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TITOLO TESI:

Analysis and optimal control for the phase-field transition system with non-homogeneous Cauchy-Neumann boundary conditions

Presentata da: Tommaso Benincasa

Coordinatore del Dottorato:
Chiar.mo Prof. Alberto Parmeggiani

Relatore:
Chiar.mo Prof. Angelo Favini

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¡Ka wo, kabiye si ile, Shango!
¡Emperador victorioso en
todos los combates!

a nonna Rosa

Introduction

A phase transition is a natural physical process. It has the characteristic of taking a given medium with given properties and transforming some or all of that medium, into a new medium with new properties. Phase transitions occur frequently and are found everywhere in the natural world. In thermodynamics, a phase transition is the transformation of a thermodynamic system from one phase to another. There are several of phase transitions. Some examples are:

- The transitions between the solid, liquid, and gaseous phases of a single component, due to the effects of temperature and/or pressure.
- The transition between the ferromagnetic and paramagnetic phases of magnetic materials at the Curie point.
- Changes in the crystallographic structure such as between ferrite and austenite of iron.
- The emergence of superconductivity in certain metals when cooled below a critical temperature.

Phase transitions happen when the free energy of a system is not sufficiently smooth for some choice of thermodynamic variables, this generally stems from the interactions of an extremely large number of particles in a system, and does not appear in systems that are too small. The first attempt at classifying phase transitions was the Ehrenfest classification scheme. Under this scheme, phase transitions were labeled by the lowest derivative of the free energy that is discontinuous at the transition. *First-order phase transitions* exhibit a discontinuity in the first derivative of the free energy with a thermodynamic variable. The various solid/liquid/gas transitions are classified as first-order phase transition because they involve a discontinuous change in density (which is the first derivative of the free energy with respect to chemical potential). *Second-order phase transitions* exhibit discontinuity in a second derivative of the free energy. These include the ferromagnetic phase transition in materials such as iron, where the magnetization, which is the first derivative of the free energy with the applied magnetic field strength, increases continuously from zero as the temperature is lowered below the Curie temperature. The magnetic susceptibility, the second derivative of the free energy with the field, changes discontinuously. Under the Ehrenfest classification scheme, there could in principle be third, fourth, and higher-order phase transitions. The Ehrenfest scheme is an inaccurate method of classifying phase transitions, for it does not take into account the case where a derivative of free energy diverges. For instance, in the ferromagnetic transition, the heat capacity diverges to infinity.

In the modern classification scheme the *first-order phase transitions* are those that involve a latent heat. During such a transition, a system either absorbs or releases a fixed amount of energy. During this process, the temperature of the system will stay constant as heat is added. Because energy cannot be instantaneously transferred between the system and its environment, first-order transitions are associated with "mixed-phase regimes" in which some parts of the system have completed the transition and others have not. They are difficult to study, because their dynamics are violent and hard to control. However, many important phase transitions fall in this category, including the solid/liquid/gas transitions. The *second-order phase transitions* have no associated latent heat. Examples of second-order phase transitions are the ferromagnetic transition, superconductor and the superfluid transition.

Phase transitions often take place between phases with different symmetry. Consider, for example, the transition between a fluid (i.e. liquid or gas) and a crystalline solid. In a fluid, which is composed of atoms arranged in a disordered but homogeneous manner, each point inside has the same properties as any other point. A crystalline solid, on the other hand, is made up of atoms arranged in a regular lattice. Each point in the solid is not similar to other points, unless those points are displaced by an amount equal to some lattice spacing. Generally, we may speak of one phase as being more symmetrical than the other. The transition from the more symmetrical phase to the less symmetrical one is a symmetry-breaking process. The presence of symmetry-breaking is important to the behavior of phase transitions. It was pointed out by Landau that, given any state of a system, one may unequivocally say whether or not it possesses a given symmetry. Therefore, it cannot be possible to analytically deform a state in one phase into a phase possessing a different symmetry. The order parameter is normally a quantity which is 0 in one phase, and non-zero in the other. It characterizes the onset of order at the phase transition. It can be understood as a measure of the degree of order in a system; the extreme values are 0 for total disorder and 1 for complete order.

Contents

Introduction **i**

1 The phase-field transition system **1**

1.1 Stefan problem 1

1.2 Phase-field equations (Caginalp model) 4

1.3 Asymptotic limits of the phase-field equations 7

2 Existence and regularity of a solution **15**

2.1 A model for continuous casting process 15

2.2 An auxiliary equation 21

2.3 Proof of Theorem 2.1.1 30

3 Fractional steps scheme **39**

3.1 Convergence of the approximating scheme 39

3.2 A numerical algorithm and numerical results. 53

4 Optimal control problem **57**

4.1 Functional costs $j(w)$ and $j^{\varepsilon}(w)$. 57

4.2 The existence in problem (P) and (P^{ε}) 59

4.3 The convergence of problem (P^{ε}) 63

4.4 Necessary optimality conditions in (P^{ε}) 66

5 An inverse problem, 1D-Case **73**

5.1 An inverse problem 73

5.2 The convergence of problem (P^{ε}) 77

5.3 Necessary optimality conditions in (P^{ε}) 79
5.4 Numerical experiments 86

Conclusion **97**

A Linear parabolic partial differential equations **99**

B Compute the integral value on Q_0 **103**

C Implementation of the algorithm InvPHT1D **105**

Bibliography **121**

List of Figures

1.1 solid and liquid regions 2
1.2 finite thickness ξ 4
1.3 several scaling limits 12

2.1 continuous casting process 16

3.1 The triangulation over Ω=[0,1300]x[0,220] 55
3.2 The approximate temperature u^{20} 55
3.3 The approximate temperature u^{40} 56

5.1 Geometrical image of the elements in inverse problem (P_{inv}) . 74
5.2 The discrete form of $Q_0,\ \Sigma_0$, and Σ in (P_{inv}) 87
5.3 The mesh Ω_{t_i} 88
5.4 The initial condition φ_0 92
5.5 The solution $z(\varepsilon,\cdot)$ of Cauchy problem (5.9) 93
5.6 The approximate solution u^0 and u^M 93
5.7 The approximate solution φ^M 94
5.8 The initial control w^0 and optimal control w^{10} 94
5.9 The values of functional $j^\varepsilon(w)$ 95

Chapter 1

The phase-field transition system

1.1 Stefan problem

The Stefan problem is probably the simplest mathematical model of a phenomenon of change of phase. It is named after Jozef Stefan, the Slovene physicist who introduced the general class of such problems around 1890, in relation to problems of ice formation. This question had been considered earlier, in 1831, by Lamé and Clapeyron. When a change of phase takes place, a *latent heat* is either absorbed or released, while the temperature of material changing its phase remains constant. Physically, one defines the melting temperature at equilibrium, $T_M \in I\!R$, as the value of the temperature at which solid and liquid may coexist in equilibrium separated by a planar interface. If a material occupies a region Ω existing in two phases, liquid (Ω_1) and solid (Ω_2), one could define the function $u(t,x) = T(t,x) - T_M$ be a reduced temperature, where $T(t,x),\ (t,x) \in [0,T] \times \Omega$ is the temperature of the material in the point x at the time t.

In the classical Stefan problem the temperature at interface, Γ, between solid and liquid region is assumed to be T_M, i.e., $u = 0$, hence

$$\Gamma(t) = \{x \in \Omega : \ u(t,x) = 0\}. \tag{1.1}$$

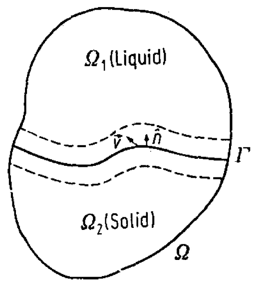

Figure 1.1: solid and liquid regions

Moreover, if $u(t,x) > 0$, the point lies in the liquid region, $x \in \Omega_1$, while $u(t,x) < 0$ implies $x \in \Omega_2$ the point be in solid phase.

The reduced temperature $u(t,x)$ must then satisfy the heat diffusion equation

$$u_t = k\Delta u \tag{1.2}$$

in both Ω_1 and Ω_2, to simplify the thermal diffusivity k is assumed to be the same constant for the two region. If one defines the unit normal $\hat{n}$ at each point of Γ (in the direction solid to liquid), across the interface, the latent heat of fusion l must be balanced by the heat flux

$$l\bar{v} \cdot \hat{n} = k(\nabla u_S - \nabla u_L) \cdot \hat{n}, \qquad x \in \Gamma, \tag{1.3}$$

where $k(\nabla u_S - \nabla u_L) \cdot \hat{n}$ is the jump in the normal component of the temperature times the thermal conductivity, $\bar{v}(t,x)$ is the local velocity and the density factor on the left side (the latent heat equation) has been set equal to one. To solve the mathematical problem one needs initial and boundary conditions for $u(t,x)$

$$u(0,x) = u_0(x), \tag{1.4}$$

$$u(t,x) = u_\partial(t,x) \qquad x \in \partial\Omega, \quad t > 0. \tag{1.5}$$

The mathematical problem to find u and $\Gamma(t)$ in suitable spaces satisfying the above equations is called Stefan problem. The interface $\Gamma(t)$ is often called the free boundary. We stress the fact the Stefan condition is merely a law of energetic balance. Oleinik in [22] studied the classical Stefan problem introducing the enthalpy or H-method. The idea is to define the function $H(u)$ by

$$H(u) \equiv u + \frac{l}{2}\varphi, \qquad \varphi \begin{cases} +1 & u > 0 \\ \\ -1 & u < 0 \end{cases} \tag{1.6}$$

The balance of heat equation can be write in the single equation

$$\frac{\partial}{\partial t} H(u) = k\Delta u, \tag{1.7}$$

and then we can find the heat-diffusion and the latent heat equations as weak formulation. This model is not able to describe deeply the physics of the phase transition, indeed it can not observe the existence of *supercooling* or *superheating* phenomenons. The phenomenon by which liquid is below its freezing point, without it becoming a solid, is called *supercooling*. The analogous phenomenon for a solid is called *superheating*. Supercooling and superheating are equilibrium phenomenons and are not merely a transient effects, their origin for a pure substance is in the effect of finite size of the interface between the solid and liquid. The classical Stefan problem neglects the thickness of the interface and treats the physics at a purely continuum level. In order that a molecule through the solid region and it moves in liquid region, a certain amount of energy is required, indeed the molecule, at surface, must overcome the binding energy of the crystal lattice and become part of liquid with lower binding energy. Thus the amount of energy required to produce this transition depends on the number of nearest neighbors in the crystal structure and on the number of nearest neighbors of an atom on the surface. In the case of a curved interface, the molecule on the surface has fewer nearest neighbors, since some are missing due to curvature. Hence, the transition will require less energy. It is clear physically that the equilibrium temperature, u, between solid and liquid, must be proportional to the curvature of interface, with proportionality constant involving the surface tension. From statistical mechanics the above idea leads to the Gibbs-Thompson re-

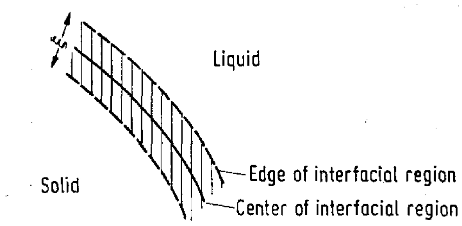

Figure 1.2: finite thickness ξ

lation, expressed by the following formula

$$u(t,x) = -\frac{\sigma}{\Delta s}\chi \quad (t,x) \in \Gamma, \tag{1.8}$$

whenever one has an curved interface between two phases in equilibrium, where Δs is the difference in entropy per unit volume, χ is the sum of the principal curvatures. The Gibbs-Thompson relation arises specifically from the finite thickness of the interface. This means that the change of phase is occurring continuously within a finite range. Thus, if the interface is moving, one cannot expect the heat equation for a homogeneous medium to hold exactly within this region as it would for a sharp interface. That, from enthalpy method, means one has a phase function φ which is a step function for a sharp interface, while, for a interface with finite thickness, the phase function should be a smooth function from the value -1 (solid) to +1 (liquid). Then the equation (1.6) should be adapted with a smooth function and not a step function φ, in this case, the phase function is essentially a local average of the phase.

1.2 Phase-field equations (Caginalp model)

A mean field theory is a model, in statistical mechanics, in which atoms are assumed to interact with a mean field created by other atoms. One of the most important field theory, to describe the phase transition, is the Landau-Ginzburg theory [16]. Lev Landau was a prominent Soviet physicist who made fundamental contributions to many areas of theoretical physics.

His accomplishments include the co-discovery of the density matrix method in quantum mechanics, the quantum mechanical theory of diamagnetism, the theory of superfluidity, the theory of second order phase transitions, the Ginzburg-Landau theory of superconductivity, the explanation of Landau damping in plasma physics, the Landau pole in quantum electrodynamics, and the two-component theory of neutrinos. He received the 1962 Nobel Prize in Physics for his development of a mathematical theory of superfluidity. Vitaly Lazarevich Ginzburg was a Russian theoretical physicist, astrophysicist, Nobel laureate, a member of the Russian Academy of Sciences and one of the fathers of Soviet hydrogen bomb. In their theory the free energy may be written as

$$\int \{\frac{1}{2}\xi^2(\nabla\varphi)^2 + \frac{1}{8}(\varphi^2-1)^2 - 2u\varphi\}dx \tag{1.9}$$

where the interaction term involves $(\nabla\varphi)^2$, ξ is a length scale and, from a molecular point of view, is a measure of the strength of the binding, $\frac{1}{8}(\varphi^2-1)^2$ is a prototype double well potential common to quantum field theory models. This double well potential indicates a lower free energy associated with the values $\varphi = \pm 1$ (pure solid or liquid) than the intermediate values corresponding to transitional states. It is possible to modify the potential in many ways to incorporate different physics. The term $-2u\varphi$, in the above equation, may be understood as the part of energy corresponds to the temperature times change in entropy. From the free energy, the Euler-Lagrange equations imply the identity

$$0 = \xi^2\Delta\varphi + \frac{1}{2}(\varphi - \varphi^3) + 2u \tag{1.10}$$

and combining this equation with the time independent heat balance equation and with appropriate boundary conditions, the following model may describe a system in equilibrium

$$\begin{cases} 0 = \xi^2 \Delta\varphi + \frac{1}{2}(\varphi - \varphi^3) + 2u, \\ \\ 0 = \Delta u, \\ \\ u(x) = u_\partial, \quad x \in \partial\Omega, \\ \\ \varphi(x) = \varphi_\partial, \quad x \in \partial\Omega. \end{cases} \tag{1.11}$$

For the non-equilibrium or time dependent situation φ will not minimize the free energy but will be differ by a term proportional to φ_t for a coefficient τ, it means the relaxation time. For the heat balance the above equation will be coupled with (1.7) and adding also initial conditions we have the phase field system, known better like Caginalp model

$$\begin{cases} \tau\varphi_t = \xi^2 \Delta\varphi + \frac{1}{2}(\varphi - \varphi^3) + 2u, \\ \\ u_t + \frac{l}{2}\varphi_t = k\Delta u, \\ \\ u(x) = u_\partial, \quad x \in \partial\Omega, \qquad \varphi(x) = \varphi_\partial, \quad x \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \Omega, \qquad \varphi(0,x) = \varphi_0(x), \quad x \in \Omega. \end{cases} \tag{1.12}$$

These equations have been studied by numerical computation, with physical reasonable results. The effect of surface tension is to act to as a stabilizing force, it is proportional to ξ. Then by adjusting ξ one may observe the competition between supercooling which tends to promote instabilities and surface tension which tends to suppress them.

Caginalp, in paper [10], built and studied the model using invariant set theory to obtain a priori bounds on $sup|u|$ and $sup|\varphi|$ in system (1.12) for suitable values of τ and ξ. When combined with classical methods, this leads to a global existence of solution. The basic idea is to examinate the flow in (u, φ) space as a function of time. The aim is to find regions such that the flow at the boundaries of the region is directed inward. The ideas about invariant sets also provide insight into the physical situation and the numerical calculations. They provide a criterion for determining the interface region, and indicate values of $\tau,\ \xi$ for which various values of φ will be stable points. The invariant regions depend on the ratio $\frac{\xi^2}{\tau}$, but not on ξ ans τ individually. One of the questions of interest is the behavior of these equations in the limit of small ξ ans τ but with ratio bounded as

$$C_1 \leq \frac{\xi^2}{\tau} \leq C_2,$$

where $C_1,\ C_2$ are two constants.

1.3 Asymptotic limits of the phase-field equations

One can obtain any of major sharp-interface models as limiting cases of a particular continuous representation of phase transitions which is based on microscopic considerations. Furthermore, the distinctions in the macroscopic sharp-interface models arise from the scaling relationships in the microscopic parameters of the continuous or phase-field model. The classical Stefan model neglects the physical effect of surface tension as stabilizing factor, as noted by Gibbs it acts by changing the temperature according the relation (1.8). Coupling the Stefan model with the (1.8), the interface is no longer defined simply by (1.1) but must be "tracked". In some applications, such as linear stability analysis, this is quite convenient; in others such as numerical computations it present difficulties. In addition to the surface tension effect, metallurgist observed that the temperature at the interface should be reduced beyond the "supercooling" exhibited by (1.8). The most relevant model to understand this phenomenon has been the linear velocity dependence

$$\Delta s[u(t,x)] = -\sigma\chi(t,x) - \alpha\sigma v(t,x), \tag{1.13}$$

where α is an adjustable parameter. The Stefan model coupled with the (1.13) equation is called modified Stefan problem, it can be studied to obtain a more realistic description of interface. The most interesting aspect of the differences between classical Stefan model, alternative model [(1.2), (1.3), (1.8)] and modified Stefan model is manifested in the stability properties of the interface. The interface in the classical problem is notoriously unstable. The Gibbs-Thompson condition restricts the magnitude of the curvature and thereby the extent of the instability, but, if the temperature is restricted via initial and boundary conditions, a large surface tension is not compatible with a large curvature. Also the parameter α, in the last equations has, in some sense, a stabilizing influence, indeed it reduces the amplitude of unstable modes. Thus, it is clear that the three problems posed will lead to very different behavior of the interface. A reliable analysis (numerical or analytical) for a particular material is only possible if the appropriate choice of the models made. This in turn depends mostly on parameters in the models. If one accept the idea that temperature, at the interface, need not to be zero, then there is the problem how one could distinguishes the two phases. The macroscopically measurable quantity that differs in two phases, φ, is called order parameter or phase field. However, the key question is determining it. One needs the implementation of a second equation that coupling with the inhomogeneous heat equation could be describe the physics of the problem. Using the Landau-Ginzburg theory of phase transitions one can obtains the system

$$\begin{cases} u_t + \frac{l}{2}\varphi_t = k\Delta u, \\ \\ \alpha\xi^2\varphi_t = \xi^2\Delta\varphi + \frac{1}{a}g(\varphi) + 2u, \\ \\ u(0,x) = u_0(x), \qquad \varphi(0,x) = \varphi_0(x), \qquad x \in \Omega, \end{cases} \tag{1.14}$$

where g is the non linear function $g(\varphi) = \frac{1}{2}(\varphi - \varphi^3)$, derivative of symmetric double-well potential, its minimum are at ± 1. For the above problem one could have several kind of boundary conditions. The asymptotic analysis, Caginalp used in [11] to understand the role of parameters and system behavior with respect to their different scales, does not depend crucially on the boundary conditions, which one needs add to the above system to solve it. The interface for (1.14) is specified as

$$\Gamma(t) = \{x \in \Omega : \ \varphi(t,x) = 0\}. \tag{1.15}$$

Within this formulation interfacial conditions can be derived from the model as a consequence of the microscopic physics built.

We note that under rather general conditions there exist a unique global solution to (1.14) in arbitrary dimension, the situation is quite different for the other problems, for examples the existence theory for the classical Stefan problem is limited to one or two dimensions and for the modified problem there is no existence theory. Caginalp showed that all sharp interface problems discussed in [11] arise as particular limits to phase-field system (1.14) in the asymptotic analysis as ξ, a and α approach zero. He, however, showed that scaling of the parameters (particularly a) is crucial in limiting behavior of the equations. In particular, one obtains distinct limits with very different behavior as a consequence due to the physical implications of this scaling. Hence, one can use a single set of equations to study (numerically) such diverse phenomena as fluid interface and solidifications problems.

It is useful to indicate, in general, the essential strategy to study the asymptotic limits.

Let r be the coordinate normal to the interface Γ (r is the distance to the interface if it is in the liquid region, negative if it is in the solid region) and let $\varphi_\pm$ the largest and the smallest roots, respectively, of $\frac{1}{a}g(\varphi)+2u=0$. Suppose that φ varies much more rapidly across the interface than u and it attains φ_+ a short distance toward the liquid side and φ_- on the solid side and φ be in the form $\varphi(r-vt)$. Under the following conditions:

$$\varepsilon^2 \equiv \xi^2 a, \qquad \alpha \ \text{ fixed}, \qquad \xi,\ a \to 0, \qquad \rho \equiv r/\varepsilon,$$

we may write the second equation in (1.14) using the prototype $g(\phi) = \frac{1}{2}(\phi - \phi^3)$, as

$$-\alpha v \in \phi_\rho \cong \phi_{\rho\rho} + \varepsilon k \phi_\rho + \ldots + \frac{1}{2}(\phi - \phi^3) + 2au, \tag{1.16}$$

where terms of order ε^2 in this expansion have been omitted. If there exists an expansion of the form $\phi = \phi^0 + \varepsilon\phi^1 + \ldots$, then the above relation implies that the $O(1)$ balance is

$$\phi^0_{\rho\rho} + \frac{1}{2}(\phi^0 - (\phi^0)^3) = 0, \tag{1.17}$$

subtracting the two relation one has the $O(\varepsilon)$ equation (provided $a\varepsilon^{-1}$ is $O(1)$ or smaller)

$$L\phi^1 \equiv \phi^1_{\rho\rho} + \frac{1}{2}(1 - 3(\phi^0)^2)\phi^1 \cong \varepsilon(-\alpha v\phi^0_\rho - k\phi^0_\rho - 2(a/\varepsilon)u) \equiv F, \tag{1.18}$$

noting that the derivative of the $O(1)$ solution satisfies the homogeneous equation, $L\phi^0_\rho = 0$, one has the solvability condition

$$0 = (F, \phi^0_\rho) = \varepsilon \int_{-\infty}^{\infty} \phi^0_\rho(-\alpha v\phi^0_\rho - k\phi^0_\rho - 2(a/\varepsilon)u)d\rho, \tag{1.19}$$

since $\int_{-\infty}^{\infty} \phi^0_\rho = \phi_+ = \phi_- = 2$, one has from the above equation the identity

$$4u(t,x) = -\frac{\varepsilon}{a}\sigma_0 k + \frac{\varepsilon}{a}\alpha\sigma_0 v \tag{1.20}$$

on Γ, where $\sigma_0 \int_{-\infty}^{\infty}(\phi^0_\rho)^2 d\rho = \frac{2}{3}$, hence $\varepsilon a^{-1} = \xi a^{-1/2}$ is a important scaling factor. The second equation in (1.14) was derived from $\tau\varphi_t = \delta\mathcal{F}/\delta\varphi$, where $\mathcal{F}$ is the free energy given by

$$\mathcal{F} = \int_\Omega \{\frac{1}{2}\xi^2(\nabla\varphi)^2 + \frac{1}{8a}(\varphi^2 - 1)^2 - 2u\varphi\}dx, \tag{1.21}$$

The surface tension σ is given by

$$\sigma \equiv \frac{\mathcal{F}(\varphi) - \frac{1}{2}\mathcal{F}(\varphi_+) - \frac{1}{2}\mathcal{F}(\varphi_-)}{A} \cong \frac{\mathcal{F}(\varphi_0)}{A}, \tag{1.22}$$

where A is the area of interface. To calculate this to first order, one multiplies (1.17) by ϕ^0_r and integrating

$$0 = \int_{-\infty}^{r} [\xi^2\varphi^0_r\varphi_{rr} + \frac{1}{2}\varphi^0_r(\varphi^0 - (\varphi^0)^3)]dr. \tag{1.23}$$

Then one may approximate the free energy in the surface tension and obtain

$$\sigma \cong \frac{\mathcal{F}(\varphi_0)}{A} = \int_{-\infty}^{\infty} \xi^2(\varphi^0_r)^2 dr = \frac{\varepsilon}{a}\int_{-\infty}^{\infty}(\phi^0_\rho)^2 d\rho. \tag{1.24}$$

Thus, being, the difference in entropy density,

$$\Delta s \equiv \frac{-\frac{\partial \mathcal{F}(\varphi_+)}{\partial u} + \frac{\partial \mathcal{F}(\varphi_-)}{\partial u}}{V} \cong 4, \tag{1.25}$$

where V is the volume the relation (1.13) follows as an $O(1)$ statement within this heuristic derivation, provided $\varepsilon a^{-1} = \xi a^{-1/2} = O(1)$ or smaller, however, if $\xi a^{-1/2}$ approaches zero, then one obtains the usual $u = 0$ Stefan condition (on Γ) and of α approaches zero while $\xi a^{-1/2} = O(1)$ one has the limit (1.8). In each of the scaling, the interface width is ε and the solution φ is approximated by (1.17). Hence far from the interface φ is constant for each ε so that the heat equation is valid. Across the interface, as ε approaches zero, one obtains, as a result of integration, the latent heat condition (1.3). It is clear that there is a crucial interplay between $\varepsilon = \xi a^{1/2}$ and $\varepsilon a^{-1} = \xi a^{-1/2}$ in the roles of interface thickness and interface tension. At a deeper level of physics, one has a competition between the atomic forces, represented by ξ and the well depth, represented by a^{-1}. One can regard it as a representation of the energy barrier between the two phases which depends on the particular microscopic considered. On a more fundamental level, the Landau-Ginzburg free energy incorporates the subtle concept of the correlation length, which is a measure of the distance within which atoms influence one another on a probabilistic basis. The correlation length concept provides yet another approach to understanding the different macroscopic limits. The heuristic calculations above suggest the following limit which one may verify explicit (see [11]).

Proposition 1.3.1. *In the limit $\xi,\ a \to 0$ with α and $\xi a^{-1/2}$ fixed, there exists a formal asymptotic solution of the phase-field model (1.14) which is governed by the modified Stefan problem (1.2), (1.3) and (1.13).*

It is possible to verify this by setting

$$c_1 \equiv \xi a^{-1/2}, \quad \varepsilon_1 = \xi^2, \quad f(\varphi) = c_1^2 g(\varphi)$$

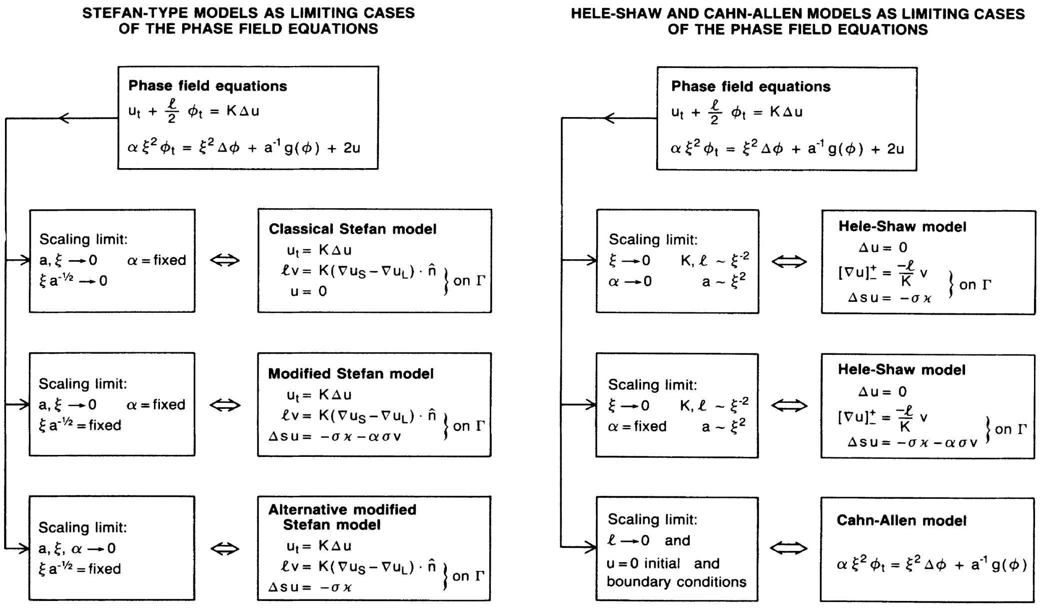

Figure 1.3: several scaling limits

and rewriting the system as

$$\begin{cases} u_t + \frac{l}{2}\varphi_t = k\Delta u, \\ \\ \alpha\varepsilon_1^2\varphi_t = \varepsilon_1^2\Delta + f(\varphi) + 2u\varepsilon_1, \end{cases}$$

Proposition 1.3.2. *In the limit ξ, a, $\alpha \to 0$ with $\xi a^{-1/2}$ fixed, there exists a formal asymptotic solution of the phase-field model (1.14) which is governed by the alternative modified Stefan problem (1.2), (1.3) and (1.8).*

We consider the same basic limit as in previous proposition but allow α to approach zero. Physically this means that the (dimensionless) relaxation time τ is small in comparison with ξ^2.

Proposition 1.3.3. *In the limit ξ, a, $\xi a^{-1/2} \to 0$ there exists a formal asymptotic solution of the phase-field model (1.14) which is governed by the classical Stefan problem (1.2), (1.3) and (1.1).*

In this case we'll set

$$\alpha = 0, \quad a \equiv \xi c_0^{-2}, \quad \xi \to 0$$

and multiplying by ξ, setting $\bar{\varepsilon}^2 = \xi$ and $f(\varphi) = c_0^2 g(\varphi)$

$$\begin{cases} u_t + \frac{l}{2}\varphi_t = k\Delta u, \\ \\ \alpha\bar{\varepsilon}^6\varphi_t = \bar{\varepsilon}^2\Delta + f(\varphi) + 2u\bar{\varepsilon}^2, \end{cases}$$

Using the parameter $\varepsilon_h = \xi^2$ we rewriting the system as

$$\begin{cases} \varepsilon_h u_t + \frac{c_2^2}{2}\varphi_t = \Delta u, \\ \\ \alpha\varepsilon_h^2\varphi_t = \varepsilon_h^2\Delta + f(\varphi) + 2u\bar{\varepsilon}_h, \end{cases}$$

one may obtain also Hele-Shaw type problems (see figure 1.3 [11]), as expressed in the next proposition

Proposition 1.3.4. *In the limit $\xi \to 0$ with $a \equiv \xi^2 c_1^{-2},\ k \equiv \xi^{-2},\ l \equiv \xi^{-2}c_2^{-2}$ and $\alpha,\ c_1$ fixed, there exists a formal asymptotic solution of the phase-field model (1.14) which is governed by the Hele-Shaw models.*

Chapter 2

Existence and regularity of a solution

2.1 A model for continuous casting process

Our goal in this work is to develop a model to study the industrial process, called continuous casting. Continuous casting is the process whereby molten metal is solidified into a "semi-finished" billet, bloom, or slab for subsequent rolling in the finishing mills. This process is used most frequently to cast steel, aluminum and copper. After undergoing any ladle treatments, and arriving at the correct temperature, molten metal is transported to the top of the casting machine. Metal is drained into the top of an open-base copper mold. The mold is water-cooled and oscillates vertically (or in a near vertical curved path) to prevent the metal sticking to the mold walls. In the mold, a thin shell of metal next to the mold walls solidifies before the metal section, now called a strand, exits from the base of the mold into a spray-chamber. The bulk of metal within the walls of the strand is still molten. The strand is immediately supported by closely-spaced, water cooled rollers. To increase the rate of solidification, the strand is also sprayed with large amounts of water as it passes through the spray-chamber. Final solidification of the strand may take place after the strand has exited the spray-chamber.

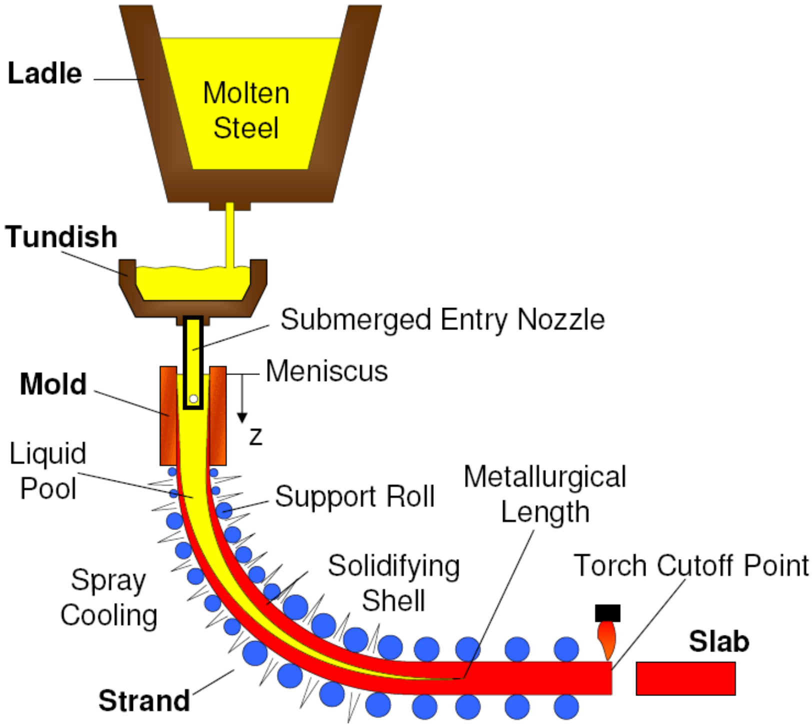

Figure 2.1: continuous casting process

That is well illustrated in the figure 2.1.

The phase-field model is the right tool to analyze this industrial process, it is able to understand the full complexity of phenomenon, to observe the *supercooling* and the *superheating* effects in the strand, indeed. And using non-homogeneous Cauchy-Neumann boundary conditions $w(t, x)$, depending on two variables, we are able to include a broad class of complex phenomena at $\partial\Omega$, untreated until now in literature. Indeed this new case can be involved as *boundary control*, like in studying the effects of the cooling spray on the solidification process, in a wide variety of industrial applications.

In literature, Caginalp's model is studied mainly with homogeneous Neumann-Neumann boundary conditions, so the first step will be define the model and deduce the existence and regularity of solution in our case.

On a bounded domain $\Omega \subset I\!R^n$, $n = 1,\ 2,\ 3$, with a C^2 boundary $\partial\Omega$ and for a time $T > 0$, setting $Q := [0, T] \times \Omega$ and $\Sigma := [0, T] \times \partial\Omega$, we will consider the phase-field transition system in the following form

$$\begin{cases} \rho c\dfrac{\partial u}{\partial t} + \dfrac{\ell}{2}\dfrac{\partial \varphi}{\partial t} = k\Delta u & \text{on } \ Q, \\ \\ \tau\dfrac{\partial \varphi}{\partial t} = \xi^2\Delta\varphi + \dfrac{1}{2a}(\varphi - \varphi^3) + 2u & \text{on } \ Q, \end{cases} \tag{2.1}$$

with the non-homogeneous Cauchy-Neumann boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial \nu} + hu = w(t,x) & \text{on } \ \Sigma, \\ \\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \ \Sigma, \end{cases} \tag{2.2}$$

and the initial conditions

$$\begin{cases} u(0,x) = u_0(x) & \text{on } \ \Omega, \\ \varphi(0,x) = \varphi_0(x) & \text{on } \ \Omega, \end{cases} \tag{2.3}$$

where

- u represents the *reduced temperature* distribution on Q,
- φ is the *phase function* used to distinguish between the states of a material which occupies the region Ω at each time $t \in (0,T]$,
- $f \in L^p(Q)$, $g \in L^q(Q)$ are given functions; also can be interpreted as *distributed control*,
- $w \in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)$ is a given function: the temperature of the surrounding at $\partial\Omega$ for each time $t \in (0,T]$ (*the boundary control*),
- $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $\varphi_0 \in W_q^{2-\frac{2}{q}}(\Omega)$, provided $\dfrac{\partial u_0}{\partial \nu} + hu_0 = w(0,x)$ and $\dfrac{\partial \varphi_0}{\partial \nu} = 0$ on $\partial\Omega$,
- p, q are given numbers which satisfy

$$q \geq p \geq 2. \tag{2.4}$$

And the positive parameters $\rho,\ c,\ \tau,\ \xi,\ \ell,\ k,\ h,\ a$ have a physical meaning, namely:

- ρ is the *density*,
- c is the *casting speed*,
- τ is the *relaxation time*,
- ξ is the *length scale of the interface*,
- ℓ denotes the *latent heat*,
- k the *heat conductivity*,
- h the *heat transfer coefficient*,
- a is a *probabilistic measure* on the individual atoms.

Basic tools in our approach are the Leray-Schauder degree theory and properties of the Nemytskij operator [12], as well as the L^p-theory of linear and quasi-linear parabolic equations [15]. We also use the Lions and Peetre embedding theorem [17] to ensure the existence of a continuous embedding $W_p^{2,1}(Q) \subset L^\mu(Q)$, where

$$\mu = \begin{cases} \infty & \text{if } \frac{1}{p} - \frac{2}{n+2} < 0, \\[2ex] \text{any number } \geq 3p & \text{if } \frac{1}{p} - \frac{2}{n+2} = 0, \\[2ex] \dfrac{p\,(n+2)}{n+2-2p} & \text{if } \frac{1}{p} - \frac{2}{n+2} > 0. \end{cases} \tag{2.5}$$

For a given positive integer k and $1 \leq p \leq \infty$ we denote by $W_p^{2k,k}(Q)$ the Sobolev space on Q

$$W_p^{2k,k}(Q) = \left\{ y \in L^p(Q) : \ \frac{\partial^r}{\partial t^r} \frac{\partial^s}{\partial x^s}\, y \in L^p(Q), \quad \text{for } 2r+s \leq k \right\}.$$

We shall use the Sobolev spaces $W_p^l(Q)$, $W_p^{l,l/2}(Q)$ with non integral l for the initial and boundary conditions, respectively (see [15, p. 70 and p. 81]). Our main results in studying the existence of solution in problem (2.1)-(2.3) is the following

Theorem 2.1.1. *Problem (2.1)-(2.3) has a unique solution* (u,φ) *with* $u \in W_p^{2,1}(Q)$ *and* $\varphi \in W_\nu^{2,1}(Q)$, *where* $\nu = min\{p,\mu\}$. *In addition* (u,φ) *satisfies*

$$\begin{aligned} \|u\|_{W_p^{2,1}(Q)} &+ \|\varphi\|_{W_\nu^{2,1}(Q)} \le C\left\{1 + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\varphi_0\|^{3-\frac{2}{q}}_{W_q^{2-\frac{2}{q}}(\Omega)}\right. \\ &+ \left.\|w\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)} + \|f\|_{L^p(Q)} + \|g\|_{L^q(Q)}\right\}, \end{aligned} \tag{2.6}$$

where the constant C *depends on* $|\Omega|$ *(the measure of* Ω*),* T, n, p, q *and physical parameters.*

Moreover, given any number $M > 0$, *if* (u_1,φ_1) *and* (u_2,φ_2) *are solutions of (2.1) for the same initial conditions, corresponding to the dates* (f_1,g_1,w_1), $(f_2,g_2,w_2) \in L^p(Q) \times L^q(Q) \times W_p^{2-1/p,1-1/2p}(\Sigma)$, *respectively, such that* $\|\varphi_1\|_{L^\nu(Q)}$, $\|\varphi_2\|_{L^\nu(Q)} \le M$, *then the estimate below holds*

$$\begin{aligned} \|u_1 - u_2\|_{W_p^{2,1}(Q)} &+ \|\varphi_1 - \varphi_2\|_{W_\nu^{2,1}(Q)} \le C\left\{\|f_1 - f_2\|_{L^p(Q)}\right. \\ &+ \left.\|g_1 - g_2\|_{L^q(Q)} + \|w_1 - w_2\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\right\}, \end{aligned} \tag{2.7}$$

where the constant C *depends on* $|\Omega|$, T, M, n, p, q *and physical parameters.*

The nonlinear part in (2.1), $\frac{1}{2a}(\varphi - \varphi^3)$, verifies the assumptions (H_0)-(H_2) in [20], precisely:

$H_0)$ $(\varphi - \varphi^3)|\varphi|^{3p-4}\varphi \le 1 + |\varphi|^{3p-1} - |\varphi|^{3p}$.

$H_1)$ There is a constant $a_0 \in I\!R$ such that

$$\left[(\varphi_1 - \varphi_1^3) - (\varphi_2 - \varphi_2^3)\right](\varphi_1 - \varphi_2) \le \alpha_1(\varphi_1 - \varphi_2)^2, \quad \forall \varphi_1, \varphi_2 \in I\!R.$$

$H_2)$ There are a function $\bar{F} : Q \times I\!R^2 \to I\!R$ and a constant $b_0 > 0$ verifying, $\forall (t,x) \in Q$, $\varphi_1, \varphi_2 \in I\!R$, the relations:

$$\left((\varphi_1 - \varphi_1^3) - (\varphi_2 - \varphi_2^3)\right)^2 \le \bar{F}(t,x,\varphi_1,\varphi_2)(\varphi_1 - \varphi_2)^2,$$

$$\bar{F}(t,x,\varphi_1,\varphi_2) \le b_0(1 + |\varphi_1|^4 + |\varphi_2|^4).$$

To verify (H_0)-(H_2), we note that $\dfrac{1}{2a}(\varphi-\varphi^3)$ could be obtainded in the next example (see Example 1.1. in [20]) considering $r=3$, $\psi_1=0$, $\psi_2=\dfrac{1}{2a}z$, $\alpha=\dfrac{1}{2a}$, i.e. $F(z)=\dfrac{1}{2a}(z-z^3)$.

Example 2.1. Fix a number p with

$$p \geq \max\{2,\ \frac{n+2}{4}\} \tag{2.8}$$

Let $F : I\!R \to I\!R$ be any function of the following form

$$F(z)=\psi(z)-\alpha|z|^{r-1}z, \qquad \forall z \in I\!R, \tag{2.9}$$

with costant $\alpha>0$, $r \geq 2$ satisfying

$$r<\frac{n+2}{n+2-2p}, \qquad \text{if} \ \ n+2-2p>0,$$

and $\psi \in C^1(I\!R)$ which fulfils the properties

$$|\psi'(z)| \leq \beta(1+|z|^{r-2}), \qquad \forall z \in I\!R, \tag{2.10}$$

$$\Big(\psi(z_1)-\psi(z_2)\Big)(z_1-z_2) \leq \gamma(z_1-z_2)^2, \qquad \forall z_1,\ z_2 \in I\!R \tag{2.11}$$

for costants $\beta,\ \gamma>0$. There exist numbers r as required because due to (2.8), $2<(n+2)/(n+2-2p)$ whenever $n+2-2p>0$.

We note that (2.10) implies by integration that ψ satisfies the growth condition

$$|\psi(z)| \leq \gamma_0(1+|z|^{r-1}), \qquad \forall z \in I\!R, \tag{2.12}$$

with a constant $\gamma_0>0$. By (2.9) and (2.12), one finds some α_0 such that (H_0) is satisfied with $\beta=\alpha$. Using (2.9), (2.11) and the fact that function $f(t)=|t|^{r-1}t$ is increasing, we see that (H_1) is verified for $\alpha_1=\gamma$. Finally, from (2.9), (2.10) and Mean Value Theorem we have

$$\Big(F(z_1) \quad - \quad F(z_2)\Big)^2 \geq \frac{1}{2}\big(\psi(z_1)-\psi(z_2)\big)^2+\frac{\alpha}{2}\cdot$$
$$\cdot \quad \Big(|z_1|^{r-1}z_1-|z_2|^{r-1}z_2\Big)^2 \leq \bar{F}(z_1,z_2)(z_1-z_2)^2, \qquad \forall z_1,\ z_2 \in I\!R$$

where

$$
\begin{aligned}
\bar{F}(z_1,z_2) &= \frac{1}{2}\sup_{t\in[0,1]}\Big(\psi'((1-t)z_1-tz_2)\Big)^2+\frac{\alpha^2r^2}{2}\sup_{t\in[0,1]}\Big((1-t)z_1-tz_2\Big)^{2(r-1)} \\
&\le \frac{\beta^2}{2}(1+|z_1|^{r-2}+|z_2|^{r-2})^2+\frac{\alpha^2r^2}{2}(|z_1|+|z_2|)^{2(r-1)} \\
&\le \beta_0(1+|z_1|^{2(r-1)}+|z_2|^{2(r-1)}), \qquad \forall z_1,\ z_2\in I\!R
\end{aligned}
$$

for some costant $\beta_0>0$, which ensures that (H_2) is satisfied. A possible choice in (2.9) is $\psi(z)=\psi_1(z)+\psi_2(z)$, $\forall z\in I\!R$, with $\psi_1\in C^1(I\!R)$ non increasing, satisfying (2.10), and $\psi_2\in C^1(I\!R)$ Lipschitzian on $I\!R$.

The result in Theorem 2.1.1 was proved in more general nonlinear case by Moroşanu and Motreanu [20] with homogeneous boundary conditions $\dfrac{\partial u_0}{\partial\nu}=\dfrac{\partial\varphi_0}{\partial\nu}=0$. The idea of the proof is the same. We treat separately the non-linear equation in φ and linear equation in u in system (2.1). Defining an homotopy for each problem, and thanks to L^p-theory and Leray-Schauder degree theory one get the existence of the solution φ in the first problem. After, following the same way, and using the estimates got in φ, we deduce the existence and the esimates for u, too.

In the following we will to denote by C several positive constants.

2.2 An auxiliary equation

We consider the nonlinear equation in Caginalp's model:

$$
\begin{cases}
\tau\dfrac{\partial\varphi}{\partial t}-\xi^2\Delta\varphi=\dfrac{1}{2a}(\varphi-\varphi^3)+\bar{g}(t,x) & \text{on } Q,\\[2ex]
\dfrac{\partial\varphi}{\partial\nu}=0 & \text{on } \Sigma,\\[2ex]
\varphi(0,x)=\varphi_0(x) & \text{on } \Omega,
\end{cases}
\tag{2.13}
$$

where $\bar{g}\in L^p(Q)$ and $\varphi_0\in W_q^{2-\frac{2}{q}}(\Omega)$ verifies $\dfrac{\partial\varphi_0}{\partial\nu}=0$ on $\partial\Omega$.

Theorem 2.2.1. *There exists a unique solution $\varphi \in W_p^{2,1}(Q)$ for (2.13) and φ satisfies*

$$\|\varphi\|_{W_p^{2,1}(Q)} \leq C \left\{1 + \|\varphi_0\|^{3-\frac{2}{p}}_{W_q^{2-\frac{2}{q}}(\Omega)} + \|\bar{g}\|_{L^p(Q)}\right\}, \tag{2.14}$$

where the constant C depends on $|\Omega|$, T, n, p, q and physical parameters.

If φ_1, φ_2 are two solutions of (2.13) corresponding to φ_0^1, $\varphi_0^2 \in W_q^{2-\frac{2}{q}}(\Omega)$ and $\bar{g}_1$, $\bar{g}_2$, respectively, such that

$$\|\varphi_1\|_{W_p^{2,1}(Q)} \leq M, \quad \|\varphi_2\|_{W_p^{2,1}(Q)} \leq M, \tag{2.15}$$

then

$$\|\varphi_1 - \varphi_2\|_{W_p^{2,1}(Q)} \leq C \left\{\|\varphi_0^1 - \varphi_0^2\|_{W_q^{2-\frac{2}{q}}(\Omega)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)}\right\}, \tag{2.16}$$

where the constant C depends on $|\Omega|$, T, M, n, p, q and physical parameters.

Proof. We will apply the Leray-Schauder degree theory. To this end let us define the nonlinear operator $T : L^{3p}(Q) \times [0,1] \to L^{3p}(Q)$ as

$$T(v, \lambda) = \varphi = \varphi(v, \lambda) \quad \forall\ v \in L^{3p}(Q),\ \forall\ \lambda \in [0,1], \tag{2.17}$$

where φ is the solution of the linear problem

$$\begin{cases} \tau \dfrac{\partial \varphi}{\partial t} - \xi^2 \Delta \varphi = \lambda \left[\dfrac{1}{2a}(v - v^3) + \bar{g}(t,x)\right] & \text{on } Q, \\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \lambda \varphi_0(x) & \text{on } \Omega. \end{cases} \tag{2.18}$$

According to **Hadamard's well-posedness conditions** we must check that:

i. a solution exists,

ii. the solution is unique,

iii. the solution depends continuously on the data, in particular the initial and boundary values.

First, let us verify that T is well-defined (problem (2.18) has a solution). We can derive that $\forall v \in L^{3p}(Q)$, $\frac{1}{2a}(v - v^3) \in L^p(Q)$. Then $\frac{1}{2a}(v - v^3) + \bar{g}(t,x) \in L^p(Q)$. Using now L^p-theory of parabolic equations [15, p. 341-342], the solution φ to problem (2.18) exists and is unique with

$$\varphi = \varphi(v, \lambda) \in W_p^{2,1}(Q) \quad \forall\, v \in L^{3p}(Q),\ \forall\, \lambda \in [0,1]. \tag{2.19}$$

Since $3p \leq \dfrac{p\,(n+2)}{n+2-2p}$ if $\dfrac{1}{p} - \dfrac{2}{n+2} > 0$ according to (2.5), we can take $\mu > 3p$ in any case; indeed $3p \geq n+2$ (see (2.4) and $\Omega \subset I\!\!R^n$, $n \in \{1,\ 2,\ 3\}$). Consequently, we have the continuous inclusions

$$W_p^{2,1}(Q) \subset L^{\mu}(Q) \subset L^{3p}(Q). \tag{2.20}$$

Summing up the latest results it means that the operator T is well defined and it maps $L^{3p}(Q)$ into $L^{3p}(Q)$.

We now check the continuity of T. Let $v_n \to v$ in $L^{3p}(Q)$ and $\lambda_n \to \lambda$ in $[0,1]$. Denote $\varphi_n^{\lambda_n} = T(v_n, \lambda_n)$, $\varphi_n^{\lambda} = T(v_n, \lambda)$ and $\varphi^{\lambda} = T(v, \lambda)$. From (2.17) and (2.18) we obtain

$$\begin{cases} \tau \dfrac{\partial}{\partial t}\,(\varphi_n^{\lambda_n} - \varphi_n^{\lambda}) - \xi^2 \Delta(\varphi_n^{\lambda_n} - \varphi_n^{\lambda}) & \\ \qquad\qquad = (\lambda_n - \lambda)\left[\frac{1}{2a}(v_n - v_n^3) + \bar{g}(t,x)\right] & \text{on } Q, \\ \dfrac{\partial}{\partial \nu}\,(\varphi_n^{\lambda_n} - \varphi_n^{\lambda}) = 0 & \text{on } \Sigma, \\ (\varphi_n^{\lambda_n} - \varphi_n^{\lambda})(0,x) = (\lambda_n - \lambda)\varphi_0(x) & \text{on } \Omega. \end{cases} \tag{2.21}$$

From L^p-theory and (2.4), we have

$$\begin{aligned} \|\varphi_n^{\lambda_n} \;&-\; \varphi_n^{\lambda}\|_{W_p^{2,1}(Q)} \leq C|\lambda_n - \lambda| \Big\{ \|\varphi_0\|_{W_q^{2-\frac{2}{q}}(\Omega)} \\ &+\; \|v_n - v_n^3\|_{L^p(Q)} + \|\bar{g}\|_{L^p(Q)} \Big\}. \end{aligned} \tag{2.22}$$

Having v_n bounded in $L^{3p}(Q)$, we derive that $v_n - v_n^3$ is bounded in $L^p(Q)$ (see [12]). So we get $\|\varphi_n^{\lambda_n} - \varphi_n^{\lambda}\|_{W_p^{2,1}(Q)} \to 0$ as $n \to \infty$. Again from (2.17) and (2.18) we obtain

$$\begin{cases} \tau \dfrac{\partial}{\partial t}\,(\varphi_n^{\lambda} - \varphi^{\lambda}) - \xi^2 \Delta(\varphi_n^{\lambda} - \varphi^{\lambda}) \\ \qquad\qquad = \lambda\Big\{\frac{1}{2a}[(v_n - v) + (v^3 - v_n^3)]\Big\} & \text{on } \ Q, \\ \dfrac{\partial}{\partial \nu}\,(\varphi_n^{\lambda} - \varphi^{\lambda}) = 0 & \text{on } \ \Sigma, \\ (\varphi_n^{\lambda} - \varphi^{\lambda})(0,x) = 0 & \text{on } \ \Omega. \end{cases} \tag{2.23}$$

As above, the L^p-theory gives the estimate

$$\|\varphi_n^{\lambda} - \varphi^{\lambda}\|_{W_p^{2,1}(Q)} \leq C\Big\{\|(v_n - v) + (v^3 - v_n^3)\|_{L^p(Q)}\Big\}, \tag{2.24}$$

and the continuity of Nemytskij operator (see [12]) allows to conclude that

$$\|\varphi_n^{\lambda} - \varphi^{\lambda}\|_{W_p^{2,1}(Q)} \to 0 \qquad \text{as } \ n \to \infty.$$

Using the continuous embedding (2.20) and (2.22)-(2.24), we derive the continuity of the mapping T defined in (2.17).

Furthermore, the mapping T is compact, we need the compactness to say that the Leray-Scauder degree of the map is invariant in λ. This can be seen by writing it as the composition

$$T : L^{3p}(Q) \times [0,1] \to W_p^{2,1}(Q) \hookrightarrow L^{3p}(Q),$$

where the second inclusion is compact since $\mu > 3p$ [17, p. 24].

We show now that there exists $\delta > 0$ such that (see (2.17))

$$(\varphi, \lambda) \in L^{3p}(Q) \times [0,1], \ \varphi = T(\varphi, \lambda) \quad \Longrightarrow \quad \|\varphi\|_{L^{3p}(Q)} < \delta. \tag{2.25}$$

Let $\varphi \in L^{3p}(Q)$ solving the problem

$$\begin{cases} \tau \dfrac{\partial}{\partial t}\,\varphi - \xi^2 \Delta\varphi = \lambda\Big\{\dfrac{1}{2a}(\varphi - \varphi^3) + \bar{g}(t,x)\Big\} & \text{on } \ Q, \\ \dfrac{\partial}{\partial \nu}\,\varphi = 0 & \text{on } \ \Sigma, \\ \varphi(0,x) = \lambda\varphi_0(x) & \text{on } \ \Omega. \end{cases} \tag{2.26}$$

Multiplying the first equation in (2.26) by $|\varphi|^{3p-4}\varphi$, integrating over $Q_t := (0,t)\times\Omega$, $t\in(0,T]$ and using Young's inequality and Green's Theorem, we get

$$\begin{aligned} &\frac{1}{3p-2}\int\limits_{\Omega} |\varphi(t,x)|^{3p-2}dx + 3(p-1)\int\limits_{Q_t} |\nabla\varphi|^2|\varphi|^{3p-4}\; dsdx \\ &\quad\le \frac{1}{3p-2}\int\limits_{\Omega} |\varphi_0(x)|^{3p-2}\; dx + \frac{\lambda}{2a}\int\limits_{Q_t}(\varphi-\varphi^3)|\varphi|^{3p-4}\varphi\; dsdx \\ &\qquad + \lambda\frac{1}{p}\varepsilon^{-p}\|\bar{g}\|^p_{L^p(Q)} + \lambda\frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\int\limits_{Q_t}|\varphi|^{3p}\; dsdx. \end{aligned} \tag{2.27}$$

By H_0 and Young's inequality, from (2.27) we obtain

$$\begin{aligned} &\frac{\lambda}{2a}\int\limits_{Q_t} |\varphi|^{3p}\; dsdx + \frac{1}{3p-2}\int\limits_{\Omega}|\varphi(t,x)|^{3p-2}dx \\ &\quad+\; 3(p-1)\int\limits_{Q_t}|\nabla\varphi|^2|\varphi|^{3p-4}\; dsdx \le \frac{1}{3p-2}\int\limits_{\Omega}|\varphi_0(x)|^{3p-2}dx \\ &\quad+\; \lambda\left(\frac{1}{2a}|\Omega|T + \frac{1}{3p}\varepsilon^{-3p}\frac{1}{2a}|\Omega|T + \frac{1}{p}\varepsilon^{-p}\|\bar{g}\|^p_{L^p(Q)}\right) \\ &\quad+\; \lambda\frac{1}{2a}\frac{3p-1}{3p}\varepsilon^{\frac{3p}{3p-1}}\int\limits_{Q_t}|\varphi|^{3p}\; dsdx + \lambda\frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\int\limits_{Q_t}|\varphi|^{3p}\; dsdx. \end{aligned} \tag{2.28}$$

Taking ε small enough, inequality (2.28) yields

$$\lambda\||\varphi|^3\|^p_{L^p(Q)} \le C(|\Omega|, T, n, p, a)\left(1 + \|\varphi_0\|^{3p-2}_{L^{3p-2}(\Omega)} + \|\bar{g}\|^p_{L^p(Q)}\right). \tag{2.29}$$

Applying L^p-theory to problem (2.26) and the embedding $W_q^{2-2/q}(\Omega) \subset W_p^{2-2/p}(\Omega)$ (see (2.4)), we see that

$$\begin{aligned} \|\varphi\|_{W_p^{2,1}(Q)} &\le C(|\Omega|, T, M, p, q) \times \\ &\times \left(\|\varphi_0\|_{W_q^{2-2/q}(\Omega)} + \lambda \|\varphi - \varphi^3\|_{L^p(Q)} + \|\bar{g}\|_{L^p(Q)} \right). \end{aligned} \tag{2.30}$$

Taking into account Lemma 1.1 in [20] and (2.29), we deduce that

$$\begin{aligned} \lambda \|\varphi - \varphi^3\|^p_{L^p(Q)} &\le \lambda C^p \int_Q (1 + |\varphi|^3)^p \, dtdx \le 2^{p-1} C^p (|\Omega| T + \lambda \| |\varphi|^3 \|^p_{L^p(Q)}) \\ &\le C(|\Omega|, T, n, p, a) \left(1 + \|\varphi_0\|^{3p-2}_{L^{3p-2}(\Omega)} + \|\bar{g}\|^p_{L^p(Q)} \right). \end{aligned}$$

Then (2.30) becomes

$$\begin{aligned} \|\varphi\|_{W_p^{2,1}(Q)} &\le C(|\Omega|, T, M, n, p, q, a) \times \\ &\times \left(1 + \|\varphi_0\|_{W_q^{2-2/q}(\Omega)} + \|\varphi_0\|^{3-\frac{2}{p}}_{L^{3p-2}(\Omega)} + \|\bar{g}\|_{L^p(Q)} \right). \end{aligned} \tag{2.31}$$

The continuous embedding in (2.20) ensures that

$$\|\varphi\|_{L^{3p}(Q)} \le C \|v\|_{W_p^{2,1}(Q)}. \tag{2.32}$$

Combining (2.31) and (2.32) we see that the claim in (2.25) holds true.

Denoting

$$B_\delta := \left\{ \varphi \in L^{3p}(Q) : \ \|\varphi\|_{L^{3p}(Q)} < \delta \right\},$$

relation (2.25) ensures that

$$T(\varphi, \lambda) \ne \varphi \qquad \forall \varphi \in \partial B_\delta, \ \ \forall \lambda \in [0, 1], \tag{2.33}$$

provided that $\delta > 0$ is sufficiently large.

Property (2.33) and thanks $T(\cdot, \lambda) : L^{3p}(Q) \to L^{3p}(Q)$ to be compact, allow to consider the Leray-Schauder degree [12]

$$\deg \left(Id_{L^{3p}(Q)} - T, B_\delta, 0 \right) \qquad \forall \lambda \in [0, 1], \tag{2.34}$$

where Id_B represents the identity of the space B (see [23, p. 15]). The homotopy invariance of the Leray-Schauder degree enables us to write the equality

$$\deg\Big(Id_{L^{3p}(Q)} - T(\cdot,0), B_\delta, 0\Big) = \deg\Big(Id_{L^{3p}(Q)} - T(\cdot,1), B_\delta, 0\Big). \tag{2.35}$$

Choosing $\delta > 0$ large enough so that the ball B_δ contain the unique solution of the linear equation $\varphi - T(\varphi, 0) = 0$, it follows that

$$\deg\Big(Id_{L^{3p}(Q)} - T(\cdot,0), B_\delta, 0\Big) = 1. \tag{2.36}$$

From (2.35) and (2.36) we can conclude that problem (2.13) has a solution $\varphi \in W_p^{2,1}(Q)$. The estimate (2.14) is a consequence of (2.31).

Next we will establish the stability result (2.16) which gives us the uniqueness of the solution of (2.13) as a corollary. By hypothesis, φ_1, $\varphi_2 \in W_p^{2,1}(Q)$ solve problem (2.13) corresponding to $\bar{g}_1$, $\bar{g}_2$ and φ_0^1, φ_0^2, respectively. Thus $\varphi_1 - \varphi_2 \in W_p^{2,1}(Q)$ and it satisfies

$$\begin{cases} \tau\dfrac{\partial}{\partial t}(\varphi_1-\varphi_2) - \xi^2\Delta(\varphi_1-\varphi_2) & \\ \qquad = \Big\{\frac{1}{2a}\big[(\varphi_1-\varphi_2)-(\varphi_1^3-\varphi_2^3)\big] + (\bar{g}_1-\bar{g}_2)\Big\} & \text{on } Q, \\ \dfrac{\partial}{\partial\nu}(\varphi_1-\varphi_2) = 0 & \text{on } \Sigma, \\ (\varphi_1-\varphi_2)(0,x) = \varphi_0^1-\varphi_0^2 & \text{on } \Omega. \end{cases} \tag{2.37}$$

Multiplying (2.37).1 by $|\varphi_1-\varphi_2|^{p-2}(\varphi_1-\varphi_2)$, integrating over Q_t, $t \in (0,T]$, and using Green's formula and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &\frac{1}{p}\int_\Omega |\varphi_1(x,t)-\varphi_2(x,t)|^p\,dx + (p-1)\int_{Q_t} |\nabla(\varphi_1-\varphi_2)|^2|\varphi_1-\varphi_2|^{p-2}\,dsdx \\ &\quad \le \frac{1}{p}\|\varphi_0^1-\varphi_0^2\|_{L^p(\Omega)}^p + \frac{2^p}{p}\|\bar{g}_1-\bar{g}_2\|_{L^p(Q)}^p + \frac{p-1}{p}\frac{1}{2^{\frac{p}{p-1}}}\int_{Q_t}|\varphi_1-\varphi_2|^p\,dsdx \\ &\quad + \int_{Q_t}\Big[(\varphi_1-\varphi_2)-(\varphi_1^3-\varphi_2^3)\Big]|\varphi_1-\varphi_2|^{p-2}(\varphi_1-\varphi_2)\,dsdx, \quad \forall t \in (0,T]. \end{aligned}$$

Due to assumption (H1) and by means of Gronwall's inequality, it results

$$\|\varphi_1 - \varphi_2\|^p_{L^p(\Omega)} \leq C(T,p)\left(\|\varphi_0^1 - \varphi_0^2\|^p_{L^p(\Omega)} + \|\bar{g}_1 - \bar{g}_2\|^p_{L^p(Q)}\right). \tag{2.38}$$

According to (2.20), we have $\varphi_1,\ \varphi_2 \in W^{2,1}_p(Q) \subset L^\mu(Q) \subset L^{3p}(Q)$, which yields that $\frac{1}{2a}\big[(\varphi_1 - \varphi_2) - (\varphi_1^3 - \varphi_2^3)\big] \in L^p(Q)$. So we may apply the L^p-theory to the linear problem (2.37) which, in conjunction with the embedding $W_q^{2-2/q}(\Omega) \subset W_p^{2-2/p}(\Omega)$ (see (2.4)), gives the estimate

$$\begin{aligned} \|\varphi_1 \ &- \ \varphi_2\|^p_{W^{2,1}_p(Q)} \leq C(|\Omega|,T,n)\Big(\|\varphi_0^1 - \varphi_0^2\|^p_{W_q^{2-2/q}(\Omega)} \\ &+ \ \|(\varphi_1 - \varphi_2) - (\varphi_1^3 - \varphi_2^3)\|^p_{L^p(Q)} + \|\bar{g}_1 - \bar{g}_2\|^p_{L^p(Q)}\Big). \end{aligned} \tag{2.39}$$

The inequality $\mu \geq 3p$ allows us to fix a number m such that

$$2 \leq p \leq \frac{\mu p}{\mu + p - 3p} \leq 3p \leq m \leq \mu. \tag{2.40}$$

Consequently, the next sequence of embeddings holds

$$W^{2,1}_p(Q) \subset L^\mu(Q) \subset L^m(Q) \subset L^{3p}(Q) \subset L^p(Q) \subset L^2(Q). \tag{2.41}$$

From (H2), (2.41) and Hölder's inequality it is seen that

$$\begin{aligned} \|(\varphi_1 \ &- \ \varphi_2) - (\varphi_1^3 - \varphi_2^3)\|_{L^p(Q)} \leq \|\bar{F}(\varphi_1,\varphi_2)^{1/2}|\varphi_1 - \varphi_2|\|_{L^p(Q)} \\ &= \ \Big(\int_Q \bar{F}(\varphi_1,\varphi_2)^{\frac{p}{2}}|\varphi_1 - \varphi_2|^p \, dtdx\Big)^{\frac{1}{p}} \\ &\leq \ \Big(\int_Q \bar{F}(\varphi_1,\varphi_2)^{\frac{n_0}{2}} \, dtdx\Big)^{\frac{1}{n_0}} \|\varphi_1 - \varphi_2\|_{L^m(Q)}, \end{aligned} \tag{2.42}$$

where we denoted $n_0 := mp/(m-p)$. The computation above makes sense because $\bar{F}(\varphi_1,\varphi_2)^{\frac{n_0}{2}} \in L^1(Q)$. Indeed, taking into account the growth condition in (H2), $\bar{F}(\varphi_1,\varphi_2) \in L^{\frac{1}{2}\frac{\mu}{2}}(Q)$ whenever $\varphi_1,\varphi_2 \in L^\mu(Q)$, and by (2.41) it is true that

$$\frac{\mu}{2} > n_0 = \frac{mp}{(m-p)} > 2. \tag{2.43}$$

Then, the inequalities in (2.43) lead to the claim above. Combining (2.39) and (2.42), we arrive at

$$\begin{aligned} \|\varphi_1 - \varphi_2\|_{W_p^{2,1}(Q)} &\leq C(|\Omega|, T, n)(\|\varphi_0^1 - \varphi_0^2\|_{W_q^{2-2/q}(\Omega)} \\ &+ \|\bar{F}(\varphi_1, \varphi_2)\|^{1/2}_{L^{\frac{n_0}{2}}(Q)} \|\varphi_1 - \varphi_2\|_{L^m(Q)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)}). \end{aligned} \tag{2.44}$$

In addition, we have (see (H_2))

$$\Big(1 + |\varphi_1|^4 + |\varphi_2|^4\Big)^{\frac{n_0}{2}} \leq C(p)\Big(1 + |\varphi_1)|^{2n_0} + |\varphi_2|^{2n_0}\Big). \tag{2.45}$$

The relation above, inequality (2.43) and estimate (2.44) imply

$$\begin{aligned} \|\varphi_1 &- \varphi_2\|_{W_p^{2,1}(Q)} \leq C(|\Omega|, T, n)\Big[\|\varphi_0^1 - \varphi_0^2\|_{W_q^{2-2/q}(\Omega)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)} \\ &+ C(|\Omega|, T, p)\Big(1 + \|\varphi_1\|^2_{L^{2n_0}(Q)} + \|\varphi_2\|^2_{L^{2n_0}(Q)}\Big)\|\varphi_1 - \varphi_2\|_{L^m(Q)}\Big] \\ &\leq C(|\Omega|, T, n, p)\Big[\|\varphi_0^1 - \varphi_0^2\|_{W_q^{2-2/q}(\Omega)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)} \\ &+ \Big(1 + \|\varphi_1\|^2_{W_p^{2,1}(Q)} + \|\varphi_2\|^2_{W_p^{2,1}(Q)}\Big)\|\varphi_1 - \varphi_2\|_{L^m(Q)}\Big] \\ &\leq C(|\Omega|, T, n, p)(1 + 2M^2) \\ &\times (\|\varphi_0^1 - \varphi_0^2\|_{W_q^{2-2/q}(\Omega)} + \|\varphi_1 - \varphi_2\|_{L^m(Q)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)}). \end{aligned} \tag{2.46}$$

By the embeddings in (2.41), the interpolation inequality (see Lions [17, p. 58]) yields that $\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that

$$\|v\|_{L^m(Q)} \leq \varepsilon\|v\|_{W_p^{2,1}(Q)} + C(\varepsilon)\|v\|_{L^p(Q)}, \quad \forall v \in W_p^{2,1}(Q). \tag{2.47}$$

From (2.46), (2.47) and (2.38), we derive that

$$\begin{aligned} \big(1 &- \varepsilon C(|\Omega|, T, n, M, p)\big)\|\varphi_1 - \varphi_2\|_{W_p^{2,1}(Q)} \\ &\leq C(|\Omega|, T, n, M, p)\Big(\|\varphi_0^1 - \varphi_0^2\|_{L^p(\Omega)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)} \\ &+ C(\varepsilon)C(|\Omega|, T, n, M, p)\left(\|\varphi_0^1 - \varphi_0^2\|_{W_q^{2-2/q}(\Omega)} + \|\bar{g}_1 - \bar{g}_2\|_{L^p(Q)}\right). \end{aligned} \tag{2.48}$$

For $\varepsilon > 0$ with $1 - \varepsilon C(|\Omega|, T, n, M, p) > 0$, (2.48) implies estimate (2.16) and thus the Theorem 2.2.1 is proved. □

2.3 Proof of Theorem 2.1.1

We introduce the homotopy $H : L^p(Q) \times [0,1] \to L^p(Q)$ as follows

$$H(v,\lambda) = u, \qquad \forall\ (v,\lambda) \in L^p(Q) \times [0,1] \tag{2.49}$$

where u is the unique solution of the linear problem

$$\begin{cases} \rho c \dfrac{\partial u}{\partial t} - k\Delta u = \lambda \Big\{ -\dfrac{\ell}{2}\dfrac{\partial \varphi}{\partial t} + f(t,x) \Big\} & \text{on } \ Q, \\[2ex] \dfrac{\partial u}{\partial \nu} + hu = w(t,x) & \text{on } \ \Sigma, \\[2ex] u(0,x) = u_0(x) & \text{on } \ \Omega, \end{cases} \tag{2.50}$$

with φ representing the unique solution of the nonlinear parabolic boundary value problem

$$\begin{cases} \tau \dfrac{\partial \varphi}{\partial t} - \xi^2 \Delta \varphi = \dfrac{1}{2a}(\varphi - \varphi^3) + 2v(t,x) + g(t,x) & \text{on } \ Q, \\[2ex] \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \ \Sigma, \\[2ex] \varphi(0,x) = \varphi_0(x) & \text{on } \ \Omega. \end{cases} \tag{2.51}$$

We recall that $f \in L^p(Q)$ and $g \in L^q(Q)$ are given functions. Since $q \geq p$ by hypothesis (see (2.4)), then, taking into account (2.49), we derive that $v + g \in L^p(Q)$. Using Theorem 2.2.1 from preview section we have that there exists a unique solution $\varphi \in W_p^{2,1}(Q)$ to problem (2.51). Thus $-\frac{\ell}{2}\varphi_t + f(t,x) \in L^p(Q)$. The L^p-theory guarantees that the linear parabolic problem (2.50) has a unique solution $u \in W_p^{2,1}(Q)$. Hence the mapping introduced in (2.49) is well defined.

We shall prove now the following technical lemmas

Lemma 2.3.1. *The mapping $H : L^p(Q) \times [0,1] \to L^p(Q)$ in (2.49) has the following properties:*

i) *$H(\cdot, \lambda) : L^p(Q) \to L^p(Q)$ is compact for every $\lambda \in [0,1]$;*

ii) *for every $\varepsilon > 0$ and every bounded set $A \subset L^p(Q)$ there exists $\delta > 0$ such that*

$$\|H(v, \lambda_1) - H(v, \lambda_2)\|_{L^p(Q)} < \varepsilon$$

whenever $v \in A$ and $|\lambda_1 - \lambda_2| < \delta$.

Proof. i) To check the continuity of $H(\cdot, \lambda)$, $\forall \lambda \in [0,1]$ at the point $\bar{v} \in L^p(Q)$, we consider $\bar{u} = H(\bar{v}, \lambda)$ and $u = H(v, \lambda)$ for any $v \in L^p(Q)$. Relation (2.49) and problem (2.50) allows to write

$$\begin{cases} \rho c \dfrac{\partial}{\partial t}(u - \bar{u}) - k\Delta(u - \bar{u}) = -\lambda \dfrac{\ell}{2} \dfrac{\partial}{\partial t}(\varphi - \bar{\varphi}) & \text{on } Q, \\ \dfrac{\partial}{\partial \nu}(u - \bar{u}) + h(u - \bar{u}) = 0 & \text{on } \Sigma, \\ (u - \bar{u})(0, x) = 0 & \text{on } \Omega. \end{cases} \tag{2.52}$$

L^p-theory applied to the linear problem (2.52) says us that there is a constant $C > 0$ such that

$$\|u - \bar{u}\|_{W_p^{2,1}(Q)} \leq C \left\| \frac{\partial}{\partial t}(\varphi - \bar{\varphi}) \right\|_{L^p(Q)}. \tag{2.53}$$

Applying Theorem 2.2.1 to problem (2.51) choosing $\varphi_1 = \varphi$, $\varphi_2 = \bar{\varphi}$, $\varphi_0^1 = \varphi_0^2 = \varphi_0$ and $\bar{g}_1 = v + g$, $\bar{g}_2 = \bar{v} + g$, we get

$$\|\varphi - \bar{\varphi}\|_{W_p^{2,1}(Q)} \leq C\|v - \bar{v}\|_{L^p(Q)}. \tag{2.54}$$

By (2.53) and (2.54) we obtain the estimate

$$\|u - \bar{u}\|_{L^p(Q)} \leq C\|v - \bar{v}\|_{L^p(Q)}. \tag{2.55}$$

From the above inequality we can derive the continuity of the map $H(\cdot, \lambda)$ at $\bar{u}$, for each $\lambda \in [0,1]$. We have that $H(\cdot, \lambda)$ is expressed as the composition

$$L^p(Q) \to W_p^{2,1}(Q) \subset L^\mu(Q) \subset L^p(Q), \tag{2.56}$$

where the second map is a compact inclusion by Lions-Peetre embedding Theorem [17]. Therefore the map $H(\cdot,\lambda)$ is compact.

ii) Fix $\varepsilon > 0$ and a bounded set $A \subset L^p(Q)$. Consider (u_1,λ_1), (u_2,λ_2) solving (2.50) where in (2.51) we take any $v \in A$. Then we have

$$\begin{cases} \rho c \dfrac{\partial}{\partial t}(u_1-u_2) - k\Delta(u_1-u_2) \\ \qquad = -\dfrac{\ell}{2}\dfrac{\partial}{\partial t}(\lambda_1\varphi_1 - \lambda_2\varphi_2) + (\lambda_1-\lambda_2)f(t,x) & \text{on } Q, \\ \dfrac{\partial}{\partial \nu}(u_1-u_2) + h(u_1-u_2) = 0 & \text{on } \Sigma, \\ (u_1-u_2)(0,x) = 0 & \text{on } \Omega. \end{cases} \tag{2.57}$$

Applying now Theorem 2.2.1 to problem (2.51) choosing $\varphi_0^1 = \varphi_0^2 = \varphi_0$ and $\bar{g}_1 = \lambda_1(v+g)$, $\bar{g}_2 = \lambda_2(v+g)$, we obtain

$$\|\varphi_1 - \varphi_2\|_{W_p^{2,1}(Q)} \le C|\lambda_1-\lambda_2|\|v+g\|_{L^p(Q)} \le C(A)|\lambda_1-\lambda_2|. \tag{2.58}$$

By L^p-theory applied to the linear problem (2.57) we get

$$\begin{aligned} \|u_1 \; - \; u_2\|_{W_p^{2,1}(Q)} &\le C\Big\{|\lambda_1-\lambda_2|\Big(\Big\|\frac{\partial}{\partial t}\varphi_1\Big\|_{L^p(Q)} + \|f\|_{L^p(Q)}\Big) \\ &+ \; \lambda_2\Big\|\frac{\partial}{\partial t}(\varphi_1-\varphi_2)\Big\|_{L^p(Q)}\Big\}. \end{aligned} \tag{2.59}$$

Estimate (2.14) assures that $\|\frac{\partial}{\partial t}\varphi_1\|_{L^p(Q)}$ is uniformly bounded with respect to v, because A is bounded. Then, from (2.58) and (2.59) we can conclude

$$\|u_1-u_2\|_{W_p^{2,1}(Q)} \le C(A)|\lambda_1-\lambda_2| \tag{2.60}$$

and so the assertion *ii)* is verified. □

Lemma 2.3.2. *There exists a number $\delta > 0$ such that*

$$H(u,\lambda) = u \quad \Longrightarrow \quad \|u\|_{L^p(Q)} < \delta. \tag{2.61}$$

Proof. The assertion is equivalent to (2.50) where φ is the unique solution of the nonlinear parabolic boundary value problem (or, the solution of problem (2.51) with $v = u$)

$$\begin{cases} \tau \dfrac{\partial \varphi}{\partial t} - \xi^2 \Delta \varphi = \dfrac{1}{2a}(\varphi - \varphi^3) + 2u + g(t,x) & \text{on } Q, \\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \varphi_0(x) & \text{on } \Omega. \end{cases} \tag{2.62}$$

Analyzing the nonlinear problem (2.13) we see that $\bar{g} = 2u + g \in L^p(Q)$. Applying Theorem 2.2.1 to problem (2.62) we have that a unique solution $\varphi \in W_p^{2,1}(Q)$ exists. In addition, estimate (2.14) is valid for $\bar{g} = u + g$. Consequently, the continuous inclusion $W_p^{2,1}(Q) \subset L^p(Q)$ implies that

$$\begin{aligned} \|\varphi_t\|_{L^p(Q)} &\leq C\|\varphi\|_{W_p^{2,1}(Q)} \leq C\Big\{1 + \|\varphi_0\|^{3-\frac{2}{p}}_{W^{2-\frac{2}{q}}(\Omega)} \\ &+ \|u\|_{L^p(Q)} + \|g\|_{L^p(Q)}\Big\}, \end{aligned} \tag{2.63}$$

where $C > 0$ is a constant which depends on $|\Omega|$, T, n, p and physical parameters.

Multiplying $(2.50)_1$ by $|u|^{p-2}u$ and integrating over Q_t, Fubini's Theorem, Green's formula and Young's inequality lead to

$$\begin{aligned} &\frac{\rho c}{p} \int_\Omega |u|^p \, dx + k(p-1) \int_{Q_t} |\nabla u|^2 |u|^{p-2} \, dsdx + kh \int_{\Sigma_t} |u|^p \, dsd\gamma \\ &\leq \frac{\rho c}{p} \|u_0\|^p_{L^p(\Omega)} + \frac{\ell}{2p} \int_{Q_t} |\varphi_t|^p \, dsdx + \frac{2(p-1)}{p} \int_{Q_t} |u|^p \, dsdx \\ &+ \frac{1}{p} \int_{Q_t} |f|^p \, dsdx + k \int_{\Sigma_t} w|u|^{p-2}u \, dsd\gamma, \qquad \textit{for all } t \in (0,T]. \end{aligned} \tag{2.64}$$

The Hölder's and Cauchy's inequality, applied to the last term in (2.64), give us

$$k \int_{\Sigma_t} w|u|^{p-2}u \, dsd\gamma \leq \frac{kh}{p} \int_{\Sigma_t} |u|^p \, dsd\gamma + \frac{p-1}{ph} \int_{\Sigma_t} |w|^p \, dsd\gamma. \tag{2.65}$$

Combining (2.64) with (2.63) and (2.65), it turns out that

$$\begin{aligned}\frac{\rho c}{p} & \int_\Omega |u|^p \, dx + k(p-1)\int_{Q_t} |\nabla u|^2|u|^{p-2} \, dsdx \\ + & \; kh(1-\frac{1}{p})\int_{\Sigma_t} |u|^p \, dsd\gamma \le \frac{\rho c}{p}\|u_0\|^p_{L^p(\Omega)} \\ + & \; \frac{\ell}{2p}C\Big\{1+\|\varphi_0\|^{3-\frac{2}{p}}_{W_p^{2-\frac{2}{q}}(\Omega)} + \|g\|_{L^p(Q)}\Big\} + \Big[\frac{\ell}{2p}C + \frac{2(p-1)}{p}\Big] \times \\ \times & \int_{Q_t} u^p \, dsdx + C\|f\|^p_{L^p(Q_t)} + C\|w\|^p_{L^p(\Sigma_t)}, \qquad \forall t \in (0,T].\end{aligned} \tag{2.66}$$

By Gronwall's lemma, 2.66 and (2.1) we derive the estimate

$$\begin{aligned}\|u\|_{L^p(Q)} & \le C\Big\{1 + \|u_0\|_{L^p(\Omega)} + \|\varphi_0\|^{3-\frac{2}{p}}_{W_q^{2-\frac{2}{q}}(\Omega)} \\ & + \|f\|_{L^p(Q)} + \|g\|_{L^q(Q)} + \|w\|_{L^p(\Sigma)}\Big\}\end{aligned} \tag{2.67}$$

which ensures that a constant $\delta > 0$ can be found such that the property expressed in (2.61) is true. □

Proof. of Theorem 2.1.1 Denoting $B_\delta := \Big\{u \in L^p(Q) : \|u\|_{L^p(Q)} < \delta\Big\}$, Lemma (2.3.2) ensures that there exists $\delta > 0$ such that

$$H(u,\lambda) \ne u \qquad \forall u \in \partial B_\delta, \;\; \forall \lambda \in [0,1]. \tag{2.68}$$

Lemma (2.3.1) allows to consider the Leray-Schauder degree (see [12])

$$\deg\Big(Id_{L^p(Q)} - H(\cdot,\lambda), B_\delta, 0\Big) \qquad \forall \lambda \in [0,1]. \tag{2.69}$$

The homotopy invariance of the Leray-Schauder degree enables us to write the equality

$$\deg\Big(Id_{L^p(Q)} - H(\cdot,0), B_\delta, 0\Big) = \deg\Big(Id_{L^p(Q)} - H(\cdot,1), B_\delta, 0\Big). \tag{2.70}$$

Observing that $H(\cdot,0) = 0$, it follows that

$$\deg\Big(Id_{L^p(Q)} - H(\cdot,0), B_\delta, 0\Big) = \deg\Big(Id_{L^p(Q)}, B_\delta, 0\Big) = 1. \tag{2.71}$$

From (2.70) and (2.71) we conclude that problem (2.50) has a solution $u \in W_p^{2,1}(Q)$. This solution is determined by the unique solution φ of (2.62).

Since one has $W_p^{2,1}(Q) \subset L^\mu(Q)$, we can apply Theorem 2.2.1 for $\bar{g} = u + g \in L^\nu(Q)$, $\nu = \min\{p, \mu\}$. This ensures the existence of a solution $(u, \varphi) \in W_p^{2,1}(Q) \times W_\nu^{2,1}(Q)$ of problem (2.1).

Moreover, estimate (2.14) in Theorem 2.2.1 yields that

$$\|\varphi\|_{W_\nu^{2,1}(Q)} \le C \left\{1 + \|\varphi_0\|^{3-\frac{2}{q}}_{W_q^{2-\frac{2}{q}}(\Omega)} + \|u\|_{L^\nu(Q)} + \|g\|_{L^q(Q)}\right\}. \tag{2.72}$$

In writing (2.72) we used the embedding $W_q^{2-\frac{2}{q}}(\Omega) \subset W_\nu^{2-\frac{2}{\nu}}(\Omega)$. The L^p-theory applied to (2.1) (unknown u), combined with the estimates (2.14) (expressed by relation (2.63)), implies the estimate

$$\begin{aligned} \|u\|_{W_p^{2,1}(Q)} &\le C \left\{1 + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\right. \\ &+ \left. \frac{\ell}{2}\|\varphi_t\|_{L^p(Q)} + \|f\|_{L^p(Q)} + \|w\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\right\} \\ &\le C \left\{1 + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\varphi_0\|^{3-\frac{2}{p}}_{W_q^{2-\frac{2}{q}}(\Omega)} + \|u\|_{L^p(Q)}\right. \\ &+ \left. \|f\|_{L^p(Q)} + \|g\|_{L^q(Q)} + \|w\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\right\}. \end{aligned} \tag{2.73}$$

On the basis of interpolation inequality for the embeddings

$$W_p^{2,1}(Q) \subset L^\nu(Q) \subset L^p(Q) \tag{2.74}$$

and relations (2.4), (2.67), (2.72), (2.73), we deduce the estimate (2.6).

Finally, let us verify the estimate (2.7). To this end we consider (u_1, φ_1), (u_2, φ_2) as in statement of Theorem 2.1.1. We already established that $u_1,\ u_2 \in W_p^{2,1}(Q)$ and $\varphi_1,\ \varphi_2 \in W_\nu^{2,1}(Q)$. Subtracting the equations in (2.1) corresponding to (u_1, φ_1), (u_2, φ_2) and using (2.4), (2.16), we find (see (2.63))

$$\begin{aligned} \|(\varphi_1 - \varphi_2)_t\|_{L^p(Q)} &\le C\|\varphi_1 - \varphi_2\|_{W_\nu^{2,1}(Q)} \\ &\le C\Big\{\|u_1 - u_2\|_{L^\nu(Q)} + \|g_1 - g_2\|_{L^q(Q)}\Big\}. \end{aligned} \tag{2.75}$$

The L^p-theory combined with the above relation shows that

$$\begin{aligned} \|u_1 \;-\; & u_2\|_{W_p^{2,1}(Q)} \le C\Big\{\|(\varphi_1-\varphi_2)_t\|_{L^p(Q)} \\ +\;& \|f_1-f_2\|_{L^p(Q)} + \|w_1-w_2\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\Big\} \\ \le\;& C\Big\{\|u_1-u_2\|_{L^\nu(Q)} + \|f_1-f_2\|_{L^p(Q)} \\ +\;& \|g_1-g_2\|_{L^q(Q)} + \|w_1-w_2\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\Big\}. \end{aligned} \tag{2.76}$$

On the basis of the interpolation inequality (see (2.74)), we obtain the following relation ($\varepsilon > 0$)

$$\|u_1-u_2\|_{L^\nu(Q)} \le \varepsilon\|u_1-u_2\|_{W_p^{2,1}(Q)} + C_\varepsilon\|u_1-u_2\|_{L^p(Q)}. \tag{2.77}$$

From (2.75)-(2.77) one deduces that

$$\begin{aligned} (1 \;-\; & 2\varepsilon)\|u_1-u_2\|_{W_p^{2,1}(Q)} + \|\varphi_1-\varphi_2\|_{W_\nu^{2,1}(Q)} \\ \le\;& C\Big\{\|u_1-u_2\|_{L^p(Q)} + \|f_1-f_2\|_{L^p(Q)} \\ +\;& \|g_1-g_2\|_{L^q(Q)} + \|w_1-w_2\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\Big\}. \end{aligned} \tag{2.78}$$

In order to estimate $\|u_1-u_2\|_{L^p(Q)}$ in (2.78), we note that u_1-u_2 solves the problem

$$\begin{cases} \rho c\dfrac{\partial}{\partial t}(u_1-u_2) - k\Delta(u_1-u_2) \\ \qquad\qquad = -\dfrac{\ell}{2}\dfrac{\partial}{\partial t}(\varphi_1-\varphi_2) + (f_1-f_2) & \text{on } Q, \\ \dfrac{\partial}{\partial\nu}(u_1-u_2) + h(u_1-u_2) = w_1-w_2 & \text{on } \Sigma, \\ (u_1-u_2)(0,x) = 0 & \text{on } \Omega. \end{cases} \tag{2.79}$$

Using 2.16, from 2.79 we obtain (see 2.67)

$$\|u_1-u_2\|_{W_p^{2,1}(Q)} \le C\Big\{\|f_1-f_2\|_{L^p(Q)} + \|g_1-g_2\|_{L^q(Q)} + \|w_1-w_2\|_{L^p(\Sigma)}\Big\}. \tag{2.80}$$

Taking $\varepsilon > 0$ small enough, from (2.78) and (2.80) we get (2.7). The uniqueness of solution (u, φ) follows from relation (2.7) by taking $f_1 = f_2$, $g_1 = g_2$, $w_1 = w_2$. Thus we got the proof to the Theorem 2.1.1. $\square$

Chapter 3

Fractional steps scheme

3.1 Convergence of the approximating scheme

We'll prove now the convergence and the weak stability of an iterative scheme of fractional step type for the phase-field transition system. The advantage of such method consists in simplifying the numerical computation necessary to be done in order to approximate the solution of a nonlinear parabolic system. This kind of approximating scheme was studied for the phase field system in [18] with homogeneous boundary conditions and was extended in [9] to the non homogeneous case. In order to approximate the nonlinear problem, with $f \equiv g \equiv 0$, for every $\varepsilon > 0$ let's associate to (2.1)-(2.3), the following approximating scheme:

$$\begin{cases} \rho c\, u_t^\varepsilon + \dfrac{\ell}{2}\varphi_t^\varepsilon = k\Delta u^\varepsilon & \text{in } \; Q_i^\varepsilon = (i\varepsilon, (i+1)\varepsilon) \times \Omega, \\ \\ \tau\varphi_t^\varepsilon = \xi^2 \Delta\varphi^\varepsilon + \dfrac{1}{2a}\varphi^\varepsilon + 2u^\varepsilon & \text{in } \; Q_i^\varepsilon, \end{cases} \tag{3.1}$$

with the boundary conditions

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial \nu} + hu^\varepsilon = w(t,x) & \text{in } \; \Sigma_i^\varepsilon = (i\varepsilon, (i+1)\varepsilon) \times \partial\Omega, \\ \\ \dfrac{\partial \varphi^\varepsilon}{\partial \nu} = 0 & \text{in } \; \Sigma_i^\varepsilon, \end{cases} \tag{3.2}$$

and with the following initial conditions

$$\begin{cases} u_+^\varepsilon(i\varepsilon,x)=u_-^\varepsilon(i\varepsilon,x) \quad u_-^\varepsilon(0,x)=u_0(x) & \text{in } \Omega \\ \\ \varphi_+^\varepsilon(i\varepsilon,x)=z((i+1)\varepsilon,\varphi_-^\varepsilon(i\varepsilon,x)) & \text{in } \Omega \end{cases} \tag{3.3}$$

where $z(\cdot,\varphi_-^\varepsilon(i\varepsilon,x))$ is the solution of Cauchy problem:

$$\begin{cases} z'(s)+\dfrac{1}{2a}z^3(s)=0 & s\in(i\varepsilon,(i+1)\varepsilon), \\ \\ z(i\varepsilon)=\varphi_-^\varepsilon(i\varepsilon,x) \quad \varphi_-^\varepsilon(0,x)=\varphi_0(x), \end{cases} \tag{3.4}$$

for $i=0,1,\cdots,M_\varepsilon-1$, with $M_\varepsilon=\left[\frac{T}{\varepsilon}\right]$, $Q_{M_\varepsilon-1}^\varepsilon=[(M_\varepsilon-1)\varepsilon,T]\times\Omega$, $\varphi_+^\varepsilon(i\varepsilon,x)=\lim\limits_{t\downarrow i\varepsilon}\varphi^\varepsilon(t,x)$ and $\varphi_-^\varepsilon(i\varepsilon,x)=\lim\limits_{t\uparrow i\varepsilon}\varphi^\varepsilon(t,x)$.

Also this time, we can deduce the existence of solution for problem (3.1)-(3.3) from L^p-theory (see Appendix A). In the next proposition, we recall an existence and regularity result regarding the following case

Proposition 3.1.1. *Let $u_0,\varphi_0\in W^1_\infty(\Omega)$ satisfying $\dfrac{\partial u_0}{\partial\nu}+hu_0=w(0,x)$, $\dfrac{\partial\varphi_0}{\partial\nu}=0$ and $w\in W^1([0,T];L^2(\partial\Omega))$. Then the linear system (3.1)-(3.3) has a unique solution $u^\varepsilon,\varphi^\varepsilon\in W^{1,2}(Q_i^\varepsilon)\cap L^\infty(Q_i^\varepsilon)$ on every time interval $[i\varepsilon,(i+1)\varepsilon]$, $i=0,1,...,M_\varepsilon-1$.*

The sketch proof of Proposition (3.1.1) can be found in [18].

The convergence of the iterative scheme will be proved in the following theorem

Theorem 3.1.2. *Assume that $u_0,\ \varphi_0\in W^1_\infty(\Omega)$ and $w\in W^1([0,T];L^2(\partial\Omega))$ satisfying $\dfrac{\partial u_0}{\partial\nu}+hu_0=w(0,x)$, $\dfrac{\partial\varphi_0}{\partial\nu}=0$. Furthermore, $\Omega\subset I\!R^n$ $(n=1,2,3)$ is a bounded domain with a smooth boundary. Let $(u^\varepsilon,\varphi^\varepsilon)$ be the solution of the approximating scheme (3.1)-(3.3). Then, for $\varepsilon\to0$, one has*

$$(u^\varepsilon(t),\varphi^\varepsilon(t))\to(u^*(t),\varphi^*(t)) \qquad \textit{strongly in } L^2(\Omega) \textit{ for any } t\in(0,T], \tag{3.5}$$

where $u^*,\varphi^*\in W^{1,2}([0,T];L^2(\Omega))\cap L^2([0,T];H^2(\Omega))$ is the solution of the nonlinear system (2.1)-(2.3), with $f\equiv g\equiv0$.

We shall prove some lemmas concerning the Cauchy problem (3.4).

Lemma 3.1.3. *If $\varphi_-^\varepsilon(i\varepsilon, x) \in L^\infty(\Omega)$, $i = 0, 1, ..., M_\varepsilon$, then $z(i\varepsilon, x) \in L^\infty(\Omega)$.*

Proof. We observe that the problem (3.4) can be solved directly, by separation of variables. Indeed, we write it in the form $(1/2z^2)' = 1/2a$ and, integrating on $(0, \varepsilon)$, we obtain

$$z^2(\varepsilon, x) = \frac{\big(\varphi_-^\varepsilon(i\varepsilon, x)\big)^2}{1 + \frac{\varepsilon}{a}\big(\varphi_-^\varepsilon(i\varepsilon, x)\big)^2} \tag{3.6}$$

and therefore

$$z^2(\varepsilon, x) < \big(\varphi_-^\varepsilon(i\varepsilon, x)\big)^2, \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

Hence $z(\varepsilon, x) \in L^\infty(\Omega)$ as claimed. □

Lemma 3.1.4. *For $i = 0, 1, ..., M_\varepsilon - 1$, the estimate below holds*

$$\|\nabla\varphi_+^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \le \|\nabla\varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \tag{3.8}$$

Proof. Denote $\theta(t, x) = \nabla z(t, x)$. From (3.4) we obtain

$$\theta_t + \frac{3}{2a}\theta z = 0, \quad \text{on}(0, \varepsilon), \tag{3.9}$$

$$\theta(0) = \nabla\varphi_-^\varepsilon(i\varepsilon, x). \tag{3.10}$$

The solution of (3.9)-(3.10) is given by

$$\theta(\varepsilon) = e^{-\int_0^\varepsilon \frac{3}{2a} z^2(t,\cdot)dt}\theta(0)$$

from which we easily get (3.8). □

Lemma 3.1.5. *The following estimate holds*

$$\|z(\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \le \varepsilon L, \tag{3.11}$$

where $L > 0$ is a constant depending on Ω, $\|\varphi_-^\varepsilon(i\varepsilon, x)\|_{L^\infty(\Omega)}$ and on the parameter a.

Proof. By (3.4), using $(1/2a)z^3(t)(z(t)-z(0))-(1/2a)z^3(0)(z(t)-z(0)) \geq 0$, we have

$$\frac{1}{2}\frac{d}{dt}|z(t)-z(0)|^2 \leq -\frac{1}{2a}z^3(0)\big(z(t)-z(0)\big). \tag{3.12}$$

Integrating over $(0,\varepsilon)$ we obtain

$$|z(\varepsilon)-z(0)| \leq \frac{\varepsilon}{2a}|z^3(0)| \tag{3.13}$$

and therefore the lemmas was proved. □

Proof. of theorem 3.1.2. We observe first that Lemma 3.1.3 and Lemma 3.1.4 ensure that $z(\varepsilon,x) \in W^1_\infty(\Omega)$. Applying now Proposition 3.1.1 to the problem (3.1)-(3.4) for $i=0$ we obtain the existence of a solution u^ε, $\varphi^\varepsilon \in W^{2,1}(Q_0^\varepsilon) \cap L^\infty(Q_0^\varepsilon)$. Reasoning iteratively after i, we may conclude that $\varphi^\varepsilon_-(i\varepsilon,x) \in L^\infty(\Omega)$, $i=1,2,...,M_\varepsilon-1$, and that the problem (3.1)-(3.4) has the solution u^ε, $\varphi^\varepsilon \in W^{2,1}(Q_i^\varepsilon) \cap L^\infty(Q_i^\varepsilon)$, $i=0,1,...,M_\varepsilon-1$.

Let us next establish *a priori* estimates for the solution u^ε, φ^ε on to each Q_i^ε, $i=0,1,...,M_\varepsilon-1$. It is useful derive them directly also if we could obtain them from *a priori* estimates to linear parabolic equations in L^p-theory (see Appendix A).

Multiplying $(3.1)_1$ by $\frac{4}{\ell}u^\varepsilon$ and $(3.1)_2$ by φ_t^ε, using integration by parts, Green's formula, yields

$$\frac{2\rho c}{\ell}\frac{\partial}{\partial t}\Big(\int_\Omega (u^\varepsilon)^2 dx\Big) + 2\int_\Omega u^\varepsilon \varphi_t^\varepsilon dx + \frac{4k}{\ell}\int_\Omega |\nabla u^\varepsilon|^2 dx \tag{3.14}$$
$$+\frac{4kh}{\ell}\int_{\partial\Omega}(u^\varepsilon)^2 d\gamma = \frac{4k}{\ell}\int_{\partial\Omega} u^\varepsilon w(t,x)d\gamma,$$

$$\tau\int_\Omega (\varphi_t^\varepsilon)^2 dx + \frac{\xi^2}{2}\frac{\partial}{\partial t}\Big(\int_\Omega |\nabla\varphi^\varepsilon|^2 dx\Big) - 2\int_\Omega u^\varepsilon\varphi_t^\varepsilon dx = \frac{1}{2a}\int_\Omega \varphi^\varepsilon\varphi_t^\varepsilon dx. \tag{3.15}$$

The Hölder's and Cauchy's inequalities applied to the term $\displaystyle\frac{4k}{\ell}\int_{\partial\Omega} u^\varepsilon w d\gamma$ in (3.14) give us:

$$\frac{4k}{\ell}\int_{\partial\Omega} u^\varepsilon w(t,x)d\gamma \leq \frac{4kh}{\ell}\int_{\partial\Omega}(u^\varepsilon)^2 d\gamma + \frac{k}{\ell h}\int_{\partial\Omega} w^2(t,x)d\gamma.$$

Thus, adding (3.14)-(3.15) and making use of the last inequality, after some simple calculus we obtain:

$$\frac{2\rho c}{\ell}\frac{\partial}{\partial t}\Big(\int\limits_{\Omega}(u^{\varepsilon})^2dx\Big)+\frac{4k}{\ell}\int\limits_{\Omega}|\nabla u^{\varepsilon}|^2dx+\tau\int\limits_{\Omega}(\varphi^{\varepsilon}_t)^2dx \tag{3.16}$$
$$+\frac{\xi^2}{2}\frac{\partial}{\partial t}\Big(\int\limits_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx\Big)\le\frac{k}{\ell h}\int\limits_{\partial\Omega}w^2(t,x)d\gamma+\frac{1}{2a}\int\limits_{\Omega}\varphi^{\varepsilon}\varphi^{\varepsilon}_t dx.$$

If we now multiply $(3.1)_2$ by φ^{ε} and then we integrate over Ω, by Green's formula we get

$$\frac{\tau}{2}\frac{\partial}{\partial t}\Big(\int\limits_{\Omega}(\varphi^{\varepsilon})^2dx\Big)+\xi^2\int\limits_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx=2\int\limits_{\Omega}u^{\varepsilon}\varphi^{\varepsilon}dx+\frac{1}{2a}\int\limits_{\Omega}(\varphi^{\varepsilon})^2dx. \tag{3.17}$$

Adding the relations (3.16)-(3.17) and performing same computations implying Buniakovsky-Schwarz's and Cauchy's inequalities, we deduce that:

$$\begin{aligned}\frac{2\rho c}{\ell}\ &\frac{\partial}{\partial t}\ \Big(\int\limits_{\Omega}(u^{\varepsilon})^2dx\Big)+\frac{\tau}{2}\frac{\partial}{\partial t}\Big(\int\limits_{\Omega}(\varphi^{\varepsilon})^2dx\Big)\\ &+\ \frac{4k}{\ell}\int\limits_{\Omega}|\nabla u^{\varepsilon}|^2dx+\xi^2\int\limits_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx\\ &+\ \frac{\tau}{2}\int\limits_{\Omega}(\varphi^{\varepsilon}_t)^2dx+\frac{\xi^2}{2}\frac{\partial}{\partial t}\Big(\int\limits_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx\Big)\\ &\le\ \int\limits_{\Omega}(u^{\varepsilon})^2dx+\Big(1+\frac{1}{2a}+\frac{1}{4a\tau}\Big)\int\limits_{\Omega}(\varphi^{\varepsilon})^2dx+\frac{k}{\ell h}\int\limits_{\partial\Omega}w^2(t,x)d\gamma.\end{aligned} \tag{3.18}$$

Integration over $(0,\varepsilon)$ and by parts of (3.18) gives now

$$\begin{aligned}\frac{2\rho c}{\ell}\ &\|u^{\varepsilon}(\varepsilon)\|^2_{L^2(\Omega)}+\frac{\tau}{2}\|\varphi^{\varepsilon}_{-}(\varepsilon)\|^2_{L^2(\Omega)}+\frac{4k}{\ell}\int\limits_0^{\varepsilon}\|\nabla u^{\varepsilon}\|^2_{L^2(\Omega)}ds\\ &+\ \xi^2\int\limits_0^{\varepsilon}\|\nabla\varphi^{\varepsilon}\|^2_{L^2(\Omega)}ds+\frac{\tau}{2}\int\limits_0^{\varepsilon}\|\varphi^{\varepsilon}_t\|^2_{L^2(\Omega)}ds+\frac{\xi^2}{2}\|\nabla\varphi^{\varepsilon}_{-}(\varepsilon)\|^2_{L^2(\Omega)}\\ &\le\ \frac{2\rho c}{\ell}\|u_0\|^2_{L^2(\Omega)}+\frac{\tau}{2}\|\varphi_0\|^2_{L^2(\Omega)}+\frac{\xi^2}{2}\|\nabla\varphi_0\|^2_{L^2(\Omega)}\\ &+\ C\int\limits_0^{\varepsilon}(\|u^{\varepsilon}(s)\|^2_{L^2(\Omega)}+\|\varphi^{\varepsilon}(s)\|^2_{L^2(\Omega)})ds+C\|w\|^2_{L^2(\Sigma^{\varepsilon}_0)}.\end{aligned} \tag{3.19}$$

Similarly, for $Q_i^\varepsilon,\ i=1,2,...,M_\varepsilon-2$, we obtain from (3.18):

$$\begin{aligned}\frac{2\rho c}{\ell}\|u^\varepsilon((i &+ 1)\varepsilon)\|^2_{L^2(\Omega)}+\frac{\tau}{2}\|\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)} \\ &+ \frac{4k}{\ell}\int\limits_{i\varepsilon}^{(i+1)\varepsilon}\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds+\xi^2\int\limits_{i\varepsilon}^{(i+1)\varepsilon}\|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds \\ &+ \frac{\tau}{2}\int\limits_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds+\frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)} \\ &\le \frac{2\rho c}{\ell}\|u^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)}+\frac{\tau}{2}\|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)}+\frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)} \\ &+ C\int\limits_{i\varepsilon}^{(i+1)\varepsilon}(\|u^\varepsilon(s)\|^2_{L^2(\Omega)}+\|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)})ds+C\|w\|^2_{L^2(\Sigma_i^\varepsilon)}.\end{aligned}\tag{3.20}$$

and for $Q^\varepsilon_{M_\varepsilon-1}$, we get

$$\begin{aligned}\frac{2\rho c}{\ell}\|u^\varepsilon(T)\|^2_{L^2(\Omega)} &+ \frac{\tau}{2}\|\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)} \\ &+ \frac{4k}{\ell}\int\limits_{(M_\varepsilon-1)\varepsilon}^{T}\|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds+\xi^2\int\limits_{(M_\varepsilon-1)\varepsilon}^{T}\|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds \\ &+ \frac{\tau}{2}\int\limits_{(M_\varepsilon-1)\varepsilon}^{T}\|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds+\frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)} \\ &\le \frac{2\rho c}{\ell}\|u^\varepsilon((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)}+\frac{\tau}{2}\|\varphi^\varepsilon_+((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)} \\ &+ \frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_+((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)}+C\|w\|^2_{L^2(\Sigma^\varepsilon_{M_\varepsilon-1})}. \\ &+ C\int\limits_{(M_\varepsilon-1)\varepsilon}^{T}\Big(\|u^\varepsilon(s)\|^2_{L^2(\Omega)}+\|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}\Big)ds\end{aligned}\tag{3.21}$$

Let's remember at this point that, corresponding to Cauchy problem (3.4), we have the following inequality (see proof of lemma 3.1.3):

$$\frac{\tau}{2}\|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)}\le\frac{\tau}{2}\|\varphi^\varepsilon_-(i\varepsilon)\|^2_{L^2(\Omega)},\tag{3.22}$$

for $i=1,2,\cdots,M_\varepsilon-1$.

If we now consider the inequalities given by (3.8) and (3.22), from (3.20) we constant that (for $i = 1, 2, \cdots, M_\varepsilon - 2$):

$$\begin{aligned} B^1_{i\varepsilon} &+ \frac{\tau}{2}\|\varphi^\varepsilon_+((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_+((i+1)\varepsilon)\|^2_{L^2(\Omega)} \\ &\leq B^1_{i\varepsilon} + \frac{\tau}{2}\|\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_-((i+1)\varepsilon)\|^2_{L^2(\Omega)} \\ &\leq B^2_{i\varepsilon} + \frac{\tau}{2}\|\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)} + \frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_+(i\varepsilon)\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} B^1_{i\varepsilon} &= \frac{2\rho c}{\ell}\|u^\varepsilon((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{4k}{\ell}\int\limits_{i\varepsilon}^{(i+1)\varepsilon} \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds \\ &\quad +\xi^2 \int\limits_{i\varepsilon}^{(i+1)\varepsilon} \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds + \frac{\tau}{2}\int\limits_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds, \\ B^2_{i\varepsilon} &= C\int\limits_{i\varepsilon}^{(i+1)\varepsilon} \Big\{\|u^\varepsilon(s)\|^2_{L^2(\Omega)} + \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}\Big\}ds \\ &\quad +\frac{2\rho c}{\ell}\|u^\varepsilon(i\varepsilon)\|^2_{L^2(\Omega)} + C\|w\|^2_{L^2(\Sigma^\varepsilon_i)}. \end{aligned}$$

Adding (3.19), (3.21), (3.23) and performing some simply calculus we derive ($M_\varepsilon \cdot \varepsilon = T$):

$$\begin{aligned} \frac{2\rho c}{\ell} &\cdot \|u^\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{\tau}{2}\|\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)} + \frac{4k}{\ell}\int\limits_0^T \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds \\ &+ \xi^2\int\limits_0^T \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds + \frac{\tau}{2}\sum_{i=0}^{M_\varepsilon-1}\int\limits_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi^\varepsilon_t\|^2_{L^2(\Omega)}ds + \frac{\xi^2}{2}\|\nabla\varphi^\varepsilon_-(T)\|^2_{L^2(\Omega)} \\ &\leq \frac{2\rho c}{\ell}\|u_0\|^2_{L^2(\Omega)} + \frac{\tau}{2}\|\varphi_0\|^2_{L^2(\Omega)} + \frac{\xi^2}{2}\|\nabla\varphi_0\|^2_{L^2(\Omega)} \\ &+ C\int\limits_0^T\Big(\|u^\varepsilon(s)\|^2_{L^2(\Omega)} + \|\varphi^\varepsilon(s)\|^2_{L^2(\Omega)}\Big)ds + C\|w\|^2_{L^2(\Sigma)}. \end{aligned}$$

Continuing by applying Gronwall-Bellman's inequality, we finally obtain:

$$\frac{\tau}{2}\int_0^T \|\varphi_t^\varepsilon\|^2_{L^2(\Omega)}ds + \frac{4k}{\ell}\int_0^T \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}ds + \xi^2\int_0^T \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)}ds \le C, \tag{3.24}$$

for all $\varepsilon > 0$, where C **does not depend** on M_ε and ε.

Multiplying now $(3.1)_1$ by u_t^ε, integrating over $[i\varepsilon, (i+1)\varepsilon]$, $i = 0, 1, \cdots, M_\varepsilon - 1$, and, using Hölder's inequality, Cauchy's inequality as well as Green's formula, we get the estimate:

$$\rho c \int_{Q_i^\varepsilon} (u_t^\varepsilon)^2 dxds + \frac{k}{2}\int_\Omega |\nabla u^\varepsilon|^2 dx + \frac{kh}{2}\int_{\partial\Omega} (u^\varepsilon)^2 d\gamma$$

$$\le k\int_{\Sigma_i^\varepsilon} u_t^\varepsilon w(t,x) d\gamma ds + \frac{\ell^2}{8\rho c}\int_{Q_i^\varepsilon} (\varphi_t^\varepsilon)^2\ dxds + \frac{\rho c}{2}\int_{Q_i^\varepsilon} (u_t^\varepsilon)^2\ dxds.$$

Summing after i and making use of (3.24), the last inequality leads to:

$$\begin{aligned} \frac{\rho c}{2} \int_Q (u_t^\varepsilon)^2\ dxds + \frac{k}{2}\int_\Omega |\nabla u^\varepsilon|^2 dx + \frac{kh}{2}\int_{\partial\Omega} (u^\varepsilon)^2 d\gamma \\ \le\ C + k\int_\Sigma u_t^\varepsilon w(t,x) d\gamma ds. \end{aligned} \tag{3.25}$$

But

$$k\int_\Sigma u_t^\varepsilon w d\gamma ds = k\int_\Sigma \frac{\partial}{\partial t}(u^\varepsilon w) d\gamma ds - k\int_\Sigma u^\varepsilon w' d\gamma ds$$

and (using Cauchy-Schwartz's and Hölder's inequalities)

$$k\int_{\partial\Omega} u^\varepsilon w d\gamma \le \frac{kh}{4}\int_{\partial\Omega} (u^\varepsilon)^2 d\gamma + \frac{k}{h}\int_{\partial\Omega} w^2 d\gamma,$$

$$k\int_\Sigma u^\varepsilon w' d\gamma ds \le \frac{k}{2}\int_\Sigma (u^\varepsilon)^2 d\gamma ds + \frac{k}{2}\int_\Sigma (w')^2 d\gamma ds.$$

Using now the Gronwall-Bellman's inequality, from (3.25) we find the estimate:

$$\frac{\rho c}{2}\int_0^t \int_\Omega (u_t^\varepsilon)^2 dxds + \frac{k}{2}\int_\Omega |\nabla u^\varepsilon|^2 dx + \frac{kh}{4}\int_{\partial\Omega} (u^\varepsilon)^2 d\gamma \le C, \tag{3.26}$$

for all $\varepsilon > 0$, where the constant $C > 0$ **does not depend** on M_ε and ε.

By virtue of estimate (3.11), summarizing for $i = 0, 1, \cdots, M_\varepsilon - 1$, we get

$$\sum_{i=0}^{M_\varepsilon - 1} \|\varphi_+^\varepsilon(i\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \leq TL = C, \tag{3.27}$$

where C **does not depend** on M_ε and ε.

Combining (3.24) and (3.26)-(3.27), we get

$$\begin{aligned} \overset{T}{\underset{0}{V}} \; \varphi^\varepsilon \; &+ \int_0^T \|u_t^\varepsilon(t)\|_{L^2(\Omega)}^2 dt + \int_0^T \|\varphi_t^\varepsilon(t)\|_{L^2(\Omega)}^2 dt \\ &+ \int_Q \Big(|\nabla u^\varepsilon(t, x)|^2 + |\nabla \varphi^\varepsilon(t, x)|^2 \Big) dx dt \leq C \quad \forall\, \varepsilon > 0, \end{aligned} \tag{3.28}$$

where $\overset{T}{\underset{0}{V}} \varphi^\varepsilon$ stands for the variation of $\varphi^\varepsilon : [0, T] \to L_2(\Omega)$. Since the injection of $L_2(\Omega)$ into $H^{-1}(\Omega)$ is compact and the set $\{\varphi_t^\varepsilon(t)\}$ is bounded in $L_2(\Omega)$ for every $t \in [0, T]$, by an infinite dimensional version of Helly-Foiaş theorem (see for instance [6], Remark 3.2, p. 60), we conclude that there exists a bounded variation function $\varphi^*(t) \in BV([0, T]; H^{-1}(\Omega))$ such that, on a subsequence also denoted $\varphi^\varepsilon(t)$, we have

$$\varphi^\varepsilon(t) \to \varphi^*(t) \;\; \text{strongly in } H^{-1}(\Omega) \quad \text{for every } t \in [0, T]. \tag{3.29}$$

From (3.28) we may assume that

$$\varphi^\varepsilon \to \varphi^* \qquad \text{weakly in } L^2(0, T; H^1(\Omega)). \tag{3.30}$$

Now, since the embedding of $H^1(\Omega)$ into $L_2(\Omega)$ is compact, for every $\lambda > 0$ there exists $C(\lambda) > 0$ such that (see references in [19] on this topic)

$$\|\varphi^\varepsilon(t) - \varphi^*(t)\|_{L^2(\Omega)} \leq \lambda \|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^1(\Omega)} + C(\lambda)\|\varphi^\varepsilon(t) - \varphi^*(t)\|_{H^{-1}(\Omega)},$$

$\forall\, \varepsilon > 0$ and $\forall\, t \in [0, T]$, where $C(\lambda) \to 0$ as $\lambda \to 0$.

Together with (3.29) and (3.30), this last inequality leads to

$$\varphi^\varepsilon \to \varphi^* \qquad \text{strongly in } L^2(\Omega) \qquad \text{for any } t \in [0, T]. \tag{3.31}$$

From $(3.1)_1$, $(3.1)_2$ and (3.28) we have, respectively:

$$\int_0^t \int_\Omega (\Delta u^\varepsilon(s,x))^2 dxds \leq C \qquad \forall t \in (0,T],$$

$$\int_0^t \int_\Omega (\Delta \varphi^\varepsilon(s,x))^2 dxds \leq C \qquad \forall t \in (0,T].$$

Then we obtain the estimates:

$$\|u^\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C, \tag{3.32}$$

$$\|\varphi^\varepsilon\|_{L^2([0,T];H^2(\Omega))} \leq C. \tag{3.33}$$

Thus, because the embedding $H^2(\Omega) \subset H^1(\Omega)$ is compact, it turns out that the sequence $\{u^\varepsilon\}$ is compact in $L^2(0,T;H^1(\Omega))$. Therefore, on a subsequence, again denoted u^ε, from (3.28) and (3.32) we have

$$\begin{aligned} u^\varepsilon \to u^* \quad & \text{strongly in } L^2([0,T];H^1(\Omega)), \\ & \text{weakly in } L^2([0,T];H^2(\Omega)), \\ u_t^\varepsilon \to u_t^* \quad & \text{weakly in } L^2([0,T];L^2(\Omega)), \end{aligned} \tag{3.34}$$

and, by the Ascoli-Arzelà theorem

$$u^\varepsilon \to u^* \quad \text{strongly in } C([0,T];L^2(\Omega)). \tag{3.35}$$

(3.31)-(3.34) cleary also implies that

$$\Delta u_n \to \Delta u^* \qquad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.36}$$

$$\Delta \varphi_n \to \Delta \varphi^* \qquad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.37}$$

From (3.31) and (3.35) we may conclude that (3.5) holds and so the Theorem (3.1.2) was proved. □

We verify now that u^*, φ^* satisfy the phase-field system (2.1)-(2.3). Let $s < t$ be two arbitrary points of $[0, T]$ such that $i\varepsilon \leq s \leq (i+1)\varepsilon < \ldots < j\varepsilon \leq t$. Consider the problem

$$\begin{cases} \tau\varphi_t^\varepsilon - \xi^2\Delta\varphi^\varepsilon - \frac{1}{2a}\varphi^\varepsilon = 2u^\varepsilon & \text{on } \ Q_k^\varepsilon, \\ \dfrac{\partial\varphi^\varepsilon}{\partial\nu} = 0 & \text{on } \ \Sigma_k^\varepsilon \\ \varphi^\varepsilon = \varphi_0 & \text{on } \ \Omega. \end{cases} \tag{3.38}$$

In a usual way, from the above relation we obtain

$$\begin{aligned} \int_\Omega |\varphi_-^\varepsilon((k \ &+ \ 1)\varepsilon) - \varphi_+^\varepsilon(k\varepsilon)|^2 \, dx + \frac{\xi^2}{2}\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega |\nabla\varphi^\varepsilon|^2 \, dtdx \\ &\leq \ \frac{\xi^2}{2}\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega |\nabla\varphi_+^\varepsilon|^2 \, dxdt + C\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega (\varphi_+^\varepsilon)^2 \, dtdx \\ &+ \ C\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega \{(u^\varepsilon)^2 + (\varphi_+^\varepsilon)^2\} \, dtdx. \end{aligned} \tag{3.39}$$

Taking into account Lemma 3.1.3 and Lemma 3.1.4, the last inequality becomes

$$\begin{aligned} \sum_{k=i}^{j-1} \|\varphi_-^\varepsilon((k+1)\varepsilon) \ &- \ \varphi_+^\varepsilon(k\varepsilon)\|_{L^2(\Omega)}^2 \leq C \cdot \\ &\cdot \ \sum_{k=i}^{j-1}\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega \{(u^\varepsilon)^2 + (\varphi_+^\varepsilon)^2\} \, dtdx. \end{aligned} \tag{3.40}$$

on the other hand, using lemma 3.1.5 we get the estimate

$$\sum_{k=i}^{j-1} \|\varphi_+^\varepsilon(k\varepsilon) - \varphi_-^\varepsilon(k\varepsilon)\|_{L^2(\Omega)}^2 \leq C(j-1)\varepsilon L. \tag{3.41}$$

Hence

$$\begin{aligned} \|\varphi^\varepsilon(t) \ &- \ \varphi^\varepsilon(s)\|_{L^2(\Omega)} \leq \|\varphi^\varepsilon(s) - \varphi_-^\varepsilon(i\varepsilon)\|_{L^2(\Omega)} \\ &+ \ \|\varphi_-^\varepsilon(j\varepsilon) - \varphi^\varepsilon(t)\|_{L^2(\Omega)} + \sum_{k=i}^{j-1} \|\varphi_+^\varepsilon(k\varepsilon) - \varphi_-^\varepsilon(k\varepsilon)\|_{L^2(\Omega)}^2 \\ &+ \ \sum_{k=i}^{j-1} \|\varphi_-^\varepsilon((k+1)\varepsilon) - \varphi_+^\varepsilon(k\varepsilon)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.42}$$

Along with (3.40) and (3.41) the last inequality implies

$$\begin{aligned}\|\varphi^\varepsilon(t) &- \varphi^\varepsilon(s)\|_{L^2(\Omega)} \le C \cdot \\ &\cdot \left\{|t-s| + |t-s|^{1/2}\left(\int_s^t \int_\Omega \{(u^\varepsilon)^2 + (\varphi_+^\varepsilon)^2\}\ dtdx\right)^{1/2}\right\}\end{aligned} \tag{3.43}$$

and therefore $\varphi^* : [0,T] \to L^2(\Omega)$ is absolutely continuous and consequently almost everywhere differentiable on $[0,T]$. Hence $\varphi_t^*(t)$ exists a.e. on $(0,T)$.

Let $v \in L^6(\Omega)$ be an element arbitrary but fixed. By (3.39) we have

$$\begin{aligned}\tau(\varphi^\varepsilon(t) &- \varphi^\varepsilon(s), \varphi^\varepsilon(s) - v) + \tau \sum_{k=i+1}^{j} (\varphi_-^\varepsilon((k\varepsilon) - \varphi_+^\varepsilon(k\varepsilon), \varphi_+^\varepsilon(k\varepsilon) - v) \\ &\le \xi^2 \int_s^t (\Delta\varphi^\varepsilon, \varphi^\varepsilon - v)\ d\tau + \frac{1}{2a}\int_s^t (\varphi^\varepsilon, \varphi^\varepsilon - v)\ d\tau \\ &+ \int_s^t (2u^\varepsilon, \varphi^\varepsilon - v)\ d\tau, \qquad \forall v \in L^6(\Omega),\end{aligned} \tag{3.44}$$

where $(\cdot,\cdot)$ stand for the inner product of $L^2(\Omega)$ and also for the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

We denote by $F : D(F) = L^6(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ the operator $z \to -\frac{1}{2a}z^3$. Then $-F = \partial[\frac{1}{8a}\int_\Omega z^4 dx]$ is m-accretive, so that F is m-dissipative. Denote by e^{-Ft} the semigroup generated by $-F$ [18]. Then (see 3.4)

$$z(t) = e^{-Ft}\varphi_-^\varepsilon(i\varepsilon) \ \text{ for } \ t \in (0,T), \quad i = 0,1,\ldots,M_\varepsilon - 1. \tag{3.45}$$

Thus

$$\begin{aligned}&(\varphi_-^\varepsilon(k\varepsilon) - \varphi_+^\varepsilon(k\varepsilon), \varphi_+^\varepsilon(k\varepsilon) - v) = \\ &= (\varphi_-^\varepsilon(k\varepsilon) - e^{-F\varepsilon}\varphi_-^\varepsilon(k\varepsilon), \varphi_-^\varepsilon(k\varepsilon) - v) + \\ &+ (\varphi_-^\varepsilon(k\varepsilon) - e^{-F\varepsilon}\varphi_-^\varepsilon(k\varepsilon), e^{-F\varepsilon}\varphi_-^\varepsilon(k\varepsilon) - \varphi_-^\varepsilon(k\varepsilon)).\end{aligned}$$

We set $S_\varepsilon y := y - e^{-F\varepsilon}y$ and then (3.45), taking into account the above relation and the monotonicity of S_ε, becomes

$$\begin{aligned}
\tau(\varphi^\varepsilon(t) &- \varphi^\varepsilon(s), \varphi^\varepsilon(s) - v) + \tau \sum_{k=i+1}^{j} (S_\varepsilon), \varphi_-^\varepsilon(k\varepsilon) - v) \qquad (3.46)\\
&\le \tau \sum_{k=i+1}^{j} \|S_\varepsilon \varphi_-^\varepsilon(k\varepsilon)\|^2_{L^2(\Omega)} + \xi^2 \int_s^t (\Delta\varphi^\varepsilon, \varphi^\varepsilon - v)\ d\tau + \\
&+ \frac{1}{2a} \int_s^t (\varphi^\varepsilon, \varphi^\varepsilon - v)\ d\tau \\
&+ \int_s^t (2u^\varepsilon, \varphi^\varepsilon - v)\ d\tau, \quad \forall v \in L^2(\Omega). \qquad (3.47)
\end{aligned}$$

By lemma 3.1.5 we have $\|S_\varepsilon \varphi_-^\varepsilon(k\varepsilon)\|^2_{L^2(\Omega)} \le \varepsilon^2 L^2$ and therefore

$$\sum_{k=i+1}^{j} \|S_\varepsilon \varphi_-^\varepsilon(k\varepsilon)\|^2_{L^2(\Omega)} \le \varepsilon^2 (j-i) L^2 = \delta_\varepsilon,$$

where $\delta_\varepsilon \to 0$ for $\varepsilon \to 0$.

Now, we define

$$\bar\varphi^\varepsilon(\tau) = \varphi_-^\varepsilon(k\varepsilon). \quad \text{for } \ \tau \in (k\varepsilon, (k+1)\varepsilon).$$

Then

$$\varepsilon \sum_{k=i+1}^{j} \left(\frac{S_\varepsilon v}{\varepsilon}, \varphi_-^\varepsilon(k\varepsilon) - v \right) = \int_s^t \left(\frac{S_\varepsilon v}{\varepsilon}, \bar\varphi^\varepsilon(\tau) - v \right)\ d\tau.$$

Since $\|\varphi_-^\varepsilon(k\varepsilon)\|_{L^2(\Omega)} \le C$, then $\|\bar\varphi^\varepsilon(\tau)\|_{L^2(\Omega)} \le C$ and therefore the above integral is well defined. Using theorem 3.1.2 we obtain

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{k=i+1}^{j} \left(\frac{S_\varepsilon v}{\varepsilon}, \varphi_-^\varepsilon(k\varepsilon) - v \right) = \int_s^t (-Fv, \varphi^*(\tau) - v)\ d\tau, \qquad (3.48)$$

$\forall v \in L^6(\Omega)$. By (3.29) and (3.37) we have

$$\begin{aligned}
&\textstyle\int_0^t (-\Delta\varphi^\varepsilon, \varphi^\varepsilon - v)\ d\tau \to \int_0^t (-\Delta\varphi^*(\tau), \varphi^*(\tau) - v)\ d\tau \\
&\textstyle\int_0^t (\varphi^\varepsilon, \varphi^\varepsilon - v)\ d\tau \to \int_0^t (\varphi^*, \varphi^*(\tau) - v)\ d\tau
\end{aligned} \qquad (3.49)$$

Taking into account (3.48), (3.49) and passing to the limit for $\varepsilon \to 0$ in (3.46) we have

$$\begin{aligned}
\tau(\varphi^*(t) &- \varphi^*(s), \varphi^*(s) - v) + \tau \int_s^t (-Fv), \varphi^*(\tau) - v)\ d\tau \qquad (3.50)\\
&- \xi^2 \int_s^t (\Delta\varphi^*(\tau), \varphi^*(\tau) - v)\ d\tau \\
&+ \frac{1}{2a} \int_s^t (\varphi^*(\tau), \varphi^*(\tau) - v)\ d\tau \\
&\leq \int_s^t (2u^*(\tau) + g, \varphi^*(\tau) - v)\ d\tau, \qquad \forall v \in L^6(\Omega). \qquad (3.51)
\end{aligned}$$

Dividing (3.50) by $t - s$ and letting s tend to zero we see that

$$\begin{aligned}
\Big(\tau \frac{\partial}{\partial t}\varphi^*(t) &- \xi^2 \Delta\varphi^*(t) - \frac{1}{2a}\varphi^*(t) - Fv, \varphi^*(t) - v\Big) \qquad (3.52)\\
&\leq (2u^*(t) + g(t), \varphi^*(t) - v)\,, \quad \text{a.e. } t \in [0,T], \quad \forall v \in L^6(\Omega).
\end{aligned}$$

Using now the maximal monotonicity of F, we infer from the above relation that

$$\tau \frac{\partial}{\partial t}\varphi^*(t) - \xi^2 \Delta\varphi^*(t) - \frac{1}{2a}\big(\varphi^*(t) - (\varphi^*(t))^3\big) - 2u^*(t) = g(t) \quad \text{a.e. } t \in [0,T]$$

Hence $\varphi^*(t,x)$ satisfies $(2.1)_2$, $(2.2)_2$ a.e. $t \in [0,T]$.

By 3.1_1 we get

$$\frac{l}{2}\int_0^T (\psi(t), d\varphi^\varepsilon(t)) + \int_Q (u_t^\varepsilon - k\Delta u^\varepsilon)\psi\ dxdt = \int_Q f\psi\ dxdt,$$

$\forall \psi \in L^2(0,T;H^1(\Omega))$. Taking into account (2.6) ($p = 2$), (3.36), we may pass now to the limit in the last equality, for $\varepsilon \to 0$, and, By Helly's theorem

$$\frac{l}{2}\int_0^T (\psi(t), d\varphi^*(t)) + \int_Q (u_t^* - k\Delta u^*)\psi\ dxdt = \int_Q f\psi\ dxdt$$

$\forall \psi \in L^2(0,T;H^1(\Omega))$. Since φ^* is absolutely continuous from $[0,T]$ to $L^2(\Omega)$ (3.43) the first Stieltjes integral can by written as $\int_0^T (\psi(t), \varphi_t^*(t))\ dt$ and so the above relation yields

$$u_t^*(t) + \frac{l}{2}\varphi_t^*(t) - k\Delta u^*(t) = f \quad \text{ a.e } t \in [0,T]. \qquad (3.53)$$

By trace theorem (the map $u^\varepsilon \to u^\varepsilon|_{\partial\Omega}$ is continuous from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$) we may conclude from the last equality that $u^*(t,x)$ satisfies $(2.1)_1$, $(2.2)_1$ a.e. $t \in [0,T]$. Therefore (u^*, φ^*) is a strong solution to (2.1)-(2.3). Particular, again by trace theorem and because we actually proved the convergence of solution in $H^2(\Omega)$, $\forall t \in [0,T]$, we obtain that boundary conditions are satisifed, too.

3.2 A numerical algorithm and numerical results.

In this Section we are concerned with the numerical approximation of the solution corresponding to (3.1) by finite element method (**fem**). The finite element method is a general method for approximating the solution of boundary value problems for partial differential equations. This method is derived from the Ritz (or Galerkin) method, characteristic for the finite element method being the chose of the finite dimensional space, namely, the *span* of a set of finite element basis functions. The steps in solving a boundary value problem using **fem** are:

P0. (D) The direct formulation of the problem;

P1. (V) A variational formulation for problem (D);

P2. The construction of a finite element mesh (triangulation);

P3. The construction of the finite dimensional space of test function, *called finite element basis functions*;

P4. (V_{nn}) A discrete analogous of (V);

P5. Assembly the linear system of equations;

P6. Solve the system obtained in P5.

Let $\varepsilon = T/M$ be the time step size ($M_\varepsilon \equiv M$). Using an implicit (backward) finite difference scheme in time and **fem** for $\Omega \subset I\!R^2$, the corresponding discrete equations are (see [9]):

$$\begin{cases} Ru_l^i + \frac{\ell}{2}B\varphi_l^i + \varepsilon khFRu_l^i = B(\rho V u_l^{i-1} + \frac{\ell}{2}\varphi_l^{i-1} + \varepsilon k w^{i-1}), \\ \\ S\varphi_l^i - 2\varepsilon B u_l^i = B\tau\varphi_l^{i-1}, \end{cases} \tag{3.54}$$

where u_l^i, φ_l^i, $i = \overline{1, M}$, $l = \overline{1, nn}$, are unknown vectors for time level i.

From the initial conditions we have

$$u_l^0 = u_0(N_l), \qquad \varphi_l^0 = \varphi_0(N_l) \qquad l = \overline{1, nn}, \tag{3.55}$$

and therefore the **numerical algorithm** to compute the approximate solution by fractional steps method can be obtained from the following sequence (i denotes the time level)

`Begin` **frac_fem2D**

  $i := 0 \ \rightarrow$ Compute u_l^0, φ_l^0, $l = \overline{1, nn}$ from (3.55)

  `For` $i := 1$ `to` M `do`

    Compute $z_l = z(\cdot, N_l)$, $l = \overline{1, nn}$ from (3.4);

    $\varphi^{i-1} := z_l$, $l = \overline{1, nn}$;

    Compute u_l^i, φ_l^i, $l = \overline{1, nn}$, solving the linear system (3.54);

  `End-for`;

`End`.

The convergence result established by Theorem 3.1.2 guarantee that the approximate solution $(u^\varepsilon, \varphi^\varepsilon)$ computed by the conceptual algorithm **frac_fem2D** is in fact the approximate solution of nonlinear parabolic system (2.1)-(2.3).

We shall present now the numerical experiments implementing the conceptual algorithm **frac_fem2D**. In Figure 3.1 it can be seen the mesh in the x_1 and x_2 - axis directions of a rectangular profile.

Figures 3.2-3.3 represents the approximate solutions u^M, $M = 20$ and $M = 40$, respectively. The shape of the graphs shows the numerical stability

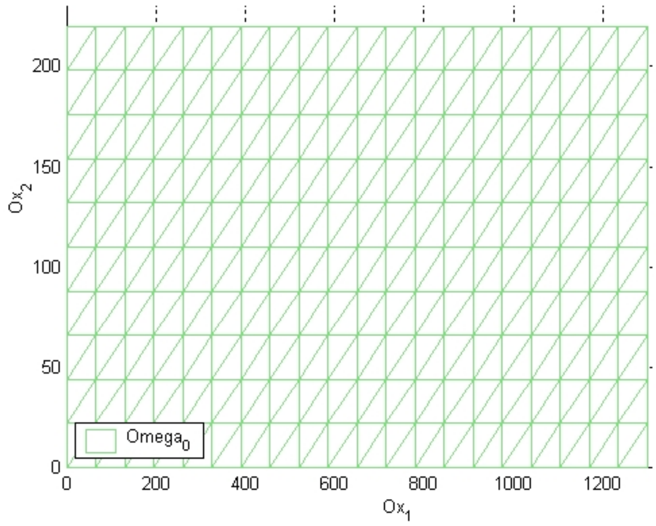

Figure 3.1: The triangulation over Ω=[0,1300]x[0,220]

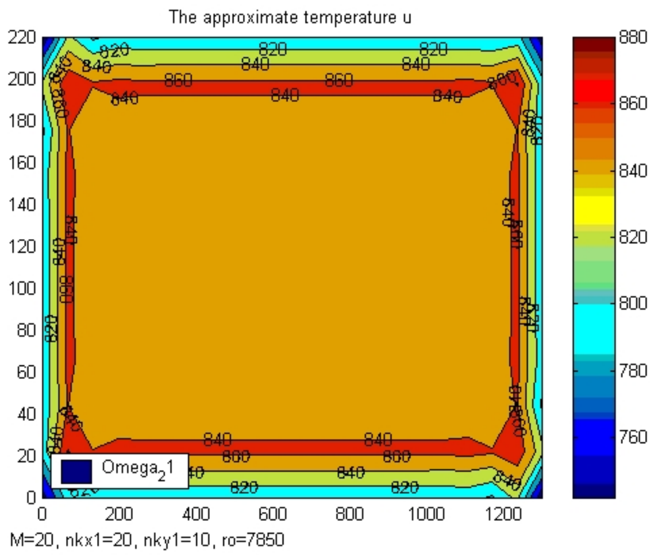

Figure 3.2: The approximate temperature u^{20}

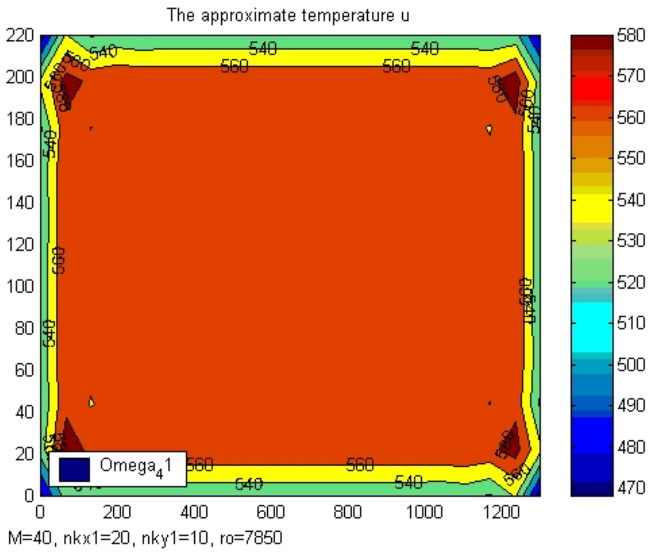

Figure 3.3: The approximate temperature u^{40}

and accuracy of the results obtained by implementing the fractional steps method (3.1)-(3.3); the most interesting aspect that we can observe analyzing the Figure 3.3 are the presence of *supercooling* and *superheating* phenomena (presence of solid fractions in the liquid, for example). Figures 3.2-3.3 may represent and "simulate" the cross section of a billet exiting from the copper mold's base, in the casting machine.

Chapter 4

Optimal control problem

4.1 Functional costs $j(w)$ and $j^\varepsilon(w)$.

Let $\Omega \subset I\!R^n$ $(n = 1,\ 2,\ 3)$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$ and let $T > 0$. Consider the following nonlinear optimal control problem ($Q = (0, T] \times \Omega$, $\Sigma = (0, T] \times \partial\Omega$):

(P) *Minimize*

$$\begin{aligned} j(w) &= \frac{1}{2}\int\limits_Q (\varphi(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0}\ dtdx \\ &+ \frac{1}{2}\int\limits_Q \left((u(t,x) - \delta_2)^+\right)^2 \cdot \chi_{Q_0}\ dtdx + \frac{1}{2}\int\limits_\Sigma w^2(t,x)\ dtd\gamma, \end{aligned}$$

on all (u, φ) *satisfying the system (2.1)-(2.3), with* ($f \equiv g \equiv 0$). δ_1, δ_2 in problem (P) are positive numbers that can be used to distinguish between the states of the material Ω at each time $t \in (0, T]$, namely:

- the *pure liquid region* $\{u(t,x) \geq \delta_2$ and $\varphi(t,x) \geq 1 + \delta_1\}$,
- the *pure solid region* $\{u(t,x) \leq -\delta_2$ and $\varphi(t,x) \leq -1 - \delta_1\}$,
- the *separating region* given by

$$\Omega_t = \{x \in \Omega;\ |u(t,x)| \leq \delta_2,\ |\varphi(t,x)| \leq 1 + \delta_1,\quad \delta_1, \delta_2 \geq 0\}.$$

So, the positive numbers in (P) come from above definitions of *pure liquid*, *pure solid* and *separating* regions. Moreover, this kind of cost functional can be involved in the numerical investigations concerning the solidification process, as: compute the minimum effort (the optimal value $w(t,x) \in \mathcal{U}$) to guide, in T time, the region $Q_0 \subset Q$, which at beginning contains the both states, to a pure solid state.

It is known, from the second chapter, that for each $u_0,\ \varphi_0 \in W^1_\infty(\Omega)$ satisfying $\dfrac{\partial u_0}{\partial \nu} + hu_0 = w(0,x)$, $\dfrac{\partial \varphi_0}{\partial \nu} = 0$ and $w \in W^1_\infty(\Sigma)$, system (2.1)-(2.3) has a unique solution $u,\ \varphi \in W$, where

$$W = W^{2,1}([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^2(\Omega)).$$

A similar problem where the boundary control structure is $w(t)g(x)$ was analyzed in [19]. Works devoted to the distributed optimal control problem governed by the phase field transition system with the distributed control acting on Ω (without state constraint) are due to Heinkenschloss and Tröltzsch [13] and Hoffmann and Jiang [14]. An optimal control problem with the distributed control acting on a subset $\omega \subset \Omega$ and with the state constraint, also governed by the phase field transition system, was discussed in the paper [21].

One of the great difficulties regarding the numerical approach of problem (P) comes from the fact that the state is governed by a nonlinear law. In order to remove this inconvenience, a new numerical method to approximate the state was developed in the work [9]. On the basis of this idea, we can associate to the nonlinear system (2.1)-(2.3) the approximating scheme (3.1)-(3.3).

Then, corresponding to problem (P), we consider the approximating optimal control problem:

(P$^\varepsilon$) *Minimize*

$$\begin{aligned} j^\varepsilon(w) &= \frac{1}{2}\int_Q (\varphi^\varepsilon(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0} dt dx \\ &+ \frac{1}{2}\int_Q \left((u^\varepsilon(t,x) - \delta_2)^+\right)^2 \cdot \chi_{Q_0} dt dx + \frac{1}{2}\int_\Sigma w^2(t,x) dt d\gamma \end{aligned}$$

The main result in the present chapter says that problem (P) can be approximated for $\varepsilon \to 0$ by the sequence of optimal control problems (P^ε) and so the computation of the boundary control $w(t,x)$ can be substituted by computation of an approximate control of (P^ε) (see Theorem 4.3.1). Besides the existence of an optimal control in problem (P^ε), necessary optimality conditions for this problem, needful for numerical approach, will be proved in the last Section and will be used in next chapter to develop a software in 1D-case.

4.2 The existence in problem (P) and (P^ε)

The existence of an optimal control in problem (P) and (P^ε) are proved in the present section. First we shall recall a result about the existence of a solution (u, φ) to the non linear parabolic system (2.1)-(2.3) and we'll introduce some estimates, for later use.

In the second chapter, we studied system (2.1)-(2.3) and we got that a solution (u, φ) exists and is unique, with $u \in W_p^{2,1}(Q)$ and $\varphi \in W_\nu^{2,1}(Q)$, In addition (u, φ) satisfies the estimate (2.6).

Now we consider the case $p = 2$ and $u_0,\ \varphi_0 \in W_\infty^1(\Omega)$. So, regarding the existence, we have:

Proposition 4.2.1. *Let* $u_0, \varphi_0 \in W_\infty^1(\Omega)$ *satisfying* $\dfrac{\partial u_0}{\partial \nu} + hu_0 = w(0,x)$, $\dfrac{\partial \varphi_0}{\partial \nu} = 0$ *and given* $w(t,x) \in W^1([0,T]; L^2(\partial\Omega))$. *Then system (2.1)-(2.3) has a unique solution* $(u,\varphi) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ *which fulfills the following* *estimates:*

$$\begin{aligned}\|u\|_{W_2^{2,1}(Q)} &+ \|\varphi\|_{W_2^{2,1}(Q)} \le C \Big\{ \|u_0\|_{W^1_\infty(\Omega)} + \|\varphi_0\|_{W^1_\infty(\Omega)} \\ &+ \|w\|_{W^1([0,T];L^2(\partial\Omega))} \Big\}.\end{aligned} \tag{4.1}$$

Proof. Taking into account $W^1_\infty(\Omega) \subset W^1_2(\Omega)$, Proposition 4.2.1 is an immediate consequence of Theorem 2.1.1 □

From the above inequality we can soon deduce the following estimates:

$$\|\varphi_t\|^2_{L^2([0,T];L^2(\Omega))} + \|u_t\|^2_{L^2([0,T];L^2(\Omega))} \le C, \tag{4.2}$$

$$\|\varphi(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{H^1(\Omega)} \le C \quad \forall t \in [0,T], \tag{4.3}$$

$$\|\varphi\|^2_{L^2([0,T];H^2(\Omega))} + \|u\|^2_{L^2([0,T];H^2(\Omega))} \le C. \tag{4.4}$$

Proposition 4.2.2. *For u_0, φ_0 and p as in proposition 4.2.1, problem (P) has at least one solution (u^*, φ^*, w^*).*

Proof. Consider $j(w) : \widehat{W} \to I\!R$ given by

$$\begin{aligned} j(w) &= \frac{1}{2}\int_Q (\varphi^w(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0} \; dtdx \\ &+ \frac{1}{2}\int_Q \big((u^w(t,x) - \delta_2)^+\big)^2 \cdot \chi_{Q_0} \; dtdx + \frac{1}{2}\int_\Sigma w^2(t,x) \; dtd\gamma, \end{aligned} \tag{4.5}$$

where (u^w, φ^w) is the solution of (2.1)-(2.3) corresponding to $w \in \widehat{W}$,

$$\widehat{W} = W^1([0,T]; L^2(\partial\Omega)) \cap L^\infty(\Sigma).$$

Let $0 \le d = \inf \; \{j(w), \; w \in \widehat{W}\}$ and let $\{w_n\} \subset \widehat{W}$ be such that

$$d \le j(w_n) \le d + \frac{1}{n}. \tag{4.6}$$

It is obvious that the sequence $\{w_n\}$ is bounded in $\widehat{W}$ because it is a subset of $L^\infty(\Sigma)$. Therefore, we can select subsequences of $\{w_n\}$, still denoted by itself, such that

$$w_n \to w^* \;\; \text{weak star in} \;\; \widehat{W}.$$

Let (u_n, φ_n) be the solution of problem (P) corresponding to w_n, i.e.,

$$\begin{cases} \rho c \dfrac{\partial u_n}{\partial t} + \dfrac{\ell}{2}\dfrac{\partial \varphi_n}{\partial t} = k\Delta u_n & \text{in } Q, \\ \tau \dfrac{\partial \varphi_n}{\partial t} = \xi^2 \Delta \varphi_n + \dfrac{1}{2a}(\varphi_n - \varphi_n^3) + 2u_n & \text{in } Q, \\ \dfrac{\partial u_n}{\partial \nu} + hu_n = w_n(t,x) & \text{in } \Sigma, \\ \dfrac{\partial \varphi_n}{\partial \nu} = 0 & \text{in } \Sigma, \\ u_n(0,x) = u_0(x) \quad \varphi_n(0,x) = \varphi_0(x) & \text{on } \Omega. \end{cases} \tag{4.7}$$

Taking into account (4.2)-(4.4) and the Sobolev embedding theorem, we have (on a subsequence, again denoted $\{n\}$)

$$\begin{array}{lll} u_n \to u^* & \text{weakly in} & L^2(0,T;H^2(\Omega)), \\ & \text{and strongly in} & L^2((0,T);H^1(\Omega)), \\ \dfrac{\partial u_n}{\partial t} \to \dfrac{\partial u^*}{\partial t} & \text{weakly in} & L^2(0,T;L^2(\Omega)), \\ \Delta u_n \to \Delta u^* & \text{weakly in} & L^2(0,T;L^2(\Omega)), \\ \varphi_n \to \varphi^* & \text{weakly in} & L^2(0,T;H^2(\Omega)), \\ & \text{and strongly in} & L^2((0,T);H^1(\Omega)), \\ \dfrac{\partial \varphi_n}{\partial t} \to \dfrac{\partial \varphi^*}{\partial t} & \text{weakly in} & L^2(0,T;L^2(\Omega)), \\ \Delta \varphi_n \to \Delta \varphi^* & \text{weakly in} & L^2(0,T;L^2(\Omega)), \end{array}$$

and, by the Ascoli-Arzéla theorem

$$\begin{array}{l} u_n \to u^* \text{ strongly in } C([0,T];L^2(\Omega)), \\ \varphi_n \to \varphi^* \text{ strongly in } C([0,T];L^2(\Omega)). \end{array}$$

By Sobolev's embedding theorem ($H^1(\Omega) \subset L^6(\Omega)$, $\Omega \subset I\!R^n$, $n \leq 3$) and thanks to the relations of convergence established above, we obtain

$$\|\varphi_n^2 + \varphi_n \varphi^* + (\varphi^*)^2\|_{L^2((0,T);L^3(\Omega))} \leq C,$$

which implies

$$\begin{aligned}\int_Q (\varphi_n^3 &- (\varphi^*)^3)^2 dxdt \\ &\leq \Big[\int_Q (\varphi_n - \varphi^*)^6 dxdt\Big]^{1/3}\Big[\int_Q |\varphi_n^2 + \varphi_n\varphi^* + (\varphi^*)^2|^2 dxdt\Big]^{2/3} \\ &\leq C\int_0^T \|\varphi_n - \varphi^*\|^2_{L^6(\Omega)} dt.\end{aligned}$$

This indicates that

$$\varphi_n^3 \to (\varphi^*)^3 \quad \text{strongly in } L^2((0,T);L^2(\Omega)).$$

So, letting n tend to $+\infty$ in (4.7) we get

$$\begin{cases} \rho c\dfrac{\partial u^*}{\partial t} + \dfrac{\ell}{2}\dfrac{\partial \varphi^*}{\partial t} = k\Delta u^* & \text{in } \ Q, \\ \tau\dfrac{\partial \varphi^*}{\partial t} = \xi^2\Delta\varphi^* + \dfrac{1}{2a}(\varphi^* - (\varphi^*)^3) + 2u^* & \text{in } \ Q, \\ \dfrac{\partial u^*}{\partial \nu} + hu^* = w^*(t,x) & \text{in } \ \Sigma, \\ \dfrac{\partial \varphi^*}{\partial \nu} = 0 & \text{in } \ \Sigma, \\ u^*(0,x) = u_0(x) \quad \varphi^*(0,x) = \varphi_0(x) & \text{on } \ \Omega. \end{cases} \tag{4.8}$$

Thus, the uniqueness of solution for (4.8) implies that (u^*, φ^*) is the solution of problem (2.1) corresponding to $w^* \in \widehat{W}$. Since j is continuous, then by (4.6), we see that $d = j(w^*)$ and the proof of Proposition 4.2.2 is completed. □

As regards the existence in problem (P^ε), we have

Proposition 4.2.3. *For u_0, φ_0 and w as in Proposition 4.2.1, problem (P^ε) has at least one solution $(u_\varepsilon^*, \varphi_\varepsilon^*, w_\varepsilon^*)$.*

Proof. In the proof of Proposition 4.2.3 (see [19]) one uses the estimates established in [9], the Sobolev embedding theorem, the Ascoli-Arzéla theorem and the continuity of $j^\varepsilon(w)$. We don't give the details. □

4.3 The convergence of problem (P^ε)

The convergence result, proved in the next theorem, of the optimal solution of problem (P^ε) to the optimal solution of problem (P), as $\varepsilon \to 0$, is the main result of this chapter.

Theorem 4.3.1. *Let $\{w_\varepsilon^*\}$ be a sequence of optimal controllers for problems (P^ε). Then*

$$\liminf_{\varepsilon \to 0} j^\varepsilon(w^*) = j(w^*) \tag{4.9}$$

and

$$\lim_{\varepsilon \to 0} j(w_\varepsilon^*) = j(w^*). \tag{4.10}$$

Moreover, every weak limit point of $\{w_\varepsilon^*\}$ is an optimal controller for problem (P).

Theorem 4.3.1 amounts to saying that (P^ε) approximates problem (P) and an optimal controller $\{w_\varepsilon^*\}$ of (P^ε) is a suboptimal controller for problem (P).

The main ingredient in the proof to Theorem 4.3.1 is the following lemma.

Lemma 4.3.2. *If $\{w_\varepsilon^*\}$ is a sequence of optimal controllers for problems (P^ε) then there exists $\{\varepsilon_n\} \to 0$ such that*

$$
\begin{array}{llll}
w_{\varepsilon_n}^* \to w^* & \textit{weak star in} & L^\infty(\Sigma), \\
\varphi_{\varepsilon_n}^* \to \varphi^* & \textit{strongly in} & L^2((0,T);H^1(\Omega)), \\
u_{\varepsilon_n}^* \to u^* & \textit{strongly in} & L^2((0,T);H^1(\Omega)),
\end{array}
$$

where $(u_{\varepsilon_n}^*, \varphi_{\varepsilon_n}^*, w_{\varepsilon_n}^*) = (u_{\varepsilon_n}^{w_{\varepsilon_n}^*}, \varphi_{\varepsilon_n}^{w_{\varepsilon_n}^*}, w_{\varepsilon_n}^*)$ and $(u^*, \varphi^*, w^*) = (u^{w^*}, \varphi^{w^*}, w^*)$ are the solutions to (3.1) corresponding to $w = w_{\varepsilon_n}^*$ and to (2.1) corresponding to $w = w^*$, respectively.

Proof. For $\{w_\varepsilon^*\}$ independent of ε this lemma was proved in [9]. The extension in the lemma is standard. $\square$

Proof. of Theorem 4.3.1. Let $\{w_\varepsilon^*\}$ be an optimal controller for problem (P^ε) and let $(u_\varepsilon^*, \varphi_\varepsilon^*, w_\varepsilon^*)$ be the corresponding solution of (3.1) with $w = w_\varepsilon^*$. By virtue of Lemma 4.3.2 it results that there exist $w^* \in L^\infty(\Sigma)$ and $\{\varepsilon_n\}$ such that

$$
\begin{array}{lll}
w_{\varepsilon_n}^* \to w^* & \text{weak star in} & L^\infty(\Sigma), \\[2ex]
\varphi_{\varepsilon_n}^{w_{\varepsilon_n}^*} \to \varphi^{w^*} & \text{strongly in} & L^2((0,T);H^1(\Omega)), \\[2ex]
u_{\varepsilon_n}^{w_{\varepsilon_n}^*} \to u^{w^*} & \text{strongly in} & L^2((0,T);H^1(\Omega)),
\end{array}
$$

where $(u_{\varepsilon_n}^{w_{\varepsilon_n}^*}, \varphi_{\varepsilon_n}^{w_{\varepsilon_n}^*}, w_{\varepsilon_n}^*)$ is the solution to (3.1) corresponding to $w = w_{\varepsilon_n}^*$ and $(u^{w^*}, \varphi^{w^*}, w^*)$ is the solution to (2.1) corresponding to $w = w^*$. Since:

$$
\begin{aligned}
\varphi &\to \frac{1}{2}\int_Q (\varphi(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0}\, dtdx, \\
u &\to \frac{1}{2}\int_Q \left((u(t,x) - \delta_2)^+\right)^2 \cdot \chi_{Q_0}\, dtdx, \\
w &\to \frac{1}{2}\int_\Sigma w^2(t,x)\, dtd\gamma
\end{aligned}
$$

are convex continuous functions, it follows that these are weakly lower semi-continuous functions (from $L^2(Q) \to I\!R$, $L^2(Q) \to I\!R$ and $L^2(\Sigma) \to I\!R$, respectively). Hence

$$
j(w^*) \le \liminf_{n\to\infty} j^{\varepsilon_n}(w_{\varepsilon_n}^*). \tag{4.11}
$$

Let $\tilde{w}^*$ be an optimal controller for problem (P). Since $w_{\varepsilon_n}^*$ is an optimal controller for problem (P^{ε_n}) it follows that

$$
j^{\varepsilon_n}(w_{\varepsilon_n}^*) \le j^{\varepsilon_n}(\tilde{w}^*).
$$

But $\varphi_{\varepsilon_n}^{\tilde{w}^*} \to \varphi^{\tilde{w}^*}$, $u_{\varepsilon_n}^{\tilde{w}^*} \to u^{\tilde{w}^*}$ strongly in $L^2((0,T);H^1(\Omega))$ and so the term in the right side give us

$$
\lim_{n\to\infty} j^{\varepsilon_n}(\tilde{w}^*) = j(\tilde{w}^*). \tag{4.12}
$$

From (4.11) and (4.12) we obtain

$$
j(w^*) \le \liminf_{n\to\infty} j^{\varepsilon_n}(w_{\varepsilon_n}^*) \le j^{\varepsilon_n}(w_{\varepsilon_n}^*) \le j^{\varepsilon_n}(\tilde{w}^*) \le \limsup_{n\to\infty} j^{\varepsilon_n}(\tilde{w}^*) = j(\tilde{w}^*).
$$

Hence

$$\liminf_{\varepsilon_n \to 0} j^{\varepsilon_n}(w^*_{\varepsilon_n}) = j(\tilde{w}^*) = j(w^*)$$

and then (4.9) holds.

To prove (4.10) we set: $\tilde{u}_\varepsilon = u^{w^*_\varepsilon}$, $\tilde{\varphi}_\varepsilon = \varphi^{w^*_\varepsilon}$ (w^*_ε optimal in (P^ε)). We have on a subsequence $\{\varepsilon_n\}$

$$\begin{array}{llll} w^*_{\varepsilon_n} \to w^0 & \text{weakly} & & L^\infty(\Sigma), \\ \tilde{\varphi}_{\varepsilon_n} \to \varphi & \text{strongly} & \text{in} & L^2((0,T); H^1(\Omega)), \\ \tilde{u}_{\varepsilon_n} \to u & \text{strongly} & \text{in} & L^2((0,T); H^1(\Omega)), \end{array}$$

where (u, φ, w^0) satisfy (2.1), i.e., $(u, \varphi) = (u^{w^0}, \varphi^{w^0})$. We have therefore

$$j(w^0) \le \inf P$$

and, since $\{\varepsilon_n\}$ was arbitrary choose, (4.10) follows.

Now, since w^*_ε is an optimal controller for problem (P^ε) it follows that

$$j^\varepsilon(w^*_\varepsilon) \le j^\varepsilon(w) \quad \forall\, w \in \mathcal{U}.$$

But, as we seen above (relation (4.11)),

$$j(w^*) \le \liminf_{\varepsilon \to 0} j^\varepsilon(w^*_\varepsilon)$$

and thus, along with above inequality, we may conclude that

$$j(w^*) \le \lim_{\varepsilon \to 0} j^\varepsilon(w) \quad \forall\, w \in \mathcal{U}.$$

Hence

$$j(w^*) \le j(w) \quad \forall\, w \in \mathcal{U}$$

i.e., the weak limit point w^* is a suboptimal controller for problem (P). This completes the proof of Theorem 4.3.1. $\square$

4.4 Necessary optimality conditions in (P^ε)

Let $(u^\varepsilon, \varphi^\varepsilon, w)$ be the solution of (3.1) and let $\tilde{w} \in L^\infty(\Sigma)$ be arbitrary but fixed and $\lambda > 0$. Set $w^\lambda = w + \lambda\tilde{w}$ and let $(u^{\lambda,\varepsilon},\ \varphi^{\lambda,\varepsilon})$ be the solution of (3.1) corresponding to w^λ. First of all we easily can derive from above that

$$w^\lambda \to w \qquad \text{strongly in} \qquad L^2(\Sigma) \qquad \text{as } \lambda \to 0.$$

Corresponding to $(u^{\lambda,\varepsilon}, \varphi^{\lambda,\varepsilon}, w^\lambda)$, for $i = 0, 1, \cdots, M_\varepsilon - 1$, we have

$$\begin{cases} \rho c u_t^{\lambda,\varepsilon} + \dfrac{\ell}{2}\varphi_t^{\lambda,\varepsilon} = k\Delta u^{\lambda,\varepsilon} & \text{in } Q_i^\varepsilon, \\ \tau\varphi_t^{\lambda,\varepsilon} = \xi^2\Delta\varphi^{\lambda,\varepsilon} + \dfrac{1}{2a}\varphi^{\lambda,\varepsilon} + 2u^{\lambda,\varepsilon} & \text{in } Q_i^\varepsilon, \end{cases} \tag{4.13}$$

with the boundary conditions

$$\begin{cases} \dfrac{\partial u^{\lambda,\varepsilon}}{\partial\nu} + hu^{\lambda,\varepsilon} = w^\lambda & \text{on } \Sigma_i^\varepsilon, \\ \dfrac{\partial\varphi^{\lambda,\varepsilon}}{\partial\nu} = 0 & \text{on } \Sigma_i^\varepsilon, \end{cases} \tag{4.14}$$

and the initial conditions

$$\begin{cases} u_+^{\lambda,\varepsilon}(i\varepsilon, x) = u_-^{\lambda,\varepsilon}(i\varepsilon, x), \quad u_-^{\lambda,\varepsilon}(0, x) = u_0(x) & \text{on } \Omega, \\ \varphi_+^{\lambda,\varepsilon}(i\varepsilon, x) = z^\lambda(\varepsilon, \varphi_-^{\lambda,\varepsilon}(i\varepsilon, x)) & \text{on } \Omega, \end{cases} \tag{4.15}$$

where z^λ is the solution of the Cauchy problem:

$$\begin{cases} \big(z^\lambda(s)\big)' + \dfrac{1}{2a}\big(z^\lambda(s)\big)^3 = 0 & s \in (i\varepsilon, (i+1)\varepsilon), \\ z^\lambda(i\varepsilon) = \varphi_-^{\lambda,\varepsilon}(i\varepsilon, x) & \varphi_-^{\lambda,\varepsilon}(0, x) = \varphi_0(x). \end{cases} \tag{4.16}$$

Subtracting (3.1) from (4.13)-(4.15) and dividing by $\lambda > 0$, we get

$$\begin{cases} \rho c\left(\frac{u^{\lambda,\varepsilon}-u^\varepsilon}{\lambda}\right)_t + \frac{\ell}{2}\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_t \\ \qquad = k\Delta\left(\frac{u^{\lambda,\varepsilon}-u^\varepsilon}{\lambda}\right) & \text{on } Q_i^\varepsilon, \\ \tau\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_t = \xi^2\Delta\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right) + \frac{1}{2a}\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right) \\ \qquad + 2\left(\frac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right) & \text{on } Q_i^\varepsilon, \end{cases} \tag{4.17}$$

satisfying the boundary conditions

$$\begin{cases} \dfrac{\partial\left(\frac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right)}{\partial\nu} + h\left(\dfrac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right) = \dfrac{w^\lambda - w}{\lambda} & \text{on } \Sigma_i^\varepsilon, \\ \dfrac{\partial\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)}{\partial\nu} = 0 & \text{on } \Sigma_i^\varepsilon, \end{cases} \tag{4.18}$$

and the initial conditions

$$\begin{cases} \frac{u_+^{\lambda,\varepsilon}(i\varepsilon,x)-u_+^\varepsilon(i\varepsilon,x)}{\lambda} = \frac{u_-^{\lambda,\varepsilon}(i\varepsilon,x)-u_-^\varepsilon(i\varepsilon,x)}{\lambda}, & \text{on } \Omega, \\ u_-^{\lambda,\varepsilon}(0,x)-u^\varepsilon(0,x) = 0 & \text{on } \Omega, \\ \frac{z^\lambda(\varepsilon,\varphi_-^{\lambda,\varepsilon}(i\varepsilon,x))-z(\varepsilon,\varphi_-^{\lambda,\varepsilon}(i\varepsilon,x))}{\lambda} = \frac{\varphi_+^{\lambda,\varepsilon}-\varphi_+^e}{\lambda} & \text{on } \Omega, \end{cases} \tag{4.19}$$

for $i = 0, 1, \cdots, M_\varepsilon - 1$. Letting λ tend to zero in (4.17)-(4.19) we get the system in variation (4.20)-(4.22) below

$$\begin{cases} \rho c\tilde{u}_t^\varepsilon + \frac{\ell}{2}\tilde{\varphi}_t^\varepsilon = k\Delta\tilde{u}^\varepsilon & \text{on } Q_i^\varepsilon, \\ \tau\tilde{\varphi}_t^\varepsilon = \xi^2\Delta\tilde{\varphi}^\varepsilon + \frac{1}{2a}\tilde{\varphi}^\varepsilon + 2\tilde{u}^\varepsilon & \text{on } Q_i^\varepsilon, \end{cases} \tag{4.20}$$

satisfying the boundary conditions

$$\begin{cases} \dfrac{\partial\tilde{u}^\varepsilon}{\partial\nu} + h\tilde{u}^\varepsilon = \tilde{w} & \text{on } \Sigma_i^\varepsilon, \\ \dfrac{\partial\tilde{\varphi}^\varepsilon}{\partial\nu} = 0 & \text{on } \Sigma_i^\varepsilon, \end{cases} \tag{4.21}$$

and the initial conditions

$$\begin{cases} \tilde{u}^\varepsilon_+(i\varepsilon, x) = \tilde{u}^\varepsilon_-(i\varepsilon, x) & \tilde{u}^\varepsilon(0,x) = 0 \quad \text{on } \ \Omega, \\ \\ \tilde{\varphi}^\varepsilon_+(i\varepsilon, x) = \eta((i+1)\varepsilon, x) & \quad \text{on } \ \Omega, \end{cases} \tag{4.22}$$

for $i = 0, 1, \cdots, M_\varepsilon - 1$, where

$$\tilde{u}^\varepsilon = \lim_{\lambda \to 0} \frac{u^{\lambda,\varepsilon} - u^\varepsilon}{\lambda}, \qquad \tilde{\varphi}^\varepsilon = \lim_{\lambda \to 0} \frac{\varphi^{\lambda,\varepsilon} - \varphi^\varepsilon}{\lambda},$$

$$\tilde{w} = \lim_{\lambda \to 0} \frac{w^\lambda - w}{\lambda}, \qquad \tilde{z} = \lim_{\lambda \to 0} \frac{z^\lambda - z}{\lambda},$$

$$\tilde{\varphi}^\varepsilon_+ = \lim_{\lambda \to 0} \frac{\varphi^{\lambda,\varepsilon}_+ - \varphi^\varepsilon_+}{\lambda}, \qquad \tilde{u}^\varepsilon_+ = \lim_{\lambda \to 0} \frac{u^{\lambda,\varepsilon}_+ - u^\varepsilon_+}{\lambda},$$

and

$$\begin{aligned} \eta((i+1)\varepsilon, x) &= \lim_{\lambda \to 0} \frac{z^\lambda(\varepsilon, \varphi^{\lambda,\varepsilon}_-(i\varepsilon, x)) - z(\varepsilon, \varphi^\varepsilon_-(i\varepsilon, x))}{\lambda} \\ &= z(\varepsilon, \varphi^\varepsilon_-(i\varepsilon, x))\tilde{\varphi}^\varepsilon_-(i\varepsilon, x) + \tilde{z}(\varepsilon, \varphi^\varepsilon_-(i\varepsilon, x)), \end{aligned}$$

with $\eta(\cdot)$ the solution of Cauchy problem

$$\begin{cases} \eta'(s) + \dfrac{3}{2a} z^2 \eta(s) = 0 & s \in (i\varepsilon, (i+1)\varepsilon), \\ \\ \eta(i\varepsilon) = \tilde{\varphi}^\varepsilon_-(i\varepsilon, x) & \tilde{\varphi}^\varepsilon_-(0, x) = 0, \end{cases} \tag{4.23}$$

that is

$$\eta((i+1)\varepsilon) = \exp\Big(- \int_{i\varepsilon}^{(i+1)\varepsilon} \frac{3}{2a} z(t, \cdot)^2 dt \Big) \tilde{\varphi}^\varepsilon_-(i\varepsilon, x). \tag{4.24}$$

We now introduce the adjoint state system. For this, the equations (4.20) can be written in the form:

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{u}^\varepsilon \\ \tilde{\varphi}^\varepsilon \end{pmatrix} = A \begin{pmatrix} \tilde{u}^\varepsilon \\ \tilde{\varphi}^\varepsilon \end{pmatrix} \qquad \text{in } \ Q^\varepsilon_i$$

where

$$A = \begin{pmatrix} \frac{1}{\rho c}\Big(k\Delta - \dfrac{\ell}{\tau}\Big) & -\dfrac{\ell}{2\tau\rho c}\Big(\xi^2\Delta + \dfrac{1}{2a}\Big) \\ \\ \dfrac{2}{\tau} & \dfrac{1}{\tau}\Big(\xi^2\Delta + \dfrac{1}{2a}\Big) \end{pmatrix},$$

$D(A) = \Big\{(\psi,\gamma) \in H^2(\Omega)\times H^2(\Omega);\ \dfrac{\partial\psi}{\partial\nu} + h\psi \in L^2(\partial\Omega),\ \dfrac{\partial\gamma}{\partial\nu} = 0\Big\}$. Then, we can deduce

$$A^* = \begin{pmatrix} \frac{1}{\rho c}\Big(k\Delta - \dfrac{\ell}{\tau}\Big) & \dfrac{2}{\tau} \\ \\ -\dfrac{\ell}{2\tau\rho c}\Big(\xi^2\Delta + \dfrac{1}{2a}\Big) & \dfrac{1}{\tau}\Big(\xi^2\Delta + \dfrac{1}{2a}\Big) \end{pmatrix},$$

$D(A^*) = \Big\{(\psi,\gamma) \in H^2(\Omega)\times H^2(\Omega);\ \dfrac{\partial\psi}{\partial\nu} + h\psi = 0,\ \dfrac{\partial\gamma}{\partial\nu} = \dfrac{2\rho c}{\ell}\dfrac{\partial\psi}{\partial\nu}\Big\}$. Thus the adjoint state system is

$$\frac{\partial}{\partial t}\begin{pmatrix} p^\varepsilon \\ q^\varepsilon \end{pmatrix} = -A^*\begin{pmatrix} p^\varepsilon \\ q^\varepsilon \end{pmatrix} + \begin{pmatrix} \dfrac{\partial j^\varepsilon(w)}{\partial u^\varepsilon} \\ \\ \dfrac{\partial j^\varepsilon(w)}{\partial \varphi^\varepsilon} \end{pmatrix}$$

i.e.

$$\begin{cases} p_t^\varepsilon + \dfrac{k}{\rho c}\Delta p^\varepsilon - \dfrac{\ell}{\tau\rho c}p^\varepsilon + \dfrac{2}{\tau}q^\varepsilon = (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0} & \text{in } Q_i^\varepsilon, \\ \\ \dfrac{\partial p^\varepsilon}{\partial\nu} + hp^\varepsilon = 0 & \text{on } \Sigma_i^\varepsilon, \\ \\ p_-^\varepsilon((i+1)\varepsilon, x) = 0, \qquad p_-^\varepsilon(T,x) = 0 & \text{on } \Omega, \end{cases} \tag{4.25}$$

$$
\begin{cases}
q_t^\varepsilon - \dfrac{\ell\xi^2}{2\tau\rho c}\Delta p^\varepsilon - \dfrac{\ell}{4a\tau\rho c}p^\varepsilon \\[2ex]
\qquad + \dfrac{\xi^2}{\tau}\Delta q^\varepsilon + \dfrac{1}{2a\tau}q^\varepsilon = (\varphi^\varepsilon - 1 - \delta_1)\cdot\chi_{Q_0} \qquad \text{in } \; Q_i^\varepsilon, \\[2ex]
\dfrac{\partial q^\varepsilon}{\partial\nu} = \dfrac{2\rho c}{\ell}\dfrac{\partial p^\varepsilon}{\partial\nu} \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \; \Sigma_i^\varepsilon, \\[2ex]
q_-^\varepsilon((i+1)\varepsilon, x) = \exp\Big(\displaystyle\int_{i\varepsilon}^{(i+1)\varepsilon} \frac{3}{2a}z^2(t,\cdot)dt\Big) \\[2ex]
\qquad \times \; q_+^\varepsilon((i+1)\varepsilon, x), \qquad q_-^\varepsilon(T,x) = 0, \qquad \text{on } \; \Omega,
\end{cases}
\tag{4.26}
$$

for $i = M_\varepsilon - 2, M_\varepsilon - 3, ..., 1, 0$, where $z(t,\cdot)$ is the solution of (3.4).

Let us introduce the cost functional

$$
j_1^\varepsilon(w) = j^\varepsilon(w) + \frac{1}{2}I_{\mathcal{U}}(w)
$$

where $I_{\mathcal{U}}(w)$ is the indicator function of the set $\mathcal{U}$. If w_ε^* is an optimal controller of problem (P^ε) then

$$
\frac{j_1^\varepsilon(w_\varepsilon^* + \lambda\tilde{w}) - j_1^\varepsilon(w_\varepsilon^*)}{\lambda} \geq 0, \quad \forall\, \lambda > 0.
$$

Letting λ tend to zero in above inequality, we get

$$
\begin{aligned}
\int_Q (\varphi^\varepsilon - 1 - \delta_1) \quad &\cdot \quad \chi_{Q_0}\, \tilde{\varphi}^\varepsilon \; dtdx + \int_Q (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\, \tilde{u}^\varepsilon \; dtdx \\
&+ \int_\Sigma w_\varepsilon^* \, \tilde{w} \; dtd\gamma + I'_{\mathcal{U}}(w_\varepsilon^*, \tilde{w}) \geq 0, \qquad \forall\, \tilde{w} \in T_{\mathcal{U}}(w_\varepsilon^*).
\end{aligned}
\tag{4.27}
$$

Multiplying $(4.25)_1$ by $\tilde{u}^\varepsilon$ and $(4.26)_1$ by $\tilde{\varphi}^\varepsilon$, using integration by parts and Green's formula, we derive

$$\begin{aligned}\int\limits_{Q_i^\varepsilon} p_t^\varepsilon \tilde{u}^\varepsilon dt\ dx \quad &+ \quad \frac{k}{\rho c}\int\limits_{Q_i^\varepsilon} p^\varepsilon \Delta\tilde{u}^\varepsilon dt\ dx - \frac{\ell}{\tau\rho c}\int\limits_{Q_i^\varepsilon} p^\varepsilon \tilde{u}^\varepsilon dt\ dx \\ &+ \quad \frac{2}{\tau}\int\limits_{Q_i^\varepsilon} q^\varepsilon \tilde{u}^\varepsilon dt\ dx + \frac{k}{\rho V}\int\limits_{\Sigma_i^\varepsilon}\Big(p^\varepsilon\frac{\partial\tilde{u}^\varepsilon}{\partial\nu} - \frac{\partial p^\varepsilon}{\partial\nu}\tilde{u}^\varepsilon\Big)dt\ d \\ &= \quad \int\limits_{Q_i^\varepsilon}(u^\varepsilon - \delta_2)^+\cdot\chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\int\limits_{Q_i^\varepsilon} q_t^\varepsilon \tilde{\varphi}^\varepsilon dt\ dx \quad &- \quad \frac{\ell\xi^2}{2\tau\rho c}\int\limits_{Q_i^\varepsilon} p^\varepsilon \Delta\tilde{\varphi}^\varepsilon dt\ dx \\ &- \quad \frac{\ell}{4a\tau\rho c}\int\limits_{Q_i^\varepsilon} p^\varepsilon\tilde{\varphi}^\varepsilon dt\ dx + \frac{\ell\xi^2}{2\tau\rho c}\int\limits_{\Sigma_i^\varepsilon}\Big(\frac{\partial p^\varepsilon}{\partial\nu}\tilde{\varphi}^\varepsilon - p^\varepsilon\frac{\partial\tilde{\varphi}^\varepsilon}{\partial\nu}\Big)dt\ d\gamma \\ &+ \quad \frac{\xi^2}{\tau}\int\limits_{\Sigma_i^\varepsilon}\Big(q^\varepsilon\frac{\partial\tilde{\varphi}^\varepsilon}{\partial\nu} - \tilde{\varphi}^\varepsilon\frac{\partial q^\varepsilon}{\partial\nu}\Big)dt\ d\gamma + \frac{\xi^2}{\tau}\int\limits_{Q_i^\varepsilon} q^\varepsilon\Delta\tilde{\varphi}^\varepsilon dt\ dx \\ &+ \quad \frac{1}{2a\tau}\int\limits_{Q_i^\varepsilon} q^\varepsilon\tilde{\varphi}^\varepsilon dt\ dx = \int\limits_{Q_i^\varepsilon}(\varphi^\varepsilon - 1 - \delta_1)\cdot\chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx.\end{aligned} \tag{4.29}$$

Now we multiply $(4.21)_1$ by p^ε, $(4.25)_2$ by $\tilde{u}^\varepsilon$ and, by subtraction we get

$$\frac{\partial p^\varepsilon}{\partial\nu}\tilde{u}^\varepsilon - \frac{\partial\tilde{u}^\varepsilon}{\partial\nu}p^\varepsilon = -p^\varepsilon\tilde{w}. \tag{4.30}$$

Adding (4.28)-(4.29) and taking into account $(4.21)_2$, $(4.22)_2$, (4.30), we obtain

$$\int\limits_{Q_i^\varepsilon} p_t^\varepsilon\tilde{u}^\varepsilon dt\ dx + \int\limits_{Q_i^\varepsilon} q_t^\varepsilon\tilde{\varphi}^\varepsilon dt\ dx + \frac{k}{\rho c}\int\limits_{\Sigma_i^\varepsilon} p^\varepsilon\tilde{w} dt\ d\gamma$$

$$+\int\limits_{Q_i^\varepsilon} p^\varepsilon\Big[\frac{k}{\rho V}\Delta\tilde{u}^\varepsilon - \frac{\ell\xi^2}{2\tau\rho c}\Delta\tilde{\varphi}^\varepsilon - \frac{\ell}{4a\tau\rho c}\tilde{\varphi}^\varepsilon - \frac{\ell}{\tau\rho c}\tilde{u}^\varepsilon\Big]dt\ dx$$

$$+\int\limits_{Q_i^\varepsilon} q^\varepsilon\Big[\frac{\xi^2}{\tau}\Delta\tilde{\varphi}^\varepsilon + \frac{1}{2a\tau}\tilde{\varphi}^\varepsilon + \frac{2}{\tau}\tilde{u}^\varepsilon\Big]dt\ dx,$$

$$= \int\limits_{Q_i^\varepsilon} (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx + \int\limits_{Q_i^\varepsilon} (\varphi^\varepsilon - 1 - \delta_1) \cdot \chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx,$$

i.e., on the basis of equations in (4.20), from the last equality we can derive that

$$\int\limits_{Q_i^\varepsilon} \big(p_t^\varepsilon \tilde{u}^\varepsilon + p^\varepsilon \tilde{u}_t^\varepsilon + q_t^\varepsilon \tilde{\varphi}^\varepsilon + q^\varepsilon \tilde{\varphi}_t^\varepsilon\big) dt\ dx + \frac{k}{\rho c} \int\limits_{\Sigma_i^\varepsilon} p^\varepsilon \tilde{w} dt\ d\gamma$$

$$= \int\limits_{Q_i^\varepsilon} (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx + \int\limits_{Q_i^\varepsilon} (\varphi^\varepsilon - 1 - \delta_1) \cdot \chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx.$$

By Fubini's theorem, integration by parts formula, $\int\limits_0^T (fg' + f'g)dt = fg\ |_0^T$, and relations (4.22), (4.25), the latter leads to

$$\frac{k}{\rho c} \int\limits_{\Sigma_i^\varepsilon} p^\varepsilon \tilde{w} dt\ d\gamma = \int\limits_{Q_i^\varepsilon} (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx + \int\limits_{Q_i^\varepsilon} (\varphi^\varepsilon - 1 - \delta_1) \cdot \chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx$$

and then (4.27) becomes

$$\frac{k}{\rho c} \int\limits_{\Sigma} p^\varepsilon \tilde{w}\ dtd\gamma + \int\limits_{\Sigma} w_\varepsilon^* \tilde{w}\ dtd\gamma + I'_{\mathcal{U}}(w_\varepsilon^*, \tilde{w}) \geq 0 \qquad \forall\, \tilde{w} \in T_{\mathcal{U}}(w_\varepsilon^*)$$

or

$$\big(\frac{k}{\rho c} p^\varepsilon + w_\varepsilon^*, \tilde{w}\big)_{L^2(\Sigma)\times L^2(\Sigma)} + I'_{\mathcal{U}}(w_\varepsilon^*, \tilde{w}) \geq 0 \qquad \forall\, \tilde{w} \in T_{\mathcal{U}}(w_\varepsilon^*).$$

The last inequality is equivalent to

$$-r(t,x) \in \partial I_{\mathcal{U}}(w_\varepsilon^*) \qquad \text{a.p.t.}\ \ (t,x) \in \Sigma,$$

where $r(t,x) = \dfrac{k}{\rho c} p^\varepsilon(t,x) + w(t,x)$, and thus we can conclude that

$$w_\varepsilon^*(t,x) = \begin{cases} R, & \text{if}\ \ r(t,x) > 0, \\ \\ 0, & \text{if}\ \ r(t,x) < 0. \end{cases} \tag{4.31}$$

Summing up, we have proved the following maximum principle for problem (P^ε)

Theorem 4.4.1. *Let $(u^{*,\varepsilon}, \varphi^{*,\varepsilon}, w_\varepsilon^*)$ be optimal in problem (P^ε). Then the optimal control is given by (4.31) where $(p^\varepsilon, q^\varepsilon)$ satisfy along with $u^{*,\varepsilon}, \varphi^{*,\varepsilon}$ the dual system (4.25)-(4.26).*

Chapter 5

An inverse problem governed by a phase-field transition system. Case 1D

5.1 An inverse problem

Denote by $\Omega = (0, b_1) \subset I\!R$, $0 < b_1 < +\infty$. Let $T > 0$ and (see Figure 5.1):

$$
\begin{aligned}
&Q_0 = \{(t,x) \in Q = (0,T) \times \Omega, \;\; \sigma(t) < x < b_1\},\\
&\Sigma_0 = \{(t,x) \in Q, \; t = \sigma^{-1}(x)\},\\
&\Sigma \;= (0,T) \times \{b_1\}.
\end{aligned}
$$

Consider the following nonlinear parabolic system in one space dimension:

$$
\begin{cases}
\rho c \; u_t + \frac{\ell}{2}\varphi_t = k u_{xx} & \text{in } \; Q_0,\\
\tau \varphi_t = \xi^2 \varphi_{xx} + \frac{1}{2a}(\varphi - \varphi^3) + 2u & \text{in } \; Q_0,
\end{cases}
\tag{5.1}
$$

subject to non-homogeneous Cauchy-Neumann boundary conditions:

$$
u_x + hu = w(t), \qquad \varphi_x = 0 \quad \text{on } \; \Sigma, \tag{5.2}
$$

$$
u_x = 0, \qquad \varphi_x = 0 \quad \text{on } \; \Sigma_0, \tag{5.3}
$$

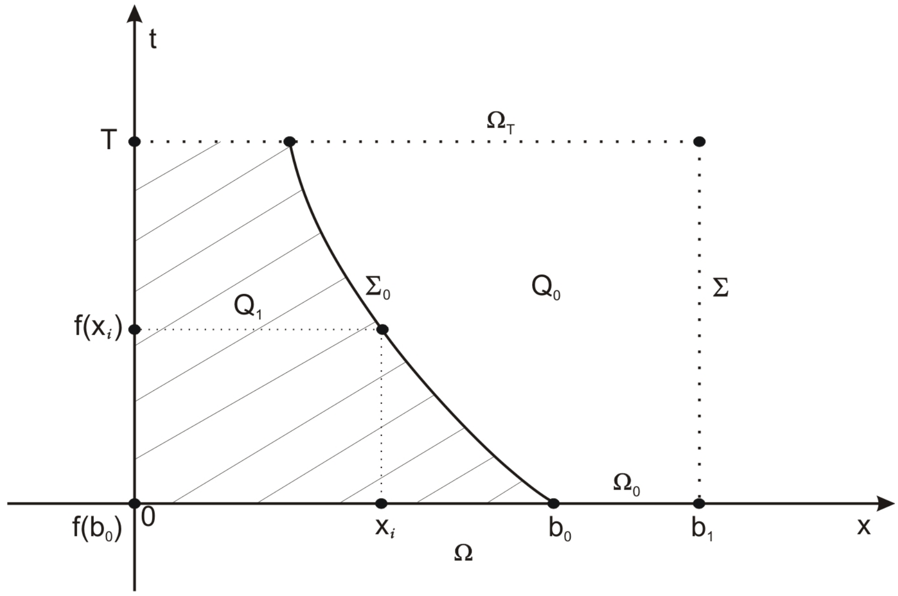

Figure 5.1: Geometrical image of the elements in inverse problem (P_{inv})

and initial conditions:

$$u(0,x) = u_0(x), \quad \varphi(0,x) = \varphi_0(x) \qquad \text{on} \;\; \Omega_0 = [b_0, b_1], \tag{5.4}$$

where, as usual, u is the *reduced temperature* distribution, φ is the *phase function* used to distinguish between the phase of Ω, u_0, $\varphi_0 : \Omega \to I\!R$ are given functions, $w : [0,T] \to I\!R$ is the *boundary control* (the temperature surrounding at $x = b_1$),

$$w \in \mathcal{U} := \{v \in L^\infty([0,T]), \quad -R \leq v(t) \leq 0 \;\; a.e. \;\; t \in [0,T]\},$$

and the positive parameters ρ, c, τ, ξ, ℓ, k, h, a, have a physical meaning (see page 18).

Assume that *separating region* between solid and liquid at the moment t is given by the equation $x = \sigma(t)$ (denoted by $t = \sigma^{-1}(x) = f(x)$) that is a function of class $C^2(\bar{\Omega}_t)$ such that (see Figure 5.1) $\Omega_t = \{x \in \Omega, \; f(x) < t\}$ is increasing in t, $|\nabla f(x)| \neq 0$ for all $x \in \Sigma_0$, $\Delta f(x) > 0$ and, $f(b_0) = 0, \; 0 < b_0 \leq b_1$.

As regards the existence, we proved in this work that under appropriate conditions on u_0, φ_0 and w, the state system (5.1)-(5.4) has a unique solution $u,\ \varphi \in W = W_p^{2,1}(Q) \cap L^\infty(Q)$.

Consider the following inverse problem:

(P_{inv}) *Given Σ_0 find the boundary control $w \in L^\infty([0,T])$ such that Q_0 is in the liquid region, $Q_1 = Q \setminus Q_0$ is in the solid region and a neighborhood of Σ_0 is the separating region between the liquid and the solid region.*

where we setted

$$Q_0 = \big\{(t,x) \in Q,\ f(x) \le t \le T\big\}.$$

This inverse problem is in general *"ill posed"* and a common way to treat it is to reformulate it as an optimal control problem with an appropriate cost functional. Consequently, we will concern in this section with an optimal control problem associated to the inverse problem (P_{inv}), namely:

(P) *Minimize*

$$\begin{aligned} j(w) &= \frac{1}{2}\int_Q (\varphi(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0}\ dtdx \\ &+ \frac{\beta}{2}\int_Q \big((u(t,x) - \delta_2)^+\big)^2 \cdot \chi_{Q_0}\ dtdx + \frac{1}{2}\int_0^T w^2(t)\ dt, \end{aligned}$$

on all (u,φ) solution of the system (5.1)-(5.4) and for all $w \in \mathcal{U}$. $\beta > 0$ is a given constant

In the statement above we denoted by u^+ the positive part of u, i.e.

$$u^+ = \begin{cases} u, & \text{if } \ u > 0, \\ 0, & \text{if } \ u < 0, \end{cases}$$

We point out that problem (P) is an optimal problem with boundary control $w(t)$ depending on time variable $t \in [0,T]$, we studied it, in previous chapter, in a more general case $(w(t,x),\ n \in \{1,\ 2,\ 3\})$. The study about this particular case is dictated by the industrial experiment that we want simulate in the last section of this chapter (*casting wire*).

We associate to the nonlinear system (5.1)-(5.4) the following approximating scheme ($\varepsilon > 0$):

$$\begin{cases} \rho c u_t^\varepsilon + \frac{\ell}{2}\varphi_t^\varepsilon = k u_{xx}^\varepsilon & \text{in } Q_0^\varepsilon = \big\{(t,x) \in Q, \quad \varepsilon \le t \le T\big\}, \\ \tau \varphi_t^\varepsilon = \xi^2 \varphi_{xx}^\varepsilon + \frac{1}{2a}\varphi^\varepsilon + 2u^\varepsilon & \text{in } Q_0^\varepsilon, \end{cases}, \tag{5.5}$$

under the boundary and initial conditions:

$$u_x^\varepsilon + h u^\varepsilon = w(t), \quad \varphi_x^\varepsilon = 0 \quad \text{on } \Sigma^\varepsilon = [\varepsilon, T] \times \{b_1\}, \tag{5.6}$$

$$u_x^\varepsilon = 0, \qquad \varphi_x^\varepsilon = 0 \quad \text{on } \Sigma_0^\varepsilon = \{(t,x) \in Q, \ \varepsilon \le t \le T\}, \tag{5.7}$$

$$u^\varepsilon(\varepsilon, x) = u_0(x) \qquad \varphi_+^\varepsilon(\varepsilon, x) = z(\varepsilon, \varphi_-^\varepsilon(\varepsilon, x)) \qquad \text{on } \Omega_\varepsilon. \tag{5.8}$$

where $z(\varepsilon, \varphi_-^\varepsilon(\varepsilon, x))$ is the solution of the Cauchy problem:

$$\begin{cases} z'(s) + \dfrac{1}{2a} z^3(s) = 0 & s \in (0, \varepsilon), \\ z(0) = \varphi_-^\varepsilon(\varepsilon, x) \qquad \varphi_-^\varepsilon(0, x) = \varphi_0(x), \end{cases} \tag{5.9}$$

and $\varphi_+^\varepsilon(\varepsilon, x) = \lim_{t \downarrow \varepsilon} \varphi^\varepsilon(t, x), \quad \varphi_-^\varepsilon(\varepsilon, x) = \lim_{t \uparrow \varepsilon} \varphi^\varepsilon(t, x)$.

The convergence and weak stability of the approximating scheme (5.5)-(5.4), in a more general case ($w(t,x)$ in place of $w(t)$), was studied in the third chapter.

Corresponding to the approximating scheme (5.5)-(5.4), we will consider the approximating optimal control problem:

$$(P^\varepsilon) \quad \textit{Minimize} \quad L_0^\varepsilon(w) = \frac{\beta}{2} \int_Q \left[(u^\varepsilon(t,x) - \delta_2)^+\right]^2 \cdot \chi_{Q_0} \, dt dx$$

$$+ \frac{1}{2} \int_Q (\varphi^\varepsilon(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0} \, dt dx + \frac{1}{2} \int_0^T w^2(t) \, dt,$$

on all $(u^\varepsilon, \varphi^\varepsilon)$ *solution of (5.5)-(5.4) corresponding to* $w \in \mathcal{U}$.

As in the previous chapter, problem (P) can be approximated for $\varepsilon \to 0$ by the sequence of optimal control problems (P^ε) and so the computation of the approximate boundary control $w(t)$ can be substituted by computation of an approximate control of (P^ε).

5.2 The convergence of problem (P^ε)

Theorem 5.2.1. *et $\{w_\varepsilon^*\}$ be a sequence of optimal controllers for problem (P^ε). Then*

$$\lim_{\varepsilon\to 0} \ \inf \ L_0^\varepsilon(w) = \inf \quad \{L_0(w); \quad w \in \mathcal{U}\} \tag{5.10}$$

and

$$\lim_{\varepsilon\to 0} L_0(w_\varepsilon^*) = \inf \quad \{L_0(w); \quad w \in \mathcal{U}\}. \tag{5.11}$$

Moreover, every weak limit point of $\{w_\varepsilon^*\}$ is an optimal controller for problem (P).

Remark. Theorem 5.2.1 amounts to saying that (P^ε) approximates problem (P) and, an optimal controller $\{w_\varepsilon^*\}$ of (P^ε) is a suboptimal controller for problem (P).

The main ingredient in the proof of the Theorem 5.2.1 is the following Lemma.

Lemma 5.2.2. *If $\{w_\varepsilon^*\}$ is a sequence of optimal controllers for problems (P^ε) then there exists $\{\varepsilon_n\} \to 0$ such that*

$$\begin{aligned} w_{\varepsilon_n}^* &\to w^* \quad \textit{weak star in} \quad L^\infty(\Sigma), && (5.12)\\ u_{\varepsilon_n}^* &\to u^* \quad \textit{strongly in} \quad L^2((0,T); H^1(\Omega)), && (5.13)\\ \varphi_{\varepsilon_n}^* &\to \varphi^* \quad \textit{strongly in} \quad L^2((0,T); H^1(\Omega)), && (5.14) \end{aligned}$$

where $(u_{\varepsilon_n}^*, \varphi_{\varepsilon_n}^*, w_{\varepsilon_n}^*) = (u_{\varepsilon_n}^{w_{\varepsilon_n}^*}, \varphi_{\varepsilon_n}^{w_{\varepsilon_n}^*}, w_{\varepsilon_n}^*)$ is the solution to (5.5)-(5.8) corresponding to $w = w_{\varepsilon_n}^*$ and $(u^*, \varphi^*, w^*) = (u^{w^*}, \varphi^{w^*}, w^*)$ is the solution to (5.1)-(5.4) corresponding to $w = w^*$.

Proof. of Theorem 5.2.1. Let $\{w_\varepsilon^*\}$ be an optimal controller for problem (P^ε) and let $(u_\varepsilon^*, \varphi_\varepsilon^*, w_\varepsilon^*)$ be the corresponding solution of (5.5)-(5.8) with $w = w_\varepsilon^*$. Lemma 5.2.2 above allows us to conclude that there exist $w^* \in L^\infty([0,T])$ and $\{\varepsilon_n\}$ such that relations (5.12)-(5.14) are valid.

Since:

$$u \to \frac{\beta}{2} \int\limits_Q \left((u(t,x) - \delta_2)^+\right)^2 \cdot \chi_{Q_0}\ dtdx,$$

$$\varphi \to \frac{1}{2} \int\limits_Q (\varphi(t,x) - 1 - \delta_1)^2 \cdot \chi_{Q_0}\ dtdx,$$

$$w \to \frac{1}{2} \int\limits_0^T w^2(t)\ dt$$

are convex continuous functions, it follows that these are weakly lower semi-continuous functions. Hence

$$L_0(w^*) \le \liminf_{n\to\infty} L_0^{\varepsilon_n}(w_{\varepsilon_n}^*). \tag{5.15}$$

Let $\bar{w}^*$ be an optimal controller for problem (P). Since $w_{\varepsilon_n}^*$ is an optimal controller for problem (P^{ε_n}) it follows that

$$L_0^{\varepsilon_n}(w_{\varepsilon_n}^*) \le L_0(\bar{w}^\varepsilon).$$

But (see (5.13) and (5.14)) $u_{\varepsilon_n}^{\bar{w}^*} \to u^{\bar{w}^*}$ $\varphi_{\varepsilon_n}^{\bar{w}^*} \to \varphi^{\bar{w}^*}$ strongly in $L^2((0,T);H^1(\Omega))$ and so, the latter inequalities implies

$$\lim_{n\to\infty} L_0^{\varepsilon_n}(\bar{w}^\varepsilon) \le L_0(\bar{w}^\varepsilon). \tag{5.16}$$

From (5.15)-(5.16) we get

$$L_0(w^*) \le \liminf_{n\to\infty} L_0^{\varepsilon_n}(w_{\varepsilon_n}^*) \le \limsup_{n\to\infty} L_0^{\varepsilon_n}(w_{\varepsilon_n}^*) \le L_0(\bar{w}^*).$$

Hence

$$\liminf_{\varepsilon_n\to 0} L_0^{\varepsilon_n}(w_{\varepsilon_n}^*) = L_0(\bar{w}^*) = \inf\{L_0(w),\ w \in \mathcal{U}\}$$

and then (5.10) holds.

To prove (5.11) we set: $\bar{u}_{\varepsilon} = u^{w_{\varepsilon}^*}$, $\bar{\varphi}_{\varepsilon} = \varphi^{w_{\varepsilon}^*}$ (we recall that w_{ε}^* is choose to be optimal in (P^{ε})). On a subsequence $\{\varepsilon_n\}$ we have

$$\begin{array}{lll} w_{\varepsilon}^* \to w^0 & \text{weak star in} & L^{\infty}([0,T]), \\ \bar{u}_{\varepsilon_n} \to u & \text{strongly in} & L^2((0,T),H^1(\Omega)), \\ \bar{\varphi}_{\varepsilon_n} \to \varphi & \text{strongly in} & L^2((0,T),H^1(\Omega)), \end{array}$$

where (u,φ,w^0) satisfy (5.1)-(5.4), i.e., $(u,\varphi) = (u^{w^0},\varphi^{w^0})$. Therefore, we derive

$$L_0(w^0) \leq \inf P$$

and, because $\{\varepsilon_n\}$ was choose arbitrarily, (5.11) follows.

Now, taking into account that w_{ε}^* is an optimal controller for problem (P^{ε}), it follows that

$$L_0^{\varepsilon}(w_{\varepsilon}^*) \leq L_0^{\varepsilon}(w) \quad \forall\, w \in \mathcal{U}.$$

On the other part, on the basis of relation (5.15), we can put

$$L_0(w^*) \leq \liminf_{\varepsilon\to 0} L_0^{\varepsilon}(w_{\varepsilon}^*)$$

and thus, along with previous inequality, we may conclude that

$$L_0(w^*) \leq \lim_{\varepsilon\to 0} L_0^{\varepsilon}(w) \quad \forall\, w \in \mathcal{U}.$$

Consequently

$$L_0(w^*) \leq L_0(w) \quad \forall\, w \in \mathcal{U}$$

i.e., the weak limit point w^* is a suboptimal controller for problem (P). This completes the proof of Theorem 5.2.1. □

5.3 Necessary optimality conditions in (P^{ε})

Let $(u^{\varepsilon},\varphi^{e},w)$ be the solution of (5.5)-(5.8) and let $\tilde{w} \in L^{\infty}[0,T])$ be arbitrary but fixed and $\lambda > 0$. Set $w^{\lambda} = w + \lambda\tilde{w}$ and let $(u^{\lambda,\varepsilon},\varphi^{\lambda,\varepsilon})$ be the solution of (5.5)-(5.8) corresponding to w^{λ}, that is:

$$\begin{cases} \rho c u_t^{\lambda,\varepsilon} + \frac{\ell}{2}\varphi_t^{\lambda,\varepsilon} = k u_{xx}^{\lambda,\varepsilon} & \text{in } Q_0^{\varepsilon}, \\ \tau\varphi_t^{\lambda,\varepsilon} = \xi^2\varphi_{xx}^{\lambda,\varepsilon} + \frac{1}{2a}\varphi^{\lambda,\varepsilon} + 2u^{\lambda,\varepsilon} & \text{in } Q_0^{\varepsilon}, \end{cases} \tag{5.17}$$

subject to non-homogeneous Cauchy-Neumann boundary conditions:

$$u_x^{\lambda,\varepsilon} + hu^{\lambda,\varepsilon} = w^\lambda, \qquad \varphi_x^{\lambda,\varepsilon} = 0 \qquad \text{on } \Sigma^\varepsilon, \tag{5.18}$$

$$u_x^{\lambda,\varepsilon} = 0, \qquad \varphi_x^{\lambda,\varepsilon} = 0 \qquad \text{on } \Sigma_0^\varepsilon, \tag{5.19}$$

and initial conditions:

$$u^{\lambda,\varepsilon}(\varepsilon, x) = u_0(x), \quad \varphi_+^{\lambda,\varepsilon}(\varepsilon, x) = z^\lambda(\varepsilon, \varphi_0(x)) \qquad \text{on } \Omega_\varepsilon, \tag{5.20}$$

where $z^\lambda(\varepsilon, \varphi_0(x))$ is the solution of the Cauchy problem:

$$\begin{cases} \big(z^\lambda(s)\big)' + \dfrac{1}{2a}\big(z^\lambda(s)\big)^3 = 0 & s \in (0, \varepsilon), \\ \\ z^\lambda(0) = \tilde{\varphi}_-^{\lambda,\varepsilon}(\varepsilon, x) & \tilde{\varphi}_-^{\lambda,\varepsilon}(0, x) = \varphi_0(x). \end{cases} \tag{5.21}$$

Subtracting (5.5)-(5.8) from (5.17)-(5.20) and dividing by $\lambda > 0$, we get

$$\begin{cases} \rho c\left(\frac{u^{\lambda,\varepsilon}-u^\varepsilon}{\lambda}\right)_t + \frac{\ell}{2}\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_t = k\left(\frac{u^{\lambda,\varepsilon}-u^\varepsilon}{\lambda}\right)_{xx} & \text{in } Q_0^\varepsilon, \\ \\ \tau\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_t = \xi^2\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_{xx} & \\ \\ \qquad + \frac{1}{2a}\left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right) + 2\left(\frac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right) & \text{in } Q_0^\varepsilon, \end{cases} \tag{5.22}$$

with the boundary conditions on Σ^ε

$$\begin{cases} \left(\frac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right)_x + h\left(\frac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right) = \frac{w^\lambda - w}{\lambda} & \text{on } \Sigma^\varepsilon, \\ \left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_x = 0 & \text{on } \Sigma^\varepsilon, \end{cases} \tag{5.23}$$

on Σ_0^ε

$$\begin{cases} \left(\frac{u^{\lambda,\varepsilon}-u^e}{\lambda}\right)_x = 0 & \text{on } \Sigma_0^\varepsilon, \\ \left(\frac{\varphi^{\lambda,\varepsilon}-\varphi^e}{\lambda}\right)_x = 0 & \text{on } \Sigma_0^\varepsilon, \end{cases} \tag{5.24}$$

and the initial conditions

$$\begin{cases} \frac{u^{\lambda,\varepsilon}(\varepsilon,x)-u^\varepsilon(\varepsilon,x)}{\lambda} = 0 & \text{on } \Omega_\varepsilon, \\ \frac{\varphi_+^{\lambda,\varepsilon}(\varepsilon,x)-\varphi_+^e(\varepsilon,x)}{\lambda} = \frac{z^\lambda(\varepsilon,\varphi_0(x))-z(\varepsilon,\varphi_0(x))}{\lambda} & \text{on } \Omega_\varepsilon. \end{cases} \tag{5.25}$$

Letting λ tend to zero in (5.22)-(5.25) we get the system in variation (5.26)-(5.29) below

$$\begin{cases} \rho c\tilde{u}^\varepsilon_t + \frac{\ell}{2}\tilde{\varphi}^\varepsilon_t = k\tilde{u}^\varepsilon_{xx} & \text{in } Q^\varepsilon_0, \\ \\ \tau\tilde{\varphi}^\varepsilon_t = \xi^2\tilde{\varphi}^\varepsilon_{xx} + \frac{1}{2a}\tilde{\varphi}^\varepsilon + 2\tilde{u}^\varepsilon & \text{in } Q^\varepsilon_0, \end{cases} \tag{5.26}$$

$$\begin{cases} \tilde{u}^\varepsilon_x + h\tilde{u}^\varepsilon = \tilde{w} & \text{on } \Sigma^\varepsilon, \\ \tilde{\varphi}^\varepsilon_x = 0 & \text{on } \Sigma^\varepsilon, \end{cases} \tag{5.27}$$

$$\begin{cases} \tilde{u}^\varepsilon_x = 0 & \text{on } \Sigma^\varepsilon_0, \\ \tilde{\varphi}^\varepsilon_x = 0 & \text{on } \Sigma^\varepsilon_0, \end{cases} \tag{5.28}$$

$$\begin{cases} \tilde{u}^\varepsilon(\varepsilon, x) = 0 & \text{on } \Omega_\varepsilon \\ \tilde{\varphi}^\varepsilon_+(\varepsilon, x) = \eta(\varepsilon, x) & \text{on } \Omega_\varepsilon. \end{cases} \tag{5.29}$$

where $\tilde{u}^\varepsilon = \lim\limits_{\lambda\to 0} \dfrac{u^{\lambda,\varepsilon} - u^\varepsilon}{\lambda}$, etc., and

$$\begin{aligned} \eta(\varepsilon, x) &= \lim_{\lambda\to 0} \frac{z^\lambda(\varepsilon, \varphi_0(x)) - z(\varepsilon, \varphi_0(x))}{\lambda} = \\ &= z^\lambda(\varepsilon, \varphi_0(x)) \cdot \tilde{\varphi}^\varepsilon_-(\varepsilon, x) + \tilde{z}(\varepsilon, \varphi_0(x)) = \tilde{z}(\varepsilon, \varphi_0(x)) \end{aligned}$$

with $\eta(\cdot, x)$ the solution of Cauchy problem

$$\begin{cases} \eta'(s, x) + \dfrac{3}{2a} z^2(s, x)\eta(s, x) = 0 & s \in (0, \varepsilon), \\ \eta(0, x) = \tilde{\varphi}^\varepsilon_-(\varepsilon, x), \end{cases} \tag{5.30}$$

that is

$$\eta(s, x) = \exp\Big(-\int_0^\varepsilon \frac{3}{2a} z(t, \cdot)^2 dt\Big)\tilde{\varphi}^\varepsilon_-(\varepsilon, x). \tag{5.31}$$

We now introduce the adjoint state system. For this, the system (5.26) can be written in the form:

$$\begin{pmatrix} \tilde{u}^\varepsilon \\ \tilde{\varphi}^\varepsilon \end{pmatrix}_t = A\begin{pmatrix} \tilde{u}^\varepsilon \\ \tilde{\varphi}^\varepsilon \end{pmatrix} \qquad \text{in } Q^\varepsilon_0$$

where

$$A=\begin{pmatrix} \frac{1}{\rho c}\Big(k\Delta-\dfrac{\ell}{\tau}\Big) & -\dfrac{\ell}{2\tau\rho c}\Big(\xi^2\Delta+\dfrac{1}{2a}\Big) \\ \\ \dfrac{2}{\tau} & \dfrac{1}{\tau}\Big(\xi^2\Delta+\dfrac{1}{2a}\Big) \end{pmatrix},$$

(here $\Delta\varphi=\varphi_{xx}$). Then

$$A^*=\begin{pmatrix} \frac{1}{\rho c}\Big(k\Delta-\dfrac{\ell}{\tau}\Big) & \dfrac{2}{\tau} \\ \\ -\dfrac{\ell}{2\tau\rho c}\Big(\xi^2\Delta+\dfrac{1}{2a}\Big) & \dfrac{1}{\tau}\Big(\xi^2\Delta+\dfrac{1}{2a}\Big) \end{pmatrix}$$

and thus, the adjoint state system is

$$\begin{pmatrix} p^\varepsilon \\ q^\varepsilon \end{pmatrix}_t=-A^*\begin{pmatrix} p^\varepsilon \\ q^\varepsilon \end{pmatrix}+\begin{pmatrix} \frac{\partial}{\partial u^\varepsilon}L_0^\varepsilon(w) \\ \frac{\partial}{\partial \varphi^\varepsilon}L_0^\varepsilon(w) \end{pmatrix}$$

i.e.

$$\begin{cases} p_t^\varepsilon+\dfrac{k}{\rho c}p_{xx}^\varepsilon-\dfrac{\ell}{\tau\rho c}p^\varepsilon+\dfrac{2}{\tau}q^\varepsilon=\beta(u^\varepsilon-\delta_2)^+\cdot\chi_{Q_0}, & \text{in } Q_0^\varepsilon, \\ p_x^\varepsilon+hp^\varepsilon=0, & \text{on } \Sigma^\varepsilon, \\ p_-^\varepsilon(\varepsilon,x)=0, & x\in\Omega_T, \\ p_-^\varepsilon(T,x)=0, & x\in\Omega_T, \end{cases} \tag{5.32}$$

$$\begin{cases} q_t^\varepsilon-\dfrac{\ell\xi^2}{2\tau\rho c}p_{xx}^\varepsilon-\dfrac{\ell}{4a\tau\rho c}p^\varepsilon+\dfrac{\xi^2}{\tau}q_{xx}^\varepsilon \\ \qquad\qquad +\dfrac{1}{2a\tau}q^\varepsilon=(\varphi^\varepsilon-1-\delta_1)\cdot\chi_{Q_0}, & \text{in } Q_0^\varepsilon, \\ q_x^\varepsilon=\dfrac{\ell}{2\rho c}p_x^\varepsilon, & \text{on } \Sigma^\varepsilon, \\ q_x^\varepsilon=0, & \text{on } \Sigma_0^\varepsilon, \\ q_-^\varepsilon(\varepsilon,x)=\exp\Big(\int\limits_0^\varepsilon \frac{3}{2a}z^2(t,\cdot)dt\Big)q_+^\varepsilon(\varepsilon,x), \\ \qquad q_-^\varepsilon(T,x)=0, & x\in\Omega_T, \end{cases} \tag{5.33}$$

Let us introduce the cost functional

$$L_1^\varepsilon(w)=L_0^\varepsilon(w)+\frac{1}{2}I_{\mathcal{U}}(w)$$

where, as usually, $I_{\mathcal{U}}(w)$ is the indicator function of the set $\mathcal{U}$.

If w^* is an optimal controller of problem (P^ε), then

$$\frac{L_1^\varepsilon(w^*+\lambda\tilde{w})-L_1^\varepsilon(w^*)}{\lambda}\geq 0 \quad \forall\,\lambda>0.$$

that leads to (letting λ tend to zero)

$$\beta\int\limits_Q \tilde{u}^\varepsilon\,(u^\varepsilon - \delta_2)^+\cdot\chi_{Q_0}\,dtdx+\int\limits_Q \tilde{\varphi}^\varepsilon\,(\varphi^\varepsilon-1-\delta_1)\cdot\chi_{Q_0}\,dtdx \tag{5.34}$$
$$+\int\limits_0^T w^*\,\tilde{w}\,dt+I'_{\mathcal{U}}(w^*,\tilde{w})\geq 0 \quad \forall\,\tilde{w}\in T_{\mathcal{U}}(w^*).$$

Multiplying $(5.32)_1$ by $\tilde{u}^\varepsilon$ and $(5.33)_1$ by $\tilde{\varphi}^\varepsilon$, using integration by parts and Green's formula, we get

$$\begin{aligned}\int\limits_{Q_0^\varepsilon} p_t^\varepsilon\tilde{u}^\varepsilon dt\,dx &+ \frac{k}{\rho c}\int\limits_{Q_0^\varepsilon} p^\varepsilon\tilde{u}^\varepsilon_{xx}dt\,dx-\frac{\ell}{\tau\rho c}\int\limits_{Q_0^\varepsilon} p^\varepsilon\tilde{u}^\varepsilon dt\,dx \\ &+ \frac{2}{\tau}\int\limits_{Q_0^\varepsilon} q^\varepsilon\tilde{u}^\varepsilon dt\,dx+\frac{k}{\rho c}\int\limits_{\Sigma^\varepsilon}\Big(p^\varepsilon\tilde{u}^\varepsilon_x-p^\varepsilon_x\tilde{u}^\varepsilon\Big)dt\,d\gamma \\ &= \beta\int\limits_{Q_0^\varepsilon}(u^\varepsilon-\delta_2)^+\cdot\chi_{Q_0}\,\tilde{u}^\varepsilon\,dtdx,\end{aligned} \tag{5.35}$$

$$\begin{aligned}\int\limits_{Q_0^\varepsilon} q_t^\varepsilon\tilde{\varphi}^\varepsilon dt\,dx &- \frac{\ell\xi^2}{2\tau\rho c}\int\limits_{Q_0^\varepsilon} p^\varepsilon\tilde{\varphi}^\varepsilon_{xx}dt\,dx-\frac{\ell}{4a\tau\rho c}\int\limits_{Q_0^\varepsilon} p^\varepsilon\tilde{\varphi}^\varepsilon dt\,dx \\ &+ \frac{\ell\xi^2}{2\tau\rho c}\int\limits_{\Sigma^\varepsilon}\Big(p^\varepsilon_x\tilde{\varphi}^\varepsilon-p^\varepsilon\tilde{\varphi}^\varepsilon_x\Big)dt\,d\gamma+\frac{\xi^2}{\tau}\int\limits_{\Sigma^\varepsilon}\Big(q^\varepsilon\tilde{\varphi}^\varepsilon_x-\tilde{\varphi}^\varepsilon q^\varepsilon_x\Big)\,dtd\gamma \\ &+ \frac{\xi^2}{\tau}\int\limits_{Q_0^\varepsilon} q^\varepsilon\tilde{\varphi}^\varepsilon_{xx}dt\,dx+\frac{1}{2a\tau}\int\limits_{Q_0^\varepsilon} q^\varepsilon\tilde{\varphi}^\varepsilon dt\,dx \\ &= \int\limits_{Q_0^\varepsilon}(\varphi^\varepsilon-1-\delta_1)\cdot\chi_{Q_0}\,\tilde{\varphi}^\varepsilon\,dtdx.\end{aligned} \tag{5.36}$$

Now we multiply (5.27) by p^ε, $(5.32)_2$ by $\tilde{u}^\varepsilon$, by subtraction we get

$$p^\varepsilon_x \tilde{u}^\varepsilon - \tilde{u}^\varepsilon_x p^\varepsilon = -p^\varepsilon \tilde{w}. \tag{5.37}$$

Adding (5.35)-(5.36) and taking into account $(5.27)_2$, $(5.33)_2$, (5.37), we obtain

$$\begin{aligned}
\int\limits_{Q_0^\varepsilon} p^\varepsilon_t \tilde{u}^\varepsilon dt\ dx \quad &+ \quad \int\limits_{Q_0^\varepsilon} q^\varepsilon_t \tilde{\varphi}^\varepsilon dt\ dx + \frac{k}{\rho c}\int\limits_{\Sigma^\varepsilon} p^\varepsilon \tilde{w} dt\ d\gamma \\
&+ \quad \int\limits_{Q_0^\varepsilon} p^\varepsilon \Big[\frac{k}{\rho c}\tilde{u}^\varepsilon_{xx} - \frac{\ell\xi^2}{2\tau\rho c}\tilde{\varphi}^\varepsilon_{xx} - \frac{\ell}{4a\tau\rho c}\tilde{\varphi}^\varepsilon - \frac{\ell}{\tau\rho c}\tilde{u}^\varepsilon\Big] dt\ dx \\
&+ \quad \int\limits_{Q_0^\varepsilon} q^\varepsilon \Big[\frac{\xi^2}{\tau}\tilde{\varphi}^\varepsilon_{xx} + \frac{1}{2a\tau}\tilde{\varphi}^\varepsilon + \frac{2}{\tau}\tilde{u}^\varepsilon\Big] dt \\
&= \quad \beta \int\limits_{Q_0^\varepsilon} (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx \\
&+ \quad \int\limits_{Q_0^\varepsilon} (\varphi^\varepsilon - 1 - \delta_1) \cdot \chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx,
\end{aligned}$$

i.e., making use of equations in (5.26), the last relation leads to

$$\int\limits_{Q_0^\varepsilon} \big(p^\varepsilon_t \tilde{u}^\varepsilon + p^\varepsilon \tilde{u}^\varepsilon_t + q^\varepsilon_t \tilde{\varphi}^\varepsilon + q^\varepsilon \tilde{\varphi}^\varepsilon_t\big) dt\ dx + \frac{k}{\rho c}\int\limits_{\Sigma^\varepsilon} p^\varepsilon \tilde{w} dt\ d\gamma =$$

$$= \beta \int\limits_{Q_0^\varepsilon} (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx + \int\limits_{Q_0^\varepsilon} (\varphi^\varepsilon - 1 - \delta_1) \cdot \chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx,$$

By Fubini's theorem and definition of distributional derivative, the latter relation give us

$$\frac{k}{\rho c}\int\limits_{\Sigma^\varepsilon} p^\varepsilon \tilde{w} dt\ d\gamma = \beta \int\limits_{Q_0^\varepsilon} (u^\varepsilon - \delta_2)^+ \cdot \chi_{Q_0}\ \tilde{u}^\varepsilon\ dtdx + \int\limits_{Q_0^\varepsilon} (\varphi^\varepsilon - 1 - \delta_1) \cdot \chi_{Q_0}\ \tilde{\varphi}^\varepsilon\ dtdx,$$

and then (5.34) becomes

$$\frac{k}{\rho c}\int\limits_{\Sigma^\varepsilon} p^\varepsilon \tilde{w}\ dtd\gamma + \int\limits_0^T w^* \tilde{w}\ dt + I'_{\mathcal{U}}(w^*, \tilde{w}) \geq 0 \qquad \forall\, \tilde{w} \in T_{\mathcal{U}}(w^*)$$

or

$$\int_0^T \big[\frac{k}{\rho c} p^\varepsilon(s,b_1) + w^*(s)\big]\tilde{w}(s)\ ds + I'_{\mathcal{U}}(w^*,\tilde{w}) \geq 0 \qquad \forall\, \tilde{w} \in T_{\mathcal{U}}(w^*).$$

The last inequality is equivalent to

$$-r(t) \in \partial I_{\mathcal{U}}(w^*) \qquad \text{a.p.t.} \quad (t,x) \in [0,T],$$

where $r(t) = \frac{k}{\rho c} p^\varepsilon(t,b_1) + w(t)$, and thus we can conclude that

$$w^*(t) = \begin{cases} 0, & \text{if } \ r(t) > 0 \\ -R, & \text{if } \ r(t) < 0. \end{cases} \tag{5.38}$$

Summing up, we have proved the following maximum principle for problem (P^ε)

Theorem 5.3.1. *Let $(u^{*,\varepsilon}, \varphi^{*,\varepsilon}, w^*)$ be optimal in problem (P^ε). Then the optimal control is given by (5.38) where $(p^\varepsilon, q^\varepsilon)$ satisfy along with $u^{*,\varepsilon}, \varphi^{*,\varepsilon}$ the dual system (5.32)-(5.33).*

Now we will present a numerical algorithm of gradient type in order to compute the approximating optimal control stated by Theorem 5.3.1.

Algorithm**InvPHT1D** (**Inv**ers **PH**ase **T**ransition case **1D**)

P0. Choose $w^{(0)} \in \mathcal{U}$ and set iter$=0$; Choose $\varepsilon > 0$;

P1. Compute $z(\varepsilon,\cdot)$ from (5.9);

P2. Compute $(u^{\varepsilon,iter}, \varphi^{\varepsilon,iter})$ from (5.5)-(5.8);

P3. Compute $(p^{\varepsilon,iter}, q^{\varepsilon,iter})$ from (5.32)-(5.33);

P4. For $t \in [0,T]$, compute

$$r^{iter}(t) = \frac{k}{\rho c} \cdot p^{\varepsilon,iter}(t,b_1) + w^{iter};$$

P5. Set

$$\tilde{w}^{iter}(t) = \begin{cases} R & \text{if } \; r^{iter}(t) > 0, \\ \\ 0 & \text{if } \; r^{iter}(t) < 0. \end{cases}$$

P6. Compute $\lambda_{iter} \in [0,1]$ solution of the minimization process:

$$\min \{L_0^\varepsilon(\lambda w^{iter} + (1-\lambda)\tilde{w}^{iter}, \quad \lambda \in [0,1];$$

Set $w^{iter+1} = \lambda_{iter} w^{iter} + (1-\lambda_{iter})\tilde{w}^{iter}$;

P7. If $\| \; w^{iter+1} - w^{iter} \; \| \leq \eta$ /* the stopping criterion */ then STOP else iter:= iter+1; Go to **P1.**

The "stopping criterion" in **P7.** may be replaced with the following one:

$$\| \; L_0^\varepsilon(w^{iter+1}) - L_0^\varepsilon(w^{iter}) \; \| \leq \eta$$

where η is a prescribed precision.

5.4 Numerical experiments

Given the values ε, T ($\varepsilon << T$) and considering M as the number of equidistant nodes in which is divided $[\varepsilon, T]$, we set

$$t_i = \varepsilon + (i-1) \cdot dt \quad i = 1, 2, \ldots, M, \quad dt = (T-\varepsilon)/(M-1).$$

Let $b_0 = \sqrt{T} = \sqrt{t_M}$ and choose $jb_0 = M+1$. We consider now $\Omega_0 = [b_0, b_1] \subset I\!R^+$ ($b_1 >> T$) and we introduce over it the grid with N equidistant nodes

$$x_j = b_0 + (j-1) \cdot dx \quad j = 1, 2, \ldots, N, \quad dx = (b_1 - b_0)/(N-1).$$

On the subinterval $[0, b_0)$ we take the set of nodes obtained as follows

$$y_k = b_0 - \sqrt{t_i}, \quad k = M - i + 1, \;\; i = 1, 2, \ldots, M.$$

Thus, on the entire domain $\Omega = [0, b_1]$ we will consider in our numerical experiments the following set of points:

$$[y_1, y_2, \ldots, y_M, x_1 = x_{jb_0}, x_2, \ldots, x_N],$$

i.e.

$$[0 = b_0 - \sqrt{t_M}, b_0 - \sqrt{t_{M-1}}, \ldots, b_0 - \sqrt{\varepsilon}, b_0, b_0 + dx, \ldots, b_1].$$

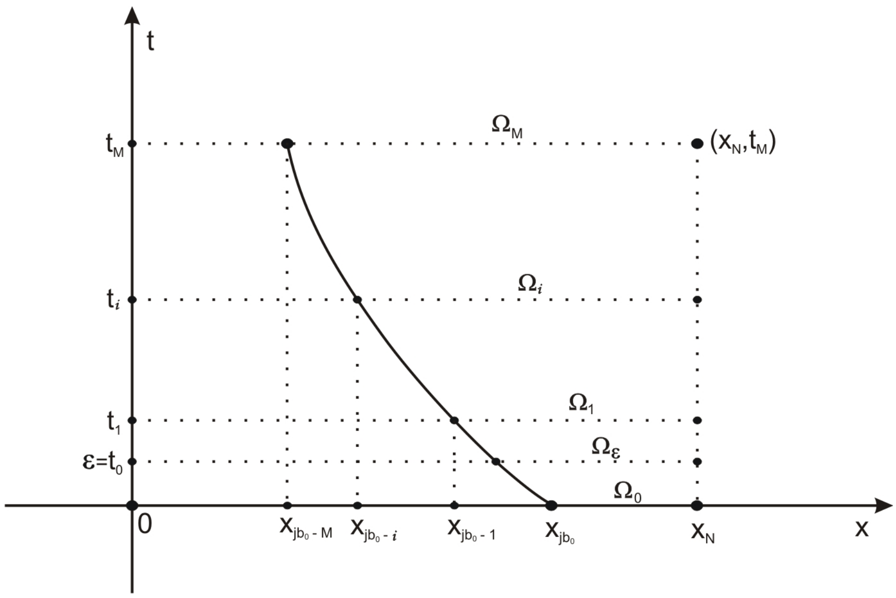

Figure 5.2: The discrete form of Q_0, Σ_0, and Σ in (P_{inv})

The Figure 5.2 show the discrete form of Q_0, Σ_0 and Σ, originally introduced in Figure 5.1.

For each time-level i, the mesh's of Ω_{t_i}, $i = \overline{1, M}$, consists in $(\Omega_{t_1} = \Omega_\varepsilon)$: and then the unknowns vectors, denoted by $\bar{u}^i$, $\bar{\varphi}^i$, are

$$\bar{u}^i = (u^i_{jb_0-i}, u^i_{jb_0-i+1}, \ldots, u^i_{jb_0}, u^i_{jb_0+1}, \ldots, u^i_N)$$

$$\bar{\varphi}^i = (\varphi^i_{jb_0-i}, \varphi^i_{jb_0-i+1}, \ldots, \varphi^i_{jb_0}, \varphi^i_{jb_0+1}, \ldots, \varphi^i_N).$$

The state $(u^\varepsilon, \varphi^\varepsilon)$ in (5.5)-(5.8) will be approximated by the matrix $\begin{pmatrix} u^i_j & \varphi^i_j \end{pmatrix}^*$, where

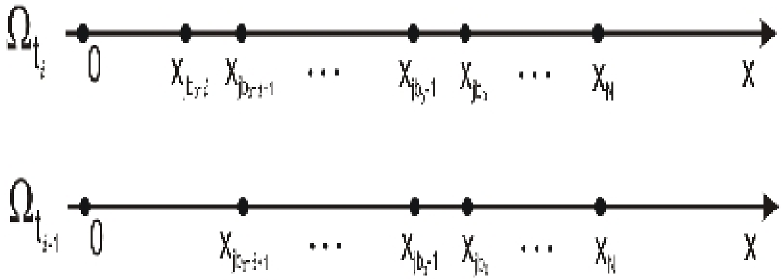

Figure 5.3: The mesh Ω_{t_i}

$$\begin{aligned} u_j^i &= u^\varepsilon(t_i, x_j) \\ \varphi_j^i &= \varphi^\varepsilon(t_i, x_j) \end{aligned} \qquad i = \overline{1, M},\ j = \overline{jb_0 - i, N}.$$

Using a standard implicit scheme, the system (5.5) is discretized as

$$\begin{aligned} -k\frac{dt}{(dx)^2} \cdot u_{j+1}^i \quad &+ \quad (\rho c + 2k\frac{dt}{(dx)^2}) \cdot u_j^i - k\frac{dt}{(dx)^2} \cdot u_{j-1}^i + \frac{\ell}{2}\varphi_j^i \qquad (5.39) \\ &= \quad \rho c \cdot u_j^{i-1} + \frac{\ell}{2} \cdot \varphi_j^{i-1} \quad i = \overline{1, M}, \quad j = \overline{jb_0 - i, N}, \end{aligned}$$

$$\begin{aligned} -2dt \cdot u_j^i \quad &- \quad \xi^2\frac{dt}{(dx)^2} \cdot \varphi_{j+1}^i + (\tau + 2\xi^2\frac{dt}{(dx)^2} - \frac{dt}{2a}) \cdot \varphi_j^i \qquad (5.40) \\ &- \quad \xi^2\frac{dt}{(dx)^2} \cdot \varphi_{j-1}^i = \tau \cdot \varphi_j^{i-1} \quad i = \overline{1, M}, \quad j = \overline{jb_0 - i, N}. \end{aligned}$$

From (5.6)-(5.3), using central differences and taking $u_{N+1}^i = u_N^i$, we obtain, for $i = \overline{1, M}$:

$$u_{N-1}^i = (1 + 2 \cdot dx \cdot h)u_N^i - 2 \cdot dx \cdot w^i \qquad (\text{on } \Sigma^\varepsilon), \qquad (5.41)$$

$$\varphi_{N+1}^i = \varphi_N^i, \quad \varphi_{jb_0-i-1}^i = \varphi_{jb_0-i}^i \qquad (\text{on } \Sigma^\varepsilon \text{ and } \Sigma_0^\varepsilon), \qquad (5.42)$$

$$u_{jb_0-i-1}^i = u_{jb_0-i}^i \qquad (\text{on } \Sigma_0^\varepsilon), \qquad (5.43)$$

where $w^i = w(t_i),\ i = 1, 2, \ldots, M$.

From the initial conditions (5.8) we have

$$u_j^0 = u_0(x_j) \quad j = \overline{x_{jb_0}, N},$$

and, involving the separation of variables method to solve the Cauchy problem (5.9), we derive

$$\varphi_j^0 = z(\varepsilon, \varphi_-^\varepsilon(0, x_j)) = \varphi_-^\varepsilon(0, x_j)\sqrt{\frac{a\tau}{a\tau + \varepsilon \cdot \varphi_-^\varepsilon(0, x_j)}}.$$

Replacing (5.41)-(5.43) in (5.39)-(5.40) and setting

$$c_1 = -k \cdot \frac{dt}{(dx)^2}, \qquad c_2 = \rho c - 2c_1, \qquad c_3 = \frac{\ell}{2},$$

$$c_4 = -2 \cdot dt, \qquad c_5 = -\xi^2 \cdot \frac{dt}{(dx)^2}, \qquad c_6 = \tau - 2c_5 - \frac{dt}{2a},$$

the system (5.39)-(5.40) can be rewritten, in the matrix form, as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \bar{u}^i \\ \bar{\varphi}^i \end{pmatrix} \begin{pmatrix} bu^{i-1} \\ b\varphi^{i-1} \end{pmatrix} \qquad i = \overline{1, M}, \tag{5.44}$$

where A_{11}, A_{12}, A_{21}, A_{22}, having the same dimension: $M + N\text{-}jb_0{+}i + 1$, are given by

$$A_{11} = \begin{pmatrix} \bar{c}_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & c_2 & c_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & c_2 & c_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 + c_1 & \bar{c}_{1N} \end{pmatrix} \quad A_{12} = \begin{pmatrix} c_3 & 0 & \cdots & 0 \\ 0 & c_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_3 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} c_4 & 0 & \cdots & 0 & 0 \\ 0 & c_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_4 & 0 \\ 0 & 0 & \cdots & 0 & c_4 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} \bar{c}_5 & c_5 & 0 & \cdots & 0 & 0 & 0 \\ c_5 & c_6 & c_5 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_5 & c_6 & c_5 \\ 0 & 0 & 0 & \cdots & 0 & c_5 & \bar{c}_5 \end{pmatrix}$$

where $\bar{c}_1 = c_1 + c_2$, $\bar{c}_{1N} = \bar{c}_1 - (1 + 2 \cdot dx \cdot h)$, $\bar{c}_5 = c_5 + c_6$, and

$$bu^{i-1} = \begin{cases} \rho \cdot c \cdot u_j^{i-1} + c_3 \cdot \varphi_j^{i-1}, & j = \overline{jb_0 - i, N-1}, \\ \rho \cdot c \cdot u_N^{i-1} + c_3 \cdot \varphi_N^{i-1} + 2 \cdot dx \cdot w^i, \end{cases}$$

$$b\varphi^{i-1} = \tau \cdot \varphi_j^{i-1}, \quad j = \overline{jb_0 - i, N}.$$

The adjoint state $(p^\varepsilon, q^\varepsilon)$ in $(5.32)_1$ and $(5.33)_1$ is approximated by the matrix $\begin{pmatrix} p_j^i & q_j^i \end{pmatrix}^*$, where

$$\begin{array}{l} p_j^i = p^\varepsilon(t_i, x_j) \\ q_j^i = q^\varepsilon(t_i, x_j) \end{array} \qquad i = \overline{1, M}, \quad j = \overline{jb_0 - i, N}\ .$$

Following the same way as in the case of the state system, the discrete systems corresponding to equations $(5.32)_1$ and $(5.33)_1$ are

$$\begin{aligned} \frac{k}{\rho c}\frac{dt}{(dx)^2} &\cdot \ p_{j+1}^i - \Big(\frac{\ell}{\rho c\tau}dt + 2\frac{k}{\rho c}\frac{dt}{(dx)^2} + 1\Big) \cdot p_j^i \\ &+ \ \frac{k}{\rho c}\frac{dt}{(dx)^2} \cdot p_{j-1}^i + \frac{2}{\tau}dt \cdot q_j^i \\ &= \ -p_j^{i+1} + \beta dt (u_j^{i+1} - \delta_2)^+ \cdot \chi_{Q_0} \end{aligned} \tag{5.45}$$

$$\begin{aligned} -\frac{\ell\xi^2}{2\rho c\tau}\frac{dt}{(dx)^2} &\cdot \ p_{j+1}^i + \Big(2\frac{\ell\xi^2}{2\rho c\tau}\frac{dt}{(dx)^2} - \frac{\ell}{4\rho c\tau a}dt\Big) \cdot p_j^i \\ &- \ \frac{\ell\xi^2}{2\rho c\tau}\frac{dt}{(dx)^2} \cdot p_{j-1}^i + \frac{\xi}{\tau}\frac{dt}{(dx)^2} \cdot q_{j+1}^i \\ &- \ \Big(2\frac{\xi^2}{\tau}\frac{dt}{(dx)^2} - \frac{1}{2\tau a}dt + 1\Big) \cdot q_j^i \\ &+ \ \frac{\xi^2}{\tau}\frac{dt}{(dx)^2} \cdot q_{j-1}^i = -q_j^{i+1} + (\varphi_j^{i+1} - 1 - \delta_1) \cdot \chi_{Q_0} \end{aligned} \tag{5.46}$$

for $i = \overline{M-1, 0}$, $j = \overline{jb_0, N}$, where $(u_j^{i+1})^+ = (u^\varepsilon)^+(t_{i+1}, x_j)$.

From $(5.32)_2$ we obtain

$$(\Sigma^\varepsilon): \qquad p_N^i - p_{N-1}^i + dx \cdot h \cdot p_N^i = 0 \qquad i = M-1, M-2, \ldots, 1, 0, \tag{5.47}$$

and from $(5.33)_2$ we derive

$$(\Sigma^\varepsilon): \qquad \frac{\ell}{2\rho c}(q_N^i - q_{N-1}^i) = p_N^i - p_{N-1}^i \qquad i = M-1, M-2, \ldots, 1, 0. \tag{5.48}$$

Implying now $(5.32)_3$ and $(5.33)_3$, we get

$$(\Sigma_0^\varepsilon): \qquad p^i_{jb_0-i-1} = p^i_{jb_0-i}, \quad q^i_{jb_0-i-1} = q^i_{jb_0-i} \qquad i = M-1,\dots,0. \tag{5.49}$$

From the final conditions $(5.32)_4$ and $(5.33)_4$ we have

$$p_j^M = 0 \qquad q_j^M = 0, \qquad\qquad j = \overline{jb_0 - M, M}, \tag{5.50}$$

and, for $i = M-1, M-2, \dots, 1, 0,$

$$q^i_{-,j} = \exp\Big(\int_0^\varepsilon \frac{3}{2a} z^2(t,\cdot)dt \Big) q^i_{+,j}, \qquad q^M_{+,j} = q^M_j. \tag{5.51}$$

Replacing (5.47)-(5.49) in (5.45)-(5.46) and setting

$$\begin{aligned}
&c_{11} = k \cdot dt/(\rho c \cdot (dx)^2), && c_{12} = 2\cdot dt/\tau,\\
&c_{10} = \ell \cdot dt/(\rho c\tau) + 2\cdot c_{11} + 1, &&\\
&c_{13} = \xi^2 \cdot dt/(\tau \cdot (dx)^2, && c_{14} = -\ell \cdot c_{13}/(2\rho c),\\
&c_{15} = -2\cdot c_{14} - \ell\cdot dt/(4\rho c\tau a) && c_{16} = -2\cdot c_{13} + dt/(2\tau a) + 1.
\end{aligned}$$

then (5.45)-(5.46) can be rewritten, in matrix form, as

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \bar{p}^i \\ \bar{q}^i \end{pmatrix} = \begin{pmatrix} bp^{i+1} \\ bq^{i+1} \end{pmatrix} \tag{5.52}$$

where $B_{11},\ B_{12},\ B_{21},\ B_{22}$, (having the same dimension $M + N - jb_0 + i + 1$,) are given by

$$B_{11} = \begin{pmatrix}
c_{10}+c_{11} & c_{11} & 0 & \cdots & 0 & 0 & 0\\
c_{11} & c_{10} & c_{11} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & c_{11} & c_{10} & c_{11}\\
0 & 0 & 0 & \cdots & 0 & -1+c_{11} & c_{10}+c11+dx\cdot h+1
\end{pmatrix}$$

$$B_{12} = \begin{pmatrix}
c_{12} & 0 & \cdots & 0 & 0\\
0 & c_{12} & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & c_{12} & 0\\
0 & 0 & \cdots & 0 & c_{12}
\end{pmatrix}$$

$$B_{21}=\begin{pmatrix} c_{14}+c_{15} & c_{14} & 0 & \cdots & 0 & 0 & 0\\ c_{14} & c_{15} & c_{14} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & c_{14} & c_{15} & c_{14}\\ 0 & 0 & 0 & \cdots & 0 & c_{14}-1 & c_{14}+c_{15}+dx\cdot h+1 \end{pmatrix}$$

$$B_{22}=\begin{pmatrix} c_{13}+c_{16} & c_{13} & 0 & \cdots & 0 & 0 & 0\\ c_{13} & c_{16} & c_{13} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & c_{13} & c_{16} & c_{13}\\ 0 & 0 & 0 & \cdots & 0 & c_{13}-\frac{\ell}{2\rho c} & c_{13}+c16+\frac{\ell}{2\rho c}+dx\cdot h \end{pmatrix}$$

and

$$bp^{i+1}=-p_j^{i+1}+\beta dt(u_j^{i+1}-\delta_2)^+\cdot\chi_{Q_0},\qquad j=\overline{jb_0-i,N},$$

$$bq^{i+1}=\begin{cases}-q_j^{i+1}+(\varphi_j^{i+1}-\delta_1-1)\cdot\chi_{Q_0}, & j=\overline{jb_0-i,N-1},\\ -q_N^{i+1}+(\varphi_N^{i+1}-\delta_1-1)\cdot\chi_{Q_0}-dx\cdot h\cdot p_N^{i+1}.\end{cases}$$

We illustrate now some pictures, about numerical experiments, got through the implementation in MATLAB of the algorithm InvPHT1D.

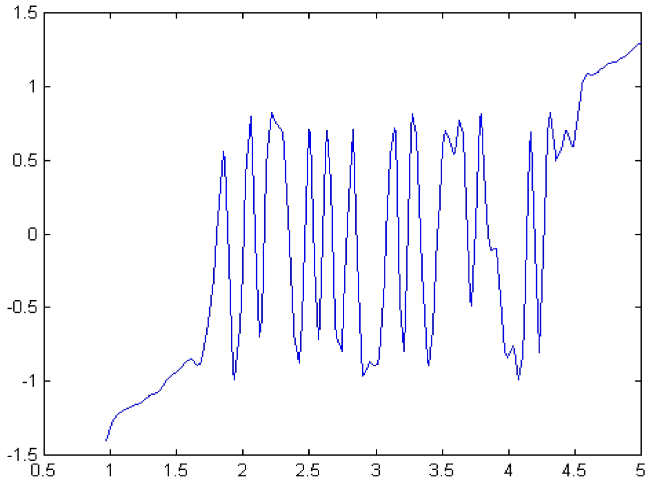

Figure 5.4: The initial condition φ_0

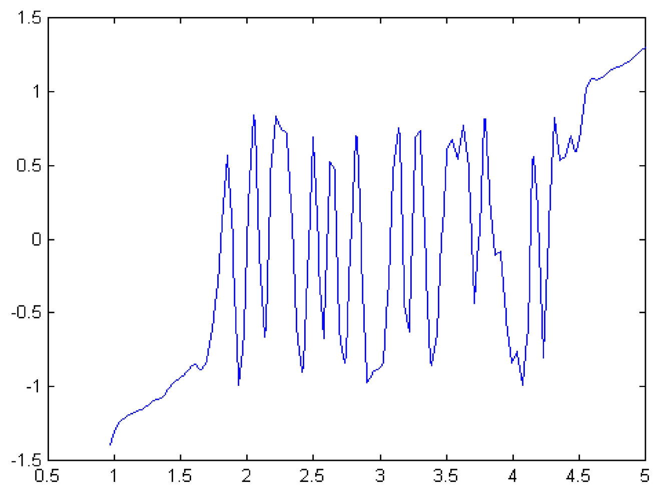

Figure 5.5: The solution $z(\varepsilon, \cdot)$ of Cauchy problem (5.9)

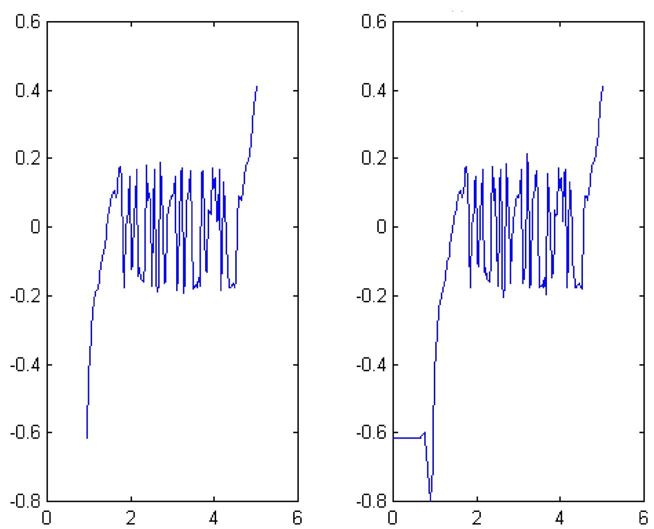

Figure 5.6: The approximate solution u^0 and u^M

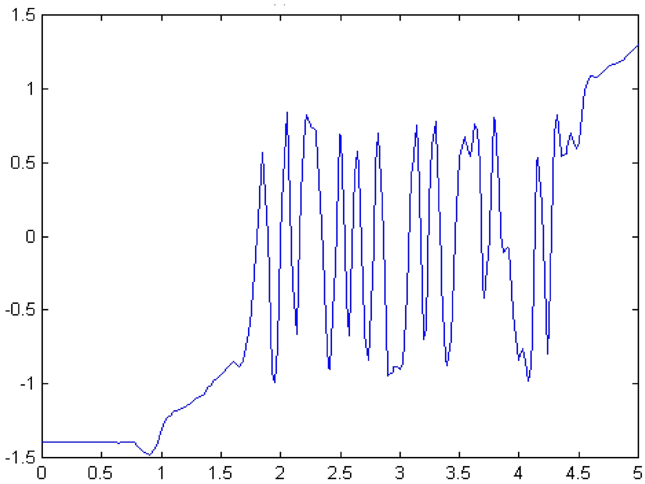

Figure 5.7: The approximate solution φ^M

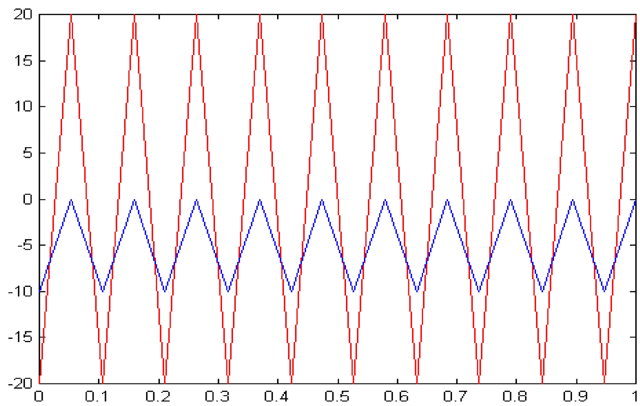

Figure 5.8: The initial control w^0 and optimal control w^{10}

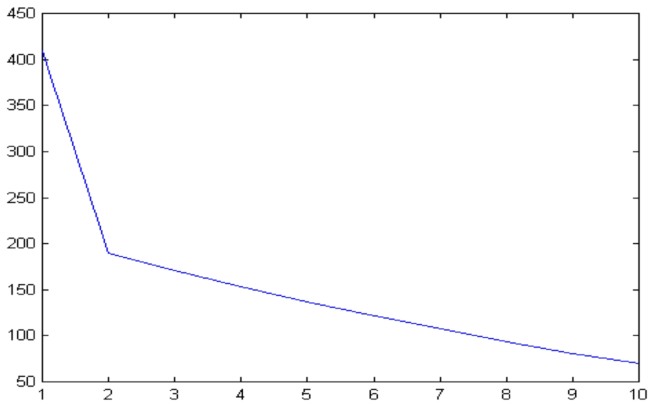

Figure 5.9: The values of functional $j^{\varepsilon}(w)$

Conclusions

Our goal, in this work, was to study an application of the Caginalp model to an industrial solidification process of a molten metal, called countiuous casting. This application pushed us to develop the theory of the phase-field transition system for non-homogeneous Cauchy-Neumann boundary conditions, so that we developed a result about the existence and the regularity of the solution in this case. The non homogenous boundary condition could be understood as boundary controls. To calculate the minimum effort to drive in T time a material, occupying a region Q, which at the initial moment is in both the states, to a pure solid state, we coupled the system $(2.1) - (2.3)$ with a functional cost j. A great difficulty to solve problem (P) comes to the fact that the state is governed by a non linear law. We used a product formula approach to solve this inconvenience. Thus we associated to the non linear system (2.1)-(2.3) an approximating scheme of fractional steps type (3.1)-(3.3) and involving its solution in the functional cost j we got an approximating problem (P^ε) for (P). We proved that an optimal control for (P^ε) is a suboptimal control for problem (P). We got also necessary optimality conditions for (P^ε) and we used them to develop a MATLAB software that simulate the 1D-case of the solidification process, (*casting wire*). The possible applications suggest us many way in which we could develop this work, as:

- Develop the software in the 2D-case,
- Implement the Fast Fuorier Transform in the software to calculate the approximate solution $(u^\varepsilon, \varphi^\varepsilon, w^\varepsilon)$,

- Calculate the minimum time T to get the full solidification process,
- Implement, for several applications, as nonlinear boundary conditions, a Stefan-Boltzmann law $\dfrac{\partial u}{\partial \nu} + \alpha(u^4 - \bar{u}^4) = 0$.

Appendix A

Linear parabolic partial differential equations

For the case of a second-order parabolic equation[1], by a non-singular change of variables, it can be reduced to the form

$$u_t - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au = f \tag{A.1}$$

with a positive-definite form $\sum a_{ij}\xi_i\xi_j$. A typical representative of a parabolic equation is the *heat equation*

$$u_t - \sum_{i,j=1}^{n} u_{x_i x_j} = 0 \tag{A.2}$$

the main properties of which are preserved for a general parabolic equations. The following problem are fundamental in studing the equation of parabolic type (A.1).

The *Cauchy-Dirichelet problem*: To find a function $u(t,x)$ that satisfies (A.1) for $x \in I\!R^n,\ t > 0$ and satisfies the initial condition

$$u|_{t=0} = \phi(x), \qquad x \in I\!R^n$$

[1]Springer Online Reference Works
http://eom.springer.de/l/l059380.htm

The *first boundary value problem* in which (A.1) is specified in a cylinder

$$Q_t := [0,T] \times \Omega,$$

where $\Omega \subset I\!R^n$. It is required to find a function u satisfying the initial condition

$$u|_{t=0} = \phi(x), \qquad x \in \Omega,$$

and the boundary condition

$$u|_{x\in\partial\Omega} = \psi(t,x) \qquad t \in [0,T]. \tag{A.3}$$

The second and third boundary value problems differ from the first only in condition (A.3), which is replaced by the *second boundary value condition*

$$\left.\frac{\partial u}{\partial \nu}\right|_{x\in\partial\Omega} \equiv \sum_{i,j=1}^{n} a_{ij} u_{x_i} \nu_i = \psi(t,x), \qquad t \in [0,T], \tag{A.4}$$

or the *third*

$$\left.\Big(\frac{\partial u}{\partial \nu} + \sigma u\Big)\right|_{x\in\partial\Omega} = \psi(t,x), \qquad t \in [0,T]. \tag{A.5}$$

where ν_i, $1 \le i \le n$ are the componet of the outward normal ν.

The classical formulation of these problems requires that the solution is continuous in the closed domain, that the derivatives with respect to the spatial variables up to the second order are continuous inside the domain, and in the case of the second and third boundary value problems that the first derivatives are continuous up to the lateral surface of the region Ω. For the Cauchy-Dirichelet problem, or if Ω is unbounded for the boundary value problems, it is also required that the growth of the solution u is specified, in a suitable way, as $|x| \to \infty$. For example it could be bounded.

Suppose that equation (A.1) is uniformly parabolic ($\sup \mathrm{Re}\ \lambda_m < -\delta$) and that the coefficients of the equation, the initial and boundary conditions and the boundary domain are smooth enough. Then the solution of the Cauchy-Dirichelet problem and the first boundary value problem exist and are unique. If $a \le 0$, $\sigma > 0$ and if some necessary compatibility conditions are satisfied, then a similar result also hold for the second and the third boundary value problems.

Uniqueness in these problems follows from the *maximum principle*. Suppose that the coefficients in (A.1) are continuous in $\bar{Q}_T$ and that Ω is bounded, let:

$$\Gamma := \partial Q_T \setminus \{(t,x) : \ x \in \Omega, \ t = T\}$$

and let

$$M = \max_{\bar{Q}_T} a, \qquad N = \max_{\bar{Q}_T} f.$$

Then for any solution of (A.1)

$$u \in C(\bar{Q}_T) \cup C^2(\bar{Q}_T \setminus \Gamma)$$

the following estimate holds

$$|u(t,x)| \leq e^{Mt}\{Nt + \max_{\Gamma} |u|\}, \qquad (t,x) \in Q_T.$$

For the boundary value problems we recall now an existence result. If $p > 1$, the coefficients a_{ij} are bounded continuous functions in Q, while the coefficients a_i and a have finite norms $\|a_i\|^{loc}_{L^r(Q)}$ and $\|a\|^{loc}_{L^s(Q)}$ (see [15, p. 343]). Suppose, further, that the quantities $\|a_i\|^{loc}_{L^r(Q_{t,t+\tau})}$ and $\|a\|^{loc}_{L^s(Q_{t,t+\tau})}$ tend to zero for $\tau \to 0$. Let $\partial\Omega \in C^2$. Then for any $f \in L^p(Q)$, $\phi \in W_p^{2-\frac{2}{p}}(\Omega)$ and $\psi \in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)$, satisfying the compatibility condition

$$\phi|_{\Omega} = \psi|_{t=0}.$$

the boundary value problem has a unique solution $u \in W_p^{2,1}(Q)$. It satisfies the estimate (see [15])

$$\|u\|_{W_p^{2,1}(Q)} \leq C\,\{\|\,f\|_{L^p(Q)} + \|\phi\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\psi\|_{W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(\Sigma)}\}.$$

Appendix B

Compute the integral value on Q_0

```
% addpath d:\Benincasa\LUCRARE_3\Programe_Matlab

J = 0; L1 = 0;

J1=0; J2=0; J3=0;

dt = (T-eps)/(M-1);

L1 = .5*up(1,1).^2 + .5*up(1,N+1).^2 ...
   +.5*fip(1,1).^2 + .5*fip(1,N+1).^2  ...
   + sum(up(1,2:N).^2) + sum(fip(1,2:N).^2);
L1 = L1*.5;

for i=2:M-1
   Ni = MpN - jb0 + i + 1;
   L1 = L1 + .5*up(i,1).^2 + .5*up(i,Ni).^2 ...
           + .5*fip(i,1).^2 + .5*fip(i,Ni).^2  ...
           + sum(up(i,2:Ni-1).^2) + sum(fip(i,2:Ni-1).^2);
```

```
   L1 = L1*dx;
end

L1 = L1 + (.5*up(M,1).^2 + .5*up(M,MpN).^2 ...
        +.5*fip(M,1).^2 + .5*fip(M,MpN).^2  ...
        + sum(up(M,2:MpN-1).^2) + sum(fip(M,2:MpN-1).^2))*.5;
L1 = L1*dt;

J = J + .5*dt*witer(1)^2 + .5*dt*witer(M)^2;
J = J + sum(dt.*witer(2:M-1).^2);

J = J + L1;
```

Appendix C

Implementation of the algorithm InvPHT1D

```
% Implementation of the algorithm InvPHT1D
%
% addpath d:\Benincasa\LUCRARE_3\Programe_Matlab; format short e
%

% P0. Initial values

  % P00. Parameters in functional cost

  clear

    R = 20;
    niu   = .1;
    beta  = 1.;
    delta1= .1;
    delta2= 1.;

  % P01. Physical parameters
```

```
  prho= 7850;
  pc  = 12.5;
  pl  = 65.28;
  pcsi= .3;
  ptau= 1.e+6*pcsi^2;
  ph  = 32.012;
  pk  = 7.8e-6;

% P02. Construction of the time and space meshs

  T=input('T=');
  eps=input('eps=');
  M=input('M=');

  dtM = (T-eps)/(M-1);

  t=eps:dtM:T;

  N=input('N=');

  b0=sqrt(T);

  b1=input('(b1>sqrt(T)) b1=');

  dx=(b1-b0)/(N-1);

  x=b0:dx:b1;

  for i=1:M
    l=M-i+1;
```

```
    xt(l)=b0-sqrt(t(i));
  end

  xMN = [xt x];

  for j=1:M-1
    dxt(j) = xt(j+1) - xt(j);
  end
    dxt(M) = x(1) - xt(M);

  MpN = M + N;
  jb0 = M + 1;

  % store the maximum space for: u, up, fi, fip, p, q

  u = zeros(1,MpN);
  up= zeros(M,MpN);

  fi= zeros(1,MpN);
  fip= zeros(M,MpN);

  p = zeros(M,MpN);
  q = zeros(M,MpN);

% P1. Compute z(eps,.) from (1.9)

  dx1 = (b1-xt(M))/63;

  x1 = xt(M):dx1:b1;

  fi0=[-1.4 -1.25 -1.2 -1.17 -1.15 -1.1 -1.08 -1.0 -.95 -.9 -.85 ...
```

```
      -.88 -.6 .0 .5 -.92 -.25 .8 -.7 .58 .75 .58 -.63 -.59 .69 -.72...
      .7 -.59 -.5 .7 -.79 -.87 -.88 .0 .72 -.8 .81 .0 -.89 .0 .7 .55 ...
      .68 -.49 .79 .0 -.1 -.8 -.78 -.83 .69 -.8 .68 .5 .7 .59 1. 1.08 1.1 ...
      1.15 1.17 1.2 1.25 1.3];

    figure(1),
%    subplot(1,2,2);
    fi0cs = csapi(x1,fi0');
    fnplt(fi0cs)
    title(' The initial condition fi_0');

    pa = sqrt(pcsi);  % .8
    pa4= 1./(4*pa);

    for j=1:N+1
      fihat0(j) = fnval(fi0cs,xMN(M+j-1));
      uhat0(j)  = (fihat0(j)*fihat0(j)*fihat0(j) - fihat0(j))*pa4;
    end

    %uhat0

  % Solve the Cauchy problem on [0,eps]

    z_eps = fihat0.*sqrt(pa./(pa + eps.*fihat0.^2));


    %subplot(1,2,2);
    figure(2),
    z_cs = csapi([xt(M) x],z_eps');
    fnplt(z_cs)
    title(' The solution of Cauchy problem (1.9)');
```

```
    for j=1:N+1
      fip(1,j)= z_eps(j);
      up(1,j) = uhat0(j);
 end

  % P1.1. Choose w^(0)(t) from [0,R], t\in [0,T]

    w    = zeros(1,M);
    witer= zeros(1,M);

    %w(1) = 1.;
    %for i=2:M
    %  w(i) = w(1) + (-1)^i*250;
    %end

    for i=1:M
      w(i) = 0. + (-1)^i*R;
    end

    witer = w;

%     gettime(&tt);
%     printf("\n timpul curent : %d,%d,%d,%4d\n",tt.ti_hour,
%     tt.ti_min,tt.ti_sec,tt.ti_hund);
%    getch();

    itmax = input('itmax:');

    vj = zeros(1,itmax);
```

```
for iter = 1:itmax

  for j=1:N+1
    fip(1,j)= z_eps(j);
    up(1,j) = uhat0(j);
  end

  % P2. SOLVE THE LINEAR SYSTEM (1.5)-(1.8)

    c3 = pl/2;
    c4 = -2*dtM;

    for i = 2:M

      Ni = MpN-jb0+i+1;  % number of nodes for Omega_i: MpN-(jb0-i-1)

      d = zeros(1,2*Ni);
      aa= zeros(2*Ni,2*Ni);

      im1 = i-1;
      ip1 = i+1;

      % the right side in (5.6); j=jb0-i:MpN

      for j=2:Ni
        d(j) = prho*pc*up(im1,j-1)+c3*fip(im1,j-1);
      end
      %d(Ni) = prho*pc*up(im1,Ni-i)+c3*fip(im1,Ni-i)+2*dx*w(im1);
      d(1)  = d(2)-1.5;

      for j=2:Ni
```

```
          jj = Ni + j;
          d(jj) = ptau*fip(im1,j-1);
        end
        d(Ni+1) = d(Ni+2)-.6;
        %d
pa=28.;
          c1 = -pk*dtM/(dxt(M-i+1)*dxt(M-i+1));      % dxt(M-1)!
          c2 = prho*pc - 2*c1;
          c5 = - pcsi^2*dtM/(dxt(M-i+1)*dxt(M-i+1));
          c6 = ptau-2*c5-dtM/(2*pa);

          aa(1,1) = c1 + c2;
          aa(1,2) = c1;

          aa(1,Ni+1) = c3;
          aa(Ni+1,1) = c4;

          aa(Ni+1,Ni+1) = c5 + c6;
          aa(Ni+1,Ni+2) = c5;

          for j=2:ip1 % matrices A_11, ... in (5.6) for j=jb0-i:jb0;  dxt !

            c1 = -pk*dtM/(dxt(M+j-i-1)*dxt(M+j-i-1));      % dxt(j)!
            c2 = prho*pc - 2*c1;
            c5 = - pcsi^2*dtM/(dxt(M+j-i-1)*dxt(M+j-i-1));
            c6 = ptau-2*c5-dtM/(2*pa);

            aa(j,j-1) = c1;
            aa(j,j)   = c2;
            aa(j,j+1) = c1;
```

```
    aa(j,Ni+j)=c3;
    aa(Ni+j,j)=c4;

    aa(Ni+j,Ni+j-1) = c5;
    aa(Ni+j,Ni+j)   = c6;
    aa(Ni+j,Ni+j+1) = c5;
  end

% matrices A_11, ... in (5.6) for j=jb0+1:MpN;  dx !

  c1 = -pk*dtM/(dx*dx);         % dx!
  c2 = prho*pc - 2*c1;
  c5 = - pcsi^2*dtM/(dx*dx);
  c6 = ptau-2*c5-dtM/(2*pa);

  aa(Ni,Ni-1)= 1 + c1;
  aa(Ni,Ni)  = c1 + c2 - 1 - 2*dx*ph;

  aa(Ni,2*Ni)= c3;
  aa(2*Ni,Ni)= c4;

  aa(2*Ni,2*Ni-1)= c5;
  aa(2*Ni,2*Ni)  = c5 + c6;

  for j=ip1+1:Ni-1
    aa(j,j-1) = c1;
    aa(j,j)   = c2;
    aa(j,j+1) = c1;

    aa(j,Ni+j)=c3;
    aa(Ni+j,j)=c4;
```

```
      aa(Ni+j,Ni+j-1) = c5;
      aa(Ni+j,Ni+j)   = c6;
      aa(Ni+j,Ni+j+1) = c5;
    end
    %aa

  % Compute the solution of (5.6) system

    z = zeros(1,2*Ni);

    [L,U]= lu(aa);
    z = U\(L\d');

    %z = inv(aa)*d';
    %z = aa\d';

    %rcond(aa) %size(aa), cond(aa),

  % u^i, fi^i;

    for j=1:Ni
      up(i,j) = z(j);
      fip(i,j)= z(Ni+j);
    end

end % next i - primal system

   % Grafic up_M,  fip_M

    viz = 0;
```

```
 viz=input('Reprezentati grafic up & fip ? (y=1,n=0):');
 if viz == 1
   grafic_up;
   grafic_fip;
   viz = 0;
 end

% up = positive part of u(i,:)

for i=1:M
  Ni = MpN - jb0 + i + 1;
  for j=1:Ni
    if up(i,j)-delta2+delta1 > 0
    up(i,j) = up(i,j)-delta2+delta1;
  else
       up(i,j) = 0.;
    end
  end
end

fip = fip - delta1 - 1;

% Compute the integral value on Qo !!!!

int_Jeps;

vj(iter) = J;

% P3. Solve the dual system (4.16)-(4.17)

pwork = zeros(1,M);
```

```
wwork = zeros(1,M);

pwork(M) = eps;

% the final conditions

p(M,1:MpN) = 0;  q(M,1:MpN) = 0;
p(M,MpN) = eps;

c12 = 2*dtM/ptau;

for i = 2:M
  l  = M-i+1;
  lp1= l+1;

Nl = MpN-jb0+lp1;  % number of nodes for Omega_l: MpN-(jb0-l-1)

  d = zeros(1,2*Nl);
  aa= zeros(2*Nl,2*Nl);

  % the right side in (5.14); j=jb0-l:MpN

  for j=1:Nl
    d(j) = -p(lp1,j+1)+beta*dtM*up(lp1,j+1);
  end

  for j=1:Nl-1
    jj = Nl + j;
    d(jj) = -q(lp1,j+1)+fip(lp1,j+1);
  end
```

```
d(2*Nl) = -q(lp1,Nl+1)+fip(lp1,Nl+1)-dx*ph*p(lp1,Nl+1);

c11 = pk*dtM/(prho*pc*dxt(M-l+1)*dxt(M-l+1));      % dxt(1)!
c10 = pl*dtM/(prho*pc*ptau)+2*c11+1;

c13 = pcsi^2*dtM/(ptau*dxt(M-l+1)*dxt(M-l+1));
c14 = -pl*c13/(2*prho*pc);
c15 = -2*c14-pl*dtM/(4*prho*pc*ptau*pa);
c16 = -2*c13+dtM/(2*ptau*pa)+1;

aa(1,1) = c10 + c11;
aa(1,2) = c11;

aa(1,Nl+1) = c12;

aa(Nl+1,1) = c14 + c15;
aa(Nl+1,2) = c14;

aa(Nl+1,Nl+1) = c13 + c16;
aa(Nl+1,Nl+2) = c13;

for j=2:lp1 % matrices B_11, ... in (5.14) for j=jb0-l:jb0; dxt !

  c11 = pk*dtM/(prho*pc*dxt(M+j-l-1)*dxt(M+j-l-1));      % dxt(j)!
  c10 = pl*dtM/(prho*pc*ptau)+2*c11+1;

  c13 = pcsi^2*dtM/(ptau*dxt(M+j-l-1)*dxt(M+j-l-1));
  c14 = -pl*c13/(2*prho*pc);
  c15 = -2*c14-pl*dtM/(4*prho*pc*ptau*pa);
  c16 = -2*c13+dtM/(2*ptau*pa)+1;
```

```
      aa(j,j-1) = c11;
      aa(j,j) = c10;
      aa(j,j+1) = c11;

      aa(j,Nl+j)=c12;

      aa(Nl+j,j-1) = c14;
      aa(Nl+j,j)   = c15;
      aa(Nl+j,j+1) = c14;

      aa(Nl+j,Nl+j-1) = c13;
      aa(Nl+j,Nl+j)   = c16;
      aa(Nl+j,Nl+j+1) = c13;
    end

    % matrices B_11, ... in (5.14) for j=jb0+1:MpN;  dx !

      c11 = pk*dtM/(prho*pc*dx*dx);      % dx!
      c10 = pl*dtM/(prho*pc*ptau)+2*c11+1;

      c13 = pcsi^2*dtM/(ptau*dx*dx);
      c14 = -pl*c13/(2*prho*pc);
      c15 = -2*c14-pl*dtM/(4*prho*pc*ptau*pa);
      c16 = -2*c13+dtM/(2*ptau*pa)+1;

for j=lp1+1:Nl-1 % matrices B_11, ... in (5.14) for j=lp1+1:Nl; dxt !

      aa(j,j-1) = c11;
      aa(j,j)   = c10;
      aa(j,j+1) = c11;
```

```
  aa(j,Nl+j)=c12;

  aa(Nl+j,j-1) = c14;
  aa(Nl+j,j)   = c15;
  aa(Nl+j,j+1) = c14;

  aa(Nl+j,Nl+j-1) = c13;
  aa(Nl+j,Nl+j)   = c16;
  aa(Nl+j,Nl+j+1) = c13;
end

  aa(Nl,Nl-1) = -1 + c11;
  aa(Nl,Nl)   = c10+c11+dx*ph+1;

  aa(Nl,2*Nl) = c12;

  aa(2*Nl,Nl-1)= c14-1;
  aa(2*Nl,Nl)  = c14+c15+dx*ph+1;

  aa(2*Nl,2*Nl-1) = c13 - pl/(2*prho*pc);
  aa(2*Nl,2*Nl) = c13+c16+pl/(2*prho*pc)+dx*ph;
  %aa

% Compute the solution of (5.14) system

  z = zeros(1,2*Nl);

  [L,U]= lu(aa);
  z = U\(L\d');
  %z = inv(aa)*d';
  %z = aa\d';
```

```
              %rcond(aa) %size(aa), cond(aa),

             % u^i, fi^i;

             for j=1:Nl
               p(i,j) = z(j);
               q(i,j) = z(Nl+j);
             end

           pwork(l) = p(i,Nl);

         end % next i - dual system
         %pwork

    % P4. Compute new boundary control

         % compute r^iter

         r = zeros(1,M);
         for i=1:M
           r(i) = (pk/(prho*pc))*pwork(i) + witer(i);
         end
         %r
         % compute wwork

         for i = 1:M
           if r(i) > 0
              wwork(i) = 0;%R/iter;
            else
              wwork(i) = -R/iter;
           end
```

```
        end
        %wwork

        % compute witer

        for i = 1:M
          if rem(i,2) == 0
             witer(i) = 0; %R-iter; %R-.1;
           else
             witer(i) = -R+iter;
          end
        end

   % P5. Stopping criterion

        %werr = norm(wwork-witer);
        %if werr <= niu
        %   break;
        % else
        %   witer = wwork;
        %end

end    % next iter
```

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