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TESI DI DOTTORATO IN FISICA

CLASSICAL AND QUANTUM FEATURES OF THE INHOMOGENEOUS MIXMASTER MODEL

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Introduzione

Il Modello Cosmologico Standard è sicuramente fra i risultati più importanti in fisica moderna. Le ipotesi di omogeneità ed isotropia della struttura dell'Universo su larga scala sono supportate sia da notevoli evidenze osservative, che da ottimi riscontri fra predizioni teoriche e dati sperimentali. Fra le prime, la più importante è certamente la quasi assoluta omogeneità ed isotropia della Radiazione Cosmica di Fondo; tra le seconde, si segnala l'ottimo accordo fra le predizioni della nucleosintesi primordiale e l'attuale abbondanza degli elementi leggeri nell'Universo. Per questi, e per altre ragioni, il dominio di validità temporale di questo schema può essere esteso da oggi fino a circa 10^{-32} secondi dal Big Bang (o, equivalentemente, ad energie dell'ordine di 10^{16} GeV).

Il Modello Standard, altresì, soffre di notevoli problemi interni: il paradosso degli orizzonti e la mancanza di un meccanismo che spieghi la formazione delle strutture a larga scala sono due fra gli esempi più rilevanti. Se alcuni di questi problemi possono essere risolti inserendo nella storia evolutiva dell'Universo una fase inflattiva, ci sono forti motivazioni che spingono a cercare l'estensione della cosmologia a modelli più generici di quelli di Friedmann-Robertson-Walker (FRW). Le dinamiche omogenee ed isotrope dei modelli FRW sono, per esempio, instabili, durante la fase di collasso, rispetto a perturbazioni tensoriali tipiche delle onde gravitazionali; oltretutto, si suppone che l'attuale struttura classica dell'Universo sia il risultato di una precedente fase quantistica della gravità. Quest'ultimo fatto limita fortemente la possibilità di simmetrie globali: infatti, per tempi dell'ordine di quello di Planck, l'Universo era composto da più orizzonti causalmente sconnessi, e le fluttuazioni topologiche, che caratterizzano una simile fase, suggeriscono che l'Universo classico sia iniziato in condizioni di assoluta generalità e senza alcuna simmetria.

Lo studio di una soluzione cosmologica generica (generale), quindi, è di assoluto interesse, sia per i motivi legati ai problemi del Modello Standard, sia come problema teorico a se stante.

Negli anni '70, V.A. Belinsky, I.M. Khalatnikov ed E.M. Lifshitz dimostrarono che è possibile costruire una soluzione cosmologica generale delle equazioni di Einstein nell'intorno della singolarità iniziale; per soluzione generale, si intende una soluzione che ha a disposizione tutti i gradi di libertà, e che quindi permette di assegnare un problema di Cauchy su una generica ipersuperficie spaziale. Questo modello è caratterizzato da una omogeneità locale e dalla presenza di una singolarità non eliminabile, e la dinamica risulta causalmente non connessa su scale che superano quella dell'orizzonte; all'interno dei singoli orizzonti, si sviluppa un'evoluzione tipica dei modelli omogenei dei tipi VIII e IX della classificazione di Bianchi, evoluzione che esibisce un regime di avvicinamento alla singolarità dal carattere fortemente oscillatorio che può essere descritto in maniera appropriata solo con strumenti statistici. Questo peculiare disaccoppiamento dei punti spaziali della varietà, associata al regime caotico, riduce la struttura dello spazio-tempo classico a quello di una "schiuma" attraverso un meccanismo di frammentazione.

La soluzione generica propone una visione, che per quanto sofisticata ed affascinante, resta comunque assolutamente classica. Il suo limite maggiore è legato alle scale temporali ed energetiche alle quali l'Universo dovrebbe essere descritto da una simile dinamica: a quei regimi, stimabili in un intervallo compreso fra i 10^{16} e i 10^{19} GeV, è attendibile che l'Universo non possa più essere descritto da una teoria classica, ma entri in un regime in cui le correzioni quantistiche siano importanti. Il fenomeno che mappa la fase quantistica in quella classica è l'espansione del volume dell'Universo; poiché la comparsa di una dinamica isotropa ed omogenea è prevista alla fine della fase oscillatoria, quando le anisotropie sono ridotte a piccole perturbazioni, ha senso ritenere che l'inizio della fase oscillatoria si sovrapponga a quella Planckiana.

L'analisi della dinamica quantistica della soluzione generale è di assoluto interesse, sia per quello che riguarda il Modello Standard, ma anche perché è uno scenario in cui poter studiare la natura e le proprietà della singolarità iniziale e, di conseguenza, i meccanismi che hanno portato alla nascita dell'Universo in cui viviamo.

Il lavoro presentato in questa dissertazione di dottorato può essere diviso in due parti, e la prima si inserisce in questa linea di ricerca, con contributi originali inerenti alle proprietà classiche, semi-classiche e quantistiche della soluzione cosmologica generale, nota anche come modello Mixmaster inomogeneo.

Per quanto concerne il regime classico, è stato effettuato uno studio dettagliato del regime caotico che ne caratterizza la dinamica; fra gli altri risultati, il più importante è quello che riguarda il problema di una caratterizzazione covariante del caos nella soluzione cosmologica generale, e questo è stato risolto calcolando un indicatore, utilizzato in teoria ergodica e noto come esponente di Lyapunov, in maniera assolutamente indipendente dal sistema di riferimento introdotto, ed, in particolare, dalla variabile temporale.

Per quanto riguarda invece i regimi semi-classici e quantistici del Mixmaster, sono stati ottenuti una collezione di risultati che riguardano l'approccio alla Hamilton-Jacobi alla dinamica, la descrizione meccanico-statistica del modello, il limite semi-classico alla WKB, ed un'analisi dettagliata del regime quantistico. Fra tutti, i risultati di maggior rilievo sono l'ordinamento operatoriale ottenuto dal confronto delle dinamiche classica e semi-classica, e la forma esplicita dello spettro (discreto) dei livelli energetici del modello Mixmaster quantistico.

La seconda parte di questa dissertazione riguarda invece il problema del caos gravitazionale nell'intorno della singolarità come fenomeno dipendente dal numero delle dimensioni del modello, e dalla materia di cui questo è riempito. I modelli multi-dimensionali, omogenei e non, esibiscono proprietà diverse: i primi possono essere caotici solo in tre dimensioni spaziali, i secondi mantengono questa proprietà, in generale, fino a dieci dimensioni spaziali. Anche la materia influenza questo aspetto in maniera decisiva: l'inserimento di un campo scalare inibisce il comportamento oscillatorio in ogni numero di dimensioni. Questi meccanismi possono però essere re-innescati se nella dinamica multi-dimensionale vengono inserite delle p-forme.

Il contributo originale qui discusso riguarda lo studio di un generico modello omogeneo accoppiato ad un campo Abeliano non massivo. L'effetto di questo campo elettromagnetico multi-dimensionale è quello di restaurare il caos in ogni numero di dimensioni; ancora più interessante è che la mappa associata esibisce una forte dipendenza dal numero di dimensioni, e che si riduce a quella del modello Mixmaster del caso tridimensionale.

Questi studi sono stato l'oggetto di tre pubblicazioni su riviste internazionali, una apparsa nel 2004 su *Physical Review D*, e due, nel 2005 e nel 2007 rispettivamente, apparse su *Classical and Quantum Gravity*.

L'esposizione del materiale è organizzata in quattro capitoli.

Nel capitolo 1, dopo aver classificato gli spazi omogenei seguendo lo schema classico di L. Bianchi, vengono discusse in dettaglio le soluzioni dei modelli del tipo I, o di Kasner, e quella del tipo IX, o Mixmaster. L'analisi della dinamica di quest'ultimo viene poi riproposta nel formalismo Hamiltoniano per evidenziare l'analogia esistente fra questi spazi ed i sistemi meccanici bidimensionali. L'ultima parte del capitolo è dedicata alla soluzione cosmologica generale: sono discussi in dettaglio la costruzione del regime oscillatorio a partire dalla soluzione di Kasner generalizzata, l'effetto delle disomogeneità, il fenomeno della rotazione degli assi e quello della frammentazione. Tre appendici di carattere tecnico chiudono la discussione.

Il capitolo 2 è centrato sul problema della covarianza del caos nella soluzione cosmologica generale. Il primo paragrafo ripercorre, attraverso la discussione dei risultati più importanti, le fasi del dibattito ventennale attorno alla possibilità di utilizzare l'esponente di Lyapunov come indicatore caotico in sistemi relativistici, con particolare interesse per il Mixmaster. Segue quindi il materiale originale, ed è così organizzato: l'approccio Hamiltoniano è discusso nel paragrafo 2, la riduzione della dinamica ai gradi fisici di libertà e la struttura asintotica dello scalare di curvatura tridimensionale sono discussi nei paragrafi 3-5; l'equivalenza con il biliardo ed il calcolo esplicito dell'esponente di Lyapunov sono proposti nel paragrafo 6, la misura invariante discussa nel 7, mentre al ruolo delle pareti e alla covarianza della descrizione proposta sono dedicati i paragrafi 8 e 9. Nell'ultimo paragrafo sono presentate le conclusioni.

Il capitolo 3 è dedicato allo studio delle proprietà semi-classiche e quantistiche del modello Mixmaster inomogeneo. Nei paragrafi 1 e 2 sono discussi due schemi diversi per la quantizzazione della gravità, l'equazione di Wheeler-DeWitt e l'approccio alla Multi-Time; nel paragrafo 3 viene risolta l'equazione di Hamilton-Jacobi e discussa la dinamica dal punto di vista meccanico-statistico, mentre nel paragrafo 4 viene analizzato il limite semi-classico alla WKB. Il modello viene quantizzato nel paragrafo 5, ed il problema della non-località dell'operatore Hamiltoniano affrontato nel paragrafo 6. Gli ultimi due paragrafi sono dedicati al cosiddetto "quantum chaos" e ad alcune conclusioni. Un'appendice su alcuni aspetti dell'analisi armonica sul piano di Lobachevsky chiude il capitolo.

Il capitolo 4 è centrato sul rapporto esistente fra caos, numero di dimensioni e materia. Le generalizzazioni a spazi multi-dimensionali, omogenei e non, della dinamica del Mixmaster nel vuoto sono rapidamente esposte nel paragrafo 1, mentre nel paragrafo 2 discussi gli effetti dei campi di materia. Il lavoro originale comincia nel paragrafo 3, dove viene formulato, in approccio Hamiltoniano, il problema dinamico di un generico modello omogeneo multi-dimensionale accoppiato ad un campo Abeliano. Il regime di tipo Kasneriano generalizzato e la mappa sono derivati nel paragrafo 4 e 5 rispettivamente. Nel paragrafo 6 sono presenti delle conclusioni finali, e in appendice è proposta la classificazione di Fee delle varietà omogenee quadri-dimensionali.

L'ultimo capitolo è dedicato alle conclusioni e a possibili sviluppi futuri.

In appendice sono inserite le copie delle pubblicazioni e degli atti dei congressi, tutte conformi agli originali.

Introduction

The Standard Cosmological Model is an outstanding result in modern physics. The homogeneity and isotropy hypotheses on the Universe large scale structure are supported both by observational evidences and by very good agreement between theoretical predictions and experimental data. Among the first, the most important is the almost absolute homogeneity and isotropy of the Cosmic Microwave Background Radiation; among the latter, we remember the very good agreement between predictions of the primordial nucleosynthesis and the actual abundance of light elements in the Universe. For these and other reasons, the temporal extension of this scheme can be extended from today up to about 10^{-32} seconds far from the Big Bang (or, equivalently, energies up to 10^{16} GeV).

The Standard Model, however, exhibits some important internal problems: the horizon paradox and the lack of a formation mechanism for large scale structure are two of the most relevant example. Even though some of these problems can be solved by adding an inflationary stage during Universe evolution, strong reasons exist for a generalization of cosmology to models more generic than Friedmann-Robertson-Walker (FRW) ones. FRW homogeneous and isotropic dynamics are, for example, unstable during the collapse phase with respect to tensorial perturbations, like that of gravitational waves; furthermore, it is supposed that the actual classical structure of the Universe is the result of a previous quantum one. This is a severe restriction on the possibility of global symmetries: indeed, at Planck time-scales, many causally not correlated horizons were present in the Universe, and the topological fluctuations suggest that the classical phase of the Universe began in conditions of maximal generality and without any symmetry.

The study of the generic (general) cosmological solution is, then, of outstanding importance, both for the problems of the Standard Model, and as a question on its own.

In the 70's, V.A. Belinsky, I.M. Khalatnikov and E.M. Lifshitz showed that it is possible to construct a generic cosmological solution of the Einstein equations in the neighbourhood of the initial singularity, *i.e.* a solution that has all its degrees of freedom available, and thus allowing to specify a Cauchy problem on a generic space-like hypersurface. This model is characterized by local homogeneity and by the presence of a non eliminable singularity, and the dynamics results to be causally not connected on scales greater than the horizon size; inside each horizon, an evolution, resembling that of the homogenous spaces of the type VIII and IX of the Bianchi classification, takes place; such an evolution exhibits an oscillatory behaviour while reaching the singularity, that can be well described only within a statistical framework. This particular de-coupling of the spatial points of the manifold, together with the oscillatory behaviour, reduces space-time structure to that of a "foam" by a fragmentation process.

The generic cosmological solution proposes a picture that, even though sophisticated and fascinating, is, however, a classical one. The most important problem is what energy and time-scale the Universe should be described by such a framework: at that energy, in a range of about $10^{16} - 10^{19}$ GeV, the Universe should be badly described by a classical theory, and quantum corrections ought to be relevant. The phenomenon that maps the classic into the quantum phase is the volume of the Universe; as soon as an isotropic and homogeneous dynamics is attended at the end of the oscillatory phase, *i.e.*, when the anisotropy are only small corrections, it is believable that the begin of the oscillatory phase overlaps the Planckian one.

The quantum analysis of the generic cosmological solution is of absolute interest, not only for the Standard Model, but also for the analysis of the properties of the initial singularity, and, consequently, of the mechanisms at the ground of the birth of our Universe.

The work presented in this PhD thesis can be divided in two parts, and the first is inserted in this line of research, with original contributions about classical, semi-classical and quantum properties of the generic cosmological solution, also known as inhomogeneous Mixmaster model.

In the classical framework, a complete study of the chaotic dynamics has been performed; among the others, the most important result deals with the problem of the covariant characterization of chaos in the generic solution, and this has been faced by a direct, time and reference frame-independent calculation of an indicator, used in ergodic theory and known as the Lyapunov exponent.

In the semi-classical and quantum framework, a collection of results has been obtained about the Hamilton-Jacobi formulation of the dynamics, the statistical-mechanical description of the model, WKB semi-classical limit, and the quantum regime. Among the others, the most relevant results are the operator-ordering, deduced from a comparison of the classical and the semi-classical dynamics, and the explicit shape of the (discrete) energy spectrum of the quantum Mixmaster model.

The second half of this dissertation is about the problem of gravitational chaos in the neighbourhood of the singularity as a phenomenon dependent on the number of dimensions and on matter fields. Multi-dimensional homogeneous models exhibits different properties from the inhomogeneous ones: the first can be chaotic only in three space-like dimensions, the latter show this property, in general, up to ten space-like dimensions. Matter has also an important influences on this feature: a scalar field inhibits the oscillatory behaviour in any number of dimensions. This mechanism can be restated as soon as p-forms are inserted in the multi-dimensional dynamics.

The original contribution here discussed deals with the study of an homogeneous multi-dimensional model coupled to a mass-less Abelian field. The main effect of this field is to restore chaos in any number of dimensions; a much more interesting feature is that the associated map exhibits a strong dependence on the number of dimensions, and reduces to that of the three-dimensional Mixmaster model in vacuum.

These results were the subject of three publications on international journals, one appeared in 2004 on *Physical Review D*, and two, in 2005 and 2007 respectively, on *Classical and Quantum Gravity*.

The subject is organized in four chapter.

In chapter 1, after Bianchi's classification of homogeneous spaces, the solution of type I or Kasner model and that of type IX or Mixmaster, are discussed in detail. The Hamiltonian formulation of the latter is also illustrated to put in evidence the existent analogy with two-dimensional mechanical system. The last part of the chapter is about the generic cosmological solution: we discuss the construction of the oscillatory behaviour from the generalized Kasner solution, the effects of inhomogeneity, axis rotation phenomenon and the fragmentation. Three technical appendices close the chapter.

Chapter 2 is centred around the question of the covariance of chaos in the generic cosmological solution. First chapter proposes the most important phases of a debate, lasted for twenty years, around the use of the Lyapunov exponent in relativistic dynamical system, with particular attention to the Mixmaster. Then the original contribution follows: Hamiltonian formulation is in section 2, the reduction of the dynamics to the physical degrees of freedom and the asymptotic structure of three-dimensional Ricci scalar are presented in sections 3-5; the equivalence with billiard description and the explicit calculation of the Lyapunov exponent are discussed in section 6, the invariant measure in section 7, and sections 8 and 9 are about the role of the potential walls and on the covariance of this scheme. Concluding remarks follow in the last section.

In Chapter 3, a detailed investigation of the semi-classical and quantum properties of the inhomogeneous Mixmaster model is presented. Two different schemes of quantization, the Wheleer-DeWitt equation and the Multi-Time formalism, are elucidated in sections 1 and 2; in section 3 the Hamilton-Jacobi equation is solved and the dynamics is discussed from the statistical-mechanical point of view, while the WKB limit is presented in section 4. The full quantization of the model is proposed in section 5, and non-locality of the Hamiltonian operator is faced in section 6. The last two sections are about the so-called "quantum chaos" and on some concluding remark. An appendix, that deals with harmonic analysis on the Lobachevsky plane, closes the chapter.

Chapter 4 deals with the existing link between chaos, number of dimensions, and matter fields. Generalization to higher-dimensional, homogeneous and inhomogeneous, models in vacuum of the Mixmaster dynamics is addressed in section 1, while section 2 deals with the effects of matter. The original contribution is presented starting in section 3, where the dynamical problem of an homogeneous multi-dimensional space coupled to an Abelian vector field is formulated in Hamiltonian framework. The generalized Kasner-like behaviour and the associated map are derived in sections 4 and 5 respectively. Concluding remarks follow in section 6, and an appendix, on Fee's classification of homogeneous five-dimensional space-times, closes the chapter.

Conclusion and outlooks are presented in the last chapter.

Copies of publications and of proceedings of congresses are appended at the end of this dissertation.

1 The Mixmaster Model and the Generic Cosmological Solution

The observational data on the Cosmic Microwave Background Radiation [dB$^+$00, S$^+$03] suggest that the Universe, on large scales, is homogeneous and isotropic. These two features completely determine the metric tensor in FRW models, and the sign of the curvature is the only free parameter; in the meanwhile, the only homogeneity hypothesis, without further symmetries, leaves a much greater arbitrariness and allows a richer behaviour in Einstein dynamics. These homogeneous spaces were classified [Bia98] at the end of the XIX century, but only in the 60's and 70's the underlying deep cosmological meaning was clarified by the works of Belinsky, Khalatnikov and Lifshitz (BKL) [BKL70, BKL82]: the types VIII and IX homogeneous models of Bianchi classification, indeed, provide the prototypes for the construction of the general (non-homogeneous) solution of the Einstein equations in the neighbourhood of the singularity.

In this chapter, we review the main existing results on the dynamics of the type IX model of the Bianchi classification (also known as *Mixmaster* [Mis69b]), and on the behaviour of the generic cosmological solution near the Big Bang.
In particular, in Section 1, we classify the homogeneous spaces, discuss their dynamics from the field equations point of view, and then the oscillatory approach to the singularity of the type IX model [BKL70, BKL82]. The Hamiltonian framework [Mis69a] and the reduction of the dynamics of the Mixmaster to that of a two-dimensional particle moving in a potential is addressed (Section 2).
The role of matter is discussed in Section 3, where it is shown that behaves as a test fluid.
The generic cosmological solution of Einstein equations in the neighbourhood of the singularity is then constructed and its dynamics analyzed in details in Section 4.
The chapter is closed by three appendices, where the tetradic formalism (1A), the

property of synchronous reference frame (1B) and the Arnowitt-Deser-Misner [ADM62] Hamiltonian formulation of General Relativity (1C) are briefly discussed.

1.1 Homogeneous 4-dimensional models and the Mixmaster

Homogeneity means that the metric properties are the same in each point of the three-manifold. More precisely, a generic space is homogeneous if it admits a group of movements that maps the space onto itself; since we deal with 3-dimensional spaces, the group has 3 independent parameters.
Let's introduce a synchronous frame of reference (see Appendix 1B), and consider a spatial slice at a fixed instant t; the action of such a group leaves unchanged the spatial metric tensor $h_{\alpha\beta}$, and, furthermore, must leave unchanged the three independent one-forms $e^{(a)}_{\alpha}dx^{\alpha}$ (here the latin index a denotes the three one-forms). In the tetradic picture (see Appendix 1A), the homogeneity can be imposed directly on vectors $e^{(a)}$, resulting in the following condition

$$\left(\frac{\partial e^{(c)}_{\alpha}}{\partial x^{\beta}} - \frac{\partial e^{(c)}_{\beta}}{\partial x^{\alpha}}\right) e^{\alpha}_{(a)} e^{\beta}_{(b)} = C^{c}_{ab} \ . \tag{1.1}$$

The constants C^{c}_{ab} are called *structure constants*, and their values completely determine the structure and the properties of the space [1]; they are anti-symmetric in the lower indexes and obey the Jacobi identity [LL71].
The standard classification of the Bianchi spaces is based on the *dual* of the C's with two indexes defined as

$$C^{c}_{ab} = e_{abd}C^{dc} \ , \tag{1.2}$$

($e_{abc} = e^{abc}$ is the unitary completely antisymmetric tensor, with $e_{123} = 1$).
The resulting classification[2] is summarized in Table (1.1). The type I denotes the euclidean space, and contains both the standard galilean space and the Kasner one; the type V contains as particular case the 3-sphere, while the type IX contains the

[1]This name is taken from the theory of Lie groups: an equivalent derivation of the homogeneous spaces exists, and it is based on this theory, where the generators are proportional to the Killing vectors of the 3 manifold, and the structure constants are exactly the same obtained with this description.

[2]The quantities a, n_1, n_2, n_3 are related to the C^{ab} by the relation $C^{ab} = n^{a}\delta_{ab}e^{ab1}a$, $a \geq 0$.

Type	**a**	$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$
I	0	0	0	0
II	0	1	0	0
VII	0	1	1	0
VI	0	1	-1	0
IX	0	1	1	1
VIII	0	1	1	-1
V	1	0	0	0
IV	1	0	0	1
VII	a	0	1	1
III ($a = 1$) VI ($a \neq 1$)	a	0	1	-1

Table 1.1:

Standard Bianchi classification of the homogeneous spaces with the corresponding structure constants [Bia98].

constant negative curvature case.

1.1.1 The Kasner solution

The first anisotropic solution was found by Kasner in 1922 while studying the dynamics of type I model. He obtained a particular cosmological solution describing a flat homogeneous space where the volumes grow linearly in time, but where the linear distances grow differently along different directions: they grows along two, and decrease along the last. This metric is called *Kasner metric*, and can be summarized as

$$ds^2 = dt^2 - t^{2p_1}dx^2 - t^{2p_2}dy^2 - t^{2p_3}dz^2 \ , \tag{1.3}$$

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \ . \tag{1.4}$$

From (1.3) and (1.4), the determinant of the 4-metric, $-g = t^2$, follows. Relations (1.4) constrain the *Kasner exponents* p_i's to be all different; only two sets of values, $(0, 0, 1)$ and $(-1/3, 2/3, 2/3)$, have an index equal to another. In general the p_i's can vary in the following intervals (if ordered as $p_1 \leqslant p_2 \leqslant p_3$)

$$-\frac{1}{3} \leqslant p_1 \leqslant 0, \qquad 0 \leqslant p_2 \leqslant \frac{2}{3}, \qquad \frac{2}{3} \leqslant p_3 \leqslant 1\ . \tag{1.5}$$

By (1.5), one and only one is negative; this determines the anisotropic evolution of the axes.

These indexes can be expressed as functions of a parameter u:

$$p_1(u) = \frac{-u}{1+u+u^2}\ , \quad p_2(u) = \frac{1+u}{1+u+u^2}\ , \quad p_3(u) = \frac{u(1+u)}{1+u+u^2}\ . \tag{1.6}$$

In Fig (1.1), the functions $p_i(u)$ are shown.

All the admissible values for the p_i's are obtainable as the parameter u runs in the

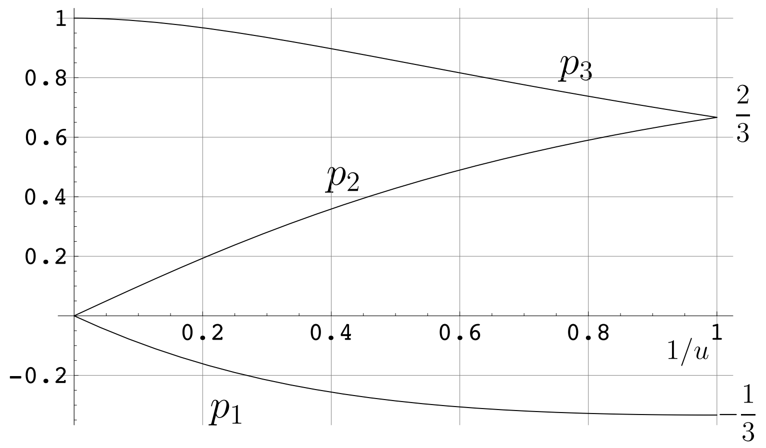

Figure 1.1:

The three curves show all the admissible values for the three p_i's. The figure clearly shows that one and only one can be negative

region $u \geq 1$; the values $u < 1$ are in the same interval because the inversion property holds, *i.e.*,

$$p_1\left(\frac{1}{u}\right) = p_1(u)\ , \qquad p_2\left(\frac{1}{u}\right) = p_3(u)\ , \qquad p_3\left(\frac{1}{u}\right) = p_2(u)\ . \tag{1.7}$$

The instant $t = 0$ is the only singular point; the metric has a singularity that cannot be eliminated with any coordinates transformation, and the invariants of the four-dimensional Ricci tensor become infinite. The only exception is the case $(0, 0, 1)$, that describes the standard flat space-time; the Galilean metric can be obtained as the coordinate transformation $t \sinh z = \zeta, t \cosh z = \tau$ is considered.

1.1.2 The type IX model

Among all the homogeneous models, the type IX stands for its very peculiar dynamics[3].

Let's introduce three spatial vectors l, m, n, and let's take the matrix $\eta_{ab}(t)$ (see Appendix 1A) diagonal; the spatial metric can be recast in the form

$$h_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta \ , \tag{1.8}$$

where all the dependence on the time variable is contained only in the functions a, b, c. Even though (1.8) is not the most general case, it already contains the most important features we want to discuss.

For the type IX model, the structure constants with two indixes take the values

$$C^{11} = C^{22} = C^{33} = 1 \ . \tag{1.9}$$

The Einstein equations assume the form

$$\begin{aligned} \frac{(\dot{a}bc)^{\cdot}}{abc} &= \frac{1}{2a^2b^2c^2}\left[(b^2-c^2)^2 - a^4\right] \ , \\ \frac{(a\dot{b}c)^{\cdot}}{abc} &= \frac{1}{2a^2b^2c^2}\left[(a^2-c^2)^2 - b^4\right] \ , \\ \frac{(ab\dot{c})^{\cdot}}{abc} &= \frac{1}{2a^2b^2c^2}\left[(a^2-b^2)^2 - c^4\right] \ , \end{aligned} \tag{1.10}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0 \ ; \tag{1.11}$$

where the dot denotes the derivative with respect to the time variable.

Let's introduce the logarithm of the functions a, b, c, and the logarithmic time variable τ

$$a = e^\alpha \ , \quad b = e^\beta \ , \quad c = e^\gamma \ , \tag{1.12}$$

$$dt = \Lambda \ d\tau = abc \ d\tau \ . \tag{1.13}$$

Then, (1.10) and (1.11) become

$$\begin{cases} 2\alpha_{,\tau,\tau} = (b^2-c^2)^2 - a^4 \ , \\ 2\beta_{,\tau,\tau} = (a^2-c^2)^2 - b^4 \ , \\ 2\gamma_{,\tau,\tau} = (a^2-b^2)^2 - c^4 \ , \end{cases} \tag{1.14}$$

[3]Most of the considerations we will develop, apply to the type VIII also (see Section 2)

$$\frac{1}{2}(\alpha+\beta+\gamma)_{,\tau,\tau} = \alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \beta_{,\tau}\gamma_{,\tau} \ , \tag{1.15}$$

from which, the following constant of the motion can be easily obtained

$$\alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \beta_{,\tau}\gamma_{,\tau} = \frac{1}{4}\left(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2\right) \ . \tag{1.16}$$

The dynamical scheme (1.10)-(1.11) cannot be solved exactly, but a detailed study of the dynamics near the singular point can be performed, and it will unravel the BKL mechanism.

The left-hand sides of (1.10) (or equivalently of (1.14)) play the role of a potential term; if we assume them to be negligible at a certain instant of time τ, then a Kasner like dynamics takes place

$$a \sim t^{p_l} \ , \quad b \sim t^{p_m} \ , \quad c \sim t^{p_n} \ , \tag{1.17}$$

where p_l, p_m and p_n satisfy the Kasner relations

$$p_l + p_m + p_n = p_l^2 + p_m^2 + p_n^2 = 1 \ . \tag{1.18}$$

The Kasner dynamics cannot last forever because the right-hand sides of (1.14) exponentially grow as the singularity is approached ($t \rightarrow 0$).

Let's assume that, for some interval of time, the regime is Kasner-like, and that $p_l = p_1 < 0$. As $t \rightarrow 0$, functions $b(t)$ and $c(t)$ become smaller and smaller, while $a(t)$ grows monotonically. The potential term in (1.14) cannot be neglected forever, and we have to retain at least the growing terms

$$\begin{aligned} \alpha_{,\tau,\tau} &= -\frac{1}{2}e^{4\alpha} \ , \\ \beta_{,\tau,\tau} = \gamma_{,\tau,\tau} &= \frac{1}{2}e^{4\alpha} \ . \end{aligned} \tag{1.19}$$

The solution to this set of equations has to describe the evolution of a metric that starts from a Kasner-like behaviour, and then is modified by the potential term; as this "initial condition" is imposed, another Kasner-like solution is obtained, where the new set of Kasner indexes (p'_l, p'_m, p'_n) is linked to the previous one by the BKL map

$$p'_l = \frac{|p_1|}{1-2|p_1|} \ , \quad p'_m = -\frac{2|p_1|-p_2}{1-2|p_1|} \ , p'_n = \frac{p_3-2|p_1|}{1-2|p_1|} \ . \tag{1.20}$$

The perturbation, *i.e.*, the right-hand sides of (1.14), induces a transition from a Kasner-regime to another where the negative index passes from the l direction to the

m one: if p_l was negative, now $p'_m < 0$. During this exchange process, the function $a(t)$ reaches a maximum, and $b(t)$ a minimum; then a starts decreasing, b starts growing, and c continues to decrease.
The result is that the initial perturbation $a^4(t)$ becomes smaller and smaller until it is negligible, the new Kasner-regime continues until the new perturbation $b^4(t)$ is relevant. Then a new transition occurs with a new set of indexes, and the phenomenon is repeated infinitely many times up to the singularity.

A suitable way to represent the replacement rule is the following: if

$$p_l = p_1(u) \ , \quad p_m = p_2(u) \ , \quad p_n = p_3(u) \ ,$$

then

$$p'_l = p_2(u-1) \ , \quad p'_m = p_1(u-1) \ , \quad p'_n = p_3(u-1) \ . \tag{1.21}$$

After each replacement, the negative index becomes positive, while the minor among the others, negative.
It is worth noting how the effects of map (1.21) on the dynamics can be analyzed in terms of *continuos fractions* in the following sense. Let's assume that the initial value of the parameter u is $u_0 = k_0 + x_0$, where k_0 is an integer and $0 < x_0 < 1$. Then, the length of the first series of oscillations is equal to k_0, and the initial value of the successive era is $u_1 = 1/x_0 = k_1 + x_1$. It's very easy to realize that the lengths of the following eras are given by the elements k_0, k_1, k_2, ... coming from the infinite continuos fraction

$$u_0 + k_0 + \cfrac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \dots}}} \ . \tag{1.22}$$

An analytical description of the properties of such a dynamics is impossible as the number of the eras increases, and the properties can be studied only by statistical techniques.

Let's summarize the dynamics of the type IX model while reaching the singularity. The continuos exchange (1.21), where the negative value passes from direction l to direction m, goes on until $u < 1$. This value becomes $u > 1$ with the law (1.7); now p_n is the smallest positive index, and the subsequent exchanges will be among l-n or m-n. It can be shown that, for arbitrary initial values of u, the process will last forever.

For the case of an exact solution, the p_i's lose their literal meaning, and this gives them some "indetermination": it makes no sense to consider particular values for the parameter u, and only generic arbitrary values have to be studied.
All this scheme means that an infinite series of oscillations occurs as the Big Bang is reached. During each of these oscillations, the linear distances along two axes oscillate and monotonically decrease along the third. The volume element decreases proportionally to t. Passing from a series to the successive one, the non-oscillating direction changes to another axis. As time goes by, this process becomes quite random (see (1.22)), and the length of the eras changes duration too.
The point $t = 0$ is the only singularity; furthermore, it is also an accumulation point for the series of oscillations.

On the non-diagonal case

We have studied in details the dynamics of the diagonal Bianchi type IX, showing how the BKL mechanism takes place because of the behaviour of the Ricci scalar in the neighbourhood of the singular point. Even though it describes as-well the oscillating regime while approaching the Big-Bang, this is not the most generic case, and new features appear as we relax the diagonal condition. It has to be said that the off-diagonal components identically vanish in the empty case because of the 0α Einstein equations; these new features indeed are typical only of a matter-filled space-time.

In the non diagonal case, the Kasner-like evolution takes place along three directions that do not coincide with the frame vectors $e^{(a)}$; these directions are called *Kasner axes*, and we will denote them with the familiar notation l, m, n[4]. The projections of the Kasner axes along the frame vectors gives us a set of constant parameters L_a, M_a, N_a

$$l = L_a e^{(a)}\,, \qquad m = M_a e^{(a)}\,, \qquad n = N_a e^{(a)}\,. \tag{1.23}$$

Using all the arbitrariness that one has, we can fix the value of L, M, N as

$$\begin{aligned} L &= (1,\, 0,\, 0)\,, \\ M &= (\cot\theta_m,\, 1,\, 0)\,, \\ N &= (\cot\theta_n/\sin\phi_n,\, \cot\phi_n,\, 1)\,, \end{aligned} \tag{1.24}$$

[4]indeed in the diagonal case the Kasner axes can be taken as frame vectors.

with

$$\begin{aligned}
\tan\theta_m &= \Lambda\frac{p_m - p_l}{C_3}\,,\\
\tan\phi_n &= \Lambda\frac{p_n - p_m}{C_1}\\
\cot\theta_n &= \frac{C_2\sin\phi_n - C_3\cos\phi_n}{\Lambda(p_l - p_n)}\,.
\end{aligned}\tag{1.25}$$

Here, the C_i's are three constants of the motion equal to $C_c = \sqrt{h}\chi_a^b C_{cb}^a$, and the latter C's are the structure constants. In terms of these angles here above introduced, we can summarize the full Kasner map, comprehensive of rotation of the axes,

$$p_l = p_1(u) \quad , \quad p_m = p_2(u)\,, \quad p_n = p_3(u)\,,$$

$$p'_l = p_2(u-1)\,, \quad p'_m = p_1(u-1)\,, \quad p'_n = p_3(u-1)\,,$$

$$\begin{gathered}
\frac{\tan\theta'_m}{\tan\theta_m} = -\frac{2u-1}{2u+1}\,, \quad \phi'_n = \phi_n\,, \quad \frac{\tan\theta'_n}{\tan\theta_n} = -\frac{u-2}{u+2}\,,\\
l' = l\,, \quad m' = m - l\frac{4p_1\cot\theta_m}{p_2 + 3p_1}\,, \quad n' = n - l\frac{4p_1}{p_3 + 3p_1}\frac{\cot\theta_n}{\sin\phi_n}\,.
\end{gathered}\tag{1.26}$$

We see that $|\theta'_m/\theta_m| < 1$ and $|\theta'_n/\theta_n| < 1$. This means that, with each replacement of the epochs, the Kasner axes approach each other. The asymptotic behaviour of the general model of type IX displays thus the same properties as the diagonal case, but a new feature is also added, a gradual drawing together of the Kasner axes.

We will not discuss the effects of matter, as it does not play any significant role, as it is shown in the Kasner case in the next paragraph; rotation of Kasner axes is the only important feature that it adds to the Bianchi type IX dynamics. For further details, see [BKL70, BLK71, BKL82] and the references therein.

1.2 The role of matter

Here we discuss the time evolution of a uniform distribution of matter in the Bianchi type I space near the singularity; it will result that it behaves as a test fluid, and, thus, it does not affect the properties of the solution.
Let's take a uniform distribution of matter, and let's assume that we can neglect the

influence of matter on the gravitational field. The hydrodynamics equations describe the evolution [LL71]

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\sigma u^i\right) = 0 \,, \tag{1.27}$$

$$(p+\epsilon)u^k\left(\frac{\partial u_i}{\partial x^k} - \frac{1}{2}u^l\frac{\partial g_{kl}}{\partial x^i}\right) = -\frac{\partial p}{\partial x^i} - u_i u^k\frac{\partial p}{\partial x^k} \,. \tag{1.28}$$

Here u^i is the four-velocity, and σ is the entropy density; in the neighbourhood of the singularity, it is necessary to use ultra-relativistic equation of state $p = \epsilon/3$, and we have, then, $\sigma \sim \epsilon^{3/4}$.
As soon as all the quantities are functions of time, we have

$$\frac{d}{dt}\left(abcu_0\epsilon^{3/4}\right) = 0\,, \qquad 4\epsilon\frac{du_\alpha}{dt} + u_\alpha\frac{d\epsilon}{dt} = 0 \,. \tag{1.29}$$

From (1.29), we get the following two integrals of motion:

$$abcu_0\epsilon^{3/4} = const, \qquad u_\alpha\epsilon^{1/4} = const \,. \tag{1.30}$$

From (1.30), all the covariant components u_α are all of the same order. Among the controvariant ones, the greatest is $u^3 = u_3$ (as $t \to 0$). Retaining only the dominant contribution in the identity $u_i u^i = 1$, we obtain that $u_0^2 \approx u_3^2$, and, from (1.30)

$$\epsilon \sim \frac{1}{a^2b^2}, \qquad u_\alpha \sim \sqrt{ab}\,, \tag{1.31}$$

or, equivalently,

$$\epsilon \sim t^{-2(p_1+p_2)} = t^{-2(1-p_3)}\,, \qquad u_\alpha \sim t^{(1-p_3)/2} \,. \tag{1.32}$$

As expected, ϵ diverges as $t \to 0$, for all the values of p_3, except $p_3 = 1$ (this is due to the non-physical character of the singularity in this case).
The validity of the approximation is verified from a direct evaluation of the components of the energy-momentum T_i^k; their dominant terms are

$$T_0^0 \sim \epsilon u_0^2 \sim t^{-(1+p_3)}\,, \qquad T_1^1 \sim \epsilon \sim t^{-2(1-p_3)}\,, \tag{1.33}$$

$$T_2^2 \sim \epsilon u_2 u^2 \sim t^{-(1+2p_2-p_3)}\,, \qquad T_3^3 \sim \epsilon u_3 u^3 \sim t^{-(1+p_3)} \,. \tag{1.34}$$

As $t \to 0$, all the components grow slower than t^{-2}, that is the behaviour of the dominant terms in the Kasner analysis. Thus they can be disregarded.

1.3 Hamiltonian formulation

A different insight on the Bianchi type IX model is that of Misner. In [Mis69a, Mis69b], he described the complex dynamics in terms of a particle motion in a potential, adopting the ADM formalism (see Appendix 1C).

In the ADM formalism, the homogeneous dynamics are described by the following line element

$$ds^2 = N^2 d\Omega^2 - \frac{1}{4} R^2(\Omega) \left(e^{2\beta}\right)_{ij} \sigma_i \sigma_j \,. \tag{1.35}$$

Here the time variable Ω is related to the cosmic time t of an observer by the relation $dt = -N d\Omega$, thus the singular point appears in the limit $\Omega \to \infty$. β_{ij} is a diagonal traceless matrix, so $e^{2\beta} = \mathrm{diag}\left(e^{2\beta_{11}}, e^{2\beta_{22}}, e^{2\beta_{33}}\right)$ and $\det(e^{2\beta}) = e^{2\mathrm{Tr}\beta} = 1$. Thus $R^3(\Omega)$ is proportional to the volume of the Universe at time Ω. The one-forms σ_i are so defined

$$\begin{aligned}
\sigma_1 &= \sin\psi d\theta - \cos\psi \sin\theta d\phi \,, \\
\sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \,, \\
\sigma_3 &= -\left(d\psi + \cos\theta d\phi\right) \,.
\end{aligned} \tag{1.36}$$

These one-forms reflect the invariance of the metric under the group of unit quaternions, which is the covering group of the rotation group $SO(3)$. The coordinates $(\psi,\, \theta,\, \phi)$ are Euler angle coordinates on $SO(3)$, taken over to covering group by letting ψ have the range $0 \leqslant \psi < 4\pi$, while $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$ as usual. The numeric factor $1/4$ is chosen so that, when $\beta_{ij} = 0$, the space part of the metric is just the standard metric for a three-sphere of radius R and circumference $2\pi R$. Thus for $\beta = 0$ this metric is the Robertson-Walker positive curvature metric.
The three-dimensional Ricci scalar can be evaluated and results to be equal to

$${}^3R = \frac{6}{R^2}(1 - V) \,, \tag{1.37}$$

$$\begin{aligned}
V(\beta_+,\, \beta_-) &= \frac{1}{3}\mathrm{Tr}\left(e^{4\beta} - 2e^{-2\beta} + 1\right) = \\
&= \frac{2}{3} e^{4\beta_+} \left(\cosh 4\sqrt{3}\beta_- - 1\right) + 1 - \frac{4}{3}\cosh 2\sqrt{3}\beta_- + \frac{1}{3} e^{-8\beta_+} \,.
\end{aligned} \tag{1.38}$$

Under these assumptions, the super-Hamiltonian constraint (1C.7) becomes

$$H = -p_\Omega + p_+^2 + p_-^2 + e^{-4\Omega}(V - 1) = 0 \,. \tag{1.39}$$

We take the following condition on the lapse function N and on the shift vector N^α

$$N = \frac{1}{H} e^{-3\Omega} \left(\frac{2}{3\pi}\right)^{1/2} , \quad N^\alpha = 0 \, . \tag{1.40}$$

To avoid numerical factors in the calculation we set

$$R = \left(\frac{2}{3\pi}\right)^{1/2} e^{-\Omega} \, . \tag{1.41}$$

The metric is, therefore,

$$ds^2 = \left(\frac{2}{3\pi}\right) H^{-2} e^{-6\Omega} d\Omega^2 - \frac{1}{6\pi} e^{-2\Omega} \left(e^{2\beta}\right)_{ij} \sigma_i \sigma_j \, . \tag{1.42}$$

As soon as we solve the super-Hamiltonian constraint with respect to p_Ω, the dynamics of the Bianchi type IX model results to be equal to that of a two-dimensional particle moving in a time-dependent potential (see Fig(1.2)) described by the following variational principle

$$I = \int \left(p_\pm d\beta_\pm - N H_{ADM}\right) d\Omega \, , \tag{1.43}$$

$$H_{ADM} = \left[p_+^2 + p_-^2 + e^{-4\Omega}(V-1)\right]^{1/2} \, . \tag{1.44}$$

The system described by (1.43) and (1.44) cannot be analytically integrated, but it unravels interesting features in the neighbourhood of the singularity ($\Omega \to +\infty$). The Hamilton equations $d\beta_\pm/d\Omega = p_\pm/H_{ADM}$ can be used to evaluate the "velocity" $|\beta'|$ of the point Universe, obtaining

$${\beta_+'}^2 + {\beta_-'}^2 + H_{ADM}^{-2} e^{-4\Omega}(V-1) = 1 \, . \tag{1.45}$$

If we analyse the time variation of H_{ADM}

$$\frac{dH_{ADM}}{d\Omega} = \frac{\partial H_{ADM}}{\partial \Omega} \quad \Rightarrow \quad \frac{d \ln H_{ADM}^2}{d\Omega} = -4(1-\beta'^2) \, , \tag{1.46}$$

we easily recognize that, towards the singular point, in first approximation, $H \simeq$ const and $\beta'^2 \simeq 1$. A closer inspection shows that these approximations are valid only for finite Ω intervals, roughly while $|\beta| < \frac{1}{2}\Omega$, interrupted by epochs where the potential V does play a role. The anisotropy potential $V(\beta_\pm)$ is positive definite with $V \approx 8(\beta_+^2 + \beta_-^2)$ near $\beta = 0$. The potential walls rise steeply away from $\beta = 0$,

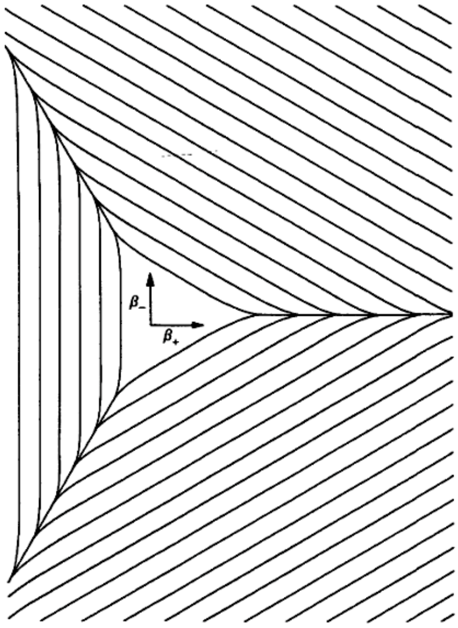

Figure 1.2:

The potential in which the point Mixmaster Universe moves. The lines represent the same equipotential at different time Ω. From this figure, it can be easily seen how the potential term behaves as a infinite well. The three corners represent three asymptotic directions by which the Universe could escape, but the trajectories result to be unstable. The symmetry under a rotation of 120° is manifest (from [Mis69a]).

with the equipotentials forming equilateral triangles in the $(\beta_+, 0, \beta_-)$ plane.
One of the three equivalent sides of the triangle is described by the asymptotic form

$$V \sim \frac{1}{3} e^{-8\beta_+} , \quad \beta_+ \to \infty , \tag{1.47}$$

which is valid in the sector $|\beta_-| < -\sqrt{3}\beta_+$. As $\Omega \to \infty$, the space curvature terms $e^{-4\Omega}(V-1)$ in H_{ADM} can only play a role if $V \gg 1$, so we will use the asymptotic form (1.47). The condition that V be important is easily seen from (1.45) to be $H^{-2}_{ADM} e^{-4\Omega} V \approx 1$ or $\exp[-4(\Omega + 2\beta_+)] \approx 3H^2_{ADM}$ or

$$\beta_+ \approx \beta_{wall} = -\frac{1}{2}\Omega - \frac{1}{8}\ln(3H^2_{ADM}) . \tag{1.48}$$

Thus β_{wall} defines an equipotential in the β plane bounding the region in which the potential terms are significant. When β is well inside this equipotential, one has $|\beta'| = 1$ and $H_{ADM} = \text{const}$, and, consequently, from (1.48) $|\beta'_{wall}| = 1/2$. Thus the

β point moves twice as fast as the receding potential wall, and at finite intervals as $\Omega \to \infty$, the $\beta(\Omega)$ trajectory will collide with the potential wall and be deflected from one straight line motion to another.

This is exactly what happens in the BKL description: the straight line motion is the Kasner-like evolution, while the bounces against the potential walls induce the BKL map.

On the dynamics of type VIII model

Up to now, we have discussed in details the very peculiar dynamics of the type IX model; now we want to discuss the properties of another homogeneous space, those of the type VIII. This model possesses most of the features we have outlined in the previous sections, as it can be easily seen in the ADM framework.
This model is characterized by the following structure constants

$$C^{11} = C^{22} = -C^{33} = 1 \,, \tag{1.49}$$

and its dynamics can be described, like those of all the homogeneous models, as that of a two-dimensional particle moving in a potential well. The analogues of (1.36) now read

$$\begin{aligned} \sigma_1 &= -\sinh\psi \sinh\theta d\phi + \cosh\psi d\theta \,, \\ \sigma_2 &= -\cosh\psi \sinh\theta d\phi + \sinh\psi d\theta \,, \\ \sigma_3 &= \cosh\theta d\phi + d\psi \,, \end{aligned} \tag{1.50}$$

and the potential structure is very similar to that of type IX, as it can be seen in Fig(1.3). As soon as we analyze the asymptotic structure of the potential term, we easily realize that the two dynamics are equivalent in the neighbourhood of the singularity; indeed, we have (1.38) can be rewritten

$$V = \frac{1}{2}\left(D^{4Q_1} + D^{4Q_2} + D^{4Q_3}\right) - D^{2Q_1+2Q_2} \pm D^{2Q_2+2Q_3} \pm D^{2Q_3+2Q_1} \,, \tag{1.51}$$

where $D \equiv \det e^{-\Omega+\beta_{ij}} = e^{-3\Omega}$ is the determinant of the three-metric, $(+)$ and $(-)$ refers respectively to Bianchi type VIII and IX, and the anisotropy parameters Q_i ($i =$

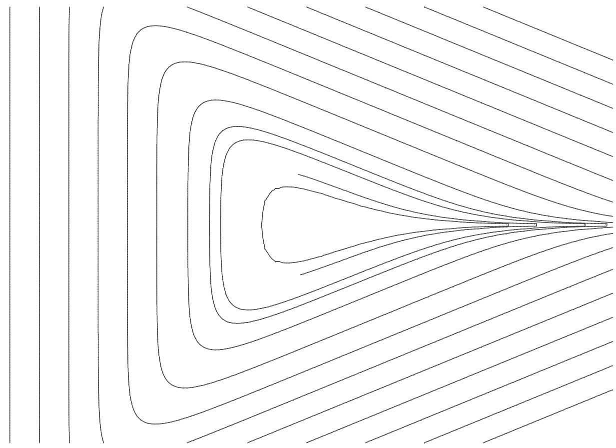

Figure 1.3:

The potential in Misner variables in which the point Bianchi type VIII Universe moves. The lines represents equipotential curves. Like in the type IX model, the walls exponentially grow in the neighbourhood of the singular point, and cut a closed domain.

$1, 2, 3)$ denote the functions [KM97a]

$$
\begin{aligned}
Q_1 &= \frac{1}{3} - \frac{\beta_+ + \sqrt{3}\beta_-}{3\Omega}\,, \\
Q_2 &= \frac{1}{3} - \frac{\beta_+ - \sqrt{3}\beta_-}{3\Omega}\,, \\
Q_3 &= \frac{1}{3} + \frac{2\beta_+}{3\Omega}\,.
\end{aligned}
\tag{1.52}
$$

In the limit $D \rightarrow 0$, the singularity appears, and the second three terms of the above potential turn out to be negligible with respect to the first ones; this shows the equivalence of the dynamics of the two models in the neighbourhood of the Big Bang.

1.4 The Generic Cosmological Solution

The homogeneous models of types VIII and IX provide the prototypes for the construction of the general (non-homogeneous) solution of the Einstein equations in the neighbourhood of the singularity. Here, with "general solution", we mean a solution that possesses the right number of "physical arbitrary functions". This number must allow to specify arbitrary initial conditions on a generic space-like hyper-surface at

a fixed instant t, thus a solution is general if it possesses four arbitrary functions in vacuum, or eight if some kind of matter is present [LL71].
In [LK63], it is shown that the Kasner solution can be generalized to the inhomogeneous case and near the singularity, as

$$\begin{cases} dl^2 = h_{\alpha\beta}dx^\alpha dx^\beta \ , \\ h_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta \ , \end{cases} \tag{1.53}$$

where

$$a \sim t^{p_l} \ , \qquad b \sim t^{p_m} \ , \qquad c \sim t^{p_n} \ , \tag{1.54}$$

and $p_{l,}$, p_m, p_n are functions of spatial coordinates subjected to the conditions

$$p_l(x^\gamma) + p_m(x^\gamma) + p_n(x^\gamma) = p_l^2(x^\gamma) + p_m^2(x^\gamma) + p_n^2(x^\gamma) = 1 \ . \tag{1.55}$$

In distinction of the homogeneous models, the frame vectors l, m, n are now arbitrary functions of the coordinates (more exactly, subjected only to the conditions imposed by the 0α of the Einstein equations [LK63]).
The behaviour (1.53) cannot last up to the singularity, unless a further condition is imposed on the vector l (*i.e.*, the one that correspond to the negative index p_1), *i.e.*,

$$l \cdot \mathrm{curl} l = 0 \ . \tag{1.56}$$

This condition reduces the number of arbitrary functions to three, one less than the general solution: in fact, (1.53) possesses twelve arbitrary functions of the coordinates (nine components of the Kasner axes and three indexes $p_i(x^\gamma)$), and is subjected to the two Kasner relations (1.55), the three 0α Einstein equations, the three conditions from the invariance under three-dimensional coordinate invariance of the theory in the synchronous frame of reference, and (1.56).
Starting from (1.55), however, it is possible to construct the general cosmological solution, and this is achieved in the following steps:

- construction of the general solution for the individual Kasner epoch;
- a general description of the alternation of two successive epochs.

The answer to the first question is given by the generalized Kasner solution (1.53); there is no need to impose in this solution any additional condition, since the individual Kasner epoch lasts only through a finite interval of time, and hence contains the entire set of four arbitrary functions in vacuum (eight if the space is matter-filled). The answer to the second question is given below; we shall see that the replacements of epochs in the general solution does indeed occur in close analogy to the replacement in the homogeneous model[5].

We will assume that the factors that determine the order of magnitude of the components of the spatial metric tensor (1.53) be included in the functions a, b, c. These functions change with time according to the laws (1.54), *i.e.*, the vectors l, m, n also define the directions of the Kasner axes. For non homogeneous spaces there is, of course, no reason to introduce a fixed set of frame vectors, which would be independent of Kasner axes.

The (time) interval of applicability of solution (1.53) is determined by conditions which follow from the Einstein equations. Near the singularity, the matter energy-momentum tensor in the 00- and $\alpha\beta$-components of these equations may be neglected

$$R_0^0 = \frac{1}{2}\dot{\chi}_\alpha^\alpha + \frac{1}{4}\chi_\beta^\alpha\chi_\alpha^\beta = 0 \, , \tag{1.57}$$

$$R_\alpha^\beta = \frac{1}{2\sqrt{h}}\frac{\partial}{\partial t}\left(\sqrt{h}\chi_\alpha^\beta\right) + P_\alpha^\beta = 0 \, . \tag{1.58}$$

Solution (1.53)-(1.55) is obtained neglecting the three-dimensional Ricci tensor P_α^β in (1.58). The condition for such neglect to be valid is easy formulated in terms of the projections of the tensors along the directions l, m, n [LK63]. The diagonal projections of the Ricci tensor must satisfy the conditions

$$P_l^l,\, P_m^m,\, P_n^n \ll t^{-2} \, , \quad P_l^l \gg P_m^m,\, P_m^n \, . \tag{1.59}$$

The off-diagonal projections of (1.58) determine the off-diagonal projections of the metric tensor (h_{lm}, h_{ln}, h_{mn}), which should be only small corrections to the leading terms of the metric, as given by (1.53). In the latter, the only non-vanishing projections are the diagonal (h_{ll}, h_{mm}, h_{nn}), and the fact that the off-diagonal projections are small means that

$$h_{lm} \ll \sqrt{h_{ll}h_{mm}} \, , \quad h_{ln} \ll \sqrt{h_{ll}h_{nn}} \, , \quad h_{mn} \ll \sqrt{h_{mm}h_{nn}} \, . \tag{1.60}$$

[5] We also remind that the regular evolution of the homogeneous model may be violated by the occurrence of a series of small oscillations [BLK72]. Although the likelihood of occurrence of such violations tends to zero as $t \to 0$ [LLK71], a construction of the general solution must also include this case; the answer to this part of the problem is in [BKL70]

This leads to the following conditions on the Ricci tensor:

$$P_{lm} \ll ab/t^2\,, \quad P_{ln} \ll ac/t^2\,, \quad P_{mn} \ll bc/t^2\,. \tag{1.61}$$

If these conditions are satisfied, the off-diagonal components of (1.58) may be completely disregarded (in the leading order). For the metric (1.53), the Ricci tensor P_α^β is given in Appendix D of [LK63].

The diagonal projections P_l^l, P_m^m, P_n^n contain the terms

$$\frac{1}{2}\left(\frac{a\,l\mathrm{curl}(a\,l)}{a\,b\,c(l\cdot[m\times n])}\right) \sim \frac{k^2a^2}{b^2c^2} = \frac{k^2a^4}{\Lambda^2t^2}\,, \tag{1.62}$$

and analogous terms with $a\,l$ replaced by $b\,m$ and $c\,n$ ($1/k$ denotes the order of magnitude of spatial distances over which the metric changes substantially, and Λ is the same as in (1.13)). In (1.62), all the vectorial operations are intended to be performed as in the Euclidean case.

The requirement that these terms be small leads, according to (1.59), to the inequalities

$$a\sqrt{k/\Lambda} \ll 1\,, \quad b\sqrt{k/\Lambda} \ll 1\,, \quad c\sqrt{k/\Lambda} \ll 1\,. \tag{1.63}$$

It is remarkable that these inequalities are not only necessary, but also sufficient conditions for the existence of the generalized Kasner solution. In other words, after conditions (1.63) are satisfied, all other terms in P_l^l, P_m^m, P_n^n, as well as all terms in P_{lm}, P_{ln}, P_{mn}, satisfy conditions (1.59) and (1.61) automatically.

An estimate of these terms leads to the conditions

$$\frac{k^2}{\Lambda^2}\left(a^2b^2\,,\ldots,\,a^3b\,,\ldots,\,a^2bc\,,..\right) \ll 1\,. \tag{1.64}$$

Inequalities (1.64) contain on the left the products of powers of two or three of the quantities which enter the inequalities (1.63), and therefore are *a fortiori* true if the latter are satisfied.

Inequalities (1.64) represent a natural generalization of those which already appear in the construction of the oscillatory regime in the homogeneous case: the Kasner regime arises when the second terms in (1.10) can be neglected compared to the first ones; this requires a^2, b^2, $c^2 \ll 1$.

As t decreases, an instant t_{tr} may eventually occur when one of the conditions (1.63) may be violated (the case when two of these conditions are violated simultaneously, and this can happen when the exponents p_1 and p_2 are close to zero, corresponds to the above mentioned small oscillations). Thus, if during a given Kasner epoch the negative exponent refers to the function $a(t)$, *i.e.*, $p_l = p_1$, then, at the instant t_{tr}, we will have

$$a_{tr} = \sqrt{\frac{k}{\Lambda}} \sim 1\,. \tag{1.65}$$

Since during that epoch the functions $b(t)$ and $c(t)$ decrease with the decrease of t, the other two inequalities of (1.63) remain valid and at $t \sim t_{tr}$ we shall have

$$b_{tr} \ll a_{tr}\,, \quad c_{tr} \ll a_{tr}\,. \tag{1.66}$$

It is remarkable that at the same time all the conditions (1.64) continue to hold. This means that all off-diagonal projections of (1.58) may be disregarded as before. In the diagonal projections (1.62), only terms of one type become important, terms which contain a^4/t. We draw attention to the fact that in these remaining terms we have

$$(a\,l \cdot \mathrm{curl}\,(a\,l)) = a\,(l \cdot [\nabla a \times l]) + a^2\,(l \cdot \mathrm{curl}\,l) = a^2\,(l \cdot \mathrm{curl}\,l)\ , \tag{1.67}$$

i.e., the spatial derivatives of a drop out.

As a result the following equations are obtained for the process of replacement of two Kasner epochs

$$\begin{aligned}
-\,R_l^l &= \frac{(\dot{a}bc)^{\cdot}}{abc} + \lambda^2 \frac{a^2}{2b^2c^2} = 0\,,\\
-\,R_m^m &= \frac{(a\dot{b}c)^{\cdot}}{abc} - \lambda^2 \frac{a^2}{2b^2c^2} = 0\,,\\
-\,R_l^l &= \frac{(ab\dot{c})^{\cdot}}{abc} - \lambda^2 \frac{a^2}{2b^2c^2} = 0\,,\\
-\,R_0^0 &= \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0\,.
\end{aligned} \tag{1.68}$$

These equations differ from the corresponding equations of the homogeneous model (1.10) only through the fact that the quantity

$$\lambda = \frac{l \cdot \mathrm{curl}\ l}{l \cdot [m \times n]}\ , \tag{1.69}$$

is no longer a constant, but a function of the space coordinates. Since, however, (1.68) is a system of ordinary differential equations with respect to time, this difference does not affect at all the solution of the equations and the map which follows from this solution. Thus the law of alternation of exponents derived for homogeneous models remains valid in the general inhomogeneous case.

1.4.1 Rotation of the Kasner axes

We pass now onto the question of the rotation of the Kasner axes with the replacement of epochs.

If in the initial epoch the spatial metric was given by (1.53), then in the final epoch we shall have

$$h_{\alpha\beta} = a^2 l'_\alpha l'_\beta + b^2 m'_\alpha m'_\beta + c^2 n'_\alpha n'_\beta \,, \tag{1.70}$$

with the new functions a, b, c given by a new set of Kasner indexes, and some new vectors l', m', n'. If we project all tensors (including $h_{\alpha\beta}$) in both epochs onto the same directions l, m, n, it is clear that the turning of the Kasner axes can be described as the appearance, in the final epoch, of off-diagonal projections h_{lm}, h_{ln}, h_{mn}, which behave in time like linear combinations of the functions a^2, b^2, c^2. It is possible to show that such projections do indeed appear [BKL82], and that these projections induces the rotation of the Kasner axes. Without going deeper in these calculations, we go directly to the conclusions.

The main effects can be reduced to a rotation of the second Kasner axis by a large (~ 1) angle, and a rotation of the first axis by a small angle ($\sim b_{tr}^2/a_{tr}^2 \ll 1$). Taking only the big rotations into account, we find that the new Kasner axes are related to the old ones by relations of the form

$$l' = l \,, \qquad m' = m + \sigma_m l \,, \qquad n' = n + \sigma_n l \,, \tag{1.71}$$

where the $\sigma_m \,, \sigma_n$ are of the order of the unity, and are given by the following expressions

$$\begin{aligned} \sigma_m &= -\frac{2}{p_2 + 3p_1}\left\{[l \times m] \cdot \nabla \frac{p_1}{\lambda} + \frac{2p_1}{\lambda} m \cdot \mathrm{curl}\; l\right\} \frac{1}{l \cdot [m \times n]} \,, \\ \sigma_n &= \frac{2}{p_2 + 3p_1}\left\{[n \times l] \cdot \nabla \frac{p_1}{\lambda} - \frac{2p_1}{\lambda} n \cdot \mathrm{curl}\; l\right\} \frac{1}{l \cdot [m \times n]} \,, \end{aligned} \tag{1.72}$$

that reduces to those of the homogeneous case (1.26).

It's quite important to underline that the rotation of the Kasner axes (that appears only for a matter-filled homogeneous space) is inherent in the inhomogeneous solution already in the vacuum case. The role played by the matter energy-momentum tensor can be imitated, in the Einstein equations, by the terms due to inhomogeneity of the spatial metric. Here again (as in the generalized Kasner solution), in the general inhomogeneous approach to the singularity the presence of matter is manifested only in the relations between the arbitrary spatial functions which appear in the solution.

1.4.2 Fragmentation

We will now qualitatively discuss a further mechanism that takes place in the inhomogeneous Mixmaster model in the limit towards the singular point: the so called *"fragmentation"* process [KK87, Kir93b, Mon95].

The extension of the BKL mechanism to the general inhomogenous case contains an important physical restriction which consists of a "local homogeneity" hypothesis. In fact, the general derivation of this behaviour is based on the assumption that the spatial variation of all the spatial metric components possess the same "characteristic length", described by a unique parameter k, which can be regarded as an average wavenumber. There are reliable indications that such local homogeneity ceases to be valid as a natural consequence of the asymptotic evolution of the system towards the singularity.

We start by noting that conditions (1.55) do not require that the functions $p_a(x^\gamma)$'s have the same ordering in all the points of space. Indeed, such functions can vary their order throughout space even an infinite number of times without violating the conditions (1.55), in agreement with an oscillatory-like behaviour of their spatial dependence.

Furthermore, the most important property of the BKL map is the strong dependence on initial values, a feature which produces an exponential divergence of the trajectories which results from its iteration.

Given a generic initial condition $p^0_a(x^\gamma)$, continuity of the three-manifold requires that, at two points of the space very close to each other, the Kasner index functions assume correspondingly close values. But because of the previously mentioned property, the trajectories emerging from these two values diverge exponentially, and, since the functions $p_a(x^\gamma)$ can vary only within the interval $[-1/3; 1]$, we conclude that spatial dependence effectively acquires an increasingly oscillatory like behaviour.

Let's illustrate some more details of this process by reviewing the simplest case.

Let's assume that, at a fixed instant of time t_0, all the points of the manifold can be described by a generalized Kasner metric, that the Kasner index functions have the same order point by point, and that the precise values $p_1(x^\gamma)$, $p_2(x^\gamma)$ and $p_3(x^\gamma)$ can be described, for all the points, by a narrow interval of u-values, let's say $u \in [K, K+1]$ for a generic integer K. We will refer to such a situation saying that the manifold is composed by one "island".

We introduce the remainder part of the function $u(x^\gamma)$

$$X(x^\gamma) = u(x^\gamma) - [u(x^\gamma)] \in [0,\, 1), \tag{1.73}$$

where the square-brackets indicate the integer part of the expression enclosed by them. Then the values of the narrow interval we chose can be written $u^0(x^\gamma) = K^0 + X^0(x^\gamma)$. As the evolution takes place, the BKL mechanism induces a transition from an epoch to another; the n-th epoch is characterized by a narrow interval $[K-n,\, K-n+1]$, until $K-n=0$; then the era comes to an end, and a new era begins. What now happens, is that the new $u^1(x^\gamma)$ starts from $u^1(x^\gamma) \in 1/X^0(x^\gamma)$, *i.e.*, takes value in the much wider interval $[0,\, \infty)$. Only points that are very close can still be in the same "island" of u values; distant points in space will now be described by very different integer K^1, and will experience eras of very different duration. As the singular point is reached, many and many eras take place, causing the formation of a greater and greater number of smaller and smaller "islands". This is the so called "fragmentation" process. The reason why such a process is interesting is that the value of the parameter k, which describes the characteristic length, increases as the islands get smaller. This means the progressive increase of the spatial gradients, and could, at least in principle, destroy the BKL mechanism.

It can be argued that this is not the case, and a qualitative explanation is the following: the progressive increase of the spatial gradients produces the same qualitative effects on all the terms present in the three-dimensional Ricci tensor, including the dominant ones. In other words, for each single value of k, in any island, a condition of the form

$$\frac{\text{inhomogeneous term}}{\text{dominant term}} \sim \frac{k^2 t^{2K_i} f(t)}{k^2 t^{4p_1}} \ll 1\ , \tag{1.74}$$

$$K_i = 1 - p_i \geq 0, \qquad f(t) = \mathscr{O}(\ln t,\, ln^2 t), \qquad t \ll 1\ , \tag{1.75}$$

is still valid, where we call inhomogeneous those terms which contain spatial gradients of the scale factors, since they are evidently absent from the dynamics of the homogeneous cosmological models.

From this point of view, the fragmentation process does not produce any behaviour capable of stopping the iterative scheme of the oscillatory regime.

Appendix 1A

Vierbein representation of Einstein equations

Let's introduce four linearly independent vectors $e^{\mu}_{(a)}$ (a is the vector index) that satisfy the only condition:

$$e^{\mu}_{(a)} e_{(b)\mu} = \eta_{ab} \; , \tag{1A.1}$$

where η_{ab} is a given, symmetric, constant matrix with signature $(+, -, -, -)$; let η^{ab} be the inverse of η_{ab}. Furthermore, let $e^{(a)\mu}$ be the reciprocal vectors, such that $e^{(a)}_{\mu} e^{\mu}_{(b)} = \delta^a_b$. Relations $e^{(a)}_{\mu} e^{\nu}_{(a)} = \delta^{\nu}_{\mu}$ are also true.
This way, the vector index is lowered and raised by the matrix η_{ab}, and the metric tensor $g_{\mu\nu}$ assumes the form

$$g_{\mu\nu} = \eta_{ab} e^{(a)}_{\mu} e^{(b)}_{\nu} dx^{\mu} dx^{\nu} \; . \tag{1A.2}$$

We denote with "vierbein components" of the vector A^{μ} the projections along the four $e_{(a)}$:

$$A_{(a)} = e^{\mu}_{(a)} A_{\mu} \; , \qquad A^{(a)} = e^{(a)}_{\mu} = \eta^{ab} A_{(b)} \; . \tag{1A.3}$$

The generalization to a tensor of any number of covariant or controvariant indexes is quite natural.
We introduce now a couple of quantities which will appear directly in the Einstein equations, γ_{acb}, and their linear combinations, λ_{abc} ($;$ denotes the covariant derivative)

$$\gamma_{acb} = e_{(a)i;k} e^{i}_{(b)} e^{k}_{(c)} \; , \tag{1A.4}$$

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb} \; . \tag{1A.5}$$

With some little algebra, the Riemann and the Ricci tensors can be easily calculated with the aid of the γ_{abc} and of the Ricci's coefficients λ_{abc}

$$R_{(a)(b)(c)(d)} = \gamma_{abc,d} - \gamma_{abd,c} + \gamma_{abf}(\gamma^{f}{}_{cd} - \gamma^{f}{}_{dc}) + \gamma_{afc}\gamma^{f}{}_{bd} - \gamma_{afd}\gamma^{f}{}_{bc} \; , \tag{1A.6}$$

$$\begin{aligned} R_{(a)(b)} \;\; = \;\; & -\frac{1}{2}\Big(\lambda_{ab,}{}^{c}{}_{c} + \lambda_{ba,}{}^{c}{}_{c} + \lambda^{c}{}_{ca,b} + \lambda^{c}{}_{cb,a} + \\ & + \lambda^{cd}{}_{b}\lambda_{cda} + \lambda^{cd}{}_{b}\lambda_{cda} - \frac{1}{2}\lambda_{b}{}^{cd}\lambda_{acd} + \lambda^{c}{}_{cd}\lambda_{ab}{}^{d} + \lambda^{c}{}_{cd}\lambda_{ba}{}^{d}\Big) \; . \end{aligned} \tag{1A.7}$$

Appendix 1B

Einstein equations in the synchronous frame

In this appendix, it is shown how to separate the time-like from the space-like components in the field equations by the introduction of a synchronous frame.
Let us define $\chi_{\alpha\beta}$

$$\chi_{\alpha\beta} = \frac{\partial h_{\alpha\beta}}{\partial t} \ , \tag{1B.1}$$

where $h_{\alpha\beta}$ is the 3-dimensional metric tensor such that $ds^2 = dt^2 - h_{\alpha\beta}dx^\alpha dx^\beta$. The standard lowering and raising operations for the tensor $\chi_{\alpha\beta}$ are achieved by the metric $h_{\alpha\beta}$.
Here below, we list the expressions of the Christoffel symbols, and of the components of the Ricci tensor.

$$\Gamma^0_{00} = \Gamma^\alpha_{00} = \Gamma^0_{0\alpha} = 0 \ , \tag{1B.2}$$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\chi_{\alpha\beta} \ , \qquad \Gamma^\alpha_{0\beta} = \frac{1}{2}\chi^\alpha_\beta \ , \qquad \Gamma^\alpha_{\beta\gamma} = \lambda^\alpha_{\beta\gamma} \ , \tag{1B.3}$$

$$R_{00} = -\frac{1}{2}\frac{\partial}{\partial t}\chi^\alpha_\alpha - \frac{1}{4}\chi^\beta_\alpha\chi^\alpha_\beta, \qquad R_{0\alpha} = \frac{1}{2}(\chi^\beta_{\alpha;\beta} - \chi^\beta_{\beta;\alpha}), \tag{1B.4}$$

$$R_{\alpha\beta} = P_{\alpha\beta} + \frac{1}{2}\frac{\partial}{\partial t}\chi_{\alpha\beta} + \frac{1}{4}(\chi_{\alpha\beta}\chi^\gamma_\gamma - 2\chi^\gamma_\alpha\chi_{\beta\gamma}) \ , \tag{1B.5}$$

where $\lambda^\alpha_{\beta\gamma}$ are the three-dimensional Christoffel symbols, and $P_{\alpha\beta}$ is the three-dimensional Ricci tensor, both constructed from the metric $h_{\alpha\beta}$

$$\begin{aligned} P^{(a)}_{(b)} =& \frac{1}{2\eta}\{2C^{bd}C_{ad} + C^{db}C_{ad} + C^{bd}C_{da} + \\ & - {C^d}_d({C^b}_a + {C_a}^b) + \delta^b_a\left[({C^d}_d)^2 - 2C^{df}C_{df}\right] \ . \end{aligned} \tag{1B.6}$$

Finally, the structure for the field equations is

$$R^0_0 = -\frac{1}{2}\frac{\partial}{\partial t}\chi^\alpha_\alpha - \frac{1}{4}\chi^\alpha_\beta\chi^\beta_\alpha = \frac{2}{k}(T^0_0 - \frac{1}{2}T), \tag{1B.7}$$

$$R^0_\alpha = \frac{1}{2}(\chi^\beta_{\alpha;\beta} - \chi^\beta_{\beta;\alpha}) = \frac{2}{k}T^0_\alpha, \tag{1B.8}$$

$$R^\beta_\alpha = -P^\beta_\alpha - \frac{1}{2\sqrt{h}}\frac{\partial}{\partial t}(\sqrt{h}\chi^\beta_\alpha) = \frac{2}{k}(T^\beta_\alpha - \frac{1}{2}\delta^\beta_\alpha T) \ , \tag{1B.9}$$

where k is a constant such that $k = (16\pi G)^{-1}$.

Appendix 1C
The Arnowitt-Deser-Misner formalism and the Hamiltonian formulation of General Relativity

In this appendix, a short review of the ADM (Arnowitt-Deser-Misner) formalism [ADM62] is given (this topic is discussed in much more details in standard textbooks, like [MTW73] or [Wal84]).

In the usual Lagrangian formulation of General Relativity, the fundamental field is the metric tensor $g_{\mu\nu}$, and the formulation is manifestly space-time covariant. The Einstein-Hilbert action is usually expressed as

$$S_{E-H} = -k \int d^4x \sqrt{-g} R(g) \ , \tag{1C.1}$$

where $R(g)$ is the Ricci scalar, which is function of $g_{\mu\nu}$ and of its first and second derivatives. The Einstein equations follow from requiring the action to be stationary under variations of the metric.
In contrast, the Hamiltonian formulation is not manifestly space-time covariant: it requires a 3+1 splitting of the metric, and the dynamical degrees of freedom are the spatial components of the metric. The usual 3+1 splitting is accomplished through the ADM decomposition of space-time metric $g_{\mu\nu}$ in terms of a lapse function N, a shift vector N_i, and an induced spatial metric h_{ij}.

Let Σ_t be a family of hypersurfaces labeled by a parameter t such that the whole space-time M is the direct product $M = \Sigma \otimes \Re$. The spatial components of $g_{\mu\nu}$ on Σ_t induce a spatial metric (*i.e.*, a three-dimensional Riemannian metric), given by $h_{ij} = -g_{ij} + n_i n_j$, where n_μ is the vector field normal to the hypersurface.
The construction of the lapse function N and of the shift vector N_i is shown in Fig (1.4), and, in what follows, refer to it for notation.

Pick a point P_1 on Σ_t and construct the normal at that point. The normal intersects Σ_{t+dt} at a point P_2. The proper distance between points P_1 and P_2 defines the lapse function $N: \ d\tau = N dt$. In general, the normal vector will take the spatial coordinates x^i on Σ_t to some other spatial coordinates on Σ_{t+dt}. Let P_3 denote the point on Σ_{t+dt} with the same spatial coordinates as the point P_1 on Σ_t. The vector from P_2 to P_3 defines the shift vector N^i.
This way, the space-time metric can be constructed from N, N^i and h_{ij}. Take some

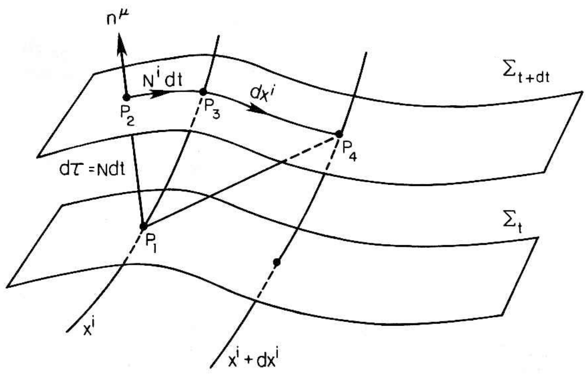

Figure 1.4:

The construction of the lapse function N and of the shift vector N_i(from [KT90]).

other point, P_4, on Σ_{t+dt} with spatial coordinates $x^i + dx^i$. The proper length from P_1 to P_4 is defined in terms of the space-time metric $g_{\mu\nu}$, that in terms of these new quantities is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (Ndt)^2 - h_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \ . \tag{1C.2}$$

From (1C.2) it is easy to obtain the form of the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N_i N_j h^{ij} & -N_j \\ -N_i & -h_{ij} \end{pmatrix} \ , \quad g^{\mu\nu} = \begin{pmatrix} \dfrac{1}{N^2} & -\dfrac{N_j}{N^2} \\ -\dfrac{N_i}{N^2} & \dfrac{N^i N^j}{N^2} - h^{ij} \end{pmatrix} \ . \tag{1C.3}$$

Here h^{ij} is the inverse of h_{ij}, and $N_i = h_{ij}N^j$; moreover $\sqrt{-g} = N\sqrt{h}$.

One way to express the Ricci scalar in term of the lapse function, of the shift vector and of the induced metric is to use the Gauss-Codacci formula [Wal84]

$$R = K^2 - K_{ij}K^{ij} -^3 R \ , \tag{1C.4}$$

where ${}^3R$ is the 3-dimensional Ricci scalar and K_{ij} is the *extrinsic curvature* tensor [MTW73]

$$K_{ij} \equiv \frac{1}{2N}\left[N_{i;j} + N_{j;i} - \frac{\partial h_{ij}}{\partial t}\right] \ . \tag{1C.5}$$

The standard Legendré transformation yields the Hamiltonian principle.
The momenta conjugate to N and N^i vanish, and these relations are referred as the "primary" constraints; much more, it is clear that the lapse and the shift are not

dynamical variables.

The final result is (π^{ij} being the momentum conjugate of h_{ij})

$$\mathcal{H} = \int d^3x \left(\pi^{ij}\partial_t h_{ij} - NH - N_i H^i\right) \ , \tag{1C.6}$$

where the "super-Hamiltonian" H and the "super-momentum" H^i read explicitly

$$\begin{aligned} H &= \frac{1}{2k\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)\pi^{ij}\pi^{kl} - k\sqrt{h}^3 R \equiv \\ &\equiv -k^{-1}G_{ijkl}\pi^{ij}\pi^{kl} - k\sqrt{h}^3 R \ , \qquad (1C.7) \\ H^i &= -2k\pi^{ij}_{;j} \ . \qquad (1C.8) \end{aligned}$$

G_{ijkl} is also referred as the *DeWitt super-metric* [DeW67].

2 Covariant Characterization of the Chaos in the Generic Cosmological Solution

From the beginning of the 80's till the end of the last century, a puzzling question attracted the interest of many scientists: is the Mixmaster dynamics really chaotic? In this chapter, we focus our attention on this topic generalized to a generic cosmological model after a quite wide review of the principal works on the problem of Mixmaster chaos in Section 1. Then the original work of this PhD thesis starts in Section 2, where the dynamics of a generic cosmological model is formulated in a Hamiltonian framework.
In Sections 3, 4 and 5 the reduction to the four physical degrees of freedom is performed together with a detailed discussion on the asymptotic behaviour to the singular point of the three-dimensional Ricci scalar. The characterization of the chaotic dynamics is addressed in Sections 6, 7 and 8 by using the Lyapunov exponent as chaos indicator, by deriving the invariant measure associated to such a model and by studying the effects of the boundary on chaos.
A detailed discussion of the covariance of this scheme then follows in Section 9.
In the last section some concluding remarks are presented.

This work was published on the international journal *Physical Review D* in the late 2004 [BM04]; the publication is appended at end of this PhD thesis (Attachment 1).

2.1 Is the Mixmaster dynamics chaotic?

The challenge over the relativistic chaos is well demonstrated in the very intense debate started at the beginning of the 80's over the chaoticity of the Mixmaster Universe. In the first works by Barrow [Bar81, Bar82b, CB83], the chaotic properties of a discrete approximation to the full dynamics, known as the Gauss map, are studied, and the question of the chaos by using some special indicator as the Lyapunov exponent, the metric entropy and the topological entropy is faced. In such works, the properties of the dynamics is reduced to those of the following map:

$$
\begin{aligned}
&\alpha = 0\,, && \frac{\partial\alpha}{\partial\tau} = sp_2(u) = \frac{s(1+u)}{1+u+u^2}\,,\\
&\beta = \frac{\Sigma x}{(x+1+u)}\,, && \frac{\partial\beta}{\partial\tau} = sp_1(u) = -\frac{su}{(1+u+u^2)}\,,\\
&\gamma = \frac{\Sigma(1+u)}{x+1+u}\,, && \frac{\partial\gamma}{\partial\tau} = sp_3(u) = \frac{su}{(1+u+u^2)}\,,
\end{aligned}
\tag{2.1}
$$

$$
(u',\ x') = \begin{cases} (u-1, x/(1+x)) \ \text{ if } \infty > u > 1 \quad \text{oscillations} \\ (1/u-1, 1+1/x) \ \text{if} \quad 1 > u > 0 \quad \text{bounces}\,. \end{cases}
\tag{2.2}
$$

Map (2.1)-(2.2) is a simple generalization of the well-studied Baker transformation (which is given as $(w',\ y') = (2w - [2w], (y + [2w])/2)$ for $w, y \in (0,\ 1)$ [AA68]): this means that it has dense periodic orbits and that there are no integral of the motion; furthermore, it is ergodic and strongly mixing. This is a strong evidence that the exact equations of motion have a very complex dynamics.

The dispute began when numerical experiments, run in different coordinate systems, revealed that the Lyapunov exponents vanish [FM88, Rug90, Ber91, HBWS91]. Furthermore, the Gauss map itself corresponds to a specific time slicing. The ambiguity of time was manifest.

In [HBWS91], the vanishing behaviour of the Lyapunov exponent for the Bianchi type IX in vacuum under certain conditions and for a finite interval of time is shown, as well as generation of entropy. Their limited results show that the application of numerical methods to the Einstein equations often requires a careful demonstration of consistency with known analytic results.

In [Ber91], the dependence of the Lyapunov exponent on the choice of the time variable is demonstrated. Numerical simulations evaluated the Lyapunov exponents along

some trajectories in the (β_+, β_-) plane for different choices of the time variable; more precisely, three different time variables, τ (BKL), Ω (Misner) and λ , the "mini-superspace" time variable($d\lambda = |-p_\Omega^2 + p_+^2 + p_-^2|^{1/2} d\tau$) were adopted to evolve the trajectories indicated in Fig.(2.1).
It is emphasized that the same trajectory, which clearly gives zero Lyapunov exponent for τ or Ω-time, clearly fails to give zero for λ-time (even if the time duration of the simulation is limited in extension). This can be seen in Fig.(2.2).
More recently, further efforts [BT93, HBC94, KM97a, CL97a, CL97b, WIB98, IM01,

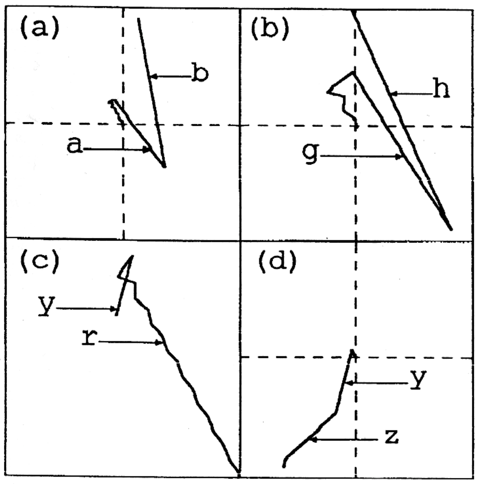

Figure 2.1:

The trajectory in the anisotropy plane (β_+, 0, β_-) evolving towards the singularity. Figures (a) - (d) represent the same trajectory with the scale increased in each case. In (a), the origin is at the center with the range on $\beta\pm$ equal to ± 20. The trajectory begins at the maximum of expansion (very near the origin). In (b), the origin is at the centre with the scale ± 800. Each straight line segment is an epoch with the labelled epochs indicating the sense of the trajectory. In (c), only the upper left quadrant of the anisotropy plane is shown. The length of each axis is 5×10^5. In (d), the entire anisotropy plane is again shown with the scale $\pm 7 \times 10^6$ (from [Ber91]).

ML01, Mot03] were made in order to decide and characterize by some covariant (time-independent) indicator if the Mixmaster model were chaotic or not.
Two convincing arguments, appeared in [CL97a, CL97b] and [IM01], support the idea that the Mixmaster chaos remains valid in any system of coordinates.

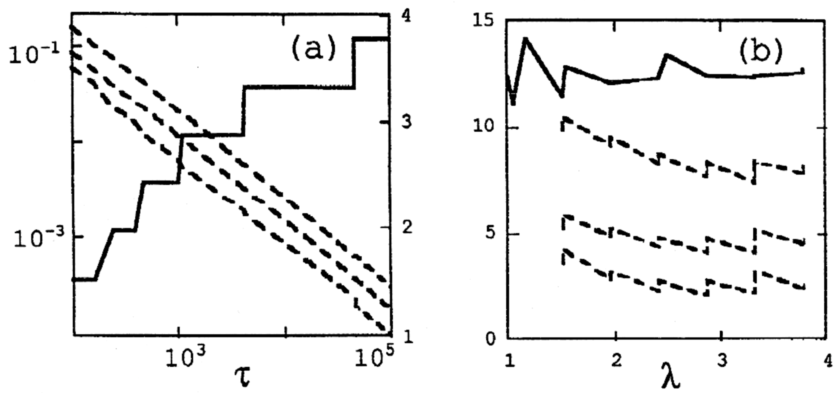

Figure 2.2:

Lyapunov exponents depend on time variable. (a) The solid line indicates mini-superspace proper time λ vs. coordinate time τ for epochs a through h of the trajectory displayed in Fig.(2.1). The dashed line indicates the three positive Lyapunov exponents and are shown in a log-log plot vs. the coordinate time τ used to compute them for the same epochs. (b) The same exponents computed with mini-superspace proper time λ are shown by the dashed line (from [Ber91]).

In [CL97a, CL97b], Cornish and Levin used coordinate-independent fractal methods to show that the Mixmaster Universe is indeed chaotic. By exploiting techniques originally developed to study chaotic scattering, they gained a new perspective on the evolution of the Mixmaster cosmology. They found a fractal structure, namely *the strange repellor* (see Fig.(2.3)), that describes the chaos well. The strange repellor is the collection of all Universes periodic in $(u,\ v)$. A typical, aperiodic Universe will experience a transient age of chaos if it brushes against the repellor. The fractal was exposed in both the exact Einstein equations and in the discrete map that they did use to approximate the solution. The most important feature of their work is that their fractal approach is independent of the time coordinate used. Thus the chaos reflected in the fractal weave of Mixmaster Universes is unambiguous.

In [IM01], the issue of the dependence of the Lyapunov exponent on the time variable is faced: addressing an Hamiltonian point of view in treating the dynamical equations of the Bianchi type IX model, they succeeded in defining it in a covariant way. The key point of their work relies on the choice of a generic function as time variable, avoiding this way all the problems related to the definition of the Lyapunov

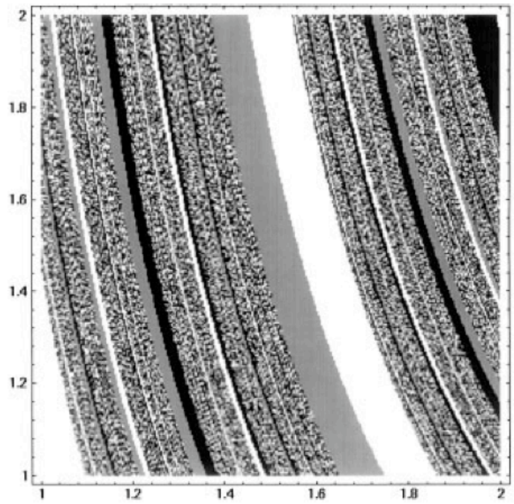

Figure 2.3:

The numerically generated basin boundaries in the $(u,\ v)$ plane are built of Universes which ride the repellor for many orbits before being thrown off. Similar fractal basins can be found by viewing alternative slices through the phase space, such as the $(\beta,\ \dot{\beta})$ plane. The overall morphology of the basins is altered little by demanding more strongly anisotropic outcomes. (From [CL97b])

exponent.

Our work [BM04] is inserted in this line of research.

2.2 Hamiltonian formulation for a generic cosmological model

A generic cosmological solution is represented by a gravitational field having available all its degrees of freedom and, therefore, allowing to specify a generic Cauchy problem. As we have already discussed in the previous sections, it must contain four physical arbitrary functions of the coordinates (in the field equations formulation), that means 8 independent functions in the phase-space; the real physical degrees of freedom are two (as the possible polarizations of a gravitational wave), that means four in the phase space.
In the ADM formalism, the metric tensor corresponding to such a generic model takes

the form

$$d\Gamma^2 = N^2 dt^2 - h_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt) \ , \tag{2.3}$$

where N and N^α denote the lapse function and the shift-vector respectively, and $h_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ the 3-metric tensor of the spatial hyper-surfaces Σ^3 for which $t = const.$

The spatial metric possesses six degrees of freedom, and we choose the following parametric representation:

$$h_{\alpha\beta} = e^{q_a}\delta_{ad}O^a_b O^d_c \partial_\alpha y^b \partial_\beta y^c, \quad a, b, c, d, \alpha, \beta = 1, 2, 3 \ . \tag{2.4}$$

Here $q^a = q^a(x, t)$ and $y^b = y^b(x, t)$ are six scalar functions, and $O^a_b = O^a_b(x)$ is a non-dynamical $SO(3)$ matrix.

By the metric tensor (2.4) and (1C.7)-(1C.8), the action for the gravitational field is

$$S = \int_{\Sigma^{(3)}\times\Re} dtd^3x \left(p_a \partial_t q^a + \Pi_d \partial_t y^d - NH - N^\alpha H_\alpha\right) \ , \tag{2.5}$$

where

$$H = \frac{1}{\sqrt{h}} \left[\sum_a (p_a)^2 - \frac{1}{2}(\sum_b p_b)^2 - h^{(3)}R\right] \ , \tag{2.6}$$

$$H_\alpha = \Pi_c \partial_\alpha y^c + p_a \partial_\alpha q^a + 2p_a (O^{-1})^b_a \partial_\alpha O^a_b \ . \tag{2.7}$$

In (2.6) and (2.7), p_a and Π_d are the momenta conjugate to the variable q^a and y^b respectively, and ${}^{(3)}R$ is the Ricci 3-scalar which plays the role of a potential term.

2.3 Solution to the super-momentum constraint

From (2.7), we note that the super-momentum constraint can be diagonalized and explicitly solved by choosing the functions y^a's as special coordinates; in fact, as soon as we take the transformation

$$\begin{cases} \eta = t \ , \\ y^a = y^a(t, x) \ , \end{cases} \tag{2.8}$$

we get the expression

$$\Pi_b = -p_a \frac{\partial q^a}{\partial y^b} - 2p_a (O^{-1})^c_a \frac{\partial O^a_c}{\partial y^b} \ . \tag{2.9}$$

This way, the solution to this constraint is trivial. Let's study more in detail this transformation: starting by (2.6) and (2.7), the following transformation law holds

$$\begin{cases} q^a(t,x) \rightarrow q^a(\eta, y) \ , \\ p_a(t,x) \rightarrow p'_a(\eta, y) = p_a(\eta, y)/|J| \ , \\ \dfrac{\partial}{\partial t} \rightarrow \dfrac{\partial y^b}{\partial t}\dfrac{\partial}{\partial y^b} + \dfrac{\partial}{\partial \eta} \ , \\ \dfrac{\partial}{\partial x^\alpha} \rightarrow \dfrac{\partial y^b}{\partial x^\alpha}\dfrac{\partial}{\partial y^b} \ , \end{cases} \tag{2.10}$$

where $|J|$ denotes the Jacobian of the transformation. The first relation holds in general for all the scalar quantities, while the second one for all the scalar densities. From direct substitution of (2.9) in action (2.5), we get

$$S = \int_{\Sigma^{(3)} \times \Re} d\eta d^3y \left(p_a \partial_\eta q^a + 2 p_a (O^{-1})^c_a \partial_\eta O^a_c - NH \right) \ . \tag{2.11}$$

The change (2.8) not only allows to easily solve the super-momentum constraint, but also "promotes" the $SO(3)$ matrices (which can be expressed, for example, with the Euler angles) to dynamical variables. This is not surprising at all: indeed $H_\alpha = 0$ are 3 relations that can be used to eliminate three variables (Π_d) from the dynamics, but, this way, we "kill" the three components of the shift-vector N^α also. In some sense, this phenomenon corresponds to the conservation of the number of dynamical variables.

2.4 The Ricci scalar near the singularity

Before going on in the dynamical analysis of the Einstein equation, let's have a look at the behaviour of the Ricci scalar near the singular point. This term plays the role of a potential term, and its analytical structure unravels the peculiar BKL dynamics. From direct but quite tedious calculation, the exact form of the Ricci scalar can be obtained:

$$
\begin{aligned}
-h\ {}^{(3)}R = &\sum_l \frac{1}{2}e^{2q^l}\sum_{kj}(O^l_j\vec{\nabla}O^l_k\vec{T}^{kj})^2 - \sum_{lm,mn,nl} e^{q^l+q^m}[\sum_{k,\xi,\sigma,\theta}(O^l_\xi\vec{\nabla}O^l_k\vec{T}^{k\xi})+\\
&- 2O^l_\xi O^m_\theta(\vec{\nabla}O^m_k\vec{T}^{k\xi})(\vec{\nabla}O^l_\sigma\vec{T}^{\sigma\theta}) - O^l_\xi O^m_\theta[(\vec{\nabla}O^m_k\vec{T}^{k\xi})(O^l_\sigma\vec{\nabla}q^l T^{\sigma\theta})+\\
&+ (\vec{\nabla}O^l_\sigma\vec{T}^{\sigma\theta})(O^m_k\vec{\nabla}q^m\vec{T}^{k\xi})] - \frac{1}{2}O^l_\xi O^m_\theta O^m_k O^l_\sigma(\vec{\nabla}q^m\vec{T}^{k\xi})(\vec{\nabla}q^l\vec{T}^{\sigma\theta})]+\\
&+ 2\sum_{lm,mn,nl}(\vec{\nabla}q^l\vec{\nabla}q^m - \frac{1}{2}\frac{\nabla h}{h}e^{q^l+q^m})[\sum_{k,\xi}[(O^l_\xi\vec{\nabla}O^m_k - O^m_\xi\vec{\nabla}O^l_k) + \frac{1}{2}(O^l_\xi O^m_k\vec{\nabla}q^m+\\
&+ O^l_k O^m_\xi\vec{\nabla}q^l)]\vec{T}^{k\xi}][\sum_{\theta\sigma}O^l_\sigma O^m_\theta\vec{T}^{\sigma\theta}] + 2\sum_{lm,mn,nl}e^{q^l+q^m}[\sum_{k,\xi\sigma\theta}[(\vec{\nabla}O^l_\xi\vec{\nabla}O^m_k - \vec{\nabla}O^m_\xi\vec{\nabla}O^l_k+\\
&+ O^l_\xi\nabla^2O^m_k - O^m_\xi\nabla^2O^l_k) + \frac{1}{2}[(\vec{\nabla}O^l_\xi O^m_k + O^l_\xi\vec{\nabla}O^m_k)\vec{\nabla}q^m + (\vec{\nabla}O^l_k O^m_\xi + O^l_k\vec{\nabla}O^m_\xi)\vec{\nabla}q^l+\\
&O^l_\xi O^m_k\nabla^2q^m + O^l_k O^m_\xi\nabla^2q^l]]\vec{T}^{k\xi}O^l_\sigma O^m_\theta\vec{T}^{\theta\sigma} + [\sum_{k,\xi}[(O^l_\xi\vec{\nabla}O^m_k - O^m_\xi\vec{\nabla}O^l_k) + \frac{1}{2}(O^l_\xi O^m_k\vec{\nabla}q^m+\\
&+ O^m_\xi O^l_k\vec{\nabla}q^l)]\vec{T}^{k\xi}][\sum_{\sigma\theta}(O^m_\theta\vec{\nabla}O^l_\sigma + O^l_\sigma\vec{\nabla}O^m_\theta)\vec{T}^{\sigma\theta}]]\ . \qquad (2.12)
\end{aligned}
$$

Here a symbolic notation is adopted in order to reduce the extension of the analytical expression; in (2.12), vectorial operations are to be performed like in the Euclidean case, all the indexes run from 1 to 3, and

$$\vec{T}^{kj} = \vec{\nabla}y^k \wedge \vec{\nabla}y^j \qquad (2.13)$$

is the Euclidean external product of the vectors $\vec{\nabla}y^k$ and $\vec{\nabla}y^j$.
This is quite a complicated expression, but, as soon as we restrict our analysis to the neighbourhood of the singular point, we can recognize the leading terms. In fact, we can model (2.12) near the Big-Bang as

$$U = \frac{D}{|J|^2}\ {}^{(3)}R = \sum_a \lambda_a^2 D^{2Q_a} + \sum_{b\neq c} D^{Q_b+Q_c}\mathcal{O}\left(\partial q, (\partial q)^2, y, \partial y\right)\ , \qquad (2.14)$$

$$D \equiv \exp\sum_a q^a, \qquad (2.15)$$

$$Q_a \equiv \frac{q^a}{\sum_b q^b} \qquad (2.16)$$

$$\lambda_a^2 \equiv \sum_{kj}\left(O^a_b\vec{\nabla}O^a_c(\vec{\nabla}y^c \wedge \vec{\nabla}y^b)^2\right)\ . \qquad (2.17)$$

Assuming the functions $y^a(t,x)$ smooth enough, (2.19), then all the gradients appearing in the potential U are regular. This assumption is justified by the following two reasons:

1. in [Kir93b], it is shown that the spatial gradients increase logarithmically in the proper time along the billiard geodesic, and therefore result to be of higher order

2. as it can be seen from the variations with respect to p_a, Π_a in the action (2.5), the functions y^a are linked to the system of co-ordinates. These provides six relations that can be re-arranged in order to relate the variables y^d with the lapse function and the shift vector

$$\partial_t y^d = N^\alpha \partial_\alpha y^d \ , \tag{2.18}$$

$$N = \frac{\sqrt{h}}{\sum_a p_a} \left(N^\alpha \partial_\alpha \sum_b q^b - \partial_t \sum_b q^b \right) . \tag{2.19}$$

   This implies, for example, that if we impose the condition $N_\alpha = 0$, fixing this way a synchronous frame of reference, than $\partial_t y^d = 0$, *i.e.*, they are frozen variables with no dynamics: the request of the smoothness corresponds to requiring regularity of the frame of reference.

It's worth noting that the quantity D is proportional to the determinant of the three-metric, being $\det(h_{\alpha\beta}) = |J| \exp \sum_a q^a = |J| D$, and controls its vanishing behaviour; thus, as $D \to 0$, the spatial curvature ${}^{(3)}R$ diverges and the cosmological singularity appears. In this limit, the first terms of the potential U dominates all the remaining ones and can be modeled by the potential wall

$$U = \sum_a \Theta(Q_a) \ , \qquad \Theta(x) = \begin{cases} +\infty & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{2.20}$$

This means that the Universe dynamics evolves independently in each space point because they are no more related among them by the spatial gradients; furthermore, this potential cuts a dynamically-closed domain in the configuration space (Γ_Q, see figure (2.4)) that corresponds to the following conditions on the so-called *anisotropy parameters* Q_a

$$Q_a \geq 0 \, , \qquad a = 1, 2, 3. \tag{2.21}$$

Since in Γ_Q the potential U vanishes asymptotically, near the singularity we have

$$\frac{\partial p_a}{\partial \eta} = 0 \ ; \tag{2.22}$$

thus the term $2p_a (O^{-1})^c_a \partial_\eta O^a_c$ in (2.11) behaves as an exact time-derivative and can be ruled out of the variational principle.

2.5 Dynamics of the physical degrees of freedom

We will now solve the super-Hamiltonian constrain $H = 0$ by introducing the so-called Misner-Chitré like variables [Chi72, Kir93b, Mon00, IM01] as

$$\begin{cases} q^l = e^\tau[\sqrt{\xi^2-1}(\cos\theta+\sqrt{3}\sin\theta)-\xi]\ , \\ q^m = e^\tau[\sqrt{\xi^2-1}(\cos\theta-\sqrt{3}\sin\theta)-\xi]\ , \\ q^n = -e^\tau(\xi+2\sqrt{\xi^2-1}\cos\theta)\ . \end{cases} \tag{2.23}$$

The reason way those variables are so suitable in treating the Mixmaster model relies in the τ-independent expressions for the Q_a's (2.16)

$$\begin{aligned} Q_1 &= \frac{1}{3}-\frac{\sqrt{\xi^2-1}}{3\xi}(\cos\theta+\sqrt{3}\sin\theta)\ . \\ Q_2 &= \frac{1}{3}-\frac{\sqrt{\xi^2-1}}{3\xi}(\cos\theta-\sqrt{3}\sin\theta)\ , \\ Q_3 &= \frac{1}{3}+\frac{2\sqrt{\xi^2-1}}{3\xi}\cos\theta\ . \end{aligned} \tag{2.24}$$

The quantity D, defined in (2.15), that controls the potential well behaviour of the Ricci scalar now explicitly reads

$$D = \exp[-3\xi e^\tau]\ . \tag{2.25}$$

We remember that the singularity appears in the limit $D \to 0$, and by (2.25), it happens as τ approach infinite values.

The super-Hamiltonian constraint can be solved in the domain Γ_Q:

$$-p_\tau \equiv H_{ADM} \equiv \epsilon = \sqrt{(\xi^2-1)p_\xi^2+\frac{p_\theta^2}{\xi^2-1}}\ , \tag{2.26}$$

and the reduced action reads as

$$\delta S_{\Gamma_Q} = \delta\int d\eta d^3y(p_\xi\partial_\eta\xi+p_\theta\partial_\eta\theta-\epsilon\partial_\eta\tau) = 0\,. \tag{2.27}$$

Before starting the analysis of the dynamical system (2.27), we would like to stress the feature of ϵ: by the asymptotic limit (2.20) and the Hamilton equations associated with (2.27), ϵ is a constant of motion, *i.e.*,

$$\frac{d\epsilon}{d\eta} = \frac{\partial\epsilon}{\partial\eta} = 0 \quad\Rightarrow\quad \epsilon = E(y^a)\,. \tag{2.28}$$

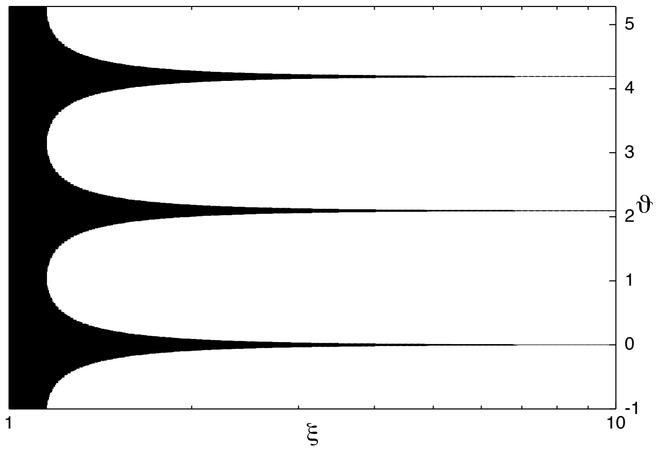

Figure 2.4:

The limited region $\Gamma_Q(\xi, \theta)$ of the configuration space where the dynamics of the point Universe is restricted by mean of the curvature term.

It corresponds to a infinite potential well, and it is the region where the conditions $Q_a \geq 0$ are fulfilled.

2.6 The Lyapunov exponent

We will now perform the analysis of the dynamics of the generic cosmological solution, first using the technique of the Jacobi metric [Arn78] to reduce the Hamilton equations to a geodesic problem on a Riemannian surface, and then showing the positiveness of the Lyapunov exponent for such a system.

The Jacobi metric

Being ϵ a constant of motion, the term $\epsilon\partial_\eta\tau = E(y^a)\partial_\eta\tau$ in (2.27) behaves like an exact time derivative; hence the variational principle rewrites:

$$\delta \int d^3y(p_\xi d\xi + p_\theta d\theta) = 0\,, \tag{2.29}$$

coupled with the constraint (2.26). As a consequence of the asymptotic behaviour of Ricci scalar, the variational principle (2.29) is the direct product of infinite independent dynamical systems, one for each point of the space; for this reason, we will be interested in only one of them.

In order to derive the Jacobi line element, we set $x^{a\prime} \equiv g^{ab}p_b$, and by the Hamiltonian equations

$$
\begin{aligned}
\xi' &= \frac{\partial}{\partial p_\xi}\left(\partial_\eta \tau H_{ADM}\right) = \frac{\partial_\eta \tau}{H_{ADM}}\left(\xi^2-1\right) p_\xi \;, \\
\theta' &= \frac{\partial}{\partial p_\theta}\left(\partial_\eta \tau H_{ADM}\right) = \frac{\partial_\eta \tau}{H_{ADM}} \frac{p_\theta}{\left(\xi^2-1\right)} \;,
\end{aligned}
\tag{2.30}
$$

we obtain the metric

$$
\begin{aligned}
g^{\xi\xi} &= \frac{\partial_\eta \tau}{E(y^a)}\left(\xi^2-1\right) \;, \\
g^{\theta\theta} &= \frac{\partial_\eta \tau}{E(y^a)} \frac{1}{\xi^2-1} \;.
\end{aligned}
\tag{2.31}
$$

By these and by the fundamental constraint relation

$$
\left(\xi^2-1\right) {p_\xi}^2 + \frac{{p_\theta}^2}{\xi^2-1} = E(y^a)^2 \,, \tag{2.32}
$$

we get

$$
g_{ab} x^{a\prime} x^{b\prime} = \frac{\partial_\eta \tau}{E(y^a)} \left\{ \left(\xi^2-1\right) {p_\xi}^2 + \frac{{p_\theta}^2}{\xi^2-1} \right\} = E(y^a) \partial_\eta \tau \;. \tag{2.33}
$$

By the definition $x^{a\prime} = \frac{dx^a}{ds}\frac{ds}{d\eta} \equiv u^a \frac{ds}{d\eta}$, (2.33) rewrites as

$$
g_{ab} u^a u^b \left(\frac{ds}{d\eta}\right)^2 = E(y^a) \partial_\eta \tau \,, \tag{2.34}
$$

which leads to the key relation

$$
d\eta = \sqrt{\frac{g_{ab} u^a u^b}{E(y^a)\partial_\eta \tau}}\; ds \,. \tag{2.35}
$$

Indeed, expression (2.35), together with $p_\xi \xi' + p_\theta \theta' = E(y^a)\partial_\eta \tau$, allows us to put the variational principle (2.29) in geodesic form:

$$
\delta \int \partial_\eta \tau E \; d\eta = \delta \int \sqrt{g_{ab} u^a u^b E(y^a) \partial_\eta \tau}\; ds = \delta \int \sqrt{G_{ab} u^a u^b}\; ds = 0 \;, \tag{2.36}
$$

where the metric $G_{ab} \equiv E(y^a)\partial_\eta \tau g_{ab}$ satisfies the normalization condition $G_{ab} u^a u^b = 1$ and therefore [1]

$$
\frac{\partial s}{\partial \eta} = E(y^a) \partial_\eta \tau \,. \tag{2.37}
$$

[1] We take the positive root since we require that the curvilinear coordinate s increase monotonically with increasing value of τ, *i.e.*, when approaching the initial cosmological singularity.

Summarizing, in the region Γ_Q, the considered dynamical problem reduces to a geodesic flow on a two dimensional Riemannian manifold described by the line element

$$ds^2 = (E(y^a)\partial_\eta \tau)^2 \left[\frac{d\xi^2}{\xi^2 - 1} + \left(\xi^2 - 1\right) d\theta^2\right] . \tag{2.38}$$

Now, it is easy to check that the metric above has negative curvature, since the associated curvature scalar reads $R = -2/\left(E(y^a)\partial_\eta \tau\right)^2$; therefore the point-universe moves over a negatively-curved bidimensional space on which the potential wall (2.20) cuts the region Γ_Q. Independently of the temporal gauge, we gave a satisfactory representation of the system as isomorphic to a billiard on a Lobachevsky plane [Arn78]. The so-obtained Jacobi metric is valid independently of the point, as the space coordinates behave like external parameters and the evolution is spatially not-correlated.

Fermi basis

Let us construct a Fermi basis $\{u^\mu, w^\mu\}, (\mu = 1, 2)$, *i.e.*, a parallel transported frame of reference. The geodesic vector u^μ can be taken equal to

$$u^\mu = \left(\frac{d\xi}{ds}, \frac{d\theta}{ds}\right) = \left(\frac{\sqrt{\xi^2 - 1}}{E(y^a)\partial_\eta \tau} \cos \phi(s), \frac{\sin \phi(s)}{E(y^a)\partial_\eta \tau \sqrt{\xi^2 - 1}}\right) , \tag{2.39}$$

where s denotes the curvilinear coordinate, while $\phi(s)$ is an angular variable ($0 \leq \phi < 2\pi$), whose dynamics is obtained by requiring that the geodesic equation hold

$$\frac{d\phi(s)}{ds} = -\frac{\xi}{E(y^a)\partial_\eta \tau \sqrt{\xi^2 - 1}} \sin \phi(s) . \tag{2.40}$$

The vector w^μ is determined by the properties of the Fermi basis; it is orthonormal, and it explicitly reads

$$w^\mu = \left(-\frac{\sqrt{\xi^2 - 1}}{E(y^a)\partial_\eta \tau} \sin \phi; \frac{\cos \phi}{E(y^a)\partial_\eta \tau \sqrt{\xi^2 - 1}}\right) . \tag{2.41}$$

Let Z^μ be the connecting vector between a generic couple of close geodesics; let us project it over the Fermi basis defined above

$$Z^\mu = Z_u(s) u^\mu + Z_w(s) w^\mu , \tag{2.42}$$

The dynamics of Z^μ is given by the geodesic deviation equation, that results in a couple of differential equations, one for each component of the vector

$$\begin{cases} \dfrac{d^2 Z_u}{ds^2} = 0\ , \\ \dfrac{d^2 Z_w}{ds^2} = \dfrac{Z_w}{\left(E(y^a)\partial_\eta \tau\right)^2}\ . \end{cases} \tag{2.43}$$

The solution to system (2.43) is given by

$$\begin{cases} Z_u = As + B, \qquad A, B = \text{const}\ , \\ Z_w = c_1 e^{\frac{s}{E(y^a)\partial_\eta \tau}} + c_2 e^{-\frac{s}{E(y^a)\partial_\eta \tau}}\ , \qquad c_1, c_2 = \text{const}\ . \end{cases} \tag{2.44}$$

The value of E given by the constraint (2.26), involved in the line element (2.38), is determined by the initial conditions and cannot vanish. By the first of solutions (2.44), no geodesic deviation takes place along the geodesic vector (as expected); on the contrary, from the second solution, we get a non-zero Lyapunov exponent of the form [Pes77]

$$\lambda(y^a) = \limsup_{s \to \infty} \frac{\ln(Z_w^2 + (dZ_w/ds)^2)}{2s} = \frac{1}{E(y^a)\partial_\eta \tau} > 0\ . \tag{2.45}$$

For the validity of this analysis, we have to verify that, in the limit to the initial singularity, the curvilinear coordinate s approaches infinity. This can be easily seen from (2.37),

$$s = E(y^a)\tau + f(y^a) \Rightarrow \lim_{\tau \to \infty} s = \infty\ , \tag{2.46}$$

$f(y^a)$ being a generic function of the space coordinates. This ensures that the curvilinear coordinate s has the requested behaviour towards the singularity (we remember that the singularity appears as and $D = \exp[-3\xi e^\tau] \to 0 \Rightarrow \tau \to \infty$).

2.7 Invariant measures

To better characterize the chaoticity of the obtained billiard, we show that, in each point of the space, this system admits an invariant measure, which, in the present variables, is uniform over the admissible phase space.

From statistical-mechanical point of view, such a system admits, point by point in space, an "energy-like" constant of motion which corresponds to the kinetic part of the ADM Hamiltonian $\epsilon = E(y^a)$. The point-universe randomizes within the closed domain Γ_Q, and it is represented by a dynamics which allows for an ensemble representation; in view of the existence of the "energy-like" constant of motion, the system evolution is appropriately described by a *microcanonical ensemble*. Therefore, the stochasticity of this system is governed by the Liouville invariant measure

$$d\rho \propto \delta\left(E(y^a) - \epsilon\right) d\xi d\theta dp_\xi dp_\theta \ , \tag{2.47}$$

where $\delta\left(x\right)$ denotes the Dirac functional.
Since the particular value taken by the function ϵ ($\epsilon = E(y^a)$) cannot influence the stochastic property of the system and must be fixed by the initial conditions, then we must integrate (in the functional sense) over all admissible form of ϵ. To do this, it is convenient to introduce the natural variables (ϵ, φ) instead of (p_ξ, p_θ) by

$$\begin{aligned} p_\xi &= \frac{\epsilon}{\sqrt{\xi^2 - 1}} \cos\varphi \ , \\ p_\theta &= \epsilon\sqrt{\xi^2 - 1}\sin\varphi \, , \end{aligned} \tag{2.48}$$

where $0 \leqslant \varphi < 2\pi$. By integrating over all functional forms of ϵ, we remove the Dirac delta functional, which leads in each point of space to the uniform normalized invariant measure

$$d\mu(y^a) = d\xi d\theta d\varphi \frac{1}{8\pi^2} \ . \tag{2.49}$$

Since the space points are dynamically decoupled, the whole invariant measure of the system corresponds to the infinite product of (2.49) through the space domain.
Let us now consider the following choice of coordinates on the 2-surface:

$$\vec{y} = (y_1, y_2) = \frac{(1+\xi)}{\sqrt{\xi^2 - 1}}(\cos\theta, \sin\theta) \ . \tag{2.50}$$

On the basis of (2.50), the new line element and the anisotropy parameters read

$$ds^2 = \frac{4E^2 d\vec{y}^2}{(1 - y^2)^2} \ , \quad y < 1 \ , \tag{2.51}$$

$$Q_a = \left[(\vec{y}^2 + A_a)^2 + 1 - (A_a)^2\right] \ , \ \ a = 1, 2, 3 \ , \tag{2.52}$$

being $A^1_a(-\sqrt{3}, \sqrt{3}, 0)$ and $A^2_a(1, 1, -2)$. We introduce the Poincaré model of the Lobachevsky plane in the form of the upper-half plane by using well-known Poincaré variables

$$\vec{\eta} = 2\frac{\vec{y} + \vec{b}}{(\vec{y} + \vec{b})^2} - \vec{b} \ , \tag{2.53}$$

where $\vec{b}$ denotes a point on the absolute ($b^2 = 1$). In terms of these new variables, the metric (2.50) takes the form

$$ds^2 = \frac{(d\vec{\eta})^2}{(\vec{\eta}\cdot\vec{b})^2}\ , \tag{2.54}$$

and the available domain $|y| \leq 1$ transforms into the half-plane $(\vec{\eta}\cdot\vec{b}) \geq 0$, while the absolute represents the line $(\vec{\eta}\cdot\vec{b}) = 0$. Now, if we write

$$\vec{\eta} = \frac{2}{\sqrt{3}}\left[\left(u+\frac{1}{2}\right)\vec{b}^{\perp} + v\vec{b}\right]\ , \qquad v \geq 0\ , \tag{2.55}$$

where $\vec{b} = (0,1), \vec{b}^{\perp} = (1,0)$, then it is easy to verify that, in term of these coordinates, the anisotropy functions have the form

$$\begin{aligned} Q_1(u,v) &= -u/d\ , \\ Q_2(u,v) &= (1+u)/d\ , \\ Q_3(u,v) &= (u(u+1)+v^2)/d\ , \end{aligned} \tag{2.56}$$

$$d = u^2 + u + 1 + v^2\ .$$

This is a very suitable expression for the boundary; in fact geodesic on this half-plane are semicircles having centers on the absolute (*i.e.* $u(s) = A + R\cos s,\ \ v(s) = R\sin s$ being $(A,0)$ the coordinate of the semicircle centre and R the corresponding radius) and rays perpendicular to the absolute; the billiard is bounded by the geodesic triangle $u = 0, u = -1$, and $(u+1/2)^2 + v^2 = 1/4$; the new domain is shown in Fig.(2.5). The billiard has a finite measure, and its open region at infinity, together with the two points on the absolute $(0,0)$ and $(-1,0)$, correspond to the three cuspids of the potential in Fig.(2.4).

It is quite easy to show, that in the u, v plane, (2.49) becomes

$$d\mu = \frac{1}{\pi}\frac{du\,dv}{v^2}\frac{d\phi}{2\pi} \tag{2.57}$$

It is worth stressing that, with the use of the invariant measure here introduced, and the Artins theorem [Art65], the complete equivalence between the BKL piece-wise description and the Misner-Chitré continuos one can be shown [KM97a].

The existence of the stationary probability distribution in Γ_Q (2.57), outlines the chaotic properties associated to the point-like billiard resulting from this analysis.

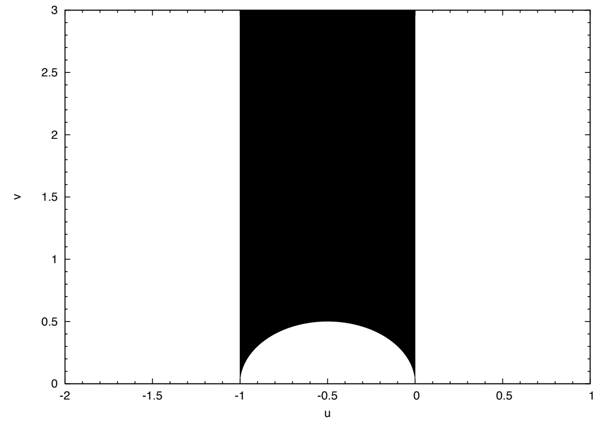

Figure 2.5:

$\Gamma_Q(u, v)$ is the available portion of the configuration space in the Poincaré upper half-plane. It is bounded by three geodesic $u = 0, u = -1$, and $(u + 1/2)^2 + v^2 = 1/4$, it has a finite measure $\mu = \pi$ (it is "dynamically" bounded).

2.8 Role of the potential walls

Up to now, we have involved the role of the potential walls in order to cut a billiard on the Lobachevsky plane. Here, we discuss a notion of Lyapunov exponents, which include both the feature of the geodesic flow, and the structure of the bounding potential walls, and we arrive to show that even in this more complete framework this system is a chaotic one. To this end, we will show that our system meets all the hypothesis at the ground of the Wojtkowski Theorem (see [Woj85]):
Theorem: *Let* (τ, A) *be a measurable cocycle with values in* $\mathscr{S}\mathscr{F}$*, i.e.* $A : X \to \mathscr{S}\mathscr{F}$*. Then the maximal Lyapunov exponent is positive almost everywhere.*

Here (X, μ) is a probability space, $\tau : X \to X$ a measure-preserving transformation, and $A : X \to \mathscr{S}\mathscr{F}$ a measurable mapping to a special set of the $n \times n$ real matrices.
We can easily construct in the (u, v) variables the cocycle and verify that this theorem applies: we identify the transformation τ with the geodesic flow and the map A with the matrix of the dynamics M defined starting from the tangent field to the geodesics. The tangent field to the geodesic flow takes the explicit form:

$$t^\mu \equiv (-v, u - A) \, , \tag{2.58}$$

and the associated matrix of the dynamics $M_\mu^\nu \equiv \partial_\mu t^\nu$ is constant and orthonormal. Both the geodesic flow and the matrix of the dynamics M satisfy the hypothesis of the theorem.

The second condition is given by the construction of an invariant bundle of sectors and by a "dispersing" profile of the boundary of the billiard (see [Woj85] for a detailed discussion). This is possible because the following two properties hold:

1. because of the constant negative curvature of the surface, a one-parameter family of geodesics with negative curvature is invariant under the evolution;

2. the bounding potential walls consist of two straight lines and a semicircle of negative curvature; the first ones do not affect the structure of the cones during the bounces of the geodesic, while the latter one, as a dispersing profile, ensures that, after reflection against it, the cones will evolve in themselves.

After this discussion, we can claim that the largest Lyapunov exponent has positive sign almost everywhere.

2.9 On the covariance of chaos

Our dynamical scheme relies on the use of Misner-Chitré-like variables and therefore the covariance of the Lyapunov exponent is invariant with respect to space-time coordinates; however it could be sensible of the choice of configurational variables, and thus the result here obtained calls attention to be extended to any choice of the configurational variables.
In this respect, we compare our result with the analysis presented in [Mot03], according to which, given a dynamical system of the form

$$d\mathbf{x}/dt = \mathbf{F}(x) \ , \tag{2.59}$$

then the positiveness of the associated Lyapunov exponents are invariant under the following diffeomorphism: $\mathbf{y} = \phi(\mathbf{x}, t), d\tau = \lambda(\mathbf{x}, t)dt$, as soon as the four hypotheses hold:

1. the system is autonomous

2. the relevant part of the phase space is bounded

3. the invariant measure is normalizable
4. the domain of the time parameter is infinite

To show that such a covariance criterion is here fulfilled, we observe that the variables $\mathbf{x}$ can be identified with τ, ξ, θ, and the time variable with our curvilinear coordinate s. On the other hand, the diffeomorphism above in its time-independent form can match a phase-space coordinate transformation; then we underline also that:

1. in the considered asymptotic limit, our dynamical system is autonomous because the Hamiltonian coincides with the constant of motion $\epsilon = E(y^a)$ and the potential walls are fixed in time.
2. apart from sets of zero-measure which cannot be explored by the system [BKL70], the phase-space Γ_Q is a compact domain.
3. the system admits, in each space point, a normalized invariant measure over the phase space.
4. the curvilinear coordinate s admits an infinite domain because $\tau \in (-\infty, \infty)$ (2.46).

Thus, on the base of [Mot03], we can claim that the Lyapunov exponent calculated in (2.45) provides an appropriate chaos indicator only when the effects of the boundary are taken into account in agreement with our discussion. Furthermore, such an indicator is covariant with respect to any configurational coordinate transformation which preserves requirements 1-4. Strikingly, we have to stress that, if we adopt Misner-like variables, the Lyapunov exponent (2.45) is no longer a good indicator; in fact, the anisotropy parameters (2.16) in Misner-like variables depend not only on β_+, β_-, but also on Ω, and therefore, after the ADM reduction, on the curvilinear coordinate s. As a consequence, the conditions 1 and 3 are no longer fullfilled for this choice, because the potential walls move with time. However, for any generic transformation of coordinates, which involves only ξ and θ, the chaoticity of the Mixmaster model is preserved. From the analysis developed in the previous sections, the chaoticity of the inhomogeneous Mixmaster dynamics is ensured if Γ_Q is a closed domain[2], and follows from the independence of the Lyapunov exponent of the lapse

[2]Indeed the compact phase space is constituted by $\Gamma_Q \otimes S^1_\phi$, being S^1_ϕ the unit ϕ-circle

function and the shift vector. In fact, N and N_α are fixed (in turn) by choosing the form of the quantities y^a, and the latter can be generic functions subjected only to the condition to be smooth enough.
The covariance of this picture is equivalent to the covariance of the inhomogeneous Mixmaster chaos because it is well known [IM01, KM97a] that the obtained billiard has stochastic properties (see also [Bar82a]), and this is mainly due to the negative curvature of the Lobachevsky plane that makes unstable the geodesic flow. The potential walls, furthermore, have the role of replacing a given geodesic with a different one (whose tangent vector is related to the previous one by a reflection rule [Bar82a]), and, as we showed, their structure influences the chaotic properties of the system dynamics. Eq.(2.45) provides the form of the Lyapunov exponent in the whole space domain, but we stress how its value depends on the choice of Misner-Chitré like variables; the independence of this scheme on the shift vector is ensured by the asymptotic behaviour of the potential term, but, to get ϵ as constant of motion, allowing the Jacobi metric representation, Misner-Chitré like variables are needed.

2.10 Concluding remarks

The main issue of the present chapter consists in the proof that the chaotic behaviour, singled out by a generic inhomogeneous model near a singularity, has a covariant nature. This result has been obtained by a "gauge" independent ADM reduction of the dynamics to the physical degrees of freedom, which, for the Universe, correspond to the anisotropy degrees, *i.e.*, to the functions ξ and θ.
We describe the evolution as independent of the lapse function and the shift vector form by adopting the variables y^a as the new spatial coordinates. However their degrees of freedom do not disappear from the problem because they are transferred to the $SO(3)$ matrices O^a_b which acquire a dependence on time in the new variables; such a dependence is then eliminated from the dynamics by solving the super-momentum constraint and using some implications deriving by the approximation of $\sqrt{h}^{(3)}R$ as an infinite potential wall.
The potential behaviour (2.20) is crucial for the existence of an "energy-like" constant of motion $\epsilon \equiv E(y^a)$, and therefore is at the basis of the chaos description.
From this point of view, the approximation is naturally induced for $D \to 0$, due to the potential structure; the only assumption required is that the functions y^a's and

$O^a_b(y^c)$ be smooth, in the sense that their presence does not affect the asymptotic behaviour of the potential term. This restriction is natural (having a good degree of generality) because the smoothness of the functions y^a is ensured by the smoothness of the lapse function and the shift vector (see (2.18)-(2.19)), *i.e.*, by the choice of a regular reference frame. The initial smoothness of the matrices O^a_b (when are taken on the spatial coordinates x^a) is preserved by the coordinate transformation.

Concluding, the cosmological meaning of this concept corresponds to independent "horizons"; neglecting in (2.26) the potential term with respect to the value of "ϵ" is equivalent to requiring that the scale of the inhomogeneities be super-horizon sized (see [BKL82, Kir93b, Mon95]).

3 Semi-classical and Quantum Properties of the Inhomogeneous Mixmaster Model

In the previous chapters, we have discussed in details the classical aspects of the inhomogeneous Mixmaster model in vacuum. However, this classical description is in conflict with the requirement of a quantum behaviour of the Universe through the Planck era; in fact, there are reliable indications [KM97b] that the Mixmaster dynamics overlaps the quantum Universe evolution, requiring an appropriate analysis of the transition between these two different regimes.
The dynamics of the very early Universe corresponds, indeed, to a very peculiar situation with respect to the link existing between the classical and quantum regimes, because the expansion of the Universe is the crucial phenomenon which maps into each other these two stages of the evolution. As shown in [KM97b], the appearance of a classical background takes place essentially at the end of the Mixmaster phase, when the anisotropy degrees of freedom can be treated as small perturbations. This result indicates that the oscillatory regime takes place almost during the Planck era, and, therefore, it is a problem of quantum dynamics. However, the end of the Mixmaster (and in principle the *quantum to classical* transition phase) is fixed by the initial conditions on the system, and, in particular, it takes place when the cosmological horizon reaches the inhomogeneity scale of the model. Therefore, the question of an appropriate treatment for the semiclassical behaviour arises when the inhomogeneity scale is so larger than the Planck scale that the horizon can approach it only in the classical limit.

The chapter is organized as follows: we review the Wheeler-DeWitt equation

[DeW67] and some aspects of the quantum cosmology in Section 1. The analysis of the Dirac scheme of quantization for gravity, and the "Multi-time" formalism [Ish92] is presented in Section 2. The original contribution begins in Section 3, with a quite complete analysis of the semi-classical properties of the inhomogeneous Mixmaster model: we will illustrate the mechanical statistical description of the dynamics, and solve the Hamilton-Jacobi functional for the model. In Section 4 the WKB limit to the quantum dynamics is analyzed; the main point we want to stress right now is that we are able to fix a proper ordering for the quantum operators by comparing the dynamics in the classical regime and in the quasi-classical one.
The complete multi-time quantization is performed in Section 5. We derive the eigenfunctions and the spectrum of the Mixmaster model; boundary conditions are taken into account in some approximations. Other results on the quantum dynamics, both analytical and numerical, are also presented. The problem of non-locality of the Hamilton operator is then discussed in Section 6, by analyzing the effects of the square root on the time evolution; some comments on the "quantum chaos" are in Section 7. A discussion on the role of the inhomogeneity can be found in Section 8, together with some concluding remark.
In Appendix 3A, a brief mathematical review on the *automorphic functions* and on the harmonic analysis on the Poincaré plane is presented.

The material presented here is published on the international journal *Classical and Quantum Gravity* [BM07]. The publication is appended at the end of this PhD Thesis (Attachment 3).

3.1 Quantum Cosmology and the Wheeler-DeWitt equation

How the Universe did begin is one of the fundamental cosmological question, and in order to to answer one must treat the gravitational degrees of freedom and interactions quantum mechanically. A huge literature exists on this and related topics, and very different points of view, like that of string theory [Pol98a, Pol98b] and loop quantum gravity [AL04]. We limit our discussion here to the quantum cosmology point of view.

Several promising and not-unrelated approaches to quantizing the gravitational degrees of freedom fall under this general category. One direction that has been pursued, is to derive, using canonical quantum procedure, the analogy of a Schroedinger equation (more accurately, a Klein-Gordon equation) for the *wave function of the Universe*, a wave function governing both the matter fields and the space-time geometry. The equation governing the wave function of the Universe is known as the Wheeler-DeWitt equation [DeW67].

Another aspect of quantum cosmology is that of third quantization. Third quantized operators create and destroy complete Universes with second quantized fields. While today a single-Universe approximation seems to be quite adeguate, the quantum effects of emitting or absorbing small Universes could play an important role in determining the fundamental parameters that we can measure in our Universe, *i.e.*, the value of the cosmological constant, particle masses, and coupling constants [Col88].

We will now develop the theory of the WDW equation.

The starting point is the Hamiltonian framework for gravity; these are the fundamental results we need (from Appendix 3A):

$$\mathcal{H} = \int d^3x \left(\pi^{ij}\partial_t h_{ij} - NH - N_i H^i\right) \ , \tag{3.1}$$

$$\begin{aligned} H &= \frac{1}{2k\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)\pi^{ij}\pi^{kl} - k\sqrt{h}\,{}^3R \equiv \\ &\equiv k^{-1}G_{ijkl}\pi^{ij}\pi^{kl} - k\sqrt{h}\,{}^3R \ , \qquad (3.2) \\ H^i &= -2k\pi^{ij}_{;j} \ , \qquad (3.3) \end{aligned}$$

where G_{ijkl} is often referred to as the *super-metric*.

Identifying the Hamiltonian constraint $H = 0$ as the zero-energy Schroedinger equation: $H[\pi_{ij},\ h_{ij}]\Psi[h_{ij}] = 0$, where the state vector Ψ is the wave function of the Universe, is the first step towards quantization.

The second step is the canonical quantization procedure in which the momenta are replaced by derivatives of their corresponding coordinates:

$$\pi^{ij} \to -ik^{3/2}\frac{\delta}{\delta h_{ij}} \ . \tag{3.4}$$

Following the canonical procedure, we obtain the Wheeler-DeWitt equation [DeW67]

$$\left[k^2 G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + k\sqrt{h}\ {}^3R\right]\Psi[h_{ij}] = 0 \ . \tag{3.5}$$

The generalization of the Wheeler-DeWitt equation with cosmological constant Λ and matter field (denoted generically by ϕ) is straightforward:

$$\left[k^2 G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + k\sqrt{h}(\,{}^3R - 2\Lambda) - \mathcal{T}\right]\Psi[h_{ij},\ \phi] = 0\ , \tag{3.6}$$

where $\mathcal{T} = T_0^0(\phi,\ -i\partial/\partial\phi)$, and $T_{\mu\nu}$ is the stress-energy tensor of the matter fields. There are many subtle questions about this technique, first of all the question and the effect of the factor-ordering of quantum operators associated with $\pi^{ij}h_{ij}$ terms. For many choices of the factor-ordering, the effect can be parametrized by a constant a, and the corresponding Hamiltonian is obtained by the substitution

$$\pi^2 \to -q^{-a}\left[\frac{\partial}{\partial q}q^a\frac{\partial}{\partial q}\right]\ . \tag{3.7}$$

The independence of time of the wave functions is a much more puzzling and interesting feature. In fact Ψ depends upon only the three-geometry h_{ij} and the matter field content ϕ. Because of this, the interpretation of Ψ is a matter of intense debate [Har88]. One possible interpretation is that $\Psi[h_{ij},\ \phi]$ represents a measure of the probabilistic correlation between h_{ij} and ϕ. In this way, one can imagine to use some function of ϕ as a surrogate of the time variable (a similar point of view is that of the *relational time* used in recent years in loop quantum cosmology [APS06]). The question of time or even whether time has a role to play in quantum cosmology is an issue far from being settled.
The only statement that most practitioners would agree with is that, when $\Psi \sim \exp[iS_{E-H}/\hbar]$, classical behaviour ought to pertain.

It should also be emphasized that having an equation for the wave function of the Universe no more resolves the issues of the quantum evolution of the Universe than the Schroedinger equation for an electron resolves the issue of the quantum evolution of the said electron. To be specific about the quantum evolution, one needs information about the initial quantum state, and this is quite puzzling in quantum cosmology.

3.1.1 The wave function of the Universe

The wave function $\Psi[h_{ij},\ \phi]$ is defined in an infinite-dimensional space of all possible three-geometries and matter field configurations, known as *superspace* [MTW73]. The number of degrees of freedom is infinite, and this makes the problem untreatable in general; the only way to face the problem is to freeze out all but a small number of degrees of freedom. The resulting finite-dimensional superspace is known as *mini-superspace*. An example is the following: let's take in consideration a FRW model with line element $ds^2 = dt^2 - R^2(t)d\Omega_3^2$, $d\Omega_3^2$ being the line element for the unit 3-sphere, and R the radius of the Universe; in this model, the shift vector N_i vanishes, and the equation follows from the canonical quantization of $H = 0$. Our mini-superspace model is governed by the following Hamiltonian (here we consider also a cosmological constant term)

$$H = -\frac{G}{3\pi R}\pi_R^2 + \frac{3\pi}{4G}R(1 - R^2\frac{\Lambda}{3}) \ , \tag{3.8}$$

here π_R is the momentum conjugate to R, and they are the only two degrees of freedom for such a system. Following the canonical quantization prescription, $\pi_R \to i\partial/\partial R$, the WDW $H\Psi(R) = 0$ equation becomes

$$\left[\frac{\partial^2}{\partial R^2} - \frac{9\pi^2}{4G^2}\left(R^2 - \frac{\Lambda}{3}R^4\right)\right] = 0 \ . \tag{3.9}$$

The factor ordering here chosen is $a = 0$, and the equation resembles the one-dimensional Schroedinger equation for a particle with zero total energy moving in the potential

$$U(R) = \frac{9\pi^2 R_0^2}{4G^2}\left[\left(\frac{R}{R_0}\right)^2 - \left(\frac{R}{R_0}\right)^4\right] \ . \tag{3.10}$$

The potential $U(R)$ admits a classicaly forbidden region $0 < R < R_0$, and an allowed one $R > R_0$; R_0 is a turning point. From the quantum-mechanical point of view, there is a probability that the "particle" can tunnel through the barrier and emerge at the classical turning point, beyond which it evolves classically.

Let's now discuss the effects of the boundary condition on the Wheleer-DeWitt equation:

a) purely expanding solution [Vil88] *i.e.* $i\Psi^{-1}\partial\Psi/\partial R > 0$, corresponding to a purely outgoing wave; this solution is referred as "creation of the Universe from nothing", where "nothing" corresponds to the initial state $R = 0$:

$$\Psi^V(R) = Ai[z(R)]Ai[z(R_0)] + iBi[z(R)]Bi[z(R_0)], \tag{3.11}$$

b) "no-boundary" boundary condition [HH83]:

$$\Psi^{HH}(R) = Ai[z(R)] \ . \tag{3.12}$$

The potential $U(R)$ and these two solution are shown in Fig.(3.1).

By this simple application, the initial conditions have shown their importance in

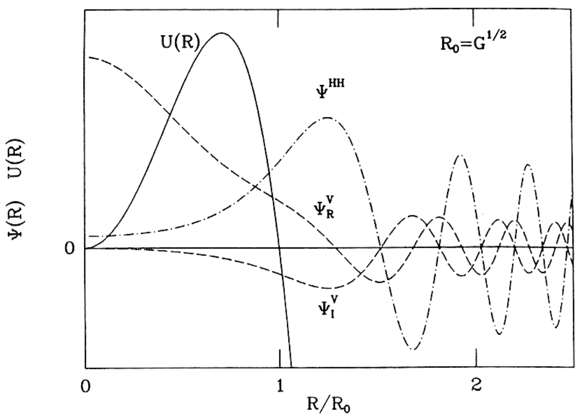

Figure 3.1:

The potential $U(R)$ and the two solution (3.11)-(3.12) are plotted. Solution (3.11) is complex, the two dashed curves represents the real and the imaginary part. Solution (3.12), instead, is a real function (from [KT90]).

the context of quantum cosmology and, in particular, in the WDW approach to the problem.

3.2 The Canonical Quantization and the Multi-Time formalism

We briefly discuss here another point of view on the possible way to a quantized version of General Relativity, *i.e.*, the multi-time formalism [Kuc72, Ish92].

The multi-time formalism is based on the idea that many gravitational degrees of freedom appearing in the classical geometrodynamics have to be not quantized because are not real physical ones; indeed we have to do with $10\ \times\infty^3$ variables, i. e., the values of the functions $N\, N^i\, h_{ij}$ in each point of the hypersurface Σ^3, but it is well-known that the gravitational field possesses only $4\times\infty^3$ physical degrees of freedom (in fact the gravitational waves have, in each point of the space, only two independent polarizations and satisfy second order equations).
The first step is therefore to extract the real canonical variables by the transformation

$$\left\{h_{ij}\, \pi^{ij}\right\} \rightarrow \left\{\xi^{\mu}\, \pi_{\mu}\right\},\left\{H_r\, P^r\right\}, \qquad \mu=0,1,2,3 \quad r=1,2 \tag{3.13}$$

Here H_r, P^r are the four real degrees of freedom, while ξ^μ π_μ play the role of embedding variables.
In terms of this new set of canonical variables, action (3.1) rewrites

$$S^{g\phi} = \int_{\mathcal{M}^4} \left\{ \pi_\mu \partial_t \xi^\mu + P^r \partial_t H_r - NH - N^i H_i \right\} d^3x dt \;, \tag{3.14}$$

where $H = H(\xi^\mu, \pi_\mu, H_r, P^r)$, and $H_i = H_i(\xi^\mu, \pi_\mu, H_r, P^r)$.
Now we provide an ADM reduction of the dynamical problem by solving the Hamiltonian constraint for the momenta π_μ

$$\pi_\mu + h_\mu(\xi^\mu, H_r, P^r, \phi, \pi_\phi) = 0 \;. \tag{3.15}$$

Hence the above action takes the reduced form

$$S = \int_{\mathcal{M}^4} \left\{ P^r \partial_t H_r - h_\mu \partial_t \xi^\mu \right\} d^3x dt \;. \tag{3.16}$$

Finally, the lapse function and the shift vector are fixed by the Hamiltonian equations lost with the ADM reduction, as soon as, the functions $\partial_t \xi^\mu$ are assigned. A choice of particular relevance is to set $\partial_t \xi^\mu = \delta^\mu_0$ which leads to

$$S = \int_{\mathcal{M}^4} \left\{ P^r \partial_t H_r - h_0 \right\} d^3x dt \;. \tag{3.17}$$

The canonical quantization of the model follows by replacing all the Poisson brackets with the corresponding commutators; if we assume that the states of the quantum system are represented by a wave functional $\Psi = \Psi(\xi^\mu, H_r)$, then the evolution is described by the equations

$$i\hbar \frac{\delta \Psi}{\delta \xi^\mu} = \hat{h}_\mu \Psi \;, \tag{3.18}$$

where $\hat{h}_\mu$ are the operator version of the classical Hamiltonian densities.

We will now apply this formalism in the case of the Bianchi type IX Universe.
To set up the multi-time approach, we have to preliminarily perform an ADM reduction, as in (1.44); we find the relation

$$-p_\Omega \equiv h_{ADM} = \sqrt{p^2_{\beta_+} + p^2_{\beta_-} + V} \quad . \tag{3.19}$$

Therefore action (1.43) rewrites as

$$S = \int \left\{ p_{\beta_+} \dot{\beta}_+ + p_{\beta_-} \dot{\beta}_- - \dot{\Omega} h_{ADM} \right\} dt \;. \tag{3.20}$$

Thus we see how Ω plays the role of an embedding variable (indeed it is related to the Universe volume), while $\beta_\pm$ are the real gravitational degrees of freedom (they describe the Universe anisotropy).
By one of the Hamiltonian equation lost in the ADM reduction, we get

$$\dot{\Omega} = -2Ne^{3\Omega}p_\Omega = 2Ne^{-3\Omega}h_{ADM} \,. \tag{3.21}$$

Hence, by setting $\dot{\Omega} = 1$, we fix the lapse function as

$$N = \frac{e^{-3\Omega}}{2h_{ADM}} \,. \tag{3.22}$$

The quantum dynamics in the multi-time approach is summarized by the equation

$$i\hbar\partial_\Omega\Psi = \sqrt{-\hbar^2(\partial^2_{\beta_+} + \partial^2_{\beta_-} + V)}\Psi \,, \qquad \Psi = \Psi(\Omega, \beta_\pm) \,. \tag{3.23}$$

We stress that, in this approach, the variable Ω, *i.e.*, the volume of the Universe, behaves as a "time"-coordinate, and therefore the quantum dynamics can not avoid the Universe reaches the cosmological singularity ($\Omega \rightarrow +\infty$).

3.3 Classical properties of the inhomogeneous Mixmaster model

3.3.1 Hamiltonian formulation

The starting point of our analysis is the classical framework developed in the previous chapter.
We have seen that the generic cosmological model in vacuum can be described in a generic gauge by the following variational principle

$$\delta S_{\Gamma_Q} = \delta \int d\eta d^3y (p_\xi \partial_\eta \xi + p_\theta \partial_\eta \theta - \epsilon \partial_\eta \tau) = 0 \,. \tag{3.24}$$

Here Γ_Q (Fig.(2.5)) is a reduced portion of the configuration space; it is a bi-dimensional surface characterized by constant negative curvature. (3.24) implies that the generic cosmological solution, in the neighbourhood of the singular point, is the sum of infinite not-correlated point Universes, which move on a closed portion of the Lobachevsky space.
We choose the gauge $\partial_\eta \tau = 1$, as soon as we are no more interested in a covariant

description in what follows.

Among the possible representations for Γ_Q, we choose the so-called Poincaré model in the complex upper half-plane [KM97a] that can be introduced with the following coordinate transformation

$$\begin{cases} \xi = \dfrac{1+u+u^2+v^2}{\sqrt{3}v} \; , \\ \theta = -\arctan \dfrac{\sqrt{3}(1+2u)}{-1+2u+2u^2+2v^2} \; . \end{cases} \tag{3.25}$$

The line element for this 2-dimensional surface reads

$$ds^2 = \frac{du^2+dv^2}{v^2} \; . \tag{3.26}$$

In the (u,v) scheme, the ADM Hamiltonian assumes the expression

$$H_{ADM} \equiv \epsilon = v\sqrt{p_u^2+p_v^2} \; . \tag{3.27}$$

3.3.2 Hamilton-Jacobi approach

We shall now derive the Hamilton-Jacobi (HJ) [Arn78] equations for the system (3.27) and we will solve them. The HJ prescribes to change the momenta into derivatives of a functional $\mathcal{S}$ with respect to the configurational variable: implementing the HJ technique, the Hamiltonian relation (3.27) leads to the functional differential equation in Γ_Q

$$-\frac{\delta\mathcal{S}}{\delta\tau} = v\sqrt{\left(\frac{\delta\mathcal{S}}{\delta u}\right)^2+\left(\frac{\delta\mathcal{S}}{\delta v}\right)^2} \; . \tag{3.28}$$

Because of the infinite-walls structure of the potential, the Hamiltonian (3.27) is "time-independent"; thus, we look for a solution of the type

$$\mathcal{S} = \mathcal{S}_0(u,v) - \left(\int d^3y\epsilon(y^a)\right)\tau \; , \tag{3.29}$$

where $\mathcal{S}_0$ satisfies the equation

$$\epsilon^2 = v^2\left(\left(\frac{\delta\mathcal{S}_0}{\delta u}\right)^2+\left(\frac{\delta\mathcal{S}_0}{\delta v}\right)^2\right) \; . \tag{3.30}$$

Equation (3.30) can be solved by separation of constant, *i.e.* by splitting it in a couple of equations

$$\begin{cases} \dfrac{\delta\mathcal{S}_0}{\delta u} = k \; , \\ \dfrac{\delta\mathcal{S}_0}{\delta u} = \sqrt{\dfrac{\epsilon^2}{v^2}-k^2} \; . \end{cases} \tag{3.31}$$

This is the Hamilton-Jacobi function for the point Universe

$$\mathcal{S}_0(u,v) = k(y^a)u + \sqrt{\epsilon^2 - k^2(y^a)v^2} - \epsilon \ln\left(2\frac{\epsilon + \sqrt{\epsilon^2 - k^2(y^a)v^2}}{\epsilon^2 v}\right) + c(y^a)\ . \quad (3.32)$$

In (3.32), $k(y^a)$ is the separation constant, and $c(y^a)$ is the integration constant. The expression (3.29), together with (3.32) and the features of the potential wall (2.56), summarizes the classical dynamics of a generic inhomogeneous Universe.

3.3.3 The statistical Mechanical description

We pointed out that, for the Mixmaster inhomogeneous dynamics, the spatial points decouple while approaching the singularity, and that an energy-like constant of motion appears. As the point Universe randomizes within a closed domain at fixed energy, we can treat it from the statistical mechanical point of view, characterizing its dynamics as a microcanonical ensemble.

The physical properties of a stationary ensemble are described by a distribution function $\rho = \rho(u, v, p_u, p_v)$ [HR]; this function represents the probability of finding the system in a precise infinitesimal interval of the phase-space (u, v, p_u, p_v), and it obeys the continuity equation

$$\frac{\partial(\dot{u}\rho)}{\partial u} + \frac{\partial(\dot{v}\rho)}{\partial v} + \frac{\partial(\dot{p_u}\rho)}{\partial p_u} + \frac{\partial(\dot{p_v}\rho)}{\partial p_v} = 0\ . \quad (3.33)$$

Here the dot denotes the time derivative and these are given by the Hamilton equations associated to (3.27), *i.e.*,

$$\begin{aligned} \dot{u} &\equiv \frac{\partial u}{\partial \tau} = \frac{\partial H_{ADM}}{\partial p_u} = \frac{v^2}{\epsilon} p_u\ , \qquad & \dot{p_u} &\equiv \frac{\partial p_u}{\partial \tau} = -\frac{\partial H_{ADM}}{\partial u} = 0\ , \\ \dot{v} &\equiv \frac{\partial v}{\partial \tau} = \frac{\partial H_{ADM}}{\partial p_v} = \frac{v^2}{\epsilon} p_v\ , \qquad & \dot{p_v} &\equiv \frac{\partial p_v}{\partial \tau} = -\frac{\partial H_{ADM}}{\partial v} = -\frac{\epsilon}{v}\ . \end{aligned} \quad (3.34)$$

From (3.33) and (3.34), we obtain

$$\frac{v^2 p_u}{\epsilon}\frac{\partial \rho}{\partial u} + \frac{v^2 p_v}{\epsilon}\frac{\partial \rho}{\partial v} - \frac{\epsilon}{v}\frac{\partial \rho}{\partial p_v} = 0\ . \quad (3.35)$$

We stress how the continuity equation provides an appropriate representation for the ensemble associated to the Mixmaster only when we are sufficiently close to the initial singularity, and, therefore, the infinite-potential-wall approximation works. Such a model for the potential term, indeed, corresponds to deal with the energy-like constant of the motion, and fixes the microcanonical nature of the ensemble. From a dynamical point of view, this picture arises naturally because the Universe volume element vanishes monotonically (for non stationary correction to this scheme in the Misner-Chitré-like variables, see [Mon01]).

We are interested in studying the distribution function in the (u, v) space, and, thus, we will reduce the dependence on the momenta by integrating $\rho(u, v, p_u, p_v)$ in the momentum space. By using the HJ solution (3.32), and assuming ρ to be a regular, vanishing at infinity in the phase-space, limited function of its arguments, we can integrate over (3.35) in the momentum space, and work out the following equation for $\tilde{w} = \tilde{w}(u, v; k) = \int \rho(u, v, p_u, p_v) dp_u dp_v$

$$\frac{\partial \tilde{w}}{\partial u} + \sqrt{\left(\frac{E}{kv}\right)^2 - 1}\frac{\partial \tilde{w}}{\partial v} + \frac{E^2 - 2k^2v^2}{kv^2}\frac{\tilde{w}}{\sqrt{E^2 - (kv)^2}} = 0 \; . \tag{3.36}$$

In this new equation, a constant k appears; this is due to the analytic expression of the HJ solution, and expresses in some way the role of the initial conditions in that solution. However, we are dealing with a distribution function that can not take in considerations these initial conditions, and we have to eliminate it from the final result.

Eq. (3.36) is a linear partial differential equation of the first order, and the solution can be obtained with the aid of the *integrals of motion*, solving an "associated" dynamical problem (see, for example [BRS93] for further details on solving such kind of equations). We obtain the following solution in term of a generic function g

$$\tilde{w}(u, v; k) = \frac{g\left(u + v\sqrt{\frac{E^2}{k^2v^2} - 1}\right)}{v\sqrt{E^2 - k^2v^2}} \; . \tag{3.37}$$

As we have said, the distribution function cannot contain the constant k, and the final result is obtained after the integration over the constant k. Therefore we define the reduced distribution $w(u, v)$ as

$$w(u, v) \equiv \int_A \tilde{w}(u, v; k) dk \; , \tag{3.38}$$

where the integration is taken over the classical available domain for $p_u \equiv k$

$$A \equiv \left[-\frac{E}{v}, \frac{E}{v}\right] \; . \tag{3.39}$$

To obtain more precise information, we have to make some considerations on the expected distribution of the model, and compare them with our result. We demonstrated in the second chapter, (2.57), that the measure associated to such a domain is the Liouville measure ; it is easy to verify that this measure w_{mc} (after integration over the admissible values of ϕ) corresponds to the case $g = const$

$$w_{mc}(u,v) = \int_{-\frac{E}{v}}^{\frac{E}{v}} \frac{1}{kv^2\sqrt{\frac{E^2}{k^2v^2} - 1}} dk = \frac{\pi}{v^2} \ . \tag{3.40}$$

Summarizing, we have derived the generic expression of the distribution function for this model, fixing its form for the microcanonical ensemble which, in view of the energy-like constant of the motion ϵ, is the most appropriate to describe the Mixmaster system when the picture is restricted to the configuration space.

3.4 Quasi-classical limit of the quantum regime

In this section, we will fix the proper operator-ordering [Kuc81] by comparing the classical and the semi-classical dynamics; this will allow us to face the problem of the full quantization of the Mixmaster model in the next section.

If we consider the WKB limit for $\hbar \to 0$, the coincidence between the classical distribution function $w_{mc}(u,v)$ and the quasi-classical probability function $r(u,v)$ takes place for a precise choice of the operator-ordering only [IM06] (for a connected topic, see [KM97b], and, for a general discussion, see [Gut90]).
We implement the canonical variables into operators

$$\hat{v} \to v \ , \ \ \hat{u} \to u \ ,$$

$$\hat{p}_v \to -i\hbar\frac{\partial}{\partial v} \ , \quad \hat{p}_u \to -i\hbar\frac{\partial}{\partial u} \ , \quad \hat{p}_\tau \to -i\hbar\frac{\partial}{\partial \tau} \ .$$

The quantum dynamics for the state function $\Phi = \Phi(u, v, \tau)$ obeys, in each point of the space, the Schrödinger equation

$$i\hbar\frac{\partial \Phi}{\partial \tau} = \hat{H}_{ADM}\Phi = \hbar\sqrt{-v^2\frac{\partial^2}{\partial u^2} - v^{2-a}\frac{\partial}{\partial v}\left(v^a\frac{\partial}{\partial v}\right)}\Phi \ , \tag{3.41}$$

where we have adopted a generic operator-ordering for the position and momentum operators parametrized by the constant a [KT90].

In the above equation, the Hamiltonian operator has a non-local character as a consequence of the square-root function; in principle, this constitutes a subtle question about the quantum implementation though. As we will demonstrate later, the presence of the square root does not produce non-local effects (see [Puz94]); thus, we will assume that the operators $\hat{H}_{ADM}$ and $\hat{H}^2_{ADM}$ have the same set of eigenfunctions with eigenvalues E and E^2, respectively[1]

If we take the following integral representation for the wave function Φ

$$\Phi(u,v,\tau) = \int_{-\infty}^{\infty} \Psi(u,v,E) e^{-iE\tau/\hbar} dE \ , \tag{3.42}$$

the eigenvalues problem reduces to

$$\hat{H}^2\Psi = \hbar^2 \left[-v^2 \frac{\partial^2}{\partial u^2} - v^{2-a} \frac{\partial}{\partial v} \left(v^a \frac{\partial}{\partial v} \right) \right] \Psi = E^2 \Psi \ , \tag{3.43}$$

where $\Psi = \Psi(u,v,E)$ In order to study the WKB limit of equation (3.43), we separate the wave function into its phase and modulus

$$\Psi(u,v,E) = \sqrt{r(u,v,E)} e^{i\sigma(u,v,E)/\hbar} \ . \tag{3.44}$$

In this scheme, the function $r(u,v)$ represents the probability density, and the quasi-classical regime appears as we take the limit for $\hbar \to 0$; substituting (3.44) in (3.43), and retaining only the lowest order in $\hbar$, we obtain the system

$$\begin{cases} v^2 \left[\left(\frac{\partial \sigma}{\partial u} \right)^2 + \left(\frac{\partial \sigma}{\partial v} \right)^2 \right] = E^2 \ , \\ \frac{\partial r}{\partial u} \frac{\partial \sigma}{\partial u} + \frac{\partial r}{\partial v} \frac{\partial \sigma}{\partial v} + r \left(\frac{a}{v} \frac{\partial \sigma}{\partial v} + \frac{\partial^2 \sigma}{\partial v^2} + \frac{\partial^2 \sigma}{\partial u^2} \right) = 0 \ . \end{cases} \tag{3.45}$$

In view of the HJ equation, and of Hamiltonian (3.27), we can identify the phase σ to the functional S_0 of the HJ approach.

Now we turn our attention to the equation for $r = r(u,v)$. Taking (3.32) into account, second of (3.45) reduces to

$$k \frac{\partial r}{\partial u} + \sqrt{\left(\frac{E}{v} \right)^2 - k^2} \frac{\partial r}{\partial v} + \frac{a(E^2 - k^2 v^2) - E^2}{v^2 \sqrt{E^2 - k^2 v^2}} r = 0 \ . \tag{3.46}$$

Comparing (3.46) with (3.36), we easily see that they coincide (as expected) for $a = 2$ only.

[1] The problems discussed in this respect by [KMV06] do not arise here because in the domain Γ_Q our ADM Hamiltonian has a positive sign (the potential vanishes asymptotically)

It is worth noting that this correspondence is expectable once a suitable choice for the configurational variables is taken; however, here it is remarkable that it arises only if the above operator-ordering is addressed. It is just in this result the importance of this correspondence, whose request fixes a particular quantum dynamics for the system.
Summarizing, we have demonstrated from our study that it is possible to get a WKB correspondence between the quasi-classical regime and the ensemble dynamics in the configuration space, and we provided the operator-ordering when quantizing the inhomogeneous Mixmaster model

$$\hat{v}^2\hat{p}_v^2 \;\rightarrow\; -\hbar^2\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right) . \tag{3.47}$$

3.5 Quantum properties of the inhomogeneous Mixmaster model

3.5.1 Schroedinger quantization and the eigenfunctions of the model

Our starting point is the point-like eigenvalue equation (3.43), which, together with the boundary conditions, completely describes the quantum features of the model, *i.e.*,

$$\left[v^2\frac{\partial^2}{\partial u^2}+v^2\frac{\partial^2}{\partial v^2}+2v\frac{\partial}{\partial v}+\left(\frac{E}{\hbar}\right)^2\right]\Psi(u,v,E)=0 ,$$
$$\Psi(\partial\Gamma_Q)=0 . \tag{3.48}$$

In equation (3.48) we can recognize a well known operator: by redefining $\Psi(u,v,E) = \psi(u,v,E)/v$, we can reduce (3.48) to the eigenvalue problem for the Laplace-Beltrami operator in the Poincaré plane [Ter85]

$$\nabla_{LB}\psi(u,v,E)\equiv v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\psi(u,v,E)=E_s\psi(u,v,E) , \tag{3.49}$$

which is central in the harmonic analysis on symmetric spaces and has been widely investigated in terms of its invariance under $SL(2,C)$ (see Appendix 3A).
Its eigenstates and eigenvalues are described as

$$\psi_s(u,v)=av^s+bv^{1-s}+\sqrt{v}\textstyle\sum_{n\neq 0}a_nK_{s-1/2}(2\pi|n|v)e^{2\pi inu},\;\; a,b,a_n\in C ,$$

$$\nabla_{LB}\psi_s(u,v) = s(s-1)\psi_s(u,v) \ , \tag{3.50}$$

where $K_{s-1/2}(2\pi nv)$ are the modified Bessel functions of the third kind [AS65], and s denotes the index of the eigenfunction. This is a continuous spectrum, and the summation runs over every real value of n.
The eigenfunctions for our model, then, read

$$\Psi(u,v,E) = av^{s-1} + bv^{-s} + \sum_{n\neq 0} a_n \frac{K_{s-1/2}(2\pi|n|v)}{\sqrt{v}} e^{2\pi inu} \ , \tag{3.51}$$

with eigenvalues

$$E^2 = s(1-s)\hbar^2 \ . \tag{3.52}$$

3.5.2 Boundary conditions and the energy spectrum

The spectrum and the explicit eigenfunctions are obtained by imposing the boundary conditions (3.48); in particular we require Dirichlet boundary conditions, *i.e.*, we require that equation (3.51) vanish on the edges of the geodesic triangle of Fig.(2.5). Since, from our analysis, no way arose to impose exact boundary conditions, we approximate the domain with the simpler one in Fig.(3.2); the value for the horizontal line $y = 1/\pi$ provides the same measure for the exact and the approximate domain:

$$\int_{\Gamma_Q} \frac{dudv}{v^2} = \int_{\text{Approximate domain}} \frac{dudv}{v^2} = \pi \ . \tag{3.53}$$

The reason why we did not succeeded in imposing the exact boundary conditions relies in the very sophisticated number theory which is linked to these functions; furthermore, the circle, that bounds from below the domain, mixes solutions with different indexes s.
We note that the Laplace-Belatrami operator and the exact boundary conditions are invariant under the parity transformation $u \to -u-1$; however, it is clear that the full symmetry group is C_{3v}. C_{3v} has two one-dimensional irreducible representations and one two-dimensional representation. Those eigenstates transforming according to one of the two dimensional representations are twofold degenerate, while the other are non-degenerate. These non-degenerate eigenstates can be divided in two classes. Either they satisfy Neumann boundary conditions, or Dirichlet ones instead. We will be interested only in the latter case. The choice of the line $v = 1/\pi$ approximates

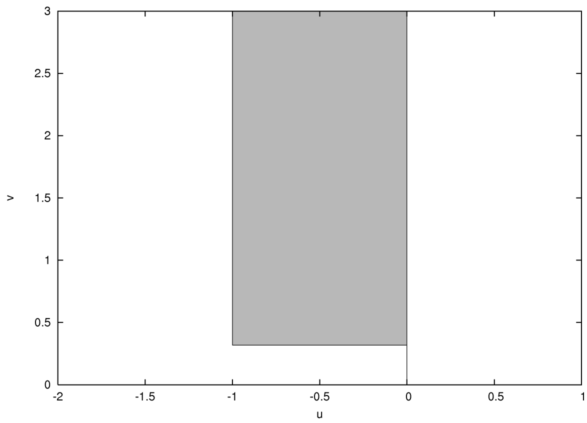

Figure 3.2:

The approximate domain where we impose the boundary conditions.

The choice $v = 1/\pi$ for the straight line preserves the measure $\mu = \pi$.

symmetry lines of the original billiard and corresponds to one-dimensional irreducible representations.

The conditions on the vertical lines $u = 0$, $u = -1$ require to disregard the first two terms in (3.51) ($a = b = 0$); furthermore, we get the condition on the last term

$$\sum_{n\neq 0} e^{2\pi i n u} \rightarrow \sum_{n=1}^{\infty} \sin(\pi n u) \ ,$$

n being an integer. As soon as we restrict to only one of the two one-dimensional representations, we get

$$\sum_{n\neq 0} e^{2\pi i n u} \rightarrow \sum_{n=1}^{\infty} \sin(2\pi n u) \ ,$$

while the condition on the horizontal line implies

$$\sum_{n>0} a_n K_{s-1/2}(2n) \sin(2n\pi u) = 0 \ , \quad \forall u \in [-1, 0] \ , \tag{3.54}$$

which is satisfied by requiring $K_{s-1/2}(2n) = 0$ only. This last condition, and the form of the spectrum (3.52), ensure the discreteness of the energy levels, because of the following result: the zeros of the Bessel functions are a discrete set.

The functions $K_\nu(x)$ are real and positive for real argument and real index, therefore the index must be imaginary, *i.e.*, $s = \dfrac{1}{2} + it$. In this case, these functions have (only) real zeros, and the corresponding eigenvalues turn out to be real and positive.

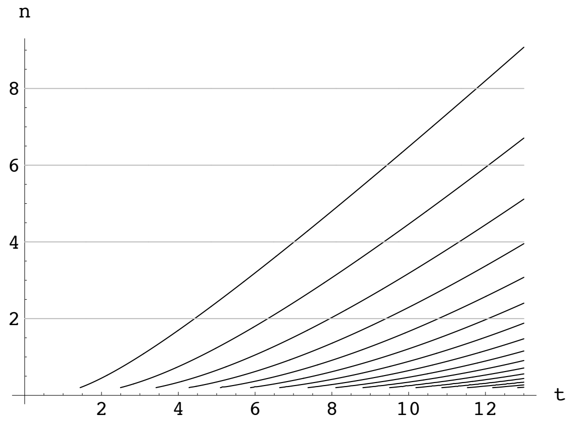

Figure 3.3:

The intersections between the straight lines and the curves represents the roots of the equation $K_{it}(n) = 0$, where K is the modified Bessel function.

n=2	**n=4**	**n=6**	**n=8**
4.425	7.016	9.434	11.768
6.333	9.353	12.076	14.655
7.947	11.313	14.283	17.059
9.410	16.264	16.263	19.212
10.774	17.742	18.096	21.203

Table 3.1:

The first roots for the first value of n are indicated.

$$(E/\hbar)^2 = t^2 + 1/4\,. \tag{3.55}$$

We remark that eigenfunctions (3.51) exponentially vanish as infinite values of v are approached.

The conditions (3.54) cannot be solved analytically for all the values of n and t, and the roots must be worked out numerically for each n. There are several results on their distribution that allow us to find at least the first levels: a theorem on the zeros of these functions states that $K_{i\nu}(\nu x) = 0 \Leftrightarrow 0 < x < 1$ (for a proof see [Pal57]); furthermore, the energy levels (3.55) depends monotonically on the zeros. These two observations allow us to search the lowest levels by solving equation (3.54) for the first n. In Fig.(3.3) and Tab. (3.1), we plot the first roots, and, in Fig.(3.4) and in

$\left(\frac{E}{\hbar}\right)^2 = t^2 + \frac{1}{4}$
19.831
40.357
49.474
63.405
87.729
89.250
116.329
128.234
138.739
146.080

Table 3.2:

The first ten energy eigenvalues, ordered from the lowest one.

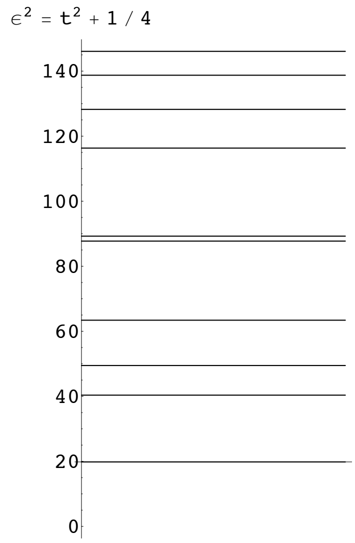

Figure 3.4:

The lines represent in a typical spectral manner the first levels.

Tab.(3.2), we list the first ten "energy" levels[2].

Completeness of the spectrum.

The problem of completeness can be faced by studying first the sinus functions, and then the Bessel ones.

$\sin(2\pi nu)$ is not a complete basis on the interval $[-1, 0]$; as soon as we request the wave function to satisfy the symmetry of the problem, we obtain that this is a complete basis.

Pick a value for $n > 0$; functions (3.51) have the form $\Phi(u, v) = \sin(2\pi nu)g(v)$. Substituting into the eigenfunction equation,$v^2\left(\partial_v^2 + (2\pi n)^2)\right)g(v) = s(1-s)g(v)$. The solutions to this equation are exactly the Bessel functions .

This property, plus the condition on the line $v = 1/\pi$, form a Sturm-Liouville problem; thus this set is complete.

[2]for a detailed numerical investigation of the energy spectrum of the standard Laplace-Beltrami operator, especially with respect to the high-energy levels, see[CGS91, GHSV91], where it is also numerically analyzed the effects on the level spacing of deforming the circular boundary condition towards the straight line

Scalar product.
The found eigenfunctions define a function space where we can introduce a scalar product. This is naturally induced by the metric of the Poincaré plane

$$(\psi, \phi) = \int \psi(x, y)\phi^*(x, y)\frac{dxdy}{y^2}\ , \tag{3.56}$$

where * denotes complex conjugation.

These last observations complete the discussion on the general properties of the discrete spectrum of the inhomogeneous Mixmaster model.

3.5.3 The ground state

Let us describe the properties of the ground state level with a major accuracy, starting from the result of its existence with a non-zero energy E_0, *i.e.*, $E_0^2 = 19.831\hbar^2$. In Fig.(3.5) we plot the wave function Ψ_{gs} in the $(u,\ 0,\ v)$ plane, and in Fig.(3.6) the corresponding probability distribution $|\Psi_{gs}|^2$, (the normalization constant is equal to $N = 739.466$). The knowledge of the ground-state eigenfunction allows us to estimate

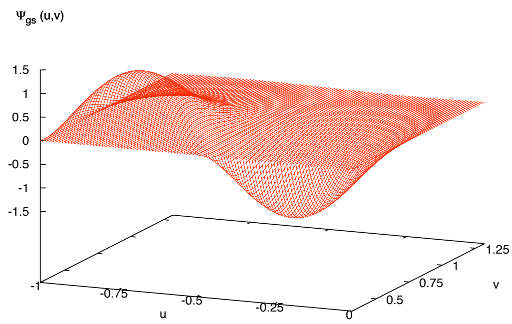

Figure 3.5:

The ground state wave function of the Mixmaster model is sketched.

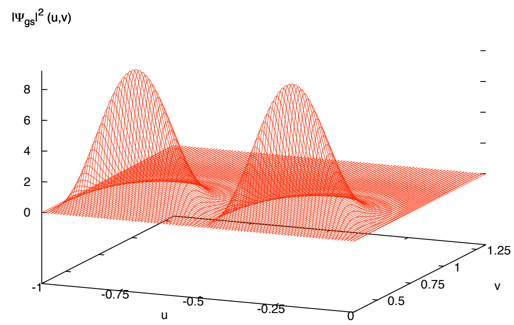

Figure 3.6:

The probability distribution in correspondence to the ground state of the theory, according to the boundary conditions.

the average values of the anisotropy variables u, v and the corresponding root mean square, *i.e.*,

$$\langle u \rangle = -1/2\ , \qquad \langle v \rangle = 0.497\ , \tag{3.57}$$

$$\Delta u = 0.266\ , \qquad \Delta v = 0.077\ . \tag{3.58}$$

This result tells us that in the ground state the Universe is not exactly an isotropic one, and it fluctuates around the line of the Misner plane $\beta_- = 0$. However, we observe that it remains localized in the center of the Misner space far from the corner at $v \to \infty$ (the other two equivalent corners were cut out from our domain by the approximation we considered on the boundary conditions, but, because of the potential symmetry, they have to be unaccessible too). Thus, we can conclude that the Universe, approaching the minimal energy configuration, conserves a certain degree of anisotropy and lives in the region where the latter can be treated as a small correction to the full isotropy. Such a behaviour is a consequence of the zero-point energy associated to the ground state which prevents the absence of oscillation modes. This feature has been numerically derived, but it can be inferred on the basis of general considerations about the Hamiltonian structure. The Hamiltonian, indeed, contains a term $\widehat{v^2 p_v^2}$ which has positive definite spectrum and cannot admit vanishing eigenvalue.

3.5.4 Asymptotic expansions

To study the distribution of the highest energy levels, let us take into account the asymptotic behavior of the zeros for the modified Bessel functions of the third kind. We will discuss asymptotic regions of the plane $(t,\ 0,\ n)$ in both the cases i) $t \gg n$ and ii) $t \simeq n \gg 1$.

i) For $t \gg n$, the Bessel functions admit the following representation:

$$K_{it}(n) = \frac{\sqrt{2\pi}e^{-t\pi/2}}{(t^2-n^2)^{1/4}}\Big[\sin a\sum_{k=0}^{\infty}\frac{(-1)^k}{t^{2k}}u_{2k}\Big(\frac{1}{\sqrt{1-p^2}}\Big)+ \\ + \cos a\sum_{k=0}^{\infty}\frac{(-1)^k}{t^{2k+1}}u_{2k+1}\Big(\frac{1}{\sqrt{1-p^2}}\Big)\Big]\ . \tag{3.59}$$

where $a = \pi/4 - \sqrt{t^2-n^2} + t\,\mathrm{arccosh}(t/n)$, $p \equiv n/t$ and u_k are the following polynomials

$$\begin{cases} u_0(t) = 1\ , \\ u_{k+1}(t) = \dfrac{1}{2}t^2(1-t^2)u_k'(t) + \dfrac{1}{8}\displaystyle\int_0^1 (1-5t^2)u_k(t)dt\ . \end{cases} \tag{3.60}$$

Retaining in the expression above only those terms of $\mathsf{o}(\frac{n}{t})$, the zeros are fixed by the following relation

$$\sin\Big(\frac{\pi}{4} - t + t(\log(2)-\log(p))\Big) - \frac{1}{12t}\cos\Big(\frac{\pi}{4} - t + t(\log(2)-\log(p))\Big) = 0\ . \tag{3.61}$$

In the limit $n/t \ll 1$, (3.61) can be recast as follows

$$t\log(t/n) = l\pi \;\Rightarrow\; t = \frac{l\pi}{\mathrm{productlog}(\frac{l\pi}{n})}\,, \tag{3.62}$$

where productlog(z) is a generalized function that gives the solution of the equation $z = we^w$, and, for real and positive domain, is a monotonic function of its argument. In (3.62), l is an integer number that must be taken much greater than 1 in order to verify the initial assumptions $n/t \ll 1$.

ii) In case the difference between $2n$ and t is $\mathsf{o}(n^{1/3})$ $(t, n \gg 1)$, we can evaluate the first zeros $k_{s,\nu}$ by this relations [Bal67]

$$k_{s,\nu} \sim \nu + \sum_{r=0}^{\infty}(-1)^r s_r(a_s)\left(\frac{\nu}{2}\right)^{-(2r-1)/3}\,, \tag{3.63}$$

where a_s is the s-th zero of $\mathrm{Ai}\big((2/z)^{1/3}\big)$, and s_i are some polynomials. From this expansion it results that, at the lowest order,

$$t = 2n + 0.030n^{1/3}\,. \tag{3.64}$$

(3.64) provides the lowest zero (and therefore the energy) for a fixed value of n and then the relation for the eigenvalues for high occupation numbers

$$\left(\frac{E}{\hbar}\right)^2 \sim 4n^2 + 0.12n^{4/3}\,. \tag{3.65}$$

In this region of the spectrum, the condition $t \sim 2n$, reduces the 2 quantum numbers to a single one, and the whole wave function is then assigned by n.

The analysis above shows that, as an effect of Dirichlet boundary condition and in the limit of high occupation numbers, analytical expressions for the discrete structure of the spectrum can be obtained. In fact, for large values of t, it was possible to give analytical representations for the position of the zeros, but we emphasize that the request of dealing with these approximations causes the loss of a large number of levels, and prevents a complete discussion of the quantum chaos associated to the model. However, the expressions above are of interest because allow us to compare these results with the corresponding spectrum provided by Misner in his original work [Mis69a]. Of course, such a comparison of the two results can take place only on a qualitative level; in fact, between Misner $(\Omega,\ \beta_+,\ \beta_-)$ and Misner-Chitré-like $(\tau,\ u,\ v)$ variables, a crucial difference exists, and it has to be recognized into the correspondingly different behaviour of the potential walls. In the Misner scheme, the domain available to the point Universe increases as $\Omega \to \infty$, and, therefore, we deal with a non stationary infinite well; the Misner-Chitré-like variables allow to fix the infinite potential walls into a time-independent configuration in the $(u,\, 0,\, v)$ plane. However, we can at least compare the behaviour of the energy eigenvalues with respect to the occupation number n.

In his original treatment, Misner replaces, for fixed Ω values (indeed he makes use of an adiabatic approximation, see [IM06]), the triangular well by a square of equal measure and determines the energy spectrum in the form

$$\frac{E_n^M}{\hbar} = \frac{\pi}{\sqrt{S}}\sqrt{n_+^2 + n_-^2} \equiv \frac{A}{\Omega}|n|\ , \tag{3.66}$$

where S denotes the triangular well measure $S = \Omega^2/A^2$, A being a numerical factor. Above, $n_\pm$ denote the occupation numbers relative to $\beta_\pm$ respectively.

In our approach, for sufficiently large n, we get the dominant behaviors

$$\frac{E_n^{MC}}{\hbar} \sim \begin{cases} \text{i)}\frac{l\pi}{\text{productlog}(\frac{l\pi}{n})}\ , \\ \text{ii)}2n + \mathcal{O}(n^{2/3})\ . \end{cases} \tag{3.67}$$

Thus we see that, in case ii), apart from a numerical factor (which is different because of the different approximation made on the real domains), the only significant difference relies on the term Ω^{-1}; in the case i) the situation is a bit different because 2 quantum numbers explicitly remain, and we get a linear relation as far as we require $l \propto n$ by a factor much greater than the unity (indeed the function productlog$(l\pi/n)$ provides a smooth contributions in the considered region $l \gg n$). The difference of the factor Ω^{-1} with respect to Misner case can be easily accounted as soon as we observe that the following relations hold:

$$\beta_\pm = \Omega b_\pm(u,\ v)\ , \tag{3.68}$$

where the functions $b_\pm$ can be calculated from the coordinates transformations

$$\begin{cases} \Omega = -e^\tau \dfrac{1+u+u^2+v^2}{\sqrt{3}v}\ , \\ \beta_+ = e^\tau \dfrac{-1+2u+2u^2+2v^2}{2\sqrt{3}v}\ , \\ \beta_- = -e^\tau \dfrac{1+2u}{2v}\ , \end{cases} \tag{3.69}$$

but their form is not relevant in what follows. On the base of (3.69), the measure of a domain D in the $\beta_\pm$ plane reads

$$\int_D d\beta_+ d\beta_- = \Omega^2 \int_{D'} |J(u,v)| du dv \ , \tag{3.70}$$

J being the Jacobian of the transformation associated to $b_\pm$, while D' is the image of D onto the $(u,\ 0,\ v)$ plane. As a consequence, we see that between a measure s in the $\beta_\pm$ plane and a similar one (even not exactly its image), there is a difference for a factor Ω^2 which immediately provides an explanation for the difference in the two energy spectra.

3.6 On the effects of the square-root on the quantum dynamics

Now we will discuss if the presence of a non-local function, like the square-root of a differential operator, can give rise to non-local phenomena. In particular, we will study the case of a wavepacket which is non zero in a finite region of the domain and far from the infinite, and we will let the true operator act on such a function. We will show that the wave packet fails to run out to infinity in a finite time, *i.e.*, the probability exponentially goes to zero. We will develop the following analysis for the approximate domain. Since the difference between the model and the exact domain lies in the region around $v = 1/\pi$, it should have little or no effect on the behaviour at infinity.

To that end, let $\Psi(u,v)$ be a function which vanishes when $v > M$ for some constant $M > 1$. Then we can expand Ψ in a Fourier series over the u,

$$\Psi(u,v) = \sum_{k=0}^{\infty} \psi_k(v) \sin(2\pi k u) \ . \tag{3.71}$$

Let $\hat{\psi}_k(q)$ denote the Fourier transform of $\psi_k(v)$ with respect to v; then we have that the whole wave-function can be expressed as

$$\Psi(u,\ v) = \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \sin(2\pi k u) e^{iqv} \hat{\psi}_k(q) dq \ . \tag{3.72}$$

If all the functions satisfy some regularity conditions, then we can invert the expansion

$$\hat{\psi}_k(q) = \int_{-1}^{0} \int_{1/\pi}^{+\infty} \Psi(u,\ v) \sin(2\pi k u) e^{-iqv} dv du \ . \tag{3.73}$$

Let's find an upper bound to the modulus of $\hat{\psi}_k(q)$; as we choose the wave-function Ψ to be different from zero only in a region $v < M$, then we have

$$\begin{aligned}|\hat{\psi}_k(q)| \leqslant & \int_{-1}^{0}\int_{1/\pi}^{\infty} |\Psi(u,\, v)||e^{-iqv}||\sin(2\pi k u)|dvdu \leqslant \\ \leqslant & \sup\,\Psi \int_{-1}^{0}\int_{1/\pi}^{M} dvdu = \\ = & \left(M - \frac{1}{\pi}\right)\sup\,\Psi \leqslant M\sup\,\Psi\ .\end{aligned} \tag{3.74}$$

Now let's study the action of the square-root:

$$v\sqrt{\partial_u^2+\partial_v^2}\Psi(u,v) = \sum_{k=0}^{\infty}\sin(2\pi k u)\int_{-\infty}^{+\infty}\sqrt{-q^2-k^2}\hat{\psi}_k(q)e^{iqv}dq\ . \tag{3.75}$$

To evaluate this integral, we pass from the real q to the complex q', and we close the integration path from above (for the convergence of the exponential). We define a branch of the square root by a cut from ik to infinity and a cut from $-ik$ to infinity. The contour of integration can be deformed to wrap about the cut from ik to infinity, thus giving

$$\begin{aligned}v\sqrt{\partial_u^2+\partial_v^2}\Psi(u,v) = & 2\sum_{k=0}^{\infty}\sin(2\pi k u)\int_{ik}^{i\infty} e^{iq'v}v\sqrt{-k^2-q^2}\hat{\psi}_k(q')dq' = \\ = & 2i\sum_{k=0}^{\infty}\sin(2\pi k u)\int_{k}^{\infty} e^{-\xi v}v\sqrt{\xi^2-k^2}\hat{\psi}_k(i\xi)d\xi\ .\end{aligned} \tag{3.76}$$

Now we can estimate from above the value that the wave-function assumes under the action of the square-root:

$$\begin{aligned}|v\sqrt{\partial_u^2+\partial_v^2}\Psi(u,v)| \leqslant & 2\,M\sup\,\Psi\sum_{k=0}^{\infty}\int_{k}^{\infty} e^{-\xi v}v\sqrt{\xi^2-k^2}d\xi = \\ = & 2\,M\sup\,\Psi\sum_{k=0}^{\infty} kK_1(k\,v)\ .\end{aligned} \tag{3.77}$$

The Bessel functions of the third kind are positive monotonic decreasing functions and go to zero faster than an exponential, thus the series absolutely converges to a finite value for every $v > 1/\pi$.

We will calculate an upper bound on the probability of finding this new function in the region $v > M$. For great values of v, we have

$$K_1(k\,v) \sim e^{-kv}\left[\sqrt{\frac{\pi}{2k\,v}} + \sqrt{\frac{\pi}{2}}\frac{3}{8(vk)^{3/2}} + O\left(\frac{1}{v^2}\right)\right] < \sqrt{\frac{\pi}{2}}e^{-kv}\ . \tag{3.78}$$

Now we can sum the series

$$\sum_{k=0}^{\infty}\sqrt{\frac{\pi}{2}}ke^{-kv}=\sqrt{\frac{\pi}{2}}\frac{e^{v}}{(-1+e^{v})^2}\sim\sqrt{\frac{\pi}{2}}e^{-v}\ . \tag{3.79}$$

Finally, we can calculate the probability of finding the packet far away, $P(v>M)$:

$$\begin{aligned}P(v>M)&=\int_{-1}^{0}\int_{M}^{\infty}|v\sqrt{\partial_u^2+\partial_v^2}\Psi(u,v)|^2\frac{dv\,du}{v^2}<\\&<4\,M^2\sqrt{\frac{\pi}{2}}\,(\sup\,\Psi)^2\int_{-1}^{0}\int_{M}^{\infty}e^{-2v}\frac{dv\,du}{v^2}=\\&=4\,M^2\sqrt{\frac{\pi}{2}}\,(\sup\,\Psi)^2\left(\frac{e^{-2M}}{M}+\mathrm{Ei}(-2M)\right)<\\&<4\sqrt{\frac{\pi}{2}}\,(\sup\,\Psi)^2\,Me^{-2M}\end{aligned} \tag{3.80}$$

($\mathrm{Ei}(z)=-\int_{-z}^{\infty}e^{-t}/tdt$ being the exponential integral function). It can be seen that the functions that are localized in a finite region $v<M$ decay at infinity exponentially fast when hit with the square-root operator. Thus, these functions fall off more rapidly than any exponential.
We have succeeded in showing that the probability of finding a wavepacket which starts out localized will decay at infinity as $v\to\infty$. This allows us to conclude that nevertheless the square root is a non-local function, non-local phenomena don't appear (like a wavepacket that starting from a localized zone fall out to infinity).

3.7 The quantum chaos in the Mixmaster model

In [Gra94], the structure of the high energy levels is also connected to the so-called *quantum chaos* of the Mixmaster model, which was studied from the wave functional point of view in [Fur86] and from the the path integral one in [Ber89] using Misner variables. Our analysis implicitly contains the information about the quantum chaos of the considered dynamics; in fact we can take a generic state of the system $\xi(\tau,\ u,\ v)$ in the form

$$\xi(\tau,\ u,\ v)=\int dt\sum_{n}c_{t,n}\Psi_n(\tau,\ u,\ v)\delta(K_{it}(2n))e^{-i\sqrt{\frac{1}{4}+t^2}\tau}\ , \tag{3.81}$$

and its evolution from a generic initial condition $\xi_0(u,\ v)\equiv\xi(\tau_0,\ u,\ v)$ at an initial instant τ_0 provides all the quantum properties of the system. The quantum chaos is

recognized in [Fur86] by numerically integrating the Wheeler-DeWitt equation from a gaussian like initial packet and outlining the appearance of a fractal structure in the profile of the resulting wave function; in this approach (as the infinite potential walls approximation works), the dynamics is provided by (3.81) and after the evolution of the dynamics from an initial localized wave packet, the quantum chaos has to arise.
About the information on the quantum chaos emerging from the high-energy spectrum, we emphasize the following two important points: i) the this-stationary corrections due to the real potential term are expected to be simply small perturbation to our result rapidly decaying as the singularity is approached and ii) the analytic expressions we provided for large values of t cannot be used to fix the existence of the quantum chaos because they explore limited regions of the plain $(t,\ 0,\ n)$ only, but they are very useful to clarify the morphology of the spectrum and its dependence on two different quantum numbers.

3.8 The Inhomogeneous picture and conclusions

At the end of our analysis, we wish to bring the reader's attention to some physical aspects of the inhomogeneous Mixmaster.
First of all, the obtained dynamics regime is indeed a generic one. In a synchronous reference (for which $\partial_t y^a = 0$), indeed, the inhomogeneous Mixmaster contains four independent (physically) arbitrary functions of the spatial coordinates ($y^a(x^i)$ and $E(x^i)$), available for the Cauchy data on a non-singular hyper-surface.
Let us now come back to the full inhomogeneous problem, in order to understand the structure of the quantum space-time near the cosmological singularity. Since the spatial gradients of the configurational variables play no relevant dynamical role in the asymptotic limit $\tau \to \infty$ (indeed the spatial curvature simply behaves as a potential well), then the quantum evolution takes place independently in each space point, and the total wave function of the Universe can be represented as follows

$$\Xi(\tau,\ u,\ v) = \Pi_{x_i} \xi_{x_i}(\tau,\ u,\ v)\ , \tag{3.82}$$

where the product is (heuristically) taken over all the points of the spatial hypersurface.
However, it is worth recalling that, in the spirit of the "long-wavelength approximation" adopted here, the physical meaning of a space point must be recovered on the

notion of a cosmological horizon; in fact, we are dealing with regions over which the inhomogeneity effects are negligible, and this statement corresponds to super-horizon sized spatial gradients. On a classical point of view, this request is at the ground of the BKL approximation and it is well confirmed on the statistical level (see [Kir93b]); however, on a quantum level, it can acquire a precise meaning if we refer the dynamics to a kind of lattice space-time in which the spatial gradients of the configurational variables become real potential terms. In this respect, it is important to observe that the geometry of the space-time is expected to acquire a discrete structure on the Planck scale, and we believe that a regularization of this approach could arrive from a "loop quantum gravity" treatment [Boj03].
Despite this local homogeneous framework of investigation, the appearance near the singularity of high spatial gradients and of a space-time foam (like outlined in the classical dynamics, see subsection 1.3.2 on the fragmentation process) can be easily recognized in the above quantum picture too. In fact, the probability that in n different space points (horizons) the variables u and v take values within the same narrow interval, decrease with n as p^n, p being the probability in a single point; in fact, these probabilities are all identical to each other and no interference phenomenon takes place. From a physical point of view, this simple consideration indicates that a smooth picture of the large scale Universe is forbidden on a probabilistic level, and different causal regions are expected to be completely disconnected from each other during their quantum evolution. Therefore, if we start with a strongly correlated initial wave function $\Xi_0(u,\ v) \equiv \Xi(\tau_0,\ u,\ v)$, its evolution towards the singularity induces increasingly irregular distributions, approaching (3.82) in the asymptotic limit $\tau \to \infty$.
The main result of this presentation can be recognized in the clear correspondence established between the classical space-time foam and the quantum one. We have outlined how this link takes place naturally for a precise choice of the operator-ordering only and how the "energy spectrum" is a discrete one, due to the billiard structure of the point-like Hamiltonian. Finally, we fixed as a new feature the zero-point "energy" for the ground state associated to the anisotropy degrees.

Appendix 3A

The Poincaré plane and the automorphic functions

In this section, the basic notions and the most important results on a particular mathematics topic, the *automorphic functions* (for a wider and more than exhaustive introduction, see [Ter85]), will be reviewed.
Automorphic functions are the eigenfunctions of the Laplace operator on the Poincaré plane, also known as Laplace-Beltrami operator

$$\nabla_{LB}\psi(x,y) \equiv v^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) = \lambda_s \psi(x,y)\,. \tag{3A.1}$$

A quite important feature of these functions is that they are invariant under Moebius transformations. Let's introduce the complex variable $z = x + iy$; than it is easy to prove that, if $\psi(z)$ is an automorphic function, than $\psi(z')$ is an automorphic function too, where $z' = \gamma z = (az + b)/(cz + d)$, a, b, c, d being integer numbers, and γ is an element of the Moebius group. This allows one to study their properties only in what is called a *fundamental domain* of the symmetry group, *i.e.*, a portion of the Poincaré upper half-plane that can be mapped *via* the element of the groups onto the whole plane. In the Fig (3.7) it is indicated the fundamental domain of $SL_2(Z)$. For this reason they are also called "modular invariant" functions.

For these function can be defined an inner product, that naturally induced by the Poincaré metric; that is

$$(\psi, \phi) = \int \psi(x,y)\phi^*(x,y)\frac{dxdy}{y^2}\,. \tag{3A.2}$$

Let's now discuss the eigenvalues of the Laplace-Beltrami operator. First, the spectrum has both a discrete and a continuous part. If the eigenvalue equation is written in the form $\nabla_{LB}\psi(x,y) = s(1-s)\psi(x,y)$ with $s = \frac{1}{2} + it$, t real, then, for every value of t, there exists an eigenfunction $G_s(x,y)$ called Eisenstein series, while for certain discrete values of s there can also be a second eigenfunction, called a *cusp form*, which we shall denote by $v_s(x,y)$.
The Eisenstein series can be defined by the series

$$G_s(z) = \sum_{\gamma \in SL_2(Z)/\Gamma} (\Im(\gamma z))^s\,, \tag{3A.3}$$

where Γ stands for the translation subgroup of $SL_2(Z)$ (*i.e.*, the group of linear fractional maps of the form $z \to z\pm$ n). Although the expression above converges absolutely and uniformly in s, and makes it clear that $G_s(z)$ is indeed a modular invariant eigenfunction with eigenvalue $s(1-s)$, a different expression for $G_s(z)$ turns out to be more useful practically. It is

$$\begin{aligned} G_s(x,y) =& y^s + \frac{\Lambda(1-s)}{\Lambda(s)} y^{1-s} + \\ &+ \frac{2}{\Lambda(s)} \sum_{n\neq 0} |n|^{s-1/2} \sigma_{1-2s}(n) \sqrt{y} K_{s-1/2}(2\pi|n|y) e^{2\pi i n x} \;, \end{aligned} \tag{3A.4}$$

$$\sigma_s(n) = \sum_{divisors\, n} d^s \;, \tag{3A.5}$$

$$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s) \;, \tag{3A.6}$$

where K is the Bessel function, and ζ is the Riemann zeta function. It should be mentioned that $\Lambda(s)G_s(z) = \Lambda(1-s)G_{1-s}(z)$, so there is indeed only one linearly independent Eisenstein series for each eigenvalue. Also, the above expression shows that, for large y, these functions grow as a power of y. Because of this growth, these functions are not square integrable.

As mentioned earlier, for certain discrete values of s, there may appear other eigenfunctions, known as cusp forms. Unlike the Eisenstein series, for which one has the explicit formulas given above, there is no explicit formula for the cusp forms. However, if a cusp form is in the form of a series

$$v_s(z) = \sum_{n\neq 0} a_n K_{s-1/2}(2\pi|n|y) e^{2\pi i n x} \;, \tag{3A.7}$$

then the coefficients a_n will satisfy the inequality $a_n = \mathcal{O}(\sqrt{n})$. The expansion above also reveals that, as $y \to \infty$, $v_s \sim \infty$. Although there is no exact formula for the location of the eigenvalues $s(1-s)$, it is known that, for large X, the number of linearly independent cusp forms with eigenvalues $s(1-s) < X$ goes as $X/12$ (this last result can be seen via the Selberg trace formula [Gut90]). Finally, it is possible to numerically compute the eigenvalues of the cusp forms.

There is an important last result we want to bring to attention: there exists an expansion of any modular invariant function which is square integrable on the fundamental domain of $SL_2(Z)$, known as the Roelcke-Selberg formula:

$$\Psi(x,y) = \sum_{n\geq 0} (f, v_n) + \frac{1}{4\pi i} \int_{\Re(s)=1/2} (f, G_s)\, G_s(x,y) ds \;, \tag{3A.8}$$

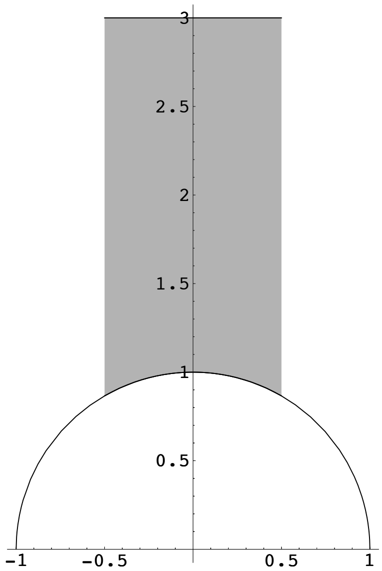

Figure 3.7:

The dashed region (open above) represents the fundamental domain of the modular group $SL_2(Z)$ in the Poincaré upper half-plane. The boundaries are three geodesics that are mapped one onto each other by a Möbius transformation. By these symmetries, all the Poincaré plane can be covered starting from this domain.

$$(\psi,\phi) := \int_{\text{fundamental domain}} \psi(x,y)\phi^*(x,y)\frac{dxdy}{y^2}\ , \tag{3A.9}$$

where $(,)$ is just the Hermitian L^2 inner product.

4 Multi-dimensional Mixmaster and the role of matter fields

We turn our attention now to the chaotic properties that the generalizations of Mixmaster show in the higher-dimensional cases and in presence of matter. We will briefly describe how chaos is a dimension-number sensitive phenomenon, and how even the presence of spatial inhomogeneity or matter fields can change this picture.
A short review of homogeneous and inhomogeneous higher dimensional models in vacuum is presented in Section 1. We will see that chaos disappears in the homogeneous case as the number of spatial dimensions is greater than three, and how it characterizes the dynamics of inhomogeneous models till nine spatial dimensions.
Multi-dimensional matter-filled space-times are then discussed in Section 2, with particular attention to the dumping effects of a scalar field; then very little is said about the more general case of the p-forms.

The original contribution constitutes the remaining of the chapter.
We consider a generic n-dimensional homogeneous model coupled to an Abelian vector field: through an ADM analysis (Section 3), a generalized Kasner behaviour is worked out as first approximation to the full dynamics (Section 4).
The standard BKL analysis can be reproduced, and, as soon as the growing terms in the potential are taken in consideration, the billiard representation is obtained (Section 5), together with both the map for the generalized Kasner exponents, both the rotation law of the Kasner axes.
Concluding remarks and one appendix follow.

The results here described were published on the international Journal *Classical and Quantum Gravity* during 2005 [BKM05]; the publication is appended at end of this PhD thesis (Attachment 2).

4.1 Mixmaster in higher-dimensional models

Homogeneous models

In a number of spatial-dimensions greater than three, homogeneous models lose their chaotic dynamics, as can be seen in the four dimensional case already (see Appendix 4A for notations and more).
The question of chaos in higher dimensional cosmologies has been widely investigated, and many authors [BSS85, Fur86, Hal03, Hal02] showed that none of higher-dimensional extensions of the Bianchi IX model possesses chaotic features. The main difference relies in the finite number of oscillations characterising the dynamics near the singularity.
Among the five-dimensional homogeneous space-times, $G13$ is the analogous of the Bianchi type IX: the structure constants are the same for both the models.
Einstein equations can be written down, and they read

$$\begin{aligned} 2\alpha_{,\tau,\tau} &= \left[(b^2-c^2)^2-a^4\right]d^2\ , \\ 2\beta_{,\tau,\tau} &= \left[(a^2-c^2)^2-b^4\right]d^2\ , \\ 2\gamma_{,\tau,\tau} &= \left[(b^2-a^2)^2-c^4\right]d^2\ , \\ \delta_{,\tau,\tau} &= 0\ , \end{aligned} \tag{4.1}$$

$$\begin{aligned} \alpha_{,\tau,\tau}+\beta_{,\tau,\tau}+\gamma_{,\tau,\tau}+\delta_{,\tau,\tau} &= 2\alpha_{,\tau}\beta_{,\tau}+2\alpha_{,\tau}\gamma_{,\tau}+ \\ &= 2\alpha_{,\tau}\delta_{,\tau}+2\beta_{,\tau}\gamma_{,\tau}+2\beta_{,\tau}\delta_{,\tau}+2\gamma_{,\tau}\delta_{,\tau}\ . \end{aligned} \tag{4.2}$$

If we assume that the BKL approximation is valid, *i.e.*, that the right-hand sides of eqns (4.1) are negligible, then the asymptotic solution for $\tau\to\infty$ is five-dimensional Kasner-like

$$ds^2 = dt^2-\sum_{i=1}^{4} t^{2p_i}(dx^i)^2\ , \tag{4.3}$$

with the Kasner exponents p_i that satisfy the generalized Kasner relations

$$\sum_{i=1}^{4} p_i = \sum_{i=1}^{4} p_i^2 = 1\ . \tag{4.4}$$

This regime can hold only until the BKL approximation works; however, as τ approaches the singularity, one or more of the terms may increase. Let's assume p_1 to be the smallest index; then $a = \exp(\alpha)$ is the greatest contribution and we can neglect all the other terms, obtaining

$$\begin{aligned}
\alpha_{,\tau,\tau} &= -\frac{1}{2}\exp(4\alpha+2\delta)\ , \\
\beta_{,\tau,\tau} &= \gamma_{,\tau,\tau} = \frac{1}{2}\exp(4\alpha+2\delta)\ , \\
\delta_{,\tau,\tau} &= 0\ .
\end{aligned} \tag{4.5}$$

As soon as the asymptotic limits for $\tau \to \pm\infty$ are considered, the following map follows from solution to (4.5)

$$p'_1 = -\frac{p_1+p_4}{1+2p_1+p_4}\ , \qquad p'_2 = \frac{p_2+2p_1+p_4}{1+2p_1+p_4}\ ,$$

$$p'_3 = \frac{p_3+2p_1+p_4}{1+2p_1+p_4}\ , \qquad p'_4 = \frac{p_4}{1+2p_1+p_4}\ ; \tag{4.6}$$

$$abcd = \Lambda' t\ , \qquad \Lambda' = (1+2p_1+p_4)\ .\Lambda \tag{4.7}$$

What makes this dynamical scheme different from the four dimensional case are the conditions needed to undergo a transition: a quite simple analysis of the behaviour of the potential terms in (4.1) shows that two of the four parameters must satisfy

$$1 - 3p_1^2 - 3p_2^2 - 2p_1p_2 + 2p_1 + 2p_2 \geq 0\ , \tag{4.8}$$

and one of the following

$$\begin{aligned}
&3p_1^2 + p_2^2 + p_1 - p_2 - p_1p_2 < 0\ , \\
&3p_2^2 + p_1^2 + p_2 - p_1 - p_1p_2 < 0\ , \\
&3p_1^2 + p_2^2 - 5p_1 - 5p_2 + 5p_1p_2 + 2 < 0\ ,
\end{aligned} \tag{4.9}$$

for a transition to occur. In Fig. (4.1), the existence of a region where (4.8) is satisfied but none of (4.9) is shown. This way, the Universe undergoes a certain number of transitions and Kasner epochs and eras; as soon as the Kasner indexes p_1, p_2 take values in the shaded region, then no more transitions take place, and the evolution remains Kasner like until the singular point is reached.

Type $G14$ is quite similar to $G13$: for this model, the structure constants are the same as Bianchi type VIII, and, under the same hypothesis, only a finite sequence of epochs occurs.

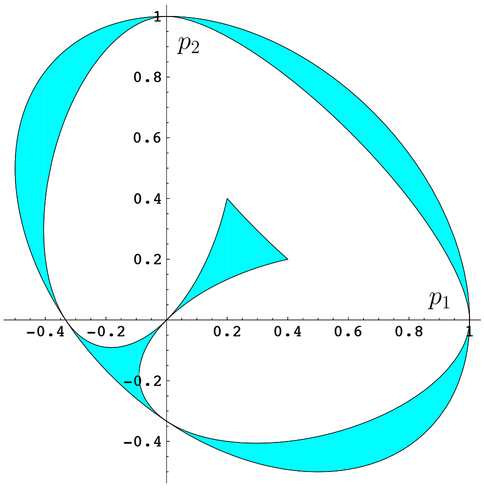

Figure 4.1:

The shaded region corresponds to all the couples (p_1, p_2) that do not satisfy (4.9): as soon as a couple (p_1, p_2) takes value in that portion, then the BKL mechanism breaks down, and the Universe experiences a last Kasner epoch till the singular point

Inhomogeneous models.

Let us consider a $(d+1)$-dimensional space-time ($d \geq 3$), whose associated metric tensor obeys to a dynamics described by a generalized Einstein-Hilbert action

$$^{(d+1)}R_{ik} = 0 \ , \qquad (i, k = 0, 1, ..., d) \ , \tag{4.10}$$

where the $(d+1)$-dimensional Ricci tensor takes its natural form in terms of the metric components $g_{ik}(x^l)$.

In [DHH$^+$86], it is shown that, within the framework of the Einstein theory, the inhomogeneous Mixmaster behaviour finds a direct generalization in correspondence to any value of d. It is also shown that, in correspondence to $d > 9$, the generalized Kasner solution acquires a generality character (in the sense of the correct number of arbitrary functions)

In a synchronous reference (described by usual coordinates (t, x^γ)), the time-evolution of the d-dimensional spatial metric tensor $h_{\alpha\beta}(t, x^\gamma)$ singles out, near the cosmological singularity ($t = 0$), an iterative structure. Each single stage consists of intervals of time (Kasner epochs) during which tensor $h_{\alpha\beta}$ takes the generalized Kasner form

$$h_{\alpha\beta}(t, x^\gamma) = \sum_{i=1}^{d} t^{2p_i} l^i_\alpha l^i_\beta \ , \tag{4.11}$$

where the Kasner index functions $p_i(x^\gamma)$'s have to satisfy the conditions

$$\sum_{i=1}^{d} p_i(x^\gamma) \;=\sum_{i=1}^{d} p_i^2(x^\gamma) \;= 1 \;, \tag{4.12}$$

and $l^1(x^\gamma), ..., l^d(x^\gamma)$ denote d linear independent forms $l^i = m^i_j dx^j$, whose components are arbitrary functions of the spatial coordinates.
In each point of the space, the conditions (4.12) define a set of ordered indexes $\{p_i\}$ ($p_1 \leq p_2 \leq ... \leq p_d$) which, from a geometrical point of view, fixes one point in R^d, lying on a connected portion of a $(d-2)$-dimensional sphere. We note that the validity of the conditions (4.12) requires $p_1 \leq 0, p_{d-1} \geq 0$, where the equality takes place only for the values $p_1 = ... = p_{d-1} = 0, p_d = 1$.

As shown in [DHS85, HGJSS87] (see also [Kir93a, KM95]), each single step of this iterative solution results to be stable, in a given point of the space, if

$$\lim_{t\to 0} t^2 \; {}^dR = 0 \; . \tag{4.13}$$

Eq. (4.13) is a sufficient condition that allows us to disregard the spatial gradients in the Einstein equations.
An elementary computation indicates that the only "dangerous terms" in $t^2 \; {}^dR$ contain the powers $t^{2\alpha_{ijk}}$, where the exponents α_{ijk} are simply related to the Kasner exponents through

$$\alpha_{ijk} = 2p_i + \sum_{l\neq i,j,k} p_l \;, \qquad (i \neq j, i \neq k, j \neq k), \qquad (i, j, k = 1, ..., d) \; . \tag{4.14}$$

For generic functions l, all possible powers $t^{2\alpha_{ijk}}$ appear in $t^2 \; {}^dR$.
This leaves two possibilities for $t^2 R^a_b$ to vanish as $t \to 0$. Either the Kasner exponents can be chosen in an open region of the Kasner sphere defined above (4.12), so as to make α_{ijk} positive for all triples i, j, k. Or the conditions

$$\forall (x^1, ..., x^d) : \alpha_{ijk}(x^\gamma) > 0 \;\; (i \neq j, i \neq k, j \neq k) \;\; (i, j, k = 1, ..., d) \; , \tag{4.15}$$

are in contradiction with (4.12), and one must impose "extra" conditions on the functions l and their derivatives. As we discussed in details in previous chapters, the second possibility occurs when $d = 4$, since α_{ijk} is then given by $2p_i$, and one Kasner exponent is always negative. This implies that (4.12) is a solution of the vacuum

Einstein equations to leading order if and only if the vector l^1, associated with the negative Kasner exponent p_1, obeys the "extra" condition

$$l_1 \cdot \text{ curl } l_1 = 0 \,, \tag{4.16}$$

and this kills one arbitrary function.

It can be shown [DHS85, HGJSS87] that, for $3 \leq d \leq 9$, at least the smallest of the quantities (4.14), i.e. $\alpha_{1,d-1,d}$ results to be always negative (excluding isolated points $\{p_i\}$ in which it vanishes); for $d \geq 10$, on the contrary, there exists an open region of the $(d-2)$-dimensional Kasner sphere where this same quantity takes positive values, the so-called Kasner Stability Region (KSR).

As a consequence, for $3 \leq d \leq 9$, the evolution of the system to the singularity consists of an infinite number of Kasner epochs; instead for $d \geq 10$, the existence of the KSR, implies a profound modification in the asymptotic dynamics. In fact, the indications, presented in [DHH$^+$86, KM95] in favor of the "attractivity" of the KSR, imply that in each space point (excluding sets of zero measure) a final stable Kasner-like regime appears.

Finally we stress that, in correspondence to any value of d, the considered iterative scheme contains the right number of $(d+1)(d-2)$ physically arbitrary functions of the spatial coordinates, required to specify generic initial conditions (on a non-singular spacelike hypersurface). This calculation is very simple: we have d^2 functions from the d vectors l and $d-2$ Kasner exponents; from these functions we have to eliminate d, because of invariance under spatial reparametrizations, and d because of the $0a$ Einstein equations. Therefore, this piecewise solution describes the asymptotic evolution of a generic inhomogeneous multidimensional cosmological model.

4.2 Matter-filled spaces

The case of the scalar field

Let ds^2 be the line element of the diagonal Bianchi type IX cosmology, that in Misner variables $(\Omega,\ \beta_+,\ \beta_-)$ reads

$$ds^2 = -dt^2 + e^{2\Omega}(e^{2\beta})_{ij}\sigma^i\sigma^j \,. \tag{4.17}$$

Furthermore, let $\phi(t)$ be a scalar field minimally coupled to gravity, and $\mathcal{V}(\phi)$ a generic potential. All the dynamics is contained in super-Hamiltonian H, that explicitly reads

$$2H = -p_\Omega^2 + p_+^2 + p_-^2 + p_\phi^2 + V(\Omega, \beta_\pm) + e^{6\Omega}\mathcal{V}(\phi) \ . \tag{4.18}$$

Here $V(\Omega, \beta_\pm)$ is proportional to the curvature: $V(\Omega, \beta_\pm) = \ {}^3h\ {}^3R$, and p_i are the momenta conjugate to the generalized positions i, $i = \beta_\pm, \Omega, \phi$.
The singularity occurs as $\Omega \to -\infty$; this implies that the potential $\mathcal{V}(\phi)$ is not important during the asymptotic phase, unless it depends exponentially on Ω. We will assume that it can be ignored, and we will retain only the curvature term.
As soon as we make the BKL assumptions on the curvature term, the solution to this model can be easily found, and the metric tensor so obtained is Kasner-like, even if the indexes p_i's satisfy different properties

$$ds^2 = dt^2 - \sum_{i=1}^{3} t^{2p_i}(dx^i)^2 \ , \tag{4.19}$$

$$p_1 + p_2 + p_3 = 1 \ , \qquad p_1^2 + p_2^2 + p_3^2 = 1 - q^2 \ , \tag{4.20}$$

where q is related to the evolution of the scalar field: $\phi(t) = q \ln(t) + \phi_0$.
Even if the difference is very small, the effect is really interesting: the insertion of q in the second Kasner relation completely destroys the Mixmaster dynamics. This can be easily realized as soon as the possible values for the indexes p_i's are studied. In order to perform this analysis, we introduce a parametric representation [BK73]

$$\begin{aligned} p_1 &= \frac{-u}{1+u+u^2} \ , \\ p_2 &= \frac{1+u}{1+u+u^2}\left[u - \frac{u-1}{2}\left(1 - \sqrt{1-\beta^2}\right)\right] \ , \\ p_3 &= \frac{1+u}{1+u+u^2}\left[1 - \frac{u-1}{2}\left(1 - \sqrt{1-\beta^2}\right)\right] \ , \end{aligned} \tag{4.21}$$

$$\beta \equiv \sqrt{2}\frac{(1+u+u^2)\,q}{(u^2-1)} \ . \tag{4.22}$$

The usual indexes (1.4) are obtained as q is taken equal to zero. We observe also that the inversion property is conserved

$$p_1\left(\frac{1}{u}\right) = p_1(u) \ , \quad p_2\left(\frac{1}{u}\right) = p_3(u) \ , \quad p_3\left(\frac{1}{u}\right) = p_2(u) \ ; \tag{4.23}$$

thus the analysis can be restricted to region $-1 \leqslant u \leqslant 1$.
From (4.20) we obtain that the admissible values for u, q are to be taken in the region defined by

$$\beta^2 = 2\frac{(1+u+u^2)^2 q^2}{(u^2-1)^2} \leqslant 1 \; . \tag{4.24}$$

As it was discussed in the first chapter, the BKL mechanism is based on the structure of the curvature term: it grows like $\sim t^{2p_i}$ as $t \to 0$, and because one of the three Kasner exponents is negative, there is always a growing term. As a scalar field is inserted in the dynamics, this last result does not hold anymore.
The region defined by the relations $p_1 \geq 0,\ p_2 \geq 0,\ p_3 \geq 0$ is shown in Fig (4.2): as soon as the map takes value in the dashed region, then the Kasner era lasts until the singularity is reached . The existence of this region inhibits the BKL dynamics from

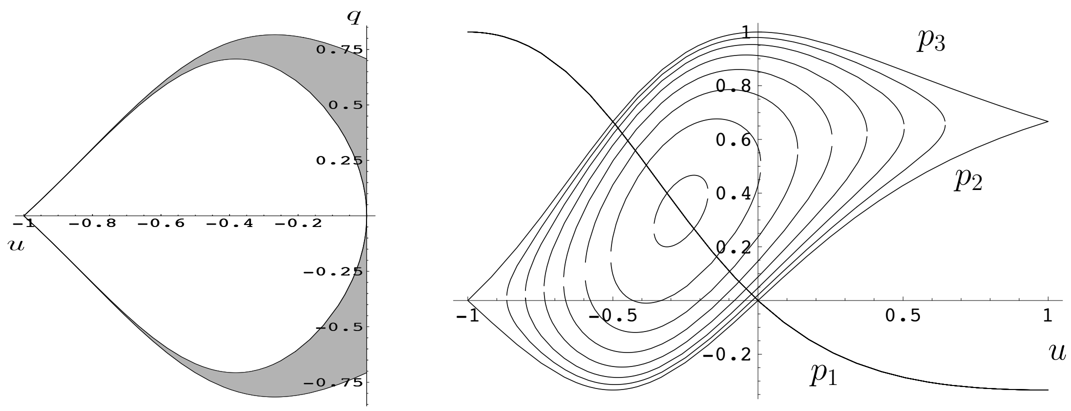

Figure 4.2:

In the graphic on the left it is shown the existence of a region where all the p_i's are greater than zero. A Kasner era, characterized by a couple $(u,\, v)$ in that region, lasts until the singularity is reached.

On the right instead the different curves of the Kasner indexes are plotted for several values of the parameter q.

lasting forever. The Universe undergoes changes until the Kasner indexes take values out of this domain; as the p_i's are mapped in the dashed region, the potential stops to grow and the following Kasner era results to be stable.
A different point of view on the question of chaos is that of "consistent potential (MCP)", originally developed in [GcvacM93] and used in [BGS97, Ber99] to explain how a minimally coupled classical scalar field can suppress Mixmaster oscillations in the approaching to the singularity of generic cosmological spacetimes. It is possible

to retain the Mixmaster behaviour, according to the MCP, if $\mathcal{V}(\phi)$ is such that

$$e^{6\Omega}\mathcal{V}(\phi) = a^2 e^{\alpha\phi} + b^2 e^{-\zeta\phi} \; , \tag{4.25}$$

where α, $\zeta > 0$. Coupling between scalar field and gravitational degrees of freedom in $e^{6\Omega}\mathcal{V}(\phi)$ could lead to very complicated behaviour (potentials like (4.25) can arise in string theory [LP96]).

The case of the p-forms

We complete this review on the effects of matter in higher-dimensional models with a brief discussion about the insertion of the p-forms and dilatons in the gravitational dynamics. This topic is quite wide, and a good review is, for example, [DHN03].

The inclusion of massless p-forms, in a generic multi-dimensional model [DH00a, DH00b], can restore chaos when it is otherwise suppressed. In particular, even though pure gravity is non-chaotic in 11 spacetime dimensions, the three-form of $D = 11$ supergravity renders the system chaotic (those p-forms are part of the low-energy bosonic sector of superstring/M-theory models).

We will not enter in any detail, but we want to stress that the billiard description we gave in the four dimensional case is quite general, and can be extended to higher spacetime dimensions, with p-forms and dilatons [IMK94, DH01]. If $d \equiv D - 1$ is the number of spatial dimensions, and if there are n dilatons, the billiard is a region of the hyperbolic space H_{d+n-1}, each dilaton being equivalent, in the Hamiltonian, to the logarithm of a new scale factor.

The other ingredients, that enter billiard definition, are the walls that bound it, that in these case can be of different types: symmetry walls related to the off-diagonal components of the spatial metric, gravitational walls related to the spatial curvature, and p-form walls (electric and magnetic); all these walls are hyperplanar, and the billiard is a convex polyhedron with finitely many vertices, some of which are at infinity.

4.3 n-dimensional homogeneous models coupled to an Abelian vector field

We will discuss this model in an ADM framework; let's set down the basic equations. Let us consider a vector field $A_\mu = (\varphi, A_\alpha)$, $(\alpha = 1, 2, \ldots, n)$, and let's adopt for the metric the standard ADM representation generalized to n-dimensional manifold

$$ds^2 = N^2 dt^2 - h_{\alpha\beta}\left(dx^\alpha + N^\alpha dt\right)\left(dx^\beta + N^\beta dt\right). \tag{4.26}$$

Lagrangian of this vector-tensorial system is the sum of the "electromagnetic" term plus the gravitational part

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{EM} \ , \tag{4.27}$$

$$\mathcal{L}_G = -\int d^n x dt \sqrt{-g}\ {}^{n+1}R \ , \tag{4.28}$$

$$\mathcal{L}_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ , \qquad \mu,\ \nu = 0, .., n \ , \tag{4.29}$$

$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the electromagnetic n-dimensional tensor.
A standard Legendre transformation yields the Hamiltonian of the model

$$I = \int d^n x dt \left\{ \pi^{\alpha\beta} \frac{\partial}{\partial t} h_{\alpha\beta} + \pi^\alpha \frac{\partial}{\partial t} A_\alpha + \varphi D_\alpha \pi^\alpha - NH - N^\alpha H_\alpha \right\} . \tag{4.30}$$

Here,

$$H = \frac{1}{\sqrt{h}} \left\{ \pi^\alpha_\beta \pi^\beta_\alpha - \frac{1}{n-1} \left(\pi^\alpha_\alpha\right)^2 + \frac{1}{2} h_{\alpha\beta} \pi^\alpha \pi^\beta + h \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - R \right) \right\} \ , \tag{4.31}$$

$$H_\alpha = -\nabla_\beta \pi^\beta_\alpha + \pi^\beta F_{\alpha\beta} \ , \tag{4.32}$$

denote, respectively, the super-Hamiltonian and super-momentum; $F_{\alpha\beta}$ is the spatial electromagnetic tensor, $h \equiv \det(h_{\alpha\beta})$ is the determinant of the n-metric, R is the n-scalar of curvature constructed by the metric $h_{\alpha\beta}$, and $D_\alpha \equiv \partial_\alpha + A_\alpha$. π^α and $\pi^{\alpha\beta}$ are the momenta conjugate to the electromagnetic field and to the n-metric, respectively; they result to be a vector and a tensorial density of weight 1/2, because their expressions contain the square root of the determinant of the spatial metric

$$\pi^{\alpha\beta} = \frac{\sqrt{h}}{N} \left(K^{\alpha\beta} - h^{\alpha\beta} \mathrm{Tr}(K)\right) , \tag{4.33}$$

$$\pi^\alpha = \frac{\sqrt{h}}{N} \left(\frac{\partial A_\beta}{\partial t} h^{\alpha\beta} - N^\beta F_\beta{}^\alpha \right) \ , \tag{4.34}$$

(here $K^{\alpha\beta}$ denotes the n-dimensional extrinsic curvature tensor in the synchronous frame). Variation with respect to the lapse function N yields the Super-Hamiltonian constraint

$$H = 0\,, \tag{4.35}$$

while its variation with respect to φ provides the constraint $\partial_\alpha \pi^\alpha = 0$.
We will deal with a source-less Abelian vector field; it is then enough to consider only the transverse (or Lorentz) components for A_α and π^α. Therefore, we take the gauge conditions $\varphi = 0$ and $D_\alpha \pi^\alpha = 0$, and this will be enough to exclude the longitudinal parts of the vector field from the action.
It is worth noting how, in the general case, *i.e.*, either in presence of the sources, or in the case of non-abelian vector fields, this simplification can no longer take place in such explicit form, and the terms $\varphi(\partial_\alpha + A_\alpha)\pi^\alpha$ must be considered in the action principle.

4.3.1 Homogeneous cosmological models

As homogeneity is considered, the whole spatial dependence of the model can be integrated out from the action, and the dynamical variables become only time dependent. In general, neglecting in the total Hamiltonian the spatial derivatives contained in $F_{\alpha\beta}$ and R, is equivalent to the generalized Kasner approximation.
When going over the homogeneous case, we choose the gauge $N^\alpha = 0$ and, within the Kasner approximation, we get equations of motion for the vector field having the form

$$E_\alpha = \frac{\partial}{\partial t} A_\alpha = \frac{N}{\sqrt{h}} h_{\alpha\beta} \pi^\alpha, \tag{4.36}$$

$$\frac{\partial}{\partial t} \pi^\alpha = 0. \tag{4.37}$$

The field equations which describe the n-metric dynamics read as follows

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \frac{2N}{\sqrt{h}} \left\{ \pi_{\alpha\beta} - \frac{1}{n-1} h_{\alpha\beta} \pi^\gamma_\gamma \right\}, \tag{4.38}$$

$$\frac{\partial}{\partial t} \pi^\alpha_\beta = -\frac{N}{2\sqrt{h}} \pi^\alpha \pi_\beta. \tag{4.39}$$

This dynamical scheme is completed by adding to the above Hamiltonian equations the super-Hamiltonian constraint (4.35).

4.4 Kasner-like behaviour

We propose a generalized Kasner-like treatment of the model by introducing a spatial vielbein l_α, and projecting all vectors and tensorial quantities along them

$$h_{\alpha\beta} = \delta_{ab} l^a_\alpha l^b_\beta, \qquad \pi_{\alpha\beta} = p_{ab} l^a_\alpha l^b_\beta \,. \tag{4.40}$$

The vielbein can be chosen in such a way that the matrix p_{ab} is diagonal:
$p_{ab} = \text{diag } (p_1, \dots, p_n)$.
A dual basis L_α can be defined by requiring

$$\begin{aligned} L^\alpha_a &= h^{\alpha\beta} l^a_\beta \,, \\ L^\alpha_a l^b_\alpha &= \delta^b_a \,, \\ L^\alpha_a l^a_\beta &= \delta^\alpha_\beta \,. \end{aligned} \tag{4.41}$$

Let's project (4.38) along the Kasner vectors defined in (4.40) and (4.41); in this way we get the following dynamical system:

$$\left(L_a \frac{\partial}{\partial t} l_a \right) = \frac{N}{\sqrt{h}} \left(p_a - \frac{1}{n-1} \sum_b p_b \right) , \tag{4.42}$$

$$\left(L_a \frac{\partial}{\partial t} l_b \right) + \left(L_b \frac{\partial}{\partial t} l_a \right) = 0, \qquad a \neq b. \tag{4.43}$$

Here $(L_a l_b) = L^\alpha_a l_{b\alpha}$ denotes the ordinary vector product, treating the vector components as Euclidean ones.
In close analogy with above, from (4.39), we find the additional equations

$$\frac{\partial}{\partial t} p_a = -\frac{N}{2\sqrt{h}} \lambda_a^2 \,, \qquad \lambda_a = \left(\pi^\alpha l^a_\alpha \right) \,, \tag{4.44}$$

$$\left(L_b \frac{\partial}{\partial t} l_a \right) = -\frac{N}{2\sqrt{h}} \frac{\lambda_a \lambda_b}{p_a - p_b} \,, \qquad a \neq b. \tag{4.45}$$

In particular, we see that (4.45) is anti-simmetric under permutation of the indexes; thus it already contains (4.43).
By combining together both systems, we obtain (4.44) as the first independent equation; the equation for the Kasner vectors takes place in the form (for the sake of simplicity, we neglect the vector index α)

$$\frac{\partial}{\partial t} l_a = \frac{N}{\sqrt{h}} \left\{ \left(p_a - \frac{1}{n-1} \sum_b p_b \right) l_a - \frac{1}{2} \sum_{b \neq a.} \frac{\lambda_a \lambda_b}{p_a - p_b} l_b \right\} . \tag{4.46}$$

Kasner dynamics is characterized by an anisotropic evolution of the linear distances along different directions; in order to unravel that behaviour, let us distinguish scale functions q^a in the vielbein

$$l_a = \exp\left(q^a/2\right)\ell_a\,, \tag{4.47}$$

$$L_a = \exp\left(-q^a/2\right)\mathcal{L}_a\,. \tag{4.48}$$

The vectors ℓ_a will be the Kasner axes.
Using this new set of dynamical variables, (4.44) and (4.46) become

$$\frac{\partial}{\partial t}p_a = -\frac{N}{2\sqrt{h}}\widetilde{\lambda}_a^2\exp\left(q^a\right)\,,\qquad \widetilde{\lambda}_a = \left(\pi^\alpha \ell_\alpha^a\right), \tag{4.49}$$

$$\frac{\partial}{\partial t}q^a = \frac{2N}{\sqrt{h}}\left(p_a - \frac{1}{n-1}\sum_b p_b\right), \tag{4.50}$$

$$\frac{\partial}{\partial t}\ell_a = -\frac{N}{2\sqrt{h}}\sum_{b\neq a.}\frac{\widetilde{\lambda}_a\widetilde{\lambda}_b}{p_a-p_b}\exp\left(q^b\right)\ell_b = \sum_{b\neq a.}\frac{\widetilde{\lambda}_a\left(\frac{\partial}{\partial t}p_b\right)}{\widetilde{\lambda}_b\left(p_a-p_b\right)}\ell_b\,. \tag{4.51}$$

The last two equations are obtained by substituting variables (4.47) and then projecting (4.46) on the vectors L_c^β; we get (4.50) as soon as the index c is taken equal to a, otherwise we get (4.51). We have to stress that the second equation is an approximated one: we have in fact ignored an higher-order term with respect to the Kasner approximation.
The dynamical scheme is complete as the Hamiltonian constraint is considered

$$H = 0 = \frac{N}{\sqrt{h}}\left\{\sum p_a^2 - \frac{1}{n-1}\left(\sum p_a\right)^2 + \frac{1}{2}\sum e^{q_a}\widetilde{\lambda}_a^2\right\}. \tag{4.52}$$

Requiring the quantities $\widetilde{\lambda}_a$ be constants implies that vectors ℓ^a do not depend on time in turn; therefore no Kasner vectors rotation can take place. Such a situation corresponds exactly to the n-dimensional Kasner behavior [DHHS85, DHH+86]. Since the rotation of Kasner vectors is induced by an higher order term, for $h \to 0$, with respect to the pure Kasner dynamics, we can assume that $\widetilde{\lambda}_a$ are near to be constant, and, therefore, near the singularity, (4.49), (4.50) give the billiards on $(n-1)$ −dimensional Lobachevsky space, exactly like in the 3-dimensional Mixmaster case.
By other words, in this scheme, the evolution is not simply Kasner like, but we get a dynamical picture in which the point Universe moves according to a piecewise Kasner

solution, as we will see later.

In order to analyze the time-evolution of Kasner vectors, let's project them on the time independent quantities π^{α}, and split them in two parts

$$\vec{\ell}_a = \vec{\ell}_{a\|} + \vec{\ell}_{a\perp} \ ,$$

$$\vec{\ell}_{a\|} = \frac{\widetilde{\lambda}_a}{\pi^2}\vec{\pi} \, , \qquad \left(\vec{\pi}\vec{\ell}_{a\perp}\right) = 0 \ . \tag{4.53}$$

Here, for the sake of simplicity, we use vector notation, which is useful when treating the components of these vectors as Euclidean ones.

Hence, we can split (4.51) into the two independent components

$$\frac{\partial}{\partial t}\widetilde{\lambda}_a = \sum_{b\neq a.} \frac{\left(\frac{\partial}{\partial t}p_b\right)\widetilde{\lambda}_a}{\widetilde{\lambda}_b\left(p_a - p_b\right)}\widetilde{\lambda}_b \ , \tag{4.54}$$

$$\frac{\partial}{\partial t}\vec{\ell}_{a\perp} = \sum_{b\neq a.} \frac{\left(\frac{\partial}{\partial t}p_b\right)\widetilde{\lambda}_a}{\widetilde{\lambda}_b\left(p_a - p_b\right)}\vec{\ell}_{b\perp} = A^a_b\left(t\right)\vec{\ell}_{b\perp} \ . \tag{4.55}$$

Matrix A^a_b does not depend on $\vec{\ell}_{a\perp}$; then the following formal solution holds

$$\vec{\ell}_{a\perp}\left(t\right) = T\exp\left\{\int\limits_{t_0}^{t} A^a_b\left(t'\right)dt'\right\}\vec{\ell}_{b\perp}\left(t_0\right) \ . \tag{4.56}$$

The remaining equations (4.54), together with (4.49) and (4.50), provide a self-consistent dynamical system.

We assume that is possible to neglect the contributions of the n-dimensional Ricci scalar to the dynamics. This means that the limit in which all the terms $\exp\left(q^a\right)$ become of higher order has to be taken. The dynamics is then described by the following simplified system

$$\begin{gathered}
p_a = const, \\
\widetilde{\lambda}_a = const, \\
\vec{\ell}_{a\perp} = const, \\
\frac{\partial}{\partial t}q_a = \frac{2N}{\sqrt{h}}\left(p_a - \frac{1}{n-1}\sum_b p_b\right), \\
\sum p_a^2 - \frac{1}{n-1}\left(\sum p_a\right)^2 + \frac{1}{2}\sum e^{q_a}\widetilde{\lambda}_a^2 = 0 \ .
\end{gathered} \tag{4.57}$$

We choose the gauge $N = 1$. In the neighbourhood of the singular point, the solution of (4.57) is Kasner-like

$$\begin{cases} h_{\alpha\beta} = \sum_a t^{2s_a} \ell^a_\alpha \ell^a_\beta \,, \\ s_a = 1 - (n-1)\dfrac{p_a}{\sum_b p_b} \,, \end{cases} \tag{4.58}$$

where the Kasner indexes s_a satisfy the generalized Kasner-like identities

$$\sum_{a=1}^{n} s_a = \sum_{a=1}^{n} s_a^2 = 1. \tag{4.59}$$

4.5 Billiard representation: the return map and the rotation of Kasner vectors

We will now unravel the BKL mechanism in this n-dimensional homogeneous model. Let's take the Kasner indexes in the increasing order

$$s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n \,. \tag{4.60}$$

This implies that $s_1 < 0$ and $s_n \geq s_{n-1} \geq 0$. The singular point appears as soon as the limit for $t \to 0$ is taken. Like in the standard four-dimensional Mixmaster model, there exists an instant t_{tr} from which we cannot neglect anymore all $\exp(q_a)$, but we have to retain in (4.49)-(4.51) at least the greatest term, *i.e.*, t^{2s_1}

The field equations (4.49) and (4.50) rewrite

$$\begin{cases} \dfrac{\partial}{\partial t}\widetilde{\lambda}_1 = 0 \,, \\ \dfrac{\partial}{\partial t}\widetilde{\lambda}_a = \dfrac{\left(\dfrac{\partial}{\partial t}p_1\right)\widetilde{\lambda}_a}{(p_a - p_1)} \,, \quad a \neq 1 \,, \\ \dfrac{\partial}{\partial t}p_a = 0 \,, \quad a \neq 1, \\ \dfrac{\partial}{\partial t}p_1 = -\dfrac{N}{2\sqrt{h}}\widetilde{\lambda}_1^2 \exp\left(q^1\right) \,, \\ \dfrac{\partial}{\partial t}q_a = \dfrac{2N}{\sqrt{h}}\left(p_a - \dfrac{1}{n-1}\displaystyle\sum_b p_b\right) . \end{cases} \tag{4.61}$$

The first of equations (4.61) gives $\widetilde{\lambda}_1 = const$, while the second admits the solution

$$\widetilde{\lambda}_a\,(p_a - p_1) = const. \tag{4.62}$$

The remaining part of the dynamical system allows us to determine the return map governing the replacements of Kasner epochs. This is obtained in the standard way, *i.e.*, imposing a Kasner-like evolution on the solution of system (4.61) and matching the asymptotic behaviour. This procedure yields the following map

$$\begin{cases} s_1' = \dfrac{-s_1}{1+\frac{2}{n-2}s_1} \,, \\ s_a' = \dfrac{s_a+\frac{2}{n-2}s_1}{1+\frac{2}{n-2}s_1} \,, \\ \widetilde{\lambda}_1' = \widetilde{\lambda}_1 \,, \\ \widetilde{\lambda}_a' = \widetilde{\lambda}_a \left(1 - 2\dfrac{(n-1)\,s_1}{(n-2)\,s_a + ns_1}\right) \,. \end{cases} \tag{4.63}$$

We definine the quantities $k_a = p_a - \dfrac{1}{n-1}\sum p_b$, and find for them the following iteration law

$$k_1' = -k_1 \,, \tag{4.64}$$

$$k_a' = k_a + \frac{2}{n-2}k_1 \,, \tag{4.65}$$

$$\sum_{i=1}^{n} k_i' = \sum_{i=1}^{n} k_i + \frac{2}{n-2}k_1 \,. \tag{4.66}$$

Our analysis is completed by investigating the rotation of Kasner vectors $\vec{\ell}_a$ through the epochs replacements; by (4.55) we get the following system

$$\begin{cases} \dfrac{\partial}{\partial t}\vec{\ell}_{1\perp} = 0 \,, \\ \dfrac{\partial}{\partial t}\vec{\ell}_{a\perp} = \left(\dfrac{\partial}{\partial t}p_1\right)\dfrac{\widetilde{\lambda}_a}{\widetilde{\lambda}_1\,(p_a - p_1)}\vec{\ell}_{1\perp}, \end{cases} \tag{4.67}$$

admitting the integral

$$\widetilde{\lambda}_1\vec{\ell}_{a\perp} - \widetilde{\lambda}_a\vec{\ell}_{1\perp} = const. \tag{4.68}$$

Putting together (4.53), (4.63) and (4.67), we arrive to the final iteration law

$$\vec{\ell}_a' = \vec{\ell}_a + \sigma_a\vec{\ell}_1 \,,$$

$$\sigma_a = \frac{\widetilde{\lambda}_a' - \widetilde{\lambda}_a}{\widetilde{\lambda}_1} = -2\frac{(n-1)\,s_1}{(n-2)\,s_a + ns_1}\frac{\widetilde{\lambda}_a}{\widetilde{\lambda}_1} \,. \tag{4.69}$$

which completes our dynamical scheme.

Thus the homogeneous Universe here discussed approaches the initial singularity being described by a metric tensor with oscillating scale factors and rotating Kasner vectors. Passing from one Kasner epoch to another one, the negative Kasner index s_1 is exchanged between different directions (for istance $\vec{\ell}_1$ and $\vec{\ell}_2$) and, at the same time, these directions rotate in the space according to the law (4.69). The presence of a vector field is crucial because, independently of the considered model, it induces a (dynamically [BM04]) closed domain on the configuration space.
In correspondence to these oscillations of the scale factors, the Kasner vectors $\vec{\ell}_a$ rotate, and, at the lowest order in q^a, the quantities σ_a remain constant along a Kasner epoch; in this sense, the vanishing behaviour of the determinant of the spatial metric tensor h approaching the singularity, does not affect significantly the rotation law (4.69).

4.6 Concluding remarks

The analysis here presented shows how chaos is restored in multi-dimensional homogeneous spaces as soon as an Abelian vector field is coupled to gravity. Furthermore, the presence of chaos is not a dimensional phenomenon, in this case.
This result is a consequence of the capability that a vector field has to generate a billiard configuration in the asymptotic evolution; such a billiard-ball representation of the Universe dynamics coincides with the BKL piecewise approach only in the 4-dimensional space-time, while, in higher dimensions, new features appear. In particular, the obtained oscillatory regime characterizes all the homogeneous models, disregarding their potential term; furthermore, the map (4.63), acquires a direct dependence on the number of dimensions.
It has to be remarked once more that the validity of the Kasner approximation is based on the possibility to neglect the n-dimensional Ricci scalar with respect to the terms containing time derivatives. In the Kasner solution this picture holds along the whole system evolution to the initial singularity. In the piecewise Kasner solution, on the contrary, the n-dimensional Ricci tensor is negligible only for finite time intervals, ending with a bounce against the potential terms arising from the spatial curvature.
The scheme here proposed relies on the choice of a synchronous (or Gaussian) reference frame, which corresponds to have $N = 1$ and $N^\alpha = 0$; however, the results

obtained have to remain valid for other gauge choices in view of the general covariance.

A valuable issue of this Hamiltonian dynamics relies on fixing the rule of the Kasner vectors rotation. (4.69) is relevant to connect the Cauchy problem with later stages of the system dynamics. More precisely, rotation of the Kasner vectors is sensitive to the boundary conditions on the "matter" fields, and has to be taken into account when using the studied homogeneous models within a cosmological "picture".

We infer that the results here presented can be extended to a generic inhomogeneous cosmological model, in the same spirit as the Bianchi type VIII and IX model oscillatory regime is upgraded in four dimensions, as soon as the spatial gradients are taken into account. Such an extension will reliably show how, in the presence of a vector field, the generic cosmological solution is described by an oscillatory approach to the Big-Bang in any number of dimensions.

Appendix 4A

Homogeneous 5-dimensional models

The Fee's classification [Fee79] divides the five-dimensional homogeneous spaces in 15 types, called $G0 - G14$; it is based on the analysis of the four dimensional Lie groups and it is reported below in Tab. (4.6) (the right- and left-invariant vector fields and forms can be found in Fee's master thesis):

Table 4.1: Four-dimensional Lie algebras

$G0$		All zero		
$G1$	$[x, z] = w$			
$G2$	$[x, z] = w$	$[y, z] = w$		
$G3$	$[x, z] = w$	$[y, z] = y$		
$G4$	$[w, y] = w$	$[x, z] = x$		
$G5$	$[w, y] = -x$	$[w, z] = w$	$[x, y] = w$	$[x, z] = w$
$G6$	$[w, z] = w$	$[x, z] = w + x$	$[y, z] = x + y$	
$G7$	$[w, z] = w$	$[x, z] = w + x$	$[y, z] = py$	
$G8$	$[w, z] = w$	$[x, z] = px$	$[y, z] = qy$	
$G9$	$[w, z] = pw + x$	$[x, z] = -w + px$	$[y, z] = qy$	
$G10$	$[w, z] = 2w$	$[x, y] = w$	$[x, z] = x$	$[y, z] = x + y$
$G11$	$[w, z] = (1 + p)w$	$[x, y] = w$	$[x, z] = x$	$[y, z] = py$
$G12$	$[w, z] = 2pw$	$[x, y] = w$	$[x, z] = px + y$	$[y, z] = -x + py$
$G13$	$[w, x] = y$	$[w, y] = -x$	$[x, y] = w$	
$G14$	$[w, x] = y$	$[w, y] = -x$	$[x, y] = -w$	

The line element can be written using the Cartan basis of left-invariant forms, and reads explicitly

$$ds^2 = dt^2 - g_{ij}\omega^i\omega^j \ , \quad g_{ij} = g_{ij}(t) \ . \tag{4A.1}$$

The one-forms ω^i obey the relation $d\omega^i = \frac{1}{2}C^i{}_{st}\omega^s \wedge \omega^t$, where the $C^i{}_{st}$ are the structure constants of the considered group. We will limit our discussion, without loss of generality, to the diagonal metric case, *i.e.*,

$$g_{ij} = \mathrm{diag}(a^2, b^2, c^2, d^2) \ . \tag{4A.2}$$

The Einstein equations are easily obtained:

$$R^0_0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\ddot{d}}{d} = 0 \ , \tag{4A.3}$$

$$R^1_1 = \frac{(\dot{a}bcd)^{\cdot}}{abcd} + S^1{}_1 = 0 \ , \tag{4A.4}$$

$$R^2_2 = \frac{(a\dot{b}cd)^{\cdot}}{abcd} + S^2{}_2 = 0 \ , \tag{4A.5}$$

$$R^3_3 = \frac{(ab\dot{c}d)^{\cdot}}{abcd} + S^3{}_3 = 0 \ , \tag{4A.6}$$

$$R^4_4 = \frac{(abc\dot{d})^{\cdot}}{abcd} + S^4{}_4 = 0 \ , \tag{4A.7}$$

$$R^0_n = \left(\frac{\dot{x}_n}{x_n} - \frac{\dot{x}_m}{x_m} \right) C^m{}_{mn} = 0 \ ; \tag{4A.8}$$

here x_n $(n = 1, 2, 3, 4)$ denotes the scale factors a, b, c, d. The $S^n{}_n$ are functions of the scale functions and of the structure constants.

The analysis developed for the Bianchi type IX model can be straightforwardly generalised to the 5 dimensional case, obtaining the following set of equations

$$\begin{aligned} \alpha_{,\tau,\tau} &= -\Lambda^2 S^1{}_1 \ , \\ \beta_{,\tau,\tau} &= -\Lambda^2 S^2{}_2 \ , \\ \gamma_{,\tau,\tau} &= -\Lambda^2 S^3{}_3 \ , \\ \delta_{,\tau,\tau} &= -\Lambda^2 S^4{}_4 \ , \end{aligned} \tag{4A.9}$$

$$\begin{aligned} \alpha_{,\tau,\tau} + \beta_{,\tau,\tau} + \gamma_{,\tau,\tau} + \delta_{,\tau,\tau} =& 2\alpha_{,\tau}\beta_{,\tau} + 2\alpha_{,\tau}\gamma_{,\tau} + \\ &+ 2\alpha_{,\tau}\delta_{,\tau} + 2\beta_{,\tau}\gamma_{,\tau} + 2\beta_{,\tau}\delta_{,\tau} + 2\gamma_{,\tau}\delta_{,\tau} \ , \end{aligned} \tag{4A.10}$$

and

$$R^0_n = 0 \ . \tag{4A.11}$$

These dynamical scheme (4A.9)-(4A.11) is valid for any of the 15 models as soon as the corresponding $S^n{}_n$ are taken into account.

Exact homogeneous 5- dimensional solutions

The simpler group to consider is the $G0$; for this model the functions $S^n{}_n$ are all equal to zero and the solution simply generalises to 4 spatial dimensions the Kasner dynamics.

Equations (4A.9) for the $G1$ model become

$$\begin{aligned} -\alpha_{,\tau,\tau} = \beta_{,\tau,\tau} &= \gamma_{,\tau,\tau} = \frac{1}{2}\exp 4\alpha + 2\delta \ , \\ \delta_{,\tau,\tau} &= 0 \ . \end{aligned} \tag{4A.12}$$

The solution is easily obtained, and reads

$$\begin{aligned} a^2 &= A\omega \mathrm{sech}(\omega\tau)\exp(-k_1\tau) \ , \quad b^2 = B\cosh(\omega\tau)\exp(k_2\tau) \ , \\ c^2 &= C\cosh(\omega\tau)\exp(k_3\tau) \ , \qquad d^2 = 1/A^2\exp(2k_1\tau) \ , \end{aligned} \tag{4A.13}$$

where $\omega^2 = -2k_1^2 + k_1k_2 + k_1k_3 + k_2k_3$ and A, B, C are constants; we observe that the asymptotic behaviour of (4A.13) also is a generalised Kasner like.

Quite interesting is the $G3$ dynamics in that it does not admit four nonzero scale factors; the functions $S^n{}_n$ are $S^1{}_1 = -\dfrac{b^2}{2a^4c^4} = -S^4{}_4; S^2{}_2 = -(a^2b^2c^4d^2)^{-1}; S^3{}_3 = S^1{}_1 + S^2{}_2$; its solution is the four dimensional Kasner behaviour:

$$a = At^p \ , $$

$$b = c = Bt^q \ , \tag{4A.14}$$

$$d = 0 \ , \tag{4A.15}$$

with $p = -\dfrac{1}{3}$ and $q = \dfrac{2}{3}$.

$G12$ is the last model of the Fee's classification we will discuss. It has a different structure and no four-dimensional analogue, but it undergoes the same "fate" of an asymptotic Kasner evolution after a first phase of BKL transitions.

Conclusions and Outlooks

The central object of my studies has been the inhomogeneous Mixmaster model. I've been interested in both the classical and the quantum properties that this dynamical system exhibits in the neighbourhood of the initial singularity. To conclude this PhD thesis, I want to underline once more the main results discussed, and illustrate possible developments.

On the classical level, the most important achieved result is the proof that the chaotic behaviour of the dynamics near the Big Bang can be characterized in a covariant way. The proof is based on the reduction of the dynamics to the physical degrees of freedom, that was obtained solving the super-Hamiltonian and the supermomentum constraints without imposing any gauge condition on the lapse function or on the shift vector. Then the dynamics of the generic cosmological solution was reduced to the sum of infinite not-correlated identical dynamical system, one for each point of the space. Finally, chaos was quantified by direct calculation of the Lyapunov exponent, and the chaotic properties granted by the shape of the potential walls and by the finite measure of the domain. I want to underline once more that the mathematical concept of "point Universe" corresponds to the much more physical concept of cosmological horizon, and this is due to the hypothesis of local homogeneity at the ground of the construction of the generic cosmological solution.

On the quantum level, instead, two are the results that I'd like to remember here. First of all, the correct operator-ordering. Indeed, the choice of an ordering when facing the quantization of a dynamical system is, most of the times, a very puzzling question; in the scheme I discussed, the correct choice was not imposed *ab initio*, but obtained requiring that the classical phase overlap the semi-classical one.
Then, the wide analysis of the quantum dynamics. I derived the eigenfunctions of the model, and, even if in some approximations, I imposed Dirichlet boundary condition, obtaining this way the energy spectrum for the quantized Mixmaster model; this

spectrum exhibits two very interesting properties, *i.e.*, it is a discrete spectrum, and the ground-state energy is greater than zero.

The last result I will discuss here concerns with the multi-dimensional generalization of the Mixmaster model. I was able to show that, in a generic n-dimensional homogeneous model, chaos is restored as soon as an Abelian vector field is coupled to the gravitational dynamics; the key result absolutely is the BKL-like map that characterizes the dynamics, and that links the Kasner exponents and the Kasner vectors of two successive epochs. This map is valid for any model, and exhibits a very peculiar dependence on the number of dimensions; furthermore, it reduces to the standard Mixmaster map in vacuum and in three spatial dimensions.

Starting from these results, there are many things that can still be done.
First of all, a complete characterization of the quantum chaos of the Mixmaster model: starting from the eigenfunctions, it is possible to perform numerical simulations of the evolution of a generic gaussian wave-packet, in order to verify if some fractal structure arises in the shape of the wave, or not.
An interesting possibility is that of quantization *a la* Wigner: this is a quantization scheme in the whole phase-space, and yields, if one succeeds, a "quasi distribution function". This would be very interesting in order to improve the analogy between classical and semi-classical dynamics.
Another possible idea is that of the use Ashtekar connection variables in the billiard framework. In Ashtekar description, the super-Hamiltonian constrain is a polynomial one without any potential term, and this, in principle, means a billiard without any wall. Always in this scheme, a complete discussion of the role of inhomogeneity would be really interesting.
In the multi-dimensional case, there are at least two way to improve and generalize my results. One is the inclusion of inhomogeneity, and the consequent discussion on the generic cosmological solution coupled to an electromagnetic field. The other is the generalization to a non-Abelian vector field, and this would be really interesting because fundamental interactions are described by this kind of fields. Further developments are surely related to String theory and super-gravity, or to some Kaluza-Klein cosmological model.

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Attachments

Attachment 1

PHYSICAL REVIEW D, VOLUME 70, 103527

Frame independence of the inhomogeneous mixmaster chaos via Misner-Chitré-like variables

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We outline the covariant nature, with respect to the choice of a reference frame, of the chaos characterizing the generic cosmological solution near the initial singularity, i.e., the so-called inhomogeneous mixmaster model. Our analysis is based on a gauge independent Arnowitt-Deser-Misner reduction of the dynamics to the physical degrees of freedom. The resulting picture shows how the inhomogeneous mixmaster model is isomorphic point by point in space to a billiard on a Lobachevsky plane. Indeed, the existence of an asymptotic (energylike) constant of the motion allows one to construct the Jacobi metric associated with the geodesic flow and to calculate a nonzero Lyapunov exponent in each space point. The chaos covariance emerges from the independence of our scheme with respect to the form of the lapse function and the shift vector; the origin of this result relies on the dynamical decoupling of the space points which takes place near the singularity, due to the asymptotic approach of the potential term to infinite walls. At the ground of the obtained dynamical scheme is the choice of Misner-Chitré-like variables which allows one to fix the billiard potential walls.

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I. BASIC STATEMENTS

The homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) metric provides a valuable framework to describe the history of the Universe up to its very early stages of evolution. Indeed the very good agreement between the light element abundances predicted for the primordial nucleosynthesis and the observed one allows one to extrapolate backward in time for the FLRW dynamics up to 10^{-2}–10^{-3} s [1]; furthermore, recent observations of the cosmic microwave background radiation [2,3] suggest that an inflationary scenario took place and that the Universe was homogeneous and isotropic on the horizon scale up to $\mathcal{O}(10^{-32})s$. In spite of such an experimental evidence in favor of the FLRW model, there are mainly two well-grounded reasons to believe that, when the Universe temperature was above the grand unification scale [$\mathcal{O}(10^{15}$ GeV)] and below the Planck mass [$\mathcal{O}(10^{19}$ GeV)], the Universe was appropriately described by a generic inhomogeneous cosmological model [4–6]; indeed we have to stress that

(1) as shown in [7] the Universe in an expanding picture is stable with respect to tensorial perturbations; in fact the amplitude of a gravitational wave decays like the inverse of the scale factor [8,9]. Therefore reversing the expanding behavior into a collapsing one, the homogeneity and the isotropy of the Universe become unstable with respect to wavelike perturbations. In particular in [9] is shown how a Bianchi type IX model far from the singularity can be represented (in terms of exact solution) by a closed FLRW model plus small gravitational ripples; instead, close to the singularity this model has a fully developed anisotropy which results into a chaotic behavior. This issue is valuable because the Bianchi type IX cosmology presents features extensible (point by point in space) to the generic cosmological solution.

(2) It is commonly believed [1,10] that the actual Universe came from a quantum regime which was fully developed during the Planckian era and approached a classical limit (see [11]) only in a later stage of evolution. Since the early Universe contained more than a single horizon (see also [12]), then during the Planckian era the metric and topology quantum fluctuations had to take place independently over two causally disconnected regions, and hence the survival of any global symmetry is prevented. In view of this, we infer that the quantum behavior of the early Universe has to be properly analyzed only in terms of a generic inhomogeneous model. With respect to these two points, it is worth noting that the bridge between the generic evolution and the FLRW dynamics is provided on the horizon scale just by an inflationary scenario, as outlined in [13].

As is well known, the generic cosmological solution [4] is characterized near the big bang by a chaotic evolution which reduces the space-time to the structure of a foam [5,6]. When approaching the cosmological singularity, the space points dynamically decouple and a time evolution resembling the so-called mixmaster dynamics of the Bianchi VIII and IX models takes place independently in each of them [14,15] (for further discussions on inhomogeneous cosmological models see [16,17]); from a physical point of view, a space neighborhood is here

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considered at a horizon size (for recent discussions on mixmaster covariance see [18–20]).

In recent years, the chaoticity of the homogeneous mixmaster model has been widely studied in the literature (see [18–23]) in view of understanding the features of its covariant nature.

Two convincing arguments, appearing in [20,22], support the idea that the mixmaster chaos (described by the invariant measure introduced in [23,24]) remains valid in any system of coordinates.

The main issue of the present work is to show that the property of space-time covariance can be extended to the inhomogeneous mixmaster model. In Sec. II we provide a gauge independent analysis (i.e., independent of the choice of the lapse function as well as of the shift vector) for the dynamics of the gravitational degrees of freedom. In Sec. III we discuss the asymptotic behavior of the potential term associated with the Ricci 3-scalar, showing how it can be modeled in terms of a potential wall surrounding a (dynamically) closed domain. In Sec. IV in view of the existence of an energylike constant of motion, we construct the Jacobi metric associated with the resulting billiard on a two-dimensional Lobachevsky plane point by point in space, and finally we calculate the covariant nonzero Lyapunov exponent. The concluding remarks presented in Sec. V are devoted to outline how the approximation used for the potential term is completely self-consistent being dynamically induced. It is worth noting how we show that for any choice of lapse function and the shift vector, taking Misner-Chitré-like variables, the system results to be chaotic; however, in this sense, our analysis states the independence of the chaos on the choice of the space-time coordinates, but the questions of covariance remains open when using different configuration coordinates because the Lyapunov exponent is not a good indicator for all of them.

II. HAMILTONIAN FORMULATION

A generic cosmological solution is represented by a gravitational field having available all its degrees of freedom and, therefore, allowing one to specify a generic Cauchy problem. In the Arnowitt-Deser-Misner (ADM) formalism, the metric tensor corresponding to such a generic model takes the form

$$d\Gamma^2 = N^2dt^2 - \gamma_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt), \quad (1)$$

where N and N^α denote (respectively) the lapse function and the shift vector, $\gamma_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ the 3-metric tensor of the spatial hypersurfaces Σ^3 for which $t =$ const, being

$$\begin{aligned} \gamma_{\alpha\beta} &= e^{q_a}\delta_{ad}O^a_bO^d_c\partial_\alpha y^b\partial_\beta y^c, \\ a, b, c, d, \alpha, \beta &= 1, 2, 3, \end{aligned} \quad (2)$$

while $q^a = q^a(x, t)$ and $y^b = y^b(x, t)$ six scalar functions, and $O^a_b = O^a_b(x)$ a SO(3) matrix. By the metric tensor (2), the action for the gravitational field is

$$S = \int_{\Sigma^{(3)}\times\Re} dtd^3x(p_a\partial_tq^a + \Pi_d\partial_ty^d - NH - N^\alpha H_\alpha), \quad (3)$$

where

$$H = \frac{1}{\sqrt{\gamma}}\left[\sum_a(p_a)^2 - \frac{1}{2}\left(\sum_b p_b\right)^2 - \gamma^{(3)}R\right], \quad (4)$$

$$H_\alpha = \Pi_c\partial_\alpha y^c + p_a\partial_\alpha q^a + 2p_a(O^{-1})^b_a\partial_\alpha O^a_b. \quad (5)$$

In (4) and (5) p_a and Π_d are the conjugate momenta of the variables q^a and y^b respectively, and the $(3)R$ is the Ricci 3-scalar which plays the role of a potential term.

The ten independent components of a generic metric tensor are represented by the three scale factors q^a, the 3 degrees of freedom y^a, the lapse function N, and the three components of the shift vector N^a; it is worth noting that, by the variation of the variables p_a, Π_a in the action (3), the relations:

$$\partial_ty^d = N^\alpha\partial_\alpha y^d, \quad (6)$$

$$N = \frac{\sqrt{\gamma}}{\sum_a p_a}\left(N^\alpha\partial_\alpha\sum_b q^b - \partial_t\sum_b q^b\right) \quad (7)$$

take place.

III. ADM REDUCTION OF THE DYNAMICS

We use the Hamiltonian constraints $H = H_\alpha = 0$ for the reduction of the dynamics to the physical degrees of freedom; from (4) and (5), we note that the supermomentum constraints can be diagonalized and explicitly solved by choosing the function y^a as special coordinates, i.e., taking the transformation $\eta = t$, and $y^a = y^a(t, x)$. In fact starting by (4) and (5) we get the expression

$$\Pi_b = -p_a\frac{\partial q^a}{\partial y^b} - 2p_a(O^{-1})^c_a\frac{\partial O^a_c}{\partial y^b}. \quad (8)$$

Furthermore, in the new coordinates we have

$$\begin{cases} q^a(t, x) \to q^a(\eta, y), \\ p_a(t, x) \to p'_a(\eta, y) = p_a(\eta, y)/|J|, \\ \frac{\partial}{\partial t} \to \frac{\partial y^b}{\partial t}\frac{\partial}{\partial y^b} + \frac{\partial}{\partial\eta} \\ \frac{\partial}{\partial x^\alpha} \to \frac{\partial y^b}{\partial x^\alpha}\frac{\partial}{\partial y^b}, \end{cases} \quad (9)$$

where $|J|$ denotes the Jacobian of the transformation. The first relation holds in general for all the scalar quantities, while the second one for all the scalar densities; hence the action (3) rewrites as

$$S = \int_{\Sigma^{(3)}\times\Re} d\eta d^3y[p_a\partial_\eta q^a + 2p_a(O^{-1})^c_a\partial_\eta O^a_c - NH]. \quad (10)$$

IV. THE POTENTIAL WALL AND THE REDUCED VARIATIONAL PRINCIPLE

The potential term appearing in the super-Hamiltonian reads, in obvious notation, as

$$U = \frac{D}{|J|^2} R = \sum_a \lambda_a^2 D^{2Q_a} + \sum_{b\neq c} D^{Q_b+Q_c} \mathcal{O}(\partial q, (\partial q)^2, y, \eta), \tag{11}$$

where

$$D \equiv \exp \sum_a q^a, \tag{12}$$

$$Q_a \equiv \frac{q^a}{\sum_b q^b}, \tag{13}$$

$$\lambda_a^2 \equiv \sum_{kj} (O_b^a \vec{\nabla} O_c^a (\vec{\nabla} y^c \wedge \vec{\nabla} y^b)^2). \tag{14}$$

Assuming the functions $y^a(t, x)$ smooth enough (which implies by (6) and (7) that the coordinates system is smooth "itself"), then all the gradients appearing in the potential U are regular. Indeed this notion of regularity is not to be intended in absolute sense. In fact what really matters here is not that the gradient increases but simply that their behavior is not so strongly divergent to destroy the billiard representation (see next paragraph). In [5] it was shown that the spatial gradients increase logarithmically in the proper time along the billiard's geodesic and therefore result to be of higher order. Thus, as $D \to 0$ [25] the spatial curvature $(3)R$ diverges and the cosmological singularity appears; in this limit, the first term of the potential U dominates all the remaining ones and can be modeled by the potential wall

$$U = \sum_a \Theta(Q_a), \tag{15}$$

being

$$\Theta(x) = \begin{cases} +\infty & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{16}$$

By (15) the Universe dynamics evolves independently in each space point; the point Universe can move only within a dynamically closed domain Γ_Q (see Fig. 1) (indeed the three corners at infinity are open but they correspond to a set of zero measure in the space of initial conditions). Since in Γ_Q the potential U asymptotically vanishes, near the singularity we have $\partial p_a/\partial \eta = 0$. Then the term $2p_a(O^{-1})_a^c \partial_\eta O_c^a$ in (10) behaves as an exact time derivative and can be ruled out of the variational principle. The ADM reduction is completed by introducing the so-called Misner-Chitré-like variables [5,22,26,27] as

$$\begin{cases} q^l = e^\tau[\sqrt{\xi^2 - 1}(\cos\theta + \sqrt{3}\sin\theta) - \xi], \\ q^m = e^\tau[\sqrt{\xi^2 - 1}(\cos\theta - \sqrt{3}\sin\theta) - \xi], \\ q^n = -e^\tau(\xi + 2\sqrt{\xi^2 - 1}\cos\theta). \end{cases} \tag{17}$$

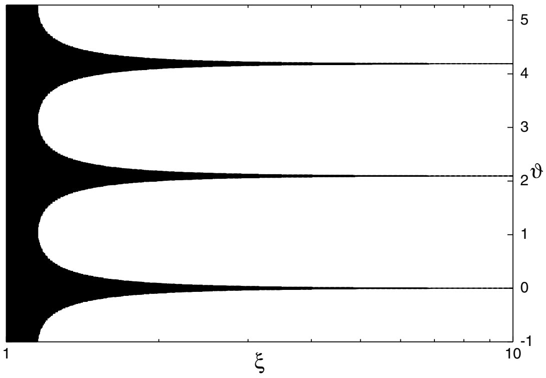

FIG. 1. $\Gamma_Q(\xi, \theta)$.

The way in which the anisotropy parameters Q_a (13) are rewritten is an important feature of these variables since we easily get the τ-independent expressions:

$$\begin{aligned} Q_1 &= \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi}(\cos\theta + \sqrt{3}\sin\theta), \\ Q_2 &= \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi}(\cos\theta - \sqrt{3}\sin\theta), \\ Q_3 &= \frac{1}{3} + \frac{2\sqrt{\xi^2 - 1}}{3\xi}\cos\theta. \end{aligned} \tag{18}$$

When expressed in terms of such variables the super-Hamiltonian constraint can be solved in the domain Γ_Q:

$$-p_\tau \equiv \epsilon = \sqrt{(\xi^2 - 1)p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1}} \tag{19}$$

and the reduced action reads as

$$\delta S_{\Gamma_Q} = \delta \int d\eta d^3 y (p_\xi \partial_\eta \xi + p_\theta \partial_\eta \theta - \epsilon \partial_\eta \tau) = 0. \tag{20}$$

By the asymptotic limit (15) and the Hamilton equations associated with (20) it follows that ϵ is a constant of motion, i.e., $d\epsilon/d\eta = \partial\epsilon/\partial\eta = 0 \Rightarrow \epsilon = E(y^a)$.

V. THE JACOBI METRIC AND THE LYAPUNOV EXPONENT

Being ϵ a constant of motion, the term $\epsilon\partial_\eta\tau = E(y^a)\partial_\eta\tau$ in (20) behaves as an exact time derivative; hence the variational principle rewrites

$$\delta \int d^3 y (p_\xi d\xi + p_\theta d\theta) = 0, \tag{21}$$

coupled with the constraint (19). This dynamical scheme allows one to construct the Hamilton-Jacobi metric [28] corresponding to the dynamical flow. Indeed for each point of the space it can be reproduced the same as the analysis developed in [22] for the homogeneous mixmaster model; in particular, all the spatial gradients are dumped and the space points dynamically decouple in the asymptotic limit to the singularity. In fact, in each space point, the system dynamics is replaced by a geodesic flow $\delta \int ds = 0$, with

$$ds^2 = E^2(y^a)\left[\frac{d\xi^2}{(\xi^2-1)} + (\xi^2-1)d\theta^2\right], \tag{22}$$

corresponding to the Jacobi line element. The Jacobi metric is valid independently in each of such point (here the space coordinates behave like external parameter) since the evolution is spatially uncorrelated. The Ricci scalar takes the value $R = -2/E^2$; hence such a metric describes a two-dimensional Lobachevsky space. The role of the potential wall (15) consists of cutting a closed domain Γ_Q on such a negative curved surface. Thus, summarizing, the system obtained is isomorphic to a billiard on a Lobachevsky plane.

A precise information about the dynamical stability of the geodesic flow associated with the line element (22) arises by the calculation of the Lyapunov exponent. Let us project the connecting vector Z^μ between two close geodesics on a Fermi basis $\{u^\mu, w^\mu\}$, $(\mu = 1, 2)$; the geodesic vector u^μ is taken in the form

$$u^\mu = \left(\frac{d\xi}{ds}, \frac{d\theta}{ds}\right) = \left[\frac{\sqrt{\xi^2-1}}{E}\cos\phi(s), \frac{\sin\phi(s)}{E\sqrt{\xi^2-1}}\right], \tag{23}$$

where s denotes the curvilinear coordinate, while $\phi(s)$ is an angular variable $(0 \leq \phi < 2\pi)$ whose dynamics is obtained by requiring that the geodesic equation is verified, i.e.,

$$\frac{d\phi(s)}{ds} = -\frac{\xi}{E\sqrt{\xi^2-1}}\sin\phi(s). \tag{24}$$

The vector w^μ is determined by the property of the Fermi basis to be orthonormal, and it reads explicitly

$$w^\mu = \left(-\frac{\sqrt{\xi^2-1}}{E}\sin\phi; \frac{\cos\phi}{E\sqrt{\xi^2-1}}\right). \tag{25}$$

Projecting the connecting vector Z^μ over the Fermi basis defined above

$$Z^\mu = Z_u(s)u^\mu + Z_w(s)w^\mu, \tag{26}$$

from the geodesic deviation equation we get that the dynamics is described by the following equations:

$$\begin{cases} \frac{d^2Z_u}{ds^2} = 0, \\ \frac{d^2Z_w}{ds^2} = \frac{Z_w}{E^2}. \end{cases} \tag{27}$$

The solution for the system (27) is given by

$$\begin{cases} Z_u = As + B, & A, B = \text{const}, \\ Z_w = c_1 e^{s/E} + c_2 e^{-s/E}, & c_1, c_2 = \text{const}. \end{cases} \tag{28}$$

The value of E given by the constraint (19) and involved in the line element (22) is determined by the initial conditions and cannot vanish. By the first of solutions (28) no geodesic deviation takes place along the geodesic vector (as expected); instead from the second solution we get a nonzero Lyapunov exponent of the form

$$\lambda(y^a) = \limsup_{s\to\infty} \frac{\ln[Z_w^2 + (dZ_w/ds)^2]}{2s} = \frac{1}{E(y^a)} > 0. \tag{29}$$

For the validity of this analysis we have to verify that in the limit to the initial singularity the curvilinear coordinate s approaches infinity. Repeating the same procedure of [22], formulas (31) appearing there, in our inhomogeneous case is replaced by

$$\frac{\partial s}{\partial \eta} = E(y^a)\partial_\eta \tau, \tag{30}$$

i.e., [being $f(y^a)$ a generic function of the space coordinates]

$$s = E(y^a)\tau + f(y^a) \Rightarrow \lim_{\tau\to\infty} s = \infty. \tag{31}$$

This ensures that the curvilinear coordinate s behaves in the appropriate manner in the limit toward the singularity $(\tau \to \infty)$.

Hence the chaoticity of the inhomogeneous mixmaster dynamics is ensured by Γ_Q to be a closed domain [29], and the covariance of such a description follows from the independence of the Lyapunov exponent with respect to the lapse function and the shift vector. In fact, in (6) and (7) N and N_α are fixed (in turn) by choosing the form of the quantities y^a and the latter can be generic functions subjected only to the condition to be smooth enough. Equation (29) provides the form of the Lyapunov exponent in the whole space domain but we stress how its value depends on the choice of Misner-Chitré-like variables. The independence of our scheme on the shift vector is ensured by the asymptotic behavior of the potential term, but to get ϵ as a constant of motion, allowing the Jacobi metric representation, we need a Misner-Chitré-like variable.

The covariance of our picture is equivalent to the covariance of the inhomogeneous mixmaster chaos because it is well known [22,23] that the obtained billiard has stochastic properties (see also [30]). In fact the negative curvature of the Lobachevsky plane makes the geodesic flow unstable; the potential walls have the role of replacing a given geodesic with a different one (whose tangent vector is related to the previous one by a reflection rule [30]), and as we will show in Sec. VI their structure will influence the chaotic properties of the system dynamics.

To better characterize the chaoticity of the obtained billiard, we show that in each point of the space our system admits an invariant measure which in the present variable is uniform over the admissible phase space.

From the point of view of statistical mechanics, such a system admits, point by point in space, an "energylike" constant of motion which corresponds to the kinetic part of the ADM Hamiltonian $\epsilon = E(y^a)$. The point Universe, randomizing within the closed domain Γ_Q, is represented by a dynamics which allows for an ensemble representation. In view of the existence of the energylike constant of motion, the system evolution is appropriately described by a *microcanonical ensemble*. Therefore the stochasticity of this system is governed by the Liouville invariant measure

$$d\rho \propto \delta(E(y^a) - \epsilon) d\xi d\theta dp_\xi dp_\theta, \tag{32}$$

where $\delta(x)$ denotes the Dirac functional.

Since the particular value taken by the function ϵ [$\epsilon = E(y^a)$] cannot influence the stochastic property of the system and must be fixed by the initial conditions, then we must integrate (in functional sense) over all admissible forms of ϵ. To do this it is convenient to introduce the natural variables (ϵ, φ) in place of (p_ξ, p_θ) by

$$p_\xi = \frac{\epsilon}{\sqrt{\xi^2 - 1}} \cos\varphi, \qquad p_\theta = \epsilon\sqrt{\xi^2 - 1} \sin\varphi, \tag{33}$$

where $0 \leq \varphi < 2\pi$. By integrating over all functional forms of ϵ, we remove the Dirac delta functional, which leads in each point of space to the uniform normalized invariant measure [31]

$$d\mu(y^a) = d\xi d\theta d\varphi \frac{1}{8\pi^2}. \tag{34}$$

The existence of the above stationary probability distribution in Γ_Q outlines the chaotic properties associated with the pointlike billiard resulting from our analysis.

As we stressed, our dynamical scheme relies on the use of Misner-Chitré-like variables and therefore the covariance of the Lyapunov exponent is invariant with respect to space-time coordinates, but it could be sensitive to the choice of configurational variables. Thus the result here obtained calls attention to be extended to any choice of the configurational variables. In this respect we compare our result with the analysis presented in [19] according to which, given a dynamical system of the form,

$$d\mathbf{x}/dt = \mathbf{F}(x). \tag{35}$$

Then the positiveness of the associated Lyapunov exponents is invariant under the following diffeomorphism: $\mathbf{y} = \phi(\mathbf{x}, t)$, $d\tau = \lambda(\mathbf{x}, t)dt$, as soon as the four requirements hold:

(1) the system is autonomous,
(2) the relevant part of the phase space is bounded,
(3) the invariant measure is normalizable,
(4) the domain of the time parameter is infinite.

To show that such a covariance criterion is here fulfilled, we observe that the variables $\mathbf{x}$ can be identified with τ, ξ, θ, and φ, and the time variable with our curvilinear coordinate s. On the other hand the above diffeomorphism relation in its time independent form can match a phase-space coordinates transformation; then we underline also that

(1) in the considered asymptotic limit our dynamical system is autonomous because its Hamiltonian coincides with the constant of motion $\epsilon = E(y^a)$ and the potential walls are fixed in time.
(2) Apart from sets of zero measure which cannot be explored by the system [14], the phase space Γ_Q is a compact domain.
(3) As shown by the above analysis which leads to (34) the system admits, in each space point, a normalized invariant measure over the phase space.
(4) We showed by (31) that the curvilinear coordinate s admits an infinite domain because the variables $\tau \in (-\infty, \infty)$.

Thus, on the base of [19], we can claim that the Lyapunov exponent calculated in (29) provides an appropriate chaos indicator only when the effects of the boundary are taken into account in agreement with our discussion of the next section. Furthermore such an indicator is covariant with respect to any configurational coordinate transformation which preserves the requirements 1–4. Strikingly, we have to stress that if we adopt Misner-like variables [15] the Lyapunov exponent (29) is no longer a good indicator; in fact the anisotropy parameters (13) in Misner-like variables depend not only on β_+, β_-, but also on α, and therefore after the ADM reduction on the curvilinear coordinate s. As a consequence, the conditions 1 and 3 are no longer fulfilled for this choice because the potential walls move with time. However for any generic transformation of coordinates which involves only ξ and θ, the chaoticity of the mixmaster model is preserved. We conclude this section by observing that if in different configurational coordinates the inhomogeneous mixmaster would not appear as chaotic, then the stochasticity has to be transferred to the coordinate transformation which links them to the Misner-Chitré-like variable.

VI. THE ROLE OF THE POTENTIAL WALLS

Now we involve the role of the potential walls in order to cut a billiard on the Lobachevsky plane. Here we discuss a notion of Lyapunov exponents which include either the feature of the geodesic flow or the structure of the bounding potential walls, and we arrive to show that even in this more complete framework our system is a chaotic one. To this end we are showing below that our system meets all the hypothesis at the ground of the Wojtkowski Theorem; see [32]. Let us consider the following new choice of coordinates on the 2-surface:

$$\vec{y} = (y_1, y_2) = \frac{(1-\xi)}{\sqrt{\xi^2 - 1}}(\cos\theta, \sin\theta). \tag{36}$$

On the basis of (36) the new line element and the anisotropy parameters read as

$$ds^2 = \frac{4E^2 d\vec{y}^2}{(1-y^2)^2}; \qquad y < 1; \tag{37}$$

$$Q_a = [(\vec{y}^2 + A_a)^2 + 1 - (A_a)^2]; \qquad a = 1, 2, 3 \tag{38}$$

being $A_a^1(-\sqrt{3}, \sqrt{3}, 0)$ and $A_a^2(1, 1, -2)$

Since here we introduce the Poincaré model of the Lobachevsky plane in the form of the upper-half plane, this is reached by using the well-known Poincaré variables

$$\vec{\eta} = 2\frac{\vec{y} + \vec{b}}{(\vec{y} + \vec{b})^2} - \vec{b}, \tag{39}$$

where $\vec{b}$ denotes a point on the absolute ($b^2 = 1$). In terms of these new variables the metric (36) takes the form

$$ds^2 = \frac{(d\vec{\eta})^2}{(\vec{\eta} \cdot \vec{b})^2} \tag{40}$$

and the available domain $|y| \leq 1$ transforms into the half-plane $(\vec{\eta} \cdot \vec{b}) \geq 0$ while the absolute represents the line $(\vec{\eta} \cdot \vec{b}) = 0$. Now if we write

$$\vec{\eta} = \frac{2}{\sqrt{3}}\left[\left(u + \frac{1}{2}\right)\vec{b}^{\perp} + v\vec{b}\right]v \geq 0, \tag{41}$$

where $\vec{b} = (0, 1)$, $\vec{b}^{\perp} = (1, 0)$, then it is easy to verify that in terms of these coordinates the anisotropy functions have the form

$$\begin{cases} Q_1(u, v) = \frac{-u}{u^2+u+1+v^2}, \\ Q_2(u, v) = \frac{1+u}{u^2+u+1+v^2}, \\ Q_3(u, v) = \frac{u(u+1)+v^2}{u^2+u+1+v^2}. \end{cases} \tag{42}$$

This is a very suitable expression for the boundary; in fact geodesic on this half-plane are semicircles having centers on the absolute [i.e., $u(s) = A + R\cos s$, $v(s) = R\sin s$ being $(A, 0)$ the coordinate of the semicircle center and R the corresponding radius] and rays being perpendicular to the absolute. The billiard is bounded by the geodesic triangle $u = 0$, $u = -1$, and $(u + 1/2)^2 + v^2 = 1/4$. The new domain is shown in Fig. 2.

The billiard has a finite measure, and its open region at infinity together with the two points on the absolute $(0, 0)$ and $(-1, 0)$ correspond to the three cuspids of the potential in Fig. 1. The tangent field to the geodesic flow takes the explicit form:

$$t^{\mu} \equiv (-v, u - A) \tag{43}$$

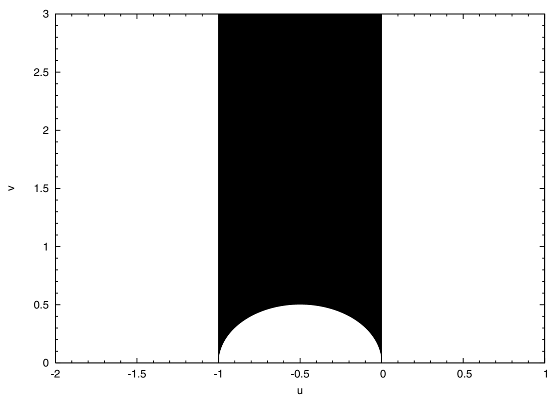

FIG. 2. $\Gamma_Q(u, v)$.

and the associated matrix of the dynamics $M_{\mu}^{\nu} \equiv \partial_{\mu} t^{\nu}$ is constant and orthonormal.

Referring to the notation of [32] we can easily construct in this variables the cocycle of the dynamics and verify that theorem 2.2 applies. Since the manifold is connected, has finite volume, and the matrix of the cocycle is constant, we have only to find an invariant bundle of sectors. This is possible because the following two properties hold:

(1) because of the constant negative curvature of the surface, a one-parameter family of geodesics with negative curvature is invariant under the evolution,
(2) the bounding potential walls are constituted by two straight lines and a semicircle of negative curvature; the former ones do not affect the structure of the cones during the bounces of the geodesic, while the latter one, being a dispersing profile, ensures that after reflection against it, the cones will evolve in themselves.

After this discussion we can claim that the largest Lyapunov exponent has a positive sign almost everywhere. The covariance of this result is ensured in view of the discussion developed in the previous section based on [19]. The analysis of this section completes our analysis about the covariant nature of the chaos associated to the billiard representation of the mixmaster model.

VII. CONCLUDING REMARKS

The main issue of the present work consists of the proof that the chaotic behavior singled out by a generic inhomogeneous model near a singularity has a covariant nature. This result has been obtained by a "gauge" independent ADM reduction of the dynamics to the physical degrees of freedom, which for the Universe corresponds to the anisotropy degrees, i.e., to the functions ξ and θ.

We describe the evolution as independent of the lapse function and the shift-vector form by adopting the variables y^a as the new spatial coordinates. However their degrees of freedom do not disappear from the problem because they are transferred to the SO(3) matrices O_b^a which acquire a dependence on time in the new variables. Such a dependence is then eliminated from the dynamics by solving the supermomentum constraint and using some implications deriving by the approximation of $\sqrt{g}R(3)$ as an infinite potential wall. The potential behavior (15) is crucial for the existence of an energylike constant of motion $\epsilon \equiv E(y^a)$, and therefore is at the basis of the chaos description. In this view, this approximation is naturally induced for $D \rightarrow 0$, due to the potential structure; the only assumption required is for the functions y^a and $O_b^a(y^c)$ to be smooth ones, in the sense that their presence does not affect the asymptotic behavior of the potential term. This restriction is a natural one having a good degree of generality, because the smoothness of the functions y^a is ensured by the smoothness of the lapse function and the shift vector [see (6) and (7)], i.e., by the choice of a regular reference frame. The initial smoothness of the matrices O_b^a (when taken on the spatial coordinates x^a) is preserved by the coordinates transformation. Concluding, the cosmological meaning of this concept corresponds to deal with independent "horizons"; the approximation neglecting in (19) the potential term with respect to the value of "ϵ" is equivalent to require the scale of the inhomogeneities to be superhorizon sized (see [4–6]).

ACKNOWLEDGMENTS

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Attachment 2

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Oscillatory regime in the multidimensional homogeneous cosmological models induced by a vector field

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Abstract
We show that in multidimensional gravity, vector fields completely determine the structure and properties of singularity. It turns out that in the presence of a vector field the oscillatory regime exists in all spatial dimensions and for all homogeneous models. By analysing the Hamiltonian equations we derive the Poincaré return map associated with the Kasner indexes and fix the rules according to which the Kasner vectors rotate. In correspondence to a four-dimensional spacetime, the oscillatory regime here constructed overlaps the usual Belinski–Khalatnikov–Liftshitz one.

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1. Introduction

The wide interest attracted by the homogeneous cosmological models of the Bianchi classification [1] relies over all in allowing for their anisotropic dynamics. Among them, types VIII and IX stand because of their chaotic evolution towards the initial singularity [2]. The stochastic properties outlined by such models were widely studied in order to describe their detailed features [3–6] and in recent years a new line of research has been developed to establish if the chaos is covariant and survives in each system of spacetime coordinates [7–11] (see also references therein). For approaches via Ashtekar variables which, providing a non-negative Hamiltonian function, could have interesting applications in the analysis of chaotic anisotropic cosmologies, see the formulation presented in [12].

The cosmological interest in the Bianchi type VIII and IX universe dynamics comes out because they correspond to the maximum degree of generality allowed by the homogeneity constraint; as a consequence it was shown [13–17] that the generic cosmological solution can be described properly, near the big bang, in terms of the homogeneous chaotic dynamics as referred to each cosmological horizon (recent interest in a special homogeneous model such as the Taub one arose because, in [18], it was shown that it admits an accelerating stage in the dynamics as the actual universe seems to do).

It is just the dynamical decoupling of the space points which takes place asymptotically that allows such an inhomogeneous extension; indeed the increase in the spatial gradients is controlled by the oscillatory behaviour of their time dependence, so that their effects are negligible at horizon size [14, 15]. However, the correspondence existing between the homogeneous dynamics and the generic inhomogeneous one holds only in four spacetime dimensions. In fact, a generic cosmological inhomogeneous model remains characterized by chaos near the big bang up to a ten-dimensional spacetime [19–23] while the homogeneous models show a regular (chaos free) dynamics beyond four dimensions [24, 25].

Here we address a Hamiltonian point of view and show how the homogeneous models (of each type) single out, near the singularity, an oscillatory regime in correspondence to any number of dimensions, as soon as an electromagnetic field is included in the dynamics (for a study of the dynamics induced by an electromagnetic field in the framework of the quasi-isotropic solution, see [26]). In fact the presence of an electromagnetic field restores a closed potential domain for the point-universe evolution; thus, even if the considered model does not possess a sufficient numbers of potential walls to determine the oscillatory regime, its asymptotic evolution is dominated by the matter field and randomizes within the corresponding billiard (for a connected analysis which studies the disappearance of chaos when a scalar field is included in the dynamics, though in the presence of a p-form, see [27]).

We stress how the electromagnetic billiard and its associated Poincaré return map coincide with usual ones in four dimensions. However, for higher dimensional spaces they differ from the corresponding vacuum dynamics; in particular, the map we derive outlines an interesting dependence on the number of dimensions. A relevant outcome of our analysis consists in fixing the law for the rotation of the n-independent oscillating directions.

The procedure followed here calls for attention to be extended to more general contexts such as the studies on the appearance of chaos in superstring dynamics [28, 29]. More precisely, in section 2 we review the Hamiltonian formalism for general relativity; in section 3 we derive the Kasner solution for homogeneous models. In section 4 we introduce the Kasner parametrization in our dynamical scheme. Finally, in section 5, we derive the solution in the Kasner approximation, and in section 6 we show how this dynamical system can be described in terms of the return map for the Kasner indexes and the law for Kasner vector rotation. In section 7 brief concluding remarks follow.

2. Hamiltonian formulation

We start by reviewing the dynamical framework of our analysis.

Let us consider a vector field $A_\mu = (\varphi, A_\alpha)(\alpha = 1, 2, \ldots, n)$ and adopt for the metric the standard ADM representation [30, 31]

$$\mathrm{d}s^2 = N^2\,\mathrm{d}t^2 - g_{\alpha\beta}(\mathrm{d}x^\alpha + N^\alpha\,\mathrm{d}t)(\mathrm{d}x^\beta + N^\beta\,\mathrm{d}t). \tag{1}$$

Then the action which describes the dynamics of the model takes the form

$$I = \int \mathrm{d}^n x\,\mathrm{d}t \left\{ \Pi^{\alpha\beta}\frac{\partial}{\partial t}g_{\alpha\beta} + \pi^\alpha\frac{\partial}{\partial t}A_\alpha + \varphi D_\alpha\pi^\alpha - NH_0 - N^\alpha H_\alpha \right\}, \tag{2}$$

where

$$H_0 = \frac{1}{\sqrt{g}}\left\{ \Pi^\alpha_\beta\Pi^\beta_\alpha - \frac{1}{n-1}\left(\Pi^\alpha_\alpha\right)^2 + \frac{1}{2}g_{\alpha\beta}\pi^\alpha\pi^\beta + g\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - R\right)\right\}, \tag{3}$$

$$H_\alpha = -\nabla_\beta \Pi^\beta_\alpha + \pi^\beta F_{\alpha\beta}, \tag{4}$$

denote respectively the super-Hamiltonian and super-momentum; here $F_{\alpha\beta} \equiv \partial_\beta A_\alpha - \partial_\alpha A_\beta$ is the electromagnetic tensor, $g \equiv \det(g_{\alpha\beta})$ is the determinant of the n-metric, R is the n-scalar of curvature constructed by the metric $g_{\alpha\beta}$ and $D_\alpha \equiv \partial_\alpha + A_\alpha$.

π^α and $\Pi^{\alpha\beta}$ are the conjugate momenta of the electromagnetic field and the n-metric respectively; they result in a vector and a tensorial density respectively of weight $1/2$ because their expressions contain $\sqrt{g}$, and are defined via the relations

$$\Pi^{\alpha\beta} = \frac{\sqrt{-g}}{N}(K^{\alpha\beta} - g^{\alpha\beta}\,\mathrm{Tr}(K)) \tag{5}$$

$$\pi^\alpha = \frac{\sqrt{-g}}{N}\left(\frac{\partial A_\beta}{\partial t} g^{\alpha\beta} - N^\beta F_\beta{}^\alpha\right), \tag{6}$$

where $K^{\alpha\beta}$ denotes the extrinsic curvature in the synchronous frame.

When varying this action with respect to the lapse function N we obtain the super-Hamiltonian constraint

$$H_0 = 0, \tag{7}$$

while its variation with respect to φ provides the constraint $\partial_\alpha \pi^\alpha = 0$. Since we are dealing with an Abelian vector field (i.e. corresponding to an Abelian group of symmetry as an electromagnetic field does) whose sources (charged particles) are absent, it is enough to consider only the transverse (or Lorentz) components for A_α and π^α. Therefore, we take the gauge conditions $\varphi = 0$ and $D_\alpha \pi^\alpha = 0$ and this will be enough to exclude the longitudinal parts of the vector field from the action.

It is worth noting how, in the general case, i.e. in the presence of the sources, or in the case of non-Abelian vector fields, this simplification can no longer take place in such explicit form and, therefore, we have to retain the term $\varphi(\partial_\alpha + A_\alpha)\pi^\alpha$ in the action principle.

In what follows we consider the behaviour of homogeneous cosmological models in the asymptotic limit towards the initial singularity.

3. Homogeneous cosmological models: basic equations

When considering the homogeneity constraint, the whole spatial dependence of the models can be integrated out from the action and the dynamical variables become only time dependent. In general neglecting the total Hamiltonian the spatial derivatives contained in $F_{\alpha\beta}$ and R is equivalent to the so-called (generalized) Kasner approximation.

When going over the homogeneous case, we choose the gauge $N^\alpha = 0$ and, within the Kasner approximation, we get equations of motion for the vector field having the form

$$E_\alpha = \frac{\partial}{\partial t} A_\alpha = \frac{N}{\sqrt{g}} g_{\alpha\beta}\pi^\alpha, \tag{8}$$

$$\frac{\partial}{\partial t}\pi^\alpha = 0. \tag{9}$$

The field equations which describe the n-metric dynamics read

$$\frac{\partial}{\partial t} g_{\alpha\beta} = \frac{2N}{\sqrt{g}}\left\{\Pi_{\alpha\beta} - \frac{1}{n-1} g_{\alpha\beta}\Pi^\gamma_\gamma\right\}, \tag{10}$$

$$\frac{\partial}{\partial t}\Pi^\alpha_\beta = -\frac{N}{2\sqrt{g}}\pi^\alpha \pi_\beta. \tag{11}$$

This dynamical scheme is completed by adding to the above Hamiltonian equations the super-Hamiltonian constraint (7).

4. Kasner parametrization

To develop the analysis below, it turns out very convenient to adopt the so-called Kasner parametrization which is based on the metric and conjugate momentum decomposition along the spatial n-bein:

$$g_{\alpha\beta} = \delta_{ab} l^a_\alpha l^b_\beta, \qquad \Pi_{\alpha\beta} = p_{ab} l^a_\alpha l^b_\beta, \tag{12}$$

where the n-bein is chosen in such a way that the matrix $p_{ab} = \mathrm{diag}\,(p_1, \ldots, p_n)$ (for a discussion of this Hamiltonian structure in correspondence to an inhomogeneous multidimensional model see [32]); this diagonal form of the conjugate momenta is a consequence of requiring the canonical nature of the adopted transformation. We also define a dual basis $L^\alpha_a = g^{\alpha\beta} l^a_\beta$, such that $L^\alpha_a l^b_\alpha = \delta^b_a$ and $L^\alpha_a l^a_\beta = \delta^\alpha_\beta$.

To face our goal we have to project (10) along the Kasner vectors defined in (12); in this way we get the following dynamical system:

$$\left(L_a \frac{\partial}{\partial t} l_a\right) = \frac{N}{\sqrt{g}} \left(p_a - \frac{1}{n-1} \sum_b p_b\right), \tag{13}$$

$$\left(L_a \frac{\partial}{\partial t} l_b\right) + \left(L_b \frac{\partial}{\partial t} l_a\right) = 0, \qquad a \neq b. \tag{14}$$

Here $(L_a l_b) = L^\alpha_a l_{b\alpha}$ denotes the ordinary vector product, treating the vector components as Euclidean ones.

In close analogy with the above, from equation (11), we find the additional equations

$$\frac{\partial}{\partial t} p_a = -\frac{N}{2\sqrt{g}} \lambda_a^2, \qquad \lambda_a = \left(\pi^\alpha l^a_\alpha\right) \tag{15}$$

$$\left(L_b \frac{\partial}{\partial t} l_a\right) = -\frac{N}{2\sqrt{g}} \frac{\lambda_a \lambda_b}{p_a - p_b}, \qquad a \neq b. \tag{16}$$

In particular, we see that (16) already contains (14).

By combining together both such systems, we obtain (15) as the first independent equation; the equation for the Kasner vectors takes place in the form (for the sake of simplicity we neglect the vector index α)

$$\frac{\partial}{\partial t} l_a = \frac{N}{\sqrt{g}} \left\{ \left(p_a - \frac{1}{n-1} \sum_b p_b\right) l_a - \frac{1}{2} \sum_{b \neq a.} \frac{\lambda_a \lambda_b}{p_a - p_b} l_b \right\}. \tag{17}$$

We want to put in evidence the oscillatory regime that the bein vectors possess; to better analyse this dynamics let us distinguish scale functions in the following way:

$$l_a = \exp(q^a/2)\ell_a. \tag{18}$$

$$L_a = \exp(-q^a/2)\mathcal{L}_a, \tag{19}$$

ℓ_a being the so-called Kasner vectors.

Thus instead of (15) and (17) we rewrite

$$\frac{\partial}{\partial t} p_a = -\frac{N}{2\sqrt{g}} \widetilde{\lambda}_a^2 \exp(q^a), \qquad \widetilde{\lambda}_a = \left(\pi^\alpha \ell^a_\alpha\right), \tag{20}$$

$$\frac{\partial}{\partial t}q^a = \frac{2N}{\sqrt{g}}\left(p_a - \frac{1}{n-1}\sum_b p_b\right), \tag{21}$$

$$\frac{\partial}{\partial t}\ell_a = -\frac{N}{2\sqrt{g}}\sum_{b\neq a}\frac{\widetilde{\lambda}_a\widetilde{\lambda}_b}{p_a - p_b}\exp(q^b)\ell_b = \sum_{b\neq a}\frac{\widetilde{\lambda}_a\left(\frac{\partial}{\partial t}p_b\right)}{\widetilde{\lambda}_b\,(p_a - p_b)}\ell_b. \tag{22}$$

The last two equations are obtained by substituting variables (18) and then projecting (17) on the vectors L_c^β; when the index c is equal to a, we get (21), otherwise we get (22) (in which we neglected a term vanishing in the exact Kasner-like solution, and therefore, in our analysis, of higher order).

To this system we should also add the Hamiltonian constraint

$$H_0 = 0 = \frac{N}{\sqrt{g}}\left\{\sum p_a^2 - \frac{1}{n-1}\left(\sum p_a\right)^2 + \frac{1}{2}\sum \mathrm{e}^{q_a}\widetilde{\lambda}_a^2\right\}. \tag{23}$$

To require that the quantities $\widetilde{\lambda}_a$ are constants implies that vectors ℓ^a do not depend on time in turn and therefore no Kasner vector rotation takes place (see section 6); such a situation corresponds exactly to the n-dimensional Kasner behaviour [20, 21, 33]. Since the rotation of Kasner vectors is induced by a higher order term for $g \to 0$ with respect to the pure Kasner dynamics, we can assume that $\widetilde{\lambda}_a$ are near to constant and therefore near the singularity (20), (21) give, in the asymptotic limit to the cosmological singularity, the billiards [10, 16, 3] on $(n-1)$-dimensional Lobachevsky space, exactly as in the three-dimensional mixmaster case [34]. In other words, in our scheme the evolution is not simply Kasner-like, but we get a dynamical picture in which the *point universe* moves according to a piecewise Kasner solution. The features predicted by equations (20) and (21) will be discussed in sections 5 and 6.

We remark that the validity of the Kasner approximation is based on the possibility of neglecting the n-dimensional Ricci scalar with respect to the term containing time derivatives. In the Kasner solution this picture holds along the whole system evolution to the initial singularity [33]; instead, in the piecewise Kasner solution, the n-dimensional Ricci tensor is negligible only for finite time intervals ending with a bounce against the potential terms arising from the spatial curvature [2]. However, in both these cases the validity of the Kasner approximation is confirmed by a large number of theoretical and numerical investigations, concerning homogeneous and inhomogeneous cosmological models near the singularity [2–6, 10, 13, 14, 16, 17, 34]

Our scheme relies on the choice of a synchronous (or Gaussian) reference frame which corresponds to having $N = 1$ and $N^\alpha = 0$. This system has many interesting properties, among which outstands its geodesic nature [33]; in fact the line $x^\alpha =$ constant results in a geodesic of the manifold and, since the normal to spatial hypersurfaces reads $n^\mu = (1, \vec{0})$, free particles are co-moving to this system. From a cosmological point of view the Gaussian frame is relevant because the galaxies are (almost) free particles in the actual universe and therefore our phenomenology relies on a synchronous gauge. However, the results obtained in this work have to remain valid for other gauge choices in view of the general covariance.

5. Kasner solution

To analyse the time evolution of the Kasner vectors and to obtain their rotation law, it is convenient to project them on the time-independent quantities π^α and to decompose them into the two parts:

$$\vec{\ell}_a = \vec{\ell}_{a\parallel} + \vec{\ell}_{a\perp}; \qquad \vec{\ell}_{a\parallel} = \frac{\widetilde{\lambda}_a}{\pi^2}\vec{\pi}, \qquad (\vec{\pi}\,\vec{\ell}_{a\perp}) = 0. \tag{24}$$

Here, for the sake of simplicity, we use the vector notation which is useful when treating the components of these vectors as Euclidean ones.

Hence we can split (22) into the two independent components

$$\frac{\partial}{\partial t}\widetilde{\lambda}_a = \sum_{b\neq a.} \frac{\left(\frac{\partial}{\partial t}p_b\right)\widetilde{\lambda}_a}{\widetilde{\lambda}_b\,(p_a - p_b)}\widetilde{\lambda}_b, \tag{25}$$

$$\frac{\partial}{\partial t}\vec{\ell}_{a\perp} = \sum_{b\neq a.} \frac{\left(\frac{\partial}{\partial t}p_b\right)\widetilde{\lambda}_a}{\widetilde{\lambda}_b\,(p_a - p_b)}\vec{\ell}_{b\perp} = A^a_b\,(t)\,\vec{\ell}_{b\perp}. \tag{26}$$

Since the above matrix A^a_b does not depend on $\vec{\ell}_{a\perp}$, then we get the formal solution

$$\vec{\ell}_{a\perp}\,(t) = T\,\exp\left\{\int_{t_0}^{t} A^a_b\left(t'\right)\mathrm{d}t'\right\}\vec{\ell}_{b\perp}(t_0). \tag{27}$$

The remaining equations (25) together with (20), (21) and (23) provide a self-consistent dynamical system.

Since the Kasner solution corresponds to neglecting the contributions of the n-dimensional Ricci scalar to the dynamics, then to get such behaviour we have to take the limit in which all the terms $\exp(q^a)$ become of higher order; under this assumption we get the following simplified dynamical system:

$$\begin{gathered} p_a = \text{const}, \qquad \widetilde{\lambda}_a = \text{const}, \qquad \vec{\ell}_{a\perp} = \text{const}, \\ \frac{\partial}{\partial t}q_a = \frac{2N}{\sqrt{g}}\left(p_a - \frac{1}{n-1}\sum_b p_b\right), \\ \sum p_a^2 - \frac{1}{n-1}\left(\sum p_a\right)^2 + \frac{1}{2}\sum \mathrm{e}^{q_a}\widetilde{\lambda}_a^2 = 0, \end{gathered} \tag{28}$$

whose solution, in the gauge $N = 1$ and towards the cosmological singularity ($g \to 0$), takes the form

$$g_{\alpha\beta} = \sum_a t^{2s_a}\ell^a_\alpha \ell^a_\beta, \qquad s_a = 1 - (n-1)\frac{p_a}{\sum_b p_b}, \tag{29}$$

where the Kasner indexes s_a satisfy the identities

$$\sum s_a = \sum s_a^2 = 1. \tag{30}$$

Let us take the Kasner indexes in the increasing order

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n. \tag{31}$$

In this way we always have $s_1 < 0$ and $s_n \geqslant s_{n-1} \geqslant 0$. Therefore the largest increasing term (as $t \to 0\; t^{s_1} \to \infty$) among the neglected ones comes from s_1 and it is to be taken into account to construct the oscillatory regime towards the cosmological singularity.

6. Billiard representation: the return map and the rotation of Kasner vectors

To construct the oscillatory regime we retain just the leading term corresponding to $\exp(q^1)$, so the field equations (20) and (21) can be rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial t}\widetilde{\lambda}_1 = 0,\\
&\frac{\partial}{\partial t}\widetilde{\lambda}_a = \frac{\left(\frac{\partial}{\partial t}p_1\right)\widetilde{\lambda}_a}{(p_a - p_1)}, \qquad a \neq 1,\\
&\frac{\partial}{\partial t}p_a = 0, \qquad a \neq 1,\\
&\frac{\partial}{\partial t}p_1 = -\frac{N}{2\sqrt{g}}\widetilde{\lambda}_1^2 \exp(q^1),\\
&\frac{\partial}{\partial t}q_a = \frac{2N}{\sqrt{g}}\left(p_a - \frac{1}{n-1}\sum_b p_b\right).
\end{aligned} \tag{32}$$

The first of equations (32) gives $\widetilde{\lambda}_1 = \text{const}$, while the second admits the solution

$$\widetilde{\lambda}_a\,(p_a - p_1) = \text{const}. \tag{33}$$

The remaining part of the dynamical system allows us to determine the return map governing the replacements of Kasner epochs (i.e. intervals of time during which the evolution is Kasner-like)

$$s_1' = \frac{-s_1}{1 + \frac{2}{n-2}s_1}, \qquad s_a' = \frac{s_a + \frac{2}{n-2}s_1}{1 + \frac{2}{n-2}s_1}, \tag{34}$$

$$\widetilde{\lambda}_1' = \widetilde{\lambda}_1, \qquad \widetilde{\lambda}_a' = \widetilde{\lambda}_a\left(1 - 2\frac{(n-1)\,s_1}{(n-2)\,s_a + ns_1}\right), \tag{35}$$

Defining the quantities $k_a = p_a - \frac{1}{n-1}\sum p_b$ we find for them the following iteration law:

$$k_1' = -k_1 \tag{36}$$

$$k_a' = k_a + \frac{2}{n-2}k_1 \tag{37}$$

$$\sum k' = \sum k + \frac{2}{n-2}k_1. \tag{38}$$

Our analysis is completed by investigating the rotation of Kasner vectors $\vec{\ell}_a$ through the epoch replacements; by (26) we get the system

$$\frac{\partial}{\partial t}\vec{\ell}_{a\perp} = \frac{\left(\frac{\partial}{\partial t}p_1\right)\widetilde{\lambda}_a}{\widetilde{\lambda}_1\,(p_a - p_1)}\vec{\ell}_{1\perp}, \qquad \frac{\partial}{\partial t}\vec{\ell}_{1\perp} = 0, \tag{39}$$

admitting the integral

$$\widetilde{\lambda}_1\vec{\ell}_{a\perp} - \widetilde{\lambda}_a\vec{\ell}_{1\perp} = \text{const}. \tag{40}$$

Putting together (24), (35) and (39) we arrive at the final iteration law

$$\vec{\ell}_a' = \vec{\ell}_a + \sigma_a\vec{\ell}_1, \qquad \sigma_a = \frac{\widetilde{\lambda}_a' - \widetilde{\lambda}_a}{\widetilde{\lambda}_1} = -2\frac{(n-1)\,s_1}{(n-2)\,s_a + ns_1}\frac{\widetilde{\lambda}_a}{\widetilde{\lambda}_1}. \tag{41}$$

which completes our dynamical scheme.

Thus the homogeneous universes discussed here approach the initial singularity being described by a metric tensor with oscillating scale factors and rotating Kasner vectors. Passing from one Kasner epoch to another, the negative Kasner index s_1 is exchanged between different directions (for instance $\vec{\ell}_1$ and $\vec{\ell}_2$) and at the same time these directions rotate in the space according to the law (41). The presence of a vector field is crucial because, independently of the considered model, it induces a (dynamically [16]) closed domain on the configuration space. The amplitudes of q^a-oscillations increase approaching the initial singularity and their minimum value approaches $-\infty$.

In correspondence with this oscillation of the scale factor the Kasner vectors $\vec{\ell}_a$ rotate and, at the lowest order in q^a, the quantities σ_a remain constant along a Kasner epoch; in this sense the vanishing behaviour of the determinant g approaching the singularity does not affect significantly the rotation law (41). We conclude by stressing that the duration of a Kasner epoch decrease as the singularity is approached in this scheme.

7. Concluding remarks

By the study developed above, we have shown how any multidimensional homogeneous cosmological model acquires, near the singularity, an oscillatory regime when an electromagnetic field is included in the dynamics. This result is a consequence of the capability of a vector field to generate a billiard configuration in the asymptotic evolution; such a 'billiard-ball' representation of the universe dynamics coincides with the BKL (Belinski–Khalatnikov–Liftshitz) piecewise approach only in the four-dimensional spacetime while in higher dimensions new features appear. In particular, the oscillatory regime obtained characterizes all the homogeneous models, disregarding their potential term; furthermore, the map by which the Kasner indexes evolve acquires a direct dependence on the dimension number.

A valuable issue of our Hamiltonian dynamics relies on fixing the rule of the Kasner vector rotation. Such a law of rotation is relevant to connect the Cauchy problem with later stages of the system dynamics. More precisely, the rotation of the Kasner vectors is sensitive to the boundary conditions on the 'matter' fields and has to be taken into account when using the studied homogeneous models within a cosmological 'picture'.

Once the spatial gradients are taken into account, our result can be extended to a generic inhomogeneous cosmological model, in the same spirit as the Bianchi type VIII and IX model oscillatory regime is upgraded in four dimensions.

Such an extension will reliably show how, in the presence of a vector field, the generic cosmological solution is described, in correspondence with any dimension number, by an oscillatory approach to the big bang.

Investigations in this direction, as well as those extended to the more general frameworks of superstrings [28, 29] and brane dynamics [35], will be a subject for further development.

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Attachment 3

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Inhomogeneous quantum Mixmaster: from classical towards quantum mechanics

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Abstract
Starting from the Hamiltonian formulation for the inhomogeneous Mixmaster dynamics, we approach its quantum features through the link of the quasi-classical limit. We fix the proper operator-ordering which ensures that the WKB continuity equation overlaps the Liouville theorem as restricted to the configuration space. We describe the full quantum dynamics of the model in some detail, providing a characterization of the (discrete) spectrum with analytic expressions for the limit of high occupation number. One of the main achievements of our analysis relies on the description of the ground state morphology, showing how it is characterized by a non-vanishing zero-point energy associated with the universe anisotropy degrees of freedom.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Einstein's equations, despite their nonlinearity, admit a generic solution asymptotically to the cosmological singularity having a piece-wise analytic expression. This dynamical regime was derived by Belinski, Khalatnikov and Lifshitz (BKL) [1, 2] and it has an oscillatory-like behaviour which extends point by point in space the homogeneous Mixmaster dynamics of the Bianchi type IX model [3]. When a generic initial condition is evolved towards the initial singularity, a spacetime foam [4, 5] appears: this is a direct consequence of the dynamical decoupling which characterizes space points close enough to the singularity and which can be appropriately described by the Mixmaster point like measure [4, 6] (a wide literature exists about the chaotic properties of the homogeneous Mixmaster model; see, for instance, [1, 7–10] and the references therein).

However, this classical description is in conflict with the requirement of a quantum behaviour of the universe through the Planck era; in fact there are reliable indications [8] that the Mixmaster dynamics overlap the quantum universe evolution, requiring an appropriate analysis of the transition between these two different regimes.

Indeed the dynamics of the very early universe correspond to a very peculiar situation with respect to the link existing between the classical and quantum regimes, because the expansion of the universe is the crucial phenomenon which maps these two stages of the evolution into each other. As shown in [8] the appearance of a classical background takes place essentially at the end of the Mixmaster phase when the anisotropy degrees of freedom can be treated as small perturbations; this result indicates that the oscillatory regime takes place almost during the Planck era and therefore it is a problem of quantum dynamics. However the end of the Mixmaster (and in principle the *quantum to classical* transition phase) is fixed by the initial conditions on the system and in particular it takes place when the cosmological horizon reaches the inhomogeneity scale of the model; therefore the question of an appropriate treatment for the semiclassical behaviour arises when the inhomogeneity scale is so larger than the Planck scale that the horizon can approach it only in the classical limit. Some interesting features of the quantum Mixmaster have been developed by [11–14] and [15, 16] for the homogeneous and the inhomogeneous cases, respectively.

Here we refine this analysis both connecting some properties of the quantum behaviour to the ensemble representation of the model and describing the precise effect of the boundary conditions on the structure of the quantum states.

We start by an Arnowitt–Deser–Misner (ADM) [17] representation of the system dynamics which allows us to disregard the contributions of the spatial gradients relative to the configurational variables, thus reducing the dynamics to a number of ∞^3 independent point-like Mixmaster models. Such representation can take place independently of the gauge choice [18], nevertheless here we require one of the Misner–Chitré-like variables to play the role of time for the system. In this way, we can describe the whole evolution in terms of a triangular billiard on the Poincaré upper half-plane where the point universe randomizes.

Then we apply the Liouville theorem restricted to the configuration space using the Hamilton–Jacobi solution of the model, and we require the continuity equation to match the WKB limit. Such a requirement leads us to determine the proper operator-ordering associated with the ADM operator in a unique way. We emphasize that the existence of an energy-like constant of motion not only provides a microcanonical measure for the statistical dynamics but also induces a quantum dynamics completely described by a time-independent Schrödinger equation on the Poincaré plane. However, the effective Hamiltonian associated with such an eigenvalue problem is non-local, characterized by the presence of a square root function; in agreement with the analysis developed by [19], we make the well-grounded hypothesis that the eigenfunctions' form is independent of the presence of the square root, since its removal implies the square of the eigenvalues only.

A crucial step relies on recognizing how our squared equation can be restated as the Laplace–Beltrami eigenvalue problem. Hence a characterization of the 'energy' spectrum comes out from imposing the triangular boundary conditions (the semicircle is replaced by a straight line preserving the measure of the domain).

We outline that the spectrum is discrete and that admits a point-zero energy which implies an intrinsic anisotropy of the quantum Mixmaster universe. We numerically investigate the first 'energy' levels and finally provide an analytic expressions for high occupation numbers.

The structure of the presentation is the following: section 2 is devoted to fixing the classical Hamiltonian formulation of the system dynamics, based on the use of Misner–Chitré-like variables which allow the ADM reduction of the degrees of freedom. In section 3 we fix the solution of the Hamilton–Jacobi equation within the domain allowed by the vanishing of the potential. Such a solution is at the ground level of the analysis presented in section 4; in fact here we restrict the Liouville theorem to the configuration space, eliminating the momentum dependence by virtue of an integration along the Hamilton–Jacobi dynamics. In section 5 we describe the Schrödinger equation for the Mixmaster model and construct the WKB limit; by the comparison of this semiclassical behaviour with the equation induced by the Liouville theorem, we get a unique choice for the operator-ordering of the super-Hamiltonian kinetic term.

Finally, in section 6 we provide a detailed description of the Mixmaster spectrum and eigenfunctions; particular attention is devoted to the ground state in view of the idea that it is the expected universe configuration during the Planck era. Section 7 is devoted to concluding remarks on the physical issues of our discussion, and particular attention is devoted to fix the full inhomogeneous quantum picture.

Over the whole paper we adopt units such that $c = 16\pi G = 1$.

2. Hamiltonian formulation

In this section we introduce the Hamiltonian formalism for a generic inhomogeneous model in vacuum in terms of the Misner–Chitré-like variables [3, 20].

In the ADM approach, the line element associated with such a model can be written in the form

$$\mathrm{d}s^2 = N^2\,\mathrm{d}\eta^2 - \gamma_{\alpha\beta}(\mathrm{d}x^\alpha + N^\alpha\,\mathrm{d}\eta)(\mathrm{d}x^\beta + N^\beta\,\mathrm{d}\eta), \tag{1}$$

where N and N^α denote the lapse function and the shift vector, respectively, while $\gamma_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ is the 3-metric tensor of the spatial hyper-surfaces $\Sigma^3\eta = \text{const}$, and we take [4]

$$\gamma_{\alpha\beta} = \mathrm{e}^{q^a}\delta_{ad}O^a_bO^d_c\frac{\partial y^b}{\partial x^\alpha}\frac{\partial y^c}{\partial x^\beta}, \qquad a, b, c, d, \alpha, \beta = 1, 2, 3, \tag{2}$$

where $q^a = q^a(x, \eta)$ and $y^b = y^b(x, \eta)$ are six scalar functions, and $O^a_b = O^a_b(x)$ an $SO(3)$ matrix (repeated indexes are summed).

Using the metric tensor (2), the action for the gravitational field reads

$$S = \int_{\Sigma^{(3)}\times\Re}\mathrm{d}\eta\,\mathrm{d}^3x\,(p_a\partial_\eta q^a + \Pi_d\partial_\eta y^d - NH - N^\alpha H_\alpha), \tag{3}$$

with

$$H = \frac{1}{\sqrt{\gamma}}\left[\sum_a(p_a)^2 - \frac{1}{2}\left(\sum_b p_b\right)^2 - \gamma\,^{(3)}R\right] \tag{4}$$

$$H_\alpha = \Pi_c\partial_\alpha y^c + p_a\partial_\alpha q^a + 2p_a(O^{-1})^b_a\partial_\alpha O^a_b; \tag{5}$$

H and H_α being the super-Hamiltonian and the super-momentum, respectively.

In (4) and (5) p_a and Π_c play the role of conjugate momenta to the variables q^a and y^b, respectively, and $^{(3)}R$ is the Ricci 3-scalar which behaves like a potential term.

No sooner should we adopt y^a like new spatial variables, than the super-momentum constraint $H_\alpha = 0$ (provided by the variation of (3) with respect to N^α) can be solved, and the action rewrites as [18]

$$S = \int_{\Sigma^{(3)}\times\Re} \mathrm{d}\eta\, \mathrm{d}^3y \left(p_a \partial_\eta q^a + 2p_a (O^{-1})^c_a \partial_\eta O^a_c - NH\right). \tag{6}$$

Furthermore the potential term appearing in the super-Hamiltonian (4) can be approximated towards the singularity as

$$U = \sum_a \Theta(Q_a), \tag{7}$$

where

$$\Theta(x) = \begin{cases} +\infty & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{8}$$

$$Q_a = \frac{q^a}{\sum_a q^a}. \tag{9}$$

This picture arises from the vanishing of the metric tensor determinant close to the singularity, and the quantities Q_a, known in the literature as *anisotropy parameters*, cut a closed domain Γ_Q to which the dynamics is restricted.

By virtue of the potential structure (7), the universe dynamics evolves independently in each space point inducing a corresponding factorization for the phase space of the model.

Since in Γ_Q the potential U asymptotically vanishes, close to the singularity the relation $\partial p_a/\partial\eta = 0$ holds and then the term $2p_a(O^{-1})^c_a\partial_\eta O^a_c$ in (6) behaves like an exact time-derivative and can be ruled out of the variational principle.

The ADM reduction is completed by introducing the so-called Misner–Chitré-like variables [4, 10, 20, 21], whose relevance consists in making the anisotropy parameters independent of τ, which will behave as the time variable:

$$\begin{cases} q^1 = \mathrm{e}^\tau\left[\sqrt{\xi^2-1}(\cos\theta + \sqrt{3}\sin\theta) - \xi\right] \\ q^2 = \mathrm{e}^\tau\left[\sqrt{\xi^2-1}(\cos\theta - \sqrt{3}\sin\theta) - \xi\right] \\ q^3 = -\mathrm{e}^\tau\left(\xi + 2\sqrt{\xi^2-1}\cos\theta\right). \end{cases} \tag{10}$$

In terms of these new variables the super-Hamiltonian rewrites

$$H = -p_\tau^2 + p_\xi^2(\xi^2-1) + \frac{p_\theta^2}{\xi^2-1} \tag{11}$$

and the Q_a become

$$\begin{aligned} Q_1 &= \frac{1}{3} - \frac{\sqrt{\xi^2-1}}{3\xi}(\cos\theta + \sqrt{3}\sin\theta) \\ Q_2 &= \frac{1}{3} - \frac{\sqrt{\xi^2-1}}{3\xi}(\cos\theta - \sqrt{3}\sin\theta) \\ Q_3 &= \frac{1}{3} + \frac{2\sqrt{\xi^2-1}}{3\xi}\cos\theta. \end{aligned} \tag{12}$$

Let us solve the constraint $H = 0$ (obtained variating N in (3)) with respect to p_τ in the domain Γ_Q in order to perform the ADM reduction, thus obtaining

$$-p_\tau \equiv \epsilon = \sqrt{(\xi^2-1)p_\xi^2 + \frac{p_\theta^2}{\xi^2-1}}. \tag{13}$$

Then, taking the time gauge $\partial_\eta \tau = 1$, the reduced action explicitly reads as

$$S_{\Gamma_Q} = \int \mathrm{d}\tau\, \mathrm{d}^3 y\, \left(p_\xi \partial_\tau \xi + p_\theta \partial_\tau \theta - \epsilon\right). \tag{14}$$

As we approach the singularity, ϵ behaves like a constant of motion, i.e. $\mathrm{d}\epsilon/\mathrm{d}\tau = \partial\epsilon/\partial\tau = 0 \Rightarrow \epsilon = E(y^a)$.

The dynamics of such a model is equivalent to (that of) a billiard ball on a Lobatchevsky plane [7, 14, 18, 22]; this can be shown by the use of the Jacobi metric, i.e. a geometric approach that reduces the dynamical equations of a generic system to a geodesic problem on a manifold. Such a technique applied to (14) produces a geodesic equation corresponding to the line element

$$\mathrm{d}l^2 = E^2(y^a)\left(\frac{\mathrm{d}\xi^2}{(\xi^2 - 1)} + (\xi^2 - 1)\, \mathrm{d}\theta^2\right). \tag{15}$$

The manifold described by (15) turns out to have a constant negative curvature, where the Ricci scalar is given by $R = -2/E^2$: the complex dynamics of the generic inhomogeneous model results in a collection of decoupled dynamical systems, one for each point of the space, and all of them equivalent to a billiard problem on a Lobatchevsky plane.

Among the possible representations for it, we choose the so-called Poincaré model in the complex upper half-plane [6] that can be introduced with the following coordinate transformation:

$$\begin{cases} \xi = \dfrac{1 + u + u^2 + v^2}{\sqrt{3}v} \\ \theta = -\arctan \dfrac{\sqrt{3}(1 + 2u)}{-1 + 2u + 2u^2 + 2v^2}. \end{cases} \tag{16}$$

The line element for this two-dimensional surface now reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}u^2 + \mathrm{d}v^2}{v^2}. \tag{17}$$

Figure 1 shows how, in terms of these coordinates, the anisotropy functions cut a stripe in the plane $(u, 0, v)$ characterized by a finite measure $\mu = \pi$ and three unstable directions to asymptotically escape.

The boundary is given by the set of points where one of the anisotropy parameters is equal to zero; it is a geodesic triangle, where the edges are given by the vanishing of the numerators of Q_a, now reading

$$\begin{cases} Q_1(u, v) = -u/\delta \\ Q_2(u, v) = (1 + u)/\delta \\ Q_3(u, v) = (u^2 + u + v^2)/\delta \end{cases} \tag{18}$$
$$\delta = u^2 + u + 1 + v^2.$$

In the (u, v) scheme, the ADM Hamiltonian (13) assumes the expression

$$H_{\mathrm{ADM}} = \epsilon = v\sqrt{p_u^2 + p_v^2}, \tag{19}$$

which will be the starting point of our analysis.

We conclude by observing how [1, 2] the present representation of the Lobatchevsky plane is suitable to link this Hamiltonian formulation to the piece-wise representation provided by Belinski, Khalatnikov and Lifshitz map, as it is easy to recognize that for $v = 0$, the functions Q_a reduce to the familiar Kasner indexes.

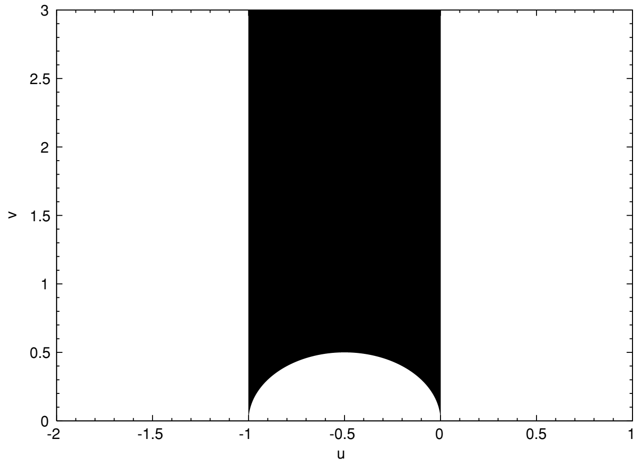

Figure 1. The domain $\Gamma_Q(u, v)$ in the Poincaré upper half-plane where the dynamics of the space-point universe is asymptotically restricted by the potential. The measure is finite and equal to π.

3. Hamilton–Jacobi approach

We shall now derive the Hamilton–Jacobi (HJ) [23] equations for the system (19) in view of the following developments.

The HJ prescribes to change the momentum variables into the derivatives of a functional $\mathcal{S}$ with respect to the configurational variable: implementing the HJ technique, the Hamiltonian relation (19) leads to the functional differential equation in Γ_Q:

$$-\frac{\delta \mathcal{S}}{\delta \tau} = v\sqrt{\left(\frac{\delta \mathcal{S}}{\delta u}\right)^2 + \left(\frac{\delta \mathcal{S}}{\delta v}\right)^2}. \tag{20}$$

To obtain the solution of this dynamical equation, we take $\mathcal{S}$ in the form

$$\mathcal{S} = \mathcal{S}_0(u, v) - \left(\int \mathrm{d}^3 y\, \epsilon(y^a)\right)\tau, \tag{21}$$

where $\mathcal{S}_0$ satisfies the equation

$$\epsilon^2 = v^2\left(\left(\frac{\delta \mathcal{S}_0}{\delta u}\right)^2 + \left(\frac{\delta \mathcal{S}_0}{\delta v}\right)^2\right) \tag{22}$$

and it is therefore provided by

$$\mathcal{S}_0(u, v) = k(y^a)u + \sqrt{\epsilon^2 - k^2(y^a)v^2} - \epsilon \ln\left(2\frac{\epsilon + \sqrt{\epsilon^2 - k^2(y^a)v^2}}{\epsilon^2 v}\right) + c(y^a), \tag{23}$$

where $k(y^a)$, $c(y^a)$ are integration constants.

The expression (21), together with (23) and the features of the potential wall (18), summarizes the classical dynamics of a generic inhomogeneous universe.

4. The statistical mechanic description

In section 2 we pointed out that, for the Mixmaster inhomogeneous dynamics, the spatial points decouple approaching the singularity and an energy-like constant of motion appears; let us discuss the problem from a statistical mechanics point of view by treating the system as a microcanonical ensemble.

The physical properties of a stationary ensemble are described by a distribution function $\rho = \rho(u, v, p_u, p_v)$, obeying the continuity equation defined in the phase space (u, v, p_u, p_v) as

$$\frac{\partial(\dot{u}\rho)}{\partial u} + \frac{\partial(\dot{v}\rho)}{\partial v} + \frac{\partial(\dot{p}_u\rho)}{\partial p_u} + \frac{\partial(\dot{p}_v\rho)}{\partial p_v} = 0, \tag{24}$$

where the dot denotes the time derivative

$$\begin{aligned} \dot{u} &\equiv \frac{\partial u}{\partial \tau} = \frac{\partial H_{\mathrm{ADM}}}{\partial p_u} = \frac{v^2}{\epsilon} p_u & \qquad \dot{p}_u &\equiv \frac{\partial p_u}{\partial \tau} = -\frac{\partial H_{\mathrm{ADM}}}{\partial u} = 0 \\ \dot{v} &\equiv \frac{\partial v}{\partial \tau} = \frac{\partial H_{\mathrm{ADM}}}{\partial p_v} = \frac{v^2}{\epsilon} p_v & \qquad \dot{p}_v &\equiv \frac{\partial p_v}{\partial \tau} = -\frac{\partial H_{\mathrm{ADM}}}{\partial v} = -\frac{\epsilon}{v}. \end{aligned} \tag{25}$$

We stress how the above continuity equation provides an appropriate representation for the ensemble associated with the Mixmaster only when we are sufficiently close to the initial singularity and therefore the infinite-potential wall approximation works; in fact, such a model for the potential term corresponds to deal with the energy-like constant of the motion and fixes the microcanonical nature of the ensemble. From a dynamical point of view this picture arises naturally because the universe volume element vanishes monotonically (for non-stationary correction to this scheme in the Misner–Chitré-like variables see [24]).

We are interested in studying the distribution function in the (u, v) space, and thus we will reduce the dependence on the momentum variables by integrating $\rho(u, v, p_u, p_v)$ in the momenta space and using the informations contained in the HJ approach. If we assume ρ to be a regular, vanishing at the infinity of the phase space, limited function of its arguments, and if we use (23), we work out the following equation for $\tilde{w}(u, v; k) = \int \rho(u, v, p_u, p_v)\, \mathrm{d}p_u\, \mathrm{d}p_v$:

$$\frac{\partial \tilde{w}(u, v; k)}{\partial u} + \sqrt{\left(\frac{E}{kv}\right)^2 - 1}\,\frac{\partial \tilde{w}(u, v; k)}{\partial v} + \frac{E^2 - 2k^2v^2}{kv^2}\frac{\tilde{w}(u, v; k)}{\sqrt{E^2 - (kv)^2}} = 0 \tag{26}$$

admitting a solution in terms of a generic function g of the form

$$\tilde{w}(u, v) = \frac{g\left(u + v\sqrt{\frac{E^2}{k^2v^2} - 1}\right)}{v\sqrt{E^2 - k^2v^2}}. \tag{27}$$

The distribution function in (u, v) is obtained after the integration over the constant k. Indeed, this constant expresses the freedom of choosing the initial conditions (25), which cannot affect the chaotic properties of the model. Therefore we define the reduced distribution $w(u, v)$ as

$$w(u, v) \equiv \int_A \tilde{w}(u, v; k)\, \mathrm{d}k, \tag{28}$$

where the integration is over the classical available domain for $p_u \equiv k$:

$$A \equiv \left[-\frac{E}{v}, \frac{E}{v}\right]. \tag{29}$$

It is easy to verify that the microcanonical Liouville measure [1, 6, 7, 10, 18, 24, 25] w_{mc} (after integration over the admissible values of ϵ) corresponds to the case $g = \text{const}$, i.e. we get the normalized distribution

$$w_{mc}(u, v) = \int_{-\frac{E}{v}}^{\frac{E}{v}} \frac{1}{kv^2\sqrt{\frac{E^2}{k^2v^2} - 1}}\,\mathrm{d}k = \frac{\pi}{v^2}. \tag{30}$$

Summarizing, we have derived the generic expression of the distribution function for our model, fixing its form for the microcanonical ensemble which, in view of the energy-like constant of the motion ϵ, is the most appropriate to describe the Mixmaster system when the picture is restricted to the configuration space.

5. Quasi-classical limit of the quantum regime

Let us underline some common features between the classical and the semi-classical dynamics with the aim of fixing the proper operator-ordering [26] in treating the quantum approach.

In fact, considering the WKB limit for $\hbar \to 0$, the coincidence between the classical distribution function $w_{mc}(u, v)$ and the quasi-classical probability function $r(u, v)$ takes place for a precise choice of the operator-ordering [16] only (for a connected topic see [8] and for a general discussion see [27]).

As soon as we implement the canonical variables into operators, as in the canonical quantization, i.e.

$$\hat{v} \to v, \quad \hat{u} \to u$$
$$\hat{p}_v \to -\mathrm{i}\hbar\frac{\partial}{\partial v}, \quad \hat{p}_u \to -\mathrm{i}\hbar\frac{\partial}{\partial u}, \quad \hat{p}_\tau \to -\mathrm{i}\hbar\frac{\partial}{\partial \tau},$$

the quantum dynamics for the state function $\Phi(u, v, \tau)$ obeys, in each point of the space, the Scrhödinger equation

$$\mathrm{i}\hbar\frac{\partial\Phi(u, v, \tau)}{\partial\tau} = \hat{H}_{\mathrm{ADM}}\Phi(u, v, \tau) = \hbar\sqrt{-v^2\frac{\partial^2}{\partial u^2} - v^{2-a}\frac{\partial}{\partial v}\left(v^a\frac{\partial}{\partial v}\right)}\Phi(u, v, \tau), \tag{31}$$

where we have adopted a generic operator-ordering for the position and momentum operators parametrized by the constant a [28].

In the above equation the Hamiltonian operator has a non-local character as a consequence of the square root function; in principle this constitutes a subtle question about the quantum implementation though. We will make the well-grounded ansatz [19] that the operators $\hat{H}_{\mathrm{ADM}}$ and $\hat{H}^2_{\mathrm{ADM}}$ have the same set of eigenfunctions with eigenvalues E and E^2, respectively[4].

If we take the following integral representation for the wavefunction Φ,

$$\Phi(u, v, \tau) = \int_{-\infty}^{\infty} \Psi(u, v, E)\,\mathrm{e}^{-\mathrm{i}E\tau/\hbar}\,\mathrm{d}E, \tag{32}$$

the eigenvalues problem reduces to

$$\hat{H}^2\Psi(u, v, E) = \hbar^2\left[-v^2\frac{\partial^2}{\partial u^2} - v^{2-a}\frac{\partial}{\partial v}\left(v^a\frac{\partial}{\partial v}\right)\right]\Psi(u, v, E) = E^2\Psi(u, v, E). \tag{33}$$

In order to study the WKB limit of equation (33), we separate the wavefunction into its phase and modulus

$$\Psi(u, v, E) = \sqrt{r(u, v, E)}\,\mathrm{e}^{\mathrm{i}\sigma(u,v,E)/\hbar}. \tag{34}$$

[4] The problems discussed in this respect by [29] do not arise here because in the domain Γ_Q our ADM Hamiltonian has a positive sign (the potential vanishes asymptotically).

In this scheme the function $r(u, v)$ represents the probability density, and the quasi-classical regime appears as we take the limit for $\hbar \to 0$; substituting (34) in (33) and retaining only the lower order in $\hbar$, we obtain the system

$$\begin{cases} v^2\left[\left(\frac{\partial\sigma}{\partial u}\right)^2+\left(\frac{\partial\sigma}{\partial v}\right)^2\right]=E^2 \\ \frac{\partial r}{\partial u}\frac{\partial\sigma}{\partial u}+\frac{\partial r}{\partial v}\frac{\partial\sigma}{\partial v}+r\left(\frac{a}{v}\frac{\partial\sigma}{\partial v}+\frac{\partial^2\sigma}{\partial v^2}+\frac{\partial^2\sigma}{\partial u^2}\right)=0. \end{cases} \tag{35}$$

In view of the HJ equation and of the Hamiltonian (19), we can identify the phase σ to the functional S_0 of the HJ approach.

Now we turn our attention to the equation for $r(u, v)$; taking (23) into account it reduces to

$$k\frac{\partial r(u,v,E)}{\partial u}+\sqrt{\left(\frac{E}{v}\right)^2-k^2}\frac{\partial r(u,v,E)}{\partial v}+\frac{a(E^2-k^2v^2)-E^2}{v^2\sqrt{E^2-k^2v^2}}r(u,v,E)=0. \tag{36}$$

Comparing (36) with (26), we easily see that they coincide (as expected) for $a = 2$ only.

It is worth noting that this correspondence is expectable once a suitable choice for the configurational variables is taken; however here it is remarkable that it arises only if the above operator-ordering is addressed. It is just in this result the importance of this correspondence whose request fixes a particular quantum dynamics for the system.

Summarizing, we have demonstrated from our study that it is possible to get a WKB correspondence between the quasi-classical regime and the ensemble dynamics in the configuration space, and we provided the operator-ordering when quantizing the inhomogeneous Mixmaster model to be [28]

$$\hat{v}^2\hat{p}_v^2 \to -\hbar^2\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right). \tag{37}$$

By means of these results, we now face the full quantum dynamical approach.

6. Canonical quantization and the energy spectrum

Our starting point is the point-like eigenvalue equation (33) (for a discussion of this same eigenvalue problem in the Misner variables see [3]) which, together with the boundary conditions, completely describes the quantum features of the model, i.e.

$$\begin{aligned} &\left[v^2\frac{\partial^2}{\partial u^2}+v^2\frac{\partial^2}{\partial v^2}+2v\frac{\partial}{\partial v}+\left(\frac{E}{\hbar}\right)^2\right]\Psi(u,v,E)=0 \\ &\Psi(\partial\Gamma_Q)=0. \end{aligned} \tag{38}$$

In equation (38) we can recognize a well-known operator: by redefining $\Psi(u, v, E) = \psi(u, v, E)/v$, we can reduce (38) to the eigenvalue problem for the Laplace–Beltrami operator in the Poincaré plane

$$\nabla_{\rm LB}\psi(u,v,E)\equiv v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\psi(u,v,E)=E_s\psi(u,v,E), \tag{39}$$

which is central in the harmonic analysis on symmetric spaces and has been widely investigated in terms of its invariance under $SL(2, C)$(for a detailed discussion, see [30] and the bibliography therein).

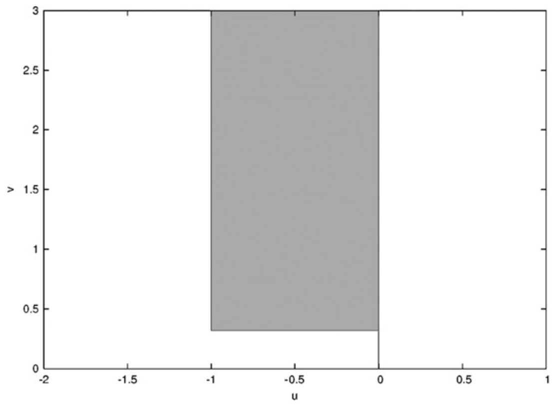

Figure 2. The approximate domain where we impose the boundary conditions. The choice $v = 1/\pi$ for the straight line preserves the measure $\mu = \pi$.

Its eigenstates and eigenvalues are described as

$$\psi_s(u,v) = av^s + bv^{1-s} + \sqrt{v}\sum_{n\neq 0} a_n K_{s-1/2}(2\pi|n|v)\,\mathrm{e}^{2\pi \mathrm{i} nu}, \qquad a, b, a_n \in C \tag{40}$$
$$\nabla_{\mathrm{LB}}\psi_s(u,v) = s(s-1)\psi_s(u,v),$$

where $K_{s-1/2}(2\pi n v)$ are the modified Bessel functions of the third kind, and s denotes the index of the eigenfunction. This is a continuous spectrum, and the summation runs over every real value of n.

The eigenfunctions for our model read as

$$\Psi(u,v,E) = av^{s-1} + bv^{-s} + \sum_{n\neq 0} a_n \frac{K_{s-1/2}(2\pi|n|v)}{\sqrt{v}}\,\mathrm{e}^{2\pi \mathrm{i} nu} \tag{41}$$

with eigenvalue

$$E^2 = s(1-s)\hbar^2. \tag{42}$$

The spectrum and the explicit eigenfunctions are obtained by imposing the boundary conditions (38), i.e. requiring that equation (41) vanishes on the edges of the geodesic triangle of figure 1.

Since from our analysis no way arose to impose exact boundary conditions we approximate the domain with the simpler one in figure 2; the value for the horizontal line $y = 1/\pi$ provides the same measure for the exact and the approximate domain.

The conditions on the vertical lines $u = 0$, $u = -1$ require to disregard the first two terms in (41) ($a = b = 0$); furthermore we get the condition on the last term

$$\sum_{n\neq 0} \mathrm{e}^{2\pi \mathrm{i} nu} \to \sum_{n=1}^{\infty} \sin(2\pi n u),$$

n being an integer, while the condition on the horizontal line implies

$$\sum_{n>0} a_n K_{s-1/2}(2n)\sin(2n\pi u) = 0, \qquad \forall u \in [-1, 0], \tag{43}$$

which is satisfied by requiring $K_{s-1/2}(2n) = 0$ only.

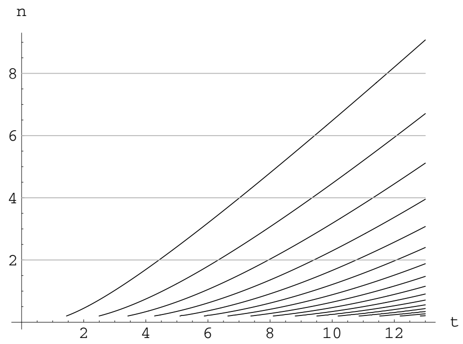

Figure 3. The intersections between the straight lines and the curves represent the roots of the equation $K_{it}(n) = 0$, where K is the modified Bessel function.

Table 1. We numerically investigated the roots of $K_{it}(n) = 0$ with respect to t for different values of n. The values in the columns are the first t's in the above figure.

$n = 2$	$n = 4$	$n = 6$	$n = 8$
4.425	7.016	9.434	11.768
6.333	9.353	12.076	14.655
7.947	11.313	14.283	17.059
9.410	16.264	16.263	19.212
10.774	17.742	18.096	21.203

The functions $K_\nu(x)$ are real and positive for real argument and real index, therefore the index must be imaginary, i.e. $s = \frac{1}{2} + \mathrm{i}t$. In this case these functions have (only) real zeros, and the corresponding eigenvalues turn out to be real and positive

$$(E/\hbar)^2 = t^2 + 1/4. \tag{44}$$

We remark that our eigenfunctions naturally vanish as infinite values of v are approached.

The conditions (43) cannot be solved analytically for all the values of n and t, and the roots must be worked out numerically for each n; there are several results on their distribution that allow us to find at least the first levels. A theorem on the zeros of these functions states that $K_{i\nu}(\nu x) = 0 \Leftrightarrow 0 < x < 1$ (for a proof see [33]); by this theorem and the monotonic dependence of the energy (44) on the zeros, we can easily search the lowest levels by solving equation (43) for the first n. In figure 3 and table 1 we plot the first roots, and in figure 4 and in table 2 we list the first ten 'energy' levels[5].

Below we will provide an analytical treatment of the high-energy levels associated with our operator in correspondence to certain region of the plane $(t, 0, n)$. In [14] the structure of the high-energy levels is also connected to the so-called *quantum chaos* of the Mixmaster model, which was studied from the wavefunctional point of view in [12] and from the path integral one in [13] using Misner variables. Our analysis implicitly contains the information about the quantum chaos of the considered dynamics; in fact we can take a generic state of

[5] For a detailed numerical investigation of the energy spectrum of the standard Laplace–Beltrami operator, especially with respect to the high-energy levels, see [31, 32], where the effects on the level spacing of deforming the circular boundary condition towards the straight line are also numerically analysed.

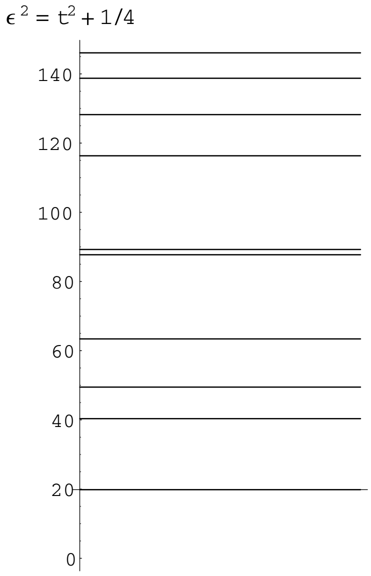

Figure 4. The lines represent in a typical spectral manner the first levels.

Table 2. The first ten energy eigenvalues, ordered from the lowest one.

$\left(\frac{E}{\hbar}\right)^2 = t^2 + \frac{1}{4}$
19.831
40.357
49.474
63.405
87.729
89.250
116.329
128.234
138.739
146.080

the system $\xi(\tau, u, v)$ in the form

$$\xi(\tau, u, v) = \int \mathrm{d}t \sum_n c_{t,n} \Psi_n(\tau, u, v) \delta(K_{it}(2n)) \, \mathrm{e}^{-\mathrm{i}\sqrt{\frac{1}{4}+t^2}\tau} \tag{45}$$

and its evolution from a generic initial condition $\xi_0(u, v) \equiv \xi(\tau_0, u, v)$ at an initial instant τ_0 provides all the quantum properties of the system. The quantum chaos is recognized in [12] by numerically integrating the Wheeler-De Witt equation from a gaussian like initial packet and outlining the appearance of a fractal structure in the profile of the resulting wavefunction; in our approach (as the infinite potential walls approximation works) the dynamics is provided by (45) and evolving it from an initial localized wave packet the quantum chaos has to arise.

About the information on the quantum chaos emerging from the high-energy spectrum we emphasize the following two important points: (i) the non-stationary corrections due to the

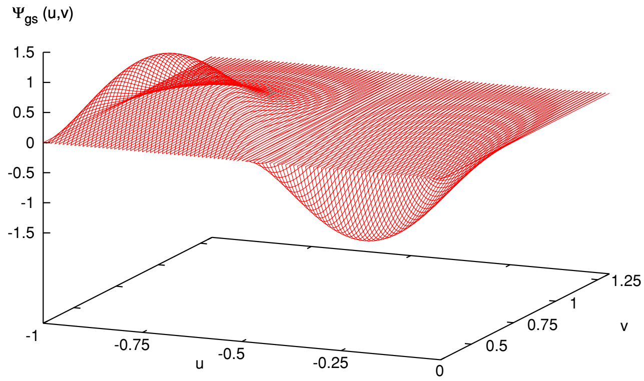

Figure 5. The ground state wavefunction is sketched.

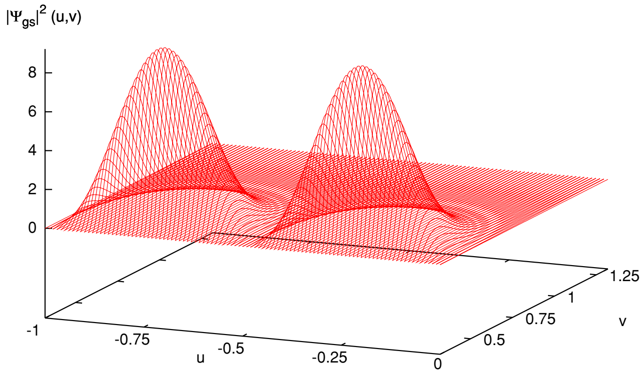

Figure 6. We show the probability distribution in correspondence to the ground state of the theory, according to the boundary conditions.

real potential term are expected to be simply small perturbation to our result rapidly decaying as the singularity is approached; (ii) the analytic expressions we are going to provide for large values of t cannot be used to fix the existence of the quantum chaos because they explore limited regions of the plane $(t, 0, n)$ only, but they are very useful to clarify the morphology of the spectrum and its dependence on two different quantum numbers.

6.1. The ground state

Let us describe the properties of the ground state level with a major accuracy, starting from the result of its existence with a non-zero energy E_0, i.e. $E_0^2 = 19.831\hbar^2$.

In figure 5 we plot the wavefunction Ψ_{gs} in the $(u, 0, v)$ plane and in figure 6 the corresponding probability distribution $|\Psi_{gs}|^2$, whose normalization constant is equal to $N = 739.466$. The knowledge of the ground state eigenfunction allows us to estimate the average values of the anisotropy variables u, v and the corresponding root mean square, i.e.

$$\langle u \rangle = -1/2, \qquad \langle v \rangle = 0.497 \tag{46}$$

$$\Delta u = 0.266, \qquad \Delta v = 0.077. \tag{47}$$

This result tells us that in the ground state the universe is not exactly an isotropic one, and it fluctuates around the line of the Misner plane $\beta_- = 0$. However we observe that it remains localized in the centre of the Misner space far from the corner at $v \to \infty$ (the other two equivalent corners were cut out from our domain by the approximation we considered on the boundary conditions, but, because of the potential symmetry, they have to be unaccessible too). Thus we can conclude that the universe, approaching the minimal energy configuration, conserves a certain degree of anisotropy but lives in the region where the latter can be treated as a small correction to the full isotropy. Such a behaviour is a consequence of the zero-point energy associated with the ground state which prevents the absence of oscillation modes concerning the anisotropy degrees of freedom; we numerically derived this feature but it can be inferred on the basis of general considerations about the Hamiltonian structure. In fact the Hamiltonian contains a term $v^2\hat{p}_v^2$ which has positive definite spectrum and cannot admit vanishing eigenvalue.

6.2. Asymptotic expansions

In order to study the distribution of the highest energy levels, let us take into account the asymptotic behaviour of the zeros for the modified Bessel functions of the third kind.

We will discuss asymptotic regions of the plane $(t, 0, n)$ in both the cases (i) $t \gg n$ and (ii) $t \simeq n \gg 1$.

(i) For $t \gg n$ the Bessel functions admit the following representation:

$$K_{it}(n) = \frac{\sqrt{2\pi}\,\mathrm{e}^{-t\pi/2}}{(t^2-n^2)^{1/4}}\left[\sin a \sum_{k=0}^{\infty}\frac{(-1)^k}{t^{2k}}u_{2k}\left(\frac{1}{\sqrt{1-p^2}}\right) + \cos a \sum_{k=0}^{\infty}\frac{(-1)^k}{t^{2k+1}}u_{2k+1}\left(\frac{1}{\sqrt{1-p^2}}\right)\right] \tag{48}$$

where $a = \pi/4 - \sqrt{t^2-n^2} + t\,\mathrm{arccosh}(t/n)$, $p \equiv n/t$ and u_k are the following polynomials:

$$\begin{cases} u_0(t) = 1 \\ u_{k+1}(t) = \dfrac{1}{2}t^2(1-t^2)u_k'(t) + \dfrac{1}{8}\displaystyle\int_0^1 (1-5t^2)u_k(t)\,\mathrm{d}t. \end{cases} \tag{49}$$

Retaining in the above expression only those terms of $\mathrm{o}\left(\frac{n}{t}\right)$, the zeros are fixed by the following relation:

$$\sin\left(\frac{\pi}{4} - t + t(\log(2) - \log(p))\right) - \frac{1}{12t}\cos\left(\frac{\pi}{4} - t + t(\log(2) - \log(p))\right) = 0, \tag{50}$$

that in the limit for $n/t \ll 1$, can be recast as follows:

$$t\log(t/n) = l\pi \Rightarrow t = \frac{l\pi}{\mathrm{productlog}\left(\frac{l\pi}{n}\right)}, \tag{51}$$

where productlog(z) is a generalized function that gives the solution of the equation $z = w\,\mathrm{e}^w$ and for real and positive domain, it is a monotonic function of its argument. In (51) l is an integer number that must be taken much greater than 1 in order to verify the initial assumptions $n/t \ll 1$.

(ii) In case the difference between $2n$ and t is $\mathsf{o}(n^{1/3})(t, n \gg 1)$, we can evaluate the first zeros by the expansion (worked out from formula (9) of [34])

$$t = 2n + 0.030n^{1/3}, \tag{52}$$

providing the lowest zero (and therefore the energy) for a fixed value of n and then the relation for the eigenvalues for high occupation numbers as

$$\left(\frac{E}{\hbar}\right)^2 \sim 4n^2 + 0.12n^{4/3}. \tag{53}$$

In this region of the spectrum the condition $t \sim 2n$, reduces the two quantum numbers characterizing the system to a single one, the whole wavefunction becoming assigned by n.

The above analysis shows that, as effect of Dirichlet boundary condition and in the limit of high occupation numbers, we get an analytical expressions for the discrete structure of the spectrum. In fact for large values of t it was possible to give analytical representations for the position of the zeros, but we emphasize that the request to deal with these approximations causes the loss of a large number of levels and prevents a complete discussion of the quantum chaos associated with the model. However the above expressions are of interest because they allow us to compare these results with the corresponding spectrum provided by Misner in his original work [11]. Of course, such a comparison of the two results can take place only on a qualitative level; in fact, between Misner $(\alpha, \beta_+, \beta_-)$ and Misner–Chitré like (τ, u, v) variables a crucial difference exists and it has to be recognized into the correspondingly different behaviour of the potential walls. In the Misner scheme the domain available to the point universe increases as $\alpha \to \infty$ (α being the isotropic Misner variable) and, therefore, we deal with a non-stationary infinite well; the Misner–Chitré like variables allow us to fix the infinite potential walls into a time-independent configuration in the $(u, 0, v)$ plane. However we can at least compare the behaviour of the energy eigenvalues with respect to the occupation number n.

In his original treatment, Misner replaces, for fixed α values (indeed he makes use of an adiabatic approximation, see [16]), the triangular well by a square of equal measure and determines the energy spectrum in the form

$$\frac{E_n^M}{\hbar} = \frac{\pi}{\sqrt{S}}\sqrt{n_+^2 + n_-^2} \equiv \frac{A}{\alpha}|n|, \tag{54}$$

where S denotes the triangular well measure $S = \alpha^2/A^2$, A being a numerical factor; $n_\pm$ denote the occupation numbers relative to $\beta_\pm$ respectively.

In our approach, for sufficiently large n, we get the dominant behaviours

$$\frac{E_n^{MC}}{\hbar} \sim \begin{cases} \text{(i) } \dfrac{l\pi}{\text{productlog}\left(\frac{l\pi}{n}\right)} \\ \text{(ii) } 2n + \mathcal{O}(n^{2/3}). \end{cases} \tag{55}$$

Thus we see that, in case (ii), a part from a numerical factor (which is different because of the different approximation made on the real domains), the only significant difference relies on the term α^{-1}; in the case (i) instead the situation is a bit different because two quantum numbers explicitly remain and we get a linear relation as far as we require $l \propto n$ by a factor much greater than the unity (indeed the function productlog$(l\pi/n)$ provides a smooth contributions in the considered region $l \gg n$). The difference of the factor α^{-1} with respect to the Misner case can be easily accounted as soon as we observe that the following relations hold:

$$\beta_\pm = \alpha b_\pm(u, v), \tag{56}$$

where the functions $b_{\pm}$ can be calculated from the coordinates transformations

$$\begin{cases} \alpha = -\mathrm{e}^{\tau}\dfrac{1+u+u^2+v^2}{\sqrt{3}v} \\ \beta_+ = \mathrm{e}^{\tau}\dfrac{-1+2u+2u^2+2v^2}{2\sqrt{3}v} \\ \beta_- = -\mathrm{e}^{\tau}\dfrac{1+2u}{2v} \end{cases} \tag{57}$$

but their form is not relevant in what follows. On the basis of (56), the measure of a domain D in the $\beta_{\pm}$ plane reads

$$\int_D \mathrm{d}\beta_+\,\mathrm{d}\beta_- = \alpha^2 \int_{D'} |J(u,v)|\,\mathrm{d}u\,\mathrm{d}v, \tag{58}$$

J being the Jacobian of the transformation associated with $b_{\pm}$, while D' is the image of D onto the $(u, 0, v)$ plane. As a consequence we see that between a measure s in the $\beta_{\pm}$ plane and a similar one (even not exactly its image), there is a difference for a factor α^2 which immediately provides an explanation for the difference in the two energy spectra.

7. The inhomogeneous picture and conclusions

At the end of our analysis, we wish to bring the reader's attention to some physical aspects of the inhomogeneous Mixmaster.

First of all, the obtained dynamics regime is indeed a generic one; in fact, in a synchronous reference (for which $\partial_t y^a = 0$), the inhomogeneous Mixmaster contains four independent (physically) arbitrary functions of the spatial coordinates ($y^a(x^i)$ and $E(x^i)$), available for the Cauchy data on a non-singular hyper-surface.

Let us now come back to the full inhomogeneous problem, in order to understand the structure of the quantum spacetime near the cosmological singularity. Since the spatial gradients of the configurational variables play no relevant dynamical role in the asymptotic limit $\tau \to \infty$ (indeed the spatial curvature simply behaves as a potential well) then the quantum evolution above outlined takes place independently in each space point and the total wavefunction of the universe can be represented as follows:

$$\Xi(\tau, u, v) = \Pi_{x_i}\xi_{x_i}(\tau, u, v) \tag{59}$$

where the product is (heuristically) taken over all the points of the spatial hypersurface.

However, it is worth to recall that, in the spirit of the 'long-wavelength approximation' adopted here, the physical meaning of a space point must be recovered on the notion of a cosmological horizon; in fact we are dealing with regions over which the inhomogeneity effects are negligible and this statement corresponds to super-horizon sized spatial gradients. On a classical point of view, this request is at the ground of the BKL approximation and it is well confirmed on the statistical level (see [4]); however on a quantum level it can acquire a precise meaning if we refer the dynamics to a kind of lattice spacetime in which the spatial gradients of the configurational variables become real potential terms. In this respect it is important to observe that the geometry of the spacetime is expected to acquire a discrete structure on the Planck scale and we believe that a regularization of our approach could arrive from a 'loop quantum gravity' treatment [35].

Despite this local homogeneous framework of investigation, the appearance near the singularity of high spatial gradients and of a spacetime foam (like outlined in the classical dynamics see [4, 5]) can be easily recognized in the above quantum picture too. In fact the probability that in n different space points (horizons) the variables u and v take values within the same narrow interval, decrease with n as p^n, p being the probability in a single point; in fact, these probabilities are all identical to each other and no interference phenomenon takes place. From a physical point of view, this simple consideration indicates that a smooth picture of the large scale universe is forbidden on a probabilistic level and different causal regions are expected completely disconnected from each other during their quantum evolution. Therefore, if we start with a strongly correlated initial wavefunction $\Xi_0(u, v) \equiv \Xi(\tau_0, u, v)$, its evolution towards the singularity induces increasingly irregular distributions, approaching (59) in the asymptotic limit $\tau \to \infty$.

The main result of our presentation can be recognized in the clear correspondence established between the classical spacetime foam and the quantum one. We have outlined how this link takes place naturally for a precise choice of the operator-ordering only and how the 'energy spectrum' is a discrete one, due to the billiard structure of the point-like Hamiltonian. Finally, we fixed as a new feature the zero-point 'energy' for the ground state associated with the anisotropy degrees.

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Attachment 4

Covariant Description of the Inhomogeneous Mixmaster Chaos

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We outline the covariant nature of the chaos characterizing the generic cosmological solution near the initial singularity. Our analysis is based on a "gauge" independent ADM-reduction of the dynamics to the physical degrees of freedom, and shows that the dynamics is isomorphic point by point in space to a billiard on a Lobachevsky plane. The Jacobi metric associated to the geodesic flow is constructed and a non-zero Lyapunov exponent is explicitly calculated. The chaos covariance emerges from the independence of the form of the lapse function and the shift vector.

In recent years, the chaoticity of the homogeneous Mixmaster model has been widely studied in the literature (see[1–4]) in view of understanding the features of its covariant nature. Two convincing arguments, appeared in,[1,4] support the idea that the Mixmaster chaos (described by the invariant measure introduced in[3,5]) remains valid in any system of coordinates.

The main issue of the present work is to show that the property of space-time covariance can be extended to the inhomogeneous Mixmaster model.

A generic cosmological solution is represented by a gravitational field having available all its degrees of freedom and, therefore, allowing to specify a generic Cauchy problem. In the Arnowitt-Deser-Misner (ADM) formalism, the metric tensor corresponding to such a generic model takes the form

$$d\Gamma^2 = N^2 dt^2 - \gamma_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt) \tag{1}$$

where N and N^α denote (respectively) the lapse function and the shift-vector, $\gamma_{\alpha\beta}$ the 3-metric tensor of the spatial hyper-surfaces Σ^3 for which $t = const$, being[6]

$$\gamma_{\alpha\beta} = e^{q_a}\delta_{ad}O^a_b O^d_c \partial_\alpha y^b \partial_\beta y^c \quad a,b,c,d,\alpha,\beta = 1,2,3; \tag{2}$$

$q^a = q^a(x,t)$ and $y^b = y^b(x,t)$ are six scalar functions, and $O^a_b = O^a_b(x)$ a $SO(3)$ matrix. By the metric tensor (2), the action for the gravitational field is

$$S = \int_{\Sigma^{(3)}\times\Re} dt d^3x \left(p_a\partial_t q^a + \Pi_d \partial_t y^d - NH - N^\alpha H_\alpha\right) , \tag{3}$$

$$H = \frac{1}{\sqrt{\gamma}}[\sum_a (p_a)^2 - \frac{1}{2}(\sum_b p_b)^2 - \gamma^{(3)} R] \tag{4}$$

$$H_\alpha = \Pi_c \partial_\alpha y^c + p_a \partial_\alpha q^a + 2 p_a (O^{-1})^b_a \partial_\alpha O^a_b; \tag{5}$$

in (4) and (5) p_a and Π_d are the conjugate momenta to the variables q^a and y^b respectively, and the ${}^{(3)}R$ is the Ricci 3-scalar which plays the role of a potential term. We use the Hamiltonian constraints $H = H_\alpha = 0$ to reduce the dynamics to the physical degrees of freedom; the super-momentum constraints can be diagonalized and explicitly solved by choosing the function y^a as special coordinates:

$$\Pi_b = -p_a \frac{\partial q^a}{\partial y^b} - 2 p_a (O^{-1})^c_a \frac{\partial O^a_c}{\partial y^b} \,. \tag{6}$$

$$S = \int_{\Sigma^{(3)} \times \Re} d\eta d^3 y \left(p_a \partial_\eta q^a + 2 p_a (O^{-1})^c_a \partial_\eta O^a_c - NH \right) \,. \tag{7}$$

The potential term appearing in the super-Hamiltonian behaves as an infinite potential wall as the determinant of the 3-metric goes to zero, and we can model it as follows

$$U = \sqrt{\gamma}^{(3)} R = \sum_a \Theta(Q_a) \qquad \Theta(x) = \begin{cases} +\infty & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{8}$$

where the Q_a's are called *anisotropy parameters*. By (8) the Universe dynamics evolves independently in each space point and the point-Universe can move within a dynamically-closed domain Γ_Q only[6] (see figure (1)). Since in Γ_Q the potential U asymptotically vanishes, near the singularity we have $\partial p_a / \partial \eta = 0$; then the term $2 p_a (O^{-1})^c_a \partial_\eta O^a_c$ in (7) behaves as an exact time-derivative that can be ruled out of the variational principle.

The ADM reduction is completed by introducing the so-called Misner-Chitré like (τ, ξ, θ) variables,[1] in terms of which the anisotropy parameters Q_a become τ independent: When expressed in terms of such variables the super-Hamiltonian con-

$$Q_1 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi}(\cos\theta + \sqrt{3}\sin\theta)$$

$$Q_2 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi}(\cos\theta - \sqrt{3}\sin\theta) \tag{9}$$

$$Q_3 = \frac{1}{3} + \frac{2\sqrt{\xi^2 - 1}}{3\xi}\cos\theta$$

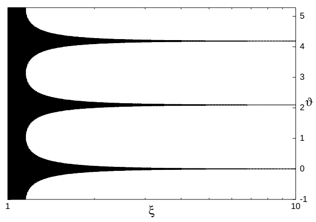

Fig. 1. $\Gamma_Q(\xi, \theta)$

straint can be solved in the domain Γ_Q according to the ADM prescription:

$$-p_\tau \equiv \epsilon = \sqrt{(\xi^2 - 1)p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1}} \tag{10}$$

$$\delta S_{\Gamma_Q} = \delta \int d\eta d^3y (p_\xi \partial_\eta \xi + p_\theta \partial_\eta \theta - \epsilon \partial_\eta \tau) = 0\,. \tag{11}$$

By virtue of the asymptotic limit (8) and the Hamilton equations associated with (11) it follows that ϵ is a constant of motion, i.e. $d\epsilon/d\eta = \partial\epsilon/\partial\eta = 0 \Rightarrow \epsilon = E(y^a)$; furthermore $\epsilon\partial_\eta\tau$ behaves as an exact time derivative.
This dynamical scheme allows us to construct the Jacobi metric corresponding to the dynamical flow, and the line element reads

$$ds^2 = E^2(y^a)\left(\frac{d\xi^2}{(\xi^2 - 1)} + (\xi^2 - 1)d\theta^2\right)\,. \tag{12}$$

Here the space coordinates behaves like external parameters since the evolution is spatially uncorrelated. The Ricci scalar takes the value $R = -2/E^2$, so that (12) describes a two-dimensional Lobachevsky space; the role of the potential wall (8) consists of cutting a closed domain Γ_Q on such a negative curved surface.
In this scheme the Lyapunov exponent associated to the geodesic deviation along the direction orthonormal to the geodesic 4-velocity can be evaluated, and it results to be equal to

$$\lambda(y^a) = \frac{1}{\epsilon(y^a)} > 0\,. \tag{13}$$

Hence the chaotic behavior of the inhomogeneous Mixmaster model can be described in a generic gauge (*i.e.* without assigning the form of the lapse function and of the shift vector) as soon as Misner-Chitré like variables are adopted. The value (13) is a positive definite function of the space point, making this calculation extendible point by point to the whole space.

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Attachment 5

VECTOR FIELD INDUCED CHAOS IN MULTI-DIMENSIONAL HOMOGENEOUS COSMOLOGIES

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We show that in multidimensional gravity vector fields completely determine the structure and properties of singularity. It turns out that in the presence of a vector field the oscillatory regime exists for any number of spatial dimensions and for all homogeneous models. We derive the Poincaré return map associated to the Kasner indexes and fix the rules according to which the Kasner vectors rotate. In correspondence to a 4-dimensional space time, the oscillatory regime here constructed overlap the usual Belinski-Khalatnikov-Liftshitz one.

1. Introduction

The wide interest attracted by the homogeneous cosmological models of the Bianchi classification relies over all in the allowance for their anisotropic dynamics; among them the types VIII and IX stand because of their chaotic evolution toward the initial singularity[1] that correspond to the maximum degree of generality allowed by the homogeneity constraint; as a consequence it was shown[2–4] that the generic cosmological solution can be described properly, near the Big-Bang, in terms of the homogeneous chaotic dynamics as referred to each cosmological horizon. However the correspondence existing between the homogeneous dynamics and the generic inhomogeneous one holds only in four space-time dimensions. In fact a generic cosmological inhomogeneous model remains characterized by chaos near the Big-Bang up to a ten dimensional space-time[5–7] while the homogeneous models show a regular (chaos free) dynamics beyond four dimensions.[8,9]
Here we address an Hamiltonian point of view showing how the homogeneous models (of each type) perform, near the singularity, an oscillatory regime in correspondence to any number of dimensions, as soon as an electromagnetic field is included in the dynamics.

2. The Standard Kasner Dynamics

Let us consider the standard $n+1$-dimensional vector-tensor theory in the ADM representation:

$$I = \int d^n x dt \left\{ \Pi^{\alpha\beta} \frac{\partial}{\partial t} g_{\alpha\beta} + \pi^\alpha \frac{\partial}{\partial t} A_\alpha + \varphi D_\alpha \pi^\alpha - N H_0 - N^\alpha H_\alpha \right\}, \tag{1}$$

$$H_0 = \frac{1}{\sqrt{g}} \left\{ \Pi^\alpha_\beta \Pi^\beta_\alpha - \frac{1}{n-1} \left(\Pi^\alpha_\alpha\right)^2 + \frac{1}{2} g_{\alpha\beta} \pi^\alpha \pi^\beta + g \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - R \right) \right\}, \tag{2}$$

$$H_\alpha = -\nabla_\beta \Pi^\beta_\alpha + \pi^\beta F_{\alpha\beta}, \tag{3}$$

Here H_0 and H_α denote respectively the super-Hamiltonian and super-momentum, $F_{\alpha\beta} \equiv \partial_\beta A_\alpha - \partial_\alpha A_\beta$ is the electromagnetic tensor, $g \equiv det(g_{\alpha\beta})$ is the determinant of the n-metric, R is the n-scalar of curvature and $D_\alpha \equiv \partial_\alpha + A_\alpha$.
Since the sources are absent, it is enough to consider only the transverse components for A_α and π^α; therefore, we take the gauge conditions $\varphi = 0$ and $D_\alpha \pi^\alpha = 0$. When going over the homogeneous case, we choose the gauge $N = 1$ and $N^\alpha = 0$.
Let's adopt the Kasner parameterization,that is based on the metric and conjugate momentum decomposition along spatial n-bein:

$$g_{\alpha\beta} = \delta_{ab} l^a_\alpha l^b_\beta, \qquad \Pi_{\alpha\beta} = p_{ab} l^a_\alpha l^b_\beta, \tag{4}$$

We also define a dual basis $L^\alpha_a = g^{\alpha\beta} l^a_\beta$, such that $L^\alpha_a l^b_\alpha = \delta^b_a$ and $L^\alpha_a l^a_\beta = \delta^\alpha_\beta$.
We want to put in evidence the oscillatory regime that the bein vectors possess and so we distinguish scale functions and the parallel from the transverse component $(\widetilde{\lambda}_a = (\pi^\alpha \ell^a_\alpha))$

$$l_a = \exp\left(q^a/2\right) \ell_a \,, \qquad L_a = \exp\left(-q^a/2\right) \mathcal{L}_a. \tag{5}$$

$$\vec{\ell}_a = \vec{\ell}_{a\parallel} + \vec{\ell}_{a\perp}; \qquad \vec{\ell}_{a\parallel} = \frac{\widetilde{\lambda}_a}{\pi^2} \vec{\pi}, \qquad \left(\vec{\pi} \vec{\ell}_{a\perp} \right) = 0. \tag{6}$$

The standard Kasner solution is obtained as soon as the limit in which all the terms $\exp(q^a)$ become of higher order is taken

$$\begin{gathered} p_a = const, \qquad \widetilde{\lambda}_a = const, \qquad \vec{\ell}_{a\perp} = const, \\ \tfrac{\partial}{\partial t} q_a = \tfrac{2N}{\sqrt{g}} \left(p_a - \tfrac{1}{n-1} \textstyle\sum_b p_b \right), \\ \textstyle\sum p_a^2 - \tfrac{1}{n-1} \left(\sum p_a\right)^2 + \tfrac{1}{2} \sum e^{q_a} \widetilde{\lambda}_a^2 = 0 \,, \end{gathered} \tag{7}$$

$$g_{\alpha\beta} = \sum_a t^{2s_a} \ell^a_\alpha \ell^a_\beta \,, \qquad s_a = 1 - (n-1) \frac{p_a}{\sum_b p_b} \,, \tag{8}$$

The Kasner indexes s_a satisfy the identities $\sum s_a = \sum s_a^2 = 1$.

3. Billiard representation: the return map and the rotation of Kasner vectors

If we order the s_a's, the largest increasing term (as $t \to 0$ $t^{s_1} \to \infty$) among the neglected ones comes from s_1 and it is to be taken into account to construct the oscillatory regime toward the cosmological singularity.

$$\begin{aligned} &\frac{\partial}{\partial t}\widetilde{\lambda}_1 = 0\,, && \frac{\partial}{\partial t}\widetilde{\lambda}_a = \frac{\left(\frac{\partial}{\partial t}p_1\right)\widetilde{\lambda}_a}{(p_a - p_1)} \\ &\frac{\partial}{\partial t}p_1 = -\frac{N}{2\sqrt{g}}\widetilde{\lambda}_1^2 \exp\left(q^1\right)\,, && \frac{\partial}{\partial t}p_a = 0, \\ &\frac{\partial}{\partial t}q_a = \frac{2N}{\sqrt{g}}\left(p_a - \frac{1}{n-1}\sum_b p_b\right). \end{aligned} \tag{9}$$

The first of equations (9) gives $\widetilde{\lambda}_1 = const$, while the second admits the solution

$$\widetilde{\lambda}_a\,(p_a - p_1) = const. \tag{10}$$

The remaining part of the dynamical system allows us to determine the return map governing the replacements of Kasner epochs and the rotation of Kasner vectors $\vec{\ell}_a$ through these epochs

$$s'_1 = \frac{-s_1}{1 + \frac{2}{n-2}s_1}, \qquad s'_a = \frac{s_a + \frac{2}{n-2}s_1}{1 + \frac{2}{n-2}s_1}, \tag{11}$$

$$\widetilde{\lambda}'_1 = \widetilde{\lambda}_1, \qquad \widetilde{\lambda}'_a = \widetilde{\lambda}_a\left(1 - 2\frac{(n-1)\,s_1}{(n-2)\,s_a + ns_1}\right), \tag{12}$$

$$\vec{\ell}'_a = \vec{\ell}_a + \sigma_a\vec{\ell}_1, \qquad \sigma_a = \frac{\widetilde{\lambda}'_a - \widetilde{\lambda}_a}{\widetilde{\lambda}_1} = -2\frac{(n-1)\,s_1}{(n-2)\,s_a + ns_1}\frac{\widetilde{\lambda}_a}{\widetilde{\lambda}_1}. \tag{13}$$

Thus the homogeneous Universes here discussed approaches the initial singularity being described by a metric tensor with oscillating scale factors and rotating Kasner vectors. The presence of a vector field is crucial because, independently on the considered model, it induces a closed domain on the configuration space.

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Attachment 6

Classical and Quantum Aspects of the Inhomogeneous Mixmaster Chaoticity

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We refine Misner's analysis of the classical and quantum Mixmaster in the fully inhomogeneous picture; we both connect the quantum behavior to the ensemble representation, both describe the precise effect of the boundary conditions on the structure of the quantum states.

Near the cosmological singularity, the dynamics of a generic inhomogeneous cosmological model is reduced by an ADM procedure to the evolution of the physical degrees of freedom, *i.e.* the anisotropies of the Universe. In fact asymptotically to the Big-Bang the space points dynamically decouple[1] because the spatial gradients of the dynamical variables become of higher order[2] and we can model the inhomogeneous Mixmaster via the reduced action:

$$I = \int d^3x d\tau \left(p_u \partial_\tau u + p_v \partial_\tau v - \epsilon\right), \qquad \epsilon = v\sqrt{p_u^2 + p_v^2}\,, \tag{1}$$

The dynamics of such a model is equivalent to (the one of) a billiard-ball on a Lobachevsky plane; this can be shown by the use of the Jacobi metric.[1] The manifold described turns out to have a constant negative curvature, where the Ricci scalar is given by $R = -2/E^2$: the complex dynamics of the generic inhomogeneous model results in a collection of decoupled dynamical systems, one for each point of the space, and all of them equivalent to a billiard problem on a Lobachevsky plane. We want to investigate the relation existing between the classical and the semiclassical dynamics, and from this analysis we will derive the correct operator ordering to be used when quantizing the system.
Let's write down the Hamilton-Jacobi equation for the system

$$\epsilon^2 = v^2\left(\left(\frac{\delta \mathcal{S}_0}{\delta u}\right)^2 + \left(\frac{\delta \mathcal{S}_0}{\delta v}\right)^2\right) \tag{2}$$

This can be explicitly solved by separation of constants

$$\mathcal{S}_0(u,v) = k(y^a)u + \sqrt{\epsilon^2 - k^2(y^a)v^2} - \epsilon \ln\left(2\frac{\epsilon + \sqrt{\epsilon^2 - k^2(y^a)v^2}}{\epsilon^2 v}\right) + c(y^a)\,. \quad (3)$$

The semiclassical analysis can be developed furthermore, and the stationary continuity equation for the distribution function can be worked in order to obtain informations about the statistical properties of the model; as soon as we restrict the dynamics to the configuration space, we get the following equation for the distribution function $\tilde{w}$

$$\frac{\partial\tilde{w}(u,v;k)}{\partial u} + \sqrt{\left(\frac{E}{kv}\right)^2 - 1}\frac{\partial\tilde{w}(u,v;k)}{\partial v} + \frac{E^2 - 2k^2v^2}{kv^2}\frac{\tilde{w}(u,v;k)}{\sqrt{E^2-(kv)^2}} = 0 \quad (4)$$

This can be solved, and the exact distribution function can be obtained as soon as we eliminate by integration the constant k

$$\tilde{w}(u,v) = \int_{-\frac{E}{v}}^{\frac{E}{v}} \frac{g\left(u + v\sqrt{\frac{E^2}{k^2v^2} - 1}\right)}{v\sqrt{E^2 - k^2v^2}}dk \quad (5)$$

It is worth nothing how in the case $g = const$, the microcanonical Liouville measure $w_{mc}(u,v) = \frac{\pi}{v^2}$ is recovered.
We expect that the distribution function $\tilde{w}(u,v)$ is re-obtained as soon as the quantum dynamics is investigated to the first order in $\hbar$. This can be easily done as a WKB approximation to the quantum dynamics is constructed; as soon as we retain only the lower order in $\hbar$, we obtain that: i) the phase $S(u,v)$ coincides with the Hamilton-Jacobi function, and ii) the probability density function $r(u,v)$ obeys the following equation:

$$k\frac{\partial r(u,v)}{\partial u} + \sqrt{\left(\frac{E}{v}\right)^2 - k^2}\frac{\partial r(u,v)}{\partial v} + \frac{a(E^2 - k^2v^2) - E^2}{v^2\sqrt{E^2 - k^2v^2}}r(u,v) = 0\,. \quad (6)$$

that coincides with (4) for a particular choice for the operator-ordering only, *i.e.*

$$\hat{v}^2\hat{p}_v^2 \;\rightarrow\; -\hbar^2\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right)\,. \quad (7)$$

With this result, the problem of the full quantization of the system can be taken in consideration. The main problem is the presence of the root square function in the definition of the Hamiltonian (1), but well grounded motivations exist[3] to assume that the real Hamiltonian and the squared one have same eigenfunctions and squared eigenvalues. This way the solution of the eigenvalue equation can be obtained

$$\Psi(u,v) = \sum_{n>0} a_n K_{s-1/2}(2n\pi v)\sin(2n\pi u) \quad (8)$$

The spectrum is obtained as Dirichelet boundary conditions on the domain Γ_Q (see Fig.1) are taken into account. The condition on the vertical lines can be imposed

exactly, but the one on the semicircle cannot be solved exactly; so we approximate it as in Fig.2 with a straight line $v = 1/\pi$ (it preserves the domain area $\mu = \pi$).
All these imply that $s = 1/2 + t$, with $t \in \Re$, and that the spectrum assumes the

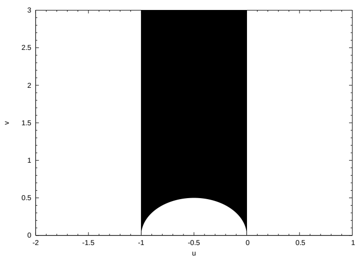

Fig. 1. The real domain Γ_Q

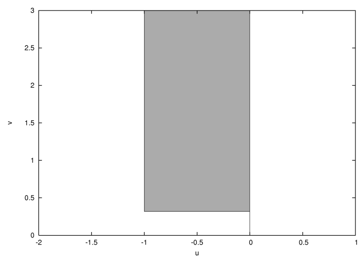

Fig. 2. The approximate domain

following form

$$E_t^2 = (\frac{1}{4} + t^2)\hbar^2 \qquad (9)$$

The values of the real parameter t have to be numerically evaluated by solving the equation $K_{it}(2n) = 0$ for every natural n. This condition implies a discrete but quite complicated shape for the spectrum.
Asymptotic expansions for high occupation numbers can be derived for different regions of the parameters $(t,\ n)$[4]

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Attachment 7

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Multi-Time approach to the Generic Quantum Cosmology

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Summary. — Starting from the Hamiltonian formulation for the inhomogeneous Mixmaster dynamics, we approach its quantum features through the link of the quasi-classical limit. We fix the proper operator-ordering which ensures that the WKB continuity equation overlaps the Liouville theorem as restricted to the configuration space. We describe the full quantum dynamics of the model within the multi-time scheme, providing a characterization of the (discrete) spectrum.

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1. – Classical Dynamics

Hamiltonian Formulation.
Near the cosmological singularity, the dynamics of a generic inhomogeneous cosmological model is reduced by an Arnowitt-Deser-Misner (ADM) procedure to the evolution of the physical degrees of freedom, *i.e.* the anisotropies of the Universe. In fact asymptotically to the Big-Bang the space points dynamically decouple [1] because the spatial gradients of the dynamical variables become of higher order [2] and we can model the inhomogeneous Mixmaster[3, 4] via the reduced action:

$$I = \int_{\Gamma_Q} d^3x d\tau \left(p_u \partial_\tau u + p_v \partial_\tau v - \epsilon\right), \qquad \epsilon = v\sqrt{p_u^2 + p_v^2}\,, \tag{1}$$

where Γ_Q is only a limited portion (see Fig.1) of the configuration-space because the asymptotic behaviour of the curvature term is like the one of a infinite potential walls. The dynamics of such a model is equivalent to (the one of) a billiard-ball on a Lobachevsky plane; this can be shown

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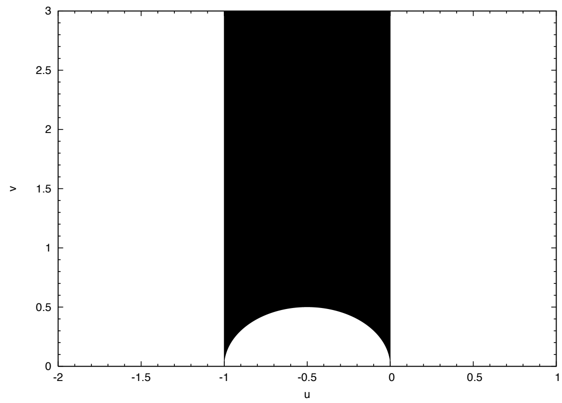

Fig. 1.: The domain $\Gamma_Q(u,v)$ in the Poincaré upper half-plane where the dynamics of the space-point Universe is asymptotically restricted by the potential.
The measure is finite and equal to π.

by the use of the Jacobi metric [1]. The manifold described turns out to have a constant negative curvature, where the Ricci scalar is given by $R = -2/E^2$, E being a fixed value of ϵ: the subtle dynamics of the generic inhomogeneous model results in a collection of decoupled dynamical systems, one for each point of the space, and all of them equivalent to a billiard problem on a Lobachevsky plane.

Hamilton-Jacobi approach.
The Hamilton-Jacobi (HJ) function in a fixed space point can be analytically obtained in this scheme; let's write down and solve the equation, *i.e.*

(2)
$$\epsilon^2 = v^2\left(\left(\frac{\partial \mathcal{S}_0}{\partial u}\right)^2 + \left(\frac{\partial \mathcal{S}_0}{\partial v}\right)^2\right)$$

(3)
$$\mathcal{S}_0(u,v) = k(y^a)u + \sqrt{\epsilon^2 - k^2(y^a)v^2} - \epsilon \ln\left(2\frac{\epsilon + \sqrt{\epsilon^2 - k^2(y^a)v^2}}{\epsilon^2 v}\right) + c(y^a)\,.$$

This result will be relevant to analize the semi-classical analysis below presented.

The statistical mechanics point of view .
Let us discuss the problem from a statistical mechanics point of view by treating the system as a microcanonical ensemble. The physical properties of a stationary ensemble are described by a distribution function $\rho = \rho(u, v, p_u, p_v)$, obeying the continuity equation defined in the phase space (u, v, p_u, p_v).
We stress how this equation provides an appropriate representation for the ensemble associated to the Mixmaster only when we are sufficiently close to the initial singularity and therefore the infinite-potential wall approximation works; in fact, such a model for the potential term corresponds to deal with the energy-like constant of the motion and fixes the microcanonical

nature of the ensemble. From a dynamical point of view this picture arises naturally because the Universe volume element vanishes monotonically .
We are interested in studying the distribution function in the (u, v) space, and thus we will reduce the dependence on the momentum variables by integrating $\rho(u, v, p_u, p_v)$ on the momenta space. If we assume ρ to be a regular, limited function and, using (3), then we work out the following equation for $\tilde{w}(u, v; k) = \int \rho(u, v, p_u, p_v) dp_u dp_v$

$$\frac{\partial \tilde{w}(u,v;k)}{\partial u} + \sqrt{\left(\frac{E}{kv}\right)^2 - 1}\frac{\partial \tilde{w}(u,v;k)}{\partial v} + \frac{E^2 - 2k^2v^2}{kv^2}\frac{\tilde{w}(u,v;k)}{\sqrt{E^2-(kv)^2}} = 0 \tag{4}$$

admitting a solution in terms of a generic function g of the form

$$\tilde{w}(u,v) = \frac{g\left(u + v\sqrt{\frac{E^2}{k^2v^2} - 1}\right)}{v\sqrt{E^2 - k^2v^2}} \tag{5}$$

The distribution function in (u, v) is obtained after the integration over the constant k. Indeed, this constant expresses the freedom of choosing the initial conditions, which cannot affect the chaotic properties of the model. Therefore we define the reduced distribution $w(u, v)$ as

$$w(u,v) \equiv \int_A \tilde{w}(u,v;k)dk\,, \qquad A \equiv \left[-\frac{E}{v}, \frac{E}{v}\right]\,. \tag{6}$$

It is easy to verify that the microcanonical Liouville measure w_{mc} corresponds to the case $g = const$, *i.e.* we get the normalized distribution

$$w_{mc}(u,v) = \int_{-\frac{E}{v}}^{\frac{E}{v}} \frac{1}{kv^2\sqrt{\frac{E^2}{k^2v^2}-1}}dk = \frac{\pi}{v^2}\,. \tag{7}$$

2. – Semi-Classical Analysis

Let us underline some common features between the classical and the semi-classical dynamics with the aim of fixing the proper operator-ordering in treating the quantum approach.
In fact, considering the WKB limit for $\hbar \to 0$, the coincidence between the classical distribution function $w_{mc}(u, v)$ and the quasi-classical probability function $r(u, v)$ takes place for a precise choice of the operator-ordering only.
As soon as we implement the canonical variables into operators, the quantum dynamics for the state function $\Phi(u, v, \tau)$ obeys, in each point of the space, the Scrhödinger equation

$$i\hbar\frac{\partial\Phi(u,v,\tau)}{\partial\tau} = \hat{H}_{ADM}\Phi(u,v,\tau) = \hbar\sqrt{-v^2\frac{\partial^2}{\partial u^2} - v^{2-a}\frac{\partial}{\partial v}\left(v^a\frac{\partial}{\partial v}\right)}\Phi(u,v,\tau)\,, \tag{8}$$

where we have adopted a generic operator-ordering for the position and momentum operators parametrized by the constant a.
In the above equation the Hamiltonian operator has a non-local character as a consequence of the square root function; we will make the well-grounded ansatz[5] that the operators $\hat{H}_{ADM}$ and $\hat{H}^2_{ADM}$ have the same set of eigenfunctions with eigenvalues E and E^2, respectively.

If we take a generalised Fourier transform $\Phi(u,v,\tau)$ for the wave function $\Psi(u,v,E)$ the eigenvalues problem reduces to

$$\hat{H}^2\Psi(u,v,E) = \hbar^2\left[-v^2\frac{\partial^2}{\partial u^2} - v^{2-a}\frac{\partial}{\partial v}\left(v^a\frac{\partial}{\partial v}\right)\right]\Psi(u,v,E) = E^2\Psi(u,v,E)\,. \tag{9}$$

In order to study the WKB limit of equation (9), we separate the wave function into its phase and modulus $\Psi(u,v,E) = \sqrt{r(u,v,E)}e^{i\sigma(u,v,E)/\hbar}$ (the function $r(u,v)$ representing the probability density). The quasi-classical regime then appears as we take the limit for $\hbar \to 0$; substituting in (9) and retaining only the lower order in $\hbar$, we obtain the system

$$\begin{cases} v^2\left[\left(\dfrac{\partial\sigma}{\partial u}\right)^2 + \left(\dfrac{\partial\sigma}{\partial v}\right)^2\right] = E^2 \\ \dfrac{\partial r}{\partial u}\dfrac{\partial\sigma}{\partial u} + \dfrac{\partial r}{\partial v}\dfrac{\partial\sigma}{\partial v} + r\left(\dfrac{a}{v}\dfrac{\partial\sigma}{\partial v} + \dfrac{\partial^2\sigma}{\partial v^2} + \dfrac{\partial^2\sigma}{\partial u^2}\right) = 0\,. \end{cases} \tag{10}$$

Thus we can identify the phase σ with the function S_0 of the HJ approach.
Now we turn our attention to the equation for $r(u,v)$; taking (3) into account it reduces to

$$k\frac{\partial r(u,v,E)}{\partial u} + \sqrt{\left(\frac{E}{v}\right)^2 - k^2}\frac{\partial r(u,v,E)}{\partial v} + \frac{a(E^2-k^2v^2)-E^2}{v^2\sqrt{E^2-k^2v^2}}r(u,v,E) = 0\,. \tag{11}$$

Comparing (11) with (4), we easily see that they coincide for $a = 2$ only.

$$\hat{v}^2\hat{p}_v^2 \;\to\; -\hbar^2\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right)\,. \tag{12}$$

This unique choice of the operator-ordering allows us to face the canonical quantization of the model which matches, in the semi-classical limit, the statistical behaviour of the Mixmaster.

3. – Canonical Quantization

The Laplace-Beltrami operator and the eigenfunctions.
Our starting point is the point-like eigenvalue equation (9) with $a = 2$ which, together with the boundary conditions, completely describes the quantum features of the model.
By redefining $\Psi(u,v,E) = \psi(u,v,E)/v$, we can reduce (9) to the eigenvalue problem for the Laplace-Beltrami operator in the Poincaré plane

$$\nabla_{LB}\psi(u,v,E) \equiv v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\psi(u,v,E) = E_s\psi(u,v,E)\,, \tag{13}$$

($K_\nu(x)$ are Bessel function of the third order). From the general theory of this operator, we get the eigenfunctions and eigenvalues for our model in the form

$$\Psi(u,v,E) = a\;\; v^{s-1} + bv^{-s} + \sum_{n\neq 0} a_n\frac{K_{s-1/2}(2\pi|n|v)}{\sqrt{v}}e^{2\pi inu}\,, \tag{14}$$

$$E^2 \;= s(1-s)\hbar^2\,. \tag{15}$$

The boundary conditions and the energy spectrum.
The spectrum and the explicit eigenfunctions are obtained by imposing the boundary conditions, *i.e.* requiring that (14) vanishes on the boundaries of the domain Γ_Q in Fig.1.
Since from our analysis no analytical way arose to impose exact boundary conditions we approximate the semicircle with the horizontal line $y = 1/\pi$.
The conditions on the vertical lines $u = 0$, $u = -1$ require to disregard the first two terms in (14) ($a = b = 0$); furthermore we get the condition on the last term (here we disregard the odd part because of symmetry reasons [7]) $\sum_{n\neq 0} e^{2\pi i n u} \to \sum_{n=1}^{\infty} \sin(2\pi n u)$, n being an integer. Instead the condition on the horizontal line $y = 1/\pi$ implies

(16) $$\sum_{n>0} a_n K_{s-1/2}(2n) \sin(2n\pi u) = 0, \forall u \in [-1, 0]\,,$$

which is satisfied by requiring $K_{s-1/2}(2n) = 0$ only. The functions $K_\nu(x)$ are real and positive for real argument and real index, therefore we take $s = \frac{1}{2} + it$. In this case these functions have (only) real zeros, and the corresponding eigenvalues turn out to be real and positive

(17) $$(E/\hbar)^2 = t^2 + 1/4\,.$$

From (17) the energy of the ground state comes out to be always greater than zero. We remark that our eigenfunctions naturally vanish as infinite values of v are approached.
The conditions (16) cannot be solved analytically for generic the values of n and t, and the roots must be worked out numerically for each n. We conclude listing the first ten eigenvalues and plotting the wave function of the ground- state.

$\left(\frac{E}{\hbar}\right)^2 = t^2 + \frac{1}{4}$
19.831
40.357
49.474
63.405
87.729
89.250
116.329
128.234
138.739
146.080

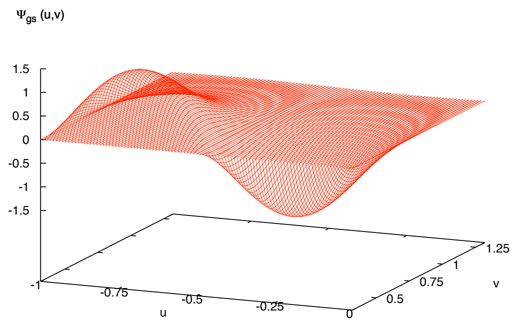

Fig. 2.: The ground state wave function

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