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Complex higher spins, Weyl invariance and tractors

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Introduction

Higher spin (HS) theories have attracted a great attention among theoretical physicists since the early days of relativistic field theories. In the spirit of quantum field theory (QFT), a free propagating particle is represented as a quantum state of the Fock space of the theory. This state is associated to a classical solution of the corresponding relativistic wave equation and it is characterized, in flat spacetime, by a unitary irreducible representation of the Poincaré group[1]. In four dimensions, these irreducible representations are uniquely individuated by the spin s and the mass m of the field. It was thus natural to study generalizations of the known Klein-Gordon, Dirac and Maxwell-Proca equations to describe the relativistic propagation of particles of arbitrary spin. This pioneering research was pursued, for example, by Dirac [1], Fierz and Pauli [2], Bargmann and Wigner [3, 4], Rarita and Schwinger [5], and already in 1939 Fierz and Pauli found the correct field equation for free massive HS fields. They realized that a spin s physical boson with mass squared m^2, should be represented as a symmetric, traceless and divergenceless, rank s tensor field $\Phi_{\mu_1 \ldots \mu_s}$, obeying a massive Klein-Gordon equation: $(\Box - m^2)\Phi_{\mu_1 \ldots \mu_s} = 0$. Analogously, a physical massive fermion of spin $s + \frac{1}{2}$ had to be represented by a symmetric, γ-traceless and divergenceless spinor-tensor $\Psi_{\mu_1 \ldots \mu_s}$ obeying a massive Dirac equation.

Even if the massive equations are the obvious generalizations of the Proca system, with the additional trace constraint removing the non irreducible parts of the field, a lagrangian formulation was provided by Singh and Hagen [6, 7] only in 1974, by adding to the model a bunch of auxiliary fields of decreasing spin, from $s - 2$ down to zero. The massless limit of the Singh-Hagen lagrangian was studied in 1978 by Fronsdal [8] for bosons and by Fang and Fronsdal [9] for fermions. Focussing on the bosonic case, it was seen that when m^2 tends to zero, all the auxiliaries but the spin $s - 2$ one decouple, while the latter combines with the traceless spin s field to give a

[1]The Poincaré group $ISO(D-1,1)$ is the global isometry group of flat Minkowski space. The very notion of particle in curved spacetimes is related to their isometry group. For example, particles in AdS space are labeled by $SO(D,2)$ representations.

double traceless, massless field $\varphi_{\mu_1 \dots \mu_s}$. The resulting Fronsdal equations are the straightforward HS generalization of Maxwell and linearized Einstein's equations, and share with them the most important feature of all known field theories, namely, gauge symmetry.

Since massless irreducible representations carry less degrees of freedom than the massive ones, it is well known that the covariant field content is redundant, and gauge freedom precisely allow to remove all the unphysical polarizations. Moreover, gauge symmetry elevated to a fundamental principle strongly constrains the form of the allowed lagrangians and field equations. This led, as the most striking successes, to the formulation of the Standard Model as a $SU(3) \times SU(2) \times U(1)$ gauge theory, and General Relativity, which showed an enormous predictive power.

However, non-abelian gauge symmetry and diffeomorphism invariance turned out to be so powerful in that they determine the structure of interactions, while Fronsdal's HS gauge symmetry is an abelian symmetry enjoyed by free fields. In fact, finding a consistent theory of interacting HS's is still a really tough challenge. Subtleties start to arise already with spin $\frac{3}{2}$ fields: the massless Rarita-Schwinger equation correctly describes free spin $\frac{3}{2}$ particles, but it was realized that consistent interactions could be achieved only in the framework of supergravity. The Rarita-Schwinger field is included as superpartner of the metric, *i.e.* a gravitino, and the associated fermionic gauge symmetry is then interpreted as local supersymmetry. Real obstacles in finding consistent interactions for HS arise thus from spin $\frac{5}{2}$, and nowadays we do not have a theory of interacting HS involving a finite number of fields. Vasiliev's theory is indeed a full non-linear theory of HS in interaction, see *e.g.* [10, 11] for a review, but it does involve an infinite number of fields, with arbitrary large spin. Moreover, it turns out that such HS interactions are higher derivative and, being non-analytic in the cosmological constant Λ, they cannot be formulated in flat spacetime.

Although we do not observe HS particles, a better understanding of their dynamics is desirable. String theory, which is one of the leading candidates for a quantum theory of gravity, is related to HS theories, and this relationship is not fully uncovered yet. Since the fundamental strings are extended objects, they allow infinitely many vibrational states with increasing spin and mass; hence, the presence of infinite states with arbitrary spin is inherent to string theory. Actually, since the order of magnitude of their mass squared is the string tension, they can be safely discarded in a low energy effective theory. Nonetheless, in order to ensure the UV finiteness of the theory the full string spectrum, including higher spins, have to play a role. Moreover, in the tensionless limit of string theory this infinite tower becomes massless and should reveal its huge gauge structure [12, 13]. Deeper insight into this feature comes from the AdS/CFT correspondence. The standard conjecture involves type IIB superstring theory on $AdS_5 \times S^5$ background, that is claimed to be dual to $\mathcal{N} = 4$ super Yang-Mills (SYM) defined on the four-dimensional boundary of AdS_5. The correspondence is at the level of generating functionals, and relates strong and weak coupling regimes on the two sides. Usually, the conjecture is tested in the regime where string theory is safely treated as low energy supergravity, and the dual superconformal field theory is strongly coupled. On the other hand, when the boundary theory is weakly coupled, or even free, an infinite number of conserved currents of arbitrary spin appear, while on the string side the tensionless limit should be realized, with the appearance of infinitely many massless HS [14, 15].

Spinning particles, first quantized approach to QFT

Having sketched the very basic concepts of HS theories, we can now turn to the main subject of this thesis work, namely, the study of worldline quantum mechanical (QM) models that allow to treat several QFT aspects in first quantization. Of course, among the field theories that can be described by using such worldline methods, we will be mainly interested in higher spin theories and eventual generalizations. Quantum mechanical models have been very useful, at first, to compute in an efficient and quite simple way chiral anomalies, as it was shown by Alvarez-Gaumé and Witten [16, 17], and the method was later generalized to include trace anomalies by Bastianelli and van Nieuwenhuizen [18, 19]. Basically, following the Fujikawa approach, the anomaly is represented as a trace over an infinite dimensional jacobian $\mathcal{J}$ that requires to be regularized. Having chosen a suitable regulator operator $\mathcal{R}$, the anomaly is given by

$$An = \lim_{\beta \to 0} \mathrm{Tr}\, \mathcal{J}\, e^{-\beta \mathcal{R}} \ .$$

The worldline model comes into the game when one interprets the regulator as a quantum mechanical hamiltonian, the jacobian is written in terms of QM operators, and the trace is taken over the Hilbert space of the particle model. This is a sensible choice since, usually, the regulator's leading term is a covariantized laplacian: $\mathcal{R} = -\nabla^2 + \text{stuff}$ that can be converted into $H = \frac{p^2}{2} + \text{stuff}$. More generally, the worldline quantization of a relativistic particle theory describes field propagation in spacetime, and we will be interested in the so called spinning particle theories, that enjoy extended *local* worldine supersymmetries. In fact, it is known that a spinning particle with N local supersymmetries, that is an extended supergravity in one dimension, describes spin $\frac{N}{2}$ fields in four dimensional spacetime [20, 21]. This is a really interesting and flexible model, since just by modifying the number of fermionic worldline symmetries, one can deal different spins in field theory.

It will turn out that the key features of the worldline canonical quantization are the gauge symmetries of the particle theory. They will produce, as usual in the canonical quantization of gauge theories, constraints on the Hilbert space that select the physical subspace, in the very same way the Virasoro constraints do in string theory. In order to get a more precise idea of the relation between the particle quantization and the spacetime field propagation, let us briefly sketch the strategy in the most simple case, *i.e.* the scalar particle. The worldline evolution of a relativistic particle is described by its spacetime coordinates $x^\mu(\tau)$. The usual form of its lagrangian is $\mathcal{L} = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu}$, and is invariant under local reparametrizations of the worldline: $\tau \to \tau' = f(\tau)$. In order to avoid the cumbersome square root in the action, it is useful to introduce an intrinsic einbein of the worldline $e(\tau)$, playing the role of a gauge field for time reparametrizations. This is exactly the same reasoning that leads in string theory from the Nambu-Goto geometric action to the quadratic Polyakov action, by adding the intrinsic worldsheet metric $h_{\alpha\beta}$. In order to perform canonical quantization, one introduces the particle momenta p_μ, and the phase space action reads

$$S[p, x, e] = \int d\tau \Big[p_\mu \dot{x}^\mu - \frac{e}{2} (p^2 + m^2) \Big] ,$$

and we note thats the massless limit is now completely smooth. In one dimension the gauge field e does not have any dynamics, and its equation of motion enforces the classical constraint $H = p^2 + m^2 = 0$. At the classical level this restricts the unphysical covariant phase space to the physical mass-shell sector. Upon canonical quantization, the Hilbert space of the particle consists on functions of the spacetime coordinates $\phi(x)$, *i.e.* scalar fields, where the momentum acts as $p_\mu = -i\partial_\mu$. At this juncture, the classical constraint $H = 0$ enters the game and translates, as usual, in a constraint over the physical states of the theory: $H|\phi\rangle = 0$, that is nothing but a massive Klein-Gordon equation

$$(\Box - m^2)\phi = 0 .$$

We have thus shown that a first quantized free relativistic particle describes the propagation of a free scalar field in spacetime. Following the same strategy, one can achieve higher spin propagation by quantizing a particle with an arbitrary number of local supersymmetries.

The present discussion concerned free particles only but, although free HS propagation is an interesting subject in its own, spinning particle models become much richer tools when coupled to nontrivial backgrounds. If the spinning particle is consistently coupled[2] to an external field, its quantization reproduces the propagation of the corresponding quantum field interacting with the background of interest. This allows for many interesting applications, such as the computation of one-loop effective actions, propagators and so on, by using quantum mechanical path integrals [22, 23, 24, 25, 26, 27, 28]. One can achieve, for instance, interactions with scalars, gauge fields and gravity, and we will be mostly interested in the last case, due to the universality of the gravitational coupling and the intriguing feature of defining HS theories on curved backgrounds. Again, to be more concrete, we briefly present the method for the simple case of scalar electrodynamics. In order to make the scalar particle interact with an external electromagnetic field $A_\mu(x)$, it is sufficient to replace the momentum p_μ in the hamiltonian with the covariant one $p_\mu - qA_\mu$. The action for the massless scalar particle becomes in configuration space

$$S = \int_0^T d\tau \Big[\frac{1}{2e}\dot{x}^\mu \dot{x}_\mu + q\, A_\mu(x)\dot{x}^\mu\Big] \,.$$

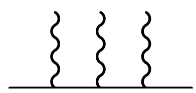
If the electromagnetic coupling is small, the path integral can be cast as a perturbative expansion in powers of q and the interactions appear as insertions of the vertex operator

$$V_A = iq \int_0^T d\tau\, A_\mu(x)\dot{x}^\mu$$

in the worldline free propagation, exactly as it happens in string perturbation theory, where vertex operators are inserted in the free worldsheet. The path integral in fact becomes

$$\int \frac{DxDe}{\text{Vol Gauge}}\, e^{iS_2} \Big[\sum_{N=0}^{\infty} \frac{1}{N!}\,(V_A)^N\Big] \,,$$

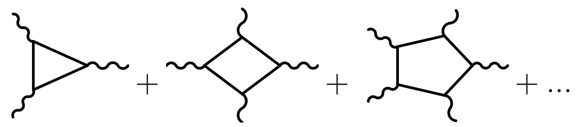
where S_2 is the free action and we schematically took into account gauge redundancy by dividing the volume of the gauge group. By modifying the geometry of our worldlines we can produce various QFT quantities. For instance, performing the functional integral over paths with fixed endpoints one obtains the propagator in the electromagnetic background:

$$\Delta(x,y)_A = \int_{x(0)=x}^{x(T)=y} \frac{DxDe}{\text{Vol Gauge}}\, e^{iS} \sim$$ 

[2]The worldline couplings have to respect the gauge symmetries of the particle, properly deformed by the interactions.

while a worldline representation of the one-loop effective action due to a charged scalar loop is obtained by quantizing the particle on a circle. It is worth noticing that in the factorization procedure of the gauge group volume by the Faddeev Popov method, one is often left with finite dimensional integrals over moduli of the worldline gauge field. The omnipresent modulus on the circle is the proper time T associated to the closed loop, that comes from the gauge fixed einbein $e(\tau)|_{\text{fixed}} = T$. From the field theory viewpoint, it results that the limit $T \to 0$ is the ultraviolet limit of the QFT. In many applications one indeed studies the QM path integral as a perturbation series in T, and the resulting coefficients in the QFT effective action are the so called Seeley-DeWitt coefficients that investigate the UV structure of the theory [29].

$$\Gamma[A] \propto \int_{\mathbb{T}^1} \frac{DxDe}{\text{Vol Gauge}} e^{iS} \sim$$



A notable feature of the worldline representation of effective actions is that the spin of the field running in the loop can be changed by simply modifying the number of fermionic species ψ_i^μ in the particle theory. Of course new kind of worldline vertices can appear but, once the two basic propagators ar given for quantum mechanical fields, roughly speaking $\langle xx \rangle$ and $\langle \psi\psi \rangle$, the perturbative structure is exactly the same, and one can easily keep track of the contributions coming from various spins. Moreover, one has not to deal with the complications of divergent momentum loop integrals, the QM perturbation theory being always finite. Indeed, we already mentioned that the UV divergencies of the quantum field theory are encoded in certain integration regions of the moduli space of the particle, exactly as it happens to be in string theory.

This brief description was carried on using scalar electrodynamics as a simple example, but we anticipated that our main focus will be on interactions with gravity and on higher spin fields. It is in fact obvious that we can make a scalar field interact with any desirable background, but this is not true anymore once we increase the spin of the field beyond the higher spin barrier. This is indeed clear from the spinning particle viewpoint: if we deform the QM operators of interest by coupling with a certain background, we must retain consistency with the gauge symmetries of the system. In general, the free generators of the worldline gauge transformations T^A will form a Lie superalgebra: $[T^A, T^B\} = f_C^{AB} T^C$. The closure of the superalgebra is essential to the consistency of the theory, but it can be broken by couplings with nontrivial backgrounds, as it happens. In the scalar particle case no problems can arise since the only gauge generator is the hamiltonian H. Let us, instead, consider a spinning particle with N local supersymmetries generated by N real supercharges Q_i. Its quantization was studied in flat space in [30] and in (A)dS in [31, 32, 33]. In the free theory the fundamental anti-commutator closes as $\{Q_i, Q_j\} = 2\delta_{ij} H$, where H is the hamiltonian. When this system is coupled to gravity, the algebra still closes on any background manifold, with covariantized operators, only for $N = 0, 1, 2$ that is for scalars, spin $\frac{1}{2}$ fermions and vectors. For higher spin fields, *i.e.* for larger N, the algebra ceases to be first class, since new operators appear coupled to the curvature:

$$\{Q_i, Q_j\} \sim 2\delta_{ij} H + R_{\mu\nu\lambda\sigma} \psi_i^\mu \psi_j^\nu \psi^\lambda \cdot \psi^\sigma \ ,$$

and the quantization on general backgrounds is inconsistent. The problem can be overcome by restricting the allowed background spacetime: for instance, on conformally flat manifolds, among which special importance is held by maximally symmetric spaces, the above superalgebra is again first class [32], and a consistent quantization is allowed. More specific aspects and subtleties of the worldline formalism we will be concerned with will be briefly discussed in the forthcoming description of the thesis content.

Let us now outline the structure and content of this thesis work. The first part, consisting in five chapters, is devoted to the research work, conducted at Bologna University, on higher spin fields from a worldline perspective, with special interest on a novel class of complex HS fields related to the so called $U(N)$ spinning particles. The second part of the thesis, instead, consists on the two last chapters and concerns the development of Weyl invariant field theories by means of a particular Weyl covariant formalism, called tractor calculus, that was carried on at UC Davis in collaboration with the Mathematics Department.

In the first, introductory, chapter we give a brief description of HS theories for the simplest case of completely symmetric tensors, deriving Fronsdal equations for massless HS in flat Minkowski space, in the constrained formulation. We then turn to the worldline formalism, and to get familiarity with the first quantized approach, we warm up with the two simplest examples of the $N = 1$ and the $O(2)$ spinning particles, describing Dirac spinors and differential p-forms. We proceed then by reviewing the canonical quantization of the $O(N)$ spinning particles in flat space. These particle models possess indeed N local worldline supersymmetries with $O(N)$ as R-symmetry group, and will describe HS conformal fields with the symmetry of a rectangular Young tableau, that reduce to the Fronsdal's completely symmetric tensors only in four dimensions. We show, following [32, 33], that their Dirac quantization produces HS equations in the geometrical formulation, which are first order equations on the Freedman-de Wit generalized curvatures. Integrating a subset of these equations one introduces the gauge fields, and readily recovers Fronsdal-Labastida and Fang-Fronsdal equations, for bosons and fermions respectively, in the compensator formalism of Francia and Sagnotti [34, 35, 36]. Moreover, in the worldline language, algebraic and differential operators of interest are naturally encoded in a compact form in the QM operators and all the manipulations and equations show up in an elegant and concise manner, the meaning of which becomes clear keeping in mind the dictionary that relates worldline operators to the usual tensor formalism.

In the second chapter we pursue the coupling of such models to background gravity. In order to build $O(N)$ spinning particle models, it is often useful to start with a particle model enjoying N *global* supersymmetries, as well as *global* time translation symmetry and $O(N)$ symmetry, to which we will refer as $O(N)$ SUSY quantum mechanics. In a second step, if the algebra of the symmetry generators is first class, one can couple such generators to gauge fields and obtain the $O(N)$ spinning particle[3]. The study of the $O(N)$ quantum mechanics is by the way an important issue: it provides the algebra of the symmetry generators, allowing to recognize which kind of spinning particles can be constructed. Moreover, it is tightly related to the gauge fixed version of the latter, to be used in quantum computation. For this last reason, the $O(N)$ SUSY quantum mechanics, and precisely its heat kernel coefficients, represents a milestone in performing path integral calculations for spinning particles. Given this state of things, we start the second chapter by presenting the $O(N)$ quantum mechanics and its $OSp(N|2M)$ generalization, that is constructed by adding $2M$ bosonic "supersymmetries" which enlarge the R-symmetry to the orthosymplectic supergroup. The models are then coupled to a general background metric $g_{\mu\nu}(x)$ by introducing suitable covariant versions of the (super)-symmetry generators, and their quantum algebra is computed. We show that the superalgebra is first class only for $N = 0, 1, 2$ and $M = 0$, the corresponding spinning particles thus reducing to the known cases of scalars, spin one half fermions and p-forms coupled to gravity. As already mentioned for the $O(N)$ model, at generic N and M the restriction to maximally symmetric spaces makes the superalgebra first class and allows the construction of the corresponding spinning particle, although the superalgebra is not anymore a Lie superalgebra but a quadratic deformation thereof. Having individuated all the operators of interest, par-

[3]It is possible, to obtain different models, to gauge only a subset of generators, provided they form a first class subalgebra

ticularly the hamiltonian H, and their algebra, we then focus on the more familiar $O(N)$ SUSY quantum mechanics and start studying its heat kernel. By explicitly using the fundamental (anti)-commutation relations, we compute the matrix elements of the evolution operator, or transition amplitude, in euclidian time for all N:

$$\langle x\bar{\lambda}|e^{-\beta H}|y\eta\rangle$$

as a perturbative expansion in β, that indeed defines the heat kernel coefficients. The result, given at first perturbative order, is well defined and can thus be used as a bench mark for the correct definition of the corresponding path integral, that is pursued later on in the chapter. In fact, it is well known that the definition of quantum mechanical path integrals in curved backgrounds involves some subtleties: carrying on perturbative calculations on the worldline one has to face with ill defined products of distributions[4], so that a regularization scheme is needed in order to produce a non-ambiguous result. Moreover, once the scheme is chosen, still the naive classical action does not reproduce the correct amplitude for the desired hamiltonian, and a suitable local counterterm $V_{CT}(x)$ is needed in the path integral action. For a detailed treatment of quantum mechanical path integrals on curved spaces see [37]. The amplitude for the $O(N)$ quantum mechanics is computed by path integral methods in the three most used regularization schemes, namely time slicing (TS), mode regularization (MR) and dimensional regularization (DR), and the corresponding counterterms are recognized by comparing path integral results for the amplitude with the well defined operatorial one. The proper definition of the functional integral is a first milestone for future applications in studying quantum properties of the $O(N)$ spinning particles.

In the third chapter we turn to the analysis of a different class of spinning particle models that will lead to novel HS equations on complex spaces. The so called $U(N)$ spinning particles are a complex generalization of the $O(N)$ models. They naturally live on Kähler backgrounds and enjoy N *complex* local supersymmetries, with $U(N)$ as R-symmetry group. For the lowest $N = 1, 2$ they were introduced in [38], while the general model was presented in [39]. Here we present both their canonical and path integral quantization in flat complex space $\mathbb{C}^d$, and find new HS equations obeyed by the corresponding complex higher spin fields [40]. They are very reminiscent of Fronsdal-Labastida equations, except that the usual trace constraints are replaced in this framework by differential ones. The model is then coupled to an arbitrary Kähler manifold with metric $g_{\mu\bar{\nu}}$, and the constraint superalgebra is studied. We find that the quantization is allowed, the algebra being

[4] They arise since the covariant kinetic term $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ contains double derivative interactions.

first class, on every background only for $N = 1, 2$, as it was already known. On the other hand, in analogy with the real models, it results that one can quantize the model at generic N on Kähler spaces with constant holomorphic curvature. These manifolds are characterized by a Riemann tensor of the form

$$R_{\mu\bar{\nu}\lambda\bar{\sigma}} = k \left(g_{\mu\bar{\nu}} g_{\lambda\bar{\sigma}} + g_{\mu\bar{\sigma}} g_{\lambda\bar{\nu}} \right) , \quad k = \text{constant}$$

and are the Kähler analogue of maximally symmetric spaces.

Having in mind that it is possible to quantize the $U(1)$ and $U(2)$ spinning particles on every Kähler space, and even the generic $U(N)$ on nontrivial backgrounds, we turn in the fourth chapter to study the associated $U(N)$ quantum mechanics[5]. The model is first generalized to the superunitary $U(N|M)$ SUSY quantum mechanics by adding new bosonic "supersymmetries", and the larger symmetry algebra is studied on generic Kähler spaces. Having chosen a suitable hamiltonian, we evaluate the corresponding transition amplitude by operator methods, result that again will allow to set up the path integral in a non-ambiguous way [41].

With the transition amplitude at hand, we are able to properly set up the functional integral. Indeed, in the fifth chapter we quantize the $U(1)$ spinning particle on an arbitrary Kähler background. In its standard form, that demands to gauge both supersymmetries Q and $\bar{Q}$, the hamiltonian H and the $U(1)$ generator J, the model describes massless holomorphic p-forms but, by considering different gaugings, we are able to give a worldline representation of the one-loop effective action for massless holomorphic p-forms, Dirac fermions and "non gauge" differential forms[6]. This is done in practice by very simple manipulations on the moduli space of the basic transition amplitude, and is indeed a good example of the aforementioned flexibility of worldline models. One can also show exact dualities at the level of partition functions in analogous way to what was done for the $O(2)$ particle in [28].

In the second part of the thesis we will explore the tractor formalism, a mathematical machinery that allows to construct Weyl invariant field theories in a nice geometric way that makes Weyl covariance manifest at each step. Let us see this in more detail.

Weyl invariance and tractor formalism

[5]We refer here to "quantum mechanics" as opposed to "spinning particle" in the very same way as we explained before.

[6]We mean that these differential forms do not come from integrating a gauge invariant field strength, and have a generalized Laplacian as kinetic operator.

The history of Weyl invariance [42] as a principle for developing physical theories is a long one. Notable early examples include Dirac's formulation of conformally invariant four-dimensional wave equations in six dimensions [43] and Zumino's work relating Weyl transformations to the conformal group [44] and the introduction of Weyl compensator fields by Deser and Zumino [44, 45]. Under a Weyl transformation the metric undergoes the following local scaling:

$$g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu}$$

and we say that it has Weyl weight $w = 2$, while the matter fields can have in principle any transformation rule, that commonly reduces to a simple power law for so called weight w_i fields: $\Phi_i \to \Omega^{w_i}(x)\,\Phi_i$. Usually Weyl invariance, that is local symmetry in the choice of unit systems, is regarded as a symmetry principle obeyed by special classes of field theories, among which notable examples[7] include four-dimensional Maxwell theory, conformally improved scalars, the massless Dirac equation and Weyl-squared gravity. We will follow, instead, a different perspective, where local unit invariance is required as a principle for *all* field theories [46], and the usual systems are recovered as gauge fixed versions of Weyl invariant ones.

Global scale invariance is obeyed by every reasonable field theory as the freedom of choosing an arbitrary unit system, and gives the usual scaling dimensions of fields and coupling constants. To promote it to a local symmetry we need in general to add a gauge field $\sigma(x)$, that behaves as a spacetime varying scale. This field has weight one, so that $\sigma \to \Omega(x)\sigma$, and can be coupled to any physical system to ensure Weyl invariance, deserving its common name of Weyl compensator. Once a field theory is presented in a Weyl invariant way, the canonical choice is a constant unit system over spacetime that translates in the gauge fixing choice $\sigma(x) = \sigma_0$. In general the gauge fixing procedure breaks Weyl symmetry, and one is left with the familiar non invariant theory. In special cases, however, the scale field decouples. When this happens, Weyl symmetry cannot be spoiled by any gauge fixing, and the resulting theory is invariant in its own in the usual sense.

Restoring Weyl symmetry by directly compensating the familiar field theories can be rather painful. If the local choices of unit systems could not possibly change the outcome of any physical measurement [46], therefore there should exist a formulation of physics that makes this symmetry manifest. For instance, in formulating general relativity it would be extremely hard to predict gravitational couplings by requiring at hand diffeomorphism invariance. Obviously, we know that the correct framework to do this is Rie-

[7] We refer here to the classical symmetry, that can be eventually spoiled at the quantum level by the presence of trace anomalies.

mannian geometry. If, in addition to diffeomorphism invariance, we require also invariance under Weyl rescalings, the physical information of the metric is encoded in the double equivalence class

$$[\sigma, g_{\mu\nu}] = [\Omega\sigma, \Omega^2 g_{\mu\nu}]$$

telling us that the relevant geometry is the so called conformal geometry, that deals with conformal classes of metrics. Without a tensor calculus for rapidly constructing Weyl invariant quantities, the above local unit invariance principle would not be particularly enlightening. Fortunately, such a calculus already exists in the mathematical literature and goes under the name "tractor calculus" [47, 48, 49, 50, 51]. It is the mathematical machinery required to replace Riemannian geometry with conformal geometry as the underpinning of physics. A particularly appealing implication is that in a description of physics that manifests local unit invariance, masses are replaced by Weyl weights which measure the response of physical fields to changes of unit systems. This will allow, for instance, to deal with massive, massless and even partially massless theories in a single stroke using tractors, just by tuning Weyl weights. Tractor fields are to conformal geometry what tensor fields are to Riemannan geometry: relativistic covariance is made manifest by grouping physical and unphysical fields into Lorentz tensors[8] $T^{m_1 \dots m_n}$, where $m = 0, ..., D-1$ is an $SO(D-1,1)$ index. To achieve manifest Weyl covariance, tractor fields in D dimensions have to be $(D+2)$-dimensional multiplets $T^{M_1 \dots M_n}$ with $SO(D,2)$ indices $M = (+, m, -)$. This is not surprising, since we know that the D-dimensional conformal group acts nicely in $D+2$ dimensions, and we will see that in the so called "ambient approach" tractor fields naturally live as tensor fields of a $(D+2)$-dimensional ambient space.

The basic ingredients of tractor calculus are presented in chapter 6, together with the tractor construction of scalar and vector theories, along the lines of [46]. In the rest of the chapter we describe the tractor formulation of Einstein's equations coupled to matter, and start to address tractor back-reaction [52].

Finally, in the last chapter we explore the interplay between a six dimensional quantum mechanical model, presented by Bars in the framework of two times physics [53], and four dimensional tractor gravity. The analysis is performed by heavily using the ambient tractor approach, that nicely maps six dimensional geometry to four dimensional conformal geometry.

We conclude the present discussion by stressing that all the tractor analysis carried on in the last two chapters is purely classical. Quantum attempts

[8]Let us restrict the discussion to bosonic theories. Moreover, we think to spacetime indices as flattened by means of the vielbein e_μ^m.

to tractor theories seem to be really appealing, especially for what concerns trace anomalies and holographic renormalization in AdS/CFT. We refer to the conclusions for some speculative ideas for future research.

Contents

Chapter 1

Higher spins and $O(N)$ spinning particles in flat spacetime

It is well known that gauge theories involving massless fields of spin not greater than two can handle all the observed elementary particles and forces in nature. Despite of this, in the last decades a great amount of theoretical work was devoted to the study of higher spin field theories (For an introduction see for example [54, 55]). Pioneering works date back to Dirac [1], Fierz, Pauli [2], Bargmann and Wigner [3, 4], who tried to generalize to particles of arbitrary spin the relativistic wave equations already known. Four dimensional Lorentz invariant lagrangians for massless fields of arbitrary integer spin were introduced by Fronsdal [8], then generalized to half integer spins by Fang and Fronsdal [9] and to arbitrary dimensions and mixed symmetry tensors by Labastida [56]. These theories of free massless spin s fields enjoy a gauge invariance that naturally generalizes the Maxwell and linearized Einstein cases, and therefore are interesting in their own. Moreover, String Theory exhibits in its spectrum an infinite tower of massive states of increasing spin and mass, that become massless in the limit of tensionless string. This observation led, for example, to the theoretical conjecture that the tension-full string could be viewed as a broken phase of a huge higher spin gauge symmetry [12, 13]. Another link between String Theory and higher spins is in the context of AdS/CFT, where massless higher spin excitations emerge in the limit of large AdS curvature [14, 15].

Finally, a hard theoretical challenge is to develop a consistent theory of interacting higher spins. So far, we do not have a consistent full interacting theory of higher spins involving a finite number of fields. The full non-linear Vasiliev's theory, that generalizes spin one Yang-Mills and spin two Einstein gauge symmetries, involves indeed an infinite number of fields of arbitrary high spin, see *e.g.* [10, 11]

In this chapter we will first give a brief introduction to Fronsdal's formulation of higher spin equations in terms of both constrained and unconstrained gauge parameters and fields, following the Francia Sagnotti compensator formalism [34, 35, 36].

The main part of the chapter will be devoted to the description and canonical quantization of the $O(N)$ spinning particle, a worldline model that in flat spacetime produces Fronsdal-Labastida equations in first quantization. The extension to the $U(N)$ spinning particle on complex manifolds, and the related complex higher spin equations, that are the principal focus of my PhD, will be presented in the subsequent chapters.

1.1 Fronsdal's equations for symmetric tensors

In the present section we will obtain Fronsdal's equations for symmetric tensors starting from Maxwell's equations for spin one, and trying to generalize the gauge principle for higher rank tensors, following the treatment of [55]. For the spin two case one recovers the linearized version of Einsteins's equations in vacuum, and for $s > 2$ one has the Fronsdal's equations for higher spin fields. It will be immediately clear that the new gauge invariance holds only for constrained, traceless, parameters, and further analysis reveals that the Lagrangian is invariant only for double traceless fields. We will restrict here to the simpler case of symmetric tensors; this is exhaustive only in four spacetime dimensions, though, it is sufficient to display all the main features of bosonic massless higher spin fields. Fermionic higher spin equations will be outlined directly in the context of first quantized spinning particles in the following.

Let us start with the well known Maxwell equations in vacuum: they were firstly discovered in terms of the physically relevant field strength $F_{\mu\nu}$ as

$$\partial_{[\mu}F_{\nu\lambda]} = 0 \ , \quad \partial^\mu F_{\mu\nu} = 0 \ ,$$

so that one solves the integrability condition setting $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$[1], and the massless spin one equation reduces to

$$\Box A_\mu - \partial_\mu \partial \cdot A = 0 \ ,$$

[1]We will use everywhere square and round brackets for weighted anti-symmetrization and symmetrization, respectively. For example, $A_{(\mu_1 \dots \mu_n)} = \frac{1}{n!}(A_{\mu_1 \dots \mu_n} +$ all permutations)

enjoying the gauge invariance under the $U(1)$ transformation $\delta A_\mu = \partial_\mu \Lambda$. Taking gauge invariance as a fundamental guide, one can recover Maxwell's equations from another perspective, that will lead us to Fronsdal's equations. Let us start with the most general linear differential equation, of second order in derivatives, and Lorentz covariant as a D-vector; it reads:

$$\Box A_\mu + \alpha\, \partial_\mu \partial \cdot A = 0 \, . \tag{1.1}$$

If now one pretend (1.1) to be invariant under the $U(1)$ gauge transformation $\delta A_\mu = \partial_\mu \Lambda$, the parameter α is uniquely fixed, $\alpha = -1$, and (1.1) reduces to Maxwell's equation. As a further example we are now able to write down free field equations for massless spin two. The most general equation for a symmetric tensor $h_{\mu\nu}$ reads:

$$\Box h_{\mu\nu} + 2\alpha_1\, \partial_{(\mu} \partial \cdot h_{\nu)} + \alpha_2\, \partial_\mu \partial_\nu h' = 0 \, , \tag{1.2}$$

where $h' = h^\alpha{}_\alpha$. We now generalize the gauge principle to spin two requiring invariance under $\delta h_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$, that is the linearized version of diffeomorphism invariance. Again, the two free parameters are fixed to be $\alpha_2 = -\alpha_1 = 1$, and (1.2) becomes

$$\Box h_{\mu\nu} - \partial_\mu \partial \cdot h_\nu - \partial_\nu \partial \cdot h_\nu + \partial_\mu \partial_\nu h' = 0 \, . \tag{1.3}$$

This is nothing but the linearized Einstein's equation for the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and (1.3) is the linearized Ricci tensor.

Now we consider a symmetric, spin s, tensor field: $\varphi_{\mu_1 \ldots \mu_s}$. Following the same procedure of spin one and two, we claim the gauge invariance under the transformation

$$\delta \varphi_{\mu_1 \ldots \mu_s} = s\, \partial_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)} \, , \tag{1.4}$$

where the gauge parameter is a symmetric $s-1$ tensor. The general equation for spin s, that we denote $\mathcal{F}_{\mu_1 \ldots \mu_s} = 0$, will be the straightforward generalization of (1.2), but one cannot find any value of α_1 and α_2 for which $\mathcal{F}_{\mu_1 \ldots \mu_s}$ is completely gauge invariant under (1.4). If one does not want to introduce other ingredients into the game, full gauge invariance under (1.4) cannot be achieved. The best one can obtain is for $\alpha_1 = s$ and $\alpha_2 = \frac{s(s-1)}{2}$, where the gauge variation of the equations of motion reduces to

$$\delta \mathcal{F}_{\mu_1 \ldots \mu_s} = 3 \binom{s}{3} \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Lambda'_{\mu_4 \ldots \mu_s)} \, .$$

Fronsdal's equations for massless symmetric tensor fields then read:

$$\mathcal{F}_{\mu_1 \ldots \mu_s} = \Box \varphi_{\mu_1 \ldots \mu_s} - s\, \partial_{(\mu_1} \partial \cdot \varphi_{\mu_2 \ldots \mu_s)} + \frac{s(s-1)}{2}\, \partial_{(\mu_1} \partial_{\mu_2} \varphi'_{\mu_3 \ldots \mu_s)} = 0 \, , \tag{1.5}$$

where we denoted the traces $\Lambda'_{\mu_3\ldots\mu_{s-1}} = \Lambda^{\alpha}{}_{\alpha\mu_3\ldots\mu_{s-1}}$ and $\varphi'_{\mu_3\ldots\mu_s} = \varphi^{\alpha}{}_{\alpha\mu_3\ldots\mu_s}$. We can notice the first significant difference with respect to the lower spin situation: for spin greater than two, the field equations are no longer gauge invariant for arbitrary gauge parameters; instead, the simple Fronsdal's formulation of higher spin equations is gauge invariant only for traceless gauge parameters:

$$\Lambda^{\alpha}{}_{\alpha\mu_3\ldots\mu_{s-1}} = 0 \; . \tag{1.6}$$

A second constraint arises when one studies Bianchi identities for Fronsdal's equations, or considering the lagrangian formulation. To derive them, let us differentiate (1.5) and its trace:

$$\begin{aligned}
\partial\cdot\mathcal{F}_{\mu_2\ldots\mu_s} &= (s-1)\left(\Box\partial_{(\mu_2}\varphi'_{\mu_3\ldots\mu_s)} - \partial_{(\mu_2}\partial\cdot\partial\cdot\varphi_{\mu_3\ldots\mu_s)}\right) \\
&\quad + \tfrac{(s-1)(s-2)}{2}\,\partial_{(\mu_2}\partial_{\mu_3}\partial\cdot\varphi'_{\mu_4\ldots\mu_s)} \\
\mathcal{F}'_{\mu_3\ldots\mu_s} &= 2\Box\varphi'_{\mu_3\ldots\mu_s} - 2\partial\cdot\partial\cdot\varphi_{\mu_3\ldots\mu_s} + (s-2)\partial_{(\mu_3}\partial\cdot\varphi'_{\mu_4\ldots\mu_s)} \\
&\quad + \tfrac{(s-2)(s-3)}{2}\,\partial_{(\mu_3}\partial_{\mu_4}\varphi''_{\mu_5\ldots\mu_s)} \\
\partial_{(\mu_2}\mathcal{F}'_{\mu_3\ldots\mu_s)} &= 2\Box\partial_{(\mu_2}\varphi'_{\mu_3\ldots\mu_s)} - 2\partial_{(\mu_2}\partial\cdot\partial\cdot\varphi_{\mu_3\ldots\mu_s)} + (s-2)\,\partial_{(\mu_2}\partial_{\mu_3}\partial\cdot\varphi'_{\mu_4\ldots\mu_s)} \\
&\quad + \tfrac{(s-2)(s-3)}{2}\,\partial_{(\mu_2}\partial_{\mu_3}\partial_{\mu_4}\varphi''_{\mu_5\ldots\mu_s)} \; ,
\end{aligned}$$

putting the various pieces together we obtain the so called "anomalous" Bianchi identities:

$$\partial\cdot\mathcal{F}_{\mu_2\ldots\mu_s} - \frac{s-1}{2}\partial_{(\mu_2}\mathcal{F}'_{\mu_3\ldots\mu_s)} + \frac{3}{2}\binom{s-1}{3}\partial_{(\mu_2}\partial_{\mu_3}\partial_{\mu_4}\varphi''_{\mu_5\ldots\mu_s)} \equiv 0 \tag{1.7}$$

We see that, if the field equations $\mathcal{F}_{\mu_1\ldots\mu_s} = 0$ hold, the gauge field is constrained to be double traceless:

$$\varphi^{\alpha}{}_{\alpha}{}^{\beta}{}_{\beta\mu_5\ldots\mu_s} = 0 \; , \tag{1.8}$$

a feature that comes into the game starting from spin four. This restriction as well is related to gauge invariance: if the gauge parameter is not traceless the field equations themselves are not invariant; gauge invariance of (1.5) does not require any constraint on φ, but gauge invariance of the Lagrangian does, so let us briefly investigate it.

To construct an action principle leading to Fronsdal's field equations one could start from the naive Lagrangian $\mathcal{L}_0 = \varphi^{\mu_1\cdots\mu_s}\mathcal{F}_{\mu_1\ldots\mu_s}$, but this is not enough, since the Fronsdal operator, defined by $\mathcal{F} = \hat{F}\varphi$ is not self adjoint, that is:

$$\int d^Dx\,\varphi\cdot\delta\mathcal{F} \neq \int d^Dx\,\delta\varphi\cdot\mathcal{F} \; .$$

We need therefore a self adjoint modification of $\mathcal{F}$, just as in general relativity the Einstein tensor is the self adjoint modification of the Ricci tensor. In fact for the spin two case the Fronsdal operator is just the linearized Ricci tensor: $\mathcal{F}_{\mu\nu} = R^{\text{lin}}_{\mu\nu}$, so that $\mathcal{F}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{F}'$ is the linearized Einstein tensor. The generalization to spin s is straightforward, and the Einstein-like operator reads:

$$\mathcal{G}_{\mu_1\ldots\mu_s} = \mathcal{F}_{\mu_1\ldots\mu_s} - \frac{s(s-1)}{4}\,\eta_{(\mu_1\mu_2}\,\mathcal{F}'_{\mu_3\ldots\mu_s)}\ , \tag{1.9}$$

and inherits the corresponding anomalous Bianchi identity:

$$\partial\cdot\mathcal{G}_{\mu_2\ldots\mu_s} \equiv -\frac{3}{2}\binom{s-1}{3}\,\partial_{(\mu_2}\partial_{\mu_3}\partial_{\mu_4}\varphi''_{\mu_5\ldots\mu_s)} - \frac{(s-1)(s-2)}{4}\,\eta_{(\mu_2\mu_3}\partial\cdot\mathcal{F}'_{\mu_4\ldots\mu_s)}\ . \tag{1.10}$$

By means of $\mathcal{G}$ the Fronsdal Lagrangian is readily recovered in the form

$$\mathcal{L} = -\frac{1}{2}\,\varphi^{\mu_1\ldots\mu_s}\,\mathcal{G}_{\mu_1\ldots\mu_s}\ , \tag{1.11}$$

that produces the field equations $\mathcal{G}_{\mu_1\ldots\mu_s} = 0$. The form $\mathcal{F}_{\mu_1\ldots\mu_s} = 0$ is obtained combining the former equation of motion with its traces, just as in Einstein's case in vacuum.

Given the Lagrangian (1.11), we can check its gauge invariance performing the variation (1.4) with a traceless parameter. The action varies as

$$\delta S = \int d^D x\,\Big[-\,\delta\varphi^{\mu_1\ldots\mu_s}\,\mathcal{G}_{\mu_1\ldots\mu_s}\Big] = -s\int d^D x\,\Big[\partial^{\mu_1}\Lambda^{\mu_2\ldots\mu_s}\,\mathcal{G}_{\mu_1\ldots\mu_s}\Big]\ ;$$

integrating by parts and using (1.10) we find that the action is gauge invariant, provided that Λ is traceless, only for double traceless φ:

$$\delta S = -\frac{3s}{2}\binom{s-1}{3}\int d^D x\,\Big[\Lambda^{\mu_2\ldots\mu_s}\,\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4}\varphi''_{\mu_5\ldots\mu_s}\Big]\ .$$

We can resume the Fronsdal's formulation for completely symmetric tensors, omitting all the indices[2] that are to be intended as totally symmetrized. The field equations, Einstein like tensor and the action in this compact notation simply read:

$$\begin{aligned} \mathcal{F} &= \Box\varphi - s\,\partial\partial\cdot\varphi + \tfrac{s(s-1)}{2}\,\partial^2\varphi' = 0\ ,\\ \mathcal{G} &= \mathcal{F} - \frac{s(s-1)}{4}\,\eta\,\mathcal{F}'\ ,\\ S &= \int d^D x\,\Big[-\frac{1}{2}\,\varphi\cdot\mathcal{G}\Big]\ . \end{aligned} \tag{1.12}$$

[2]For example $\varphi\cdot\mathcal{F} \equiv \varphi^{\mu_1\ldots\mu_s}\mathcal{F}_{\mu_1\ldots\mu_s}$, $\partial\partial\cdot\varphi \equiv \partial_{(\mu_1}\partial^{\alpha}\varphi_{\alpha\mu_2\ldots\mu_s)}$, and $\partial^2\varphi \equiv \partial_{(\mu_1}\partial_{\mu_2}\varphi_{\mu_3\ldots\mu_{s+2})}$

They are invariant under the gauge transformation

$$\delta\varphi = \partial\Lambda \ , \tag{1.13}$$

provided the Fronsdal's constraints:

$$\Lambda' = 0 \ , \quad \varphi'' = 0 \ . \tag{1.14}$$

The present formulation is quite simple, although these constraints seem to be rather unnatural. Indeed, they can be overcome following different routes. One approach is to extend the Fronsdal operator with non-local terms that can cancel the gauge variation for arbitrary gauge parameters [34]: $\mathcal{F}_{\rm new} = \mathcal{F} + \mathcal{F}_{\rm NL}$, with $\delta\mathcal{F}_{\rm new} = 0$. This amounts to demand $\delta\mathcal{F}_{\rm NL} = -3\binom{s}{3}\,\partial^3\Lambda'$. One can then define a new Einstein like operator $\mathcal{G}_{\rm new}$ that enjoys an ordinary Bianchi identity: $\partial \cdot \mathcal{G}_{\rm new} = 0$, and ensures therefore the invariance of the action for unconstrained field.

An alternative to the non-local extension of Fronsdal's equations is the introduction of auxiliary fields $\rho_{\mu_1 \ldots \mu_{s-3}}$, called compensators [34, 35, 36], that transform as the trace of the gauge parameter, $\delta\rho \sim \Lambda'$ and modify the field equations like

$$\mathcal{F} = \partial^3 \rho \ .$$

Gauge fixing the compensator fields to zero one recovers, with the residue gauge invariance, the constrained Fronsdal formulation. The compensator formalism is naturally introduced studying the higher spin equations in their geometric formulation, that starts from the generalized curvatures, as will emerge from the quantization of the $O(N)$ spinning particle in the next section. Recalling general relativity one can think of the Riemann tensor or, better, its linearized version, as the gauge invariant curvature to start with. In fact, the Einstein's field equations amount to setting to zero its trace, and the linearized Riemann tensor can be expressed in terms of the metric fluctuation $h_{\mu\nu}$ in order to solve the Bianchi integrability equation $\partial_{[\mu} R_{\nu\lambda]\sigma\rho} = 0$. Following this reasoning one can introduce the so called de Wit-Freedman generalized curvatures R, invariant under unconstrained gauge transformations and obeying generalized Bianchi integrability conditions. The field equations then read as $\mathrm{Tr} R = 0$ and solving Bianchi's by expressing the curvatures in terms of gauge potentials one recovers the Fronsdal formulation with compensators, as we will describe in the next section.

1.2 $O(N)$ spinning particles and higher spin fields

It is known [20, 21, 31] that the quantization of the worldline action of a spinning particle, enjoying N-extended local supersymmetries, leads to field equations for spin $N/2$ fields in the geometrical formulation. This is seen by recalling that gauge symmetries of the worldline theory give rise to first class constraints that select physical states from the Hilbert space. In flat space the constraints of the $O(N)$ spinning particle produce equations of motion written in terms of tensors that are interpreted as generalized curvatures describing higher spin fields. Gauge potentials, as mentioned before, can be introduced by solving Bianchi integrability conditions.

We will present the $O(N)$ spinning particle and its canonical quantization in flat space, following [32] for the case of even N (*i.e.* bosonic higher spins), while the odd N case will be sketched along the recent work [33]. Spinning particle models others than the $O(N)$'s are related as well to higher spin field equations, and generalizations of the $O(N)$ spinning particle can be introduced, see for example [57, 58, 59, 60]. In the following only the $OSp(N|2M)$ generalization will be discussed in detail. To warm up our notation and get closer to the interplay between the supersymmetric spinning particle on the worldline and the spacetime field it describes, we will start our discussion by investigating the well known $N = 1$ and $O(2)$ models, describing Dirac spinors and spin one (or more generally p-form) fields, respectively.

1.2.1 $N = 1$ supersymmetry, the Dirac field

As we mentioned in the introduction, in order to write down a worldline action enjoing local supersymmetry, it is simpler to start with a theory invariant under rigid symmetries that will be gauged in a second step. Let us consider a particle moving in flat Minkowski space, with metric $\eta_{\mu\nu}$, with coordinates $x^\mu(\tau)$, being τ the worldline affine parameter. The particle is dressed in its motion with fermionic degrees of freedom described by the worldline Majorana fermions (just real Grassmann variables in one dimension) $\psi^\mu(\tau)$. The phase space action describing its motion is

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + \frac{i}{2}\, \psi_\mu \dot{\psi}^\mu - \frac{1}{2}\, p_\mu p^\mu \right] , \tag{1.15}$$

from which we read off the fundamental Poisson brackets[3]:

$$\{x^\mu, p_\nu\}_{\rm pb} = \delta^\mu{}_\nu \ , \quad \{\psi^\mu, \psi^\nu\}_{\rm pb} = -i\eta^{\mu\nu} \ . \tag{1.16}$$

The symmetries we want to gauge in the following are: time translations, generated by the Hamiltonian $H = \frac{1}{2}p^2$, and supersymmetry, generated by the supercharge $Q = \psi \cdot p$. The above action is indeed invariant under the transformations

$$\delta x^\mu = \xi\, p^\mu + i\,\epsilon\,\psi^\mu \ , \quad \delta p_\mu = 0 \ , \quad \delta\psi^\mu = -\epsilon\, p^\mu$$

for constant time translations ξ and Grassmann supersymmetry parameter ϵ. The Hamiltonian and the supercharge form a first class algebra:

$$\{Q, Q\}_{\rm pb} = -2i\, H \ , \quad \{Q, H\}_{\rm pb} = 0 \ , \tag{1.17}$$

so that we can gauge it and make the symmetries local. To this aim one introduces gauge fields: an einbein $e(\tau)$ and a real gravitino $\chi(\tau)$, to be coupled to the Hamiltonian and the supercharge respectively. The final action, enjoing $N = 1$ local supersymmetry on the worldline is thus [61, 27]:

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + \frac{i}{2}\,\psi_\mu \dot{\psi}^\mu - i\,\chi\, Q - e\, H \right] \ . \tag{1.18}$$

We stress that the equations of motion of the gauge fields force the Noether charges to vanish: $Q = H = 0$. This classical statement will be turned in the quantum theory to the constraints that the physical sector of the Hilbert space has to satisfy.

Let us perform the canonical quantization of the above model. All the phase space variables become operators, and the fundamental Poisson brakets turn into the (anti)-commutators:

$$[x^\mu, p_\nu] = i\,\delta^\mu{}_\nu \ , \quad \{\psi^\mu, \psi^\nu\} = \eta^{\mu\nu} \ , \tag{1.19}$$

while the supersymmetry algebra becomes

$$\{Q, Q\} = 2H \ , \quad [Q, H] = 0 \ . \tag{1.20}$$

The classical constraints $H = Q = 0$ do not hold at the operator level, but, just as in the case of Virasoro constraints in string theory, they select the physical Hilbert space:

$$|\phi\rangle \in \mathcal{H}_{\rm phys} \quad \Leftrightarrow \quad Q|\phi\rangle = H|\phi\rangle = 0 \ . \tag{1.21}$$

[3] Actuallt, they are Dirac brakets, since the fermionic phase space is subject to the second class constraint $\pi^\mu = -\frac{i}{2}\psi^\mu$

Since $H|\phi\rangle = 0$, the system does not evolve in the worldline time τ, and the whole dynamics is encoded in (1.21). As usual, x and p operators are represented as multiplication by x^μ and derivative $-i\partial_\mu$, acting on functions belonging to $L^2(\mathbb{R}^D)$. The ψ algebra, on the other hand, is nothing but a Clifford algebra, and we shall represent $\psi^\mu = \frac{1}{\sqrt{2}}\gamma^\mu$ acting on a Dirac spinor. The whole wave function is thus a Dirac field: $|\phi\rangle \sim \phi_\alpha(x)$, and the operators Q and H are just the massless Dirac operator and d'Alembertian: $Q = -\frac{i}{\sqrt{2}}\not{\partial}$, $H = -\frac{1}{2}\Box$. Since the relation $H = Q^2$ holds, the only nontrivial dynamical equation among (1.21) is

$$Q|\phi\rangle = 0 \quad \Rightarrow \quad \not{\partial}\,\phi = 0\ . \tag{1.22}$$

We have showed that the worldline quantization of a spinning particle enjoying $N = 1$ local supersymmetry describes a spacetime spinor field obeying the massless Dirac equation.[4] In this sketch the spacetime was the flat Minkowski one, but this model can be consistently quantized on any curved background, as will be discussed in the next chapter.

1.2.2 $N = 2$ supersymmetry, Maxwell field and p-forms

To build up the action for the $N = 2$ spinning particle [62, 28], one has to add to the x and ψ variables a second fermionic species, so that we get the $O(2)$ doublet $\psi_i^\mu \equiv (\psi_1^\mu, \psi_2^\mu)$. It is more convenient to work in a $U(1)$ complex basis so we introduce complex Dirac fermions

$$\psi^\mu = \frac{1}{\sqrt{2}}(\psi_1^\mu + i\,\psi_2^\mu)\ , \quad \bar{\psi}^\mu = \frac{1}{\sqrt{2}}(\psi_1^\mu - i\,\psi_2^\mu)\ .$$

The action with rigid $N = 2$ supersymmetry in complex basis reads:

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + i\,\bar{\psi}_\mu\dot{\psi}^\mu - \frac{1}{2}\,p_\mu p^\mu\right]\ , \tag{1.23}$$

where the $\{x, p\}_{\text{pb}}$ Poisson brackets are the common ones and $\{\psi^\mu, \bar{\psi}^\nu\}_{\text{pb}} = -i\eta^{\mu\nu}$. The charges generating the extended $N = 2$ rigid supersymmetry, *i.e.* the Hamiltonian $H = \frac{p^2}{2}$, the two supercharges $Q = \psi\cdot p$ and $\bar{Q} = \bar{\psi}\cdot p$, and the $U(1)$ R-symmetry generator $J = \psi\cdot\bar{\psi}$, close under Poisson brackets and will be gauged. Following the same route we displayed for the $N = 1$ model, we introduce the gauge fields $G = (e, \chi, \bar{\chi}, a)$: the einbein and the two complex gravitini to gauge time translations and the two supersymmetries,

[4]To add mass in this formulation a simple technique is to add a spacetime dimension and perform a Kaluza-Klein reduction.

and the gauge field a for the $U(1)$ symmetry. The complete action enjoys local $O(2)$-extended supersymmetry and reads:

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + i\,\bar{\psi}_\mu \dot{\psi}^\mu - i\,\bar{\chi}\,Q - i\chi\bar{Q} - e\,H - a\,(J-q) \right] , \tag{1.24}$$

where we added a Chern-Simons piece $S_{\rm CS} = q\int d\tau\, a$ invariant by itself. Just as for the previous model, the gauge fields' equations of motion impose the vanishing of the $N=2$ charges: $H = Q = \bar{Q} = J - q = 0$, that in the quantum theory translate to the constraint equations:

$$|\phi\rangle \in \mathcal{H}_{\rm phys} \quad \Leftrightarrow \quad Q|\phi\rangle = \bar{Q}|\phi\rangle = H|\phi\rangle = (J-q)|\phi\rangle = 0 \ , \tag{1.25}$$

that are consistent at the quantum level since the charges form a first class algebra, namely:

$$\{Q,\bar{Q}\} = 2H \ , \quad [Q,\tilde{J}] = -Q \ , \quad [\bar{Q},\tilde{J}] = \bar{Q} \ , \tag{1.26}$$

where $\tilde{J} = J - q$, and the other commutators vanish. In the definition of the quantum J there is an ordering ambiguity, not modifying the algebra though, that we fix by a graded symmetric ordering prescription:

$$J = \frac{1}{2}[\psi^\mu,\bar{\psi}_\mu] = \psi\cdot\bar{\psi} - \frac{D}{2} \ .$$

Let us turn to the representation of the fundamental quantum algebra: the $[x,p]$ algebra is treated in the usual manner, while we can see that the fermionic anti-commutator $\{\psi^\mu,\bar{\psi}^\nu\} = \eta^{\mu\nu}$ is a creation-annihilation algebra. We use a Schrödinger like basis for fermions, so that the ψ operator is represented by multiplication by the Grassmann classical variable ψ^μ, and the $\bar{\psi}$ operator by fermionic derivative $\frac{\partial}{\partial\psi_\mu}$. The wave function can be expanded in a finite power series in ψ, that is:

$$|\phi\rangle = \sum_{k=0}^{D} \phi_{\mu_1\ldots\mu_k}(x)\,\psi^{\mu_1}\ldots\psi^{\mu_k} \ .$$

Since the ψ^μ are anti-commuting, they can be effectively thought as basis one-forms dx^μ endowed with the exterior product, and we get for the wave function and all the operators an enlightening geometric interpretation. The Hilbert space consist in a collection of differential forms[5]:

$$|F\rangle = \sum_{k=0}^{D} F_{\mu_1\ldots\mu_k}(x)\,dx^{\mu}\wedge\ldots\wedge dx^{\mu_k} \ , \tag{1.27}$$

[5]We use the letter F to stress the analogy with electromagnetism, and we will soon drop the wedge symbol that will have to be understood in multiplicating dx's.

while the supercharges and the Hamiltonian are represented by the exterior derivative, divergence and d'Alembertian:

$$\begin{aligned} i\,Q &= dx^\mu \partial_\mu = d\ , \quad i\,\bar{Q} = \frac{\partial}{\partial dx^\mu}\partial^\mu = d^\dagger \\ -2H &= \partial^\mu \partial_\mu = \Box = \{d, d^\dagger\}\ . \end{aligned} \tag{1.28}$$

Consider now the $U(1)$ generator: the operator $\psi \cdot \bar{\psi} = dx \cdot \frac{\partial}{\partial dx} = \mathbf{N}$ is the number operator that counts the number of dx's, *i.e.* the form degree. Its spectrum is therefore integer and this constrains the Chern-Simons coupling: $q = p + 1 - \frac{D}{2}$, where p is an integer. We are now ready to impose the independent dynamical equations. Let us start from the J equation, that is solved fixing the form degree to be $p+1$:

$$\begin{aligned} &(J - q)|F\rangle = (\mathbf{N} - p - 1)|F\rangle = 0 \quad \Rightarrow \\ &|F\rangle = F_{\mu_1 \ldots \mu_{p+1}}\, dx^{\mu_1} \ldots dx^{\mu_{p+1}} = F_{(p+1)}\ . \end{aligned} \tag{1.29}$$

Now, the only independent equations left are $Q|F\rangle = \bar{Q}|F\rangle = 0$, that are precisely Bianchi and Maxwell equations for a $(p+1)$-form field strength:

$$\begin{aligned} &\partial_{[\mu} F_{\mu_1 \ldots \mu_{p+1}]} = 0\ , \quad \partial^\mu F_{\mu\mu_2 \ldots \mu_{p+1}} = 0 \quad \text{or} \\ &dF_{(p+1)} = 0\ , \quad d^\dagger F_{(p+1)} = 0\ . \end{aligned} \tag{1.30}$$

For $p = 1$, (1.30) reduce to usual electromagnetism, but they are as well the spin one version of the geometric equations involving higher spin curvatures that we will encounter in the following. Using the Poincaré lemma, we can integrate the first of (1.30) in each topologically trivial patch and write the field strength in term of a p-form gauge potential:

$$F_{\mu_1 \ldots \mu_{p+1}} = \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \quad \text{or} \quad F_{(p+1)} = dA_p\ . \tag{1.31}$$

The physical field strength is left unchanged by a gauge transformation of the potential:

$$\delta A_{\mu_1 \ldots \mu_p} = \partial_{[\mu_1} \Lambda_{\mu_2 \ldots \mu_p]} \quad \text{or} \quad \delta A_p = d\Lambda_{(p-1)}\ , \tag{1.32}$$

where Λ_{p-1} is a $(p-1)$-form, and the field equations, that read[6]

$$\begin{aligned} &\Box A_{\mu_1 \ldots \mu_p} - p\,\partial_{[\mu_1} \partial \cdot A_{\mu_2 \ldots \mu_p]} = 0 \quad \text{or} \\ &d^\dagger d A_p = \Box A_p - d d^\dagger A_p = 0\ , \end{aligned} \tag{1.33}$$

[6]The dot denotes contraction with the first index.

are invariant as well. As for the $N = 1$ spinning particle, also the $O(2)$ model can be coupled to any background metric. In fact, its canonical quantization shows that the $N = 2$ spinning particle describes the propagation of gauge p-forms in spacetime, and we know that the description in terms of exterior calculus does not rely on any use of the metric, and can thus be extended to any curved spacetime.

1.2.3 $N = 2s$, bosonic higher spins

After having displayed how the canonical quantization of spinning particle models produces free field equations for spacetime spin $\frac{1}{2}$ and 1 fields we are ready to generalize the same concepts to arbitrary spin. The gauging of extended worldline supergravities will lead to geometrical field equations for generalized curvatures that will be integrated in terms of gauge potentials. One ends up with higher derivative equations that can be modified and reduced to second order equations that are nothing but the Fronsdal-Labastida field equations in the compensator formalism.

To produce the $O(N)$ spinning particle model we first generalize the $O(2)$ particle previously described introducing a N-multiplet of Majorana worldline fermions: $\psi_i^\mu(\tau)$, where $i = 1, ..., N$ is a $O(N)$ index. The Hamiltonian is left unchanged, and supercharges and $SO(N)$ generators are readily constructed:

$$J_{ij} = \frac{i}{2}\,[\psi_i^\mu, \psi_{\mu j}]\;, \quad Q_i = \psi_i \cdot p\;, \quad H = \frac{p^2}{2}\;, \tag{1.34}$$

where, thinking to the quantized model, we already prescribed the ordering in defining J_{ij}. The classical action for the $O(N)$ spinning particle is the natural extension of the $N = 2$ model presented before:

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + \frac{i}{2}\,\psi_\mu^i \dot{\psi}_i^\mu - i\,\chi^i\,Q_i - e\,H - \frac{1}{2}\,a^{ij}\,J_{ij}\right]\,, \tag{1.35}$$

where $O(N)$ indices are raised and lowered with the unit metric δ_{ij} and its inverse, and a $SO(N)$ gauge field $a^{ij}(\tau)$ and N real gravitini $\chi^i(\tau)$ have been introduced.

The action is invariant under N-extended local supersymmetry transformations with time translation parameter ξ, supersymmetry parameters ϵ_i and $SO(N)$ parameters α_{ij}, that read:

$$\begin{aligned} \delta x^\mu &= \xi\, p^\mu + i\epsilon^i\, \psi_i^\mu\;, \quad \delta p_\mu = 0 \\ \delta\psi_i^\mu &= -\epsilon_i\, p^\mu + \alpha_{ij}\, \psi^{\mu j} \end{aligned} \tag{1.36}$$

for the "matter" fields, and

$$\begin{aligned} \delta e &= \dot{\xi} + 2i\chi^i \epsilon_i \ , \quad \delta\chi_i = \dot{\epsilon}_i - a_{ij}\,\epsilon^j + \alpha_{ij}\,\chi^j \\ \delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im}\,a^m{}_j + \alpha_{jm}\,a_i{}^m \end{aligned} \tag{1.37}$$

for the gauge fields. The transformation rules for the $N=1$ and $N=2$ models are just particular cases of the ones above. The symmetry generators form the $O(N)$ extended superalgebra, that we present already in the quantum form:

$$\begin{aligned} \{Q_i, Q_j\} &= 2\,\delta_{ij}\,H \ , \\ [J_{ij}, Q_k] &= i\,\delta_{jk}\,Q_i - i\,\delta_{ik}\,Q_j \ , \\ [J_{ij}, J_{kl}] &= i\,\delta_{jk}\,J_{il} - i\,\delta_{ik}\,J_{jl} + i\,\delta_{il}\,J_{jk} - i\,\delta_{jl}\,J_{ik} \ . \end{aligned} \tag{1.38}$$

It is a first class algebra indeed, and it is thus consistent to impose the constraints coming from the gauge fields' equations of motion, but in order to make contact more directly with the standard field theory, it is better to switch again to a complex basis in order to represent our quantum operators. As already done in the previous section, since we have an even number of ψ's, we introduce Dirac fermions[7]:

$$\psi^\mu_I = \frac{1}{\sqrt{2}}\,(\psi^\mu_i + i\,\psi^\mu_{i+s}) \ , \quad \bar{\psi}^{\mu I} = \frac{1}{\sqrt{2}}\,(\psi^\mu_i - i\,\psi^\mu_{i+s}) \ ,$$

where $I = 1, ..., s$ is a $U(s)$ index. The fermionic anti-commutators become

$$\{\psi^\mu_I, \bar{\psi}^{\nu J}\} = \eta^{\mu\nu}\,\delta_I^J \ , \quad \{\psi^\mu_I, \psi^\nu_J\} = 0 \ , \quad \{\bar{\psi}^{\mu I}, \bar{\psi}^{\nu J}\} = 0$$

and are s copies of the creation-annihilation algebra. Accordingly, the supercharges and $SO(N)$ generators split, and produce:

$$\begin{aligned} Q_i \ &\rightarrow \left\{ \begin{array}{l} Q_I = \psi_I \cdot p \\ \bar{Q}^I = \bar{\psi}^I \cdot p \end{array} \right. \\ J_{ij} \ &\rightarrow \left\{ \begin{array}{l} J^I_J = \psi_J \cdot \bar{\psi}^I - \frac{D}{2}\delta^I_J \\ K_{IJ} = \psi_I \cdot \psi_J \\ \bar{K}^{IJ} = \bar{\psi}^I \cdot \bar{\psi}^J \end{array} \right. \end{aligned} \tag{1.39}$$

As for the $O(2)$ model, we shall represent ψ's as multiplication by ψ^μ_I, and their momenta $\bar{\psi}$'s as $\frac{\partial}{\partial\psi^\mu_I}$, and the wave function can be expressed as a power

[7] In this formula the index i runs only form 1 to $s = N/2$.

series in the Grassmann variables. The general wave function can thus be written in this rather involved form:

$$|R\rangle = \sum_{k_i=0}^{D} R_{\mu_1^1\ldots\mu_{k_1}^1,\ldots,\mu_1^s\ldots\mu_{k_s}^s}(x)\,\psi_1^{\mu_1^1}\ldots\psi_1^{\mu_{k_1}^1},\ldots,\psi_s^{\mu_1^s}\ldots\psi_s^{\mu_{k_s}^s} \sim \sum_{k_i=0}^{D} R_{[k_1][k_2]\ldots[k_s]} \tag{1.40}$$

and represents a bunch of multi-form tensors with s blocks of k_i indices each, with every k_i ranging from 0 to D.

Our operators, in a geometrical language, appear as:

$$\begin{aligned} -2H &= \partial^\mu\partial_\mu = \Box\ , \\ i\,Q_I &= \psi_I \cdot \partial = {}_{(I)}d\ , \quad i\,\bar{Q}^I = \frac{\partial}{\partial\psi_I}\cdot\partial = {}_{(I)}d^\dagger\ , \\ J^I_J &= \psi_J \cdot \frac{\partial}{\partial\psi_I} - \frac{D}{2}\delta^I_J\ , \quad K_{IJ} = \psi_I\cdot\psi_J = \mathbf{g}_{(IJ)}\ , \\ \bar{K}^{IJ} &= \frac{\partial}{\partial\psi_I}\cdot\frac{\partial}{\partial\psi_I} = \mathbf{Tr}_{(IJ)}\ , \end{aligned} \tag{1.41}$$

where we denote with ${}_{(I)}d$ and ${}_{(I)}d^\dagger$ the operators acting as exterior derivative and divergence on the I-th block of indices, while $\mathbf{g}_{(IJ)}$ acts by multiplication by the metric tensor $\eta_{\mu\nu}$, where the two indices $\mu\nu$ have to be intended as belonging to the I-th and J-th blocks of the whole tensor and finally, $\mathbf{Tr}_{(IJ)}$ takes the trace over one index from block I and one from block J.

The superalgebra (1.38) splits as well in complex basis (look at [32]) and as a result the independent dynamical equations to be imposed are[8]:

$$J^I_J|R\rangle = Q_I|R\rangle = \bar{K}^{IJ}|R\rangle = 0\ , \tag{1.42}$$

and will be examined in the order presented above. First of all, let us consider the J^I_J constraint for $I = J$. The corresponding $SO(N)$ generator is a shifted number operator (I is fixed and not summed): $J^I_I = \mathbf{N}_{(I)} - \frac{D}{2}$ and one immediately see that, since $\mathbf{N}_{(I)}$ has integer eigenvalues for each I, the model is empty for an odd number of spacetime dimensions. Restricting thus to $D = 2d$, the equation $J^I_I|R\rangle = 0$ is solved restricting the physical field to a multi-form of s blocks of d antisymmetric indices each:

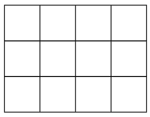
$$|R\rangle = R_{\mu_1\ldots\mu_d,\ldots,\nu_1\ldots\nu_d}(x)\,\psi_1^{\mu_1}\ldots\psi_1^{\mu_d},\ldots,\psi_s^{\nu_1}\ldots\psi_s^{\nu_d} \sim R_{[d]_1\ldots[d]_s}\ . \tag{1.43}$$

Requiring the remaining J^I_J equations one has algebraic Bianchi identities involving the I-th and J-th blocks, *e.g.*:

$$J^2_1|R\rangle = 0 \quad \Rightarrow \quad R_{[\mu_1\ldots\mu_d,\mu]\nu_2\ldots\nu_d,\ldots,\lambda_1\ldots\lambda_d} = 0 \tag{1.44}$$

[8]The K_{IJ} constraint does not appear to be dependent from the algebra, but it acts as $\bar{K}^{IJ}$ on the Hodge dual curvature, see [32].

and analogous relations for the other blocks. This restricts the R tensor to belong to the rectangular $GL(D)$ Young tableau with d rows and s columns:

$$R_{[d]_1 \dots [d]_s} \quad \sim$$ 

that implies the multi-form is symmetric under the exchange of two blocks.

The Q_I equation enforces a Bianchi integrability condition involving the I-th block: ${}_{(I)}dR = 0$ that reads

$$Q_I|R\rangle = 0 \quad \Rightarrow \quad \partial_{[\mu} R_{\mu_1 \dots \mu_d], \dots, \nu_1 \dots \nu_d} = 0 \tag{1.45}$$

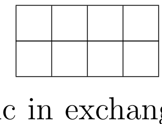
where, using symmetry in block exchange, we moved the I-th block of indices in first position. To solve the Bianchi equation for the curvature, we have to express it as a derivative operator, that should annihilate the Q_I, acting on a potential: $|R\rangle = q|\phi\rangle$. Using the anti-commutation relation $\{Q_I, Q_J\} = 0$ we see that a suitable operator is

$$q = Q_1 Q_2 \dots Q_s = \frac{1}{s!}\, \epsilon^{I_1 \dots I_s}\, Q_{I_1} \dots Q_{I_s} \ ,$$

and the integrability condition is solved in terms of the potential ϕ:

$$Q_I|R\rangle = 0 \quad \Rightarrow \quad |R\rangle = q|\phi\rangle \ . \tag{1.46}$$

Since $[J^I_J, q] = q\delta^I_J$, the J constraints on R are still satisfied provided that ϕ is described by the Young tableau with $d-1$ rows and s columns:

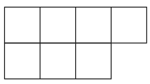
$$\phi_{[d-1]_1 \dots [d-1]_s} \ \sim$$  $$, \qquad R_{[d]_1 \dots [d]_s} = d_{(1)} d_{(2)} \dots d_{(s)} \phi_{[d-1]_1 \dots [d-1]_s} \ ,$$

and ϕ is symmetric in exchanging blocks of indices as well. The operator q is of s-th order in derivatives and so the generalized curvature is expressed as s derivatives of the potential and will produce an higher derivative field equation for ϕ for spin greater than two.

At this stage we can introduce the gauge invariance of the theory: having solved $|R\rangle = q|\phi\rangle$, we see that the physical curvature is left unchanged if we transform the gauge field as $\delta|\phi\rangle = Q\dots$. By means of an arbitrary vector field V^μ, we define

$$|\Lambda^I\rangle = V \cdot \frac{\partial}{\partial \psi_I}\, |\Lambda\rangle$$

where Λ is a tensor with the same Young tableau of ϕ. The operator $V \cdot \frac{\partial}{\partial \psi_I}$ saturates one index in the I-th block with the vector V, and thus the gauge parameter Λ^I has the Young tableau

$$\Lambda^I_{[d-1]_1 \dots [d-2]_I \dots [d-1]_s} \quad \sim$$  .

The gauge transformation that leaves the curvature invariant can be written as

$$\delta|\phi\rangle = Q_I|\Lambda^I\rangle \tag{1.47}$$

The remaining dynamical equation is $\bar{K}^{IJ}|R\rangle = 0$ *i.e.* traceless curvature, that in the gravitational case is Ricci-flatness. The equation of motion

$$\bar{K}^{IJ}|R\rangle = \bar{K}^{IJ}\, q|\phi\rangle = 0 \tag{1.48}$$

is of order s in derivatives. We want to isolate a second order differential operator and, using the extended superalgebra, on can see [32] that $\bar{K}^{IJ}q = q^{IJ}G$, where q^{IJ} is the product of all but the I-th and J-th Q's

$$q^{IJ} = Q_1 ... Q_{I-1}Q_{I+1} ... Q_{J-1}Q_{J+1} ... Q_s \ ,$$

while G is the second order operator we were looking for, the Fronsdal-Labastida operator:

$$G = -2H + Q_I\bar{Q}^I + \frac{1}{2}\, Q_IQ_J\bar{K}^{IJ} \tag{1.49}$$

that in differential operators language reads

$$G \sim \Box - \sum_I {}_{(I)}d\, {}_{(I)}d^\dagger - \frac{1}{2}\sum_{IJ} {}_{(I)}d\, {}_{(J)}d\, \mathbf{Tr}_{(IJ)} \ ,$$

and is $U(s)$ invariant, since $[J^I_J, G] = 0$. However, the field equation is still in a not suitable form: $q^{IJ}G|\phi\rangle = 0$. Since q^{IJ} contains $s-2$ Q's, and the product of $s+1$ Q's is forced to vanish, we can parameterize the kernel of q^{IJ} using the product of three supercharges. In analogy to what we said for the gauge parameter, we introduce the new field ρ, with the same Young tableau as ϕ and Λ and define the compensator fields:

$$|\rho^{IJK}\rangle = W \cdot \frac{\partial}{\partial\psi_I}W \cdot \frac{\partial}{\partial\psi_J}W \cdot \frac{\partial}{\partial\psi_K}|\rho\rangle \ ,$$

where W^μ is a generic vector field, with the gun-like Young tableau:

$\rho^{IJK}_{[d-1]_1 ... [d-2]_I .. [d-2]_J .. [d-2]_K ... [d-1]_s} \quad \sim$  .

One can now solve the equation $q^{IJ}G|\phi\rangle = 0$ by demanding:

$$G|\phi\rangle = Q_IQ_JQ_K|\rho^{IJK}\rangle \ , \tag{1.50}$$

that is fully gauge invariant under (1.47), provided that the compensators transform as[9]

$$\delta|\rho^{IJK}\rangle = -\frac{1}{2}\bar{V}^{[K}\bar{K}^{JI]}|\Lambda\rangle \tag{1.51}$$

in order to cancel the gauge variation $G\delta|\phi\rangle$.

What we described in the language of our quantum mechanical operators are the Fronsdal-Labastida equations for spin s fields in $D = 2d$ spacetime dimensions. The gauge field

$$\phi_{\mu_1 \dots \mu_{d-1}, \dots, \nu_1 \dots \nu_{d-1}}$$

obeys the field equations with compensators:

$$\begin{aligned} G|\phi\rangle &= Q_I Q_J Q_K |\rho^{IJK}\rangle \ , \\ G &= -2H + Q_I \bar{Q}^I + \frac{1}{2} Q_I Q_J \bar{K}^{IJ} \ , \end{aligned} \tag{1.52}$$

that are gauge invariant under the combined transformations:

$$\delta|\phi\rangle = Q_I \, \bar{V}^I |\Lambda\rangle \ , \quad \delta|\rho^{IJK}\rangle = -\frac{1}{2}\bar{V}^{[K}\bar{K}^{JI]}|\Lambda\rangle \ . \tag{1.53}$$

One can easily recover the original form of the Fronsdal-Labastida equations with constrained gauge parameters by partially gauge fixing: using the transformation rule (1.51), the compensators can be gauge fixed to zero, $\rho^{IJK} = 0$; the gauge freedom left is then subject to the tracelessness condition $\bar{K}^{IJ}|\Lambda\rangle = 0$. The field equations and gauge invariance take the form

$$\begin{aligned} G|\phi\rangle &= 0 \ , \\ \delta|\phi\rangle &= Q_I \, \bar{V}^I |\Lambda\rangle \ , \quad \bar{K}^{IJ}|\Lambda\rangle = 0 \ . \end{aligned} \tag{1.54}$$

In the gauge fixed formulation a constraint on the gauge field appears as well. In fact, an anomalous Bianchi identity holds:

$$\left(\bar{Q}^I - \frac{1}{2} Q_J \bar{K}^{IJ}\right) G|\phi\rangle = -\frac{1}{4} Q_J Q_K Q_L \bar{K}^{IJ} \bar{K}^{KL}|\phi\rangle \tag{1.55}$$

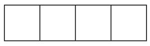
which, to be consistent with the gauge fixed field equation $G|\phi\rangle = 0$, requires

$$\bar{K}^{IJ}\bar{K}^{KL}|\phi\rangle = 0 \tag{1.56}$$

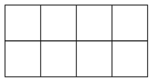
i.e. that the gauge field ϕ must be double traceless.

[9]We denote $\bar{V}^K = V \cdot \frac{\partial}{\partial \psi_K}$

In four spacetime dimensions things simplifies: the gauge field is a completely symmetric tensor

$$\phi_{\mu_1\dots\mu_s} \quad \sim$$  ,

while the curvature has s symmetric couples of antisymmetric indices:

$$R_{\mu_1\nu_1\dots\mu_s\nu_s} = \partial_{\mu_1}\dots\partial_{\mu_s}\phi_{\nu_1\dots\nu_s} \quad \sim$$  ,

where antisymmetrization is understood in the couples (μ_i, ν_i). The gauge parameter and compensator reduce to a rank $s-1$ and $s-3$ symmetric tensors respectively, and the field equations become the compensated Fronsdal's equations:

$$\Box\phi_{\mu_1\dots\mu_s} - s\,\partial_{(\mu_1}\partial\cdot\phi_{\mu_2\dots\mu_s)} + \tfrac{s(s-1)}{2}\,\partial_{(\mu_1}\partial_{\mu_2}\mathrm{Tr}\,\phi_{\mu_3\dots\mu_s)} = \partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\rho_{\mu_4\dots\mu_s)}$$

or, using the compact notation introduced in the first section, the equations and gauge transformations read

$$\begin{aligned} &\Box\phi - s\,\partial\partial\cdot\phi + \tfrac{s(s-1)}{2}\,\partial^2\phi' = \partial^3\rho\ , \\ &\delta\phi = \partial\Lambda\ , \quad \delta\rho = \Lambda'\ , \end{aligned} \tag{1.57}$$

and, after gauge fixing the compensator to zero,

$$\begin{aligned} &\Box\phi - s\,\partial\partial\cdot\phi + \tfrac{s(s-1)}{2}\,\partial^2\phi' = 0\ , \\ &\delta\phi = \partial\Lambda\ , \quad \Lambda' = 0\ , \quad \phi'' = 0\ . \end{aligned} \tag{1.58}$$

The coupling of the $O(N)$ spinning particle to curved spacetimes reflects the troubles one encounters in field theory in trying to make higher spin fields interact with gravity, and will be studied in the next chapter. Let us now turn to a brief description of half integer higher spin in the worldline formalism we developed so far.

1.2.4 $N = 2s+1$, fermionic higher spins

All the discussion of the previous part on $O(2s)$ spinning particles is valid until the introduction of the complex basis. In the present case, we will isolate the very last fermion ψ^μ_N, and treat the other $2s$ exactly as before, constructing the s Dirac fermions ψ^μ_I and $\bar\psi^{\mu I}$. The non vanishing anticommutators read

$$\{\psi^\mu_I, \bar\psi^{\nu J}\} = \eta^{\mu\nu}\,\delta^J_I\ , \quad \{\psi^\mu_N, \psi^\nu_N\} = \eta^{\mu\nu}\ ,$$

that are precisely s copies of fermionic oscillator relations plus a Clifford algebra. We rescale the fermion left alone in order to match with gamma matrices normalization: $\psi^\mu_N = \frac{1}{\sqrt{2}}\gamma^\mu$, and the Hilbert space basis will be the tensor product of $L^2(\mathbb{R}^D)$, the fermionic oscillator basis, and a spinor space. The wave function will be a multi-form spinor-tensor

$$|\mathcal{R}\rangle = \sum_{k_i=0}^{D} \mathcal{R}^a_{\mu^1_1\ldots\mu^1_{k_1},\ldots,\mu^s_1\ldots\mu^s_{k_s}}(x)\, \psi_1^{\mu^1_1}\ldots\psi_1^{\mu^1_{k_1}}, \ldots, \psi_s^{\mu^s_1}\ldots\psi_s^{\mu^s_{k_s}}|a\rangle \sim \sum_{k_i=0}^{D} \mathcal{R}_{[k_1][k_2]\ldots[k_s]} \tag{1.59}$$

where in the last compact notation we left implicit the spinor index a.

The extended superalgebra has the same form as before if we use $SO(N)$ indices, but using the $U(s)$ indices I and gamma matrices for ψ^μ_{2s+1}, the charges split further:

$$\begin{array}{l} Q_i \;\rightarrow \left\{ \begin{array}{l} Q_I = \psi_I \cdot p \\ \bar{Q}^I = \bar{\psi}^I \cdot p \\ \not{P} \;= \gamma \cdot p \end{array} \right. \\ J_{ij} \;\rightarrow \left\{ \begin{array}{l} J^I_J = \psi_J \cdot \bar{\psi}^I - \frac{D}{2}\delta^I_J \\ K_{IJ} = \psi_I \cdot \psi_J \\ \bar{K}^{IJ} = \bar{\psi}^I \cdot \bar{\psi}^J \\ L_I = \gamma \cdot \psi_I \\ \bar{L}^I = \gamma \cdot \bar{\psi}^I \end{array} \right. \end{array} . \tag{1.60}$$

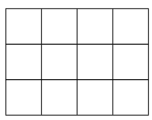
The set of operators already encountered in the $O(2s)$ case have the same geometric interpretation. The new ones act as:

$$\begin{aligned} & i\not{P} \;= \not{\partial} \\ & L_I = {}_{(I)}\gamma \;, \quad \bar{L}^I = {}_{(I)}\gamma\cdot \;, \end{aligned} \tag{1.61}$$

denoting the Dirac operator, the product with γ_μ and anti-symmetrization in the I-th block, and γ-trace in the I-th block, respectively. The superalgebra (1.38) splits again among these new operators, see [33], and as a result the independent dynamical equations now read[10]:

$$J^I_J|\mathcal{R}\rangle = Q_I|\mathcal{R}\rangle = \bar{L}^I|\mathcal{R}\rangle = 0\;. \tag{1.62}$$

Exactly as in the bosonic case, solving $J^I_J|\mathcal{R}\rangle = 0$ restricts the tensor structure to the rectangular $d \times s$ Young tableau:

$\mathcal{R}_{[d]_1\ldots[d]_s} \sim$  ,

[10]For the K_{IJ} constraint the same remark mentioned in the previous section holds.

as well, the Q_I equation imposes Bianchi conditions:

$$\partial_{[\mu}\mathcal{R}^a_{\mu_1\ldots\mu_d],\ldots,\nu_1\ldots\nu_d} = 0 \tag{1.63}$$

and the $\bar{L}^I$ finally entails γ-tracelessness on the curvature:

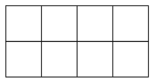
$$\gamma^\mu\mathcal{R}^a_{\mu\mu_2\ldots\mu_d,\ldots,\nu_1\ldots\nu_d} = 0 \;, \tag{1.64}$$

where in both equations we moved the I-th block of indices to the first place thanks to the exchange symmetry.

Following exactly the same strategy used for integer spins one solves the Bianchi constraint by means of the q operator:

$$Q_I|\mathcal{R}\rangle = 0 \quad \Rightarrow \quad |\mathcal{R}\rangle = q|\Psi\rangle \;, \tag{1.65}$$

for the spin $s+\frac{1}{2}$ spinor-tensor gauge field $\Psi^a_{\mu_1\ldots\mu_{d-1},\ldots,\nu_1\ldots\nu_{d-1}}$, with rectangular Young tableau

$\Psi_{[d-1]_1\ldots[d-1]_s}$ $\sim$  , $\mathcal{R}_{[d]_1\ldots[d]_s} = d_{(1)}d_{(2)}\ldots d_{(s)}\Psi_{[d-1]_1\ldots[d-1]_s}$.

Gauge invariance of the $\mathcal{R}$ curvature and of the field equations has the same form as before:

$$\delta|\Psi\rangle = Q_I\bar{V}^I|\Xi\rangle \;,$$

where Ξ is a spinor-tensor of the same species of Ψ. To deal with the final equation of motion $\bar{L}^I|\mathcal{R}\rangle = 0$ one has to use the superalgebra in the new basis to find:

$$\bar{L}^I q|\Psi\rangle = q^I\mathcal{G}|\Psi\rangle = 0 \;, \tag{1.66}$$

where $q^I = Q_1..Q_{I-1}Q_{I+1}..Q_s$, and the $\mathcal{G}$ is the Fang-Fronsdal operator written in terms of the supercharges:

$$\mathcal{G} = (-)^{s-1}\left(\not{P} \; + Q_K\bar{L}^K\right) \;. \tag{1.67}$$

To get rid of the q^I operator we introduce compensator fields

$$|\rho^{IJ}\rangle = \bar{W}^I\bar{W}^J|\rho\rangle \;,$$

where ρ has the same structure as Ψ, and we obtain the Fang-Fronsdal equations for spin $s+\frac{1}{2}$ fields in $D = 2d$ dimensional spacetime:

$$\mathcal{G}|\Psi\rangle = Q_IQ_J|\rho^{JI}\rangle \;, \tag{1.68}$$

gauge invariant under the transformations:

$$\delta|\Psi\rangle = Q_I\,\bar{V}^I|\Xi\rangle \;, \quad \delta|\rho^{IJ}\rangle = (-)^{s-1}\bar{V}^{[I}\bar{L}^{J]}|\Xi\rangle \;. \tag{1.69}$$

Using part of the gauge invariance one can set the compensators to zero, and the residual gauge freedom is constrained to γ-traceless parameters:

$$\begin{aligned} &\mathcal{G}|\Psi\rangle = 0 \ , \\ &\delta|\Psi\rangle = Q_I\,\bar{V}^I|\Xi\rangle \ , \quad \bar{L}^I|\Xi\rangle = 0 \ . \end{aligned} \tag{1.70}$$

The gauge fixed theory satisfy a fermionic version of the anomalous Bianchi identity, see [33], and therefore inherits an algebraic constraint on the gauge field, namely triple γ-tracelessness:

$$\bar{L}^I\bar{L}^J\bar{L}^K|\Psi\rangle = 0 \ . \tag{1.71}$$

We finish here the outline of the canonical quantization of $O(N)$ spinning particle in Minkowski spacetime. We have reviewed how the worldline models are related to the propagation in spacetime of spin $\frac{N}{2}$ fields, and we will pursue the analysis of the coupling to curved backgrounds in the next chapter, where we will focus also on the definition of the functional integral for these classes of quantum mechanical models.

Chapter 2

Transition amplitudes and path integrals

Quantum mechanical models with extended supersymmetry find useful applications in the worldline description of relativistic field theories. Indeed fields with spin S can be described in four dimensions by quantizing particle actions with $N = 2S$ extended local supersymmetry on the worldline [20, 21]. While complete in four dimensions, these models describe only conformal invariant fields in other dimensions [63]. They can be consistenly defined not only in Minkowski space, but also in maximally symmetric spacetimes [31] and, more generally, in conformally flat spacetimes [32]. Upon gauge fixing, the worldline actions of the spinning particles moving on such spaces give rise to an interesting class of one-dimensional nonlinear sigma models possessing extended rigid supersymmetries. The goal of this chapter is to analyze the corresponding quantum mechanics.

In particular, we are interested in computing the transition amplitude for the $O(N)$ extended supersymmetric nonlinear sigma models in a euclidean short time expansion since, in applications to higher spin field theory, the transition amplitude can be used to study ultraviolet properties of propagators and one-loop effective actions [29]. We will achieve this result using two different methods. The first method employs canonical quantization, and starting from the operatorial definition of the transition amplitude we compute it perturbatively in the euclidean time β by using the commutation properties of the various operators. The final result yields a perturbative solution to the heat equation (the Schrödinger equation with imaginary time) and identifies a benchmark for equivalent calculations in terms of path integrals. This canonical approach has been already employed in [64] for the $N = 0, 1, 2$ supersymmetric quantum mechanics, that can be defined on any curved manifold (see [37] for a review of the method in the bosonic case). We extend that computation to arbitrary N for the $O(N)$ extended supersymmetric nonlinear sigma models. For $N > 2$ extended supersymmetry may be broken by a generic curved target space, though it can be maintained on locally symmetric spaces. Nevertheless, we use an arbitrary target space geometry since for the present purposes we do not need to gauge supersymmetry, and can view the latter just as an accidental symmetry emerging on particular backgrounds.

The second method we employ for computing the transition amplitude makes use of path integrals. Our reason for considering this approach is that, in typical first-quantized applications, canonical methods are first used to identify the precise operators of interest and path integrals are next used to perform more extensive calculations and manipulations. A classical example is the calculation of chiral anomalies and the proof of index theorems [16, 17, 65]. It is therefore important to be able to reproduce the transition amplitude with path integrals. Path integrals for particles moving on curved spaces can be quite subtle, and their consistent definition needs the specification of a regularization scheme which carry finite counterterms to guarantee scheme independence, see [37] for an extensive treatment and [66] for a short discussion. We will use three different schemes, for completeness and because each one carries its own advantages. The first scheme, time slicing (TS), can be deduced directly by using operatorial methods [67, 68] and can be related to the lattice approximation of the propagation time. A second scheme, mode regularization (MR), is related to a momentum cut-off (or, more properly, energy cut-off in quantum mechanics) and allows the introduction of a regulated functional space to integrate over [18, 19, 69, 70, 71]. Finally, dimensional regularization (DR) is defined as a purely perturbative regularization but has the advantage of carrying only covariant counterterms [72, 73, 27, 28]. For each of the three regularization schemes we find the corresponding counterterms that ensure scheme independence, making them ready for extending the worldline approach to higher spin fields initiated in [30] to curved backgrounds.

Let us now describe the precise form of the (supersymmetric) quantum mechanics we are interested in. We consider a particle moving in a curved space $\mathcal{M}$ of dimensions D and metric $g_{\mu\nu}$. It is described in phase space by bosonic coordinates and momenta (x^μ, p_μ), where $\mu = 1, \dots, D$ is a curved spacetime index, and by N fermionic Majorana variables ψ_i^a, where $i = 1, \dots, N$ is a flavour index and $a = 1, \dots, D$ is a flat spacetime index related to curved indices by the vielbein e_μ^a, defined by $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$ with η_{ab} the flat tangent space metric (which can be taken either minkowskian or euclidean, according to the desired applications). Quantum mechanically the bosonic variables satisfy the usual commutation relations $[x^\mu, p_\nu] = i\delta_\nu^\mu$, and the fermionic ones give rise to a multi Clifford algebra $\{\psi_i^a, \psi_j^b\} = \eta^{ab}\delta_{ij}$. The general hamiltonian operator we wish to consider involves a free covariant kinetic term H_0, a four-Fermi interaction depending on the Riemann tensor and carrying a coupling constant α, and a contribution from an arbitrary scalar potential V which depends only on the spacetime coordinates x^μ (and which in most applications is proportional to the curvature scalar R)

$$H = H_0 + \alpha R_{abcd}\, \psi_i^a \psi_i^b \psi_j^c \psi_j^d + V \tag{2.1}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}\Big(\pi^a - i\omega_b{}^{ba}\Big)\pi_a \\ \pi_a &= e_a^\mu \pi_\mu\ , \quad \pi_\mu = g^{1/4} p_\mu g^{-1/4} - \frac{i}{2}\omega_{\mu ab}\psi_i^a \psi_i^b \end{aligned} \tag{2.2}$$

with $\omega_{\mu ab}$ the spin connection and $\omega_{abc} = e_a^\mu \omega_{\mu bc}$. Note the appearance of the spin connection, required by covariance, and the powers of g next to the momentum operator that ensure hermiticity of the hamiltonian. This hamiltonian is general enough to allow future applications to the first-quantized theory of higher spin fields on curved backgrounds, and we will see how it emerges by studying the extended susy algebra in the following.

The corresponding euclidian classical action in configuration space is given by

$$S = \int_0^\beta d\tau \left[\frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{1}{2}\psi_{ai} D_\tau \psi_i^a + \alpha R_{abcd}\psi_i^a \psi_i^b \psi_j^c \psi_j^d + V\right] \tag{2.3}$$

where $D_\tau \psi_i^a = \partial_\tau \psi_i^a + \dot{x}^\mu \omega_\mu{}^a{}_b \psi_i^b$. It describes the particle propagation for an euclidean time β and will be used in the path integral quantization. For notational simplicity we do not make explicit distinction between quantum operators and classical variables, as it will be clear from the context which one is used.

Having defined the model, we proceed and start describing the $O(N)$ extended superalgebra, that allows to fix a suitable hamiltonian. We move next to the computation of the transition amplitude. In section 2.2 we use operatorial methods. In section 2.3 we consider path integrals in various regularization schemes, namely TS, MR and DR, calculating the corresponding counterterms and finding complete agreement with the expression of the transition amplitude found in the previous section.

2.1 $O(N)$ superalgebra in curved space

In this section we will review the $O(N)$ extended supersymmetry algebra on curved space, in order to see all the restrictions to the allowed backgrounds in defining higher spin fields, as well as to fix a suitable hamiltonian operator to be used in the following.

First, let us recall the $O(N)$ extended supersymmetry algebra in flat space, that was already presented in the previous chapter. The hermitian quantum symmetry charges entering the symmetry algebra are the hamiltonian H, the supersymmetry charges Q_i and the $SO(N)$ generators J_{ij}

$$H = \tfrac{1}{2} p_\mu p^\mu \ , \qquad Q_i = \psi_i^\mu p_\mu \ , \qquad J_{ij} = \frac{i}{2}[\psi_i^\mu, \psi_{\mu j}] \ , \tag{2.4}$$

and satisfy the quantum algebra

$$\begin{aligned} \{Q_i, Q_j\} &= 2\,\delta_{ij} H \ , \\ [J_{ij}, Q_k] &= i\delta_{jk} Q_i - i\delta_{ik} Q_j \ , \\ [J_{ij}, J_{kl}] &= i\delta_{jk} J_{il} - i\delta_{ik} J_{jl} - i\delta_{jl} J_{ik} + i\delta_{il} J_{jk} \ , \end{aligned} \tag{2.5}$$

where other (anti)-commutators vanish.

In curved space, we use the vielbein $e_\mu^a(x)$ to flatten indices on fermions $\psi_i^a = e_\mu^a \psi_i^\mu$ which we now use as independent variables that commute with x and p. The algebra of the $SO(N)$ hermitian currents $J_{ij} = \frac{i}{2}[\psi_i^a, \psi_{aj}]$ is then unchanged. In order to build Lorentz covariant momenta, it is also useful to introduce the $SO(D)$ Lorentz generators

$$M^{ab} = \frac{1}{2}[\psi_i^a, \psi_i^b] = \psi^a \cdot \psi^b - \frac{N}{2}\eta^{ab} \ , \tag{2.6}$$

where a dot denotes contraction of the $SO(N)$ indices, obeying the algebra

$$\begin{aligned} [M^{ab}, M^{cd}] &= \eta^{bc} M^{ad} - \eta^{ac} M^{bd} + \eta^{ad} M^{bc} - \eta^{bd} M^{ac} \\ [M^{ab}, J_{ij}] &= 0 \ . \end{aligned} \tag{2.7}$$

Then one introduces the spin connection $\omega_{\mu ab}$ and Lorentz covariant momenta[1]

$$\pi_\mu = g^{1/4}(p_\mu - \frac{i}{2}\omega_{\mu ab} M^{ab}) g^{-1/4} \ , \qquad \pi_a = e_a^\mu \pi_\mu \tag{2.8}$$

which satisfy

$$\begin{aligned} [\pi_\mu, \pi_\nu] &= -\frac{1}{2} R_{\mu\nu ab} M^{ab} \\ [\pi_a, \pi_b] &= -\frac{1}{2} R_{abcd} M^{cd} + i(\omega_{ab}{}^c - \omega_{ba}{}^c)\pi_c\Big) \ , \end{aligned} \tag{2.9}$$

[1]The powers of the metric determinant g ensure hermiticity of the supercharges: $(Q_i)^\dagger = Q_i$ and hence of the hamiltonian.

where we denoted $\omega_{abc} = e_a^\mu \omega_{\mu bc}$. With covariant momenta at hand, the supercharges are uniquely covariantized as

$$Q_i = \psi_i^a \pi_a \tag{2.10}$$

and we can see that their algebra does not close anymore on the desired operators:

$$\{Q_i, Q_j\} = 2\delta_{ij} H_0 - \frac{1}{2}\psi_i^a \psi_j^b R_{abcd} M^{cd} \tag{2.11}$$

where the minimally covariantized hamiltonian reads[2]

$$H_0 = \frac{1}{2}(\pi^a - i\omega_b{}^{ba})\pi_a \ . \tag{2.12}$$

Hence, we see that on generally curved manifolds the superalgebra is not first class. Moreover, the evolution generated by the minimal H_0 does not conserve the supercharges, and the supersymmetry is broken by the target space geometry:

$$[Q_i, H_0] = -\frac{1}{2} R_{abcd} \psi_i^a M^{cd} \pi^b - \frac{i}{2} \nabla_b R_{ca} \psi_i^a M^{bc} + \frac{1}{2} R_{ab} \psi_i^a \pi^b \ . \tag{2.13}$$

For these reasons a redefinition of the hamiltonian may be useful in showing that the susy charges can be conserved on any space for $N = 1, 2$ and on locally symmetric spaces for arbitrary N. A tentative redefinition is of the form

$$H = H_0 + \alpha R_{abcd} M^{ab} M^{cd} + V \ . \tag{2.14}$$

It is precisely the hamiltonian (2.1) we wish to study in the following, as one can easily see looking at the form of Lorentz generators. We kept as general as possible the scalar potential $V(x)$, that in most applications will be proportional to the scalar curvature R. The non minimal hamiltonian H gives the modified commutator:

$$\begin{aligned}
[Q_i, H] = & - i\alpha \nabla_e R_{abcd} \psi_i^e M^{ab} M^{cd} - \Big(\frac{1}{2} + 4\alpha\Big) R_{abcd} \psi_i^a M^{cd} \pi^b \\
& - \frac{i}{2} \nabla_b R_{ca} \psi_i^a M^{bc} + \Big(\frac{1}{2} + 4\alpha\Big) R_{ab} \psi_i^a \pi^b \\
& - i \nabla_a V \psi_i^a \ .
\end{aligned} \tag{2.15}$$

[2] Reminding that Majorana fermions give rise to a multi Clifford algebra, so that $\psi_i^a = \frac{1}{\sqrt{2}}\gamma_{(i)}^a$, the covariant momentum acts as a Lorentz covariant derivative on a multi spinor: $\pi_\mu = -i\nabla_\mu^{(L)}$, and H_0 acts indeed as the covariant laplacian $H_0 = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \nabla_\nu$. Let us stress that, acting on a multispinor, the covariant derivatives appearing in the last expression are *fully* covariant, and the first ∇_μ does contain a Christoffel piece.

This result shows that for $\alpha = -\frac{1}{8}$ the susy charges are conserved on locally symmetric spaces[3], though the QQ commutator gives rise to a non standard susy algebra with additional conserved charges appearing on the right hand side together with the hamiltonian. These results hold for arbitrary N, but stronger results can be proven for lower values of N. In fact, it is well known that for $N = 1$ and $V = \frac{1}{16}R$ one obtains a standard susy algebra on any curved manifold, in which case the supercharge describes the Dirac operator $\nabla\!\!\!\!/$. Similarly, for $N = 2$ and $V = 0$ the superalgebra closes on any curved manifold and the supercharges describe the exterior derivative d and its adjoint $d^\dagger$.

Let us summarize: we collect here the algebra of the charges J_{ij}, Q_i, H. For H we take at the moment $\alpha = -\frac{1}{8}$ and $V = \frac{\delta_{N,1}}{16}R$ (i.e. a nonvanishing V is present only for $N = 1$). Apart from the obvious $SO(N)$ relations the relevant part of the algebra reads

$$\begin{aligned}
\{Q_i, Q_j\} &= 2\delta_{ij}H - \frac{1}{2}R_{abcd}\big(\psi_i^a\psi_j^b - \frac{1}{2}\delta_{ij}M^{ab}\big)M^{cd} - \frac{1}{8}\delta_{N,1}\delta_{ij}R \\
[Q_i, H] &= \frac{i}{8}\nabla_e R_{abcd}\psi_i^e M^{ab}M^{cd} - \frac{i}{2}\nabla_b R_{ca}\psi_i^a M^{bc} \\
&\quad - \frac{i}{16}\delta_{N,1}\nabla_a R\,\psi_i^a\ .
\end{aligned} \tag{2.16}$$

For $N = 1, 2$ the algebra reduces to

$$\{Q_i, Q_j\} = 2\delta_{ij}H\ , \qquad [Q_i, H] = 0$$

thanks to the properties of the Riemann tensor. For arbitrary N on locally symmetric spaces ($\nabla_a R_{bcde} = 0$) one has

$$\begin{aligned}
\{Q_i, Q_j\} &= 2\delta_{ij}H - \frac{1}{2}R_{abcd}\big(\psi_i^a\psi_j^b - \frac{1}{2}\delta_{ij}M^{ab}\big)M^{cd} - \frac{1}{8}\delta_{N,1}\delta_{ij}R \\
[Q_i, H] &= 0\ .
\end{aligned}$$

However, when considering the transition amplitude we will be more general, and will find the result for the hamiltonian (2.1), that is for arbitrary α and a generic scalar potential $V(x)$.

We stressed several times that the extended superalgebra is still first class on maximally symmetric spaces for generic N, allowing thus to define the spinning particle. Hence, let us briefly recall the form of the relevant part of the algebra on these very interesting backgrounds. The Riemann tensor is given by

$$R_{\mu\nu\lambda\sigma} = \Lambda\left(g_{\mu\lambda}g_{\nu\sigma} - g_{\nu\lambda}g_{\mu\sigma}\right)\ , \tag{2.17}$$

[3]Such manifolds are indeed defined by $\nabla_\mu R_{\nu\lambda\sigma\rho} = 0$.

where Λ is the cosmological constant in a suitable normalization. One can see that, for such a curvature tensor, all the operators appearing on the righthand sides of the (anti)-commutators give rise to J_{ij} $SO(N)$ generators, and one obtains a quadratic deformation proportional to Λ of the flat Lie superalgebra [32]:

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij} H + \frac{\Lambda}{2} \Big(J_{ik} J_{jk} + J_{jk} J_{ik} - \delta_{ij} J_{kl} J_{kl} \Big) , \\ [Q_i, H] &= 0 , \\ \text{where} \quad H &= H_0 + \frac{\Lambda}{4} J_{ij} J_{ij} + \frac{\Lambda D}{8} (D + N - 2) . \end{aligned} \tag{2.18}$$

Actually, as was showed in [32], more general backgrounds seem to be allowed. If one considers conformally flat manifolds, the curvature tensor can be expressed in terms of the metric and the Ricci tensor:

$$\begin{aligned} R_{\mu\nu\lambda\sigma} = &\frac{1}{D-2} \Big(g_{\mu\lambda} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\lambda} - g_{\nu\lambda} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\lambda} \Big) \\ &- \frac{R}{(D-2)(D-1)} \Big(g_{\mu\lambda} g_{\nu\sigma} - g_{\nu\lambda} g_{\mu\sigma} \Big) , \end{aligned} \tag{2.19}$$

and the algebra turns out to be again first class, although becoming a complicate functional deformation of a Lie superalgebra.

2.2 Transition amplitude with operator methods

The aim of this section is the explicit computation of the transition amplitude for an euclidean time β determined by the quantum mechanical hamiltonian H given in (2.1). We will compute the matrix elements of the evolution operator $e^{-\beta H}$ between position eigenstates for the bosonic variables and suitable coherent states for the fermionic ones. We consider a short time expansion, and using systematically the fundamental (anti)-commutation relations of the basic operators x, p and ψ we obtain the final perturbative answer. The calculation is tailored after similar ones performed in [64] for the $N = 0, 1, 2$ supersymmetric quantum mechanics, explained carefully in [37] for the bosonic case, and in [41] for a class of supersymmetric sigma models on Kähler spaces. As typical in semiclassical approximations, the result can be cast as the product of three terms: i) the exponential of the classical action evaluated on the (perturbative) classical solution, ii) a standard leading prefactor depending on the propagation time as $\beta^{-\frac{D}{2}}$, usually arranged in the so-called Van Vleck determinant, iii) a power series in positive powers of the propagation time, which identify the heat-kernel coefficients [29]. In our calculation we keep the approximation up to the first non trivial heat kernel coefficient, i.e. up to order β in the power series, keeping in mind that the bosonic displacement $z^\mu \equiv y^\mu - x^\mu$ can be considered of order $\sqrt{\beta}$, as will be explained later on.

In this section we restrict the calculation to even $N \equiv 2S$, with S an integer, which allows us to introduce complex combinations of the fermionic operators and consider the corresponding coherent states as a complete basis of the associated Hilbert space. This is appropriate for applications to fields of integer spin. The method can be easily extended to the case of odd N, appropriate for applications to fields with half-integer spin: one way is to introduce another set of auxiliary Majorana fermions ("doubling trick") to be able to consider complex (Dirac) fermions and their coherent states. However, we refrain in doing that at this stage, as we employ it in the path integral approach of the following section.

Thus, we consider $N = 2S$ and introduce S Dirac worldline fermions

$$\Psi^a_I = \frac{1}{\sqrt{2}}\,(\psi^a_i + i\psi^a_{i+S})\ , \quad \bar\Psi^{aI} = \frac{1}{\sqrt{2}}\,(\psi^a_i - i\psi^a_{i+S})\,, \quad i, I = 1,\ldots,S \tag{2.20}$$

so that the new variables display the usual creation-annihilation algebra

$$\{\Psi^a_I, \bar\Psi^{bJ}\} = \eta^{ab}\delta^J_I\ .$$

With these complex fields at hand we can readily construct coherent states such that $\Psi^a_I|\eta\rangle = \eta^a_I|\eta\rangle$ and $\langle\bar\lambda|\bar\Psi^{aI} = \langle\bar\lambda|\bar\lambda^{aI}$, with usual normalization $\langle\bar\lambda|\eta\rangle = e^{\bar\lambda\cdot\eta}$ (other properties of fermionic coherent states can be found in appendix A). Denoting by $|y\,\eta\rangle \equiv |y\rangle\otimes|\eta\rangle$ where $|y\rangle$ is the usual position eigenstate, the euclidean heat kernel we want to compute is

$$\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle \tag{2.21}$$

where the quantum hamiltonian (2.1) and the covariant momentum, written in terms of the new complex variables, read

$$\begin{aligned} H &= H_0 + 4\alpha R_{abcd}\,\bar\Psi^a\cdot\Psi^b\,\bar\Psi^c\cdot\Psi^d + V \\ \pi_\mu &= g^{1/4}(p_\mu - i\omega_{\mu ab}\bar\Psi^a\cdot\Psi^b)g^{-1/4} \end{aligned} \tag{2.22}$$

with $\bar\Psi^a\cdot\Psi^b \equiv \bar\Psi^{aI}\Psi^b_I$. In close analogy with the procedure employed in [64] we first focus on the mixed amplitude containing momentum eigenstates on the right hand side

$$\langle x\,\bar\lambda|e^{-\beta H}|p\,\eta\rangle = \sum_{k=0}^{\infty}\frac{(-)^k}{k!}\,\beta^k\langle x\,\bar\lambda|H^k|p\,\eta\rangle\ . \tag{2.23}$$

From the above expansion in β one could naively think to retain only the linear term: $e^{-\beta H} = 1-\beta H+\mathcal{O}(\beta^2)$ to get the answer up to the desired order. This is not the case for nonlinear sigma models, as well-known (see [64]). In fact it is necessary to take into account contributions for every order k, the only truncation being the number of (anti)-commutators to keep track of. In order to evaluate (2.23), we push all p's and Ψ's in each factor H^k to the right and all x's and $\bar{\Psi}$'s to the left, taking into account all (anti)-commutators, and substitute with the corresponding eigenvalues. Since the hamiltonian is quadratic in momenta, the matrix element of H^k is a polinomial of degree $2k$ in p, so that in general one finds structures of the form

$$\langle x\,\bar{\lambda}|H^k|p\,\eta\rangle = \sum_{l=0}^{2k} B_l^k(x,\eta,\bar{\lambda})p^l\langle x|p\rangle\langle\bar{\lambda}|\eta\rangle \ , \qquad (2.24)$$

where p^l stands for the part of such polinomial with precisely l-th powers of p eigenvalues, and the coefficients B_l^k are computed in appendix B. For plane waves we use the normalization

$$\langle x|p\rangle = (2\pi)^{-D/2}g^{-1/4}(x)\,e^{ip\cdot x}$$

so, inserting in (4.41) a momentum completeness relation: $\mathbb{1} = \int d^Dp|p\rangle\,\langle p|$, and rescaling momenta by $p_\mu = \frac{1}{\sqrt{\beta}}\,q_\mu$, we obtain for the transition amplitude

$$\begin{aligned}\langle x\,\bar{\lambda}|e^{-\beta H}|y\,\eta\rangle &= (4\pi^2\beta)^{-D/2}[g(x)g(y)]^{-1/4}\int d^Dq\,e^{\frac{i}{\sqrt{\beta}}q\cdot(x-y)}e^{\bar{\lambda}\cdot\eta} \\ &\times\sum_{k=0}^{\infty}\frac{(-)^k}{k!}\sum_{l=0}^{2k}\beta^{k-l/2}B_l^k(x,\eta,\bar{\lambda})\,q^l\ .\end{aligned} \qquad (2.25)$$

Let us look at this formula: it is well-known that the leading free particle contribution to (4.41) contains the exponential $\exp\{-\frac{(x-y)^2}{2\beta}\}$, that gives the effective order of magnitude $(x-y)\sim\sqrt{\beta}$. Therefore, we see that $q\sim\beta^0$ and it is immediate to realize that, for all k, only the terms B_{2k}^k, B_{2k-1}^k and B_{2k-2}^k will contribute up to order β. Taking for B_l^k the expressions given in appendix B, it is immediate to sum the series in k, and one gets

$$\begin{aligned}&\langle x\,\bar{\lambda}|e^{-\beta H}|y\,\eta\rangle = (4\pi^2\beta)^{-D/2}[g(x)g(y)]^{-1/4}\int d^Dq\,e^{-\frac{1}{2}g^{\mu\nu}q_\mu q_\nu+\frac{i}{\sqrt{\beta}}q\cdot(x-y)}e^{\bar{\lambda}\cdot\eta}\times \\ &\Big\{1+\sqrt{\beta}\Big[\frac{i}{2}g^\mu q_\mu-\frac{i}{4}g^{\mu\nu\lambda}\,q_\lambda\,q_\mu\,q_\nu+ig^{\mu\nu}\omega_\mu q_\nu\Big]+\beta\Big[-\frac{1}{32}\partial_\mu\ln g\partial^\mu\ln g \\ &-\frac{1}{8}\partial_\mu\partial^\mu\ln g-\frac{1}{8}g^\mu\partial_\mu\ln g-\Big(\frac{1}{4}\partial^\mu g^\nu+\frac{1}{8}g^\mu g^\nu+\frac{1}{8}g^\sigma g_\sigma^{\mu\nu}+\frac{1}{8}g_\sigma^{\mu\nu\sigma}\Big)\,q_\mu\,q_\nu\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{12}g^{\mu\nu\sigma\lambda}+\frac{1}{8}g^{\sigma\lambda\mu}g^{\nu}+\frac{1}{12}g^{\kappa\sigma\lambda}g^{\mu\nu}_{\kappa}+\frac{1}{24}g^{\sigma\lambda}_{\kappa}g^{\mu\nu\kappa}\Big)\, q_\sigma\, q_\lambda\, q_\mu\, q_\nu \\
&-\Big(\frac{1}{32}g^{\mu\nu\lambda}g^{\sigma\rho\kappa}\Big)\, q_\lambda\, q_\mu\, q_\nu\, q_\kappa\, q_\sigma\, q_\rho-\frac{1}{2}\partial^\nu(g^{\lambda\sigma}\omega_\lambda)q_\nu\, q_\sigma-\frac{1}{4}g^{\lambda\sigma\mu}\omega_\mu\, q_\lambda q_\sigma \\
&-\frac{1}{2}g^{\mu\nu}\omega_\mu\, g^{\sigma}\, q_\nu q_\sigma+\frac{1}{4}g^{\mu\nu}\omega_\mu\, g^{\alpha\beta\sigma}\, q_\nu q_\sigma q_\alpha q_\beta-\frac{1}{4}g^{\mu\nu}\omega_\mu\partial_\nu\ln g \\
&+\frac{1}{2\sqrt{g}}\partial_\mu[\sqrt{g}g^{\mu\nu}\omega_\nu]+\frac{1}{2}(g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd}-8\alpha\, R_{abcd})(\bar\lambda^a\!\cdot\eta^d\eta^{bc}+\bar\lambda^a\!\cdot\eta^b\bar\lambda^c\!\cdot\eta^d) \\
&-V-\frac{1}{2}g^{\mu\nu}\omega_{\mu ab}g^{\lambda\sigma}\omega_{\lambda cd}(\bar\lambda^a\!\cdot\eta^d\eta^{bc}+\bar\lambda^a\!\cdot\eta^b\bar\lambda^c\!\cdot\eta^d)q_\nu q_\sigma\Big]\Big\}\ , \qquad (2.26)
\end{aligned}
$$

where we remind that all the geometric quantities are evaluated at the final point x, unless otherwise specified, and we used the following compact notations:

$$
\begin{aligned}
&\omega_\mu=\omega_{\mu ab}\bar\lambda^a\cdot\eta^b\ , \quad \partial_\mu...\partial_\lambda g^{\nu\sigma}=g^{\nu\sigma}_{\mu...\lambda}\ , \quad g^{\mu\nu}g^{\lambda\sigma}_{\nu}=g^{\lambda\sigma\mu}\ , \\
&g^{\mu\nu}_{\nu}=g^{\mu}\ , \quad g^{\mu\nu}\partial_\nu g^{\lambda\sigma}_{\sigma}=\partial^\mu g^\lambda\ .
\end{aligned}
$$

Now we are ready to perform the q integration, that reduces to a bunch of gaussian integrals. Defining the coordinate displacement as $z^\mu\equiv y^\mu-x^\mu$, one obtains the transition amplitude, expanded up to order β, in a cumbersome form that hides manifest covariance

$$
\begin{aligned}
&\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle=(2\pi\beta)^{-D/2}\left[\frac{g(x)}{g(y)}\right]^{1/4}e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu}\,e^{\bar\lambda_a\cdot\eta^a}\Big\{1+z^\mu\, g^{-1/4}\,\partial_\mu\, g^{1/4} \\
&-\frac{1}{4\beta}\,\partial_\lambda g_{\mu\nu}\, z^\mu z^\nu z^\lambda+\frac{1}{2}\,z^\mu z^\nu\, g^{-1/4}\,\partial_\mu\partial_\nu g^{1/4}-\frac{1}{4\beta}\,z^\mu g^{-1/4}\partial_\mu g^{1/4}\partial_\lambda g_{\sigma\rho}\, z^\lambda z^\sigma z^\rho \\
&+\frac{1}{2}\left[\frac{1}{4\beta}\,\partial_\lambda g_{\mu\nu}\, z^\mu z^\nu z^\lambda\right]^2-\frac{1}{12\beta}\left[\partial_\lambda\partial_\sigma g_{\mu\nu}-\frac{1}{2}g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\right]z^\mu z^\nu z^\lambda z^\sigma \\
&+\frac{1}{12}\,R_{\mu\nu}\, z^\mu z^\nu+z^\mu\omega_\mu+\frac{1}{2}z^\mu z^\nu\partial_\mu\omega_\nu+\frac{1}{4}z^\mu\omega_\mu\, z^\nu g^{\lambda\sigma}\partial_\nu g_{\lambda\sigma} \\
&+\frac{1}{2}z^\mu z^\nu\omega_{\mu ab}\omega_\nu{}^b{}_c\bar\lambda^a\!\cdot\eta^c+\frac{1}{2}(z^\mu\omega_\mu)^2-z^\mu\omega_\mu\left(\frac{1}{4\beta}z^\nu z^\lambda z^\sigma\partial_\nu g_{\lambda\sigma}\right) \\
&+\beta\Big[-4\alpha\, R_{abcd}\bar\lambda^a\!\cdot\eta^b\bar\lambda^c\!\cdot\eta^d+4\alpha\, R_{ab}\bar\lambda^a\!\cdot\eta^b+\frac{1}{12}R-V\Big]\Big\}\ . \qquad (2.27)
\end{aligned}
$$

This form is quite explicit. It can be simplified and cast in alternative and more suggestive forms. Keeping in mind that the exponent of the on-shell action should appear in the final result, one can factorize (2.27) in the following way[4]

$$\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle = (2\pi\beta)^{-D/2}\exp\Big\{-\frac{1}{\beta}\Big[\frac{1}{2}\,g_{\mu\nu}\,z^\mu z^\nu + \frac{1}{4}\,\partial_\mu g_{\nu\lambda}\,z^\mu z^\nu z^\lambda$$
$$+\frac{1}{12}\Big(\partial_\mu\partial_\nu g_{\lambda\sigma} - \frac{1}{2}\,g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\Big)\,z^\mu z^\nu z^\lambda z^\sigma\Big]\Big\}\exp\Big\{\bar\lambda^{a\cdot}\,\eta_a + z^\mu\omega_{\mu ab}\bar\lambda^{a\cdot}\,\eta^b$$
$$+\frac{1}{2}z^\mu z^\nu\Big(\partial_\mu\omega_{\nu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b\Big)\bar\lambda^{a\cdot}\,\eta^b - 4\alpha\beta\,R_{abcd}\bar\lambda^{a\cdot}\,\eta^b\bar\lambda^{c\cdot}\,\eta^d\Big\}$$
$$\times\Big[1 + \frac{1}{12}R_{\mu\nu}z^\mu z^\nu + \beta\Big(4\alpha R_{ab}\bar\lambda^{a\cdot}\,\eta^b + \frac{1}{12}R - V\Big)\Big]\,. \tag{2.28}$$

The first exponential is the expansion up to order β of the on-shell bosonic action evaluated with the boundary conditions $x^\mu(0) = y^\mu$ and $x^\mu(\beta) = x^\mu$

$$S_x = \int_0^\beta d\tau\,\Big[\frac{1}{2}g_{\mu\nu}\dot x^\mu\dot x^\nu\Big]_{\text{on shell}}$$
$$= \frac{1}{\beta}\Big[\frac{1}{2}\,g_{\mu\nu}\,z^\mu z^\nu + \frac{1}{4}\,\partial_\mu g_{\nu\lambda}\,z^\mu z^\nu z^\lambda + \frac{1}{12}\Big(\partial_\mu\partial_\nu g_{\lambda\sigma} - \frac{1}{2}\,g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\Big)\,z^\mu z^\nu z^\lambda z^\sigma\Big] \tag{2.29}$$

where all functions in the second line are evaluated at the point x. Similarly, one can see that the second exponential in (4.50) is the expansion, up to order β, of the fermionic action evaluated on-shell, with boundary conditions $\Psi^a_I(0) = \eta^a_I$ and $\bar\Psi^{aI}(\beta) = \bar\lambda^{aI}$, and with the usual boundary term added

$$S_\Psi = \Big(\int_0^\beta d\tau\,\Big[\bar\Psi^I_a D_\tau\Psi^a_I + 4\alpha\,R_{abcd}\bar\Psi^{a\cdot}\,\Psi^b\bar\Psi^{c\cdot}\,\Psi^d\Big] - \bar\Psi_a(\beta)\cdot\Psi^a(\beta)\Big)\Big|_{\text{on shell}}$$
$$= -\Big\{\bar\lambda^{a\cdot}\,\eta_a + z^\mu\omega_{\mu ab}\bar\lambda^{a\cdot}\,\eta^b + \frac{1}{2}z^\mu z^\nu\Big(\partial_\mu\omega_{\nu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b\Big)\bar\lambda^{a\cdot}\,\eta^b$$
$$- 4\alpha\beta\,R_{abcd}\bar\lambda^{a\cdot}\,\eta^b\bar\lambda^{c\cdot}\,\eta^d\Big\} \tag{2.30}$$

where the covariant time derivative reads $D_\tau\Psi^a_I = \dot\Psi^a_I + \dot x^\mu\omega_\mu{}^a{}_b\,\Psi^b_I$. Similar calculations of on-shell actions can be found, for instance, in [64]. Once the expansions of the on-shell classical actions have been recognized, the transition amplitude can be cast in the following covariant form

$$\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle = \frac{1}{(2\pi\beta)^{D/2}}\,\exp\{-(S_x + S_\Psi)\}$$
$$\times\Big[1 + \frac{1}{12}R_{\mu\nu}z^\mu z^\nu + \beta\Big(4\alpha R_{ab}\bar\lambda^{a\cdot}\,\eta^b + \frac{1}{12}R - V\Big) + \mathcal{O}(\beta^2)\Big] \tag{2.31}$$

[4] The factors $[g(x)/g(y)]^{1/4}$ of the metric determinants cancel against their Taylor expansion around x, that can be factored out from (2.27).

with all functions evaluated at point x. This is our final result for the transition amplitude.

For comparison purposes, it may be useful to present the result after tracing over the fermionic Hilbert space

$$\langle x|\mathrm{Tr}_\Psi(e^{-\beta H})|y\rangle = \frac{2^{\frac{ND}{2}}}{(2\pi\beta)^{D/2}}\,\exp\{-S_x\}\,\Big[1-\frac{N}{16}z^\mu z^\nu\omega_{\mu ab}\omega_\nu{}^{ab}\Big] \\ \times\Big[1+\frac{1}{12}R_{\mu\nu}z^\mu z^\nu+\beta\Big(\frac{1+6\alpha N}{12}R-V\Big)+\mathcal{O}(\beta^2)\Big]\,. \tag{2.32}$$

Evaluated at coinciding points ($z^\mu=0$) it reads as

$$\langle x|\mathrm{Tr}_\Psi(e^{-\beta H})|x\rangle = \frac{2^{\frac{ND}{2}}}{(2\pi\beta)^{D/2}}\Big[1+\beta\Big(\frac{1+6\alpha N}{12}R-V\Big)+\mathcal{O}(\beta^2)\Big] \tag{2.33}$$

which shows the first heat kernel coefficient at coinciding points for our model. As we shall see the last two formulas remain valid also for odd N.

2.3 Transition amplitude from path integrals

In the present section we compute the transition amplitude by making use of path integral methods. To define the path integrals we fix a suitable regularization scheme, and compute the transition amplitude. Then, by comparing with the previous section or by imposing the Schrödinger equation, we identify the corresponding counterterms that ensure scheme independence.

Unlike the previous section here we treat together both the cases with even and odd numbers of Majorana variables. In order to do so, we found it convenient to use the so-called "doubling trick" that consists in doubling the number of fermionic variables by adding "spectator" Majorana fermions $\psi_i'^a$ that satisfy free anticommutation relations. These new fermions are spectators in that they do not enter the interactions. With the help of these new variables one can consider Dirac fermions

$$\Psi_i^a=\frac{1}{\sqrt{2}}\Big(\psi_i^a+i\psi_i'^a\Big)\,,\quad \bar\Psi_i^a=\frac{1}{\sqrt{2}}\Big(\psi_i^a-i\psi_i'^a\Big) \tag{2.34}$$

that satisfy

$$\{\Psi_i^a,\bar\Psi_j^b\}=\eta^{ab}\delta_{ij} \tag{2.35}$$

giving rise to a set of fermionic harmonic oscillators, whose Hilbert space can be represented in terms of bra and ket coherent states

$$\Psi_i^a|\eta\rangle=\eta_i^a|\eta\rangle\,,\quad \langle\bar\lambda|\bar\Psi_i^a=\langle\bar\lambda|\bar\lambda_i^a \tag{2.36}$$

and whose properties are briefly recalled in appendix A. The wave function of the system $\Phi(x,\bar\lambda) = \langle x\,\bar\lambda|\Phi\rangle$, with $\langle x\,\bar\lambda| \equiv \langle x| \otimes \langle\bar\lambda|$, evolves in time as

$$\begin{aligned}\Phi(x,\bar\lambda;\beta) &= \langle x\,\bar\lambda|e^{-\beta H}|\Phi\rangle \\ &= \int d^Dy\sqrt{g(y)}\int d\bar\zeta d\eta\; e^{-\bar\zeta\cdot\eta}\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle\Phi(y,\bar\zeta;0)\end{aligned} \tag{2.37}$$

where we have used the spectral decomposition of unity for the bosonic and fermionic sectors

$$\mathbb{1}_b = \int d^Dy\sqrt{g(y)}|y\rangle\langle y| \quad \mathbb{1}_f = \int d\bar\zeta d\eta\; e^{-\bar\zeta\cdot\eta}|\eta\rangle\langle\bar\zeta|\;. \tag{2.38}$$

The evolution is generated by the basic transition amplitude that can be written in terms of a path integral as

$$\begin{aligned}&\langle x\,\bar\lambda|e^{-\beta H}|y\,\eta\rangle = \langle x\,\bar\lambda;\beta|y\,\eta;0\rangle \\ &= \int_{x(-1)=y}^{x(0)=x} DxDaDbDc\int_{\Psi(-1)=\sqrt{\beta}\eta}^{\bar\Psi(0)=\sqrt{\beta}\bar\lambda} D\bar\Psi D\Psi\; e^{-S[x,a,b,c,\Psi,\bar\Psi]}\end{aligned} \tag{2.39}$$

where

$$\begin{aligned}S[x,a,b,c,\Psi,\bar\Psi] &= \frac{1}{2\beta}\int_{-1}^{0} d\tau\, g_{\mu\nu}(x(\tau))(\dot x^\mu\dot x^\nu + a^\mu a^\nu + b^\mu c^\nu) \\ &+\frac{1}{\beta}\int_{-1}^{0} d\tau\,\bar\Psi_{ai}\dot\Psi^a_i - \frac{1}{\beta}\bar\Psi_{ai}(0)\Psi^a_i(0) + \frac{1}{2\beta}\int_{-1}^{0} d\tau\,\dot x^\mu\omega_{\mu ab}(x(\tau))\psi^a_i\psi^b_i \\ &+\frac{\alpha}{\beta}\int_{-1}^{0} d\tau\, R_{abcd}(x(\tau))\psi^a_i\psi^b_i\psi^c_j\psi^d_j + \beta\int_{-1}^{0} d\tau\, V(x(\tau))\;.\end{aligned} \tag{2.40}$$

Here, and in the following, we use a shifted and rescaled euclidean time $\tau \in [-1,0]$ to make comparison with the literature easier. We use bosonic (a^μ) and fermionic (b^μ, c^μ) ghosts to reproduce the reparametrization invariant measure $\mathcal{D}x = \prod_\tau \sqrt{g(x(\tau))}d^Dx(\tau)$. We also rescaled fermionic "coordinates" and ghosts so that all propagators are of order β, and added a fermionic boundary term to be able to set boundary conditions at initial time for Ψ and at final time for $\bar\Psi$. Finally, let us note that the arbitrary potential V will eventually be modified by the addition of a counterterm V_{CT} related to the regularization chosen. Apart form these modifications, this is the same action given in eq. (2.3).

We are interested in the short-time perturbative expansion of the transition amplitude. Thus we expand all geometric expressions about the final point x^μ, and split the action into a quadratic part plus interactions,

$S = S_2 + S_{int}$, with

$$S_2 = \frac{1}{2\beta} g_{\mu\nu} \int_{-1}^{0} d\tau \, (\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + \frac{1}{\beta} \int_{-1}^{0} d\tau \, \bar{\Psi}_{ai} \dot{\Psi}_i^a - \frac{1}{\beta} \bar{\Psi}_{ai}(0) \Psi_i^a(0) \; . \tag{2.41}$$

Here and in the following, geometric quantities with no explicit functional dependence are meant to be evaluated at the final point $x(0) = x$, such as $g_{\mu\nu} = g_{\mu\nu}(x)$. We can thus split the fields into backgrounds, satisfying the free equations of motion with corresponding boundary conditions, and quantum fluctuations; namely

$$x^\mu(\tau) = \tilde{x}^\mu(\tau) + q^\mu(\tau) \, , \qquad \tilde{x}^\mu(\tau) = x^\mu - z^\mu \tau \, , \quad q^\mu(0) = q^\mu(-1) = 0 \tag{2.42}$$

$$\Psi_i^a(\tau) = \tilde{\Psi}_i^a(\tau) + Q_i^a(\tau) \, , \quad \tilde{\Psi}_i^a(\tau) = \sqrt{\beta} \eta_i^a \qquad Q_i^a(-1) = 0 \tag{2.43}$$

$$\bar{\Psi}_i^a(\tau) = \tilde{\bar{\Psi}}_i^a(\tau) + \bar{Q}_i^a(\tau) \, , \quad \tilde{\bar{\Psi}}_i^a(\tau) = \sqrt{\beta} \bar{\lambda}_i^a \, , \qquad \bar{Q}_i^a(0) = 0 \tag{2.44}$$

where $z^\mu \equiv y^\mu - x^\nu$. The free on-shell classical action reads (henceforth we use a dot to indicate the contraction of whatever type of free flat indices)

$$\tilde{S}_2 = \frac{1}{2\beta} g_{\mu\nu} z^\mu z^\nu - \bar{\lambda} \cdot \eta \tag{2.45}$$

and the free propagators for the quantum fields are

$$\langle q^\mu(\tau) q^\nu(\sigma) \rangle \; = \; -\beta g^{\mu\nu} \Delta(\tau, \sigma) = \quad \text{————} \tag{2.46}$$

$$\langle a^\mu(\tau) a^\nu(\sigma) \rangle \; = \; \beta g^{\mu\nu} \Delta_{gh}(\tau, \sigma) = \quad \text{--------} \tag{2.47}$$

$$\langle b^\mu(\tau) c^\nu(\sigma) \rangle \; = \; -2\beta g^{\mu\nu} \Delta_{gh}(\tau, \sigma) = \quad \text{--------} \tag{2.48}$$

$$\langle \bar{Q}_i^a(\tau) Q_j^b(\sigma) \rangle \; = \; \beta \eta^{ab} \delta_{ij} G(\tau, \sigma) \tag{2.49}$$

where the right-hand sides are given, at the unregulated level, by the following distributions

$$\begin{aligned} \Delta(\tau, \sigma) &= \tau(\sigma + 1)\theta(\tau - \sigma) + \sigma(\tau + 1)\theta(\sigma - \tau) \\ \Delta_{gh}(\tau, \sigma) &= \delta(\tau - \sigma) \\ G(\tau, \sigma) &= -\theta(\sigma - \tau) \end{aligned} \tag{2.50}$$

which obey the Green equations ${}^{\bullet\bullet}\Delta(\tau, \sigma) = \Delta_{gh}(\tau, \sigma) = \delta(\tau - \sigma)$ and ${}^{\bullet}G(\tau, \sigma) = \delta(\tau - \sigma)$, with boundary conditions $\Delta(0, \sigma) = \Delta(\tau, 0) = \Delta(-1, \sigma) = \Delta(\tau, -1) = 0$ and $G(0, \sigma) = G(\tau, -1) = 0$. Dots to the left (right) indicate derivatives with respect to the first (second) variable. These propagators have to be regulated in order to deal with divergences and ambiguities present in some diagrams. However, for each regularization scheme, one is able to cast all expressions in a way that can be unambiguously computed by directly using the expressions (2.50).

Since fermions enter the interactions only through the combination $\psi_i^a = \frac{1}{\sqrt{2}}(\Psi_i^a + \bar{\Psi}_i^a)$ it is convenient to write backgrounds and propagators for these fields as well. We split them as $\psi_i^a(\tau) = \tilde{\psi}_i^a + \chi_i^a(\tau)$ with

$$\tilde{\psi}_i^a = \sqrt{\frac{\beta}{2}}(\eta_i^a + \bar{\lambda}_i^a) \tag{2.51}$$

and

$$\langle \chi_i^a(\tau)\chi_j^b(\sigma)\rangle \;=\; \beta\eta^{ab}\delta_{ij}\Delta_F(\tau,\sigma) = \quad \cdots\cdots\cdots\cdots \tag{2.52}$$

satisfying ${}^\bullet\Delta_F(\tau,\sigma) = -\Delta_F^\bullet(\tau,\sigma) = \delta(\tau-\sigma)$ and given at the unregulated level by

$$\Delta_F(\tau,\sigma) = \frac{1}{2}(\theta(\tau-\sigma) - \theta(\sigma-\tau)) = \frac{1}{2}\epsilon(\tau-\sigma)\;. \tag{2.53}$$

We only wrote propagators for unprimed fermionic fields since only they enter the interactions. The primed fields instead only contribute to the one-loop semiclassical factor that normalizes the path integral and drop out of the computation immediately.

The interaction vertices that enter the perturbative expansion can be obtained by Taylor expanding the action (2.40) about the final point $x(0) = x$ and read $S_{int} = \sum_{n=3}^{\infty} S_n$, with

$$\begin{aligned}
S_3 &= \frac{1}{2\beta}\omega_{\mu ab}\int_{-1}^{0} d\tau(\dot{q}^\mu - z^\mu)(\tilde{\psi}^a + \chi^a)\cdot(\tilde{\psi}^b + \chi^b)\\
&+ \frac{1}{2\beta}\partial_\lambda g_{\mu\nu}\int_{-1}^{0} d\tau(q^\lambda - z^\lambda\tau)\Big((\dot{q}^\mu - z^\mu)(\dot{q}^\nu - z^\nu) + a^\mu a^\nu + b^\mu c^\nu\Big)\\
S_4 &= \beta V + \frac{1}{2\beta}\partial_\lambda\omega_{\mu ab}\int_{-1}^{0} d\tau(q^\lambda - z^\lambda\tau)(\dot{q}^\mu - z^\mu)(\tilde{\psi}^a + \chi^a)\cdot(\tilde{\psi}^b + \chi^b)\\
&+ \frac{\alpha}{\beta}R_{abcd}\int_{-1}^{0} d\tau(\tilde{\psi}^a + \chi^a)\cdot(\tilde{\psi}^b + \chi^b)(\tilde{\psi}^c + \chi^c)\cdot(\tilde{\psi}^d + \chi^d)\\
&+ \frac{1}{4\beta}\partial_\lambda\partial_\sigma g_{\mu\nu}\int_{-1}^{0} d\tau(q^\lambda - z^\lambda\tau)(q^\sigma - z^\sigma\tau)\Big((\dot{q}^\mu - z^\mu)(\dot{q}^\nu - z^\nu) + a^\mu a^\nu + b^\mu c^\nu\Big)
\end{aligned} \tag{2.54}$$

Higher order terms are not reported because we compute the transition amplitude to order β, for which only the previous two terms are needed; all fields, classical and quantum, count as $\beta^{1/2}$.

The transition amplitude can now be computed perturbatively using Wick-contractions of the quantum fields

$$\begin{aligned}\langle x,\bar\lambda;\beta|y,\eta;0\rangle &= A\ e^{-\tilde S_2}\Big\langle e^{-S_{int}}\Big\rangle \\ &= A\ e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu+\bar\lambda\cdot\eta}\exp\Big[-\langle S_3\rangle-\langle S_4\rangle+\frac{1}{2}\langle S_3^2\rangle_c\Big]\end{aligned} \tag{2.55}$$

with the suffix c indicating connected diagrams only and with the prefactor A soon to be commented upon. The various Wick-contractions produce

$$\begin{aligned}&\exp\Big[-\langle S_3\rangle-\langle S_4\rangle+\frac{1}{2}\langle S_3^2\rangle_c\Big] = \exp\Bigg[-\frac{1}{4\beta}\partial_\lambda g_{\mu\nu}z^\lambda z^\mu z^\nu+\frac{1}{2\beta}\omega_{\lambda ab}z^\lambda\tilde\psi^a\cdot\tilde\psi^b \\ &-z^\lambda\left(\frac{1}{2}\partial_\lambda g\,\mathbf{I}_1+g_\lambda\,\mathbf{I}_2\right)-\frac{1}{12\beta}\partial_\lambda\partial_\sigma g_{\mu\nu}z^\lambda z^\sigma z^\mu z^\nu-\frac{\beta}{4}\partial^2 g\,\mathbf{I}_3-\frac{\beta}{2}\partial^\lambda\partial^\sigma g_{\lambda\sigma}\,\mathbf{I}_4 \\ &+\frac{1}{4}\partial_\lambda\partial_\sigma g z^\lambda z^\sigma\,\mathbf{I}_5+\frac{1}{4}\partial^2 g_{\lambda\sigma}z^\lambda z^\sigma\,\mathbf{I}_6+\partial_\lambda g_\sigma z^\lambda z^\sigma\,\mathbf{I}_7+\frac{1}{4\beta}\partial_\lambda\omega_{\sigma ab}\tilde\psi^a\cdot\tilde\psi^b z^\lambda z^\sigma \\ &+\frac{1}{2}\partial^\lambda\omega_{\lambda ab}\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_2-\frac{\alpha}{\beta}R_{abcd}\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d-\beta V \\ &-\frac{\beta}{4}(\partial_\lambda g_{\mu\nu})^2\,\mathbf{I}_9-\frac{\beta}{2}\partial_\lambda g_{\mu\nu}\partial_\mu g_{\lambda\nu}\,\mathbf{I}_{10}-\frac{\beta}{8}(\partial_\lambda g)^2\,\mathbf{I}_{11}-\frac{\beta}{2}g_\lambda\partial_\lambda g\,\mathbf{I}_{12}-\frac{\beta}{2}g_\lambda^2\,\mathbf{I}_{13} \\ &+\frac{1}{2}\partial_\lambda g_{\mu\nu}\partial_\lambda g_{\mu'\nu}z^\mu z^{\mu'}\,\mathbf{I}_{14}+\frac{1}{2}\partial_\lambda g_{\mu\nu}\partial_\nu g_{\mu'\lambda}z^\mu z^{\mu'}\,\mathbf{I}_{15}+\frac{1}{4}\partial_\lambda g_{\mu\nu}\partial_{\lambda'}g_{\mu\nu}z^\lambda z^{\lambda'}\,\mathbf{I}_{16} \\ &+\partial_\lambda g_{\mu\nu}\partial_\mu g_{\mu'\nu}z^\lambda z^{\mu'}\,\mathbf{I}_{17}+\frac{1}{4}\partial_\lambda g_{\mu\nu}\partial_\lambda g z^\mu z^\nu\,\mathbf{I}_{18}+\frac{1}{2}\partial_\lambda g_{\mu\nu}g_\lambda z^\mu z^\nu\,\mathbf{I}_{19} \\ &+\frac{1}{2}\partial_\lambda g_{\mu\nu}\partial_\mu g z^\lambda z^\nu\,\mathbf{I}_{20}+\partial_\lambda g_{\mu\nu}g_\mu z^\lambda z^\nu\,\mathbf{I}_{21}-\frac{1}{8\beta}\partial_\lambda g_{\mu\nu}\partial_\lambda g_{\mu'\nu'}z^\mu z^\nu z^{\mu'}z^{\nu'}\,\mathbf{I}_{22} \\ &-\frac{1}{2\beta}\partial_\lambda g_{\mu\nu}\partial_{\lambda'}g_{\lambda\nu'}z^\mu z^\nu z^{\lambda'}z^{\nu'}\,\mathbf{I}_{23}-\frac{1}{2\beta}\partial_\lambda g_{\mu\nu}\partial_{\lambda'}g_{\mu\nu'}z^\lambda z^\nu z^{\lambda'}z^{\nu'}\,\mathbf{I}_{24} \\ &+\frac{\beta N}{4}\omega_{\lambda ab}^2\,\mathbf{I}_{25}-\frac{N}{4}\omega_{\lambda ab}\,\omega_\sigma{}^{ab}z^\lambda z^\sigma\,\mathbf{I}_{26}+\frac{1}{2}\omega_{\lambda ac}\,\omega_{\lambda b}{}^c\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_{27} \\ &-\frac{1}{2\beta}\omega_{\lambda ac}\,\omega_{\sigma b}{}^c z^\lambda z^\sigma\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_{28}-\frac{1}{8\beta}\omega_{\lambda ab}\,\omega_{\lambda cd}\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d\,\mathbf{I}_{29} \\ &+\frac{1}{4}\omega_{\lambda ab}\partial_\lambda g\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_{30}+\frac{1}{2}\omega_{\lambda ab}g_\lambda\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_{31}-\frac{1}{4\beta}\omega_{\lambda ab}\partial_\lambda g_{\mu\nu}z^\mu z^\nu\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_{32} \\ &-\frac{1}{2\beta}\omega_{\mu ab}\partial_\lambda g_{\mu\nu}z^\lambda z^\nu\tilde\psi^a\cdot\tilde\psi^b\,\mathbf{I}_{33}\Bigg]\end{aligned} \tag{2.56}$$

where we made use of several shortcut notations, including $g_\mu = g^{\lambda\nu}\partial_\lambda g_{\mu\nu}$, $\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$, $\partial_\lambda\partial_\sigma g = g^{\mu\nu}\partial_\lambda\partial_\sigma g_{\mu\nu}$ and $\partial^2 = g^{\lambda\sigma}\partial_\lambda\partial_\sigma$. The integrals $\mathbf{I}_k$ are reported in appendix C along with the pictorial representation of the diagrams they belong to (an integral named $\mathbf{I}_8$ is absent, but we kept the same notation as in [37], where such an integral arose from the coupling to external gauge fields, to allow easy comparison). We compute them in the following subsections, using the different regularization schemes discussed in the introduction. The purely bosonic contributions ($k \leq 24$) are well-known from many previous computations, see [37] for example; the remaining ones have been computed, for $N \leq 2$, in the time-slicing regularization technique [68], in dimensional regularization [27, 28], and in mode regularization [71]. However, in the last two schemes fermions were traced out to compute directly heat kernel coefficients and trace anomalies. In the present case we are interested in the full transition amplitude and wish to keep N generic, so that we need to reconsider all such integrals with fermionic contributions. Finally, the prefactor A can be fixed by requiring that, in the limit $\beta \to 0$, the transition amplitude reduces to $\langle x, \bar\lambda; \beta | y, \eta; 0\rangle \to \delta(x-y)\, e^{\bar\lambda\cdot\eta}$, which in MR and DR gives

$$A = \frac{1}{(2\pi\beta)^{D/2}} \,. \tag{2.57}$$

In TS such prefactor can be deduced as well starting from operatorial methods, and reads

$$A = \left[\frac{g(x)}{g(y)}\right]^{1/4} \frac{1}{(2\pi\beta)^{D/2}} \,. \tag{2.58}$$

We are now ready to consider the various regularization schemes.

2.3.1 Time-slicing regularization

Time-sliced path integrals in curved space were extensively discussed in [67, 68]. In essence time-slicing regularization consists in studying the discretized version of the path integral, as derived form the operatorial picture by using Weyl ordering and midpoint rule, and recognizing the action with the correct counterterms and the rules how to compute Feynman graphs that are to be used directly in the continuum limit. These rules state that the Dirac delta functions should always be implemented as if they were Kronecker delta's, using the value $\theta(0) = \frac{1}{2}$ for the discontinuous step function. We do not need to repeat here many computations, namely the bosonic ones, though one can easily compute them using the explicit expressions collected in appendix C, or extract them directly from [37]. It is enough to focus on the graphs containing fermionic lines, i.e. $\mathbf{I}_k$ with $k \geq 25$.

Thus, let us start with $\mathbf{I}_{25}$, that is the only diagram that depends heavily on the regularization chosen. Following the TS prescriptions, we have

$$\begin{aligned} \mathbf{I}_{25} &= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, {}^\bullet\Delta^\bullet\, \Delta_F^2 = \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\Big(1-\delta(\tau-\sigma)\Big)\Big(\frac{1}{2}\epsilon(\tau-\sigma)\Big)^2 \\ &= \frac{1}{4}\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\, \epsilon^2(\tau-\sigma) = \frac{1}{4}\int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma = \frac{1}{4} \end{aligned} \tag{2.59}$$

since ${}^\bullet\Delta^\bullet = 1-\delta(\tau-\sigma)$ and $\epsilon(0)=0$, as follows form $\theta(0)=\frac{1}{2}$ and eq. (2.53). The regular $\mathbf{I}_{26}$ does not need any prescription and corresponds to the last line above, giving $\mathbf{I}_{26}=\frac{1}{4}$. Similarly, one finds that the integrals $\mathbf{I}_k$ with $k\geq 27$ yield a vanishing result (those with $k\geq 28$ actually contain only bosonic propagators, but depend on the external fermionic backgrounds).

The transition amplitude, in an obvious notation, finally reads

$$\begin{aligned} \langle x\,\bar\lambda;\beta|y\,\eta;0\rangle &= \frac{e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu+\bar\lambda\cdot\eta}}{(2\pi\beta)^{D/2}}\exp\Big[\langle\text{bosonic}\rangle_{TS}+\frac{1}{2\beta}\omega_{\lambda ab}z^\lambda\tilde\psi^a\cdot\tilde\psi^b \\ &+\frac{1}{4\beta}\partial_\lambda\omega_{\sigma ab}z^\lambda z^\sigma\tilde\psi^a\cdot\tilde\psi^b-\frac{\alpha}{\beta}R_{abcd}\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d-\frac{N}{16}\omega_{\lambda ab}\omega_{\sigma ab}z^\lambda z^\sigma \\ &+\frac{\beta N}{16}\omega^2_{\lambda ab}\Big] \end{aligned} \tag{2.60}$$

where $\langle\text{bosonic}\rangle_{TS}$ contains the purely bosonic contributions contained in (2.56), including the metric factors appearing in (2.58). It can be extracted from the literature, or easily recomputed with the set-up described above, and reads

$$\begin{aligned} \langle\text{bosonic}\rangle_{TS} = &-\frac{1}{4\beta}\,\partial_\mu g_{\nu\lambda}\,z^\mu z^\nu z^\lambda-\frac{1}{12\beta}\Big(\partial_\mu\partial_\nu g_{\lambda\sigma}-\frac{1}{2}\,g_{\rho\tau}\,\Gamma^\rho_{\mu\nu}\,\Gamma^\tau_{\lambda\sigma}\Big)\,z^\mu z^\nu z^\lambda z^\sigma \\ &+\frac{1}{12}R_{\mu\nu}z^\mu z^\nu+\beta\Big(\frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}-\frac{1}{24}R-V\Big)\,. \end{aligned} \tag{2.61}$$

This transition amplitude is the one computed with a TS regularization of the Feynman diagrams, and in general it differs form those computed with other regularizations if no counterterms are introduced. In particular, eq. (2.60) satisfies a Schrödinger equation with a non-covariant hamiltonian that differs from the one given in eq. (2.1). One expects different regularizations to be related by finite local counterterms, so we need to identify the correct counterterm that make sure that we are discussing the same quantum theory as the one associated to the hamiltonian H in eq. (2.1). To achieve this we can either compare with the transition amplitude obtained by operatorial methods, or compute directly the hamiltonian associated with the transition amplitude above. We shall follow both methods as a check of our calculations.

As it stands the transition amplitude calculated above cannot be compared directly with the result from canonical methods in eq. (2.31): the latter is valid for even N and is written in terms of fermionic coherent states that correspond to the physical fermions only, while the present result, valid for any integer N, contains also the Majorana spectator fields introduced in the fermion doubling trick. To overcome these differences, we can take a trace over the fermionic Hilbert space and eliminate the contribution of the decoupled spectator variables by subtracting their degrees of freedom. Thus, let us compute the trace in the fermionic sector of eq. (2.60) by integrating over the Grassmann variables with measure $\int d\eta d\bar{\lambda}\, e^{\bar{\lambda}\cdot\eta}$, see appendix A. The final result can be obtained, for instance, using the standard Wick-contractions associated to the gaussian integral $\int d\eta d\bar{\lambda}\, e^{2\bar{\lambda}\cdot\eta}$ (note the factor 2 in the exponent arising form the trace measure and the leading part of (2.60)), which gives the following propagators

$$\langle \bar{\lambda}_i^a \eta_j^b \rangle = \frac{1}{2}\eta^{ab}\delta_{ij} \qquad \rightarrow \qquad \langle \tilde{\psi}_i^a \tilde{\psi}_j^b \rangle = 0 \tag{2.62}$$

where we used (2.51). Note that the normalization factor $\int d\eta d\bar{\lambda}\, e^{2\bar{\lambda}\cdot\eta} = 2^{ND}$ has to be renormalized to $2^{\frac{ND}{2}}$ to undo the fermion doubling. Taking all this into account, and setting $z \equiv y - x = 0$ for simplicity, we obtain

$$\begin{aligned} & 2^{-\frac{ND}{2}} \int d\eta d\bar{\lambda}\; e^{\bar{\lambda}\cdot\eta} \langle x\, \bar{\lambda}; \beta | x\, \eta; 0\rangle = \\ & \frac{2^{\frac{ND}{2}}}{(2\pi\beta)^{D/2}} \Big[1 + \beta\Big(\frac{1}{8} g^{\mu\nu}\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \frac{1}{24}R - V\Big) + \frac{\beta N}{16}\omega^2_{\lambda ab} + \mathcal{O}(\beta^2)\Big] \end{aligned} \tag{2.63}$$

where the first term in round brackets is due to the purely bosonic contributions. Comparing with (2.33) one recognizes the counterterm that needs to be added to the potential V in the path integral action to achieve equality

$$V_{TS}^{(N)} \;=\; -\Big(\frac{1}{8} + \frac{\alpha N}{2}\Big) R + \frac{1}{8} g^{\mu\nu}\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \frac{N}{16}\omega_{\mu ab}\omega^{\mu ab}\;. \tag{2.64}$$

Power counting shows that no higher order corrections to the counterterm are to be expected.

Alternatively, one may compute the hamiltonian appearing in the Schrödinger equation satisfied by the amplitude (2.60). To do this we insert the latter into (2.37), Taylor-expand to order β all terms in the right hand side about the final point and identify the Schrödinger equation associated to it. Comparing with the Schrödinger equation due to the hamiltonian (2.1) one deduces eventual counterterms. Let us describe this computation. We perform the Gaussian integrals over $d^D y = d^D z$ and the integrals over the fermionic coherent states using the properties summarized in appendix A. The purely bosonic contributions of the diagrammatic expansion yield the standard TS result

$$\Phi(x,\bar\lambda;\beta) \;=\; \Big(1-\beta\partial_t-\beta H_B+O(\beta^2)\Big)\Phi(x,\bar\lambda;\beta) \tag{2.65}$$

with

$$H_B=-\frac{1}{2\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu+V+\frac{1}{8}\Big(R-g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\Big)\,. \tag{2.66}$$

It requires the addition of the familiar counterterm$V^{(0)}_{TS}=-\frac{1}{8}(R-g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma})$ in order to get the covariant $H=-\frac{1}{2\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu+V$. We need to dress this result with the fermionic contributions. One way to perform the integrals over the fermionic coherent states is to use the procedure briefly explained at the end of section A. By defining the full hamiltonian H_F as

$$H_F=H_B+\Delta \tag{2.67}$$

and using (2.65), we obtain

$$\begin{aligned}\Delta=&-\frac{1}{4}\partial^\lambda\omega_{\lambda ab}M^{ab}+\frac{1}{4}g^{\lambda\sigma}\Gamma^\mu_{\lambda\sigma}\omega_{\mu ab}M^{ab}-\frac{1}{2}\omega^\mu{}_{ab}M^{ab}\partial_\mu\\&-\frac{1}{8}\omega^\lambda{}_{ab}\omega_{\lambda cd}M^{ab}M^{cd}+\alpha R_{abcd}M^{ab}M^{cd}+\frac{\alpha N}{2}R-\frac{N}{16}(\omega_{\lambda ab})^2\end{aligned} \tag{2.68}$$

with $M^{ab}=\frac{1}{2}\Big(\bar\lambda^a\cdot\bar\lambda^b+\bar\lambda^a\cdot\frac{\partial}{\partial\bar\lambda_b}-\bar\lambda^b\cdot\frac{\partial}{\partial\bar\lambda_a}+\frac{\partial}{\partial\bar\lambda_a}\cdot\frac{\partial}{\partial\bar\lambda_b}\Big)$ being Lorentz generators. Finally we observe that the noncovariant terms in the line above are those necessary to render the scalar laplacian completely covariant, namely

$$\begin{aligned}H_F\;=\;&H_B+\Delta=-\frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu+\alpha R_{abcd}M^{ab}M^{cd}+V\\&+\Big(\frac{1}{8}+\frac{\alpha N}{2}\Big)R-\frac{1}{8}g^{\lambda\sigma}\Gamma^\tau_{\lambda\rho}\Gamma^\rho_{\sigma\tau}-\frac{N}{16}(\omega_{\lambda ab})^2\end{aligned} \tag{2.69}$$

so that in order to have[5]

$$H\;=\;-\frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu+\alpha R_{abcd}\psi^a\cdot\psi^b\psi^c\cdot\psi^d+V \tag{2.70}$$

one needs to add to the path integral the same counterterm found before in eq. (2.64). Thus we found complete agreement.

[5]Here we use $\langle\bar\lambda|\psi^{ai}|\Phi\rangle=\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\bar\lambda^i_a}+\bar\lambda^{ai}\Big)\Phi(\bar\lambda)$, so that the Lorentz generators can be written as $M^{ab}=\frac{1}{2}\left(\psi^a\cdot\psi^b-\psi^b\cdot\psi^a\right)$.

The above expression for the counterterm matches all the previously known cases [67, 68]: the purely bosonic case ($N = 0$) is obviously reproduced. For $N = 1$ supersymmetry fixes $\alpha = -1/4$ and $V = 0$ so that $V_{TS}^{(1)} = \frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \frac{1}{16}(\omega_{\mu ab})^2$ comes out correctly. Note that in this case the relation $R_{abcd}\psi^a\psi^b\psi^c\psi^d = -\frac{1}{2}R$ allows to use the notation previously employed: $\alpha = -\frac{1}{8}$ and $V = \frac{1}{16}R$ as well to identify the same supersymmetric hamitonian. For $N = 2$, supersymmetry requires $\alpha = -\frac{1}{8}$ and $V = 0$, so that again $V_{TS}^{(2)} = \frac{1}{8}g^{\mu\nu}\Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \frac{1}{8}(\omega_{\mu ab})^2$ is correctly reproduced.

2.3.2 Mode regularization

In this section we approach the previous computation using mode regularization (MR). It can be considered as the worldline equivalent of a cut-off regularization in momentum space of usual quantum field theories. Mode regularization starts by expanding all fields in Fourier sums, thus identifying a suitable functional space to integrate over in the path integral. The regularization is achieved by truncating the Fourier sums at a fixed mode M, so that all distributions appearing in Feynman graphs become well-behaved functions. Eventually one takes the limit $M \to \infty$ to obtain a unique (finite) result. Often one can proceed faster: performing manipulations that are guaranteed to be valid at the regulated level one may try to cast the integrands in alternative forms that can be computed directly in the continuum limit, without encountering any ambiguity.

The boundary conditions for the bosonic variables are as in (2.42), so that the bosonic quantum fluctuations, as well as the ghosts, are naturally expanded in a Fourier sine series

$$q^\mu(\tau) = \sum_{m=1}^{M} q^\mu_m \sin(\pi m \tau) \tag{2.71}$$

where the mode M is the regulating cut-off that is eventually sent to infinity, as in [18]. This choice preserves the boundary conditions imposed at initial and final times. On the other hand, fermionic fields satisfy first order differential equation and carry boundary conditions only at initial or final times, but not at both, see eqs. (2.43) and (2.44). Thus we find it useful to expand the fermionic quantum fields in half integers modes as follows

$$\bar{Q}^{ai}(\tau) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} \bar{Q}^{ai}_r \sin(\pi r\tau) \;, \qquad Q^{ai}(\tau) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} Q^{ai}_r \cos(\pi r\tau) \;. \tag{2.72}$$

This choice preserves the boundary conditions imposed and provides us with a regulated functional space to integrate over also for the fermions. In addition, it produces a simple regulated kinetic term that is easily inverted to obtain the propagators. Finally, the path integral is defined as a regulated integration over the Fourier coefficients of the various fields.

Perturbatively, the propagators are as in eqs. (2.46)–(2.49) and are regulated as follows

$$
\begin{aligned}
\Delta(\tau,\sigma) &= -\sum_{m=1}^{M} \frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) && (2.73) \\
\Delta_{gh}(\tau,\sigma) &= \sum_{m=1}^{M} 2 \sin(\pi m \tau) \sin(\pi m \sigma) && (2.74) \\
G(\tau,\sigma) &= \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} \frac{2}{\pi r} \sin(\pi r \tau) \cos(\pi r \sigma) \, . && (2.75)
\end{aligned}
$$

As a consequence

$$
\Delta_F(\tau,\sigma) = \sum_{r=\frac{1}{2}}^{M-\frac{1}{2}} \frac{1}{\pi r} \sin(\pi r(\tau-\sigma)) \qquad (2.76)
$$

for (2.52) that turns out to be translational invariant. Note that in the limit $M \to \infty$ the previous formulas reproduce eqs. (2.50) and (2.53).

We are now ready to compute the Feynman integrals with fermionic contributions in MR (the purely bosonic ones are standard and can be extracted from [37]). Using the regularized expressions one obtains $\mathbf{I}_{27} = \mathbf{I}_{28} = ... = \mathbf{I}_{33} = 0$. Also, one finds $\mathbf{I}_{26} = \frac{1}{4}$, as it is unambiguous and gives the same result in all regularization schemes. The only tricky integral is $\mathbf{I}_{25}$. It turns out that it is pretty hard to calculate it analytically in mode regularization. Following quite a long detour, it was done in [71] with antiperiodic fermions, yielding $\mathbf{I}_{25} = \frac{1}{6}$. With the present boundary conditions we are still working on it and, although numerical approximations seem to hint at the same result $\mathbf{I}_{25} = \frac{1}{6}$, its careful computation deserves further investigation and will be presented in a forthcoming publication.

With the preliminary result $\mathbf{I}_{25} = \frac{1}{6}$ the transition amplitude computed in mode regularization would read

$$\langle x\,\bar\lambda;\beta|y\,\eta;0\rangle = \frac{e^{-\frac{1}{2\beta}g_{\mu\nu}z^\mu z^\nu+\bar\lambda\cdot\eta}}{(2\pi\beta)^{D/2}}\exp\Big[\langle\text{bosonic}\rangle_{_{MR}} + \frac{1}{2\beta}\omega_{\mu ab}\,z^\mu\,\tilde\psi^a\cdot\tilde\psi^b$$
$$+\frac{1}{4\beta}\,\partial_\mu\omega_{\nu ab}\,z^\mu z^\nu\,\tilde\psi^a\cdot\tilde\psi^b - \frac{\alpha}{\beta}\,R_{abcd}\,\tilde\psi^a\cdot\tilde\psi^b\tilde\psi^c\cdot\tilde\psi^d - \frac{N}{16}\,\omega_{\mu ab}\,\omega_\nu{}^{ab}\,z^\mu z^\nu$$
$$+\frac{\beta N}{24}\,\omega_{\mu ab}^2\Big]\ . \tag{2.77}$$

where $\langle\text{bosonic}\rangle_{_{MR}}$ is the purely bosonic contribution that can be found in [37]. Comparing the above result with the TS one given in eq. (2.60), we notice that the only difference coming from the fermionic sector sits in the coefficients of the last ω^2 term, which is due to $\mathbf{I}_{25}$. As the bosonic part of the MR calculation requires the counterterm

$$V_{MR}^{(0)} = -\frac{1}{8}R - \frac{1}{24}\,(\Gamma^\mu_{\nu\lambda})^2$$

to reproduce the heat kernel for $H = -\frac{1}{2\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu$, where $(\Gamma^\mu_{\nu\lambda})^2 \equiv g^{\nu\nu'}g^{\lambda\lambda'}g_{\mu\mu'}\Gamma^\mu_{\nu\lambda}\Gamma^{\mu'}_{\nu'\lambda'}$, we see that, in order to obtain the correct amplitude for (2.1), we need the counterterm

$$V_{\rm MR}^{(N)} = -\Big(\frac{1}{8}+\frac{\alpha N}{2}\Big)R - \frac{1}{24}\left(\Gamma^\mu_{\nu\lambda}\right)^2 + \frac{N}{24}\omega_{\mu ab}\omega^{\mu ab} \tag{2.78}$$

that again would match all the known results valid for $N = 0, 1, 2$ [69, 71].

2.3.3 Dimensional regularization

Finally, we reconsider the previous calculations using dimensional regularization. This is a perturbative regularization that uses an adaptation of standard dimensional regularization to regulate the distributions defined on the one dimensional compact space $I = [0, \beta]$. One adds d extra infinite dimensions and perform all computations of ambiguous Feynman graphs in $d + 1$ dimensions. Extra dimensions act as a regulator when d is extended analytically to the complex plane, as in the usual QFT case. After evaluation of the various integrals one should take the $d \to 0$ limit. This is quite difficult in general, since the compact space I produces sums over discrete momenta, and standard formulas of dimensional regularization do not include that situation. However one may use manipulations valid at the regulated level, like differential equations satisfied by the Green functions and partial integration, to cast the integrand in equivalent forms that can be unambiguously computed in the $d \to 0$ limit. While purely perturbative, this method carries a covariant counterterm that simplify extensive calculations, as the one performed in [74] to obtain trace anomalies for a scalar field in six dimensions using worldline methods.

In DR the computation turns out to be quite simple for most diagrams: $\mathbf{I}_{27} = \mathbf{I}_{28} = 0$ because the integrand is odd, whereas $\mathbf{I}_{29} = \cdots = \mathbf{I}_{33} = 0$ as can be shown by integrating by parts. Also $\mathbf{I}_{26} = \frac{1}{4}$, as it is regular and can be safely evaluated by using the unregulated expression for the propagator. As usual, more care is needed to compute $\mathbf{I}_{25}$ since the integral is ambiguous (products of distributions). By dimensionally extending the cubic vertex

$$\dot{x}^\mu \psi_i^a \psi_i^b \quad \to \quad \partial_A x^\mu \bar{\psi}_i^a \gamma^A \psi_i^b \,, \quad A = 1, \ldots, d+1 \,, \tag{2.79}$$

where the bar denotes Majorana conjugation, the above integral becomes

$$\begin{aligned} \mathbf{I}_{25} &= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; \Delta_F^2 \\ &\to \int\int dt dt' \; {}_A\Delta_B \; \mathrm{tr}[\gamma^A \Delta_F \gamma^B \Delta_F] \end{aligned} \tag{2.80}$$

where $\Delta(t,t') = -\beta^{-1}\langle q(t) q(t')\rangle$ and $\Delta_F(t,t') = \beta^{-1}\langle \chi(t)\bar{\chi}(t')\rangle$ are the dimensionally regulated propagators where $\gamma^A \partial_A \Delta_F = -\Delta_F \gamma^A \overleftarrow{\partial'_A} = \delta(t,t')$, and ${}_A\Delta_B(t,t') = \partial_A \partial'_B \Delta(t,t')$. One can partially integrate the above integral without picking boundary terms since, in the compact direction bosonic fields vanish at the boundary, and in the extended directions Poincaré invariance allows partial integration as usual. Hence the above expression becomes

$$\begin{aligned} &-\int\int dt dt' \; \Delta_B \; \partial_A \; \mathrm{tr}\gamma^A \Delta_F \gamma^B \Delta_F \\ &= -2\int\int dt dt' \; \Delta_B \; \mathrm{tr}(\gamma^A \partial_A \Delta_F)\gamma^B \Delta_F \\ &= -2\int dt \; \Delta_B \; \mathrm{tr}\gamma^B \Delta_F(t,t) \quad \to \quad \mathbf{I}_{25} = 2\int_{-1}^{0} d\tau \; {}^\bullet\Delta \; \Delta_F|_\tau = 0 \end{aligned} \tag{2.81}$$

where we have used the regulated Green equation and used that Δ_F vanishes for coinciding points. The transition amplitude in DR thus reads

$$\begin{aligned} \langle x\, \bar{\lambda}; \beta | y\, \eta; 0\rangle &= \frac{e^{-\frac{1}{2\beta} g_{\mu\nu} z^\mu z^\nu + \bar{\lambda}\cdot\eta}}{(2\pi\beta)^{D/2}} \exp\Big[\langle \mathrm{bosonic}\rangle_{DR} + \frac{1}{2\beta}\omega_{\lambda ab} z^\lambda \tilde{\psi}^a \cdot \tilde{\psi}^b \\ &+ \frac{1}{4\beta}\partial_\lambda \omega_{\sigma ab} z^\lambda z^\sigma \tilde{\psi}^a \cdot \tilde{\psi}^b - \frac{\alpha}{\beta} R_{abcd} \tilde{\psi}^a \cdot \tilde{\psi}^b \tilde{\psi}^c \cdot \tilde{\psi}^d - \frac{N}{16}\omega_{\lambda ab}\omega_{\sigma ab} z^\lambda z^\sigma \Big] \,. \end{aligned} \tag{2.82}$$

where the bosonic contribution $\langle\text{bosonic}\rangle_{DR}$ can be extracted from [37]. Comparing this result with the ones obtained before in other regularizations we note that, thanks to the vanishing of $\mathbf{I}_{25}$, the diagrammatic expansion does not produce the term proportional to ω^2 which previously had to be canceled by the counterterms. Thus, the standard bosonic counterterm $V_{DR}^{(0)} = -\frac{1}{8}R$ is dressed up to

$$V_{DR}^{(N)} = -\Big(\frac{1}{8} + \frac{\alpha N}{2}\Big) R \; . \tag{2.83}$$

This matches known results for $N = 0, 1, 2$ [73, 27, 28].

In this chapter we have studied the quantum mechanics of one dimensional nonlinear sigma models possessing a $O(N)$ extended supersymmetry on suitable target space backgrounds. In particular, we have computed the transition amplitude for short propagation time using both canonical and path integrals methods, obtaining in the latter case the correct counterterms associated to various regularization schemes needed to define the path integrals.

A possible use of our result may be in the discussion of higher spin fields in a first quantized picture. Worldline approaches are useful in finding efficient way of computing amplitudes for relativistic processes both in flat space [25] and curved space [26], and the quantum mechanical nonlinear sigma model discussed here arose precisely in an attempt to use worldlines methods to describe one loop effective action due to higher spin fields in a curved background [30, 32]. In future works we plan indeed to use the path integrals constructed here to study effective actions induced by higher spin fields and compute corresponding heat kernel coefficients.

Chapter 3

$U(N)$ spinning particles and complex higher spins

In this chapter, based on [40], we analyze spinning particles with gauged $U(N)$-extended susy on the worldline and use them to derive gauge invariant higher spin equations on certain complex manifolds. The $U(N)$ particles for $N = 1, 2$ were originally introduced in [38] as a dimensional reduction of the $N = 2$ string, and generalized to arbitrary N in [39]. Their Dirac quantization introduces constraints on the particle Hilbert space that are interpreted as equations of motion for certain tensor fields with holomorphic indices satisfying the symmetries of a rectangular Young tableau [39]. We analyze these equations on the flat complex space $\mathbb{C}^d$. By integrating a subset of them in terms of gauge potentials we are led to gauge invariant field equations which are quite similar in form to the higher spin equations introduced by Fronsdal [8].

An example of these equations is that of a gauge field $\varphi_{\mu_1...\mu_N}$ with N symmetric holomorphic indices (we use complex coordinates $x^\mu, \bar{x}^{\bar{\mu}}$ of $\mathbb{C}^d$; tensor indices are raised and lowered with the flat hermitian metric $\delta_{\mu\bar{\nu}}$). It satisfies the equation

$$\partial_\alpha \bar{\partial}^\alpha \varphi_{\mu_1...\mu_N} - \sum_{i=1}^{N} \partial_{\mu_i} \bar{\partial}^\alpha \varphi_{\mu_1..\alpha..\mu_N} = 0 \tag{3.1}$$

where the α index in the second term is located in i-th position. The gauge invariance is given by

$$\delta\varphi_{\mu_1...\mu_N} = \partial_{\mu_1}\lambda_{\mu_2...\mu_N} + \textit{ cyclic perm.} \tag{3.2}$$

where the gauge parameter $\lambda_{\mu_2...\mu_N}$ has $N-1$ symmetric holomorphic indices and is constrained by $\bar{\partial}^\alpha \lambda_{\alpha\mu_3...\mu_N} = 0$. For consistency the gauge field must also satisfy a differential constraint $\bar{\partial}^\alpha \bar{\partial}^\beta \varphi_{\alpha\beta\mu_3\ldots\mu_N} = 0$. These equations are very much reminiscent of Fronsdal's equations. Since there is no invariant concept of taking traces on holomorphic indices, the usual trace constraints that appear in Fronsdal's formulation are naturally substituted here by differential constraints.

The constraints on gauge fields and on gauge parameters can be relaxed by adding compensator fields. For example in the above case with $N = 2$, one can introduce a single compensator field ρ and the equation reads

$$\partial_\alpha \bar{\partial}^\alpha \varphi_{\mu\nu} - \partial_\mu \bar{\partial}^\alpha \varphi_{\alpha\nu} - \partial_\nu \bar{\partial}^\alpha \varphi_{\mu\alpha} = \partial_\mu \partial_\nu \rho \tag{3.3}$$

with gauge symmetry

$$\delta\varphi_{\mu\nu} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu \ , \qquad \delta\rho = -2\bar{\partial}^\alpha \lambda_\alpha \ . \tag{3.4}$$

This is reminiscent of the Francia-Sagnotti construction [34] for relaxing the trace constraints of standard higher spin gauge theories using compensator fields.

We derive equations also for more general tensor fields with the symmetry type of a rectangular Young tableaux with p rows and N columns. We do so by using the compact notation provided by the quantum mechanical operators of the spinning particle.

The equations just discussed are defined on a flat complex space, viewed as a Kähler manifold, but it is interesting to study if they can be extended to more general Kähler spaces. While it is known that the $U(N)$ particles for $N = 1, 2$ can be coupled to any Kähler manifold [38], it was thought that for $N > 2$ these particles could only be consistent on flat manifolds, as the standard susy transformation rules do not leave the particle action invariant on a curved space [39]. We can actually show that a coupling is still possible for Kähler manifolds with constant holomorphic curvature. To achieve this result we use a hamiltonian approach and notice that the first class algebra defining the model closes on Kähler manifolds with constant holomorphic curvature, though in a nonlinear way. In fact we obtain a quadratic first class algebra that, quite interestingly, is seen to coincide with the zero mode sector of two dimensional nonlinear $U(N)$ superconformal algebras, introduced sometimes ago by Bershadsky and Knizhnik [75, 76]. This result is consistent with [39] in that the susy transformations rules associated to a nonlinear algebra differ from the one employed in [39]. The corresponding gauge invariant differential equations can similarly be defined on such complex spaces.

Having understood that $U(N)$ particles and related gauge invariant field equations can be defined on a non trivial class of curved spaces, it is interesting to study their quantum properties. We begin this analysis using a first quantized path integral description. This worldline approach is quite flexible and efficient, and by using closed worldlines one can study directly the one loop effective action associated to the field equations described above. To construct the path integral is necessary to gauge-fix the particle action and identify the correct measure over the moduli space of inequivalent gauge choices. We start considering a flat target space and compute the physical degrees of freedom. This gives a check on the path integral measure, which can be used to compute more general observables.

While the complex nature of target space does not suggest us an immediate physical application of these higher spin equations (either the target space has an even number of times, or no time direction at all) they might still be useful to describe properties of complex manifolds or for developing additional intuition on the standard theory of higher spin fields. From this point of view it would be quite interesting to search for consistent nonlinear extensions of the free equations described here.

3.1 The $U(N)$ spinning particle in flat space

We consider an even dimensional flat space, viewed as the flat Kähler manifold $\mathbb{C}^d$, with $D = 2d$ real dimensions; the bosonic fields $x^M(\tau)$, interpreted as space-time coordinates, split into complex components $x^\mu(\tau)$ and $\bar{x}^{\bar{\mu}}(\tau)$, with $\mu = 1 \dots d$. They are paired with fermionic superpartners $\psi_i^\mu(\tau)$ and $\bar{\psi}^{\bar{\mu}i}(\tau)$, $i = 1 \dots N$, belonging to the $\mathbf{N}$ and $\bar{\mathbf{N}}$ of $U(N)$, respectively. The flat metric in complex coordinates is simply $\delta_{\mu\bar{\nu}}$, the other components being zero. With these ingredients the phase space action

$$S = \int_0^1 d\tau \left[p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i \bar{\psi}_\mu^i \dot{\psi}_i^\mu - p_\mu \bar{p}^\mu \right] \tag{3.5}$$

describes the motion of a free particle with a pseudoclassical spin associated to the Grassmann coordinates. This system enjoys various conserved quantities, including those corresponding to the $U(N)$-extended supersymmetry on the worldline

$$H = p_\mu \bar{p}^\mu \ , \quad Q_i = \psi_i^\mu p_\mu \ , \quad \bar{Q}^i = \bar{\psi}^{\bar{\mu}i} \bar{p}_{\bar{\mu}} \ , \quad J_i^j = \psi_i^\mu \bar{\psi}_\mu^j \tag{3.6}$$

where indices are lowered and raised using the $\delta_{\mu\bar{\nu}}$ metric and its inverse. We have chosen normalizations so that H is real, $(Q_i)^* = \bar{Q}^i$, and $(J_i^j)^* = J_j^i$, so that J_i^i is real for any fixed i. The fundamental Poisson brackets are easily read off from the symplectic term of the action

$$\{x^\mu, p_\nu\}_{PB} = \delta_\nu^\mu \ , \quad \{\bar{x}^{\bar{\mu}}, \bar{p}_{\bar{\nu}}\}_{PB} = \delta_{\bar{\nu}}^{\bar{\mu}} \ , \quad \{\psi_i^\mu, \bar{\psi}^{\bar{\nu}j}\}_{PB} = -i\delta^{\mu\bar{\nu}} \delta_i^j \tag{3.7}$$

and the above conserved charges generate symmetry transformations through Poisson brackets (using $\delta z = \{z, \mathcal{G}\}_{_{PB}}$ with $\mathcal{G} \equiv \xi H + i\bar{\epsilon}^i Q_i + i\epsilon_i \bar{Q}^i + \alpha_i^j J_j^i$)

$$\begin{aligned}
&\delta x^\mu = \xi \bar{p}^\mu + i\bar{\epsilon}^i \psi_i^\mu \ , && \delta \bar{x}^{\bar{\mu}} = \xi p^{\bar{\mu}} + i\epsilon_i \bar{\psi}^{\bar{\mu} i} \\
&\delta \psi_i^\mu = -\epsilon_i \bar{p}^\mu + i\alpha_i^j \psi_j^\mu \ , && \delta \bar{\psi}^{\bar{\mu} i} = -\bar{\epsilon}^i p^{\bar{\mu}} - i\alpha_j^i \bar{\psi}^{\bar{\mu} j} \\
&\delta p_\mu = 0 \ , && \delta \bar{p}_{\bar{\mu}} = 0 \ ,
\end{aligned} \tag{3.8}$$

which correspond to rigid time translations with parameter ξ, N complex supersymmetries with grassmannian parameters ϵ_i and $\bar{\epsilon}^i$, and $U(N)$ rotations parametrized by α_j^i. The explicit $U(N)$-extended supersymmetry algebra is easily computed

$$\begin{aligned}
&\{Q_i, \bar{Q}^j\}_{_{PB}} = -i\delta_i^j H \\
&\{J_i^j, Q_k\}_{_{PB}} = -i\delta_k^j Q_i \ , \qquad \{J_i^j, \bar{Q}^k\}_{_{PB}} = i\delta_i^k \bar{Q}^j \\
&\{J_i^j, J_k^l\}_{_{PB}} = i\delta_i^l J_k^j - i\delta_k^j J_i^l
\end{aligned} \tag{3.9}$$

with other independent Poisson brackets vanishing.

The model we are interested in is obtained by gauging this first class algebra through the introduction of corresponding gauge fields: an einbein $e(\tau)$ for time translations, complex gravitini $\chi_i(\tau)$ and $\bar{\chi}^i(\tau)$ for the extended supersymmetry, and a $U(N)$ gauge field $a_j^i(\tau)$ for the rotations. These fields correspond to the gauge fields of a $U(N)$-extended supergravity on the worldline, and the full action of the $U(N)$ spinning particle becomes

$$S = \int_0^1 d\tau \, \Big[p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}_\mu^i \dot{\psi}_i^\mu - e \underbrace{p_\mu \bar{p}^\mu}_{H} - i\bar{\chi}^i \underbrace{p_\mu \psi_i^\mu}_{Q_i} - i\chi_i \underbrace{\bar{p}_{\bar{\mu}} \bar{\psi}^{\bar{\mu}\, i}}_{\bar{Q}^i} - a_j^i (\underbrace{\psi_i^\mu \bar{\psi}_\mu^j}_{J_i^j} - s\delta_i^j)\Big] \tag{3.10}$$

where we have inserted also a Chern-Simons coupling s for the $U(1)$ part of the gauge group $U(N)$, since it is invariant by itself. The supergravity gauge fields turn the rigid symmetries of eqs. (3.8) into local ones and transform as follows

$$\begin{aligned}
\delta e &= \dot{\xi} + i\bar{\chi}^i \epsilon_i + i\chi_i \bar{\epsilon}^i \\
\delta \chi_i &= \dot{\epsilon}_i - ia_i^k \epsilon_k + i\alpha_i^k \chi_k = \mathcal{D}\epsilon_i + i\alpha_i^k \chi_k \\
\delta \bar{\chi}^i &= \dot{\bar{\epsilon}}^i + ia_k^i \bar{\epsilon}^k - i\alpha_k^i \bar{\chi}^k = \mathcal{D}\bar{\epsilon}^i - i\alpha_k^i \bar{\chi}^k \\
\delta a_j^i &= \dot{\alpha}_j^i - ia_j^k \alpha_k^i + ia_k^i \alpha_j^k = \mathcal{D}\alpha_j^i
\end{aligned} \tag{3.11}$$

where $\mathcal{D}$ stands for the $U(N)$ covariant derivative.
From the phase space action (3.10) it is immediate to see that the equations

of motion of the gauge fields $G \equiv (e, \chi, \bar{\chi}, a)$ constrain the Noether charges to vanish

$$\frac{\delta S}{\delta G} = 0 \qquad \Rightarrow \qquad H = Q_i = \bar{Q}^i = J_i^{\,j} - s\delta_i^j = 0 \; . \tag{3.12}$$

The Poisson brackets of these generators form the $U(N)$-extended supersymmetry algebra that, as we shall see, ceases to be first class for $N > 2$ on arbitrary curved manifolds. This hints to a fundamental obstruction in imposing the constraints listed above and is the signal, from a worldline point of view, of the difficulties that arise in coupling higher spin particles to curved spaces. We will discuss this issue in more depth in section 4.

Eliminating the momenta p and $\bar{p}$ one obtains the action in configuration space

$$S[X,G] = \int_0^1 d\tau \big[e^{-1}\big(\dot{x}^\mu - i\bar{\chi}^i\psi_i^\mu\big)\big(\dot{\bar{x}}_\mu - i\chi_j\bar{\psi}^j_\mu\big) + i\bar{\psi}^i_\mu\big(\delta_i^j\partial_\tau - ia_i^j\big)\psi_j^\mu + sa_i^i\big] \tag{3.13}$$

where $X \equiv (x, \bar{x}, \psi, \bar{\psi})$ and $G \equiv (e, \chi, \bar{\chi}, a)$. We shall use this form when constructing the path integral in section 5.

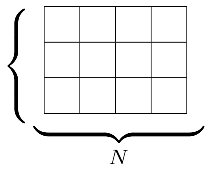
3.2 Equations of motion in flat space

We now use canonical quantization and obtain the equations of motion in flat space. From the constraint $H = 0$, we see that the system has a constant τ evolution. The dynamics of the particle is then entirely contained in the constraints $H = Q_i = \bar{Q}^i = J_i^{\,j} - s\delta_i^j = 0$: these classical statements translate, in the quantum theory, into the selection of the physical Hilbert space, which is obtained by requiring the symmetry generators to annihilate physical states, *i.e.*

$$|\Phi\rangle \in \mathcal{H}_{phys} \quad \Longleftrightarrow \quad T_a|\Phi\rangle = 0 \; , \quad T_a = (H \, , \, Q_i \, , \, \bar{Q}^i \, , \, J_i^{\,j} - s\delta_i^j) \tag{3.14}$$

where the generators T_a are now to be understood as operators. The Chern-Simons coupling s will satisfy a quantization condition that can be stated precisely once a prescription for resolving the ordering ambiguities contained in $J_i^{\,j}$ is taken care of. What we have just described is the Dirac quantization procedure, which generalizes the quantization à la Gupta-Bleuler of electrodynamics. As already discussed in [39], the particle states can be represented by generalized field strengths of the form $F_{\mu_1^1 .. \mu_m^1, \ldots, \mu_1^N .. \mu_m^N}$, where the integer m is related to the quantized Chern-Simons coupling s. In particular, the J constraints require that F is antisymmetric within each block of m indices, symmetric in exchanging entire blocks, and in addition satisfies algebraic Bianchi identities, *i.e.* it belongs to an irreducible representation of $U(d)$ with rectangular $m \times N$ Young tableau:

$$F_{\mu_1^1 .. \mu_m^1 ,..., \mu_1^N .. \mu_m^N} \quad \sim \quad m \tag{3.15}$$



The Q and $\bar{Q}$ constraints enforce generalized Maxwell equations, while the H constraint is automatically satisfied in virtue of the constraint algebra.

We now proceed in deriving the results stated above: looking at the fundamental (anti)-commutation relations, which follows from the classical Poisson brackets (3.7),

$$\left[x^\mu \, , \, p_\nu\right] = i\delta^\mu_\nu \ , \quad \left[\bar{x}^{\bar{\mu}} \, , \, \bar{p}_{\bar{\nu}}\right] = i\delta^{\bar{\mu}}_{\bar{\nu}} \ , \quad \left\{\psi^\mu_i \, , \, \bar{\psi}^{\bar{\nu}\, j}\right\} = \delta^{\mu\bar{\nu}}\delta^j_i \ , \tag{3.16}$$

one can decide to project the states of the Hilbert space onto the x^μ, $\bar{x}^{\bar{\mu}}$ and ψ^μ_i eigenstates. In this way x, $\bar{x}$ and ψ act by multiplication, while momenta p, $\bar{p}$ and $\bar{\psi}$ act as derivatives: $p_\mu \sim -i\partial_\mu$, $\bar{p}_{\bar{\mu}} \sim -i\bar{\partial}_{\bar{\mu}}$ and $\bar{\psi}^{\bar{\mu}i} \sim \frac{\partial}{\partial \psi^\mu_i}$. The states are thus represented by functions of x, $\bar{x}$ and ψ: $|F\rangle \sim \langle x, \bar{x}, \psi|F\rangle = F(x, \bar{x}, \psi)$. With this realization the symmetry generators T_a read

$$\begin{aligned} J_i^{\,j} - s\delta_i^j &= \psi_i \cdot \frac{\partial}{\partial \psi_j} - m\,\delta_i^j \\ Q_i &= -i\psi^\mu_i\,\partial_\mu \\ \bar{Q}^i &= -i\,\frac{\partial}{\partial \psi^\mu_i}\,\bar{\partial}^\mu \\ H &= -\delta^{\mu\bar{\nu}}\,\partial_\mu\bar{\partial}_{\bar{\nu}} \end{aligned} \tag{3.17}$$

where $\bar{\partial}^\nu = \delta^{\nu\bar{\mu}}\,\bar{\partial}_{\bar{\mu}}$. Ordering ambiguities are only present in the J constraint. We have resolved them by using a graded-symmetric ordering, which coincides with the natural regularization that arises form the path integral of section 5,

$$J_i^{\,j} = \frac{1}{2}(\psi^\mu_i\bar{\psi}^j_\mu - \bar{\psi}^j_\mu\psi^\mu_i) = \psi^\mu_i\bar{\psi}^j_\mu - \frac{d}{2}\delta_i^j \quad \Longrightarrow \quad J_i^{\,j} - s\delta_i^j = \psi_i \cdot \frac{\partial}{\partial \psi_j} - m\,\delta_i^j \tag{3.18}$$

where we have set $m \equiv (\frac{d}{2} + s)$. The quantum constraints satisfy a first class algebra corresponding to the quantum version of (3.9)

$$\begin{aligned} \left\{Q_i, \bar{Q}^j\right\} &= \delta_i^j H \\ \left[J_i^{\,j}, Q_k\right] &= \delta_k^j Q_i \ , \qquad\qquad \left[J_i^{\,j}, \bar{Q}^k\right] = -\delta_i^k \bar{Q}^j \\ \left[J_i^{\,j}, J_k^{\,l}\right] &= \delta_k^j J_i^{\,l} - \delta_i^l J_k^{\,j} \end{aligned} \tag{3.19}$$

while other independent graded-commutators vanish. Here we have used the simple J_i^j generators, but it is evident that the same result holds by substituting them with $J_i^j - s\delta_i^j$ since the Chern-Simons term is central and in addition it cancels on right hand sides.
Due to the grassmannian nature of the ψ variables, the states have a finite Taylor expansion in ψ's

$$|F\rangle \sim \sum_{A_i=0}^{d} F_{\mu_1^1..\,\mu_{A_1}^1,\dots,\mu_1^N..\,\mu_{A_N}^N}(x,\bar{x})\,\psi_1^{\mu_1^1}..\,\psi_1^{\mu_{A_1}^1}\dots\,\psi_N^{\mu_1^N}..\,\psi_N^{\mu_{A_N}^N} \tag{3.20}$$

and we can now study which of them survive the constraint equations.

First we consider the J_i^j constraints. The J_i^i constraint at fixed i counts fermions of i-th type and fixes them to be m in number, see (3.18). Thus m must be an integer and this, in turn, fixes the possible quantized values of the Chern-Simons coupling s. Hence, the only term of (4.10) surviving this constraint is

$$F_{\mu_1^1..\,\mu_m^1,\dots,\mu_1^N..\,\mu_m^N}\ \psi_1^{\mu_1^1}..\psi_1^{\mu_m^1}\dots\psi_N^{\mu_1^N}..\psi_N^{\mu_m^N} \tag{3.21}$$

i.e. a tensor with N blocks of m indices. In term of complex geometry, the tensor $F_{\mu_1^1..\mu_m^1,\dots,\mu_1^N..\mu_m^N}(x,\bar{x})$ can be thought of a differential multiple $(m,0)$-form: in fact each ψ_i block in (3.21) plays the role of a basis for the $(m,0)$-forms, $dx^{\mu_1}\wedge\dots\wedge dx^{\mu_m}$. The J_i^j constraint for $i\neq j$ then ensures algebraic Bianchi identities: it picks an index of the j-th block, antisymmetrizes it with those of the i-th block, and set the resulting tensor to zero. For example, the J_1^2 constraint gives

$$F_{[\mu_1^1..\,\mu_m^1,\mu_1^2]\dots,\mu_1^N..\,\mu_m^N}=0 \tag{3.22}$$

and so on. As a consequence, the tensor $F_{\mu_1^1..\,\mu_m^1,\dots,\mu_1^N..\,\mu_m^N}$ has N blocks of m antisymmetric indices and is symmetric under exchanges of blocks. The antisymmetry within each block is evident from the Grassmann nature of the ψ's, while symmetry between blocks can be understood considering particular $U(N)$ transformations. In fact, a $\frac{\pi}{2}$ rotation in the $i-j$ plane sends ψ_i in ψ_j and ψ_j in $-\psi_i$. The final effect of this $U(N)$ transformation is to exchange the i-th and j-th blocks of indices on the tensor F in (3.21) without any additional sign. Since this is a $U(N)$ transformation connected to the identity, it can be cast in the form $e^{i\alpha_j^i J_i^j}$ for some α_j^i with $i\neq j$. Requiring $J_i^j|F\rangle=0$ produces the anticipated symmetry between the i-th and j-th blocks of indices of the tensor F. All these algebraic symmetries are summarized by saying that F belongs to an irreducible representation of the group $U(d)$ described by the Young tableau in eq. (3.15). Finally, using the representation (3.17) of the operators Q_i and $\bar{Q}^i$, it is straightforward to see that their constraints impose the following generalized Maxwell equations on the curvature F

$$\partial_{[\mu} F_{\mu_1^1 .. \mu_m^1], \ldots, \mu_1^N .. \mu_m^N} = 0 \ , \tag{3.23}$$

$$\bar{\partial}^{\mu} F_{\mu .. \mu_m^1, \ldots, \mu_1^N .. \mu_m^N} = 0 \ . \tag{3.24}$$

3.2.1 Gauge fields

In analogy with Maxwell, Yang-Mills and higher spin gauge theories, we first try to solve eq. (3.23). This equation can be interpreted as an integrability condition. In the absence of topological obstructions, the closure of a form F is achieved expressing it as the exterior derivative of a gauge field: $F = d\phi \ \rightarrow \ dF = 0$. In our context we are dealing with N-multiple $(m, 0)$-forms, and we are going to show that (3.23), that is to say $Q_i|F\rangle = 0$, can be solved writing F as the multiple action (one for each block of indices) of the holomorphic Dolbeault operator ∂, that sends forms of bidegree (p, q) into $(p+1, q)$-forms. As the $\partial_{(i)}$ operator[1] in our quantum mechanical notation is simply Q_i, it is useful to define

$$q = Q_1 Q_2 \ldots Q_N = \frac{1}{N!} \, \epsilon^{i_1 \ldots i_N} Q_{i_1} \ldots Q_{i_N} \tag{3.25}$$

which is identically annihilated by the Q_i's: $qQ_i = Q_i q = 0$, due to $Q_i^2 = 0$ and to the fact that q contains already all of the Q_i's. Setting

$$|F\rangle = q|\phi\rangle \tag{3.26}$$

automatically satisfies the Q constraints and, writing down (3.26) in components, we see that $F \sim \partial_{(1)} \ldots \partial_{(N)} \phi$, where each Dolbeault operator antisymmetrizes only over the corresponding block of indices. To solve the J constraints one can take ϕ to be a N-multiple $(p, 0)$-form with $p \equiv m-1$ that forms a $U(d)$ irreducible tensor (a rectangular $p \times N$ Young tableau)

$$|\phi\rangle \quad \sim \quad \phi_{\mu_1^1 .. \mu_p^1, \ldots, \mu_1^N .. \mu_p^N}(x, \bar{x}) \quad \sim \quad p \left\{ \underbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \end{array}}_{N} \right. \ . \tag{3.27}$$

In fact, we note that $J_i^j \, q = q \, (J_i^j + \delta_i^j)$. Thus $(J_i^j - s\delta_i^j)|F\rangle = 0$ is satisfied if one requires

$$\left(J_i^j - (s-1)\delta_i^j \right) |\phi\rangle = 0 \ , \tag{3.28}$$

[1] The index i refers to the block on which the Dolbeault operator ∂ acts, while other blocks are treated as spectators.

that is, $N_i = m-1 \equiv p$ if taking $i=j$ (by $N_i \equiv \psi_i \cdot \frac{\partial}{\partial \psi_i}$ at fixed i we indicate the number operator that counts the fermions of the i-th type), while the off diagonal equations are the same as for F: they impose algebraic Bianchi identities and, in particular, symmetry between block exchanges.
Next it remains to implement the last independent constraint, $\bar{Q}^i|F\rangle = 0$. This produces generalized Maxwell equations for the gauge field. From (3.26) it is clear that $\bar{Q}^i\, q|\phi\rangle = 0$ is an higher derivative equation of motion for the gauge potential, precisely of order $N+1$. It is convenient to use some $Q, \bar{Q}$ algebra in order to factorize from the operator $\bar{Q}^i\, q$ a second order differential operator G, that will play a role analogous to the Fronsdal-Labastida operator [8, 56] for higher spin fields. Iterated use of $\{Q_i, Q_j\} = 0$ and $\{Q_i, \bar{Q}^j\} = \delta_i^j\, H$ gives (in the following equation j is fixed, not summed)

$$\begin{aligned}\bar{Q}^j\, q &= \bar{Q}^j\, Q_1 Q_2 .. Q_N = (-1)^{j-1} Q_1 .. \bar{Q}^j Q_j .. Q_N \\ &= (-1)^{j-1} \big(Q_1 .. Q_{j-1} Q_{j+1} .. Q_N\big) \bar{Q}^j Q_j \\ &= (-1)^{j-1} \big(Q_1 .. Q_{j-1} Q_{j+1} .. Q_N\big) \big(H - Q_j \bar{Q}^j\big)\ .\end{aligned}$$

At this point is possible to sum over j in $H - Q_j \bar{Q}^j$, since the extra terms vanish anyhow, and cast the equation of motion in the form

$$\bar{Q}^j q|\phi\rangle = q^j G|\phi\rangle = 0 \tag{3.29}$$

where, in an obvious notation,

$$q^j = (-1)^j Q_1 .. Q_{j-1} Q_{j+1} .. Q_N = \frac{(-1)^j}{(N-1)!} \epsilon^{j j_2 \dots j_N} Q_{j_2} \dots Q_{j_N}\ .$$

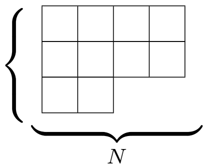
G is the second order operator we were looking for, analogous to the Fronsdal-Labastida operator without the trace term

$$G = -H + Q_i \bar{Q}^i \quad \sim \quad \partial_\alpha \bar{\partial}^\alpha - \psi_i^\alpha \frac{\partial}{\partial \psi_i^\beta}\, \partial_\alpha \bar{\partial}^\beta\ . \tag{3.30}$$

To obtain a second order equation of motion from (3.29) it is necessary to eliminate the operator q^j. One way to do this is recalling that a generic expression containing two Q's represents the kernel of q^j, that is $q^j Q_k Q_l \equiv 0$, and so a general solution of $q^i\big(G|\phi\rangle\big) = 0$ is

$$G|\phi\rangle = Q_i Q_j |\rho^{ij}\rangle \tag{3.31}$$

where $|\rho^{ij}\rangle$ are the compensator fields. One can present the compensators also in the form $|\rho^{ij}\rangle = \bar{V}^i \bar{V}^j |\rho\rangle$. This second form of writing the compensators is slightly more convenient. Here $\bar{V}^i \equiv V^\mu \bar{\psi}^i_\mu$ depends on an arbitrary vector field V^μ, and $|\rho\rangle$ is a state that must satisfy $(J_i^j - (s-1)\delta_i^j)|\rho\rangle = 0$ (because of eq. (3.28) and $[G,\, J_i^j] = 0$) and thus is represented by a tensor with the same structure and Young tableau of ϕ. The action of $\bar{V}^i$ is to eliminate one ψ from the i-th block and saturate the corresponding index of the ρ tensor with V^μ. Therefore the compensator ρ^{ij} has $N-2$ blocks with p antisymmetric indices and two blocks, the i-th and j-th ones, with $p-1$ indices. Its Young tableau has the form

$\rho^{ij} \quad \sim \quad p$  . (3.32)

The key feature of eq. (3.31) is to be a second order wave equation. The price for this is the introduction of the auxiliary fields ρ^{ij}. Of course one would like also to obtain an equation without compensators, $G|\phi\rangle = 0$. This is indeed possible using gauge symmetries. In fact, in theories where the physical field strength is expressed in terms of a potential, one expects the presence of a gauge symmetry.

Gauge symmetry

In term of forms if $F = d\phi$, the gauge transformation leaving F invariant is $\delta\phi = d\Lambda$. In our model the gauge symmetry enjoyed by the curvature F is an "higher spin" generalization of the linearized diffeomorphisms of general relativity, like the gauge transformations of standard higher spin fields. In our operator formalism, exterior holomorphic derivatives acting on the i-th block are represented by the supercharge Q_i. Thus, recalling that $|F\rangle = q|\phi\rangle$ and $qQ_i = 0$, one finds immediately an invariant way of writing down the gauge transformations that leave the F tensor invariant

$$\delta|\phi\rangle = Q_i|\Lambda^i\rangle \tag{3.33}$$

where $|\Lambda^i\rangle$ are the gauge parameters. Again a slightly more convenient way of writing the gauge parameters is in the form $|\Lambda^i\rangle = \bar{W}^i|\Lambda\rangle$, where $\bar{W}^i \equiv W^\mu \bar{\psi}^i_\mu$ with W^μ a vector field and $|\Lambda\rangle$ a state containing a tensor with the same index structure and Young tableau of $|\phi\rangle$. These gauge transformations clearly do not affect $|F\rangle = q|\phi\rangle$, but let us compute how the left hand side of (3.31) transforms. Making use of the Q, $\bar{Q}$ algebra the gauge variation can be written as

$$G\delta|\phi\rangle = -Q_i Q_j \big(\bar{Q}^i |\Lambda^j\rangle\big)\ , \tag{3.34}$$

and if we want the equations of motion to be gauge invariant, the compensator field (from this its name) has to cancel the above expression and transform as

$$\delta|\rho^{ij}\rangle = -\bar{Q}^{[i}|\Lambda^{j]}\rangle \ . \tag{3.35}$$

It is well known from higher spin field theories [54, 55, 34] that the equations of motion in the compensator formalism are invariant for general gauge transformations, but if we try to gauge fix the compensators to zero, constraints on gauge parameters and on gauge fields appear, namely the gauge parameters must be traceless and the gauge fields double traceless. In our framework there are no ways of taking the trace of completely holomorphic tensors, instead differential constraints appear on gauge parameters and on gauge fields. To see this, let us use part of the gauge freedom in (3.33) and (3.35) to make the compensators vanish: $\rho^{ij} = 0$. The residual gauge symmetry must satisfy $\bar{Q}^{[i}|\Lambda^{j]}\rangle = 0$, and this can be achieved if the gauge parameters are taken to be "divergenceless": $\bar{Q}^{i}|\Lambda^{j}\rangle = 0$ for $i \neq j$. Similarly, the gauge choice $\rho^{ij} = 0$ imposes constraints also on the gauge field ϕ. This can be seen by acting with $\bar{Q}^{k}$ on both sides of eq. (3.31) to obtain

$$Q_i \bar{Q}^k \bar{Q}^i |\phi\rangle = Q_i \left[\bar{Q}^k Q_j |\rho^{ij}\rangle - H|\rho^{ki}\rangle\right] \ . \tag{3.36}$$

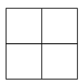
The right hand side of this equation vanishes in the partially gauge fixed theory with $\rho^{ij} = 0$. For consistency the left hand side must vanish as well, and this is guaranteed if $\bar{Q}^k \bar{Q}^i |\phi\rangle = 0$, that corresponds to setting to zero all possible double divergences. One may check that this constraint is kept invariant by gauge transformations with parameters satisfying $\bar{Q}^{i}|\Lambda^{j}\rangle = 0$ with $i \neq j$. Once the compensator fields have been eliminated, the gauge potential describing the particle satisfies the simpler second order wave equation $G|\phi\rangle = 0$ that, in tensorial language, reads

$$\partial_\alpha \bar{\partial}^\alpha \phi_{\mu_1..\mu_p,\ldots,\nu_1..\nu_p} - p\, \partial_{\mu_1} \bar{\partial}^\alpha \phi_{\alpha\mu_2..\mu_p,\ldots,\nu_1..\nu_p} - \ldots - p\, \partial_{\nu_1} \bar{\partial}^\alpha \phi_{\mu_1..\mu_p,\ldots,\alpha\nu_2..\nu_p} = 0 \tag{3.37}$$

where $p \equiv m-1$, and weighted antisymmetrization is understood on μ's, ν's and so on.

In order to clarify the meaning of our quantum mechanical notation, let us analyze in tensorial language a specific case: $N = 2$, $p = 2$. This is the simplest model where all of the issues treated so far appear in a non trivial way. The gauge field ϕ has the structure

$$\phi_{\mu_1\mu_2,\nu_1\nu_2} \sim \tag{3.38}$$



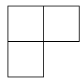
while the unique independent compensator is a symmetric tensor $\rho_{\mu\nu}$. The gauge invariant equations of motion read

$$\partial_\alpha \bar{\partial}^\alpha \phi_{\mu_1\mu_2,\nu_1\nu_2} - 2\partial_{\mu_1} \bar{\partial}^\alpha \phi_{\alpha\mu_2,\nu_1\nu_2} - 2\partial_{\nu_1} \bar{\partial}^\alpha \phi_{\mu_1\mu_2,\alpha\nu_2} = 2\partial_{\mu_1}\partial_{\nu_1}\rho_{\mu_2\nu_2} \tag{3.39}$$

with an understood weighted antisymmetrization on the μ and ν group of indices, that will be employed in all of the following equations as well. The gauge transformations for the field ϕ and the compensator are given by

$$\delta\phi_{\mu_1\mu_2,\nu_1\nu_2} = \partial_{\mu_1}\Lambda_{\nu_1\nu_2,\mu_2} + \partial_{\nu_1}\Lambda_{\mu_1\mu_2,\nu_2} \ , \quad \delta\rho_{\mu\nu} = -\bar{\partial}^\alpha\Lambda_{\alpha\mu,\nu} - \bar{\partial}^\alpha\Lambda_{\alpha\nu,\mu} \quad (3.40)$$

where a factor of $-2i$ has been absorbed in the definition of the gauge parameter, whose Young tableau is

$$\Lambda_{\mu_1\mu_2,\nu} \sim$$  $$. \quad (3.41)$$

Using part of the gauge freedom, one can fix the compensator to zero, obtaining the gauge invariant equation

$$\partial_\alpha\bar{\partial}^\alpha\phi_{\mu_1\mu_2,\nu_1\nu_2} - 2\partial_{\mu_1}\bar{\partial}^\alpha\phi_{\alpha\mu_2,\nu_1\nu_2} - 2\partial_{\nu_1}\bar{\partial}^\alpha\phi_{\mu_1\mu_2,\alpha\nu_2} = 0 \quad (3.42)$$

which is left invariant by the gauge transformations (3.40) with constrained gauge parameters

$$\bar{\partial}^\alpha\Lambda_{\alpha\mu,\nu} = 0 \ . \quad (3.43)$$

For consistency, the gauge field appearing in this equation must also satisfy a differential constraint

$$\bar{\partial}^\alpha\bar{\partial}^\beta\phi_{\alpha\mu,\beta\nu} = 0 \quad (3.44)$$

which is preserved by the gauge transformations with constrained gauge parameters.

To count the physical degrees of freedom, one has to use the remaining gauge freedom to eliminate unphysical "polarizations" from ϕ. This way one ends up with a gauge field $\phi_{m_1..m_p,\dots,n_1..n_p}$, where indices run over $d-2$ directions, *i.e.* $m,n = 1,2,\dots,d-2$. Perhaps this is best seen in the particle language, since by using the complex N supersymmetries one can eliminate the fermionic fields ψ_i^μ and their complex conjugates with the index μ pointing along two chosen directions. In this "light cone gauge" the tensor $\phi_{m_1..m_p,\dots,n_1..n_p}$ describes an irreducible representation of the little group for massless particles, $U(d-2)$, with the same Young tableau of eq. (3.27). The dimension of such representation corresponds to the number of physical degrees of freedom of the particle. Using the "factors over hook" rule it is easy to compute the dimension of this Young tableau, and the resulting degrees of freedom, for all d, N and p, are

$$Dof(d,N,p) = \prod_{j=0}^{N-1} \frac{j!(j+d-2)!}{(j+p)!(j+d-2-p)!} \quad (3.45)$$

where we recall that $p = m - 1 = \frac{d}{2} + s - 1$. We note that in the case of an odd number of complex dimensions the physical spectrum is empty unless the Chern-Simons term is added, *i.e.* $s \neq 0$. The quantization of this Chern-Simons coupling can be understood also from the requirement of cancelling gauge anomalies [77].

Let us analyze a few examples. From (3.45) one can see that in $d = 2$ (four *real* dimensions) without Chern-Simons coupling, there is always one degree of freedom for any value of N: $Dof(2, N, 0) = 1$. So, with $s = 0$, all the $U(N)$ spinning particle theories propagate only a scalar field in two complex dimensions, and share for this aspect the features of $N = 2$ superstrings, where only the scalar ground states survive at the critical dimension $d = 2$, see for example the review [78]. Another simple case is the $N = 1$ theory in arbitrary complex dimensions: the field strengths are $(p+1, 0)$-forms $F_{\mu_1..\mu_{p+1}}$, the gauge potentials are $(p, 0)$-forms $\phi_{\mu_1..\mu_p}$ and (3.45) gives $Dof(d, 1, p) = \binom{d-2}{p}$; that is the number of independent components of an antisymmetric tensor of $U(d-2)$ with p indices, $\phi_{m_1..m_p}$. In the last section we will compute the one-loop partition function for the $U(N)$ spinning particle. After covariantly gauge fixing the action (3.13) on the torus, the path integral reduces to an integral over a corresponding moduli space which computes the number of physical degrees of freedom. Indeed, we shall see that they coincide with the canonical computation just presented.

To summarize, we have described gauge invariant equations with compensators

$$G|\phi\rangle = Q_i Q_j |\rho^{ij}\rangle \tag{3.46}$$

with $G = -H + Q_i \bar{Q}^i$, and gauge symmetries given by

$$\delta|\phi\rangle = Q_i|\Lambda^i\rangle\ , \qquad \delta|\rho^{ij}\rangle = -\bar{Q}^{[i}|\Lambda^{j]}\rangle \tag{3.47}$$

where $|\rho^{ij}\rangle \equiv \bar{V}^i \bar{V}^j |\rho\rangle$, $|\Lambda^i\rangle \equiv \bar{W}^i|\Lambda\rangle$ and with $|\phi\rangle$, $|\rho\rangle$, $|\Lambda\rangle$ describing tensors with rectangular $p \times N$ Young tableaux of $U(d)$, as in (3.27).

Similarly, gauge invariant equations without compensators are given by

$$G|\phi\rangle = 0 \tag{3.48}$$

with gauge symmetry

$$\delta|\phi\rangle = Q_i|\Lambda^i\rangle \tag{3.49}$$

where $|\Lambda^i\rangle \equiv \bar{W}^i|\Lambda\rangle$, with fields and gauge parameters satisfying the differential constraints

$$\bar{Q}^i \bar{Q}^j |\phi\rangle = 0\ , \qquad \bar{Q}^i|\Lambda^j\rangle = 0 \qquad (i \neq j). \tag{3.50}$$

3.3 Supersymmetry algebra in curved Kähler manifolds

We now turn to study the supersymmetry algebra on arbitrary Kähler manifolds. It will be shown that for all N it is possible to close the algebra, though quadratically, on Kähler manifolds of constant holomorphic curvature, and so even for $N > 2$ a consistent quantization can be obtained beyond the case of flat space.

Looking at the quantum algebra (3.19), we note that the last three relations just state that J_i^j are $U(N)$ generators and that $Q_i, \bar{Q}^j$ belong to the $\mathbf{N}$, $\bar{\mathbf{N}}$ of $U(N)$, and presumably these relations should be left unchanged even in curved space. The first equation is the key ingredient of the supersymmetry algebra, and is going to be modified by a nonvanishing curvature. Our aim is to deform the algebra (3.19) introducing curvature, but keeping it first class, as necessary if we want to impose the corresponding constraints consistently.

Thus, let us consider the theory on an arbitrary Kähler manifold. The only non vanishing components of the metric are $g_{\mu\bar{\nu}}(x,\bar{x}) = g_{\bar{\nu}\mu}(x,\bar{x})$, which lead to nonvanishing Christoffel coefficients for the total holomorphic or antiholomorphic parts only: $\Gamma^{\mu}_{\nu\lambda}$, $\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\lambda}}$. In curved space we will use fermions with flat indices: ψ_i^a and $\bar{\psi}^{\bar{a}i}$; the $U(N)$ generators are essentially unchanged, being defined by $J_i^j = \frac{1}{2}\big[\psi_i^a\,, \bar{\psi}_a^j\big]$ (the flat tangent metric is simply $\delta_{a\bar{b}}$), but the supercharges need a suitable covariantization. Since the holonomy group of Kähler manifolds of real dimension $D = 2d$ is $U(d)$, the connection would be a $U(d)$ spin connection, and the covariant derivative reads

$$\nabla_\mu = \partial_\mu + \omega_{\mu a\bar{b}} M^{a\bar{b}}$$

where $M^{a\bar{b}}$ are the $U(d)$ generators. In the particle model these generators can be realized by

$$M^{a\bar{b}} = \frac{1}{2}\big[\psi_i^a\,, \bar{\psi}^{\bar{b}\,i}\big] = \psi_i^a \bar{\psi}^{\bar{b}i} - \frac{N}{2}\delta^{a\bar{b}} \tag{3.51}$$

as they satisfy indeed the Lie algebra of $U(d)$

$$\big[M^{a\bar{b}}\,, M^{c\bar{d}}\big] = \delta^{c\bar{b}} M^{a\bar{d}} - \delta^{a\bar{d}} M^{c\bar{b}}\ .$$

In this way we construct covariantized momenta[2]

$$\begin{aligned} \pi_\mu &= g^{1/2}\big(p_\mu - i\omega_{\mu a\bar{b}} M^{a\bar{b}}\big) g^{-1/2} \\ \bar{\pi}_{\bar{\mu}} &= g^{1/2}\big(\bar{p}_{\bar{\mu}} - i\omega_{\bar{\mu} a\bar{b}} M^{a\bar{b}}\big) g^{-1/2}\ , \end{aligned} \tag{3.52}$$

[2] We denote $g = \det g_{\mu\bar{\nu}}$ and the g factors ensure hermiticity.

and supercharges

$$Q_i = \psi_i^a \, e_a^\mu \, \pi_\mu \quad , \quad \bar{Q}^j = \bar{\psi}^{\bar{a}j} \, e_{\bar{a}}^{\bar{\mu}} \, \bar{\pi}_{\bar{\mu}} \; . \tag{3.53}$$

With these charges the JJ, JQ and $J\bar{Q}$ commutators are the same as before, but the $Q\bar{Q}$ anticommutator now reads

$$\big\{Q_i \, , \bar{Q}^j\big\} = \delta_i^j \, H_0 - R_{a\bar{b}c\bar{d}} \, \psi_i^a \bar{\psi}^{\bar{b}j} M^{c\bar{d}} \; , \tag{3.54}$$

where $H_0 = g^{\bar{\mu}\nu}\bar{\pi}_{\bar{\mu}}\pi_\nu$ is the minimal covariantization of the hamiltonian. As in the case of $O(N)$ supersymmetry [31, 32], we can achieve the closure of the algebra on particular manifolds, namely Kähler manifolds with constant holomorphic curvature, which admit a Riemann tensor of the form [79]

$$R_{a\bar{b}c\bar{d}} = \Lambda \left(\delta_{a\bar{b}}\delta_{c\bar{d}} + \delta_{a\bar{d}}\delta_{c\bar{b}}\right) \; , \tag{3.55}$$

with constant Λ. As for real manifolds maximally symmetric spacetimes are de Sitter, anti-de Sitter and flat Minkowski space, prototypes of Kähler manifolds with a Riemann tensor of the form (3.55) are the complex projective space $\mathbb{CP}^d$, complex hyperbolic space $\mathbb{CH}^d$ and, of course, flat complex space $\mathbb{C}^d$ viewed as a Kähler manifold. Inserting the $U(d)$ generators $M^{a\bar{b}} = \frac{1}{2}\big[\psi_i^a \, , \bar{\psi}^{\bar{b}\,i}\big]$, the $\big\{Q, \bar{Q}\big\}$ anticommutator closes quadratically (up to an obvious redefinition of the hamiltonian)

$$\big\{Q_i \, , \bar{Q}^j\big\} = \delta_i^j \left(H_0 - a\,J - b\right) - \Lambda J_i^j \, J + \tfrac{\Lambda}{2} \left\{J_i^k \, , J_k^j\right\} \; , \tag{3.56}$$

with $J = J_k^k$, $a = \Lambda\frac{d+1}{2}$ and $b = \Lambda\frac{d(N+d)}{4}$. The hamiltonian H_0 has, however, an unusual commutator with the supercharges, namely

$$\begin{aligned} \big[H_0 \, , Q_i\big] &= -\Lambda \, J_i^k \, Q_k + \Lambda \, J \, Q_i + \Lambda \tfrac{N+d}{2} \, Q_i \quad , \\ \big[H_0 \, , \bar{Q}^i\big] &= -\big[H_0 \, , Q_i\big]^\dagger \; , \end{aligned} \tag{3.57}$$

so we add to H_0 a hermitian and $U(N)$ neutral J combination in order to cancel the commutators above. We recall that, including a Chern-Simons coupling, the quantum constraint on J is $J_i^j - s\delta_i^j = 0$ and so, in order to make manifest the quadratic closure of our algebra, we set $\tilde{J}_i^j = J_i^j - s\delta_i^j$ and $\tilde{J} = \tilde{J}_i^i$, finally obtaining

$$\begin{aligned} &\big[H \, , \tilde{J}_i^j\big] = \big[H \, , Q_i\big] = \big[H \, , \bar{Q}^j\big] = 0 \\ &\big[\tilde{J}_i^j \, , \tilde{J}_k^l\big] = \delta_k^j \, \tilde{J}_i^l - \delta_i^l \, \tilde{J}_k^j \\ &\big[\tilde{J}_i^j \, , Q_k\big] = \delta_j^k \, Q_i \quad , \quad \big[\tilde{J}_i^j \, , \bar{Q}^k\big] = -\delta_i^k \, \bar{Q}^j \\ &\big\{Q_i \, , \bar{Q}^j\big\} = \delta_i^j H + \Lambda \left[\tilde{J}_i^k \, \tilde{J}_k^j - \tilde{J}_i^j \tilde{J} + h_1 \tilde{J}_i^j + \frac{1}{2}\delta_i^j \left(\tilde{J}^2 - \tilde{J}_k^l \tilde{J}_l^k + h_2 \tilde{J}\right)\right] \; , \end{aligned} \tag{3.58}$$

where the complete hamiltonian reads

$$H = H_0 + \frac{\Lambda}{2}\left[J_i^k J_k^i - J^2 - h_3 J - h_4\right] \ , \tag{3.59}$$

with the h_i being defined by

$$\begin{aligned} h_1 &= \big(2-N\big)s - \frac{N}{2} \ , \\ h_2 &= 2s\big(N-2\big)+1 \ , \qquad h_3 = d+1 \ , \\ h_4 &= \frac{d}{2}\big(N+d\big) - s^2\big(N-1\big)\big(N-2\big) \ . \end{aligned} \tag{3.60}$$

This is no more a Lie algebra but, being still first class, permits a consistent realization of the constraints $\tilde{J}_i^j = H = Q_i = \bar{Q}^j = 0$, which define higher spin equations on such curved backgrounds. As the analogous result obtained in [32] for the $O(N)$ spinning particle, the quadratic algebra (3.58) coincides with the zero mode, in the Ramond sector, of the quadratic $U(N)$ superconformal algebra found by Bershadsky and Knizhnik in [75, 76].

Up to now we have used $U(d)$ generators with the preferred ordering given in (3.51), but a quadratic closure of the supersymmetry algebra can be achieved with an arbitrary ordering, corresponding to a different coupling to the $U(1)$ part of the spin connection $\omega_\mu = \omega_{\mu a\bar{b}}\delta^{a\bar{b}}$: if in eq. (3.52) we choose as $U(d)$ generators

$$\mathcal{M}^{a\bar{b}} = \psi_i^a \bar{\psi}^{\bar{b}i} - c\delta^{a\bar{b}} \ , \tag{3.61}$$

with arbitrary c, (3.54) remains unchanged in form, and choosing the Riemann tensor as in (3.55), the quadratic algebra in (3.58) and (3.59) maintains the same structure but with different numerical coefficients $h_i \to h_i(c)$, given by

$$\begin{aligned} h_1(c) &= \big(2-N\big)s - \frac{d}{2}\big(N-2c\big) + c - N \\ h_2(c) &= \big(d+1\big)\big(N-2c\big) + 2s\big(N-2\big) + 1 \\ h_3(c) &= \big(d+1\big)\big(N-2c+1\big) \\ h_4(c) &= d\left[\frac{d}{2}\big(N-2c+1\big) + N - c\right] + s\big(N-1\big) \\ &\quad \times \left[\big(d+1\big)\big(2c-N\big) - s\big(N-2\big)\right] \ . \end{aligned} \tag{3.62}$$

To recover the previous results is sufficient to put $c = N/2$ in the above formulas.

With this constraint algebra at hand it is possible to achieve the quantization of the $U(N)$ particle, for all N, on Kähler manifolds of constant holomorphic curvature.

3.4 Partition function and degrees of freedom

In order to extract from the $U(N)$ spinning particle action (3.13) the number of physical excitations, we proceed in computing the one-loop partition function that gives, as its first Seeley-DeWitt coefficient, the number of degrees of freedom. Of course, other heat kernel coefficients vanish in flat space, but once the measure over the moduli space arising from the gauge fixing procedure is correctly identified, one could perform, in principle, more general path integral calculations to investigate the quantum properties of the field equations on the backgrounds described previously.

In order to deal with gaussian path integrals rather than oscillating ones, we perform as usual a Wick rotation on the proper time $\tau \to -i\tau$ and on the gauge field $a^i_j \to i a^i_j$. The resulting euclidean action reads

$$S[X,G] = \int_0^1 d\tau \; \big[e^{-1}\big(\dot{x}^\mu - \bar{\chi}^i\psi^\mu_i\big)\big(\dot{\bar{x}}_\mu - \chi_j\bar{\psi}^j_\mu\big) + \bar{\psi}^i_\mu\big(\delta^j_i\partial_\tau - ia^j_i\big)\psi^\mu_i - isa^i_i\big] \tag{3.63}$$

and is invariant under the supergravity transformations in euclidean time

$$\begin{aligned} \delta e &= \dot{\xi} + \bar{\chi}^i\epsilon_i + \chi_i\bar{\epsilon}^i \\ \delta\chi_i &= \dot{\epsilon}_i - ia^k_i\epsilon_k + i\alpha^k_i\chi_k \\ \delta\bar{\chi}^i &= \dot{\bar{\epsilon}}^i + ia^i_k\bar{\epsilon}^k - i\alpha^i_k\bar{\chi}^k \\ \delta a^i_j &= \dot{\alpha}^i_j - ia^k_j\alpha^i_k + ia^i_k\alpha^k_j \; . \end{aligned} \tag{3.64}$$

The partition function is obtained by performing the functional integral on a circle, taking periodic boundary conditions for the bosonic fields, and antiperiodic ones for the fermionic fields

$$Z = \int_{S^1} \frac{DXDG}{\text{Vol(Gauge)}}\, e^{-S[X,G]} \tag{3.65}$$

where, in condensed notation, $X \equiv (x, \bar{x}, \psi, \bar{\psi})$ refers to the matter fields, while $G \equiv (e, \chi, \bar{\chi}, a)$ represents the supergravity multiplet. Since our model is a gauge theory, it is necessary to divide by the volume of the gauge group. The gauge fixing procedure can be achieved with the standard Faddeev-Popov method. We select a covariant gauge by imposing gauge fixing conditions on the worldline supergravity fields. The latter can be gauged away, except for a remaining finite number of modular integrations that take into account gauge inequivalent configurations. We follow the same strategy employed in [30] for the $O(N)$ spinning particle, to which we refer for additional details.

Gauge fixing on the circle

The einbein $e(\tau)$ has periodic boundary conditions and is characterized by the gauge invariant quantity $\beta = \int_0^1 e(\tau)d\tau$, which represents the invariant length of the circle. A standard gauge for worldline reparametrizations is to fix $e(\tau) = \beta$, and the path integral over e reduces to an ordinary integral over the usual proper time β, with the familiar "one-loop" measure

$$\int_0^\infty \frac{d\beta}{\beta} \, .$$

Due to antiperiodic boundary conditions, the complex gravitini χ_i and $\bar{\chi}^i$ can be completely gauged away, $\chi_i(\tau) = \bar{\chi}^i(\tau) = 0$, leaving corresponding Faddeev-Popov determinants of the differential operators that can be extracted from (3.64). Finally, the gauge field a_i^j can have nontrivial Wilson loops around the circle, that capture the complete gauge invariant information contained in them. They can be gauge fixed to a constant hermitian $N \times N$ matrix, $a_i^j(\tau) = \theta_i^j$, that can be always diagonalized through a constant $U(N)$ gauge transformation

$$\theta_i^j \to \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_N \end{pmatrix} . \tag{3.66}$$

Recalling that a_i^j belongs to the Lie algebra of $U(N)$, we see by exponentiation that the θ_i are in fact angles ranging from 0 to 2π. Now, the path integral over x and $\bar{x}$ gives as usual $V\,(2\pi\beta)^{-d}$, where $V = i^d \int d^d x_0 d^d \bar{x}_0$ (the integral over the x zero modes) is the spacetime volume. The $D\psi D\bar{\psi}$ integral gives $\mathrm{Det}_A(\delta_i^j \partial_\tau - i\theta_i^j)^d$, while integrals over the susy ghosts give a power -2 of the same determinant. Subscripts P and A keep track of the periodic or antiperiodic boundary conditions. From the diagonalization (3.66), we see that the integration over the moduli space of a_i^j reduces to integration over the angles

$$\frac{1}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi} \, , \tag{3.67}$$

and division by $N!$ is needed to eliminate the overcounting due to the permutations of the θ's, that are all gauge equivalent. The last integration to be performed is over the ghosts for the gauge group $U(N)$, that gives $\mathrm{Det}'_P(\partial_\tau + i\theta_{\mathrm{adj}})$, *i.e.* with the zero modes removed and the gauge fixed a_i^j taken in the adjoint representation, as follows from $\delta a_i^k = \mathcal{D}\alpha_i^k$ in (3.64). Now, we use the diagonalized form (3.66), and putting together the various contributions we obtain for the partition function

$$Z \propto V \int_0^\infty \frac{d\beta}{\beta} \frac{1}{(2\pi\beta)^d} \frac{1}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{-is\theta_i} \mathrm{Det}_A(\partial_\tau - i\theta_i)^{d-2} \times \prod_{k\neq l} \mathrm{Det}_P(\partial_\tau - i(\theta_k - \theta_l)) \,. \tag{3.68}$$

These determinants are standard ones and can be computed using operator methods with simple fermionic systems. Namely, they are: $\mathrm{Det}_A(\partial_\tau - i\theta) = 2\cos\frac{\theta}{2}$ and $\mathrm{Det}_P(\partial_\tau - i\theta) = 2i\sin\frac{\theta}{2}$. Substituting in the expression for Z one finally finds

$$Z \propto V \int_0^\infty \frac{d\beta}{\beta} \frac{1}{(2\pi\beta)^d} \left[\frac{1}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{-is\theta_i} \left(2\cos\frac{\theta_i}{2}\right)^{d-2} \times \prod_{k<l} \left(2\sin\frac{\theta_k - \theta_l}{2}\right)^2 \right] . \tag{3.69}$$

The part in square brackets of the above formula gives the number of degrees of freedom of the particle, since the rest is simply the partition function for the center of mass, and so we have the following expression for the physical degrees of freedom

$$Dof(d, N; s) = \frac{1}{N!} \prod_{i=1}^N \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{-is\theta_i} \left(2\cos\frac{\theta_i}{2}\right)^{d-2} \prod_{k<l} \left(2\sin\frac{\theta_k - \theta_l}{2}\right)^2 . \tag{3.70}$$

It is normalized to $Dof(d, 0; 0) = 1$ for $N = 0$, which corresponds to a simple scalar field. It is now convenient to go to complex coordinates: $z_i = e^{i\theta_i}$. Recalling that $s = m - \frac{d}{2} = p + 1 - \frac{d}{2}$, the above expression in terms of p becomes

$$Dof(d, N, p) = \frac{1}{N!} \prod_{i=1}^N \oint \frac{dz_i}{2\pi i} \frac{1}{z_i^{p+1}} (z_i + 1)^{d-2} \prod_{k<l} |z_k - z_l|^2 \tag{3.71}$$

where the integration contour is the unit circle around the origin in $\mathbb{C}$, $\forall i$. Now, we perform a new change of variables, passing from the unit complex circle to the real line by means of stereographic projection: $z_j = \frac{i - x_j}{i + x_j}$. The integral becomes

$$Dof(d, N, p) = \frac{2^{N^2 + Nd - 3N}}{N! \pi^N} \int_{\mathbb{R}^N} d^N x\, |\Delta(x)|^2 \prod_{j=1}^N \frac{(1 - ix_j)^{(p+2-d-N)}}{(1 + ix_j)^{(N+p)}} \tag{3.72}$$

where we have recognized the square of the Van der Monde determinant

$$\Delta(x) = \prod_{i<j} (x_i - x_j) \ . \tag{3.73}$$

Written in term of the x_i variables, (3.72) is seen to belong to a wide class of Selberg's integrals, that can be computed by means of orthogonal polynomials techniques[3]. The known Selberg's integral in question, that can be found in [80], reads

$$\begin{aligned} J(a,b,\alpha,\beta,\gamma,n) &= \int_{\mathbb{R}^N} d^N x \ |\Delta(x)|^{2\gamma} \prod_{j=1}^{n} (a + i x_j)^{-\alpha} (b - i x_j)^{-\beta} \\ &= \frac{(2\pi)^n}{(a+b)^{(\alpha+\beta)n - \gamma n(n-1) - n}} \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(n+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)} \end{aligned} \tag{3.74}$$

valid for Rea, Reb, Reα, Re$\beta > 0$, Re$(\alpha+\beta) > 1$, and

$$-\frac{1}{n} < \mathrm{Re}\gamma < \min\left(\frac{\mathrm{Re}\alpha}{n-1}, \frac{\mathrm{Re}\beta}{n-1}, \frac{\mathrm{Re}(\alpha+\beta+1)}{2(n-1)}\right) \ .$$

Our eq. (3.72) corresponds to this form of the Selberg's integral with ($a = b = \gamma = 1$, $\alpha = N + p$, $\beta = d + N - p - 2$, $n = N$) so, with (3.74) at hand, after a little algebra, we obtain the final result

$$\begin{aligned} Dof(d,N,p) &= \frac{2^{N^2+Nd-3N}}{N!\pi^N} J(1,1,N+p,d+N-p-2,1,N) \\ &= \prod_{j=0}^{N-1} \frac{j!(j+d-2)!}{(j+p)!(j+d-2-p)!} \end{aligned} \tag{3.75}$$

that agrees with the dimension of the rectangular Young tableau of $U(d-2)$ with p rows and N columns, as in (3.45), thus reproducing the number of physical polarizations predicted by canonical quantization.

In conclusion, by means of Dirac canonical quantization we analyzed the $U(N)$ spinning particle models and found new higher spin equations obeyed by complex HS fields in flat space. We found the correct measure on the moduli space for the $U(N)$ extended supergravity, and we realized that such HS equations can be defined also on constant holomorphic curvature (CHC) Kähler manifolds. We will pursue the quantization of the $U(1)$ particle on

[3]Much information and many details about these techniques can be found in [80].

arbitrary curved Kähler backgrounds in chapter 5, but it would be very interesting to investigate the quantum properties of the higher spin $U(N)$ particle on CHC spaces, and an even more exciting route would be to find consistent non-linear extensions of our linear equations in the spirit of Vasiliev's theory.

Chapter 4

$U(N|M)$ quantum mechanics on Kähler manifolds

$O(N)$ spinning particles [20, 21, 31] have been useful to describe higher spin fields in first quantization [30, 81]. Similarly, $U(N)$ spinning particles [38, 39] have been instrumental to discover a new class of higher spin field equations which possess a novel type of gauge invariance [40]. To investigate the quantum properties of these equations in their worldline formulation, it is important to study the related quantum mechanics. It is the purpose of this chapter to discuss these quantum mechanics, which in the most general case take the form of nonlinear sigma models.

First we shall discuss linear sigma models, i.e. models with flat complex space $\mathbb{C}^d$ as target space. These sigma models exhibit a $U(N)$ extended supersymmetry on the worldline. They define "spinning particle" models once the extended supersymmetry is made local. It is useful, and almost effortless, to extend these models by adding extra bosonic coordinates. This extension produces $U(N|M)$ sigma models, by which we mean sigma models with a worldline extended supersymmetry characterized by supercharges transforming in the fundamental representation of $U(N|M)$ (i.e. $U(N|M)$ is the R-symmetry group of the supersymmetry algebra). This extension may be useful for constructing wider classes of spinning particles, as happened in the case of the $OSp(N|2M)$ extension [57] of the standard $O(N)$ supersymmetric quantum mechanics, used for example in [59, 60, 82, 83] to describe higher spin fields. We present these quantum mechanical models and their symmetry algebra in the first section. Next, we consider sigma models with generic Kähler manifolds as target spaces. The symmetry algebra gets modified by the geometry, so that it will not be always possible to gauge the extended supersymmetry to obtain spinning particles and corresponding higher spin equations. This signals the difficulties of coupling higher spin fields to generic backgrounds, not to mention the even more difficult problem of constructing nonlinear field equations. However, on special backgrounds one can find a deformed $U(N|M)$ susy algebra that becomes first class, so that it can be gauged to produce consistent spinning particles. An example is the case of Kähler manifolds with constant holomorphic sectional curvature. No restrictions apply to the special cases of $U(1|0)$ and $U(2|0)$, whose susy algebra can be gauged to produce nontrivial field equations on any Kähler space, in analogy with standard $N = 1$ and $N = 2$ susy quantum mechanics on arbitrary riemannian manifolds (i.e. $O(1)$ and $O(2)$ quantum mechanics in the language used above).

Nevertheless, before gauging, the $U(N|M)$ quantum mechanics here constructed are perfectly consistent on any Kähler manifold, and even posses conserved supercharges when the Riemann tensor obeys a locally symmetric space condition (again in close analogy with the riemannian case [57]). Thus, we will work with an arbitrary Kähler manifold and compute the quantum mechanical transition amplitude in euclidean time (i.e. the heat kernel) in the limit of short propagation time and using operatorial methods. This last result is going to be particularly useful for obtaining an unambiguous construction of the corresponding path integral, which is needed when considering worldline applications. This is indeed one of our future aims, namely using worldline descriptions of higher spin fields to obtain useful and computable representations of their one-loop effective actions, as done in [28] for the $O(2)$ spinning particle. In that case a worldline representation allowed to compute in a single stroke the first few heat kernel coefficients and prove various duality relations for massless and massive p-forms in arbitrary dimensions. Finally, we confine to the appendices details of our calculations.

4.1 Linear $U(N|M)$ sigma model

We introduce here the $U(N|M)$ extended supersymmetric quantum mechanics. In the most simple case it describes the motion of a particle in $\mathbb{C}^d$, the flat complex space of d complex dimensions with coordinates $(x^\mu,\, \bar{x}^{\bar{\mu}})$, $\mu = 1, ..., d$. The flat metric in these complex coordinates is simply $\delta_{\mu\bar{\nu}}$, and we use it to raise and lower indices. In addition, the particle carries extra degrees of freedom described by worldline Dirac fermions $(\psi_a^\mu,\, \bar{\psi}_\mu^a)$ and complex bosons $(z_\alpha^\mu,\, \bar{z}_\mu^\alpha)$, where $a = 1, ..., N$ and $\alpha = 1, ..., M$ are indices in the $U(N)$ and $U(M)$ subgroups of $U(N|M)$, respectively. These extra degrees of freedom can be interpreted as worldline superpartners of the coordinates $(x^\mu,\, \bar{x}^{\bar{\mu}})$. Of course, when the superpartners have bosonic character one finds a kind of "bosonic" supersymmetry, that generalizes usual concepts. With these degrees of freedom at hand the phase space lagrangian defining our model has the standard form $\mathcal{L} \sim p\dot{q} - H$, namely

$$\mathcal{L} = p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}^a_\mu \dot{\psi}^\mu_a + i\bar{z}^\alpha_\mu \dot{z}^\mu_\alpha - p_\mu \bar{p}^\mu \ . \tag{4.1}$$

This model enjoys a $U(N|M)$ extended supersymmetry, which we are going to describe directly in the quantum case

The fundamental (anti)-commutators are easily read off from (4.1)

$$\begin{aligned} &[x^\mu, p_\nu] = i\hbar\delta^\mu_\nu \ , \qquad &[\bar{x}^{\bar{\mu}}, \bar{p}_{\bar{\nu}}] = i\hbar\delta^{\bar{\mu}}_{\bar{\nu}} \\ &\{\psi^\mu_a, \bar{\psi}^b_\nu\} = \hbar\delta^b_a\delta^\mu_\nu \ , \qquad &[z^\mu_\alpha, \bar{z}^\beta_\nu] = \hbar\delta^\beta_\alpha\delta^\mu_\nu \ . \end{aligned} \tag{4.2}$$

The $U(N|M)$ charges are readily constructed from the worldline operators

$$\begin{aligned} J^a_b &= \frac{1}{2}[\bar{\psi}^a_\mu, \psi^\mu_b] - c\hbar\delta^a_b = \bar{\psi}^a_\mu\psi^\mu_b - m\hbar\delta^a_b \qquad && U(N) \text{ subgroup}, \\ J^\alpha_\beta &= \frac{1}{2}\{\bar{z}^\alpha_\mu, z^\mu_\beta\} + c\hbar\delta^a_b = \bar{z}^\alpha_\mu z^\mu_\beta + m\hbar\delta^\alpha_\beta \qquad && U(M) \text{ subgroup}, \\ J^\alpha_b &= \bar{z}^\alpha_\mu\psi^\mu_b \ , \qquad J^a_\beta = \bar{\psi}^a_\mu z^\mu_\beta \qquad && U(N|M) \text{ fermionic generators}, \end{aligned} \tag{4.3}$$

where $m = c + \frac{d}{2}$. They obey the $U(N|M)$ algebra

$$\begin{aligned} [J^a_b, J^c_d] &= \hbar\,(\delta^c_b J^a_d - \delta^a_d J^c_b) \\ [J^\alpha_\beta, J^\gamma_\delta] &= \hbar\,(\delta^\gamma_\beta J^\alpha_\delta - \delta^\alpha_\delta J^\gamma_\beta) \\ [J^a_b, J^\alpha_c] &= -\hbar\,\delta^a_c J^\alpha_b \ , \quad [J^a_b, J^c_\alpha] = \hbar\,\delta^c_b J^a_\alpha \\ [J^\alpha_\beta, J^\gamma_a] &= \hbar\,\delta^\gamma_\beta J^\alpha_a \ , \quad [J^\alpha_\beta, J^a_\gamma] = -\hbar\,\delta^\alpha_\gamma J^a_\beta \\ \{J^\alpha_a, J^b_\beta\} &= \hbar\,(\delta^b_a J^\alpha_\beta + \delta^\alpha_\beta J^b_a) \ . \end{aligned} \tag{4.4}$$

In the definition of these charges we have used a "graded symmetric" ordering prescription modified by an arbitrary central charge c that specifies possible different orderings allowed by the symmetry algebra. The possibility of inserting the central charge is related to the algebraic fact that $U(N|M) = U(1) \times SU(N|M)$. All these charges commute with the hamiltonian $H = p_\mu\bar{p}^\mu$ and are conserved.

Other conserved quantities are the supersymmetric charges involving the space momenta: there are $2N$ fermionic supercharges $Q_a = \psi^\mu_a\, p_\mu$, $\bar{Q}^a = \bar{\psi}^a_\mu\,\bar{p}^\mu$, and $2M$ bosonic charges $Q_\alpha = z^\mu_\alpha\, p_\mu$, $\bar{Q}^\alpha = \bar{z}^\alpha_\mu\,\bar{p}^\mu$. All these operators form the $U(N|M)$ extended superalgebra that, together with the $U(N|M)$

internal algebra (4.4), is given by the following relations

$$
\begin{aligned}
&[J^a_b, Q_c] = -\hbar\, \delta^a_c\, Q_b\ , && [J^a_b, \bar{Q}^c] = \hbar\, \delta^c_b\, \bar{Q}^a \\
&[J^\alpha_\beta, Q_\gamma] = -\hbar\, \delta^\alpha_\gamma\, Q_\beta\ , && [J^\alpha_\beta, \bar{Q}^\gamma] = \hbar\, \delta^\gamma_\beta\, \bar{Q}^\alpha \\
&[J^\alpha_a, Q_\beta] = -\hbar\, \delta^\alpha_\beta\, Q_a\ , && [J^b_\alpha, \bar{Q}^\beta] = \hbar\, \delta^\beta_\alpha\, \bar{Q}^b \\
&\{J^a_\alpha, Q_b\} = \hbar\, \delta^a_b\, Q_\alpha\ , && \{J^\alpha_b, \bar{Q}^c\} = \hbar\, \delta^c_b\, \bar{Q}^\alpha \\
&\{Q_a, \bar{Q}^b\} = \hbar\, \delta^b_a\, H\ , && [Q_\alpha, \bar{Q}^\beta] = \hbar\, \delta^\beta_\alpha\, H\ .
\end{aligned}
\tag{4.5}
$$

(Anti)-commutators needed to close the algebra and not explicitly reported vanish.

All these relations can be written in a more covariant way. In order to show up the full supergroup structure, let us introduce the superindex $A = (a, \alpha)$ and the $U(N|M)$ metrics

$$
\delta^A_B = \begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}\ , \qquad \epsilon^A_B = \begin{pmatrix} -\delta^a_b & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}\ . \tag{4.6}
$$

The internal fermions and bosons are grouped into the fundamental and anti-fundamental representations of the supergroup, $Z^\mu_A = (\psi^\mu_a, z^\mu_\alpha)$, $\bar{Z}^A_\mu = (\bar{\psi}^a_\mu, \bar{z}^\alpha_\mu)$. The fundamental (anti)-commutation relations can be written as $[Z^\mu_A, \bar{Z}^B_\nu\} = \hbar\, \delta^B_A\, \delta^\mu_\nu$, or equivalently as $[\bar{Z}^B_\nu, Z^\mu_A,\} = -\hbar\, \epsilon^B_A\, \delta^\mu_\nu$. Here the graded commutator is used: $[A, B\}$ is defined as anti-commutator for A and B both fermionic, and as a commutator otherwise. Then we collect all the $U(N|M)$ generators in

$$
J^A_B = \begin{pmatrix} J^a_b & J^a_\beta \\ J^\alpha_b & J^\alpha_\beta \end{pmatrix} = \bar{Z}^A_\mu Z^\mu_B + m\hbar\, \epsilon^A_B\ . \tag{4.7}
$$

With these notations at hand the entire superalgebra (4.4) is packaged into the single relation

$$
[J^A_B, J^C_D\} = \hbar\, (\delta^C_B\, J^A_D \pm \delta^A_D\, J^C_B)\ , \tag{4.8}
$$

where the plus sign refers to the case with J^A_B and J^C_D both fermionic, and the minus sign to the other possibilities.

By means of this supergroup notation, the supercharges are written as $Q_A = (Q_a, Q_\alpha)$ and $\bar{Q}^A = (\bar{Q}^a, \bar{Q}^\alpha)$, and the above superalgebra is summarized by

$$
\begin{aligned}
&[J^A_B, Q_C\} = \pm\hbar\, \delta^A_C\, Q_B\ , \qquad [J^A_B, \bar{Q}^C\} = \hbar\, \delta^C_B\, \bar{Q}^A \\
&[Q_A, \bar{Q}^B\} = \hbar\, \delta^B_A\, H\ ,
\end{aligned}
\tag{4.9}
$$

where $\pm$ stands for plus for J^A_B and Q_C both fermionic, and minus otherwise.

All these quantum mechanical operators have simple geometrical meanings in terms of differential operators living on $\mathbb{C}^d$. Let us give a brief description. Generic wave functions of the Hilbert space can be represented by functions of the coordinates $(x, \bar{x}, \psi, z)$. Expanding them in ψ^μ and z^μ shows how they contain all possible tensors with $N + M$ blocks of holomorphic indices. Each of the first N blocks of indices is totally antisymmetric, while each of the last M blocks of indices is totally symmetric. In formulae

$$\phi(x,\bar{x},\psi,z) \sim \sum_{A_i=0}^{d} \sum_{B_i=0}^{\infty} \phi_{[\mu_1^1..\,\mu_{A_1}^1],...,[\mu_1^N..\,\mu_{A_N}^N],(\nu_1^1..\,\nu_{B_1}^1),...,(\nu_1^M..\,\nu_{B_M}^M)}(x,\bar{x}) \\ \times \left(\psi_1^{\mu_1^1}..\,\psi_1^{\mu_{A_1}^1}\right)\dots\left(\psi_N^{\mu_1^N}..\,\psi_N^{\mu_{A_N}^N}\right)\left(z_1^{\nu_1^1}..\,z_1^{\nu_{B_1}^1}\right)\dots\left(z_M^{\nu_1^M}..\,z_M^{\mu_{B_M}^M}\right) . \tag{4.10}$$

The quantum mechanical operators take the form of differential operators acting on these tensors. The hamiltonian is proportional the standard laplacian $H \sim \partial_\mu \bar{\partial}^\mu = \delta^{\mu\bar{\nu}}\,\partial_\mu \bar{\partial}_{\bar{\nu}}$. The supercharge Q_a acts as the Dolbeault operator ∂ restricted to the antisymmetric indices of block "a", and $\bar{Q}^a$ as its adjoint $\partial^\dagger$. Similarly the "bosonic" supercharge Q_α is realized as a symmetrized gradient acting on the symmetric indices of block "α", and $\bar{Q}^\alpha$ is its adjoint, taking the form of a divergence. The action of the $U(N|M)$ operators, i.e. the J_B^A charges, is also amusing: they perform certain (anti)-symmetrizations on the tensors indices, and we leave it to the interested reader to work them out explicitly. The algebra of these differential/algebraic operators, as encoded in the susy algebra, is only valid in flat space. In the next section we will see how this algebra extends to generic Kähler manifolds.

4.2 Nonlinear $U(N|M)$ sigma model

We now extend the previous construction to nonlinear sigma models with generic Kähler manifolds as target spaces. On Kähler manifolds, in holomorphic coordinates, the only non vanishing components of the metric are $g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}$, and similarly $\Gamma^\mu_{\nu\lambda}$ and $\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\lambda}}$ are the only non vanishing components of the connection. We use the following conventions for curvatures

$$R^\mu{}_{\nu\bar{\sigma}\lambda} = \partial_{\bar{\sigma}}\Gamma^\mu_{\nu\lambda}\ , \quad R^\mu_\nu = -g^{\bar{\sigma}\lambda}\,R^\mu{}_{\nu\bar{\sigma}\lambda}\ , \quad R = R^\mu_\mu\ , \tag{4.11}$$

and denote by $g = \det(g_{\mu\bar{\nu}})$ the determinant of the metric, as standard in Kähler geometry.

The classical phase space lagrangian with a minimally covariantized hamiltonian becomes

$$\mathcal{L} = p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}}\dot{\bar{x}}^{\bar{\mu}} + i\bar{Z}^A_\mu \dot{Z}^\mu_A - g^{\mu\bar{\nu}}(p_\mu - i\Gamma^\lambda_{\mu\sigma}\bar{Z}^A_\lambda Z^\sigma_A)\bar{p}_{\bar{\nu}} \tag{4.12}$$

though, for future applications, it will be useful to consider more general hamiltonians. The corresponding configuration space lagrangian is the typical one for nonlinear sigma models

$$\mathcal{L} = g_{\mu\bar{\nu}} \dot{x}^\mu \dot{\bar{x}}^{\bar{\nu}} + i \bar{Z}^A_\mu \frac{DZ^\mu_A}{dt} \tag{4.13}$$

where the covariant time derivative is given by $\frac{DZ^\mu_A}{dt} = \dot{Z}^\mu_A + \dot{x}^\nu \, \Gamma^\mu_{\nu\sigma} \, Z^\sigma_A$.

In the quantum case, it will be crucial to resolve ordering ambiguities by demanding target space covariance. Before discussing the quantum operators, let us make a few comments. We treat the $\bar{Z}^A_\mu$ fields as momenta, as such they have a natural lower holomorphic curved index. In this situation there is no real advantage in introducing a vielbein, so we will avoid introducing one. Also, the holonomy group of a Kähler manifold of complex dimensions d is $U(d)$, and it will be convenient to define the $U(d)$ generators

$$M_\mu^\nu = \frac{1}{2}[\bar{\psi}^a_\mu, \psi^\nu_a] + \frac{1}{2}\{\bar{z}^\alpha_\mu, z^\nu_\alpha\} - k\hbar\delta^\nu_\mu \tag{4.14}$$

where k is a central charge parametrizing different orderings allowed by the $U(d) = U(1) \times SU(d)$ symmetry. These generators can be written as well as

$$M_\mu^\nu = \bar{Z}^A_\mu Z^\nu_A - s\hbar\delta^\nu_\mu \tag{4.15}$$

with $s = k + \frac{N-M}{2}$. They satisfy the correct $U(d)$ algebra

$$[M_\nu^\mu, M_\sigma^\rho] = \hbar \, \delta^\mu_\sigma \, M^\rho_\nu - \hbar \, \delta^\rho_\nu \, M^\mu_\sigma \ . \tag{4.16}$$

We are now ready to discuss the covariantization of the quantum operators belonging to the $U(N|M)$ extended supersymmetry algebra. As we shall see, not all of the charges generate symmetries on generic Kähler manifolds: some of them do not commute with the hamiltonian and thus are not conserved.

It is easiest to start with the generators of $U(N|M)$. They are left unchanged as the metric does not enter their definition: $J^A_B = \bar{Z}^A_\mu Z^\mu_B + m\hbar\epsilon^A_B$. They satisfy the same $U(N|M)$ symmetry algebra given in eq. (4.8).

Now we consider the Q supercharges. To covariantize them we introduce covariant momenta

$$\bar{\pi}_{\bar{\mu}} = g^{1/2} \, \bar{p}_{\bar{\mu}} \, g^{-1/2} \ , \qquad \pi_\mu = g^{1/2} \left(p_\mu - i \, \Gamma^\lambda_{\mu\sigma} \, M^\sigma_\lambda\right) g^{-1/2} \ , \tag{4.17}$$

and write down covariantized supercharges as

$$Q_A = Z^\mu_A \, \pi_\mu \ , \qquad \bar{Q}^A = \bar{Z}^A_\mu \, g^{\mu\bar{\nu}} \, \bar{\pi}_{\bar{\nu}} \ . \tag{4.18}$$

Similarly, the covariant hamiltonian operator is given by

$$H_0 = g^{\bar{\mu}\nu}\bar{\pi}_{\bar{\mu}}\pi_\nu = g^{1/2}\, g^{\bar{\mu}\nu}\bar{p}_{\bar{\mu}}\left(p_\nu - i\,\Gamma^{\lambda}_{\nu\sigma}\, M^{\sigma}_{\lambda}\right) g^{-1/2}\ . \tag{4.19}$$

At this stage it is worthwhile to spend some words on the hermiticity properties of our operators: since the $\bar{Z}^A_\mu$ fields are defined as independent variables with lower holomorphic indices, but hermitian conjugation of vector indices naturally sends holomorphic into anti-holomorphic indices, and vice versa, the natural definition of the adjoint of Z^μ_A is $(Z^\mu_A)^\dagger = \bar{Z}^A_\nu\, g^{\nu\bar{\mu}}$. In this way, hermitian conjugation of the momentum is nontrivial: if $[p_\mu, Z^\nu_A] = 0$, it must hold that $[(p_\mu)^\dagger, (Z^\nu_A)^\dagger] = [(p_\mu)^\dagger, \bar{Z}^A_\lambda\, g^{\lambda\bar{\nu}}] = 0$ as well. Requiring this property we find

$$\left(p_\mu\right)^\dagger = \bar{p}_{\bar{\mu}} - i\,\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\sigma}}\, M^{\lambda}_{\sigma}\, g^{\sigma\bar{\sigma}} g_{\lambda\bar{\lambda}}\ . \tag{4.20}$$

Now, if we define the supercharges in the natural way written above, namely $Q_A = Z^\mu_A\,\pi_\mu$ and $\bar{Q}^A = \bar{Z}^A_\mu\, g^{\mu\bar{\nu}}\,\bar{\pi}_{\bar{\nu}}$, then it results that $(Q_A)^\dagger = \bar{Q}^A$ and $H_0^\dagger = H_0$. Note that the power of the metric determinant entering the various operators is necessary for verifying the hermiticity properties.

Let us now consider their algebra. The first line of (4.9) simply states that Q_A and $\bar{Q}^A$ belong to the fundamental and anti-fundamental representation of $U(N|M)$, and one can check that these relations remain unchanged even in curved space,

$$[J^A_B, Q_C\} = \pm\hbar\,\delta^A_C\, Q_B\ , \qquad [J^A_B, \bar{Q}^C\} = \hbar\,\delta^C_B\,\bar{Q}^A\ . \tag{4.21}$$

On the other hand the last relation becomes

$$[Q_A, \bar{Q}^B\} = \hbar\,\delta^B_A\, H_0 + \hbar\, Z^\mu_A \bar{Z}^B_\nu\, R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma} M^{\sigma}_{\lambda}\ . \tag{4.22}$$

The minimal covariant hamiltonian H_0, emerging from this commutator as the term multiplying δ^B_A and already given in (4.19), does not conserve the supercharges except than in flat space; in fact the commutator between H_0 and Q does not vanish and reads

$$\begin{aligned} &[Q_A, H_0] = \hbar\, Z^\mu_A\, R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma}\, M^{\sigma}_{\lambda}\,\pi_\nu + \hbar^2\, Z^\mu_A\, R^\nu_\mu\,\pi_\nu \\ &[\bar{Q}^A, H_0] \equiv -[Q_A, H_0]^\dagger\ . \end{aligned} \tag{4.23}$$

H_0 is a central operator only in flat space. Finally, it is simple to verify that

$$[Q_A, Q_B\} = [\bar{Q}^A, \bar{Q}^B\} = 0\ . \tag{4.24}$$

Relations (4.21), (4.22), (4.23) and (7.20), together with (4.8), describe the deformation of the $U(N|M)$ supersymmetry algebra realized by our quantum nonlinear sigma model on a Kähler manifold. Supersymmetry is broken as the supercharges are not conserved. Only on flat spaces the hamiltonian H_0 becomes central and the supercharges get conserved.

Given this state of affairs, one may try to redefine the hamiltonian in an attempt to make it central on more general backgrounds, thus recovering conserved supercharges. For this purpose, we add to H_0 several non minimal couplings

$$H = H_0 + c_1 R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma}\, M^{\mu}_{\nu}\, M^{\sigma}_{\lambda} + c_2\, \hbar\, R^{\mu}_{\nu}\, M^{\nu}_{\mu} + c_3\, \hbar^2 R\ . \tag{4.25}$$

With these generic couplings (4.23) becomes

$$\begin{aligned}[][Q_A, H] = {} & \hbar\,(1+2c_1)\, Z^{\mu}_A\, R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma}\, M^{\sigma}_{\lambda}\, \pi_\nu + \hbar^2\,(1+c_1+c_2)\, Z^{\mu}_A\, R^{\nu}_{\mu}\, \pi_\nu \\ & - i\hbar c_1 Z^{\rho}_A \nabla_\rho R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma}\, M^{\mu}_{\nu}\, M^{\sigma}_{\lambda} - i\hbar^2 c_2\, Z^{\sigma}_A \nabla_\sigma R^{\mu}_{\nu}\, M^{\nu}_{\mu} - i\hbar^3 c_3\, Z^{\mu}_A \nabla_\mu R\ . \end{aligned} \tag{4.26}$$

We see that for the choice $c_1 = -\frac{1}{2}$, $c_2 = -\frac{1}{2}$ and generic c_3, the terms in the first line proportional to the covariant momentum π_ν vanish and, choosing $c_3 = 0$ for simplicity, we identify a canonical hamiltonian $H_{(c)}$ so that eq. (4.26) reduces to

$$[Q_A, H_{(c)}] = \frac{i\hbar}{2}\, Z^{\rho}_A\, \nabla_\rho R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma}\, M^{\mu}_{\nu}\, M^{\sigma}_{\lambda} + \frac{i\hbar^2}{2}\, Z^{\sigma}_A\, \nabla_\sigma R^{\mu}_{\nu}\, M^{\nu}_{\mu}\ , \tag{4.27}$$

showing that $H_{(c)}$ is central on locally symmetric spaces. Of course, also the graded commutator (4.22) changes and becomes

$$[Q_A, \bar{Q}^B\} = \hbar\, \delta^B_A\, H_{(c)} + \hbar\, R^{\nu}{}_{\mu}{}^{\lambda}{}_{\sigma} \left(Z^{\mu}_A \bar{Z}^B_\nu + \frac{1}{2}\, \delta^B_A\, M^{\mu}_{\nu} \right) M^{\sigma}_{\lambda} + \frac{1}{2}\, \hbar^2\, \delta^B_A\, R^{\mu}_{\nu}\, M^{\nu}_{\mu}\ . \tag{4.28}$$

Thus one concludes that with the redefinition of the hamiltonian given above the supercharges are conserved on locally symmetric Kähler manifolds.

One of the most interesting applications of the nonlinear sigma models discussed so far is to use them to construct spinning particles and related higher spin equations. This is achieved by gauging the extended susy algebra identified by the charges $(H, Q_A, \bar{Q}^A, J^B_A)$, possibly with a suitable redefinition of the hamiltonian. Unfortunately, we see that on generic Kähler manifolds the $U(N|M)$ extended susy algebra is not first class, as additional independent operators appear on the right hand sides, as evident for example in eqs. (4.27) and (4.28). However, there are special cases, namely the $U(1|0)$ and $U(2|0)$ quantum mechanics, which generate first class superalgebras with a central hamiltonian on any Kähler background. In fact, for the $U(1|0) \equiv U(1)$ model the algebra reduces to

$$\{Q, \bar{Q}\} = \hbar\, H\ , \quad [Q, H] = 0 \tag{4.29}$$

where the hamiltonian is now defined by

$$H = H_0 - \frac{\hbar}{2} R^{\mu}_{\nu} M^{\nu}_{\mu} + \frac{\hbar^2}{4} R = H_0^{sym} + \frac{\hbar^2}{4} R \,, \tag{4.30}$$

with $H_0^{sym} = \frac{1}{2} g^{\mu\bar\nu}(\pi_\mu \bar\pi_{\bar\nu} + \bar\pi_{\bar\nu}\pi_\mu)$. For the $U(2|0) \equiv U(2)$ model the choice of the hamiltonian is the canonical one, *i.e.* the one in (4.25) with $c_1 = c_2 = -\frac{1}{2}$ and $c_3 = 0$, and the superalgebra closes as

$$\{Q_a, \bar Q^b\} = \delta_a^b\, H \,, \quad [Q_a, H] = 0 \,. \tag{4.31}$$

For the general $U(N|M)$ extended susy algebras one cannot achieve such generality. Nevertheless, one may look for special backgrounds that make (4.27) and (4.28) first class. A nontrivial class of Kähler manifolds where the first class property can be achieved is that of manifolds with constant holomorphic sectional curvature. On these manifolds, the Riemann and Ricci tensors take the form

$$R_{\mu\bar\nu\sigma\bar\lambda} = -\frac{R}{d(d+1)} (g_{\mu\bar\nu} g_{\sigma\bar\lambda} + g_{\sigma\bar\nu} g_{\mu\bar\lambda}) \,, \qquad R_{\mu\bar\nu} = \frac{R}{d} g_{\mu\bar\nu} \tag{4.32}$$

where R is the constant scalar curvature. Substituting these relations into the algebra, one notices that the metric tensor gets contracted with the Z and $\bar Z$ operators, producing additional charges J_A^B on the right hand side, so that with a suitable redefinition of the hamiltonian one obtains a first class algebra for generic m, s, c_1 and c_2, while c_3 gets fixed to a unique value. There is no loss of generality in choosing c_1 and c_2 equal to their canonical values, $c_1 = c_2 = -\frac{1}{2}$, when using the algebra as a first class constraint algebra. In this case

$$\begin{aligned} c_3 = &-\frac{m}{2d(d+1)} \Big((N-M)^2 + (N-M)(4d - 3m - 2s + 1) + 2(m-d) \Big) \\ &+ \frac{s}{2} \Big(1 + \frac{2(d-m)}{d} - \frac{s}{d+1} \Big) \end{aligned} \tag{4.33}$$

and the algebra can be casted in the following form

$$\begin{aligned} [Q_A, \bar Q^B\} &= \hbar \delta_A^B\, H - \frac{\hbar R}{d(d+1)} \Big\{ (-)^{(A+B)C} J_A^C J_C^B + (-)^{AB} J_A^B J + (-)^{AB} \hbar k_1 J_A^B \\ &\quad + \delta_A^B \Big(\frac{1}{2} J_D^C \epsilon_E^D J_C^E + \frac{1}{2} J^2 + \hbar k_2 J \Big) \Big\} \,, \\ [Q_A, H] &= 0 \end{aligned} \tag{4.34}$$

where

$$\begin{aligned} k_1 &= d - s(d+1) + m(N-M-2) \\ k_2 &= d - s(d+1) - \Big(m + \frac{1}{2}\Big)(N-M) + \frac{1}{2}\ . \end{aligned} \tag{4.35}$$

We denoted $J \equiv J_A^A$ and used the notation $(-)^A$ with $A = 0$ for a bosonic index and $A = 1$ for a fermionic one. Gauging this first class algebra produces "$U(N|M)$ spinning particles" on Kähler manifolds with constant holomorphic curvature, in a way analogous to the coupling of standard "$O(N)$ spinning particles" to (A)dS spaces constructed in [31].

One may recall that Kähler spaces with constant holomorphic sectional curvature are a subclass of spaces with vanishing Bochner tensor. The latter is a sort of complex analogue of the riemannian Weyl tensor, introduced in [84] and defined by

$$\begin{aligned} B_{\mu\bar\nu\sigma\bar\lambda} &= R_{\mu\bar\nu\sigma\bar\lambda} + \frac{1}{d+2}(g_{\mu\bar\nu}R_{\sigma\bar\lambda} + g_{\sigma\bar\lambda}R_{\mu\bar\nu} + g_{\sigma\bar\nu}R_{\mu\bar\lambda} + g_{\mu\bar\lambda}R_{\sigma\bar\nu}) \\ &\quad - \frac{R}{(d+1)(d+2)}(g_{\mu\bar\nu}g_{\sigma\bar\lambda} + g_{\sigma\bar\nu}g_{\mu\bar\lambda})\ . \end{aligned} \tag{4.36}$$

It satisfies the nice property of being traceless, $g^{\mu\bar\nu}B_{\mu\bar\nu\sigma\bar\lambda} = 0$. It seems likely that on spaces with vanishing Bochner tensor one may obtain a first class algebra, indeed it is relatively easy to verify it at the classical level, but we do not wish to pursue the detailed quantum analysis here.

4.3 Transition amplitude

Up to now we have discussed nonlinear sigma models with $U(N|M)$ extended supersymmetry, broken at times by the target space geometry, and used them to analyze algebraic properties of differential operators defined on Kähler manifolds. The aim of this section is the explicit computation of the transition amplitude in euclidean time, that is $\langle x\,\bar\eta|e^{-\frac{\beta}{\hbar}H}|y\,\xi\rangle$, in the limit of short propagation time and using operatorial methods. Such a calculation was presented for standard nonlinear sigma models with one, two or no supersymmetries in [64], see also [37], with the main purpose of identifying a benchmark to which compare path integral evaluations of the same heat kernel. As we wish to be able to master path integrals for $U(N|M)$ sigma models, and eventually use them to address quantum properties of higher spin equations on Kähler manifolds, we compute here the heat kernel using the operatorial formulation of quantum mechanics. To achieve sufficient generality and allow diverse applications, we compute the heat kernel for the general hamiltonian (4.25) containing three arbitrary couplings (c_1, c_2, c_3) to the background curvature plus a fourth one, the charge s, hidden in the $U(1)$ part of the connection, see eq. (4.15).

Before starting the actual computation, we shall review our set up. We work on a $2d$ real dimensional Kähler manifold as target space. Holomorphic and anti-holomorphic vector indices will be often grouped into a riemannian index $i = (\mu, \bar{\mu})$ for sake of brevity. The metric in holomorphic coordinates factorizes as follows

$$g_{ij} = \begin{pmatrix} 0 & g_{\mu\bar{\nu}} \\ g_{\bar{\mu}\nu} & 0 \end{pmatrix} . \tag{4.37}$$

For determinants we use the conventions $g = \det(g_{\mu\bar{\nu}})$ and $G = |\det(g_{ij})| = |g|^2$. The dynamical variables of the $U(N|M)$ supersymmetric quantum mechanics consist of the following operators: target space coordinates $(x^\mu, \bar{x}^{\bar{\mu}}) = x^i$, conjugate momenta p_i, and graded vectors Z^μ_A and $\bar{Z}^A_\nu$. Their fundamental (anti)-commutation relations are given in (4.2). For computational advantages we recast the full quantum hamiltonian (4.25) in a way that directly shows the dependence on the Z operators

$$\begin{aligned} H &= H_0 + \Delta H \quad \text{with} \\ H_0 &= g^{\bar{\mu}\nu}\, g^{1/2}\, \bar{p}_{\bar{\mu}} \left(p_\nu - i\,\Gamma^\lambda_{\nu\sigma}\, M^\sigma_\lambda\right) g^{-1/2} \\ \Delta H &= a_1\, R_\mu{}^\nu{}_\rho{}^\sigma\, \bar{Z}_\nu \cdot Z^\mu\, \bar{Z}_\sigma \cdot Z^\rho + a_2\, \hbar\, R^\mu_\nu\, \bar{Z}_\mu \cdot Z^\nu + a_3\, \hbar^2\, R\,, \end{aligned} \tag{4.38}$$

where the a couplings are related to the c couplings by

$$a_1 = c_1\,, \quad a_2 = c_2 + 2sc_1\,, \quad a_3 = c_3 - sc_2 - s^2 c_1\,. \tag{4.39}$$

Finally, it useful to recall that the final answer for the heat kernel will contain the exponent of the classical action, suitably Wick-rotated to euclidean time τ $(t \to -i\tau)$, which in phase space takes the form

$$S = \int_{-\beta}^{0} d\tau \Big[- ip_\mu \dot{x}^\mu - i\bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + \bar{Z}^A_\mu \dot{Z}^\mu_A + H_{cl} \Big] \tag{4.40}$$

where H_{cl} is the classical hamiltonian, a function, modified by suitable quantum corrections depending on $\hbar$.

Now we are ready for the explicit computation of the transition amplitude, through order β (up to the leading free particle propagator), between position eigenstates and coherent states for the internal degrees of freedom, *i.e.*

$$\langle x\, \bar{\eta}| e^{-\frac{\beta}{\hbar} H} |y\, \xi\rangle\,, \tag{4.41}$$

where $Z^\mu_A|\xi\rangle = \xi^\mu_A|\xi\rangle$ and $\langle\bar{\eta}|\bar{Z}^A_\mu = \langle\bar{\eta}|\bar{\eta}^A_\mu$. Of course, $|x\rangle$ and $|y\rangle$ denote eigenvectors of the position operator x^i as usual, $|y\,\xi\rangle \equiv |y\rangle \otimes |\xi\rangle$, and so on. For convenience in the normalization of the coherent states, from now on we rescale the Z fields by a factor of $\sqrt{\hbar}$, so that $[Z^\mu_A, \bar{Z}^B_\nu\} = \delta^\nu_\mu\, \delta^B_A$. We are going to insert in (4.41) a complete set of momentum eigenstates, and as an intermediate stage we need to compute

$$\langle x\, \bar{\eta}| e^{-\frac{\beta}{\hbar}H} |p\, \xi\rangle \;, \tag{4.42}$$

pushing all p's and Z's to the right, all x's and $\bar{Z}$'s to the left, taking into account all (anti)-commutators and then substituting these operators with the corresponding eigenvalues. Let us focus on the evaluation of (4.42); clearly we have

$$\langle x\, \bar{\eta}| e^{-\frac{\beta}{\hbar}H} |p\, \xi\rangle = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \left(\frac{\beta}{\hbar}\right)^k \langle x\, \bar{\eta}| H^k |p\, \xi\rangle \;. \tag{4.43}$$

It is well known that, in the case of a nonlinear sigma model, it is not sufficient to expand the exponent to first order, *i.e.* $e^{-\beta H/\hbar} \sim 1 - \frac{\beta}{\hbar}H$, to obtain the correct transition amplitude to order β, see [64, 37]. Contributions for all k must be retained in the sum (4.43), but taking into account at most two $[x, p]$ commutators. Let us see this in more detail. In a factor of H^k, pushing all p's to the right by repeated use of the $[x, p]$ commutator, one obtains, remembering that each H can give at most two p eigenvalues,

$$\langle x\, \bar{\eta}| H^k |p\, \xi\rangle = \sum_{l=0}^{2k} B^k_l(x, \bar{\eta}, \xi)\, p^l\, \langle x\, \bar{\eta}|p\, \xi\rangle \;, \tag{4.44}$$

where p^l stands for a homogeneous polynomial in p of degree l. For the position eigenstates we use the normalization: $\langle x|x'\rangle = g^{-1/2}(x)\delta^{2d}(x - x')$, while the standard normalization is employed for p-eigenstates. In this way the completeness relations read

$$\mathbb{1} = \int d^{2d}p\, |p\rangle\langle p| \quad , \quad \mathbb{1} = \int d^{2d}x\, g\, |x\rangle\langle x| \;, \tag{4.45}$$

while the plane waves are given by: $\langle x|p\rangle = (2\pi\hbar)^{-d} g^{-1/2}(x) e^{ip\cdot x}$, with $p\cdot x \equiv p_i x^i = p_\mu x^\mu + \bar{p}_{\bar{\mu}} \bar{x}^{\bar{\mu}}$. Finally, coherent states are normalized as $\langle\bar{\eta}|\xi\rangle = e^{\bar{\eta}\cdot\xi}$. Having set our normalizations, we expand the transition amplitude as follows

$$\begin{aligned}
\langle x\, \bar{\eta}| e^{-\frac{\beta}{\hbar}H} |y\, \xi\rangle &= (2\pi\hbar)^{-d}\, g^{-1/2}(y) \int d^{2d}p\, e^{-\frac{i}{\hbar}p\cdot y}\, \langle x, \bar{\eta}| e^{-\beta H/\hbar} |p\, \xi\rangle \\
&= \tfrac{1}{(2\pi\hbar)^{2d}\, [g(x)g(y)]^{1/2}} \int d^{2d}p\, e^{\frac{i}{\hbar}p\cdot(x-y)}\, e^{\bar{\eta}_\mu\cdot\xi^\mu} \sum_{k=0}^{\infty} \left(-\frac{\beta}{\hbar}\right)^k \frac{1}{k!} \sum_{l=0}^{2k} B^k_l(x, \bar{\eta}, \xi)\, p^l \;.
\end{aligned} \tag{4.46}$$

Now, to make the β dependence explicit, we rescale momenta as $p_i = \sqrt{\hbar/\beta}\, q_i$ and obtain

$$\langle x\,\bar\eta| e^{-\frac{\beta}{\hbar}H} |y\,\xi\rangle = (4\pi^2\hbar\beta)^{-d}[g(x)g(y)]^{-1/2} e^{\bar\eta_\mu\cdot\xi^\mu} \int d^{2d}q\, e^{iq\cdot(x-y)/\sqrt{\beta\hbar}} \times \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \sum_{l=0}^{2k} \left(\frac{\beta}{\hbar}\right)^{k-l/2} B_l^k(x,\bar\eta,\xi)\, q^l \,. \tag{4.47}$$

After momentum integration, in configuration space the leading term in $(x-y)$ will be of the form $\exp[-(x-y)^2/2\beta\hbar]$, showing that effectively $(x-y)\sim \mathcal{O}(\beta^{1/2})$. Then, looking at (4.47), we see that $q\sim\mathcal{O}(\beta^0)$ and so in the sum over l only B^k_{2k}, B^k_{2k-1} and B^k_{2k-2} will contribute, for all k, to the order β amplitude, as anticipated[1].

The B_l^k coefficients are explicitly derived in appendix D, and inserting (D.3) and (D.4) into (4.47), one can see that the sum in k can be immediately performed, producing the gaussian exponential $\exp[-q^2/2]$. The transition amplitude (4.47) then becomes

$$\begin{aligned}
&\langle x\,\bar\eta| e^{-\frac{\beta}{\hbar}H} |y\,\xi\rangle = (4\pi^2\hbar\beta)^{-d}[g(x)g(y)]^{-1/2} e^{\bar\eta_\mu\cdot\xi^\mu} \int d^{2d}q\, e^{-q^2/2 - iq\cdot z/\sqrt{\beta\hbar}} \\
&\times \Big\{ 1 + \sqrt{\beta\hbar}\Big[\frac{i}{2}g^j q_j - \frac{i}{4}g^{klj}\, q_j\, q_k\, q_l + i g^{\bar\mu\nu}\,\Gamma^\lambda_{\nu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)'\bar q_{\bar\mu}\Big] \\
&+ \beta\hbar\Big[-\frac{1}{32}\ln G_i \ln G^i - \frac{1}{8}\ln G^i_i - \frac{1}{8}g^i \ln G_i \\
&- \Big(\frac{1}{4}\partial^j g^l + \frac{1}{8}g^j g^l + \frac{1}{8}g^k g^{jl}_k + \frac{1}{8}g^{jlk}_k\Big)\, q_j\, q_l + \Big(\frac{1}{12}g^{mnkl} + \frac{1}{8}g^{klm}g^n + \frac{1}{12}g^{ikl}g^{mn}_i \\
&+ \frac{1}{24}g^{kl}_i g^{mni}\Big)\, q_k\, q_l\, q_m\, q_n - \Big(\frac{1}{32}g^{klj}g^{pqm}\Big)\, q_j\, q_k\, q_l\, q_m\, q_p\, q_q \\
&- \frac{1}{2}g^{ij}\partial_j\Big(g^{\mu\bar\nu}\,\Gamma^\lambda_{\mu\sigma}\Big)(\bar\eta_\lambda\cdot\xi^\sigma)' q_i\, \bar q_{\bar\nu} \\
&- \frac{1}{2}g^{\bar\mu\nu}\,\Gamma^\rho_{\nu\sigma}\,(\bar\eta_\rho\cdot\xi^\sigma)'\Big(\partial_{\bar\mu}g^{\lambda\bar\sigma}\, q_\lambda\, \bar q_{\bar\sigma} + g^j q_j\, \bar q_{\bar\mu} - \frac{1}{2}g^{klj}\, q_j\, q_k\, q_l\, \bar q_{\bar\mu} + g^{\lambda\bar\sigma}\partial_{\bar\mu} g_{\lambda\bar\sigma}\Big) \\
&- a_1\, R_\mu{}^\nu{}_\rho{}^\sigma\, \bar\eta_\nu\cdot\xi^\mu \bar\eta_\sigma\cdot\xi^\rho - (a_2 - a_1 + 1)\, R^\mu_\nu\, \bar\eta_\mu\cdot\xi^\nu - \Big(a_3 - s\Big)\, R \\
&- \frac{1}{2}g^{\bar\mu\nu}\Gamma^\mu_{\nu\tau}g^{\lambda\bar\sigma}\Gamma^\rho_{\lambda\sigma}\, \bar q_{\bar\mu}\, \bar q_{\bar\sigma}\, \big[(\bar\eta_\mu\cdot\xi^\tau)'(\bar\eta_\rho\cdot\xi^\sigma)' + \delta^\tau_\rho\, \bar\eta_\mu\cdot\xi^\sigma\big]\Big]\Big\}\,,
\end{aligned} \tag{4.48}$$

where $z^i = y^i - x^i$ and $(\bar\eta_\lambda\cdot\xi^\sigma)' = (\bar\eta_\lambda\cdot\xi^\sigma - s\,\delta^\sigma_\lambda)$. In order to lighten the

[1] Note that in B_l^k at most $2k-l$ $[x,p]$ commutators are taken into account.

formulae we have used the following compact notation

$$\partial_i ... \partial_m g^{jk} = g^{jk}_{i...m}\ , \quad g^{ij} g^{kl}_j = g^{kli}\ , \quad g^{ij}_j = g^i$$
$$g^{jk}\partial_k g^{lm}_m = \partial^j g^l\ , \quad \partial_i \ln G = \ln G_i\ , \quad g^{ij}\partial_i\partial_j \ln G = \ln G^i_i\ .$$

Now we can complete squares in the exponent of (4.48), shift integration variables and perform the gaussian integral over momenta. The transition amplitude, up to order β, is then given by

$$\begin{aligned}
&\langle x\,\bar\eta|e^{-\frac{\beta}{\hbar}H}|y\,\xi\rangle = (2\pi\hbar\beta)^{-d}\,\big[g(x)/g(y)\big]^{1/2}\,e^{-\frac{1}{2\beta\hbar}g_{ij}z^i z^j}\,e^{\bar\eta_\mu\cdot\xi^\mu}\\
&\times\Big\{1 + z^i\,g^{-1/2}\,\partial_i\,g^{1/2} - \frac{1}{4\beta\hbar}\,\partial_k g_{ij}\,z^i z^j z^k + \frac{1}{2}\,z^i z^j\,g^{-1/2}\,\partial_i\partial_j g^{1/2}\\
&-\frac{1}{4\beta\hbar}\,z^i g^{-1/2}\partial_i g^{1/2}\partial_k g_{mn}\,z^k z^m z^n + \frac{1}{2}\left[\frac{1}{4\beta\hbar}\,\partial_k g_{ij}\,z^i z^j z^k\right]^2\\
&-\frac{1}{12\beta\hbar}\left[\partial_k\partial_l g_{ij} - \frac{1}{2}g_{mn}\,\Gamma^m_{ij}\,\Gamma^n_{kl}\right]z^i z^j z^k z^l + \frac{1}{6}\,R_{\mu\bar\nu}\,z^\mu\bar z^{\bar\nu}\\
&+z^\nu\,\Gamma^\lambda_{\nu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)' + \left[z^\nu\Gamma^\lambda_{\nu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)'\right]\left[z^i g^{-1/2}\partial_i g^{1/2}\right] + \frac{1}{2}\left[z^\nu\Gamma^\lambda_{\nu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)'\right]^2\\
&-\frac{1}{4\beta\hbar}\,\partial_j g_{kl}\,z^j z^k z^l\left(z^\nu\Gamma^\mu_{\nu\sigma}\,(\bar\eta_\mu\cdot\xi^\sigma)'\right) + \frac{1}{2}z^i z^\mu\partial_i\Gamma^\lambda_{\mu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)'\\
&+\frac{1}{2}\,z^\nu z^\lambda\,\Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{\lambda\rho}\,\bar\eta_\mu\cdot\xi^\rho - a_1\,\beta\hbar\,R_\mu{}^\nu{}_\rho{}^\sigma\,\bar\eta_\nu\cdot\xi^\mu\,\bar\eta_\sigma\cdot\xi^\rho\\
&+\left(a_1 - a_2 - \frac{1}{2}\right)\beta\hbar\,R^\mu_\nu\,\bar\eta_\mu\cdot\xi^\nu + \left(\frac{1}{6} + \frac{s}{2} - a_3\right)\beta\hbar\,R + \mathcal{O}(\beta^{3/2})\Big\}\ .
\end{aligned}\tag{4.49}$$

All functions in (4.49), if not specified otherwise, are evaluated at point x. Keeping in mind that the transition amplitude is a bi-scalar, and that in a semiclassical expansion the classical action evaluated on-shell should appear in the exponent, we factorize and exponentiate, up to order β, four terms

$$\begin{aligned}
&\langle x\,\bar\eta|e^{-\frac{\beta}{\hbar}H}|y\,\xi\rangle = (2\pi\hbar\beta)^{-d}\,g(y)^{-1/2}\Big[g^{1/2} + z^i\partial_i g^{1/2} + \frac{1}{2}\,z^i z^j\partial_i\partial_j g^{1/2}\Big]\\
&\exp\Big\{-\frac{1}{\beta\hbar}\Big[\frac{1}{2}\,g_{ij}\,z^i z^j + \frac{1}{4}\,\partial_i g_{jk}\,z^i z^j z^k\\
&+\frac{1}{12}\left(\partial_k\partial_l g_{mn} - \frac{1}{2}\,g_{ij}\,\Gamma^i_{kl}\,\Gamma^j_{mn}\right)z^k z^l z^m z^n\Big]\Big\}\\
&\exp\Big\{\bar\eta_\mu\cdot\xi^\mu + z^\nu\,\Gamma^\lambda_{\nu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)' + \frac{1}{2}z^i z^\mu\partial_i\Gamma^\lambda_{\mu\sigma}\,(\bar\eta_\lambda\cdot\xi^\sigma)' + \frac{1}{2}\,z^\nu z^\lambda\,\Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{\lambda\rho}\,\bar\eta_\mu\cdot\xi^\rho\\
&-a_1\,\beta\hbar\,R_\mu{}^\nu{}_\rho{}^\sigma\,\bar\eta_\nu\cdot\xi^\mu\,\bar\eta_\sigma\cdot\xi^\rho - a_2\,\beta\hbar\,R^\mu_\nu\,\bar\eta_\mu\cdot\xi^\nu - a_3\,\beta\hbar\,R\Big\}\\
&\Big[1 + \frac{1}{6}\,R_{\mu\bar\nu}\,z^\mu\bar z^{\bar\nu} + \left(a_1 - \frac{1}{2}\right)\beta\hbar\,R^\mu_\nu\,\bar\eta_\mu\cdot\xi^\nu + \left(\frac{1}{6} + \frac{s}{2}\right)\beta\hbar\,R\Big]\ .
\end{aligned}\tag{4.50}$$

The first term contains the Taylor expansion around x of $g(y)^{1/2}$, that cancel the $g(y)^{-1/2}$ factor. The second and third terms should be the expansions of the exponential of the classical action, and the fourth is evidently covariant. The detailed study of the expansion of the on-shell action is demanded to appendix E. Comparing the result (E.11) for the classical on-shell action $\tilde{S}_{os}$ with the expansion (4.50), we see that, as expected, the transition amplitude can finally be cast in an explicitly covariant form

$$\langle x\,\bar{\eta}|e^{-\frac{\beta}{\hbar}H}|y\,\xi\rangle = (2\pi\hbar\beta)^{-d}\, e^{-\tilde{S}_{os}/\hbar}\Big[1+\frac{1}{6}\,R_{\mu\bar{\nu}}\,z^{\mu}\bar{z}^{\bar{\nu}}+\Big(a_1-\frac{1}{2}\Big)\,\beta\hbar\,R^{\mu}_{\nu}\,\bar{\eta}_{\mu}\cdot\xi^{\nu} + \Big(\frac{1}{6}+\frac{s}{2}\Big)\,\beta\hbar\,R+\mathcal{O}(\beta^2)\Big] \tag{4.51}$$

where the coordinate displacements z^{μ} are considered of order $\sqrt{\beta}$.

The transition amplitude computed here can be used in order to define functional integrals for the corresponding spinning particle models. For example, it will be useful in the next chapter to fix the correct counterterm needed in performing the path integral for the $U(1)$ spinning particle on general Kähler backgrounds. More generally, it is the fundamental starting point for every future calculation involving the complex HS related to $U(N|M)$ spinning particles.

Chapter 5

Quantum theory of massless $(p,0)$-forms

In the previous chapters we have analyzed complex spinning particles with $U(N)$ extended supersymmetries on the worldline. We noticed that the $U(1)$ spinning particle describes massless $(p,0)$-forms, it can be consistently quantized on arbitrary Kähler backgrounds and we individuated its quantum hamiltonian. Indeed, in this chapter we are going to perform its path integral quantization in order to obtain a worldline representation of the one-loop effective action of the $(p,0)$-form. This representation allows to discuss exact duality relations and address a related topological mismatch. We also compute the first few heat kernel coefficients appearing in the local expansion of the effective action.

5.1 Free particles and canonical quantization

In this section we review the free particle model and its Dirac quantization to describe how the Maxwell equations for a $(p,0)$-form in flat complex space $\mathbb{C}^d$ emerge naturally from first-quantizing a particle system. We first consider a supersymmetric particle that produces a Hilbert space $\mathcal{H}$ formed by the $(p,0)$-forms with arbitrary p

$$\mathcal{H} = \bigoplus_{p=0}^{d} \Lambda^{p,0}(\mathbb{C}^d)$$

where by $\Lambda^{p,q}$ we indicate the space of (p,q)-forms. This mechanical model contains conserved supercharges Q and $\bar{Q}$ realized on the Hilbert space by the Dolbeault operator ∂ and its hermitian conjugate $\partial^\dagger$. The supercharges belong to a multiplet of conserved charges containing also the hamiltonian H and a U(1) charge J. This multiplet satisfies a $U(1)$ extended supersymmetry algebra. Gauging all these charges produces the action of the U(1) spinning particle that leads to the quantum theory of a $(p,0)$-form obeying holomorphic Maxwell equations. The details are as follows.

Let us consider a particle moving in flat complex space $\mathbb{C}^d$ and described by the complex coordinates $(x^\mu, \bar{x}^{\bar{\mu}})$, with $\mu = 1,..,d$. The particle carries additional degrees of freedom associated to the Grassmann variable ψ^μ and complex conjugate $\bar{\psi}^{\bar{\mu}}$. Indices are lowered and raised with the flat metric $\delta_{\mu\bar{\nu}}$ and its inverse. With these ingredients, the ungauged model is identified by the phase space action

$$S = \int_0^1 d\tau \left[p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}_\mu \dot{\psi}^\mu - p_\mu \bar{p}^\mu\right] \tag{5.1}$$

that indeed describes the motion of a free particle on $\mathbb{C}^d$. The conserved charges

$$H = p_\mu \bar{p}^\mu \ , \quad Q = \psi^\mu p_\mu \ , \quad \bar{Q} = \bar{\psi}^{\bar{\mu}} \bar{p}_{\bar{\mu}} \ , \quad J = \psi^\mu \bar{\psi}_\mu \tag{5.2}$$

guarantee the existence of a U(1)-extended supersymmetry on the worldline. Canonical quantization shows immediately that the corresponding Hilbert space can be realized by the set of all $(p,0)$-forms with $p = 0,1,..,d$. In fact, the elementary commutation relations obtained from the classical Poisson brackets read

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \ , \qquad [\bar{x}^{\bar{\mu}}, \bar{p}_{\bar{\nu}}] = i\delta^{\bar{\mu}}_{\bar{\nu}} \ , \qquad \{\psi^\mu, \bar{\psi}_\nu\} = \delta^\mu_\nu \ . \tag{5.3}$$

By considering $(x^\mu, \bar{x}^{\bar{\mu}}, \psi^\mu)$ as coordinates and $(p_\mu, \bar{p}_{\bar{\mu}}, \bar{\psi}_\mu)$ as momenta, one can realize the latter as differential operators with respect to the former,

$$p_\mu = -i\partial_\mu \ , \qquad \bar{p}_{\bar{\mu}} = -i\bar{\partial}_{\bar{\mu}} \ , \qquad \bar{\psi}_\mu = \frac{\partial}{\partial \psi^\mu}$$

(we use left derivative for Grassmann variables), so that a generic wave function $\phi(x, \bar{x}, \psi)$ has a finite expansion with respect to the Grassmann variables and contains all differential $(p,0)$-forms up to $p = d$

$$\phi(x, \bar{x}, \psi) = \sum_{k=0}^{d} \frac{1}{k!} F_{\mu_1 \ldots \mu_k}(x, \bar{x})\, \psi^{\mu_1} \ldots \psi^{\mu_k} \tag{5.4}$$

There are a total of 2^d independent components, which equals the number of the independent components of a Dirac fermion. This is not a coincidence, as it is known that on Kähler manifolds the space of all $(p,0)$-forms is equivalent to the Hilbert space of a Dirac fermion [85]. The Hilbert space metric is the one that emerges naturally by considering coherent states for worldline fermions, and takes the following schematic form

$$\langle\chi|\phi\rangle = \int dx d\bar{x} d\psi d\bar{\psi}\ \mathrm{e}^{\bar{\psi}\psi}\ \overline{\chi(x,\bar{x},\psi)}\ \phi(x,\bar{x},\psi)\ . \tag{5.5}$$

On the Hilbert space thus constructed the quantized conserved charges are represented by differential operators. In particular, the operator $Q = -i\psi^\mu\partial_\mu$ naturally acts as the Dolbeault operator $\partial = dx^\mu \wedge \partial_\mu$ on $(p,0)$-forms. Then, $\bar{Q} = -i\bar{\partial}^\mu \frac{\partial}{\partial\psi^\mu}$ corresponds to the adjoint of Q and is realized by the $\partial^\dagger$ operator acting on $(p,0)$-forms. The Hamiltonian is given by the laplacian $H = -\bar{\partial}^\mu\partial_\mu$. Finally, the U(1) charge operator $J = \psi^\mu\frac{\partial}{\partial\psi^\mu}$ counts the rank p of a $(p,0)$-form, up to a normal ordering ambiguity that we shall discuss in a short while.

The $U(1)$ extended supersymmetry algebra satisfied by these operators is easily computed and reads

$$\{Q,\bar{Q}\} = H\ , \qquad [J,Q] = Q\ , \qquad [J,\bar{Q}] = -\bar{Q} \tag{5.6}$$

while other (anti)-commutators vanish.

The $U(1)$ spinning particle we shall consider is obtained by gauging all of the symmetries generated by the charges in (5.2). The emerging model has a $U(1)$ extended local supersymmetry on the worldline, and it is characterized by the phase space action

$$S = \int dt\ \Big[p_\mu\dot{x}^\mu + \bar{p}_{\bar{\mu}}\dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}_\mu\dot{\psi}^\mu - eH - i\chi\bar{Q} - i\bar{\chi}Q + a(J-s)\Big] \tag{5.7}$$

where $G = (e,\chi,\bar{\chi},a)$ are the worldline gauge fields that make local the symmetries generated by the constraints $T = (H,Q,\bar{Q},J-s)$. The coupling s in (5.7) can be considered as a Chern-Simons coupling (note that its redefinition can take into account different ordering prescriptions that may be chosen when constructing the operator J in canonical quantization). It is crucial for obtaining a non-empty model, and for this purpose it must be quantized to integer values. In a Dirac quantization scheme, one can gauge-fix the worldline gauge fields to predetermined values, and require the constraints to annihilate physical states: $T|\phi_{phys}\rangle = 0$. The constraint $J - s = 0$ selects $(s,0)$-forms

$$\phi_{phys}(x,\bar{x},\psi) = \frac{1}{s!}F_{\mu_1..\mu_s}(x,\bar{x})\psi^{\mu_1}..\psi^{\mu_s}\ , \tag{5.8}$$

so that the model may be non-empty if the coupling s in an integer with values $0 \le s \le d$. For convenience we set $s \equiv p+1$, so that the J constraint selects the $(p+1,0)$-form $F_{(p+1,0)}$. Then the Q constraints is equivalent to

$$\partial F_{(p+1,0)} = 0 \tag{5.9}$$

which can be solved by $F_{(p+1,0)} = \partial A_{(p,0)}$, up to a gauge transformation $\delta A_{(p,0)} = \partial\lambda_{(p-1,0)}$. Finally, the $\bar{Q}$ constraint gives the remaining Maxwell equation

$$\partial^\dagger F_{(p+1,0)} = 0 \tag{5.10}$$

that reads as $\partial^\dagger\partial A_{(p,0)} = 0$ in terms of the gauge potential.

In components, the equations of motion of the field strength take the form

$$\partial_{[\mu} F_{\mu_1..\mu_{p+1}]} = 0\ , \qquad \bar{\partial}^{\mu_1} F_{\mu_1..\mu_{p+1}} = 0 \tag{5.11}$$

and are expressed in terms of the gauge potential by

$$\begin{aligned} &F_{\mu_1..\mu_{p+1}} = \partial_{\mu_1} A_{\mu_2..\mu_{p+1}} \pm \text{cyclic permutations} \\ &\bar{\partial}^\mu \partial_\mu A_{\mu_1..\mu_p} + (-1)^p p\, \bar{\partial}^\mu \partial_{[\mu_1} A_{\mu_2..\mu_p]\mu} = 0 \end{aligned} \tag{5.12}$$

with square brackets indicating weighted antisymmetrization. These equations are invariant under gauge transformations $\delta A_{(p,0)} = \partial\lambda_{(p-1,0)}$, i.e.

$$\delta A_{\mu_1..\mu_p} = \partial_{\mu_1}\lambda_{\mu_2..\mu_p} \pm \text{cyclic permutations of the indices}\ .$$

In particular, for $p=1$ one obtains the simple holomorphic Maxwell equations

$$\bar{\partial}^\mu F_{\mu\nu} = 0\ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.13}$$

with gauge symmetry $\delta A_\mu = \partial_\mu \lambda$.

Of course, different models can be obtained by gauging different subgroups of the $U(1)$ extended supermultiplet of charges. In particular, if one decides to gauge the hamiltonian H and the linear combination of the supercharges $Q+\bar{Q}$, one obtains a first quantized description of a massless Dirac field. In fact, on Kähler manifolds the Hilbert space of a fermion corresponds to the collection of all $(p,0)$-forms, and the Dirac operator corresponds to the real supercharge $Q+\bar{Q} \sim \partial + \partial^\dagger$, see [85]. Thus, its first quantization is obtained by quantizing the worldline action

$$S = \int dt\, \Big[p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}_\mu \dot{\psi}^\mu - eH - i\chi(Q+\bar{Q})\Big] \tag{5.14}$$

where χ is now a real gravitino on the worldline.

5.2 Coupling to gravity and the Dirac index

We are now going to consider the coupling to an arbitrary background Kähler metric. For curvature tensors we will employ the following Kählerian conventions:

$$R^{\mu}{}_{\nu\bar{\sigma}\lambda} = \partial_{\bar{\sigma}}\Gamma^{\mu}_{\nu\lambda}\ , \quad R^{\mu}_{\nu} = -g^{\bar{\sigma}\lambda}\, R^{\mu}{}_{\nu\bar{\sigma}\lambda}\ , \quad R = R^{\mu}_{\mu}\ , \quad \Box R = g^{\mu\bar{\nu}}\nabla_{\bar{\nu}}\nabla_{\mu}R\ . \tag{5.15}$$

It is useful to start with the ungauged version of the particle, as it provides us with a nonlinear sigma model that contains already all operators of interest. As a first application we use it to compute the Dirac index by calculating the partition function with periodic boundary conditions.

A simple way to introduce couplings to the background Kähler metric, while maintaining the $U(1)$ extended supersymmetry, is to consider the covariantization of the symmetry charges $J, Q, \bar{Q}$, and then imposing the susy algebra to obtain the correct hamiltonian H. The covariantization, at the classical level, is achieved by introducing "covariant" momenta (their Poisson bracket is proportional to the curvature tensor) in the supersymmetry charges

$$J = \psi^{\mu}\bar{\psi}_{\mu}\ , \qquad Q = \psi^{\mu}\big(p_{\mu} - i\Gamma^{\lambda}_{\mu\sigma}\bar{\psi}_{\lambda}\psi^{\sigma}\big)\ , \qquad \bar{Q} = \bar{\psi}_{\mu}g^{\mu\bar{\mu}}\bar{p}_{\bar{\mu}}\ . \tag{5.16}$$

Now one can compute their Poisson bracket algebra and check that the U(1)-extended supersymmetry algebra is realized with the hamiltonian

$$H = g^{\mu\bar{\nu}}\bar{p}_{\bar{\nu}}\big(p_{\mu} - i\Gamma^{\lambda}_{\mu\sigma}\bar{\psi}_{\lambda}\psi^{\sigma}\big)\ . \tag{5.17}$$

No non-minimal terms arise form requiring supersymmetry at the classical level. With this H the searched for model reads

$$S = \int dt\, \Big[p_{\mu}\dot{x}^{\mu} + \bar{p}_{\bar{\mu}}\dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}_{\mu}\dot{\psi}^{\mu} - H\Big]\ . \tag{5.18}$$

Eliminating the momenta $(p, \bar{p})$ one obtains the corresponding nonlinear sigma model in configuration space

$$S = \int dt\, \Big[g_{\mu\bar{\nu}}\dot{x}^{\mu}\dot{\bar{x}}^{\bar{\nu}} + i\bar{\psi}_{\mu}D_t\psi^{\mu}\Big] \tag{5.19}$$

where the covariant time derivative is given by $D_t\psi^{\mu} = \dot{\psi}^{\mu} + \dot{x}^{\nu}\Gamma^{\mu}_{\nu\lambda}\psi^{\lambda}$. The covariant supersymmetry charges are the same we introduced in chapter 4, and we already discussed their tricky hermiticity properties. Although they seem to be asymmetric the action is real, up to boundary terms. The aforementioned asymmetry between Q and $\bar{Q}$ is just due to our choice of variables, that indeed simplifies computations, but a canonical transformation

$\psi^a = e^a_\mu\, \psi^\mu$, $\bar\psi^{\bar a} = e^{\bar a \mu}\, \bar\psi_\mu$ brings back the action to a perfectly symmetric form:

$$\begin{aligned} S_{\text{fermion}} &= i \int dt\, \bar\psi_\mu \Big(\dot\psi^\mu + \dot x^\nu \Gamma^\mu_{\nu\lambda} \psi^\lambda \Big) \\ &= i \int dt\, \bar\psi_a \Big(\dot\psi^a + \dot x^\mu \omega_\mu{}^a{}_b \psi^b + \dot{\bar x}^{\bar\mu} \omega_{\bar\mu}{}^a{}_b \psi^b \Big) \end{aligned} \tag{5.20}$$

that descends from the symmetric supercharges employed in chapter 3. The two canonically related choices of variables show up in the quantum theory as states and operators related by a similarity transformation. As already outlined for the flat space case, canonical quantization produces an Hilbert space formed by the space of all $(p,0)$-forms living on the Kähler manifold M, that is $\mathcal{H} = \bigoplus_{p=0}^d \Lambda^{p,0}(M)$. One may again expect the susy charges Q and $\bar Q$ to be represented by the Dolbeault operators ∂ and $\partial^\dagger$, and the real charge $Q + \bar Q$ by the Dirac operator $\nabla\!\!\!/ \; = \gamma^\mu \nabla_\mu + \bar\gamma^{\bar\mu} \bar\nabla_{\bar\mu}$. This is indeed correct on manifolds of $SU(d)$ holonomy, where the Dirac operator satisfies $\nabla\!\!\!/ \; = \gamma^\mu \nabla_\mu + \bar\gamma^{\bar\mu} \bar\nabla_{\bar\mu} \sim \partial + \partial^\dagger$. However, on generic Kähler manifolds of $U(d)$ holonomy, one may find a nontrivial coupling to the $U(1)$ part of the $U(d) = U(1) \times SU(d)$ connection. Therefore, let us analyze in more details the operatorial realization of the susy charges in terms of differential operators to unearth the precise couplings to the $U(1)$ part of the connection, as this will be crucial for the interpretation of the path integral calculations described in the next section.

The quantum realization of the holomorphic covariant momentum, as shown in chapter 4, reads

$$\pi_\mu = g^{1/2} \Big(p_\mu - i \Gamma^\lambda_{\mu\sigma}\, M^\sigma_\lambda \Big) g^{-1/2} \tag{5.21}$$

where the $U(d)$ "Lorentz" generator can be realized as

$$M^\mu_\nu = \frac{1}{2} [\bar\psi_\nu, \psi^\mu] = -\psi^\mu \bar\psi_\nu + \frac{1}{2} \delta^\mu_\nu \, . \tag{5.22}$$

Let us stress that the ordering prescription used in defining M^μ_ν can shift the central charge by any amount. If one chooses $M^\mu_\nu = -\psi^\mu \bar\psi_\nu + q \delta^\mu_\nu$, the holomorphic charge Q will be represented by

$$iQ = \nabla_q \equiv \partial + q\, \Gamma \,,$$

where $\Gamma = \Gamma^\nu_{\nu\mu} dx^\mu$ is the $U(1)$ connection one form, obeying $\nabla_q^2 = 0$. Conversely, the anti-holomorphic charge is given by the usual adjoint Dolbeault operator

$$i\bar Q = \partial^\dagger = \frac{\partial}{\partial (dx^\mu)} g^{\mu\bar\nu} \partial_{\bar\nu} \,,$$

and the hamiltonian is represented by

$$H_q = \{Q, \bar{Q}\} = -\frac{1}{2} g^{\mu\bar{\nu}} \Big(\nabla^q_\mu \nabla_{\bar{\nu}} + \nabla_{\bar{\nu}} \nabla^q_\mu \Big) + \frac{1}{4} R \,,$$

where the holomorphic covariant derivative ∇^q_μ contains a $U(1)$ piece with coupling q that acts even on scalars as $\nabla^q_\mu \phi = (\partial_\mu + q\,\Gamma_\mu)\phi$. The precise understanding of the relation between the q charge chosen in the operatorial picture and its realization in the path integral is still under investigation; nonetheless, a previous heat kernel computation, carried on by using flat indices and the spin connection, showed that the path integral without explicit coupling to the $U(1)$ part of the connection, reproduces the "canonical" value $q = \frac{1}{2}$ given in (5.22). Hence, for the time being, we will choose no explicit $U(1)$ charge in the path integral action.

Let us now review the calculation of the Dirac index using this supersymmetric sigma model [16, 65], as it will enter subsequent discussions. The connection between index theorems and supersymmetric quantum mechanics makes use of the concept of the Witten index, defined as $\mathrm{Tr}\,(-1)^F$, where F is the fermion number and the trace is over the quantum mechanical Hilbert space. Standard reasonings show that the Witten index counts the number of bosonic zero energy states minus the number of fermionic zero energy states [86]. It is a topological invariant that computes the index of the differential operator representing the hermitian supercharge $Q + \bar{Q}$, the Dirac operator $\partial + \partial^\dagger$ in our case. In the Hilbert space of the particle system, bosonic states are given by $(p, 0)$-forms with even p, and fermionic states by forms with odd p. They correspond to positive chirality and negative chirality spinors, respectively. Thus for our quantum mechanical model the Witten index reduces to the Dirac index. Being a topological invariant it can be regulated as $\mathrm{Tr}\,(-1)^F \mathrm{e}^{-\beta H}$, where H is the hamiltonian, and computed for small β using its path integral representation

$$\mathrm{ind}(\nabla\!\!\!\!/\ ) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{-\beta H} = \int_P DxD\psi\ \mathrm{e}^{-S} \qquad (5.23)$$

where the subscript P indicates periodic boundary conditions for all bosonic and fermionic fields, and S is the Wick rotated version of the action in (5.19), namely

$$S = \int_0^\beta d\tau\, \Big[g_{\mu\bar{\nu}} \dot{x}^\mu \dot{\bar{x}}^{\bar{\nu}} + \bar{\psi}_\mu D_\tau \psi^\mu \Big]\,. \qquad (5.24)$$

To calculate (5.23) one expands all periodic fields in Fourier series with frequencies $\frac{2\pi n}{\beta}$. For small β the zero modes dominate, and one only needs to take care of the semiclassical corrections due to a bosonic determinant. It is useful to use Riemann normal coordinates adapted to the Kähler structure, scale suitably the fermionic zero mode by $\beta^{-\frac{1}{2}}$ and obtain

$$\mathrm{ind}(\not\nabla\ ) = \int \frac{d^{2d}x_0 d^d\psi_0 d^d\bar\psi_0}{(2\pi i)^d} \left[\frac{\mathrm{Det}'(-\partial_\tau^2 + \mathcal{R}\partial_\tau)}{\mathrm{Det}'(-\partial_\tau^2)}\right]^{-1} \tag{5.25}$$

where Det' indicates a functional determinant on the space of periodic fields orthogonal to the zero modes, the subscript 0 indicates zero modes, and $\mathcal{R} = R^\mu{}_{\nu\bar\lambda\sigma}\bar\psi_0^{\bar\lambda}\psi_0^\sigma$ describes a matrix valued two-form. Now one can compute the functional determinant and express it in terms of a standard $d\times d$ determinant of a function of the matrix $\mathcal{R}$

$$\frac{\mathrm{Det}'(-\partial_\tau^2 + \mathcal{R}\partial_\tau)}{\mathrm{Det}'(-\partial_\tau^2)} = \det\left(\frac{\sinh \mathcal{R}/2}{\mathcal{R}/2}\right) . \tag{5.26}$$

Berezin integration over the Grassmann variables extracts from the expansion of this determinant the contribution of the top $2d$-form only. Thus one can reabsorb the measure factor into the determinant and present the final answer as

$$\mathrm{ind}(\not\nabla\ ) = \int_M \det\left(\frac{\mathcal{R}/4\pi}{\sinh \mathcal{R}/4\pi}\right) \tag{5.27}$$

where now $\mathcal{R} = R^\mu{}_{\nu\bar\lambda\sigma} d\bar x^{\bar\lambda} dx^\sigma$.

Note that, as just mentioned, for a given Kähler manifold M only the top form from the expansion of the determinant in (5.27) contributes. But since the determinant of an even function of $\mathcal{R}$ has an expansion in terms of $\mathcal{R}^2$, the index is nonvanishing only for manifolds of even complex dimensions. The first example is for $d = 2$, where the above formula gives

$$\mathrm{ind}(\not\nabla\ ) = -\frac{1}{96\pi^2}\int_M \mathrm{tr}\ \mathcal{R}^2 = \frac{1}{96\pi^2}\int_M d^2x d^2\bar x g \left(R_{\mu\bar\nu}R^{\mu\bar\nu} - R_{\mu\bar\nu\lambda\bar\sigma}R^{\mu\bar\nu\lambda\bar\sigma}\right) . \tag{5.28}$$

5.3 Effective action of quantized $(p, 0)$-forms

We are now ready to come to the main part of the chapter, and discuss the quantization of $(p, 0)$-forms and corresponding effective actions using worldline methods. Generically one is not able to compute the effective action exactly, but here we aim at obtaining a useful worldline representation of the one-loop effective action in an arbitrary Kähler background. The effective action may be depicted by the Feynman diagram in Figure 1, where a quantum $(p, 0)$-form circulates in the loop and external lines represent the curved

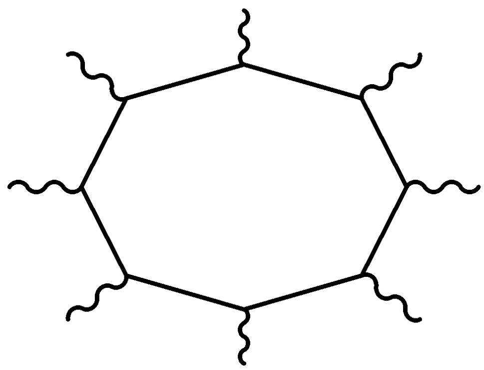


Figure 5.1: Feynman diagram for the one-loop effective action. A quantum $(p, 0)$-form circulates in the loop and external lines represent the curved background.

background. Its worldline representation allows to study various quantum properties and derive exact duality relations. One may then try to compute it in some perturbative expansion. Here, we compute the first few heat kernel coefficients appearing when considering a short time approximation.

As discussed, to obtain the Maxwell equation for a $(p, 0)$-form we need to gauge the whole $U(1)$ supersymmetry algebra carried by the ungauged model in (5.18). The suitable covariantization of the charges has been described in the previous section, see eqs. (5.16) and (5.17). The action with local symmetries is obtained by coupling gauge fields to the charges and adding a Chern-Simons coupling s. Thus one obtains an action of the same form as in (5.7), but with covariantized charges. To recover the euclidean action in configuration space we first eliminate momenta by means of their equations of motion, and then perform a Wick rotation, obtaining finally

$$S[X, G] = \int d\tau \left[e^{-1} g_{\mu\bar{\nu}} \big(\dot{x}^\mu - \bar{\chi}\psi^\mu \big) \big(\dot{\bar{x}}^{\bar{\nu}} - \chi \bar{\psi}^{\bar{\nu}} \big) + \bar{\psi}_\mu \big[D_\tau + ia \big] \psi^\mu + isa \right] , \quad (5.29)$$

where we denoted $\bar{\psi}^{\bar{\nu}} = g^{\mu\bar{\nu}} \bar{\psi}_\mu$, $X = (x, \bar{x}, \psi, \bar{\psi})$, $G = (e, \chi, \bar{\chi}, a)$, and where the covariant time derivative is given by $D_\tau \psi^\mu = \dot{\psi}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\lambda} \psi^\lambda$. Note that along with the Wick rotation $t \to -i\tau$, we have rotated also the gauge field $a \to ia$ to keep the $U(1)$ gauge group compact.

Quantization of this spinning particle model on a circle parametrized by $\tau \in [0, 1]$ gives the partition function for the holomorphic $(p, 0)$-form coupled to a Kähler curved space

$$Z[g] \propto \int \frac{\mathcal{D}X \mathcal{D}G}{\text{Vol(Gauge)}} \, \mathrm{e}^{-S} \quad (5.30)$$

and corresponds to Figure 1. A point that it is worth stressing is that we will regulate the path integral, and the related functional determinants, so that they correspond to a graded-symmetric operatorial ordering of the current J, namely $J = \frac{1}{2}(\psi^\mu\bar{\psi}_\mu - \bar{\psi}_\mu\psi^\mu) = \psi^\mu\bar{\psi}_\mu - \frac{d}{2}$, an ordering that is responsible for the standard fermionic zero point energy. Then the projection to the physical field strenght $F_{(p+1,0)}$ is obtained by using the Chern-Simons coupling $s \equiv p+1-\frac{d}{2}$ (so that $J - s = \psi\bar{\psi} - (p+1)$ as an operator).

Using the standard Fadeev-Popov procedure to get rid of gauge redundancy, we fix gauge fields to the constant values $G = (\beta, 0, 0, \phi)$, and we are left with modular integrations over β and ϕ, with the following one-loop measure that was carefully studied in [28]

$$Z[g] \propto \int_0^\infty \frac{d\beta}{\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\left(2\cos\frac{\phi}{2}\right)^{-2}\int_{\mathrm{P}}\mathcal{D}x\mathcal{D}\bar{x}\int_{\mathrm{A}} D\bar{\psi}D\psi\, \mathrm{e}^{-S_{\mathrm{gf}}} \tag{5.31}$$

with S_{gf} denoting the gauge fixed action, and P and A denoting periodic and antiperiodic boundary conditions, respectively. The integral over β is the usual proper time integral with the well known one-loop measure, while the factor $\left(2\cos\frac{\phi}{2}\right)^{-2}$ is the Fadeev-Popov determinant of the bosonic superghosts associated to χ and $\bar{\chi}$. We denote with $\mathcal{D}x$ the general coordinate invariant measure, *i.e.* $\mathcal{D}x \sim \prod_{\tau=0}^{1} d^dx(\tau)g^{1/2}(x(\tau))$, with $g = \det g_{\mu\bar{\nu}}$, while $D\psi \sim \prod_{\tau=0}^{1} d^d\psi(\tau)$ is the simple translational invariant measure.[1] Now, we choose an arbitrary x_0 as a base-point for our loop. The path integral then factorizes as $\int_{\mathrm{P}}\mathcal{D}x\mathcal{D}\bar{x} = \int d^dx_0 d^d\bar{x}_0 g(x_0)\int_{x(0)=x(1)=x_0}\mathcal{D}x\mathcal{D}\bar{x}$. It is possible then to perform background-fluctuations splitting as $x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$, with $q^\mu(0) = q^\mu(1) = 0$. Clearly the x path integral becomes $\int_{\mathrm{D}}\mathcal{D}q\mathcal{D}\bar{q}$, where D stands for Dirichlet boundary conditions, *i.e.* fields are taken to vanish at boundaries. The next step is that of getting rid of the field dependent measure $\mathcal{D}q\mathcal{D}\bar{q}$. Following the trick of [18, 19] we exponentiate the g factors with a path integral over fermionic complex ghosts b^μ and $\bar{c}^{\bar{\nu}}$: $\mathcal{D}q\mathcal{D}\bar{q} = DqD\bar{q}\int DbD\bar{c}\,\mathrm{e}^{-S_{\mathrm{gh}}}$. At this stage the gauge fixed action plus the ghost action takes the following form[2]

$$S_{\mathrm{gf}} + S_{\mathrm{gh}} = \frac{1}{\beta}\int_0^1 d\tau\left[g_{\mu\bar{\nu}}\left(\dot{q}^\mu\dot{\bar{q}}^{\bar{\nu}} + b^\mu\bar{c}^{\bar{\nu}}\right) + \bar{\psi}_\mu(D_\tau + i\phi)\psi^\mu\right] + is\phi\,. \tag{5.32}$$

In order to perform perturbative calculations we expand all background fields around the fixed point x_0. The action written above splits into a

[1]Note that, since ψ's are spacetime vectors, while $\bar{\psi}$'s are covectors, one has $\mathcal{D}\bar{\psi}\mathcal{D}\psi = D\bar{\psi}D\psi$.

[2]We rescaled fermions by $\psi \to \frac{1}{\sqrt{\beta}}\psi$ in order to extract a common β as loop counting parameter.

quadratic part S_2 giving propagators, as usual, and an interaction part. We denote as $\langle\bullet\rangle$ the quantum average weighted with the free path integral: $\langle\bullet\rangle = \frac{1}{\int \mathrm{e}^{-S_2}}\int \bullet\, \mathrm{e}^{-S_2}$. The partition function (5.31) now reads

$$Z \propto \int_0^\infty \frac{d\beta}{\beta}\int_0^{2\pi}\frac{d\phi}{2\pi}\left(2\cos\frac{\phi}{2}\right)^{d-2}\mathrm{e}^{-is\phi}\int\frac{d^dx_0 d^d\bar{x}_0}{(2\pi\beta)^d}g(x_0)\langle \mathrm{e}^{-S_{\rm int}}\rangle\ , \tag{5.33}$$

where $(2\cos\frac{\phi}{2})^d(2\pi\beta)^{-d}$ is the usual free path integral normalization, and the interaction part is

$$S_{\rm int} = \frac{1}{\beta}\int_0^1 d\tau\left[\big(g_{\mu\bar\nu}(x)-g_{\mu\bar\nu}(x_0)\big)\big(\dot q^\mu\dot{\bar q}^{\bar\nu}+b^\mu\bar c^{\bar\nu}\big)+\dot q^\nu\Gamma^\mu_{\nu\lambda}(x)\bar\psi_\mu\psi^\lambda\right]. \tag{5.34}$$

For our computation we can choose any coordinate system so, in order to have manifest covariance at each step, and at the same time to maintain holomorphic coordinates, we use Kähler normal coordinates (see [87], for example) centered at x_0. Denoting with S_n the part of $S_{\rm int}$ containing n-fields vertices, it results that, in Kähler normal coordinates, the only terms giving non vanishing contribution up to order β^2 are the following ones

$$\begin{aligned}
S_4 &= \frac{1}{\beta}\int d\tau\left[R_{\mu\bar\nu\lambda\bar\sigma}q^\lambda\bar q^{\bar\sigma}\big(\dot q^\mu\dot{\bar q}^{\bar\nu}+b^\mu\bar c^{\bar\nu}\big)+R^\lambda{}_{\sigma\bar\nu\mu}\dot q^\mu\bar q^{\bar\nu}\bar\psi_\lambda\psi^\sigma\right]\\
S_6 &= \frac{1}{\beta}\int d\tau\left[\frac{1}{4}\big[\nabla_{(\bar\sigma}\nabla_\lambda R_{\mu\bar\nu\rho\bar\kappa)}+3R^{\bar\tau}{}_{(\bar\nu\lambda\bar\kappa}R_{\mu\bar\sigma\rho)\bar\tau}\big]q^\lambda\bar q^{\bar\sigma}q^\rho\bar q^{\bar\kappa}\big(\dot q^\mu\dot{\bar q}^{\bar\nu}+b^\mu\bar c^{\bar\nu}\big)\right.\\
&\left.\quad+\frac{1}{2}\big[\nabla_{(\bar\sigma}\nabla_\lambda R_{\mu\bar\nu\rho\bar\kappa)}+3R^{\bar\tau}{}_{(\bar\nu\lambda\bar\kappa}R_{\mu\bar\sigma\rho)\bar\tau}-2R^{\bar\tau}{}_{\bar\nu\lambda\bar\kappa}R_{\mu\bar\sigma\rho\bar\tau}\big]g^{\nu\bar\nu}\dot q^\mu\bar q^{\bar\sigma}q^\lambda\bar q^{\bar\kappa}\bar\psi_\nu\psi^\rho\right]
\end{aligned} \tag{5.35}$$

where all tensors are calculated at x_0 and round brackets denote weighted symmetrization, separately among holomorphic and anti-holomorphic indices, *i.e.* $A_{(\mu_1\ldots\mu_n\bar\nu_1\ldots\bar\nu_m)} \equiv A_{(\mu_1\ldots\mu_n)(\nu_1\ldots\nu_m)}$. From the quadratic action $S_2 = \frac{1}{\beta}\int[g_{\mu\bar\nu}(x_0)(\dot q^\mu\dot{\bar q}^{\bar\nu}+b^\mu\bar c^{\bar\nu})+\bar\psi_\mu(\partial_\tau+i\phi)\psi^\mu]$ one extracts the following two point functions

$$\begin{aligned}
&\langle q^\mu(\tau)\bar q^{\bar\nu}(\sigma)\rangle = -\beta g^{\mu\bar\nu}(x_0)\Delta(\tau,\sigma)\ ,\quad \langle b^\mu(\tau)\bar c^{\bar\nu}(\sigma)\rangle = -\beta g^{\mu\bar\nu}(x_0)\delta(\tau,\sigma)\ ,\\
&\langle\bar\psi_\nu(\tau)\psi^\mu(\sigma)\rangle = -\beta\delta^\mu_\nu\Delta_f(\sigma-\tau,\phi)
\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}
\Delta(\tau,\sigma) &= (\tau-1)\sigma\theta(\tau-\sigma)+(\sigma-1)\tau\theta(\sigma-\tau)\ ,\\
\Delta_f(x,\phi) &= \frac{\mathrm{e}^{-i\phi x}}{2\cos\frac{\phi}{2}}\big[\mathrm{e}^{i\frac{\phi}{2}}\theta(x)-\mathrm{e}^{-i\frac{\phi}{2}}\theta(-x)\big]
\end{aligned} \tag{5.37}$$

with $\theta(x)$ the step function and $\delta(\tau,\sigma)$ the Dirac delta acting on functions vanishing at the boundaries. We note that in performing perturbative calculations one encounters products and derivatives of such distributions, that are ill defined. To resolve this ambiguity we will use Time Slicing (TS) regularization [67, 37], that gives well known prescriptions on how to handle such products of distributions and necessitates no counterterms (the standard TS counterterm vanish on Kähler manifolds). The rules are as follows: when computing the various Feynmann diagrams all delta functions should be implemented with the prescription of considering $\theta(0) = \frac{1}{2}$ for the step function, while the ghost system guarantees that no products of delta functions can ever arise.

Looking at (5.36) we immediately see that each piece S_n of $S_{\rm int}$ gives a contribution of order $\beta^{n/2-1}$. Therefore, our quantum average can be written explicitly as

$$\langle {\rm e}^{-S_{\rm int}} \rangle = 1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2}\langle S_4^2 \rangle \; . \tag{5.38}$$

Using the expressions given in (5.35) and Time Slicing prescriptions in calculating Feynman diagrams, one finally obtains

$$\begin{aligned}\langle {\rm e}^{-S_{\rm int}} \rangle = 1 - \frac{\beta}{12} R + \beta^2 \Big[\Big(\frac{1}{180} - \frac{1}{96} \cos^{-2}\frac{\phi}{2} \Big) R_{\mu\bar\nu\lambda\bar\sigma} R^{\mu\bar\nu\lambda\bar\sigma} + \Big(- \frac{19}{1440} \\ + \frac{1}{96} \cos^{-2}\frac{\phi}{2} \Big) R_{\mu\bar\nu} R^{\mu\bar\nu} + \frac{1}{288} R^2 - \frac{1}{120} \Box R \Big] \; .\end{aligned} \tag{5.39}$$

Plugging this result in the partition function (5.31) one can perform the ϕ integral, taking care of the possible pole arising at $\phi = \pi$. Switching to the Wilson loop variable $w = e^{i\phi}$ one has a contour integral on the unit circle surrounding the origin, with a possible pole on the integration path at $w = -1$. If we slightly deform our path in a way that excludes the pole, as in Figure 2, we finally obtain for our partition function

$$\begin{aligned} Z \propto \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^d x_0 d^d \bar x_0 g(x_0)}{(2\pi\beta)^d} \binom{d-2}{p} \Big\{ 1 - \frac{\beta}{12} R \\ + \beta^2 \Big[\Big(\frac{1}{180} - \frac{p(d-p-2)}{24(d-2)(d-3)} \Big) R_{\mu\bar\nu\lambda\bar\sigma} R^{\mu\bar\nu\lambda\bar\sigma} \\ + \Big(\frac{p(d-p-2)}{24(d-2)(d-3)} - \frac{19}{1440} \Big) R_{\mu\bar\nu} R^{\mu\bar\nu} + \frac{1}{288} R^2 - \frac{1}{120} \Box R \Big] \Big\} \end{aligned} \tag{5.40}$$

The present formula is valid for $d > 3$, while for lower d's one has to take care of the pole arising at $w = -1$. From this answer one can easily extract the first few heat kernel coefficients associated to the quantum theory of

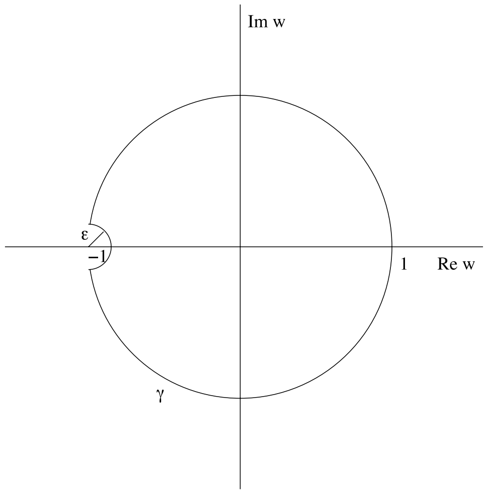


Figure 5.2: Contour modular integration excluding the pole at $w = -1$.

the gauge $(p, 0)$-form. As we see, the answer is manifestly dual under the exchange $p \leftrightarrow (d - p - 2)$, expressing the duality between a $(p, 0)$-form and a $(d - p - 2, 0)$-form. If one tries to set $p = 0$ in the above formula, it is easy to see that does not recover the well known result for a scalar in a curved background. This is due to the aforementioned $U(1)$ piece of the holonomy, that gives rise to an "electromagnetic" coupling to the Ricci tensor $R_{\mu\bar{\nu}}$. Topological mismatches in the duality will show up, for $d > 3$, in higher order terms in the expansion. On the other hand, if we take, for instance, $d = 3$, the model propagates only $p = 0, 1$ forms. Restricting thus to $p = 0, 1$ and excluding the pole as depicted in Figure 2 one obtains, in $d = 3$:

$$\begin{aligned} Z \propto & \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^3x_0 d^3\bar{x}_0 g(x_0)}{(2\pi\beta)^3} \Big\{ 1 - \frac{\beta}{12} R \\ & + \beta^2 \Big[\Big(\frac{1}{180} - \frac{p}{24} \Big) R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} \\ & + \Big(\frac{p}{24} - \frac{19}{1440} \Big) R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + \frac{1}{288} R^2 - \frac{1}{120} \Box R \Big] \Big\} \end{aligned} \tag{5.41}$$

that indeed is not invariant upon $p \leftrightarrow 1 - p$ because of the topological mismatch we will analyze in the following.

We now turn to different gauging choices of our $U(1)$ extended supergravity. For instance, if we choose not to gauge the $U(1)$ part of the first class algebra, we do not have a modular integration over ϕ any more. Then, the result for this new model is obtained for free by setting $\phi = 0$ in the above formulas, giving

$$
\begin{aligned}
Z \propto \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^d x_0 d^d \bar{x}_0 g(x_0)}{(2\pi\beta)^d} \Big\{ 1 - \frac{\beta}{12} R \\
+ \beta^2 \Big[- \frac{7}{1440} R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} - \frac{1}{360} R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + \frac{1}{288} R^2 - \frac{1}{120} \Box R \Big] \Big\} .
\end{aligned}
\tag{5.42}
$$

It corresponds to the quantum theory of the sum of all $(p,0)$-forms with dynamics dictated by the Maxwell equations. As a check of this result, one may observe that this effective action must be proportional to the one-loop effective action of a Dirac field. In fact, the path integral over the complex gravitino present in (5.42) can at most change the overall normalization of the partition function if compared with the path integral over the real gravitino needed for the Dirac field, recall eq. (5.14). Indeed, one may check that fixing suitably the overall normalization of (5.42), one recovers the heat kernel coefficients of a Dirac spinor, compare for example with [88, 27].

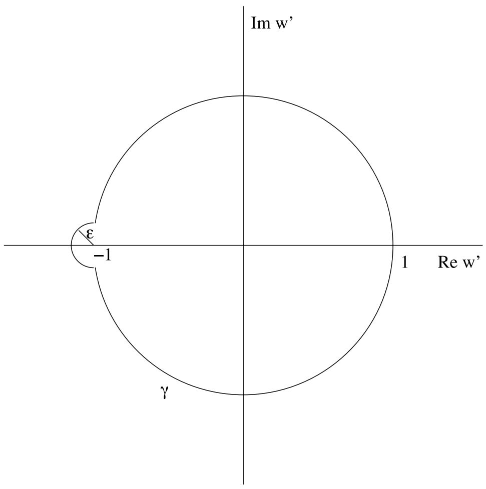


Figure 5.3: Contour encircling the pole in the dual variable.

Finally, one might wish not to gauge the two supersymmetries at all, but gauge the $U(1)$ charge instead. This produce the effective action of a single $(p,0)$-form, but now with the dynamics dictated by the hamiltonian H only, namely a $(p,0)$-form A_p without any gauge invariance but with dynamical equation $(\partial\partial^\dagger + \partial^\dagger\partial)A_p = 0$. To achieve this, we only need to drop from (5.31) the Faddeev-Popov determinat $\left(2\cos\frac{\phi}{2}\right)^{-2}$ due to the gauge fixing of the gravitini, fix the Chern-Simons coupling $s = p - d/2$, and obtain

$$
\begin{aligned}
Z \propto & \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^d x_0 d^d \bar{x}_0 g(x_0)}{(2\pi\beta)^d} \binom{d}{p} \Big\{ 1 - \frac{\beta}{12} R \\
& + \beta^2 \Big[\Big(\frac{1}{180} - \frac{p(d-p)}{24\, d(d-1)} \Big) R_{\mu\bar{\nu}\lambda\bar{\sigma}} R^{\mu\bar{\nu}\lambda\bar{\sigma}} \\
& + \Big(\frac{p(d-p)}{24\, d(d-1)} - \frac{19}{1440} \Big) R_{\mu\bar{\nu}} R^{\mu\bar{\nu}} + \frac{1}{288} R^2 - \frac{1}{120} \Box R \Big] \Big\}
\end{aligned}
\tag{5.43}
$$

5.4 Dualities

Let us now discuss in more depth the issue of duality and prove some exact relations between dual formulations. It is useful to start from the classical particle actions in (5.7) and (5.29), which are characterized by the Chern-Simons coupling s. One may start noticing that the model with coupling $-s$ is equivalent to the model with coupling s. In fact, one obtains the latter from the former by a suitable transformation of the dynamical variables: one needs to exchange $\psi \leftrightarrow \bar{\psi}$ and change the sign of the $U(1)$ gauge field $a \to -a$. The equivalence of these two models corresponds at the quantum level to a duality between different forms. In fact, the constraint selecting the physical form degree reads as $(J-s)|F\rangle = 0$ that, recalling that $s = p+1-\frac{d}{2}$, amounts to $(\mathbf{N} - (p+1))|F\rangle = 0$. This constraint selects a $(p+1, 0)$ field strength $F_{(p+1,0)}$ of a $(p, 0)$-form $A_{(p,0)}$. Sending $s \to -s$ switches the constraint to $(\mathbf{N} - (d-p-1))|F\rangle = 0$ that reduces to a $(d-p-2, 0)$-form $A_{(d-p-2,0)}$.

To discuss duality issues, it is useful to switch to an operatorial picture and cast the effective action (5.33) as follows

$$
\begin{aligned}
Z &\propto \int_0^\infty \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\phi}{2\pi} \left(2\cos\frac{\phi}{2} \right)^{-2} \mathrm{Tr}\, [\mathrm{e}^{i\phi(J-s)} \mathrm{e}^{-\beta H}] \\
&= \int_0^\infty \frac{d\beta}{\beta} \oint \frac{dw}{2\pi i w} \frac{w}{(1+w)^2} \mathrm{Tr}\, [w^{J-s} \mathrm{e}^{-\beta H}] \\
&= \int_0^\infty \frac{d\beta}{\beta} \underbrace{\oint \frac{dw}{2\pi i w} \frac{w}{(1+w)^2} \mathrm{Tr}\, [w^{\mathbf{N}-(p+1)} \mathrm{e}^{-\beta H}]}_{z_s(\beta)} \ .
\end{aligned}
\tag{5.44}
$$

where we have used different notations to be able to underline various properties. Here J and H are quantum operators, and in the last two expressions we have employed the Wilson loop variable $w = \mathrm{e}^{i\phi}$, and the countour integral is along the unit circle $|w| = 1$. In the last expression we have explicitated the fermion number operator $\mathbf{N} = \psi\bar{\psi}$, used in the Dirac index computation. As $J = \frac{1}{2}(\psi\bar{\psi} - \bar{\psi}\psi) = \psi\bar{\psi} - \frac{d}{2}$ and $s = p + 1 - \frac{d}{2}$, one may check that $J - s = \mathbf{N} - (p+1)$.

Let us now analyze these formulas in various cases.

1) If susy is not gauged, the corresponding ghost term $\frac{w}{(1+w)^2}$ is absent and one obtains using a similar notation as in (5.44)

$$
\begin{aligned}
z_s^{\text{no susy}}(\beta) &= \oint \frac{dw}{2\pi i w} \mathrm{Tr}\,[w^{\mathbf{N}-(p+1)} \mathrm{e}^{-\beta H}] \\
&= \oint \frac{dw}{2\pi i w} \sum_{n=0}^{d} w^{n-(p+1)} t_n(\beta) = t_{p+1}(\beta)
\end{aligned}
\tag{5.45}
$$

where $t_n(\beta)$ arises from the trace restricted to the Hilbert space with fermion number $\mathbf{N} = n$. No poles are present along the contour $|w| = 1$, and the integral extracts the contibution due to a $p+1$ form. It corresponds to the quantum theory of a $(p+1, 0)$-form with kinetic term given by the Dolbeault laplacian and satisfying no other constraints (thus there is no gauge invariance for the QFT in question). The classical equivalence $s \to -s$ corresponds to the equivalence of the $(p, 0)$-form and $(d-p, 0)$-form effective actions, with $0 \le p \le d$. Indeed, one may compute

$$
\begin{aligned}
z_{-s}^{\text{no susy}}(\beta) &= \oint_\gamma \frac{dw}{2\pi i w} \mathrm{Tr}\,[w^{J+s} \mathrm{e}^{-\beta H}] \\
&= \oint_{-\gamma} \frac{dw'}{2\pi i w'} \mathrm{Tr}\,[w'^{(-J+s)} \mathrm{e}^{-\beta H}] \\
&= \oint_\gamma \frac{dw}{2\pi i w} \mathrm{Tr}\,[w^{J-s} \mathrm{e}^{-\beta H}] = z_s^{\text{no susy}}(\beta)
\end{aligned}
\tag{5.46}
$$

where we have first changed $J \to -J$ to take into account the role exchange of ψ and $\bar{\psi}$ and used $w \to w' = \frac{1}{w}$ to take into account the sign change of the gauge field $\phi \to -\phi$. Note that in terms of the coordinate w' the countour γ is traced with the reverse orientation. Then, returning to the coordinate $w = \frac{1}{w'}$ by means of a change of variables reproduces the result of the model with CS coupling s. This proves the Hodge duality between $(p, 0)$-form and $(d-p, 0)$-form at the quantum level, namely $t_p(\beta) = t_{d-p}(\beta)$.

2) If susy is gauged, the ghost term $\frac{w}{(1+w)^2}$ is present and one must use a prescription to integrate over w. We exclude the pole and should check that this is the correct prescription. Duality is again obtained by $s \to -s$, but now the mapping $w \to w' = \frac{1}{w}$ sends a contour that excludes the pole, like in Figure 2, to one that encircles it, and this results in a mismatch proportional

to the residue

$$\mathrm{Res}\left[\frac{1}{(1+w)^2}\mathrm{Tr}\,[w^{\mathbf{N}-(p+1)}\mathrm{e}^{-\beta H}],\ w=-1\right]$$

from which, using the same manipulations of [28], we get

$$z_p(\beta) = z_{d-p-2}(\beta) + (-1)^p(p+1)\ \mathrm{ind}(\nabla\!\!\!\!/\ ) + (-1)^p z_{d-1}(\beta) \tag{5.47}$$

where the last term is due to a topological $(d-1)$-form which carries no degrees of freedom. The precise relative coefficients of the above relation are still under study but, as a promising check, we can take $d=2$ in (5.47). The only propagating field is then $p=0$ and one gets

$$z_1 = -\mathrm{ind}(\nabla\!\!\!\!/\ )\ .$$

Excluding the pole in $w=-1$, one plugs (5.39) in (5.33) for $d=2$ and $p=1$, and finds for the topological $(1,0)$-form

$$z_1 \propto \int d^2x d^2\bar{x}\, g\, \frac{1}{96\pi^2}\left[R_{\mu\bar{\nu}}R^{\mu\bar{\nu}} - R_{\mu\bar{\nu}\lambda\bar{\sigma}}R^{\mu\bar{\nu}\lambda\bar{\sigma}}\right]\ ,$$

that perfectly matches the result for the Dirac index given in (5.28).

Chapter 6

Weyl invariant field theories and Tractors

The history of Weyl invariance [42] as a principle for developing physical theories is a long one. Notable early examples include Dirac's formulation of conformally invariant four-dimensional wave equations in six dimensions [43] and Zumino's work relating Weyl transformations to the conformal group [44] and the introduction of Weyl compensator fields by Deser and Zumino [44, 45]. If the local choices of unit systems, that is local unit invariance, could not possibly change the outcome of any physical measurement [46], therefore there should exist a formulation of physics that makes this symmetry manifest. A similar line of reasoning led Einstein to postulate a theory of gravitation in terms of (pseudo)-Riemannian geometry in order to manifest a local coordinate invariance. This simple idea led Weyl to a study of local metric transformations of the form [89]

$$g_{\mu\nu} \mapsto \Omega^2(x) g_{\mu\nu}$$

which are by now called Weyl transformations. There exist relatively few physical models that exhibit this symmetry, notable examples include four-dimensional Maxwell theory, conformally improved scalars, the massless Dirac equation and Weyl-squared gravity. In general, just as general coordinate invariant theories typically require the introduction of a metric $g_{\mu\nu}(x)$, local unit invariant theories require a new gauge field called the scale $\sigma(x)$, whose value at differing spacetime points reflects the ratio of unit systems at those points. It can also be viewed as a spacetime dependent Newton constant. In the physics literature, the scale $\sigma(x)$ is often called a dilaton or Weyl compensator [44, 45] . Under local changes of unit systems, the scale transforms as

$$\sigma \mapsto \Omega(x) \sigma$$

In mathematical terms, the symmetry under the above transformation implies that physics can be formulated in terms of conformal geometry, the theory of conformal classes of metrics $[g_{\mu\nu}] = [\Omega^2 g_{\mu\nu}]$. It is important to note that the choice of gauge (local Weyl frame)

$$\sigma(x) = \kappa^{\frac{2}{d-2}} = M_{\rm Pl}^{-1}$$

both yields the standard description of physics with a constant value of the Newton constant or equivalently, Planck mass, and selects a distinguished or "canonical" metric from the double equivalence class $[g_{\mu\nu}, \sigma] = [\Omega^2 g_{\mu\nu}, \Omega\sigma]$. In other words the scale is precisely the field that defines the gravitational coupling. Without a tensor calculus for rapidly constructing Weyl invariant quantities, the above local unit invariance principle would not be particularly enlightening. Fortunately, such a calculus already exists in the mathematical literature and goes under the name "tractor calculus" [47, 48, 49, 50, 51]. It is the mathematical machinery required to replace Riemannian geometry with conformal geometry as the underpinning of physics. A particularly appealing implication is that in a description of physics that manifests local unit invariance, masses are replaced by Weyl weights which measure the response of physical fields to changes of unit systems. In particular, mass terms become spacetime dependent in general gravitational backgrounds, yet constant when these backgrounds are Einstein. We will describe in the present chapter the basics of the Tractor formalism following [46], and the construction of Weyl invariant massive and massless field theories for scalars and vectors, and we will conclude by providing the tractor formulation of Einstein's gravity coupled to matter

6.1 The Tractor formalism

The main idea underlying the tractor philosophy is to develop a machinery that allows to keep under control Weyl invariance in a manifest covariant way. Under a Weyl transformation, the metric undergoes the following local scaling:

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \,. \tag{6.1}$$

In the context of conformal geometry and for the tractor calculus, it is useful to define a trace-adjusted version of the Ricci tensor: the so called Schouten tensor $\mathsf{P}_{\mu\nu}$,

$$\mathsf{P}_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} - \frac{1}{2}\frac{1}{d-1} g_{\mu\nu} R\right) \,, \quad \mathsf{P} \equiv \mathsf{P}^{\mu}{}_{\mu} = \frac{R}{2(d-1)}$$

that allows to express the Riemann curvature as

$$R_{\mu\nu\lambda\sigma} = W_{\mu\nu\lambda\sigma} + 2\mathsf{P}_{\lambda[\mu}\, g_{\nu]\sigma} - 2\mathsf{P}_{\sigma[\mu}\, g_{\nu]\lambda}$$

where $W_{\mu\nu\lambda\sigma}$ is the traceless Weyl tensor, that in $d > 3$ represents the obstruction to conformal flatness.

Let us denote

$$\Upsilon_\mu \equiv \omega^{-1}\partial_\mu\Omega = \partial_\mu \log\Omega \ .$$

We define a weight w tractor scalar ϕ a field that under (6.1) transforms as

$$\phi \to \Omega^w(x)\phi \ .$$

It is well known that the conformal group of a d-dimensional spacetime with minkowskian signature is $SO(d,2)$, and it acts nicely as the Lorentz group on a $d+2$ dimensional spacetime with an extra time-like direction. It is therefore not surprising that the natural tensor objects of the tractor calculus, whose aim is to make Weyl transformations geometrically clear, are $SO(d,2)$ multiplets. The simplest example is the tractor scalar already mentioned. Let us consider now a d-dimensional vector T^m with a flat Lorentz index, and two spacetime scalars that we will denote T^+ and T^-. If they transform under a Weyl transformation (6.1) as

$$\begin{aligned} &T^+ \to \Omega^{w+1}T^+ \\ &T^m \to \Omega^w(T^m + \Upsilon^m T^+) \\ &T^- \to \Omega^{w-1}(T^- - \Upsilon_n T^n - \tfrac{1}{2}\Upsilon_n\Upsilon^n T^+) \ , \end{aligned} \tag{6.2}$$

they can be packaged in a weight w tractor vector T^M, where $M = (+, m, -)$ is an $SO(d,2)$ index, that indeed transforms covariantly under Weyl rescalings:

$$T^M \to \Omega^w\, U^M{}_N\, T^N \ . \tag{6.3}$$

The matrix $U(\Omega)$ belongs to a parabolic subgroup of $SO(d,2)$ and is called a tractor gauge transformation. In components it reads

$$U^M{}_N(\Omega) = \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \delta^m_n & 0 \\ -\frac{1}{2}\Omega^{-1}\Upsilon^2 & -\Omega^{-1}\Upsilon_n & 0 \end{pmatrix} \ , \tag{6.4}$$

and allows to define the tractor gauge transformation for every tractor tensor, following the usual tensor product rule. Just as in the Yang-Mills construction, once we have the gauge transformation, we need to find a covariant derivative that determines a connection on the gauge bundle. In the case at hand we have a weighted $SO(d,2)$ bundle, the so called tractor bundle. In fact, with the help of the Schouten tensor $\mathsf{P}_{\mu\nu}$ and of the vielbein e_μ^m, we construct the tractor covariant derivative ${\mathcal{D}_\mu}$[1]

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu = \begin{pmatrix} \partial_\mu & -e_{\mu n} & 0 \\ \mathsf{P}_\mu{}^m & \nabla_\mu{}^m{}_n & e_\mu{}^m \\ 0 & -\mathsf{P}_{\mu n} & \partial_\mu \end{pmatrix} , \tag{6.5}$$

that transforms as

$$\mathcal{D}_\mu \to U \mathcal{D}_\mu U^{-1}$$

when the metric undergoes (6.1). On a tractor scalar it acts as a simple derivative: $\mathcal{D}_\mu \phi = \partial_\mu \phi$ while, for instance, on a tractor vector it produces

$$\mathcal{D}_\mu T^M = \begin{pmatrix} \partial_\mu T^+ - T_\mu \\ \nabla_\mu T^m + \mathsf{P}_\mu{}^m T^+ + e_\mu{}^m T^- \\ \partial_\mu T^- - \mathsf{P}_{\mu\nu} T^\nu \end{pmatrix}$$

where obviously $T_\mu = e_{\mu m} T^m$. At this stage it is useful to introduce an $SO(d,2)$ flat metric, the tractor metric

$$\eta_{MN} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta_{mn} & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.6}$$

that is a weight zero, symmetric, rank two tractor tensor that is parallel with respect to $\mathcal{D}_\mu$. Using the last fact it is safe in calculations to use the tractor metric and its inverse to raise and lower tractor indices $M, N, ...$ in the usual fashion, and this we shall do without further mention. Along similar lines, it is easy to see that

$$X^M = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.7}$$

is a (Weyl invariant) weight one tractor vector, often called "canonical tractor". Note that, by contraction, X^M may be used to project out the top slot of a tractor vector field, *i.e.* $X^M T_M = T^+$.

[1] This connection was first encountered in a study of conformal gravity [90] undertaken as a stepping stone to theories of conformal supergravity. Indeed this approach is part of a general program for gauging spacetime algebras [91, 92, 93]. For a detailed derivation see also [94]

The covariant derivative presented in (6.5), is useful to maintain tractor covariance but, on the other hand, it does not map tractors into tractors. It is, however, the building block of a genuine tractor operator that will be of fundamental importance in what follows. This is the Thomas D-operator, that acts on weight w tractors as

$$D^M = \begin{pmatrix} (d+2w-2)w \\ (d+2w-2)\mathcal{D}^m \\ -(\mathcal{D}^\mu \mathcal{D}_\mu + w\mathsf{P}) \end{pmatrix} . \qquad (6.8)$$

This is a weight -1 tractor vector, it is nilpotent

$$D^M D_M = 0 ,$$

and it acts on its argument by tensor product

$$D^M T^N = \begin{pmatrix} (d+2w-2)w\, T^N \\ (d+2w-2)\mathcal{D}^m T^N \\ -(\mathcal{D}^\mu \mathcal{D}_\mu + w\mathsf{P})T^N \end{pmatrix} .$$

As we can see by inspecting the bottom slot D^-, a second order operator, the tractor laplacian $\mathcal{D}^2$, appears in the definition of the Thomas D-operator. Therefore it should not be confused with a generalized covariant derivative and, in fact, it is not leibnitzian. In spite of this, it can often be employed to similar effect.

Let us give a look at the Thomas D acting on a scalar φ. The tractor laplacian $\mathcal{D}^2 = g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu$ reduces on scalars to the usual curved space laplacian that we denote $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$, and if we take the scalar field to have weight $1-\frac{d}{2}$ we get:

$$D^M\varphi = \begin{pmatrix} 0 \\ 0 \\ -\Big(\Delta - \frac{d-2}{2}\mathsf{P}\Big)\varphi \end{pmatrix} . \qquad (6.9)$$

Recalling that $\mathsf{P} = \frac{1}{2(d-1)}R$ we can see that the bottom slot is nothing but the conformally improved scalar wave operator, so that the improved wave equation reads in tractor language as $D^M\varphi = 0$. This is valid only for a scalar of weight $w = 1-\frac{d}{2}$, but it will be extremely important in looking for a general scalar tractor equation.

In order to construct tractor actions out of the Thomas D, we need a rule to integrate it by parts. It is not obvious, since it contains a second order operator, that such a rule should exists. Nonetheless, at commensurate weights, an integration by parts formula does hold. For example, if V^M is a weight w tractor vector and φ is a weight $1-d-w$ scalar then (see *e.g.* [95])

$$\int \sqrt{-g}\, V_M D^M \varphi = \int \sqrt{-g}\, \varphi D^M V_M \,. \tag{6.10}$$

Notice, there is no sign flip in this formula and that the integral itself is Weyl invariant because the metric determinant carries Weyl weight d. An analogous formula holds for tractor tensors.

The final piece of the tractor formalism we need, is a method that can handle the choice of a scale without spoiling the tractor covariance of the system. The answer will be the introduction of a scale field, *i.e.* a space-time dependent scale that can compensate Weyl transformations. Let us consider a conformal class of metrics $[g_{\mu\nu}] = [\Omega^2 g_{\mu\nu}]$. We now introduce a non-vanishing weight one scalar field $\sigma(x)$ with equivalence

$$[g_{\mu\nu}, \sigma] = [\Omega^2 g_{\mu\nu}, \Omega\sigma]\,, \tag{6.11}$$

we can use σ to uniquely (up to an overall constant factor) pick a metric $g^0_{\mu\nu}$ from this equivalence class by requiring the accompanying representative scalar σ_0 is constant for that choice.

The Thomas D-operator acting on the scale σ plays an extremely important and distinguished role. In fact, it defines the scale tractor

$$I^M = \frac{1}{d} D^M \sigma = \begin{pmatrix} \sigma \\ \nabla^m \sigma \\ -\frac{1}{d}\Big[\Delta + \mathsf{P}\Big]\sigma \end{pmatrix}\,, \tag{6.12}$$

that controls in a Weyl covariant way the coupling of matter with the scale. When the physical gauge $\sigma = \sigma_0$ is chosen, the scale tractor parametrizes the breaking of Weyl symmetry. On the other hand, I^M is closely related to gravity itself: remarkably, the gravity-dilaton action, that we will describe in the following chapters, can be written entirely in terms of the scale tractor. Moreover, we can see that a tractor-parallel scale tractor, *i.e.* $\mathcal{D}_\mu I^M = 0\,$, amounts to Einstein's equations in vacuum. To see this we explicitly compute the tractor derivative of I^M that, once evaluated at the choice of constant scale $\sigma = \sigma_0$, reads

$$\mathcal{D}_\mu I^M|_{\sigma=\sigma_0} = \sigma_0 \begin{pmatrix} 0 \\ \mathsf{P}_\mu{}^m - \frac{1}{d} e_\mu{}^m \mathsf{P} \\ -\frac{1}{d}\partial_\mu \mathsf{P} \end{pmatrix}\,. \tag{6.13}$$

Setting this to zero says $R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R$ and $R = constant$, so that $g_{\mu\nu}$ is precisely an Einstein manifold. This happens at the choice of scale $\sigma = \sigma_0$, so we can say that the scale tractor is parallel when the metric is conformally Einstein. In addition, when the scale tractor is tractor-parallel, one has

$$I^2 = \text{const} , \quad [D^M, I^N] = 0 .$$

We have assembled all the ingredients needed to the formulation of physically relevant models using tractors. Hence, we will review the construction of scalar and vector models following [46].

6.2 Scalar fields

Let us now review the construction of a tractor scalar theory. We have seen that, when the Weyl weight of the scalar field is $1 - \frac{d}{2}$, the improved wave equation is produced by the tractor equation $D^M \varphi = 0$. For generic weights this will not be true anymore. The scalar field will transform under (6.1) as

$$\varphi \to \Omega^w \varphi ,$$

and it will couple to the scale σ to ensure Weyl symmetry. We have noticed that a coupling to the scale respecting Weyl covariance and the tractor framework is accomplished via the scale tractor I^M. In fact, we will demand that in the field equations the scale couples to the matter system *only* by means of I^M. The natural field equation arising for the scalar field is thus:

$$I_M D^M \varphi = 0 . \tag{6.14}$$

In components it reads[2]

$$\Big[\Delta + \frac{2\mathsf{P}}{d}\, w(d + w - 1)\Big]\varphi + (d + 2w - 2)\Big[\frac{w}{d}(b^2 + \nabla.b) - b.\nabla\Big]\varphi = 0 ,$$

where $b_\mu \equiv \sigma^{-1}\partial_\mu \sigma$. This equation is completely Weyl invariant when the metric $g_{\mu\nu}$, the scale σ and the scalar φ are transformed simultaneously. Choosing the scale to be constant for the background metric we are interested in, we see that in the physical gauge $\sigma = \sigma_0$, we have $b_\mu = 0$ and the field equation (6.14) reduces to

$$\Big(\Delta + \frac{2\mathsf{P}}{d}\, w(d + w - 1)\Big)\varphi = 0 . \tag{6.15}$$

[2] We denote contraction of d-dimensional indices by a dot "." while contractions of tractor indices are given by a slightly higher dot "·"

First of all, we notice that at the special weight $w = 1 - \frac{d}{2}$, (6.15) becomes

$$\Big(\Delta - \frac{1}{4}\frac{d-2}{d-1}\,R\Big)\varphi = 0\;, \tag{6.16}$$

that is precisely the conformally improved scalar equation. We know that the improved scalar theory is Weyl invariant, and so it does not need the introduction of any scale. This can be interpreted nicely in the present context: looking at (6.9), we know that at the special weight $w = 1 - \frac{d}{2}$ the only non-zero component of $D^M\varphi$ is precisely the left-hand side of (6.16). Hence, we can say that at the conformal weight $1 - \frac{d}{2}$ the scale tractor I^M decouples, and Weyl invariance is ensured without the introduction of any scale.

$$I \cdot D\varphi = 0 \quad \text{at } w = 1 - \tfrac{d}{2} \quad \Leftrightarrow \quad D^M\varphi = 0$$

It is simple to write down a Weyl invariant action that produces (6.14) as field equation. It reads

$$S[g_{\mu\nu}, \sigma, \varphi] = \frac{1}{2}\int \sqrt{-g}\,\sigma^{1-d-2w}\varphi I_M D^M\varphi\;, \tag{6.17}$$

and it is immediate to see that for $w = 1 - \frac{d}{2}$ the above action does not depend on σ.

For generic weights, we can consider the equation (6.15) as a massive scalar equation on constant curvature backgrounds, where $\mathsf{P} = \frac{\Lambda d}{(d-1)(d-2)}$, and the mass-Weyl weight relation reads:

$$m^2 = \frac{2\mathsf{P}}{d}\Big[\Big(\frac{d-1}{2}\Big)^2 - \Big(w + \frac{d-1}{2}\Big)^2\Big]\;. \tag{6.18}$$

In summary, we reviewed that the Weyl covariant equation $I \cdot D\varphi = 0$ describes massive propagation for generic weights on constant curvature backgrounds. At the conformal weight $w = 1 - \frac{d}{2}$ the scale decouples and one can demand the equation $D^M\varphi = 0$, that is invariant in its own.

Looking at the mass formula (6.18), we see that for every real weight w, one has the following lower bound on negative curvature spaces:

$$m^2 \geq \frac{\mathsf{P}}{2d}\,(d-1)^2\;, \tag{6.19}$$

that is, in the present formulation, the Breitenlohner–Freedman bound [96, 97] for stable scalar propagation in Anti de Sitter space. It is saturated by setting the second term in (6.18) to zero, so that $w = \frac{1}{2} - \frac{d}{2}$.

6.3 Vector fields

In this section we are going to describe the tractor construction of Weyl invariant vector theories. First of all, in order to deal with a vector field V^μ within the tractor formalism, it is necessary to add two auxiliary scalar fields V^+ and V^- that, together with $V^m \equiv e_\mu{}^m V^\mu$, form the weight w tractor vector

$$V^M = \begin{pmatrix} V^+ \\ V^m \\ V^- \end{pmatrix} .$$

According to the tractor gauge transformation (7.5), the above fields have to transform under Weyl rescalings as [3]

$$\begin{aligned} V^+ &\to \Omega^{w+1} V^+ , \\ V_\mu &\to \Omega^{w+1}(V_\mu + \Upsilon_\mu V^+) , \\ V^- &\to \Omega^{w-1}(V^- - \Upsilon^\mu [V_\mu + \frac{1}{2}\Upsilon_\mu V^+]) . \end{aligned} \tag{6.20}$$

These are rather unfamiliar transformations: recall, for example, that in the four dimensional Weyl invariant Maxwell theory the vector is inert under Weyl transformations. It will turn out that the present field content is too large, although necessary to make Weyl covariance manifest, and one auxiliary can be removed by adding a Weyl covariant constraint. For special weights both the auxiliaries will decouple and can be set to zero consistently; for generic weights, one auxiliary remains and will be a Stückelberg field enabling us to deal with massive fields in a gauge invariant way. To reduce the number of independent fields we can impose the Weyl covariant constraint

$$D_M V^M = 0 , \tag{6.21}$$

that can be solved algebraically for V^-, giving

$$V^M = \begin{pmatrix} V^+ \\ V^m \\ -\frac{1}{d+w-1}\Big(\nabla . V - \frac{1}{d+2w}\Big[\Delta - (d+w-1)\mathsf{P}\Big] V^+\Big) \end{pmatrix} , \tag{6.22}$$

for $w \neq -\frac{d}{2}, 1-d$, although these two weights have nothing special for the theory, see [46]. Having removed the auxiliary field V^- we still have to treat

[3] notice that the vielbein transforms as $e_\mu{}^m \to \Omega e_\mu{}^m$.

V^+. To do so we rely, as usual in physics, on gauge invariance. Therefore we require invariance under the local transformation

$$\delta V^M = D^M \xi \,, \tag{6.23}$$

where ξ is a weight $w+1$ tractor scalar. We see that, due to the nilpotence of the Thomas D-operator, the constraint (6.21) is gauge invariant and hence consistent. The two independent components explicitly transform as

$$\begin{aligned} \delta V^+ &= (d+2w)(w+1)\,\xi \,, \\ \delta V_\mu &= (d+2w)\nabla_\mu \xi \,. \end{aligned} \tag{6.24}$$

Interestingly, the middle slot of our tractor vector undergoes the usual Maxwell gauge transformation for $w \neq -\frac{d}{2}$, while the auxiliary V^+ exhibits, for $w \neq -\frac{d}{2}, -1$, a Stückelberg shift symmetry. Special values for w that seem to appear from (6.24) will be described in the following.

In order to write down a consistent field equation, we look for gauge invariant and Weyl covariant quantities built from V^M. The nice-looking analogue of the Maxwell curvature is a strong candidate:

$$\mathcal{F}^{MN} = D^M V^N - D^N V^M \,, \tag{6.25}$$

which we shall call the *tractor Maxwell curvature*, even though it is not the curvature of a connection. Its gauge invariance is manifest because Thomas D-operators commute acting on scalars (for any conformal class of metrics).

For $w \neq -\frac{d}{2}, -1$, it is possible to build a new gauge invariant object out of V^M. This is a new tractor vector $\widetilde{V}^M$, where the V^+ dependence has been absorbed in the middle and bottom slots. To clarify, let us write down it both in tractor form and in components:

$$\widetilde{V}^M = V^M - \frac{1}{(d+2w)(w+1)} D^M X \cdot V = \begin{pmatrix} 0 \\ \widetilde{V}^m \\ -\frac{1}{d+w-1} \nabla . \widetilde{V} \end{pmatrix} \,, \tag{6.26}$$

where

$$\widetilde{V}_\mu = V_\mu - \frac{1}{w+1} \nabla_\mu V^+ \,,$$

is the usual Stückelberg combination, invariant under (6.24). Moreover, let us stress that the tractor Maxwell curvature can be built out of V^M or $\widetilde{V}^M$ indifferently, since Thomas D's commute acting on scalars.

Once we have all the kinematic building blocks, we are ready to give a dynamical equation. Generalizing the scalar case, we couple $\mathcal{F}^{MN}$ to the scale via the scale tractor, and propose as field equations [46]

$$I_M \mathcal{F}^{MN} \equiv G^N = 0 \,, \tag{6.27}$$

that generalize the scalar one, as we can see expressing them in terms of V^M:

$$I \cdot D\, V^N - I_M D^N V^M = 0 \,. \tag{6.28}$$

With these Weyl covariant field equations at hand, it is now time to study the physics they describe. The gauge transformations (6.24) point out that the weights $w = -\frac{d}{2}, -1$ can play special roles. The $w = -\frac{d}{2}$ case is misleading. In fact, it describes exactly the same Proca system as the generic w values do, as it is shown in the original work. Conversely, the $w = -1$ value is far more interesting. At this special weight, the V^+ auxiliary field is inert under the gauge transformation (6.24), and we can consistently impose the further constraint $X^M V_M = V^+ = 0$, that is gauge invariant at $w = -1$. Having constrained V^+ to vanish, it is easy to see that the tractor Maxwell curvature at $w = -1$ reads:

$$\mathcal{F}^{MN} = \begin{pmatrix} 0 & 0 & 0 \\ \text{a/s} & (d-4)F^{mn} & \nabla_r F^{rm} \\ \text{a/s} & \text{a/s} & 0 \end{pmatrix}$$

where $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$ is the usual Maxwell curvature. At constant σ it is the immediate to see that the field equations $I_M \mathcal{F}^{MN} = 0$ reduce to

$$-\sigma\, \nabla_m F^{mn} = 0 \,.$$

We can see thus that at $w = -1$ we have just the familiar Maxwell equations in curved space as it was expected, since at this weight we have only one independent vector field V_μ endowed with the Maxwell gauge invariance. Concerning Weyl invariance, notice that in order to obtain Maxwell equations at $w = -1$, we do couple the theory to scale:

$$I_M \mathcal{F}^{MN} = 0 \,, \quad \text{at } w = -1 \quad \Leftrightarrow \quad \nabla_m F^{mn} = 0 \,,$$

and in fact the Maxwell theory is *not* Weyl invariant in arbitrary dimensions. We notice further that, having set $V^+ = 0$ and at $w = -1$, according to (6.20) the vector V_μ is inert under Weyl, as it happens in the usual Maxwell theory.

To proceed in analyzing the tractor vector theory, let us compute, for arbitrary weight, the tractor Maxwell curvature, satisfying the constraint (6.21):

$$\mathcal{F}^{MN} = \begin{pmatrix} 0 & (d+2w-2)(w+1)\widetilde{V}^n & -\frac{(d+2w-2)(w+1)}{d+w-1}\,\nabla.\widetilde{V} \\ \text{a/s} & (d+2w-2)F^{mn} & \nabla_r F^{rm} - (w+1)[2\mathsf{P}^m_r\widetilde{V}^r - \mathsf{P}\widetilde{V}^m + \frac{1}{d+w-1}\nabla^m\nabla.\widetilde{V}] \\ \text{a/s} & \text{a/s} & 0 \end{pmatrix} . \tag{6.29}$$

We can see that, at the special weight $w = 1 - \frac{d}{2}$, that is the engineering dimension of a field in d-dimensional spacetime, the only non-vanishing component of $\mathcal{F}^{MN}$ is $\mathcal{F}^{m-}$, and the field equation $I_M\mathcal{F}^{MN} = 0$ is equivalent to $\mathcal{F}^{MN} = 0$. This implies that at this particular weight, as it happened for the scalar field, the theory decouples from scale and is Weyl invariant in its own. One can get rid of the auxiliary V^+ using the gauge transformation (6.24), and define the only independent field as $A_\mu \equiv \widetilde{V}_\mu|_{V^+=0}$, that transforms as

$$A_\mu \to \Omega^{-\frac{d-4}{2}}\, A_\mu \, .$$

The Weyl covariant field equations now read

$$\Delta A_\mu - \frac{4}{d}\,\nabla^\nu\nabla_\mu A_\nu + \frac{d-4}{d}\Big(2\,\mathsf{P}^\nu_\mu A_\nu - \frac{d+2}{2}\,\mathsf{P} A_\mu\Big) = 0 \, . \tag{6.30}$$

These rather unfamiliar equations are the Weyl invariant, though not gauge invariant, vector equations introduced by Deser and Nepomechie [98]. In tractor language they simply arise, with all the correct couplings to curvature, from the general equation $I_M\mathcal{F}^{MN} = 0$ when the scale decouples at $w = 1-\frac{d}{2}$. In fact, at this weight, the equation (6.27) is equivalent to

$$\mathcal{F}^{MN} = 0 \, .$$

These Weyl invariant vector equations turn into ordinary Maxwell equations at $d = 4$. Maxwell equations are indeed Weyl invariant only in four dimensions, and the Maxwell branch $w = -1$ intersects the conformal branch $w = 1 - \frac{d}{2}$ precisely at $d = 4$, giving an elegant and unifying viewpoint within the tractor framework.

In order to deal with arbitrary weights w, let us specialize to a conformally Einstein background. The scale tractor is then tractor parallel: $\mathcal{D}_\mu I^M = 0$, and commutes with the Thomas D-operator. At constant scale one has

$$\mathsf{P}_{\mu\nu} = \frac{1}{d}\, g_{\mu\nu}\,\mathsf{P} \, , \quad \nabla_\mu \mathsf{P} = 0 \, .$$

The proposed field equations (6.27) obey the Bianchi-like identity and the constraint $I \cdot G = 0$ and $D \cdot G = 0$ and explicitly read, at the choice of constant scale

$$0 = G^M = \sigma \begin{pmatrix} \frac{(d+2w-2)(w+1)}{d+w-1} \nabla . \widetilde{V} \\ -\nabla_n F^{nm} - \frac{2\mathsf{P}}{d} (w+1)(d+w-2)\widetilde{V}^m + \frac{w+1}{d+w-1} \nabla^m \nabla . \widetilde{V} \\ \frac{\mathsf{P}}{d} \frac{(d+2w-2)(w+1)}{d+w-1} \nabla . \widetilde{V} \end{pmatrix} . \tag{6.31}$$

Combining the top and middle slots of the above equation one recovers it in a more suggestive form:

$$G^m - \frac{1}{d+2w-2} \nabla^m G^+ = -\sigma \Big(\nabla_n F^{nm} + \frac{2\mathsf{P}}{d}(w+1)(d+w-2)\widetilde{V}^m \Big) \equiv \mathcal{G}^m . \tag{6.32}$$

At $w = -1$ the mass term vanishes, and one gets back Maxwell equations[4], but for generic weights (6.32) gives the gauge invariant Stückelberg formulation of a Proca massive field. Gauge invariance is manifest, since only the F^{mn} and $\widetilde{V}^m$ combinations appear. One can use the Stückelberg gauge invariance to get rid of V^+, and is left with the original vector field V_μ. Taking the divergence of the equation of motion (6.32), we obtain the usual constraint $\nabla_\mu V^\mu = 0$, and the Proca equation for a massive, divergence free vector field:

$$\begin{cases} \Big(\Delta + \frac{2\mathsf{P}}{d} [w(w+d-1)-1] \Big) V_\mu = 0 , \\ \nabla^\mu V_\mu = 0 . \end{cases} \tag{6.33}$$

The ordinary Proca mass is the coefficient of V_μ in (6.32): $\nabla_\mu F^{\mu\nu} - m^2 V^\nu = 0$ and it turns out to be

$$m^2 = \frac{2\mathsf{P}}{d} \Big[\Big(\frac{d-3}{2} \Big)^2 - \Big(w + \frac{d-1}{2} \Big)^2 \Big] . \tag{6.34}$$

This result implies a Breitenlohner–Freedman bound

$$m^2 \geq \frac{2\mathsf{P}}{d} \Big(\frac{d-3}{2} \Big)^2 , \tag{6.35}$$

for the massive vector. Instead of the standard definition of the mass given above, one can define a mass parameter μ^2 as the eigenvalue of the Laplacian, in order to mimic the scalar case:

$$\Delta V_\mu = \mu^2 V_\mu , \tag{6.36}$$

[4] After having constrained $V^+ = 0$.

then the mass-Weyl weight relation is slightly modified, but can unify the spin $s = 0$ scalar and $s = 1$ Proca vector in

$$\mu^2 = \frac{2\mathsf{P}}{d}\Big[\Big(\frac{d-1}{2}\Big)^2 - \Big(w + \frac{d-1}{2}\Big)^2 + s\Big]\,. \tag{6.37}$$

This result holds at arbitrary spin s (see [46]), and predicts an arbitrary spin Breitenlohner–Freedman bound

$$\mu^2 \geq \frac{2\mathsf{P}}{d}\,\Big[\Big(\frac{d-1}{2}\Big)^2 + s\Big]\,. \tag{6.38}$$

As for the scalar theory, it is possible to write down a Weyl invariant tractor action for the Maxwell-Proca system, but we do not want to lengthen the discussion, and refer to the original work for this construction.

We have seen how the tractor formalism allows to represent in a unified and elegant way massive and massless fields, keeping explicitly Weyl covariance. The tractor construction can be extended to spin two and further to arbitrary spin, as described in [46]. Starting from spin two, a new branch of partially massless fields arise, that again are treated in a single stroke within the tractor field equations. In order to avoid a lengthy discussion, we prefer to skip the tractor higher spin theories, and start to describe in the following chapters the tractor formulation of gravity.

6.4 Tractor formulation of Einstein's gravity coupled to scalars

In the previous sections we introduced the tractor formalism and described the construction of tractor theories for scalar and vector fields. With no more than this tractor machinery, we can describe gravity and bosonic theories in a manifestly local unit invariant way [52].

Let us remind the form of the scale tractor:

$$I^M = \frac{1}{d}D^M\sigma = \sigma\begin{pmatrix} 1 \\ b^m \\ -\frac{1}{d}\Big[\mathsf{P} + \nabla.b + b.b\Big]\end{pmatrix}\,, \qquad b_\mu = \sigma^{-1}\partial_\mu\sigma\,.$$

The scale tractor controls the coupling of matter to scale, but we can see that its length squared, once evaluated at constant choice of scale, is just proportional to the scalar curvature $R = 2(d-1)\mathsf{P}$. This allows to rephrase the Einstein Hilbert action principle in terms of I^M. In fact the action for cosmological Einstein gravity coupled to a massive scalar field is given by [5]

$$\begin{aligned} S = -\frac{d(d-1)}{2}\int\frac{\sqrt{-g}}{\sigma^d}\Big[I\cdot I+\frac{2\lambda}{d(d-1)}\Big] \\ -\frac{1}{2}\int\frac{\sqrt{-g}}{\sigma^{d+2w-1}}\,\varphi\, I\cdot D\,\varphi\,. \end{aligned} \tag{6.39}$$

Here we have introduced the dimensionless combination of the cosmological constant and Newton's constant $\lambda=\kappa^{\frac{4}{d-2}}\Lambda$. We recall that the scale tractor is tractor parallel

$$\mathcal{D}_\mu I^M = 0\,,$$

exactly when the metric $g_{\mu\nu}$ is conformally Einstein. Its length $I\cdot I$ is therefore constant in this case. In general however, the length of the scale tractor is Weyl invariant but spacetime dependent.

The action S above is Weyl invariant precisely when the scalar φ has weight w

$$S[g_{\mu\nu},\sigma,\varphi]=S[\Omega^2 g_{\mu\nu},\Omega\sigma,\Omega^w\varphi]\,. \tag{6.40}$$

Moreover the scalar equation of motion $I\cdot D\,\varphi=0$ explicitly says

$$\Big[-\sigma^2 g^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu+m^2_{\rm grav}\Big]\varphi=0\,.$$

The first term is the Weyl compensated scalar wave equation where the $\widetilde{\nabla}_\mu$ denotes the Weyl compensated covariant derivative (it acts on scalars as $\widetilde{\nabla}_\mu=\nabla_\mu-wb_\mu$). The second term in the scalar equation of motion is a "gravitational mass" term

$$m^2_{\rm grav}=w(d+w-1)\,I\cdot I\ . \tag{6.41}$$

If the gravitational background is taken on-shell and back-reaction is ignored, then the square of the scale tractor $I\cdot I$ is constant and the above "mass-Weyl weight relationship" relates masses to weights of tractors. In fact, it generalizes to all higher spins, and allows mass to be reinterpreted as weight [46]. However, since the constant of proportionality is of order of the cosmological constant this introduces an unnatural tuning of weights to masses. We explore further this situation in the next sections by accounting for back reaction.

[5] Remind that we denote the inner product of two tractors with the tractor metric by a dot, so $A\cdot B\equiv A^M\eta_{MN}B^N$ (not to be confused with a period for the dot product of four vectors $a.b=a^\mu b_\mu$).

6.4.1 Tractor Stress Tensors

In the previous sections, we saw that tractors were the natural language in which to formulate physics in a way manifestly independent of local choices of unit systems. Once matter systems are coupled to gravity, the natural way to present the field equations is a tractor generalization of Einstein's equations, along the lines of

$$\mathfrak{G}^{MN} + \lambda\,\eta^{MN} = \mathfrak{T}^{MN}\,, \tag{6.42}$$

where $\mathfrak{G}^{MN}$ and $\mathfrak{T}^{MN}$ are symmetric tractor tensors built from the usual stress energy tensor and Einstein tensor. Our first step is to construct a "tractor stress tensor" $\mathfrak{T}^{MN}$, transforming correctly under Weyl rescalings. A locally unit invariant description of the matter sector of a theory yields, in general, a Weyl invariant action for the matter fields ϕ_i:

$$S_{\rm Matter}[g,\sigma,\phi_i] = S_{\rm Matter}[\Omega^2 g, \Omega\sigma, \mathcal{F}_i(\Omega,\phi_i)]\,.$$

In this formula the transformation of the matter fields $\phi_i \mapsto \mathcal{F}_i(\Omega,\phi_i)$ is denoted by some function $\mathcal{F}_i$, which can often be easily determined using tractor technology along the lines of [46]. For the following arguments, its precise form is irrelevant.

The standard stress energy tensor of general relativity is given by

$$T_{\mu\nu}(g,\sigma,\phi_i) = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm Matter}}{\delta g^{\mu\nu}}\,.$$

Clearly, under Weyl transformations, it obeys

$$T_{\mu\nu} \mapsto \Omega^{2-d}T_{\mu\nu}\,.$$

Thus, the most natural ingredient of our construction will be the Weyl invariant stress tensor:

$$\mathfrak{T}^{mn}(g,\sigma,\phi_i) = -\frac{2}{\sqrt{-g}}\sigma^d e^{\mu m}e^{\nu n}\frac{\delta S_{\rm Matter}}{\delta g^{\mu\nu}}\,. \tag{6.43}$$

It is worthwhile noting that the scale σ enters in $\mathfrak{T}^{mn}$ in a non trivial way, and that the usual stress tensor (apart from some powers of κ) is recovered once one chooses the canonical constant scale $\sigma = \kappa^{\frac{2}{d-2}}$.

The tractor stress tensor we want to construct should contain various physical objects related to $\mathfrak{T}^{mn}$, its trace $\mathfrak{T} = \mathfrak{T}^m_m$, divergence $\nabla.\mathfrak{T}^m$ and so on. The only requirement needed to completely fix $\mathfrak{T}^{MN}$ is Weyl covariance, namely

$$\mathfrak{T}^{MN} \mapsto U^M{}_R\, U^N{}_S\, \mathfrak{T}^{RS}\,, \tag{6.44}$$

where the tractor gauge transformation is given by (7.5). Imposing this transformation law, we are able to package $\mathfrak{T}^{mn}$, its trace and derivatives, into the following tractor stress tensor:

$$\mathfrak{T}^{MN} = \begin{pmatrix} 0 & 0 & \frac{1}{d}\mathfrak{T} \\ 0 & \mathfrak{T}^{mn} & -\frac{1}{d}\nabla.\mathfrak{T}^m \\ \frac{1}{d}\mathfrak{T} & -\frac{1}{d}\nabla.\mathfrak{T}^m & \mathfrak{T}^{--} \end{pmatrix} \,, \tag{6.45}$$

with

$$\mathfrak{T}^{--} = \frac{1}{d(d-1)}\Big[\nabla.\nabla.\mathfrak{T} + \frac{\Delta\mathfrak{T}}{d-2} + \mathsf{P}_{mn}\Big(d\,\mathfrak{T}^{mn} - \eta^{mn}\mathfrak{T}\Big)\Big]\,.$$

These couplings of the stress tensor to the curvature are precisely those required to ensure that $\mathfrak{T}^{MN}$ is a rank 2, weight zero, symmetric tractor tensor.

Now, let us turn to the gravity side. Of course, exactly the same procedure as for the matter sector applies. One starts by making any choice of gravitational theory with action principle built from the metric. Then, by introducing the scale σ, one rewrites the theory in a locally invariant way:

$$S_{\rm Gravity}[g,\sigma] = S_{\rm Gravity}[\Omega^2 g, \Omega\sigma]\,.$$

Then one computes the tensor $\mathfrak{T}^{mn}_{\rm Gravity}$ by exactly the same procedure as above with $S_{\rm Gravity}$ in the place of $S_{\rm Matter}$ and forms the tractor tensor $\mathfrak{T}^{MN}_{\rm Gravity}$. The equations of motion are then simply $\mathfrak{T}^{MN}_{\rm Gravity} + \mathfrak{T}^{MN} = 0$. Specializing to the case of the Einstein-Hilbert action, the gravity stress tensor is minus the Einstein tensor, and the same applies to its tractor generalization in (6.42). An explicit computation (by varying the action $S = -\frac{d(d-1)}{2}\int \frac{\sqrt{-g}}{\sigma^d} I\cdot I$) yields

$$\begin{aligned} \mathfrak{G}^{mn} &= \sigma^2\, G^{mn}(\sigma^{-2}g) \\ &= \sigma^2\,\Big\{G^{mn}(g) + (d-2)\Big[\nabla^m b^n + b^m b^n - \eta^{mn}\big(\nabla.b - \frac{d-3}{2}\,b.b\big)\Big]\Big\}\,. \end{aligned}$$

where $G_{\mu\nu}$ is the standard Einstein tensor and the first line, of course, is the invariant tensor obtained from it by Weyl compensating. The tractor Einstein tensor is therefore

$$\mathfrak{G}^{MN} = \begin{pmatrix} 0 & 0 & \frac{1}{d}\mathfrak{G} \\ 0 & \mathfrak{G}^{mn} & -\frac{1}{d}\nabla.\mathfrak{G}^m \\ \frac{1}{d}\mathfrak{G} & -\frac{1}{d}\nabla.\mathfrak{G}^m & \mathfrak{G}^{--} \end{pmatrix} \,,$$

with $\mathfrak{G}^{--}$ given by the analogous formula to $\mathfrak{T}^{--}$ in (6.45). At the canonical choice of scale $\sigma = \kappa^{\frac{2}{d-2}}$ we find

$$\mathfrak{G}^{MN}(g, \kappa^{\frac{2}{d-2}}) = \kappa^{\frac{4}{d-2}} \begin{pmatrix} 0 & 0 & \frac{1}{d}\, G \\ 0 & G^{mn} & 0 \\ \frac{1}{d}\, G & 0 & \mathfrak{G}^{--}(g, \kappa^{\frac{2}{d-2}}) \end{pmatrix}, \qquad (6.46)$$

where

$$\mathfrak{G}^{--}(g, \kappa^{\frac{2}{d-2}}) = \frac{d-2}{d-1}\, \mathsf{P}_{mn}(\mathsf{P}^{mn} - \frac{1}{d}\eta^{mn}\mathsf{P}) - \frac{1}{d}\Delta \mathsf{P}\,.$$

Vanishing of $\mathfrak{G}^{MN}(g, \kappa^{\frac{2}{d-2}})$ is exactly Einstein's equations in vacuum. For general choices of local unit systems, $\mathfrak{G}^{MN} = 0$, implies that the metric is conformally Ricci flat. Adding the cosmological term, so that $\mathfrak{G}^{MN} + \lambda\eta^{MN} = 0$, says the metric is conformally Einstein.

It is worth mentioning that the conformally Einstein condition can also be simply expressed as

$$D^M I^N = 0\,.$$

Indeed, this relationship holds because the tractor Einstein tensor can be written as

$$\begin{aligned} \mathfrak{G}^{MN} = \sigma D^M I^N - X^{(M} D^{N)} I \cdot I + \frac{(d-1)(d-2)}{2} \eta^{MN} I \cdot I \\ - \frac{1}{(d-1)(d-2)^2} X^M X^N \Big(D_R I_S\; D^R I^S \Big)\,, \end{aligned} \qquad (6.47)$$

where $X^M = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the Weyl-invariant weight one "canonical tractor."

Finally, we have so far discussed variations with respect to the metric, but in a locally unit invariant description of physics, one generally incorporates the scale field σ which must also be varied. However, for any locally unit invariant action $S[g, \sigma] = S[\Omega^2 g, \Omega\sigma]$ we have

$$\frac{\delta S[g,\sigma]}{\delta \sigma} = \frac{\delta S[\sigma^{-2} g, 1]}{\delta \sigma} = -\frac{\sqrt{-g}}{\sigma} T^\mu_\mu\,.$$

Examining (6.45), we see therefore that the σ field equation is not a new relation but simply implies[6] $\mathfrak{T}^M_{M\,\rm Gravity} + \mathfrak{T}^M_M = 0$. Specializing to cosmological Einstein gravity coupled to matter, this field equation reads

$$\mathfrak{G}^M_M + (d+2)\lambda = \mathfrak{T}^M_M\,, \qquad (6.48)$$

and will play a special *rôle* in our treatment of back-reaction in the next section.

[6] One might be tempted to think that the Weyl invariance of $S[g, \sigma]$ would imply vanishing of the stress energy trace as identity (*i.e.*, off-shell). However, since this invariance is achieved via compensating, in fact no new identity follows.

6.4.2 Tractor Back-reaction

We showed how local unit invariance implied that mass could be replaced with the more fundamental geometric notion of weight. (To be precise the weight of a tractor field.) This has the advantage that masses no longer need be defined in terms of quadratic Casimirs of the spacetime isometries, but rather in terms of the geometric field content of the theory. This is extremely appealing, because no recourse to backgrounds with special isometries is needed. The drawback of this approach is the unnaturalness of the relationship between masses and weights (6.41), relying on the cosmological constant to set fundamental mass scales. We now turn to set up the tractor back-reaction to have further insight into the problem.

Consider the simplest case of a massive scalar field coupled to cosmological Einstein gravity (whose tractor version of the standard action principle is given in (6.39)). The full equations of motion, including back-reaction are

$$\begin{aligned} I \cdot D\,\varphi &= 0\,, \\ \mathfrak{G}^{MN} + \lambda\eta^{MN} &= \mathfrak{T}^{MN}\,. \end{aligned}$$

In the lowest approximation, ignoring back-reaction, the solution of the second equation is an Einstein metric with constant scalar curvature, or in tractor language, constant length scale tractor, $I \cdot I$ = constant. As explained in section 6.2, the first equation describes a standard massive scalar field in an Einstein background with constant mass determined by the weight w of φ by (6.41). Once we include back-reaction, we are faced with the above coupled set of equations, whose novel feature is that the mass term for the scalar field is proportional to the spacetime dependent (but Weyl invariant) quantity $I \cdot I$. Explicitly (transcribing (6.40)) this reads

$$\sigma^2 g^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\,\varphi = w(d+w-1)I \cdot I\,\varphi\,. \tag{6.49}$$

However, in vacua, the length of the scale tractor is set by the trace of the tractor Einstein tensor (or in commonplace language, the trace of the Einstein tensor is proportional to the scalar curvature). We must therefore correct this relationship to include backreaction. To be precise, tracing (6.47) with the tractor metric and using (6.48,6.45) we obtain

$$\frac{1}{2}(d-1)(d-2)(d+2)I \cdot I + (d+2)\lambda = \mathfrak{T}^M_M\,.$$

Hence

$$\sigma^2 g^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\,\varphi = -\frac{2w(d+w-1)}{(d-1)(d-2)}\left[\,\lambda - \frac{1}{d}\mathfrak{T}\,\right]\varphi\,.$$

In the canonical choice of scale we have

$$\Delta\varphi = -\frac{2w(d+w-1)}{(d-1)(d-2)}\,\Big[\Lambda - \frac{\kappa^{-\frac{4w}{d-2}}}{d}\,T(g,\varphi)\Big]\,\varphi\,, \tag{6.50}$$

where $T(g,\varphi)$ is the trace of the standard stress tensor for this system. In a theory with a constant mass term, the right hand side of (6.49) would not have been modified as it is above by back-reaction. The tractor theory, on the other hand, coincides with the standard theory in the absence of back-reaction, but replaces the cosmological constant Λ by

$$\Lambda \longrightarrow \Lambda - \frac{\kappa^{-\frac{4w}{d-2}}}{d}\,T$$

in the back-reacting mass term.

Now let us recompute the mass-Weyl weight relationship to include back-reaction. For that, there are two routes we could take. The first route is to insert the trace of the matter stress tensor computed from the classical matter action. There is no way for this computation to know about scales where particle physics takes place, so it is unlikely that it can solve our naturalness problem. Nonetheless, the classical computation does generate a non trivial non-linear potential for the scalar field. The second route is a phenomenological approach taking into account quantum effects. Once we introduce a particle physics scale into the problem, either by cutting off divergences at some characteristic scale, or augmenting the toy-scalar field model with standard model fields, the expectation of the trace of the stress tensor will be of the same scale. To investigate the leading order quantum effects of integrating out the gravity modes, therefore, we must replace the trace of the stress tensor with its vacuum expectation value and find

$$m^2_{\rm Back-reaction} = -\frac{2w(d+w-1)}{(d-1)(d-2)}\Big[\Lambda - \frac{\kappa^{-\frac{4w}{d-2}}}{d}\,\langle T\rangle\Big]\,.$$

No longer does the cosmological constant alone set the scale of the mass term, but instead it appears in combination with a particle physics scale object, the trace of the stress tensor.

We close this section by noting that the above mechanism is germane to particles of arbitrary spin. The idea is no different to above. In the work of [46], it was shown that massive, massless and partially massless [98, 99, 100] fields of arbitrary spin s could be written in a manifestly unit covariant way using symmetric, weight w tractors $\varphi^{M_1\cdots M_s}$ of rank s. The equations of motion are simply

$$\begin{aligned} I\cdot D\varphi^{M_1\dots M_s} &= 0\,,\\ D\cdot\varphi^{M_2\dots M_s} &= 0\,,\\ I\cdot\varphi^{M_2\dots M_s} &= 0\,,\\ \varphi_M^{MM_3\dots M_s} &= 0\,. \end{aligned} \tag{6.51}$$

The last three equations are constraints (or generalized Feynman-type gauges for massless theories) ensuring the correct propagating degrees of freedom. The first equation implies a Klein–Gordon equation for the physical modes $\varphi^{\mu_1\dots\mu_s}$ (subject to $\nabla_\mu\varphi^{\mu\mu_2\dots\mu_s}=0=\varphi_\mu^{\mu\mu_3\dots\mu_s}$)

$$\sigma^2 g^{\mu\nu}\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu\varphi^{\mu_1\dots\mu_s}=[w(w+d-1)-s]\,I\cdot I\,\varphi^{\mu_1\dots\mu_s}\,.$$

This brings us to the starting point (6.49) for our discussion of the back-reacting massive scalar field. From here the discussion proceeds exactly as above. Moreover, an analogous discussion for Fermi fields can be carried out, since a Weitzenbock type identity for those models yields again an equation of the above Klein–Gordon type (see [101] for details). Finally, we make note that our analysis applies equally well to massless and partially massless theories since they are described also by (6.51): Strictly massless theories are obtained by setting the weight w to $w=s-2$ while depth t partially massless theories (which enjoy higher derivative gauge symmetries of order t) occur at $w=s-t-1$ for $2\le t\le s$.

6.4.3 Higher Order Classical Back-reaction

In this section we compute the higher order classical corrections to the equation of motion for a massive scalar with a unit-covariant, scale tractor coupling to cosmological Einstein gravity. Starting from the central equation (6.50), we must first compute the trace of the stress energy tensor. We find

$$T(g,\varphi)=\frac{(d+2w)(d+2w-2)}{2d}\Big[-\partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi+\frac{2w(d+w-1)}{d}\mathsf{P}\varphi^2\Big]\,.$$

Firstly notice that the latter vanishes for $w=1-\frac{d}{2},-\frac{d}{2}$. These weights correspond to having a scalar field conformally coupled to gravity $\int\varphi\big(\Delta-\frac{(d-2)}{2}\mathsf{P}\big)\varphi$. The first special weight is the engineering dimension for the conformal scalar field in d-dimensions, whereas the second simply amounts to a trivial field redefinition of the conformal scalar $\varphi\to\varphi/M_{\rm Pl}$. For generic

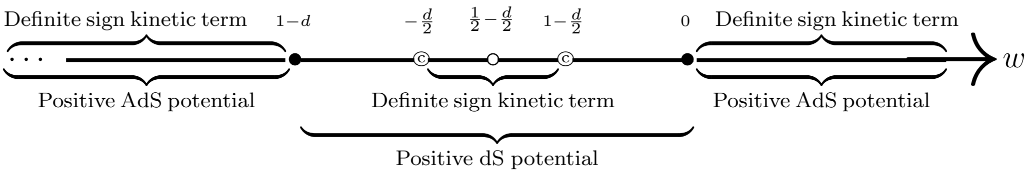


Figure 6.1: Behavior of kinetic and potential terms as a function of w. The circle $\circ$ denotes the value of w corresponding to a mass saturating the Breitenlohner–Freedman bound. The value corresponding to a conformally improved scalar field is denoted by ©.

weights the trace of the Schouten tensor P can be computed in terms of φ from (6.48) which yields

$$T(g,\varphi) = \frac{(d+2w)(d+2w-2)}{2d}\,\frac{c_1\Lambda\varphi^2 - \partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi}{1 + c_2\kappa^{-\frac{4w}{d-2}}\varphi^2}\,, \tag{6.52}$$

where $c_1 = \frac{2w(d+w-1)}{(d-1)(d-2)}$ and $c_2 = \frac{w(d+2w)(d+w-1)(d+2w-2)}{d^2(d-1)(d-2)}$. Note that both c_1 and c_2 are invariant under $w \to 1-w-d$.

Putting classical gravity on-shell (*i.e.* tree-level graviton exchange[7]) generates effectively non-linear scalar self interactions since the scalar equation (6.50) now takes the explicit form:

$$\Delta\varphi + c_1\Lambda\varphi + c_2\kappa^{-\frac{4w}{d-2}}\,\frac{\partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi}{1 + c_2\kappa^{-\frac{4w}{d-2}}\,\varphi^2} - c_1c_2\kappa^{-\frac{4w}{d-2}}\,\frac{\Lambda\,\varphi^3}{1 + c_2\kappa^{-\frac{4w}{d-2}}\,\varphi^2} = 0\,. \tag{6.53}$$

However this equation in fact follows by varying the simple non-linear sigma model action:

$$S = \int \sqrt{-g}\Big[-\frac{1}{2}\,G(\varphi)\,\partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi - U(\varphi)\Big]\,, \tag{6.54}$$

with sigma model metric and potential given by

$$G(\varphi) = 1 + c_2\kappa^{-\frac{4w}{d-2}}\,\varphi^2\,, \quad U(\varphi) = -\frac{c_1}{2}\,\Lambda\,\varphi^2\,.$$

Classical back-reaction has thus generated scalar interactions that are encoded in the above lagrangian. By means of a field redefinition

$$G(\varphi)d\varphi^2 = d\chi^2\,, \tag{6.55}$$

[7] Actually we are putting $R(g)$ on-shell, but the as-yet unknown, dynamical, metric $g_{\mu\nu}$ still resides in the terms $\partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi$ and $\Delta\varphi$.

one can generically put (6.54) in the form of a linear sigma model with a non trivial potential, *i.e.*

$$S = \int \sqrt{-g}\Big[-\frac{1}{2}\,\partial_\mu\chi\, g^{\mu\nu}\partial_\nu\chi + \frac{c_1}{2}\Lambda\varphi^2(\chi)\Big]\,.$$

The above field redefinition comes with a *caveat*: the kinetic term must have definite sign for all field configurations. This is true only when c_2 is positive which only holds for certain values of w. Another interesting feature of the model is that the potential is only positive when $-\Lambda c_1 > 0$ which again holds only for certain ranges of w. These ranges are depicted in figure 6.1. We do not perform a detailed phenomenological analysis of this model here, but note that it is rather interesting to see the rich structure introduced by this simple toy model.

Chapter 7

Gravity, two times physics and tractors

Theories with extra dimensions have been heavily scrutinized since the time of Kaluza and Klein [102]. The terminus of this train of thought is String Theory which attempts to encode the couplings of four dimensional theories in the geometry of hidden higher dimensions. A simpler and more generic rationale for further dimensions, however, might follow a line of reasoning similar to Einstein's original identification of time as an additional coordinate, along with a gauge principle—general coordinate invariance—guiding the construction of physical theories in terms of Riemannian geometry.

In this chapter, following [103], we focus on two fairly recent suggestions that physics is inherently six dimensional. Firstly, motivated by duality and holographic arguments, Bars observed that many seemingly different four dimensional particle models could be regarded as gauge fixed versions of a single underlying six dimensional model. In fact the idea of using six dimensions to describe four dimensional physics dates back to Dirac [43]. What is notable about Bars' "two times physics" [104] (see [105] for an overview) is that it aims ultimately to describe *any* physical system, whereas Dirac's work pertained only to models with conformal symmetry[1].

The second approach relies on replacing Riemannian geometry with conformal geometry so that physics is described by conformal classes of metrics and all equations are manifestly locally Weyl invariant. This is achieved by utilizing the simple physical principle that no physical quantity can depend

[1]In fact there is a extensive literature on the handling of four dimensional conformal theories using six dimensional methods. Pertinent contributions include Boulanger's conformal tensor calculus [106], the conformal space method of [107], the conformal higher spin studies [108], the BRST conformal parent action method of [109], and the application to scattering amplitudes in [110].

on local choices of unit system which implies there must exist a way to write any physical system in a Weyl invariant way [46, 101, 94]. Weyl invariance is intimately related to conformal symmetry, and for reasons very similar to those first observed by Dirac, manifest Weyl invariance can be achieved by grouping existing four dimensional physical quantities in six dimensional multiplets known as "tractors". This approach relies heavily on tractor calculus [47, 111, 50], a mathematical machinery designed for efficiently handling conformal geometries. Not only does the tractor approach identify a simple gauge principle—local unit invariance—for constructing models, it also identifies the additional timelike coordinate in two times physics as the choice of scale.

We map out the relationship between the two times and tractor approaches, since they are in fact highly complementary, and in doing so present seven different formulations of four dimensional Einstein gravity[2], several of which are novel. Of these, the action (7.27) can be viewed as a parent action depending on infinitely many fields living in a six dimensional spacetime while all other theories are gauge fixed versions of this parent action. This starting point was first proposed by Bars as part of his two times description of physics although not precisely as a four dimensional theory of gravity [53]. This action comes from a BRST quantization of the worldline conformal group gauge symmetries of a two times particle model[3]. The operators generating local worldline conformal transformations form the gravity multiplet of the model. Bars' action couples this gravity multiplet to a scalar multiplet which can be viewed as a dilaton. This fits extremely well with the tractor description of gravity in terms of a conformal class of metrics coupled to a scale field—the gauge field for local changes of unit systems.

There is an alternative proposal for a two times description of four dimensional gravity due to Bars [112]. It has the advantage that at least part of the equations for the generators of worldline conformal transformations follow from an action principle. On the other hand, unlike the action (7.27), it does not make the worldline conformal group $\mathfrak{sp}(2)$ symmetry—a central component of the two times set-up—manifest. It turns out that the two approaches are in fact equivalent, a fact that follows rapidly using tractor technology.

The tractor approach takes standard four dimensional physical quanti-

[2]Our results are valid for any spacetime dimensionality, and all formulæ will be presented as functions of d, the spacetime dimension. We will, however often use the shorthand "four" to stand for d-dimensional and "six" to stand for $(d+2)$-dimensional.

[3]Massless four dimensional spinning particles were obtained earlier from six dimensions by Siegel in [63] and further studied in [31].

ties and groups them in Weyl-multiplets labeled by $SO(d,2)$ representations[4] known as tractors. These tractors are functions of four dimensional spacetime. In particular, from the scale field σ (the spacetime dependent Planck's constant), one builds a tractor vector I^M known as the scale tractor. Like any tractor, under Weyl transformations it undergoes a tractor gauge transformation which in turn defines a covariant derivative known as the tractor connection[5] [111]. The beauty of this approach is that the Einstein condition amounts to the scale tractor being parallel with respect to this connection. The length of the scale tractor is therefore parallel for physical geometries and in fact measures the cosmological constant. Upon coupling to matter, it also provides a massive coupling constant. Remarkably, even though the small size of the cosmological constant might seem to make the length of the scale tractor inappropriate for setting particle physics mass scales, including backreaction immediately solves this "cosmological constant hierachy problem" [52]. In fact, parallel scale tractors form the first part of a link between the tractor and two times descriptions of gravity.

The link between two times physics and tractors is completed by the ambient formulation of tractor calculus developed by [51, 50, 114]. The main idea underlying ambient tractors relies on the Fefferman–Graham description of four dimensional conformal geometries in terms of six dimensional Ricci flat geometries admitting a closed homothety [115]. The latter condition implies that the six dimensional ambient geometry enjoys a curved null cone with a dilation-like vector field. This allows four dimensional conformal geometries to be realized as rays in this ambient lightcone. Bars' $\mathfrak{sp}(2)$ triplet of worldline conformal group Noether charges can be viewed, respectively, as the defining function for the ambient null cone, dilation generator and the harmonic condition obeyed by the Weyl tensor for a Ricci flat geometry. Essentially taking the old Fefferman–Graham ambient metric construction, alongside with the idea of describing unit invariant four dimensional physics with conformal geometry leads one directly to Bars' two times physics program. Needless to say, this confluence of mathematical and physical technologies is likely to lead to major advances in both fields.

The present chapter is organized as follows: In section 7.1 we recall how Einstein gravity can be recovered in the tractor framework as a parallel condition on the scale tractor. In section 7.2, we set out the ambient description

[4]For example, for a relativistic particle, from the four-velocity v_μ, the component of the four-acceleration a^μ and the vanishing function, one can build a tractor "six-velocity" $V_M = (\frac{v.a}{v.v}, e^\mu{}_m v_\mu, 0)$ transforming as a multiplet under Weyl transformations according to (7.5).

[5]In fact, the tractor connection also appears in the Yang–Mills-like construction of conformal supergravity [113].

of tractors and introduce the triplet of $\mathfrak{sp}(2)$ operators underlying the two times approach. We discuss the latter in detail in section 7.3, where we introduce the most general deformation of the flat $\mathfrak{sp}(2)$ algebra which contains an infinite tower of background fields. In section 7.4 we give the main new results based on a detailed analysis of Bars' BRST parent field theory action. By careful gauge choices and identification of the dilaton field we produce the slew of new descriptions of four dimensional gravity mentioned above as well as establishing the link between tractor and two times approaches. In section 7.4.1 we give a succinct tractor analysis of Bars' alternate proposal for a two times gravity theory. We conclude discussing the six dimensional quantum mechanical origin of four dimensional gravity, a candidate master theory generating the $\mathfrak{sp}(2)$ and dilaton dynamics, a frame-like formulation of two times physics and the relation between the towers of auxiliary fields of the two times approach and an unfolding of the full (non-linear) four dimensional Einstein's equations.

7.1 Gravity and Parallel Scale Tractors

It is well known that the Einstein–Hilbert gravitational action can be viewed as the gauge fixed version of a conformally improved scalar field theory [44, 45]

$$S[\varphi, g] = -\tfrac{4(d-1)}{(d-2)} \int d^d x \sqrt{-g} \Big[\frac{1}{2} (\nabla\varphi)^2 + \frac{1}{8} \frac{d-2}{d-1} R \, \varphi^2 \Big] \,, \qquad (7.1)$$

which is invariant under local Weyl rescalings $\Omega(x)$, transforming $\varphi \mapsto \Omega^{\frac{2-d}{2}} \varphi$ and

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu} \,. \qquad (7.2)$$

On the one hand this seems a rather trivial observation because choosing the gauge in which φ is constant and equal to κ^{-1}, one recovers the usual gravity action $S(g, \kappa^{-1}) = -\frac{1}{2\kappa^2} \int d^d x \sqrt{-g} R$. To see that this is in fact a statement of fundamental importance, first note that the Weyl transformation (7.2) defines the equivalence class relation $g_{\mu\nu} \sim \Omega^2 g_{\mu\nu}$ of a conformal class of metrics $[g_{\mu\nu}]$, so that physics can be cast in terms of conformal, rather than Riemannian geometry. Secondly, note that the Weyl transformation (7.2) amounts to making local redefinitions of unit systems, which along with general coordinate invariance, is a symmetry that any formulation of physics must enjoy.

So far there is no hint of any six dimensional quantities. To see these, we attempt to write the Weyl invariant formulation (7.1) of Einstein–Hilbert gravity as the square of a single vector I^M

$$S[g,\sigma] = \tfrac{d(d-1)}{2} \int d^d x \frac{\sqrt{-g}}{\sigma^d}\, I^M I_M\,. \tag{7.3}$$

The six component vector

$$I^M = \begin{pmatrix} \sigma \\ \nabla^m \sigma \\ -\tfrac{1}{d}\Big[\Delta + \mathsf{P}\Big]\sigma \end{pmatrix}\,, \tag{7.4}$$

is the scale tractor we defined in the previous chapter and is distinguished by its transformation properties under Weyl transformations. Here the scalar $\sigma = \varphi^{\frac{2}{2-d}}$ is simply a relabeling of the dilaton φ so that it has unit Weyl weight

$$\sigma \mapsto \Omega\,\sigma\,.$$

The field σ is nothing but the previously encountered scale field, that measures the relative choice of unit system from point to point in spacetime. Also, we recall the Schouten tensor $\mathsf{P}_{\mu\nu}$ which is the trace adjusted Ricci-type tensor, defined by

$$\mathsf{P}_{\mu\nu} = \frac{1}{d-2}\Big(R_{\mu\nu} - \frac{1}{2(d-1)} g_{\mu\nu} R\Big)\,,$$

and its trace is denoted $\mathsf{P} = \mathsf{P}_\mu^\mu$.

The main features of the action (7.3) are

- It depends on conformal classes of metrics, embedded in the double equivalence class $[g_{\mu\nu},\sigma] \sim [\Omega^2 g_{\mu\nu}, \Omega\sigma]$. This allows for manifest Weyl invariance while still specifying a canonical metric $g^0_{\mu\nu}$ in the conformal class satisfying $[g_{\mu\nu},\sigma] \sim [g^0_{\mu\nu}, \kappa^{\frac{2}{d-2}}]$.
- The measure $\sqrt{-g}\,\sigma^{-d}$ is separately Weyl invariant, as is also the square of the scale tractor I^2. This holds because the scale tractor I^M transforms under particular local $SO(d,2)$ transformations known as tractor gauge transformations.
- Einstein's equations amount to the scale tractor being parallel with respect to the tractor connection, exactly the covariant derivative implied by tractor gauge transformations.
- The "length" of the scale tractor measures the cosmological constant. Hence Ricci flatness implies a lightlike scale tractor.

We already set up the tractor formalism in the previous chapter from a purely four-dimensional viewpoint. Let us briefly remind the main ingredients that will be useful in the six-dimensional description we will pursue in the following.

From the four dimensional viewpoint, a six-component multiplet (V^+, V^m, V^-) with $m = 0, .., d-1$, forms a weight w tractor vector V^M, $M = (+, m, -)$, if under Weyl transformations it obeys the tractor gauge transformation :

$$V^M \mapsto \Omega^w U^M{}_N V^N \,, \qquad U^M{}_N = \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \delta^m_n & 0 \\ -\frac{\Upsilon^2}{2\Omega} & -\frac{\Upsilon_n}{\Omega} & \frac{1}{\Omega} \end{pmatrix} \,, \tag{7.5}$$

where $\Upsilon_\mu = e_\mu{}^m \Upsilon_m = \Omega^{-1}\partial_\mu \Omega$. In section 7.2 we will see that tractors naturally live as six-vectors in a six dimensional, signature $(4,2)$ spacetime endowed with a curved light-cone structure. The reduction to four dimensions induces a tractor-covariant connection:

$$\mathcal{D}_\mu = \begin{pmatrix} \partial_\mu & -e_{\mu n} & 0 \\ \mathsf{P}_\mu{}^m & \nabla_\mu{}^m{}_n & e_\mu{}^m \\ 0 & -\mathsf{P}_{\mu n} & \partial_\mu \end{pmatrix} \,, \tag{7.6}$$

such that

$$\mathcal{D}_\mu V^M \mapsto \Omega^w U^M{}_N \big[\mathcal{D}_\mu + w\Upsilon_\mu\big] V^N \,.$$

By means of the tractor connection one can construct a weight -1 tractor-vector operator, the so called "Thomas D-operator", which acting on weight w tractors reads:

$$D^M = \begin{pmatrix} w(d+2w-2) \\ (d+2w-2)\mathcal{D}^m \\ -(\mathcal{D}_\mu \mathcal{D}^\mu + w\mathsf{P}) \end{pmatrix} \,. \tag{7.7}$$

Acting with the Thomas D-operator on the scale σ, we obtain a weight 0 tractor-vector, the scale tractor

$$I^M = \frac{1}{d} D^M \sigma \,,$$

which has components exactly given by (7.4).

The scale tractor's main importance is twofold: first, in tractor theories it controls the coupling of matter to scale in a Weyl-covariant way [101], parametrizing the breaking of local scale invariance in the $\sigma = constant$

physical gauge. On the other hand, I^M is closely related to gravity itself: remarkably, the gravity-dilaton action (7.1), can be written entirely in terms of the scale tractor as in (7.3) where tractor indices are raised and lowered with the $SO(d,2)$ invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta_{mn} & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

To see that a tractor-parallel scale tractor, *i.e.* $\mathcal{D}_\mu I^M = 0$, amounts to Einstein's equations we explicitly compute the tractor derivative of I^M that, once evaluated at the choice of constant scale $\sigma = \sigma_0$, reads

$$\mathcal{D}_\mu I^M|_{\sigma=\sigma_0} = \sigma_0 \begin{pmatrix} 0 \\ \mathsf{P}_\mu{}^m - \frac{1}{d} e_\mu{}^m \mathsf{P} \\ -\frac{1}{d}\partial_\mu \mathsf{P} \end{pmatrix} . \tag{7.8}$$

Setting this to zero says $R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R$ and $R = \textit{constant}$, so that $g_{\mu\nu}$ is precisely an Einstein manifold. This happens at the choice of scale $\sigma = \sigma_0$, so we can say that the scale tractor is parallel when the metric is conformally Einstein:

$$\mathcal{D}_\mu I^M = 0 \quad \Leftrightarrow \quad g_{\mu\nu} = \Omega^2 g^0_{\mu\nu} \quad \text{with} \quad R_{\mu\nu}(g^0) \propto g^0_{\mu\nu} .$$

Moreover, if the scale tractor is parallel then its length squared $I^2 \equiv I^M I_M$ is constant, and proportional to the cosmological constant.

Geometrically the scale tractor can be viewed as coming from a vector perpendicular to a hypersurface in six dimensions. The intersection of that hypersurface with a (curved) lightcone defines a conformal class of metrics on the four dimensional intersection. This picture relies on a six dimensional ambient description of tractors which we describe in the next section. Given the significance of the scale tractor I^M, it would be extremely interesting to formulate four dimensional gravity in terms of an independent six component vector field. That result is obtained by combining ambient tractors with Bars' two times physics proposal and is given in section 7.4.

7.2 Ambient Tractors

The importance of six-dimensional spacetimes for describing conformally invariant four-dimensional theories has been clear since the work of Dirac [43].

(Perhaps the simplest motivation for this is that the Minkowski space conformal group $SO(4,2)$ acts naturally on the flat Lorentzian space $\mathbb{R}^{4,2}$.) Weyl invariance ensures rigid conformal symmetry whenever the metric enjoys conformal isometries; this suggests that four-dimensional conformal geometries can be studied in terms of six-dimensional Riemannian geometries. This was shown to be the case by Fefferman and Graham [115] who formulated the problem of constructing conformal invariants in terms of a six-dimensional ambient metric. This idea was extended to the tractor calculus description of conformal geometry in the series of articles [51, 50] (see also [114]).

Based on duality and holographic arguments, the two times approach of Bars advocates that four dimensional physics (irrespective of whether it enjoys rigid conformal symmetry or not) can be described using a six dimensional spacetime. The tractor approach of Gover *et al* uses the simple principle of invariance under local choices of unit system to argue that four dimensional physics should be formulated in terms of conformal geometry. Since the latter, in turn, enjoys an ambient six dimensional formulation, local unit invariance and tractors also support a formulation of four dimensional physics using a six dimensional spacetime. In this section we give the main ingredients of the six dimensional ambient description of tractor calculus.

A four dimensional conformal manifold equipped with an equivalence class of metrics $[g_{\mu\nu}]$, with equivalence defined by local Weyl transformations

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}\,,$$

can be viewed as the space of rays in a five dimensional null hypersurface embedded in a six dimensional Riemannian ambient space with metric G_{MN}. Specializing to the conformally flat case, consider the ambient space $\mathbb{R}^{4,2}$ with the standard flat Lorentzian metric $dX^M \eta_{MN} dX^N$, which enjoys a closed (and therefore hypersurface orthogonal) homothety given by the dilation/Euler operator $X^M \frac{\partial}{\partial X^M}$. The zero locus of the homothetic potential $X^M X_M \equiv X^2$ defines a five dimensional null cone so the space of null rays ξ^M subject to the equivalence relation $\xi^M \sim \Omega\,\xi^M$ (where $\Omega \in \mathbb{R}^+$) is four dimensional and determines a (conformally flat) four dimensional conformal structure. The conformal class of metrics follows by letting $\xi^M(x)$ be a section of the null cone. The ambient metric then pulls back to a four dimensional metric $ds^2 = d\xi^M d\xi_M$. Choosing a different section $\xi^M(x)$ results in a conformally related metric. For example, in the conformally flat setting, de Sitter, Minkowski and anti de Sitter space all inhabit the same conformal class. In this case the tractor connection of (7.6) is the pullback of the Cartan–Maurer form of $SO(4,2)$ to the conformally flat four dimensional space time described as a coset $SO(4,2)/P$ where P stabilizes a lightlike ray.

The above flat model of conformal geometry, as the space of lightlike rays in a six dimension ambient space, extends to curved spaces and general conformal structures as follows: A four dimensional conformal structure determines a Fefferman–Graham ambient metric which admits a hypersurface orthogonal homothety. In the flat case this homothety is generated by the Euler vector field whose components coincide with the standard Cartesian coordinates. In the curved ambient construction, the corresponding homothetic vector field will still be denoted by X^M (which are *not* generally coordinates for which we reserve the notation Y^M). The key identity is then the equation

$$G_{MN} = \nabla_M X_N \,, \tag{7.9}$$

where G_{MN} is the ambient metric and ∇ is its Levi-Civita covariant derivative. This condition already suffices to uniquely determine a four dimensional conformal structure. The symmetric part of (7.9) implies the homothetic conformal Killing equation while its antisymmetric part says that the one-form dual to X^M is closed. Indeed this one form is exact

$$X_M = \frac{1}{2}\nabla_M X^2 \,.$$

Clearly, the ambient metric is the double gradient of the homothetic potential $G_{MN} = \frac{1}{2}\nabla_M \partial_N X^2$. The zero locus of the potential X^2 defines a curved cone, a quotient of which recovers the four dimensional conformal manifold. Observe that the above identities for the ambient metric imply

$$X^M R_{MNRS} = 0 = (X^T \nabla_T + 2) R_{MN}{}^R{}_S \,.$$

To ensure uniqueness of the ambient metric for a given four dimensional structure, Fefferman and Graham require that the ambient metric is formally Ricci flat in any odd dimension (to all orders), and Ricci flat to finite order in the defining function X^2 in even dimensions greater than or equaling four. For our purposes, uniqueness of the underlying four dimensional conformal structure is all we need, so we will typically work with six dimensional ambient metrics subject to (7.9) but need not impose six dimensional Ricci flatness.

The Rosetta Stone between six dimensional ambient space operators and the Thomas D-tractor operator (7.7) on a four dimensional conformal manifold was first given in [50] and simply reads

$$D_M \quad \equiv \quad \nabla_M (d + 2X^N \nabla_N - 2) - X_M \, \Delta \,. \tag{7.10}$$

The canonical tractor of [111] corresponds to the vector field X^M while tractor weights are eigenvalues of the operator $X^M \nabla_M$. (In [114], it was realized that these operators are related to a momentum space representation of the ambient space conformal group.) Tractor tensors $T^{M_1 \ldots M_s}(x)$ (sections of weighted tractor tensor bundles over four dimensional spacetime) can then be viewed as equivalence classes of six dimensional ambient space tensors

$$T^{M_1 \ldots M_s}(Y) \sim T^{M_1 \ldots M_s}(Y) + X^2\, U^{M_1 \ldots M_s}(Y)\,, \tag{7.11}$$

subject to a weight constraint

$$X^M \nabla_M T^{M_1 \ldots M_s} = w\, T^{M_1 \ldots M_s}\,. \tag{7.12}$$

The equivalence relation can also be handled by working with weight $w-2$ ambient space tensors of the form

$$\delta(X^2)\, T^{M_1 \ldots M_s}\,,$$

subject to the constraint $X^2 = 0$. It is not difficult to check that the ambient operator (7.10) is well defined on equivalence classes defined by the cone condition (7.11).

The equivalence relation (7.11) and weight constraint (7.12) do not define a unique extension of a four dimensional tractor to the six dimensional ambient space. For that, one needs to "fix a gauge" for the equivalence relation. A convenient choice is to require that six dimensional quantities are harmonic. The first example of this is the Ricci flat condition of Fefferman–Graham (because the remaining Weyl part of the ambient Riemann curvature is then harmonic). In fact, it is easily verified that the triplet of operators

$$\{X^2\,, X^M \nabla_M + \frac{d+2}{2}\,, \Delta\}\,, \tag{7.13}$$

obey an $\mathfrak{sp}(2)$ Lie algebra. This algebraic fact underlies Bars' two times approach described in the next section.

7.3 Two Times Physics

A simple starting point for understanding two times physics, is the Howe dual pair [116]

$$\mathfrak{sp}\big(2(d+2)\big) \supset \mathfrak{sp}(2) \oplus \mathfrak{so}(d,2)\,. \tag{7.14}$$

This Lie algebra statement—namely that $\mathfrak{sp}(2)$ and $\mathfrak{so}(d,2)$ are maximal cocommutants in $\mathfrak{sp}\big(2(d+2)\big)$—says that imposing as constraints an $\mathfrak{sp}(2)$ subalgebra of the natural $\mathfrak{sp}\big(2(d+2)\big)$ algebra acting on a $d+2$ dimensional phase space, leaves a residual $\mathfrak{so}(d,2)$ global symmetry algebra. This latter algebra generates the conformal isometries of d-dimensional Minkowski (or more generally conformally flat) spacetime.

Consider, for example, Bars' approach to the relativistic particle [117, 118]. Instead of requiring worldline reparameterization invariance and therefore a four dimensional Hamiltonian constraint, Bars requires local worldline conformal invariance under $\mathfrak{so}(2,1) \cong \mathfrak{sp}(2)$ which imposes a triplet of first class constraints. In four dimensions a three dimensional constraint algebra would be too constraining, but as is clear from the Fefferman–Graham ambient space construction described above, if this constraint algebra acts in six dimensions as in (7.13), the null cone and weight constraints perform the reduction to four dimensions leaving a single Hamiltonian constraint just as in the standard approach. By making different gauge choices for the local $\mathfrak{sp}(2)$ symmetry, one can obtain a plethora of four dimensional models—"holographic shadows"—all encompassed by a single six dimensional one [119].

The above discussion pertains to single particle models propagating in fixed backgrounds. Our chief interest is a description of four dimensional field theories and in particular four dimensional gravity. For that, two main ingredients are required. Firstly we must quantize the underlying particle model so that, in turn, quantum mechanical wave functions can be reinterpreted as quantum fields. Secondly we need to write equations of motion for the background fields. Both steps can be achieved in a unified way by working with quantum mechanical operators. (An alternative approach employed heavily by Bars [118, 120] is to employ phase space quantization technology [121], but we find working directly with quantum mechanical operators to be more direct.)

Our model, described in detail in the next section, will be built from two multiplets, the first "gravity multiplet" will describe ambiently a conformal class of metrics along with an additional vector field intimately related to the scale tractor of section 7.1. The second "dilaton multiplet" describes the dilaton or scale field (or in other words a spacetime-varying Planck's constant). Equations of motion for the gravity multiplet have already been proposed by Bars [122]. Classically they amount to a triplet of Hamiltonians $Q_{ij} = Q_{ji}$ $(i, j = 1, 2)$ on a $2(d + 2)$ dimensional phase space subject to an $\mathfrak{sp}(2)$ algebra under Poisson brackets

$$\{Q_{ij}, Q_{kl}\} = \varepsilon_{kj} Q_{il} + \varepsilon_{ki} Q_{jl} + \varepsilon_{lj} Q_{ik} + \varepsilon_{li} Q_{jk}\,. \tag{7.15}$$

Here one must solve for the Q_{ij} modulo gauge transformations corresponding to canonical transformations

$$Q_{ij} \mapsto Q_{ij} + \{\epsilon, Q_{ij}\}\,. \tag{7.16}$$

An elegant solution has been found by Bars [122] by choosing Darboux coordinates $\{P_M, Y^N\} = \delta_M^N$, expanding in powers of the momentum P_M shifted by some vector field $A_M(Y)$, and then partially fixing the gauge invariance (7.15) so that

$$Q = \begin{pmatrix} X^M G_{MN}(Y) X^N & X^M \widetilde{P}_M \\ X^M \widetilde{P}_M & \Sigma(Y) + \widetilde{P}_M G^{MN}(Y) \widetilde{P}_N + H(\widetilde{P}, Y) \end{pmatrix}, \qquad (7.17)$$

where

$$\begin{aligned} \widetilde{P}_M &\equiv P_M + A_M(Y)\,, \\ H(\widetilde{P}, Y) &\equiv \sum_{k=2}^{\infty} H^{M_1 \ldots M_k}(Y) \widetilde{P}_{M_1} \cdots \widetilde{P}_{M_k}\,. \end{aligned}$$

In addition, this result is intimately connected to ambient tractors, because the algebra (7.15) requires the metric G_{MN} appearing in (7.17) to obey the closed homethety condition (7.9). Moreover the vector field A_M appearing in $\widetilde{P}^M$ obeys

$$X^M F_{MN} \equiv (\pounds_X + 1) A_N - \nabla_N (X^M A_M) = 0\,, \qquad (7.18)$$

and the scalar Σ and totally symmetric tensors $H^{M_1 \ldots M_k}$ are subject to weight conditions

$$\begin{aligned} (\pounds_X + 2)\, \Sigma &\equiv (X^M \nabla_M + 2)\; \Sigma = 0\,, \\ (\pounds_X + 2) H^{M_1 \ldots M_k} &\equiv (X^M \nabla_M + 2 - k) H^{M_1 \ldots M_k} = 0\,. \qquad (7.19) \end{aligned}$$

Classically the tensors $H^{M_1 \ldots M_s}$ must also be transverse to the homothetic vector field X^M. The above solution still enjoys residual gauge symmetries of the form (7.16). The beauty of Bars' solution is that these residual transformations amount to diffeomorphisms of the tensors X^M, G_{MN}, A_M, Σ and $H^{M_1 \ldots M_k}$, abelian Maxwell gauge transformations of A_M, as well as a certain class of higher rank symmetries of the symmetric tensors $H^{M_1 \ldots M_k}$ which we will discuss in detail later.

To quantize the Hamiltonians Q_{ij}, we look for operators acting on wavefunctions depending on coordinates Y^M. We express these as expansions in the covariant derivatives $\widetilde{\nabla}_M = \nabla_M + A_M$. This amounts to a choice of quantum orderings for a basis of all operators acting on wavefunctions. More precisely, momenta P_M act on wavefunctions as derivatives ∂_M, but we add subleading ordering terms to higher powers of momenta in order to maintain covariance. We then require that the quantum commutator of the Q_{ij}'s obeys the $\mathfrak{sp}(2)$ algebra

$$[Q_{ij}, Q_{kl}] = \varepsilon_{kj} Q_{il} + \varepsilon_{ki} Q_{jl} + \varepsilon_{lj} Q_{ik} + \varepsilon_{li} Q_{jk}\,, \tag{7.20}$$

modulo the quantum symmetry

$$Q_{ij} \mapsto Q_{ij} + [\epsilon, Q_{ij}]\,, \tag{7.21}$$

whose parameter ϵ is now itself an operator. This system of equations has been proposed by Bars in an equivalent phase space and star product quantization [122]. Quantization necessitates a slight modification of Bars' classical solution to

$$Q = \begin{pmatrix} X^2 & X^M \widetilde{\nabla}_M + \frac{d+2}{2} \\ X^M \widetilde{\nabla}_M + \frac{d+2}{2} & \Sigma + \widetilde{\nabla}^2 + H(\widetilde{\nabla}, Y) \end{pmatrix}\,, \tag{7.22}$$

with

$$H(\widetilde{\nabla}, Y) \equiv \sum_{k=2}^{\infty} H^{M_1 \ldots M_k}(Y) \widetilde{\nabla}_{M_1} \cdots \widetilde{\nabla}_{M_k}\,.$$

Here the closed homothety, curvature and weight conditions are unaltered from their classical counterparts (7.9,7.18,7.19), but the transverse conditions on the *symmetric* tensors $H^{M_1 \ldots M_k}$ are modified to read

$$2 X_M H^{M M_2 \ldots M_k} + (k+1) H_M{}^{M M_2 \ldots M_k} = 0\,. \tag{7.23}$$

From this we learn iteratively that the trace of H^{MN} vanishes, the trace of H^{MNR} is the part of H^{MN} parallel to X^M *etcetera*. More succinctly, the condition (7.23) just says

$$[X^2, H(\widetilde{\nabla}, Y)] = 0\,.$$

But now let us examine which gauge symmetries respect the quantum solution (7.22). Firstly, expanding the gauge parameter in powers of $\widetilde{\nabla}_M$

$$\epsilon(\widetilde{\nabla}, Y) = -\alpha(Y) + \xi^M(Y) \widetilde{\nabla}_M + \varepsilon(\widetilde{\nabla}, Y)\,,$$

where all terms of quadratic order and higher are stored in ε, it is easy to verify that the zeroth and first order terms generate abelian gauge transformations

$$A_M \mapsto A_M + \nabla_M \alpha\,,$$

and diffeomorphisms with parameter ξ^M. These are desirable symmetries, so we do not want to gauge fix them at this juncture. We still have the higher order gauge freedoms in ε, although these are not completely arbitrary: Requiring $Q_{11} = X^2$ to be inert, the gauge parameter ε obeys the same commutation relation with the homothetic potential as H

$$[X^2, \epsilon] = 0\,. \tag{7.24}$$

Furthermore, invariance of Q_{12} implies that

$$[X^M \widetilde{\nabla}_M, \varepsilon] = 0\,.$$

It follows that $\delta Q_{22} \equiv [\varepsilon, \Sigma + \widetilde{\nabla}^2 + H]$ obeys the same conditions as H, namely

$$[X^2, \delta Q_{22}] = 0 = [X^M \widetilde{\nabla}_M, Q_{22}] + 2Q_{22}\,.$$

Now we define a vector

$$U_M \equiv \nabla_M \Sigma\,,$$

and note that

$$[\varepsilon, \Sigma] = \frac{1}{2}\varepsilon^{MN} \pounds_U G_{MN} + \varepsilon^{MN} U_M \widetilde{\nabla}_N + \sum_{k=3}^{\infty} k\, \varepsilon^{M_1 \ldots M_k} (U_{M_1} \widetilde{\nabla}_{M_2} \cdots \widetilde{\nabla}_{M_k})_{\rm W}\,, \tag{7.25}$$

where $(\bullet)_{\rm W}$ denotes Weyl ordering in the symbols $(U, \widetilde{\nabla})$.

We now make the assumption that the vector U_M is non-vanishing. Certainly, the set of vanishing U_M is measure zero (a situation similar to non-invertible metrics among the space of 4×4 matrices). Bars has suggested that models with vanishing U_M might describe a novel "higher spin branch", but we do not pursue this line of argument any further here. With U_M non-vanishing the space of rank two and higher symmetric tensors $U_M \varepsilon^{M M_1 \ldots M_k}$ appearing in the summation in formula (7.25) suffices to gauge away the operators $H(\widetilde{\nabla}, Y)$. One might worry that this reintroduces new contributions to Q_{22} at order zero and one in $\widetilde{\nabla}$, but we have as yet not used the freedom to choose the first two terms in (7.25). Clearly, when $U_M \neq 0$, we can choose $\varepsilon^{MN} U_M$ to ensure that Q_{22} has no term linear in $\widetilde{\nabla}$. Finally, when U_M is not a conformal Killing vector (notice that (7.24) implies that ε^{MN} is trace-free) we can try to use the first term in (7.25) to remove Σ. A generic choice of metric G_{MN} will not admit conformal Killing vectors so we may safely[6] pick a gauge for which $\Sigma = 0$.

[6]It is possible that Σ can still be gauged away even if the metric G_{MN} admits conformal Killing vectors $U^M = \nabla^M \Sigma$. We have not analyzed this issue in detail, but it interesting to note that the condition $\nabla_{(M} U_{N)} \propto G_{MN}$ along with the weight condition (7.19) for Σ implies that Σ is an eigenstate of the quadratic Casimir of the triplet of operators (7.13).

Thus, we arrive at our final solution for the quantum equations (7.20)

$$Q(G_{MN}, A_M) = \begin{pmatrix} X^2 & X^M \widetilde{\nabla}_M + \frac{d+2}{2} \\ X^M \widetilde{\nabla}_M + \frac{d+2}{2} & \widetilde{\nabla}^2 \end{pmatrix} . \tag{7.26}$$

It is parameterized, modulo diffeomorphisms and $SO(1,1)$ gauge transformations by a metric G_{MN} and abelian gauge field A_M subject to the closed homothety and transverse curvature requirements in equations (7.9) and (7.18), respectively. This is the gravity mulitplet of our model. It describes spacetime geometry but does not describe gravitational dynamics. From the tractor viewpoint, that requires coupling to scale. Or in other words, a dilaton. Therefore, we now describe the coupling of the gravity multiplet to the dilaton multiplet.

7.4 Main Results: Gravity

In section 7.1 we saw that instead of formulating gravity in terms of an Einstein–Hilbert action functional depending on four-metrics, one could build from the square of the scale tractor I^M an equivalent action depending on the scale (or dilaton) σ and a conformal class of four dimensional metrics $[g_{\mu\nu}]$. The operator Q of the previous section depended on (i) a six dimensional metric G_{MN} with closed homothety and (ii) a six dimensional vector A_M. Since the metric G_{MN} encodes a four dimensional conformal class of metrics $[g_{\mu\nu}]$ one can hope that the vector A_M is somehow related to the scale tractor and so that a theory built from the operator Q could amount to a tractor description of Einstein–Hilbert gravity. For this proposal to work, we still need to couple to a dilaton field, or in other words scalar matter. From a two times physics perspective this coupling should respect the gauge symmetry (7.21) as well as the $\mathfrak{sp}(2)$ gauge symmetry generated by the operators Q. A coupling to scalars with exactly these symmetries has been computed by Bars using first quantized BRST techniques [53] and reads

$$S(Q, \Omega, \Theta, \Lambda, \Psi) = \tfrac{2(d-1)}{d-2} \int d^{d+2}Y \sqrt{G} \left[\Omega\, Q_{22} + \Theta\, Q_{12} + \Lambda\, Q_{11}\right] \Psi\,. \tag{7.27}$$

Our claim is that this action principle, along with the conditions (7.20) on the operator Q amounts to the tractor description of four dimensional Einstein–Hilbert gravity.

The action (7.27) depends (from a six dimensional viewpoint) on an infinite set of fields through the operator Q. However it also enjoys infinitely many local symmetries generated by an operator parameter ϵ as well as a local $\mathfrak{sp}(2)$ invariance with local parameters $(\lambda(Y), \theta(Y), \omega(Y))$

$$\begin{array}{rcl} Q & \mapsto & Q + [\epsilon, Q]\,, \\ \Psi & \mapsto & \Psi + \epsilon\Psi\,, \\ \Omega & \mapsto & \Omega - \epsilon^\dagger\Omega - Q_{11}^\dagger\theta + [Q_{12}^\dagger + 2]\omega\,, \\ \Theta & \mapsto & \Theta - \epsilon^\dagger\Theta + Q_{11}^\dagger\lambda - Q_{22}^\dagger\omega - 4\,\theta\,, \\ \Lambda & \mapsto & \Lambda - \epsilon^\dagger\Lambda + Q_{22}^\dagger\theta - [Q_{12}^\dagger - 2]\lambda\,. \end{array}$$

Here the dagger operation is the standard adjoint with respect to the six dimensional measure appearing in (7.27). We are now ready to verify our claim that (7.27) is the theory of gravity.

The first step is use the gauge freedom ϵ to reach the gauge (7.26) for the operator Q. This yields a standard, generally covariant, six dimensional action depending only on finitely many fields $(G_{MN}, A_M, \Omega, \Theta, \Lambda)$

$$S = \tfrac{2(d-1)}{d-2} \int d^{d+2}Y \sqrt{G}\, \Big[\Omega\, \widetilde{\nabla}^2 + \Theta\big(X^M \widetilde{\nabla}_M + \frac{d+2}{2}\big) + \Lambda X^2\Big]\Psi\,, \quad (7.28)$$

with gauge invariance

$$\begin{array}{rcl} A_M & \mapsto & A_M + \nabla_M \alpha\,, \\ \Psi & \mapsto & \Psi - \alpha\,\Psi\,, \\ \Omega & \mapsto & \Omega + \alpha\,\Omega - X^2\theta - \big(X^M\widetilde{\nabla}_M + \frac{d+2}{2} - 2\big)\omega\,, \\ \Theta & \mapsto & \Theta + \alpha\,\Theta + X^2\lambda - \widetilde{\nabla}^2\omega - 4\,\theta\,, \\ \Lambda & \mapsto & \Lambda + \alpha\,\Lambda + \widetilde{\nabla}^2\theta + \big(X^M\widetilde{\nabla}_M + \frac{d+2}{2} + 2\big)\lambda\,. \end{array} \quad (7.29)$$

The action (7.28) is four dimensional gravity wearing a six dimensional disguise. To disrobe it further, we use the $SO(1,1)$ gauge symmetry α to choose a gauge

$$X^M A_M = -w \qquad \text{which implies} \qquad X^N \nabla_N A_M = -A_M\,. \quad (7.30)$$

Here w is an arbitrary real number. We could equally well have chosen $w = 0$, but we prefer the above since it will imply the most general assignments of tractor weights to the scalar fields. In any case, w will drop out at the end of our computation, and thereby serves as a check on our algebra. Notice that using (7.18), the potential A_M now has weight -1 with respect to the weight operator $X^M\nabla_M$. Note that the vector A_M still enjoys residual abelian gauge transformations with weight zero gauge parameter $X^M\nabla_M\alpha = 0$.

We now integrate out the Lagrange multipliers (Θ, Λ) which imposes constraints

$$X^M\nabla_M\Psi = \big(w - \frac{d}{2} - 1\big)\Psi\,, \qquad X^2\,\Psi = 0\,.$$

Solving the latter constraint via

$$\Psi = \delta(X^2)\phi\,, \qquad \phi \sim \phi + X^2\chi\,,$$

and comparing with (7.11) and (7.12), we see that ϕ is a weight $w - \frac{d}{2} + 1$ tractor scalar.

There is still the freedom using the gauge parameter ω to gauge away Ω save for gauge transformations ω in the kernel of $X^M\nabla_M + w + \frac{d}{2} - 1$. Hence all that remains is the part of Ω of weight $-w - \frac{d}{2} + 1$. The remaining field content along with their weights are summarized in the following table

Field	Weight
Ω	$-w - \frac{d}{2} + 1$
ϕ	$w - \frac{d}{2} + 1$
A_M	-1

Integrating by parts to ensure no derivatives act on the delta function in Ψ, the action now takes the extremely simple form

$$S = \tfrac{2(d-1)}{d-2} \int d^{d+2}Y\sqrt{G}\,\delta(X^2)\;T\,, \tag{7.31}$$

where

$$T = \phi(\nabla^M - A^M)(\nabla_M - A_M)\Omega\,. \tag{7.32}$$

Since $T \sim T + X^2\,U$, it is a tractor scalar with weight $-d$ (see the above table). We would like to express the action (7.31) as a four dimensional integral over tractor-valued objects[7]. To that end we need to express (7.32) in terms of ambient tractor operators: Using the ambient expression (7.10)

[7] Bars handles delta-function valued ambient space integrals by developing a calculus for derivative of delta functions [112]. The simple tractor analysis given here, obviates the need for such methods.

for the Thomas D-operator, we easily derive the following ambient tractor identities

$$\Delta\Omega - 2A^M\nabla_M\Omega = \frac{1}{w}A^M D_M\Omega\,,$$
$$\nabla^M A_M = \frac{1}{d-2}\,D_M A^M\,. \qquad (7.33)$$

(There is no pole at $w = 0$ in the first identity, as can be easily verified by using the four dimensional component expression (7.7) for the Thomas D-operator.) Hence

$$T = \phi\Big(\frac{1}{w}A^M D_M - \frac{1}{d-2}(D_M A^M) + A^2\Big)\Omega\,.$$

The beauty of this expression is that $\delta(X^2)T$ now only depends on equivalence classes $A_M \sim A_M + X^2 B_M$, $\Omega \sim \Omega + X^2\,\Xi$. Therefore *all* fields are now tractor valued. Hence we may replace the ambient space integral (7.31), with a four dimensional integral depending on tractors (ϕ, Ω, A_M)

$$S = \tfrac{2(d-1)}{d-2}\int d^dx\sqrt{-g}\,\phi\Big[\frac{1}{w}A^M D_M - \frac{1}{d-2}(D_M A^M) + A^2\Big]\Omega\,. \qquad (7.34)$$

Note that the integrand has weight $-d$ while the metric determinant has weight d under Weyl transformations so this action principle is now manifestly Weyl invariant. Our claim is now that this tractor action is equivalent to the formulation of the Einstein–Hilbert action in terms of the square of the scale tractor (7.3).

To verify our final claim we must examine the remaining $SO(1,1)$ gauge symmetry

$$\begin{aligned} A_M &\mapsto A_M + \frac{1}{d-2}D_M\alpha\,,\\ \Omega &\mapsto \Omega + \alpha\,\Omega\,,\\ \phi &\mapsto \phi - \alpha\,\phi\,, \end{aligned} \qquad (7.35)$$

where the gauge parameter α is a weight zero tractor scalar. Notice that the gauge transformation of A_M respects the condition $X^M A_M = -w$. Now observe that the action depends only algebraically on the $SO(1,1)$ gauge field A_M and the pair of fields (ϕ, Ω) form a doublet under this symmetry. Hence, we expect that upon integrating out A_M, only the gauge invariant combination $\phi\,\Omega$ should survive. This computation can be performed either using component expressions for the tractor quantities in (7.34) or directly using tractors. In components, one finds that the bottom slot A^- of the gauge field decouples completely from the action and that integrating out the middle slot of A_M sets it equal to the $SO(1,1)$ current $\frac{1}{2}\nabla_m \log(\Omega/\phi)$. This yields the four dimensional action for a conformally improved scalar field

$$S = \tfrac{2(d-1)}{d-2} \int d^d x \sqrt{-g}\, \varphi \Big[\Delta - \frac{d-2}{2}\, \mathsf{P}\Big] \varphi\,,$$

where φ is the weight $1 - \frac{d}{2}$ scalar field defined by

$$\varphi^2 = \phi\,\Omega\,.$$

In other words it is the *dilaton.* Using the relationship between the dilaton and scale, $\varphi = \sigma^{1-\frac{d}{2}}$, we obtain as explained in section 7.1 the tractor version of the Einstein–Hilbert action in terms of the square of the scale tractor

$$S = \tfrac{d(d-1)}{2} \int d^d x \frac{\sqrt{-g}}{\sigma^d}\, I^M I_M\,. \tag{7.36}$$

This completes our demonstration that the $\mathfrak{sp}(2)$ invariant theory (7.27) amounts to a theory of four dimensional gravity. We conclude this section by briefly recalling the alternative six dimensional gravitational model proposed by Bars in [112].

7.4.1 An Alternative Six Dimensional Formulation of Gravity

In [112] Bars proposed the following six dimensional field theory model for gravity coupled to scalar field

$$\begin{aligned} S \;\; = \;\; -\frac{1}{2} \int d^{d+2} Y \sqrt{G}\, \Big[\delta(W) \Big(R(G) \varphi^2 + \alpha\, (\nabla\varphi)^2 - \lambda \varphi^{\frac{2d}{d-2}} \Big) \\ - \,\delta'(W) \Big((\Delta W - 4)\, \varphi^2 - \nabla_M W\; \nabla^M \varphi^2 \Big) \Big]\,, \end{aligned} \tag{7.37}$$

with $\alpha = \frac{4(d-1)}{d-2}$ and for some λ playing the *rôle* of the cosmological constant. A distinguishing feature of this action is that the homothetic condition and the weight condition on φ follow from its equations of motion; they indeed arise from the field equations for G_{MN} and φ instead of requiring closure of the $\mathfrak{sp}(2)$ algebra. Partially solving those equations, one obtains the following set of relations

$$W = X^2, \quad G_{MN} = \nabla_M X_N, \quad X^M \nabla_M \varphi = \big(1 - \frac{d}{2}\big) \varphi\;.$$

Plugging these back in (7.37) we get the following model

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}\,\delta(X^2)\Big[R(G)\varphi^2 - \alpha\,\varphi\Delta\varphi - \lambda\,\varphi^{\frac{2d}{d-2}}\Big]\,. \tag{7.38}$$

Now note that, introducing the scale tractor I^M constructed from $\sigma = \varphi^{\frac{2}{2-d}}$ in the usual way (see section 7.1), the action (7.38) becomes

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}\,\delta(X^2)\Big[R(G)\varphi^2 + \frac{\alpha}{\sigma}\,\varphi\,I^M D_M\,\varphi - \lambda\,\varphi^{\frac{2d}{d-2}}\Big]\,,$$

that in turn, by using the relation $I^M D_M \sigma^k = k(d+k-1)\sigma^{k-1}I^2$, can be rewritten as

$$S = -\frac{1}{2}\int d^{d+2}Y\sqrt{G}\,\delta(X^2)\,\frac{1}{\sigma^d}\Big[R(G)\sigma^2 - d(d-1)I^2 - \lambda\,\Big]\,. \tag{7.39}$$

Let us observe at this point that, as was shown by Fefferman and Graham in [115], a conformal class of d-dimensional metrics $[g_{\mu\nu}]$ determines a Ricci flat ambient space if d is odd, and a Ricci flat ambient space modulo $(X^2)^{\frac{d-2}{4}}$. Hence, since the action (7.39) depends only on the conformal class of metrics $[g_{\mu\nu}]$ and includes the delta function $\delta(X^2)$, we can set to zero the curvature term in (7.39). In fact, another way to see this, is that we could have chosen a gauge in section 7.3 where $\Sigma = R(G)$.

Now that the model is completely written in terms of tractor objects it may be directly written in four dimensional language as

$$S = \tfrac{d(d-1)}{2}\int d^dx\frac{\sqrt{-g}}{\sigma^d}\left[I^M I_M + \frac{\lambda}{d(d-1)}\right]\,. \tag{7.40}$$

When $\lambda = 0$, this model coincides with (7.36) demonstrating the equivalence of these two models in that case. The formulation (7.37) has the advantage that it includes a cosmological constant and partially imposes the relations (7.20) as equations of motion coming from a variational principle. Its disadvantage is that the manifest $\mathfrak{sp}(2)$ symmetry is lost.

7.5 Outlook

In this chapter [103], we formulated the Einstein–Hilbert action as a trace

$$S = \mathrm{tr}\,Q\,P \tag{7.41}$$

over quantum mechanical operators Q (as in (7.26)) and

$$P = \begin{pmatrix} |\Psi\rangle\langle\Lambda| & \frac{1}{2}|\Psi\rangle\langle\Theta| \\ \frac{1}{2}|\Psi\rangle\langle\Theta| & |\Psi\rangle\langle\Omega| \end{pmatrix} .$$

In this formulation, second quantization amounts to integrating over the space of operators Q and P in the path integral. This leads one to wonder whether quantum field theory effects, such as Weyl anomalies, can be understood from this six-dimensional quantum mechanical picture. An advantage of this two times approach is that it formulates gravity in terms of a very limited field content: the three components of Q viewed as functions of a twelve dimensional phase space. Weyl and diffeomorphism symmetries are neatly encoded in the algebra (7.20) and its gauge invariance (7.21). A pressing question therefore is to compute anomalies in the $\mathfrak{sp}(2)$ symmetry.

Another benefit of the two times starting point (7.27) is that it yields a new tractor formulation of the conformally Einstein condition (see the action (7.34)). At the very least, this should have implications for conformal geometry; the triplet of tractor fields (ϕ, Ω, A_M) underly the scale tractor I^M. This observation deserves further investigation.

Another interesting avenue for further research is whether there exists a framelike formulation of two times physics. This is based on the simple observation that the operator (7.26) can be factorized as

$$Q = \left[\begin{pmatrix} X_M \\ \widetilde{\nabla}_M \end{pmatrix} \begin{pmatrix} X^M & \widetilde{\nabla}^M \end{pmatrix} \right]_W .$$

The operator $V_i^M = (X^M \;\; \widetilde{\nabla}^M)$ can then be interpreted as a two times frame field, so one could try to impose the Howe dual pair (7.14) decomposition as equations of motion for fundamental fields V_i^M. This might be particularly interesting when one considers the interpretation of the infinite tower of six dimensional auxiliary fields appearing in the parent action (7.27). In particular, one wonders whether these fields solve the problem posed, and partially solved in [123], of finding an unfolding of the full nonlinear Einstein's equations. The relation between these two approaches may be clearer in a framelike formulation, since (unlike unfolding constructions) two times models are typically constructed in a metric formulation.

Finally, a gravitational two times action principle that simultaneously incorporates the benefits of both actions (7.27) and (7.37)—namely producing the $\mathfrak{sp}(2)$ algebra as equations of motion while maintaining manifest $\mathfrak{sp}(2)$ symmetry—would be very desirable. In fact, once we understand that our work implies that the coupling of the gravity multiplet (built from $\mathfrak{sp}(2)$

generators) to scalars really amounts to a gravity-dilaton coupling, then we can identify yet another action principle proposed by Bars as a candidate model for cosmological four dimensional Einstein gravity. Bars' proposal is to produce the equations of motion for the operator Q from a Chern–Simons action [120]

$$S_{\rm CS} = \int \big[Q \star Q + Q \star Q \star Q\big]\,,$$

(where the Moyal star product $\star$ is employed to produce operator equations of motion from phase space valued fields). Hence the sum of this action plus the BRST action $S_{\rm BRST}$ in (7.27)

$$S = S_{\rm CS} + \lambda S_{\rm BRST}\,, \tag{7.42}$$

deforms the $\mathfrak{sp}(2)$ relations by dilaton dependent terms (see [120] for explicit formulæ). A simple conjecture, therefore, is that these produce the cosmological constant coupling missing from the action (7.27). In particular, the relative coefficient λ in the total action (7.42) could be identified with the cosmological constant.

Conclusions

The main subject of my PhD research work has been the study of worldline supersymmetric models in curved backgrounds. $O(N)$ spinning particles describe conformal HS fields in first quantization. Their worldline description has been pursued in flat space in [30] and has been reviewed in the first chapter of this thesis, while the canonical quantization in $(A)dS$ was performed in [32, 33]. As it is explained in chapter 2, we computed the transition amplitude for the related N-extended SUSY quantum mechanics in curved space by both operatorial and functional methods, and this allowed us to find the proper counterterms that are needed in defining the path integral. With these counterterms at hand, as a future interesting work, one can perform the functional quantization of $O(N)$ spinning particles an $(A)dS$ backgrounds, or on even more general conformally flat manifolds, and study the quantum properties of the related HS fields.

The less familiar $U(N)$ spinning particles instead, describe holomorphic HS on Kähler spaces. They have been quantized in flat complex space in [40], and it was shown that they describe a novel class of HS fields obeying gauge invariant equations similar to Fronsdal-Labastida equations, that are presented in the third chapter of this thesis. We noticed that they can be consistently quantized on Kähler manifolds with constant holomorphic curvature (CHC), a sort of complex version of maximally symmetric spaces. In chapter 4 we showed how the model has been extended to a $U(N|M)$ supersymmetric quantum mechanics, whose algebra has been studied on general Kähler backgrounds. This allowed to recognize a suitable hamiltonian operator for the model, and the corresponding transition amplitude was computed by using operatorial methods [41]. The non-ambiguous result for the transition amplitude is the bench mark for path integral calculations. In fact, in the fifth chapter we set up the functional integral for the $U(1)$ spinning particle on a generic Kähler background, and studied the quantum properties of holomorphic p-forms. Namely, we obtained a worldline representation of the one-loop effective action for holomorphic differential forms as well as for Dirac fermions and “non gauge” forms. The quantum mechanical representation allowed, moreover, to establish duality relations including topological mismatches, in close analogy to the $O(2)$ particle case of [28]. An interesting future development would be the quantization of the $U(2)$ particle on general backgrounds, as well as the "higher spin" $U(N)$ spinning particle on CHC backgrounds. Finally, an even more intriguing subject would be the search for non-linear extensions of our complex HS equations, in the spirit of Vasiliev's theory.

The two last chapters of the thesis are devoted, instead, to the second research line I followed during my PhD, namely the study and application of tractor formalism in field theory. In chapter 6 we presented the basic ingredients of the tractor machinery as well as the construction of tractor scalar and vector theories following [46]. Already in describing such simple field theories, the tractor formalism turns out to be a powerful and unifying tool. Indeed, one is able to cast in a unique tractor action massless, massive and conformally improved scalars, as well as massive, massless and Weyl invariant vectors, and one can switch between the different branches just by tuning the Weyl weights of the fields. The rest of the chapter is devoted to the study of cosmological Einstein's gravity coupled to matter in the tractor framework. The tractor theory of free spin two fields is known [46], but here we find the tractor formulation of the complete non-linear Einstein theory, and set up tractor backreaction [52].

Finally, in the last chapter we analyzed the relationship between six dimensional quantum mechanical models presented by Bars in the "two times physics" approach, ambient tractors and gravity. We have seen that Einstein's equations are equivalent to the existence of a parallel scale tractor (a six component vector subject to a certain first order covariant constancy condition at every point in four dimensional spacetime). These results suggest a six dimensional description of four dimensional physics, a viewpoint promulgated by the two times physics program of Bars, see *e.g.* [104, 105]. We recast four dimensional gravity in terms of six dimensional quantum mechanics by melding the two times and tractor approaches [103]. This "parent" formulation of gravity is built from an infinite set of six dimensional fields. Successively integrating out these fields yields various novel descriptions of gravity including a new four dimensional one built from a scalar doublet, a tractor vector multiplet and a conformal class of metrics.

So far we have concerned ourselves with an essentially classical, but manifestly unit invariant, analysis of field theories. However, since Weyl and scale invariances are typically anomalous, it is extremely interesting to apply our ideas to quantum field theories. Among the more stunning implications of Maldacena's AdS/CFT correspondence [124] is the formulation of renormalization group flows of boundary quantum field theories in terms of geometry of a bulk theory of one higher dimension [125]. In particular, boundary Weyl anomalies appear as coefficients of logarithms of the radial coordinate when attempting to perform expansion of the bulk metric away from the boundary characterized by a conformal class of boundary metrics [126]. An intriguing future research route would be to undertake the holographic renormalization program from a tractor perspective. It would be advantageous in that the tractor formulation manifests scale invariance at each step, and the key idea would be to substitute the AdS radial coordinate with the scale field itself, that controls the breaking of Weyl invariance, to expand bulk quantities. An ambient space scalar σ subject to a unit weight condition, amounts to a canonical choice of metric in the D-dimensional conformal class of metrics (the field σ is the ambient version of the scale $\sigma(x)$). In particular, choosing σ so that the scale tractor is parallel and non-null yields an Einstein manifold with non-zero cosmological constant. Then identifying this manifold with the bulk theory one can try to construct a correspondence with a $(D-1)$-dimensional boundary conformal field theory. Since the canonical bulk metric appears at constant values of the scale σ, it makes sense to express bulk quantities as power series in σ. Again, one could study obstructions in this expansion to obtain boundary Weyl anomalies.

Appendix A

Fermionic coherent states

The even-dimensional Clifford algebra

$$\{\psi^M, \psi^N\} = \delta^{MN}, \quad M, N = 1, \ldots, 2l \tag{A.1}$$

can be written as a set of l fermionic harmonic oscillators (the index M may collectively denote a set of indices that may involve internal indices as well as a space-time index), by simply taking complex combinations of the previous operators

$$a^m = \frac{1}{\sqrt{2}}\left(\psi^m + i\psi^{m+l}\right) \tag{A.2}$$

$$a^\dagger_m = \frac{1}{\sqrt{2}}\left(\psi^m - i\psi^{m+l}\right), \qquad m = 1, \ldots, l \tag{A.3}$$

$$\{a^m, a^\dagger_n\} = \delta^m_n \tag{A.4}$$

and it can be thus represented in the vector space spanned by the 2^l orthonormal states $|\mathbf{k}\rangle = \prod_m (a^\dagger_m)^{k_m}|0\rangle$ with $a_m|0\rangle = 0$ and the vector $\mathbf{k}$ with elements taking only two possible values, $k_m = 0, 1$. This basis (often called spin-basis) yields a standard representation of the Clifford algebra, i.e. of the gamma matrices.

An alternative overcomplete basis is given by the coherent states that are eigenstates of creation and annihilation operators.

$$|\xi\rangle = e^{a^\dagger_m \xi^m}|0\rangle \quad \to \quad a^m|\xi\rangle = \xi^m|\xi\rangle = |\xi\rangle\xi^m \tag{A.5}$$

$$\langle\bar\eta| = \langle 0|e^{\bar\eta_m a^m} \quad \to \quad \langle\bar\eta|a^\dagger_m = \langle\bar\eta|\bar\eta_m = \bar\eta_m\langle\bar\eta| \,. \tag{A.6}$$

Below we list some of the useful properties satisfied by these states. Using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^X e^{[X,Y]}$, valid if $[X,Y] = c$-number, one finds

$$\langle\bar\eta|\xi\rangle = e^{\bar\eta\cdot\xi} \tag{A.7}$$

that in turn implies

$$\langle\bar{\eta}|a^m|\xi\rangle = \xi^m\langle\bar{\eta}|\xi\rangle = \frac{\partial}{\partial\bar{\eta}_m}\langle\bar{\eta}|\xi\rangle \tag{A.8}$$

$$\langle\bar{\eta}|a^\dagger_m|\xi\rangle = \bar{\eta}_m\langle\bar{\eta}|\xi\rangle \tag{A.9}$$

so that $\{\frac{\partial}{\partial\bar{\eta}_m},\bar{\eta}_n\} = \delta^m_n$. Defining

$$d\bar{\eta} = d\bar{\eta}_l\cdots d\bar{\eta}_1\ , \qquad d\xi = d\xi^1\cdots d\xi^l \tag{A.10}$$

so that $d\bar{\eta}d\xi = d\bar{\eta}_1 d\xi^1 d\bar{\eta}_2 d\xi^2\cdots d\bar{\eta}_l d\xi^l$, we have the following relations

$$\int d\bar{\eta}d\xi\ e^{-\bar{\eta}\cdot\xi} = 1 \tag{A.11}$$

$$\int d\bar{\eta}d\xi\ e^{-\bar{\eta}\cdot\xi}\ |\xi\rangle\langle\bar{\eta}| = \mathbb{1} \tag{A.12}$$

where $\mathbb{1}$ is the identity in the Fock space. One can also define a fermionic delta function with respect to the measure (A.10) by

$$\int d\xi\ e^{(\bar{\lambda}-\bar{\eta})\cdot\xi} = (\bar{\eta}^1-\bar{\lambda}^1)\cdots(\bar{\eta}^l-\bar{\lambda}^l)\equiv\delta(\bar{\eta}-\bar{\lambda})\ . \tag{A.13}$$

Finally, the trace of an arbitrary operator can be written as

$$\mathrm{Tr}\,A = \int d\bar{\eta}d\xi\ e^{-\bar{\eta}\cdot\xi}\langle-\bar{\eta}|A|\xi\rangle = \int d\xi d\bar{\eta}\ e^{\bar{\eta}\cdot\xi}\langle\bar{\eta}|A|\xi\rangle\ . \tag{A.14}$$

As a check one may compute the trace of the identity

$$\mathrm{Tr}\,\mathbb{1} \ = \ \int d\xi d\bar{\eta}\ e^{\bar{\eta}\cdot\xi}\langle\bar{\eta}|\xi\rangle = \int d\xi d\bar{\eta}\ e^{2\bar{\eta}\cdot\xi} = 2^l\ . \tag{A.15}$$

Let us end this section by listing a few expressions that are helpful in the computation of section 2.3.1 (here fermions are labelled by two indices, a tangent space index a and an SO(N) R-symmetry index i)

$$\begin{aligned}\int d\bar{\zeta}d\eta\ e^{(\bar{\lambda}-\bar{\zeta})\cdot\eta}\tilde{\psi}^a_i \ &= \ \sqrt{\frac{\beta}{2}}\Big(\frac{\partial}{\partial\bar{\lambda}^i_a}+\bar{\lambda}^a_i\Big)\int d\bar{\zeta}d\eta\ e^{(\bar{\lambda}-\bar{\zeta})\cdot\eta}\\ &= \ \sqrt{\frac{\beta}{2}}\Big(\frac{\partial}{\partial\bar{\rho}^i_a}+\bar{\lambda}^a_i\Big)\int d\bar{\zeta}\ \delta(\bar{\zeta}-\bar{\rho})\Big|_{\rho=\lambda}\end{aligned} \tag{A.16}$$

where the above apparently baroque notation is used because the fermionic derivatives only act upon the delta-function and not on eventual $\bar{\lambda}$-dependent expression that may appear next to it. We thus have

$$\int d\bar{\zeta}d\eta\ e^{(\bar{\lambda}-\bar{\zeta})\cdot\eta}\tilde{\psi}^a\cdot\tilde{\psi}^b \ = \ \beta\tilde{M}^{ab}\int d\bar{\zeta}\ \delta(\bar{\zeta}-\bar{\rho})\Big|_{\rho=\lambda} \tag{A.17}$$

with

$$\tilde{M}^{ab} = \frac{1}{2}\Big(\bar{\lambda}^a \cdot \bar{\lambda}^b + \bar{\lambda}^a \cdot \frac{\partial}{\partial \bar{\rho}_b} - \bar{\lambda}^b \cdot \frac{\partial}{\partial \bar{\rho}_a} + \frac{\partial}{\partial \bar{\rho}_a} \cdot \frac{\partial}{\partial \bar{\rho}_b}\Big) \tag{A.18}$$

and one can then eventually switch to the Lorentz generators

$$M^{ab} = \frac{1}{2}\Big(\bar{\lambda}^a \cdot \bar{\lambda}^b + \bar{\lambda}^a \cdot \frac{\partial}{\partial \bar{\lambda}_b} - \bar{\lambda}^b \cdot \frac{\partial}{\partial \bar{\lambda}_a} + \frac{\partial}{\partial \bar{\lambda}_a} \cdot \frac{\partial}{\partial \bar{\lambda}_b}\Big) \tag{A.19}$$

by suitably subtracting those terms that appear when derivatives on $\bar{\lambda}$ act on the neighbor M^{cd}. In the computation of section 2.3.1 it tuns out that such additional terms are provided in the transition amplitude.

Appendix B

B coefficients for the $O(N)$ quantum mechanics

The coefficients $B^k_l(x,\bar\eta,\xi)$, defined in (2.24), can be computed following the strategy described in detail in [64] for $N = 0, 1, 2$, and in [41] for the complex $U(N|M)$ sigma model. First of all we divide the hamiltonian (2.22) into three pieces contributing at most two, one or none p eigenvalues:

$$\begin{aligned} H &= H_B + H_1 + H_2\ , \quad \text{where} \\ H_B &= \frac{1}{2} g^{-1/4} p_\mu g^{1/2} g^{\mu\nu} p_\nu g^{-1/4} \\ H_1 &= -i\, g^{\mu\nu}\omega_{\mu ab}\bar\Psi^a\cdot\Psi^b \left(g^{1/4}p_\nu g^{-1/4}\right) \\ H_2 &= -\frac{1}{2}\, g^{-1/2}\partial_\mu\left(g^{1/2}g^{\mu\nu}\omega_{\nu ab}\right)\bar\Psi^a\cdot\Psi^b \\ &\quad -\frac{1}{2}\left(g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd} - 8\alpha\, R_{abcd}\right)\bar\Psi^a\cdot\Psi^b\bar\Psi^c\cdot\Psi^d + V\ . \end{aligned} \tag{B.1}$$

First of all, notice that H_B is precisely the usual bosonic quantum hamiltonian, carefully studied in the literature [64, 37]. Let us start with B^k_{2k}: the only way to have $2k$ p eigenvalues is from k factors of H_B and no commutators taken into account, giving simply

$$B^k_{2k}\, p^{2k} = \left(\frac{p^2}{2}\right)^k\ . \tag{B.2}$$

For B^k_{2k-1} we can have two kinds of terms. The first comes from k factors of H_B with one p acting as a derivative; this gives the corresponding B^k_{2k-1} of the purely bosonic model, whose computation is explained in detail in [64, 37]. The other term comes from $k-1$ factors of H_B and one H_1, by substituting all operators with the corresponding eigenvalues. Putting things together we obtain

$$\begin{aligned} B^k_{2k-1}\, p^{2k-1} &= -\frac{ik}{2}\left(\frac{p^2}{2}\right)^{k-1} g^{\mu}\, p_\mu - i\binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\frac{1}{2} g^{\nu\lambda\mu}\, p_\mu\, p_\nu\, p_\lambda \\ &\quad - i\,k\left(\frac{p^2}{2}\right)^{k-1} g^{\mu\nu}\omega_\mu\, p_\nu\,. \end{aligned} \tag{B.3}$$

For B^k_{2k-2} four types of term contribute: *i)* k factors of H_B, giving the coefficient of the corresponding bosonic model, *ii)* $k-1$ factors of H_B and one H_1, with one p acting as a derivative. This contribution gives four terms: the derivative acting from one H_B to H_1, from H_1 to one H_B, within H_1 or within the $k-1$ H_B's. *iii)* $k-1$ factors of H_B and one H_2, substituting all operators with their eigenvalues, and *iv)* $k-2$ factors of H_B and two H_1, substituting all with eigenvalues. Remember that in *iii)* and *iv)* $\{\Psi,\bar\Psi\}$ anticommutators have to be taken into account in order to obtain eigenvalues on the coherent states. Altogether it results in

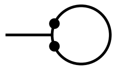
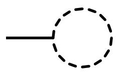
$$\begin{aligned} B^k_{2k-2}\, p^{2k-2} &= k\left(\frac{p^2}{2}\right)^{k-1}\left[\frac{1}{32}\partial_\mu \ln g\,\partial^\mu \ln g + \frac{1}{8}\partial_\mu\partial^\mu \ln g + \frac{1}{8} g^\mu \partial_\mu \ln g\right] \\ &- \binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\left[\frac{1}{2}\partial^\mu g^\nu + \frac{1}{4} g^\mu g^\nu + \frac{1}{4} g^\lambda g^{\mu\nu}_\lambda + \frac{1}{4} g^{\mu\nu\lambda}_\lambda\right] p_\mu\, p_\nu \\ &- \binom{k}{3}\left(\frac{p^2}{2}\right)^{k-3}\left[\frac{1}{2} g^{\lambda\sigma\mu\nu} + \frac{3}{4} g^{\mu\nu\lambda} g^\sigma + \frac{1}{2} g^{\rho\mu\nu} g^{\lambda\sigma}_\rho + \frac{1}{4} g^{\mu\nu}_\rho g^{\lambda\sigma\rho}\right] p_\mu\, p_\nu\, p_\lambda\, p_\sigma \\ &- \binom{k}{4}\left(\frac{p^2}{2}\right)^{k-4}\left[\frac{3}{4} g^{\nu\lambda\mu} g^{\rho\tau\sigma}\right] p_\mu\, p_\nu\, p_\lambda\, p_\sigma\, p_\rho\, p_\tau \\ &- \binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\left[\partial^\mu\left(g^{\nu\lambda}\omega_\lambda\right) - \frac{1}{2} g^{\mu\nu\lambda}\omega_\lambda\right] p_\mu\, p_\nu - k\left[\frac{1}{2}(k-1)\left(\frac{p^2}{2}\right)^{k-2} g^\mu p_\mu \right. \\ &+ \left.\binom{k-1}{2}\left(\frac{p^2}{2}\right)^{k-3}\frac{1}{2} g^{\nu\lambda\mu}\, p_\mu\, p_\nu\, p_\lambda\right] g^{\sigma\rho}\omega_\sigma\, p_\rho + \frac{1}{4}k\left(\frac{p^2}{2}\right)^{k-1} g^{\mu\nu}\omega_\mu \partial_\nu \ln g \\ &- \frac{1}{2}k\left(\frac{p^2}{2}\right)^{k-1}\Big\{ g^{-1/2}\partial_\mu\left(g^{1/2} g^{\mu\nu}\omega_\nu\right) + \Big(g^{\mu\nu}\omega_{\mu ab}\omega_{\nu cd} - 8\alpha\, R_{abcd}\Big)\Big[\bar\eta^a\cdot\, \xi^d \eta^{bc} \\ &+ \bar\eta^a\cdot\, \xi^b \bar\eta^c\cdot\, \xi^d\Big] - 2V\Big\} \\ &- \binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2} g^{\mu\nu}\omega_{\mu ab} g^{\lambda\sigma}\omega_{\lambda cd}\Big[\bar\eta^a\cdot\, \xi^d\eta^{bc} + \bar\eta^a\cdot\, \xi^b\bar\eta^c\cdot\, \xi^d\Big] p_\nu\, p_\sigma \end{aligned} \tag{B.4}$$

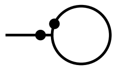
where again we use the compact notations for tensors introduced below eq. (2.26).

Appendix C

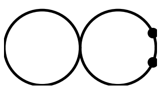
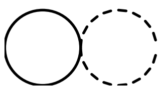
Feynman diagrams

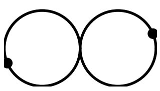
We list the set of Feynman diagrams and the associated worldline integrals that enter the computation of the transition amplitude to order β in section 2.3. We are not reporting here those diagrams that involve fermionic self-contractions as with the rules of section 2.3 such self-contrations are trivial. Hence, the a priori non-trivial diagrams entering the contribution $\langle S_3 \rangle$ are

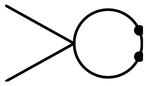
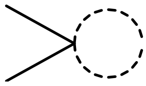
$\mathbf{I}_1 =$  $+$  $= \int_{-1}^{0} d\tau\ \tau ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau$ (C.1)

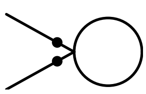
$\mathbf{I}_2 =$  $= \int_{-1}^{0} d\tau\ {}^\bullet\Delta|_\tau\ .$ (C.2)

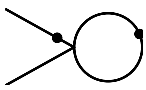
Those contributing to $\langle S_4 \rangle$ are

$\mathbf{I}_3 =$  $+$  $= \int_{-1}^{0} d\tau\ \Delta|_\tau ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau$ (C.3)

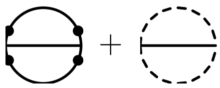
$\mathbf{I}_4 =$  $= \int_{-1}^{0} d\tau\ {}^\bullet\Delta^2|_\tau$ (C.4)

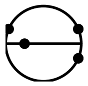
$\mathbf{I}_5 =$  $+$  $= \int_{-1}^{0} d\tau\ \tau^2 ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau$ (C.5)

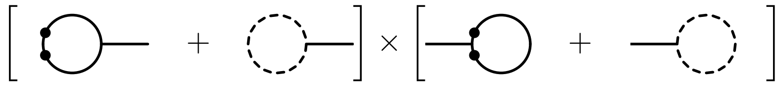
$\mathbf{I}_6 =$  $= \int_{-1}^{0} d\tau\ \Delta|_\tau$ (C.6)

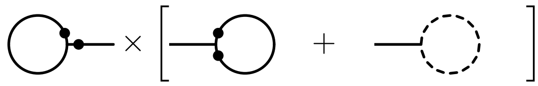
$\mathbf{I}_7 =$  $= \int_{-1}^{0} d\tau\ \tau\Delta^\bullet|_\tau\ .$ (C.7)

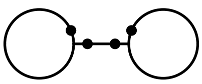
The remaining ones contributing to $\langle S_3^2 \rangle_c$ can be divided into purely bosonic contributions

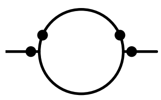
$$\mathbf{I}_9 =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \Delta({}^\bullet\Delta^{\bullet 2} - {}^{\bullet\bullet}\Delta^2) \tag{C.8}$$

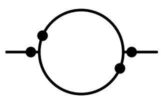
$$\mathbf{I}_{10} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet\Delta^\bullet\Delta^\bullet \tag{C.9}$$

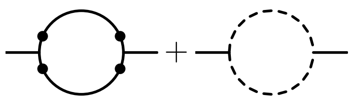
$$\mathbf{I}_{11} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\tau \; \Delta \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma \tag{C.10}$$

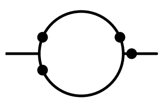
$$\mathbf{I}_{12} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta|_\tau \; {}^\bullet\Delta \; ({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)|_\sigma \tag{C.11}$$

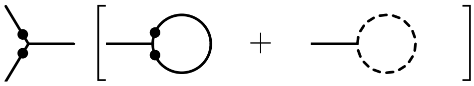
$$\mathbf{I}_{13} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta|_\tau \; {}^\bullet\Delta^\bullet \; \Delta^\bullet|_\sigma \tag{C.12}$$

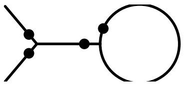
$$\mathbf{I}_{14} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; \Delta \tag{C.13}$$

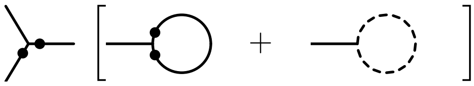
$$\mathbf{I}_{15} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta \; \Delta^\bullet \tag{C.14}$$

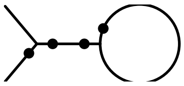
$$\mathbf{I}_{16} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \tau\sigma({}^\bullet\Delta^{\bullet 2} - {}^{\bullet\bullet}\Delta^2) \tag{C.15}$$

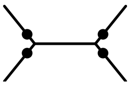
$$\mathbf{I}_{17} =$$  $$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; {}^\bullet\Delta \tag{C.16}$$

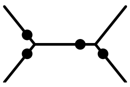
$\mathbf{I}_{18} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \Delta\ ({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_{\sigma}$ (C.17)

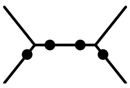
$\mathbf{I}_{19} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \Delta^{\bullet}\ \Delta^{\bullet}|_{\sigma}$ (C.18)

$\mathbf{I}_{20} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \tau\ {}^{\bullet}\Delta\ ({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_{\sigma}$ (C.19)

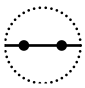
$\mathbf{I}_{21} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \tau\ {}^{\bullet}\Delta^{\bullet}\ \Delta^{\bullet}|_{\sigma}$ (C.20)

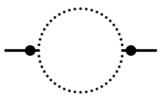
$\mathbf{I}_{22} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \Delta$ (C.21)

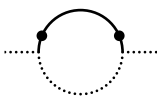
$\mathbf{I}_{23} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \sigma\ \Delta^{\bullet}$ (C.22)

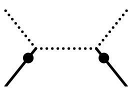
$\mathbf{I}_{24} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \tau\sigma\ {}^{\bullet}\Delta^{\bullet}$ (C.23)

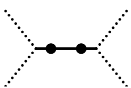
and those with mixed bosonic-fermionic contributions

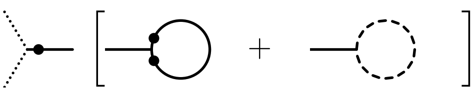
$\mathbf{I}_{25} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^{\bullet}\Delta^{\bullet}\ \Delta_F^2$ (C.24)

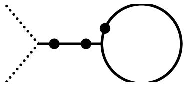
$\mathbf{I}_{26} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \Delta_F^2$ (C.25)

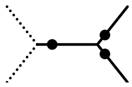
$\mathbf{I}_{27} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^{\bullet}\Delta^{\bullet}\ \Delta_F$ (C.26)

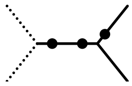
$\mathbf{I}_{28} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ \Delta_F$ (C.27)

$\mathbf{I}_{29} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^{\bullet}\Delta^{\bullet}$ (C.28)

$\mathbf{I}_{30} =$  $= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma\ {}^{\bullet}\Delta\ ({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)|_{\sigma}$ (C.29)

$$\mathbf{I}_{31} = $$

$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta^\bullet \; \Delta^\bullet|_\sigma \tag{C.30}$$

$$\mathbf{I}_{32} = $$

$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; {}^\bullet\Delta \tag{C.31}$$

$$\mathbf{I}_{33} = $$

$$= \int_{-1}^{0}\int_{-1}^{0} d\tau d\sigma \; \sigma \; {}^\bullet\Delta^\bullet \tag{C.32}$$

Appendix D

B coefficients for the $U(N|M)$ quantum mechanics

In order to compute the B coefficients defined in eq. (4.44) we follow the strategy explained in [64], and divide the hamiltonian (4.38) in three pieces, contributing at most two, one or no p eigenvalues, respectively[1]

$$\begin{aligned}
H &= H_B + H_1 + H_2 \qquad \text{where} \\
H_B &= g^{\bar{\mu}\nu}\, g^{1/2}\, \bar{p}_{\bar{\mu}}\, p_\nu\, g^{-1/2} = \frac{1}{2}\, G^{-1/4}\, p_i\, G^{1/2}\, g^{ij}\, p_j\, G^{-1/4}\ , \\
H_1 &= -i\hbar\, g^{\bar{\mu}\nu}\, \Gamma^\lambda_{\nu\sigma}\, \left(\bar{Z}_\lambda \cdot Z^\sigma - s\, \delta^\sigma_\lambda\right)\, g^{1/2}\, \bar{p}_{\bar{\mu}}\, g^{-1/2}\ , \\
H_2 &= a_1 \hbar^2\, R_\mu{}^\nu{}_\rho{}^\sigma\, \bar{Z}_\nu \cdot Z^\mu \bar{Z}_\sigma \cdot Z^\rho + (a_2+1)\hbar^2\, R^\nu_\mu\, \bar{Z}_\nu \cdot Z^\mu + (a_3 - s)\, \hbar^2\, R\ .
\end{aligned} \tag{D.1}$$

First of all, notice that H_B is precisely the usual bosonic quantum hamiltonian, carefully studied in the literature [64, 37]. Let us start with B^k_{2k}: the only way to have $2k$ p eigenvalues is k factors of H_B and no commutators taken into account, giving simply

$$B^k_{2k}\, p^{2k} = \left(\frac{p^2}{2}\right)^k\ , \tag{D.2}$$

where we use the notation $p^2 = g^{ij} p_i p_j = 2 g^{\mu\bar{\nu}} p_\mu \bar{p}_{\bar{\nu}}$. For B^k_{2k-1} we can have two terms. The first term comes from k factors of H_B with one p acting as a derivative; this gives the corresponding B^k_{2k-1} coefficient, that we call A^k_{2k-1}, of the purely bosonic model, whose computation is explained in detail in [64, 37]. The other term comes from $k-1$ factors of H_B and one H_1, by

[1] Remember that we are using rescaled Z's.

substituting all operators with the corresponding eigenvalues. Putting things together we obtain

$$
\begin{aligned}
B^k_{2k-1}\, p^{2k-1} &= A^k_{2k-1}\, p^{2k-1} - i\hbar k\, \left(\frac{p^2}{2}\right)^{k-1} \Gamma^\lambda_{\nu\sigma}\, \left(\bar\eta_\lambda \cdot \xi^\sigma\right)'\, g^{\bar\mu\nu}\, \bar p_{\bar\mu} \\
&= -\frac{i\hbar k}{2}\, \left(\frac{p^2}{2}\right)^{k-1} g^j\, p_j - i\hbar \binom{k}{2} \left(\frac{p^2}{2}\right)^{k-2} \frac{1}{2} g^{klj}\, p_j\, p_k\, p_l \\
&- i\hbar k\, \left(\frac{p^2}{2}\right)^{k-1} \Gamma^\lambda_{\nu\sigma}\, \left(\bar\eta_\lambda \cdot \xi^\sigma\right)'\, g^{\bar\mu\nu}\, \bar p_{\bar\mu}\ ,
\end{aligned}
\tag{D.3}
$$

where we denoted $(\bar\eta_\lambda \cdot \xi^\sigma)' = (\bar\eta_\lambda \cdot \xi^\sigma - s\,\delta^\sigma_\lambda)$, $g^j = \partial_i g^{ij}$ and $g^{ijk} = g^{kl}\,\partial_l g^{ij}$. For B^k_{2k-2} four types of term contribute: *i)* k factors of H_B, giving the corresponding coefficient A^k_{2k-2}, *ii)* $k-1$ factors of H_B and one H_1, with one p acting as a derivative. This contribution gives four terms: the derivative acting from one H_B to H_1, from H_1 to one H_B, within H_1 or within the $k-1$ H_B's. *iii)* $k-1$ factors of H_B and one H_2, substituting all operators with their eigenvalues, and *iv)* $k-2$ factors of H_B and two H_1, substituting all with eigenvalues. Remember that in *iii)* and *iv)* $[Z, \bar Z\}$ (anti)-commutators have to be taken into account in order to obtain eigenvalues on the coherent states. Altogether it results in

$$
\begin{aligned}
&B^k_{2k-2}\, p^{2k-2} = A^k_{2k-2}\, p^{2k-2} - \hbar^2 \binom{k}{2} \left(\frac{p^2}{2}\right)^{k-2} g^{ij}\partial_j\Big(g^{\bar\mu\nu}\Gamma^\lambda_{\nu\sigma}\Big)\big(\bar\eta_\lambda\cdot\xi^\sigma\big)' p_i\, \bar p_{\bar\mu} \\
&- \hbar^2 \binom{k}{2} \left(\frac{p^2}{2}\right)^{k-2} g^{\bar\mu\nu}\Gamma^\lambda_{\nu\sigma}\big(\bar\eta_\lambda\cdot\xi^\sigma\big)'\partial_{\bar\mu} g^{\rho\bar\sigma}\, p_\rho\, \bar p_{\bar\sigma} \\
&+ \frac{1}{2}\,\hbar^2 k \left(\frac{p^2}{2}\right)^{k-1} g^{\bar\mu\nu}\Gamma^\lambda_{\nu\sigma}\big(\bar\eta_\lambda\cdot\xi^\sigma\big)' g^{\rho\bar\sigma}\partial_{\bar\mu} g_{\rho\bar\sigma} \\
&- i\hbar k\, A^{k-1}_{2k-3}\, p^{2k-3}\, g^{\bar\mu\nu}\Gamma^\lambda_{\nu\sigma}\big(\bar\eta_\lambda\cdot\xi^\sigma\big)'\bar p_{\bar\mu} + \hbar^2 k \left(\frac{p^2}{2}\right)^{k-1} \Big(a_1\, R_{\mu\ \rho}^{\ \nu\ \sigma}\, \bar\eta_\nu\cdot\xi^\mu \bar\eta_\sigma\cdot\xi^\rho \\
&+ (a_2 - a_1 + 1)\, R^\mu_\nu\, \bar\eta_\mu\cdot\xi^\nu + \Big(a_3 - s\Big)\, R\Big) - \hbar^2 \binom{k}{2} \left(\frac{p^2}{2}\right)^{k-2} g^{\bar\mu\nu}\Gamma^\tau_{\nu\sigma}\, g^{\lambda\bar\sigma}\Gamma^\rho_{\lambda\mu}\, \bar p_{\bar\mu}\, \bar p_{\bar\sigma} \\
&\times \Big[\big(\bar\eta_\tau\cdot\xi^\sigma\big)'\big(\bar\eta_\rho\cdot\xi^\mu\big)' + \delta^\sigma_\rho\, \bar\eta_\tau\cdot\xi^\mu\Big]\ .
\end{aligned}
\tag{D.4}
$$

In the formulae above the bosonic coefficients are given by

$$
\begin{aligned}
A^{k-1}_{2k-3}\, p^{2k-3} &= -\frac{i\hbar}{2}\,(k-1)\left(\frac{p^2}{2}\right)^{k-2} g^j p_j - \frac{i\hbar}{2}\binom{k-1}{2}\left(\frac{p^2}{2}\right)^{k-3} g^{klj}\, p_j\, p_k\, p_l\ ,\\
A^{k}_{2k-2}\, p^{2k-2} &= \hbar^2 k\left(\frac{p^2}{2}\right)^{k-1}\left[\frac{1}{32}\ln G_i \ln G^i + \frac{1}{8}\ln G^i_i + \frac{1}{8} g^j \ln G_j\right]\\
&- \hbar^2\binom{k}{2}\left(\frac{p^2}{2}\right)^{k-2}\left[\frac{1}{2}\partial^j g^l + \frac{1}{4} g^j g^l + \frac{1}{4} g^k g^{jl}_k + \frac{1}{4} g^{jlk}_k\right] p_j\, p_l\\
&- \hbar^2\binom{k}{3}\left(\frac{p^2}{2}\right)^{k-3}\left[\frac{1}{2} g^{mnkl} + \frac{3}{4} g^{klm} g^n + \frac{1}{2} g^{ikl} g^{mn}_i + \frac{1}{4} g^{kl}_i g^{mni}\right] p_k\, p_l\, p_m\, p_n\\
&- \hbar^2\binom{k}{4}\left(\frac{p^2}{2}\right)^{k-4}\left[\frac{3}{4} g^{klj} g^{pqm}\right] p_j\, p_k\, p_l\, p_m\, p_p\, p_q \qquad \text{(D.5)}
\end{aligned}
$$

and we recall that the following compact notation was employed

$$
\begin{aligned}
\partial_i \ldots \partial_m g^{jk} = g^{jk}_{i\ldots m}\ ,\quad g^{ij} g^{kl}_j = g^{kli}\ ,\quad g^{ij}_j = g^i\\
g^{jk}\partial_k g^{lm}_m = \partial^j g^l\ ,\quad \partial_i \ln G = \ln G_i\ ,\quad g^{ij}\partial_i\partial_j \ln G = \ln G^i_i\ .
\end{aligned}
$$

Appendix E

The on-shell action

The euclidean action generated by the hamiltonian (4.38) is given by[1]

$$S = \int_{-\beta}^{0} d\tau \left[g_{\mu\bar{\nu}}\, \dot{x}^{\mu}\dot{\bar{x}}^{\bar{\nu}} + \hbar\, \bar{Z}_{\mu} \cdot \frac{DZ^{\mu}}{d\tau} - s\hbar\, \Gamma_{\mu}\dot{x}^{\mu} + \hbar^2\, \Delta H \right] , \tag{E.1}$$

where $\Gamma_{\mu} \equiv \Gamma^{\nu}_{\nu\mu}$ is the $U(1)$ piece of the Kähler connection and s plays the role of an additional $U(1)$ coupling. The additional piece

$$\Delta H = a_1\, R_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}\, \bar{Z}_{\nu} \cdot Z^{\mu} \bar{Z}_{\sigma} \cdot Z^{\rho} + a_2\, R^{\mu}_{\nu}\, \bar{Z}_{\mu} \cdot Z^{\nu} + a_3\, R \tag{E.2}$$

contains the generalized couplings to curvatures, and the covariant time derivative on Z fields reads

$$\frac{DZ^{\mu}_{A}}{d\tau} = \dot{Z}^{\mu}_{A} + \dot{x}^{\nu}\, \Gamma^{\mu}_{\nu\sigma}\, Z^{\sigma}_{A} \,, \qquad \frac{D\bar{Z}^{A}_{\sigma}}{d\tau} = \dot{\bar{Z}}^{A}_{\sigma} - \dot{x}^{\nu}\, \Gamma^{\mu}_{\nu\sigma}\, Z^{A}_{\mu} \,. \tag{E.3}$$

From the action (E.1) the following equations of motion arise

$$\begin{aligned}
&\ddot{x}^{i} + \Gamma^{i}_{jk}\, \dot{x}^{j}\dot{x}^{k} - \hbar\, R^{i}{}_{j}{}^{\mu}{}_{\nu}\, \dot{x}^{j} \left(\bar{Z}_{\mu} \cdot Z^{\nu} - s\, \delta^{\nu}_{\mu} \right) - \hbar^2\, g^{ik}\partial_{k}\Delta H = 0 \\
&\dot{Z}^{\mu}_{A} + \Gamma^{\mu}_{\nu\sigma}\, \dot{x}^{\nu}\, Z^{\sigma}_{A} + 2a_1\hbar\, R_{\nu}{}^{\mu}{}_{\lambda}{}^{\sigma}\, Z^{\nu}_{A}\bar{Z}_{\sigma} \cdot Z^{\lambda} + a_2\hbar\, R^{\mu}_{\nu}\, Z^{\nu}_{A} = 0 \\
&\dot{\bar{Z}}^{A}_{\nu} - \Gamma^{\mu}_{\nu\sigma}\, \dot{x}^{\sigma}\, Z^{A}_{\mu} - 2a_1\hbar\, R_{\nu}{}^{\mu}{}_{\lambda}{}^{\sigma}\, \bar{Z}^{A}_{\mu}\bar{Z}_{\sigma} \cdot Z^{\lambda} - a_2\hbar\, R^{\mu}_{\nu}\, Z^{A}_{\mu} = 0 \,.
\end{aligned} \tag{E.4}$$

Now, we have to expand the action (E.1) up to order β, with the fields obeying (E.4), with boundary conditions: $x^{i}(-\beta) = y^{i}$, $x^{i}(0) = x^{i}$, $Z^{\mu}_{A}(-\beta) = \xi^{\mu}_{A}$ and $\bar{Z}^{A}_{\mu}(0) = \bar{\eta}^{A}_{\mu}$. Expanding fields in a Taylor series around $\tau = 0$, we will see that, for small β, we have

$$\frac{d^n x^i}{d\tau^n} \sim \frac{d^n Z^{\mu}_{A}}{d\tau^n} \sim \beta^{-n/2} \,.$$

[1] Remember that we are using rescaled Z's.

Expanding also the on-shell lagrangian, we can write

$$S_{os} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \, (-)^n \, \frac{d^n \mathcal{L}_{os}}{d\tau^n}\bigg|_{\tau=0} \beta^{n+1} \; , \tag{E.5}$$

and one notices that, for all pieces of the lagrangian but the $U(1)$ one, it is sufficient to keep the order zero: $S_{os} = \beta\, \mathcal{L}_{os}(0)$. For the $U(1)$ piece, it is necessary the next order: $\beta\, \mathcal{L}_{os}(0) - \frac{1}{2}\beta^2 \dot{\mathcal{L}}_{os}(0)$. Let us begin with the x's: we expand in Taylor series and obtain

$$x^i(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \, \frac{d^n x^i}{d\tau^n}(0) \; , \tag{E.6}$$

setting $\tau = -\beta$ and using the boundary conditions we have

$$y^i = x^i - \beta\, \dot{x}^i(0) + \frac{\beta^2}{2}\, \ddot{x}^i(0) - \frac{\beta^3}{6}\, \dddot{x}^i(0) + ... \tag{E.7}$$

and similarly for the first derivative

$$\dot{x}^i(0) = -\frac{z^i}{\beta} + \frac{\beta}{2}\, \ddot{x}^i(0) - \frac{\beta^2}{6}\, \frac{d}{d\tau}\, \ddot{x}^i(0) + ... \; . \tag{E.8}$$

Now one uses the equations of motion (E.4) and solves by iteration. To order $\beta^{1/2}$ we obtain

$$\dot{x}^i(0) = -\frac{z^i}{\beta} - \frac{1}{2\beta}\, \Gamma^i_{jk}\, z^j z^k - \frac{1}{6\beta} \left(\partial_l \Gamma^i_{jk} + \Gamma^i_{js}\, \Gamma^s_{kl}\right) z^j z^k z^l - \frac{z^j}{2}\, \hbar\, R^i{}_j{}^\mu{}_\nu\, (\bar{\eta}_\mu \cdot \xi^\nu)' \; . \tag{E.9}$$

Adopting the same procedure for Z and $\dot{Z}$ we have

$$\begin{aligned} Z^\mu_A(0) &= \xi^\mu_A + \Gamma^\mu_{\nu\lambda}\, z^\nu \xi^\lambda_A \; , \\ \dot{Z}^\mu_A(0) &= \frac{Z^\mu_A(0) - \xi^\mu_A}{\beta} - \frac{1}{2\beta}\, \partial_j \Gamma^\mu_{\nu\lambda}\, z^j z^\nu \xi^\lambda_A + \frac{1}{2\beta}\, \Gamma^\mu_{\nu\lambda}\, \Gamma^\nu_{\sigma\rho}\, z^\sigma z^\rho \xi^\lambda_A \\ &\quad + \frac{1}{2\beta}\, \Gamma^\mu_{\nu\lambda}\, \Gamma^\nu_{\sigma\rho}\, z^\sigma z^\lambda \xi^\rho_A \; . \end{aligned} \tag{E.10}$$

Now we can substitute the above expansions in $\beta\, \mathcal{L}_{os}(0)$ and in $-\frac{\beta^2}{2} \dot{\mathcal{L}}^{U(1)}_{os}(0)$. Remembering that in fermionic actions one needs also a boundary term, it is convenient to use the modified action $\tilde{S} = S - \hbar \bar{Z}_\mu(0) \cdot Z^\mu(0)$, and using (E.1) for S we finally arrive at the following expansion

$$\begin{aligned} \tilde{S}_{os} &= \frac{1}{2\beta}\, g_{ij}\, z^i z^j + \frac{1}{4\beta}\, \partial_i g_{jk}\, z^i z^j z^k + \frac{1}{12\beta} \left(\partial_k \partial_l g_{mn} - \frac{1}{2}\, g_{ij}\, \Gamma^i_{kl}\, \Gamma^j_{mn}\right) z^k z^l z^m z^n \\ &\quad - \hbar\, \bar{\eta}_\mu \cdot \xi^\mu - \hbar\, z^\nu\, \Gamma^\lambda_{\nu\sigma}\, (\bar{\eta}_\lambda \cdot \xi^\sigma)' - \frac{\hbar}{2} z^i z^\mu \partial_i \Gamma^\lambda_{\mu\sigma}\, (\bar{\eta}_\lambda \cdot \xi^\sigma)' - \frac{\hbar}{2}\, z^\nu z^\lambda\, \Gamma^\mu_{\nu\sigma} \Gamma^\sigma_{\lambda\rho}\, \bar{\eta}_\mu \cdot \xi^\rho \\ &\quad + a_1\, \beta \hbar^2\, R_\mu{}^\nu{}_\rho{}^\sigma\, \bar{\eta}_\nu \cdot \xi^\mu\, \bar{\eta}_\sigma \cdot \xi^\rho + a_2\, \beta \hbar^2\, R^\mu_\nu\, \bar{\eta}_\mu \cdot \xi^\nu + a_3\, \beta \hbar^2\, R + ... \end{aligned} \tag{E.11}$$

which appears in the final result (4.51).

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