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Preface

This thesis contains most of my works during my Ph.D. study at the Operations Research Group, University of Bologna, and serves as documentation of my work done during the years from 2007 until 2010. This work has been partially supported by MIUR (Ministero Istruzione, Università e Ricerca), Italy. This support is gratefully acknowledged.

Combinatorial Optimization is a branch of optimization that deals with the problems where the set of feasible solutions is discrete. Routing problem is a well studied branch of Combinatorial Optimization that concerns the process of deciding the best way of visiting the nodes (customers) in a network. Routing problems appear in many real world applications including: Transportation, Telephone or Electronic data Networks.

During the years, many solution procedures have been introduced for the solution of different Routing problems. Some of them are based on exact approaches to solve the problems to optimality and some others are based on heuristic or metaheuristic search to find optimal or near optimal solutions. There is also a less studied method, which combines both heuristic and exact approaches to face different problems including those in the Combinatorial Optimization area.

The aim of this dissertation is to develop some solution procedures based on the combination of heuristic and Integer Linear Programming (ILP) techniques for some important problems in Routing Optimization. In this approach, given an initial feasible solution to be possibly improved, the method follows a destruct-and-repair paradigm, where the given solution is randomly destroyed (i.e., customers are removed in a random way) and repaired by solving an ILP model, in an attempt to find a new improved solution.

This thesis contains four chapters. There are two other chapters and since the works of these chapters are in progress, we have not reported them in this dissertation. The results of this thesis have been presented in five research papers submitted or published in International journals.

In the first chapter we focus on the Open Vehicle Routing Problem. This problem is a variant of the "classical" (Capacitated and Distance Constrained) Vehicle Routing Problem (VRP) in which the vehicles are not required to return to the depot after completing their service. We present an ILP-based improvement procedure for the OVRP based on ILP techniques.

In Chapter 2 we apply the latter mentioned ILP-based improvement technique to the solution of the Capacitated m-Ring Star Problem. We have also used this approach for the solution of the Covering Salesman Problem and of the Median Cycle Problem successfully, but, since they are Works-in-Progress, the results have not been reported in this thesis.

Chapter 3 concerns the Label Constrained Minimum Labelling Spanning Tree Problem that occurs in Telephone Networks and is motivated from applications in the communications sector.

Finally in Chapter 4 we study the Covering Salesman Problem and some of its generalizations occurring in the routing of Transportation Networks.

Chapter 1

An ILP Improvement Procedure for the Open Vehicle Routing Problem

1.1 Introduction

In this chapter[1] we address the Open Vehicle Routing Problem (OVRP), a variant of the "classical" (Capacitated and Distance Constrained) Vehicle Routing Problem (VRP) in which the vehicles are not required to return to the depot after completing their service. OVRP can be formally stated as follows. We are given a central *depot* and a set of n *customers*, which are associated with the nodes of a complete undirected graph $G = (V, E)$ (where $V = \{0, 1, \ldots, n\}$, node 0 represents the depot and $V \setminus \{0\}$ is the set of customers). Each edge $e \in E$ has an associated finite *cost* $c_e \geq 0$ and each customer $v \in V \setminus \{0\}$ has a *demand* $q_v > 0$ (with $q_0 = 0$). A fleet of m identical *vehicles* is located at the depot, each one with a *fixed cost* F, a *capacity* Q and a *total distance-traveled (duration)* limit D. The customers must be served by at most m Hamiltonian paths (*open routes*), each path associated with one vehicle, starting at the depot and ending at one of the customers. Each route must have a duration (computed as the sum of the edge costs in the route) not exceeding the given limit D of the vehicles, and can visit a subset S of customers whose total demand $\sum_{v \in S} q_v$ does not exceed the given capacity Q. The problem consists of finding a feasible solution covering (i.e., visiting) exactly once all the customers and having a minimum overall cost, computed as the sum of the traveled edge costs plus the fixed costs associated with the vehicles used to serve the customers. OVRP is known to be $\mathcal{NP}$-hard in the strong sense, as it generalizes the Bin Packing Problem and the Hamiltonian Path Problem.

[1]The results of this chapter appear in: Salari M., Toth P., and Tramontani A.: "An ILP improvement procedure for the open vehicle routing problem". Computers & Operations Research, To appear [68].

In this paper we present a heuristic improvement procedure for OVRP based on Integer Linear Programming (ILP) techniques. Given an initial feasible solution to be possibly improved, the procedure iteratively performs the following steps: (a) randomly select several customers from the current solution, and build the restricted solution obtained from the current one by extracting (i.e., short-cutting) the selected customers; (b) reallocate the extracted customers to the restricted solution by solving an ILP problem, in the attempt of finding a new improved feasible solution. This method has been proposed by De Franceschi et al. [20] and deeply investigated by Toth and Tramontani [77] in the context of the classical VRP. The method follows a destruct-and-repair paradigm, where the current solution is randomly destroyed (i.e., customers are removed in a random way) and repaired by following ILP techniques. Hence, the overall procedure can be considered as a general framework which could be extended to cover other variants of Vehicle Routing Problems.

The notion of using ILP techniques to improve a feasible solution of a combinatorial optimization problem has emerged in several papers in the last few years. Addressing the split delivery VRP, Archetti et al. [2] developed a heuristic algorithm that integrates tabu search with ILP by solving integer programs to explore promising parts of the solution space identified by a tabu search heuristic. A similar approach has been presented by Archetti et al. [1] for an inventory routing problem. Hewitt et al. [37] proposed to solve the capacitated fixed charge network flow problem by combining exact and heuristic approaches. In this case as well a key ingredient of the method is to use ILP to improve feasible solutions found during the search. Finally, the idea of exploiting ILP to explore promising neighborhoods of feasible solutions has been also investigated in the context of general purpose integer programs; see, e.g., Fischetti and Lodi [26] and Danna et al. [18]. The methods presented in [18] and in [26] are currently embedded in the commercial mixed integer programming solver Cplex [39].

The chapter is organized as follows. Section 1.2 recalls the main works proposed in the literature for OVRP. In Section 1.3 we describe a neighborhood for OVRP and the ILP model which allows to implicitly define and explore the presented neighborhood. The implementation of the heuristic improvement procedure is given in Section 1.4, while Section 1.5 reports the computational experiments on benchmark capacitated OVRP instances from the literature (with/without distance constraints), comparing the presented method with the most effective metaheuristic techniques proposed for OVRP. Some conclusions are finally drawn in Section 1.6.

1.2 Literature review

The classical VRP is a fundamental combinatorial optimization problem which has been widely studied in the literature (see, e.g., Toth and Vigo [78] and Cordeau et al. [15]). At first glance, having open routes instead of closed ones looks like a minor change, and in fact OVRP can be also formulated as a VRP on a directed graph, by fixing to 0 the cost of each arc entering the depot. However, if the undirected case is considered, the open version turns out to be more general than the closed one. Indeed, as shown by Letchford et al. [45], any closed VRP on n customers in a complete undirected graph can be transformed into an OVRP on n customers, but there is no transformation in the reverse direction. Further, there are many practical applications in which OVRP naturally arises. This happens, of course, when a company does not own a vehicle fleet, and hence customers are served by hired vehicles which are not required to come back to the depot (see, e.g., Tarantilis et al. [75]). But the *open model* also arises in pick-up and delivery applications, where each vehicle starts at the depot, delivers to a set of customers and then it is required to visit the same customers in reverse order, picking up items that have to be backhauled to the depot. An application of this type is described in Schrage [70]. Further areas of application, involving the planning of train services and of school bus routes, are reported by Fu et al. [31].

OVRP has recently received an increasing attention in the literature. Exact branch-and-cut and branch-cut-and-price approaches have been proposed, respectively, by Letchford et al. [45] and Pessoa et al. [56], addressing the capacitated problem with no distance constraints and no empty routes allowed (i.e., $D = \infty$ and exactly m vehicles must be used). Heuristic and metaheuristic algorithms usually take into account both capacity and distance constraints, and consider the number of routes as a decision variable. In particular, an unlimited number of vehicles is supposed to be available (i.e., $m = \infty$) and the objective function is generally to minimize the number of used vehicles first and the traveling cost second, assuming that the fixed cost of an additional vehicle always exceeds any traveling cost that could be saved by its use (i.e., considering $F = \infty$). However, several authors address as well the variant in which there are no fixed costs associated with the vehicles (i.e., $F = 0$) and hence the objective function is to minimize the total traveling cost with no attention to the number of used vehicles (see, e.g., Tarantilis et al. [75]). Considering capacity constraints only (i.e., taking $D = \infty$), Sariklis and Powell [69] propose a two-phase heuristic which first assigns customers to clusters and then builds a Hamiltonian path for each cluster, Tarantilis et al. [73] describe a population-based heuristic, while Tarantilis et al. [74, 75] present threshold accepting metaheuristics. Taking into account both capacity and distance constraints, Brandão [7], Fu et al. [31, 32] and Derigs and Reuter [21] propose tabu search heuristics, Li et al. [46] describe a record-to-record travel heuristic, Pisinger and Ropke [58] present an adaptive large neighborhood search heuristic which follows a destruct-and-repair paradigm, while Fleszar et al. [30] propose a variable neighborhood search heuristic.

1.3 Reallocation Model

Let z be a feasible solution of the OVRP defined on G. For any given node subset $\mathcal{F} \subset V \setminus \{0\}$, we define $z(\mathcal{F})$ as the *restricted solution* obtained from z by *extracting* (i.e., by short-cutting) all the nodes $v \in \mathcal{F}$. Let $\mathcal{R}$ be the set of routes in the restricted solution, $\mathcal{I} = \mathcal{I}(z, \mathcal{F})$ the set of all the edges in $z(\mathcal{F})$, and $S = S(\mathcal{F})$ the set of all the *sequences* which can be obtained through the recombination of nodes in $\mathcal{F}$ (i.e., the set of all the elementary paths in $\mathcal{F}$). Each edge $i \in \mathcal{I}$ is viewed as a potential *insertion point* which can allocate one or more nodes in $\mathcal{F}$ through at most one sequence $s \in S$. We say that the insertion point $i = (a, b) \in \mathcal{I}$ allocates the nodes $\{v_j \in \mathcal{F} : j = 1, \ldots, h\}$ through the sequence $s = (v_1, v_2, \ldots, v_h) \in S$, if the edge (a, b) in the restricted solution is replaced by the edges $(a, v_1), (v_1, v_2), \ldots, (v_h, b)$ in the new feasible solution. Since the restricted routes, as well as the final ones, are open paths starting at the depot, in addition to the edges of the restricted solution we also consider the insertion points (called *appending insertion points* in the following) $i = (p_r, 0)$, where p_r denotes the last customer visited by route $r \in \mathcal{R}$, which allow to append any sequence to the last customer of any restricted route. Further, empty routes in the restricted solution are associated with insertion points $(0, 0)$.

For each sequence $s \in S$, $c(s)$ and $q(s)$ denote, respectively, the cost of the elementary path corresponding to s and the sum of the demands of the nodes in s. For each insertion point $i = (a, b) \in \mathcal{I}$ and for each sequence $s = (v_1, v_2, \ldots, v_h) \in S$, γ_{si} denotes the extra-cost (i.e., the extra-distance) for assigning sequence s to insertion point i in its best possible orientation (i.e., $\gamma_{si} := c(s) - c_{ab} + \min\{c_{av_1} + c_{v_h b}, c_{av_h} + c_{v_1 b}\}$). Note that, for the appending insertion points $i = (p_r, 0)$, γ_{si} is computed as $c(s) + \min\{c_{p_r v_1}, c_{p_r v_h}\}$. The extra-cost for assigning the sequence s to the insertion point $i = (0, 0)$ associated with an empty route is simply $c(s) + \min\{c_{0v_1}, c_{0v_h}\}$. For each route $r \in \mathcal{R}$, $\mathcal{I}(r)$ denotes the set of insertion points associated with r, while $\tilde{q}(r)$ and $\tilde{c}(r)$ denote, respectively, the total demand and the total distance computed for route r, still in the restricted solution.

For each $i \in \mathcal{I}$, $S_i \subseteq S$ denotes a sequence subset containing the sequences which can be allocated to the specific insertion point i. The definition of S_i will be discussed later in this section. Then, a neighborhood of the given solution z can be formulated (and explored) by solving an ILP problem (denoted as the *Reallocation Model*) based on the decision variables

$$x_{si} = \begin{cases} 1 & \text{if sequence } s \in S_i \text{ is allocated to insertion point } i \in I, \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

which reads as follows:

$$\sum_{r \in R} \tilde{c}(r) + \min \sum_{i \in I} \sum_{s \in S_i} \gamma_{si} x_{si} \tag{1.2}$$

subject to:

$$\sum_{i \in I} \sum_{s \in S_i(v)} x_{si} = 1 \qquad v \in \mathcal{F}, \tag{1.3}$$

$$\sum_{s \in S_i} x_{si} \leq 1 \qquad i \in I, \tag{1.4}$$

$$\sum_{i \in I(r)} \sum_{s \in S_i} q(s) x_{si} \leq Q - \tilde{q}(r) \qquad r \in \mathcal{R}, \tag{1.5}$$

$$\sum_{i \in I(r)} \sum_{s \in S_i} \gamma_{si} x_{si} \leq D - \tilde{c}(r) \qquad r \in \mathcal{R}, \tag{1.6}$$

$$x_{si} \in \{0, 1\} \qquad i \in I, \;\; s \in S_i, \tag{1.7}$$

where, for any $i \in I$ and $v \in F$, $S_i(v) \subseteq S_i$ denotes the set of sequences covering customer v which can be allocated to insertion point i. The objective function (1.2), to be minimized, gives the traveling cost of the final OVRP solution. Constraints (1.3) impose that each extracted node belongs to exactly one of the selected sequences, i.e., that it is covered exactly once in the final solution. Constraints (1.4) avoid to allocate two or more sequences to the same insertion point. Finally, constraints (1.5) and (1.6) impose that each route in the final solution fulfills the capacity and distance restrictions, respectively. Note that, if there is a non-null fixed cost F associated with the vehicles, it can be taken into account by simply adding F to the cost of the edges incident at the depot node.

The Reallocation Model (1.2)–(1.7) defines a neighborhood of a given solution z which depends on the extracted nodes $\mathcal{F}$ and on the subsets S_i ($i \in \mathcal{I}$). In particular, for any given $\mathcal{F}$, the choice of S_i is a key factor in order to allow an effective exploration of the solution space in the neighborhood of the given solution. The subsets S_i are built by following a column generation approach: we initialize the Linear Programming (LP) relaxation of the Reallocation Model (LP-RM) with a subsets of variables with small insertion cost, and afterwards we iteratively solve the column generation problem associated with LP-RM, adding other variables with *small* reduced cost. The overall procedure for building the subsets S_i can be described as follows.

1. (Initialization) For each insertion point $i = (a_i, b_i) \in \mathcal{I}$, initialize subset S_i with the *basic* sequence extracted from i (i.e., the, possibly empty, sequence of nodes connecting node a_i and b_i in the given solution z) plus the feasible singleton sequence with the minimum insertion cost (i.e., the sequence (v), with $v \in \mathcal{F}$, with the minimum extra-cost among all the singleton sequences which can be allocated to i without violating the capacity and distance restrictions for the restricted route containing i). Initialize LP-RM with the initial set of variables corresponding to the current subsets S_i, and solve LP-RM.

2. (Column generation) For each insertion point $i \in \mathcal{I}$, solve the *column generation problem* associated with i, adding to S_i all the sequences s corresponding to elementary paths in $\mathcal{F}$, whose associated variables x_{si} have a reduced cost rc_{si} under a given threshold RC_{max} (i.e., variables x_{si} such that $rc_{si} \leq RC_{max}$). If at least one sequence/variable has been added, solve the new LP-RM and repeat step 2. Otherwise terminate.

For any fixed insertion point $i \in \mathcal{I}$, the column generation problem associated with i in LP-RM is a Resource Constrained Elementary Shortest Path Problem (RCESPP), which usually arises in the Set Partitioning formulation of the classical VRP (see, e.g., Feillet et al. [24] and Righini and Salani [62]). Here, for each insertion point $i \in \mathcal{I}$, we solve the corresponding RCESPP through a simple greedy heuristic, with the aim of finding as many variables with small reduced cost as possible. Hashing techniques are used to avoid the generation of duplicated variables.

Note that each subset S_i contains the *basic* sequence extracted from insertion point i, and hence the current solution can always be obtained as a new feasible solution of the Reallocation Model.

1.3.1 Column generation for the Reallocation Model

Let π_v^1, π_i^2, π_r^3 and π_r^4 be the dual variables associated, respectively, with constraints (1.3), (1.4), (1.5) and (1.6) in LP-RM, where $v \in \mathcal{F}$, $i \in \mathcal{I}$ and $r \in \mathcal{R}$, and denote with $\tilde{\pi} = (\tilde{\pi}_v^1, \tilde{\pi}_i^2, \tilde{\pi}_r^3, \tilde{\pi}_r^4)$ the optimal dual solution of LP-RM. For any fixed $i = (a_i, b_i) \in \mathcal{I}$, consider the directed graph $\tilde{G}(i, \tilde{\pi}) = (V_i, A_i)$, with $V_i := \{a_i, b_i\} \cup \mathcal{F}$ and $A_i := \{(v, w) : v \in V_i, w \in V_i\} \backslash \{(a_i, b_i), (b_i, a_i)\}$. Associate with each arc $a = (v, w) \in A_i, w \neq 0$, a weight θ_a equal to the cost of the corresponding edge $e = (v, w)$ in the graph G, while set $\theta_a := 0$ for each arc $a = (v, 0) \in A_i$, if $0 \in V_i$. Associate with each arc $a \in A_i$ a cost $c'_a = \theta_a(1 - \tilde{\pi}_{r_i}^4)$, and associate with each node $v \in \mathcal{F}$ a weight q_v and a cost $q'_v = -(\tilde{\pi}^1_v + q_v\tilde{\pi}^3_{r_i})$. Then, let $P = (V_P, A_P)$ be an elementary path $(a_i, v_1, \ldots, v_h, b_i)$ connecting nodes a_i and b_i in $\tilde{G}(i, \tilde{\pi})$, where $V_P := \{v_1, \ldots, v_h\} \subseteq V_i$ and $A_P := \{(a_i, v_1), \ldots, (v_h, b_i)\} \subseteq A_i$. We say that P is a feasible path if

$$\sum_{v \in V_P} q_v \leq Q - \tilde{d}(r_i),$$

$$\sum_{a \in A_P} \theta_a \leq D - \tilde{c}(r_i) + c_i,$$

where c_i denotes the cost of insertion point $i = (a_i, b_i)$, while the cost of the path is

$$c'(P) = \sum_{a \in A_P} c'_a + \sum_{v \in V_P} q'_v.$$

Any sequence $s = (v_1, \ldots, v_h) \in S$ is clearly associated with the elementary path $(a_i, v_1, \ldots, v_h, b_i)$ in $\tilde{G}(i, \tilde{\pi})$. The reduced cost rc_{si} of variable x_{si} in LP-RM is defined by

$$rc_{si} := \gamma_{si} - \sum_{v \in V_P} \tilde{\pi}^1_v - \tilde{\pi}^2_i - q(s)\tilde{\pi}^3_{r_i} - \gamma_{si}\tilde{\pi}^4_{r_i}$$

and can easily be rewritten as

$$rc_{si} := -\tilde{\pi}^2_i - c_i(1 - \tilde{\pi}^4_{r_i}) + \sum_{a \in A_P} c'_a + \sum_{v \in V_P} q'_v.$$

Hence, the following proposition holds:

Proposition 1 *For any $i = (a_i, b_i) \in \mathcal{I}$, the column generation problem associated with i in LP-RM is the problem of finding an elementary feasible path P from a_i to b_i in $\tilde{G}(i, \tilde{\pi})$, with cost $c'(P) < \tilde{\pi}^2_i + c_i(1 - \tilde{\pi}^4_{r_i})$.*

As described above, the column generation problem for LP-RM associated with any insertion point $i \in \mathcal{I}$ is a Resource Constrained Elementary Shortest Path Problem (RCESPP) defined on graph $\tilde{G}(i, \tilde{\pi})$, whose size strictly depends on $|\mathcal{F}|$. The orientation of $\tilde{G}(i, \tilde{\pi})$ is required only when the considered $i = (a_i, b_i) \in \mathcal{I}$ is an appending insertion point (i.e., b_i is the depot node). Even in this case, the column generation problem could be addressed on a mixed graph, where only the edges incident at the depot are replaced by directed arcs (of different cost and weight) entering and leaving the depot. In the general case, $\tilde{G}(i, \tilde{\pi})$ contains negative cycles (i.e., cycles in which the sum of the costs c'_a associated with the arcs and the costs q'_v associated with the nodes is negative): indeed, while dual variables $\pi^2_i, \pi^3_r, \pi^4_r$ are non positive, dual variables π^1_v are free and usually assume positive values. Positive values of variables π^1_v can lead to negative node costs q'_v and to negative cycles in graph $\tilde{G}(i, \tilde{\pi})$. Therefore, the column generation problem in LP-RM is strongly $\mathcal{NP}$-hard.

In order to find a promising set of variables for the Reallocation Model in a short computing time, we solve the RCESPP associated with each insertion point through a simple heuristic. We say that a node $v \in \mathcal{F}$ is *feasible* for $i \in \mathcal{I}$ if the singleton sequence (v) can be allocated to i without violating the capacity and distance restrictions on the restricted route r_i. For any given insertion point $i = (a_i, b_i) \in \mathcal{I}$, we first build a *reduced* graph $\tilde{G}(i, \tilde{\pi})$, obtained by considering only nodes a_i, b_i and the nf feasible nodes of $\mathcal{F}$ with smallest insertion cost (i.e., the nf feasible nodes $v_k \in \mathcal{F}, k = 1, \ldots, nf$, whose associated singleton sequences (v_k) have the smallest extra-cost for i). At each iteration of the column generation step described in Section 1.3, nf is uniformly randomly generated in $[nf_{min}, nf_{max}]$. Then, on the reduced graph $\tilde{G}(i, \tilde{\pi})$, we apply the following simple heuristic:

1. Find an initial feasible path $P = (a_i, v, b_i)$, in $\tilde{G}(i, \tilde{\pi})$.

2. Evaluate all the 1-1 *feasible* exchanges between each node $w \in V_i \setminus V_P$ and each node $v \in V_P$, and select the best one (with respect to the cost of the corresponding path); if this exchange leads to an improvement, perform it and repeat step 2.

3. Evaluate all the *feasible* insertions of each node $w \in V_i \setminus V_P$ in each arc $(v_1, v_2) \in A_P$ and select the best one; if no feasible insertion exists, terminate; otherwise, force such an insertion even if it leads to a worse path and repeat step 2.

Whenever a new path in $\tilde{G}(i, \tilde{\pi})$ is generated , the corresponding sequence is added to S_i if the reduced cost of x_{si} is smaller than a given threshold RC_{max}.

1.4 Heuristic Improvement Procedure

The Reallocation Model described in the previous section allows for exploring a neighborhood of a given feasible solution, depending on the choice of the extracted customers in $\mathcal{F}$. We propose a heuristic improvement procedure for OVRP, based on model (1.2)–(1.7), which iteratively explores different neighborhoods of the current solution. Given an initial feasible solution z_0 for OVRP (taken from the literature or found by any heuristic method), the procedure works as follows.

1. (Initialization) Set $kt := 0$ and $kp := 0$. Take z_0 as the incumbent solution and initialize the current solution z_c as $z_c := z_0$.

2. (Node selection) Build set $\mathcal{F}$ by selecting each customer with a probability p.

3. (Node extraction) Extract the nodes selected in the previous step from the current solution z_c and construct the corresponding restricted OVRP solution $z_c(\mathcal{F})$, obtained by short-cutting the extracted nodes.

4. (Reallocation) Define the subsets S_i ($i \in \mathcal{I}(z_c, \mathcal{F})$) as described in Section 1.3. Build the corresponding Reallocation Model (1.2)–(1.7) and solve the model by using a general-purpose ILP solver. Once an optimal ILP solution has been found, construct the corresponding new OVRP solution and possibly update z_c and z_0.

5. (Termination) Set $kt := kt + 1$. If $kt = KT_{max}$, terminate.

6. (Perturbation) If z_c has been improved in the last iteration, set $kp := 0$; otherwise set $kp := kp + 1$. If $kp = KP_{max}$, "perturb" the current solution z_c and set $kp := 0$. In any case, repeat step 2.

The procedure performs KT_{max} iterations and at each iteration explores a randomly generated neighborhood of the current solution z_c. However, if z_c is not improved for KP_{max} consecutive iterations, we introduce a random perturbation (see Step 6) in order to move to a different area of the solution space, so as to enforce the diversification of the search. In particular, when performing a Perturbation Step, we randomly extract np customers from z_c (with np uniformly randomly chosen in $[np_{min}, np_{max}]$ and with each customer having the same probability to be extracted), and reinsert each extracted customer, in turn, in its best feasible position. If a customer cannot be inserted in any currently non-empty route (due to the capacity and/or distance restrictions), a new route is created to allocate the customer. In general, when performing the Perturbation Step, several customers cannot be inserted in the non-empty routes of the current solution, and hence the new perturbed solution can use more vehicles than the current one.

1.5 Computational Results

The performance of the Heuristic Improvement Procedure (HIP) described in the previous sections was evaluated on the 16 benchmark instances usually addressed in the literature, taken from Christofides et al. [12] (instances `C1`–`C14`) and from Fisher [29] (instances `F11`–`F12`), and on the 8 large scale benchmark instances proposed by Li et al. [46], and also addressed by Derigs and Reuter [21] (instances `O1`–`O8`). The number of customers of `C1`–`C14` and `F11`–`F12` ranges from 50 to 199. `C1`–`C5`, `C11`–`C12` and `F11`–`F12` have only capacity constraints, while `C6`–`C10` and `C13`–`C14` are the same instances as `C1`–`C5` and `C11`–`C12`, respectively, but with both capacity and distance constraints. Instances `O1`–`O8` have no distance restrictions and a number of customers varying from 200 to 480. As usual, for the problems with distance constraints, the route duration limit D is taken as the original value for the classical VRP multiplied by 0.9.

HIP needs an initial solution to be given, which in principle could be computed through any available constructive heuristic algorithm. We decided to run HIP starting from an extremely-good feasible solution available from the literature (in several cases, the best known solution reported in the literature), with the aim of attempting to improve it (this is of course impossible if the initial solution is provably optimal, as it is the case for some of them). In particular, we considered as initial solutions the ones obtained by Fu et al. [31, 32], Pisinger and Ropke [58], Derigs and Reuter [21] and Fleszar et al. [30].

HIP has been tested on a Pentium IV 3.4 GHz with 1 GByte RAM, running under Microsoft Windows XP Operative System, and has been coded in C++ with Microsoft Visual C++ 6.0 compiler. The ILP solver used in the experiments is ILOG Cplex 10.0 [39]. HIP setting depends on the parameters RC_{max}, p, nf_{min}, nf_{max}, np_{min}, np_{max}, and on the number of iterations KP_{max} and KT_{max}. Although these parameters could be tuned considering the edge costs and the particular characteristics of each tested instance, we preferred to run all the experiments with a fixed set of parameters: $RC_{max} = 1$, $p = 0.5$ (i.e., 50% of the customers are selected on average), $nf_{min} = 15$, $nf_{max} = 25$, $np_{min} = 15$, $np_{max} = 25$, $KP_{max} = 50$ and $KT_{max} = 5,000$ (i.e., we perform globally 5,000 iterations, and the current solution is perturbed if it cannot be improved for 50 consecutive iterations). Further, since several authors address the problem considering as objective function the minimization of the number of vehicles first and of the traveling cost second (i.e., assuming $F = \infty$), while other authors considered as objective function the minimization of the traveling cost (i.e., $F = 0$), we decided to run HIP without allowing to change the number of vehicles used in the initial solution. However, as stated in Section 1.4, the Perturbation Step often requires additional routes to be created (to preserve the feasibility of the solution). In such cases, we add a small penalty θ to the cost of the edges incident at the depot, in order to force HIP to "recover" the solution in the following iterations. After some preliminary tests, we decided to fix $\theta = 12$ for the considered instances. Finally, HIP is a randomized algorithm and hence the computational results may depend on the randomization. For each tested instance (and each initial solution), we considered 5 runs of the algorithm corresponding to 5 different seeds for generating the random numbers.

The computational results are reported in Tables 1.1–1.3. All the CPU times are expressed in seconds, and all the solution costs have been computed in double precision.

Table 1.1 reports the computational results on the 16 instances C1–C14 and F11–F12 obtained by starting from the solutions provided by Fu, Eglese and Li and obtained through the algorithm proposed in [31]. In some cases, several solutions are provided for the same instance, obtained by using slightly different versions of their algorithm, with the same number of routes and different traveling cost. Among the different solutions for the same instance, we considered as initial solution for HIP the best one provided. For instances `C1` and `F11`, all the solutions available from [31, 32] are provably optimal (see, e.g., Letchford et al. [45]) and cannot be further improved. Thus, these instances were not considered in this set of experiments. The upper part of the table reports the solutions found by HIP. The first column gives the instance name (*Pb*). Columns 2–3 report the number of vehicles used in the initial solution (m) and the cost of the best known solution using the same number of vehicles (*P.best*). Columns 4–5 report the cost of the initial solution (*cost*) and the corresponding percentage deviation w.r.t. the best known value (*%dev*), computed as 100*(*cost-P.best*)/*P.best*. Then, for each of the 5 runs of the algorithm, we report the final solution cost provided by HIP and the corresponding percentage deviation (again computed w.r.t. the best known value). When HIP was not able to improve on the initial solution, we mark with a "—" the final solution cost. Finally, we report the best, the worst and the average result out of the 5 different runs. Final solution costs equal to the previously best known ones are underlined, new best solutions are in bold face, while provably optimal solutions, taken from Letchford et al. [45], are marked with an *. The lower part of the table gives the computing times. First, we report the overall CPU time of the algorithm corresponding to the initial solution, obtained on a Pentium IV 3 GHz. These times have been taken from [32]. However, the cost of the initial solution for instance `C8` is better than the ones reported in [32], and hence for this initial solution we did not report the corresponding computing time. Then, for each run of the algorithm, we report the overall computing time required to perform all the 5,000 iterations (*t.time*) and the CPU time required to reach the final solution (*b.time*). For a "fair" calculation of the average values, when HIP was not able to improve on the initial solution we considered *b.time* equal to the overall computing time. Finally, the last two columns give the average CPU times (i.e., average *t.time* and average *b.time*) out of the 5 different runs.

Table 1.2 reports the computational results on the same instances by starting from the best available solutions among the ones obtained by Fu et al. [31, 32], Pisinger and Ropke [58], Derigs and Reuter [21] and Fleszar et al. [30]. The table has the same structure as Table 1.1, but column 2 in the lower part of the table reports the source of the initial solution used in the experiments. For instances `C5`, `C7`, `C9`, `C13` and `C14`, the best available solutions for the case $F = \infty$ and the case $F = 0$ are different. In such cases, we considered both the solutions as initial solutions for HIP. For instances `C1`, `C3`, `C12` and `F11`, all the solutions available from [21], [30] and [58] are provably optimal and hence these instances were not considered in this set of experiments.

Finally, Table 1.3 reports the computational results on the 8 large scale instances `O1`–`O8` by starting from the solutions provided by Derigs and Reuter [21]. The table has the same structure as Table 1.1, but the CPU time related to the initial solution (column 2 in the lower part of the table) was obtained on a Pentium IV 2.8 GHz.

The tables show that HIP is able to improve even extremely-good quality solutions, obtained by some of the most effective metaheuristic techniques proposed for OVRP. It is worth noting that the solutions and the CPU times provided by Fu et al. [31, 32] and reported in Table 1.1 are the best ones from among 20 runs of the corresponding randomized algorithm with different seeds. Hence, taking into account the different performance of the processors used for testing the different algorithms, the overall computing time required by HIP is comparable with the others reported in the tables, and in several cases the final improved solution is found very quickly. Our test-bed concerns in practice 35 different, non provably optimal, initial solutions which could be possibly improved, corresponding to 22 different instances. By considering the best result from among the 5 different runs executed for each of these 35 initial solutions, HIP improves on the initial solution in 22 cases. For these cases, HIP reaches 6 times the previously best known solution (provably optimal in 2 cases), while finds 12 times a new best solution. Considering the 13 initial solutions which HIP does not improve, it is worth noting that all these solutions are the best known ones in the literature (for the case $F = \infty$ or $F = 0$). Looking at the different runs executed for each initial solution, we can note that in some cases the results depend on the seed used for the random generator. However, the method is overall quite consistent since, by considering all the tested initial solutions, the average computing time and the average final percentage deviation are only slightly affected by the choice of the seed.

In order to look for possible better solutions, we performed some additional experiments. In particular, after the first 5,000 iterations, we ran HIP for 2,000 more iterations with a slightly different parameter setting. Starting from the solutions provided by Fu et al. [32], for instance `C5` with 17 vehicles, after 5220 iterations and 237.4 seconds HIP found a solution of cost 868.44 that corresponds to a further improvement on the previous best known solution. Finally, still starting from the solutions by Fu et al. [32], we ran HIP with a different tuning of parameter p, to investigate how the neighborhood size affects the overall performance of the method, both in terms of quality of the solutions found and of CPU time. Let $z_{avg}(\bar{p})$ be the average final solution cost obtained on the 14 instances `C2`–`C14` and `F12` with $p = \bar{p}$, and let $ttime_{avg}(\bar{p})$ be the corresponding average CPU time in seconds. With $p = 0.3$, $p = 0.5$ and $p = 0.7$ we obtained the following results: $z_{avg}(0.3) = 684.55$ and $ttime_{avg}(0.3) = 71.9$, $z_{avg}(0.5) = 681.94$ and $ttime_{avg}(0.5) = 262.8$, $z_{avg}(0.7) = 683.32$ and $ttime_{avg}(0.7) = 460.0$. As expected, the average CPU time consistently increases with the number of extracted customers, while the best solution costs are

Table 1.1: Computational results on the "classical" 16 benchmark instances starting from the solutions by Fu et al. [31, 32]. CPU times are expressed in seconds.

			Initial		Run 1		Run 2		Run 3		Run 4		Run 5		Best		Worst		Average	
Pb	m	P.best	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev
C2	10	567.14	567.14	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	567.14	0.00	567.14	0.00	567.14	0.00
C3	8	*639.74	641.88	0.33	*639.74	0.00	640.42	0.11	640.42	0.11	640.42	0.11	640.42	0.11	*639.74	0.00	640.42	0.11	640.28	0.08
C4	12	733.13	738.94	0.79	733.13	0.00	733.13	0.00	733.13	0.00	733.13	0.00	733.13	0.00	733.13	0.00	733.13	0.00	733.13	0.00
C5	17	869.25	878.95	1.12	**868.81**	-0.05	**868.81**	-0.05	**868.81**	-0.05	**868.81**	-0.05	**868.81**	-0.05	**868.81**	-0.05	**868.81**	-0.05	**868.81**	-0.05
C6	6	412.96	412.96	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	412.96	0.00	412.96	0.00	412.96	0.00
C7	11	568.49	568.49	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	568.49	0.00	568.49	0.00	568.49	0.00
C8	9	644.63	646.31	0.26	644.63	0.00	644.63	0.00	644.63	0.00	644.63	0.00	644.63	0.00	644.63	0.00	644.63	0.00	644.63	0.00
C9	14	756.14	761.28	0.68	756.14	0.00	756.14	0.00	756.38	0.03	756.38	0.03	756.14	0.00	756.14	0.00	756.38	0.03	756.24	0.01
C10	17	875.07	903.10	3.20	878.54	0.40	879.13	0.46	877.47	0.27	880.25	0.59	879.68	0.53	877.47	0.27	880.25	0.59	879.01	0.45
C11	7	682.12	717.15	5.14	683.64	0.22	685.20	0.45	685.20	0.45	682.83	0.10	682.83	0.10	682.83	0.10	685.20	0.45	683.94	0.27
C12	10	*534.24	534.71	0.09	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00
C13	12	896.50	917.90	2.39	**894.19**	-0.26	897.37	0.10	896.66	0.02	897.37	0.10	**896.14**	-0.04	**894.19**	-0.26	897.37	0.10	**896.35**	-0.02
C14	11	591.87	600.66	1.49	591.87	0.00	591.87	0.00	591.87	0.00	591.87	0.00	591.87	0.00	591.87	0.00	591.87	0.00	591.87	0.00
F12	7	769.66	777.07	0.96	**769.55**	-0.01	770.38	0.09	**769.55**	-0.01	770.38	0.09	770.38	0.09	**769.55**	-0.01	770.38	0.09	770.05	0.05
avg.				1.18		0.02		0.08		0.06		0.07		0.05		0.00		0.09		0.06

	Initial	Run 1		Run 2		Run 3		Run 4		Run 5		Average	
Pb	t.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time
C2	7.8	84.2	*84.2*	86.5	*86.5*	80.2	*80.2*	81.9	*81.9*	89.6	*89.6*	84.5	84.5
C3	23.2	119.9	106.0	110.9	74.7	117.9	56.8	139.9	132.0	118.8	88.3	121.5	91.6
C4	6.8	156.6	21.2	157.8	0.6	154.7	0.4	198.1	0.8	164.1	0.7	166.3	4.7
C5	61.9	220.3	10.3	220.0	11.6	228.5	32.7	277.8	12.9	225.5	12.5	234.4	16.0
C6	0.6	45.1	*45.1*	43.2	*43.2*	49.0	*49.0*	44.7	*44.7*	42.2	*42.2*	44.8	44.8
C7	6.0	83.1	*83.1*	87.7	*87.7*	83.2	*83.2*	79.8	*79.8*	79.5	*79.5*	82.7	82.7
C8		136.2	0.1	135.9	0.1	144.9	3.2	177.7	2.6	144.7	0.1	147.9	1.2
C9	46.6	255.5	102.7	265.5	7.9	247.7	195.7	312.1	299.6	259.2	18.3	268.0	124.8
C10	51.9	460.2	323.9	477.7	35.9	513.0	453.1	505.9	474.5	497.8	366.5	490.9	330.8
C11	23.1	198.8	165.8	199.8	40.4	229.8	225.7	329.6	204.2	201.4	200.0	231.9	167.2
C12	4.2	94.0	1.6	98.2	73.2	99.1	89.5	106.2	16.6	107.8	92.1	101.1	54.6
C13	82.1	1165.3	475.0	1004.9	201.4	1180.5	685.0	1146.5	725.2	1396.5	1230.9	1178.7	663.5
C14	2.5	354.7	293.8	339.2	88.6	321.3	33.7	366.6	337.3	453.9	435.8	367.1	237.8
F12	28.4	148.2	77.8	158.0	74.9	154.8	70.7	169.3	30.9	168.2	67.5	159.7	64.4
avg.	24.7	251.6	127.9	241.8	59.1	257.5	147.1	281.2	174.5	282.1	194.6	262.8	140.6

Table 1.2: Computational results on the "classical" 16 benchmark instances starting from the best available solutions. CPU times are expressed in seconds.

			Initial		Run 1		Run 2		Run 3		Run 4		Run 5		Best		Worst		Average	
Pb	m	P.best	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev
C2	10	567.14	567.14	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	567.14	0.00	567.14	0.00	567.14	0.00
C4	12	733.13	733.13	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	733.13	0.00	733.13	0.00	733.13	0.00
C5	16	879.37	896.08	1.90	892.37	1.48	892.37	1.48	892.37	1.48	892.37	1.48	892.37	1.48	892.37	1.48	892.37	1.48	892.37	1.48
C5	17	869.24	869.24	0.00	**868.93**	-0.04	**868.93**	-0.04	**869.00**	-0.03	**868.93**	-0.04	**868.93**	-0.04	**868.93**	-0.04	**869.00**	-0.03	**868.94**	-0.03
C6	6	412.96	412.96	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	412.96	0.00	412.96	0.00	412.96	0.00
C7	10	583.19	583.19	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	583.19	0.00	583.19	0.00	583.19	0.00
C7	11	568.49	568.49	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	568.49	0.00	568.49	0.00	568.49	0.00
C8	9	644.63	644.63	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	644.63	0.00	644.63	0.00	644.63	0.00
C9	13	757.84	757.84	0.00	**757.73**	-0.01	**757.69**	-0.02	**757.70**	-0.02	**757.73**	-0.01	**757.73**	-0.01	**757.69**	-0.02	**757.73**	-0.01	**757.72**	-0.02
C9	14	756.14	756.14	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	756.14	0.00	756.14	0.00	756.14	0.00
C10	17	875.07	875.07	0.00	**874.71**	-0.04	**874.71**	-0.04	**874.71**	-0.04	**874.71**	-0.04	**874.71**	-0.04	**874.71**	-0.04	**874.71**	-0.04	**874.71**	-0.04
C11	7	682.12	682.12	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	682.12	0.00	682.12	0.00	682.12	0.00
C13	11	904.04	904.04	0.00	**899.16**	-0.54	**899.16**	-0.54	**899.16**	-0.54	**899.16**	-0.54	**899.16**	-0.54	**899.16**	-0.54	**899.16**	-0.54	**899.16**	-0.54
C13	12	896.50	917.90	2.39	**894.19**	-0.26	897.37	0.10	896.66	0.02	897.37	0.10	**896.14**	-0.04	**894.19**	-0.26	897.37	0.10	**896.35**	-0.02
C14	11	591.87	591.87	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	591.87	0.00	591.87	0.00	591.87	0.00
C14	12	581.81	581.81	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	581.81	0.00	581.81	0.00	581.81	0.00
F12	7	769.66	769.66	0.00	**769.55**	-0.01	**769.55**	-0.01	—	0.00	**769.55**	-0.01	—	0.00	**769.55**	-0.01	769.66	0.00	**769.59**	-0.01
avg.				0.25		0.03		0.05		0.05		0.06		0.05		0.03		0.06		0.05

Pb	source	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time
C2	[30, 32, 58]	84.2	*84.2*	86.5	*86.5*	80.2	*80.2*	81.9	*81.9*	89.6	*89.6*	84.5	84.5
C4	[30, 58]	151.2	*151.2*	137.5	*137.5*	123.2	*123.2*	164.5	*164.5*	153.5	*153.5*	146.0	146.0
C5	[58]	450.2	276.5	480.4	237.5	463.4	237.5	434.0	237.5	518.3	237.5	469.3	245.3
C5	[21]	275.2	130.2	247.5	195.4	278.1	21.4	260.5	205.0	255.8	166.9	263.4	143.8
C6	[30, 32, 58]	45.1	*45.1*	43.2	*43.2*	49.0	*49.0*	44.7	*44.7*	42.2	*42.2*	44.8	44.8
C7	[58]	80.6	*80.6*	86.1	*86.1*	73.1	*73.1*	81.2	*81.2*	85.2	*85.2*	81.2	81.2
C7	[32]	83.1	*83.1*	87.7	*87.7*	83.2	*83.2*	79.8	*79.8*	79.5	*79.5*	82.7	82.7
C8	[30]	136.8	*136.8*	142.2	*142.2*	137.2	*137.2*	130.0	*130.0*	143.9	*143.9*	138.0	138.0
C9	[58]	412.5	33.9	404.6	330.6	372.9	0.2	355.4	11.0	413.0	6.9	391.7	76.5
C9	[21]	243.4	*243.4*	213.9	*213.9*	221.9	*221.9*	227.6	*227.6*	267.1	*267.1*	234.8	234.8
C10	[21]	454.7	2.6	390.1	2.4	344.0	1.7	395.6	4.2	387.6	0.7	394.4	2.3
C11	[30, 58]	178.3	*178.3*	183.4	*183.4*	181.0	*181.0*	183.4	*183.4*	165.3	*165.3*	178.3	178.3
C13	[30]	959.3	6.3	1022.0	133.3	1030.3	6.6	1027.2	4.6	980.5	78.6	1003.9	45.9
C13	[32]	1165.3	475.0	1004.9	201.4	1180.5	685.0	1146.5	725.2	1396.5	1230.9	1178.7	663.5
C14	[30, 58]	276.3	*276.3*	293.8	*293.8*	263.6	*263.6*	294.4	*294.4*	301.1	*301.1*	285.8	285.8
C14	[21]	364.2	*364.2*	304.0	*304.0*	310.5	*310.5*	290.4	*290.4*	354.1	*354.1*	324.6	324.6
F12	[30]	142.2	56.6	157.2	103.4	143.9	*143.9*	137.2	64.2	129.7	*129.7*	142.0	99.6
avg.		323.7	154.4	310.9	163.7	313.9	154.1	313.8	166.4	339.0	207.8	320.2	169.3

Table 1.3: Computational results on the 8 large scale benchmark instances starting from the solutions by Derigs and Reuter [21]. CPU times are expressed in seconds.

			Initial		Run 1		Run 2		Run 3		Run 4		Run 5		Best		Worst		Average	
Pb	m	P.best	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev
O1	5	6018.52	6018.52	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	6018.52	0.00	6018.52	0.00	6018.52	0.00
O2	9	4584.55	4584.69	0.00	**4573.53**	-0.24	**4573.53**	-0.24	**4573.53**	-0.24	**4573.53**	-0.24	**4573.53**	-0.24	**4573.53**	-0.24	**4573.53**	-0.24	**4573.53**	-0.24
O3	7	7731.46	7731.46	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	7731.46	0.00	7731.46	0.00	7731.46	0.00
O4	10	7260.59	7260.59	0.00	**7259.81**	-0.01	**7253.91**	-0.09	**7253.91**	-0.09	**7253.20**	-0.10	**7251.74**	-0.12	**7251.74**	-0.12	**7259.81**	-0.01	**7254.51**	-0.08
O5	9	9167.19	9167.19	0.00	**9165.40**	-0.02	**9156.74**	-0.11	**9157.42**	-0.11	**9159.22**	-0.09	**9159.22**	-0.09	**9156.74**	-0.11	**9165.40**	-0.02	**9159.6**	-0.08
O6	9	9803.80	9805.45	0.02	—	0.02	—	0.02	—	0.02	—	0.02	9804.25	0.00	9804.25	0.00	9805.45	0.02	9805.21	0.01
O7	10	10348.57	10348.57	0.00	**10344.37**	-0.04	**10344.37**	-0.04	**10344.37**	-0.04	**10344.37**	-0.04	**10344.37**	-0.04	**10344.37**	-0.04	**10344.37**	-0.04	**10344.37**	-0.04
O8	10	12420.16	12420.16	0.00	—	0.00	—	0.00	—	0.00	—	0.00	—	0.00	12420.16	0.00	12420.16	0.00	12420.16	0.00
avg.				0.00		-0.04		-0.06		-0.06		-0.06		-0.06		-0.06		-0.04		-0.05

Pb	Initial t.time	Run 1 t.time	Run 1 b.time	Run 2 t.time	Run 2 b.time	Run 3 t.time	Run 3 b.time	Run 4 t.time	Run 4 b.time	Run 5 t.time	Run 5 b.time	Average t.time	Average b.time
O1	467.0	182.2	*182.2*	191.5	*191.5*	174.0	*174.0*	168.5	*168.5*	175.9	*175.9*	178.4	178.4
O2	467.0	284.0	34.6	298.6	233.0	302.8	89.2	313.0	63.3	395.5	360.6	318.8	156.1
O3	4047.0	304.6	*304.6*	279.6	*279.6*	300.6	*300.6*	295.5	*295.5*	296.8	*296.8*	295.4	295.4
O4	927.0	438.9	34.4	405.5	387.5	437.6	72.6	421.7	219.9	406.3	385.4	422.0	220.0
O5	1186.0	499.6	41.4	479.1	270.8	513.9	496.5	550.3	173.6	479.5	210.5	504.5	238.6
O6	1231.0	581.3	*581.3*	590.4	*590.4*	637.4	*637.4*	620.1	*620.1*	590.8	361.1	604.0	558.1
O7	3190.0	653.0	8.6	631.8	23.8	661.6	420.9	743.5	306.6	619.7	387.2	661.9	229.4
O8	1969.0	623.6	*623.6*	635.2	*635.2*	668.9	*668.9*	653.7	*653.7*	647.9	*647.9*	645.9	645.9
avg.	1685.5	445.9	226.3	438.9	326.5	462.1	357.5	470.8	312.7	451.6	353.2	453.9	315.2

obtained with the default setting of p (i.e., $p = 0.5$), thus indicating that extracting too many customers leads in general to worse solutions (i.e., $z_{avg}(0.7) > z_{avg}(0.5)$). This is not completely surprising, and it is essentially due to the column generation heuristic, which falls in troubles in finding good variables for the Reallocation Model when the current solution has been almost completely "destroyed" by the removal of too many customers.

As previously seen, the proposed algorithm is able to improve on high-quality initial solutions. However, a natural question concerns the effectiveness of the method if the initial solution is instead a "bad-quality" solution. To answer this question, we implemented a modified version of the tabu search algorithm proposed by Fu et al. [31] (we refer the reader to [31] for a detailed description of this algorithm). More precisely, we first computed an initial random (and typically infeasible) solution, and then we applied only 200 iterations of the tabu search algorithm, with the aim of quickly finding a feasible solution, possibly "far" from the good ones. The computational results provided by HIP on the 16 instances `C1`–`C14` and `F11`–`F12` when starting from such initial solutions are reported in Table 1.4.

The table has the same structure as Table 1.1 and shows that HIP is quite effective even when the initial solution is not a good-quality solution. First, we can note that all the solutions are improved by all the 5 different runs. Further, even in this case the method is quite consistent, as all the 5 different runs provide on average very similar results, both in terms of quality of the solutions found and of CPU time. Finally, considering all the instances and all the different runs, the average behavior of the algorithm is satisfactory: starting from a set of initial solutions with an average percentage deviation (w.r.t. the best known value) of 19.67, HIP finds a set of final solutions with an average percentage deviation of 0.70 in an average overall computing time of 257.4 seconds.

The current best known solution costs for the tested instances are given in summary in Table 1.5, where we also report the number of customers n and the route duration limit D associated with the vehicles. Solution costs are given both for the case $F = \infty$ (i.e., when the objective is to minimize the number of used vehicles first and the traveling cost second) and the case $F = 0$ (i.e., when the objective is to minimize the traveling cost). As usual, the best known solution cost for the case $F = 0$ is reported only if the traveling cost is smaller than the corresponding one for the case $F = \infty$. For each instance whose best known solution was not improved by HIP we report the algorithms providing the corresponding best known costs. Previously best known solution costs reached also by HIP (starting from a worse solution) are underlined, while new best solution costs found by HIP are in bold face. For the capacitated instances, in the case $F = \infty$, we also report the best known lower bound LB taken from [45] and [56].

Table 1.4: Computational results on the “classical” 16 benchmark instances starting from “bad initial solutions”. CPU times are expressed in seconds.

			Initial		Run 1		Run 2		Run 3		Run 4		Run 5		Best		Worst		Average	
Pb	m	P.best	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev	cost	%dev
C1	5	*416.06	467.80	12.44	417.37	0.31	417.36	0.31	*416.06	0.00	*416.06	0.00	417.37	0.31	*416.06	0.00	417.37	0.31	416.84	0.19
C2	11	564.06	657.07	16.49	564.06	0.00	564.06	0.00	564.06	0.00	564.06	0.00	564.06	0.00	564.06	0.00	564.06	0.00	564.06	0.00
C3	8	*639.74	768.93	20.19	642.14	0.38	642.14	0.38	642.98	0.51	642.14	0.38	643.75	0.63	642.14	0.38	643.75	0.63	642.63	0.45
C4	12	733.13	1069.38	45.86	738.05	0.67	741.75	1.18	748.63	2.11	742.11	1.22	744.15	1.50	738.05	0.67	748.63	2.11	742.94	1.34
C5	17	869.24	1449.20	66.72	887.40	2.09	879.89	1.23	882.12	1.48	887.48	2.10	887.85	2.14	879.89	1.23	887.85	2.14	884.95	1.81
C6	6	412.96	444.98	7.75	416.84	0.94	412.96	0.00	416.85	0.94	416.84	0.94	412.96	0.00	412.96	0.00	416.85	0.94	415.29	0.56
C7	11	568.49	654.27	15.09	568.49	0.00	568.49	0.00	568.49	0.00	568.49	0.00	569.51	0.18	568.49	0.00	569.51	0.18	568.69	0.04
C8	9	644.63	752.98	16.81	647.56	0.45	645.16	0.08	645.16	0.08	645.16	0.08	645.16	0.08	645.16	0.08	647.56	0.45	645.64	0.16
C9	14	756.14	896.61	18.58	756.81	0.09	756.81	0.09	757.78	0.22	756.38	0.03	759.60	0.46	756.38	0.03	759.60	0.46	757.48	0.18
C10	17	875.07	983.97	12.44	901.18	2.98	898.16	2.64	897.99	2.62	886.75	1.33	887.69	1.44	886.75	1.33	901.18	2.98	894.35	2.20
C11	7	682.12	835.93	22.55	690.83	1.28	689.24	1.04	691.10	1.32	692.63	1.54	691.36	1.35	689.24	1.04	692.63	1.54	691.03	1.31
C12	10	*534.24	545.25	2.06	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00	*534.24	0.00
C13	12	896.50	1025.11	14.35	902.87	0.71	912.53	1.79	904.17	0.86	905.14	0.96	905.79	1.04	902.87	0.71	912.53	1.79	906.10	1.07
C14	12	581.81	641.66	10.29	581.92	0.02	581.92	0.02	581.92	0.02	581.92	0.02	581.92	0.02	581.92	0.02	581.92	0.02	581.92	0.02
F11	4	*177.00	201.27	13.71	*177.00	0.00	*177.00	0.00	*177.00	0.00	*177.00	0.00	*177.00	0.00	*177.00	0.00	*177.00	0.00	*177.00	0.00
F12	7	769.66	919.22	19.43	783.41	1.79	784.12	1.88	782.66	1.69	785.35	2.04	784.12	1.88	782.66	1.69	785.35	2.04	783.93	1.85
avg.				19.67		0.73		0.66		0.74		0.67		0.69		0.45		0.98		0.70

Pb	t.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time	t.time	b.time
C1	0.0	70.2	20.6	72.3	14.9	65.8	33.6	61.9	14.8	61.4	25.0	66.3	21.8
C2	0.1	86.2	62.2	78.2	28.7	94.3	32.0	83.0	58.5	77.2	52.1	83.8	46.7
C3	0.1	136.4	69.1	110.7	98.4	110.3	37.6	119.8	68.3	112.0	36.0	117.8	61.9
C4	0.2	177.8	127.5	159.2	88.3	188.9	148.5	186.4	64.3	177.5	79.1	178.0	101.5
C5	0.4	271.1	269.7	269.1	224.8	291.6	173.5	262.1	152.7	268.2	238.2	272.4	211.8
C6	0.0	48.5	0.6	44.0	26.8	51.1	0.6	50.6	0.5	45.9	10.7	48.0	7.8
C7	0.1	85.0	68.3	75.6	15.5	81.3	13.7	75.3	43.2	75.7	23.0	78.6	32.7
C8	0.1	153.1	87.3	160.8	123.6	169.6	29.2	153.3	51.3	151.7	110.5	157.7	80.4
C9	0.2	295.2	138.5	298.2	200.5	317.3	250.9	281.9	258.4	304.3	260.2	299.4	221.7
C10	0.4	705.2	665.0	721.8	524.1	729.1	719.1	828.4	678.8	584.2	556.8	713.7	628.8
C11	0.1	219.8	145.9	176.5	45.1	248.5	217.7	227.1	145.8	227.3	194.7	219.8	149.8
C12	0.1	99.6	23.0	89.8	61.3	96.2	12.5	102.2	74.2	96.6	27.9	96.9	39.8
C13	0.1	1113.8	393.4	1359.0	1270.8	1105.3	787.1	845.0	603.9	1244.1	562.0	1133.4	723.4
C14	0.1	363.1	213.6	327.1	124.8	486.7	421.3	305.6	159.2	452.1	325.2	386.9	248.8
F11	0.1	97.9	59.1	79.8	74.7	88.9	25.7	91.1	53.0	88.2	31.6	89.2	48.8
F12	0.2	190.9	36.4	176.7	80.3	152.2	43.5	178.8	140.4	181.7	43.9	176.1	68.9
avg.	0.1	257.1	148.8	262.4	187.7	267.3	184.2	240.8	160.5	259.3	161.1	257.4	168.4

Table 1.5: Current best known solution costs for the tested OVRP benchmark instances.

Inst.	n	D	Best known solution						
			$F=\infty$				$F=0$		
			m	LB	cost	best heuristics	m	cost	best heursitiscs
C1	50		5	416.1	*416.06	[7], [21], [30], [31,32], [46], [58]	6	412.96	[73], [74], [75]
C2	75		10	559.62	567.14	[21], [30], [31,32], [46], [58]	11	564.06	[73], [74], [75]
C3	100		8	639.7	*639.74	[21], [30], [46]	9	639.57	[75]
C4	150		12	730.2	733.13	[21], [30], [46], [58]			
C5	199		16	848.5	879.37	[74]	17	**868.44**	
C6	50	180	6		412.96	[7], [21], [30], [31,32], [46], [58]			
C7	75	144	10		583.19	[58]	11	568.49	[21], [31,32], [46]
C8	100	207	9		644.63	[7], [21], [30], [46]			
C9	150	180	13		**757.69**		14	756.14	[21]
C10	199	180	17		**874.71**				
C11	120		7	657.1	682.12	[21], [30], [58]	10	678.54	[75]
C12	100		10	534.2	*534.24	[21], [30], [46], [58], [73], [74], [75]			
C13	120	648	11		**899.16**		12	**894.19**	
C14	100	936	11		591.87	[21], [30], [46], [58]	12	581.81	[21]
F11	71		4	177.0	*177.00	[21], [31,32], [46], [58]			
F12	134		7	762.9	**769.55**				
O1	200		5		6018.52	[21], [46]			
O2	240		9		**4573.53**				
O3	280		7		7731.46	[21]			
O4	320		10		**7251.74**				
O5	360		8		9197.61	[46]	9	**9156.74**	
O6	400		9		9803.80	[46]			
O7	440		10		**10344.37**				
O8	480		10		12420.16	[21]			

1.6 Conclusions and Future Directions

We addressed the Open Vehicle Routing Problem (OVRP), a variant of the "classical" Vehicle Routing Problem (VRP) in which the vehicles are not required to return to the depot after completing their service. OVRP has recently received an increasing attention in the literature, and several heuristic and metaheuristic algorithms have been proposed for this problem, as well as exact approaches.

We presented a heuristic improvement procedure for OVRP based on Integer Linear Programming (ILP) techniques. Given an initial solution to be possibly improved, the method follows a destruct-and-repair paradigm, where the given solution is randomly destroyed (i.e., customers are removed in a random way) and repaired by solving an ILP model, in the attempt of finding a new improved solution.

Computational results on 24 benchmark instances from the literature showed that the proposed improvement method can be used as a profitable tool for finding good-quality OVRP solutions, and that even extremely-good quality solutions found by the most effective metaheuristic techniques proposed for OVRP can be improved. Out of 30 best known solutions which are not provably optimal, in 10 cases the proposed method was able to improve on the best known solution reported in the literature.

Future directions of work could involve more sophisticated criteria for removing customers from the current solution, as well as more sophisticated algorithms for solving the column generation problem related to the ILP model. On the other side, the overall procedure can be considered as a general framework and it could be extended to cover other variants of Vehicle Routing Problems, as, for example, Vehicle Routing Problems with heterogenous vehicles and multi-depot Vehicle Routing Problems.

Chapter 2

An Integer Linear Programming Based Heuristic Approach for the Capacitated m-Ring-Star Problem

2.1 Introduction

Introduced by Baldacci et al. [6] in 2007, the Capacitated m-Ring-Star Problem (CmRSP), has many applications in telecommunication networks, in particular in the fiber optic communication networks (see, e.g. Baldacci et al. [6])[1].

The CmRSP can be described as follows: a mixed graph $G = (V, E \cup A)$ is given, where V is the set of nodes, $E = \{\{i, j\} : i, j \in V, i \neq j\}$ is the set of edges (undirected arcs) and A is the set of arcs. The node set V is defined as $V = \{0\} \cup U \cup W$, where node 0, U and W represent, respectively, the central depot, the set of customers and the set of Steiner nodes and for each customer $i \in U$, $C_i \subset U \cup W$ denotes the subset of nodes to which customer i can be connected. The arc set A is defined as $A = \{(i, j) : i \in U, j \in C_i\}$. Each edge $e \in E$ has a non negative *visiting* cost c_e and a non negative *allocation* cost d_{ij}.

We refer to a simple cycle consisting of a subset of nodes and the depot as a *ring* and if a customer is visited by the ring or allocated to a node of the ring, it is *assigned* to that ring. Two input parameters m and Q are the number of rings and the capacity of each ring, respectively, and we have $mQ \geq |U|$.

A solution of the CmRSP is feasible if each customer is assigned to exactly one ring, no Steiner node is used more than once, and the number of customers assigned to a ring

[1]The results of this chapter appear in: Naji-Azimi Z., Salari M., and Toth P.: "An Integer Linear Programming Based Heuristic Approach for the Capacitated m-Ring-Star Problem". Technical Report. DEIS, University of Bologna, 2010 [53].

is less than or equal to the capacity Q. The goal of the CmRSP is to find m rings with the minimum global *visiting* and *allocation* costs. The CmRSP is known to be $\mathcal{NP}$-hard since it is the generalization of the Symmetric Traveling Salesman Problem (TSP), arising when $m = 1,\ Q = |U|,\ W = \phi,\ A = \phi$.

Two Integer Linear Programming (ILP) formulations and a Branch-and-Cut (BC) approach have been proposed by Baldacci et al. [6] in 2007, but only the small-size instances were solved to optimality within a reasonable time. Moreover, Baldacci et al. [6] proposed two heuristic procedures (H1 and H2), for the solution of the CmRSP. The first heuristic, H1, is based on the algorithm proposed for the multi-depot CmRSP [5] and executed at the root node. The other heuristic, H2, is based on the information obtained by the solution of the Linear Programming (LP) relaxation of the proposed ILP formulations to construct a CmRSP solution and is executed on a given set of nodes of the enumeration tree.

A hybrid metaheuristic approach, which combines *GRASP* and *Tabu Search* algorithms, has been proposed by Mauttone et al. [49] in 2007.

An ILP formulation, based on a Set Covering model, and a Branch-and-Price (BP) algorithm have been developed for the CmRSP by Hoshino and de Souza [38] in 2008. Comparing the Branch-and-Cut (BC) and the Branch-and-Price (BP) approaches, shows that their performance are approximately the same and both of these methods fail, in many cases, in finding the optimal solutions, within two hours of computing time, for instances with more than 50 nodes.

Finally in 2009, Salari et al. [65] proposed a heuristic approach for the solution of the CmRSP. The proposed algorithm, after a construction phase, applies iteratively a series of different local search procedures. A comparison of this heuristic with algorithms H1 and H2 proposed by Baldacci et al. [6] and the hybrid metaheuristic approach proposed by Mauttone et al. [49] indicates that the heuristic outperforms the latter methods.

Studies on different variants of CmRSP, arising in telecommunication networks, have been published in the literature and described in Baldacci et al. [6] and Labbé et al. [42,43].

In this work, considering the general scheme of the Variable Neighborhood Search (VNS) approach, we incorporate an ILP based improvement method to enhance the quality of the current solution. The proposed algorithm is able to obtain, within reasonable times, most of the optimal solutions and to improve some of the best known results proposed in the literature for the CmRSP.

The utilization of using ILP techniques to improve a feasible solution of a combinatorial optimization problem has been considered in recent years. Fischetti and Lodi [26] and Danna et al. [18] used the idea of exploiting ILP to explore promising neighborhoods of feasible solutions in Mixed Integer Programs. The methods presented in [18] and [26] are currently embedded in the commercial mixed integer programming solver Cplex [39]. This approach has been also used by De Franceschi et al. [20] and investigated by Toth and Tramontani [77] in the context of the classical VRP. Archetti et al. [2] developed a heuristic algorithm for the Split Delivery Vehicle Routing Problem that integrates Tabu Search with ILP to explore promising parts of the solution space identified by the Tabu Search. They also used a similar approach for an Inventory Routing Problem [1]. Hewitt et al. [37] combined exact and heuristic approaches to solve the Capacitated Fixed Charge Network Flow Problem by using ILP to improve feasible solutions found during the search. Finally, Salari et al. [68] used ILP techniques to improve the solution of the Open Vehicle Routing Problem.

The reminder of this paper is organized as follows. Section 2.2 describes the proposed ILP embedded VNS method for the CmRSP. Experimental results on benchmark instances from the literature and on a new set of large instances are provided in Section 2.3. Section 2.4 contains concluding remarks.

A preliminary version of this paper was presented at the International Symposium on Combinatorial Optimization in Hammamet, Tunesia [64].

2.2 Description of the proposed algorithm

In this section, we develop a VNS algorithm, called NST, strengthened with an ILP based improvement method to the solution of CmRSP. VNS is a metaheuristic approach, proposed by Mladenovic and Hansen [50], which applies a strategy based on the dynamic change of the neighborhood structures. The basic idea of the VNS approach is to choose a set of neighborhood structures that vary in size. After constructing an initial solution, the algorithm iterates through different neighborhood structures to improve the solution, until a stopping criterion is met [50]. Considering the VNS approach as our basic framework, we start by constructing an initial solution. We then improve upon this initial solution, using a *Local Search Procedure*. Inside the VNS scheme, the improvement of this solution continues in a loop until a termination criterion is reached. This loop contains a *Shaking Procedure* and a *Local Search Procedure*. The *Shaking Procedure* follows the VNS paradigm. It considers a specifically designed neighborhood and makes random changes to the current solution so as to explore neighborhoods farther away from the current solution. The *Local Search Procedure* considers a more restricted neighborhood and tries to improve upon the quality of a given solution. Following the VNS scheme, in case of having an improvement in the solution cost we return to the first neighborhood size; otherwise we increase the size of the neighborhood to try to find a possible better solution by calling the *Local Search Procedure*. Moreover, we have found useful the utilization of the threshold accepting criterion in updating the current solution at the end of each iteration of the VNS algorithm. In particular, we accept a worse solution as the current one if its cost is not greater than a given percentage P of the cost of the best known solution or if the number J of consecutive iterations without any improvement does not exceed a given value $JMAX$ (where P and $JMAX$ are two input parameters). The framework of the proposed method is given in Algorithm 1, while in the following sections we describe each procedure separately.

Beside some minor changes in the *Shaking Procedure* and threshold accepting idea, the main differences of this approach with respect to the method proposed by Salari et al. [65], are the ideas of using ILP techniques to improve the solutions and applying all of these procedures in a VNS framework that leads to the better results.

```
CurrentSolution:=Initialization();
BestCost:=Cost(CurrentSolution) and BestSolution:=CurrentSolution;
Local Search Procedure(CurrentSolution);
Update CurrentSolution, BestSolution and BestCost;
J:=0;
while the time limit is not reached do
    K:=Initial_K;
    while K ≤ |U| /2 do
        ShakingProcedure(CurrentSolution,K);
        Local Search Procedure(CurrentSolution);
        if Cost(CurrentSolution) < BestCost then
            K:=Initial_K;
            J:=0;
        end
        else
            K:=K+1;
            J:=J+1;
        end
    end
    if (Cost(CurrentSolution) > P*BestCost) or (J > JMAX) then
        Set the best known solution as the current one;
    end
end
```

Algorithm 1: The general framework of the proposed heuristic for CmRSP

2.2.1 Initialization Procedure

Similar to what has been done for the heuristic method proposed by Salari et al. [65], the initial solution is created by using the clustering algorithm proposed by Fischetti et al. [27] for the Generalized Traveling Salesman Problem (GTSP). In this procedure, the goal is to construct a feasible solution that only uses customers in the structure of the rings. To do so, we pick the depot and m customers as far as possible one from each other. Following this step we have m rings by connecting the customers to the depot. Then each of the remaining customers are allocated or visited in their best feasible position, i.e., the position that results in the minimum *visiting* or *allocation* cost.

2.2.2 Local Search Procedure

The *Local Search Procedure* consists of four major parts: *Improvement Procedure*, *ILP-based Procedure*, solving the *Generalized Assignement Problem* over the star part of the solution, i.e. for the customers assigned to the nodes of the rings, and finally, using the *Lin-Kernighan* TSP solver (see Lin and Kernighan [47] and Helsgaum [36]).

The *Improvement Procedure* follows the major ideas of the analogous procedure considered in the heuristic proposed by Salari et al. [65] and includes the *Swap* and the *Extraction-Assignment* procedures. The *ILP-based Procedure*, which is one of the major differences between this method and the heuristic proposed in [65], has been designed to further improve the solution and is based on Integer Linear Programming (ILP) techniques. Moreover, since the objective function consists of minimizing two different costs, i.e. *allocation* and *visiting* costs, we use the *Generalized Assignement Problem* and *Lin-Kernighan* TSP procedure [36, 47], to check any possible improvement in the *allocation* and *visiting* costs, respectively. Algorithm 2 describes the outline of the *Local Search Procedure.*

In this procedure we start by calling the *Improvement Procedure* and as long as the solution can be improved, we iterate to enhance the quality of the solution. After calling the *Improvement Procedure* and in the case of having an improvement in the solution cost, we apply the *ILP-Improvement Procedure*, for a given number of iterations, to try to improve the current solution. Followed by the *ILP-based Procedure*, in the case of having an improvement in the solution cost, the algorithm looks for possible further improvement of the solution by solving the *Generalized Assignement* problem. To do so, we extract all the customers allocated to visited nodes in the current solution and reallocate them in their best feasible positions by solving the *Generalized Assignement* problem. Following these steps, we call the *Lin-Kernighan* procedure to find a possible better order of visited nodes in the rings. Whenever applying the *ILP-based Procedure* followed by the *Generalized Assignement* problem and *Lin-Kernighan* procedure results in a better solution, we update the best known solution and call the *Improvement Procedure* as long as the solution can be improved. All of these steps continues in a loop until the solution cannot be improved any more.

```
while CurrentSolution can be improved do
    ImprovementProcedure(CurrentSolution);
end
if Cost(CurrentSolution) < BestCost then
    while Cost(CurrentSolution) < BestCost do
        Update BestCost and BestSolution;
        for i=1,···, ILP-Iter do
            ILP_based Procedure(CurrentSolution);
            if i=1 or the CurrentSolution has been improved by calling the ILP_based
            Procedure then
                Extract all the allocated customers to those visited in the tours and
                reallocate them by solving the Generalized Assignment Problem to
                optimality;
            end
        end
        For each ring, call the Lin-Kernighan procedure to improve the CurrentSolution;
        if Cost(CurrentSolution) < BestCost then
            Update BestCost and BestSolution;
            while CurrentSolution can be improved do
                ImprovementProcedure(CurrentSolution);
            end
        end
    end
end
```

Algorithm 2: Local Search Procedure

Improvement Procedure

The improvement of the current solution starts with the *Swap Procedure*. As soon as the solution cannot be further improved by using the moves of the *Swap Procedure*, the *Improvement Procedure* continues by calling the *Extraction-Assignment Procedure*. In this section we briefly describe these procedures. For more details on the *Swap* and *Extraction-Assignment* procedures, one can refer to Salari et al. [65].

Swap Procedure In this procedure, for each customer in the current solution and in a random order, we test all the possible ways to swap this customer with another visited or allocated node which is near to the selected one, starting from the first nearest node up to the T^{th} nearest one (where T is an input parameter).

Extraction-Assignment Procedure In this procedure, we extract, in a random order, each customer from its current position, and reassign it to a possibly better feasible position. To this aim, we consider T nodes nearest to the extracted customer and check all the possibilities for allocating the customer to one of these nodes or visiting the customer in a ring, consecuently before or after one of the T nearest nodes. We then select the best feasible movement, i.e. the one that results in the least cost solution.

ILP-based Procedure

In this section we present a heuristic improvement procedure based on ILP techniques. Given an initial feasible solution to be possibly improved, the method follows a destruct-and-repair paradigm, where the given solution is destroyed (i.e., some nodes are removed from the solution) and repaired by solving an ILP model (called *Reallocation Model*), in an attempt to find an improved feasible solution. A similar approach has been used by De Franceschi et al. [20] and Toth and Tramontani [77] in the context of the classical Vehicle Routing Problem (VRP), and by Salari et al. [68] for the solution of the Open Vehicle Routing Problem.

Let z be the current feasible solution of the CmRSP and F a subset of customers or Steiner nodes visited in the current solution. We define $z(F)$ as the restricted solution obtained from z by extracting (i.e., by shortcutting) all the nodes in F. We then add to F all the customers allocated to the nodes in F and the Steiner nodes not visited by z. We can partition F into two subsets, i.e. $F = F1 \cup F2$, where $F1$ and $F2$ are the set of customers and the set of Steiner nodes, respectively. Now, let $R = R(z, F)$ be the set of rings in the restricted solution. An *insertion point* is a potential location in the restricted solution which can be used to insert a sequence or a node subset or to allocate one or more customers. We have to notice that a sequence consists of a set of customers which can be inserted between two visited nodes and a node subset consists of a single customer which can be allocated to the nodes visited by the rings or it is a customer or Steiner node with at least one customer allocated to that.

We use the notations $I = I(z, F)$ to define the set of insertion points corresponding to edges in the restricted solution that can be used to insert a sequence or a node subset, and $J = J(z, F)$ to define the set of customers or Steiner nodes visited in the restricted solution, which can be considered as insertion points to allocate one or more customers.

Moreover, we define $I(r)$ and $J(r)$ as the set of those insertion points belonging to ring $r \in R$. Let $S = S(F)$ be the set of all the feasible sequences or node subsets which can be obtained by recombining the nodes in F plus all the singleton customers in F that can be allocated to the insertion points in J. We define $q(s)$ as the number of customers of the sequence or node subset $s \in S$. For each insertion point $i \in I$ let $S_i \subseteq S$ be the set of all feasible sequences or node subsets which can be inserted into i, and for each insertion point $j \in J$ let $S'_j \subseteq S$ be the set of singleton customers that can be allocated to j. Moreover, for each $i \in I$ and $w \in F$ we define $S_i(w) \subseteq S_i$ as the set of sequences or node subsets including node w which can be inserted into insertion point i. Let γ_{si} be the *visiting* or *allocation* cost for inserting sequence or node subset $s \in S$ into insertion point $i \in I$, and d_{vj} the *allocation cost* for allocating customer $v \in S'_j$ to the insertion point $j \in J$. We also define $\tilde{q}(r)$ and $\tilde{c}(r)$ as the number of customers and the cost of ring r in the restricted solution $z(F)$, respectively. The *Reallocation Model* (RM) corresponding to z and F can be defined as follows:

$$\sum_{r \in R} \tilde{c}(r) + \min \sum_{i \in I} \sum_{s \in S_i} \gamma_{si} x_{si} + \sum_{j \in J} \sum_{v \in S'_j} d_{vj} y_{vj} \tag{2.1}$$

subject to:

$$\sum_{i \in I} \sum_{s \in S_i(v)} x_{si} + \sum_{j \in J} y_{vj} = 1 \qquad v \in F1, \tag{2.2}$$

$$\sum_{i \in I} \sum_{s \in S_i(w)} x_{si} \leq 1 \qquad w \in F2, \tag{2.3}$$

$$\sum_{s \in S_i} x_{si} \leq 1 \qquad i \in I, \tag{2.4}$$

$$\sum_{i \in I(r)} \sum_{s \in S_i} q(s) x_{si} + \sum_{j \in J(r)} \sum_{v \in S'_j} y_{vj} \leq Q - \tilde{q}(r) \qquad r \in R, \tag{2.5}$$

$$x_{si}, y_{vj} \in \{0, 1\} \quad i \in I, j \in J, s \in S_i, v \in F1. \tag{2.6}$$

where the decision variables x_{si} and y_{vj} are defined as follows:

$$x_{si} = \begin{cases} 1 & \text{if sequence or node subset } s \in S_i \text{ is inserted into insertion point } i \in I, \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

and

$$y_{vj} = \begin{cases} 1 & \text{if customer } v \in F1 \text{ is allocated to insertion point } j \in J, \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

The objective function (2.1), to be minimized, gives the total cost of the rings which consists of both *visiting* and *allocation* costs. Constraints (2.2) impose that each extracted customer v to be visited or allocated exactly once, while constraints (2.3) force each Steiner node to be used at most once. Constraints (2.4) impose that for each insertion point $i \in I$ we can insert at most one sequence or node subset. We have to note that for the insertion points $j \in J$, we do not impose such a restriction. Finally, constraints (2.5) impose that each ring in the final solution fulfills the capacity constraint.

Now, the main steps of the *ILP-based Procedure* can be expressed as follows:

- **Selection Phase:** Build set F by selecting from each ring of the current solution z, with the same probability, all the visited nodes in odd or even positions.

- **Extraction Phase:** Extract from z the nodes selected in the previous step and build the restricted solution $z(F)$. Add also to F all the customers currently allocated to nodes in F and all the Steiner nodes not visited by z.

- **Initialization Phase:** For each insertion point $i \in I$, initialize S_i with the possible *basic* sequence or node subset extracted from i in the *Extraction Phase*, plus the singleton sequence (consisting of one customer belonging to F) having the minimum *visiting* cost. For each insertion point $i \in J$, initialize S'_j with the customer belonging to F having the minimum *allocation* cost. Initialize the Linear Programming (LP) relaxation of the *Reallocation Model* (LP-RM) by considering the initial subsets $S_i(i \in I)$ and $S'_j(j \in J)$ and solve it.

- **Column Generation Phase:** For each insertion point $i \in I$ (or $j \in J$), solve the corresponding column generation problem by means of the *Heuristic Column Generation Procedure*, described in the next section, and add to S_i (or to S'_j) all the sequences or node subsets (or customers belonging to $F1$) such that the associated variables x_{si} (or y_{vj}) have a reduced cost under a given threshold RC_{max}.

- **Reallocation Phase:** Using the sequences and node subsets generated up to this step, build the corresponding reallocation model and solve it to optimality by using an ILP solver.

Heuristic Column Generation Procedure For each insertion point we use a heuristic approach to solve the corresponding column generation phase. In particular, for each insertion point $j \in J$ and for each customer $v \in F1$, we consider the possible allocation of $\{v\}$ to j. If the reduced cost corresponding to this allocation (i.e., $s = \{v\}$ to j) is less than the given threshold, RC_{max}, we add v to S'_j. For each insertion point $i \in I$, say $i = (a, b)$, we first generate all the sequences s consisting of one or two customers belonging to $F1$ and if the reduced cost corresponding to the insertion of s into i (obtained by substituting (a, b) with (a, s, b)), is less than the given threshold, RC_{max}, we add sequence s to S_i. In the next step, we generate all the node subsets s obtained by allocating from one to five customers belonging to $F1$ to a node belonging to F. Then if the resulted reduced cost, obtained by substituting (a, b) with (a, s, b), is less than the given threshold, we add the node subset s to S_i. We note that for each insertion point of type I or J, we just work with a limited percentage of the nodes (RP) nearest to that insertion point to generate the sequences or node subsets.

2.2.3 Shaking Procedure

Following the *Local Search Procedure*, discussed in the previous section, we use *Shaking Procedure*. The *Shaking Procedure* follows the VNS paradigm and dynamically expands the neighborhood search area. In this step, we produce a solution that is in the neighborhood size K of the *Current Solution*, i.e. $N_K(CurrentSolution)$. Specifically, $N_K(CurrentSolution)$ is defined as the set of solutions that can be obtained from the *Current Solution* by removing K random nodes, along with their possible allocated customers, from the *Current Solution* and then by assigning each of them, in a random order and once at a time, to its best feasible position (i.e. the position that generates the minimum *visiting* or *allocation* cost). In this procedure, we do not reinsert the possible extracted Steiner nodes to the solution.

2.3 Computational Results

In this section we report on our computational results. To test the performance of the proposed heuristic, we have used the datasets proposed by Baldacci et al. [6] as well as a set of large instances that we have developed for this problem. The instances proposed in [6], are varying in size from 26 to 101 nodes. There are two classes of instances (A and B). The topology of the underlying graphs, i.e. the coordinates of the nodes and the number of customer or Steiner nodes, in both classes are exactly the same and the difference is in the structure of the distance matrices. In particular, in the first class of instances, class A, the *visiting* and the *allocation* costs (c_{ij} and d_{ij}) are the same as the euclidean distance between the considered nodes while in the second class of instances, class B, the *visiting* and *allocation* costs are $c_{ij} = \lceil 7e_{ij} \rceil$ and $d_{ij} = \lceil 3e_{ij} \rceil$, respectively, where e_{ij} is the Euclidean distance between the pair of nodes i and j.

Beside considering classes A and B, we have also designed a set of 48 larger datasets, derived from two instances of TSPLIB library [60], KroA150 and KroA200, containing 150 and 200 nodes, respectively. To design these instances, we have followed the rules proposed by Baldacci et al. [6] for the smaller instances (i.e. instances from 26 to 101 nodes). Also, there are 6 additional real world instances used in the paper by Baldacci et al. [6], which are not available.

The proposed heuristic, NST, has been implemented in C and tested on a Pentium IV PC running at 3.4 GHz with 1 GB of RAM.

The list of parameters which should be determined for the proposed method includes P, $Initial\_K$, $JMAX$, ILP_Iter, T, RP and RC_max (see section 2.2). The termination criterion of the overall algorithm is considered as a given number of iterations of the main loop without any improvement. After testing different values for the parameters we fixed them as follows: $P = 1.05$, $Initial\_K = 5$, $JMAX = 100$, $ILP\_Iter = 10$, $T = 0.20 * NoVertices$, $RP = 0.30 * NoVertices$ and $RC_{max} = 0.005 * BestCost$, in which $NoVertices$ and $BestCost$ are the number of vertices and the cost of the best known solution for each tested instance, respectively. Finally, the termination criterion used to run the experiments is 150 consecutive iterations of the main VNS loop (see Algorithm 1), without an improvement.

The results of the proposed algorithm in comparison to heuristics H1 and H2 [6], HP [65], and the exact algorithm BC [6], are given in Tables 2.1 to 2.3. For each instance we have performed 5 independent executions of the proposed algorithm using 5 different seeds for initializing the random number generator. The best (B.Cost) and the average solution cost (Avg.Cost) among these runs have been reported in the tables. For each instance the reported total time (T.T) is related to the total execution time corresponding to different runs. All computing times are expressed in seconds.

Since the results of the large instances, reported in Table 2.3, can be improved by increasing the number of runs, we have also reported the results of the NST and HP [65] with 20 independent runs. It should be mentioned that increasing the number of independent executions, more than 5 in the small instances and more than 20 in large ones, would not cause any significant improvement in the solutions cost, so we ignored to report more results.

In each table, the name of each instance (Data), the number of nodes (n), the number of rings (m), the number of customers ($|U|$) and the capacity of each ring (Q) are given in columns 1 to 5, respectively. The results of heuristics H1 and H2 proposed by Baldacci et al. [6] together with their computing time are given in columns 6-7 and 8-9, respectively, while the results of BC algorithm [6] are reported in column 10. The time limit for the BC algorithm is two hours of CPU time and for those instances whose optimal solutions can not be obtained during this time, the best solutions found during the BC process have been reported. The corresponding computing times of the BC algorithm are given in column 11. In columns 12 to 14, the best and the average solution costs obtained during 5 different runs of HP [65] and the corresponding total computing time are reported, respectively. In Table 2.3, the results of the execution of heuristic HP [65] with 20 independent runs are given as well. In the last 3 columns of each table the best, the average cost and the total time required for 5 and/or 20 independent runs of the NST algorithm are reported, respectively. To provide the results reported in Tables 2.1 to 2.3, we have used the original code of H1, H2 and BC algorithms (provided by Roberto Baldacci) and the original code of HP [65]. In all tables, in each row the best solutions are written in bold and the average values of each column (Avg) and the number of best solutions found by each algorithm (NB) are reported in the last two rows of the tables.

Tables 2.1 and 2.2 show that for 63 out of 90 small instances, solved to optimality, using the BC algorithm, NST could find all of the optimal solutions, while H1, H2 and HP obtained 35, 38 and 62 of the optimal solutions, respectively. For the other 27 instances whose optimal solutions are not available, NST achieved the best solution for 27 instances, while H1, H2, BC and HP generated the best solutions 0, 0, 2 and 23 times, respectively.

As it can be seen from Tables 2.1 and 2.2, considering the best performance of the heuristics, and the number of times that the proposed methods are able to reach the best known results, NST algorithm is the best by obtaining all of the best known results in both classes A and B. Moreover, the average computing time of NST algorithm in small instances of classes A and B are 10.08 and 11.42 seconds, while H1, H2, BC [6] and HP [65] take 5.4, 29.7, 2098.3 and 6.22 seconds in class A, and 5.89, 28.90, 2952.90 and 11.14 seconds in class B, respectively.

A comparison of NST algorithm with the methods H1, H2, BC and HP on the large instances is reported in Table 2.3. For the large instances, clearly the proposed NST method outperforms all the other algorithms. In these data sets since the proposed heuristic is faster than H2 and BC, beside running NST and HP algorithms with 5 runs, we prefered to execute them with 20 seeds as well. Considering only 5 independent runs of the algorithm, the average of the best solution cost obtained by NST algorithm, is clearly better than heuristic algorithms H1, H2 [6] and HP [65] and the exact algorithm BC [6]. Increasing the quality of the solutions by increasing the number of independent executions of the algorithm can be seen in this table apparently. By considering 20 different runs, the proposed algorithm can obtain the best solutions for 40 out of 48 instances. This is in comparison to H1, H2, BC [6] and HP [65] (with the same number of runs) which generate 3, 3, 11 and 17 best solutions, respectively. In terms of the computing time, H1 [6] is a bit faster than totally 5 independent runs of NST algorithm, while the speed of the proposed method is approximately equivalent with HP [65]. In any case by running the NST algorithm even for 20 times, the method is still faster than H2 and BC algorithms [6].

2.4 Conclusion

We have proposed an ILP based VNS approach for the Capacitated m-Ring-Star Problem (CmRSP). Considering the general scheme of the VNS, this method incorporates an ILP improvement method whenever the Improvement Procedure is not able to enhance the quality of the solution. We compared the proposed method with the available heuristic and exact methods for CmRSP in the literature. The results clearly show the superiority of the proposed method, especially as the instances get larger. The proposed method, within a short computing time, can obtain 66 out of 67 optimal solutions and in the remaining instances whose optimal solutions are not known, it can obtain 36 best known solutions and improve 28 of the best results obtained by other heuristics and exact method.

Table 2.1: Comparison of different solution procedures for the CmRSP problem on small instances (Class A).

Data	n	m	$\lvert U \rvert$	Q	H1		H2		BC		HP			NST		
					Cost	T.T	Cost	T.T	Cost	T.T	B.Cost	Avg.Cost	T.T	B.Cost	Avg.Cost	T.T
A01	26	3	12	5	**242**	0.3	**242**	0.1	**242**	0.1	**242**	**242.00**	0.5	**242**	**242.00**	0.9
A02	26	4	12	4	**261**	0.3	**261**	0.0	**261**	0.0	**261**	**261.00**	0.5	**261**	**261.00**	0.8
A03	26	5	12	3	**292**	0.3	**292**	0.0	**292**	0.0	**292**	**292.00**	0.4	**292**	**292.00**	0.7
A04	26	3	18	7	**301**	0.4	**301**	0.7	**301**	0.5	**301**	**301.00**	0.8	**301**	**301.00**	1.2
A05	26	4	18	5	**339**	0.4	**339**	0.4	**339**	0.3	**339**	**339.00**	1.0	**339**	**339.00**	1.9
A06	26	5	18	4	**375**	0.4	**375**	1.4	**375**	0.7	**375**	**375.00**	1.6	**375**	**375.00**	1.3
A07	26	3	25	10	333	0.8	333	1.7	**325**	3.8	**325**	**325.00**	1.5	**325**	**325.00**	1.4
A08	26	4	25	7	**362**	0.7	**362**	0.9	**362**	0.3	**362**	**362.00**	1.2	**362**	**362.00**	1.1
A09	26	5	25	6	**382**	0.6	**382**	0.6	**382**	0.2	**382**	**382.00**	2.3	**382**	**382.00**	2.0
A10	51	3	12	5	**242**	0.3	**242**	0.1	**242**	0.2	**242**	**242.00**	0.6	**242**	**242.00**	0.7
A11	51	4	12	4	**261**	0.2	**261**	0.1	**261**	0.4	**261**	**261.00**	0.7	**261**	**261.00**	0.7
A12	51	5	12	3	**286**	0.3	**286**	0.1	**286**	0.1	**286**	**286.00**	0.6	**286**	**286.00**	0.6
A13	51	3	25	10	331	0.8	**322**	1.0	**322**	2.1	**322**	**322.00**	1.7	**322**	**322.00**	2.0
A14	51	4	25	7	**360**	0.7	**360**	1.1	**360**	2.1	**360**	**360.00**	1.9	**360**	**360.00**	2.2
A15	51	5	25	6	**379**	0.6	**379**	1.7	**379**	2.3	**379**	**379.00**	2.4	**379**	**379.00**	2.6
A16	51	3	37	14	**373**	2.3	**373**	6.7	**373**	8.4	**373**	**373.00**	3.2	**373**	**373.00**	2.5
A17	51	4	37	11	408	1.6	408	7.6	**405**	41.7	**405**	**405.00**	3.5	**405**	**405.00**	3.6
A18	51	5	37	9	441	2.2	435	11.8	**432**	52.2	**432**	432.80	3.8	**432**	**432.00**	4.7
A19	51	3	50	19	459	4.8	469	14.1	**458**	182.8	**458**	458.20	5.0	**458**	**458.00**	8.1
A20	51	4	50	14	501	3.0	493	20.8	**490**	220.4	**490**	**490.00**	5.4	**490**	**490.00**	7.2
A21	51	5	50	12	521	5.3	521	19.2	**520**	6334.2	**520**	520.80	6.2	**520**	520.80	8.2
A22	76	3	18	7	**330**	0.7	**330**	2.9	**330**	48.3	**330**	**330.00**	1.7	**330**	**330.00**	1.3
A23	76	4	18	5	**385**	0.6	**385**	2.7	**385**	30.6	**385**	**385.00**	1.6	**385**	**385.00**	0.7
A24	76	5	18	4	**448**	0.8	**448**	4.2	**448**	63.7	**448**	**448.00**	2.5	**448**	**448.00**	2.1
A25	76	3	37	14	407	2.2	409	9.5	**402**	567.7	**402**	**402.00**	4.6	**402**	**402.00**	4.7
A26	76	4	37	11	462	2.3	461	16.5	460	7200.0	**457**	457.80	4.8	**457**	458.00	5.6
A27	76	5	37	9	**479**	3.1	484	21.4	**479**	509.3	**479**	**479.00**	5.2	**479**	**479.00**	6.4
A28	76	3	56	21	475	7.3	478	38.9	**471**	1584.4	**471**	**471.00**	8.0	**471**	**471.00**	14.4
A29	76	4	56	16	523	7.1	524	50.5	523	7200.0	**519**	519.80	8.0	**519**	519.60	9.8
A30	76	5	56	13	552	6.3	552	40.2	**545**	3221.3	**545**	548.00	8.8	**545**	547.40	11.5
A31	76	3	75	28	570	14.8	565	45.0	**564**	479.5	**564**	565.00	12.5	**564**	566.20	18.8
A32	76	4	75	21	617	15.3	628	57.4	606	7200.0	**602**	604.20	12.0	**602**	602.50	23.6
A33	76	5	75	17	659	13.6	654	81.7	654	7200.0	**640**	648.80	12.5	**640**	642.00	33.0
A34	101	3	25	10	**363**	0.9	**363**	3.2	**363**	8.7	**363**	**363.00**	2.9	**363**	**363.00**	2.0
A35	101	4	25	7	**415**	1.1	**415**	9.2	**415**	91.8	**415**	**415.00**	3.0	**415**	**415.00**	1.6
A36	101	5	25	6	**448**	1.5	**448**	10.8	**448**	680.4	**448**	**448.00**	4.5	**448**	**448.00**	4.9
A37	101	3	50	18	503	6.5	501	58.8	**500**	7200.0	**500**	**500.00**	7.0	**500**	**500.00**	7.4
A38	101	4	50	14	532	3.9	533	44.5	532	7200.0	**528**	**528.00**	8.3	**528**	**528.00**	10.5
A39	101	5	50	12	571	4.0	568	48.4	568	7200.0	**567**	**567.00**	7.7	**567**	**567.00**	8.8
A40	101	3	75	28	605	18.6	622	115.6	**595**	6690.1	**595**	**595.00**	14.3	**595**	595.20	22.5
A41	101	4	75	21	629	13.3	635	74.5	625	7200.0	**623**	623.20	15.8	**623**	623.60	32.1
A42	101	5	75	17	663	11.5	665	120.5	662	7200.0	**657**	658.60	13.7	**657**	657.80	24.0
A43	101	3	100	38	672	31.7	672	134.3	**646**	283.0	648	651.00	26.5	**646**	649.80	52.6
A44	101	4	100	28	702	26.5	704	109.0	680	7200.0	**679**	680.20	25.9	**679**	679.80	50.3
A45	101	5	100	23	719	24.5	717	148.7	**700**	1310.8	**700**	**700.00**	23.8	**700**	700.40	50.9
Avg.					448.40	5.4	448.82	29.7	444.62	2098.3	443.82	444.36	6.2	**443.78**	444.14	10.1
NB					21	–	21	–	36	–	44	–	–	**45**	–	–

Table 2.2: Comparison of different solution procedures for the CmRSP problem on small instances (Class B).

Data	n	m	|U|	Q	H1		H2		BC		HP			NST		
					Cost	T.T	Cost	T.T	Cost	T.T	B.Cost	Avg.Cost	T.T	B.Cost	Avg.Cost	T.T
B01	26	3	12	5	**1684**	0.3	**1684**	0.1	**1684**	0.1	**1684**	**1684.00**	0.6	**1684**	**1684.00**	0.3
B02	26	4	12	4	**1827**	0.2	**1827**	0.1	**1827**	0.1	**1827**	**1827.00**	0.5	**1827**	**1827.00**	0.8
B03	26	5	12	3	**2041**	0.3	**2041**	0.0	**2041**	0.0	**2041**	**2041.00**	0.5	**2041**	**2041.00**	0.6
B04	26	3	18	7	**2104**	0.4	**2104**	0.6	**2104**	0.5	**2104**	**2104.00**	0.9	**2104**	**2104.00**	1.1
B05	26	4	18	5	**2370**	0.4	**2370**	1.5	**2370**	0.5	**2370**	**2370.00**	1.3	**2370**	**2370.00**	2.0
B06	26	5	18	4	**2615**	0.5	**2615**	2.2	**2615**	0.7	**2615**	**2615.00**	2.3	**2615**	**2615.00**	1.2
B07	26	3	25	10	2314	0.8	**2251**	1.6	**2251**	0.4	**2251**	**2251.00**	1.2	**2251**	**2251.00**	2.4
B08	26	4	25	7	**2510**	1.1	**2510**	1.2	**2510**	0.5	**2510**	**2510.00**	1.4	**2510**	**2510.00**	1.4
B09	26	5	25	6	**2674**	0.8	**2674**	2.9	**2674**	0.8	**2674**	**2674.00**	1.9	**2674**	**2674.00**	2.1
B10	51	3	12	5	**1681**	0.3	**1681**	0.4	**1681**	0.8	**1681**	**1681.00**	0.9	**1681**	**1681.00**	0.7
B11	51	4	12	4	**1821**	0.2	**1821**	0.6	**1821**	1.5	**1821**	**1821.00**	0.9	**1821**	**1821.00**	0.8
B12	51	5	12	3	**1972**	0.3	**1972**	0.2	**1972**	0.3	**1972**	**1972.00**	1.0	**1972**	**1972.00**	0.9
B13	51	3	25	10	**2176**	1.5	**2176**	1.6	**2176**	1.1	**2176**	**2176.00**	1.8	**2176**	**2176.00**	1.5
B14	51	4	25	7	2476	1.1	2495	4.1	**2470**	7.2	**2470**	**2470.00**	2.1	**2470**	**2470.00**	2.1
B15	51	5	25	6	2596	1.0	**2579**	2.4	**2579**	4.1	**2579**	**2579.00**	2.6	**2579**	**2579.00**	2.6
B16	51	3	37	14	2507	2.3	2599	9.4	**2490**	17.9	**2490**	**2490.00**	4.2	**2490**	2496.80	2.9
B17	51	4	37	11	2772	1.9	2811	10.5	**2721**	74.9	**2721**	**2721.00**	3.9	**2721**	**2721.00**	4.1
B18	51	5	37	9	2938	2.2	2937	14.2	**2908**	145.0	**2908**	2914.60	4.8	**2908**	**2908.00**	4.9
B19	51	3	50	19	3095	4.0	3071	17.4	**3015**	296.7	**3015**	**3015.00**	9.0	**3015**	**3015.00**	8.6
B20	51	4	50	14	3365	3.6	3298	18.1	**3260**	336.6	**3260**	**3260.00**	8.4	**3260**	**3260.00**	7.0
B21	51	5	50	12	3525	5.7	3516	18.9	**3404**	6470.7	**3404**	**3404.00**	9.1	**3404**	3420.60	11.1
B22	76	3	18	7	2260	0.7	2259	2.6	**2253**	105.5	**2253**	**2253.00**	2.2	**2253**	2256.60	1.8
B23	76	4	18	5	2625	0.5	**2620**	3.3	**2620**	29.5	**2620**	**2620.00**	2.1	**2620**	**2620.00**	1.2
B24	76	5	18	4	**3059**	0.9	**3059**	3.4	**3059**	85.3	**3059**	**3059.00**	2.4	**3059**	**3059.00**	1.9
B25	76	3	37	14	2742	3.1	**2720**	14.3	**2720**	1897.6	**2720**	**2720.00**	4.9	**2720**	**2720.00**	5.3
B26	76	4	37	11	3176	2.7	3138	17.5	3138	7200.0	**3100**	3115.20	6.8	**3100**	3113.80	6.7
B27	76	5	37	9	3339	3.0	3364	23.8	3311	7200.0	**3284**	**3284.00**	5.8	**3284**	**3284.00**	5.3
B28	76	3	56	21	3112	7.1	3146	31.4	3088	7200.0	**3044**	3060.00	14.8	**3044**	3049.40	14.1
B29	76	4	56	16	3447	5.1	3496	50.3	3447	7200.0	**3415**	3438.60	16.2	**3415**	3440.80	12.0
B30	76	5	56	13	3652	4.6	3703	35.5	3648	7200.0	3636	3642.20	15.0	**3632**	3643.20	15.2
B31	76	3	75	28	3786	14.1	3820	69.1	3740	7200.0	**3652**	3687.20	26.9	**3652**	3670.20	27.3
B32	76	4	75	21	4057	13.9	4084	78.8	4026	7200.0	4003	4006.40	23.9	**3964**	4002.80	31.6
B33	76	5	75	17	4442	15.9	4288	54.5	4288	7200.0	**4217**	**4217.00**	23.5	**4217**	**4217.00**	31.4
B34	101	3	25	10	2437	0.7	2439	4.3	**2434**	24.2	**2434**	**2434.00**	3.5	**2434**	**2434.00**	2.3
B35	101	4	25	7	**2782**	1.2	2819	9.5	**2782**	115.4	**2782**	**2782.00**	3.4	**2782**	**2782.00**	1.4
B36	101	5	25	6	3043	1.0	3012	4.8	**3009**	862.4	**3009**	**3009.00**	5.1	**3009**	**3009.00**	5.2
B37	101	3	50	18	3404	6.3	3387	37.2	3332	7200.0	**3322**	**3322.00**	9.4	**3322**	**3322.00**	9.9
B38	101	4	50	14	3593	4.3	3586	32.1	**3533**	7200.0	**3533**	**3533.00**	10.2	**3533**	**3533.00**	10.7
B39	101	5	50	12	3880	4.4	3872	33.2	3872	7200.0	**3834**	3839.60	12.1	**3834**	3839.20	13.8
B40	101	3	75	28	3935	26.0	3923	260.7	3923	7200.0	**3887**	3887.80	25.9	**3887**	3888.00	35.6
B41	101	4	75	21	4190	16.0	4202	63.5	4125	7200.0	**4082**	4088.40	25.2	**4082**	4091.40	31.0
B42	101	5	75	17	4486	14.2	4458	47.9	4458	7200.0	**4358**	**4358.00**	21.9	**4358**	**4358.00**	35.2
B43	101	3	100	38	4275	35.9	4155	103.0	**4110**	7200.0	4135	4150.40	74.7	**4110**	4126.00	52.0
B44	101	4	100	28	4583	28.0	4608	97.4	4506	7200.0	4358	4377.60	56.4	**4355**	4379.80	58.0
B45	101	5	100	23	4671	26.7	4639	114.6	4632	7200.0	**4565**	4568.40	50.4	**4565**	4566.40	46.4
Avg.					3023.09	5.89	3018.42	28.90	2991.71	2952.90	2975.00	2978.50	11.1	**2973.42**	2977.82	11.4
NB					13	–	17	–	30	–	41	–	–	**45**	–	–

Table 2.3: Comparison of different solution procedures for the CmRSP problem on large datasets.

Data	n	m	|U|	Q	H1		H2		BC		HP						NST					
											5 runs			20 runs			5 runs			20 runs		
					Cost	T.T	Cost	T.T	Cost	T.T	B.Cost	Avg.Cost	T.T	B.Cost	Avg.Cost	T.T	B.Cost	Avg.Cost	T.T	B.Cost	Avg.Cost	T.T
C01	150	3	37	14	17191	4.3	17341	42.9	17163	7200.0	**17138**	17138.0	11.1	**17138**	17138.00	45.3	**17138**	17138.0	5.9	**17138**	17138	21.2
C02	150	4	37	11	18793	5.3	18793	39.3	**18782**	7200.0	**18782**	18782.0	11.0	**18782**	18782.00	45.6	**18782**	18782.0	4.3	**18782**	18782	17.3
C03	150	5	37	9	20220	3.6	**20135**	33.5	**20135**	6534.5	**20135**	20169.2	17.0	**20135**	20186.30	63.9	**20135**	20237.6	7.1	**20135**	20237.60	29.7
C04	150	3	74	28	**20741**	30.9	20902	194.1	**20741**	7200.0	**20741**	20741.0	34.3	**20741**	20741.00	134.3	**20741**	20741.0	30.8	**20741**	20741	128.6
C05	150	4	74	21	22810	16.9	23304	164.5	22810	7200.0	**22525**	22525.0	34.7	**22525**	22566.25	146.8	**22525**	22525.0	43.6	**22525**	22525	196.3
C06	150	5	74	17	24970	20.0	25557	193.6	24955	7200.0	**24949**	24957.4	32.4	**24949**	24954.50	139.1	**24949**	24957.4	34.9	**24949**	24953.20	132.4
C07	150	3	111	42	23532	57.3	23615	519.9	**23259**	2914.7	**23259**	23259.0	69.2	**23259**	23259.00	292.9	**23259**	23259.0	69.8	**23259**	23314.80	300.1
C08	150	4	111	31	25809	59.8	25887	373.2	25121	7200.0	**25006**	25006.0	60.8	**25006**	25006.00	246.1	**25006**	25006.0	82.5	**25006**	25006	285.7
C09	150	5	111	25	27749	50.7	27992	505.1	27605	7200.0	**27277**	27277.0	63.7	**27277**	27288.75	268.5	**27277**	27277.0	79.6	**27277**	27284.15	347.4
C10	150	3	149	56	28041	173.5	28190	740.7	27250	7200.0	27283	27314.4	154.0	**27233**	27311.95	625.0	27274	27339.4	200.3	27273	27326.55	725.2
C11	150	4	149	42	28665	121.5	29013	716.1	**28536**	2400.2	**28536**	28538.2	150.0	**28536**	28573.60	613.6	**28536**	28543.8	168.2	**28536**	28551.55	602.4
C12	150	5	149	34	31492	104.9	31286	752.3	31286	7200.0	30823	30844.8	136.3	30811	30857.15	563.2	30788	30824.0	191.6	**30669**	30832.95	760.0
C13	200	3	49	19	18651	8.2	18651	108.7	18614	7200.0	**18567**	18567.0	26.8	**18567**	18567.00	111.2	**18567**	18567.0	16.4	**18567**	18567	64.6
C14	200	4	49	14	20875	12.6	21099	90.7	20834	7200.0	**20650**	20680.0	31.9	**20650**	20687.50	133.8	**20650**	20737.0	28.6	**20650**	20709.25	94.0
C15	200	5	49	11	23556	9.1	24021	119.2	23510	7200.0	**23496**	23563.8	35.4	**23496**	23502.25	145.9	**23496**	23518.6	28.2	**23496**	23501.65	104.5
C16	200	3	99	37	22919	39.7	23047	371.5	22919	7200.0	**22882**	22882.0	76.0	**22882**	22882.00	311.4	**22882**	22882.0	119.1	**22882**	22882	524.9
C17	200	4	99	28	25671	56.2	25957	950.5	25660	7200.0	25485	25621.2	80.1	25485	25627.20	341.0	**25472**	25545.8	179.7	**25472**	25559.35	552.4
C18	200	5	99	22	28438	53.3	28500	677.0	28413	7200.0	28388	28363.2	89.7	**28300**	28365.20	367.2	28333	28376.4	125.0	28333	28364.65	490.9
C19	200	3	149	56	27419	154.1	27704	1292.9	27325	7200.0	26990	27097.0	182.4	26987	27054.70	766.7	26990	27057.8	323.7	**26971**	26999.85	1240.7
C20	200	4	149	42	29802	157.4	30355	2152.7	29778	7200.0	29361	29686.0	182.4	29333	29649.45	737.8	**29268**	29446.8	444.7	**29268**	29586.85	1503.2
C21	200	5	149	34	32401	138.4	33407	2368.4	32243	7200.0	31965	32079.2	167.5	**31944**	32054.85	706.5	31947	31986.0	381.5	31946	31993.70	1463.2
C22	200	3	199	74	30462	455.0	30574	3411.5	30462	7200.0	30323	30424.2	352.4	30256	30591.80	1345.4	30204	30373.6	698.7	**30181**	30351.00	2613.8
C23	200	4	199	56	32719	403.4	33052	3374.1	32463	7200.0	32318	32405.2	309.9	32233	32404.15	1285.2	32203	32397.8	528.9	**32152**	32362.25	2195.2
C24	200	5	199	45	35607	267.2	35497	3082.6	34969	7200.0	34507	34580.0	277.3	34502	34590.15	1136.0	34472	34537.4	569.5	**34455**	34524.20	2191.6
D01	150	3	37	14	111491	2.7	**110350**	30.7	**110350**	3193.5	111342	111342.0	18.6	111185	111360.75	69.9	110607	110607.0	5.3	110607	110618.00	21.1
D02	150	4	37	11	122840	3.4	122574	26.2	**121569**	7200.0	122852	122852.0	22.4	122415	122855.00	89.4	122138	123064.4	6.6	122066	123211.90	23.4
D03	150	5	37	9	130298	3.8	129882	27.0	**129540**	7200.0	130144	130237.6	26.7	129882	130224.65	108.6	**129540**	130248.0	12.4	**129540**	130045.85	42.2
D04	150	3	74	28	130349	36.9	130349	292.3	130349	7200.0	130937	131539.6	57.7	130117	130832.90	225.9	**128736**	129568.4	54.3	**128736**	130106.45	162.8
D05	150	4	74	21	144646	24.4	144646	545.8	144646	7200.0	142756	142920.8	59.0	142675	142922.10	239.2	141716	141728.6	39.3	**141680**	141796.80	152.9
D06	150	5	74	17	161128	22.5	162327	382.6	161128	7200.0	161508	161753.2	52.9	160988	161570.00	218.9	**159938**	160972.8	29.5	**159938**	161178.41	118.1
D07	150	3	111	42	**144756**	128.5	144934	2392.7	**144756**	7200.0	146479	147261.8	140.3	146479	147509.55	573.4	145257	145434.2	94.0	145257	145786.95	331.7
D08	150	4	111	31	159330	68.8	161151	913.9	159197	7200.0	159593	160281.4	118.2	159368	159951.15	491.2	157287	158122.2	81.6	**157193**	158128.75	305.4
D09	150	5	111	25	180170	61.9	181394	530.4	179727	7200.0	179341	179501.2	102.1	178790	179511.75	422.3	176899	178088.0	59.8	**176635**	177704.36	255.1
D10	150	3	149	56	171548	305.0	168893	1809.8	**163932**	7200.0	168780	169205.8	333.2	167825	169422.15	1333.3	167733	168288.6	85.6	164864	167790.30	377.1
D11	150	4	149	42	175931	156.8	180213	1362.2	174667	7200.0	176300	176595.2	249.2	175777	176683.25	1032.6	173162	175699.6	139.4	**172716**	175427.11	487.4
D12	150	5	149	34	196712	171.9	195838	1112.7	195838	7200.0	196167	196752.8	221.5	196133	197087.50	903.4	192425	193578.0	146.3	**192298**	194374.86	429.0
D13	200	3	49	19	**120704**	9.9	**120704**	292.4	**120704**	7200.0	121763	121824.2	38.3	121684	121812.20	157.9	121101	121269.0	19.0	120913	121306.10	76.3
D14	200	4	49	14	138037	9.0	138607	133.0	134630	7200.0	135652	135854.0	40.6	135276	135754.75	171.7	134602	134954.4	19.0	**134215**	134947.70	107.1
D15	200	5	49	11	153027	8.6	156385	102.3	151439	7200.0	152015	152841.4	45.7	152012	152779.35	195.8	151128	151856.2	26.2	**151125**	151772.20	103.6
D16	200	3	99	37	145308	44.3	145308	1105.9	145308	7200.0	145766	145888.2	142.4	145241	145764.45	588.0	144962	145282.2	132.8	**144895**	145513.41	516.8
D17	200	4	99	28	164322	80.9	168750	698.8	163581	7200.0	164370	164747.0	136.2	163935	165012.70	556.6	162493	164553.6	156.1	**162363**	164571.45	487.9
D18	200	5	99	22	184939	66.8	187889	975.6	183284	7200.0	184839	185418.4	131.7	183190	185295.20	539.4	**181182**	184627.2	121.3	**181182**	184612.61	535.7
D19	200	3	149	56	166409	199.2	169483	3657.3	165666	7200.0	165972	166955.8	474.3	165878	166905.50	1922.8	164654	165081.0	277.3	**164306**	165591.61	1070.9
D20	200	4	149	42	188439	261.5	186529	2586.2	185886	7200.0	186242	187632.8	347.8	185855	188008.50	1421.7	182709	184802.0	253.1	**182707**	185268.91	1001.6
D21	200	5	149	34	206528	189.2	207092	2058.1	201848	7200.0	203294	204669.0	304.4	203294	204684.15	1258.6	201648	203817.2	255.6	**201134**	203505.00	1193.7
D22	200	3	199	74	189121	573.6	188262	5577.1	183547	7200.0	186802	187390.8	902.9	186031	271502.45	3725.6	182266	184038.2	612.8	**181049**	183388.30	2007.0
D23	200	4	199	56	204024	569.7	199621	5275.0	199621	7200.0	202978	204643.6	721.9	202332	205043.65	2924.6	198516	201464.0	513.3	**197673**	201010.70	2321.4
D24	200	5	199	45	222973	453.5	221042	4678.5	218610	7200.0	223609	224061.8	294.9	221917	223632.50	2000.0	218075	220549.4	329.4	**216993**	220696.66	1265.6
Avg.					94407.56	122.0	94710.46	1225.9	93430.33	6913.4	93976.81	94264.0	157.7	93735.54	96015.98	661.3	92909.75	93536.5	163.2	**92723.91**	93551.01	624.6
NB					3	–	3	–	11	–	14	–	–	17	–	–	20	–	–	**40**	–	–

Chapter 3

Variable Neighborhood Search for the Label Constrained Minimum Spanning Tree Problem

3.1 Introduction

The Minimum Label Spanning Tree (MLST) problem was introduced by Chang and Leu [10]. In this problem, we are given an undirected graph $G = (V, E)$ with labeled edges; each edge has a single label from the set of labels L and different edges can have the same label. The objective is to find a spanning tree with the minimum number of distinct labels. The MLST is motivated from applications in the communications sector. Since communication networks sometimes include numerous different media such as fiber optics, cable, microwave or telephone lines and communication along each edge requires a specific media type, decreasing the number of different media types in the spanning tree reduces the complexity of the communication process. The MLST problem is known to be $\mathcal{NP}$-complete [10]. Several researchers have studied the MLST problem including Brüggemann et al. [8], Cerulli et al. [13], Consoli et al. [14], Krumke and Wirth [40], Wan et al. [80], and Xiong et al. [83–85].

Recently Xiong et al. [82] introduced a more realistic version of the MLST problem called the Label Constrained Minimum Spanning Tree (LCMST) problem. In contrast to the MLST problem, which completely ignores edge costs, the LCMST problem takes into account the cost or weight of edges in the network (we use the term cost and weight interchangeably in this paper). The objective of the LCMST problem is to find a minimum weight spanning tree that uses at most K labels (i.e., different types of communications media). Xiong et al. [82] describe two simple local search heuristics and a genetic algorithm for solving the LCMST problem. They also describe a Mixed Integer Programming (MIP) model to solve the problem exactly. However, the MIP models were unable to find solutions for problems with greater than 50 nodes due to excessive memory requirements.

The Cost Constrained Minimum Label Spanning Tree (CCMLST) problem is another realistic version of the MLST problem. The CCMLST problem was introduced by Xiong et al. [82]. In contrast to the LCMST problem, there is a threshold on the cost of the minimum spanning tree (MST) while minimizing the number of labels. Thus, given a graph $G = (V, E)$, where each edge (i, j) has a label from the set L and an edge weight c_{ij}, and a positive budget B, the goal of the CCMLST problem is to find a spanning tree with the fewest number of labels whose weight does not exceed the budget B. The notion is to design a tree with the fewest number of labels while ensuring that the budget for the network design is not exceeded. Xiong et al. [82] showed that both the LCMST and the CCMLST are $\mathcal{NP}$-complete. Thus, the resolution of these problems requires heuristics.

In this chapter[1], we focus on the LCMST problem. We propose a Variable Neighborhood Search (VNS) method for this problem. We then compare the VNS method to the heuristics described by Xiong et al. [82]. To do so, we consider existing data sets and also design a set of nine Euclidean large-scale datasets, derived from TSPLIB instances [60]. The VNS method performs extremely well on the LCMST problem, with respect to solution quality and computational running time.

The rest of this chapter is organized as follows. Section 3.2 describes the mathematical formulation proposed for the LCMST problem. Section 3.3 describes the VNS method that we have proposed to solve the problem. Section 3.4 reports on our computational experiments. Finally, Section 3.5 provides concluding remarks.

3.2 Mathematical Formulation

In this section, we provide two mixed integer programming (MIP) models for the LCMST problem. They are based on a singlecommodity and multicommodity network flow formulations [82].

$$e_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is used,} \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

[1]The results of this chapter appear in: Naji-Azimi Z., Salari M., Golden B., S. Raghacan, and Toth P.: “Variable Neighborhood Search for the Cost Constrained Minimum Label Spanning Tree and Label Constrained Minimum Spanning Tree Problems”. Computers & Operations Research, To appear [52].

$$y_k = \begin{cases} 1 & \text{if label } k \text{ is used,} \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

$$f_{ij} = \text{flow on edge } (i,j) \text{ from } i \text{ to } j. \tag{3.3}$$

The MIP formulation based on the single commodity flow (scf) model is as follows:

$$\min \sum_{(i,j)\in E} c_{ij} e_{ij} \tag{3.4}$$

subject to

$$\sum_{(i,j)\in E} e_{ij} = n - 1, \tag{3.5}$$

$$\sum_{i:(i,j)\in A} f_{ij} - \sum_{l:(j,l)\in A} f_{jl} = 1 \qquad j \in V \setminus \{1\}, \tag{3.6}$$

$$\sum_{i:(i,1)\in A} f_{i1} - \sum_{l:(1,l)\in A} f_{1l} = -(n-1), \tag{3.7}$$

$$f_{ij} + f_{ji} \le (n-1) \cdot e_{ij} \qquad \forall (i,j) \in E, \tag{3.8}$$

$$\sum_{(i,j)\in E_k} e_{ij} \le (n-1) \cdot y_k \qquad \forall k \in L, \tag{3.9}$$

$$\sum_{k\in L} y_k \le K, \tag{3.10}$$

$$e_{ij}, y_k \in \{0,1\} \qquad \forall\, (i,j) \in E, \forall k \in L, \tag{3.11}$$

$$f_{ij} \ge 0 \qquad \forall\, (i,j) \in A. \tag{3.12}$$

In the objective function (3.4), we want to minimize the total cost or weight of spanning tree. Constraint (3.5) ensures the tree has exactly (n-1) edges. Constraints (3.6) and (3.7) are included to ensure the graph is connected. To do so, we pick node 1 as the root node (any node of the graph may be picked for this purpose). A supply of n-1 units of flow is available at this root node. All of the other nodes have a demand of 1 unit of flow. Consequently, we need to send one unit of flow from the root node to all the other nodes. Constraint set (3.8) is a forcing constraint set. These constraints enforce the condition that if flow is sent along an edge, the edge must be included in the tree. Constraint set (3.9) is a forcing constraint set between edges and labels. It says that if an edge with label k is used, then this label must be selected. Constraint (3.10) imposes an upper bound on the number of labels, and finally the variables are defined in (3.11) and (3.12).

This single commodity flow formulation is identical to Xiong et al. [82]. However, this formulation can be considerable strengthened by using the technique of disaggregation [81] on constraint set (3.9). We replace this constraint with the stronger:

$$e_{ij} \leq y_k \qquad \forall k \in L, \forall \{i,j\} \in E_k. \tag{3.13}$$

To describe the multicommodity formulation, we define a bidirected network obtained by replacing each undirected edge (i,j) by a pair of arcs (i,j) and (j,i). Let A denote the set of arcs. The variables e_{ij} and y_k are, as before, in the single commodity flow formulation. In addition we define

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is used,} \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

$$f_{ij}^h = \text{flow of commodity } h \text{ along arc } (i,j). \tag{3.15}$$

The multicommodity flow formulation (mcf) can then be described as follows :

$$\min \sum_{(i,j)\in E} c_{ij} e_{ij} \tag{3.16}$$

subject to

$$\sum_{(i,j)\in E} e_{ij} = n-1, \tag{3.17}$$

$$\sum_{i:(i,h)\in A} f_{ih}^h - \sum_{l:(h,l)\in A} f_{hl}^h = 1 \qquad \forall h \in V \setminus \{1\}, \tag{3.18}$$

$$\sum_{i:(i,1)\in A} f_{i1}^h - \sum_{l:(1,l)\in A} f_{1l}^h = -1 \qquad \forall h \in V \setminus \{1\}, \tag{3.19}$$

$$\sum_{i:(i,j)\in A} f_{ij}^h - \sum_{l:(j,l)\in A} f_{jl}^h = 0 \qquad \forall h \in V \setminus \{1\}, \forall j \neq h, \tag{3.20}$$

$$f_{ij}^h \leq x_{ij} \qquad \forall (i,j) \in A, \forall h \in V \setminus \{1\}, \tag{3.21}$$

$$x_{ij} + x_{ji} \leq e_{ij} \qquad \forall (i,j) \in E, \tag{3.22}$$

$$e_{ij} \leq y_k \qquad \forall k \in L, \forall (i,j) \in E_k, \tag{3.23}$$

$$\sum_{k\in L} y_k \leq K, \tag{3.24}$$

$$e_{ij}, y_k \in \{0,1\} \qquad \forall (i,j) \in E, \forall k \in L, \tag{3.25}$$

$$x_{ij} \geq 0 \qquad \forall (i,j) \in A, \tag{3.26}$$

$$f_{ij}^h \geq 0 \qquad \forall (i,j) \in A, \forall h \in V \setminus \{1\}. \tag{3.27}$$

The objective function (3.16) and constraint (3.17) are identical to (3.4) and (3.5). In the multicommodity flow model, each commodity has the root node as its origin and the destination is one of the nodes in $\{2, 3, \cdots, n\}$, for a total of n-1 commodities. Each commodity has a supply of 1 unit of flow and a demand of 1 unit of flow. Thus, constraints (3.18) to (3.20) represent the flow balance constraints for these flows. Constraints (3.21) and (3.22) are forcing constraints. Constraint (3.21) ensures that if flow of a commodity is sent on an arc, the arc is selected. Constraint (3.22) ensures that either arc (i, j) or arc (j, i) can be in the solution, but not both. In fact, the tree must be directed away from the root node. Constraints (3.23) and (3.24) are as before. We note that we have used the strengthened version of the forcing constraint between the edge variables and the label variables, as opposed to Xiong et al. [82] who have used the weaker version. Constraints (3.25) define the edge and label variables as binary.

Constraints (3.26) and (3.27) define the arc and flow variables as continuous variables. We note that these variables can be defined as binary. However, it is easy to see that if the edge and label variables are binary then the arc and flow variables are automatically integral.

In our computational work, we use the multicommodity flow model to obtain lower bounds and optimal solutions on our test instances. We found that it is considerably stronger than the multicommodity flow model proposed by Xiong et al. [82] that has the aggregated version of constraint (3.23).

3.3 Variable Neighborhood Search for the LCMST Problem

In this section, we develop our Variable Neighborhood Search algorithm for the LCMST problem. Variable Neighborhood Search is a metaheuristic proposed by Mladenovic and Hansen [50], which explicitly applies a strategy based on dynamically changing neighborhood structures. The algorithm is very general and many degrees of freedom exist for designing variants.

The basic idea is to choose a set of neighborhood structures that vary in size. These neighborhoods can be arbitrarily chosen, but usually a sequence of neighborhoods with increasing cardinality is defined. In the VNS paradigm, an initial solution is generated, then the neighborhood index is initialized, and the algorithm iterates through the different neighborhood structures looking for improvements, until a stopping condition is met.

We consider VNS as a framework, and start by constructing an initial solution. We then improve upon this initial solution using *local search*. Then, the improvement of the incumbent solution (R) continues in a loop until the termination criterion is reached. This loop contains a *shaking phase* and a *local search phase*. The *shaking phase* follows the VNS paradigm. It considers a specially designed neighborhood and makes random changes to the current solution that enables us to explore neighborhoods farther away from the current solution. The *local search phase* considers a more restricted neighborhood set and attempts to improve upon the quality of a given solution.

We now make an important observation regarding the relationship between the selected labels and the associated solution. Given a set of labels, the minimum cost solution on the labels R is the minimum spanning tree computed on the graph induced by the labels in R. We denote the minimum spanning tree on the graph induced by the labels in R as MST(R) and its cost by MSTCOST(R). These two can be computed rapidly using any of the well-known minimum spanning tree algorithms [41, 59]. Consequently, our search for a solution focuses on selecting labels (as opposed to edges), and our neighborhoods as such are neighborhoods on labels. Our solutions then are described in terms of the labels they contain (as opposed to the edges they contain).

3.3.1 Initial Solution

Since having more labels in the solution results in a less MSTCOST, our procedure to construct an initial solution focuses on selecting a set of labels with the maximum number of allowed labels that result in a connected graph. Let *Components*(R) denote the number of connected components in the graph induced by the labels in R. This can easily be computed using depth first search [76]. Our procedure adds labels to our solution in a greedy fashion. The label selected for addition to the current set of labels is the one (amongst all the labels that are not in the current set of labels) that when added results in the minimum number of connected components. Ties between labels are broken randomly. In other words, we choose a label for addition to the current set of labels R randomly from the set

$$S = \{t \in (L \setminus R) : \min \; Components\,(R \cup \{t\})\}. \tag{3.28}$$

This continues until the selected labels result in a single component.

In figure 3.1, an example illustrating the initialization method is shown. Suppose there are three labels, namely a, b, and c, in the label set. Since the number of connected components after adding label c is less than for the two other labels, we add this label to the solution. However the graph is still not connected, so we go further by repeating this procedure with the remaining labels. Both labels a and b produce the same number of components, so we select one of them randomly (label b).

It is easy to observe that if R is a subset of labels, then the cost of an MST on any superset T of R is less than or equal to the cost of the MST on R. In other words, if $R \subseteq T$ then MSTCOST(R) $\geq$ MSTCOST(T). Consequently, in order to try to minimize the cost, we iteratively add labels to the initial set of labels until we have K labels. To do so we choose the label that, when added, results in the lowest cost minimum spanning tree. In other words the label to be added is selected from

$$S = \{t \in (L \setminus R) : \min \ MSTCOST(R \cup \{t\})\}. \tag{3.29}$$

and ties are broken randomly. We continue adding labels to the current solution in this fashion until we obtain a maximum number of labels.

3.3.2 Shaking Phase

The shaking phase follows the VNS paradigm and dynamically expands the neighborhood search area. Suppose R denotes the current solution (it really denotes the labels in the current solution, but as explained earlier it suffices to focus on labels). In this step, we use randomization to select a solution that is in the size k neighborhood of the solution R, i.e., $N_k(R)$. Specifically $N_k(R)$ is defined as the set of labels that can be obtained from R by performing a sequence of exactly k additions and/or deletions of labels. So $N_1(R)$ is the set of labels obtained from R by either adding exactly one label from R, or deleting exactly one label from R. $N_2(R)$ is the set of labels obtained from R by either adding exactly two labels, or deleting exactly two labels, or adding exactly one label and deleting exactly one label.

The shaking phase may result in the selection of labels that do not result in a connected graph, or result in a solution that does not meet the label constraint. If the set of labels results in a graph that is not connected, we add labels that are not in the current solution one by one, at random until the graph is connected. If the number of labels in the solution does not meet the label constraint, we iteratively delete labels from the current set of labels by choosing the label that when removed results in the lowest cost minimum spanning tree.

3.3.3 Local Search Phase

The local search phase consists of two parts. The first part adds labels to a given solution until it has K labels. The additional labels are selected iteratively, each time selecting a label that provides the greatest decrease in the cost of the minimum spanning tree, until we have K labels. In fact we select at random a label from the set of (3.29) and we add it to the solution. The local search procedure then tries to swap each of the labels in the current solution with an unused one, if it results in a lower minimum spanning tree cost. To this aim, it iteratively considers the labels in the solution and tests all possible exchanges of a given label with unused labels until it finds an exchange resulting in a lower MST cost. If we find such an exchange, we implement it (i.e., we ignore the remaining unused labels) and proceed to the next label in our solution. Obviously, a label remains in the solution if the algorithm cannot find a label swap resulting in an improvement. This is illustrated with an example in figure 3.2. Consider $A = \{b, c\}$ we have MSTCOST(A) $= 12$ and MSTCOST$(A \setminus \{b\} \cup a) = 9$. Therefore, we remove label b and add label a to the representation of our solution. The pseudocode for the VNS method for the LCMST problem is provided in Algorithm 3.

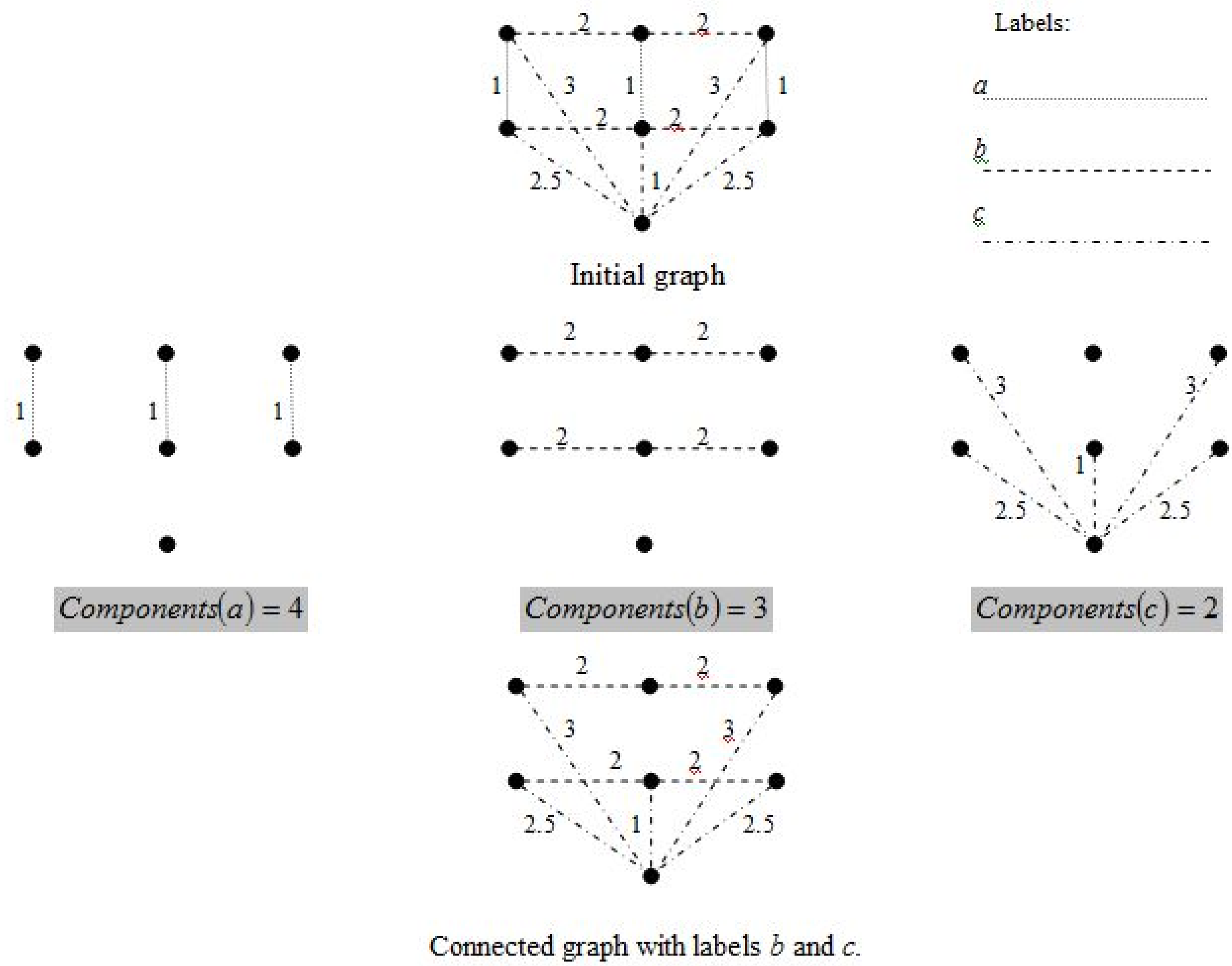


Figure 3.1: An example illustrating the selection of labels for the initial connected subgraph.

Figure 3.2: An example for the swap of used and unused labels.

3.4 Computational Results

In this section, we report on an extensive set of computational experiments on the LCMST problem. The proposed method has been tested on a Pentium IV machine with a 2.61 GHz processor and 2 GB RAM, under the Windows operating system. We also use ILOG CPLEX 10.2 to solve the MIP formulation [39].

The two parameters that are adjustable within the VNS procedure are the value of k (the size of the largest neighborhood $N_k(R)$ in the VNS method), and *Iter*, the number of iterations in which the algorithm is not able to improve the best known solution (which is the termination criterion). Increasing k, not only increases the size of the neighborhood but also increases the running time. We found that setting k=5 provides the best results without a significant increase in running time. Additionally, as the value of *Iter* is increased the running time of the algorithm is increase, though the quality of the solution improves. We found that setting *Iter*=10 provides the best results in a reasonable amount of running time. We now describe how we generated our datasets, and then discuss our computational experience on these datasets for the LCMST problem.

3.4.1 Datasets

Xiong et al. [82] created a set of test instances for the LCMST problem. These include 37 small instances with 50 nodes or less, 11 medium-sized instances that range from 100 to 200 nodes, and one large instance with 500 nodes. All of these instances are complete graphs. We also generated a set of 18 large instances that range from 500 to 1000 nodes. These were created from nine large TSP instances in TSPLIB and considered to be Euclidean (since the problems arise in the telecommunications industry, the costs of edges are generally proportional to their length). To produce a labeled graph from a TSPLIB instance, we construct a complete graph using the coordinates of the nodes in the TSPLIB instance. The number of labels in the instance is one half of the total number of nodes, and the labels are randomly assigned. Then two values for K, as the maximum allowed labels, have been considered. Specifically, we used K= 75 and K=150.

```
R=φ;
Initialization Procedure(G,R);
Local Search(G,R);
while Termination criterion not met do
    k=1;
    while k ≤ 5 do
        R̄=Shaking_Phase(G, k, R);
        while Components(R̄) > 1 do
            Select at random a label u ∈ L\R̄ and add it to R̄;
        end
        while |R̄| > K do
            S={t ∈(L\R̄): min MSTCOST(R̄\{t})};
            Select at random a label u ∈ S and delete it from R̄;
        end
        Local Search(G,R̄);
        if MSTCOST(R̄) < MSTCOST(R) then
            R̄=Ŕ and k=1;
        end
        else
            k=k+1;
        end
    end
end

Initialization Procedure(G,R):
while Components(R) > 1 do
    S={t ∈ (L\R) : min Components (R ∪ {t})};
    Select at random a label u ∈ S and add it to R;
end
while |R| < K do
    S={t ∈ (L \ R) : min MSTCOST(R ∪ {t})};
    Select at random a label u ∈ S and add it to R;
end

Shaking_Phase(G,k,R):
for i=1,...,k do
    r=random(0,1);
    If r ≤ 0.5 Then Delete at random a label from R Else Add at random a label to R;
    Return(R);
end

Local Search(G,R):
while |R| < K do
    S = {t ∈ (L\R) : min MSTCOST(R ∪ {t})};
    Select at random a label u ∈ S and add it to R;
end
Consider the labels i ∈ R one by one
Swap the label i with the first unused label that strictly lowers the MST cost;
end
```

Algorithm 3: Variable Neighborhood Search Algorithm for the LCMST Problem

3.4.2 Results

We have executed the proposed VNS method on 66 LCMST instances. The results are described in Table 3.1 for the small instances, 3.2 for the medium-sized instances and 3.3 for the large instances.

On the 37 small instances, the VNS method found the optimal solution in all 37 instances (recall that the optimal solution is known in all of these instances), while LS1, LS2, and GA generated the best solution 34, 33, and 29 times, respectively, out of the 37 instances. The average running time of the VNS method was 0.13 seconds while LS1, LS2, and GA took 0.11, 0.11, and 0.05 seconds, respectively. For the small and medium-sized instances, the termination criterion used was 10 iterations without an improvement. On the 11 medium-sized instances, the VNS method generated the best solution in all 11 instances, while LS1, LS2, and GA generated the best solution 7, 8, and 3 times, respectively, out of the 11 instances. The average running time of the VNS method was 9.82 seconds while LS1, LS2, and GA took 35.51 , 38.05, and 8.03 seconds, respectively.

This indicates that the VNS method finds better solutions in a much greater number of instances than any of the three comparative procedures. This advantage is clear on the medium-sized instances. For the large instances the termination criterion used was a specified running time which is shown in the tables with the computational results. On the 18 large instances, the VNS method generated the best solution in 12 out of the 18 instances, while LS1, LS2, and GA generated the best solution 2, 4, and 2 times, respectively, out of the 18 instances. The average running time of the VNS method was 1089 seconds, while LS1, LS2, and GA took 3418, 3371, and 1617 seconds, respectively. This suggests that the VNS method is the best method in that it rapidly finds solutions for the LCMST problem the most number of times. However, there seem to be a fair number of instances (about a third) where an alternate heuristic like LS1, LS2, or GA obtains a superior solution. On the whole, the VNS method is clearly the best amongst these four heuristics.

3.5 Conclusion

In this chapter, we considered the LCMST problem. We developed a VNS method for solving this problem. We compared the solutions obtained by the VNS method to optimal solutions for small instances and to solutions obtained by three heuristics LS1, LS2, and GA that were previously proposed for the LCMST problem [82]. We generated a set of large instances from the TSPLIB dataset. The VNS method was clearly the best heuristic for the LCMST instances. Of the 66 instances, it provided the best solution in

Table 3.1: VNS, GA, LS1, and LS2 for the LCMST problem on small datasets.

# Nodes, # Labels	K	MIP Value	LS1		LS2		GA		VNS	
			Gap	Time	Gap	Time	Gap	Time	Gap	Time
20,20	2	6491.35	0.00	0.02	0.00	0.01	0.00	0.00	0.00	0.01
	3	5013.51	0.00	0.04	0.00	0.04	0.00	0.00	0.00	0.02
	4	4534.67	0.00	0.05	0.00	0.05	0.00	0.01	0.00	0.02
	5	4142.57	0.00	0.05	0.00	0.04	0.00	0.01	0.00	0.02
	6	3846.50	0.00	0.05	0.00	0.04	0.00	0.02	0.00	0.02
	7	3598.05	0.00	0.05	0.00	0.05	0.00	0.02	0.00	0.03
	8	3436.57	0.00	0.05	0.00	0.05	0.00	0.02	0.00	0.03
	9	3281.05	0.00	0.04	0.00	0.05	0.00	0.02	0.00	0.03
	10	3152.05	0.00	0.05	0.00	0.04	0.00	0.02	0.00	0.03
	11	3034.01	0.00	0.05	0.00	0.04	0.00	0.03	0.00	0.03
30,30	3	7901.81	0.00	0.04	0.00	0.05	0.00	0.01	0.00	0.05
	4	6431.58	0.00	0.06	0.00	0.05	0.00	0.02	0.00	0.05
	5	5597.36	0.00	0.07	0.00	0.06	0.00	0.03	0.00	0.06
	6	5106.94	0.00	0.11	0.00	0.07	0.00	0.04	0.00	0.06
	7	4751.00	0.00	0.12	0.46	0.07	0.00	0.05	0.00	0.08
	8	4473.11	0.00	0.07	0.00	0.07	0.00	0.05	0.00	0.09
	9	4196.71	0.00	0.08	0.00	0.11	0.00	0.05	0.00	0.10
	10	3980.99	0.52	0.09	0.52	0.16	0.00	0.06	0.00	0.12
	11	3827.23	0.00	0.09	0.00	0.13	1.41	0.07	0.00	0.14
	12	3702.08	0.00	0.09	0.23	0.09	0.00	0.07	0.00	0.13
	13	3585.42	0.00	0.09	0.00	0.10	0.00	0.07	0.00	0.14
40,40	3	11578.61	0.00	0.05	0.00	0.06	0.00	0.02	0.00	0.11
	4	9265.42	1.72	0.06	2.56	0.07	1.72	0.03	0.00	0.09
	5	8091.45	0.00	0.09	0.00	0.10	0.75	0.03	0.00	0.15
	6	7167.27	0.00	0.11	0.00	0.11	2.53	0.07	0.00	0.20
	7	6653.23	0.13	0.12	0.00	0.13	0.13	0.05	0.00	0.23
	8	6221.63	0.00	0.29	0.00	0.20	0.00	0.09	0.00	0.28
	9	5833.39	0.00	0.26	0.00	0.22	0.48	0.10	0.00	0.25
	10	5547.08	0.00	0.16	0.00	0.24	0.00	0.10	0.00	0.33
	11	5315.92	0.00	0.18	0.00	0.20	0.00	0.15	0.00	0.31
	12	5164.14	0.00	0.35	0.00	0.21	0.00	0.09	0.00	0.31
50,50	3	14857.09	0.00	0.07	0.00	0.08	3.08	0.02	0.00	0.14
	4	12040.89	0.00	0.15	0.00	0.12	0.00	0.04	0.00	0.14
	5	10183.95	0.00	0.21	0.00	0.26	0.00	0.12	0.00	0.28
	6	9343.69	0.00	0.25	0.00	0.34	0.00	0.12	0.00	0.25
	7	8594.36	0.00	0.30	0.00	0.31	1.51	0.12	0.00	0.25
	8	7965.52	0.00	0.24	0.00	0.34	0.00	0.20	0.00	0.30

Table 3.2: VNS, GA, LS1, and LS2 for the LCMST problem on medium-sized datasets.

# Nodes, # Labels	Labels	LS1		LS2		GA		VNS	
		Cost	Time	Cost	Time	Cost	Time	Cost	Time
100, 50	20	**8308.68**	3.92	**8308.68**	3.26	8335.75	2.94	**8308.68**	1.54
100, 100	20	**10055.85**	5.89	**10055.85**	6.35	10138.27	1.70	**10055.85**	1.24
100, 100	40	7344.72	11.36	**7335.61**	12.70	**7335.61**	2.68	**7335.61**	2.44
150, 75	20	11882.62	7.47	**11846.80**	13.95	11854.17	4.49	**11846.80**	12.61
150, 75	40	**9046.71**	17.42	**9046.71**	18.57	9047.22	12.63	**9046.71**	2.83
150, 150	20	15427.54	25.33	**15398.42**	19.88	15688.78	3.82	**15398.42**	17.73
150, 150	40	**10618.58**	41.67	10627.36	55.38	10728.93	7.04	**10618.58**	18.45
200, 100	20	**14365.95**	27.19	**14365.95**	17.25	14382.65	10.64	**14365.95**	14.21
200, 100	40	**10970.94**	46.23	**10970.94**	49.26	**10970.94**	12.84	**10970.93**	5.80
200, 200	20	18951.05	44.61	18959.37	89.58	**18900.25**	13.06	**18900.25**	9.50
200, 200	40	**12931.46**	159.56	12941.85	132.40	12987.29	16.49	**12931.46**	21.64

- The best solutions are in bold.

Table 3.3: VNS, GA, LS1, and LS2 for the LCMST problem on large datasets.

# Nodes, # Labels	Labels	LS1		LS2		GA		VNS		
		Cost	Time	Cost	Time	Cost	Time	Cost	Time	Max time
532, 266	75	95671.13	397	95617.19	558	95808.91	383	**95562.78**	236	500
	150	78418.55	703	**78392.49**	1011	78400.90	627	78400.90	363	500
574, 287	75	44799.25	451	44650.40	485	44682.80	415	**44633.91**	377	500
	150	34521.44	850	34501.63	944	34502.26	753	**34457.48**	442	500
575, 287	75	8319.41	654	8309.18	706	8335.96	501	**8307.45**	335	500
	150	6683.95	892	**6683.48**	996	6683.95	822	**6683.48**	370	500
654, 327	75	34751.11	861	34809.59	1117	34795.62	551	**34722.55**	451	1200
	150	30107.18	1325	30092.77	1901	30103.64	1004	**30090.78**	1108	1200
657, 328	75	61970.46	2001	61978.58	1290	61977.86	684	**61941.07**	502	1200
	150	47355.97	2430	**47351.50**	2624	47413.26	1443	47397.51	1185	1200
666, 333	75	3509.17	706	3500.22	981	3505.82	663	**3496.59**	414	1200
	150	2821.43	1441	2821.70	2043	2820.87	1920	**2820.63**	773	1200
724, 362	75	56600.13	1374	56377.24	1435	**56245.65**	973	56510.83	700	1200
	150	42874.35	4869	42855.57	3694	42847.26	2273	**42835.26**	1060	1200
783, 391	75	12560.76	2839	12539.51	2466	**12529.96**	1039	12542.81	875	1200
	150	9509.07	2766	**9495.57**	3748	9502.96	2024	**9495.57**	1180	1200
1000, 500	75	**27246908**	9100	27280794	13092	27428732	4955	27257080	3333	7200
	150	**20196378**	27869	20216576	21587	20258258	8085	20229668	5903	7200

- The best solutions are in bold.

60 instances.

Chapter 4

The Generalized Covering Salesman Problem

4.1 Introduction

The Traveling Salesman Problem (TSP) is one of the most celebrated combinatorial optimization problems. Given a graph $G = (N, E)$, the goal is to find the minimum length tour of the nodes in N, such that the salesman, starting from a node, visits each node exactly once and returns to the starting node (see Dantzig et al., [19]). In recent years, many new variants such as the TSP with profits [24], the Clustered TSP [11], the Generalized TSP [27], the Prize Collecting TSP [28], and the Selective TSP [44] have been introduced and studied. The recent monograph by Gutin and Punnen [35] has a nice discussion of different variations of the TSP and solution procedures.

Current [16] defined and introduced a variant of the TSP called the Covering Salesman Problem (CSP). In the CSP the goal is to find a minimum length tour of a subset of n given nodes, such that every node i not on the tour is within a predefined covering distance d_i from a node on the tour. If $d_i = 0$ or $d_i < min_j c_{ij}$, where c_{ij} denotes the shortest distance between nodes i and j, the CSP reduces to TSP (thus it is $\mathcal{NP}$-hard). Current and Schilling [17] referred to several real world examples, such as routing of rural healthcare delivery teams where the assumption of visiting each city is not valid since it is sufficient for all cities to be near to some stops on the tour (the inhabitants of those cities which are not in the tour are expected to go to their nearest stop). Current and Schilling [17] also suggested a heuristic for the CSP where in the first step a Set Covering Problem (SCP) over the given nodes is solved. Specifically, to solve the related Set Covering Problem, a zero-one $n \times n$ matrix, i.e. matrix A, in which the rows and columns correspond to the nodes is considered. If node i can be covered by node j (i.e., $d_i \geq c_{ij}$) then a_{ij} is equal to

1, otherwise it is 0. Since the value of covering distance d_i varies for each node i, it should be clear that A is not a symmetric matrix, but for each node i we have $a_{ii} = 1$. We should also mention that in the CSP there is no cost associated with the nodes, so the cost of columns of matrix A are all equal to one. Therefore a unit cost Set Covering Problem is solved in the first step of this algorithm to obtain the cities visited on the tour. Then the algorithm finds the optimal TSP tour of the nodes over these cities. Since there might be multiple optimal solutions to the SCP, Current and Schilling suggest that all optimal solutions to the SCP be tried out (i.e., have an optimal TSP tour constructed over the nodes selected in the optimal SCP), and the best solution be selected. The algorithm is demonstrated on a sample problem, but no additional computational results are reported.

Arkin and Hassin [3] introduced a geometric version of the Covering Salesman Problem. In this problem each node specifies a compact set in the plane, its neighborhood, within which the salesman should meet the stop. The goal is computing the shortest length tour that intersects all of the neighborhoods and returns to the initial node. In fact, this problem generalizes the Euclidean Traveling Salesman Problem in which the neighborhoods are single points. Unlike the CSP in which each node i should be within a covering distance d_i from the nodes which are visited by the tour, in the geometric version it is sufficient for the tour to intersects the specific neighborhoods without visiting any specific node of the problem. Arkin and Hassin [3] presented simple heuristics for constructing tours for a variety of neighborhood types. They show that the heuristics provide solutions where the length of the tour is guaranteed to be within a constant factor of the length of the optimal tour.

Other than Current [16], Current and Schilling [17], and Arkin and Hassin [3] the CSP does not seem to have got much attention in the literature. However, some generalizations of the CSP have appeared in the literature. One generalization and closely related problem discussed in Gendreau et al. [34] is the Covering Tour Problem (CTP). Here, some subset of the nodes must be on the tour while the remaining nodes need not be on the tour. Like the CSP, a node i not on the tour must be within a predefined covering distance d_i from a node on the tour. When the subset of nodes that must be on the tour is empty the CTP reduces to the CSP, and when the subset of nodes that must be on the tour consists of the entire node set the CTP reduces to the TSP. Gendreau et al. [34] proposed a heuristic that combines GENIUS, a high quality heuristic for the TSP [33], with PRIMAL1, a high quality heuristic for the SCP [4].

Vogt et al. [79] considered the Single Vehicle Routing Allocation Problem (SVRAP) that further generalizes the CTP. Here, in addition to tour (routing) costs, nodes covered by the tour (that are not on it) incur an allocation cost, and nodes not covered by the tour incur a penalty cost. If the penalty costs are set high and the allocation costs are set to

0, the SVRAP reduces to the CTP. Vogt et al. [79] discussed a tabu search algorithm for the SVRAP that includes aspiration, path relinking and frequency based-diversification.

All of the earlier generalizations of the CSP assume that when a node is covered, its entire demand can be covered. However, in many real-world applications this is not necessarily the case. As an example, suppose we have a concert tour which must visit or cover several cities. Since each show has a limited number of tickets, and large metropolitan areas are likely to have ticket demand which exceeds ticket supply for a single concert, there must be concerts on several nights in each large city in order to fulfill the ticket demand. Also in the rural healthcare delivery problem, discussed in Current and Schilling [17], when we create a route for the rural medical team, on each day a limited number of people can benefit from the services, so the team should visit some places more than once. Consequently, rather than assuming that a node's demand is completely covered when either it or a node that can cover it is visited, we generalize the CSP by specifying the coverage demand k_i which denotes the number of times a node i should be covered. In other words, node i must be covered k_i times by a combination of visits to node i and visits to nodes that can cover node i. If k_i=1 for all nodes, we obtain the CSP. This generalization significantly complicates the problem, and is quite different from the earlier generalizations that effectively deal with unit coverage (i.e., k_i=1). In addition, since in many applications there is a cost for visiting a node (e.g., cost of hotel for staying in a city for one night) we include node visiting costs (for nodes on the tour) in the GCSP. In the next section, we introduce and explain in more detail three different variations that can arise in the GCSP (that deal with whether a node can be revisited or not). All these variants are strongly $\mathcal{NP}$-hard, since they contain the classical TSP as a special case.

The rest of this chapter[1] is organized as follows. In Section 4.2, we formally define the generalized covering salesman problem, and describe three variants. We also describe a mathematical model for the problem. Section 4.3 describes two local search heuristics for the GCSP. Section 4.4 discusses our computational experience on the three different variants of the GCSP, as well as the CSP and the Generlized TSP (GTSP), which are special cases of the GCSP. Section 4.5 provides concluding remarks and discusses some possible extensions of the GCSP.

[1]The results of this chapter appear in: Golden B., Naji-Azimi Z., S. Raghacan, Salari M., and Toth P.: "The Generalizaed Covering Salesman Problem". INFORMS Journal on Computing. Submitted for publication [51].

4.2 Problem Definition

In the *Generalized Covering Salesman Problem* (GCSP), we are given a graph $G = (N, E)$ with $N = \{1, 2, \cdots, n\}$ and $E = (\{i, j\} : i, j \in N, i < j)$ as the node and edge sets respectively. Without loss of generality, we assume the graph is complete with edge lengths satisfying the triangle inequality, and let c_{ij} denote the cost of edge $\{i, j\}$ (c_{ij} may be simply set to the cost of the shortest path from node i to j). Each node i can cover a subset of nodes D_i (note that $i \in D_i$, and when coverage is based on distance D_i, it can be computed easily from c_{ij}) and has a predetermined coverage demand k_i. F_i is the fixed cost associated with visiting node i, and a solution is feasible if each node i is covered at least k_i times by the nodes in the tour. The objective is to minimize the total cost which is the sum of the tour length and the fixed costs associated with the visited nodes.

We discuss three variants of the GCSP: *Binary GCSP*, *Integer GCSP without overnight* and *Integer GCSP with overnight*. In the following we explain each of these variants.

Binary Generalized Covering Salesman Problem: In this version, the tour is not allowed to visit a node more than once and after visiting a node we must satisfy the remaining coverage demand of that node by visiting other nodes that can cover it. We use the qualifier binary as this version only permits a node to be visited once.

Integer Generalized Covering Salesman Problem without Overnights: Here a node can be visited more than once, but overnight stay is not allowed. Therefore, to have a feasible solution, after visiting a node, the tour can return to this node, if necessary, after having visited at least one other node. In other words, the tour is not allowed to visit a node more than one time consecutively. We use the qualifier integer as this version allows a node to be visited multiple (or an integer number of) times.

Integer Generalized Covering Salesman Problem with Overnights: This version is similar to the previous one, but overnight stay at a node is allowed.

In the CSP, $k_i = 1$ for all nodes $i \in N$. Clearly, the CSP is a special case of the binary GCSP. When there are unit demands there is no benefit to revisiting a node, consequently the CSP can also be viewed as a special case of the integer variants of the GCSP. Thus the CSP is a special case of all three variants of the GCSP. As the TSP is a special case of the CSP, all three GCSP variants are strongly $\mathcal{NP}$-hard.

We now discuss the issue of feasibility of a given instance of the problem. For the binary GCSP, the problem is feasible if demand is covered when all nodes in the graph are visited by the tour. In other words if h_j denotes the number of nodes that can cover node j (i.e., the number of nodes i for which $j \in D_i$), then the problem is feasible if $k_j \leq h_j$. For the integer GCSP with and without overnights, the problem is always feasible, since a tour on all nodes in the graph may be repeated until all demand is covered.

We now formulate the three different variants of the GCSP. We first provide an integer programming formulation for the binary GCSP, and then an integer programming formulation for the integer GCSP. Our models are on directed graphs (for convenience, as they can easily be extended to asymmetric versions of the problem). Hence we replace the edge set E by an arc set A, where each edge $\{i, j\}$ is replaced by two arcs (i, j) and (j, i) with identical costs. Also, from the problem data we have available

$$a_{ij} = \begin{cases} 1 & \text{if node } j \text{ can cover node } i, \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

We introduce the decision variables:

$$w_i = \begin{cases} 1 & \text{if node } i \text{ is on the tour,} \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is chosen to be in the solution,} \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

The integer programming model can now be stated as:

BinaryGCSP:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{i\in N} F_i w_i \tag{4.4}$$

subject to:

$$\sum_{j:(j,i)\in A} x_{ji} = \sum_{j:(i,j)\in A} x_{ij} = w_i \qquad \forall i \in N, \tag{4.5}$$

$$\sum_{j\in N} a_{ij}w_j \geq k_i \qquad \forall i \in N, \tag{4.6}$$

$$\sum_{l\in S}\sum_{k\in N\setminus S} x_{lk} + \sum_{k\in N\setminus S}\sum_{l\in S} x_{kl} \geq 2(w_i + w_j - 1) \qquad S \subset N, 2 \leq |S| \leq n-2 \tag{4.7}$$

$$i \in S, j \in N \setminus S,$$

$$x_{ij} \in \{0, 1\} \qquad \forall (i, j) \in A, \tag{4.8}$$

$$w_i \in \{0,1\} \qquad \forall (i) \in N. \tag{4.9}$$

The objective is to minimize the sum of the tour costs and the node visiting costs. Constraint set 4.5 ensures that for each on-tour customer, we have one incoming and one outgoing arc. Constraint set 4.6 specifies that the demand of each node must be covered. Constraint set 4.7 is a connectivity constraint that ensures that there are no subtours. Note that there are an exponential number of connectivity constraints. Constraints 4.8 and 4.9 define the variables as binary.

For the integer GCSP without overnights we introduce two additional variables to represent the number of times a node is visited, and the number of times an arc is traversed in the tour.

y_i: Number of times that node i is visited by the tour.
z_{ij}: Number of times arc (i,j) is traversed by the tour.
The integer programming model can now be stated as:

IntegerGCSP:

$$\min \sum_{(i,j)\in A} c_{ij} z_{ij} + \sum_{i \in N} F_i y_i \tag{4.10}$$

subject to:

$$\sum_{j:(j,i)\in A} z_{ji} = \sum_{j:(i,j)\in A} z_{ij} = y_i \qquad \forall i \in N, \tag{4.11}$$

$$\sum_{j \in N} a_{ij} y_j \geq k_i \qquad \forall i \in N, \tag{4.12}$$

$$y_i \leq L w_i \qquad \forall i \in N, \tag{4.13}$$

$$z_{ij} \leq L x_{ij} \qquad \forall (i,j) \in A, \tag{4.14}$$

$$\sum_{l \in S} \sum_{k \in N \setminus S} x_{lk} + \sum_{k \in N \setminus S} \sum_{l \in S} x_{kl} \geq 2(w_i + w_j - 1) \qquad S \subset N, 2 \leq |S| \leq n-2 \tag{4.15}$$

$$i \in S, j \in N \setminus S,$$

$$x_{ij} \in \{0,1\}, z_{ij} \in Z^+ \qquad \forall (i,j) \in A, \tag{4.16}$$

$$w_i \in \{0,1\}, y_i \in Z^+ \qquad \forall (i) \in N. \tag{4.17}$$

where L is a sufficiently large positive value. The objective is to minimize the sum of the tour costs and the node visiting costs. Constraint set 4.11 ensures that if node i is visited y_i times, then we have y_i incoming and y_i outgoing arcs. Constraint set 4.12

specifies that the demand of each node must be covered. Constraint sets 4.13 and 4.14 are linking constraints, ensuring that w_i and x_{ij} are 1 if y_i or z_{ij} are greater than 0 (i.e., if a node is visited or an arc is traversed). Note that it suffices to set $L = max_{i \in N} \{k_i\}$. Constraint set 4.15 is a connectivity constraint that ensures that there are no subtours. Note again, that there are an exponential number of connectivity constraints. Finally, constraint sets 4.16 and 4.17 define the variables as binary and integer as appropriate. For the integer GCSP with overnights, the above integer programming model (IntegerGCSP) is valid if we augment the arc set A with self loops. Specifically, we add to A the arc set $\{(i,i) : i \in N\}$ (or $A = A \cup \{(i,i) : i \in N\}$) with c_{ii} the cost of self loop arcs (i,j) set to 0.

Note that both the binary GCSP and the integer GCSP formulations rely heavily on the integrality of the node variables. Consequently, the LP-relaxations of these models can be quite poor. Further, these models have an exponential number of constraints, implying that this type of model can only be solved in a cutting plane or a branch-and-cut framework. Thus considerable strengthening of the above formulations is necessary, before they are viable for obtaining exact solutions to the GCSP. In this paper, we focus on local search algorithms to develop high-quality solutions for the GCSP.

4.3 Local Search Algorithms

In this section we propose two local search solution procedures, and refer to them as LS1 and LS2, respectively. They are designed to be applicable to all variants of GCSP. In both algorithms, we start from a random initial solution. As we discussed in Section 4.2, assuming that a problem is feasible (which can be checked easily for the binary GCSP) any random order of the n nodes produces a feasible solution for the binary GCSP, and repeating this ordering until all demand is covered produces a feasible solution for the integer GCSP. We provide an initial solution to our local search heuristics by considering a random initial ordering of the nodes in the graph and repeat this ordering for the integer variants (if necessary) to cover all of the demand.

A solution is represented by the sequence of nodes in the tour. Thus for the binary GCSP no node may be repeated on the tour, while in the integer GCSP nodes may be repeated on the tour. For the integer GCSP with no overnights a repeated node may not be next to itself in the sequence, while in the integer GCSP with overnights a repeated node is allowed to be next to itself in the sequence. Thus $\langle 1,2,3,4,5,8,9 \rangle$, $\langle 1,2,3,4,3,2,8 \rangle$, $\langle 1,1,2,3,3,8 \rangle$ represent tour sequences that do not repeat nodes, repeat nodes but not consecutively, and repeat nodes consecutively. Observe that if the costs are non-negative, then in the integer GCSP with overnights there is no benefit to going away from a node and returning to revisit it.

4.3.1 LS1

LS1 tries to find improvements in a solution S by replacing some nodes of the current tour. It achieves this in a two step manner. First, LS1 deletes a fixed number of nodes. (The number of nodes removed from the tour is equal to a predefined parameter, *Search-magnitude*, multiplied by the number of nodes in the current tour. If this number is greater than 1 it is rounded down, otherwise it is rounded up.) It removes a node k from the current solution S with a probability that is related to the current tour and computed as:

$$P_k = C_k / \sum_{s \in S} C_s \tag{4.18}$$

where C_k is the amount of decrease in the tour cost by deleting node k from S (while keeping the rest of the tour sequence as before). Since the deletion of some nodes from the tour S may result in a tour S' that is no longer feasible, LS1 attempts to make the solution feasible by inserting new nodes into S'. We refer to this as the Feasibility Procedure. Suppose that P is the set of nodes that can be added to the current tour. For the binary GCSP P, consists of the nodes not in the tour S', while in the integer GCSP P, consists of all nodes that do not appear more than L times in S'. We select the node $k \in P$ for which

$$I_k/N_k^2 = min_{j \in P}\left(I_j/N_j^2\right) \tag{4.19}$$

Here I_k is the amount of increase in the tour cost by insering node k into its best position in the tour, while N_k is the number of uncovered nodes (or uncovered demand) which can be covered by node k. We update the calculation of N_k for all nodes in P and repeat the selection and insertion of nodes procedure until we obtain a feasible solution. After this step, LS1 checks for the possible removal of "redundant" nodes from the current tour in the *Delete_Redundant_Nodes Procedure*. A node is redundant if, by removing it, the solution remains feasible.

Next, in the case LS1 finds an improvement, i.e., the cost of S' is less than the cost of S, it tries to improve the tour length (and thus the overall cost) by applying the *Lin-Kernighan Procedure* [47] to the solution S'. We apply the Lin-Kernighan code LKH version 1.3 of Helsgaun [36] that is available for download on the web. Since the procedure is computationally expensive, we only apply it after *max_k* (a parameter) improvements over the solution S.

In order to get out locally optimum solutions, and to search through a larger set in the feasible solution space, we apply a *Mutation Procedure* whenever the algorithm is not able to increase the quality of the solution for a given number of consecutive iterations. In the mutation procedure, a node is selected randomly and if the node does not belong to the solution it is added to the solution in its best place (i.e. the place which causes the minimum increase in the tour length); otherwise it is removed from the solution. In the latter case, the algorithm calls the feasibility procedure to ensure the solution is feasible, and updates the best solution if necessary.

To add diversity to the search procedure, we allow downhill moves with respect to the best solution that LS1 has found. In other words, if the cost of the solution S' that LS1 obtains is better than $(1 + \alpha)$ times the best solution found we keep it as the current solution (over which we try and find an improvement), otherwise we use the best solution obtained so far as the current solution. The stopping criterion for LS1 is a given number of iterations that we denote by max_iter. The pseudo-code of LS1 is given in Algorithm 4. The parameters to be tuned for LS1 and their best values obtained in our computational testing are described in Table 4.1 (see Section 4.4).

4.3.2 LS2

This local search procedure tries to improve the cost of a solution by either deleting a node on the tour if the resulting solution is feasible; or by extracting a node and substituting it with a promising sequence of nodes. In contrast to LS1, this local search algorithm maintains feasibility (i.e., it only considers feasible neighbors in the local search neighborhood). LS2 mainly consists of two iterative procedures: the *Improvement Procedure* and the *Perturbation Procedure*. In the *Improvement Procedure* the algorithm considers extraction of nodes from the current tour in a round robin fashion. (In other words, given some ordering of nodes on the tour, it first tries to delete the first node on the tour, and then it tries to delete the second node on the tour, and so on, until it tries to delete the last node on the tour.) If by removing a node on the tour the solution remains feasible, the tour cost has improved and the node is deleted from the tour. On the other hand, extracting a node from the tour may cause some other nodes to lose their covering demands (meaning that their demand is no longer fully covered and the solution becomes infeasible). Consequently, in such cases we try to obtain a feasible solution by substituting the deleted node with a new subsequence of nodes. To this aim, the algorithm considers the T nodes nearest to the extracted node and generates all the promising subsequences with cardinality one or two. Then it selects the subsequence s that has the minimum insertion cost (i.e., the cost of the tour generated by substituting the deleted node by subsequence s minus the cost of tour with the deleted node). In the case of improvement in the tour cost (i.e., when the minimum insertion cost is negative) we make this substitution; otherwise, we disregard it (i.e. reinsert the deleted node back into its initial position) and continue. The improvement procedure is repeated until it cannot find any improvements (i.e., no change is found while extracting nodes from the current tour in a round robin fashion).

In the *Perturbation Phase*, LS2 tries to escape from a locally optimum solution by perturbing the solution. In the perturbation procedure we iteratively add up to K nodes to the tour. It randomly selects one node from among the nodes eligible for addition to the tour (in the binary GCSP the nodes must be selected from those out of the current tour, while for the two other GCSP variants the nodes can be selected as well from those visited in the current tour) and inserts it in the tour in its best possible position. Since the tour is feasible prior to the addition of these nodes, the tour remains feasible upon addition of these K nodes. In one iteration of the procedure the improvement phase and perturbation phase are iteratively applied J times. After one iteration, when the best solution has improved (i.e., an iteration found a solution with lower cost) we use the *Lin-Kernighan Procedure* [47], to improve the current tour length (and thus the cost of the solution). The stopping criterion for LS2 is a given number of iterations that we denote by max_iter. The pseudo-code for LS2 is given in Algorithm 5, and the parameters to be tuned for LS2 and their best values obtained in our computational testing are described in Table 4.2 (see Section 4.4).

4.4 Computational Experiments

In this section we report on our computational experience with the two local search heuristics LS1 and LS2 on the different GCSP variants. We first consider the CSP, and compare the performance of the two proposed heuristics LS1 and LS2, with that of the method proposed by [17] for the CSP. Next we compare LS1 and LS2 on a large number of GCSP instances for the three variants. We also consider a Steiner version of the GCSP, and report our experience with the two local search heuristics. Finally, in order to compare the quality of the solutions found by the two heuristics, we compare them with existing heuristics for the GTSP where there exist well studied instances in the literature. All of the experiments suggest that the heuristics are of a high quality and run very rapidly.

4.4.1 Test Problems

Since there are no test problems in the literature for the CSP (as well as the variants of the GCSP we introduce), we created datasets based on the TSP library instances (Reinelt, 1991). In particular we constructed our datasets based on 16 Euclidean TSPLIB instances

Begin
S := An initial random tour of n nodes, S^* := S and $BestCost$:= Cost(S^*);
C_S= Decrease in the tour cost by short cutting node s;
I_S= Increase in the tour cost by adding node s to its best position in the tour;
N_S= max{1, Number of uncovered nodes covered by node s};
No_Null_Iter= Number of iterations without improvement;
Set k := 0; No_Null_Iter := 0;
for i=1,...,*max_iter* **do**
    for j=1,...,*Search-magnitude*×$|S|$ **do**
        Delete node k from S according to the probability $C_k / \sum_{s \in S} C_s$;
    end
    S':= Restricted solution obtained by shortcutting the nodes deleted in the previous step;
    Apply *Feasibility Procedure*(S');
    Apply *Delete_Redundant_Nodes Procedure*(S');
    if *Cost*(S') < *Cost*(S) **then**
        If (k=*max_k*) Obtain *TSP_tour*(S') by calling *Lin-Kernighan Procedure* and set k := 0;
        Else k := k+1;
    end
    if *Cost*(S') > *BestCost* ×(1+α) **then**
        S:=S^*;
        No_Null_Iter := No_Null_Iter + 1;
    end
    else
        S := S';
        if *Cost*(S) < *BestCost* **then**
            Update S^* := S, *BestCost* := *Cost*(S) and No_Null_Iter := 0;
        end
    end
    If No_Null_Iter > $Mutation\_Parameter$ **then** apply *Mutation Procedure*(S);
end
Obtain TSP_tour(S*) by calling *Lin-Kernighan Procedure*. Output the solution S*;
End.

Feasibility Procedure(S'):
P = The set of nodes that can be entered into the solution;
while *there exist uncovered nodes* **do**
    Select node $k \in P$ such that I_k / N_k^2 = $min_{j \in P}$ (I_j / N_j^2);
    Insert node k in its best position in S';
    For each node j update the remaining coverage demand, I_j and N_j;
end

Delete_Redundant_Nodes Procedure(S'):
for $i \in S'$ **do**
    If by removing node i from S' the solution remains feasible, **then** remove node i;
end

Mutation Procedure(S);
Select a random node k from the set of nodes P;
If node $k \notin S$ **then** add node k to S in its best position;
Else remove node k from S and call *Feasibility Procedure*(S);
If Cost(S) < *BestCost* **then** update S^*:= S, *BestCost*:= *Cost*(S).

Algorithm 4: Local Search Algorithm 1 (LS1) for the GCSP whose size ranged from 51 to 200 nodes. In the datasets created, each node can cover its 7, 9 or 11 nearest nodes (resulting in 3 instances for each TSPLIB instance), and each node i must be covered k_i times, where k_i is a randomly chosen integer number between 1 and 3. We generated the datasets to ensure that a tour over all of the nodes covers the demand (i.e., we ensured that the binary GCSP instances were feasible). Although the cost for visiting a node can be different from node to node, for simplicity we consider the node visiting costs to be the same for all nodes in an instance. In fact, if we assign a high node visiting cost, the problem becomes a Set Covering Problem (as the node visiting costs dominate the routing cost) under the assumption that a tour over all the nodes covers the demand. On the other hand, if the node visiting cost is insignificant (i.e., the routing costs dominate), there is no difference between the integer GCSP with overnight and the CSP. This is because if there is no node visiting cost, a salesman will stay overnight at a node (at no additional cost) until he/she covers all the demand that can be covered from that node. After testing different values for the node visiting cost, to ensure that its effect was not to either extreme (Set Covering Problem or CSP), we fixed the node visiting cost value to 50 for all the instances (which turned out to be an appropriate amount for the different kinds of instances studied in this paper). In this fashion we constructed 48 datasets for our computational work.

After considerable experimentation on a set of small test instances, we determined the best values of the parameters to be used in both LS1 and LS2. Tables 4.1 and 4.2 show the different values that were tested for various parameters and the best value obtained for the parameters in LS1 and LS2. Both LS1 and LS2 were implemented in C and tested on a Windows Vista PC with an Intel Core Duo processor running at 1.66 GHz with 1 GB RAM. As is customary in testing the performance of randomized heuristic algorithms, we performed several independent executions of the algorithms. In particular, for each benchmark instance, 5 independent runs of the algorithms LS1 and LS2 were performed, with 5 different seeds for initializing the random number generator and the best and the average performances of the two heuristics are provided.

In all tables reporting the computational performance of the heuristics, the first column is related to the instance name which includes the number of nodes. The second column (NC) gives the number of nearest nodes that can be covered by each node. Moreover, for each method the best and the average cost, the number of nodes in the best solution (NB), the average time to best solution (Avg.TB), i.e. the average time until the best solution is found (note the local search heuristic typically continues after this point until it reaches its termination criterion), and the average time (Avg.TT) are reported (TT is the total time for one run of the local search heuristic). In all tables, in each row the best solution is written in bold and the last two rows give the average of each column (Avg)

Begin
S:= An initial random tour of n nodes, $S^* := S$ and *BestCost* := Cost(S^*);
$N(S)$ = Number of nodes in S;
for i=1,...,*max_iter* **do**
  bestimprove := false;
  for j=1,...,J **do**
    improve := *true*;
    Repeat
    Improvement Procedure(S,*improve*)
    Until (*improve* = *false*)
    if *Cost(S)* < *BestCost* **then**
      $S^* := S$;
      BestCost := Cost(S);
      bestimprove := true;
    end
    Else $S := S^*$;
    Perturbation Procedure(S);
  end
  if *bestimprove=true* **then**
    Obtain TSP_tour(S^*) by calling *Lin-Kernighan Procedure.* Output the solution S^*;
    S := S^* and *BestCost* := Cost(S^*);
  end
end
End.

Improvement Procedure(S, improve):
Begin
improve :=*false*;
r:=1;
while $r \leq N(S)$ **do**
  Extract the r^{th} node of the tour from the current solution S;
  if *the new solution (obtained from extracting the* r^{th} *node of S) is feasible* **then**
    S := new solution;
    improve := *true*;
  end
  else
    Generate all subsequences with cardinality 1 or 2, by considering the T closest nodes to the extracted node;
    Extra_Cost := Extra cost for the subsequence with the minimum insertion cost;
    if *Extra_Cost* < *0* **then**
      Update S by substituting the new subsequence for the extracted node;
      improve := *true*;
    end
  end
  $r := r+1$;
end

Perturbation Procedure(S):
Begin
for *i=1*,...,K **do**
  Randomly select an eligible node;
  Insert the node in its best feasible position in the tour;
end
End.

Algorithm 5: Local Search Algorithm 2 (LS2) for the GCSP

Table 4.1: Parameters for LS1.

Parameters	Different values tested	Best value
Search-magnitude	{0.1, 0.2, 0.3, 0.4, 0.5, 0.6}	0.2
Mutation_parameter	{5, 10, 15, 20}	10
max_k	{5, 10, 15, 20}	10
α	{0, 0.1, 0.01, 0.001}	0.001
max_iter	{1500, 3500, 5500, 7500, 8500}	3500 (CSP and Binary GCSP) 7500 (Integer GCSP)

Table 4.2: Parameters for LS2.

Parameters	Different values tested	Best value
J	{50, 100, 150, 200, 250, 300}	200
K	{5, 10, 15, 20}	10
T	{5, 10, 15}	10
max_iter	{15, 20, 25, 30, 35, 40, 45, 50, 55, 60}	25 (CSP and Binary GCSP) 50 (Integer GCSP)

and the number of best solutions found by each method (No.Best), respectively. All the computing times are expressed in seconds.

4.4.2 Comparison of LS1, LS2 and Current and Schilling's Heuristic for the CSP

Since Current and Schilling [17] introduced the CSP and proposed a heuristic for it, we compare the performance of LS1 and LS2 against their heuristic. Recall, their algorithm was described in Section 4.1. Since there are no test instances or computational experiments reported in Current and Schilling's paper, we coded their algorithm to compare the performance of the heuristics. For Current and Schilling's method, we used CPLEX 11 [39] to generate all optimal solutions of the SCP, and since solving the TSP to optimality is computationally quite expensive on these instances we use the *Lin-Kernighan Procedure* [47] to find a TSP tour for each solution. Sometimes finding all the optimal solutions of an SCP instance is quite time consuming, so we only consider those optimal solutions for the SCP that can be found in less than 10 minutes of running time.

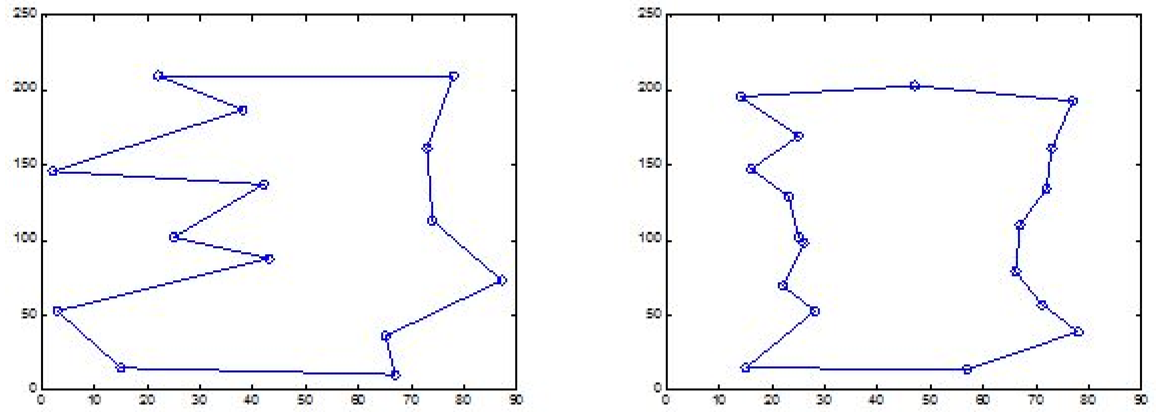
Table 4.3 reports the results obtained by LS1, LS2 and our implementation of Current and Schilling's method. In this table, the number of optimal solutions (NO) of the set covering problem is given. In Table 4.3 instances for which all the optimal solutions to the set covering problem cannot be obtained within the given time threshold are shown with an asterisk. As can be seen in Table 4.3 for the CSP, both LS1 and LS2 can obtain, in a few seconds, better solutions than Current and Schilling's method. The results of both the heuristics in all except one case (where they are tied with Current and Schilling's method) are better than Current and Schilling's method, while they are several orders

Table 4.3: Comparison of Current and Schilling's method with LS1 and LS2 for CSP.

Instance	NC	Current and Schilling					LS1					LS2				
		NO	Cost	NB	TB	TT	BestCost	Avg.Cost	NB	Avg.TB	Avg.TT	BestCost	Avg.Cost	NB	Avg.TB	Avg.TT
Eil 51	7	13	194	7	0.07	0.21	**164**	164.0	10	0,20	1,48	**164**	164.0	10	0.04	0.77
	9	309	169	6	1.92	1.97	**159**	159.0	8	0.10	1.34	**159**	159.0	9	0.03	0.61
	11	282	167	5	0.59	1.70	**147**	147.0	7	0.04	1.22	**147**	147.0	8	0.03	0.55
Berlin 52	7	2769	4019	8	19.39	21.04	**3887**	3966.2	11	0.08	1.68	**3887**	3887.0	11	0.26	0.67
	9	11478	**3430**	7	26.08	94.14	**3430**	3435.8	7	0.10	1.41	**3430**	3430.0	7	0.04	0.62
	11	11	3742	5	0.22	0.26	**3262**	3262.0	6	0.02	1.60	**3262**	3262.0	6	0.02	0.34
St 70	7	32832	297	10	232.24	454.07	**288**	288.0	11	0.11	1.86	**288**	288.0	12	0.05	1.03
	9	18587	271	9	173.87	176.00	**259**	259.0	10	0.05	1.79	**259**	259.0	10	0.05	1.22
	11	1736	269	7	13.21	13.74	**247**	247.0	10	0.16	1.98	**247**	247.0	10	0.04	0.88
Eil 76	7	241	241	11	1.15	2.46	**207**	210.6	15	0.53	2.09	**207**	207.0	15	0.17	1.11
	9	1439	193	9	7.43	13.95	186	186.8	11	0.26	1.98	**185**	185.0	11	0.05	1.13
	11	7050	180	8	30.48	78.88	**170**	176.4	11	0.05	2.14	**170**	170.0	11	0.05	1.07
Pr 76	7	26710	53255	11	54.20	170.41	**50275**	51085.0	14	0.55	1.86	**50275**	50275.0	14	0.78	1.27
	9	326703*	45792	10	6743.66	9837.36	**45348**	45348.0	12	0.27	2.01	**45348**	45348.0	12	0.26	1.12
	11	20	45955	7	0.11	0.20	**43028**	43418.4	10	0.48	1.95	**43028**	43028.0	10	0.07	1.03
Rat 99	7	3968	572	14	22.74	32.75	**486**	486.4	18	0.08	2.20	**486**	486.0	18	0.16	1.77
	9	170366	462	12	1749.66	2729.67	**455**	455.6	15	0.67	2.38	**455**	455.0	15	0.11	1.92
	11	16301	456	10	88.87	140.18	**444**	444.8	12	0.43	2.25	**444**	444.0	12	0.09	1.75
KroA 100	7	208101*	10306	15	6303.03	6475.95	**9674**	9674.0	19	0.38	2.06	**9674**	9674.0	19	0.31	2.04
	9	95770	9573	12	524.49	1365.42	**9159**	9159.0	15	0.13	2.28	**9159**	9159.0	15	0.14	1.85
	11	33444	9460	10	409.47	433.97	**8901**	8912.2	13	0.19	2.56	**8901**	8901.0	13	0.13	1.62
KroB 100	7	4068	11123	14	45.62	48.35	**9537**	9537.0	20	0.39	1.99	**9537**	9537.0	20	0.33	1.93
	9	133396	9505	12	2112.57	2623.76	**9240**	9262.2	15	0.54	2.13	**9240**	9240.0	15	0.16	1.99
	11	90000*	9049	10	1056.27	2895.35	**8842**	8842.6	13	1.34	2.62	**8842**	8842.0	13	0.09	1.83
KroC 100	7	129545*	10367	15	3391.82	4212.98	9728	9728.6	18	0.72	2.46	**9723**	9723.0	17	0.17	1.97
	9	5028	9952	12	35.91	52.25	**9171**	9184.4	13	0.12	2.45	**9171**	9171.0	13	0.19	1.91
	11	75987*	9150	10	1389.84	2482.00	**8632**	8632.0	13	0.14	2.74	**8632**	8632.0	13	0.09	1.85
KroD 100	7	1392	11085	14	10.29	15.58	**9626**	9626.0	20	1.35	2.39	**9626**	9626.0	20	0.21	1.83
	9	700	10564	11	6.18	7.74	**8885**	8903.8	13	0.75	2.38	**8885**	8885.0	13	0.12	2.04
	11	85147*	9175	10	968.39	2761.51	**8725**	8730.4	13	0.48	2.83	**8725**	8725.0	13	0.13	1.89
KroE 100	7	92414*	11323	15	1971.32	3075.58	**10150**	10154.8	19	0.14	2.48	**10150**	10150.0	19	1.06	1.84
	9	85305*	9095	12	1918.72	2764.70	8992	8992.0	13	0.33	2.69	**8991**	8991.0	14	0.16	1.90
	11	70807*	8936	10	609.81	2335.43	**8450**	8450.0	13	0.36	2.89	**8450**	8450.0	13	0.08	1.97
Rd 100	7	2520	4105	14	24.43	4196.23	**3461**	3478.2	18	0.31	2.53	**3461**	3485.6	18	0.24	1.83
	9	95242*	3414	12	1798.14	3118.93	**3194**	3211.4	16	0.91	2.65	**3194**	3194.0	16	0.25	1.76
	11	1291	3453	10	8.60	22.11	2944	2944.0	12	0.44	3.1	**2922**	2922.0	13	0.14	1.54
KroA150	7	97785*	12367	22	2252.50	3499.43	11480	11548.8	27	0.89	2.68	**11423**	11481.0	28	1.97	2.91
	9	69377*	11955	17	2454.99	2477.69	10072	10072.0	23	0.71	2.78	**10056**	10056.0	26	1.91	2.75
	11	169846*	10564	15	5483.07	5518.26	**9439**	9439.0	21	1.06	2.82	**9439**	9439.0	21	0.39	2.68
KroB 150	7	14400	12876	21	196.85	270.94	11490	11517.0	30	1.39	2.58	**11457**	11463.6	30	1.66	3.08
	9	137763*	11774	18	2760.03	4572.81	**10121**	10173.4	24	1.19	2.77	**10121**	10121.0	24	0.64	2.78
	11	1431	10968	14	26.64	46.96	**9611**	9639.8	21	0.61	2.88	**9611**	9611.0	21	0.28	2.88
KroA 200	7	53686*	14667	28	537.60	1170.37	13293	13345.2	34	1.05	3.25	**13285**	13313.8	34	3.99	4.28
	9	64763*	12683	23	1504.07	1628.36	11710	11753.6	29	1.23	2.80	**11708**	11725.0	28	3.25	3.88
	11	29668*	12736	19	398.25	671.55	**10748**	10813.4	29	1.04	3.16	**10748**	10814.8	29	3.10	3.65
KroB 200	7	107208*	14952	29	365.08	3351.89	13280	13297.6	36	0.46	2.83	**13051**	13147.4	35	2.24	4.38
	9	38218*	13679	23	637.66	805.04	**11864**	11898.6	29	1.02	2.98	**11864**	11937.8	29	2.30	4.02
	11	67896*	12265	20	493.64	1410.60	**10644**	10714.0	29	0.98	2.81	**10644**	10650.4	29	3.10	3.72
Avg		55896	9808.02	13	1017.94	1626.68	9031.4	9070.3	17	0.52	2.35	9023.6	9031.5	17	0.65	1.95
No.Best			1				38					**48**				

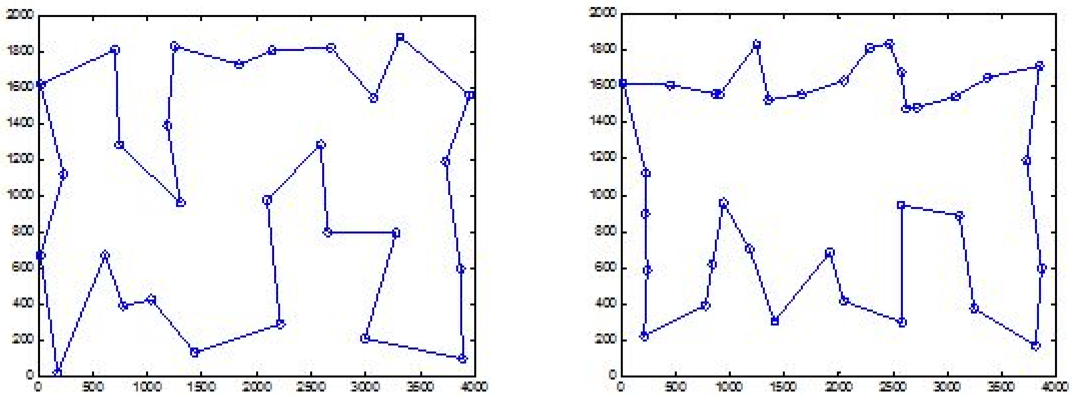
of magnitude faster than Current and Schilling's method. Between LS1 and LS2, LS2 outperforms LS1 as it obtains the best solution in all 48 instances, while LS1 only obtains the best solution in 38 out of the 48 instances.

From Table 4.3 we can make the following counter-intuitive observation. Sometimes by selecting a set of nodes with a larger cardinality, we are able to find a shorter tour length, so the optimal solution of the Set Covering Problem is not necessary a good solution for the Covering Salesman Problem. Figures 4.1 and 4.2 illustrate two examples of CSP (Rat99 and KroA200) in which, by increasing the number of nodes in the tour, the tour length is decreased.



a) Number of nodes in the tour: 14, Tour length: 572   b) Number of nodes in the tour: 18, Tour length: 486

Figure 4.1: An example of decreasing the tour length by increasing the number of nodes in Rat99 (NC=7).



a) Number of nodes in the tour: 28, Tour length: 14667   b) Number of nodes in the tour: 34, Tour length: 13285

Figure 4.2: An example of decreasing the tour length by increasing the number of nodes in KroA200 (NC=7).

4.4.3 Comparison of LS1 and LS2 on GCSP Variants

In Table 4.4 the results of the two local search heuristics on the binary GCSP are given. As can be seen in this table, for the binary GCSP the two local search heuristics are very competitive with each other. Although on average LS2 is a bit faster than LS1, in terms of the average cost, average time to best solution, and the number of best solutions found LS1 is better than LS2. Over the 48 instances, the two heuristics were tied in 22 instances. While, in 14 instances LS1 is strictly better than LS2, and in 12 instances LS2 is strictly better than LS1. Table 4.5 provides a comparison of LS1 and LS2 on the integer GCSP without overnights. Here, the table contains one additional column reporting the number of times a solution revisits cities (NR). Here, over 48 test instances, LS1 is strictly better than LS2 in 12 instances, LS2 is strictly better than LS1 in 11 instances, while they are tied in 26 instances. Again the running time of both LS1 and LS2 is extremely small, taking no more than 20 seconds even for the largest instances. Table 4.6 compares LS1 and LS2 on integer GCSP with overnights. Here, the table contains one additional column reporting the number of times a solution stays overnight at a node (ON). Here, over 48 test instances, LS1 is strictly better than LS2 in 8 instances, LS2 is strictly better than LS1 in 30 instances, and they are tied in 10 instances. However, the running time of LS1 increases significantly compared to LS2. This increase in running time appears to be due to a significant increase in the number of times LS1 calls the *Lin-Kernighan Procedure*. Overall, LS2 appears to be a better choice than LS1 for the integer GCSP with overnights. Notice that a solution to the binary GCSP is a feasible solution to the integer GCSP without overnights, and a feasible solution to the integer GCSP without overnights is a feasible solution for the integer GCSP with overnights. Hence, we should expect that the average cost of the solutions found should go down as we move from Tables 4.4-4.6. This is confirmed in our experiments.

4.4.4 GCSP with Steiner Nodes

In our earlier test instances every node had a demand. We now construct some Steiner instances, i.e., ones where some nodes have k_i set to zero (the rest of the demands remain unchanged). In these cases, a tour could contain some "Steiner nodes" (i.e., nodes without any demand) that can help satisfy the coverage demand of the surrounding (or nearby) nodes. On the other hand, if fewer nodes have demands then it is likely that fewer nodes need to be visited (in particular the earlier solutions obtained are feasible for the Steiner versions), and thus we would expect the cost of the solutions to the GCSP with Steiner nodes to decrease compared to the instances of the GCSP without Steiner nodes. Table 4.7 confirms this observation. Here we compare LS1 and LS2 on the CSP with Steiner

Table 4.4: Comparison of LS1 and LS2 on Binary GCSP.

Instance	NC	LS1					LS2				
		BestCost	Avg.Cost	NB	Avg.TB	Avg.TT	BestCost	Avg.Cost	NB	Avg.TB	Avg.TT
Eil 51	7	1224	1224	20	0.10	1.65	**1190**	1191.6	19	0.24	1.08
	9	**991**	996.2	15	0.48	1.63	**991**	993.4	15	0.21	1.10
	11	**844**	869.4	13	0.27	1.49	**844**	849.4	13	0.19	1.14
Berlin 52	7	**5429**	5429.0	17	0.06	1.65	**5429**	5514.6	17	0.11	1.11
	9	**4807**	4818.8	14	0.09	1.49	**4807**	4834.0	14	0.08	1.08
	11	**4590**	4655.0	13	0.27	1.58	**4590**	4639.6	13	0.30	0.85
St 70	7	1836	1841.6	29	0.56	2.58	**1834**	1836.4	29	0.45	1.33
	9	1461	1468.8	22	0.35	1.76	**1460**	1460.0	22	0.42	1.38
	11	**1268**	1270.2	19	0.90	1.73	**1268**	1270.2	19	0.45	1.37
Eil 76	7	**1610**	1630.8	26	0.52	2.51	**1610**	1623.0	26	0.25	1.63
	9	**1270**	1319.8	20	0.37	1.83	1296	1301.2	21	0.83	1.64
	11	**1117**	1130.8	18	0.31	1.87	**1117**	1122.2	18	1.04	1.61
Pr 76	7	66789	66850.8	28	0.66	1.46	**66455**	66887.8	29	0.83	1.71
	9	**62907**	62916.0	23	0.18	1.71	63114	63203.6	25	0.83	1.68
	11	**52175**	52527.0	19	0.25	1.58	**52175**	52175.0	19	0.32	1.47
Rat 99	7	2341	2346.0	34	0.64	3.07	**2325**	2340.2	33	0.98	2.17
	9	**1936**	1940.4	27	0.24	1.97	**1936**	1941.2	27	1.01	2.43
	11	**1686**	1714.2	23	0.31	1.80	**1686**	1691.2	23	1.21	2.39
KroA 100	7	**14660**	14660	41	0.53	2.18	**14660**	14726.6	41	1.26	2.25
	9	**12974**	12974	33	0.13	1.65	**12974**	12987.2	33	0.47	2.38
	11	11970	11977.2	28	0.42	1.57	**11942**	11942.0	29	0.41	2.38
KroB 100	7	**14415**	14451.8	44	0.87	1.91	14459	14577.6	42	0.43	2.23
	9	12222	12296.4	34	0.86	2.17	**12194**	12247.0	33	2.18	2.27
	11	**11276**	11277.2	28	1.20	2.55	**11276**	11315.2	28	0.83	2.43
KroC 100	7	**13830**	13888.8	41	0.13	2.88	**13830**	13850.2	41	2.08	2.24
	9	**12149**	12190.2	33	0.64	2.12	**12149**	12189.6	33	1.45	2.21
	11	**11032**	11032	26	0.11	2.00	**11032**	11032.0	26	1.74	2.22
KroD 100	7	**13567**	13666.4	38	0.06	2.53	13704	13857.2	38	0.31	2.42
	9	**12409**	12448.6	32	1.03	1.92	12419	12479.8	31	2.08	2.48
	11	11486	11520.8	28	0.43	1.76	**11443**	11515.6	29	1.34	2.11
KroE 100	7	**15321**	15485.0	41	0.37	2.62	15471	15700.6	41	0.30	1.99
	9	**12482**	12482	32	0.19	1.64	**12482**	12482.0	32	0.40	2.33
	11	**11425**	11452.4	30	0.66	1.48	11456	11490.6	28	2.24	2.26
Rd 100	7	6209	6210.8	37	0.30	2.20	**6170**	6251.4	37	0.70	2.33
	9	**5469**	5595.0	29	0.23	2.10	**5469**	5477.2	29	1.06	2.44
	11	**4910**	4985.6	28	0.52	1.63	**4910**	4965.2	28	1.17	2.22
KroA150	7	**17258**	17274.6	55	0.96	4.52	17270	17425.8	54	2.11	3.63
	9	**15007**	15042.6	46	1.07	3.68	**15007**	15145.4	46	2.60	4.20
	11	**13666**	13755.6	40	1.20	2.93	13762	14010.8	41	2.50	3.75
KroB 150	7	**17639**	17745.8	60	2.94	4.16	**17639**	18141.4	60	2.18	3.56
	9	**15505**	15688.0	50	0.90	3.61	15506	15854.8	50	3.71	3.76
	11	13740	13899.0	42	1.82	2.97	**13719**	13836.4	40	2.82	3.69
KroA 200	7	21388	21553.8	74	3.23	8.11	**21346**	21543.8	76	3.71	4.73
	9	**17843**	17999.4	59	2.33	5.70	17893	18103.8	60	3.09	4.91
	11	16591	16702.4	54	1.84	4.94	**16380**	16580.4	55	3.80	4.85
KroB 200	7	**20736**	20960.0	79	2.30	8.71	20882	21117.6	79	3.88	4.52
	9	**18266**	18377.4	66	1.94	6.28	18269	18500.6	67	3.05	4.73
	11	**15961**	16428.8	55	2.37	4.71	16173	16372.0	55	3.57	4.65
Avg		13035.2	13103.6	35	0.79	2.72	13041.9	13137.4	35	1.40	2.49
No.Best		**36**					34				

Table 4.5: Comparison of LS1 and LS2 on Integer GCSP without overnight.

Instance	NC	LS1						LS2					
		BestCost	Avg.Cost	NB	NR	Avg.TB	Avg.TT	BestCost	Avg.Cost	NB	NR	Avg.TB	Avg.TT
Eil 51	7	**1185**	1199.8	19	1	0.28	4.19	**1185**	1187.0	19	1	0.96	3.83
	9	**991**	992.8	15	0	1.59	4.09	**991**	996.2	15	0	0.85	3.30
	11	**843**	845.0	13	1	1.27	4.77	**843**	843.0	13	1	1.12	3.22
Berlin 52	7	**5429**	5429.0	17	0	0.14	3.99	**5429**	5429.0	17	0	0.18	3.37
	9	**4785**	4796.8	15	1	0.02	4.17	**4785**	4807.6	15	1	0.12	2.69
	11	**4590**	4651.4	13	0	0.12	3.27	**4590**	4620.0	13	0	0.66	2.51
St 70	7	**1778**	1783.4	28	3	1.21	6.56	1782	1786.0	28	1	0.48	5.10
	9	1461	1497.8	22	0	0.23	5.27	**1460**	1461.2	22	0	1.46	4.56
	11	1268	1268.0	19	0	1.33	3.31	**1241**	1264.2	18	1	1.70	4.03
Eil 76	7	**1600**	1626.6	26	1	1.69	6.67	**1600**	1619.4	26	2	2.27	5.11
	9	**1270**	1291.6	20	0	1.26	5.10	**1270**	1294.0	20	0	1.93	4.79
	11	**1117**	1121.2	18	0	0.89	4.13	**1117**	1117.0	18	0	0.48	4.63
Pr 76	7	65990	66615.8	28	1	1.47	4.73	**64111**	65560.8	29	4	1.70	5.44
	9	57147	57945.2	29	1	1.14	5.17	**54907**	55862.4	29	6	1.37	4.94
	11	51587	51650.0	20	2	1.52	4.42	**49445**	49445.0	21	3	1.02	4.22
Rat 99	7	**2311**	2315.0	33	1	0.65	7.39	**2311**	2341.2	33	1	2.70	7.61
	9	**1936**	1937.8	27	0	1.53	5.85	**1936**	1949.4	28	0	1.99	7.19
	11	**1683**	1704.4	23	0	2.07	4.55	**1683**	1701.0	23	0	1.87	6.77
KroA 100	7	**14660**	14678.8	41	0	1.52	6.35	**14660**	14784.4	41	0	2.40	8.18
	9	**12974**	12974.0	33	0	0.53	4.98	**12974**	13090.0	33	0	2.27	7.49
	11	**11737**	11737.0	29	1	0.47	4.59	**11737**	11737.0	29	1	2.03	0.30
KroB 100	7	**14246**	14394.2	45	6	3.00	6.16	14297	14316.8	43	3	3.54	8.83
	9	12200	12348.6	34	3	2.39	4.91	**12189**	12197.8	33	2	1.56	7.35
	11	**11268**	11394.2	27	2	0.30	6.38	**11268**	11378.0	27	2	0.74	6.95
KroC 100	7	**13520**	13644.0	42	5	3.42	7.49	13792	13999.8	41	1	2.55	8.42
	9	**12119**	12209.0	33	1	1.22	6.76	**12119**	12119.0	33	1	1.18	7.22
	11	**11032**	11032.0	26	0	0.57	9.82	**11032**	11074.4	26	0	0.85	6.19
KroD 100	7	**13501**	13517.6	39	2	4.97	7.61	**13501**	13635.0	39	2	1.97	8.84
	9	12261	12303.2	31	1	0.62	6.20	**12257**	12279.6	31	1	1.98	7.71
	11	11452	11534.0	29	1	2.79	9.43	**11409**	11450.2	30	1	1.65	7.30
KroE 100	7	**15308**	15386.8	42	1	2.71	7.05	15471	15767.2	41	0	4.37	8.02
	9	**12482**	12541.8	32	0	0.42	5.72	**12482**	12485.0	32	0	1.29	7.18
	11	**11344**	11417.8	30	1	2.89	7.40	**11344**	11373.6	30	1	2.92	6.68
Rd 100	7	**6078**	6182.8	37	2	0.86	6.78	**6078**	6199.6	37	2	2.05	7.45
	9	**5384**	5501.0	30	2	3.54	6.39	**5384**	5418.8	30	2	2.60	7.15
	11	**4853**	4916.6	29	1	1.73	4.14	**4853**	4867.2	29	1	1.58	6.29
KroA150	7	**16947**	16974.0	57	4	3.29	13.36	16976	17143.0	56	3	7.22	12.50
	9	15007	15158.6	46	0	2.55	9.47	**15000**	15136.8	49	3	2.40	11.80
	11	**13580**	13709.4	40	2	1.92	7.47	13683	13791.8	41	2	4.76	10.60
KroB 150	7	**17621**	17776.8	59	1	5.55	11.46	17639	18136.2	60	0	3.15	12.15
	9	**15332**	15609.8	48	3	3.90	10.68	15383	15556.0	48	3	4.83	11.78
	11	**13554**	13582.0	41	2	2.76	7.66	**13554**	13670.8	41	2	4.96	10.82
KroA 200	7	21337	21415.2	79	3	10.35	20.03	**21120**	21294.8	78	6	10.09	16.92
	9	**17812**	17927.0	62	2	6.77	14.37	17832	18186.8	64	5	10.91	14.97
	11	**16290**	16517.8	54	4	9.14	12.56	16370	16485.0	53	4	10.41	14.48
KroB 200	7	**20628**	20808.2	78	2	5.91	20.45	20862	21027.4	77	2	6.62	16.01
	9	**18247**	18387.6	67	3	4.75	14.88	18260	18448.6	68	3	8.84	14.79
	11	15888	16150.0	56	3	4.03	11.73	**15688**	15968.8	56	4	8.01	13.80
Avg		12825.7	12925.0	35	1	2.36	7.50	12706.3	12839.7	35	2	2.97	7.74
No.Best		**37**						36					

Table 4.6: Comparison of LS1 and LS2 on Integer GCSP with overnight.

Instance	NC	LS1						LS2					
		BestCost	Avg.Cost	NB	ON	Avg.TB	Avg.TT	BestCost	Avg.Cost	NB	ON	Avg.TB	Avg.TT
Eil 51	7	**1146**	1146.0	19	10	0.60	10.21	**1146**	1146.0	19	10	0.09	2.81
	9	**958**	980.8	15	5	0.77	8.26	**958**	968.4	15	5	0.70	3.15
	11	842	866.8	13	5	1.66	7.85	**827**	829.4	13	4	0.33	2.90
Berlin 52	7	4969	4981.6	19	11	5.65	9.22	**4966**	4976.0	18	9	0.66	2.87
	9	**4272**	4301.2	16	8	3.00	6.74	**4272**	4324.0	16	8	0.58	2.35
	11	**3962**	4149.2	14	8	0.10	8.09	**3962**	3962.0	14	8	0.12	2.17
St 70	7	**1654**	1656.2	27	15	2.95	17.46	1655	1655.0	27	14	0.35	4.26
	9	1442	1453.0	23	12	1.07	11.21	**1416**	1438.6	22	9	0.40	3.79
	11	**1196**	1226.8	18	6	1.71	8.30	**1196**	1213.0	18	6	1.77	3.75
Eil 76	7	**1554**	1587.0	26	10	6.00	12.37	1562	1578.4	26	10	1.06	4.63
	9	**1268**	1307.2	21	7	4.23	9.54	**1268**	1298.2	21	7	1.00	4.22
	11	**1107**	1125.6	18	3	4.99	8.01	**1107**	1110.2	18	4	0.68	4.08
Pr 76	7	53270	56065.2	29	15	1.41	16.58	**53266**	54142.0	30	16	1.97	4.61
	9	47226	49028.4	26	15	7.40	12.84	**46912**	47245.8	27	17	0.72	3.99
	11	44036	46104.0	19	8	0.73	11.47	**44028**	44029.6	20	10	0.99	3.62
Rat 99	7	**2229**	2241.8	33	10	3.43	10.24	**2229**	2259.4	33	10	3.09	6.34
	9	**1908**	1940.6	27	5	5.58	10.05	1922	1947.0	28	8	3.17	6.19
	11	1673	1697.2	24	9	3.89	10.84	**1650**	1686.6	23	6	0.73	5.71
KroA 100	7	12474	12762.4	42	24	10.20	28.73	**12006**	12322.6	43	26	1.34	6.69
	9	11671	11733.4	34	21	16.59	22.31	**11218**	11245.2	35	21	1.00	6.28
	11	10886	10931.8	29	17	6.79	19.91	**10665**	10700.8	31	17	2.20	5.64
KroB 100	7	12728	12920.6	39	18	7.31	21.13	**12273**	12530.0	43	25	3.31	7.21
	9	11176	11232.0	34	19	3.11	17.40	**11128**	11133.2	35	21	3.65	6.59
	11	**10302**	10534.6	28	15	5.28	16.54	**10302**	10409.8	28	15	1.69	6.04
KroC 100	7	12202	12401.6	41	22	11.17	32.46	**12043**	12269.2	45	27	2.52	6.63
	9	11196	11374.8	33	16	3.63	17.75	**11031**	11141.0	35	20	1.95	5.71
	11	10445	10629.2	27	15	2.00	18.27	**10299**	10406.2	28	15	1.26	5.42
KroD 100	7	11868	12115.4	38	20	5.13	22.28	**11725**	11827.8	39	20	1.70	6.56
	9	11062	11287.0	31	16	6.55	15.75	**10742**	10869.6	35	20	2.56	6.11
	11	10523	10714.0	27	13	4.35	15.30	**10404**	10469.4	29	16	1.10	5.83
KroE 100	7	13101	13332.4	42	25	6.35	24.37	**12689**	12859.6	45	28	1.83	6.01
	9	**10821**	11193.2	34	20	6.03	24.89	**10821**	10905.0	34	20	1.52	5.61
	11	**10007**	10190.6	29	17	4.59	16.30	**10007**	10136.4	29	17	0.59	5.34
Rd 100	7	5626	5834.2	37	17	4.83	20.75	**5570**	5645.6	39	20	2.72	6.55
	9	**4950**	5129.4	32	18	4.24	13.02	5037	5093.2	30	13	0.81	5.89
	11	4541	4705.8	27	13	4.72	15.30	**4514**	4581.8	27	12	2.01	5.23
KroA150	7	**15341**	15483.2	59	32	19.02	41.95	15385	15644.6	60	37	3.34	9.56
	9	13475	13714.2	49	28	6.60	30.64	**12944**	13288.6	51	28	6.32	10.07
	11	**12151**	12399.4	43	25	4.31	21.24	12215	12407.4	44	25	3.31	8.92
KroB 150	7	15825	15964.0	58	31	8.91	32.67	**15252**	15774.2	61	32	4.47	10.49
	9	13198	13415.8	52	30	21.04	36.04	**13139**	13372.0	52	31	8.28	9.78
	11	12418	12933.0	40	24	6.91	28.11	**12174**	12561.6	44	26	3.46	9.06
KroA 200	7	18093	18186.4	76	39	27.64	47.57	**17873**	18431.8	82	49	6.32	13.74
	9	**15562**	15979.4	63	37	10.07	37.55	15782	16141.6	64	35	3.48	12.36
	11	14873	14933.0	55	30	20.95	30.29	**14629**	14835.0	56	32	4.41	11.39
KroB 200	7	18119	18436.6	82	46	36.42	56.66	**17701**	18108.4	85	51	8.09	13.98
	9	16289	16395.4	68	36	28.23	40.13	**15766**	16264.4	66	37	8.69	12.27
	11	**14217**	14705.8	54	26	13.26	29.64	14360	14490.8	56	27	7.99	11.37
Avg		11246.9	11529.7	35	18	7.74	20.50	11125.8	11284.9	36	19	2.51	6.54
No.Best		18						**40**					

nodes. For each CSP instance (in Table 4.3) we select 10 percent of the nodes randomly and set their corresponding demands to zero. The behavior of LS1 and LS2 is similar to that of the earlier CSP instances. Specifically, over the 48 test instances LS1 was strictly better once, LS2 was strictly better 6 times, and the two methods were tied 41 times. Overall LS2 runs slightly faster than LS1. For brevity, we have limited the comparison to the CSP with Steiner nodes.

4.4.5 Analyzing the Quality of LS2 on the Generalized TSP

Overall, LS2 seems to be a better choice than LS1, in that it is more robust than LS1. It outperforms LS1 on the CSP and the integer GCSP with overnights, while it is tied with LS1 for the binary GCSP and integer GCSP without overnights. Further, the run time of LS2 remains fairly stable. However, since we do not have lower bounds or optimal solutions for the CSP and GCSP instances, it is hard to assess the quality of the solutions. Noting that the generalized TSP (GTSP) is a special case of the CSP (we explain how momentarily), we use some well studied GTSP instances in the literature (see Fischetti et al. [27]) and compare LS2 with eight different heuristics designed specifically for the GTSP; as well as to the optimal solutions on these instances obtained by Fischetti et al. [27] using a branch-and-cut method. In the GTSP, the set of nodes in the graph are clustered into disjoint sets and the goal is to find the minimum length tour over a subset of nodes so that at least one node from each cluster is visited by the tour. This can be formulated as a CSP, where each node has unit demand (i.e., k_i=1 for each node i) and each node in a cluster covers every other node in a cluster (and no other nodes). We executed LS2 on the benchmark GTSP dataset (see Fischetti et al. [27]) by first tuning its parameters. The tuned parameters of LS2 are configured as follows: $J = 300$, $K = 10$, $T = 10$, $max\_iter = 50$ and 10 independent runs of LS2 were performed. We compared LS2 to eight other heuristics in the literature that are described below.

- MSA: A Multi-Start Heuristic by Cacchiani et al. [9]
- mrOX: a Genetic Algorithm by Silberholz and Golden [71],
- RACS: a Reinforcing Ant Colony System by Pintea et al. [57],
- GA: a Genetic Algorithm by Snyder and Daskin [72],
- GI^3: a composite algorithm by Renaud and Boctor [61],
- NN: a Nearest Neighbor approach by Noon [54],
- *FST-Lagr* and *FST-root*: Two heuristics by Fischetti et al. [27].

Table 4.7: Comparison of LS1 and LS2 on Steiner CSP.

Instance	LS1					LS2				
	Best Cost	Avg. Cost	NB	Avg. TB	Avg. TT	Best Cost	Avg. Cost	NB	Avg. TB	Avg. TT
Eil 51	**163**	163.0	8	0.02	1.29	**163**	163.0	9	0.03	0.69
	159	159.40	8	0.38	4.90	**159**	159.0	9	0.03	0.52
	147	147.0	7	0.10	2.08	**147**	147.0	8	0.03	0.57
Berlin 52	**3470**	3483.60	10	0.09	1.45	**3470**	3470.0	10	0.33	0.68
	3097	3097.0	7	0.98	4.45	**3097**	3097.0	7	0.03	0.51
	2956	2959.60	6	0.05	2.46	**2956**	2956.0	6	0.02	0.54
St 70	288	288.0	11	0.11	1.64	**287**	287.0	12	0.08	0.89
	259	259.0	9	0.04	4.53	**259**	259.0	10	0.06	1.07
	245	245.80	10	0.46	2.71	**245**	245.0	10	0.04	0.82
Eil 76	**207**	211.40	15	0.11	1.64	**207**	210.0	15	0.25	0.91
	186	186.40	11	0.76	4.39	**185**	185.0	10	0.04	0.99
	169	170.80	10	0.19	2.63	**169**	169.0	11	0.04	0.92
Pr 76	**49773**	50566.80	13	0.18	1.61	**49773**	49773.0	13	0.18	1.15
	44889	44889.0	12	0.47	4.65	**44889**	44889.0	12	0.12	1.08
	42950	43399.20	9	0.34	2.84	**42950**	42950.0	9	0.05	0.87
Rat 99	483	483.0	17	0.19	1.91	**482**	482.0	18	0.29	1.54
	454	454.0	14	0.56	4.38	**454**	454.0	14	0.09	1.67
	444	444.40	12	0.76	3.09	**444**	444.0	12	0.08	1.47
KroA 100	**9545**	9545.0	18	0.31	2.04	**9545**	9545.0	18	0.39	1.75
	9112	9112.0	15	0.09	1.72	**9112**	9112.0	15	0.35	1.55
	8833	8841.40	13	0.23	3.42	**8833**	8833.0	13	0.08	1.32
KroB 100	**9536**	9536.0	19	0.33	1.93	**9536**	9536.0	19	0.37	1.62
	9199	9205.80	15	0.66	1.53	**9199**	9199.0	15	0.09	1.64
	8763	8763.0	11	0.50	3.55	**8763**	8763.0	11	0.12	1.61
KroC 100	9591	9591.0	15	0.17	1.97	**9590**	9590.0	16	0.14	1.72
	9171	9171.0	13	0.70	1.79	**9171**	9171.0	13	0.27	1.52
	8632	8632.0	13	0.39	3.55	**8632**	8632.0	13	0.09	1.63
KroD 100	**9526**	9526.0	19	0.21	1.83	**9526**	9526.0	19	0.15	1.51
	8885	8885.40	13	0.78	1.87	**8885**	8885.0	13	0.15	1.83
	8725	8731.40	13	0.77	3.69	**8725**	8725.0	13	0.10	1.65
KroE 100	**9800**	9800.0	16	1.06	1.84	**9800**	9800.0	16	0.16	1.54
	8987	8987.0	13	0.29	2.02	**8986**	8986.0	14	0.11	1.56
	8450	8450.0	13	0.40	3.85	**8450**	8450.0	13	0.11	1.70
Rd 100	**3412**	3412.0	18	0.24	1.83	**3412**	3434.4	18	0.25	1.60
	3194	3206.80	16	0.43	1.87	**3194**	3194.0	16	0.31	1.52
	2761	2761.0	12	0.49	3.66	**2761**	2761.0	12	0.09	1.33
KroA150	**10939**	10939.0	27	1.97	2.91	**10939**	11099.6	27	0.85	2.45
	9808	9823.20	23	0.25	2.25	**9808**	9808.0	23	0.20	2.26
	9360	9382.60	20	1.13	3.46	**9360**	9360.0	20	0.29	2.30
KroB 150	**11225**	11288.6	30	1.66	3.08	**11225**	11240.4	30	1.08	2.48
	10121	10211.40	24	1.10	2.35	**10121**	10121.0	24	0.64	2.31
	9542	9556.60	20	0.86	3.54	**9542**	9542.0	20	0.19	2.55
KroA 200	**13042**	13042.0	32	3.99	4.28	13227	13268.0	35	1.12	3.62
	11392	11429.20	27	0.22	2.42	**11392**	11424.0	27	0.83	3.18
	10527	10615.80	24	0.51	4.02	**10525**	10673.8	26	0.79	2.95
KroB 200	**13020**	13160.20	34	2.24	4.38	**13020**	13092.0	34	1.77	3.41
	11712	11788.60	28	0.79	2.63	**11712**	11837.2	28	1.89	3.39
	10614	10769.40	28	1.62	3.60	**10614**	10734.8	28	0.89	2.99
Avg	8911.73	8953.56	16	0.63	2.82	8915.4	8930.9	16	0.33	1.65
No.Best	42					**47**				

Table 4.8: Comparison of computing times of GTSP methods.

Computer	*Mflops*	r	Method
Gateway Profile 4MX	230	1.568	*GA*
Sun Sparc Station LX	4.6	0.032	GI^3, NN
HP 9000/720	2.3	0.016	FST-Lagr, FST-Root, B&C
Unknown	-	1	RACS
Dell Dimension 8400	290	2	mrOX
Pentium(R) IV, 3.4 Ghz	295	2.03	MSA
Our	145	1	LS2

In order to perform a fair comparison on the running times of the different heuristics, we scaled the running times for the different computers as indicated in [22]. The computer factors are shown in Table 4.8. The columns indicate the computer used, solution method used, *Mflops* of the computer, and r the scaling factor. Thus the reported running times in the different papers are appropriately multiplied by the scaling factor r. We note that an identical approach was taken in [9] to compare across these heuristics for the GTSP. Since no computer information is available for the RACS heuristic, we use a scaling factor of 1.

Table 4.9 reports on the comparison. For each instance we report the percentage gap with respect to the optimal solution value and the computing time (expressed in seconds and scaled according to the computer factors given in Table 4.8) for all the methods but for *B&C* (for which we report only the computing time). Some of the methods (RACS, GI^3, and NN) only reported solutions for 36 of the 41 instances. Consequently, in the last four rows of Table 4.9 we report for each algorithm, the average percentage gap and the average running time on the 36 instances tested by all the methods, as well as over all 41 instances (for all methods except RACS, GI^3, and NN). We also summarize the number of times the optimum solution was found by a method. As Table 4.9 indicates, although LS2 was not explicitly developed for the GTSP (but rather for a generalization of it), it performs quite creditably. On average it takes 2.2 seconds, finds solutions that are on average 0.08% from optimality, and found optimal solutions in 30 out of 41 benchmark GTSP instances.

4.5 Summary and Conclusions

In this chapter we considered the CSP, and introduced a generalization quite different from earlier generalizations of the CSP in the literature. Specifically, in our generalization nodes must be covered multiple times (i.e., we introduce a notion of coverage demand of a node). This may require a tour to visit a node multiple times (which is not the case in earlier generalizations), and there are also node visiting costs. We discussed three

Table 4.9: Comparison of LS2 against 8 other heuristics on benchmark GTSP instances in the literature.

Instances	LS2		MSA		mrOX		RACS	GA		GI^3		NN		FST-lagr		FST-Root		B&C
	gap	time	gap	time	gap	time	gap	gap	time	gap	time	gap	time	gap	time	gap	time	time
Att48	0	0.4	0	0	0	0.8	-	0	0	-	-	-	-	0	0	0	0	0.0
Gr48	0	0.4	0	0	0	0.6	-	0	0.8	-	-	-	-	0	0	0	0	0.0
Hk48	0	0.5	0	0	0	0.6	-	0	0.4	-	-	-	-	0	0	0	0	0.0
Eil51	0	0.5	0	0	0	0.6	0	0	0.2	0	0	0	0	0	0	0	0	0.0
Brazil58	0	0.6	0	0	0	1.6	-	0	0.4	-	-	-	-	0	0	0	0	0.0
St70	0	0.7	0	0	0	0.8	0	0	0.4	0	0	0	0	0	0	0	0.2	0.2
Eil76	0	0.8	0	0	0	0.8	0	0	0.4	0	0	0	0	0	0	0	0.2	0.2
Pr76	0	0.9	0	0	0	1.0	0	0	0.4	0	0	0	0	0	0	0	0.2	0.2
Rat99	0	1.2	0	0	0	1.0	0	0	1.0	0	0.2	0	0.2	0	0	0	0.8	0.8
KroA100	0	1.2	0	0	0	1.2	0	0	0.6	0	0.2	0	0.2	0	0	0	0.2	0.2
KroB100	0	1.3	0	0	0	1.2	0	0	0.6	0	0.2	0	0	0	0	0	0.4	0.4
KroC100	0	1.2	0	0	0	1.2	0	0	0.4	0	0.2	0	0.2	0	0	0	0.2	0.2
KroD100	0	1.2	0	0	0	1.4	0	0	0.6	0	0.2	0	0.	0	0	0	0.2	0.2
KroE100	0	1.3	0	0	0	1.2	0	0	1.2	0	0.2	0	0	0	0	0	0.2	0.2
Rd100	0	1.2	0	0	0	1.0	0	0	0.4	0.08	0.2	0.08	0.2	0.08	0	0	0.2	0.2
Eil101	0	1.1	0	0	0	1.0	0	0	0.4	0.4	0.2	0.4	0	0	0	0	0.4	0.4
Lin105	0	1.3	0	0	0	1.2	0	0	0.4	0	0.4	0	0.2	0	0	0	0.2	0.2
Pr107	0	1.2	0	0	0	1.0	0	0	0.6	0	0.2	0	0.2	0	0	0	0.2	0.2
Gr120	0	1.1	0	0	0	1.4	-	0	0.8	-	-	-	-	1.99	0	0	0.6	0.6
Pr124	0	1.5	0	0	0	1.4	0	0	1.0	0.43	0.4	0	0.4	0	0	0	0.4	0.4
Bier127	0.04	1.6	0	0	0	1.6	0	0	0.8	5.55	1.0	9.68	0.2	0	0.2	0	0.4	0.4
Pr136	0	1.8	0	0	0	1.6	0	0	0.8	1.28	0.4	5.54	0.2	0.82	0.2	0	0.6	0.6
Pr144	0	1.6	0	0	0	2.0	0	0	0.4	0	0.4	0	0.4	0	0	0	0.2	0.2
KroA150	0	2.1	0	0	0	2.0	0	0	2.0	0	0.6	0	0.6	0	0.2	0	1.4	1.5
KroB150	0	1.9	0	0	0	2.0	0	0	1.6	0	0.4	0	0.6	0	0.2	0	0.8	0.8
Pr152	0	2.0	0	0	0	2.0	0	0	2.4	0.47	0.6	1.8	0.4	0	0.2	0	0.8	1.5
U159	0	2.2	0	0	0	2.0	0.01	0	1.0	2.6	0.6	2.79	0.8	0	0.2	0	2.0	2.0
Rat195	0	2.5	0	0.2	0	2.8	0	0	1.0	0	1.2	1.29	2.6	1.87	0.2	0	3.5	3.5
D198	0.32	3.4	0	0	0	3.2	0.01	0	1.8	0.6	1.8	0.6	3.6	0.48	0.2	0	10.8	10.8
KroA200	0	2.8	0	0	0	3.4	0.01	0	4.2	0	0.8	5.25	1.6	0	0.2	0	2.6	2.6
KroB200	0	2.7	0	0	0.05	3.2	0	0	2.2	0	1.0	0	4.0	0.05	0.2	0	3.9	3.9
Ts225	0	2.9	0	4.3	0.14	3.4	0.02	0	3.9	0.61	2.6	0	3.6	0.09	0.2	0.09	18.5	538.2
Pr226	0.09	2.5	0	0	0	3.0	0.03	0	1.6	0	0.8	2.17	2.0	0	0.2	0	1.4	1.4
Gil262	0.79	3.6	0	7.1	0.45	7.2	0.22	0.79	3.0	5.03	3.5	1.88	3.6	3.75	0.2	0.89	20.5	94.2
Pr264	0.59	3.6	0	0	0	4.8	0	0	2.0	0.36	2.0	5.73	4.5	0.33	0.4	0	4.9	4.9
Pr299	0.04	4.5	0	1.4	0.05	9.2	0.24	0.02	9.7	2.23	0.2	2.01	8.5	0	0.4	0	11.6	11.6
Lin318	0.01	4.4	0	0	0	16.2	0.12	0	5.5	4.59	6.2	4.92	9.7	0.36	0.8	0.36	12.0	23.8
Rd400	0.98	6.2	0	0.6	0.58	29.2	0.87	1.37	5.5	1.23	12.2	3.98	34.7	3.16	0.8	2.97	71.5	99.9
Fl417	0.01	5.8	0	0	0.04	16.4	0.57	0.07	3.8	0.48	12.8	1.07	40.8	0.13	1.0	0	237.5	237.5
Pr439	0.09	7.1	0	3.9	0	38.2	0.79	0.23	14.4	3.52	18.4	4.02	37.8	1.42	2.0	0	76.9	77.1
Pcb442	0.16	6.9	0	1.6	0.01	46.8	0.69	1.31	16.0	5.91	17.0	0.22	25.6	4.22	1.2	0.29	76.1	835.1
Average 36	0.09	2.5	0	0.53	0.04	6.12	0.10	0.10	2.58	0.98	2.42	1.48	5.21	0.46	0.26	0.13	15.61	54.32
# Opt 36	25		36		29		24	30		19		18		23		31		36
Average 41	0.08	2.2	0	0.47	0.03	5.38		0.09	2.31					0.46	0.22	0.11	13.72	47.71
# Opt 41	30		41		34			35						27		36		41

variants of the GCSP. The binary GCSP where revisiting a node is not permitted, the integer GCSP without overnights where revisiting a node is permitted only after another node is visited, and the integer GCSP with overnights where revisiting a node is permitted without any restrictions. We designed two local search heuristics, LS1 and LS2, for these variants. Overall LS2 appears to be more robust in terms of its running time as well as its performance in terms of the number of times it found the best solutions in the different variants. When LS2 is compared to 8 benchmark heuristics for the GTSP (that were specifically designed for the GTSP), LS2 performs quite well, finding high-quality solutions rapidly. We introduced two integer programming models for the binary and integer GCSP respectively. However, both these models require considerable strengthening and embedding in a branch-and-cut framework in order to obtain exact solutions to the GCSP. This is a natural direction for research on the GCSP (as it will provide an even better assessment of the quality of heuristics for the GCSP), and we hope researchers will take up this challenge. Some natural generalizations of the GCSP (along the lines of the earlier generalizations of the CSP) may be considered in future research. The earlier generalizations of the CSP (see Vogt et al. [79]) included requirements in terms of (i) requiring some nodes to be on the tour, (ii) requiring some nodes not to be on the tour, (iii) allowing a node not to be covered at a cost (for our GCSP that would mean the covering demand of a node could be partially covered at a cost), and (iv) including a cost for allocating nodes not on the tour to the tour. These would be natural generalizations of this multi-unit coverage demand variant of the CSP that we have introduced.

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