Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
DOTTORATO DI RICERCA IN MATEMATICA, XXI CICLO

MAT 07: Fisica Matematica

An equilibrium approach to modelling social interaction

doctoral thesis

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Esame Finale anno 2009

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Abstract

The aim of this work is to put forward a statistical mechanics theory of social interaction, generalizing econometric discrete choice models. After showing the formal equivalence linking econometric multinomial logit models to equilibrium statical mechanics, a multi-population generalization of the Curie-Weiss model for ferromagnets is considered as a starting point in developing a model capable of describing sudden shifts in aggregate human behaviour.

Existence of the thermodynamic limit for the model is shown by an asymptotic sub-additivity method and factorization of correlation functions is proved almost everywhere. The exact solution of the model is provided in the thermodynamical limit by finding converging upper and lower bounds for the system's pressure, and the solution is used to prove an analytic result regarding the number of possible equilibrium states of a two-population system.

The work stresses the importance of linking regimes predicted by the model to real phenomena, and to this end it proposes two possible procedures to estimate the model's parameters starting from micro-level data. These are applied to three case studies based on census type data: though these studies are found to be ultimately inconclusive on an empirical level, considerations are drawn that encourage further refinements of the chosen modelling approach.

Contents

Chapter 1

Introduction

In recent years there has been an increasing awareness towards the problem of finding a quantitative way to study the role played by human interactions in shaping the kind of aggregate behaviour observed at a population level: reference [3] provides a comprehensive account of how ramified this field of study already is. There the author reviews efforts made by researchers from areas as diverse as psychology, economics and physics, to cite a few, in the pursuit of regularities that may characterize different kinds of aggregate human behaviour such as urban traffic, market behaviour and the internet.

The idea of characterizing society as a unitary entity, characterized by global features not dissimilar from those exhibited by physical or living systems has accompanied the development of philosophical thought since its very beginning, and one must look no further than Plato's *Republic* to find an early example of such a view. The proposal that mathematics might play a crucial role in pursuing such an idea, on the other hand, dates back at least to Thomas Hobbes's *Leviathan*, where an attempt is made to draw analogies between the laws describing mechanics, and features of society as a whole. Hobbes's work gives an inspiring outlook on the ways in which modern science might contribute to practical human affairs from an organizational point of view, as well as technological.

In later centuries, nevertheless, quantitative science has grown aware of the fact that, though a holistic view such as Hobbes's plays an important motivational role in the development of new scientific enterprises, it is only by reducing a problem to its simplest components that success is attained by empirical studies. One of the interesting sub-problems singled out by the modern approach is that of characterizing the behaviour of a large groups of people, when each individual is faced with a choice among a finite set of alternatives, and a set of motives driving the choice can be identified. Such motives might be given by the person's personal preferences, as well as by the way he interacts with other people. My thesis aims to contribute to the research effort which is currently analysing the role played by social interaction in human decision making process just described.

As early as in the nineteen-seventies the dramatic consequences of including interaction between peers into a mathematical model of choice comprising large groups of people have been recognized independently by the physical [23], economical [62] and social science [34] communities. The conclusion reached by all these studies is that mathematical models have the potential to describe several features of social behaviour, among which the sudden and dramatic shifts often observed in society trends [47], and that these are unavoidably linked to the way individual people influence each other when deciding how to behave.

The possibility of using such models as a tool of empirical investigation, however, is not found in the scientific literature until the beginning of the present decade [21]: the reason is to be found in the intrinsic difficulty of establishing a methodology of systematic measurement for social features. Confidence that such an aim might be an achievable one has been boosted by the wide consensus gained by econometrics following the Nobel prize awarded in 2000 to economist Daniel Mcfadden for his work on probabilistic models of discrete choice, and by the increasing interest of policy makers for tools enabling them to cope with the global dimension of today's society [39, 27].

This has led very recently to a number of studies confronting directly the challenge of quantitatively measuring social interaction for *bottom-up* models, that is, models deriving macroscopic phenomena from assumptions about human behaviour at an individual level [11, 61, 51, 65].

These works show an interesting interplay of methods coming from econometrics [25], statistical physics [26] and game theory [43], which reveals a substantial overlap in the basic assumptions driving these three disciplines. It must also be noted that all of these studies rely on a simplifying assumption which considers interaction working on a global uniform scale, that is on a *mean field* approach. This is due to the inability, stated in [69], of existing methods to measure social network topological structure in any detail. It is expected that it is only matter of time before technology allows to overcome this difficulty: in the meanwhile, one of the roles of today's empirical studies is to assess how much information can be derived from the existing kind of data such as that coming from surveys, polls and censuses.

This thesis considers a mean field model that highlights the possibility of using the methods of discrete choice econometrics to apply a statistical mechanical generalization of the model introduced in [21]. The approach is mainly that of mathematical-physics: this means that the main aim shall be to establish the mathematical properties of the proposed model, such as the existence of the thermodynamical limit, its factorization properties, and its solution, in a rigorous way: it is hoped that this might be used as a good building block for later more refined theories. Furthermore, since maybe the most problematic point of a mathematical study of society lies in the feasibility of measuring the relevant quantities starting from real data, two estimation procedures are put forward: one tries to mimic the econometrics approach, while the other stems directly from equilibrium statistical mechanics, by stressing the role played by fluctuations of main observable quantities. These procedures are applied to some simple case studies.

The thesis is therefore organised as follows: the first chapter reviews the theory of Multinomial Logit discrete choice models. These models are based on a probabilistic approach to the psychology of choice [48], which is chosen here as the modeling approach to human decision making. In this chapter we focus on the mathematical form of Multinomial Logit, and in particular on its equivalence to the statistical mechanics of non-interacting particles. In the second chapter we consider the Curie-Weiss model, of which we provide a treatment recently developed in the wider study of mean field spin glasses [37], which allows to give elegant rigorous proofs of the model's properties. In chapter three we generalise results from chapter two for a system partitioned into an arbitrary number of components. Since such a model corresponds to the generalization of discrete choice first considered in [21], which includes the effect of peer pressure into the process decision making, it provides a potential tool for the study of social interaction: chapter four shows an application of this to three simple case studies.

Chapter 2

Discrete choice models

In this chapter we describe the general theory of discrete choice models. These are econometric models that were first applied to the study of demand in transportation systems in the nineteen-seventies [6]. When people travel they can choose the mode of transportation between a set of distinct alternatives, such as train or automobile, and the basic tenet of these models is that such a *discrete choice* can be described by a probability distribution, and that proposals for the form of such distribution can be derived from principles established at the level of individuals. As we shall see this modus operandi is one familiar to statistical mechanics, and corresponds to what is commonly known as a *bottom-up* strategy in finance.

After describing the general scope of discrete choice analysis, in section 2.3 we describe precisely the mathematical structure of one of the most widely used discrete choice models, the Multinomial Logit model. Here we shall see how the probability distribution describing people's choices arises from the assumption that individual act trying to maximize the benefit coming from that choice, which is the common setting of neoclassical economics. Discrete choice models, in general, ignore the effect of social interaction, but we shall see in subsection 2.3.3 that the Multinomial Logit can be rephrased precisely as a statistical mechanical model, which gives an ideal starting point for extending such a model of behaviour to a context including interaction, to be considered in later chapters.

Due to his development of the theory of the Multinomial Logit model economist Daniel McFadden was awarded the Nobel Prize in Economics in 2000 [50], for bringing economics closer to quantitative scientific measurement. The purpose of discrete choice theory is to describe people's behaviour: it is an econometric technique to infer people's preferences from empirical data. In discrete choice theory the decision-maker is assumed to make choices that maximise his/her own benefit. Their 'benefit' is described by a mathematical formula,

Table 1. Prediction Success Table, Journey-to-Work

(Pre-BART Model and Post-BART Choices)

Cell Counts	Predicted Choices				
Actual Choices	**Auto Alone**	**Carpool**	**Bus**	**BART**	**Total**
Auto Alone	255.1	79.1	28.5	15.2	378
Carpool	74.7	37.7	15.7	8.9	137
Bus	12.8	16.5	42.9	4.7	77
BART	9.8	11.1	6.9	11.2	39
Total	352.4	144.5	94.0	40.0	631
Predicted Share	55.8%	22.9%	14.9%	6.3%	
(Std. Error)	*(11.4%)*	*(10.7%)*	*(3.7%)*	*(2.5%)*	
Actual Share	59.9%	21.7%	12.2%	6.2%	

98% agreement

(Source: McFadden 2001)

Figure 2.1: Discrete choice predictions against actual use of travel modes in San Francisco, 1975 (source: McFadden 2001)

a *utility function*, which is derived from data collected in surveys. This utility function includes rational preferences, but also accounts for elements that deviate from rational behaviour.

Though discrete choice models do not account for 'peer pressure'or 'herding effects', it is nonetheless a fact that the standard performance of discrete choice models is close to optimal for the analysis of many phenomena where peer influence is perhaps not a major factor in an individual's decision: Figure 2.1 shows an example of this. The table (taken from [50]) compares predictions and actual data concerning use of travel modes, before and after the introduction of new rail transport system called BART in San Francisco, 1975. We see a remarkable agreement between the predicted share of people using BART (6.3%), and the actual measured figure after the introduction of the service (6.2%).

2.1 General theory

In discrete choice each decision process is described mathematically by a *utility function*, which each individual seeks to maximize. The principle of utility maximization is one which lies at the heart of neoclassical economics: this has often been critised as too simplistic an assumption for complex human behaviour, and this criticism has been supported by the poor performance of quantitative models arising from such an assumption. It must be noted however, that if we wish to attain a quantitative description of human behaviour at all, we must do so by considering a description which is analytically treatable. There exist of course alternatives approaches (e.g. agent-based modeling), but since this field of research is still in its youth, it pays to consider possible improvements of utility maximisation before abandoning it altogether. This is indeed the view taken by discrete choice, which sees people as rational utility maximizers, but also takes into account a certain degree of irrationality, which is modeled through a random contribution to the utility function.

As an example, a binary choice could be to either cycle to work or to catch a bus. The utility function for choosing the bus may be written as:

$$U = V + \varepsilon \tag{2.1}$$

where V, the deterministic part of the utility, could be symbolically parametrised as follows

$$V = \sum_a \lambda_a x_a + \sum_a \alpha_a y_a \tag{2.2}$$

The variables x_a are assumed to be attributes regarding the choice alternatives themselves. For example, the bus fare or the journey time. On the other hand, the y_a may socio-economic variables that define the decision-maker, for example their age, gender or income. It is this latter set of parameters that allows us to zoom in on specific geographical areas or socio-economic groups. The λ_a and α_a are parameters that need to be estimated empirically, through survey data, for instance. The key property of these parameters is that they quantify the relative importance of any given attribute in a person's decision: the larger its value, the more this will affect a person's choice. For example, we may find that certain people are more affected by the journey time than the bus fare; therefore changing the fare may not influence their behaviour significantly. The next section will explain how the value of these parameters is estimated from empirical data. It is an observed fact [49, 2] that choices are not always perfectly rational. For example, someone who usually goes to work by bus may one day decide to cycle instead. This may be because it was a nice sunny day, or for no evident reason. This unpredictable component of people's choices is accounted for by the random term ε. The distribution of ε may be assumed to be of different forms, giving rise to different possible models: if, for instance, ε is assumed to be normal, the resulting model is called a *probit* model, and it doesn't admit a closed form solution. Discrete choice analysis assumes ε to be extreme-value distributed, and the resulting model is called a logit model [6]. In practice this is very convenient as it does not impose any significant restrictions on the model but simplifies it considerably from a practical point of view. In particular, it allows us to obtain a closed form solution for the probability of choosing a particular alternative, say catching a bus rather than cycling to work :

$$P = \frac{e^V}{1 + e^V}, \tag{2.3}$$

(see section 2.3 for the derivation).

In words, this describes the rational preferences of the decision maker. As will be explained later on, (2.3) is analogous to the equation describing the equilibrium state of a perfect gas of heterogeneous magnetic particles (a *Langevin paramagnet*): just like gas particles react to external forces differently depending, for instance, on their mass and charge, discrete choice describes individuals as experiencing heterogeneous influences in their decision-making, according to their own socio-economic attributes, such as gender and wealth. A question arises spontaneously: do people and gases behave in the same way? The answer to such a controversial question is that in some circumstances they might. Models are idealisations of reality, and equation (2.3) is telling us that the same equation may describe idealised aspects of both human and gas behaviour; in particular, how individual behaviour relates to macroscopic or societal variables. These issues go beyond the scope of this thesis, but it is important to note that (2.3) offers a mathematical and intuitive link between econometrics and statistical mechanics. The importance of this 'lucky coincidence' cannot be overstated, and some of the implications will be discussed later on in more detail.

2.2 Empirical estimation

Discrete choice may be seen as a purely empirical model. In order to specify the actual functional form associated with a specific group of people facing a specific choice, empirical data is needed. The actual utility function is then specified by estimating the numerical values of the parameters λ_a and α_a which appear in our definition of V given by (2.2), thus establish the choice probabilities (2.3). As mentioned earlier, these parameters quantify the relative importance of the attribute variables x_a and y_a. For example, costs are always associated with negative parameters: this means that the higher the price of an alternative, the less likely people will be to choose it. This makes intuitive sense: what discrete choice offers is a quantification of this effect. Once the data has been collected, the model parameters may be estimated by standard statistical techniques: in practice, Maximum Likelihood estimation methods are used most often (see, e.g., [6] chapter 4). We shall see in further chapters how, though optimal for standard discrete choice models, Maximum likelihood estimation seems to be unsuitable for phenomena involving interaction due to discontinuities in the probability structure. As we shall see, a valuable alternative is given by a method put forward by Joseph Berkson [7].

Discrete choice has been used to study people's preferences since the seventies [50]. Initial applications focused on transport [68, 53]. These models have been used to develop national and regional transport models around the world, including the UK, the Netherlands [24], as well as Copenhagen [54]. Since then discrete choice has also been applied to a range of social problems, for example healthcare [30, 59], telecommunications [42] and social care [60]

2.3 The Multinomial logit model

The binomial logit model which gives the probabilities (2.3) can be seen as a special case of the Multinomial Logit model introduced by R. Duncan Luce in 1959 [48] when developing a mathematical theory of choice in psychology, and was later given the utility maximization form which we describe here by Daniel Mcfadden [50].

In the following three subsections we shall describe the mathematical structure of a Multinomial Logit model. In the first subsection we shall first give information about the Gumbel extreme-distribution, which is the distribution by which the model describes the random contribution ε to a person's utility, and is chosen essentially for reasons of analytical convenience. The second subsection uses the properties of Gumbel distribution in order to derive the probability structure of the model. These two sections are an 'executive summary' of all the main things, and they can be found on any standard book on econometrics [6, 25].

The third subsection gives the statistical mechanical reformulation of the Multinomial Logit model, by showing that the same probability structure arises when we compute the pressure of a suitably chosen Hamiltonian: this leads the way for the extensions of the model that shall be considered in later chapters.

2.3.1 Properties of the Gumbel distribution

In order to implement the modelling assumption of utility maximization in a quantitative way, we need a suitable probability distribution for the random term ε.

The Multinomial Logit Model models randomness in choice by a Gumbel distribution, which has a cumulative distribution function

$$F(x) = \exp\{-e^{-\mu(x-\eta)}\}, \quad \mu > 0,$$

and probability density function

$$f(x) = \mu e^{-\mu(x-\eta)} \exp\{-\mu(x-\eta)\}.$$

We have that if $\varepsilon \stackrel{d}{=} \text{Gumbel}(\eta, \mu)$ then

$$\mathbb{E}(\varepsilon) = \eta + \frac{\gamma}{\eta}, \quad \text{Var}(\varepsilon) = \frac{\pi^2}{6\mu^2},$$

where γ is the Euler-Mascheroni constant ($\cong 0.577$).

The Gumbel distribution is a type of extreme-value distribution, which means that under suitable conditions it gives the limit distribution for the value of the extremum of a sequence of i.i.d random variables, just like the Gaussian distribution does for their average under the central limit theorem. In econometric the Gumbel distribution for mainly analytical reasons, since it has a number of interesting properties, which make it suitable as a modeling tool. As we shall see in subsection 2.3.3 the model that one obtains can be readily mapped into a statistical mechanical model, thus establishing an interesting link between economics and physics.

The following two properties regard Gumbel variables with equal variance, and hence equal μ (see [6], pag. 104).

I. If $\varepsilon' \stackrel{d}{=} \text{Gumbel}(\eta_1, \mu)$ and $\varepsilon'' \stackrel{d}{=} \text{Gumbel}(\eta_2, \mu)$ are independent random variables, then $\varepsilon = \varepsilon' - \varepsilon''$ is logistically distribute with cumulative distribution

$$F_\varepsilon(x) = \frac{1}{1 + e^{-\mu(\eta_2 - \eta_1 - x)}},$$

and probability density

$$f_\varepsilon(x) = \frac{\mu e^{-\mu(\eta_2 - \eta_1 - x)}}{(1 + e^{-\mu(\eta_2 - \eta_1 - x)})^2}.$$

II. If $\varepsilon_i \stackrel{d}{=} \text{Gumbel}(\eta_i, \mu)$ for $1 \leqslant i \leqslant k$ are independent then

$$\max_{i=1..k} \varepsilon_i \stackrel{d}{=} \text{Gumbel}\Big(\frac{1}{\mu} \ln \sum_{i=1}^{k} e^{\mu \eta_i}, \mu\Big)$$

As we said, the logit is a model which is founded on the assumption that individuals choose their behaviour trying to maximize a utility, or a "benefit" function. In the next section we shall use Property **II** to handle the probabilistic maximum of the utilities coming from many different choices, whereas Property **I** shall be used to compare probabilistically the benefits of two different choices.

2.3.2 Econometrics

We shall now derive the probability distribution for an individual l choosing between k alternatives $i = 1..k$. We have that choice i yields l a utility:

$$U_i^{(l)} = V_i^{(l)} + \varepsilon_i^{(l)}$$

We assume that l chooses the alternative with the highest utility. However, since these are random we can only compute the probability that a particular choice is made:

$$p_{l,i} = P(\text{“ } l \text{ chooses } i \text{ ”})$$

This is in fact the probability that $U_i^{(l)}$ is bigger than all other utilities, and we can write this as follows:

$$p_{l,i} = P\big(U_i^{(l)} \geqslant \max_{j \neq i} U_j^{(l)}\big) = P\big(\, V_i^{(l)} + \varepsilon_i^{(l)} \geqslant \max_{j \neq i}(\, V_j^{(l)} + \varepsilon_j^{(l)})\big)$$

Now define

$$U^* = \max_{j \neq i}(\, V_j^{(l)} + \varepsilon_j^{(l)}).$$

By property **II** of the Gumbel distribution,

$$U^* \stackrel{d}{=} \text{Gumbel}\big(\frac{1}{\mu} \ln \sum_{j \neq i} e^{\mu V_j^{(l)}}, \mu\big)$$

So, if

$$V^* = \frac{1}{\mu} \ln \sum_{j \neq l} e^{\mu V_j^{(l)}},$$

we have that $U^* = V^* + \varepsilon^*$ with $\varepsilon^* \stackrel{d}{=} \text{Gumbel}(0, \mu)$.

This in turn gives us that

$$p_{l,i} = P(V_i^{(l)} + \varepsilon_i^{/(l)} \geqslant V^* + \varepsilon^*) = P(V_i^{(l)} - V^* \geqslant \varepsilon^* - \varepsilon_i^{/(l)}) = \frac{1}{1 + e^{\mu(V^* - V_i^{(l)})}} =$$

by property **I** of the Gumbel distribution, and this can be re-expressed as

$$p_{l,i} = \frac{e^{\mu V_i^{(l)}}}{e^{\mu V_i^{(l)}} + e^{\mu V^*}} = \frac{e^{\mu V_i^{(l)}}}{\sum_{j=1}^{k} e^{\mu V_i^{(l)}}}$$

According to econometric knowledge μ is a parameter which cannot be *identified* from statistical data. From a physical perspective, this corresponds to the lack of a well defined temperature: intuitively this makes sense, since measuring temperature consists in comparing a system of interest with another system whose state we assume to know perfectly well. In physics this can be done to a high degree of precision: in social systems, however, such a concept has yet no clear meaning, and finding one will most certainly require a change in perspective about what we mean by measuring a quantity.

As a practical consequence, in this simple model we have that we can let the parameter μ be incorporated into the degrees of freedom $V_i^{(l)}$ of the various utilities, and get the choice probabilities in the following form:

$$p_{\,l,\,i} = \frac{e^{V_i^{(l)}}}{\sum_{j=1}^{k} e^{V_i^{(l)}}} \tag{2.4}$$

2.3.3 Statistical mechanics

As we have seen, the Multinomial logit model follows a *utility-maximization* approach, in that it assumes that each person behaves as to optimize his/her own benefit. From a statistical-mechanical perspective, this amounts to the community of people trying to identify its *ground state*, where some definition of self-perceived well-being, the utility, takes the role traditionally played by energy.

If there were an exact value of the utility corresponding to each behaviour, a system characterized by such maximizing principle for the ground state would identify *microcanonical ensemble* in a equilibrium statistical mechanics. This in amounts to stating that the energy of the system has an exact value, as opposed to being a random variable.

However, since the Multinomial logit defines utility itself as a Gumbel random variable in order to try and capture both the predictable and unpredictable components of human decisions, its "ground state" turns out to be a "noisy" object. Statistical mechanics models this situation by defining a so-called *canonical ensemble*, where all possible values of the energy are considered, each with a probability given by a *Gibbs distribution*, which weights energetically favourable states more than unfavourable ones. We will now see how the Gibbs distribution leads to a model which is formally equivalent to the Multinomial logit arising from the Gumbel distribution.

Assume that we have a population of N people, each of whom makes a choice

$$\sigma^{(l)} = \mathbf{e_l}$$

where vectors $\mathbf{e_i}$ form the k-dimensional canonical basis

$$\mathbf{e_1} = (1, 0, .., 0), \quad \mathbf{e_2} = (0, 1, .., 0), \quad \text{etc.}$$

We have then that a particular state of this system can be described by the following set:

$$\underset{\sim}{\sigma} = \{\sigma^{(1)}, ..., \sigma^{(N)}\}$$

Now define $v^{(l)}$ as a k-dimensional vector giving the utilities of the various choices for individual l:

$$v^{(l)} = (V_1^{(l)}, .., V_k^{(l)}).$$

We have that $V_i^{(l)}$, which is the deterministic part of the utility considered in the last section, changes from person to person, and that it can be parametrised by a person's social attributes, for instance. For the moment, however, we just consider them as different numbers, since the exact parametrization doesn't change the nature of the probability structure.

If we now denote by $v^{(l)} \cdot \sigma^{(l)}$ the scalar product between the two vectors, we may express the energy (also called *Hamiltonian*) for the Multinomial Logit Model as follows:

$$H_N(\underset{\sim}{\sigma}) = -\sum_{l=1}^{N} v^{(l)} \cdot \sigma^{(l)}.$$

Intuitively, a Hamiltonian model is one where the defines a model where the favoured states $\underset{\sim}{\sigma}$ are the ones which make the quantity H_N small, which due to the minus sign, correspond to people choosing as to maximise their utility. Most of the information contained in an equilibrium statistical mechanical model can be derived from its pressure, which is defined as

$$P_N = \ln \sum_{\underset{\sim}{\sigma}} e^{-H_N(\underset{\sim}{\sigma})},$$

which acts as a moment generating function for the Gibbs distribution

$$p(\underset{\sim}{\sigma}) = \frac{e^{-H_N(\underset{\sim}{\sigma})}}{\sum_{\underset{\sim}{\sigma}'} e^{-H_N(\underset{\sim}{\sigma}')}},$$

and can recover many of the features of the model, among which the probabilities $p_{l,i}$, as derivatives of P_N with respect to suitable parameters.

This distribution is chosen in physics since it is the one which maximises the system's entropy at a given temperature, which in turn just means that it is the most likely distribution to expect for a system which is at equilibrium. This is not to say that using such a model corresponds to accepting that society is at equilibrium, but rather to believing that some features of society might have small enough variations for a period of time long enough to allow a quantitative study. As pointed out in a later chapter, this belief has at least some quantitative backing if one considers the remarkable findings made by Émile Durkheim as early as at the end of 19^{th} century [20].

We will now show that this model is equivalent to the Multinomial Logit by computing its pressure explicitly and finding its derivatives. Indeed, since the model doesn't include interaction this is a task that can be done easily for a finite N:

$$\begin{aligned} P_N &= \ln \sum_{\underset{\sim}{\sigma}} e^{-H_N(\underset{\sim}{\sigma})} = \ln \sum_{\underset{\sim}{\sigma}} \exp\Big\{ \sum_{l=1}^{N} v^{(l)} \cdot \sigma^{(l)} \Big\} = \\ &= \ln \sum_{\sigma^{(1)}} \exp\left\{ v^{(1)} \cdot \sigma^{(1)} \right\} \ldots \sum_{\sigma^{(N)}} \exp\left\{ v^{(N)} \cdot \sigma^{(N)} \right\} = \\ &= \ln \prod_{l=1}^{N} \sum_{i=1}^{k} \exp\{V_i^{(l)}\} = \sum_{l=1}^{N} \ln \sum_{i=1}^{k} \exp\{V_i^{(l)}\}. \end{aligned}$$

Once we have the pressure P_N it's easy to find the probability $p_{i,l}$ that person l chooses alternative k, just by computing the derivative of P_N with respect to utility $V_i^{(l)}$:

$$p_{i,l} = P(\text{“}l \text{ chooses } i\text{ ”}) = \frac{\partial P_N}{\partial V_i^{(l)}} = \frac{e^{V_l^{(l)}}}{\sum_{j=1}^{k} e^{V_j^{(l)}}},$$

which is the same as (2.4).

This shows how the utility maximization principle is equivalent to a Hamiltonian model, whenever the random part of the utility is Gumbel distributed. There is a simple interpretation for this statistical mechanical model: it is a gas of N magnetic particles, each of which has k states, and the energy of these states depend on the corresponding value of the utility $V_k^{(l)}$, which therefore bears a close analogy to a magnetic field acting on the particle.

This model may seem completely uninteresting, since it is in no essential way different from a Langevin paramagnet. What is interesting, however, is how such a familiar, if trivial, model has arisen independently in the field of economics, and there are a few simple points to be made that can emphasize the change in perspective.

First, we see how for this model it makes sense to consider the pressure P_N as an extensive quantity. This is due to the fact that these models are applied to samples of data that yield information about each single individual, rather than be applied to extremely large ensembles of particles that we regard as identical, and of which we measure average quantities. Second, the availability of data about individuals (*microeconomic data*) allows us to define the vector $v^{(l)}$ which assigns a benefit value to each of the alternative that individual l has.

The main goal of an econometric model of this kind is then to find the parametrization for $v^{(l)}$ in terms of observable socio-economic features which fits micro data in an optimal way. The main goal of statistical mechanics is, on the other hand, to find a microscopic theory capable of generating laws that are observed consistently over a large number of experiments and measured with extreme precision at a macroscopic level. Since the numbers available for microeconomic data are not as high as the number of particles in a physical systems, but these that are more detailed at the level of individuals, the goal of a model of social behaviour could be seen as an interesting mixture of the above.

2.4 The role of statistical mechanics

We have see how discrete choice can be given a statistical mechanical description: in this section we consider why this is of interest to modeling social phenomena.

A key limitation of discrete choice theory is that it does not formally account for social interactions and imitation. In discrete choice each individual's decisions are based on purely personal preferences, and are not affected by other people's choices. However, there is a great deal of theoretical and empirical evidence to suggest that an individual's behaviour, attitude, identity and social decisions are influenced by that of others through vicarious experience or social influence, persuasions and sanctioning [1, 4]. These theories specifically relate to the interpersonal social environment including social networks, social support, role models and mentoring. The key insight of these theories is that individual behaviours and decisions are affected by their relationships with those around them - e.g. their parents or their peers.

Mathematical models that take into account social influence have been considered by social psychology since the '70s (see [63] for a short review). In particular, influential works by Schelling [62] and Granovetter [34] have shown how models where individuals take into account the mean behaviour of others are capable of reproducing, at least qualitatively, the dramatic opinion shifts observed in real life (for example in financial bubbles or during street riots). In other words, they observed that the interaction built into their models was unavoidably linked to the appearance of structural changes on a phenomenological level in the models themselves.

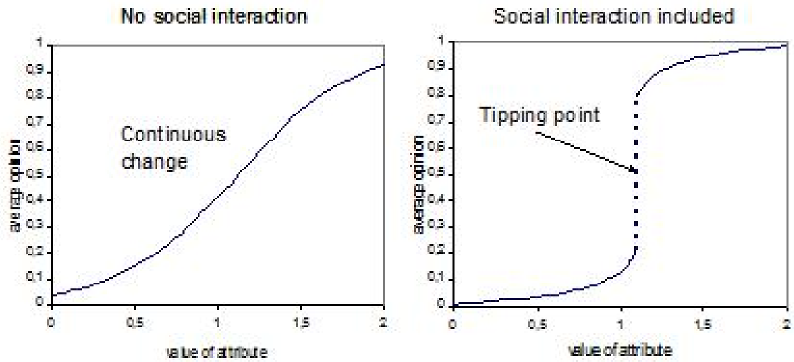

Figure 2.2: The diagram illustrates how the inclusion of social interactions (right) leads to the existence tipping points. By contrast models that do not account for social interactions cannot account for the tipping points.

Figure 2.2 compares the typical dependence of average choice with respect to an attribute parameter, such as cost, in discrete choice analysis (left), where the dependence is always a continuous one, with the typical behaviour of an interaction model of Schelling or Granovetter kind (right), where small changes in the attributes can lead to a drastic jump in the average choice, reflecting structural changes such as the disappearing of equilibria in the social context.

The research course initiated by Schelling was eventually linked to the parallel development of the discrete choice analysis framework at the end of the '90s, when Brock and Durlauf [21] suggested a direct econometric implementation of the models considered by social psychology. In order to accomplish this, Brock and Durlauf had to delve into the implications of a model where an individual takes into account the behaviour of others when making a discrete choice: this could only be done by considering a new utility function which depended on the choices of all other people.

This new utility function was built by starting from the assumptions of discrete choice analysis. The utility function reflects what an individual considers desirable: if we hold (see, e.g., [10]) that people consider desirable to conform to people they interact with, we have that, as a consequence, an individual's utility increases when he agrees with other people.

Symbolically, we can say that when an individual i makes a choice, his utility for that choice increases by an amount J_{ij} when another individual j agrees with him, thus defining a set of interaction parameters J_{ij} for all couples of individuals. The new utility function for individual i hence takes the following form:

$$U_i = \sum_j J_{ij}\tau_j + \sum_a \lambda_a x_a^{(i)} + \sum_a \alpha_a y_a^{(i)} + \varepsilon, \tag{2.5}$$

where the sum $\sum_j$ ranges over all individuals, and the symbol τ_j is equal to 1 if j agrees with i, and 0 otherwise.

Analysing the general case of such a model is a daunting task, since the choice of another individual j is itself a random variable, which in turn correlates the choices of all individuals. This problem, however, has been considered by statistical mechanics since the end of the 19^{th} century, throughout the twentieth century, until the present day. Indeed, the first success of statistical mechanics was to give a microscopic explanation of the laws governing perfect gases, and this was achieved thanks to a formalism which is strictly equivalent to the one obtained by discrete choice analysis in (5.4).

The interest of statistical mechanics eventually shifted to problems concerning interaction between particles, and as daunting as the problem described by (2.5) may be, statistical physics has been able to identify some restrictions on models of this kind to make them tractable while retaining great descriptive power as shown, e.g., in the work of Pierre Weiss [70] regarding the behaviour of magnets.

The simplest way devised by physics to deal with such a problem is called a *mean field* assumption, where interactions are assumed to be of a uniform and global kind. This leads to manageable closed form solution and a model that is consistent with the models of Schelling and Granovetter. Moreover, this assumption is also shown by Brock and Durlauf to be closely linked to the assumption of rational expectations from economic theory, which assumes that the observed behaviour of an individual must be consistent with his belief about the opinion of others.

By assuming *mean field* or *rational expectations* we can rewrite (2.5) in the tamer form

$$U_i = Jm + \sum_a \lambda_a x_a^{(i)} + \sum_a \alpha_a y_a^{(i)} + \varepsilon, \tag{2.6}$$

where m is the average opinion of a given individual, and this average value is coupled to the model parameters by a closed form formula.

If we now define V_i to be the deterministic part of the utility, similarly as before,

$$V_i = Jm + \sum_a \lambda_a x_a^{(i)} + \sum_a \alpha_a y_a^{(i)},$$

we have that the functional form of the choice probability, given by 5.4,

$$P_i = \frac{e^{V_i}}{1 + e^{V_i}}, \tag{2.7}$$

remains unchanged, allowing the empirical framework of discrete choice analysis to be used to test the theory against real data. This sets the problem as one of heterogeneous interacting particles, and we shall see in the next two chapters how such a mean-field model, just like the standard Multinomial Logit, can be given a Hamiltonian statistical mechanical form, and solved in a completely rigorous way using elementary mathematics, via methods recently developed in the context of spin glasses [37].

Though the mean field assumption might be seen as a crude approximation, since it considers a uniform and fixed kind of interaction, one should bear in mind that statistical physics has built throughout the twentieth century the expertise needed to consider a wide range of forms for the interaction parameters J_{ij}, of both deterministic and random nature, so that a partial success in the application of mean field theory might be enhanced by browsing through a rich variety of well developed, though analytically more demanding, theories.

Nevertheless, an empirical attempt to assess the actual descriptive and predictive power of such models has not been carried out to date: the natural course for such a study would be to start by empirically testing the mean field picture, as it was done for discrete choice in the seventies (see Figure 1), and to proceed by enhancing it with the help available from the econometrics, social science, and statistical physics communities. Two recent examples of empirical studies of mean-field models can be found in [65] and [29].

Chapter 3

The Curie-Weiss model

The Curie-Weiss model was first introduced in 1907 by Pierre Weiss [70] as a proposal for a phenomenological model capable of explaining the experimental observations carried out by Pierre Curie in 1895 [18], concerning the dependance on temperature of the magnetic nature for metals such as iron, nickel, and magnetite.

Iron and nickel are materials capable of retaining a degree of magnetization, which we call *spontaneous magnetization*, after having been exposed to a magnetic field: such materials are said to be *ferromagnetic*, from the Latin name for iron. However, it had been known since the day of Faraday ([18], pag. 1) that these materials tend to lose their ability to retain magnetization as their temperature increases.

Pierre Curie's experiments showed not only that the loss of the ferromagnetic property indeed occurs, but also observed that it occurs in a very peculiar fashion. For each of the materials he considered, he found a definite temperature at which spontaneous magnetization vanishes abruptly, giving rise to an irregular point in the graph plotting spontaneous magnetization versus temperature (see Figure 3.1): we now call this temperature the *Curie temperature* for the given material.

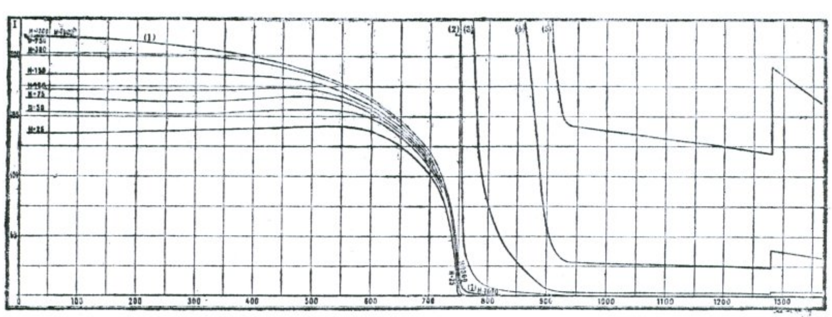

Figure 3.1: Pierre Curie's measurements in 1895

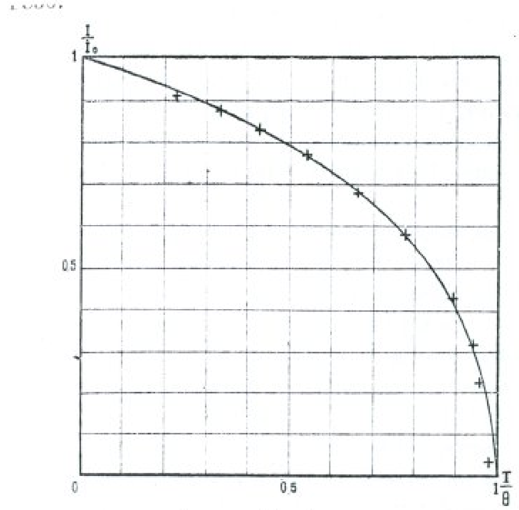

Fig. 2.

Figure 3.2: Pierre Weiss's measurements (crosses) fitted against his theoretical prediction (line) in 1907: the graph shows the dependance of spontaneous magnetization on temperature for magnetite

Weiss's model arises from physical considerations about the nature of magnetic interactions between atoms: he claims that single atoms must experience, as well as the external field, a sum of all the fields produced by all the other particles inside the material. He calls this field a "molecular field" (*champ moleculaire*), and by adding a term corresponding to this field inside the balance equation derived by Paul Langevin to describe *paramagnetic* materials (that is, magnetic materials that do not retain magnetition after exposure to a field), he formulates a balance equation for ferromagnetic materials.

In his 1907 paper Weiss shows that the theoretical predictions of his model show remarkable agreement with physical reality by fitting them against measurements, carried on by himself, on a ellipsoid made of magnetite (Figure 3.2).

Today we know that the Curie-Weiss is not completely accurate: indeed, it is well known that some physically measurable quantities for ferromagnetic materials, called *critical exponents*, are not predicted correctly by it (see [41], pag. 425). The subsequent study of more detailed models, such as the Ising model, has brought to light the reason for such a mismatch: when rewritten in the language of modern statistical mechanics, the model of Curie-Weiss readily shows to be equivalent to one where all particles are interacting with each other. This turns out to be too strong an assumption for a system where all particles sit next to each other geometrically and which interact, according to quantum mechanics, up to a very short range. On the other hand though, the Ising model, which still makes use of all of Weiss's other simplifying assumptions about interaction between particles, manages to predict critical exponents correctly, just by assuming that particles only interact with their nearest neighbours on a regular lattice, though, from a mathematical point of view, this modification implies a drastic reduction of the symmetry of the problem, which has so far proved to be analitically untreatable in more than two dimensions (see [41] pag. 341).

All objections standing, it is nevertheless worth remembering that the degree of agreement between theory and reality for the Curie-Weiss model is truly remarkable given the simplicity of the model. Today, Weiss's "molecular field" assumption is dimmed a *mean field* assumption, and scientific wisdom tells that this assumption is of great value in exploring the phase structure of a system so that, when faced with a new situation, one would try mean field first ([41], pag. 423).

3.1 The model

As a modern statistical mechanics model, the Curie-Weiss model is defined by its Hamiltonian:

$$H(\sigma) = -\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} h_i\sigma_i \ . \tag{3.1}$$

We consider Ising spins, $\sigma_i = \pm 1$, subject to a uniform magnetic field $h_i = h$ and to isotropic interactions $J_{i,j} = J/2N$, so that we have.

$$H(\sigma) = -\frac{J}{2N}\sum_{i,j=1}^{N} \sigma_i\sigma_j - h\sum_{i=1}^{N} \sigma_i \ . \tag{3.2}$$

If we now introduce the magnetization of a configuration σ as

$$m(\sigma) \ = \ \frac{1}{N}\sum_{i=1}^{N}\sigma_i$$

we can rewrite the Hamiltonian per particle as:

$$\frac{H(\sigma)}{N} = -\frac{J}{2}m(\sigma)^2 - hm(\sigma) \tag{3.3}$$

The established statistical mechanics framework defines the equilibrium value of an observable $f(\sigma)$ as the average with respect to the *Gibbs distribution* defined by the Hamiltonian. We call this average the *Gibbs state* for $f(\sigma)$, and write it explicitly as:

$$\langle\, f\,\rangle \ = \ \frac{\sum_\sigma f(\sigma)\; e^{-H(\sigma)}}{\sum_\sigma e^{-H(\sigma)}} \ .$$

The main observable for our model is the average value of a spin configuration, i.e. the *magnetization*, $m(\sigma)$, which explicitly reads:

$$m(\sigma) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i.$$

Our quantity of interest is therefore $\langle m \rangle$: to find it, as well as the moments of many other observables, statistical mechanics leads us to consider the pressure function:

$$p_N \;=\; \frac{1}{N}\log\sum_{\sigma} e^{-H(\sigma)}\;.$$

It is easy to verify that, once it's been derived exactly, the pressure is capable of generating the Gibbs state for the magnetization as

$$\langle m \rangle = \frac{\partial p_N}{\partial h}.$$

3.2 Existence of the thermodynamic limit

We show two ways of computing the existence of the thermodynamic limit in the Curie-Weiss model. The first method follows [5] in exploiting directly the convexity of the Hamiltonian in order to prove subadditivity in N for the systems's pressure.

The second method consists in a refinement of the first, and covers models for which the Hamiltonian is not necessarily convex, such as the two-population model considered in the next chapter. It is important to point out that a careful application of this method to the Sherrington-Kirkpatrick spin glass model allowed Guerra [36] to prove the twenty-years standing question concerning existence of thermodynamic limit.

3.2.1 Existence by convexity of the Hamiltonian

We consider a system of N spins defined as above. Following [5] we split the system in two subsystem of N_1 and N_2 spins, respectively, with $N_1 + N_2 = N$. For each of these systems we define partial magnetizations

$$m_1(\sigma) \;=\; \frac{1}{N_1}\sum_{i=1}^{N_1}\sigma_i \quad \text{and} \quad m_2(\sigma) \;=\; \frac{1}{N_2}\sum_{i=N_1+1}^{N}\sigma_i,$$

which allow us to define partial Hamiltonians

$$H_{N_1} = -N_1(\frac{J}{2}m_1^2 + hm_1) \quad \text{and} \quad H_{N_2} = -N_2(\frac{J}{2}m_2^2 + hm_2).$$

We have by definition that

$$m = \frac{N_1}{N}m_1 + \frac{N_2}{N}m_2 \tag{3.4}$$

and since $f(x) = x^2$ is a convex function we also have that

$$m \leqslant \frac{N_1}{N}m_1^2 + \frac{N_2}{N}m_2^2. \tag{3.5}$$

We are now ready to prove the following

Proposition 1. *There exists a function $p(J, h)$ such that*

$$\lim_{N\to\infty} p_N = p\ .$$

Proof. Relations (3.4) and (3.5) imply that

$$H_N \leqslant H_{N_1} + H_{N_2}$$

and this in turn gives

$$Z_N = \sum_\sigma e^{-H_N(\sigma)} \leqslant \sum_\sigma e^{-H_{N_1}(\sigma:1..N_1)-H_{N_1}(\sigma:N_1+1..N_2)} = Z_{N_1}Z_{N_2}$$

where $\sigma : 1..N_1 = \{\sigma_1, .., \sigma_{N_1}\}$ and $\sigma : N_1 + 1..N = \{\sigma_{N_1+1}, .., \sigma_N\}$. Hence we have the following inequality

$$Np_N \leqslant N_1p_{N_1} + N_2p_{N_2}, \quad \text{for } N_1 + N_2 = N$$

This identifies the sequence $\{Np_N\}$ as a *subadditive* sequence, for which the following holds

$$\lim_{N\to\infty} \frac{Np_N}{N} = \lim_{N\to\infty} p_N = \inf_N p_N.$$

Hence in order to verify the existence of a finite limit we need to verify that the sequence $\{p_N\}$ is bounded below, which follows from the boundedness of the intensive quantity

$$\frac{H(\sigma)}{N} = -\frac{J}{2}m^2 - hm,$$

for $-1 \leqslant m \leqslant 1$. Indeed, if $\frac{H(\sigma)}{N} \leqslant K$,

$$p_N = \frac{1}{N} \ln \sum_{\sigma} e^{-H(\sigma)} \geqslant \frac{1}{N} \ln 2^N e^{NK} = \ln 2 + K$$

so the result follows.

□

3.2.2 Existence by interpolation

We shall now prove that our model admits a thermodynamic limit by exploiting an existence theorem provided for mean field models in [8]: the result states that the existence of the pressure per particle for large volumes is guaranteed by a monotonicity condition on the equilibrium state of the Hamiltonian. We therefore prove the existence of the thermodynamic limit independently of an exact solution. Such a line of enquiry is pursued in view of the study of models, that shall possibly involve random interactions of spin glass or random graph type, and that might or might not come with an exact expression for the pressure.

Proposition 2. *There exists a function $p(J, h)$ such that*

$$\lim_{N\to\infty} p_N = p\ .$$

Proof. Theorem 1 in [8] states that given a Hamiltonian H_N such that $\frac{H_N}{N}$ is bounded in N, and its associated equilibrium state ω_N, the model admits a thermodynamic limit whenever the physical condition

$$\omega_N(H_N) \geqslant \omega_N(H_{N_1}) + \omega_N(H_{N_2}), \qquad N_1 + N_2 = N, \tag{3.6}$$

is verified.

For the Curie-Weiss model the condition is easy to verify once we define partial magnetizations

$$m_1(\sigma) \;=\; \frac{1}{N_1}\sum_{i=1}^{N}\sigma_i \quad \text{and} \quad m_2(\sigma) \;=\; \frac{1}{N_2}\sum_{i=1}^{N}\sigma_i.$$

This gives that

$$m = \frac{N_1}{N} m_1 + \frac{N_2}{N} m_2$$

so that

$$\begin{aligned} H_N - H_{N_1} - H_{N_2} &= -N(\frac{J}{2}m^2 + hm) + N_1(\frac{J}{2}m_1^2 + hm_1) + N_2(\frac{J}{2}m_2^2 + hm_2) = \\ &= -N\frac{J}{2}(m^2 - \frac{N_1}{N}m_1^2 - \frac{N_2}{N}m_2^2) - Nh(m - \frac{N_1}{N}m_1 - \frac{N_2}{N}m_2) \\ &= -N\frac{J}{2}(m^2 - \frac{N_1}{N}m_1^2 - \frac{N_2}{N}m_2^2) \geqslant 0 \end{aligned}$$

The last inequality follows from convexity of the function $f(x) = x^2$, and since it holds for every configuration σ, it also implies (3.6), proving the result.

□

3.3 Factorization properties

In this section we shall prove that the correlation functions of our model factorize completely in the thermodynamic limit, for almost every choice of parameters. This implies that all the thermodynamic properties of the system can be described by the magnetization. Indeed, the exact solution of the model to be derived in the next section comes as an equation of state which, as expected, turns out to be the same as the balance equation derived by Weiss.

Proposition 3.

$$\lim_{N\to\infty} \left(\omega_N(m^2) - \omega_N(m)^2\right) = 0$$

for almost every choice of h.

Proof. We recall the definition of the Hamiltonian per particle

$$\frac{H_N(\sigma)}{N} = -\frac{J}{2}m^2 - hm,$$

and of the pressure per particle

$$p_N = \frac{1}{N}\ln\sum_\sigma e^{-H_N(\sigma)}.$$

By taking first and second partial derivatives of p_N with respect to h we get

$$\frac{\partial p_N}{\partial h} = \frac{1}{N}\sum_\sigma Nm(\sigma)\frac{e^{-H(\sigma)}}{Z_N} = \omega_N(m), \qquad \frac{\partial^2 p_N}{\partial\, h^2} = \omega_N(m^2) - \omega_N(m)^2.$$

By using these relations we can bound above the integral with respect to h of the fluctuations of m in the Gibbs state:

$$\begin{aligned}\left|\int_{h^{(1)}}^{h^{(2)}} (\omega_N(m^2)-\omega_N(m)^2)\, dh\right| &= \frac{1}{N}\left|\int_{h^{(1)}}^{h^{(2)}} \frac{\partial^2 p_N}{\partial h^2}\, dh\right| = \frac{1}{N}\left|\frac{\partial p_N}{\partial\, h}\Big|_{h^{(1)}}^{h^{(2)}}\right| \leqslant \\ &\leqslant \frac{1}{N}\big(\big|\omega_N(m)|_{h^{(2)}}\big| + \big|\omega_N(m)|_{h^{(1)}}\big|\big) = O\big(\frac{1}{N}\big).\end{aligned} \tag{3.7}$$

On the other hand we have that

$$\omega_N(m) = \frac{\partial p_N}{\partial h},$$

and

$$\omega_N(m) = \frac{\partial p_N}{\partial J},$$

so, by convexity of the thermodynamic pressure $p = \lim_{N\to\infty} p_N$, both quantities $\frac{\partial p_N}{\partial h}$ and $\frac{\partial p_N}{\partial J}$ have well defined thermodynamic limits almost everywhere. This together with (3.7) implies that

$$\lim_{N\to\infty} (\omega_N(m^2) - \omega_N(m)^2) = 0 \quad \text{a.e. in } h. \tag{3.8}$$

□

The last proposition proves that $m(\sigma)$ is a *self-averaging* quantity, that is, a random quantity whose fluctuations vanish in the thermodynamic limit. This is indeed a powerful result, which can be exploited thanks to the following

Proposition 4. *(Cauchy-Schwartz inequality) Let X and Y be two random variables defined on a finite probability space such that $P(X_i) = P(Y_i) = p_i$. Then the following holds*

$$\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \leqslant \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$$

Proof. Let us define the following quantities:

$$\mathbb{E}(X) = \sum_i X_i p_i = \mu_X, \quad \mathrm{Var}(X) = \sigma_X^2$$

$$\mathbb{E}(Y) = \sum_i Y_i p_i = \mu_Y, \quad \mathrm{Var}(Y) = \sigma_Y^2$$

If we now define rescaled versions of X and Y:

$$\bar{X} = \frac{X - \mu_X}{\sigma_X}, \quad \text{and} \quad \bar{Y} = \frac{Y - \mu_Y}{\sigma_Y},$$

we get that $\{X_i p_i^{1/2}\}$ and $\{Y_i p_i^{1/2}\}$ are vectors of Euclidean length equal to 1. This implies

$$|\mathbb{E}(\bar{X}\bar{Y})| = |\sum_i \bar{X}_i \bar{Y}_i p_i| = |\sum_i \bar{X}_i p_i^{1/2} \bar{Y}_i p_i^{1/2}| \leqslant 1$$

where the inequality only points out that $\mathbb{E}(\bar{X}\bar{Y})$ is the projection of a unit vector against another, and therefore that its modulus is less than one.

If we now substitute back X and Y we get our result.

□

By putting together the self-avering property and the Cauchy-Schwartz inequality we get the following

Proposition 5. *Given any integer k we have that*

$$\lim_{N\to\infty} \left(\omega_N(m^k) - \omega_N(m)^k\right) = 0$$

for almost every choice of h.

Proof. Applying the Cauchy-Schartz inequality to $X = m^{k-1}$ and $Y = m$ we get that

$$|\omega_N(m^{k-1}m) - \omega_N(m^{k-1}m)| \leqslant \sqrt{\mathrm{Var}_N(m^{k-1})\mathrm{Var}_N(m)}. \tag{3.9}$$

Now self-averaging tells us that $\mathrm{Var}_N(m)$ tends to zero in the limit, and since m^{k-1} is a bounded quantity, (3.9) implies:

$$\lim_{N\to\infty} \left(\omega_N(m^k) - \omega_N(m)^{k-1}\omega_N(m)\right) = 0$$

and the rest of the proposition follows by induction on the same argument.

□

The last proposition is very important for this model, because the mean-field nature of the system allows to use the factorization of the magnetization in order to prove factorization of spin correlation functions, thus characterizing all the thermodynamics of the system.

In the following proposition we shall only prove the factorization of 2-spins: the factorization of k-spins is done in the same way.

Proposition 6.

$$\lim_{N\to\infty}\big(\omega_N(\sigma_i\sigma_j)-\omega_N(\sigma_i)\omega_N(\sigma_j)\big)=0$$

for almost every choice of h, whenever σ_i, σ_j are distinct spins.

Proof. Now we can use the self-averaging of $m(\sigma)$ the factorization of correlation functions. This is done by exploiting the translation invariance of the Gibbs measure on spins, which in turn follows from the mean-field nature of the model:

$$\begin{aligned}
\omega_N(m) &= \omega_N(\frac{1}{N}\sum_{i=1}^{N}\sigma_i)=\omega_N(\sigma_1),\\
\omega_N(m^2) &= \omega_N(\frac{1}{N^2}\sum_{i,j=1}^{N}\sigma_i\sigma_j)=\omega_N(\frac{1}{N^2}\sum_{i\neq j=1}^{N_1}\sigma_i\sigma_j)+\omega_N(\frac{1}{N^2}\sum_{i=j=1}^{N}\sigma_i\sigma_j)=\\
&= \frac{N-1}{N}\omega_N(\sigma_1\sigma_2)+\frac{1}{N}.
\end{aligned}\tag{3.10}$$

We have that (3.10) and (3.8) imply

$$\lim_{N\to\infty}\omega_N(\sigma_i\sigma_j)-\omega_N(\sigma_i)\omega_N(\sigma_j)=0,\quad \text{for a.e. h}\tag{3.11}$$

which verifies our statement for all couples of spins $i\neq j$.

□

The self-averaging of the magnetization has been proved directly here: this, however, can be seen as a consequence of the convexity of the pressure. Indeed, the second derivative of any convex function exists almost everywhere: this is a consequence of the first derivative existing almost everywhere and being monotonically increasing (se, e.g., [57]).

Therefore existence almost everywhere of $\frac{\partial^2 p}{\partial h^2}$ together with the intensivity property of the magnetization implies trivially that its fluctuations vanish in the thermodynamic limit. This also implies that, since energy per particle is another intensive quantity which is obtained by differentiating the pressure with respect J, energy per particle is a self-averaging quantity too.

As we can see from Proposition 5 factorization of spins only holds a.e. for h, and indeed it can be proved that factorization doesn't hold at $h=0$, $J>1$. However, by using the self-averaging of energy-per-particle proved above, we can similarly obtain a weaker factorization rule which covers this regime:

Proposition 7.

$$\lim_{N\to\infty} \omega_N(\sigma_i\sigma_j\sigma_k\sigma_l) - \omega_N(\sigma_i\sigma_j)\omega_N(\sigma_k\sigma_l) = 0, \quad \textit{for a.e. } J$$

for almost every choice of J*, whenever* σ_i*,* σ_j*,* σ_k*,* σ_l *are distinct spins.*

Proof. The proof follows the same argument of Proposition 5, and uses the self-averaging of the energy per particle instead of the self-averaging of the magnetization.

□

3.4 Solution of the model

We shall derive upper and lower bounds for the thermodynamic limit of the pressure. The lower bound is obtained through the standard entropic variational principle, while the upper bound is derived by a decoupling strategy.

3.4.1 Upper bound

In order to find an upper bound for the pressure we shall divide the configuration space into a partition of microstates of equal magnetization, following [19, 37, 38]. Since the system consists of N spins, its magnetization can take exactly $N+1$ values, which are the elements of the set

$$R_N = \Big\{ -1, -1 + \frac{1}{2N}, \dots, 1 - \frac{1}{2N}, 1 \Big\}.$$

Clearly for every $m(\sigma)$ we have that

$$\sum_{\bar{m}\in R_N} \delta_{m,\bar{m}} = 1,$$

where $\delta_{x,y}$ is a Kronecker delta. Therefore we have that

$$Z_N \;=\; \sum_{\sigma} \exp\big\{N(\frac{J}{2}m^2 + hm)\big\} = = \sum_{\sigma}\sum_{\bar{m}\in R_N} \delta_{m,\bar{m}} \exp\big\{N(\frac{J}{2}m^2 + hm)\big\}. \quad (3.12)$$

Thanks to the Kronecker delta symbols, we can substitute m (the average of the spins within a configuration) with the parameter $\bar{m}$ (which is not coupled to the spin configurations) in any convenient fashion. Therefore we can use the following relation in order to linearize the quadratic term appearing in the Hamiltonian

$$(m - \bar{m})^2 \;=\; 0,$$

and once we've carried out this substitution into (3.12) we are left with a function which depends only linearly on m:

$$Z_N \quad = \quad \sum_{\sigma} \sum_{\bar{m} \in R_N} \delta_{m,\bar{m}} \exp\big\{N(\frac{J}{2}(2m\bar{m} - \bar{m}^2) + hm)\big\}.$$

and bounding above the Kronecker deltas by 1 we get

$$Z_N \quad \leqslant \quad \sum_{\sigma} \sum_{\bar{m} \in R_N} \exp\big\{N(\frac{J}{2}(2m\bar{m} - \bar{m}^2) + hm)\big\}.$$

Since both sums are taken over finitely many terms, it is possible to exchange the order of the two summation symbols, in order to carry out the sum over the spin configurations, which now factorizes, thanks to the linearity of the interaction with respect to the ms. This way we get:

$$Z_N \leqslant \sum_{\bar{m} \in R_N} G(\bar{m}).$$

where

$$G(\bar{m}) \quad = \quad \exp\big\{-N\frac{1}{2}J\bar{m}^2\big\} \cdot 2^N \big(\cosh\big(J\bar{m} + h\big)\big)^N \tag{3.13}$$

Since the summation is taken over the range R_N of cardinality $N+1$ we get that the total number of summands is $N+1$. Therefore

$$Z_N \quad \leqslant \quad (N+1) \sup_{\bar{m}} G, \tag{3.14}$$

which leads to the following upper bound for p_N:

$$\begin{aligned} p_N \quad &= \quad \frac{1}{N} \ln Z_N \leqslant \frac{1}{N} \ln (N+1) \sup_{\bar{m}} G = \\ &= \quad \frac{1}{N} \ln(N+1) + \frac{1}{N} \sup_{\bar{m}} \ln G\ . \end{aligned} \tag{3.15}$$

where the last equality follows from monotonicity of the logarithm.

Now defining the N independent function

$$p_{up}(\bar{m}_1, \bar{m}_2) = \frac{1}{N} \ln G = \ln 2 - \frac{J}{2}\bar{m}^2 + \ln\cosh\big(J\bar{m} + h\big),$$

and keeping in mind that $\lim_{N\to\infty} \frac{1}{N}\ln(N+1) = 0$, in the thermodynamic limit we get:

$$\limsup_{N\to\infty} p_N \leqslant \sup_{\bar{m}} p_{UP}(\bar{m}). \tag{3.16}$$

We can summarize the previous computation into the following:

Lemma 1. *Given a Hamiltonian as defined in (3.3), and defining the pressure per particle as $p_N = \frac{1}{N}\ln Z$, given parameters J and h, the following inequality holds:*

$$\limsup_{N\to\infty} p_N \leqslant \sup_{\bar{m}} p_{up}$$

where

$$p_{up}(\bar{m}) = \ln 2 - \frac{J}{2}\bar{m}^2 + \ln\cosh\big(J\bar{m} + h\big),$$

and $\bar{m} \in [-1, 1]$.

We shall give two ways of deriving a lower bound for the pressure: indeed, it is important to keep in mind that having as many bounding tecniques as possible can be a good way of approaching more refined models.

3.4.2 Lower bound by convexity of the Hamiltonian

Proposition 8. *Given a Hamiltonian as defined in (3.3) and its associated pressure per particle $p_N = \frac{1}{N}\ln Z$, the following inequality holds for every J, h:*

$$p_N \geqslant \sup_{-1\leqslant\bar{m}\leqslant 1} p_{low}$$

where

$$p_{low}(\bar{m}) = -\frac{J}{2}\bar{m}^2 + \ln 2 + \ln\cosh(J\bar{m} + h)$$

Proof. We recall the Hamiltonian per particle written in terms of the configuration's magnetization $m(\sigma)$:

$$\frac{H(\sigma)}{N} = -\frac{J}{2}m^2 - hm.$$

Now, given any number $\bar{m} \in [-1, +1]$, the following holds:

$$(m - \bar{m})^2 \geqslant 0 \;\Rightarrow\; m^2 \geqslant 2m\bar{m} - \bar{m}^2$$

so that

$$\begin{aligned} p_N &= \frac{1}{N}\ln Z_N = \frac{1}{N}\ln\sum_\sigma \exp\{N(\frac{J}{2}m^2+hm)\} \geqslant \\ &\geqslant \frac{1}{N}\ln\sum_\sigma \exp\{N(Jm\bar{m}-\frac{J}{2}\bar{m}^2+hm)\} = \\ &= \frac{1}{N}\ln\left(\exp\{-\frac{NJ}{2}\bar{m}^2\}\sum_\sigma \exp\{N(J\bar{m}m+hm)\}\right) = \\ &= -\frac{J}{2}\bar{m}^2+\frac{1}{N}\ln\left(2^N\cosh(J\bar{m}+h)^N\right) = -\frac{J}{2}\bar{m}^2+\ln 2+\ln\cosh(J\bar{m}+h) \end{aligned}$$

This way we get new lower bound which can be expressed as

$$p_N \geqslant \sup_{-1\leqslant\bar{m}\leqslant 1} p_{low}$$

where

$$p_{low}(\bar{m}) = -\frac{J}{2}\bar{m}^2+\frac{1}{N}\ln\left(2^N\cosh(J\bar{m}+h)^N\right) = \ln 2-\frac{J}{2}\bar{m}^2+\ln\cosh(J\bar{m}+h)$$

which is the result.

□

3.4.3 Variational lower bound

The second lower bound is provided by exploiting the well-known Gibbs entropic variational principle (see [58], pag. 188). In our case, instead of considering the whole space of *ansatz* probability distributions considered in [58], we shall restrict to a much smaller one, and use the upper bound derived in the last section in order to show that the lower bound corresponding to the restricted space is sharp in the thermodynamic limit.

The mean-field nature of our Hamiltonian allows us to restrict the variational problem to a product measure with only one degree of freedom, represented by the non-interacting Hamiltonian:

$$\tilde{H} = -r\sum_{i=1}^N \sigma_i,$$

and so, given a Hamiltonian $\tilde{H}$, we define the ansatz Gibbs state corresponding to it as $f(\sigma)$ as:

$$\tilde{\omega}(f) = \frac{\sum_\sigma f(\sigma)e^{-\tilde{H}(\sigma)}}{\sum_\sigma e^{-\tilde{H}(\sigma)}}$$

In order to facilitate our task, we shall express the variational principle of [58] in the following simple form:

Proposition 9. *Let a Hamiltonian H, and its associated partition function $Z = \sum_{\sigma} e^{-H}$ be given. Consider an arbitrary trial Hamiltonian $\tilde{H}$ and its associated partition function $\tilde{Z}$. The following inequality holds:*

$$\ln Z \geqslant \ln \tilde{Z} - \tilde{\omega}(H) + \tilde{\omega}(\tilde{H}) \ . \tag{3.17}$$

Given a Hamiltonian as defined in (3.3) and its associated pressure per particle $p_N = \frac{1}{N} \ln Z$, the following inequality follows from (3.17)*:*

$$\liminf_{N \to \infty} p_N \geqslant \sup_{\bar{m}_1, \bar{m}_2} p'_{low} \tag{3.18}$$

where

$$p'_{low}(\bar{m}) \quad = \quad \frac{J}{2}\bar{m}^2 + h\bar{m} - \frac{1+\bar{m}}{2} \ln(\frac{1+\bar{m}}{2}) - \frac{1-\bar{m}}{2} \ln(\frac{1-\bar{m}}{2}). \tag{3.19}$$

and $\bar{m} \in [-1, 1]$.

Proof. The inequality (3.17) follows straightforwardly from Jensen's inequality:

$$e^{\tilde{\omega}(-H+\tilde{H})} \leq \tilde{\omega}(e^{-H+\tilde{H}}) \ . \tag{3.20}$$

We recall the Hamiltonian:

$$H(\sigma) = -\frac{J}{2N} \sum_{i,j} \sigma_i \sigma_j - h \sum_i \sigma_i, \tag{3.21}$$

so that its expectation on the trial state is

$$\tilde{\omega}(H) = -\frac{J}{2N} \sum_{i,j} \tilde{\omega}(\sigma_i \sigma_j) - h \sum_i \tilde{\omega}(\sigma_i)$$

and a standard computation for the moments of a non-interacting system (i.e. for a perfect gas) leads to

$$\tilde{\omega}(H) \quad = \quad -N(1-1/N)\frac{J}{2}(\tanh r)^2 - N\frac{J}{2} - Nh \tanh r. \tag{3.22}$$

Analogously, the trial Gibbs state of $\tilde{H}$ is:

$$\tilde{\omega}(\tilde{H}) = -Nr\ \tanh r,$$

and the non interacting partition function is:

$$\tilde{Z}_N = \sum_\sigma e^{-\tilde{H}(\sigma)} = 2^N(\cosh r)^N,$$

which implies that the non-interacting pressure gives

$$\tilde{p}_N = \frac{1}{N}\ln \tilde{Z}_N = \ln 2 + \ln \cosh r$$

So we can finally apply Proposition (3.17) in order to find a lower bound for the pressure $p_N = \dfrac{1}{N}\ln Z_N$:

$$p_N = \frac{1}{N}\ln Z_N \geqslant \frac{1}{N}\left(\ln \tilde{Z}_N - \tilde{\omega}(H) + \tilde{\omega}(\tilde{H})\right) \tag{3.23}$$

which explicitly reads:

$$\begin{aligned} p_N = \frac{1}{N}\ln Z_N \quad &\geqslant \quad \ln 2 + \ln \cosh r + \frac{J}{2}(\tanh r)^2 + h\tanh r - r\tanh r \\ &+J/2N - J(\tanh r)^2/N. \end{aligned} \tag{3.24}$$

Taking the liminf over N and the supremum in r of the left hand side we get (4.21) after performing the change of variables $\bar{m} = \tanh r$, and obtaining the following form for the right hand side:

$$p_{low}(\bar{m}) = \frac{J}{2}\bar{m}^2 + h\bar{m} - \frac{1+\bar{m}}{2}\ln(\frac{1+\bar{m}}{2}) - \frac{1-\bar{m}}{2}\ln(\frac{1-\bar{m}}{2}).$$

□

3.4.4 Exact solution of the model

We have derived two lower bounds and one upper bound to the thermodynamic pressure, which are given by the suprema w.r.t. $\bar{m}$ of the following functions:

$$\begin{aligned} p_{up}(\bar{m}) = p_{low}(\bar{m}) &= \ln 2 - \frac{J}{2}\bar{m}^2 + \ln\cosh(J\bar{m} + h) \\ p'_{low}(\bar{m}) &= \frac{J}{2}\bar{m}^2 + h\bar{m} - \frac{1+\bar{m}}{2}\ln(\frac{1+\bar{m}}{2}) - \frac{1-\bar{m}}{2}\ln(\frac{1-\bar{m}}{2}) \end{aligned} \quad (3.25)$$

Since $p_{up} = p_{low}$, the supremum of this function gives the thermodynamic value of the pressure, and thus provides the exact solution to the model. However, it is important to verify that the bounds provided by all functions coincide, since for more general cases one of the bounding arguments may fail, as indeed happens in the next chapter, where a bound of type p_{low} cannot be found due to lack of convexity in the Hamiltonian. Furthermore, p'_{low} has a direct thermodynamic interpretation, as shall be explained in the following section.

For the standard Curie-Weiss model that we are studying here the equivalence of the two bounds can be proved by way of a peculiar property of the Legendre transformation, and we will do this in this section.

Proposition 10. *The function*

$$f^*(y) = \frac{1}{J}\Big(\frac{1+y}{2}\ln\frac{1+y}{2} + \frac{1-y}{2}\ln\frac{1-y}{2} - y\,h\Big)$$

is the Legendre transform of

$$f(x) = \frac{1}{J}\ln\, 2\cosh(Jx + h)$$

Proof. The Legendre transformation is defined by

$$f^*(y) = \sup_x \big(xy - f(x)\big)$$

Since we are dealing with a convex function we can find the supremum by differentiation:

$$\frac{df}{dx} = y - \tanh(Jx + h) = 0$$

which implies

$$Jx = \text{arctanh}\; y - h,$$

so that by substituting we find that the Legendre transform of f is

$$\begin{aligned}
f^*(y) &= y\,\frac{1}{J}(\text{arctanh }y - h) - \frac{1}{J}\ln 2\cosh(\text{ arctanh }y - h + h) = \\
&= y\,\frac{1}{J}\text{arctanh }y - \frac{yh}{J} - \frac{1}{J}\ln 2\cosh\text{arctanh }y = \\
&= y\frac{1}{2J}\ln\frac{1+y}{1-y} - \frac{yh}{J} - \frac{1}{J}\ln\Big(\exp\{\frac{1}{2}\ln\frac{1+y}{1-y}\} + \exp\{\frac{1}{2}\ln\frac{1-y}{1+y}\}\Big) = \\
&= y\frac{1}{2J}\ln\frac{1+y}{1-y} - \frac{yh}{J} - \frac{1}{J}\ln\Big(\frac{1+y+1-y}{\sqrt{1-y^2}}\Big) = y\frac{1}{2J}\ln\frac{1+y}{1-y} - \frac{yh}{J} - \frac{1}{J}\ln\Big(\frac{2}{\sqrt{1-y^2}}\Big) = \\
&= \frac{1}{J}\Big(\frac{1+y}{2}\ln(1+y) + \frac{1-y}{2}\ln(1-y) - y\,h - \ln 2\Big) = \\
&= \frac{1}{J}\Big(\frac{1+y}{2}\ln\frac{1+y}{2} + \frac{1-y}{2}\ln\frac{1-y}{2} - yh\Big),
\end{aligned}$$

which is the required result.

□

We can similarly verify that the Legendre transform of $g(x) = -\frac{1}{2}x^2$ is given by the function $g^*(x) = \frac{1}{2}x^2$.

This way we see that we can write the bounding functions as:

$$\begin{aligned}
p_{up}(\bar{m}) = p_{low}(\bar{m}) &= J(f(\bar{m}) - g(\bar{m})), \\
p'_{low}(\bar{m}) &= J(g^*(\bar{m}) - f^*(\bar{m})).
\end{aligned} \tag{3.26}$$

and the following proposition tells us that all of the bounds that we have found coincide.

Proposition 11. *Let f and g be two convex functions and f^* and g^* be their Legendre transforms. Then the following is true:*

$$\sup_x f(x) - g(x) = \sup_y g^*(y) - f^*(y)$$

Proof. For a nice proof see [22], or the appendix in [40].

□

The last proposition tells us that both the variational principles we have derived provide the correct value for the thermodynamic pressure, and so the results of this section can be summarised in the following

Theorem 1. *Given a hamiltonian as defined in (3.3), and defining the pressure per particle* *as* $p_N = \dfrac{1}{N} \ln Z$*, given parameters* J *and* h*, the thermodynamic limit*

$$\lim_{N\to\infty} p_N = p$$

of the pressure exists, and can be expressed in one of the following equivalent forms:

a) $p = \sup_{\bar{m}} \; p_{up}(\bar{m}) = \sup_{\bar{m}} \; p_{low}(\bar{m})$

b) $p = \sup_{\bar{m}} \; p'_{low}(\bar{m})$

3.5 Consistency equation

In the last section we have expressed the thermodynamic pressure of the Curie-Weiss model as the supremum of two distinct functions. Indeed, more can be said about this variational principle, since even the argument of the supremum has a very important meaning: we shall see in this section that, in case there is a unique supremum for $p_{up} = p_{low}$ or p'_{low}, its argument gives the thermodynamic value of the magnetization. If there exists more than one supremum, we have a phase transition, and each argument gives a pure state for the magnetization.

First, we point out the straight-forward fact that stationary points of both $p_{up} = p_{low}$ and p'_{low} satisfy the condition:

$$\bar{m}^* = \tanh(J\bar{m}^* + h), \tag{3.27}$$

which can be found in the literature as *consistency equation*, *mean field equation*, *state equation*, *secularity equation*, and other names, depending on the context.

This equation is indeed important: since the bounding functions are smooth, and since it can be easily seen by checking derivatives that none of the admit suprema at the boundary of $[-1, 1]$, we have as a consequence that any supremum of the function satisfies this equation. It is also interesting to notice that the trivial fact that this equation has always a solution inside $[-1, 1]$ can be also seen as a consequence of the existence results of Section 3.2.

Proposition 12. *Let* J *and* h *be given so that* $p_{up} = p_{low}$ *has a unique supremum, which is attained at* $\bar{m}^*$*. Then* $\bar{m}^* = \lim_{N\to\infty} \omega_N(m) = \lim_{N\to\infty} \omega_N(\sigma_i)$*.*

Proof. The following holds at finite N, by definition of the pressure $p_N(J, h)$:

$$\frac{\partial p_N}{\partial h} = \omega_N(m_N).$$

We have proved that $\{p_N\}$ is a convergent sequence of functions which are convex (for a proof of the convexity of the pressure see [32], where convexity is proved for the free-energy in the Ising model, which is essentially the same as the pressure multiplied by -1). This implies that the limit function is also convex, and as such it is differentiable almost everywhere. As a consequence we have the following:

$$\lim_{N\to\infty} \omega_N(m) = \lim_{N\to\infty} \frac{\partial p_N}{\partial h} = \frac{\partial \sup_{\bar{m}} p_{low}}{\partial h}$$

whenever the last derivative exists (for a proof that the limit of the derivatives coincides with the derivative of the limit in this case see [22] pag. 114).

Therefore if we write $\lim_{N\to\infty} p_N = p(J, h, \bar{m}^*(J, h))$, we can write the following:

$$\frac{\partial \sup_{\bar{m}} p_{low}}{\partial h} = \frac{\partial p(J, h, \bar{m}^*(J, h)}{\partial h} = -J\frac{\partial \bar{m}^*}{\partial h}\bar{m}^* + \tanh(J\bar{m}^* + h) + J\frac{\partial \bar{m}^*}{\partial h}\tanh(J\bar{m}^* + h),$$

and by substituting (3.27) we get

$$\frac{\partial \sup_{\bar{m}} p_{low}}{\partial h} = \bar{m}^*,$$

which is our result.

□

A similar proposition can be proved analogously for p'_{low}. Let us now write

$$\omega(m) = \lim_{N\to\infty} \omega_N(m) \quad \text{and} \quad \omega(\sigma_i) = \lim_{N\to\infty} \omega_N(\sigma_i).$$

As a consequence of Proposition 12 we have that we can write

$$p'_{low}(\bar{m}^*) = S - U$$

where

$$S = -\frac{1 + \omega(\sigma_i)}{2} \ln\Big(\frac{1 + \omega(\sigma_i)}{2}\Big) - \frac{1 - \omega(\sigma_i)}{2} \ln\Big(\frac{1 - \omega(\sigma_i)}{2}\Big)$$

is the thermodynamic entropy and

$$U = \frac{J}{2}\omega(m)^2 + h\omega(m)$$

is the thermodynamic internal energy, as can be derived directly from the Gibbs distribution.

3.6 A heuristic approach

We shall now describe a heuristic procedure to obtain the consistency equation 3.27. First of all, we make the following observation about the Gibbs average $\omega_N(\sigma_N)$ of the magnetization:

$$\omega_N(m) = \omega_N(\sigma_1) = \frac{1}{Z_N} \sum_{\sigma\in\{-1,1\}^N} \sigma_1 e^{-H(\sigma)}$$

We now define the following Hamiltonian $\tilde{H}_N$:

$$\tilde{H}_N = -\frac{J}{2(N+1)} \sum_{i,j=1}^{N} \sigma_i\sigma_j - h\sum_{i=1}^{N} \sigma_i,$$

and its associated partition function

$$\tilde{Z}_N = \sum_{\sigma\in\{-1,1\}} e^{-\tilde{H}_N},$$

which allows us to write:

$$\begin{aligned}
\omega_N(\sigma_1) &= \frac{\sum_{\sigma\in\{-1,1\}^N} \sigma_1 e^{\frac{J}{N}\sum_{i=1}^N \sigma_i\sigma_N + h\sigma_N} e^{-\tilde{H}_{N-1}(\sigma)}}{\sum_{\sigma\in\{-1,1\}^N} e^{\frac{J}{N}\sum_{i=1}^N \sigma_i\sigma_N + h\sigma_N} e^{-\tilde{H}_{N-1}(\sigma)}} = \\
&= \frac{\tilde{Z}_N \sum_{\sigma\in\{-1,1\}^N} \sigma_1 e^{\frac{J}{N}\sum_{i=1}^N \sigma_i\sigma_N + h\sigma_N} e^{-\tilde{H}_{N-1}(\sigma)}}{\tilde{Z}_N \sum_{\sigma\in\{-1,1\}^N} e^{\frac{J}{N}\sum_{i=1}^N \sigma_i\sigma_N + h\sigma_N} e^{-\tilde{H}_{N-1}(\sigma)}} = \\
&= \frac{\tilde{Z}_N \sum_{\sigma\in\{-1,1\}^{N-1}} \sigma_1 \sinh(\frac{J}{N}\sum_{i=1}^{N-1} \sigma_i + h + \frac{J}{N}) e^{-\tilde{H}_{N-1}(\sigma)}}{\tilde{Z}_N \sum_{\sigma\in\{-1,1\}^{N-1}} \cosh(\frac{J}{N}\sum_{i=1}^{N-1} \sigma_i + h + \frac{J}{N}) e^{-\tilde{H}_{N-1}(\sigma)}} = \\
&= \frac{\tilde{\omega}_N(\sinh(\frac{J}{N}\sum_{i=1}^{N-1} \sigma_i + h + \frac{J}{N}))}{\tilde{\omega}_N(\cosh(\frac{J}{N}\sum_{i=1}^{N-1} \sigma_i + h + \frac{J}{N}))}
\end{aligned}$$

Now, if we assume that the last line implies

$$\lim_{N\to\infty} \omega_N(\sigma_i) = \lim_{N\to\infty} \frac{\omega_N(\sinh(Jm+h))}{\omega_N(\cosh(Jm+h))} \tag{3.28}$$

we can use the factorization properties of the model in order to derive the following.

Let us consider $\omega_N(\sinh(Jm+h))$, and write it by making the power series at the argument explicit:

$$\omega_N(\sinh(Jm+h)) = \omega_N\Big(\sum_{k=0}^{\infty}\frac{(Jm+h)^k}{(2k+1)!}\Big)$$

Now, if we consider only a partial sum up to n at the argument of the Gibbs state, and take the thermodynamic limit, the self-averaging property of the magnetization tells us that the following holds a.e. in J and h:

$$\begin{aligned}
\lim_{N\to\infty}\omega_N\Big(\sum_{k=0}^{n}\frac{(Jm+h)^k}{(2k+1)!}\Big) &= \lim_{N\to\infty}\omega_N\Big(\sum_{k=0}^{n}\frac{1}{(2k+1)!}\sum_{l=0}^{k}\binom{k}{l}(Jm)^l h^{k-l}\Big) = \\
&= \lim_{N\to\infty}\sum_{k=0}^{n}\frac{1}{(2k+1)!}\sum_{l=0}^{k}\binom{k}{l}J^l\omega_N(m)^l h^{k-l}\Big) = \\
&= \lim_{N\to\infty}\sum_{k=0}^{n}\frac{(J\omega_N(m)+h)^k}{(2k+1)!}
\end{aligned}$$

Now, disregarding convergence problems, the limit of (3.29) together with the assumption (3.28) give the following equation:

$$\bar{m}^* = \tanh(J\bar{m}^* + h),$$

where $\bar{m}^*$ is the thermodynamic magnetization. This way we have derived heuristically the consistency equation describing the most important quantity for our model just by making use of the model's factorization properties.

It is important, however, to stress that the procedure we proposed in this section is not mathematically rigorous: assumption (3.28), though sensible, hasn't been derived rigorously, and the possible convergent problems have not been considered. Nevertheless, since the procedure has provided the right answer which we have derived rigorously throughout the chapter, and since it consists simple considerations, it can be see as a way of approaching models defined on random networks instead that on the complete graph, which are not as well understood as the one treated in this chapter.

Chapter 4

The Curie-Weiss model for many populations

In this chapter we consider the problem of characterizing the equilibrium statistical mechanics of an mean field interacting system partitioned into p sets of spins. The relevance of such a problem to social modelling is that such a partition can be made to correspond to the partition into classes of people sharing the same socio-economics attributes, as described in chapter 2.

Our results can be summarised as follows. After introducing the model we show in section 3 that it is well posed by showing that its thermodynamic limit exists. The result is non-trivial, since sub-additivity is not met at finite volume. In section 4 we show that the system fulfills a factorization property for the correlation functions which reduces the equilibrium state to only n degrees of freedom the equilibrium state. The method is conceptually similar to the one developed by Guerra in [35] to derive identities for the overlap distributions in the Sherrington and Kirkpatrick model.

We also derive the pressure of the model by rigorous methods developed in the recent study of mean field spin glasses (see [37] for a review). It is interesting to notice that though very simple, our model encompasses a range of regimes that do not admit solution by the elegant interpolation method used in the celebrated existence result of the Sherrington and Kirkpatrick model [36]. This is due to the lack of positivity of the quadratic form describing the considered interaction. Nevertheless we are able to solve the model exactly in section 4.4, using the lower bound provided by the Gibbs variational principle, and thanks to a further bound given by a partitioning of the configuration space, itself originally devised in the study of spin glasses (see [37, 19, 38]).

As in the classical Curie-Weiss model, the exact solution is provided in an implicit form;

for our system, however, we find a system of equations of state, which are coupled as well as trascendental, and this makes the full characterization of all the possible regimes highly non-trivial. A simple analytic result about the number of solutions for the two-population case is proved in section 4.5.

4.1 The Model

We can generalize the Curie-Weiss model to p-populations, allowing r-body interactions with $r = 1..p$. This gives rise to the following Hamiltonian:

$$H_N = -N \sum_{r=1}^{p} \sum_{i_1,\dots,i_r=1}^{p} J_{i_1,\dots,i_r} \prod_{k=1}^{r} m_{i_k}, \tag{4.1}$$

or, equivalently, to the following Utility function for individual i:

$$U_i = \sum_{r=1}^{p} \sum_{i_1,\dots,i_{r-1}=1}^{p} J_{i_1,\dots,i_{r-1},i} \prod_{k=1}^{r-1} m_{i_k}.$$

Here $J_{i_1,\dots,i_r}$ gives the interaction coefficients corresponding to the r-body interaction among individuals coming from populations $i_1, ..., i_r$, respectively. We can also consider the external fields to be already included in this form of the model, just by setting $J_i = h_i$. So we have defined interactions by using a tensor $J_{i_1,\dots,i_r}$ of rank r for each of the r-body interactions.

4.2 Existence of the thermodynamic limit for many populations

We shall prove that our model admits a thermodynamic limit by exploiting an existence theorem provided for mean field models in [8]: the result states that the existence of the pressure per particle for large volumes is guaranteed by a monotonicity condition on the equilibrium state of the Hamiltonian. Such a result proves to be quite useful when the condition of convexity introduced by the interpolation method [36, 37] doesn't apply due to lack of positivity of the quadratic form representing the interactions. We therefore prove the existence of the thermodynamic limit independently of an exact solution. Such a line of enquiry is pursued in view of further refinements of our model, that shall possibly involve random interactions of spin glass or random graph type, and that might or might not come with an exact expression for the pressure.

Proposition 13. *There exists a function p of all the parameters $J_{i_1,\dots,i_r}$ such that*

$$\lim_{N\to\infty} p_N = p\ .$$

The previous proposition is proved with a series of lemmas. Theorem 1 in [8] states that given a Hamiltonian H_N and its associated equilibrium state ω_N the model admits a thermodynamic limit whenever the physical condition

$$\omega_N(H_N) \geqslant \omega_N(H_{N_1}) + \omega_N(H_{N_2}), \qquad N_1 + N_2 = N, \tag{4.2}$$

is verified.

We proceed by first verifying this condition for an alternative Hamiltonian $\tilde{H}_N$, and then showing that its pressure $\tilde{p}_N$ tends to our original pressure p_N as N increases. We choose $\tilde{H}_N$ in such a way that the condition (4.2) is verified as an equality.

Now, define the alternative Hamiltonian $\tilde{H}_N$ as follows:

$$\tilde{H}_N = -C\,N \prod_{l=1}^{p} \frac{(N_{i_l}-k_l)!}{N_{i_l}!} \sum_{\substack{j_k = N_{i_{k-1}}+1,\dots,N_{i_k} \\ j_k \neq j_h \text{ for } k\neq h}} \sigma_{j_1}...\sigma_{j_r}$$

where C is a real number.

Though the notation is cumbersome at this point, the new Hamiltonian simply considers products of r distinct spins, k_i of which are taken from population i (i.e. $\sum_{i=1}^{p} k_i = r$) and so the combinatorial coefficient is just dividing the sum by the correct number of terms contained in the sum itself.

Lemma 1. *There exists a function $\tilde{p}$ such that*

$$\lim_{N\to\infty} \tilde{p}_N = \tilde{p}$$

Proof. By linearity we have that

$$\omega_N(\tilde{H}_N) = -C\,N \prod_{l=1}^{p} \frac{(N_{i_l}-k_l)!}{N_{i_l}!} \sum_{\substack{j_k = N_{i_{k-1}}+1,\dots,N_{i_k} \\ j_k \neq j_h \text{ for } k\neq h}} \omega_N(\sigma_{j_1}...\sigma_{j_r}) = -C\,N\,\omega_N(\sigma_{j_1}...\sigma_{j_r}), \tag{4.3}$$

where, with a little abuse of notation, we let $\sigma_{j_1},..,\sigma_{j_r}$, after the last equality be distinct spins taken from their own respective populations. The last equality hence follows from the invariance of $\tilde{H}_N$ with respect to permutations of spins belonging to the same population.

Equation (4.2) implies trivially

$$\omega_N(\tilde{H}_N - \tilde{H}_{N_1} - \tilde{H}_{N_2}) = 0$$

for $N_1 + N_2 = N$, which verifies (4.2) as an equality.

□

The following two Lemmas show that the difference between H_N and $\tilde{H}_N$ is thermodynamically negligible and as a consequence their pressures coincide in the thermodynamic limit.

Though the notation is quite tedious, the proof is in no way different from the one described in [8]. We chose to keep full generality during this existence proof in order to show that the mean-field allows one to consider a whole range of possibilities for interaction, which might turn out useful for the modelling effort.

Lemma 2.

$$H_N = \tilde{H}_N + O(1) \tag{4.4}$$

i.e.

$$\lim_{N\to\infty} \frac{H_N}{N} = \lim_{N\to\infty} \frac{\tilde{H}_N}{N}$$

Proof. We begin the proof by rephrasing the Hamiltonian in term of the spins, as follows:

$$\begin{aligned}
H_N &= -N\sum_{r=1}^{p}\sum_{i_1,\ldots,i_r=1}^{p}\left\{J_{i_1,\ldots,i_r}\prod_{k=1}^{r}\frac{N_{i_k}}{N_{i_k}}m_{i_k}\right\} = \\
&= -\sum_{r=1}^{p}\left\{\sum_{i_1,\ldots,i_r=1}^{p} N\frac{N^r}{N^r}\prod_{k=1}^{r}\frac{1}{N_{i_k}}J_{i_1,\ldots,i_r}\prod_{k=1}^{r}N_{i_k}m_{i_k}\right\} = \\
&= -\sum_{r=1}^{p}\sum_{i_1,\ldots,i_r=1}^{p}\left\{\frac{1}{N^{r-1}}\prod_{k=1}^{r}\frac{1}{\alpha_{i_k}}J_{i_1,\ldots,i_r}\sum_{j_k=N_{i_k-1}+1,\ldots,N_{i_k}}\sigma_{j_1}\ldots\sigma_{j_r}\right\} =
\end{aligned}$$

where

$$N = \sum_{i=1}^{p} N_i, \quad \alpha_i = \frac{N_i}{N}, \quad N_0 = 0.$$

We only need to give details of the proof in the case only one of the coefficients $J_{i_1,\ldots,i_r} \neq 0$. The general case follows by summing up all the terms corresponding to non-zero interacting coefficients and noticing that, since this sum has only finitely many terms, the result still holds.

So we consider the following Hamiltonian

$$H_N = -NJ_{i_1,\dots,i_r}\prod_{k=1}^{r} m_{i_k} = \frac{1}{N^{r-1}}\prod_{k=1}^{r}\frac{1}{\alpha_{i_k}}J_{i_1,\dots,i_r}\sum_{j_k=N_{i_k-1}+1,\dots,N_{i_k}}\sigma_{j_1}\dots\sigma_{j_r},$$

and we can lighten our notation by setting $C=\frac{1}{\alpha_{i_k}}J_{i_1,\dots,i_r}$,

$$H_N = \frac{C}{N^{r-1}}\sum_{j_k=N_{i_k-1}+1,\dots,N_{i_k}}\sigma_{j_1}\dots\sigma_{j_r}.$$

Now, following [8] we divide the sum in two parts, as follows:

$$H_N = \frac{C}{N^{r-1}}\sum_{\substack{j_k=N_{i_k-1}+1,\dots,N_{i_k}\\ j_k\neq j_l \text{ for } k\neq l}}\sigma_{j_1}\dots\sigma_{j_r} + \frac{C}{N^{r-1}}\sum{}^{*}\sigma_{j_1}\dots\sigma_{j_r}.$$

The first part is a sums only over products of distinct spins, whereas $\sum^*$ is a sum of all products where at least two spins are equal. It is straightforward to show that

$$\frac{C}{N^{r-1}}\sum{}^{*}\sigma_{j_1}\dots\sigma_{j_r} = O(1),$$

so that we can rewrite H_N as follows:

$$H_N = \frac{C}{N^{r-1}}\sum_{\substack{j_k=N_{i_k-1}+1,\dots,N_{i_k}\\ j_k\neq j_l \text{ for } k\neq l}}\sigma_{j_1}\dots\sigma_{j_r} + O(1).$$

A straightforward calculation comparing H_N and $\tilde{H}_N$ can now check that

$$H_N = \tilde{H}_N + O(1),$$

which is our result.

□

Lemma 3. *Say* $p_N = \dfrac{1}{N}\ln Z_N$*, and say* $h_N(\sigma) = \dfrac{H_N(\sigma)}{N}$*. Define* $\tilde{Z}$*,* $\tilde{p}_N$ *and* $\tilde{h}_N$ *in an analogous way.*

Define

$$k_N = \|h_N - \tilde{h}_N\| = \sup_{\sigma\in\{-1,+1\}^N}\{|h_N(\sigma)-\tilde{h}_N(\sigma)|\} < \infty. \tag{4.5}$$

Then

$$|p_N - \tilde{p}_N| \leqslant \|h_N - \tilde{h}_N\| \, .$$

Proof.

$$\begin{aligned}
p_N - \tilde{p}_N &= \frac{1}{N}\ln Z_N - \frac{1}{N}\ln \tilde{Z}_N = \frac{1}{N}\ln\frac{Z_N}{\tilde{Z}_N} \\
&= \frac{1}{N}\ln\frac{\sum_\sigma e^{-H_N(\sigma)}}{\sum_\sigma e^{-\tilde{H}_N(\sigma)}} \leqslant \frac{1}{N}\ln\frac{\sum_\sigma e^{-H_N(\sigma)}}{\sum_\sigma e^{-N(h_N(\sigma)+k_N)}} = \\
&= \frac{1}{N}\ln\frac{\sum_\sigma e^{-H_N(\sigma)}}{e^{-Nk_N}\sum_\sigma e^{-Nh_N(\sigma)}} = \frac{1}{N}\ln e^{Nk_N} = k_N = \|h_N - \tilde{h}_N\|
\end{aligned}$$

where the inequality follows from the definition of k_N in (4.5) and from monotonicity of the exponential and logarithmic functions. The inequality for $\tilde{p}_N - p_N$ is obtained in a similar fashion. □

We are now ready to prove the main result for this section:

Proof of Proposition 13: The existence of the thermodynamic limit follows from our Lemmas. Indeed, since by Lemma 1 the limit for $\tilde{p}_N$ exists, Lemma 3 and Lemma 2 tell us that

$$\lim_{N\to\infty}|p_N - \tilde{p}_N| \leqslant \lim_{N\to\infty}\|h_N - \tilde{h}_N\| = 0,$$

implying our result.

□

4.3 Factorization properties

From now on we shall restrict the model to include pair interactions only. Therefore, we have a Hamiltonian of the following kind:

$$H_N = -N\sum_{i,j=1}^{p}\frac{J_{i,j}}{2}m_i m_j - N\sum_{i=1}^{p}h_i m_i, \tag{4.6}$$

In this section we shall prove that the correlation functions of our model factorize completely in the thermodynamic limit, for almost every choice of parameters. This implies that all the thermodynamic properties of the system can be described by the magnetizations m_i of the p populations defined in Section 4.1. Indeed, the exact solution of the model, to be derived in the next section, comes as p coupled equations of state for the m_i.

Proposition 14.

$$\lim_{N\to\infty}\big(\omega_N(\sigma_i\sigma_j)-\omega_N(\sigma_i)\omega_N(\sigma_j)\big)=0$$

for almost every choice of parameters, where σ_i, σ_j are any two distinct spins in the system.

Proof. We recall the definition of the Hamiltonian

$$H_N=-N\sum_{i,j=1}^{p}J_{i,j}m_im_j-N\sum_{i=1}^{p}h_im_i,$$

and of the pressure per particle

$$p_N=\frac{1}{N}\ln\sum_{\sigma}e^{-H_N(\sigma)}.$$

By taking first and second partial derivatives of p_N with respect to h_i we get

$$\frac{\partial p_N}{\partial h_i}=\frac{1}{N}\sum_{\sigma}Nm_i(\sigma)\frac{e^{-H(\sigma)}}{Z_N}=\omega_N(m_i),\qquad \frac{\partial^2 p_N}{\partial\, h_i^2}=N(\omega_N(m_i^2)-\omega_N(m_i)^2).$$

By using these relations we can bound above the integral with respect to h_i of the fluctuations of m_i in the Gibbs state:

$$\begin{aligned}\left|\int_{h_i^{(1)}}^{h_i^{(2)}}(\omega_N(m_i^2)-\omega_N(m_i)^2)\,dh_i\right| &= \frac{1}{N}\left|\int_{h_i^{(1)}}^{h_i^{(2)}}\frac{\partial^2 p_N}{\partial h_i^2}\,dh_i\right|=\frac{1}{N}\left|\int_{h_i^{(1)}}^{h_i^{(2)}}\frac{\partial p_N}{\partial\, h_i}\Big|_{h_i^{(1)}}^{h_i^{(2)}}\right|\leqslant\\ &\leqslant \frac{1}{N}\big(\big|\omega_N(m_i)|_{h_i^{(2)}}\big|+\big|\omega_N(m_i)|_{h_i^{(1)}}\big|\big)=O\big(\frac{1}{N}\big).\end{aligned}\tag{4.7}$$

On the other hand we have that

$$\omega_N(m_i)=\frac{\partial p_N}{\partial h_i},$$

and

$$\omega_N(m_i^2)=2\frac{\partial p_N}{\partial J_{i,i}},$$

so, by convexity of the thermodynamic pressure $p=\lim_{N\to\infty}p_N$, both quantities $\frac{\partial p_N}{\partial h_i}$ and $\frac{\partial p_N}{\partial J_{i,i}}$ have well defined thermodynamic limits almost everywhere. This together with (4.7) implies that

$$\lim_{N\to\infty}(\omega_N(m_i^2)-\omega_N(m_i)^2)=0\quad\text{a.e. in } h_i,\ J_{i,i}.\tag{4.8}$$

In order to prove our statement we shall write the magnetization m_i in terms of spins belonging to the i^{th} population, and then use the permutation invariance of the Gibbs measure:

$$
\begin{aligned}
\omega_N(m_i) &= \omega_N(\frac{1}{N_i - N_{i-1}} \sum_{j=N_{i-1}}^{N_i} \sigma_i) = \omega_N(\sigma_1), \\
\omega_N(m_i^2) &= \omega_N(\frac{1}{(N_i - N_{i-1})^2} \sum_{j,\ l=N_{i-1}}^{N_i} \sigma_j \sigma_l) = \\
&= \omega_N(\frac{1}{(N_i - N_{i-1})^2} \sum_{j \neq l=N_{i-1}}^{N_i} \sigma_j \sigma_l) + \omega_N(\frac{1}{(N_i - N_{i-1})^2} \sum_{j=l=N_{j-1}}^{N_j} \sigma_j \sigma_l) = \\
&= \frac{N_i - N_{i-1} - 1}{N_i - N_{i-1}} \omega_N(\sigma_1 \sigma_2) + \frac{1}{N_i - N_{i-1}}.
\end{aligned}
\tag{4.9}
$$

We have that (4.9) and (4.8) imply

$$
\lim_{N \to \infty} \omega_N(\sigma_i \sigma_j) - \omega_N(\sigma_i)\omega_N(\sigma_j) = 0, \tag{4.10}
$$

which verifies our statement for all couples of spins $i \neq j$ belonging to the same population.

Furthermore, by defining $\mathrm{Var}_N(m_i) = \left(\omega_N(m_i^2) - \omega_N(m_i)^2\right)$ for all populations i, we exploit (4.8), and use the Cauchy-Schwartz inequality to get

$$
|\omega_N(m_i m_j) - \omega_N(m_i)\omega_N(m_j)| \leqslant \sqrt{\mathrm{Var}(m_i)\mathrm{Var}(m_j)} \underset{N \to \infty}{\longrightarrow} 0 \quad \text{a.e. in } J_{i,i},\ J_{j,j},\ h_i,\ h_j \tag{4.11}
$$

By using (4.9) and (4.11) we can therefore verify statements which are analogous to (4.10), but which concern $\omega_N(\sigma_i \sigma_j)$ where σ_i and σ_j are spins belonging to different subsets.

We have thus proved our claim for any couple of spins in the global system.

□

4.4 Solution of the model

We shall derive upper and lower bounds for the thermodynamic limit of the pressure. The lower bound is obtained through the standard entropic variational principle, while the upper bound is derived by a decoupling strategy.

4.4.1 Upper bound

In order to find an upper bound for the pressure we shall divide the configuration space into a partition of microstates of equal magnetization, following [19, 37, 38]. Since each population g consists of N_g spins, its magnetization can take exactly $N_g + 1$ values, which are the elements of the set

$$R_{N_g} = \Big\{ -1, -1 + \frac{1}{2N_g}, \dots, 1 - \frac{1}{2N_g}, 1 \Big\}.$$

Clearly for every $m_g(\sigma)$ we have that

$$\sum_{\bar{m}_g \in R_{N_g}} \delta_{m_g, \bar{m}_g} = 1,$$

where $\delta_{x,y}$ is a Kronecker delta. This allows us to rewrite the partition function as follows:

$$\begin{aligned} Z_N &= \sum_\sigma \exp\big\{ \frac{N}{2} \sum_{i,j=1}^p J_{i,j} m_i m_j + N \sum_{i=1}^p h_i m_i \big\} = \\ &= \sum_\sigma \sum_{\forall g} \sum_{\bar{m}_g \in R_{N_g}} \prod_{g=1}^p \delta_{m_g, \bar{m}_g} \exp\big\{ \frac{N}{2} \sum_{i,j=1}^p J_{i,j} m_i m_j + N \sum_{i=1}^p h_i m_i \big\}. \end{aligned} \tag{4.12}$$

Thanks to the Kronecker delta symbols, we can substitute m_i (the average of the spins within a configuration) with the parameter $\bar{m}_i$ (which is not coupled to the spin configurations) in any convenient fashion.

Therefore we can use the following relations in order to linearize all quadratic terms appearing in the Hamiltonian

$$\begin{aligned} (m_i - \bar{m}_i)^2 &= 0 \ \forall i, \\ (m_i - \bar{m}_i)(m_j - \bar{m}_j) &= 0 \ \forall i \neq j, . \end{aligned}$$

Once we've carried out these substitutions into (4.12) we are left with a function which depends only linearly on the m_i:

$$\begin{aligned} Z_N &= \sum_{\sigma} \sum_{\forall g\ \bar{m}_g \in R_{N_g}} \prod_{g=1}^{p} \delta_{m_g,\bar{m}_g} \exp\big\{ \frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} m_i m_j + N \sum_{i=1}^{p} h_i m_i \big\} = \\ &= \sum_{\sigma} \sum_{\forall g\ \bar{m}_g \in R_{N_g}} \prod_{g=1}^{p} \delta_{m_g,\bar{m}_g} \exp\big\{ \frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} (m_i \bar{m}_j + \bar{m}_i m_j - \bar{m}_i \bar{m}_j) + N \sum_{i=1}^{p} h_i m_i \big\} = \\ &= \sum_{\sigma} \sum_{\forall g\ \bar{m}_g \in R_{N_g}} \prod_{g=1}^{p} \delta_{m_g,\bar{m}_g} \exp\big\{ -\frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} \bar{m}_i \bar{m}_j + \frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} (m_i \bar{m}_j + \bar{m}_i m_j) + \\ &\quad + N \sum_{i=1}^{p} h_i m_i \big\} = \end{aligned}$$

and bounding above the Kronecker deltas by 1 we get

$$Z_N \leqslant \sum_{\sigma} \sum_{\forall g\ \bar{m}_g \in R_{N_g}} \exp\big\{ \frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} \bar{m}_i \bar{m}_j + \frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} (m_i \bar{m}_j + \bar{m}_i m_j) + N \sum_{i=1}^{p} h_i m_i \big\} = \tag{4.13}$$

As observed many times by Guerra [37], since both sums are taken over finitely many terms, it is possible to exchange the order of the two summation symbols, in order to carry out the sum over the spin configurations, which now factorizes, thanks to the linearity of the interaction with respect to the m_g. This way we get:

$$Z_N \leqslant \sum_{\forall g\ \bar{m}_g \in R_{N_g}} G(\bar{m}_1, ..., \bar{m}_p).$$

where

$$G = \exp\big\{ -\frac{N}{2} \sum_{i,j=1}^{p} J_{i,j} \bar{m}_i \bar{m}_j \big\} \cdot \prod_{j=1}^{p} 2^{N_j} \big(\cosh\big(\sum_{i=1}^{p} \frac{J_{i,j} + J_{j,i}}{2\alpha_j} \bar{m}_i + \frac{h_j}{\alpha_j} \big) \big)^{N_j} \tag{4.14}$$

where

$$\alpha_j = \frac{N_j}{N}$$

Since the summation is taken over the ranges R_{N_g}, of cardinality $N_g + 1$, we get that the total number of terms is $(N_1+1)(N_2+1)$. Therefore

$$Z_N \leqslant \prod_{g=1}^{p}(N_g+1) \sup_{\bar{m}_1,\dots,\bar{m}_p} G, \tag{4.15}$$

which leads to the following upper bound for p_N:

$$p_N = \frac{1}{N}\ln Z_N \leqslant \sum_{g=1}^{p}\frac{1}{N}\ln(N_g+1) + \frac{1}{N}\ln \sup_{\bar{m}_1,\dots,\bar{m}_p} G\ . \tag{4.16}$$

Now defining the N independent function

$$p_{UP} = \frac{1}{N}\ln G = \ln 2 - \frac{1}{2}\sum_{i,j=1}^{p} J_{i,j}\bar{m}_i\bar{m}_j + \sum_{j=1}^{p}\alpha_j \ln\cosh\Big(\sum_{i=1}^{p}\frac{J_{i,j}+J_{j,i}}{2\alpha_j}\,\bar{m}_i + \frac{h_j}{\alpha_j}\Big), \tag{4.17}$$

where

$$\alpha_j = \frac{N_j}{N}$$

the thermodynamic limit gives:

$$\limsup_{N\to\infty} p_N \leqslant \sup_{\bar{m}_1,\dots,\ \bar{m}_p} p_{UP}. \tag{4.18}$$

We can summarize the previous computation into the following:

Lemma 4. *Given a Hamiltonian as defined in (4.6), and defining the pressure per particle as $p_N = \frac{1}{N}\ln Z$, given parameters $J_{i,j}$ and h_i, the following inequality holds:*

$$\limsup_{N\to\infty} p_N \leqslant \sup_{\bar{m}_1,\dots,\ \bar{m}_p} p_{UP}$$

where

$$p_{UP} = \ln 2 - \frac{1}{2}\sum_{i,j=1}^{p} J_{i,j}\bar{m}_i\bar{m}_j + \sum_{j=1}^{p}\alpha_j \ln\cosh\Big(\sum_{i=1}^{p}\frac{J_{i,j}+J_{j,i}}{2\,\alpha_j}\,\bar{m}_i + \frac{h_j}{\alpha_j}\Big), \tag{4.19}$$

and $\bar{m}_i \in [-1,1]$.

4.4.2 Lower bound

The lower bound is provided by exploiting the well-known Gibbs entropic variational principle (see [58], pag. 188). In our case, instead of considering the whole space of *ansatz* probability distributions considered in [58], we shall restrict to a much smaller one, and use the upper bound derived in the last section in order to show that the lower bound corresponding to the restricted space is sharp in the thermodynamic limit.

The mean-field nature of our Hamiltonian allows us to restrict the variational problem to a two-degrees of freedom product measures represented through the non-interacting Hamiltonian:

$$\tilde{H} = -r_1 \sum_{i=1}^{N_1} \sigma_i - r_2 \sum_{i=N_1+1}^{N_1+N_2} \sigma_i + ... - r_p \sum_{i=\sum_{i=1}^{p-1} N_i+1}^{N} \sigma_i,$$

and so, given a Hamiltonian $\tilde{H}$, we define the ansatz Gibbs state corresponding to it as $f(\sigma)$ as:

$$\tilde{\omega}(f) = \frac{\sum_\sigma f(\sigma) e^{-\tilde{H}(\sigma)}}{\sum_\sigma e^{-\tilde{H}(\sigma)}}$$

In order to facilitate our task, we shall express the variational principle of [58] in the following simple form:

Proposition 15. *Let a Hamiltonian H, and its associated partition function $Z = \sum_\sigma e^{-H}$ be given. Consider an arbitrary trial Hamiltonian $\tilde{H}$ and its associated partition function $\tilde{Z}$. The following inequality holds:*

$$\ln Z \geqslant \ln \tilde{Z} - \tilde{\omega}(H) + \tilde{\omega}(\tilde{H}) \; . \tag{4.20}$$

Given a Hamiltonian as defined in (5.1) and its associated pressure per particle $p_N = \frac{1}{N} \ln Z$, the following inequality follows from (4.20)*:*

$$\liminf_{N\to\infty} p_N \geqslant \sup_{\bar{m}_1,\dots,\bar{m}_p} p_{LOW} \tag{4.21}$$

where

$$p_{LOW} \;\; = \;\; \frac{1}{2} \sum_{g,k=1}^{p} J_{g,k} \bar{m}_g \bar{m}_k + \sum_{g=1}^{p} h_g \bar{m}_g + \sum_{g=1}^{p} \alpha_g S(\bar{m}_g), \tag{4.22}$$

the function $S(\bar{m}_g)$ being the entropy

$$S(\bar{m}_g) = -\frac{1+\bar{m}_g}{2} \ln(\frac{1+\bar{m}_g}{2}) - \frac{1-\bar{m}_g}{2} \ln(\frac{1-\bar{m}_g}{2})$$

and $\bar{m}_g \in [-1, 1]$.

Proof. The (4.20) follows straightforwardly from Jensen's inequality:

$$e^{\tilde{\omega}(-H+\tilde{H})} \leq \tilde{\omega}(e^{-H+\tilde{H}}) \,. \tag{4.23}$$

The Hamiltonian (4.6) can be written in term of spins as:

$$H(\sigma) = -\frac{1}{2N}\sum_{g,k=1}^{p}\Big\{\frac{J_{g,k}}{\alpha_g\alpha_k}\sum_{i\in P_g,\ j\in P_k}\sigma_i\sigma_j\Big\} - \sum_{g=1}^{p}\{\frac{h_g}{\alpha_g}\sum_{i\in P_g}\sigma_i\},\ ; \tag{4.24}$$

where P_g contains the labels for spins belonging to the g^{th} subpopulation, that is

$$P_g = \{\sum_{k=1}^{g-1}N_k+1, \sum_{k=1}^{g-1}N_k+2, ..., \sum_{k=1}^{g}N_k\}$$

indeed its expectation on the trial state is

$$\tilde{\omega}(H) = -\frac{1}{2N}\sum_{g,k=1}^{p}\Big\{\frac{J_{g,k}}{\alpha_g\alpha_k}\sum_{i\in P_g,\ j\in P_k}\tilde{\omega}(\sigma_i\sigma_j)\Big\} - \sum_{g=1}^{p}\{\frac{h_g}{\alpha_g}\sum_{i\in P_g}\tilde{\omega}(\sigma_i)\} \tag{4.25}$$

and a standard computation for the moments leads to

$$\begin{aligned}\tilde{\omega}(H) \;=\; & -\frac{N}{2}\sum_{g=1}^{p}(1-\frac{1}{N\alpha_g})J_{g,g}(\tanh r_g)^2 - \frac{1}{2}\sum_{g=1}^{p}\frac{1}{\alpha_g J_{g,g}} - \frac{N}{2}\sum_{g\neq k=1}^{p}J_{g,k}\tanh r_g \tanh r_k \\ & -N\sum_{g=1}^{p}h_g\tanh r_g.\end{aligned} \tag{4.26}$$

Analogously, the Gibbs state of $\tilde{H}$ is:

$$\tilde{\omega}(\tilde{H}) = -N\sum_{g=1}^{p}\alpha_g\, r_g\, \tanh r_g,$$

and the non interacting partition function is:

$$\tilde{Z}_N = \sum_{\sigma}e^{-\tilde{H}(\sigma)} = \sum_{g=1}^{p}2^{N_g}(\cosh r_g)^{N_g}$$

which implies that the non-interacting pressure gives

$$\tilde{p}_N = \frac{1}{N}\ln \tilde{Z}_N = \ln 2 + \sum_{g=1}^{p} \alpha_g \ln\cosh r_g$$

So we can finally apply Proposition (4.20) in order to find a lower bound for the pressure $p_N = \frac{1}{N}\ln Z_N$:

$$p_N = \frac{1}{N}\ln Z_N \geqslant \frac{1}{N}\left(\ln \tilde{Z}_N - \tilde{\omega}(H) + \tilde{\omega}(\tilde{H})\right) \tag{4.27}$$

which explicitly reads:

$$\begin{aligned} p_N = \frac{1}{N}\ln Z_N \quad &\geqslant \quad \ln 2 + \sum_{g=1}^{p} \alpha_g \ln\cosh r_g + &(4.28)\\ &+\frac{1}{2}\sum_{g,k=1}^{p} J_{g,k}\tanh r_g \tanh r_k + \sum_{g=1}^{p} h_g \tanh r_g &(4.29)\\ &-\sum_{g=1}^{p}\alpha_g\, r_g\, \tanh r_g + \\ &+\frac{1}{2N}\sum_{g=1}^{p}\frac{J_{g,g}}{\alpha_g}(\tanh r_g)^2 + \frac{1}{2N}\sum_{g=1}^{p}\frac{1}{\alpha_g J_{g,g}} &(4.30)\\ & &(4.31)\end{aligned}$$

Taking the lim inf over N and the supremum in the variables r_g the left hand side we get the (4.21) after performing the change of variables $\bar{m}_g = \tanh r_g$.

□

4.4.3 Exact solution of the model

Though the functions p_{LOW} and p_{UP} are different, it is easily checked that they share the same local suprema. Indeed, if we differentiate both functions with respect to parameters $\bar{m}_g$, we see that the extremality conditions are given in both cases by the Mean Field Equations:

$$\bar{m}_g = \tanh\Big(\sum_{k=1}^{p}\frac{J_{g,k}+J_{k,g}}{2\,\alpha_g}\bar{m}_k + \frac{h_g}{\alpha_g}\Big) \quad g = 1..p \tag{4.32}$$

If we now use these equations to express $\tanh^{-1} m_i$ as a function of m_i and we substitute back into p_{UP} and p_{LOW} we get the same function:

$$p = -\frac{1}{2}\sum_{g,k=1}^{p} J_{g,k}\bar{m}_g\bar{m}_k - \sum_{g=1}^{p}\alpha_g\frac{1}{2}\ln\frac{1-\bar{m}_g^2}{4}. \tag{4.33}$$

Since this function returns the value of the pressure when the vector $(\bar{m}_1, .., \bar{m}_p)$ corresponds to an extremum, and this is the same both for p_{LOW} and p_{UP}, we have proved the following:

Theorem 1. *Given a hamiltonian as defined in (4.6), and defining the pressure per particle as* $p_N = \frac{1}{N}\ln Z$, *given parameters* $J_{i,j}$ *and* h_i, *the thermodynamic limit*

$$\lim_{N\to\infty} p_N = p$$

of the pressure exists, and can be expressed in one of the following equivalent forms:

a) $p = \sup_{\bar{m}_1,..,\bar{m}_p} p_{LOW}$

b) $p = \sup_{\bar{m}_1,..,\bar{m}_p} p_{UP}$

4.5 An analytic result for a two-population model

The form we derived for the pressure can be rightfully considered a solution of the statistical mechanical model, since it expresses the thermodynamic properties of a large number of particles in terms of a finite number of parameters.

Nevertheless, the equations of state cannot be solved explicitly in terms of the parameters: indeed, even the phase diagram for the two-population case has only been characterised fully in a subset of our parameter space, in which it has been found useful for a few physical applications [13, 44, 46]. This gives us a feeling of how the mean field assumption, being simplistic from one point of view, can given rise to models exhibiting non-trivial behaviour.

In this section we shall focus on the two-population case, which is the case considered in the applications of the next chapter, and find an analytic result concerning the maximum number of equilibrium states arising from our equations of state. In particular we shall prove that, for any choice of the parameters, the total number of local maxima for the function $p(\bar{m}_1, \bar{m}_2)$ is less or equal to five.

By applying a convenient relabelling to the model's parameters, we get the mean field equations for our two-population model in the following form:

$$\begin{cases} \bar{m}_1 &= \tanh(J_{11}\alpha\bar{m}_1 + J_{12}(1-\alpha)\bar{m}_2 + h_1) \\ \bar{m}_2 &= \tanh(J_{12}\alpha\bar{m}_1 + J_{22}(1-\alpha)\bar{m}_2 + h_2) \end{cases},$$

and correspond to the stationarity conditions of $p(\bar{m}_1, \bar{m}_2)$. So, a subset of solutions to this system of equations are local maxima, and some among them correspond to the thermodynamic equilibrium.

These equations give a two-dimensional generalization of the Curie-Weiss mean field equation. Solutions of the classic Curie-Weiss model can be analysed by elementary geometry: in our case, however, the geometry is that of 2 dimensional maps, and it pays to recall that Henon's map, a simingly harmless 2 dimensional diffeomorhism of $\mathbb{R}^2$, is known to exhibit full-fledged chaos. Therefore, the parametric dependence of solutions, and in particular the number of solutions corresponding to local maxima of $p(\bar{m}_1, \bar{m}_2)$, is in no way apparent from the equations themselves.

We can, nevertheless, recover some geometric features from the analogy with one-dimensional picture. For the classic Curie-Weiss equation, continuity and the Intermediate Value Theorem from elementary calculus assure the existence of at least one solution. In higher dimensions we can resort to the analogous result, Brouwer's Fixed Point Theorem, which states that any continuous map on a topological closed ball has at least one fixed point. This theorem, applied to the smooth map R on the square $[-1,1]^2$, given by

$$\begin{cases} R_1(\bar{m}_1,\ \bar{m}_2) &= \tanh(J_{11}\alpha\bar{m}_1 + J_{12}(1-\alpha)\bar{m}_2 + h_1) \\ R_2(\bar{m}_1,\ \bar{m}_2) &= \tanh(J_{12}\alpha\bar{m}_1 + J_{22}(1-\alpha)\bar{m}_2 + h_2) \end{cases}$$

establishes the existence of at least one point of thermodynamic equilibrium.

We can gain further information by considering the precise form of the equations: by inverting the hyperbolic tangent in the first equation, we can $\bar{m}_1$ as a function of $\bar{m}_2$, and vice-versa for the second equation. Therefore, when $J_{12} \neq 0$ we can rewrite the equations in the following fashion:

$$\begin{cases} \bar{m}_2 &= \dfrac{1}{J_{12}(1-\alpha)}(\tanh^{-1}\bar{m}_1 - J_{11}\alpha\bar{m}_1 - h_1) \\ \bar{m}_1 &= \dfrac{1}{J_{12}\alpha}(\tanh^{-1}\bar{m}_2 - J_{22}(1-\alpha)\bar{m}_2 - h_2) \end{cases} \tag{4.34}$$

Consider, for example, the first equation: this defines a function $\bar{m}_2(\bar{m}_1)$, and we shall call its graph *curve* γ_1. Let's consider the second derivative of this function:

$$\frac{\partial^2 \bar{m}_2}{\partial \bar{m}_1^2} = -\frac{1}{J_{12}(1-\alpha)} \cdot \frac{2\bar{m}_1}{(1-\bar{m}_1^2)^2}.$$

We see immediately that this second derivative is strictly increasing, and that it changes sign exactly at zero. This implies that γ_1 can be divided into three monotonic pieces, each having strictly positive *third* derivative as a function of $\bar{m}_1$. The same thing holds for the second equation, which defines a function $\bar{m}_1(\bar{m}_2)$, and a corresponding *curve* γ_2. An analytical argument easily establishes that there exist at most 9 crossing points of γ_1 and γ_2 (for convenience we shall label the three monotonic pieces of γ_1 as I, II and III, from left to right): since γ_2, too, has a strictly positive third derivative, it follows that it intersects each of the three monotonic pieces of γ_1 at most three times, and this leaves the number of intersections between γ_1 and γ_2 bounded above by 9 (see an example of this in Figure 4.1).

By definition of the mean field equations, the stationary points of the pressure correspond to crossing points of γ_1 and γ_2. Furthermore, common sense tells us that not all of these stationary points can be local maxima. This is indeed true, and it is proved by the following:

Proposition 16. *The function $p(\bar{m}_1, \bar{m}_2)$ admits at most 5 maxima.*

To prove 16 we shall need the following:

Lemma 5. *Say P_1 and P_2 are two crossing points linked by a monotonic piece of one of the two functions considered above. Then at most one of them is a local maximum of the pressure $p(\bar{m}_1, \bar{m}_2)$.*

Proof of Lemma 5: The proof consists of a simple observation about the meaning of our curves. The mean field equations as stationarity conditions for the pressure, so each of γ_1 and γ_2 are made of points where one of the two components of the gradient of $p(\bar{m}_1, \bar{m}_2)$ vanishes. Without loss of generality assume that P_1 is a maximum, and that the component that vanishes on the piece of curve that links P_1 to P_2 is $\dfrac{\partial p}{\partial \bar{m}_1}$.

Since P_1 is a local maximum, $p(\bar{m}_1, \bar{m}_2)$ locally increases on the piece of curve γ. On the other hand, the directional derivative of $p(\bar{m}_1, \bar{m}_2)$ along γ is given by

$$\hat{\mathbf{t}} \cdot \nabla p$$

where $\hat{\mathbf{t}}$ is the unit tangent to γ. Now we just need to notice that by assumptions for any point in γ $\hat{\mathbf{t}}$ lies in the same quadrant, while ∇p is vertical with a definite verse. This implies that the scalar product giving directional derivative is strictly non-negative over all γ, which prevents P_2 form being a maximum.

□

Proof of Proposition 16: The proof considers two separate cases:

a) All crossing points can be joined in a chain by using monotonic pieces of curve such as the one defined in the lemma;

b) At least one crossing point is linked to the others only by non-monotonic pieces of curve.

In case a), all stationary can be joined in chain in which no two local maxima can be nearest neighbours, by the lemma. Since there are at most 9 stationary points, there can be at most 5 local maxima.

For case b) assume that there is a point, call it P, which is not linked to any other point by a monotonic piece of curve. Without loss of generality, say that P lies on I (which, we recall, is defined as the leftmost monotonic piece of γ_1). By assumption, I cannot contain other crossing points apart from P, for otherwise P would be monotonically linked to at least one of them, contradicting the assumption. On the other hand, each of II and III contain at most 3 stationary points, and, by Lemma 5, at most 2 of these are maxima. So we have at most 2 maxima on each of II and III, and and at most 1 maximum on I, which leaves the total bounded above by 5. The cases in which P lies on II, or on III, are proved analogously, giving the result.

□

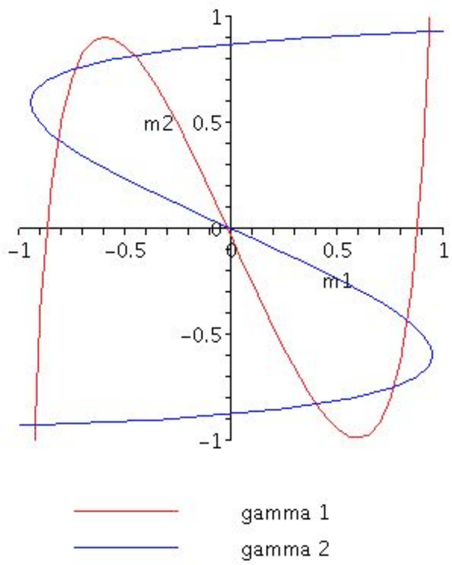

Figure 4.1: The crossing points correspond to solutions of the mean field equations

Chapter 5

Case studies

In previous chapters we defined a model which, generalizing well known tools from econometrics, provides a viable approach to study phenomena of human interaction. Its well-posedness as an equilibrium statistical mechanical model, proved in the last chapter, though supporting the idea that modelling social phenomena working from the bottom up[1] may be feasible, doesn't imply the relevance of the proposed tool to any actual scenario: indeed, for any model such relevance may only be established as a result of success in describing, and most importantly predicting events from the real world.

There are many possible instances from the social sciences to which quantitative modelling is an appealing prospective. Due to the increasingly global nature of human mobility, one particularly timely social issue is immigration. The applicability of our model to immigration matters was considered in References [16] and [17]. Reference [17] analyses how the microscopic assumptions of the model reflect the tendency of individuals to act consistently with their cultural legacy as well as with what they identify as their social group, which are both tenets in the field of social psychology. The numerical analysis carried out in Reference [16] shows how such simple assumptions are enough for the model to identify regimes in which a global change in a cultural trait is triggered by a small fraction of immigrants interacting with a large population of residents.

The descriptive power shown by the model in the case of immigration further supports the view that equilibrium statistical mechanics can play a role in a quantitative theory of social phenomena. However, though qualitatively inspiring, the immigration scenario seems ill-suited as a first quantitative case study, due to the intrinsic difficulty of finding a database that characterizes such a social issue adequately. We therefore turn to the problem of giving

[1] that is, starting from individual interactions and trying to establish patterns that might be at work on a larger scale

our model a first implementation on some "simpler" matters.

The aim of this chapter is two-fold. On one hand we are interested in assessing the simplest instance of the model considered in the last chapter, that is a mean field model where the population has been partitioned into two groups, based on their geographical residence, so that the model generalizes a discrete choice model with one binary attribute.

On the other hand, we'd like to propose two simple procedures of model estimation, that we feel might be very appealing for models at an early stage of development. The first procedure is statistical in nature, and it's based on a method developed by Berkson [7], whereas the second takes a statistical mechanical perspective by considering the role played by the fluctuations of the main observable quantities for the model.

5.1 The model

We consider a population of individuals facing with a "YES/NO" question, such as choosing between marrying through a religious or a civil ritual, or voting in favor or against of death penalty in a referendum. We index individuals by $i,\ i = 1...N$, and assign a numerical value to each individual's choice σ_i in the following way:

$$\sigma_i = \begin{cases} +1 \text{ if } i \text{ says YES} \\ -1 \text{ if } i \text{ says NO} \end{cases},$$

Consistently with the many population Curie-Weiss model analysed in the last chapter, which as we saw generalises the multinomial logit model described in chapter 2, we assume that the joint probability distribution of these choices is well approximated by a Boltzmann-Gibbs distribution corresponding to the following Hamiltonian

$$H_N(\sigma) = -\sum_{i,l=1}^{N} J_{il}\sigma_i\sigma_l - \sum_{i=1}^{N} h_i\sigma_i.$$

Heuristically, this distribution favours the agreement of people's choices σ_i with some external influence h_i which varies from person to person, and at the same time favours agreement of a couple of people whenever their interaction coefficient J_{il} is positive, whereas favors disagreement whenever J_{il} is negative.

Given the setting, the model consists of two basic steps:

1) A parametrization of quantities J_{il} and of h_i,

2) A systematic procedure allowing us to "measure" the parameters characterizing the

model, starting from statistical data (such as surveys, polls, etc).

The parametrization must be chosen to fit as well as possible the data format available, in order to define a model which is able to make good use of the increasing wealth of data available through information technologies.

5.2 Discrete choice

Let us first consider our model when it ignores interactions $J_{il} \equiv 0 \, \forall \, i,l \in (1,...,N)$, that is

$$H_N(\sigma) = -\sum_{i=1}^{N} h_i\sigma_i.$$

The model shall be applied to data coming from surveys, polls, and censuses, which means that together with the answer to our binary question, we shall have access to information characterizing individuals from a socio-economical point of view. We can formalize such further information by assigning to each person a vector of *socio-economic attributes*

$$a_i = \{a_i^{(1)}, a_i^{(2)}, ..., a_i^{(k)}\}$$

where, for instance,

$$a_i^{(1)} = \begin{cases} 1 \text{ for } i \text{ Male} \\ 0 \text{ for } i \text{ Female} \end{cases},$$

and

$$a_i^{(2)} = \begin{cases} 1 \text{ for } i \text{ Employee} \\ 0 \text{ for } i \text{ Self-employed} \end{cases},$$

etc.

As we have seen in chapter 2, the general setting of the multinomial logit allows to exploit the supplementary data by assuming that h_i (which is the "field" influencing the choice of i) is a function of the vector of attributes a_i. Since for the sake of simplicity we choose our attributes to be binary variables, so that the most general form for h_i turns out to be linear

$$h_i = \sum_{j=1}^{k} \alpha_j a_i^{(j)} + \alpha_0$$

and the model's parameters are given by the components of the vector $\alpha = \{\alpha_0, \alpha_1, ..., \alpha_k\}$. It's worth pointing out that the parameters α_j, $j = 0...k$ do not depend on the specific individual i.

We know that discrete choice theory holds that, when making a choice, each person weights out various factors such as his own gender, age, income, etc, as to maximize in probability the benefit arising from his/her decision. Parameters α tell us the relative weight (i.e. their relative importance importance) that the various socio-economic factors have when people are making a decision with respect to our binary question. The parameter α_0 does not multiply any specific attribute, and thus it is a homogeneous influence which is felt by all people in the same way, regardless of their individual characteristics. A discrete choice model is considered good when the parametrized attributes are very suitable for the specific choice, so that the parameter α_0 is found to be small in comparison to the attribute-specific ones.

We have shown in chapter 2 that elementary statistical mechanics gives us the probability of an individual i with attributes a_i answering "YES" to our question as:

$$\begin{aligned} p_i &= P(\sigma_i = 1) = \frac{e^{h_i}}{e^{h_i} + e^{-h_i}}, \\ h_i &= \sum_{j=1}^{k} \alpha_j a_i^{(j)} + \alpha_0, \end{aligned}$$

which as we saw is equivalent to the result obtained by applying economics' utility maximization principle to a random utility with Gumbel disturbances. Therefore collecting the choices made by a relevant number of people, and keeping track of their socio-economic attributes, allows us to use statistics in order to find the value of α for which our distribution best fits the real data. This in turn allows to assess the implications on aggregate behavior if we apply incentives to the population which affect specific attribute, as can be commodity prices in a market situation.

5.3 Interaction

The kind of model described in the last section has been successfully used by econometrics for the last thirty years [50], and has opened the way to the quantitative study of social phenomena. Such models, however, only apply to situations where the functional relation between the people's attributes α and the population's behavior is a smooth one: it is ever more evident, on the other hand, that behavior at a societal level can be marked by sudden jumps [51, 61, 47].

There exist many examples from linguistics, economics, and sociology where it has been observed how the global behaviour of large groups of people can change in an abrupt manner as a consequence of slight variations in the social structure (such as, for instance, a change in the pronunciation of a language due to a little immigration rate, or as a substantial decrease in crime rates due to seemingly minor action taken by the authorities) [3, 31, 47]. From a statistical mechanical point of view, these abrupt transitions may be considered as phase transitions caused by the interaction between individuals, and this is what led us to consider in this thesis the interesting mapping between discrete choice econometrics and the Curie-Weiss theory, first stated in [21].

We then go back to studying the general interacting model

$$H_N(\sigma) = -\sum_{i,l=1}^{N} J_{il}\sigma_i\sigma_l - \sum_{i=1}^{N} h_i\sigma_i, \tag{5.1}$$

while keeping

$$h_i = \sum_{j=1}^{k} \alpha_j a_i^{(j)} + \alpha_0.$$

We now need to find a suitable parametrization for the interaction coefficients J_{il}. Since each person is characterized by k binary socio-economic attributes, the population can be naturally partitioned into 2^k subgroups, so that using the mean-field assumptions allows one to rewrite the model in terms of subgroup-specific magnetizations m_g, as in the general Hamiltonian (4.1). Equation (4.1) is general enough to consider populations with different relative sizes (such as one in which residents make up a much larger share of population than immigrants): nevertheless, it turns out that the mean-field assumption implies a relation of direct proportionality between interaction coefficients and population sizes, that might be considered innatural.

The approach taken in this thesis, therefore, is to consider sub-populations of comparable size, and model them in the thermodynamic limit as having equal size. In specific, in all cases we divide the data into two geographical regions which have a similar population. This "equal size" assumption can be considered as part of the modelling process: by using it to analyze data, as we do here, we can gain insights on how to relax it in future refinements of the model. So, for the time being, let J_{il} depend explicitly on a partition of sub-populations of equal sizes. By using the mean-field assumption we can express this as follows

$$J_{il} = \frac{1}{2^k N} J_{gg'}, \text{ if } i \in g \text{ and } l \in g',$$

where g and g' are two sub-population (not necessarily distinct). This in turn allows us to rewrite (5.1) as

$$H_N(\sigma) = -\frac{N}{2^k}(\sum_{g,g'=1}^{2^k} J_{gg'} m_g m_{g'} + \sum_{g=1}^{2^k} h_g m_g)$$

where m_g is the average opinion of group g:

$$m_g = \frac{1}{2^k\, N} \sum_{i=(g-1)N/2^k+1}^{g\, N/2^k} \sigma_i.$$

We readily see how this is the many-population model considered in the previous chapter, and this gives us a solid microscopic foundation for the theory. Indeed, the results we obtained through relatively elementary mathematics establish rigourously the existence of the model's thermodynamic limit, as well as its factorization properties, and just as importantly provide us with a closed form for the thermodynamic state equations.

Therefore if we are willing to test how well the model's assumptions compare with real data, we can use these equations as the main tool for a procedure of statistical estimation. Here we shall confront the simple case where $k = 1$. This is a bipartite model which, as we know from the last chapter, can have at most five metastable equilibrium states, given by the thermodynamically stable solutions to the following equations:

$$\bar{m}_1 = \tanh(J_{11}\bar{m}_1 + J_{12}\bar{m}_2 + h_1) \tag{5.2}$$

$$\bar{m}_2 = \tanh(J_{21}\bar{m}_1 + J_{22}\bar{m}_2 + h_2) \tag{5.3}$$

Equation (4.32) which was derived from the model's exact solution shows that the equilibrium state equations for a system consisting of two parts of equal size do not carry two different parameters J_{12} and J_{21}, but that, even if these two parameters were different in the Hamiltonian, what characterizes each of the two subparts is rather their average $(J_{12} + J_{21})/2$. We keep J_{12} and J_{21} as two distinct parameters throughout the statistical application in order to use them as a consistency test: we shall be able to consider systems to be in equilibrium only if $J_{12} - J_{21} = 0$.

The state equations (5.2) allow us, in particular, to write the probability of i choosing YES in a closed form, similar to the non-interacting one:

$$p_i = P(\sigma_i = 1) = \frac{e^{U_g}}{e^{U_g} + e^{-U_g}}, \tag{5.4}$$

where

$$U_g = \sum_{g'=1}^{2} J_{g,g'} \bar{m}_{g'} + h_g.$$

This is the basic tool needed to estimate the model starting from real data. We describe the estimation procedure in the next section.

5.4 Estimation

We have seen that according to the model an individual i belonging to group g has probability of choosing "YES" equal to

$$p_i = \frac{e^{U_g}}{e^{U_g} + e^{-U_g}}$$

where

$$U_g = \sum_{g'} J_{g,g'} \bar{m}_{g'} + h_g.$$

The standard approach of statistical estimation for discrete models is to maximize the probability of observing a sample of data with respect to the parameters of the model (see e.g. [6]). This is done by maximizing the likelihood function

$$L = \prod_i p_i$$

with respect to the model's parameters, which in our case consist of the interaction matrix J and the vector α.

Our model, however, is such that p_i is a function of the equilibrium states m_g, which in turn are *discontinuous* functions of the model's parameters. This problem takes away much of the appeal of the maximum likelihood procedure, and calls for a more feasible alternative.

The natural alternative to maximum likelihood for problems of model regression is given by the least squares method [25], which simply minimizes the squared norm of the difference between observed quantities, and the model's prediction. Since in our case the observed quantities are the empirical average opinions $\tilde{m}_g$, we need to find the parameter values which minimize

$$\sum_g (\tilde{m}_g - \tanh U_g)^2, \tag{5.5}$$

which in our case correspond to satisfying as closely as possible the state equations (5.2) in squared norm. This, however, is still computationally cumbersome due to the non-linearity of the function $\tanh(U_j)$. This problem has already been encountered by Berkson back in the nineteen-fifties, when developing a statistical methodology for bioassay [7]: this is an interesting point, since this stimulus-response kind of experiment bears a close analogy to the natural kind of applications for a model of social behavior, such as linking stimula given by incentive through policy and media, to behavioral responses on part of a population.

The key observation in Berkson's paper is that, since U_g is a linear function of the model's parameters, and the function $\tanh(x)$ is invertible, a viable modification to least squares is given by minimizing the following quantity, instead:

$$\sum_g (\operatorname{arctanh} \tilde{m}_g - U_g)^2. \tag{5.6}$$

This reduces the problem to a linear least squares problem which can be handled with standard statistical software, and Berkson finds an excellent numerical agreement between this method and the standard least squares procedure.

There are nevertheless a number of issues with Berkson's approach, which are analyzed in [6], pag. 96. All the problems arising can be traced to the fact that to build (5.6), we are collecting the individual observations into subgroups, each of average opinion m_g. The problem is well exemplified by the case in which a subgroup has average opinion $m_g \equiv \pm 1$: in this case $\operatorname{arctanh} m_g = -\infty$, and the method breaks down. However the event $m_g \equiv \pm 1$ has a vanishing probability when the size of the groups increases, so that the method behaves properly for large enough samples.
The proposed measurement technique is best elucidated by showing a few simple concrete examples, which we do in the next section.

5.5 Case studies

We shall carry out the estimation program for real situations which correspond to a very simple case of our model. The data was obtained from periodical censuses carried out by Istat[2]: since census data concerns events which are recorded in official documents, for a large number of people, we find it to be an ideal testing ground for our model.

For the sake of simplicity, individuals are described by a single binary attribute characterizing their place of residence (either Northern or Southern Italy) and we chose, among the several possible case studies, the ones for which choices are likely to involve peer interaction in a major way.

[2] Italian National Institute of Statistics

The first phenomenon we choose to study concerns the share of people who chose to marry through a religious ritual, rather than through a civil one. The second case deals with divorces: here individuals are faced with the choice of a consensual/ non-consensual divorce. The last test we perform regards the study of suicidal tendencies, in particular the mode of execution.

5.5.1 Civil vs religious marriage in Italy, 2000-2006

To address this first task we use data from the annual report on the institution of marriage compiled by Istat in the seven years going from 2000 to 2006. The reason for choosing this specific social question is both a methodological and a conceptual one.

Firstly, we are motivated by the exceptional quality of the data available in this case, since it is a census which concerns a population of more than 250 thousand people per year, for seven years. This allows us some leeway from the possible issues regarding the sample size, such as the one highlighted in the last section. And just as importantly the availability of a time series of data measured at even times also allows to check the consistency of the data as well as the stability of the phenomenon.

Secondly, marriage is probably one of the few matters where a great number of individuals make a genuine choice concerning their life that gets recorded in an official document, as opposed to what happens, for example, in the case of opinion polls.

We choose to study the data with one of the simplest forms of the model: individuals are divided according to only to a binary attribute $a^{(1)}$, which takes value 1 for people from Northern Italy, and 0 for people form Southern Italy. In the formalism of Section 2, therefore, the model is defined by the Hamiltonian

$$\begin{aligned} H_N(\sigma) &= -\frac{N}{2}(J_{11}m_1^2 + (J_{12}+J_{21})m_1 m_2 + J_{22}m_2^2 + h_1 m_1 + h_2 m_2), \\ h_i &= \alpha_1 a_i^{(1)} + \alpha_0, \end{aligned}$$

and the state equations to be used for Berkson's statistical procedure are given by (5.4).

Table 5.1 shows the time evolution of the share of men choosing to marry through a religious ritual: the population is divided in two geographical classes. The first thing worth noticing is that these shares show a remarkable stability over the seven-year period: this confirms how, though arising from choices made by distinct individuals, who bear extremely different personal motivations, the aggregate behavior can be seen as an observable feature characterizing society as a whole.

	% of religious marriages, by year						
Region	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**
Northern Italy	68.35	64.98	61.97	60.90	57.91	55.95	54.64
Southern Italy	81.83	80.08	79.32	79.02	76.81	76.52	75.46

Table 5.1: Percentage of religious marriages, by year and geographical region

	4-year period			
Parameter	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**
α_0	-0.10 $\pm$ 0.42	-0.16 $\pm$ 0.15	-0.18 $\pm$ 0.10	-0.13 $\pm$ 0.01
α_1	0.20 $\pm$ 0.59	0.20 $\pm$ 0.22	0.16 $\pm$ 0.14	0.14 $\pm$ 0.01
J_1	1.16 $\pm$ 0.41	1.09 $\pm$ 0.16	1.01 $\pm$ 0.11	1.02 $\pm$ 0.01
J_2	1.29 $\pm$ 0.89	1.40 $\pm$ 0.33	1.45 $\pm$ 0.21	1.36 $\pm$ 0.01
J_{12}	-0.21 $\pm$ 0.89	-0.10 $\pm$ 0.33	0.03 $\pm$ 0.21	-0.01 $\pm$ 0.01
J_{21}	0.09 $\pm$ 0.41	0.02 $\pm$ 0.16	-0.01 $\pm$ 0.11	0.01 $\pm$ 0.01

Table 5.2: Religious vs civil marriages: estimation of the interacting model

In order to apply Berkson's method of estimation, we choose gather the data into periods of four years, starting with $2000-2003$, then $2001-2004$, etc. Now, if we label the share of men in group g choosing the religious ritual in a specific year (say in 2000) by m_g^{2000}, we have that the quantity that ought to be minimized in order to estimate the model's parameters for the first period is the following, which we label X^2:

$$\begin{aligned} X^2 &= \sum_{year=2000}^{2003} \sum_{g=1}^{2} (\operatorname{arctanh} m_g^{year} - U_g^{year})^2, \\ U_g^{year} &= \sum_{g'=1}^{2} J_{g,g'} m_{g'}^{year} + h_g, \\ h_g &= \alpha_1 a_g^{(1)} + \alpha_0. \end{aligned}$$

The results of the estimation for the four periods are shown in Table 5.2, whereas Table 5.3 shows the corresponding estimation for a discrete choice model which doesn't take into account interaction.

5.5.2 Divorces in Italy, 2000-2005

The second case study uses data from the annual report compiled by Istat in the six years going from 2000 to 2005. The data show how divorcing couples chose between a consensual

	4-year period			
Parameter	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**
α_0	0.67 ± 0.15	0.63 ± 0.03	0.61 ± 0.06	0.58 ± 0.03
α_1	-0.41 ± 0.1	-0.43 ± 0.04	-0.45 ± 0.08	-0.46 ± 0.04

Table 5.3: Religious vs civil marriages: estimation of the non-interacting model

	% of consensual divorces, by year					
Region	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**
Northern Italy	75.06	80.75	81.32	81.62	81.55	81.58
Southern Italy	58.83	72.80	71.80	72.61	72.76	72.08

Table 5.4: Percentage of consensual divorces, by year and geographical region

and a non-consensual divorce in Northern and Southern Italy. As shown in Table 5.4 here too, when looking at the ratio among consensual versus the total divorces, the data show a remarkable stability.

Again we gather the data into periods of four years and Table 5.5 presents the estimation of our model's parameters for the whole available period, while in Table 5.6 we show the corresponding fit by the non-interacting discrete choice model.

We notice that the estimated parameters have some analogies with the preceding case study in that here too the cross interactions $J_{12},\ J_{21}$ are statistically close to zero whereas the diagonal values $J_{11},\ J_{22}$ are both greater than one suggesting an interaction scenario characterized by multiple equilibria [28]. Furthermore, in both cases the attribute-specific parameter α_1 is larger than the generic parameter α_0 in the interacting model (Tables 2 and 5), as opposed to what we see in the non-interacting case (Tables 3 and 6): this suggests that by accounting for interaction we might be able to better evaluate the role played by socio-economic attributes.

5.5.3 Suicidal tendencies in Italy, 2000-2007

The last case study deals with suicidal tendencies in Italy, again following the annual report compiled by Istat in the eight years from 2000 to 2007, and we use the same geographical attribute used for the former two studies.

The data in Table 5.7 shows the percentage of deaths due to hanging as a mode of execution. The topic of suicide is of particular relevance to sociology: indeed, the very first systematic quantitative treatise in the social sciences was carried out by Émile Durkheim

	4-year period		
Parameter	**2000-2003**	**2001-2004**	**2002-2005**
α_0	0.02 ± 0.06	-0.08 ± 0.01	-0.07 ± 0.01
α_1	-0.25 ± 0.08	-0.22 ± 0.01	-0.23 ± 0.01
J_1	1.59 ± 0.14	1.64 ± 0.01	1.66 ± 0.01
J_2	1.16 ± 0.06	1.25 ± 0.01	1.25 ± 0.01
J_{12}	-0.05 ± 0.06	0.01 ± 0.01	0.00 ± 0.01
J_{21}	-0.08 ± 0.14	0.00 ± 0.01	-0.01 ± 0.01

Table 5.5: Consensual vs non-consensual divorces: estimation of the interacting model

	4-year period		
Parameter	**2000-2003**	**2001-2004**	**2002-2005**
α_0	0.41 ± 0.13	0.48 ± 0.01	0.480046 ± 0.01
α_1	0.28 ± 0.18	0.25 ± 0.02	0.261956 ± 0.01

Table 5.6: Consensual vs non-consensual divorces: estimation of the non-interacting model

[20], a founding father of the subject, who was puzzled by how a phenomenon as unnatural as suicide could arise with the astonishing regularity that he found. Such a regularity as even been dimmed the "sociology's one law" [56], and there is hope that the connection to statistical mechanics might eventually shed light on the origin of such a law.

Mirroring the two previous case studies, we present the time series in Table 5.7, whereas Table 5.8 shows the estimation results for the interacting model, and Table 5.9 are the estimation results for the discrete choice model. Again, the data agrees with the analogies found for the two previous case studies.

	% suicides by hanging							
Region	**2000**	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**
Northern Italy	34.17	37.02	35.83	34.58	35.21	36.23	33.57	38.08
Southern Italy	37.10	37.40	37.34	38.54	34.71	38.90	40.63	36.66

Table 5.7: Percentage of suicides with hanging as mode of execution, by year and geographical region

	4-year period				
Parameter	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**	**2004-2007**
α_0	0.01 ± 0	0.02 ± 0.01	0.01 ± 0.01	0.02 ± 0.01	0.02 ± 0.01
α_1	0.01 ± 0.01	0.00 ± 0.01	0.00 ± 0.01	0.00 ± 0.01	0.00 ± 0.01
J_1	1.09 ± 0.01	1.09 ± 0.01	1.09 ± 0.02	1.10 ± 0.03	1.09 ± 0.01
J_2	1.06 ± 0.01	1.08 ± 0.01	1.08 ± 0.01	1.07 ± 0.01	1.07 ± 0.01
J_{12}	0 ± 0.01	0.00 ± 0.01	0.00 ± 0.01	0.00 ± 0.01	0.00 ± 0.01
J_{21}	0 ± 0.01	0.01 ± 0.01	0.00 ± 0.02	0.01 ± 0.03	0.01 ± 0.01

Table 5.8: Suicidal tendencies: estimation of the interacting model

	4-year period				
Param.	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**	**2004-2007**
α_0	-0.25 ± 0.02	-0.27 ± 0.03	-0.26 ± 0.03	-0.24 ± 0.04	-0.25 ± 0.05
α_1	-0.05 ± 0.03	-0.03 ± 0.04	-0.04 ± 0.04	-0.07 ± 0.06	-0.04 ± 0.07

Table 5.9: Suicidal tendencies: estimation of the non-interacting model

5.6 A statistical mechanical approach to model estimation

We shall now estimate our model parameters using a different approach, which makes explicit use of the time fluctuations of our main observable quantities $\tilde{m}_i$. This approach is not econometric, but typically statistical mechanical, in that it equates fluctuations observed over time with fluctuations of a system which is in an equilibrium which is defined by an *ensemble* of states rather than by a single state. The problem of retracing a model's parameters from observable quantities in this context has been referred to in the literature as the "inverse Ising problem" (see e.g. [64]).

We start from the usual model

$$
\begin{aligned}
H_N(\sigma) &= -\frac{N}{2}(J_{11}m_1^2 + (J_{12}+J_{21})m_1m_2 + J_{22}m_2^2 + h_1m_1 + h_2m_2), \\
h_i &= \alpha_1 a_i^{(1)} + \alpha_0,
\end{aligned}
\tag{5.7}
$$

and we shall analyze the data from our three case studies again using the model's state equations

$$
\begin{aligned}
\bar{m}_1 &= \tanh(J_{11}\bar{m}_1 + J_{12}\bar{m}_2 + h_1), \\
\bar{m}_2 &= \tanh(J_{21}\bar{m}_1 + J_{22}\bar{m}_2 + h_2),
\end{aligned}
\tag{5.8}
$$

which, as we shall see, will now also provide us with the system's fluctuations as well as the average quantities. Just as in the last section, we choose to use two distinct parameters J_{12} and J_{21} inside the state equations (5.8) instead of their average $\frac{1}{2}(J_{12}+J_{21})$ in order to test for consistency.

5.6.1 Two views on susceptibility

The method presented here comes from an observation about quantity $\frac{\bar{m}_i}{\partial h_j}$, which is called m_i's *susceptibility* with respect to external field h_i in physics, or m_i's *elasticity* with respect to incentive h_i in econometrics.

The two relevant points of view that make $\frac{\bar{m}_i}{\partial h_j}$ such an interesting quantity are those of statistical mechanics and thermodynamics.

5.6.1.1 Statistical mechanics

For statistical mechanics $\frac{\partial \bar{m}_i}{\partial h_j}$ is a quantity defined internally to the system. The following formula clarifies this point: From (5.7)

$$\frac{\partial \bar{m}_i}{\partial h_j} = \frac{\partial}{\partial h_j}\Big\{\sum_{\sigma} m_i(\sigma)\frac{e^{-H_N(\sigma)}}{Z}\Big\} = \frac{N}{2}\left(\omega_N(m_i m_j) - \omega_N(m_i)\omega_N(m_j)\right) \equiv c_{ij}. \tag{5.9}$$

The quantity $\frac{\partial \bar{m}_i}{\partial h_j}$, which we shall refer to as c_{ij} for notational convenience, is thus simply the amount of fluctuations that we observe in quantities m_i: if imagine the system as a closed box, and we imagine being inside such closed box, we can in principle measure c_{ij} by studying the way m_i vary.

5.6.1.2 Thermodynamics

The second point of view is intrinsically different: for thermodynamics $\frac{\partial \bar{m}_i}{\partial h_j}$ corresponds to the response of the "closed box" mentioned in the last paragraph to an external influence given by a small change in the field h_j. Differently from statistical mechanics, thermodynamics cannot provide us with this response's value a priori from observations, since it doesn't know any details of what is going on inside the box. Thermodynamics does tell us, however, that responses of the system to different influences , if the system is to obey to the thermodynamic law identified by state equations (5.8).

These interrelations can be made explicit by considering the partial derivatives of (5.8)

$$\begin{aligned}
\frac{\partial \bar{m}_1}{\partial h_1} &= (1-\bar{m}_1^2)\Big(J_1 \frac{\partial \bar{m}_1}{\partial h_1} + J_{12}\frac{\partial \bar{m}_2}{\partial h_1} + 1\Big), \\
\frac{\partial \bar{m}_1}{\partial h_2} &= (1-\bar{m}_1^2)\Big(J_1 \frac{\partial \bar{m}_1}{\partial h_2} + J_{12}\frac{\partial \bar{m}_2}{\partial h_2}\Big), \\
\frac{\partial \bar{m}_2}{\partial h_2} &= (1-\bar{m}_2^2)\Big(J_{21} \frac{\partial \bar{m}_1}{\partial h_2} + J_{2}\frac{\partial \bar{m}_2}{\partial h_2} + 1\Big), \\
\frac{\partial \bar{m}_2}{\partial h_1} &= (1-\bar{m}_2^2)\Big(J_{21} \frac{\partial \bar{m}_2}{\partial h_1} + J_{2}\frac{\partial \bar{m}_2}{\partial h_1}\Big),
\end{aligned}$$

By relabeling $d_i = (1-\bar{m}_i^2)$ and using definition (5.9) we can rewrite this system of equations as

$$\begin{aligned}
J_1\, c_{11} + J_{12}\, c_{12} &= \frac{c_{11}}{d_1} - 1, \\
J_1\, c_{12} + J_{12}\, c_{22} &= \frac{c_{12}}{d_1}, \\
J_{21}\, c_{12} + J_2\, c_{22} &= \frac{c_{22}}{d_2} - 1, \\
J_{21}\, c_{11} + J_2\, c_{12} &= \frac{c_{12}}{d_2}.
\end{aligned}$$

This is linear in the J_{ij}, and the former two equations are independent from the latter two, so that we can easily solve for the J_{ij} using Cramer's rule. This together with the equations of state (5.8) allows us to express all the model parameters $J_{i,j}$ and h_i as functions of the observable quantities $\bar{m}_i$ and c_{ij}, as follows:

$$\begin{aligned}
J_{12} &= \frac{c_{12}}{c_{11}c_{22} - c_{12}^2} = J_{21}, \\
J_{11} &= \frac{\big(\frac{c_{11}}{d_1} - 1\big)c_{22} - \frac{c_{12}^2}{d_1}}{c_{11}c_{22} - c_{12}^2}, \\
J_{22} &= \frac{\big(\frac{c_{22}}{d_2} - 1\big)c_{11} - \frac{c_{12}^2}{d_2}}{c_{11}c_{22} - c_{12}^2}, \\
h_1 &= \operatorname{arctanh} \bar{m}_1 - J_1\, \bar{m}_1 - J_{12}\, \bar{m}_2, \\
h_2 &= \operatorname{arctanh} \bar{m}_2 - J_{12}\, \bar{m}_1 - J_2\, \bar{m}_2.
\end{aligned}$$

In this case we see the consistency condition $J_{12} = J_{21}$ fulfilled a priori. This tells us that, given a set of sub-magnetizations, together with its covariance matrix, our parametrized family contains one and only one model corresponding to it. As a consequence we can say that such model makes use of exactly the amount of information provided into the time series of standard statistics (i.e. means and covariances) of a poll-type database.

Estimators for $\bar{m}_i$ and c_{ij} from the time series data are straightforward to obtain, and we have gathered these statistics for our three case studies in Tables 5.10, 5.12 and 5.14. Given a time period T, which in our case shall correspond to a range of four consecutive years, we define estimators $\tilde{m}_i(T)$ of $\bar{m}_i$ and $\tilde{c}_{ij}(T)$ of $\bar{c}_{ij}$ corresponding to it

$$\begin{aligned} \tilde{m}_i(T) &= \frac{1}{|T|} \sum_{year \in T} \bar{m}_i^{year}, \\ \tilde{c}_{i,j}(T) &= N_T \frac{1}{|T|} \sum_{year \in T} (\bar{m}_i^{year} - \tilde{m}_i(T))(\bar{m}_j^{year} - \tilde{m}_j(T)). \end{aligned} \tag{5.10}$$

We must point out that in order to be well defined, such estimators should apply to a time series of samples which are of equal size, since susceptibility $c_{i,j}$ has indeed an explicit size dependence. Our systems, on the other hand, cannot be of equal size since they consist of people who chose to participate into an activity, and the number of these people cannot be established a priori. As stated before, however, the point of view in this thesis is that human affairs can behave following the kind of *quasi-static* processes familiar to thermodynamics. Consistently with this perspective, and with some justification coming from the considered data, we shall consider the system's population a slowly varying quantity, and use its average of small periods of time as the quantity N_T in order to define $\tilde{c}_{i,j}(T)$

$$N_T = \frac{1}{|T|} \sum_{year \in T} N^{year}.$$

We can thus use relations (5.10) in order to obtain estimates for the model parameters. By considering that

$$\alpha_0 = h_2, \tag{5.11}$$

$$\alpha_1 = h_1 - h_2, \tag{5.12}$$

we can compare the new estimates, presented in Tables 5.11, 5.13 and 5.15, with those from the preceding section.

5.6.2 Comments on results from the two estimation approaches

We can now compare Tables 5.11, 5.13 and 5.15 with their counterparts from last section, which estimated the same model for the same data coming from our three chosen case studies, using our adaptation of Berkson's method.

Such comparison can be summarised as follows: comparing Table 5.11, showing parameter estimations for the "religious vs civil marriage" case study, with Table 5.2, we find the estimated values to be definitely different, but we also see that they bear some interesting similarities, especially if we consider the confidence interval provided by the least squares method in Table 5.2. Three shared features are particularly noteworthy:

- The estimated values for J_1 and J_2 are similar in one aspect: in both cases J_2 is estimated to be consistently greater than J_1 over the years;

- J_{12} is estimated to be very close to zero in Table 5.11: J_{12} and J_{21} can be considered to be statistically zero in Table 5.2 (which is also consistent with the condition $J_{12}-J_{21} = 0$);

- α_0 and α_1 consistently estimated with equal signs by both methods: this is an essential prerequisite that any model needs to satisfy.

The agreement is not good for the two remaining case studies, however. In the "consensual vs non-consensual divorce" case study, despite estimations being consistent in the first time range (that is 2000-2003), agreement gets worse and worse in the following two periods. As for the third case study, the two estimation methods do not show any agreement whatsoever.

An important point to be made is the dependence of method agreement against population size. For the first case-study, where the population is made up of over 200 thousand people the agreement between the two methods is good. In the second case-study we have a population of roughly 40 thousand people, and we find agreement in one of the three considered time spans. The third case-study doesn't show any agreement: the population size here, however, is of only around 2000 people.

Finally, though the last point certainly motivates further enquiry, one should not be over-confident about population size being the only problem. An extremely important objection comes from the fact that wherever agreement is found, estimators $\tilde{c}_{ij}$ are found to give very high values. We must remember that we are looking at the data through a model that assumes equilibrium: such big $\tilde{c}_{ij}$ values correspond to large fluctuations, and these should cause an equilibrium model to be less precise and not more.

The failure of the two estimation methods to give consistent results in regimes with small fluctuations (that is whenever $\tilde{c}_{ij}$ are small), reveal the presented study as inconclusive on an empirical level. There are however several improvements that can be made by using the same framework established here, the most important one concerning the handling of the data. This thesis has as its goal to propose both a model, and a procedure allowing to establish the empirical relevance of the model itself. It was hence of the foremost importance to show a concrete example of such a procedure; since this was not a professional work in statistics, however, it featured several drawbacks, some of which can be described as follows:

- Though showing a remarkable temporal coherence, the time series consists of a number of measurements which is insufficient for any statistic to be reliable. In order to work on consistent groups of data, the choice was made to gather data in four-year ranges: the situation may be improved by considering a phenomenon having the same kind of temporal coherence, but for which measurements are available on a monthly basis;

- The regional separation between "Northern Italy" and "Southern Italy" is an artificial one, decided for technical reasons. The quality of the statistical study could be greatly improved by considering a partition into groups which is directly relevant to the issue under study;

- No use was made of the data regarding the relative sizes of the considered subpopulations. This, as noted before, was due to a difficulty arising from the mean-field assumption, which lead us to characterize the population as having equal size. This drawback can be amended in two ways: 1) at a fundamental level, by further considering the implications of having populations of different size for the model 2) by keeping the same model, but considering estimators for c_{ij} that make use of the information coming from the subpopulation sizes.

A final point to make concerns the model itself: very little is known about the structure of the phase diagram of a mean-field model of a multi-part system: indeed, as noted in earlier chapters, a subcase case of a two-part system considered here was studied in several occasions since the nineteen-fifties [33, 9] until recently [46], and found to be highly non-trivial. As a consequence, it is to be expected that the analysis of the features characterizing the regime that empirical data identify will need to be treated locally and numerically before any kind of global picture arises, and it is not a priori clear whether the presence of big values for the c_{ij} might characterize and interesting regime rather than just a failure of the model. It is mainly for this reason that much of the effort in this thesis has been directed towards the aim of establishing a way to link the model to data, rather than to pursue further the analytic treatment of the model on its own.

	4-year period			
Statistic	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**
$\tilde{m}_1$	0.25	0.20	0.15	0.12
$\tilde{m}_2$	0.58	0.56	0.54	0.52
$\tilde{c}_{11}$	1636.09	953.63	466.42	106.59
$\tilde{c}_{22}$	346.88	214.58	122.02	22.30
$\tilde{c}_{12}$	562.15	336.03	176.09	34.09

Table 5.10: Religious vs civil marriages: statistics

	4-year period			
Parameter	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**
α_0	-0.21	-0.18	-0.15	-0.10
α_1	0.20	0.17	0.14	0.08
J_1	1.07	1.04	1.02	1.00
J_2	1.51	1.44	1.39	1.29
J_{12}	0.00	0.00	0.01	0.03

Table 5.11: Religious vs civil marriages: estimation of the interacting model

	4-year period		
Statistic	**2000-2003**	**2001-2004**	**2002-2005**
$\tilde{m}_1$	0.59	0.63	0.63
$\tilde{m}_2$	0.38	0.45	0.45
$\tilde{c}_{11}$	74.21	1.27	0.15
$\tilde{c}_{22}$	356.27	1.76	1.68
$\tilde{c}_{12}$	120.56	-0.17	0.28

Table 5.12: Consensual vs non-consensual divorces: statistics

	4-year period		
Parameter	**2000-2003**	**2001-2004**	**2002-2005**
α_0	-0.05	0.23	-0.71
α_1	-0.17	0.01	5.81
J_1	1.51	0.85	-8.06
J_2	1.16	0.68	0.39
J_{12}	0.01	-0.07	1.61

Table 5.13: Consensual vs non-consensual divorces: estimation of the interacting model

	4-year period				
Statistic	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**	**2004-2007**
$\tilde{m}_1$	-0.29	-0.29	-0.29	-0.30	-0.28
$\tilde{m}_2$	-0.25	-0.26	-0.25	-0.24	-0.25
$\tilde{c}_{11}$	0.94	0.62	0.30	0.71	1.95
$\tilde{c}_{22}$	0.23	1.50	2.05	3.57	3.66
$\tilde{c}_{12}$	-0.08	0.00	0.11	-0.50	-0.91

Table 5.14: Suicidal tendencies: statistics

	4-year period				
Parameter	**2000-2003**	**2001-2004**	**2002-2005**	**2003-2006**	**2004-2007**
α_0	-1.21	-0.16	-0.06	-0.13	-0.11
α_1	0.81	-0.29	-0.87	-0.37	-0.08
J_1	-0.01	-0.53	-2.33	-0.45	0.51
J_2	-3.40	0.41	0.57	0.75	0.76
J_{12}	-0.39	0.00	0.19	-0.22	-0.14

Table 5.15: Suicidal tendencies: estimation of the interacting model

Acknowledgment

In first place I would like to express my gratitude to my advisor Prof. Pierluigi Contucci, for introducing me to the challenging topic of which this work gives but a glimpse, and for carefully guiding me throughout the process of my Ph.d. Special thanks go to Federico Gallo, who made me aware of the subtle connection between statistical mechanics and econometrics which is central to the thesis defended here; many thanks are due to Cristian Giardinà for his fundamental role in educating me on one of the key-points of the analytic treatment provided in this work, and to Adriano Barra for his important contribution to the empirical case studies considered. I would also like to thank Diego Grandi, who was available for many interesting discussions on physics which were of great help to the completion of this task, and Stefano Ghirlanda and Giulia Menconi with whom I was happy to collaborate in the early stage of my doctorate.

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