaced zeros and let $r_i = \lim_{n\to+\infty} r_{n,i}$ and $l_i = \lim_{n\to+\infty} \rho^-_{n,i}$. Then*

(i) For every $z \in \mathbb{C}$, we have either

$$\lim_{n\to+\infty} |\Pi_n(z)| = +\infty$$

or

$$\lim_{n\to+\infty} |\Pi_n(z)| = 0.$$

(ii) For every $i \in \mathbb{N}$ we have

$$a \in [l_i,\ r_i] \Rightarrow \lim_{n\to+\infty} |\Pi_n(a)| = 0.$$

Notice that, following the notation of Lemma 3.4.2, from *(i)* in Lemma 3.4.9 we get again the two cases obtained in Theorems 3.3.12 and 3.3.13, i.e. when λ is an eigenvalue then the Fourier coefficients converge to 0, otherwise their absolute values diverges.

We now fix the notation we will use hereafter.

Definition 3.4.10. *From now on we will denote with $\{\Pi_n\}_n$ one of the two sequences $\{b_n\}_n$, $\{c_n\}_n$ defined by (3.123), (3.124). Furthermore we will use the notation fixed in Definitions 3.4.1, 3.4.5 and in Theorem 3.4.4, recalling that either $\{\Pi_n\}_n := \{b_n\}_n$ or $\{\Pi_n\}_n := \{c_n\}_n$.*

Lemma 3.4.9 implies, recalling Theorems 3.3.12 and 3.3.13 and their proofs, that $r_i = l_i$ and that these values are exactly the eigenvalues of P_L. In particular we have the following

Corollary 3.4.11. *Using the notation of Definition 3.4.10 and of Lemma 3.4.9 we have, by Lemma 3.4.7, that* $\{\Pi_n\}_n = \{\Pi_n(\mu)\}_n$ *is an admissible sequence of polynomials in* μ *with interlaced zeros and*

$$\lim_{n\to+\infty} r_{n,i} = r_i = l_i = \lim_{n\to+\infty} \rho^-_{n,i} \qquad \forall\ i \in \mathbb{N}.$$

Furthermore the set $\{r_i,\ i \in \mathbb{N}\backslash\{0\}\}$ *coincides with the set of eigenvalues of* P_L.

Proof. By Theorems 3.3.12 and 3.3.13 we have, recalling (3.123) and (3.124), that $|\Pi_n(\mu)| \to 0$ if and only if μ is an eigenvalue of P_L. From Lemma 3.4.9 we have that $|\Pi_n(a)| \to 0$ for all $a \in [l_i, r_i]$. By Proposition 1.2.4 P_L has discrete spectrum, therefore $l_i = r_i$ and r_i is an eigenvalue of P_L. □

We next show that Corollary 3.4.11 implies that all Fourier coefficients of the eigenfunction associated with the lowest eigenvalue of P_L can not vanish.

Corollary 3.4.12. *Let* μ_0 *be the lowest eigenvalue of* P_L. *If*

$$v = \sum_{n=0}^{+\infty} v_n \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}x\right) \tag{3.128}$$

is the eigenfunction associated with μ_0 *then we have* $v_n \neq 0$ *for all* $n \in \mathbb{N}$.

Proof. Notice that v is an even eigenfunction, because, by 2) of Theorem 1.2.6, it does not vanish in the interior of I. This justify the expansion (3.128). From Corollary 3.4.11 we have that μ_0 is the limit of the sequence $\{r_{n,1}\}_n$, where $r_{n,1}$ denotes the lowest zero of $v_n = v_n(\mu)$, considered as a polynomial in μ. Since, by Lemma 3.4.2 and Theorem 3.4.3, the zeros of v_n interlace those of v_{n-1}, for all n, we have that the sequence $\{r_{n,1}\}_n$ is monotonic decreasing. As

$$\mu_0 = \lim_{n\to+\infty} r_{n,1}$$

follows immediatly that μ_0 can not be a zero for any v_n. $\square$

We now state an important result about the continued fraction

$$f = f(\mu) = \vartheta_0(\mu) + K_{n=1}^{+\infty}\left(-1/\vartheta_n(\mu)\right),$$

where $\{\vartheta_n\}_n$ represents, as established in Definition 3.2.3, one of the sequences $\{\gamma_n\}_n,\ \{\delta_n\}_n$. This function appears in the necessary and sufficient condition for the eigenvalues of P_L, stated in Remarks 3.3.22 and 3.3.23.

In particular we claim that this function is meromorphic in μ (for the proof see [13]).

Notice also that the continued fraction $K_{n=1}^{+\infty}\left(-1/\vartheta_n(\mu)\right)$ is equivalent to (see Definition 3.3.20)

$$K_{n=1}^{+\infty}\left(1/(-1)^n\vartheta_n(\mu)\right).$$

Whence, recalling (3.125), we can give the following

Definition 3.4.13. *Define the function*

$$f = f(\mu) = \beta_0(\mu) + K_{n=1}^{+\infty}\left(1/\beta_n(\mu)\right), \tag{3.129}$$

with $\{\beta_n\}_n := \{\chi_n\}_n$ *or* $\{\beta_n\}_n := \{\psi_n\}_n$ *(see (3.125)).*

We write the approximants of f (see Definition 3.3.2) as

$$f_n = \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\ddots + \cfrac{1}{\beta_n}}} = \frac{P_n}{Q_n}, \tag{3.130}$$

where $P_n,\ Q_n$ denote, respectively, the *n-th canonical numerator* and *denominator* of f (see Definition 3.3.2). From Remark 3.3.4 these sequences verify certain recurrence equations. In particular we have the following

Proposition 3.4.14. *Let* $f = \beta_0 + K_{n=1}^{+\infty}(1/\beta_n)$ *be a continued fraction and let* $f_n = \frac{P_n}{Q_n}$ *be its approximants (see Definition 3.3.2).*

Let, by definition, $P_{-1} = 1,\ \ P_0 = \beta_0,\ \ Q_{-1} = 0,\ \ Q_0 = 1$. *Then the sequences* $\{P_n\}_{n\geq -1},\ \ \{Q_n\}_{n\geq -1}$ *verify these relations:*

$$P_{n+1} = \beta_{n+1}P_n + P_{n-1}, \qquad n \in \mathbb{N}, \tag{3.131}$$

$$Q_{n+1} = \beta_{n+1}Q_n + Q_{n-1}, \qquad n \in \mathbb{N}. \tag{3.132}$$

From (3.131) we have that, when $\{\beta_n\}_n := \{\chi_n\}_n$, the sequence $\{P_n\}_n$ coincides with the sequence of coefficients of eigenfunctions $\{b_n\}_n$ (see (3.123) and (3.125)), and we have $\{P_n\}_n = \{c_n\}_n$ when $\{\beta_n\}_n := \{\psi_n\}_n$ (see (3.124) and (3.125)). Notice also that Corollary 3.4.11 shows that the eigenvalues of P_L are the limits of zeros of P_n.

We recall, for the sake of completeness, an important intermediate result, used in [13] to prove that f, defined by (3.129), is meromorphic. We denote by $\Omega_m(z)$ the functions

$$\Omega_m(z) = \cfrac{1}{\beta_{m+1}(z) + \cfrac{1}{\beta_{m+2}(z) + \ddots}}.$$

These functions are holomorphic on a certain domain of $\mathbb{C}$.

Proposition 3.4.15. *Using the notation fixed in Definition 3.4.6, let be* $\mu \in \mathbb{R}$ *and let* $m_0 \in \mathbb{N}$ *such that* $B_{m_0+1} > \mu + \frac{1}{\xi}$ *and* $m_0 \geq N_0$. *Then, for every* $m \geq m_0$, $\Omega_m(z)$ *is holomorphic on* $C_\mu = \{z \in \mathbb{C}; \mathrm{Re}(z) < \mu\}$.

From here one can prove the following

Proposition 3.4.16. *The function f, defined by (3.129), is meromorphic on $\mathbb{C}$, and it has a pole in z if*

$$\lim_{n\to+\infty}|Q_n(z)| = 0.$$

(For the proof see [13].)

If we treat $\{Q_n\}_n$ as a sequence of polynomials with interlaced zeros and if we use the notation of Definitions 3.4.1 and 3.4.5 we get two sequences converging, one from above, the other from below, to the poles of f. In particular we have the following

Proposition 3.4.17. *Let $\{Q_n\}_n$ be defined by (3.130) and let $r_{n,i}$ be the zeros of Q_n. Furthermore let $r_i = \lim_{n\to+\infty} r_{n,i}$ for all $i \in \mathbb{N}$. Then:*

i) f has a pole in r_i for every $i \in \mathbb{N}\backslash\{0\}$.

ii) The only poles of f are the r_i, for $i \in \mathbb{N}\backslash\{0\}$.

Furthermore we have

$$\lim_{n\to+\infty}\rho^-_{n,i} = r_i, \qquad i \in \mathbb{N}\backslash\{0\}.$$

3.5 Estimates for large eigenvalues

Recall that, by the notation fixed in Proposition 1.1.1, we have $\lambda = \dfrac{2\mu}{h}$ and $\mu = \dfrac{\lambda h}{2}$. In this section we will study the behaviour of eigenvalues μ, for fixed h and $\mu > C = C(h)$. In particular we will provide upper and lower bounds for these eigenvalues. In order to prove these results we will use Worpitzky's theorem (Theorem 3.3.17) about continued fractions. As usual we will analyse in the first place the eigenvalues associated with even eigenfunctions and afterwards those associated with odd eigenfunctions.

In order to apply Worpitzky's Theorem to $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n\delta_{n+1}}}{-1}\right)$ we will study the values of $|\delta_n\delta_{n+1}| = |\delta_n(\mu)\delta_{n+1}(\mu)|$ for varying μ. For this reason is useful to recall the definition of δ_n :

$$\begin{cases} \delta_0 = \delta_0(\mu) = h^2 + 1 - 8\mu h \\ \delta_n = \delta_n(\mu) = (2n+1)^2 h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases} \tag{3.133}$$

To have a better understanding of the problem, it helps using a geometric approach. More precisely we can think of the functions $\delta_n(\mu)\delta_{n+1}(\mu)$, for every n, as parabolas in the variable μ. In this way we get a sequence of parabolas $\{\delta_n(\mu)\delta_{n+1}(\mu)\}_n$ with the property that the maximum zero of $\delta_n(\mu)\delta_{n+1}(\mu)$ is the minimum zero of $\delta_{n+1}(\mu)\delta_{n+2}(\mu)$, for every $n \in \mathbb{N}$. Furthermore the sequence of the vertexes of these parabolas, for $n \geq 1$, is monotonic decreasing. These properties are straightforward consequences of (3.133). In the following results we find out sufficient conditions for these parabolas to have absolute value greater than or equal to 4. We will see that if μ is such that this last condition is fulfilled then μ can not be an eigenvalue of P.

Lemma 3.5.1. *Let n be a natural number and let μ be such that $\delta_n(\mu) < 0$ and $\delta_{n+1}(\mu) > 0$, i.e. such that*

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8}h + \frac{1}{4h}. \tag{3.134}$$

Then we have

$$|\delta_n(\mu)\delta_{n+1}(\mu)| < |\delta_{n+1}(\mu)\delta_{n+2}(\mu)|\,. \tag{3.135}$$

Proof. We prove that $|\delta_n(\mu)| < |\delta_{n+2}(\mu)|$, from which (3.135) follows immediatly.

By the definition of the δ_n (see (3.133)) and by (3.134) we have $\delta_{n+2}(\mu) > \delta_{n+1}(\mu) > 0$. Thus, to obtain (3.135), it suffices to show that $|\delta_n(\mu)| < \delta_{n+2}(\mu)$, that is

$$-\delta_{n+2}(\mu) < \delta_n(\mu) < \delta_{n+2}(\mu).$$

It is straightforward that $\delta_n(\mu) < \delta_{n+2}(\mu)$, for $\delta_{n+2}(\mu) > 0$ and $\delta_n(\mu) < 0$. We now prove that $-\delta_{n+2}(\mu) < \delta_n(\mu)$. From (3.134) it follows that $-8\mu h > -(2n+3)^2h^2 - 2$, so that we have

$$\delta_n(\mu) = (2n+1)^2h^2 + 2 - 8\mu h > (2n+1)^2h^2 + 2 - (2n+3)^2h^2 - 2 =$$

$$= (4n^2 + 4n + 1 - 4n^2 - 12n - 9)h^2 = (-8n-8)h^2. \tag{3.136}$$

In addition, from (3.134) it also follows that

$$-\delta_{n+2}(\mu) = -(2n+5)^2h^2 - 2 + 8\mu h < -(2n+5)^2h^2 - 2 + (2n+3)^2h^2 + 2 =$$

$$= (-4n^2 - 20n - 25 + 4n^2 + 12n + 9)h^2 = (-8n-16)h^2. \tag{3.137}$$

From (3.136) and (3.137), being $(-8n-8)h^2 > (-8n-16)h^2$, we get $\delta_n(\mu) > -\delta_{n+2}(\mu)$ and hence (3.135). □

In the hypothesis of Lemma 3.5.1, we now study

$$\min\{|\delta_n(\mu)\delta_{n+1}(\mu)|, |\delta_n(\mu)\delta_{n-1}(\mu)|\}.$$

Proposition 3.5.2. *Fix $n \in \mathbb{N}$. Let μ be such that*

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8}h + \frac{1}{4h}. \tag{3.138}$$

Then we have that

1) if $\mu < \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *then*

$$|\delta_n(\mu)\delta_{n-1}(\mu)| < |\delta_n(\mu)\delta_{n+1}(\mu)|\,;$$

2) if $\mu > \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *then*

$$|\delta_n(\mu)\delta_{n-1}(\mu)| > |\delta_n(\mu)\delta_{n+1}(\mu)|\,;$$

3) if $\mu = \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *then*

$$|\delta_n(\mu)\delta_{n-1}(\mu)| = |\delta_n(\mu)\delta_{n+1}(\mu)| = 16h^4(2n+1).$$

Proof. Points 1) and 2) follow from the analysis of the inequality

$$|\delta_{n-1}(\mu)| < |\delta_{n+1}(\mu)|\,. \tag{3.139}$$

In fact, notice that from (3.138) we have $\delta_{n+1}(\mu) > 0$ and $\delta_{n-1}(\mu) < 0$, so that relation (3.139) can be written as

$$-\delta_{n-1}(\mu) < \delta_{n+1}(\mu). \tag{3.140}$$

Substituting the values of $\delta_{n-1}(\mu)$, $\delta_{n+1}(\mu)$ in (3.140) yields

$$-(2n-1)^2h^2 - 2 + 8\mu h < (2n+3)^2h^2 + 2 - 8\mu h,$$

that is

$$16\mu h < \big((2n+3)^2 + (2n-1)^2\big)h^2 + 4$$

and thus

$$\mu < \frac{1}{4h} + \frac{\big((2n+1)^2+4\big)h}{8}.$$

From this we get 1) and 2).

To obtain 3), we simply replace the value of μ in $|\delta_{n-1}(\mu)\delta_n(\mu)|$ (recalling (3.133)). □

Fix n in $\mathbb{N}$. We denote by R_n the function defined by

$$R_n(\mu) = \min\{|\delta_n(\mu)\delta_{n-1}(\mu)|\,,\ |\delta_n(\mu)\delta_{n+1}(\mu)|\}. \tag{3.141}$$

We will show in the following Propositions that

$$|\delta_m(\mu)\delta_{m+1}(\mu)| > R_n(\mu)$$

for every $m \in \mathbb{N}$, with $m \neq n-1,\ n$, and for μ fulfilling the hypothesis of Proposition 3.5.2. In this way (recalling (3.141)) conditions on values of $|\delta_n(\mu)\delta_{n-1}(\mu)|$ and $|\delta_n(\mu)\delta_{n+1}(\mu)|$ give rise to conditions on all the other terms $|\delta_m(\mu)\delta_{m+1}(\mu)|$. Hereafter we occasionally denote $\delta_n(\mu)$ for short simply by δ_n.

Proposition 3.5.3. *Fix n in $\mathbb{N}$. Let μ be such that*

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8}h + \frac{1}{4h}. \tag{3.142}$$

Then

a) if $\mu < \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *we have*

$$|\delta_m(\mu)\delta_{m+1}(\mu)| > |\delta_n(\mu)\delta_{n-1}(\mu)| \quad \textit{for every } m = 0, 1, \ldots, n-2;$$

b) if $\mu > \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *we have*

$$|\delta_m(\mu)\delta_{m+1}(\mu)| > |\delta_n(\mu)\delta_{n+1}(\mu)| \quad \textit{for every } m = 0, 1, \ldots, n-2.$$

Proof. Notice that

$$|\delta_k(\mu)| > |\delta_{k+1}(\mu)| \quad \text{for every} \quad k = 0, \ldots, n-1, \tag{3.143}$$

for, being $\delta_k(\mu) < 0$ by (3.142), inequality (3.143) can be written as

$$-\delta_k > -\delta_{k+1}. \tag{3.144}$$

Substituting (3.133) in (3.144) gives

$$-(2k+1)^2 - 2 + 8\mu h > -(2k+3)^2 - 2 + 8\mu h$$

for every $k = 0, \dots, n-1$, which proves (3.143). Recalling that $\delta_k < 0$ for all $k = 0, \dots, n-1$, inequality (3.143) implies that

$$|\delta_m \delta_{m+1}| > |\delta_n \delta_{n-1}|, \quad \forall\, m = 0, \dots, n-2. \tag{3.145}$$

From this we get a). As an aside remark, recalling that $\delta_k < 0$ for all $k = 0, \dots, n-1$, we notice that

$$|\delta_k \delta_{k+1}| = \delta_k \delta_{k-1} \qquad \forall\, k = 0, \dots, n-1.$$

b) The relation (3.145) holds also in the hypothesis

$$\frac{1}{4h} + \frac{[(2n+1)^2+4]h}{8} < \mu < \frac{1}{4h} + \frac{(2n+3)^2 h}{8}.$$

Thus if we show that

$$|\delta_n \delta_{n-1}| > |\delta_n \delta_{n+1}|$$

we get immediatly b). This last inequality has already been proved in Proposition 3.5.2, 2). □

We now study the case in which $m > n$.

Proposition 3.5.4. *Let n be a fixed natural number. If*

$$\frac{1}{4h} + \frac{(2n+1)^2 h}{8} < \mu < \frac{1}{4h} + \frac{(2n+3)^2 h}{8}$$

we have

$$|\delta_m \delta_{m+1}| > |\delta_n \delta_{n+1}| \quad \forall\, m > n. \tag{3.146}$$

Proof. From Lemma 3.5.1 we get (3.146) with $m = n+1$. We obtain the assertion for $m \neq n+1$ if we notice that $\delta_k = |\delta_k| < |\delta_{k+1}| = \delta_{k+1} \quad \forall\, k = n+1, n+2, \dots$ (see (3.133)). □

From Propositions 3.5.3 and 3.5.4 it follows that, if μ satisfy (3.138), $|\delta_m \delta_{m+1}|$ is always greater than R_n (recall (3.141)), for all $m \neq n-1, n$. In other words we have the following

Corollary 3.5.5. *Let $n \in \mathbb{N}$. If*

$$\frac{(2n+1)^2}{8} h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8} h + \frac{1}{4h} \tag{3.147}$$

then

$$|\delta_m(\mu)\delta_{m+1}(\mu)| > R_n(\mu), \quad \forall\, m \neq n, n-1.$$

Proof. It is an immediate conseguence of Propositions 3.5.3 and 3.5.4. □

By Theorem 3.3.12 we have that μ is an eigenvalue of P_L if and only if it fulfills

$$1 - \frac{1}{\delta_0 \delta_1} = K_{n=1}^{+\infty} \left(\frac{-\frac{1}{\delta_n \delta_{n+1}}}{-1} \right), \tag{3.148}$$

in case $\delta_n \neq 0$ for every $n \in \mathbb{N}$. Notice that this last condition, $\delta_n(\mu) \neq 0$, is immediatly fulfilled when μ satisfies the hypotheses of Corollary 3.5.5 (see equations (3.147) and (3.133)).

Now we apply Worpitzky's Theorem (Theorem 3.3.17) to the continued fraction appearing in (3.148) to find out estimates for the eigenvalues.

Theorem 3.5.6. *Fix n in $\mathbb{N}$. If μ is such that*

$$\frac{(2n+1)^2}{8} h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8} h + \frac{1}{4h}$$

and, at the same time,

$$R_n(\mu) \geq 4 \tag{3.149}$$

(recall (3.141)) then μ is not an eigenvalue for P_L.

Proof. By Corollary 3.5.5 and by (3.149) we have $|\delta_n \delta_{n+1}| \geq 4$ for every $n \in \mathbb{N}$. Then

$$\frac{1}{|\delta_n \delta_{n+1}|} \leq \frac{1}{4}, \quad \forall\, n \in \mathbb{N} \tag{3.150}$$

whence the continued fraction $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n \delta_{n+1}}}{-1}\right)$ verifies the hypothesis of Worpitzky's Theorem. In particular we have

$$\left| K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n \delta_{n+1}}}{-1}\right) \right| \leq \frac{1}{2}. \tag{3.151}$$

Notice that $\delta_n \neq 0$ for every n, because $|\delta_n \delta_{n+1}| > 4$ for every n, hence the hypothesis of Theorem 3.3.12 are verified. Therefore in case μ is an eigenvalue for P_L, equation (3.148) is satisfied by μ and, recalling (3.151) and (3.150), we get

$$\frac{1}{2} \geq \left| K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n \delta_{n+1}}}{-1}\right) \right| = \left| 1 - \frac{1}{\delta_0 \delta_1} \right| \geq \left| 1 - \left| \frac{1}{\delta_0 \delta_1} \right| \right| \geq \frac{3}{4}$$

which is a contradiction. □

From Theorem 3.5.6 we obtain two different estimates for the eigenvalues, depending on the value of $R_n(\mu) = \min\{|\delta_n(\mu)\delta_{n-1}(\mu)|,\ |\delta_n(\mu)\delta_{n+1}(\mu)|\}$. Proposition 3.5.2 establishes that if

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{[(2n+1)^2 + 4]h}{8} + \frac{1}{4h} \tag{3.152}$$

then

$$R_n(\mu) = \min\{|\delta_n(\mu)\delta_{n-1}(\mu)|,\ |\delta_n(\mu)\delta_{n+1}(\mu)|\} = |\delta_n(\mu)\delta_{n-1}(\mu)|,$$

and if

$$\frac{[(2n+1)^2+4]h}{8}+\frac{1}{4h}<\mu<\frac{(2n+3)^2h}{8}+\frac{1}{4h}$$

then

$$R_n(\mu)=\min\{|\delta_n(\mu)\delta_{n-1}(\mu)|,\ |\delta_n(\mu)\delta_{n+1}(\mu)|\}=|\delta_n(\mu)\delta_{n+1}(\mu)|.$$

In addition we recall 3) of Proposition 3.5.2:

$$\delta_n\left(\frac{[(2n+1)^2+4]h}{8}+\frac{1}{4h}\right)\delta_{n-1}\left(\frac{[(2n+1)^2+4]h}{8}+\frac{1}{4h}\right)=16h^4(2n+1).$$

Thus, as a consequence of Theorem 3.5.6, and recalling the definition of δ_n (3.133), we have two different situations according to whether $4>16h^4(2n+1)$ or $4<16h^4(2n+1)$. In particular we have the following

Theorem 3.5.7. *Let n be a natural number such that $n\geq\frac{1}{2h^2}-1$. Then we have the following.*

1) If $4\geq 16h^4(2n+1)$, let μ be such that

$$\begin{cases}\mu\geq\dfrac{4(n+1)^2+1}{8}h+\dfrac{1}{4h}-\dfrac{\sqrt{4(n+1)^2h^4-1}}{4h}\\[2ex]\mu\leq\dfrac{4(n+1)^2+1}{8}h+\dfrac{1}{4h}+\dfrac{\sqrt{4(n+1)^2h^4-1}}{4h}.\end{cases}\tag{3.153}$$

Then μ cannot be an eigenvalue for P_L, associated to an even eigenfunction.

2) If $4<16h^4(2n+1)$, let μ be such that

$$\begin{cases}\mu\geq\dfrac{(4n^2+1)}{8}h+\dfrac{1}{4h}+\dfrac{\sqrt{4n^2h^4+1}}{4h}\\[2ex]\mu\leq\dfrac{4(n+1)^2+1}{8}h+\dfrac{1}{4h}+\dfrac{\sqrt{4(n+1)^2h^4-1}}{4h}.\end{cases}\tag{3.154}$$

Then μ cannot be an eigenvalue for P_L, associated to an even eigenfunction.

Proof. Note that the hypoteses of 1) and 2) imply that μ fulfills (3.152). Notice, furthermore, that $n \geq \frac{1}{2h^2} - 1$ is a necessary condition for the estimates (3.153) and (3.154) to make sense. In fact this condition assures that the radicand which appears in these expressions is greater than or equal to 0.

1) If $4 \geq 16h^4(2n+1)$ then the condition of Theorem 3.5.6

$$R_n(\mu) = \min\{|\delta_n(\mu)\delta_{n-1}(\mu)|,\ |\delta_n(\mu)\delta_{n+1}(\mu)|\} \geq 4$$

is equivalent to

$$|\delta_n\delta_{n+1}| > 4, \tag{3.155}$$

by Proposition 3.5.2. Relation (3.155) can be written as $-\delta_n\delta_{n+1} > 4$, because $\delta_n < 0$ and $\delta_{n+1} > 0$. Substituting (3.133) in (3.155) gives

$$(8\mu h)^2 - 8\mu h\big[(8(n+1)^2+2)h^2+4\big] + \big[4(n+1)^2-1\big]^2 h^4 + \big[16(n+1)^2+4\big]h^2 + 8 < 0$$

and thus 1).

2) Similarly to 1) by Proposition 3.5.2 we have that if $4 < 16h^4(2n+1)$ then the condition of Theorem 3.5.6

$$R_n(\mu) = \min\{|\delta_n(\mu)\delta_{n-1}(\mu)|,\ |\delta_n(\mu)\delta_{n+1}(\mu)|\} \geq 4$$

is equivalent to

$$a \leq \mu \leq b, \tag{3.156}$$

where a is the maximal solution of the equation $|\delta_n\delta_{n-1}| = 4$ and b is the maximal solution of the equation $|\delta_n\delta_{n+1}| = 4$. This follows immediatly from Proposition 3.5.2 and from (3.133). We compute a and b. By (3.133) we have $|\delta_n\delta_{n-1}| = 4$ if

$$(8\mu h)^2 - 8\mu h\big[(8n^2+2)h^2+4\big] + \left(4n^2-1\right)^2 h^4 + \left(16n^2+4\right)h^2 = 0,$$

thus

$$a = \frac{(4n^2+1)}{8}h + \frac{1}{4h} + \frac{\sqrt{4n^2h^4+1}}{4h}.$$

The computation of b has been already done in the proof of 1). Replacing the values of a and b in (3.156) we get the assertion. □

The approach of this section applies, in a similar way, to eigenvalues associated to odd eigenfunctions. In this way we get estimates similar to those stated in Theorem 3.5.7. We just state an analogous theorem, this time about odd eigenfunctions. To this purpose we recall the definition of coefficients γ_n :

$$\gamma_n = 4(n+1)^2h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}. \tag{3.157}$$

Theorem 3.5.8. *Let n be a natural number such that $n \geq \frac{1}{2h^2} - \frac{1}{2}$.*

1) If $4 \geq 32nh^4$, let μ be such that it satisfies

$$\begin{cases} \mu \geq \dfrac{n^2+(n+1)^2}{4}h + \dfrac{1}{4h} - \dfrac{\sqrt{(2n+1)^2h^4-1}}{4h} \\ \mu \leq \dfrac{n^2+(n+1)^2}{4}h + \dfrac{1}{4h} + \dfrac{\sqrt{(2n+1)^2h^4-1}}{4h}. \end{cases} \tag{3.158}$$

Then μ cannot be an eigenvalue of P_L, associated to an odd eigenfunction.

2) If $4 < 32nh^4$, let μ be such that it satisfies

$$\begin{cases} \mu \geq \dfrac{[n^2+(n-1)^2]}{4}h + \dfrac{1}{4h} + \dfrac{\sqrt{(2n+1)^2h^4+1}}{4h} \\ \mu \leq \dfrac{[n^2+(n+1)^2]}{4}h + \dfrac{1}{4h} + \dfrac{\sqrt{(2n+1)^2h^4-1}}{4h}. \end{cases} \tag{3.159}$$

Then μ cannot be an eigenvalue of P_L, associated to an odd eigenfunction.

From Theorems 3.5.7 and 3.5.8 it follows that the eigenvalues of P_L belong to the union of an infinite number of intervals. In particular we have the following

Corollary 3.5.9. *Let* $n_0 \in \mathbb{N}$ *be such that* $n_0 \geq \frac{1}{2h^2} - 1$ *and we denote with* $\mathrm{Spec}_+(P_L)$ *the set of all eigenvalues of* P_L *associated with even eigenfunctions.*

Then, upon setting

$$C_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} - \frac{\sqrt{4(n+1)^2h^4-1}}{4h},$$

$$D_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} + \frac{\sqrt{4(n+1)^2h^4-1}}{4h},$$

$$E_n = \frac{(4n^2+1)}{8}h + \frac{1}{4h} + \frac{\sqrt{4n^2h^4+1}}{4h},$$

$$F_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} + \frac{\sqrt{4(n+1)^2h^4-1}}{4h},$$

there exists $n_1 \in \mathbb{N}$ *such that*

$$\mathrm{Spec}_+(P_L) \cap [C_{n_0}, +\infty) \subset \left(\bigcup_{n=n_0}^{n_1-1} (D_n,\ C_{n+1}) \right) \cup \left(\bigcup_{n=n_1}^{+\infty} (F_n,\ E_{n+1}) \right),$$

with $n_1 \geq \frac{1}{8h^4} - \frac{1}{2}$.

Proof. It follows immediatly from Theorem 3.5.7. □

An analogous result holds for eigenvalues associated to odd eigenfunctions.

Corollary 3.5.10. *Let* $n_0 \in \mathbb{N}$ *be such that* $n_0 \geq \frac{1}{2h^2} - \frac{1}{2}$ *and denote with* $\mathrm{Spec}_-(P_L)$ *the set of all the eigenvalues of* P_L *associated to odd eigenfunctions. Then, posing*

$$G_n = \frac{n^2+(n+1)^2}{4}h + \frac{1}{4h} - \frac{\sqrt{(2n+1)^2h^4-1}}{4h},$$

$$H_n = \frac{n^2+(n+1)^2}{4}h + \frac{1}{4h} + \frac{\sqrt{(2n+1)^2h^4-1}}{4h},$$

$$L_n = \frac{n^2+(n-1)^2}{4}h + \frac{1}{4h} + \frac{\sqrt{(2n+1)^2h^4+1}}{4h},$$

$$M_n = \frac{n^2+(n+1)^2}{4}h + \frac{1}{4h} + \frac{\sqrt{(2n+1)^2h^4-1}}{4h},$$

there exists $n_1 \in \mathbb{N}$ *such that*

$$\mathrm{Spec}_-(P_L) \cap [G_{n_0}, +\infty) \subset \left(\bigcup_{n=n_0}^{n_1-1} (H_n,\ G_{n+1}) \right) \cup \left(\bigcup_{n=n_1}^{+\infty} (M_n,\ L_{n+1}) \right),$$

with $n_1 \geq \frac{1}{8h^4}$.

Now we recall classical asymptotic estimates of large eigenvalues of Sturm-Liouville problems, in order to be able to compare these estimates to those obtained in Corollaries 3.5.9 and 3.5.10. For the proof of the next statement see [22], p. 244.

Proposition 3.5.11. *Let* $B = B(x)$ *be a continuous real-valued, bounded-variation function. Let* $U = U(x)$ *be a solution for the boundary value problem (on the interval* $[0, \pi]$*)*

$$\begin{cases} U'' + (\lambda^2 - B)U = 0, \\ U(0) = U(\pi) = 0. \end{cases}$$

Then, if $\lambda > \int_0^\pi |B(t)|dt$, *we can express the n-th eigenvalue* λ_n *as*

$$\lambda_n = n + \frac{\int_0^\pi B(t)dt}{2\pi n} + \frac{\alpha(n)}{n^2},$$

where $\alpha(n)$ *is a bounded function, depending on* U*. Moreover, for the eigenfunction associated with* λ_n *we have*

$$\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) + \frac{\alpha(x,n)}{n},$$

where $\alpha(x, n)$ *is a bounded function of* x *and* n.

We will apply this proposition to our boundary value problem

$$\begin{cases} \psi'' + \left[\frac{2\mu}{h} - \frac{\sin^2\left(\frac{t}{2}\right)}{h^2} \right] \psi = 0, \\ \psi(\pm\pi) = 0. \end{cases}$$

To obtain a problem which fulfills the hypotheses of Proposition 3.5.11 we proceed as in the proof of Proposition 1.1.1, so that we can state the following

Remark 3.5.12. *Let* $\psi \in H_0^1(I) \cap H^2(I)$ *be a solution of the equation*

$$\psi'' + \left[\frac{2\mu}{h} - \frac{\sin^2\left(\frac{t}{2}\right)}{h^2}\right]\psi = 0. \tag{3.160}$$

Set $\varphi(x) = \psi(2x - \pi)$. *Then* $\varphi \in H_0^1(0,\pi) \cap H^2(0,\pi)$ *and* φ *is a solution of the equation*

$$\varphi'' + \left[\frac{8\mu}{h} - \frac{4\cos^2(x)}{h^2}\right]\varphi = 0. \tag{3.161}$$

Moreover if $\varphi \in H_0^1(0,\pi) \cap H^2(0,\pi)$ *satisfies equation (3.161) then, upon setting* $\psi(t) = \varphi\left(\frac{t+\pi}{2}\right)$, *it follows that* $\psi \in H_0^1(I) \cap H^2(I)$ *and* ψ *is a solution of (3.160).*

Thus, by applying Proposition 3.5.11 to (3.161) we get the following

Proposition 3.5.13. *Let* μ_m *be an eigenvalue of* P_L *such that* $\mu_m > \frac{\pi^2}{2h^3}$. *Then, using the notation of Proposition 3.5.11, we have the following asymptotic expansion (with respect to* $m \to +\infty$*):*

$$\mu_m = \frac{m^2h}{8} + \frac{1}{4h} + \frac{h}{8}\left[\frac{1}{m^2h^4} + \frac{\alpha^2(m,h)}{m^4} + \frac{2\alpha(m,h)}{m} + \frac{2\alpha(m,h)}{h^2m^3}\right]. \tag{3.162}$$

Notice that in this case α is a function of h, since it depends on the eigenfunction U. Besides, this dependence can not be written explicitly.

Now we can compare the expansion (3.162) with the extremes of the intervals where the eigenvalues of P_L are located, found in Corollaries 3.5.9 and 3.5.10. For example we notice that the term $\frac{1}{4h}$ is present in all the intervals of type E_n, F_n, L_n, M_n and it appears also in (3.162). Also the term $\frac{m^2h}{8}$ is common to (3.162) and E_n, F_n, L_n, M_n, by recalling that (3.162) gives all eigenvalues of P_L

so that, for $m = 2n$ we have eigenvalues associated to even eigenfunctions and for $m = 2n + 1$ we have the remaining eigenvalues. Nevertheless, Corollaries 3.5.9 and 3.5.10, proved using the continued fractions approach give a more precise result than the asymptotics (3.162). In fact, as already remarked, the function α is not easy to compute, for it depends on the eigenfunction itself, whereas the bounds C_n to F_n, G_n to M_n are quite elementary.

Chapter 4

Remarks on the asymptotic expansion of the lowest eigenvalue as $h \to 0^+$

4.1 Uniform convergence of eigenfunction coefficients

It is known (see e. g. [3], pp. 39,41) that some kind of parameter-dependent operators admit asymptotic expansions (in the same parameter) for their eigenvalues. We recall a result about these expansions and we study, using the continued fractions approach, the lowest eigenvalue of P as a function of h. At first we associate to P another operator, $\widetilde{P}$. Then, denoting by $\varpi = \varpi(h)$ the lowest eigenvalue of $\widetilde{P}$, we will prove the monotonicity of $\varpi(h)$ with respect to h, from which it will follow the existence of $\lim_{h \to 0^+} \varpi(h)$.

This section is intended to fix some notation and to prove some technical results, useful for our purposes.

Recall the definition of P :

$$P\left(h^{-2}\right) := P : D(P) \longrightarrow L^2(I),$$

with

$$(Pf)(x) = -f''(x) + V(x)f(x), \quad V(x) = \frac{1}{h^2}\sin^2\left(\frac{x}{2}\right),$$

where $D(P) = H_0^1(I) \cap H^2(I) \subset L^2(I)$ and $I = (-\pi, \pi)$. We can write the eigenvalue problem for P (see (1.7) and Proposition 1.1.1),

$$P(f) = \lambda f = \frac{2\mu}{h} f, \qquad f \in D(P), \tag{4.1}$$

as

$$-h^2 f'' + h^2 V f = \lambda h^2 f = 2\mu h f, \quad f \in D(P). \tag{4.2}$$

Recall that, by (1.1), μ represents an eigenvalue of the operator P_L.

Now we introduce the operator $\widetilde{P}$. We will apply to $\widetilde{P}$ the aforementioned result, which grants the existence of asymptotic expansions (in h) for its eigenvalues. As $\widetilde{P}$ is closely related to P we wil obtain immediatly asymptotic expansion for the eigenvalues of P.

Definition 4.1.1. *We put $\widetilde{P} = h^2 P$, $D(\widetilde{P}) = D(P)$. We set $\widetilde{V}(x) = \sin^2\left(\frac{x}{2}\right)$.*

By this definition and by (4.1) and (4.2) we get the relation between eigenvalues of $\widetilde{P}$ and P.

Remark 4.1.2. *$\widetilde{\lambda} = \widetilde{\lambda}(h)$ is an eigenvalue of $\widetilde{P}$ if and only if $\lambda = \frac{\widetilde{\lambda}}{h^2}$ is an eigenvalue of P. Moreover, recalling (1.1), (4.1) and (4.2), λ is an eigenvalue of*

P if and only if $\lambda = \frac{2\mu}{h}$, with μ eigenvalue of P_L. Hence $\widetilde{\lambda}$ is an eigenvalue of $\widetilde{P}$ if and only if $\widetilde{\lambda} = 2\mu h$, with μ eigenvalue of P_L.

Now we recall the theorem, by Helffer and Sjöstrand, that gives the asymptotic expansion of eigenvalues of $\widetilde{P}$ (we just state this result in our particular case, for the general case see [3], pp. 39, 41).

Theorem 4.1.3. *Set $\widetilde{V}_0(x) := \frac{1}{4}x^2$ and let $\widetilde{P}_0$ be the armonic oscillator*

$$\widetilde{P}_0 : D(\widetilde{P}_0) := D(P) \longrightarrow L^2(I), \quad (\widetilde{P}_0 f)(x) = -h^2 f''(x) + \widetilde{V}_0(x) f(x). \tag{4.3}$$

Let

$$\{E_n\}_{n\in\mathbb{N}} := \left\{ \frac{2n+1}{2} \right\}_{n\in\mathbb{N}}$$

be the sequence of eigenvalues of $\widetilde{P}_0$. Fix $0 < C_0 \notin \{E_0, E_1, \ldots\}$ and let $N_0 \in \mathbb{N}$ be such that $E_{N_0-1} < C_0 < E_{N_0}$.

Then there exists $h_0 > 0$ such that for $0 < h \leq h_0$, $\widetilde{P}$ has precisely N_0 eigenvalues $0 < \widetilde{\lambda}_0(h) \leq \cdots \leq \widetilde{\lambda}_{N_0-1}(h)$ in $[0, C_0 h]$. Moreover, $\widetilde{\lambda}_n$ has the asymptotic expansion

$$\widetilde{\lambda}_n(h) \sim h(E_n + a_1 h + a_2 h^2 + \ldots), \qquad a_n \in \mathbb{R}, \quad h \to 0^+. \tag{4.4}$$

Notice that $\widetilde{V}_0(x) = \frac{1}{4}x^2$ represents the first term in the Taylor's series expansion of $\widetilde{V}(x)$.

As already remarked Theorem 4.1.3 gives asymptotic expansion for the eigenvalues of a general class of operators, which contains P. It is interesting to see if this same theorem can be proved in our particular, one-dimensional case, using simpler techniques. In what follows we give a partial answer to this question, by analysing the case of the lowest eigenvalue of $\widetilde{P}$, which we will denote by ϖ. In particular we will prove that there exists $\lim_{h\to 0^+} \varpi(h)$. Before doing this we state the asymptotic expansion for $\varpi(h)$ as follows from Theorem 4.1.3. From equation (4.4) follows that

$$\varpi(h) \sim h\left(\frac{1}{2} + a_1 h + a_2 h^2 + \dots\right), \qquad a_n \in \mathbb{R}. \tag{4.5}$$

Recalling Remark 4.1.2, as $\varpi(h) = 2\mu_0(h)h$, where μ_0 is the lowest eigenvalue of P_L, we have

$$\mu_0(h) \sim \frac{1}{4} + \frac{a_1}{2}h + \frac{a_2}{2}h^2 + \dots, \qquad a_n \in \mathbb{R}.$$

Now we fix $h_0 \in \mathbb{R}$, with $h_0 > 0$. We will show that $\frac{d}{dh}\varpi(h)|_{h=h_0}$ is positive for every $h_0 > 0$, from this the monotonicity of $\varpi(h)$, with respect to h, will follow (and from here the existence of $\lim_{h\to 0^+} \varpi(h)$). Since we will analyse $\frac{d}{dh}\varpi(h)|_{h=h_0}$ we assume that $|h - h_0|$ is small, so that we can use once again the Perturbation Theory. In particular we will use Theorem 2.1.6. Through this approach we will prove uniform estimates on coefficients of the eigenfunction associated with $\varpi(h)$, for h in a complex neighbourhood of h_0. Then, using an integral equation which relates $\varpi(h)$ and its associated eigenfunction, we will get information on $\varpi(h)$.

We can write (recall (2.3))

$$\widetilde{P} = \widetilde{P}(h) = -h^2\frac{d^2}{dx^2} + \widetilde{V} = -h^2\frac{d^2}{dx^2} + h_0^2\frac{d^2}{dx^2} - h_0^2\frac{d^2}{dx^2} + \widetilde{V} =$$

$$= \widetilde{P}(h_0) + (h^2 - h_0^2)P(0). \tag{4.6}$$

From (4.6) we can use Theorem 2.1.2 to show that the $h-$dependent family of operators $\widetilde{P} = \widetilde{P}(h)$ forms an holomorphic family of type (A) in the parameter $(h^2 - h_0^2)$.

Proposition 4.1.4. *The family of operators $\widetilde{P} = \widetilde{P}(h)$ (see Definition 4.1.1) is a selfadjoint holomorphic family of type (A) in the perturbative parameter $(h^2 - h_0^2)$.*

Proof. Since, by (4.6),

$$\widetilde{P}(h) = \widetilde{P}(h_0) + (h^2 - h_0^2)P(0),$$

by Theorem 2.1.2 it suffices to prove that there exist $a, b \geq 0$ such that

$$\|P(0)f\| \leq a\|f\| + b\|\widetilde{P}(h_0)f\|.$$

We have

$$\|P(0)f\| = \| - f''\| = \frac{1}{h_0^2}\| - h_0^2 f''\| = \frac{1}{h_0^2}\| - h_0^2 f'' + \widetilde{V}f - \widetilde{V}f\| \leq$$

$$\leq \frac{1}{h_0^2}(\|P(h_0)f\| + \|f\|), \tag{4.7}$$

where the last inequality follows from

$$\max_{|x|\leq\pi} \widetilde{V}(x) = \max_{|x|\leq\pi} \left[\sin^2\left(\frac{x}{2}\right)\right] = 1.$$

In other words, from (4.7), we can set in (2.2) of Theorem 2.1.2, $c = 0$ and $a = b = \frac{1}{h_0^2}$. Thus, by the same theorem, $\widetilde{P} = \widetilde{P}(h)$ forms an holomorphic family of type (A) in $(h^2 - h_0^2)$, for $|h^2 - h_0^2| < h_0^2$.

Moreover, by recalling Definition 2.1.4, we have that $\widetilde{P}(h)$ is selfadjoint. □

By Proposition 4.1.4 and by Theorem 2.1.6, we can expand all eigenfunctions and eigenvalue of $\widetilde{P}$ in power series of the perturbative parameter $(h^2 - h_0^2)$. Notice that these series are defined for complex values of the perturbative parameter, thus we will consider, from now on, h as a *complex* parameter, varying in a neighbourhood of the real parameter h_0. From these expansion will follow uniform estimates on coefficients of the eigenfunction in the *same* complex neighbourhood of h_0.

We give the expansion for the lowest eigenvalue ϖ and its associated eigenfunction, $\widetilde{\psi}$.

Proposition 4.1.5. *Let ϖ be the lowest eigenvalue of $\widetilde{P}$ (see Definition 4.1.1). Then, for every $h \in \mathbb{C}$ such that $|h^2 - h_0^2| < h_0^2$, $\varpi = \varpi(h)$ admits the following power series expansion*

$$\varpi = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \varpi_n. \tag{4.8}$$

Let $\widetilde{\psi} = \widetilde{\psi}(h)$ be the eigenfunction associated with ϖ. We have, for every $h \in \mathbb{C}$ such that $|h^2 - h_0^2| < h_0^2$, that $\widetilde{\psi}$ admits the following expansion

$$\widetilde{\psi} = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_n,$$

with $\widetilde{\psi}_n \in L^2(I)$.

Proof. It is an immediate consequence of Proposition 4.1.4 and Theorem 2.1.6. □

We prove next some technical results which give estimates for coefficients of the expansions of $\widetilde{\psi}$ and ϖ. Later on we will write ϖ in terms of its associated eigenfunction $\widetilde{\psi}$ and we will use these estimates to obtain information about the monotonicity of $\varpi(h)$. Now we show the convergence to 0, as $n \to +\infty$, of the coefficients ϖ_n is uniform on $|h^2 - h_0^2| \leq \alpha^2$, for some $\alpha > 0$. To do this we recall a classical result on convergent power series (for the proof see e. g. [10], p. 56.)

Proposition 4.1.6. *Suppose $\sum_{n=0}^{+\infty} a_n z^n$ has a radius of convergence, $r > 0$. Then there exists a positive number C such that if $A > \frac{1}{r}$ then*

$$|a_n| \leq CA^n, \qquad \forall\, n \in \mathbb{N}.$$

Using this statement we can show the following

Lemma 4.1.7. *Fix $h_0 > 0$. Let $\varpi(h)$ be the lowest eigenvalue of $\widetilde{P}(h)$. Then there exist C, α, $\alpha_1 > 0$, with $0 < \alpha < \alpha_1 < h_0$, such that*

$$|\varpi(h)| \leq \frac{C\alpha_1^2}{\alpha_1^2 - \alpha^2},$$

for every $h \in \mathbb{C}$ in the disk $|h^2 - h_0^2| \leq \alpha^2$.

Proof. By Proposition 4.1.5 we get the expansion

$$\varpi = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \varpi_n, \tag{4.9}$$

for every h such that $|h^2 - h_0^2| < h_0^2$. Thus the radius of convergence of the series in (4.9), which we denote by ρ^2, is greater than or equal to h_0^2. Now we fix $\alpha_1 > 0$ such that $0 < \alpha_1^2 < h_0^2 \leq \rho^2$ and thus $\alpha_1^{-2} > h_0^{-2} \geq \rho^{-2}$. As $\alpha_1^{-2} > \rho^{-2}$, by Proposition 4.1.6 there exists $C > 0$ such that

$$|\varpi_n| \leq C \left|\frac{1}{\alpha_1^2}\right|^n, \qquad \forall\, n \in \mathbb{N}.$$

Therefore equation (4.9) gives

$$|\varpi(h)| \leq \sum_{n=0}^{+\infty} |h^2 - h_0^2|^n \; C \left|\frac{1}{\alpha_1^2}\right|^n. \tag{4.10}$$

Now we fix $\alpha > 0$ such that $0 < \alpha < \alpha_1$. Therefore, for all h such that $|h^2 - h_0^2| \leq \alpha^2$, we have

$$|\varpi(h)| \leq \sum_{n=0}^{+\infty} C \left(\frac{\alpha}{\alpha_1}\right)^{2n}.$$

Thus, as the last sum is a geometric series,

$$|\varpi(h)| \leq C \frac{1}{1-\left(\dfrac{\alpha}{\alpha_1}\right)^2} = C \frac{\alpha_1^2}{\alpha_1^2 - \alpha^2}.$$

□

Notice that, by 2) of Theorem 1.2.6, the eigenfunction $\widetilde{\psi}$, associated to ϖ, does not vanish on the interior of I. Therefore $\widetilde{\psi}$ must be an even function. Since $\left\{\frac{1}{\sqrt{\pi}}\cos\left(\frac{2m+1}{2}\,x\right)\right\}_{m\in\mathbb{N}}$ is an orthonormal basis of all even functions of $L^2(I)$ and as $\widetilde{\psi}$ is an analytic even function, in $h^2 - h_0^2$, (see Proposition 4.1.5), we can give the following expansion for $\widetilde{\psi}$

$$\widetilde{\psi}(h,x) = \sum_{m=0}^{+\infty}\left(\sum_{n=0}^{+\infty}(h^2-h_0^2)^n\widetilde{\psi}_{mn}\right)\frac{1}{\sqrt{\pi}}\cos\left(\frac{2m+1}{2}x\right). \tag{4.11}$$

Recalling the notation fixed in Chapter 3 (see equation (3.1)) we will write

$$\sum_{n=0}^{+\infty}(h^2-h_0^2)^n\widetilde{\psi}_{mn} = v_m = v_m(h). \tag{4.12}$$

Furthermore, as $\widetilde{\psi}$ is an eigenfunction of $\widetilde{P}$ associated to ϖ, recalling Remark 4.1.2, the same function $\widetilde{\psi}$ is an eigenfunction also of P, associated to the eigenvalue ϖ/h^2, which is the lowest eigenvalue of P. Thus, by Proposition 3.1.5 and Remark 3.1.11, using the notation fixed by (4.12), we have that

$$v_{-1} := 0 \qquad v_{n+1} = \delta_n v_n - v_{n-1}, \qquad n \in \mathbb{N}, \tag{4.13}$$

where

$$\begin{cases} \delta_0 = \delta_0\left(\dfrac{\varpi}{h^2}\right) = h^2 + 1 - 4\varpi \\ \delta_n = \delta_n\left(\dfrac{\varpi}{h^2}\right) = (2n+1)^2h^2 + 2 - 4\varpi, \qquad \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases} \tag{4.14}$$

From Lemma 4.1.7 follows an estimate on $\delta_n\left(\varpi/h^2\right)$, which we will use, exploiting relation (4.13), to estimate the Fourier coefficients $v_m(h)$ (see (4.12)).

Lemma 4.1.8. *Fix $h_0 > 0$. Let $\varpi = \varpi(h)$ be the lowest eigenvalue of $\widetilde{P}(h)$. Then there exist α, with $0 < \alpha < h_0$ and $n_0 \in \mathbb{N}$ such that (recall (4.14))*

$$\left|\delta_n\left(\frac{\varpi}{h^2}\right)\right| = |(2n+1)^2h^2 + 2 - 4\varpi| \geq 2, \tag{4.15}$$

for every $n \geq n_0$ and for every h in the disk $|h^2 - h_0^2| \leq \alpha^2$.

Proof. Note that

$$\left|(2n+1)^2h^2 - (4\varpi(h) - 2)\right| \geq \left|(2n+1)^2|h|^2 - |4\varpi(h) - 2|\right|.$$

Thus if we prove

$$(2n+1)^2|h|^2 - |4\varpi(h) - 2| \geq 2 \tag{4.16}$$

we obtain as a consequence (4.15). Moreover we have

$$2 + |4\varpi(h) - 2| \leq 4 + 4|\varpi(h)|,$$

thus, from (4.16), if we prove that

$$(2n+1)^2|h|^2 \geq 4 + 4|\varpi(h)| \tag{4.17}$$

we get (4.15). By Lemma 4.1.7 there exist $C, \alpha, \alpha_1 > 0$, with $0 < \alpha < \alpha_1 < h_0$, such that

$$|\varpi(h)| \leq \frac{C\alpha_1^2}{\alpha_1^2 - \alpha^2} \tag{4.18}$$

for every h in the disk $|h^2 - h_0^2| < \alpha^2$.

From (4.17) and (4.18) if we prove that

$$(2n+1)^2|h|^2 \geq 4 + \frac{4C\alpha_1^2}{\alpha_1^2 - \alpha^2} \tag{4.19}$$

we get (4.15). Dividing both sides of (4.19) by $|h|^2$ gives

$$(2n+1)^2 \geq \frac{1}{|h|^2}\left[4 + \frac{4C\alpha_1^2}{(\alpha_1^2 - \alpha^2)}\right].$$

Notice that for every h such that $|h^2 - h_0^2| \leq \alpha^2$ we have

$$\frac{1}{|h|^2} \leq \frac{1}{h_0^2 - \alpha^2}.$$

Therefore, if there exists $n_0 \in \mathbb{N}$ such that

$$(2n+1)^2 \geq \frac{1}{(h_0^2 - \alpha^2)} \left[4 + \frac{4C\alpha_1^2}{(\alpha_1^2 - \alpha^2)} \right], \tag{4.20}$$

for all $n \geq n_0$ the assertion follows. Such an n_0 exists, since the right-hand side of (4.20) is fixed. □

We will show the convergence to 0 of the coefficients $v_m = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_{mn}$ in (4.12), as $m \to +\infty$, uniformly for $|h^2 - h_0^2|$ in a neighborhood of 0. To this purpose we will reason as in the proof of Proposition 3.3.19. Recall that, as the eigenfunction associated to ϖ is even, we have $\{\vartheta_n\}_n := \{\delta_n\}_n$ (see Definition 3.2.3).

In the sequel we will consider the Fourier coefficients v_m in (4.12) as *complex* functions in the parameter $(h^2 - h_0^2)$, $h \in \mathbb{C}$, such that $|h^2 - h_0^2| \leq \alpha^2$, for a fixed $\alpha > 0$. To do this we just substitute a complex value of h in (4.14) and we compute the value for v_m using the recurrence relation (4.13). Then, again using (4.13), we will show, as in the proof of Proposition 3.3.19, the following formula for v_m :

$$v_{n_0+1+m} = \delta_{n_0+1} \ldots \delta_{n_0+m} z_{n_0} \ldots z_{n_0+m-1} v_{n_0+1}, \qquad \forall\, m > 0, \tag{4.21}$$

and for all *complex* h such that $|h^2 - h_0^2| \leq \alpha^2$, with

$$z_{n_0+m} = \cfrac{\cfrac{1}{\delta_{n_0+m+1}\delta_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\delta_{n_0+m+2}\delta_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\, m \in \mathbb{N}, \tag{4.22}$$

and for all *complex* h such that $|h^2 - h_0^2| \leq \alpha^2$.

We will prove in the first place that the functions $\delta_{n_0+m}\left(\frac{\varpi}{h^2}\right) z_{n_0+m-1}\left(\frac{\varpi}{h^2}\right)$ are holomorphic in $h^2 - h_0^2$. Afterwards, from relation (4.21), we will obtain an estimate on coefficients v_m, uniform with respect to $h^2 - h_0^2$. We will use, in this analysis, two classical results about holomorphic functions; we just state them (for the proof see [10], pp. 69, 156).

Proposition 4.1.9. *If f, g are analytic on U then f/g is analytic on the open subset of $\{z \in U|\ g(z) \neq 0\}$.*

Theorem 4.1.10. *Let $\{f_n\}_n$ be a sequence of holomorphic functions on an open set U. Assume that for each compact subset K of U the sequence converges uniformly on K, and let f be the limit function. Then f is holomorphic on U.*

Now we prove equation (4.21).

Proposition 4.1.11. *Let ϖ be the lowest eigenvalue of $\widetilde{P}$ and let $\widetilde{\psi}$ be the associated eigenfunction given by (4.11):*

$$\widetilde{\psi}(h,x) = \sum_{m=0}^{+\infty} v_m(h) \frac{1}{\sqrt{\pi}} \cos\left(\frac{2m+1}{2}x\right), \qquad v_m(h) = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_{mn}.$$

Then there exists $\beta \in \mathbb{R}$, $0 < \beta < h_0$, and $n_0 \in \mathbb{N}$ such that

$$v_{n_0+1+m} = \delta_{n_0+1} \dots \delta_{n_0+m} z_{n_0} \dots z_{n_0+m-1} v_{n_0+1}, \qquad \forall\ m > 0, \tag{4.23}$$

with

$$z_{n_0+m} = \cfrac{\cfrac{1}{\delta_{n_0+m+1}\delta_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\delta_{n_0+m+2}\delta_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\ m \in \mathbb{N}, \tag{4.24}$$

for all complex h such that $|h^2 - h_0^2| < \beta^2$.

Furthermore the functions $\delta_{n_0+m+1}\left(\varpi/h^2\right) z_{n_0+m}\left(\varpi/h^2\right)$ *are holomorphic on the set* $|h^2 - h_0^2| < \beta^2$*, for all* $m \in \mathbb{N}$*.*

Proof. We follow the proof of Proposition 3.3.19. Notice that, by Lemma 4.1.8 there exist α, with $0 < \alpha < h_0$, and $n_1 \in \mathbb{N}$ such that

$$\left|\delta_n\left(\frac{\varpi}{h^2}\right)\right| \geq 2, \tag{4.25}$$

for every $n \geq n_1$ and for every h in the disk $|h^2 - h_0^2| \leq \alpha^2$. Recall that, by the recurrence relation (4.13), all v_n are holomorphic in the parameter $h^2 - h_0^2$. By Corollary 3.4.12, as $\varpi(h_0)$ is the lowest eigenvalue of $\widetilde{P}$, we have that $v_m\left(h_0\right) \neq 0$ for all $m \in \mathbb{N}$. As $v_m\left(h\right)$ are holomorphic in $h^2 - h_0^2$ then

$$\lim_{h^2 \to h_0^2} v_m(h) = v_m(h_0) \neq 0, \qquad \forall\, m \in \mathbb{N}.$$

Thus there exist $n_0 > n_1$ and $0 < \beta < \alpha < h_0$ such that

$$v_{n_0}(h),\ v_{n_0+1}(h),\ v_{n_0+2}(h) \neq 0,$$

for all h such that $|h^2 - h_0^2| \leq \beta^2$. By Proposition 4.1.9, and since we chose $n_0 > n_1$ (so that $\delta_{n_0+1} \neq 0$), we have that the function appearing in Proposition 3.3.19, this time considered as complex valued,

$$1 - \frac{1}{\dfrac{v_{n_0+1}}{v_{n_0}}\delta_{n_0+1}}, \tag{4.26}$$

is holomorphic in $|h^2 - h_0^2| \leq \beta^2$. We recall equality (3.87), which holds for real h.

$$1 - \frac{1}{\dfrac{v_{n_0+1}}{v_{n_0}}\delta_{n_0+1}} = \frac{\dfrac{1}{\delta_{n_0+1}\delta_{n_0+2}}}{1 - \dfrac{\dfrac{1}{\delta_{n_0+2}\delta_{n_0+3}}}{1 - \ddots}}. \tag{4.27}$$

From what just proved the left-hand side of (4.27) makes sense also for complex value of h. The right-hand side of (4.27) makes sense too, in the same neighborhood of 0 in which the function (4.26) is holomorphic, that is for all h in $|h^2 - h_0^2| \leq \beta^2$. In fact, as $n_0 > n_1$ we have

$$\left| \frac{1}{\delta_{n_0+m+1}\delta_{n_0+m+2}} \right| < \frac{1}{4}, \qquad \forall\, m \in \mathbb{N}, \quad \forall\, h \in \mathbb{C}, \quad |h^2 - h_0^2| \leq \beta^2.$$

Thus, by Worpitzky's theorem (Theorem 3.3.17), we have that the continued fraction in the right-hand side of (4.27) converges for all h such that $|h^2 - h_0^2| \leq \beta^2$. Moreover we will show that this function is analytic on $|h^2 - h_0^2| \leq \beta^2$. Therefore, on recalling that the left-hand side of (4.27) is analytic too, in the same neighbourhood of 0, and as (4.27) holds for real h, it will follow the equality (4.27) on all $|h^2 - h_0^2| \leq \beta^2$. We set

$$z_n = \cfrac{\cfrac{1}{\delta_{n+1}\delta_{n+2}}}{1 - \cfrac{\cfrac{1}{\delta_{n+2}\delta_{n+3}}}{1 - \ddots}}. \tag{4.28}$$

By proving the analyticity of the functions z_n we will obtain also (4.23) and (4.24). In fact from Proposition 3.3.19 we know that (4.23) is true for real values of h. Furthermore, as already noticed, v_{n_0+1} is holomorphic on $|h^2 - h_0^2| \leq \beta^2$ and v_{n_0+1+m} is holomorphic in the same neighborhood, by the recurrence relation (4.13). Thus we can obtain (4.23) for all h, with $|h^2 - h_0^2| \leq \beta^2$, if we show that all $z_n(h)$ are holomorphic in the same set, for $n \geq n_0$. We prove this by showing that $\delta_{n+1} z_n$ are holomorphic for all $n \geq n_0$.

By (4.28) we have, writing an equivalent continued fraction (see Defintion

3.3.20)

$$\delta_{m+1} z_m = \cfrac{1}{\delta_{m+2} - \cfrac{1}{\delta_{m+3} - \ddots}}. \tag{4.29}$$

Let

$$f_n = \cfrac{1}{\delta_{m+2} - \cfrac{1}{\ddots - \cfrac{1}{\delta_{m+n+1}}}} = \frac{\widetilde{A}_n}{\widetilde{B}_n}$$

be the *n-th* approximant of the continued fraction in (4.29). As $\widetilde{A}_n$, $\widetilde{B}_n$ represent the *n-th* numerator and denominator for this continued fraction, from Remark 3.3.4, upon setting

$$\widetilde{A}_{-1} = 1, \quad \widetilde{A}_0 = 0, \quad \widetilde{B}_{-1} = 0, \quad \widetilde{B}_0 = 1,$$

we have

$$\begin{cases} \widetilde{A}_n = \delta_{m+1+n} \widetilde{A}_{n-1} - \widetilde{A}_{n-2}, & n \geq 1, \\ \widetilde{B}_n = \delta_{m+1+n} \widetilde{B}_{n-1} - \widetilde{B}_{n-2}, & n \geq 1. \end{cases}$$

Through these relations, on recalling (4.14) and (4.9), we obtain that $\widetilde{A}_n$, $\widetilde{B}_n$ are holomorphic functions of $h^2 - h_0^2$. We prove that $\widetilde{B}_n$ is never 0 on $|h^2 - h_0^2| \leq \beta^2$ and that $\left\{ \frac{\widetilde{A}_n}{\widetilde{B}_n} \right\}_n$ converges to $\delta_{m+1} z_m$, for all $m \geq n_0$, uniformly on every compact subset of $|h^2 - h_0^2| \leq \beta^2$. From here, by Theorem 4.1.10 it will follow that $\delta_{m+1} z_m$ are holomorphic on $|h^2 - h_0^2| < \beta$, for all $m \in \mathbb{N}$.

We recall that, as $n_0 > n_1$, for all $m \geq n_0$ and for all h such that $|h^2 - h_0^2| \leq \beta^2$ we have $\left|\delta_m \left(\frac{\varpi}{h^2}\right)\right| > 2$ (see equation (4.25)).

Notice that

$$|\widetilde{B}_1| = |\delta_{m+2} \widetilde{B}_0| = |\delta_{m+2}| > 2 > 1 = |\widetilde{B}_0|.$$

As $|\widetilde{B}_1| > |\widetilde{B}_0|$ and as $|\delta_m| > 2$, for all $m \geq n_0$, from the recurrence relation we get

$$|\widetilde{B}_2| = |\delta_{m+3}\widetilde{B}_1 - \widetilde{B}_0| \geq ||\delta_{m+3}||\widetilde{B}_1| - |\widetilde{B}_0|| \geq 2|\widetilde{B}_1| - |\widetilde{B}_0| \geq |\widetilde{B}_1|.$$

We can use the same procedure inductively, as $|\delta_m| > 2$ for all $m \geq n_0$; so we find that $|\widetilde{B}_{n+1}| > |\widetilde{B}_n|$ for all n. Again from the recurrence relation we get

$$|\widetilde{B}_n| \geq |\delta_{m+1+n}||\widetilde{B}_{n-1}| - |\widetilde{B}_{n-2}| \geq 2|\widetilde{B}_{n-1}| - |\widetilde{B}_{n-2}|.$$

This implies

$$|\widetilde{B}_n| - |\widetilde{B}_{n-1}| \geq |\widetilde{B}_{n-1}| - |\widetilde{B}_{n-2}| \geq \cdots \geq |\widetilde{B}_1| - |\widetilde{B}_0| \geq 1.$$

From $|\widetilde{B}_n| - |\widetilde{B}_{n-1}| \geq 1$ for all n we get $|\widetilde{B}_n| \geq n$. We prove the uniform convergence of $f_n = f_n(h^2 - h_0^2)$ (i.e. as functions of $h^2 - h_0^2$). If $n > j$ we have

$$|f_n - f_j| = |f_n - f_0 - (f_j - f_0)| = \left|\sum_{k=1}^{n}(f_k - f_{k-1}) - \sum_{k=1}^{j}(f_k - f_{k-1})\right| =$$

$$= \left|\sum_{k=j+1}^{n}(f_k - f_{k-1})\right| \leq \sum_{k=j+1}^{n}|f_k - f_{k-1}|. \tag{4.30}$$

Notice that by relation (3.58) we have

$$f_k - f_{k-1} = \frac{A_k}{B_k} - \frac{A_{k-1}}{B_{k-1}} = \frac{(-1)^{k-1}}{B_k B_{k-1}}\prod_{j=1}^{k} a_j.$$

From here and from (4.30) it follows

$$|f_n(h^2 - h_0^2) - f_j(h^2 - h_0^2)| \leq \sum_{k=j+1}^{n}\frac{1}{|B_k||B_{k-1}|} \leq \sum_{k=j+1}^{n}\frac{1}{k(k-1)},$$

for all $n > j$ and for every h such that $|h^2 - h_0^2| \leq \beta^2$.

As the series $\sum_{k=j}^{+\infty}\frac{1}{k(k-1)}$ converges, we have the uniform convergence on all compact set of $|h^2 - h_0^2| \leq \beta^2$. From here the assertion follows. □

Now, using (4.23) and (4.24), we obtain uniform estimates for v_n.

Proposition 4.1.12. *Let ϖ be the lowest eigenvalue of $\widetilde{P}$ and let $\widetilde{\psi}$ be the associated eigenfunction. We recall the expansion (4.11) for $\widetilde{\psi}$:*

$$\widetilde{\psi}(h,x) = \sum_{m=0}^{+\infty} v_m(h) \frac{1}{\sqrt{\pi}} \cos\left(\frac{2m+1}{2}x\right), \qquad v_m(h) = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_{mn}.$$

There exists $\alpha \in \mathbb{R}$, $0 < \alpha < h_0$, such that the coefficients $v_m(h)$ tend to zero, as $m \to +\infty$, faster than any negative power of m, uniformly in $h \in \mathbb{C}$ such that $|h^2 - h_0^2| \leq \alpha^2$.

Proof. On recalling proposition (4.1.11) and its proof we have that there exists $\beta \in \mathbb{R}$, $0 < \beta < h_0$, and $n_0 \in \mathbb{N}$ such that

$$v_{n_0+1+m} = \delta_{n_0+1} \dots \delta_{n_0+m} z_{n_0} \dots z_{n_0+m-1} v_{n_0+1}, \qquad \forall\, m > 0, \tag{4.31}$$

with

$$z_{n_0+m} = \cfrac{\cfrac{1}{\delta_{n_0+m+1}\delta_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\delta_{n_0+m+2}\delta_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\, m \in \mathbb{N}, \tag{4.32}$$

for all h such that $|h^2 - h_0^2| < \beta^2$; furthermore, for the same values of h, we have that $\delta_{n_0+m} > 2$ for all $m \in \mathbb{N}$.

Thus $\{z_m\}_m$ fulfills the hypothesis of Worpitzky's theorem (Theorem 3.3.17) and therefore we get

$$|z_m| = \left| K_{j=m+1}^{+\infty} \left(\frac{-\frac{1}{\delta_{n_0+j}\delta_{n_0+j+1}}}{-1} \right) \right| \leq \frac{1}{2}, \tag{4.33}$$

for every $m \geq n_0$ and for every $h > 0$ such that $|h^2 - h_0^2| \leq \beta^2$. From (4.33) it follows that

$$|1 - z_{m+1}| \geq \big|1 - |z_{m+1}|\big| \geq \frac{1}{2},$$

so that, by recalling (4.32),

$$|z_m| = \frac{\left|\frac{1}{\delta_{m+1}\delta_{m+2}}\right|}{|1 - z_{m+1}|} \leq 2\left|\frac{1}{\delta_{m+1}\delta_{m+2}}\right|. \tag{4.34}$$

Thus, from (4.31) we get

$$|v_{n_0+1+m}| \leq |v_{n_0+1}|\left|\frac{2}{\delta_{n_0+1}\delta_{n_0+2}}\right|\dots\left|\frac{2}{\delta_{n_0+m}\delta_{n_0+m+1}}\right||\delta_{n_0+1}\dots\delta_{n_0+m}| =$$

$$= |v_{n_0+1}|\frac{2^m}{|\delta_{n_0+2}\dots\delta_{n_0+m+1}|}. \tag{4.35}$$

As in the proof of Theorem 3.3.12, we write δ_m as

$$\delta_m = (2m+1)^2 h^2 \left(1 - \frac{\frac{1}{h^2}(4\varpi - 2)}{(2m+1)^2}\right). \tag{4.36}$$

Plugging (4.36) into (4.35) gives

$$|v_{n_0+1+m}| \leq \frac{|v_{n_0+1}|\ 2^m}{|h^2|^m\left[(2n_0+5)\dots(2n_0+2m+3)\right]^2 \prod_{k=n_0+2}^{n_0+m+1}\left|1 - \frac{\frac{1}{h^2}(4\varpi-2)}{(2k+1)^2}\right|} \leq$$

$$\leq \frac{|v_{n_0+1}|}{|2h^2|^m\left[(n_0+3)(n_0+4)\dots(n_0+m+2)\right]^2 \prod_{k=n_0+2}^{n_0+m+1}\left|1 - \frac{\frac{1}{h^2}(4\varpi-2)}{(2k+1)^2}\right|} \tag{4.37}$$

We have (upon possibly increasing n_0)

$$\prod_{k=n_0+2}^{n_0+m+1}\left|1 - \frac{\frac{1}{h^2}(4\varpi - 2)}{(2k+1)^2}\right| \geq \prod_{k=n_0+2}^{n_0+m+1}\left(1 - \frac{\frac{1}{|h^2|}|4\varpi - 2|}{(2k+1)^2}\right). \tag{4.38}$$

Lemma 4.1.7 states that there exist C, α, $\alpha_1 > 0$, with $0 < \alpha < \alpha_1 < h_0$, such that

$$|\varpi(h)| \leq \frac{C\alpha_1^2}{\alpha_1^2 - \alpha^2}, \qquad \forall\ h > 0, \quad \text{with} \quad |h^2 - h_0^2| \leq \alpha^2.$$

We assume, without loss of generality, that $\alpha < \alpha_1 < \beta$. Thus we have

$$\frac{1}{|h^2|}|4\varpi - 2| \leq \frac{1}{|h^2|}(|4\varpi| + 2) \leq \frac{1}{|h^2|}\left(\frac{4C\alpha_1^2}{\alpha_1^2 - \alpha^2} + 2\right) \leq$$

$$\leq \frac{1}{h_0^2 - \alpha^2}\left(\frac{4C\alpha_1^2}{\alpha_1^2 - \alpha^2} + 2\right) \tag{4.39}$$

From (4.38) and (4.39), by supposing that n_0 is such that

$$\frac{\frac{1}{|h^2|}|4\varpi - 2|}{(2k+1)^2} < 1, \qquad \forall\, k \geq n_0 + 2,$$

we get

$$\prod_{k=n_0+2}^{n_0+m+1} \left|1 - \frac{\frac{1}{h^2}(4\varpi - 2)}{(2k+1)^2}\right| \geq \prod_{k=n_0+2}^{n_0+m+1} \left|1 - \frac{\frac{4C\alpha_1^2}{\alpha_1^2-\alpha^2} + 2}{(h_0^2 - \alpha^2)(2k+1)^2}\right| > 0. \tag{4.40}$$

Thus, upon setting

$$\widetilde{D}_m := \prod_{k=n_0+2}^{n_0+m+1} \left|1 - \frac{\frac{4C\alpha_1^2}{\alpha_1^2-\alpha^2} + 2}{(h_0^2 - \alpha^2)(2k+1)^2}\right|, \tag{4.41}$$

we have

$$\lim_{m\to+\infty} \widetilde{D}_m = a \in \mathbb{R}_+. \tag{4.42}$$

Thus, by (4.40) and by using (4.41) in (4.37), and upon dividing and multiplying by $[(n_0+2)!]^2$, we have

$$|v_{n_0+m+1}| \leq \frac{|v_{n_0+1}|\ [(n_0+2)!]^2}{|2h^2|^m\ [(m+1)!]^2\ \widetilde{D}_m}.$$

Multiplying and dividing by $2\pi(m+1)^{2m+3}e^{-2(m+1)}$ gives

$$|v_{n_0+1+m}| \leq \frac{|v_{n_0+1}|\ [(n_0+2)!]^2\ 2\,\pi\,(m+1)^{2m+3}\,e^{-2(m+1)}}{|2h^2|^m\ [(m+1)!]^2\ \widetilde{D}_m\,2\,\pi\,(m+1)^{2m+3}\,e^{-2(m+1)}}. \tag{4.43}$$

On setting

$$\widetilde{C}_m := \frac{2\pi(m+1)^{2(m+1)+1}e^{-2(m+1)}}{[(m+1)!]^2} \tag{4.44}$$

we have, by Stirling's formula,

$$\lim_{m\to+\infty} \widetilde{C}_m = 1. \tag{4.45}$$

Thus, substituting (4.44) in (4.43) gives

$$|v_{n_0+1+m}| \leq \frac{|v_{n_0+1}|\ [(n_0+2)!]^2\,\widetilde{C}_m e^2}{\left[2\left|\frac{h\,(m+1)}{e}\right|^2\right]^m \widetilde{D}_m\,2\,\pi\,(m+1)^3}. \tag{4.46}$$

As we assumed that $|h^2 - h_0^2| \leq \alpha^2$, from (4.46) it follows that

$$|v_{n_0+1+m}| \leq \frac{|v_{n_0+1}|\ [(n_0+2)!]^2\, \widetilde{C}_m e^2}{\left[2(h_0^2 - \alpha^2)\left(\dfrac{m+1}{e}\right)^2\right]^m \widetilde{D}_m\, 2\,\pi\,(m+1)^3}. \tag{4.47}$$

Notice that, since v_m verifies the recurrence relation

$$v_{m+1} = \delta_m v_m - v_{m-1}, \qquad \forall\, m \in \mathbb{N},$$

we have that $v_{n_0+1} = v_{n_0+1}(h)$ represents an analytic function in $(h^2 - h_0^2)$. Thus, upon possibly shrinking α, the value of $|v_{n_0+1}|$ is bounded for all h such that $|h - h_0^2| \leq \alpha^2$. Thus, by (4.47), if we notice that α and h_0 are fixed and by recalling (4.42) and (4.45) the assertion follows. □

We re-write inequality (4.47) in a simpler form.

Corollary 4.1.13. *In the hypotheses of Proposition 4.1.12 there exist D, $\alpha > 0$ such that*

$$|v_m(h)| = \left|\sum_{n=0}^{+\infty}(h^2 - h_0^2)^n \widetilde{\psi}_{mn}\right| \leq \frac{D}{m^m}, \quad \forall\, h,\ |h^2 - h_0^2| \leq \alpha^2, \quad \forall\, m \in \mathbb{N}. \tag{4.48}$$

Proof. It follows immediatly from the proof of Proposition 4.1.12 and inequality (4.47). Notice that, again from the proof of Proposition 4.1.12, the constant D is independent of m. □

4.2 Monotonicity of $\varpi(h)$

Using (4.48) we will prove some other needed estimates on coefficients of the eigenfunction $\widetilde{\psi}$ and its derivative with respect to h. These estimates, together with the Picone identity, which links $\varpi(h)$ and its associated eigenfunction, will be used later on to show the monotonicity of $\varpi(h)$, with respect to h. To this purpose we recall a result on analytic functions (see [12], p. 6).

Proposition 4.2.1. *Let f be an holomorphic function in U and let $|f(z)| \leq M$ for every $z \in U$. Then for any compact set $K \subset U$ and any α we have*

$$|D^\alpha f(z)| \leq M\alpha!\delta^{-|\alpha|} \qquad \forall\ z \in K,$$

where δ is the distance of K from the boundary of U.

Now we set $\zeta = h^2 - h_0^2$ and consequently write $v_m = v_m(\zeta)$.

Using Corollary 4.1.13 and Proposition 4.2.1 we can immediatly prove an estimate on $v'_m = v'_m(\zeta) = \frac{d}{d\zeta}v_m(\zeta)$.

Proposition 4.2.2. *Using the notation of Corollary 4.1.13 let γ be such that $0 < \gamma < \alpha$. Then we have*

$$|v'(\zeta)| = \left|\frac{d}{d\zeta}\sum_{n=0}^{+\infty}\zeta^n\widetilde{\psi}_{mn}\right| \leq \frac{D}{(\alpha^2-\gamma^2)m^m}, \qquad \forall\ \zeta \in \mathbb{C}, \quad |\zeta| \leq \gamma^2.$$

Before expressing $\varpi(h)$ in terms of $\widetilde{\psi}$ and its derivative we prove one more technical lemma which gives an estimate for $|v_m(\zeta) - v_m(0)|$; this will allow us to use the mean value theorem for the v_m.

From now on we will consider again the parameter h (and thus ζ) as a *real* number.

Lemma 4.2.3. *Using notation of Corollary 4.1.13 let $0 < \gamma < \alpha$. We have, when ζ is real,*

$$|v_m(\zeta) - v_m(0)| \leq \frac{|\zeta|\ D}{(\alpha^2-\gamma^2)\,m^m} = \frac{D\ |h^2-h_0^2|}{(\alpha^2-\gamma^2)\,m^m}$$

Proof. In fact, by the mean value Theorem, we have

$$|v_m(\zeta) - v_m(0)| = |\zeta||v'(\tilde{\zeta})|, \qquad \text{with } \tilde{\zeta} \in (0, \zeta).$$

By Proposition 4.2.2 the assertion follows. □

Now we recall the Picone identity (for the proof see [22] p. 194) which will help us to express $\varpi(h)$ in terms of an integral depending on $\widetilde{\psi}(h)$, $\widetilde{\psi}(h_0)$ and their derivatives with respect to x.

Proposition 4.2.4. *Let the following differential equations be given:*

$$\frac{d}{dx}\left[\theta \frac{dy}{dx}\right] - Q(x)y(x) = 0 \tag{4.49}$$

$$\frac{d}{dx}\left[\theta_1 \frac{dz}{dx}\right] - Q_1(x)z(x) = 0 \tag{4.50}$$

and assume that the functions θ, θ', Q, θ_1, θ_1', Q_1 are real-valued and continuous on the interval $[\alpha, \beta]$, with $\theta > 0$, $\theta_1 > 0$ on $[\alpha, \beta]$. Let y, z be real-valued solutions of (4.49) and (4.50), respectively. Furthermore let $y(\alpha) = y(\beta) = 0$ and $z(x) \neq 0$ for $\alpha < x < \beta$. Then

$$0 = \int_\alpha^\beta (Q - Q_1)y^2 dx + \int_\alpha^\beta (\theta - \theta_1)(y')^2 dx + \int_\alpha^\beta \theta_1 \left[y' - \frac{yz'}{z}\right]^2 dx. \tag{4.51}$$

Now we use Picone's identity, (4.51), to get an expression of $\varpi(h)$ which we will use in computing $\frac{d}{dh}\varpi(h)|_{h=h_0}$.

Remark 4.2.5. *Let $\varpi(h)$ and $\varpi(h_0)$ represent the lowest eigenvalue of the operators $\widetilde{P}(h)$ and $\widetilde{P}(h_0)$, respectively, and let $\widetilde{\psi}(h)$, $\widetilde{\psi}(h_0)$ be the associated eigenfunctions. Assume also that these eigenfunctions are normalized, so that*

$$\|\widetilde{\psi}(h)\|^2 = \|\widetilde{\psi}(h_0)\|^2 = 1. \tag{4.52}$$

Then we have

$$\varpi(h) - \varpi(h_0) = (h^2 - h_0^2) \int_{-\pi}^{\pi} (\widetilde{\psi}'(h))^2 dx +$$

$$+h_0^2 \int_{-\pi}^{\pi} (\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0))^2 dx + h_0^2 \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx +$$

$$+2h_0^2 \int_{-\pi}^{\pi} (\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0)) \widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) dx. \qquad (4.53)$$

Proof. By hypothesis we have that $\widetilde{\psi}(h_0)$ is a solution of the problem

$$\begin{cases} \widetilde{P}(h)y = -h^2 y'' + V(x)y = \varpi(h)y \\ y(\pm\pi) = 0 \end{cases} \qquad (4.54)$$

and $\widetilde{\psi}(h_0)$ is a solution of the problem

$$\begin{cases} \widetilde{P}(h_0)z = -h_0^2 z'' + V(x)z = \varpi(h_0)z \\ z(\pm\pi) = 0. \end{cases} \qquad (4.55)$$

Then, by Picone's identity (4.51)

$$0 = \int_{-\pi}^{\pi} (\varpi(h_0) - \varpi(h)) \widetilde{\psi}(h)^2 dx + \int_{-\pi}^{\pi} (h^2 - h_0^2)(\widetilde{\psi}'(h))^2 dx +$$

$$+ \int_{-\pi}^{\pi} h_0^2 \left[\widetilde{\psi}'(h) - \frac{\widetilde{\psi}(h)\widetilde{\psi}'(h_0)}{\widetilde{\psi}(h_0)} \right]^2 dx.$$

Thus, recalling (4.52)

$$\varpi(h) - \varpi(h_0) = \int_{-\pi}^{\pi} (h^2 - h_0^2)(\widetilde{\psi}'(h))^2 dx + \int_{-\pi}^{\pi} h_0^2 \left[\widetilde{\psi}'(h) - \frac{\widetilde{\psi}(h)\widetilde{\psi}'(h_0)}{\widetilde{\psi}(h_0)} \right]^2 dx.$$

By adding and subtracting $\widetilde{\psi}'(h_0)$ in the second intergral we get

$$\varpi(h) - \varpi(h_0) =$$

$$= (h^2 - h_0^2) \int_{-\pi}^{\pi} (\widetilde{\psi}'(h))^2 dx + \int_{-\pi}^{\pi} h_0^2 \left[\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0) + \widetilde{\psi}'(h_0) - \frac{\widetilde{\psi}(h)\widetilde{\psi}'(h_0)}{\widetilde{\psi}(h_0)} \right]^2 dx$$

and from here the assertion follows. □

We will analyse

$$\frac{d}{dh}\varpi(h)\Big|_{h=h_0} = \lim_{h\to h_0}\frac{\varpi(h)-\varpi(h_0)}{h-h_0}, \quad h_0>0.$$

By (4.53) we have

$$\begin{aligned}\lim_{h\to h_0}\frac{\varpi(h)-\varpi(h_0)}{h-h_0} &= \lim_{h\to h_0}(h+h_0)\int_{-\pi}^{\pi}(\widetilde{\psi}'(h))^2dx+\\ &+\lim_{h\to h_0}\frac{h_0^2}{h-h_0}\int_{-\pi}^{\pi}(\widetilde{\psi}'(h)-\widetilde{\psi}'(h_0))^2dx+\\ &+\lim_{h\to h_0}\frac{h_0^2}{h-h_0}\int_{-\pi}^{\pi}\left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2dx+\\ &+\lim_{h\to h_0}\frac{2h_0^2}{h-h_0}\int_{-\pi}^{\pi}(\widetilde{\psi}'(h)-\widetilde{\psi}'(h_0))\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)dx.\end{aligned} \tag{4.56}$$

In particular we will prove that the limit in (4.56) is greater than 0.

Recalling the notation used up to now, by (4.11) and (4.12) we have

$$\widetilde{\psi}(h,x) = \sum_{m=0}^{+\infty} v_m\frac{1}{\sqrt{\pi}}\cos\left(\frac{2m+1}{2}x\right), \tag{4.57}$$

with

$$v_m = v_m(h) = \sum_{n=0}^{+\infty}(h^2-h_0^2)^n\widetilde{\psi}_{mn}; \tag{4.58}$$

and we have

$$\widetilde{\psi}(h_0,x) = \sum_{m=0}^{+\infty}\widetilde{\psi}_{m0}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2m+1}{2}x\right). \tag{4.59}$$

In order to compute the derivatives of $\widetilde{\psi}$, appearing in (4.53), notice that from Proposition 4.1.12 and Corollary 4.1.13 we have $v_m \to 0$, as $m \to +\infty$, faster than any negative power of m and uniformly with respect to h. Thus we can differentiate the series in (4.57) term by term. Moreover, by Theorem 3.3.12, we can differentiate term by term equation (4.59), since $\widetilde{\psi}_{m0} \to 0$, as $m \to +\infty$, faster than any negative power of m. In particular we have the following

Remark 4.2.6. *Let $\varpi(h)$ and $\varpi(h_0)$ represent the lowest eigenvalue of $\widetilde{P}(h)$ and $\widetilde{P}(h_0)$ respectively. Let $\widetilde{\psi}(h)$, $\widetilde{\psi}(h_0)$ be the associated eigenfunctions given by (4.57) and (4.59). We have*

$$\widetilde{\psi}'(h) = \sum_{m=0}^{+\infty} v_m \left(-\frac{2m+1}{2\sqrt{\pi}}\right) \sin\left(\frac{2m+1}{2}\,x\right), \tag{4.60}$$

$$\widetilde{\psi}'(h_0) = \sum_{m=0}^{+\infty} \widetilde{\psi}_{m0} \left(-\frac{2m+1}{2\sqrt{\pi}}\right) \sin\left(\frac{2m+1}{2}\,x\right). \tag{4.61}$$

Using the estimates proved up to now we show that $\varpi(h)$ is monotonic increasing with respect to h.

Theorem 4.2.7. *The eigenvalue $\varpi = \varpi(h)$ is monotone increasing as a function of h (for $h > 0$); as a consequence there exists $\lim_{h\to 0}\varpi(h)$.*

Proof. We will show that

$$\frac{d}{dh}\varpi(h)\Big|_{h=h_0} = \lim_{h\to h_0}\frac{\varpi(h)-\varpi(h_0)}{h-h_0} > 0, \qquad \forall\ h_0 > 0.$$

To do this we compute each term in equation (4.56). Consider the first term in the right-hand side of (4.56):

$$\lim_{h\to h_0}(h+h_0)\int_{-\pi}^{\pi}(\widetilde{\psi}'(h))^2 dx = 2h_0 \lim_{h\to h_0}\int_{-\pi}^{\pi}(\widetilde{\psi}'(h))^2 dx. \tag{4.62}$$

From (4.62) and (4.60) of Remark 4.2.6 we get

$$\lim_{h\to h_0}(h+h_0)\int_{-\pi}^{\pi}(\widetilde{\psi}'(h))^2 dx =$$

$$= 2h_0 \lim_{h\to h_0}\int_{-\pi}^{\pi}\left[\sum_{m=0}^{+\infty} -\frac{2m+1}{2\sqrt{\pi}}\sin\left(\frac{2m+1}{2}x\right) v_m(h)\right]^2. \tag{4.63}$$

By Corollary 4.1.13 and since the $v_m(h)$ are analytic in h we can exchange in (4.63) the limit with the integral and then with the sum, thus obtaining

$$\lim_{h\to h_0}(h+h_0)\int_{-\pi}^{\pi}(\widetilde{\psi}'(h))^2 dx =$$

$$= 2h_0 \int_{-\pi}^{\pi} \left\{ \sum_{m=0}^{+\infty} \lim_{h\to h_0} \left[-\frac{2m+1}{2\sqrt{\pi}} \sin\left(\frac{2m+1}{2} x\right) v_m(h) \right] \right\}^2 =$$

$$= 2h_0 \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} -\frac{2m+1}{2\sqrt{\pi}} \sin\left(\frac{2m+1}{2} x\right) \widetilde{\psi}_{m0} \right]^2 . \qquad (4.64)$$

Notice that, by (4.61) of Remark 4.2.6, the sum appearing in the last term of (4.64) represents the function $\widetilde{\psi}'(h_0, x)$. Therefore, by (4.64), we have

$$\lim_{h\to h_0} (h + h_0) \int_{-\pi}^{\pi} \left(\widetilde{\psi}'(h)\right)^2 dx = 2h_0 \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0)\right]^2 dx > 0.$$

We will next see that all the other terms in the right-hand side of (4.56) vanish, thus concluding the proof.

We consider the second term in (4.56). By (4.60) and (4.61) we have

$$\lim_{h\to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left(\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0)\right)^2 dx =$$

$$= \lim_{h\to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} \left(-\frac{2m+1}{2\sqrt{\pi}}\right) \sin\left(\frac{2m+1}{2}\, x\right) \left(v_m(h) - \widetilde{\psi}_{m0}\right) \right]^2 dx.$$

As $\widetilde{\psi}_{m0} = v_m(h_0)$, by Lemma 4.2.3 there exist D, α, $\gamma > 0$ such that

$$\lim_{h\to h_0} \left| \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left(\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0)\right)^2 dx \right| \leq$$

$$\leq \lim_{h\to h_0} \frac{h_0^2}{|h - h_0|} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} \left| -\frac{2m+1}{2\sqrt{\pi}} \sin\left(\frac{2m+1}{2}\, x\right) \right| \frac{|h^2 - h_0^2|\; D}{(\alpha^2 - \gamma^2)\; m^m} \right]^2 dx =$$

$$= \lim_{h\to h_0} \frac{h_0^2 |h - h_0| |h + h_0|^2 D^2}{(\alpha^2 - \gamma^2)^2} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} \frac{2m+1}{2\sqrt{\pi}} \left| \sin\left(\frac{2m+1}{2}\, x\right) \right| \frac{1}{m^m} \right]^2 dx,$$

and this is 0.

We compute the third term in the right-hand side of (4.56):

$$\lim_{h\to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx.$$

As already noticed the eigenfunctions $\widetilde{\psi}(h)$ and $\widetilde{\psi}(h_0)$ are even functions, without any zeros on $(-\pi, \pi)$. So we have

$$\lim_{h \to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx =$$

$$= 2 \lim_{h \to h_0} \frac{h_0^2}{h - h_0} \int_{0}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx. \tag{4.65}$$

Multiplying and dividing the right-hand side of (4.65) by $(x - \pi)$ gives

$$\lim_{h \to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx =$$

$$= 2 \lim_{h \to h_0} \frac{h_0^2}{h - h_0} \int_{0}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{x - \pi}}{\frac{\widetilde{\psi}(h_0)}{x - \pi}} \right) \right]^2 dx. \tag{4.66}$$

The function $\frac{\widetilde{\psi}(h_0,x)}{x-\pi}$ does not vanish on the interval $[0, \pi)$. If we prove that

$$\lim_{x \to \pi^-} \frac{\widetilde{\psi}(h_0, x)}{x - \pi} \neq 0$$

then there exists $R > 0$ such that

$$\left| \frac{\widetilde{\psi}(h_0, x)}{x - \pi} \right| > R, \qquad \forall\, x \in [0, \pi]. \tag{4.67}$$

By De L'Hospital's theorem we have

$$\lim_{x \to \pi^-} \frac{\widetilde{\psi}(h_0, x)}{x - \pi} = \lim_{x \to \pi^-} \widetilde{\psi}'(h_0, x).$$

The right-hand side of this equation is obviously different from zero, because it cannot be $\widetilde{\psi}'(h_0, \pi) = \widetilde{\psi}(h_0, \pi) = 0$, as $\widetilde{\psi}$ is a non-trivial solution of

$$\begin{cases} (\widetilde{P}(h_0) - \varpi(h_0))\widetilde{\psi}(h_0, x) = 0, & \forall\, x \in [-\pi, \pi] \\ \widetilde{\psi}(h_0, \pm\pi) = 0. \end{cases}$$

From (4.67) and (4.66) it follows

$$\left| \lim_{h \to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx \right| \leq$$

$$\leq 2 \lim_{h \to h_0} \frac{h_0^2}{|h - h_0|} \int_0^{\pi} \left[\frac{\widetilde{\psi}'(h_0)}{R} \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{x - \pi} \right) \right]^2 dx. \tag{4.68}$$

We set

$$S_m(x) = \frac{(-1)^{m+1} \sin\left(\frac{2m+1}{2}(x - \pi)\right)}{x - \pi}.$$

Thus, by (4.68), (4.57) and (4.59) and recalling Lemma 4.2.3 we have

$$\left| \lim_{h \to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx \right| \leq$$

$$\leq 2 \lim_{h \to h_0} \frac{h_0^2}{|h - h_0|} \int_0^{\pi} \left[\frac{\widetilde{\psi}'(h_0)}{R} \left(\sum_{m=0}^{+\infty} (v_m(h) - v_m(h_0)) \frac{\cos\left(\frac{2m+1}{2}x\right)}{x - \pi} \right) \right]^2 dx \leq$$

$$\leq 2 \lim_{h \to h_0} \frac{h_0^2}{|h - h_0|} \int_0^{\pi} \left[\frac{\widetilde{\psi}'(h_0)}{R} \left(\sum_{m=0}^{+\infty} \frac{|h^2 - h_0^2| \; D \; S_m(x)}{(\alpha^2 - \gamma^2) m^m} \right) \right]^2 dx =$$

$$= \lim_{h \to h_0} \frac{2h_0^2 |h - h_0| |h + h_0|^2 D^2}{(\alpha^2 - \gamma^2)^2} \int_0^{\pi} \left[\frac{\widetilde{\psi}'(h_0)}{R} \sum_{m=0}^{+\infty} \frac{S_m(x)}{m^m} \right]^2 dx.$$

Thus we get that the third term of (4.56) is 0. With analogous procedures one can prove that the limit of the last term in (4.56) is 0, concluding the proof. □

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