ALMA MATER STUDIORUM-UNIVERSITÀ DI BOLOGNA

DOTTORATO DI RICERCA IN

Scienze Statistiche

Ciclo: XXXVI

Settore Concorsuale: 13/D4 – METODI MATEMATICI DELL'ECONOMIA E DELLE SCIENZE ATTUARIALI E FINANZIARIE

Settore Scientifico Disciplinare: SECS-S/06 METODI MATEMATICI DELL'ECONOMIA E DELLE SCIENZE ATTUARIALI E FINANZIARIE

Pseudo-Operations and Pseudo-Analysis: Applications to Probability and Risk Modelling

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Esame finale anno 2024

Contents

Abstract

The aim of this thesis is to generalize some well-known concepts in probability and statistics from the standard ring to the non-idempotent semi-ring $(\mathbb{R}_+, \oplus_h, \otimes_h)$, where pseudo-sum $\oplus_h$ and $\otimes_h$ are defined by a suitable function h, called generator.
In the first part of this dissertation, we introduce the notions of pseudo-independence with respect to a pseudo-additive fuzzy measure and of pseudo-moment generating functions, showing that the classical results concerning moment generating functions of a vector of independent random variables and of their sum extend to pseudo-moment generating functions if the random variables involved are pseudo-independent. Moreover, we prove that pseudo moment generating functions, and more in general pseudo-analysis, can be particularly efficient in characterizing a new class of bivariate random vectors that we call "pseudo-Schur constant" family, which represents an extension of the well-known Schur-constant class.
In the second part of this thesis, we give a generalization of strong and weak bivariate lack-of-memory properties, substituting into their associated functional equations the standard product by the pseudo one $\otimes_h$: we call the distributions satisfying them pseudo strong and weak distributions. After characterising the pseudo weak distribution in full generality, we study the induced dependence structure of the underlying lifetimes and that of the residual ones. Moreover, we show that the distributions satisfying pseudo lack-of-memory properties coincide with the solutions of suitable generalizations of Kaminsky (1983) and Marshall and Olkin (2015) functional equations; finally, we analyse several examples of pseudo weak distributions that may be used in life insurance and we give a non-life insurance application to LOSS and ALAE modelling problem.

Chapter 1

Introduction

Pseudo-analysis is based on the structure of general semirings where a pseudo-sum $\oplus$ and a pseudo-product $\otimes$ substitute and generalize the usual sum and multiplication. In the particular case of a non-idempotent semiring, it has been proved that these pseudo-operations are identified by a continuous and invertible function through which both operations can be expressed, id est

$$x \oplus_h y = h(h^{-1}(x) + h^{-1}(y)), \tag{1.1}$$

and

$$x \otimes_h y = h(h^{-1}(x)\ h^{-1}(y)), \tag{1.2}$$

$x, y \in [0, \infty)$, where $h : [a, b] \to [0, \infty)$ is a continuous and strictly increasing function, see Mesiar and Rybárik (1995) and Pap (1995). For the main literature about pseudo-calculus and pseudo-integral, we refer to Zhang et al. (2022).

The aim of this thesis, which is divided in two parts, is to generalize some well-known concept in statistics and in probability from the standard ring to the non-idempotent semiring $(\mathbb{R}_+, \oplus_h, \otimes_h)$, where pseudo sum $\oplus$ and pseudo product $\otimes$ are given by (1.1) and (1.2) respectively.

Fuzzy measure can be used to account for uncertainty and ambiguity. In the first part of the dissertation, we consider a measurable space provided with a fuzzy measure assuming values in a given semiring and we introduce a definition of pseudo moment-generating function that is in line with the pseudo-Laplace transform introduced in Pap and Ralévic (1998), but considering a more general class of exponential functions obtained through distortions. Since the fuzzy measures we are dealing with assume values in a semiring, we introduce the concept of pseudo-independence and we show that the classical results concerning moment generating functions of a vector of independent random variables and of their sum extend to pseudo-moment

generating functions and to a more general type of aggregation of random variables if the latter are pseudo-independent.
This setup allows to characterize a new class of distributions that represent an extension of the Schur-Constant distributions. We recall that a bivariate vector (X,Y) is Schur-Constant if its joint survival function $\bar{F}_{X,Y}$ satisfies the following equation:

$$\bar{F}_{X,Y}(x,y) = S(x+y), \quad x,y > 0, \tag{1.3}$$

where S is a convex survival function (see, for instance, Barlow and Mendel, 1992, Caramellino and Spizzichino, 1994, Nelsen, 2005). A continuous Schur-constant random vector can be completely characterized in terms of $V = X+Y$ and of an uniform random variable U independent of V (see for instance Chi et al., 2009). Moreover, in Kozlova and Salminen (2004) and, later, in Ta and Van (2017), a characterization of Schur-constant random vector is provided in terms of its joint moment-generating function.
In this thesis, we consider random vectors with survival function of type (1.3) where the standard sum is replaced by the pseudo sum (1.1): we call them "pseudo-Schur constant" random vectors, showing that their dependence structure is of Archimedean type and extending the above mentioned characterizations of Schur-constant random vectors in terms of pseudo-operations and pseudo-analysis.
In the second part of the thesis, we consider a generalization of bivariate lack-of-memory properties introduced in the seminal paper by Marshall and Olkin (1967). We recall that a distribution satisfies standard lack-of-memory property in strong version if

$$\bar{F}_{X,Y}(s_1+t_1, s_2+t_2) = \bar{F}_{X,Y}(s_1,s_2)\ \bar{F}_{X,Y}(t_1,t_2), \forall s_1, s_2, t_1, t_2 \geq 0 \tag{1.4}$$

and in weak version if

$$\bar{F}_{X,Y}(s_1+t, s_2+t) = \bar{F}_{X,Y}(s_1,s_2)\ \bar{F}_{X,Y}(t,t), \forall s_1, s_2, t \geq 0. \tag{1.5}$$

Marshall and Olkin show that the unique distribution satisfying the former is $\bar{F}_{X,Y}(x,y) = e^{-\lambda_1 x - \lambda_2 y}, \lambda_1, \lambda_2 > 0$ and that the unique distribution with exponential marginal survival function for which the latter holds true is $\bar{F}_{X,Y}(x,y) = e^{-\lambda_1 x - \lambda_2 - \lambda_0 \max(x,y)}, \lambda_i > 0, i = 1,2,3$.

Muliere and Scarsini (1987) generalized the functional equation (1.5) by substituting the standard sum $+$ by the pseudo sum $\oplus$ given by (1.1). Similarly, in this thesis, we replace the standard product in (1.4) and (1.5) by the pseudo product $\otimes$ given by (1.2), obtaining the following functional equations:

$$\bar{F}_{X,Y}(s_1+t, s_2+t) = \bar{F}_{X,Y}(s_1,s_2) \otimes_h \bar{F}_{X,Y}(t_1,t_2),\ \ s_1, s_2, t_1, t_2 \geq 0 \tag{1.6}$$

and

$$\bar{F}_{X,Y}(s_1 + t, s_2 + t) = \bar{F}_{X,Y}(s_1, s_2) \otimes_h \bar{F}_{X,Y}(t, t), s_1, s_2, t \geq 0. \tag{1.7}$$

After finding sufficient and necessary conditions under which the solutions of (1.6) and (1.7) are bivariate survival functions, we focus on the latter, that we call pseudo weak distribution. We show that, for a suitable choice of the generator h and of the marginal survival functions, we recover the distribution obtained in Marshall and Olkin (2015) as the solution, with Gompertz marginal survival functions, of a functional equation that is an extension to the bivariate case of Kaminsky's functional equation.
In Ghurye and Marshall (1984), the authors show that the distribution of the vector $(X_1, \dots, X_n)$ satisfying standard weak lack-of-memory property can be completely characterised in terms of the random variables $U = \min(X_1, \dots, X_n)$ and $\bar{W} = (X_1 - U, \dots, X_n - U)$; moreover, for $n = 2$, they derive the distribution of $(U, \bar{W}, N)$, where $N = 1_{X_1 > X_2} - 1_{X_1 < X_2}$: for distributions satisfying (1.7), we find the joint distribution of the vector $(U, \bar{W}, N)$, but we do not achieve a complete characterization.
In Block and Basu (1974), in a bivariate setting, an alternative characterization of the standard weak distribution is given in terms of the random variables U and $V = X - Y$: in this case, we give a complete characterization of pseudo weak distribution in terms again of U and V.
Moreover, we investigate the dependence structure of pseudo weak distribution, determining its singularity and its Kendall distribution function in full generality and studying the upper and lower dependence coefficients for some specific choices of the marginal survival functions and of the generator h.
Then, we generalize Kaminsky (1983) and Marshall-Olkin' (2015) functional equations and we show that their solutions coincide with the solutions of the functional equations that define pseudo lack-of-memory properties. Moreover we study the dependence structure of residual lifetimes vector, investigating how it changes as the threshold t varies through the analysis of the Kendall distribution function and of the tail dependence coefficients.
After giving sufficient conditions under which a function is sub-multiplicative or super-multiplicative in the unit interval, we discuss possible applications to insurance modelling. First, we study the dependence structure of bivariate distributions obtained as distortions of sub-multiplicative functions and we'll look at the impact they have in pricing some well-known insurance contracts written on residual lifetimes. Moreover, as an explanatory example, we consider the bivariate LOSS-ALAE (Allocated Loss Adjustment Expenses) modelling problem in actuarial science, considering the dataset of 1500 general liability claims randomly chosen from late settlement lags provided by US Insurance Services Office.

Assuming that the vector of LOSS and of a scalar transformation of the corresponding ALAE follows pseudo weak distribution, we select the best joint distribution among several ones by log-likelihood optimization: it turns out that the generator obtained is super-multiplicative, meaning that probability of high joint losses is significant for the insurance company. Finally, we compute some well-known risk measures and premiums for different reinsurance contracts written on LOSS and ALAE.

Chapter 2

Preliminaries and State of Art

In this section we will introduce the main concepts and results concerning semirings, pseudo-calculus, fuzzy measures and pseudo-integrals, that will be used in the rest of the thesis; moreover, we will give some results about tail dependence of distorted copulas that will be used in Chapter 4. Definition and extension of bivariate lack-of-memory properties are also presented.

2.1 Pseudo-Calculus

In this section we will introduce the main concepts and results concerning semirings, pseudo-calculus, fuzzy measures and pseudo-integrals, that will be used in the sequel.

2.1.1 Semirings

Semirings have been widely studied and applied to many fields (see for example Golan, 1992). A binary operation $\oplus$ on the interval $[a, b] \subseteq [-\infty, \infty]$, endowed with a metric $\mathbf{d}$, is a pseudo-addition if it is commutative, non-decreasing, associative and with a neutral element denoted by $\mathbf{0} \in [a, b]$. Similarly, a binary operation $\otimes$ on $[a, b]$ is a pseudo-multiplication if it is commutative, positively non-decreasing, id est ,

$$x \leq y \rightarrow x \otimes z \leq y \otimes z \ \forall z \in [a, b],$$

associative and with unit element $\mathbf{1} \in [a, b]$. If we further assume that $\otimes$ is distributive with respect to $\oplus$, then the structure $([a, b], \oplus, \otimes)$ is called a semiring.

We can distinguish between three cases of semirings based on the fact that these pseudo-operations are idempotent or not (see, among the others, Zhang and Pap, 2020, and Zhang et al., 2022).

1. Both pseudo-operations $\oplus$ and $\otimes$ are idempotent. In this case, we may have two possible representations of these operations.

   (a)
   $$x \oplus y = \sup(x, y) \tag{2.1}$$
   and
   $$x \otimes y = \inf(x, y); \tag{2.2}$$
   in this case, the associated metric is given by $\mathbf{d}(x, y) = |\arctan(x) - \arctan(y)|$.

   (b)
   $$x \oplus y = h(\sup(h^{-1}(x), h^{-1}(y)))$$
   and
   $$x \otimes y = h(\inf(h^{-1}(x), h^{-1}(y))),$$
   where $h : [a, b] \to [0, M]$, with $M \in (0, \infty)$. In this case, the associated metric is given by $\mathbf{d}(x, y) = |h^{-1}(x) - h^{-1}(y)|$.

2. $\oplus$ is idempotent but $\otimes$ is non-idempotent:
   $$x \oplus y = \sup(x, y)$$
   and
   $$x \otimes_h y = h(h^{-1}(x)\ h^{-1}(y)),$$
   where $h : [a, b] \to [0, \infty)$. In this case, the associated metric is given by $\mathbf{d}(x, y) = |h^{-1}(x) - h^{-1}(y)|$.

3. It has been proven in Mesiar and Rybarik (1995) that the semi-ring $([a, b], \oplus, \otimes)$ is non-idempotent if and only if there exists a continuous strictly increasing bjection $h : [a, b] \to [0, \infty)$ such that
   $$x \oplus_h y = h(h^{-1}(x) + h^{-1}(y)) \tag{2.3}$$
   $$x \otimes_h y = h(h^{-1}(x)\ h^{-1}(y)). \tag{2.4}$$
   In this case, the associated metric is given by $\mathbf{d}(x, y) = |h^{-1}(x) - h^{-1}(y)|$.

Throughout the thesis, we'll work with a non-idempotent semiring as in case *3.* and we will consider the following class of functions:

$$\mathcal{G} = \{h : h \textit{ is a strictly increasing bjection of } \mathbb{R}_+ \textit{with } h(1) = 1\};$$

the elements of $\mathcal{G}$ will be called "generators".

In this chapter, we will use the notation $\oplus_h$ and $\otimes_h$ in order to specify the dependence on h. Clearly, when $h = id$ in (2.3) and in (2.4), we recover the standard sum and the standard product respectively.

Starting from the pseudo-sum defined in (2.3), the pseudo-difference $\ominus_h$ is given by

$$x \ominus_h y = h\left(h^{-1}(x) - h^{-1}(y)\right), \; x \geq y \geq 0,$$

see Jain (2022). Similarly, for pseudo-division $\oslash$, we have:

$$x \oslash_h y = h\left(h^{-1}(x) : h^{-1}(y)\right), \; \text{for } x \geq 0, \; y > 0.$$

By the same way, starting from the pseudo-product in (2.4), the pseudo-power of a number $x \in (a, b]$ is given by

$$x^{\otimes_h \alpha} = h\left([h^{-1}(x)]^\alpha\right), \; x \geq 0$$

with $\alpha > 0$ if $x \geq 0$ and with $\alpha \in \mathbb{R}$ if $x > 0$.

In the sequel we will consider also a non-commutative extension of the pseudo-sum (2.3). Given $k, k_1, k_2 \in \mathcal{G}$, we define the non-commutative sum operator (see for instance Pap and Vivona, 2000) as

$$x \;{}_{k_1}\!\overset{k}{\oplus}_{k_2}\; y = k(k_1^{-1}(x) + k_2^{-1}(y)), \quad x, y \in \mathbb{R}_+. \tag{2.5}$$

Remark 2.1.1. *Clearly, the triplet* (k, k_1, k_2) *identifies a sum operator, up to a multiplicative constant: that is, for all* $c > 0$*, if* $\hat{k}(x) = k(cx)$*,* $\hat{k}_1(x) = k_1(cx)$*,* $\hat{k}_2(x) = k_2(cx)$*, then* (k, k_1, k_2) *and* $(\hat{k}, \hat{k}_1, \hat{k}_2)$ *define the same addition operator.*

Similarly as for the commutative case, we can define the generalized pseudo-difference as

$$x \;{}_{k_1}\!\overset{k}{\ominus}_{k_2}\; y = k(k_1^{-1}(x) - k_2^{-1}(y)), \quad x \geq k_1 \circ k_2^{-1}(y), \, y \in \mathbb{R}_+. \tag{2.6}$$

2.1.2 Pseudo-Derivatives

An important tool in pseudo-calculus is the pseudo-derivative of a function (see Ralevic, 2007, among the others).

Definition 2.1.1. *Let $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}_+$ be a differentiable and non-decreasing function. Let $h \in \mathcal{G}$ be a differentiable generator in $\mathbb{R}_+$ with $h'(t) \neq 0$ for all $t \in \mathbb{R}_+$ that generates the semiring $(\bar{\mathbb{R}}_+, \oplus_h, \otimes_h)$. Then we define the pseudo-derivative of f at $x \in \mathcal{D}$ by:*

$$\frac{^{\oplus_h}d}{dx} f(x) = h\left(\frac{d}{dx}\left[h^{-1} \circ f(x)\right]\right).$$

By iteratively applying the above definition, if f and h are n times differentiable with $\frac{d^n}{d^n t}\left[h^{-1} \circ f\right](t)$ non-negative, we get an expression for the n-th pseudo derivative of f, that is

$$\frac{^{\oplus_h}d^n}{d^n x} f(x) = h\left(\frac{d^n}{d^n x}\left[h^{-1} \circ f(x)\right]\right).$$

2.1.3 Fuzzy Measures

In this section we introduce the notion of fuzzy measure (see, among the wide literature, the seminal paper of Choquet, 1953, and Teran, 2023).

Definition 2.1.2. *Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $m : \mathcal{F} \to [0, 1]$ is called a fuzzy measure if it satisfies the following properties:*

1. *$m(\varnothing) = 0$,*
2. *$m(A) \leq m(B)$, whenever $A \subseteq B$, $A, B \in \mathcal{F}$,*
3. *for every monotone sequence $\{A_i\}_{i=1,2,\ldots} \subset \mathcal{F}$, $\lim_{i\to\infty} m(A_i) = m\left(\lim_{i\to\infty} A_i\right)$.*

If $m(\Omega) = 1$, m is called "regular".

In what follows, given $h \in \mathcal{G}$ and the semiring $(\mathbb{R}_+, \oplus_h, \otimes_h)$, we will focus on a suitable sub-class of fuzzy measures that satisfies the $\sigma - \oplus_h$-additive property.

Definition 2.1.3. *Let $(\Omega, \mathcal{F})$ be a measurable space and m be a fuzzy measure. m is $\sigma - \oplus_h$-additive if*

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigoplus_{h\, n=1}^{+\infty} m(A_n)$$

for any sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$.

In order to specify the $\sigma - \oplus_h$-additivity of m, we will use the notation m_h. By Definitions 2.1.2 and 2.1.3, the set function $h^{-1} \circ m_h : \mathcal{F} \to [0,1]$ is an additive measure on the space $(\Omega, \mathcal{F})$ and a $\sigma - \oplus_h$-additive fuzzy measure is a distorted measure.

2.1.4 Pseudo-Integrals

The following construction of pseudo-integrals can be found in Zhang et al. (2022). Given a semiring $(\mathbb{R}_+, \oplus, \otimes)$ with metric d and a measure space $(\Omega, \mathcal{A}, m)$, where m is a σ-$\oplus_h$-additive measure, a function $f : X \to [0, \infty)$ is said to be measurable if $f^{-1}(B) = \{x \in X : f(x) \in B\} \in \mathcal{A}$ for each B, where B is a Borel subset of $\mathbb{R}_+$. The characteristic function of a set $A \subseteq \Omega$ with values in our semi-ring is defined by $1_A(x) = 1$ for $x \in A$ and 0 for $x \notin A$. A step measurable function is a mapping $e : \Omega \to [0, \infty)$ with the following representation

$$e = \oplus_{i=1}^{n} a_i \otimes 1_{A_i}, \tag{2.7}$$

where $a_i \in [0, \infty)$ and where $A_i \in \mathcal{A},\ i = 1, 2, ..., n$ is a sequence of pairwise disjoint sets.
The pseudo-integral of a non-negative step function $e : \Omega \to [0, \infty)$, defined in (2.7), with respect to m over $A \in \mathcal{A}$, is given by

$$\int_A^{\oplus} e \otimes dm = \ \oplus_{i=1}^{n} a_i \otimes m(A \cap A_i).$$

The pseudo-integral of a non-negative measurable function $f : \Omega \to [0, \infty)$ with respect to m over $A \in \mathcal{A}$ is defined by

$$\int_A^{\oplus} f \otimes dm = \sup \left\{ \int_A^{\oplus} e \otimes dm : e \leq f, e \in S(X) \right\}$$

where $S(X)$ is the set of all step function from Ω to $[0, \infty)$. In the case of a non-idempotent semi-ring, the pseudo integral of f with respect to a σ-$\oplus$ additive measure m_h reduces to

$$\int_\Omega^{\oplus_h} f \otimes_h dm_h = h \left(\int_\Omega h^{-1} \circ f \ d(h^{-1} \circ m_h) \right). \tag{2.8}$$

In case of a regular fuzzy measure m_h, the pseudo-expectation of a random variable X in $(\Omega, \mathcal{F})$ with non-negative values is given by

$$\tilde{E}^{m_h}[X] = \int_\Omega^{\oplus_h} X \otimes_h dm_h = h \left(E^{\mathbb{P}} \left[h^{-1} (X) \right] \right), \tag{2.9}$$

where $E^{\mathbb{P}}$ denotes the standard expectation with respect to the probability $\mathbb{P} = h^{-1} \circ m_h$, see Agahi and Dehnavi (2021).

2.2 Distorted Copulas

We briefly recall here the conditions that a copula C must satisfy.

Definition 2.2.1. *A function $C : [0,1] \times [0,1] \to [0,1]$ is called a copula if and only if:*

1. $C(0, y) = C(x, 0) = 0,\ x, y \in [0, 1]$
2. $C(x, 1) = C(1, x) = x,\ x \in [0, 1]$
3. $C(x_2, y_2) + C(x_1, y_1) \geq C(x_2, y_1) + C(x_1, y_2), 0 \leq x_1 \leq x_2 \leq 1,\ 0 \leq y_1 \leq y_2 \leq 1.$

Given a copula C and an increasing bijection $\phi : [0, 1] \to [0, 1]$, the function $C_\phi : [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$C_\phi(x, y) = \phi(C(\phi^{-1}(x), \phi^{-1}(y))), \quad x, y \in [0, 1]$$

is called distortion or transformation of a copula by means of ϕ. We define

$$\mathcal{I} = \{f : f \textit{is an increasing bjection of} [0, 1]\}; \tag{2.10}$$

moreover, we say that $\phi \in \mathcal{I}(C)$ if and only if the distortion of the copula C by means of ϕ is still a copula.
It is obvious that all the distortions of a copula C through $\phi \in \mathcal{I}$ satisfy conditions *1.* and *2.* in Definition 2.2.1, but not necessarily property *3.*: in the case in which *3.* is not satisfied, C_ϕ is a semi-copula but not a copula (see Bassan and Spizzichino, 2001, Durante and Sempi, 2005). Conditions on ϕ under which the distorted copula C_ϕ is still a copula have been widely studied in the literature. In Durante and Sempi (2005) and in Klement et al. (2005), it has been proven that convexity of ϕ is a sufficient condition to guarantee that C_ϕ is still a copula; further results have been obtained in Durante et al. (2010). In fact, therein, the authors show that, for any $\phi \in \mathcal{I}(C)$ and $\psi \in \mathcal{I}$, if $\phi \circ \psi^{-1}$ is convex, then $\psi \in \mathcal{I}(C)$. Moreover, they prove that, if $\psi \in \mathcal{I}(C)$ and $\phi \in \mathcal{I}(C)$, then the convex combination $\alpha\psi + (1 - \alpha)\phi \in \mathcal{I}(C)$, for any $\alpha \in [0, 1]$.
Statistical properties of distorted copulas C_ψ have been also analysed. Investigation of how the dependence structure of distorted copulas changes after application of a distortion can be found in Tran et al. (2016); the following results for tail dependence of distorted copulas have been proven in Durante et al. (2010).

Proposition 2.2.1. *Let C be a copula with finite lower tail dependence coefficient $\lambda_L(C)$. Moreover, let $\psi \in \mathcal{I}(C)$. Then, if*

$$\lim_{t \to 0^+} \frac{\psi(t)}{t^\alpha} = b \in (0, +\infty)$$

for some $\alpha > 0$, then $\lambda_L(C_\psi) = (\lambda_L(C))^\alpha$.

Proposition 2.2.2. *Let C be a copula with finite upper tail dependence coefficient $\lambda_U(C)$. Moreover, let $\psi \in \mathcal{I}(C)$. Then, if*

$$\lim_{t \to 1^-} \frac{1 - \psi(t)}{(1-t)^\alpha} = b \in (0, +\infty)$$

for some $\alpha > 0$, then $\lambda_U(C_\psi) = 2 - (2 - \lambda_U(C))^\alpha$.

In the case of a regularly varying distortion at 0^+, the following proposition in Sepanski (2020) also holds true.

Proposition 2.2.3. *Let C be a copula with finite lower and upper tail dependence coefficients. Let $\phi \in \mathcal{I}(C)$. Then:*

1. *If $\phi(x)$ is regularly varying at 0^+ with index $\alpha > 0$, then $\lambda_L(C_\phi) = (\lambda_L(C))^\alpha$.*
2. *If $1-\phi(1-x)$ is regularly varying at 0^+ with index $\beta > 0$, then $\lambda_U(C_\phi) = 2 - (2 - \lambda_L(C))^\beta$.*

2.3 Standard Lack-of-Memory Properties

In this section, we will focus on bivariate distributions with absolutely continuous marginal survival functions.
A survival distribution $\bar{G}$ satisfies the standard univariate lack-of-memory property if and only if

$$\bar{G}(s+t) = \bar{G}(s)\bar{G}(t), \;\; s, t \geq 0. \tag{2.11}$$

The unique survival function satisfying the above functional equation is well known and is given by $\bar{G}(x) = e^{-\lambda x}, \lambda > 0$.
Definitions of bivariate standard lack-of-memory properties have been given in the seminal paper of Marshall and Olkin (1967).
A bivariate survival function $\bar{G}$ satisfies strong bivariate lack-of-memory property if and only if

$$\bar{G}(s_1 + t_1, s_2 + t_2) = \bar{G}(s_1, s_2)\bar{G}(t_1, t_2), \;\; s_1, s_2, t_1, t_2 \geq 0 \tag{2.12}$$

and the unique survival function satisfying the above functional equation is given by

$$G(x, y) = e^{-\lambda_1 x - \lambda_2 y}, x, y \geq 0 \tag{2.13}$$

for some λ_1, $\lambda > 0$.
Similarly, a bivariate survival function $\bar{G}$ satisfies the weak bivariate lack-of-memory property if and only if

$$\bar{G}(s_1 + t, s_2 + t) = \bar{G}(s_1, s_2)\ \bar{G}(t, t),\ \ s_1, s_2, t \geq 0. \tag{2.14}$$

The solution of the above functional equation is given by

$$\bar{G}(x, y) = \begin{cases} e^{-\lambda y}\bar{G}_1(x - y), & x \geq y \\ e^{-\lambda x}\bar{G}_2(y - x), & x < y \end{cases}, \tag{2.15}$$

where $\bar{G}_1$, $\bar{G}_2$ are univariate survival functions of positive and absolutely continuous random variables and λ is a positive constant. However, the function (2.15) is a survival function if and only if

$$\begin{cases} \lambda \leq g_1(0) + g_2(0) \\ \frac{\partial \log(g_i(z))}{\partial z} \geq -\lambda,\ \forall z \geq 0,\ i = 1, 2 \end{cases}, \tag{2.16}$$

where $g_i(z) = \frac{-\partial \bar{G}_i(x)}{\partial x}$, see theorem 5.1 in Marshall and Olkin (1967). The two conditions of the system above guarantee that the probability mass on the line $x = y$ is between 0 and 1 and that the function (2.15) is absolutely continuous in the region $\{(x, y) : x \geq 0, y \geq 0, x \neq y\}$.
The unique solution of (2.14) with marginal survival exponential distributions is given by

$$\bar{G}_{X,Y}(x, y) = e^{-\gamma_1 x - \gamma_2 y - \gamma_3 \max(x,y)}, \quad x, y \geq 0 \tag{2.17}$$

with $\gamma_i > 0,\ i = 1, 2, 3$: this is the distribution of the vector

$$(X, Y) \stackrel{d}{=} (\min(T_1, T_3), \min(T_2, T_3)) \tag{2.18}$$

where T_i are independent exponentially distributed random variables. Obviously, (2.17) has constant marginal failure rates, restricting its usefulness for practical needs. As a response, other solutions of (2.14) with non-exponential marginals have been introduced (see, among the others, Block and Basu, 1974). An important contribution to the topic has been given in Kulkarni (2006): therein, the author suggests a class of bivariate distributions satisfying standard weak lack-of-memory property having increasing or/and decreasing marginal failure rates, but with same restrictions needed.

Moreover, Li and Pellerey (2011) introduce the Generalized MO model considering non-exponential independent random variables T_i, $i = 1, 2, 3$ in (2.18): however, the corresponding joint distributions do not satisfy lack-of-memory property; further extension of their model has been performed in Mulinacci (2018). Finally, in Pinto and Kolev (2015), it is assumed that individual shocks T_1 and T_2 are dependent, but still independent of the common shock: the resulting joint distribution may satisfy lack-of-memory property or not according to the value of the parameters chosen.

Distributions satisfying standard weak lack-of-memory property have been widely studied in the literature: we will present here two characterizations of these distributions which will be used in the sequel. The first characterization has been given in section 3 in Ghurye and Marshall (1984).

Proposition 2.3.1. *The survival distribution function $\bar{G}$ of the vector $\bar{X} = (X_1, X_2, \ldots, X_n)$ satisfies standard weak lack-of-memory property if and only if there exist random variables U and $\bar{W} = (W_1, W_2, \ldots, W_n)$ such that*

1. $\bar{X} = U\bar{e} + \bar{W}$, *where* $\bar{e} = (1, 1, \ldots, 1)$ *is the n-th dimensional unit vector;*
2. *U and $\bar{W}$ are independent;*
3. $P(\min(W_1, \ldots, W_n) = 0) = 1$*;*
4. *U has an exponential distribution.*

The authors actually show that $U = \min(X_1, \ldots, X_n)$; moreover, in the bivariate case, they show that the random variables $U = \min(X_1, X_2)$ and $N = 1_{X_1 > X_2} - 1_{X_1 < X_2}$ are independent.
Another characterization of the same family of distributions, in a bivariate setting, can be found in Block and Basu (1974).

Proposition 2.3.2. *Let (X, Y) be a random vector with absolutely continuous bivariate distribution $\bar{G}$. Then $\bar{G}$ satisfies standard weak lack-of-memory property if and only if, for $U = min(X, Y)$ and $V = X - Y$, there exists $\lambda > 0$ such that*

1. *U and V are independent;*
2. *U has exponential distribution with parameter λ;*

3.

$$P(V \leq t) = \begin{cases} G_1(t) + \frac{g_1(t)}{\lambda} & \textit{if } t \geq 0 \\ 1 - G_2(-t) - \frac{g_2(-t)}{\lambda} & \textit{if } t < 0 \end{cases},$$

where G_1, G_2 *are the marginal cumulative distribution functions of* X *and* Y *respectively and* $g_i(x) = G_i'(x),\ x \geq 0$.

Chapter 3

Pseudo-Moment Generating Functions: Application to Pseudo-Schur Constant Random Vectors.

In this chapter, we introduce the notions of pseudo- independence and of pseudo-moment generating functions, showing that the classical results concerning moment generating functions of a vector of independent random variables and of their sum can be extended to pseudo-moment generating functions if the random variables involved are pseudo-independent. Moreover, we prove that pseudo-moment generating functions, and more in general pseudo-analysis, can be particularly efficient in characterizing a new class of bivariate random vectors that we call "pseudo-Schur constant" family, which represents an extension of the well-known Schur-constant class.
The results of this chapter are published in an even more general setup in Mulinacci and Ricci (2024).

3.1 Random Variables and Pseudo-Independence

In this section we introduce the notion of pseudo-independence as an extension of the classical notion of independence.

3.1.1 Fuzzy Cumulative Distribution Functions

Let $(\Omega, \mathcal{F})$ be a measurable space. A function X is a random variable in $\mathbb{R}$ if and only if $X^{-1}(\Lambda) \in \mathcal{F}$ for any subset Λ of the Borel sigma-algebra of

$\mathbb{R}$. Given $(\Omega, \mathcal{F}, m)$, where m is a regular fuzzy measure, we can associate to any random variable X the function $F_X(x) = m(X \leq x)$, $\forall x \in \mathbb{R}$. The function F_X satisfies all the properties of a standard cumulative distribution function but it doesn't identify the regular fuzzy measure induced by X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If instead, we consider a regular $\sigma - \oplus_h$-additive fuzzy measure m_h with generator $h \in \mathcal{G}$, then the $\sigma - \oplus_h$-additive fuzzy measure $\mu_h^{F_X}$ induced by F_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is defined through

$$\mu_h^{F_X}((a, b]) = F_X(b) \ominus_h F_X(a)$$

on intervals of type $(a, b]$ and then extended to $\mathcal{B}(\mathbb{R})$ taking into account that

$$h^{-1} \circ \mu_h^{F_X}((a, b]) = h^{-1} \circ F_X(b) - h^{-1} \circ F_X(a)$$

and that $h^{-1} \circ \mu_h^{F_X}$ extends to a unique probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Notice that the cumulative distribution function of X with respect to the probability $\mathbb{P} = h^{-1} \circ m_h$ is $F_X^{\mathbb{P}}(x) = h^{-1} \circ F_X(x)$.

Exactly as for the univariate case, given the random vector (X, Y), if $F_{X,Y}(x, y) = m_h(X \leq x, Y \leq y)$ for $x, y \in \mathbb{R}$ is the joint fuzzy cumulative distribution function, then the joint cumulative distribution function of (X, Y) with respect to the probability $\mathbb{P} = h^{-1} \circ \mu_h$ is $F_{X,Y}^{\mathbb{P}}(x) = h^{-1} \circ F_{X,Y}(x, y)$. As for the additive case, the joint fuzzy cumulative distribution function can be written in terms of the fuzzy marginal ones and of the semi-copula that links them (see Durante and Sempi, 2005, and Durante and Spizzichino, 2010, for the notion of semi-copula and related results). In the particular case in which the fuzzy measure is σ-$\oplus_h$-additive, then the semi-copula $S(u, v)$ associated to $F_{X,Y}$ is linked to the copula $C(u, v)$ associated to $F_{X,Y}^{\mathbb{P}}$ in the following way:

$$S(u, v) = h\left(C(h^{-1}(u), h^{-1}(v))\right); \tag{3.1}$$

if h is convex, then S is indeed a copula function (see Proposition 2.8 in Durante and Spizzichino, 2010, and related comments).

3.1.2 Pseudo-Independence

The definition of pseudo-independence is obtained substituting the standard product by the pseudo one in the usual definition of independence.

Definition 3.1.1. *Let $(\Omega, \mathcal{F}, m)$ be a measurable space provided with a regular fuzzy measure m. Given $h \in \mathcal{G}$, we say that $A, B \in \mathcal{F}$ are h-independent if and only if*

$$m(A \cap B) = m(A) \otimes_h m(B). \tag{3.2}$$

Clearly, dealing with regular fuzzy measures, two different generators that coincide on the interval $[0, 1]$ define the same relation in (3.2). According to Definition 3.1.1, it is natural to introduce the notion of pseudo-independence between random variables. If X and Y are two real valued random variables defined on $(\Omega, \mathcal{F}, m)$, where m is a regular fuzzy measure and $h \in \mathcal{G}$ is a generator, they are said to be h-independent if and only if

$$m\,(X \in B_1, Y \in B_2) = m\,(X \in B_1) \otimes_h m\,(Y \in B_2) \tag{3.3}$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$.

The notion can be extended to more than two events and to more than two random variables exactly as for classical independence.

If, in addition, we assume a $\sigma - \oplus_h$-additive fuzzy measure m_h, equation (3.3) can be rewritten as

$$h^{-1} \circ m_h\,(X \in B_1, Y \in B_2) = h^{-1} \circ m_h\,(X \in B_1) \cdot h^{-1} \circ m_h\,(Y \in B_2)$$

for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$, so (3.3) holds true if and only if X and Y are independent with respect to the probability $\mathbb{P} = h^{-1} \circ m_h$. As a consequence, (3.3) is equivalent to

$$m_h\,(X \leq x, Y \leq y) = m_h\,(X \leq x) \otimes_h m_h\,(Y \leq y) \tag{3.4}$$

for all $x, y \in \mathbb{R}$. In terms of fuzzy cumulative distribution functions, (3.4) can be rewritten as

$$F_{X,Y}(x, y) = F_X(x) \otimes_h F_Y(y) \tag{3.5}$$

and $S(u, v) = u \otimes_h v$ is the associated semi-copula.

Clearly, $F_{X,Y}$ in (3.5) is a joint cumulative distribution function if and only if $u \otimes_h v$ for $u, v \in [0, 1]$ is a copula. It is well-known that an Archimedean copula can be written in multiplicative and in an additive form as

$$u \otimes_h v = \phi\left(\phi^{-1}(u) + \phi^{-1}(v)\right) \tag{3.6}$$

where $\phi(t) = h\left(e^{-t}\right)$, and that (3.6) is indeed a copula if and only if $\phi(0) = 1$, $\lim_{t\to+\infty} \phi(t) = 0$ and ϕ is continuous, non-increasing and convex on $[0, +\infty)$ (see for instance Theorem 2 in McNeil and Nešlehová, 2009, or Nelsen, 2006). Since $h \in \mathcal{G}$, we have that ϕ satisfies the required properties if and only if additionally $h(e^{-x})$ is convex.

Remark 3.1.1. *If we consider the idempotent product (2.2) instead of the pseudo product (1.2), equation (3.5) becomes*

$$F_{X,Y}(x, y) = \min(F_X(x), F_Y(y)),$$

which is the joint cumulative distribution function corresponding to perfect positive dependence between X and Y.

3.2 Pseudo-Moment Generating Function

In this section we will generalize the notion of moment generating function (both at the univariate as well as at the bivariate level) considering the more general setup introduced in Section 2.1. Given the extreme generality and flexibility of the framework that assumes a general semiring and a fuzzy measure, we start introducing a general notion of exponential function based on which the moment generating function will be defined.

3.2.1 Pseudo-Exponential Functions

The exponential function is characterized as the solution of some specific functional or differential equations. When considering the analogous functional and differential equations expressed in terms of pseudo-operations and pseudo-derivatives, we obtain different functions that, clearly, degenerate to the standard exponential function when the involved generators coincide with the identity function.

Proposition 3.2.1. *Let $h \in \mathcal{G}$, $k \in \mathcal{S}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then*

$$f(x \oplus_k y) = f(x) \otimes_h f(y) \tag{3.7}$$

for all $x, y \in \mathbb{R}_+$ if and only if $f(x) = h(a^{k^{-1}(x)}),\ a > 0$.

Proof. (3.7) is equivalent to

$$f \circ k(k^{-1}(x) + k^{-1}(y)) = h\left(h^{-1} \circ f(x) \cdot h^{-1} \circ f(y)\right),$$

from which, setting $u = k^{-1}(x)$ and $v = k^{-1}(y)$, we get

$$h^{-1} \circ f \circ k(u + v) = h^{-1} \circ f \circ k(u) \cdot h^{-1} \circ f \circ k(v),$$

which is true if and only if $h^{-1} \circ f \circ k(x) = a^x$ for some $a > 0$, and the conclusion trivially follows. □

In the setting of the semiring $(\mathbb{R}_+, \oplus_h, \otimes_h)$, that is when $k = h$, the resulting pseudo-exponential function consistent with it is the function $f(x) = h\left(a^{h^{-1}(x)}\right)$.

When $k(x) = x$ we obtain the function $f(x) = h(a^x)$, that, when $a = e$, is the solution of the following pseudo-differential equation.

Proposition 3.2.2. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a differentiable and non-decreasing function. Then*

$$f(x) = \frac{^{\oplus_h}d}{dx} f(x) \tag{3.8}$$

if and only if $f(x) = h\left(e^x\right)$.

Proof. By definition of pseudo-derivative, equation (3.8) is equivalent to

$$h^{-1}(f(x)) = \frac{d}{dx}\left[h^{-1} \circ f(x)\right]$$

that holds true if and only if $h^{-1} \circ f(x) = e^x$. □

Since the above pseudo-exponential functions will be used in the sequel in the specific case $a = e$, given $(h, k) \in \mathcal{G} \times \mathcal{S}$, we use the notation

$$\exp_{h,k}(x) = h(e^{k^{-1}(x)}). \tag{3.9}$$

3.2.2 Pseudo-Moment Generating Functions

Based on the pseudo-exponential function of type (3.9), which depends on the specific choice of $h \in \mathcal{G}$ and $k \in \mathcal{S}$, we can define a family of pseudo-moment generating functions, parametrized by the pair $(h, k) \in \mathcal{G} \times \mathcal{S}$.

Definition 3.2.1. *Let $(h, k) \in \mathcal{G} \times \mathcal{S}$ and $(\Omega, \mathcal{F}, m_h)$ be a measurable space provided with a regular σ-$\oplus_h$-additive fuzzy measure m_h. Let X be a non-negative random variable. We define the (h, k)-pseudo-moment generating function as*

$$M_X^{(h,k)}(t) = \tilde{E}^{m_h}\left[\exp_{h,k}^{\otimes_h t}(X)\right], \tag{3.10}$$

where $\exp_{h,k}^{\otimes_h t}(t) = \left(\exp_{h,k}(t)\right)^{\otimes_h t}$.

Remark 3.2.1. *When $k(x) = x$, Definition 3.2.1 corresponds to the definition of the pseudo-Laplace transform in Pap and Ralević (1998); therein, the authors introduce also the pseudo Laplace transform when the standard sum and the standard product are replaced by (2.1) and (2.2) respectively.*

By (2.9), the (h, k)-pseudo-moment generating function defined in (3.10) can be easily rewritten as

$$\begin{aligned} M_X^{(h,k)}(t) &= \tilde{E}^{m_h}[h(e^{tk^{-1}(X)})] = h(E^{\mathbb{P}}[e^{tk^{-1}(X)}]) = \\ &= h(M^{\mathbb{P}}_{k^{-1}(X)}(t)) = M^{(h,id)}_{k^{-1}(X)}(t) \end{aligned} \tag{3.11}$$

where $M^{\mathbb{P}}_{k^{-1}(X)}$ is the standard moment-generating function of $k^{-1}(X)$ with respect to the probability $\mathbb{P} = h^{-1} \circ m_h$.
So, $M_X^{(h,k)}$ and $M^{\mathbb{P}}_{k^{-1}(X)}$ share the same domain and $M_X^{(h,k)}$ identifies the σ-$\oplus_h$-additive regular fuzzy measure induced by X on $\mathbb{R}_+$.
An explicit expression for $M_X^{(h,k)}$ can be trivially recovered in the cases in which we are dealing with random variables whose transformation through k^{-1} has a known standard moment generating function.

Example 3.2.1. *If, under $P = h^{-1} \circ m_h$, $k^{-1}(X) \stackrel{d}{=} Poi(\lambda)$, then, clearly, $M_X^{(h,k)}(t) = h(e^{\lambda(e^t-1)})$, while, if, under $P = h^{-1} \circ m_h$, $k^{-1}(X) \stackrel{d}{=} \Gamma(\alpha, \mu)$, then $M_X^{(h,k)}(t) = h(\mu^\alpha(\mu - t)^{-\alpha}), t < \mu$.*

Given a non-negative random variable X, we define its n-th pseudo-moment as $\tilde{E}^{m_h}[X^{\otimes_h n}]$ (when it is finite) that can be determined using the pseudo moment-generating function via pseudo-differentiation.

Proposition 3.2.3. *Let X be a non-negative random variable on $(\Omega, \mathcal{F}, m_h)$ so that $M_X^{(h,h)}$ is defined in a neighborhood of 0. Then, for $n \in \mathbb{N}$,*

$$\frac{{}^{\oplus_h} d^n}{dt^n} M_X^{(h,h)}(t) \bigg|_{t=0} = \tilde{E}^{m_h}[X^{\otimes_h n}]. \tag{3.12}$$

Proof. By applying the n-th pseudo differentiation to (3.11) with $k = h$, we have that

$$\frac{{}^{\oplus_h} d^n}{dt^n} M_X^{(h,h)}(t) = h\left(\frac{d^n}{dt^n} M^{\mathbb{P}}_{h^{-1}(X)}(t)\right).$$

But

$$\frac{d^n}{dt^n} M^{\mathbb{P}}_{h^{-1}(X)}(t)\bigg|_{t=0} = E^{\mathbb{P}}[(h^{-1}(X))^n] = h^{-1}\left(\tilde{E}^{m_h}\left[h\left(\left(h^{-1}(X)\right)^n\right)\right]\right),$$

from which the conclusion follows. □

3.2.3 Joint Pseudo-Moment Generating Function

In this subsection we will extend the definition of pseudo-moment generating function to random vectors, in line with the classical case.

Let $(h, k_1, k_2) \in \mathcal{G} \times \mathcal{S}^2$ and (X_1, X_2) be a $\mathbb{R}_+ \times \mathbb{R}_+$-valued random vector defined on $(\Omega, \mathcal{F}, m_h)$ where m_h is a regular σ-$\oplus_h$-additive fuzzy measure.

We define

$$M^{(h,k_1,k_2)}_{X_1,X_2}(s,t) = \tilde{E}^m \left[\exp^{\otimes_h s}_{h,k_1}(X_1) \otimes_h \exp^{\otimes_h t}_{h,k_2}(X_2)\right] \tag{3.13}$$

the "(h,k_1,k_2)-pseudo joint moment generating function".
Exactly as for the one-dimensional version, we have that

$$\begin{aligned} M^{(h,k_1,k_2)}_{X_1,X_2}(s,t) &= h\left(M^{\mathbb{P}}_{k_1^{-1}(X_1),k_2^{-1}(X_2)}(s,t)\right) = \\ &= M^{(h,id,id)}_{k_1^{-1}(X_1),k_2^{-1}(X_2)}(s,t), \end{aligned} \tag{3.14}$$

where $M^{\mathbb{P}}_{U,W}(s,t)$ is the standard moment generating function of the random vector (U,W) with respect to $\mathbb{P} = h^{-1} \circ m_h$. Similarly as in the one-dimensional case, $M^{(h,k_1,k_2)}_{X_1,X_2}$ and $M^{\mathbb{P}}_{k_1^{-1}(X_1),k_2^{-1}(X_2)}$ share the same domain and $M^{(h,k_1,k_2)}_{X_1,X_2}$ identifies the σ-$\oplus_h$-additive regular fuzzy measure induced by (X_1,X_2) on $\mathbb{R}_+ \times \mathbb{R}_+$ since the dependence structure between X_1 and X_2 is the same as that of $k_1^{-1}(X_1)$ and $k_2^{-1}(X_2)$.

As the classical moment generating function of two independent random variables decomposes into the product of the marginal moment generating functions, the (h,k_1,k_2)-pseudo-moment generating function in (3.13) of two h-independent random variables can be written as the pseudo-product $\otimes_h$ of the one-dimensional corresponding (h,k_i)-pseudo-moment generating functions, $i = 1,2$, as we show in the following proposition.

Theorem 3.2.1. *Let $(k_1,k_2) \in \mathcal{S}^2$ and X_1 and X_2 be non-negative random variables on $(\Omega, \mathcal{F}, m_h)$. They are h-independent if and only if*

$$M^{(h,k_1,k_2)}_{X_1,X_2}(s,t) = M^{(h,k_1)}_{X_1}(s) \otimes_h M^{(h,k_2)}_{X_2}(t) \tag{3.15}$$

and this is true for all choices of $(k_1,k_2) \in \mathcal{S}^2$.

Proof. By (3.14) and (3.11) and the fact that h-independence is equivalent to $\mathbb{P} = h^{-1} \circ m_h$-independence, we have that

$$\begin{aligned} M^{(h,k_1,k_2)}_{X_1,X_2}(s,t) &= h\left(M^{\mathbb{P}}_{k_1^{-1}(X_1),k_2^{-1}(X_2)}(s,t)\right) = \\ &= h\left(M^{\mathbb{P}}_{k_1^{-1}(X_1)}(s) \cdot M^{\mathbb{P}}_{k_2^{-1}(X_2)}(t)\right) = \\ &= h\left(h^{-1}(M^{(h,k_1)}_{X_1}(s)) \cdot h^{-1}(M^{(h,k_2)}_{X_2}(t))\right). \end{aligned}$$

□

Example 3.2.2. *Let X_1 and X_2 be h-independent so that, under $P = h^{-1} \circ m_h$, $k_i^{-1}(X_i)$ is Bernoulli distributed with parameter π_i, $i = 1, 2$. Then, by Theorem 3.2.1,*

$$M_{X_1,X_2}^{(h,k_1,k_2)}(s,t) = h\left((e^t\pi_1 + 1 - \pi_1)(e^t\pi_2 + 1 - \pi_2)\right).$$

In the standard probability case, the moment generating function of the sum of two independent random variables decomposes into the product of the corresponding one-dimensional moment generating functions. A similar result can be obtained for a general non-commutative pseudo-sum of h-independent random variables.

Proposition 3.2.4. *Let $h \in \mathcal{G}$, $k_1, k_2 \in \mathcal{S}$ and X_1 and X_2 be two h-independent and non-negative random variables on $(\Omega, \mathcal{F}, m_h)$. Then, for any $k \in \mathcal{S}$,*

$$M^{(h,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t) = M_{X_1}^{(h,k_1)}(t) \otimes_h M_{X_2}^{(h,k_2)}(t). \tag{3.16}$$

Proof. Since

$$\begin{aligned} M_{X_1,X_2}^{(h,k_1,k_2)}(t,t) &= \tilde{E}^{m_h}\left[h\left(e^{t\left(k_1^{-1}(X_1)+k_2^{-1}(X_2)\right)}\right)\right] = \\ &= \tilde{E}^{m_h}\left[h\left(e^{tk^{-1}\left(X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2\right)}\right)\right] = \\ &= M^{(h,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t), \end{aligned}$$

the conclusion follows from Proposition 3.2.1. □

From (3.16), depending on the choice of the generators, we can obtain several interesting particular cases:

1. If $k = k_1 = k_2 = id$, we get
$$M_{X_1+X_2}^{(h,id)}(t) = M_{X_1}^{(h,id)}(t) \otimes_h M_{X_2}^{(h,id)}(t) \tag{3.17}$$
that is the equivalent of the statement of Theorem 3 in Pap and Ralević (1998) where the authors extend the relationship between Laplace transform and convolution to the pseudo-Laplace transform and the pseudo-convolution.

2. If $h = id$,
$$M^{(id,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t) = M_{X_1}^{(id,k_1)}(t) \cdot M_{X_2}^{(id,k_2)}(t)$$
that corresponds to the case in which the underlying fuzzy measure is indeed a probability.

3. In the semiring framework, that is when $h = k = k_1 = k_2 \in \mathcal{G}$, we obtain
$$M^{(h,h)}_{X_1 \oplus_h X_2}(t) = M^{(h,h)}_{X_1}(t) \otimes_h M^{(h,h)}_{X_2}(t).$$

Example 3.2.3. *Here we will provide some examples of the above cases.*

- *If, under $P = h^{-1} \circ m_h$, for $i = 1, 2$, $k_i^{-1}(X_i)$ is Poisson distributed with parameter λ_i, with X_1 and X_2 h-independent, then*
$$M^{(h,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t) = h(e^{(\lambda_1+\lambda_2)(e^t-1)}).$$
In particular, if $h(x) = \frac{e^{\theta x}-1}{e^\theta-1}, \theta \neq 0$ and $k_i = id$, $i = 1, 2$ and $k = id$ we get
$$M^{(h,id)}_{X_1+X_2}(t) = \frac{\exp\left(\theta e^{(\lambda_1+\lambda_2)(e^t-1)}\right) - 1}{e^\theta - 1}.$$
While, if, under $P = h^{-1} \circ m_h$, for $i = 1, 2$, $k_i^{-1}(X_i)$ is Gamma distributed, that is $k_i^{-1}(X_i) \overset{d}{=} \Gamma(\alpha_i, \mu)$, with X_1 and X_2 h-independent, then
$$M^{(h,k)}_{X_1 {}_{k_1}\overset{k}{\oplus}_{k_2} X_2}(t) = h(\mu^{\alpha_1+\alpha_2}(\mu - t)^{-(\alpha_1+\alpha_2)}).$$

- *Let $h(x) = \frac{e^{\theta x}-1}{e^\theta-1}, \theta \neq 0$. Let X_1 and X_2 be h-independent and assume that, under $P = h^{-1} \circ m_h$, $h^{-1}(X_i)$ is Bernoulli distributed with parameter π_i, $i = 1, 2$. Then*
$$M^{(h,h)}_{X_1 \oplus_h X_2}(t) = \frac{e^{\theta(e^t\pi_1+1-\pi_1)(e^t\pi_2+1-\pi_2)} - 1}{e^\theta - 1}.$$
The first pseudo moment of $X_1 \oplus_h X_2$, using equation (3.12), is given by
$$\tilde{E}^{m_h}[X_1 \oplus_h X_2] = \frac{e^{\theta(\pi_1+\pi_2)} - 1}{e^\theta - 1}.$$

3.2.4 Pseudo-Moment Generating Functions and Dependence Structure

As discussed at the end of subsection 3.1.1, there is a one-to-one correspondence between the semi-copula associated to a vector (X_1, X_2) with respect to the σ-$\oplus_h$-additive fuzzy measure m_h and the copula associated with respect to the probability $\mathbb{P} = h^{-1} \circ m_h$ (see formula (3.1)). Choosing in the appropriate way $k_1, k_2 \in \mathcal{S}$, we will show that we can separate, in the expression of the pseudo-joint moment generating function, the contribution of the marginal fuzzy distributions from that of the dependence structure. Let $(\Omega, \mathcal{F}, m_h)$ with m_h σ-$\oplus_h$-additive regular fuzzy measure and (X_1, X_2) a random vector with values in $\mathbb{R}_+ \times \mathbb{R}_+$. We assume that with respect to $\mathbb{P} = h^{-1} \circ m_h$, its joint cumulative distribution function is, for $x, y > 0$, $F(x, y) = C\left(F_{X_1}(x), F_{X_2}(y)\right)$, where C is a copula and F_{X_i}, $i = 1, 2$ are the corresponding marginal cumulative distributions that we assume to be strictly increasing on $\mathbb{R}_+$.

Let G be a benchmark cumulative distribution function with $G(0) = 0$ and strictly increasing on $\mathbb{R}_+$. If $k_i(x) = F_{X_i}^{-1} \circ G(x) \in \mathcal{S}$ for $i = 1, 2$, then, for every $z > 0$, $F_{k_i^{-1}(X_i)}(z) = G(z)$. It follows that, if under $\mathbb{P}$ the cumulative distribution function of the vector (V_1, V_2) is given by $F_{V_1,V_2}(x, y) = C\left(G(x), G(y)\right)$, then

$$\begin{aligned} M_{X_1,X_2}^{(h,k_1,k_2)}(s,t) &= h\left(M^P_{k_1^{-1}(X_1),k_2^{-1}(X_2)}(s,t)\right) = \\ &= h\left(M^P_{V_1,V_2}\left(s,t\right)\right) = \\ &= M_{V_1,V_2}^{(h,id,id)}\left(s,t\right). \end{aligned}$$

This way, in $M_{X_1,X_2}^{h,F_{X_1}^{-1}\circ G,F_{X_2}^{-1}\circ G}$, we have separated the marginal distributions from the dependence structure, since $M_{U,V}^{(h,id,id)}$ depends only on the latter.

Example 3.2.4. *In the particular case in which the random variables are h-independent and the reference distribution G is the exponential one with parameter equal to 1, for $k_i(x) = F_{X_i}^{-1}\left(1 - e^x\right)$, $i = 1, 2$, we get*

$$M_{X_1,X_2}^{(h,k_1,k_2)}(s,t) = M_{X_1}^{(h,k_1)}(s) \otimes_h M_{X_2}^{(h,k_2)}(t) = h\left(\frac{1}{(1-s)(1-t)}\right), s, t < 1.$$

3.3 Pseudo-Schur Constant Random Vectors

In this section we show that some characterization of Schur-constant random vectors can be extended to a more general family of random vectors with Archimedean dependence using pseudo-operations and pseudo-integrals. More precisely, we show that the tools related to pseudo-fuzzy measures can be used to provide characterization results for the underlying probability measure.

We start recalling the notion of Schur-constant bivariate random vector (see, among the others, Barlow and Mendel, 1992, Caramellino and Spizzichino, 1994, Nelsen, 2005 and Chi et al., 2009).

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let X and Y be two continuous positive random variables. Then the random vector (X, Y) is Schur-constant if and only if there exists a convex survival function S such that

$$\bar{F}_{X,Y}(x, y) = P(X > x, Y > y) = S(x + y), \ \forall (x, y) \in R_+ \times R_+. \qquad (3.18)$$

The definition implies that the random vector is exchangeable with survival marginal distribution S and survival copula of Archimedean type with generator S.

Remark 3.3.1. *Substituting into equation (3.18) the standard sum by the idempotent one (2.1), we get*

$$\bar{F}_{X,Y}(x, y) = S(\sup(x, y)),$$

which is the bivariate survival function of a random vector with perfectly positive dependent components with common marginal survival function S.

Starting from the general non-commutative pseudo-sum defined in (2.5), we generalize (3.18) considering joint survival distributions of type

$$\bar{F}_{X,Y}(x, y) = S\left(x_{k_1} \overset{k}{\oplus}_{k_2} y\right), \qquad (3.19)$$

where $k, k_1, k_2 \in \mathcal{S}$ and S is a survival function such that $S \circ k$ is convex. A random vector (X, Y) with survival distribution (3.19) will be called "pseudo-Schur-constant".

We notice that representation (3.19) is not unique. In addition to Remark 2.1.1, we have that, if $k, \hat{k}, k_1, k_2 \in \mathcal{S}$, and S and $\hat{S}$ are univariate survival distribution functions, then

$$S\left(x_{k_1} \overset{k}{\oplus}_{k_2} y\right) = \hat{S}\left(x_{k_1} \overset{\hat{k}}{\oplus}_{k_2} y\right)$$

if and only if $\hat{S}(z) = S \circ k \circ \hat{k}^{-1}(z)$. It can be easily checked that the family in (3.19) coincides with the family of survival distributions of positive random pairs (X, Y) with marginal survival distributions $\bar{F}_X(x) = S \circ k \circ k_1^{-1}(x)$ and $\bar{F}_Y(y) = S \circ k \circ k_2^{-1}(y)$, respectively, and survival copula function of Archimedean type with generator $S \circ k$.

Remark 3.3.2. *Notice that*

$$\bar{F}_{X,Y}(x, y) = S\left(x_{k_1} \overset{k}{\oplus}_{k_2} y\right) = \bar{F}_X(x) \otimes_\psi \bar{F}_Y(y)$$

with $\psi \in \mathcal{G}$ such that $\psi(t) = S \circ k(-\ln t)$ for $t \in [0, 1]$ and $\bar{F}_{X,Y}$ can be interpreted as the fuzzy survival distribution function of (X, Y) with respect to a σ-$\oplus_\psi$ additive measure μ_ψ under which X and Y are ψ-independent.

As a consequence of Remark 3.3.2 and Theorem 3.2.1 we obtain the following characterization and decomposition:

Proposition 3.3.1. (X, Y) *is a pseudo-Schur constant random vector under $\mathbb{P}$ with survival function (3.19) if and only if there exists $\psi \in \mathcal{G}$ and a σ-$\oplus_\psi$ additive measure μ_ψ that generates the same joint fuzzy survival distribution function and under which*

$$M_{X,Y}^{(\psi,k_1,k_2)}(s, t) = M_X^{(\psi,k_1)}(s) \otimes_\psi M_Y^{(\psi,k_2)}(t).$$

Proposition 5.7 in Kozlova and Salminen (2004) (see also Theorem 2.5 in Ta and Van (2017) for an alternative proof) provides a characterization of the Schur-constancy of a random vector (X, Y) in terms of the moment generating function of the sum $X + Y$. More precisely, it states that (X, Y) is Schur-constant if and only if there exists a random variable Z with the same distribution as $X + Y$ such that, for all $s, t < 0$, $s \neq t$

$$\mathbb{E}\left[e^{sX+tY}\right] = \frac{\int_t^s \mathbb{E}\left[e^{rZ}\right] dr}{s - t}.$$

This result can be generalized to pseudo-Schur-constant random vectors using the pseudo-moment generating functions defined in (3.10) and (3.13) and the pseudo-integral given in (2.8).

Theorem 3.3.1. (X, Y) *is a pseudo-Schur-constant random vector with joint survival distribution function (3.19) if and only if there exists $h \in \mathcal{G}$ such that under $m_h = h \circ \mathbb{P}$, for all $s, t < 0$,*

$$M_{X,Y}^{(h,k_1,k_2)}(s, t) = \begin{cases} \left(\int_{(s,t)}^{\oplus_h} M_Z^{(h,k)} \otimes_h d\mathcal{L}_h\right) \oslash_h h(t - s), & s < t \\ \left(\int_{(t,s)}^{\oplus_h} M_Z^{(h,k)} \otimes_h d\mathcal{L}_h\right) \oslash_h h(s - t), & s > t, \end{cases} \tag{3.20}$$

where $Z \stackrel{d}{=} X_{k_1} \overset{k}{\oplus}_{k_2} Y$ *and where* $\mathcal{L}_h = h \circ \mathcal{L}$ *with* $\mathcal{L}$ *the Lebesgue measure on the real line. If there exists* $h \in \mathcal{G}$ *for which (3.20) is true, then (3.20) holds true for all* $h \in \mathcal{G}$.

Proof. Clearly, (3.19) is equivalent to the fact that $(k_1^{-1}(X), k_2^{-1}(Y))$ is a Schur-constant random vector with survival function $S \circ k$. Thanks to the mentioned result in Kozlova and Salminen (2004) and Ta and Van (2017), we have that (3.19) is equivalent to

$$M^P_{k_1^{-1}(X),k_2^{-1}(Y)}(s,t) = \frac{1}{t-s}\int_s^t M_V^P(r)dr, \quad s,t<0,\ s \neq t,$$

where $V \stackrel{d}{=} k_1^{-1}(X) + k_2^{-1}(Y)$. Then, given any $h \in \mathcal{G}$ and considering the σ-$\oplus_h$-additive fuzzy measure $m_h = h \circ P$, by (3.14) and (3.11),

$$\begin{aligned} M^{(h,k_1,k_2)}_{X,Y}(s,t) &= h\left(\frac{1}{t-s}\int_s^t M_V^P(r)\ dr\right) = \\ &= h\left(\frac{1}{t-s}\int_s^t h^{-1}\left(M_Z^{(h,k)}(r)\right)\ dr\right) \end{aligned}$$

and the thesis trivially follows from (2.8). □

An immediate consequence of this result is that, for $v < 0$,

$$M_X^{(h,k_1)}(v) = M_Y^{(h,k_2)}(v) = \left[\int_{(v,0)}^{\oplus_h} M_Z^{(h,k)}(r) \otimes_h d\mathcal{L}_h\right] \oslash_h\ h(-v). \qquad (3.21)$$

Since the distribution of Z depends on both k_1 and k_2 while each marginal pseudo-moment generating function depends only on one of them, the marginal distributions of X and Y only depend on Z through the dependence structure between X and Y.
From (3.20) and (3.21) we have that

$$M^{(h,k_1,k_2)}_{X,Y}(s,t) = \begin{cases} h\left(\frac{-t}{t-s}\right) \otimes_h M_Y^{(h,k_2)}(t) \ominus_h\ h\left(\frac{-s}{t-s}\right) \otimes_h M_X^{(h,k_1)}(s),\ t<s \\ h\left(\frac{-s}{s-t}\right) \otimes_h M_X^{(h,k_1)}(s) \ominus_h\ h\left(\frac{-t}{s-t}\right) \otimes_h M_Y^{(h,k_2)}(t),\ t>s \end{cases}$$

and the joint pseudo-moment generating function can be written in terms of the marginal pseudo-moment generating functions.

It is well known that (X,Y) is a continuous Schur-constant random vector if and only if there exist a random variable U uniformly distributed in the interval $(0,1)$ and a positive random variable V such that

$$(X,Y) \stackrel{d}{=} (U\ V, (1-U)\ V), \qquad (3.22)$$

with U independent of V and $V \stackrel{d}{=} X + Y$ (see, for example, Theorem 2.1 in Chi et al., 2009). (3.22) can be generalized to the case in which (X, Y) is a pseudo-Schur constant random vector, by considering the non-commutative pseudo-sum in place of the classical sum and the pseudo-multiplication in place of the standard one, as we show in the following Proposition.

Proposition 3.3.2. *(X, Y) is a pseudo-Schur constant random vector with survival function (3.19) with $k_1, k_2 \in \mathcal{G}$ if and only if there exists a random variable W with values in $(0, 1)$ and cumulative distribution function k_1^{-1} such that*

$$(X, Y) \stackrel{d}{=} \left(W \otimes_{k_1} Z_1, \left(1_{k_1} \overset{k_2}{\ominus}_{k_1} W \right) \otimes_{k_2} Z_2 \right), \tag{3.23}$$

where $Z_1 \stackrel{d}{=} X_{k_1} \overset{k_1}{\oplus}_{k_2} Y$ and $Z_2 \stackrel{d}{=} X_{k_1} \overset{k_2}{\oplus}_{k_2} Y$ are both independent of W.

Proof. Thanks to (3.22), the fact that $(k_1^{-1}(X), k_2^{-1}(Y))$ is a Schur-constant random vector is equivalent to

$$\left(k_1^{-1}(X), k_2^{-1}(Y)\right) \stackrel{d}{=} (U\ V, (1 - U)\ V)\,,$$

with U uniformly distributed in $(0, 1)$ and independent of $V \stackrel{d}{=} k_1^{-1}(X) + k_2^{-1}(Y)$. Then, we have

$$(X, Y) \stackrel{d}{=} (k_1\,(U\ V), k_2\,((1 - U)\ V)))\,,$$

that is equivalent to

$$(X, Y) \stackrel{d}{=} (k_1(U) \otimes_{k_1} Z_1, k_2(1 - U) \otimes_{k_2} Z_2)\,,$$

where $Z_1 \stackrel{d}{=} X_{k_1} \overset{k_1}{\oplus}_{k_2} Y$, $Z_2 \stackrel{d}{=} X_{k_1} \overset{k_2}{\oplus}_{k_2} Y$ and the conclusion follows. □

Remark 3.3.3. *In the semiring framework, that is when only one generator $h \in \mathcal{G}$ is involved, we obtain exactly the same results as for the Schur-constant case, just substituting the standard addition and multiplication with the pseudo-ones. More precisely, if $h \in \mathcal{G}$, the following conditions are equivalent:*

- *under $\mathbb{P}$, (X, Y) is distributed according to the survival distribution function*

$$\bar{F}(x, y) = S\left(x \oplus_h y\right), x, y \geq 0 \tag{3.24}$$

or, equivalently, under $m_h = h \circ \mathbb{P}$, according to the fuzzy survival distribution

$$\bar{F}(x, y) = \hat{S}\left(x \oplus_h y\right), x, y \geq 0$$

where $\hat{S} = h \circ S$;

- *the joint pseudo-moment generating function with respect to* $m_h = h \circ \mathbb{P}$ *is of the form*

$$M_{X,Y}^{(h,h,h)}(s,t) = \begin{cases} \left[\int_{(s,t)}^{\oplus} M_Z^{(h,h)}(r) \otimes_h d\mathcal{L}_h\right] \oslash_h h(t-s), & \text{when } s < t \\ \left[\int_{(t,s)}^{\oplus} M_Z^{(h,h)}(r) \otimes_h d\mathcal{L}_h\right] \oslash_h h(s-t), & \text{when } s > t \end{cases}$$

  with $Z \stackrel{d}{=} X \oplus_h Y$ *and* $\mathcal{L}_h = h \circ \mathcal{L}$ *where* $\mathcal{L}$ *is the Lebesgue measure on the real line;*

- *under* $m_h = h \circ \mathbb{P}$*, there exist a continuous random variable* W *with fuzzy cumulative distribution function* $F_W(\omega) = \omega$*, for* $0 \leq \omega \leq 1$*, and a random variable* $Z \stackrel{d}{=} X \oplus_h Y$ *such that* W *and* Z *are* h*-independent and*

$$(X,Y) \stackrel{d}{=} (W \otimes_h Z, (1 \ominus_h W) \otimes_h Z).$$

Notice that (3.24) coincides with the time-transformed exponential model introduced in Bassan and Spizzichino (2005).

Remark 3.3.4. *Analogous characterizations to those provided in Theorem 3.3.1 and Proposition 3.3.2 can be obtained for pseudo-Schur constant discrete positive random variables by extending the corresponding results for Schur-constant discrete positive random vectors proved in Kolev and Mulinacci (2022).*

Chapter 4

A Generalization of Lack-of-Memory Properties

In this chapter, we'll give a generalization of bivariate lack-of-memory properties, in strong and weak versions: in particular, we'll analyse the dependence structure of the distribution satisfying generalised bivariate weak lack-of-memory property and we'll give two possible characterizations of it.

4.1 Pseudo Lack-of-Memory Properties

In this section, we generalize univariate and bivariate lack-of-memory properties by substituting in the associated functional equations the standard product by the pseudo one. In order to do that, we'll consider generators belonging to the class of functions $\mathcal{I}$, see (2.10). From now on, since the pseudo product will depend always on h, by $\otimes$ we'll mean $\otimes_h$: we notice that the functions $h(x)$ and $h(x^\alpha), \alpha > 0$ define the same pseudo product, as we show in the following lemma.

Lemma 4.1.1. *The functions $h : [0,1] \to [0,1]$ and $\hat{h} : [0,1] \to [0,1]$ generate the same pseudo product, id est*

$$a \otimes_h b = a \otimes_{\hat{h}} b, \ \forall a, b \in [0,1],$$

if and only if there exists $\alpha > 0$ such that $h(x) = \hat{h}(x^\alpha), \forall x \in [0,1]$. In fact,

$$a \otimes_h b = a \otimes_{\hat{h}} b, \ \forall a, b \in [0,1]$$

is equivalent to

$$\hat{h}^{-1}(a)\hat{h}^{-1}(b) = \hat{h}^{-1}(h(h^{-1}(a) \ h^{-1}(b))).$$

Setting $x = h^{-1}(a)$, $y = h^{-1}(b)$, *we have*

$$\hat{h}^{-1}(h(x))\ \hat{h}^{-1}(h(y)) = \hat{h}^{-1}(h(x\ y)). \tag{4.1}$$

Now let us define $g(x) = \hat{h}^{-1}(h(x))$, *then equation (4.1) becomes*

$$g(x)g(y) = g(xy).$$

The solution of the latter is well-known and is given by $g(x) = x^\alpha$, *for some* $\alpha \in \mathbb{R}$, *meaning that* $h(x) = \hat{h}(x^\alpha)$. *The fact that* α *must be larger than* 0 *comes from the fact that* h *and* $\hat{h}$ *must be increasing functions in* $[0, 1]$.

4.1.1 Univariate Pseudo Lack-of-Memory Property

Definition 4.1.1. *We say that a survival distribution* $\bar{F}$ *satisfies the univariate pseudo lack-of-memory property if*

$$\bar{F}(s + t) = \bar{F}(s) \otimes \bar{F}(t),\ \ s, t \geq 0, \tag{4.2}$$

where $\otimes$ *is the pseudo product given in Equation (2.4).*

It is trivial to prove that the solution of the functional equation (4.2) is given by

$$\bar{F}(x) = h(e^{-\lambda x}) = \exp_h(\lambda x), \lambda > 0. \tag{4.3}$$

A random variable with survival distribution function (4.3) will be called "pseudo exponential" random variable with parameter λ.

Remark 4.1.1. *Notice that any univariate survival distribution satisfies the pseudo lack-of-memory property with respect to a suitable pseudo product. In fact, let* $\bar{H}$ *be a univariate survival function: then it satisfies the pseudo univariate lack-of-memory property with respect to* $h(x) = \bar{H}(-\log x)$.

Remark 4.1.2. *Substituting into equation (2.11) the standard product by the idempotent product (2.2), we get the following functional equation:*

$$\bar{F}(x + t) = \inf(\bar{F}(x), \bar{F}(t)). \tag{4.4}$$

However, the solution of the function above is not a bivariate survival function. In fact, let us consider the value of $\bar{F}(t)$ *for some* $t > 0$*: after* n *iterations, we have*

$$\bar{F}(t) = \bar{F}\left(\frac{t}{2} + \frac{t}{2}\right) = \bar{F}\left(\frac{t}{2}\right) = \cdots = \bar{F}\left(\frac{t}{2^n}\right).$$

If we fix t *and we let* $n \to \infty$, $\bar{F}\left(\frac{t}{2^n}\right) \to 1$, *meaning that* $\bar{F}(t) = 1$ *for any* $t > 0$, *so* $\bar{F}$ *is not a survival function of a real-valued random variable.*

4.1.2 Bivariate Pseudo Strong Lack-of-Memory Property

Similarly, we now generalize the bivariate strong lack-of-memory property.

Definition 4.1.2. *A bivariate distribution $\bar{F}$ satisfies the bivariate pseudo strong lack-of-memory property if*

$$\bar{F}(s_1+t_1, s_2+t_2) = \bar{F}(s_1, s_2) \otimes \bar{F}(t_1, t_2), \forall s_1, s_2, t_1, t_2 \geq 0. \tag{4.5}$$

The solution of the above functional equation is given by

$$\bar{F}(s,t) = h(\exp(-\lambda_1 s - \lambda_2 t)) = \exp_h(\lambda_1 s + \lambda_2 t),\ \lambda_1, \lambda_2 > 0. \tag{4.6}$$

In fact, (4.5) is equivalent to

$$h^{-1}(\bar{F}(s_1+t_1, s_2+t_2)) = h^{-1}(\bar{F}(s_1, s_2))\ h^{-1}(\bar{F}(t_1, t_2)) :$$

setting $\bar{F}(x,y) = h(\bar{G}(x,y))$, we obtain equation (2.12), whose solution is given by (2.13). Now, let us consider the following function:

$$\begin{aligned} \bar{C}(u,v) &= \bar{F}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)) = \\ &= h(h^{-1}(u)h^{-1}(v)) = u \otimes v : \end{aligned} \tag{4.7}$$

the function (4.7) is the survival copula of the distribution (4.6) if and only if $h^{-1}(x)$ is log-concave, id est $\log(h^{-1}(x))$ is concave. Moreover, setting $\phi(t) = h(e^{-t})$, it is possible to show that $\bar{C}(u,v) = \phi(\phi^{-1}(u)+\phi^{-1}(v))$, which is a copula if and only if ϕ is convex, see Nelsen (2006): this is equivalent to the log-convexity of h^{-1}.

Remark 4.1.3. *Survival functions of type (4.6) have been identified in Proposition 3.1 in Genest and Kolev (2021), with $\psi(x) = h(e^{-x})$, according to the notation therein.*

4.1.3 Bivariate Pseudo Weak Lack-of-Memory Property

The generalization of the bivariate weak lack-of-memory property is obtained substituting into equation (2.14) the standard product by the pseudo one.

Definition 4.1.3. *A bivariate distribution $\bar{F}$ satisfies the bivariate pseudo weak lack-of-memory property if*

$$\bar{F}(s_1+t, s_2+t) = \bar{F}(s_1, s_2) \otimes \bar{F}(t,t), \forall s_1, s_2, t \geq 0. \tag{4.8}$$

It can be easily verified that the solution of the functional equation given in Definition 4.1.3 is:

$$\begin{aligned} \bar{F}(x,y) &= h(\bar{G}(x,y)) = \\ &= \begin{cases} h\left(e^{-\lambda y}\bar{G}_1(x-y)\right) & x \geq y \\ h\left(e^{-\lambda x}\bar{G}_2(y-x)\right) & x < y \end{cases} = \\ &= \begin{cases} \exp_h(\lambda y) \otimes \bar{F}_1(x-y) & x \geq y \\ \exp_h(\lambda x) \otimes \bar{F}_2(y-x) & x < y \end{cases}, \end{aligned} \tag{4.9}$$

where $\bar{G}$ satisfies standard weak-lack-of-memory property, with $\bar{G}_1$, $\bar{G}_2$, $\bar{F}_1$ and $\bar{F}_2$ marginal univariate survival functions of non-negative random variables such that $\bar{F}_i(x) = h(\bar{G}_i(x)),\ i = 1,2$.
Let us consider the following function $\bar{C}^F : [0,1] \times [0,1]$:

$$\begin{aligned} &\bar{C}^F(u,v) = \\ &= \bar{F}(\bar{F_1}^{-1}(u), \bar{F_2}^{-1}(v)) = \\ &= h(e^{-\lambda \bar{F_2}^{-1}(v)}\bar{G}_1(\bar{F_1}^{-1}(u) - \bar{F}_2^{-1}(v))1_{\bar{F_1}^{-1}(u)-\bar{F}_2^{-1}(v)\geq 0}+ \\ &+ e^{-\lambda \bar{F}_1^{-1}(u)}\bar{G}_2(\bar{F_2}^{-1}(v) - \bar{F}_1^{-1}(u))1_{\bar{F_2}^{-1}(v)-\bar{F}_1^{-1}(u)\geq 0}) = \\ &= h(\bar{C}^G(h^{-1}(u), h^{-1}(v))), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} \bar{C}^G(u,v) = \\ &= e^{-\lambda \bar{G}_2^{-1}(v)}\bar{G}_1(\bar{G}_1^{-1}(u) - \bar{G}_2^{-1}(v))1_{u<v}+ \\ &+ e^{-\lambda \bar{G}_1^{-1}(u)}\bar{G}_2(\bar{G}_2^{-1}(v) - \bar{G}_1^{-1}(u))1_{u\geq v}. \end{aligned} \tag{4.11}$$

By theorem 2.1 in Klement et al. (2005), the function above is a survival copula if $\bar{C}^G$ is a survival copula and if h is a strictly increasing and convex

bjection of the unit interval; in that case, due to Sklar theorem, the function (4.9) is a survival distribution.
However, for our purposes, we want that $\bar{F}$ is a bivariate survival function, regardless of the fact that $\bar{G}$ is a bivariate survival function or not: sufficient and necessary conditions under which $\bar{F}$ is a bivariate survival function are given in the following Proposition.

Proposition 4.1.1. *Let $\bar{F}_1$ and $\bar{F}_2$ be twice differentiable univariate marginal survival functions and let h be a twice differentiable generator with $h^{'}(x) > 0, \forall x \in [0,1]$. Then (4.9) is a survival function if and only if*

$$\begin{cases} \frac{\partial^2 \bar{F}(x,y)}{\partial x \partial y} \geq 0 \ \forall x \geq 0, \ y \geq 0 \ x \neq y \\ \lambda \leq \frac{f_1(0)+f_2(0)}{h^{'}(1)} \end{cases}, \tag{4.12}$$

where $f_i(x) = -\bar{F}_i^{'}(x), i = 1,2$. Moreover, if $\lambda < \frac{f_1(0)+f_2(0)}{h^{'}(1)}$, the distribution has a singularity along the line $x = y$ with probability mass $\frac{f_1(0)+f_2(0)}{\lambda h^{'}(1)} - 1$.

Proof. Under the considered assumptions, the second mixed derivative $\frac{\partial^2 \bar{F}}{\partial x \partial y}$ is well-defined. When $x > y$, we have $\bar{F}(x,y) = h(e^{-\lambda y} h^{-1}(\bar{F}_1(x-y)))$, so

$$\begin{aligned} &\int_0^\infty \int_0^x \frac{\partial^2 \bar{F}}{\partial x \partial y} \, dy \, dx = \int_0^\infty \frac{\partial \bar{F}(x,y)}{\partial x}\Big|_{y=0}^{x} \, dx \\ &= \int_0^\infty \left(-h'(\exp(-\lambda x)) \frac{\exp(-\lambda x)}{h'(1)} f_1(0) + f_1(x) \right) \, dx = \\ &= \frac{h(\exp(-\lambda x)) f_1(0)}{h^{'}(1)\lambda} - \bar{F}_1(x)\Big|_{x=0}^{\infty} = 1 - \frac{f_1(0)}{\lambda h^{'}(1)}; \end{aligned} \tag{4.13}$$

by the same way of reasoning, if $x < y$, we have that

$$\int_0^\infty \int_0^y \frac{\partial^2 \bar{F}}{\partial x \partial y} \, dx \, dy = 1 - \frac{f_2(0)}{\lambda h^{'}(1)}. \tag{4.14}$$

Hence, if the vector $(X,Y) \sim F$, it follows that

$$\begin{aligned} P(X = Y) &= 1 - \left(1 - \frac{f_1(0)}{\lambda h^{'}(1)}\right) - \left(1 - \frac{f_1(0)}{\lambda h^{'}(1)}\right) = \\ &= \frac{f_1(0) + f_2(0)}{\lambda h^{'}(1)} - 1, \end{aligned} \tag{4.15}$$

which is non-negative if and only if the last condition of the system (4.12) holds true: the fact that it is smaller than 1 comes from the fact that (4.13) and (4.14) are positive due to the first condition of the system (4.12). $\square$

In the rest of the thesis, we'll work under the assumptions given in Proposition 4.1.1.

Remark 4.1.4. *The singularity mass does not depend on the generator h but only on the function $\bar{G}$ satisfying functional equation (2.14) from which $\bar{F}$ is obtained as a distortion. In fact, recalling that $\bar{G}_i(x) = h^{-1}(\bar{F}_i(x)),\ i = 1,2$, it is easy to verify that $\frac{f_1(0)+f_2(0)}{\lambda h'(1)} - 1 = \frac{g_1(0)+g_2(0)}{\lambda} - 1$, where $g_i(z) = -\bar{G}_i'(z),\ i = 1,2$.*

Remark 4.1.5. *Let us suppose that the distribution of (X, Y) satisfies pseudo weak lack-of-memory property: then the distribution of the vector $(aX, aY),\ a > 0$ satisfies pseudo weak lack-of-memory property too.*

In fact,

$$\bar{F}_{aX,aY}(x,y) = P[aX > x, aY > y] = \\ = \begin{cases} h\left(e^{-\lambda(\frac{y}{a})}h^{-1}\left(\bar{F}_1(\frac{x-y}{a})\right)\right), & x \geq y \\ h\left(e^{-\lambda(\frac{x}{a})}h^{-1}\left(\bar{F}_2(\frac{y-x}{a})\right)\right), & x < y \end{cases} :$$

then it immediately follows that

$$\bar{F}_{aX,aY}(x+t, y+t) = \bar{F}_{aX,aY}(x,y) \otimes \bar{F}_{aX,aY}(t,t),$$

so $\bar{F}_{aX,aY}$ satisfies functional equation (4.8). Furthermore,

$$\frac{\partial^2 \bar{F}_{aX,aY}(x,y)}{\partial x \partial y} = \frac{1}{a^2}\frac{\partial^2 \bar{F}_{X,Y}(\hat{x},\hat{y})}{\partial x, \partial y},$$

where $\hat{x} = \frac{x}{a}$ and $\hat{y} = \frac{y}{a}$; finally, $P(aX, aY) = P(X = Y)$, so, if $\bar{F}_{X,Y}$ is a survival function, then $\bar{F}_{aX,aY}$ is a survival function too.

In the case in which $\bar{F}$ is not absolutely continuous, we can analyse how the probability mass is distributed on the line $x = y$.

Proposition 4.1.2. *Let $\bar{F}$ be a survival function satisfying Definition 4.1.3. Then, under the same assumptions of Proposition 4.1.1, we have*

$$P(X = Y, X \geq t) = \exp_h(\lambda t)\left(\frac{f_1(0) + f_2(0)}{\lambda h'(1)} - 1\right). \tag{4.16}$$

Proof. Since

$$P(X = Y, X \geq t) = \bar{F}(t,t) - P(X > Y \geq t) - P(Y > X \geq t)$$

and $\bar{F}(t,t) = \exp_h(\lambda t)$, we have

$$\begin{aligned}
P(X > Y \geq t) &= \int_t^{+\infty} \int_t^x \frac{\partial^2 \bar{F}(x,y)}{\partial x \partial y} \, dy \, dx = \\
&= \int_t^{\infty} \frac{\partial \bar{F}(x,y)}{\partial x} \Big|_{y=t}^{x} \, dx = \\
&= \int_t^{\infty} -h'(e^{-\lambda x}) e^{-\lambda x} \frac{f_1(0)}{h'(1)} \, dx + \\
&+ h'(e^{-\lambda t} h^{-1}(\bar{F}_1(x-t))) \; e^{-\lambda t} \frac{f_1(x-t)}{h'(h^{-1}(\bar{F}_1(x-t)))} \, dx = \\
&= h(e^{-\lambda t}) \left(1 - \frac{f_1(0)}{\lambda h'(1)}\right).
\end{aligned}$$

By the same way of reasoning,

$$P(Y > X \geq t) = h(e^{-\lambda t}) \left(1 - \frac{f_2(0)}{\lambda h'(1)}\right),$$

from which the conclusion follows trivially. □

We now provide an example in which $\bar{F}$ is a survival function satisfying (4.8) but $\bar{G} = h^{-1}(\bar{F})$ is not a survival function too.

Example 4.1.1. *Let* $h(x) = \left(\frac{e^{\theta x}-1}{e^{\theta}-1}\right)^{\beta}$ *and let* $\bar{F}_1(x) = \bar{F}_2(x) = h((1+x)^{-\alpha})$, *with* $\alpha > 0$, $\beta > 1$, $\theta > 0$ *and* $\max\left(\alpha + \frac{1}{\beta}, \frac{\alpha+1+\alpha\beta}{\beta+1}\right) \leq \lambda \leq \min(\alpha+1, 2\alpha)$. *Then the function* $\bar{F}$ *satisfying definition 4.1.3 with marginals* $\bar{F}_1$, $\bar{F}_2$ *is given by*

$$\begin{aligned}
&\bar{F}_{X,Y}(x,y) = \\
&\quad = \left(\frac{e^{\theta e^{-\lambda y}(1+x-y)^{-\alpha}} - 1}{e^{\theta} - 1}\right)^{\beta} 1_{x \geq y} + \\
&\quad + \left(\frac{e^{\theta e^{-\lambda x}(1+y-x)^{-\alpha}} - 1}{e^{\theta} - 1}\right)^{\beta} 1_{x < y}.
\end{aligned}$$

Moreover, let us consider the function $\bar{G}(x,y) = h^{-1}(\bar{F}_{X,Y}(x,y))$, *that is given by*

$$\bar{G}(x,y) = e^{-\lambda y}(1+x-y)^{-\alpha} 1_{x \geq y} + e^{-\lambda x}(1+y-x)^{-\alpha} 1_{x<y} :$$

then its second mixed derivative $g(x,y)$ *is given by*

$$\begin{aligned} g(x,y) = & \frac{\alpha \cdot (-\lambda y + (x+1)\,\lambda - \alpha - 1)\,\mathrm{e}^{-\lambda y}}{(-y+x+1)^{\alpha+2}} 1_{x>y} + \\ & + \frac{\alpha \cdot (-\lambda x + (y+1)\,\lambda - \alpha - 1)\,\mathrm{e}^{-\lambda x}}{(-x+y+1)^{\alpha+2}} 1_{x<y}. \end{aligned}$$

Under the conditions on the parameters stated above, we have that, when $(x,y) \to (0^+,0^+)$*,* $g(x,y) \to -\alpha(\alpha+1-\lambda) < 0$*, so* g *is not a density function.*
Now let us consider the second-mixed derivative of the function $\bar{F}$ *on the set* $\{(x,y) : x > y \geq 0\}$*: setting* $u = e^{-\lambda y}$*,* $k = x - y$*, with* $0 < u \leq 1$*,* $0 < k < \infty$*, after some algebra, we have*

$$\begin{aligned} & \frac{\partial^2 \bar{F}}{\partial x \partial y} \left(k - \frac{\log(u)}{\lambda}, -\frac{\log(u)}{\lambda} \right) = \\ & = C(u,k)\{\theta[\beta e^{\theta u (1+k)^{-\alpha}} - 1](\lambda + \lambda k - \alpha)u + \\ & + [e^{\theta u(1+k)^{-\alpha}} - 1](\lambda + \lambda k - \alpha - 1)(1+k)^{\alpha}\}, \end{aligned} \tag{4.17}$$

where $C(u,k) = \frac{\alpha\beta\theta u}{(e^{\theta}-1)^{\beta}}\, e^{\theta u(1+k)^{-\alpha}}\, [e^{u\theta(1+k)^{-\alpha}} - 1]^{\beta-2}\, (1+k)^{-2\alpha-2} > 0$*. Under the conditions on the parameters stated above, the function (4.17) is non-negative if* $k \geq \frac{\alpha+1}{\lambda} - 1, \forall u \in (0,1]$*, so let us focus on the case in which* $0 < k < \frac{\alpha+1}{\lambda} - 1$*.*
For this purpose, we define $\rho(k) = \theta(\lambda + \lambda k - \alpha)$*,* $\gamma(k) = (1+k)^{\alpha}(\lambda + \lambda k - \alpha - 1)$ *and* $z(k) = \theta(1+k)^{-\alpha}$*: since* $0 < k < \frac{\alpha+1}{\lambda} - 1$ *and* $\lambda \geq \alpha$*, we can easily see that* $\rho(k) \geq 0$*,* $z(k) > 0$ *and* $\gamma(k) < 0$*. Basically, we need that the function*

$$w(u,k) = \rho(k)(\beta e^{uz(k)} - 1)u + \gamma(k)(e^{uz(k)} - 1) \tag{4.18}$$

is non-negative in the set $\left\{(u,k) : 0 < u \leq 1, 0 < k < \frac{\alpha+1}{\lambda} - 1)\right\}$*. But its first partial derivative with respect to* u *is non-negative in that rectangle, implying that the infimum of* $w(u,k)$ *on that set is obtained as* $u \to 0^+$ *and it is equal to* 0*. Similar results hold when* $x < y$*.*
So we conclude that $\bar{F}$ *is a survival function but, under the same conditions on the parameters,* $\bar{G}$ *is not a survival function.*

In general, if $\bar{F}$ satisfies the pseudo weak bivariate lack-of-memory property with generator h, its marginal survival functions $\bar{F}_1$, $\bar{F}_2$ do not necessarily satisfy univariate pseudo lack-of-memory property with the same generator h (see Remark 4.1.1).

It is well-known that the only distribution satisfying bivariate standard weak lack-of-memory property with marginals satisfying univariate standard lack-of-memory property is the exponential Marshall-Olkin distribution with survival function (2.17). A similar result can be found for the pseudo version of the lack-of-memory properties, as we show in next Proposition.

Proposition 4.1.3. *Let $h : [0,1] \to [0,1]$ be a generator such that $h^{-1}(x)$ is log-concave. The only distribution satisfying pseudo bivariate weak lack-of-memory property, with generator h, with marginal survival functions satisfying pseudo univariate lack-of-memory property with the same generator h, is*

$$\begin{aligned}\bar{F}_{X,Y}(x,y) &= \\ &= \begin{cases} \exp_h(\lambda y) \otimes \exp_h(\gamma_1(x-y)), & x \geq y \\ \exp_h(\lambda x) \otimes \exp_h(\gamma_2(y-x)), & x < y \end{cases} = \\ &= \exp_h(\lambda_1 x + \lambda_2 y + \lambda_0 \max(x,y)),\end{aligned} \tag{4.19}$$

with $0 < \max(\gamma_1, \gamma_2) \leq \lambda \leq \gamma_1 + \gamma_2$ *and with* $\lambda_1 = \lambda - \gamma_2$, $\lambda_2 = \lambda - \gamma_1$ *and* $\lambda_0 = \gamma_1 + \gamma_2 - \lambda$.

The survival function (4.19) can be obtained from the following construction based on a shock model. In fact, let us consider three random variables Z_1,Z_2 and Z_3 with marginal survival functions $\bar{F}_{Z_i}(x) = \exp_h(\lambda_i x), i = 1,2,3$ such that

$$P[Z_1 > z_1, Z_2 > z_2, Z_3 > z_3] = \exp_h(\lambda_1 z_1) \otimes \exp_h(\lambda_2 z_2) \otimes \exp_h(\lambda_3 z_3),$$

meaning that the associated copula is Archimedean with generator $\psi(t) = h(e^{-t})$. Furthermore, let us consider the random variables $X = \min(Z_1, Z_3)$ and $Y = \min(Z_2, Z_3)$. Then

$$\begin{aligned}P[X > x, Y > y] &= P[Z_1 > x, Z_2 > y, Z_3 > \max(x,y)] = \\ &= h(\exp(-\lambda_1 x - \lambda_2 y - \lambda_3 \max(x,y)) = \\ &= \exp_h(\lambda_1 x + \lambda_2 y + \lambda_3 \max(x,y)),\end{aligned} \tag{4.20}$$

which is exactly the survival function we obtained in (4.19). Survival functions of the type (4.19) belong to a sub-class of the functions identified by equation (1) in Mulinacci (2018), with, according to the notation therein, $G(x) = \exp_h(x)$ and $H_i(x) = \lambda_i x,\ i = 1,2,3$.

Example 4.1.2. *Let* $h(x) = \frac{e^{\theta x}-1}{e^{\theta}-1}$ *and let* $\bar{F}_i(x) = e^{-\gamma_i x},\ i = 1, 2$*: then the function*

$$\begin{aligned}\bar{F}(x,y) &= \\ &= \frac{\exp\{e^{-\lambda y}\log((e^{\theta}-1)e^{-\gamma_1(x-y)}+1)\}-1}{e^{\theta}-1}\,1_{x\geq y}+ \\ &+ \frac{\exp\{e^{-\lambda x}\log((e^{\theta}-1)e^{-\gamma_2(y-x)}+1)\}-1}{e^{\theta}-1}\,1_{x<y}\end{aligned} \tag{4.21}$$

is a survival function satisfying (4.8) if

$$\max(\gamma_1,\gamma_2)\cdot\max\left(1, \frac{e^{\theta}-1}{\theta e^{\theta}}\right) \leq \lambda \leq \frac{(\gamma_1+\gamma_2)(e^{\theta}-1)}{\theta e^{\theta}}.$$

Using the conditional distribution method, see Nelsen (2006), we generate data from this distribution with parameters $\gamma_1 = 0.5$*,* $\gamma_2 = 0.6$ *and* $\lambda = 0.645$*. In Figure* 1*, we show the scatterplots from (4.21) for three different values of the parameter* θ*.*

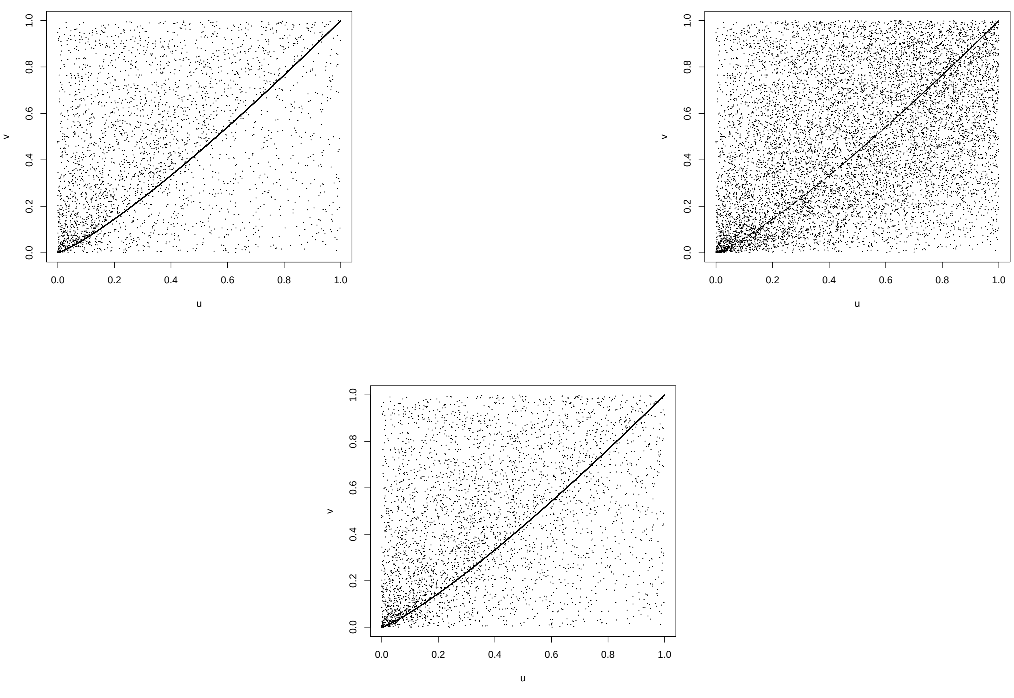


Figure 4.1: Scatterplots from (4.21). Top Left: $\theta = 0.01$. Bottom: $\theta = 0.1$. Top Right: $\theta = 1$.

4.2 Properties and Characterizations of Pseudo Weak Distribution

In the spirit of Ghurye and Marshall (1984), as for the case of pseudo lack-of-memory property, we get the following result.

Proposition 4.2.1. *Let $\bar{F}_{X,Y}$ be the bivariate survival function of (X, Y) satisfying pseudo weak lack-of-memory property; moreover, let $\bar{W} = (W_1, W_2) = (X - U, Y - U)$, where $U = \min(X, Y)$. Then:*

1. *$P(\min(W_1, W_2) = 0) = 1$;*
2. *U has a pseudo exponential distribution with parameter λ;*
3. *$N = \mathbf{1}_{X>Y} - \mathbf{1}_{X<Y}$ and U are independent;*
4. *The joint distribution of U and the vector $\bar{W}$ is given by*

$$P(U \geq u, \bar{W} \geq \bar{w}) = \\ = \begin{cases} \exp_h(\lambda u), \;\; w_1 \leq 0, w_2 \leq 0 \\ \exp_h(\lambda u) \otimes \bar{F}_1(w_1) \left(1 - \frac{r_1(w_1)}{\lambda}\right), \;\; w_1 > 0, w_2 \leq 0 \\ \exp_h(\lambda u) \otimes \bar{F}_2(w_2) \left(1 - \frac{r_2(w_2)}{\lambda}\right), \;\; w_1 \leq 0, w_2 > 0 \\ 0, \;\; w_1 > 0, w_2 > 0 \end{cases},$$

where $\bar{F}_i(x) = h(\bar{G}_i(x)), i = 1, 2$ and r_i is the hazard rate of the survival distribution $\bar{G}_i, i = 1, 2$.

Proof. 1. holds true for the same reasons given in Ghurye and Marshall (1984).
For *2.*, since $\bar{F}_{X,Y}$ satisfies pseudo weak lack-of-memory property,

$$\bar{F}_U(u) = P(X \geq u, Y \geq u) = \bar{F}_{X,Y}(u, u) = h(e^{-\lambda u}), u \geq 0.$$

Regarding *3.*, let us start with the case in which $N = 1$: then, by (4.9),

$$\begin{aligned} P(N = 1, U \geq u) = P(X > Y \geq u) &= \int_u^\infty \int_u^x \frac{\partial^2 \bar{F}}{\partial x \partial y} \, dy \, dx = \\ &= \int_u^\infty -h'(e^{-\lambda y} \bar{G}_1(x - y)) e^{-\lambda y} g_1(x - y) \Big|_{y=u}^x \, dx = \\ &= h(e^{-\lambda u}) \left(1 - \frac{g_1(0)}{\lambda}\right). \end{aligned} \tag{4.22}$$

Using equation (4.13), $P(N=1) = \left(1 - \frac{g_1(0)}{\lambda}\right)$, so (4.22) rewrites

$$P(N=1, U \geq u) = P(U \geq u)\ P(N=1).$$

Similar results hold for $N = -1$. In the case in which $N = 0$,

$$P(U \geq u, N=0) = P(X = Y \geq u) = h(e^{-\lambda u}) \left(\frac{g_1(0) + g_2(0)}{\lambda} - 1 \right),$$

thanks to (4.16); the conclusion follows taking into account that $P(N = 0) = \left(\frac{g_1(0)+g_2(0)}{\lambda} - 1\right)$ by (4.15). For part *4.*, it is trivial to show that, if $\bar{w} = (w_1, w_2)$, $\bar{F}_{(U,\bar{W})}(u, \bar{w}) = h(e^{-\lambda u})$ if $w_1, w_2 \leq 0$ and that $\bar{F}_{(U,\bar{W})}(u, \bar{w}) = 0$ if $w_1, w_2 > 0$: for this reason, we focus on the cases in which $w_1 > 0$ and $w_2 \leq 0$ and $w_1 \leq 0$ and $w_2 > 0$.
For $w_1 > 0$ and $w_2 \leq 0$, we have that:

$$\begin{aligned}
&P(U \geq u, \bar{W} \geq \bar{w}) = P(X \geq u, Y \geq u, X - Y \geq w_1) = \\
&= P(X \geq w_1 + u, u \leq Y \leq X - w_1) = \\
&= \int_{u+w_1}^{\infty} \int_{u}^{x-w_1} \frac{\partial^2 \bar{F}}{\partial x \partial y} dy\ dx = \\
&= \int_{u+w_1}^{\infty} \left(-h^{'}(e^{-\lambda y} \bar{G}_1(x-y)) g_1(x-y) e^{-\lambda y} \Big|_{y=u}^{x-w_1} \right) dx = \\
&= h(e^{-\lambda u} \bar{G}_1(w_1)) \left(1 - \frac{g_1(w_1)}{\lambda \bar{G}_1(w_1)} \right) = \\
&= \exp_h(\lambda u) \otimes \bar{F}_1(w_1) \left(1 - \frac{r_1(w_1)}{\lambda} \right).
\end{aligned} \tag{4.23}$$

Similarly, if $w_2 > 0$ and $w_1 \leq 0$, we have

$$P(U \geq u, \bar{W} \geq \bar{w}) = \exp_h(\lambda u) \otimes \bar{F}_2(w_2) \left(1 - \frac{r_2(w_2)}{\lambda} \right). \tag{4.24}$$

□

Similarly, in the next Proposition, we extend the result given in Proposition 2.3.2 given in Block and Basu (1974) to bivariate survival distributions satisfying the pseudo weak lack-of-memory property.

Proposition 4.2.2. *Let $F_{X,Y}$ be the bivariate survival function of the vector (X, Y); moreover, let $U = \min(X, Y)$ and $V = X - Y$. Then $\bar{F}_{X,Y}$ satisfies* *pseudo weak lack-of-memory property if and only if*

$$P(U \geq u, V \geq v) =$$
$$= \begin{cases} \exp_h(\lambda u) + h(e^{-\lambda u}\bar{G}_2(-v)) \left(\frac{g_2(-v)}{\lambda \bar{G}_2(-v)} - 1\right), u \geq 0, v \leq 0 \\ h(e^{-\lambda u}\bar{G}_1(v)) \left(1 - \frac{g_1(v)}{\lambda \bar{G}_1(v)}\right), u \geq 0, v > 0 \end{cases}. \tag{4.25}$$

Proof. Let us start with the case $v < 0$. We have:

$$\begin{aligned} P(U \geq u, V \geq v) &= \\ &= P(u \leq Y < X) + P(X = Y \geq u) + \\ &+ P(X \geq u, X < Y \leq X - v). \end{aligned}$$

The first probability is already known from equation (4.22), while the second one is given by (4.16), so we need to compute only the last one. We get

$$\begin{aligned} &P(X \geq u, X < Y \leq X - v) = \\ &= \int_u^\infty \int_x^{x-v} \frac{\partial^2 \bar{F}_{X,Y}(x,y)}{\partial x \partial y}\, dy = \\ &= \int_u^\infty h^{'}(e^{-\lambda x}\bar{G}_2(y-x))(-\lambda e^{-\lambda x}\bar{G}_2(y-x) + \\ &+ e^{-\lambda x} g_2(y-x))\Big|_{y=x}^{x-v}\, dx = \\ &= h(e^{-\lambda u}\bar{G}_2(-v)) \left(\frac{g_2(-v)}{\lambda \bar{G}_2(-v)} - 1\right) + h(e^{-\lambda u}) \left(1 - \frac{g_2(0)}{\lambda}\right). \end{aligned}$$

Overall,

$$P(U \geq u, V \geq v) = h(e^{-\lambda u}) + h(e^{-\lambda u}\bar{G}_2(-v)) \left(\frac{g_2(-v)}{\lambda \bar{G}_2(-v)} - 1\right).$$

If $v > 0$, by equation (4.23),

$$P(U \geq u, V \geq v) = h(e^{-\lambda u}\bar{G}_1(v)) \left(1 - \frac{g_1(v)}{\lambda \bar{G}_1(v)}\right).$$

Conversely, let us suppose that (4.25) holds true. If $0 < x \leq y$, we have that

$$\begin{aligned} \bar{F}_{X,Y}(x,y) &= \\ &= P(X > x, Y > y, X \geq Y) + P(X > x, Y > y, X < Y) = \\ &= P(U + V > x, U > y, V \geq 0) + \\ &+ P(U > x, U - V > y, V < 0) = \\ &= P(U > y, V \geq 0) + P(U > x, U - V > y, V < 0) = \\ &= P(U > y, V \geq 0) + P(x < U < y,\ x - y < V < U - y) + \\ &+ P(U > y, x - y < V < 0) + P(U > x, V \leq x - y). \end{aligned}$$

From (4.25), $P(U \geq y, V \geq 0) = h(e^{-\lambda u})\frac{g_2(0)}{\lambda}$. Moreover,

$$P(U > y, x-y < V < 0) =$$
$$= \int_y^{\infty} -\lambda e^{-\lambda u} h'\left(e^{-\lambda u}\bar{G}_2(-v)\right) \bar{G}_2(-v)\left(\frac{g_2(-v)}{\lambda\bar{G}_2(-v)} - 1\right)\Big|_{x-y}^{0} \, du =$$
$$= h(e^{-\lambda y}\bar{G}_2(y-x))\left(\frac{g_2(y-x)}{\lambda\bar{G}_2(y-x)} - 1\right) - h(e^{-\lambda y})\left(\frac{g_2(0)}{\lambda} - 1\right);$$

similarly,

$$P(x < U < y,\ x-y < V < U-y) =$$
$$= h(e^{-\lambda x}\bar{G}_2(y-x))\frac{g_2(y-x)}{\lambda\bar{G}_2(y-x)} - h(e^{-\lambda y}) +$$
$$- h(e^{-\lambda y}\bar{G}_2(y-x))\left(\frac{g_2(y-x)}{\lambda\bar{G}_2(y-x)} - 1\right).$$

Finally,

$$P(U > x, V \leq x-y) = h(e^{-\lambda x}\bar{G}_2(y-x))\left(1 - \frac{g_2(y-x)}{\lambda\bar{G}_2(y-x)}\right)$$

Summing up all the probabilities above, we get that

$$\bar{F}_{X,Y}(x,y) = h(e^{-\lambda x}\bar{G}_2(y-x)),\ 0 < x \leq y;$$

similar results hold for $x > y$. $\square$

4.3 Upper and Lower Tail Dependence Coefficients of Pseudo Weak Distribution

Given a random vector (X, Y) with copula C and marginal cumulative distribution functions F_X and F_Y, we recall that the upper tail dependence coefficient λ_U can be written as

$$\lambda_U = \lim_{u \to 1^-} P[F_X(X) > u |\ F_Y(Y) > u] = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u};$$

analogously, the lower tail dependence coefficient λ_L is given by

$$\lambda_L = \lim_{u \to 0^+} P[F_X(X) < u |\ F_Y(Y) < u] = \lim_{u \to 0^+} \frac{C(u, u)}{u}.$$

Thanks to (4.10), the survival copula associated to the survival distribution satisfying pseudo weak lack-of-memory property is $C^F(u, v) = h(C^G(h^{-1}(u), h^{-1}(v))$. Setting $\bar{G}_1 = \bar{G}_2 = \bar{G}$ in (4.11), we have

$$\begin{aligned} C^{\bar{G}}(u, v) = & \\ & = e^{-\lambda \bar{G}^{-1}(v)} \bar{G}(\bar{G}^{-1}(u) - \bar{G}^{-1}(v)) 1_{u<v} + \\ & + e^{-\lambda \bar{G}^{-1}(u)} \bar{G}(\bar{G}^{-1}(v) - \bar{G}^{-1}(u)) 1_{u \geq v}. \end{aligned} \tag{4.26}$$

By the system of inequalities (2.16), we know that this is a survival copula if and only if $\lambda \leq 2g(0)$ and $\frac{\partial \log(g(z))}{\partial z} \geq -\lambda,\ \forall z \geq 0$.

Proposition 4.3.1. *Let (X, Y) be a random vector with survival copula $C^{\bar{G}}$ of type (4.26) and let $g(x) = -\bar{G}'(x),\ x \geq 0$ such that $\lambda \leq 2g(0)$ and $\frac{\partial \log(g(z))}{\partial z} \geq -\lambda,\ \forall z \geq 0$. Then:*

1. *$\lambda_L = 0$ if $\bar{G}$ is heavy tailed, id est $\lim_{x \to \infty} \frac{\bar{G}(x)}{e^{-\lambda x}} = +\infty,\ \forall \lambda > 0$.*
2. *$\lambda_U = 2 - \frac{\lambda}{g(0)}$.*

Proof. For *1.*, we have

$$\lambda_L = \lim_{u \to 0^+} \frac{C^{\bar{G}}(u, u)}{u} = \lim_{u \to 0^+} \frac{e^{-\lambda \bar{G}^{-1}(u)}}{u}.$$

Setting $x = \bar{G}^{-1}(u)$, we have

$$\lambda_L = \lim_{x \to \infty} \frac{e^{-\lambda x}}{\bar{G}(x)} = 0.$$

Regarding *2.*, we can write

$$\lambda_U = \lim_{u\to 1^-} \frac{1-2u+\bar{C^G}(u,u)}{1-u} = 2 + \lim_{u\to 1^-} \frac{e^{-\lambda \bar{G}^{-1}(u)}-1}{1-u} =$$
$$= 2 + \lim_{x\to 0^+} \frac{e^{-\lambda x}-1}{1-\bar{G}(x)} = 2 - \lambda \lim_{x\to 0^+} \frac{x}{G(x)},$$

from which the conclusion follows. □

In the case in which the marginal distribution is light-tailed, the value of the lower tail dependence coefficient depends on the functional form of the distribution $\bar{G}$.

Using Propositions 2.2.1 and 2.2.2, we are able to find the lower and the upper tail dependence coefficients for different choices of the common marginal survival functions and of the generator h, as shown in the following Examples.

Example 4.3.1. *Let (X,Y) be a random vector satisfying bivariate pseudo-weak lack-of-memory property with marginal survival functions $\bar{F}_i(x) = \exp_h(\mu x)$, $i = 1,2$, $\mu \le \lambda \le 2\mu$.*
If $h = id$, we get the standard Marshall Olkin distribution: since $\bar{G}_i(x) = e^{-\mu_i x}$, we can show that $\lambda_L(\bar{C}^G) = 0$ if $\mu < \lambda \le 2\mu$ and that $\lambda_L(\bar{C}^G) = 1$ if $\lambda = \mu$.
Moreover, using Proposition 4.3.1, we can easily prove that $\lambda_U(\bar{C}^G) = 2-\frac{\lambda}{\mu} \in [0,1]$.
If $h(x) = 1-\left(\frac{\tan(\theta(1-x))}{\tan(\theta)}\right)^\beta, -\frac{\pi}{2} < \theta < 0,\ 0 < \beta < 1$, then h is a convex bjection of the unit interval and condition given in Proposition 2.2.1 is satisfied with $\alpha = 1$ and with $b = \frac{2\theta\beta}{\sin(2\theta)}$, implying that $\lambda_L(\bar{C}_h) = \lambda_L(\bar{C}^G)$.
Moreover, it satisfies also Proposition 2.2.2 with parameters $\alpha = \beta$ and $b = \left(\frac{\theta}{\tan(\theta)}\right)^\beta$: so $\lambda_U(\bar{C}_h) = 2 - \left(\frac{\lambda}{\mu}\right)^\beta \in [0,1]$.

Example 4.3.2. *Let (X,Y) be a random vector satisfying bivariate pseudo-weak lack-of-memory property with marginal survival functions $\bar{F}_i(x) = h((1+x)^{-\gamma})$, $i = 1,2$, $\gamma \ge 1$, $\gamma+1 \le \lambda \le 2\gamma$.*
If $h = id$, by Proposition 4.3.1, it follows that $\lambda_L(\bar{C}^G) = 0$.
Moreover, using Proposition 4.3.1, we can easily prove that $\lambda_U(\bar{C}^G) = 2-\frac{\lambda}{\gamma} \in [0,1]$. If $h(x) = 1-\left(\frac{e^{\theta(1-x)}-1}{e^\theta-1}\right)^\beta$, $\theta < 0,\ 0 < \beta < 1$, then h is a convex bjection of the unit interval $[0,1]$ and condition given in Proposition 2.2.1 is satisfied with $\alpha = 1$ and with $b = \frac{\theta\beta e^\theta}{e^\theta-1}$, implying that $\lambda_L(\bar{C}_h) = \lambda_L(\bar{C}^G) = 0$.

Moreover, it satisfies also Proposition 2.2.2 with parameters $\alpha = \beta$ *and* $b = \frac{\theta}{(e^{\theta}-1)^{\beta}}$: *so* $\lambda_U(\bar{C}_h) = 2 - \left(\frac{\lambda}{\gamma}\right)^{\beta} \in [0,1]$.
Using conditional distribution method, we simulate data with parameters $\gamma = 1.5$, $\lambda = 2.6$ *and* $\theta = -1$ *from the survival distribution function*

$$\begin{aligned} \bar{F}_{X,Y}(x,y) = & \\ & = 1 - \left(\frac{e^{\theta(1-(1+x-y)^{-\gamma})} - 1}{e^{\theta} - 1}\right)^{\beta} 1_{x \geq y} + \\ & + 1 - \left(\frac{e^{\theta(1-(1+y-x)^{-\gamma})} - 1}{e^{\theta} - 1}\right)^{\beta} 1_{x < y} : \end{aligned} \tag{4.27}$$

the scatterplots are given below for three different values of β.

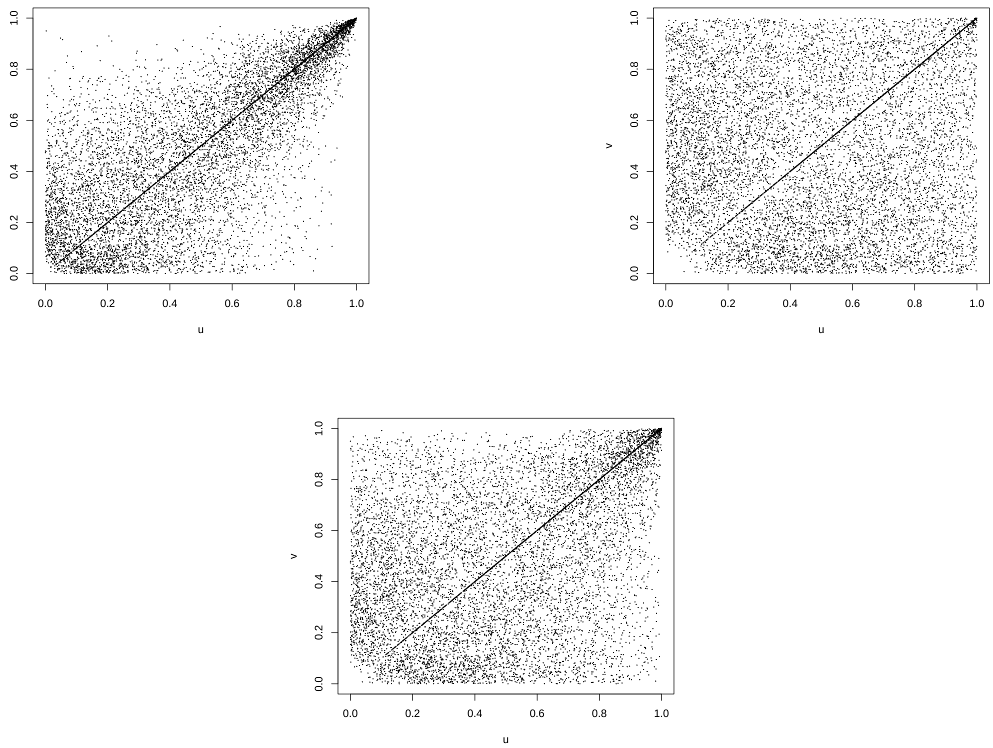


Figure 4.2: Scatterplots from (4.27). Top Left: $\beta = 0.3$. Bottom: $\beta = 0.6$. Top Right: $\beta = 0.9$.

4.4 Kendall Distribution Function of Pseudo Weak Distribution

The Kendall Distribution function of a random vector (X, Y) with cumulative distribution F is defined as

$$K(t) = P(F(X, Y) \leq t),\ t \in [0, 1].$$

It turns out to be very useful to study dependence between the components of a bivariate random vector. In the case of perfectly positive dependence, $K(t) = t, \forall t \in [0, 1]$, while, in the case of independence, $K(t) = t - t\log(t), \forall t \in (0, 1]$, see Joe (2014).
The Kendall tau is is a statistic used to measure the ordinal association between two measured quantities. More precisely, let (X, Y), (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vector with cumulative distribution function F: then the Kendall tau of (X, Y) is given by

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0).$$

The Kendall tau can be computed from the Kendall distribution function, id est:

$$\tau = 3 - 4 \int_0^1 K(t)\ dt,$$

see for example Genest and Rivest (2001). Basic properties of Kendall distribution functions are studied also in Nelsen et al. (2003).

In this section, we want to find an expression for the Kendall Distribution Function associated to the survival function of type (4.9) in terms of $\bar{G}_i, i = 1, 2$ and in terms of h.

Proposition 4.4.1. *Let K be the Kendall distribution function associated to the survival function (4.9). Let us assume that h is differentiable and that $\bar{G}_i = h^{-1}\left(\bar{F}_i\right)$ admits a density g_i for $i = 1, 2$. Then*

$$K(s) = s - H(h^{-1}(s)) \tag{4.28}$$

where

$$H(v) = h'(v)\, v \left[2\ln(v) + \frac{1}{\lambda}\left(J_1(v) + J_2(v)\right)\right]$$

with

$$J_i(v) = \int_0^{\bar{G}_i^{-1}(v)} \frac{g_i^2(z)}{\bar{G}_i^2(z)} dz,\ i = 1, 2.$$

Proof. Since $\bar{F}(x,y) = h\left(\bar{G}(x,y)\right)$ we have

$$\mathbb{P}\left(\bar{F}(X,Y) \le s\right) = \mathbb{P}\left(\bar{G}(X,Y) \le h^{-1}(s)\right).$$

Let $z_s = -\frac{1}{\lambda}\ln\left(h^{-1}(s)\right)$ be the solution of $\bar{G}(x,x) = h^{-1}(s)$ and let D_1 and D_2 be defined as

$$D_1 = \left\{(x,y) : 0 \le y \le z_s,\ z_s \le x \le y + \bar{G}_1^{-1}\left(h^{-1}(s)e^{\lambda y}\right)\right\}$$

and

$$D_2 = \left\{(x,y) : 0 \le x \le z_s,\ z_s \le y \le x + \bar{G}_2^{-1}\left(h^{-1}(s)e^{\lambda y}\right)\right\} :$$

then

$$\begin{aligned} K(s) &= \bar{F}_1(z_s) + \bar{F}_2(z_s) - \mathbb{P}\left((X,Y) \in D_1\right) - \mathbb{P}\left((X,Y) \in D_2\right) - \bar{F}(z_s,z_s) = \\ &= \bar{F}_1(z_s) + \bar{F}_2(z_s) - \mathbb{P}\left((X,Y) \in D_1\right) - \mathbb{P}\left((X,Y) \in D_2\right) - s. \end{aligned}$$

Noticing that

$$\mathbb{P}(X > x|Y = y)\ f_2(y) = -\frac{\partial \bar{F}(x,y)}{\partial y} = -h'(\bar{G}(x,y))\frac{\partial \bar{G}(x,y)}{\partial y}$$

and evaluating it in $x = y + \bar{G}_1^{-1}\left(h^{-1}(s)e^{\lambda y}\right)$, we have

$$\begin{aligned} &\mathbb{P}((X,Y) \in D_1) = \\ &\int_0^{z_s} \left[\mathbb{P}\left(X > z_s|Y = y\right) - \mathbb{P}\left(X > y + \bar{G}_1^{-1}\left(h^{-1}(s)e^{\lambda y}\right)|Y = y\right)\right] f_2(y)\, dy = \\ &= \int_0^{z_s} \left(h'(h^{-1}(s))\left[-\lambda h^{-1}(s) + e^{-\lambda y} g_1\left(\bar{G}_1^{-1}\left(h^{-1}(s)e^{\lambda y}\right)\right)\right] + \right. \\ &\left. - \frac{\partial \bar{F}(z_s,y)}{\partial y}\right) dy = -\bar{F}(z_s,z_s) + \bar{F}_1(z_s) - \lambda h'(h^{-1}(s))h^{-1}(s)z_s + \\ &+ h'(h^{-1}(s)) \int_0^{z_s} e^{-\lambda y} g_1\left(\bar{G}_1^{-1}\left(h^{-1}(s)e^{\lambda y}\right)\right)\, dy = \\ &= -s + \bar{F}_1(z_s) + h'(h^{-1}(s))h^{-1}(s)\ln\left(h^{-1}(s)\right) + \\ &+ \frac{1}{\lambda}h'(h^{-1}(s))h^{-1}(s) \int_0^{\bar{G}_1^{-1}\left(h^{-1}(s)\right)} \frac{g_i^2(z)}{\bar{G}_1^2(z)}\, dy \end{aligned},$$

where, in the last integral, we have substituted $z = \bar{G}_1^{-1}\left(h^{-1}(s)e^{\lambda y}\right)$.

The probability $\mathbb{P}((X,Y) \in D_2)$ can be obtained similarly. □

We now recover an expression for the Kendall distribution function for some choices of h and $\bar{G}_i,\ i = 1,2$: since the standard weak lack-of-memory property can be recovered from the pseudo one when $h = id$, Proposition 4.4.1 allows to recover Kendall Distribution Function also in the classical setting.

Example 4.4.1. *Let* $\bar{G}_i(x) = e^{-\alpha_i x}$, $i = 1, 2$ *and* $h(x) = x$, *with* $\max(\alpha_1, \alpha_2) \leq \lambda \leq \alpha_1 + \alpha_2$, *then it is possible to verify that*

$$K(t) = t\left(1 - \log(t)\left(2 - \frac{\alpha_1 + \alpha_2}{\lambda}\right)\right).$$

Example 4.4.2. *Let* $\bar{G}_i(x) = (1 + x)^{-\alpha_i}$, $i = 1, 2$ *and* $h(x) = x$, *with* $\max(\alpha_1, \alpha_2) + 1 \leq \lambda \leq \alpha_1 + \alpha_2$, *then it is possible to verify that*

$$K(t) = t\left(1 - 2\log(t) - \frac{\alpha_1^2(1 - t^{\frac{1}{\alpha_1}}) + \alpha_2^2(1 - t^{\frac{1}{\alpha_2}})}{\lambda}\right).$$

Example 4.4.3. *Let* $\bar{G}_i(x) = (1 + x)^{-\alpha}$, $i = 1, 2$ *and* $h(x) = e^{-\gamma(x^{-\frac{1}{\alpha}} - 1)}$, *with* $\alpha + 1 \leq \lambda \leq 2\alpha$ *and* $\gamma \geq \alpha + 1$, *then it is possible to verify that*

$$K(t) =$$
$$= t\left(1 - 2\gamma\left(1 - \frac{\log(t)}{\gamma}\right)\left(\frac{\alpha}{\lambda}\left(1 - \left(1 - \frac{\log(t)}{\gamma}\right)^{-1}\right) - \log\left(1 - \frac{\log(t)}{\gamma}\right)\right)\right).$$

Chapter 5

Kaminsky Type Functional Equations for Residual Lifetimes and Insurance Applications

In this chapter, we will generalize Kaminski (1983) and Marshall and Olkin (2015) functional equations and we will show that the solutions of the latter coincide with the survival functions satisfying pseudo lack-of-memory properties. After studying the dependence structure of residual lifetimes, we will give formulas for insurance products written on the vector of residual lifetimes when the latter follows pseudo weak distribution and we will analyse, for pure explanatory purposes, some particular examples. An application of pseudo weak lack-of-memory property to LOSS ALAE insurance modelling problem is also given.

5.1 Kaminsky and Marshall-Olkin Functional Equations

In the following, we will denote by $\bar{F}_{X_t}$ the survival function of the univariate residual lifetime $X_t = X - t|X > t$ and by $\bar{F}_{\bar{X}_t}$ the survival function of the bivariate residual lifetime $\bar{X}_t = X - t, Y - t|X > t, Y > t$ for some $t \geq 0$. Let X be a non-negative continuous random variable with Gompertz distribution, with survival function

$$\bar{F}(x) = \exp(-a(e^{bx} - 1)),\ x \geq 0, \tag{5.1}$$

for some $a, b > 0$. It is easy to check that the function $\bar{F}(x)$ satisfies the functional equation

$$\bar{F}_{X_t}(x) = [\bar{F}(x)]^{\xi(t)}, \; x \geq 0, t \geq 0 \tag{5.2}$$

with $\xi(t) = e^{bt}$. More generally, the following result given in Kaminsky (1983) holds.

Proposition 5.1.1. *Let $\bar{F} : \mathbb{R}_+ \to [0,1]$ be a non-increasing function. Then the function $\bar{F}(\cdot)$ satisfies equation (5.2) for some $b > 0$ and for some function $\xi : [0,\infty) \to [0,\infty)$ not depending on x if and only if either:*

1. *the function $\bar{F}(x) = e^{-bx}$ and $\xi(t) = 1$;*
2. *the function $\bar{F}(x)$ is the survival function of the Gompertz distribution given by the equation (5.1) and $\xi(t) = e^{bt}$;*
3. *the function $\bar{F}(x) = e^{a(e^{-bx}-1)}$ and $\xi(t) = e^{-bt}$.*

Notice that the solution of the functional equation (5.2) given by part *3.* in proposition (5.1.1), called Negative Gompertz Distribution, is not a univariate survival function since $\lim_{x\to\infty} \bar{F}(x) \neq 0$, see Kolev (2016).

In Marshall and Olkin (2015), the following bivariate version of (5.2) is considered:

$$\bar{F}_{\bar{X}_{t,s}}(x,y) = \bar{F}(x,y)^{\phi(t,s)}, \tag{5.3}$$

for some $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$. Therein, the authors prove that, if $\bar{F}$ has Gompertz marginal survival functions, then

$$\bar{F}(x,y) = e^{-\xi(e^{\lambda_1 x + \lambda_2 y}-1)},$$

with $\lambda_i > 0, \; i = 1,2$ and $\xi \geq 1$: this distribution is positively quadrant dependent, id est $\bar{F}(x,y) \geq \bar{F}(x,0)\bar{F}(0,y)$.
Moreover, in Marshall and Olkin (2015), the following weaker version of (5.3) is also considered:

$$\bar{F}_{\bar{X}_t}(x,y) = \bar{F}(x,y)^{\phi(t)}, \tag{5.4}$$

for some $\phi : [0,\infty) \to [0,\infty)$. The authors prove that, if $\bar{F}_{X,Y}$ has Gompertz marginal survival functions, then the unique solution of the latter is given by

$$\bar{F}_{X,Y}(x,y) = \begin{cases} \exp\left(-\gamma\left(e^{\lambda y}\left(1 + \frac{\eta_1(e^{\beta_1(x-y)}-1)}{\gamma}\right) - 1\right)\right) & x \geq y \\ \exp\left(-\gamma\left(e^{\lambda x}\left(1 + \frac{\eta_2(e^{\beta_2(y-x)}-1)}{\gamma}\right) - 1\right)\right) & x < y \end{cases}, \tag{5.5}$$

which is a bivariate survival function if and only if

$$\begin{cases} \lambda \geq \max(\beta_1, \beta_2) \\ \lambda(\gamma - 1) \geq \max(\beta_1(\eta_1 - 1), \beta_2(\eta_2 - 1)) \\ \lambda\gamma \leq \beta_1\eta_1 + \beta_2\eta_2 \end{cases},$$

with $\phi(t) = e^{\lambda t}$.
Moreover, they show that, if $\lambda = \beta_1 = \beta_2$, then the distribution is positevely quadrant dependent with non-negative correlation: independence is obtained if we further assume that $\gamma = \eta_1 + \eta_2$.
In the case in which $\beta_i \to 0,\ i = 1, 2$, then the marginal survival functions are exponential and $Z = \min(X, Y)$ is exponentially distributed; if we further assume that $\lambda \to 0$, then we obtain the bivariate exponential Marshall Olkin distribution, see Marshall and Olkin (1967).

5.2 Generalization of Kaminsky's Equation

In this section, we consider a generalization of univariate Kaminsky functional equation (5.2) and of Marshall and Olkin bivariate functional equations (5.3) and (5.4) and we show that the solutions of the generalized functional equations coincide with the survival functions satisfying pseudo lack-of-memory properties.

5.2.1 Univariate Case

In what follows, we consider the generalization of Kaminsky's equation

$$\bar{F}_{X_t}(x) = d_t\left(\bar{F}(x)\right), \quad t, x \geq 0 \tag{5.6}$$

where, for every $t \geq 0$, d_t is a strictly increasing bijection of $[0, 1]$ with $d_0(x) = x$. (5.6) represents a generalization of Kaminsky's equation (5.2) since the latter corresponds to the choice $d_t(x) = x^{\phi(t)}$.

We know that a survival function $\bar{F}$ satisfies univariate pseudo lack-of-memory property if $h(x) = \bar{F}(-\log x)$, see Remark 4.1.1. It can be easily verified that every survival function of a positive random variable satisfies (5.6) for a suitable time dependent function d_t.

Proposition 5.2.1. *$\bar{F}$ satisfies (5.6) if and only if it satisfies univariate pseudo lack-of-memory property with $h(z) = \bar{F}(-\ln z)$. Moreover*

$$d_t(x) = \frac{h\left(e^{-t}h^{-1}(x)\right)}{h\left(e^{-t}\right)} = \frac{\bar{F}\left(t + \bar{F}^{-1}(x)\right)}{\bar{F}(t)}.$$

5.2.2 Bivariate Strong Case

Let $\bar{X}_{s,t} = [X - s, Y - t | X > s, Y > t]$ be the residual lifetimes vector of the positive random vector (X, Y).

The Marshall and Olkin (2015) functional equation (5.3) can similarly be generalized to

$$\bar{F}_{\bar{X}_{s,t}}(x, y) = d_{s,t}\left(\bar{F}(x, y)\right), \quad t, x, s, y \geq 0 \tag{5.7}$$

where, for all $s, t \geq 0$, $d_{s,t}$ is a strictly increasing bijection of $[0, 1]$ with $d_{0,0}(x) = x$.

Following the same reasoning of the proof of Proposition 3.1 in Marshall and Olkin (2015), we get the following result:

Proposition 5.2.2. *A bivariate survival function $\bar{F}$, with marginal survival distributions $\bar{F}_1$ and $\bar{F}_2$ satisfies (5.7) if and only if it satisfies bivariate pseudo strong lack-of-memory property with generator $h(x) = \bar{F}_1(-\ln x)$. Moreover,*

$$d_{s,t}(x) = \frac{h\left(e^{-s-at} h^{-1}(x)\right)}{h\left(e^{-s-at}\right)} = \frac{\bar{F}_1\left(s + ta + \bar{F}_1^{-1}(x)\right)}{\bar{F}_1\left(s + ta\right)}, \tag{5.8}$$

where $a > 0$ is such that $\bar{F}_2(z) = \bar{F}_1(az)$.

Proof. Let $\bar{F}$ be a survival function satisfying bivariate pseudo strong lack-of-memory property. Then $\bar{F}(x, y) = h\left(e^{-x-ay}\right) = \bar{F}_1\left(x + ay\right)$, for some $a > 0$ (see Lemma 4.1.1), and since

$$\bar{F}_{\bar{X}_{s,t}}(x, y) = \frac{\bar{F}(x, y) \otimes_h \bar{F}(s, t)}{\bar{F}(s, t)} = \frac{h\left(h^{-1}\left(\bar{F}(x, y)\right) h^{-1}\left(\bar{F}(s, t)\right)\right)}{\bar{F}(s, t)},$$

(5.7) follows under (5.8).

Let us now assume that (5.7) holds true. Setting $y = t = 0$, we get

$$\bar{F}_1(x + s) = d_{s,0}\left(\bar{F}_1(x)\right) \bar{F}_1(s);$$

moreover, setting $x = s = 0$, we get

$$\bar{F}_2(y + t) = d_{0,t}\left(\bar{F}_2(y)\right) \bar{F}_2(t).$$

By Proposition 5.2.1,

$$d_{s,0}(z) = \frac{\bar{F}_1\left(s + \bar{F}_1^{-1}(z)\right)}{\bar{F}_1(s)} \quad \text{and} \quad d_{0,t}(z) = \frac{\bar{F}_2\left(t + \bar{F}_2^{-1}(z)\right)}{\bar{F}_2(t)}. \tag{5.9}$$

Now, setting $x = t = 0$ in (5.7) we get

$$\bar{F}(s, y) = d_{s,0}\left(\bar{F}_2(y)\right)\bar{F}_1(s),\ s, y \geq 0, \tag{5.10}$$

while, setting $y = s = 0$, we get

$$\bar{F}(x, t) = d_{0,t}\left(\bar{F}_1(x)\right)\bar{F}_2(t),\ x, t \geq 0. \tag{5.11}$$

Since the two expressions in (5.10) and (5.11) must coincide, we get

$$d_{s,0}\left(\bar{F}_2(t)\right)\bar{F}_1(s) = d_{0,t}\left(\bar{F}_1(s)\right)\bar{F}_2(t),\ s, t \geq 0$$

that, by (5.9), gives

$$\bar{F}_1\left(s + \bar{F}_1^{-1}\left(\bar{F}_2(t)\right)\right) = \bar{F}_2\left(t + \bar{F}_2^{-1}\left(\bar{F}_1(s)\right)\right).$$

Setting now $r = \bar{F}_2^{-1}\left(\bar{F}_1(s)\right)$, we obtain

$$\bar{F}_1^{-1}\left(\bar{F}_2(t + r)\right) = \bar{F}_1^{-1}\left(\bar{F}_2(r)\right) + \bar{F}_1^{-1}\left(\bar{F}_2(t)\right),\ s, t \geq 0,$$

from which

$$\bar{F}_1^{-1}\left(\bar{F}_2(z)\right) = az,\ z \geq 0$$

for some $a > 0$. By substituting in (5.10), we get

$$\bar{F}(x, y) = \bar{F}_1(x + ay) \tag{5.12}$$

and, setting $h(x) = \bar{F}_1(-\ln x)$, we reach the conclusion. Moreover, using (5.12), we recover that $d_{s,t}(x) = \frac{\bar{F}_1\left(s+at+\bar{F}_1^{-1}(x)\right)}{\bar{F}_1(s+at)}$.

5.2.3 Bivariate Weak Case

Let now $\bar{X}_t = [X - t, Y - t | X > t, Y > t]$.

The generalized version of the Marshall and Olkin (2015) functional equation (5.4) is

$$\bar{F}_{\bar{X}_t}(x, y) = d_t\left(\bar{F}(x, y)\right), \quad t, x, y \geq 0 \tag{5.13}$$

where, for all $t \geq 0$, d_t is a strictly increasing bijection of $[0, 1]$ with $d_0(x) = x$.

The class of bivariate survival distribution functions satisfying a functional equation of type (5.13) coincides with that of bivariate survival distribution functions satisfying the pseudo weak lack of memory property, as we show in next Proposition.

Proposition 5.2.3. *A bivariate survival function* $\bar{F}$ *satisfies (5.13) if and only if it satisfies bivariate pseudo weak lack-of-memory property with generator* $h(x) = \bar{F}\left(-\ln(x), -\ln(x)\right)$. *Moreover,*

$$d_t(x) = \frac{h\left(e^{-t}h^{-1}(x)\right)}{h\left(e^{-t}\right)} = \frac{\bar{F}\left(t - \ln(h^{-1}(x)), t - \ln(h^{-1}(x))\right)}{\bar{F}(t,t)}.$$

Proof. If $\bar{F}$ satisfies the pseudo weak lack of memory property with respect to a given generator h, then it can be easily verified that (5.13) holds true with $d_t(x) = \frac{h\left(e^{-t}h^{-1}(x)\right)}{h(e^{-t})}$.

Let us now assume that (5.13) holds true. Substituting $y = x$, we get

$$\bar{F}_{\bar{X}_t}(x,x) = d_t\left(\bar{F}(x,x)\right), \quad t, x \geq 0,$$

that, if $Z = \min(X,Y)$, is equivalent to

$$\bar{F}_{Z_t}(x) = d_t\left(\bar{F}_Z(x)\right), \quad t, x \geq 0,$$

where $\bar{F}_Z$ is the survival distribution of Z and $Z_t = [Z - t|Z > t]$. By Proposition 5.2.1, $\bar{F}_Z$ satisfies univariate pseudo lack-of-memory property with $h(x) = \bar{F}_Z\left(-\ln(x)\right) = \bar{F}\left(-\ln(x), -\ln(x)\right)$ and, necessarily, $d_t(x) = \frac{h\left(e^{-t}h^{-1}(x)\right)}{h(e^{-t})}$. Hence, substituting in (5.13), we get

$$\begin{aligned}\bar{F}(x+t, y+t) &= h\left(e^{-t}h^{-1}(\bar{F}(x,y))\right) = \\ &= \bar{F}(t,t) \otimes_h \bar{F}(x,y).\end{aligned}$$

□

As a consequence of the above result, we have that $\bar{F}$ satisfies the pseudo weak lack of memory property if and only if

$$\bar{F}_{\bar{X}_t}(x,y) = h_t\left(\bar{G}(x,y)\right), \quad t, x, y \geq 0, \tag{5.14}$$

where

$$h_t(x) = \frac{h\left(e^{-t}x\right)}{h\left(e^{-t}\right)} \tag{5.15}$$

is a generator for every $t \geq 0$.

Example 5.2.1. *In Theorem 5.1 in Mulinacci (2018) it is shown that the distributions family*

$$\begin{aligned}\bar{G}_\alpha(x,y) &= \left(\alpha_2 e^{cx} + \alpha_1 e^{cy} + (1 - \alpha_1 - \alpha_2)\, e^{c\,\max(x,y)}\right)^{-\alpha} = \\ &= \begin{cases} e^{-\alpha c\, y}\left(\alpha_1 + (1-\alpha_1)e^{c(x-y)}\right)^{-\alpha}, & x \geq y > 0 \\ e^{-\alpha c\, x}\left(\alpha_2 + (1-\alpha_2)e^{c(y-x)}\right)^{-\alpha}, & 0 < x < y \end{cases}\end{aligned} \tag{5.16}$$

with $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_1+\alpha_2 \leq 1$ and $\alpha, c > 0$ satisfies the standard weak lack of memory property.
The marginal distributions are $\bar{G}_{\alpha,i}(z) = (\alpha_i + (1-\alpha_i)e^{cz})^{-\alpha}$, $i = 1,2$ with hazard rates $r_{\alpha,i}(z) = \alpha \cdot c(1-\alpha_i)\frac{e^{cz}}{\alpha_i+(1-\alpha_i)e^{cz}}$, $i = 1,2$.

Let Z be a strictly positive random variable and (X,Y) a random vector for which

$$\mathbb{P}(X > x, Y > y | Z = \alpha) = \bar{G}_\alpha(x,y).$$

This corresponds to assuming that Z is a common multiplicative stochastic factor affecting $r_{1,i}$, for $i = 1,2$ and the parameter c. Then

$$\begin{aligned}\bar{F}(x,y) &= \mathbb{P}(X > x, Y > y) = \mathbb{E}\left[\mathbb{P}(X > x, Y > y|Z)\right] = \\ &= \mathbb{E}\left[\bar{G}_Z(x,y)\right] = \mathbb{E}\left[\bar{G}_1^Z(x,y)\right] = \\ &= \mathbb{E}\left[e^{Z\cdot\ln\left(\bar{G}_1(x,y)\right)}\right] = M_Z\left(\ln\left(\bar{G}_1(x,y)\right)\right)\end{aligned}$$

where M_Z is the moment generating function of the random variable Z. But $h(t) = M_Z(\ln(t))$, satisfies the properties of a generator and

$$\bar{F}(x,y) = h\left(\bar{G}_1(x,y)\right), \quad \alpha_1, \alpha_2 \in (0,1),\ \alpha_1+\alpha_2 \leq 1,\ c > 0$$

satisfies the pseudo weak lack of memory property with respect to the generator $h(t) = M_Z(\ln(t))$. In this case, the distortion function d_t in Kaminsky-type functional equation (5.13) is given by

$$d_t(z) = \frac{M_Z(M_Z^{-1}(z) - 1)}{M_Z(-t)}.$$

We now consider some possible distributions for Z for which the moment generating function is known in closed form.

1. *Z gamma distributed, $\Gamma(a,1)$: $M_Z(u) = (1-u)^{-a}$, $h(t) = (1-\ln(t))^{-a}$, $t \in [0,1]$ and $d_t(z) = \left(\frac{\lambda t + z^{-\frac{1}{a}}}{1+\lambda t}\right)^{-a}$, $z \in (0,1]$.*
2. *Z positive stable distributed, with parameter $a \in (0,1]$: $M_Z(u) = e^{-|u|^a}$, $h(t) = e^{-\left(\ln\left(\frac{1}{t}\right)\right)^a}$, $t \in [0,1]$ and $d_t(z) = \frac{e^{-\left(\lambda t + (-\log(z))^{\frac{1}{a}}\right)^a}}{e^{-(\lambda t)^a}}$, $z \in (0,1]$.*
3. *Z Sibuya distributed, with parameter $a \in (0,1]$: $M_Z(u) = 1-(1-e^u)^a$, $h(t) = 1-(1-t)^a$ and $d_t(z) = \frac{1-\left(1-e^{-\lambda t}+e^{-\lambda t}(1-z)^{\frac{1}{a}}\right)^a}{1-(1-e^{-\lambda t})^a}$, $z \in [0,1]$.*

4. Z distributed according to the logarithmic series distribution with parameter $a > 0$:
$M_Z(u) = -\frac{1}{a}\ln\left(1+(e^{-a}-1)e^u\right)$, $h(t) = -\frac{1}{a}\ln\left(1+(e^{-a}-1)t\right), t \in [0,1]$.
This generator can be reparametrized by setting $\theta = e^{-a}-1 \in (-1,0)$ in the form $h(t) = \frac{\ln(1+\theta t)}{\ln(\theta+1)}, t \in [0,1]$, so $d_t(z) = \frac{\log(1+e^{-\lambda t}(e^{z\log(\theta+1)}-1))}{\log(1+\theta e^{-\lambda t})}$, $z \in [0,1]$.

The dependence structure of distributions satisfying pseudo strong lack-of-memory property is of Archimedean type and it is extensively studied in the literature, so we focus on the dependence structure of distributions satisfying pseudo weak lack-of-memory property.

5.3 Dependence Structure in the Bivariate Weak Case

5.3.1 Parametric Families of Distortions

In this subsection, given a generator h, we will consider the family of generators of type

$$\tilde{h}_c(x) = \frac{h(cx)}{h(c)} \tag{5.17}$$

and we will analyze the dependence structure as a function of the parameter $c \in (0,1]$: notice that the original case of the generator h is obviously recovered when $c = 1$. Families of generators of type (5.17) arise from the analysis of residual lifetimes: in fact, (5.15) is of type (5.17) when $c = e^{-t}$.

We want to study how the lower and the upper tail dependence coefficients λ_L and λ_U change after applications of distortions of kind (5.17) in the case in which the marginal survival functions are the same.

Let us start analyzing the corresponding lower tail dependence coefficient $\lambda_L(\bar{C}_{h_c})$, where

$$\bar{C}_{h_c}(u,v) = \frac{h\left(c\, C^G\left(\frac{h^{-1}(uh(c))}{c}, \frac{h^{-1}(vh(c))}{c}\right)\right)}{h(c)}$$

and where $\bar{C^G}$ is given by Equation (4.11). If Proposition 2.2.1 is satisfied by the generator h for some $\alpha, b \in (0,+\infty)$, then

$$\lim_{t\to 0^+} \frac{\tilde{h}_c(t)}{t^\alpha} = \lim_{z\to 0^+} \frac{c^\alpha}{h(c)} \frac{h(z)}{z^\alpha} = \frac{c^\alpha}{h(c)} b \in (0,+\infty),$$

meaning that, by the same Proposition,

$$\lambda_L(\bar{C}_{\tilde{h}_c}) = \lambda_L(\bar{C}_h) = \left(\lambda_L(\bar{C}^G)\right)^{\alpha}.$$

As for the upper tail dependence coefficient, $\lambda_U(\bar{C}_{\tilde{h}_c})$, in order to apply Proposition 2.2.2, we assume that h is continuously differentiable and we analyze

$$\lim_{t\to 1^-}\frac{1-\tilde{h}_c(t)}{(1-t)^{\alpha}} = \lim_{t\to 1^-}\frac{c}{\alpha\tilde{h}(c)}\frac{h'(ct)}{(1-t)^{\alpha-1}}.$$

There are three alternatives:

1. $h'(c) \in (0,+\infty)$: then $\lim\limits_{t\to 1^-}\frac{1-\tilde{h}_c(t)}{(1-t)^{\alpha}} = b \in (0,+\infty)$ if and only if $\alpha = 1$;
2. $h'(c) = +\infty$: then, if $\lim\limits_{t\to 1^-}\frac{1-\tilde{h}_c(t)}{(1-t)^{\alpha}} = b \in (0,+\infty)$, necessarily $\alpha < 1$;
3. $h'(c) = 0$: then, if $\lim\limits_{t\to 1^-}\frac{1-\tilde{h}_c(t)}{(1-t)^{\alpha}} = b \in (0,+\infty)$, necessarily $\alpha > 1$.

So, for the upper tail dependence coefficient, the following proposition holds.

Proposition 5.3.1. *Let $\bar{C^G}$ be the copula (4.11) with upper tail dependence coefficient $\lambda_U(\bar{C^G})$ and let $h : [0,1] \to [0,1]$ be a continuously differentiable generator with $h^{'}(x) \in (0,\infty)$ $\forall x \in (0,1)$: then $\lambda_U(\bar{C}_{\tilde{h}_c}) = \lambda_U(\bar{C^G})$.*

Moreover, from Proposition 4.4.1, we know that the general expression for the Kendall's function is given by

$$K(s) = s - H_h(h^{-1}(s))$$

where

$$H_h(v) = h'(v)\,v\left[2\ln(v) + \frac{1}{\lambda}\left(J_1(v) + J_2(v)\right)\right]$$

with

$$J_i(v) = \int_0^{\bar{G}_i^{-1}(v)} \frac{g_i^2(z)}{\bar{G}_i^2(z)}dz,\ i = 1,2. \tag{5.18}$$

Similarly, the expression of the Kendall's function K_c associated to the generator $\tilde{h}_c$ will be given by

$$K_c(s) = s - H_{\tilde{h}_c}\left(\tilde{h}_c^{-1}(s)\right) = s - H_{\tilde{h}_c}\left(\frac{h^{-1}(s\ h(c))}{c}\right),$$

where

$$\begin{aligned} H_{\tilde{h}_c}(v) &= \tilde{h}_c'(v)\, v \left[2\ln(v) + \frac{1}{\lambda}\left(J_1(v) + J_2(v)\right)\right] = \\ &= \frac{ch'(cv)}{h(c)} v \left[2\ln(v) + \frac{1}{\lambda}\left(J_1(v) + J_2(v)\right)\right] \end{aligned},$$

where $J_i,\ i = 1, 2$ are given by equation (5.18).

□

5.4 Actuarial Applications

In survival analysis and in actuarial applications, we are interested in bivariate residual lifetimes observed in the same time interval. For this reason, we consider only distributions satisfying (5.13), that, we know, coincide with the class of survival functions possessing pseudo weak lack-of-memory property.

5.4.1 Submultiplicative and Supermultiplicative Generators

It is well-known that distributions used in survival analysis do not satisfy in general lack-of-memory property: in fact, the probability to survive additionally t years for a component aged x is smaller than the probability to survive t years for a new component. This is known as "new better than used" property. In the case in which "new worse than used" property holds true, the probability to survive additionally t years for a component aged x is higher than the probability to survive t years for a new component.

In order to model these two situations in the bivariate case, we consider generators h that are sub-multilplicative, id est,

$$h(xy) \leq\ h(x)\ h(y), \forall x, y \in [0, 1]$$

and super-multiplicative, id est

$$h(xy) \geq\ h(x)\ h(y), \forall x, y \in [0, 1].$$

In fact, let us assume that $\bar{F}(x, y) = h(\bar{G}(x, y))$ satisfies pseudo weak lack-of-memory property, where $\bar{G}$ satisfies standard weak lack-of-memory property. Then

$$\bar{F}_{\bar{X}_t}(x, y) \leq \bar{F}_{X,Y}(x, y)$$

if and only if h is sub-multiplicative in the interval $[0, 1]$. The opposite situation holds true in the case of supermultilplicative generators.
We now give sufficient conditions under which a function f is sub-multiplicative in the unit interval $[0, 1]$.

Proposition 5.4.1. *Let* $f : [0, 1] \rightarrow [0, 1]$ *be a strictly increasing and concave bjection such that* $f^{'''}(x) \leq 0$*. Then* f *is sub-multiplicative in* $[0, 1]$*.*

Proof. Let us define $g : [0, 1] \times [0, 1]$ as

$$g(u, v) = f(uv) - f(u)f(v) :$$

then f is sub-multiplicative in $[0, 1]$ if and only if g is non-positive. On the sides of the square $[0, 1] \times [0, 1]$, g is equal to 0, so it is sufficient to prove that there are not maximum points inside the square. The second partial derivative with respect to u is non-negative, in fact

$$\begin{aligned} &\frac{\partial^2 g(u, v)}{\partial^2 u} = \\ &= v^2 f^{''}(uv) - f(v)f^{''}(u) \geq f(v)[f^{''}(uv) - f^{''}(u)] \geq 0, \end{aligned}$$

using concavity of f, decreasingness of $f^{''}$ and noticing that a strictly increasing and concave function lies above the bisector of the first quadrant in the interval $[0, 1]$. Since $\frac{\partial^2 g}{\partial^2 u}$ is non-negative, there are no maximum points inside the square, meaning that $g(u, v) \leq 0 \ \forall (u, v) \in [0, 1] \times [0, 1]$. □

Similarly, we find sufficient conditions under which a function f is supermultiplicative in the unit interval $[0, 1]$.

Proposition 5.4.2. *Let* $f : [0, 1] \rightarrow [0, 1]$ *be a strictly increasing and convex bjection such that* $f^{'''}(x) \geq 0$ *and* $f(x) \geq x^2$*,* $\forall x \in [0, 1]$*. Then* f *is supermultiplicative in* $[0, 1]$*.*

Example 5.4.1. *Let* $f(x) = \frac{3x - x^3}{2}$*. Then* f *is a generator and it is submultiplicative since it satisfies conditions given in Proposition 5.4.1.*

Example 5.4.2. *Let* $f(x) = \frac{\sin(\theta x)}{\sin(\theta)}, 0 < \theta < \frac{\pi}{2}$*. Then* f *is a generator and it is sub-multiplicative since it satisfies conditions given in Proposition 5.4.1.*

Example 5.4.3. *Let* $f(x) = \frac{1}{4}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x$. *Then* f *is a generator and it is super-multiplicative since it satisfies conditions given in Proposition 5.4.2.*

Furthermore, we notice that a function h is sub-multiplicative (super-multiplicative) in $[0,1]$ if and only if the function $g(t) = \log(h(e^{-t})),\ t \geq 0$ is sub-additive (super-additive): this allows us to consider the following well-known lemma about super-additive and sub-additive functions.

Lemma 5.4.1. *Let* $g : R_+ \to R$ *be a continuous and convex (concave) function such that* $g(0) \leq 0$ *(*$g(0) \geq 0$*), then* g *is super-additive (sub-additive).*

We now show two applications of this lemma.

Example 5.4.4. *Let us consider the generator* $f(x) = \frac{4\arctan(x)}{\pi}$. *The function* $g(x) = \log(4) - \log(\pi) + \log(\arctan(e^{-x}))$ *is a concave function, then by lemma 5.4.1* $g(x)$ *is sub-additive, so* $f(x)$ *is sub-multiplicative.*

Example 5.4.5. *Let* $f(x) = \frac{\log(\theta x+1)}{\log(\theta+1)},\ \theta > 0$, *so* $g(x) = \log[\log(\theta e^{-x} + 1)] - \log[\log(\theta + 1)]$. *It follows that*

$$\frac{d^2 g}{d^2 x} = \frac{\theta e^x \cdot (\ln(\theta \mathrm{e}^{-x} + 1) - \theta e^{-x})}{(\mathrm{e}^x + \theta)^2 \ln^2(\theta \mathrm{e}^{-x} + 1)}$$

in fact, it can be proven that the function $z(x) = \log(\theta e^{-x} + 1) - \theta e^{-x} < 0, \forall x \in R$, *so the second derivative of* g *is negative, hence* $g(x)$ *is sub-additive and* $f(x)$ *is sub-multiplicative.*

Example 5.4.6. *Let us consider the generator* $f(x) = e^{-\gamma(x^{-1}-1)}, \gamma > 0$. *The function* $g(x) = \gamma(1 - e^x)$ *is a concave function, then by lemma 5.4.1* $g(x)$ *is sub-additive, so* $f(x)$ *is sub-multiplicative.*

Example 5.4.7. *Let us consider the generator* $f(x) = (\theta x^{-1} + 1 - \theta)^{-1}, \theta > 0$. *The function* $g(x) = -\log(\theta e^x + 1 - \theta)$ *is a concave function if and only if* $0 < \theta < 1$, *then by lemma 5.4.1* $g(x)$ *is sub-additive, so* $f(x)$ *is sub-multiplicative if* $0 < \theta < 1$. *Viceversa,* $g(x)$ *is a convex function if and only if* $\theta > 1$, *then by lemma 5.4.1* $g(x)$ *is super-additive, so* $f(x)$ *is super-multiplicative if* $\theta > 1$.

5.4.2 Examples

We study some examples of bivariate distributions that are generated by submultiplicative distortions: we will show that pseudo weak lack-of-memory can be used to model both positive and negative dependence.

Example 5.4.8. *Let (X, Y) be a random vector satisfying bivariate pseudo-weak lack-of-memory property with distorted marginal survival functions $\bar{G}_i(x) = (1+x)^{-\alpha}, \alpha > 0,\ i = 1, 2,$ and with generator $h(x) = 1 - \left(\frac{e^{\theta(1-x)}-1}{e^{\theta}-1}\right)^{\beta},\ \theta < 0,\ \beta > 0$. Then the function*

$$
\begin{aligned}
&\bar{F}_{\bar{X}_t}(x, y) = \\
&= \frac{1}{1-\left(\frac{e^{\theta(1-e^{-\lambda t})}-1}{e^{\theta}-1}\right)^{\beta}}\left(1-\left(\frac{e^{\theta(1-e^{-\lambda t}(1+x-y)^{-\alpha})}-1}{e^{\theta}-1}\right)^{\beta}\right)1_{x\geq y}+ \\
&+\frac{1}{1-\left(\frac{e^{\theta(1-e^{-\lambda t})}-1}{e^{\theta}-1}\right)^{\beta}}\left(1-\left(\frac{e^{\theta(1-e^{-\lambda t}(1+y-x)^{-\alpha})}-1}{e^{\theta}-1}\right)^{\beta}\right)1_{x<y}
\end{aligned}
\tag{5.19}
$$

is a survival function if and only if $\alpha + 1 \leq \lambda \leq 2\alpha,\ \alpha \geq 1$ and $0 < \beta \leq 1$, with $P(X = Y) = \frac{2\alpha}{\lambda} - 1$ independent of β. The plot of the Kendall distribution function of the vector of residuals $\bar{X}_t$ for different values of t and of the parameter β, setting $\alpha = 1.5$, $\lambda = 2.6$ and $\theta = -1$, is given in Figure 5.1: it can be seen that dependence decreases strongly for $t = 0$ when β increases, while the dependence for $t = 1.5$ and $t = 3$ is more stable with respect to β. Since h satisfies Proposition 2.2.1 with $\alpha = 1$, we have that the lower tail dependence coefficient of (X, Y) is the same of $\bar{X}_t$ and of the copula $\bar{C^G}$. Moreover, since h satisfies Proposition 2.2.2 with $h'(x) \in (0, \infty)\ \forall x \in (0, 1)$ and $h'(1) = +\infty$ and since $0 < \beta \leq 1$, due to Proposition 5.3.1 and results given in Example 4.3.1, $\lambda_U(\bar{C}_{\bar{X}_t}) = \lambda_U(\bar{C^G}) \leq \lambda_U(\bar{C}_h)$. Notice that the singularity along the line $x = y$ is equal to $\frac{2}{13}$.

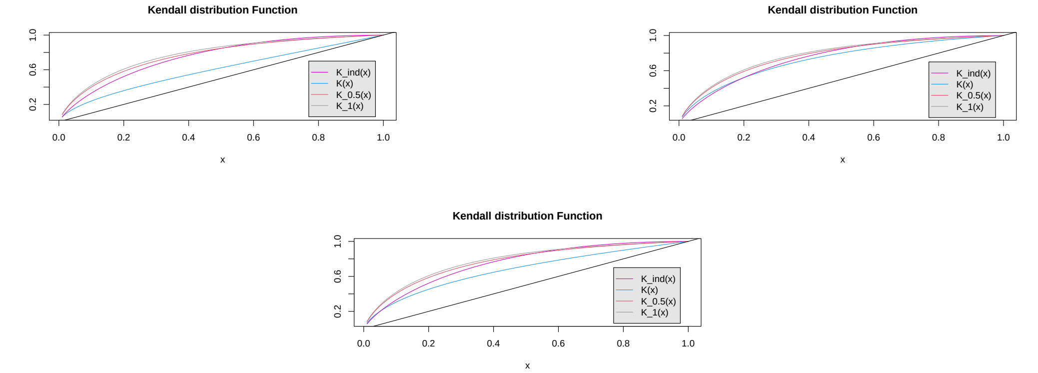


Figure 5.1: Kendall distribution function of of $\bar{X}_t$ for $t = 0$, $t = 0.5$ and $t = 1$. Top Left: $\beta = 0.3$. Bottom: $\beta = 0.6$. Top Right: $\beta = 0.9$.

The distortion functions $h_t(x)$ for different values of β and t are shown in Figure 5.2.

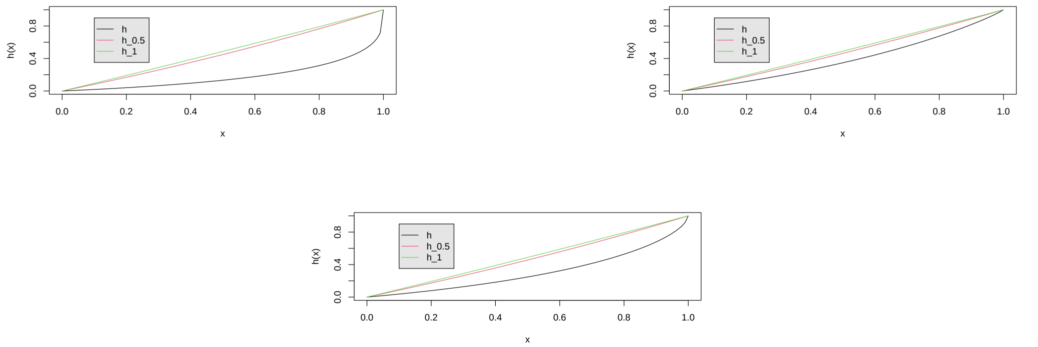


Figure 5.2: Distortion Functions $h_t(x)$ for $t = 0$, $t = 0.5$ and $t = 1$. Top Left: $\beta = 0.3$. Bottom: $\beta = 0.6$. Top Right: $\beta = 0.9$.

Moreover, in table 5.1, the values of the Kendall tau for the same values of β and t discussed before are given: it can be seen that the level of dependence decreases strongly when $t = 0.5$ or when $t = 1$ with respect to the case in which $t = 0$.

t	β	0.3	0.6	0.9
0		0.6	0.31	0.09
0.5		-0.06	-0.07	-0.08
1		-0.11	-0.12	-0.12

Table 5.1: Kendall Tau for different values of β and t

Example 5.4.9. *Let (X,Y) be a random vector satisfying bivariate pseudo-weak lack-of-memory property with distorted marginal survival functions $\bar{G}_i(x) = e^{-\mu x}$, $i = 1,2$, $\mu > 0$ and with generator $h(x) = 1 - \left(\frac{\tan(\theta(1-x))}{\tan(\theta)}\right)^{\beta}$, $-\frac{\pi}{2} < \theta < 0$, $\beta > 0$. Then the function*

$$\begin{aligned}
&\bar{F}_{\bar{X}_t}(x,y) = \\
&= \frac{1}{1-\left(\frac{\tan(\theta(1-e^{-\lambda t}))}{\tan(\theta)}\right)^{\beta}}\left(1-\left(\frac{\tan(\theta(1-e^{-\lambda t}e^{-\mu(x-y)}))}{\tan(\theta)}\right)^{\beta}\right)1_{x\geq y}+ \\
&+ \frac{1}{1-\left(\frac{\tan(\theta(1-e^{-\lambda t}))}{\tan(\theta)}\right)^{\beta}}\left(1-\left(\frac{\tan(\theta(1-e^{-\lambda t}e^{-\mu(y-x)}))}{\tan(\theta)}\right)^{\beta}\right)1_{x<y}
\end{aligned} \tag{5.20}$$

is a survival function if and only if $\mu \leq \lambda \leq 2\mu$ and $0 < \beta \leq 1$, with $P(X = Y) = \frac{2\mu}{\lambda} - 1$. The plot of the Kendall distribution function of the vector of residuals $\bar{X}_t$ for different values of t and of the parameter β is given in Figure 5.3, setting $\mu = 0.5$, $\lambda = 0.75$ and $\theta = -1$: it can be seen that dependence is positive for any values of β and of t and that the distance between the curves falls as β increases. Moreover, since h satisfies Propositions 2.2.1 and 2.2.2 with $h'(x) \in (0,\infty)$, $\forall x \in (0,1)$ and $h'(1) = +\infty$, conclusions about lower and upper tail dependence coefficients are the same as those given in Example 5.4.8. The singularity along the line $x = y$ is constant over t and over β and it is equal to $\frac{1}{3}$.

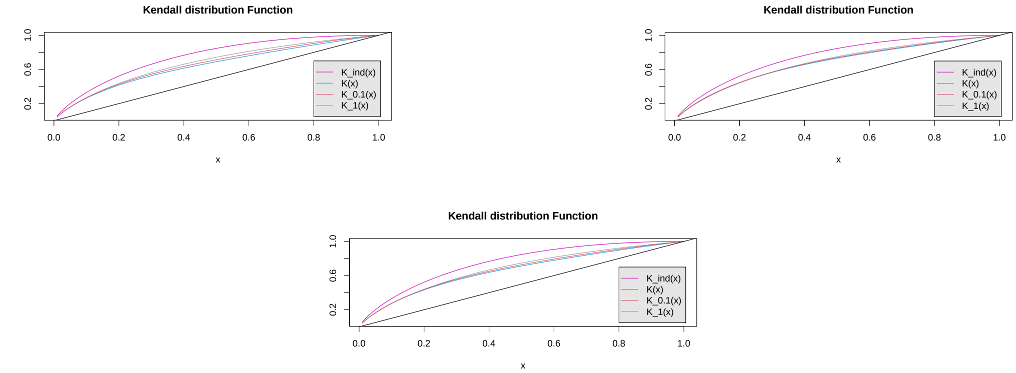


Figure 5.3: Kendall distribution function of of $\bar{X}_t$ for $t = 0$, $t = 0.1$ and $t = 1$. Top Left: $\beta = 0.7$. Bottom: $\beta = 0.8$. Top Right: $\beta = 0.9$.

The plot of the distortion functions $h_t(x)$ for different values of β and t is given in Figure 5.4.

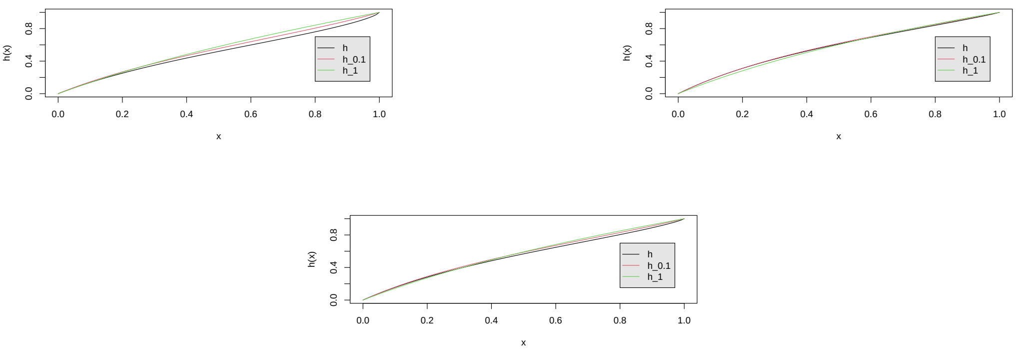


Figure 5.4: Distortion Functions $h_t(x)$ for $t = 0$, $t = 0.1$ and $t = 1$. Top Left: $\beta = 0.7$. Bottom: $\beta = 0.8$. Top Right: $\beta = 0.9$.

Moreover, in table 5.2, the values of the Kendall tau for the same values of the parameters β and t assumed before are given.

t	β	0.7	0.8	0.9
0		0.42	0.36	0.31
0.1		0.29	0.28	0.27
1		0.29	0.28	0.27

Table 5.2: Kendall Tau for different values of β and t

5.4.3 Life Insurance Pricing Formulas

In the following, we will provide formulas for the expected present value of some well-known insurance contracts written on couples of residual lifetimes X and Y, aged r and s respectively, assuming that the distribution of X and Y satisfies pseudo weak lack-of-memory property with marginal survival functions $\bar{F}_1$ and $\bar{F}_2$. In the following, we will denote by ρ the instantaneous interest rate.
First, we consider a pure endowment contract that pays 1 if X and Y are both larger than t. Its expected present value is given by

$$ {}_t\bar{E}_{r,s} = e^{-\rho t}\bar{F}_{X,Y}(t,t) = e^{-\rho t} h(e^{-\lambda t}). \tag{5.21} $$

The joint survivor annuity is an insurance contract that provides an income as long as both individuals are alive: its expected present value, in continuous time, is given by

$$ \begin{aligned} \bar{a}_{r,s} &= E\left[\int_0^\infty 1_{\{X>t,Y>t\}}\; e^{-\rho t}\; dt\right] = \\ &= \int_0^\infty \bar{F}_{X,Y}(t,t)\; e^{-\rho t}\; dt = \int_0^\infty h(e^{-\lambda t})\; e^{-\rho t}\; dt. \end{aligned} \tag{5.22} $$

In the case of joint survivor annuity deferred of u years, we have

$$ \begin{aligned} {}_{u|}\bar{a}_{r,s} &= E\left[\int_u^\infty 1_{\{X>t,Y>t\}}\; e^{-\rho t}\; dt\right] = \\ &= \int_u^\infty \bar{F}_{X,Y}(t,t)\; e^{-\rho t}\; dt = \int_0^\infty \hat{h}_u(e^{-\lambda z}) h(e^{-\lambda u})\; e^{-\rho(z+u)}\; dz = \\ &= {}_u\bar{E}_{r,s} \int_0^\infty \hat{h}_u(e^{-\lambda z})\; e^{-\rho z}\; dz = {}_u\bar{E}_{r,s}\; \bar{a}_{r+u,s+u}, \end{aligned} $$

where

$$ \hat{h}_u(x) = \frac{h(e^{-\lambda u} x)}{h(e^{-\lambda u})}. \tag{5.23} $$

The last survivor annuity is an insurance contract that provides an income to the policyholders upon the second death: its expected present value, in continuous time, is given by

$$\begin{aligned} \bar{a}_{\overline{r,s}} &= E\left[\int_0^\infty \left(1_{\{X>t\}} + 1_{\{Y>t\}} - 1_{\{X>t,Y>t\}}\right)\ e^{-\rho t}\ dt\right] = \\ &= \int_0^\infty (\bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_{X,Y}(t,t))\ e^{-\rho t}\ dt = \\ &= \int_0^\infty (\bar{F}_1(t) + \bar{F}_2(t))\ e^{-\rho t}\ dt - \bar{a}_{r,s}. \end{aligned} \tag{5.24}$$

In the case of last survivor annuity deferred of u years, we have

$$\begin{aligned} {}_{u|}\bar{a}_{\overline{r,s}} &= E\left[\int_u^\infty \left(1_{\{X>t\}} + 1_{\{Y>t\}} - 1_{\{X>t,Y>t\}}\right)\ e^{-\rho t}\ dt\right] = \\ &= \int_u^\infty (\bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_{X,Y}(t,t))\ e^{-\rho t}\ dt = \\ &= \int_0^\infty (\bar{F}_1(z+u) + \bar{F}_2(z+u))e^{-\rho(z+u)}\ dz + \\ &- {}_u\bar{E}_{r,s}\ \bar{a}_{r+u,s+u}. \end{aligned} \tag{5.25}$$

For the sake of simplicity and for pure explanatory purposes, we will analyse some fictitious examples of joint distribution satisfying pseudo weak lack-of-memory property with sub-multiplicative generators and with identical marginal survival functions of X and Y: due to Remark 4.1.5, X and Y are measured in a given unit of time. In the following, we will set the instantaneous interest rate ρ equal to 0.04.

Example 5.4.10. *Let (X, Y) be a random vector satisfying bivariate pseudo-weak lack-of-memory property with distorted marginal survival functions $\bar{G}_i(x) = (1+a(e^{bx}-1))^{-1}$, $a, b > 0$, $i = 1, 2$ and with generator $h(x) = \frac{\log(\theta x+1)}{\log(\theta+1)}$, $\theta > 0$. Then the function*

$$\bar{F}_{X,Y}(x,y) = \\ = \frac{\ln\left(\frac{\theta e^{-y\lambda}}{a\cdot\left(e^{b\cdot(x-y)}-1\right)+1}+1\right)}{\ln(\theta+1)}1_{x\geq y} + \frac{\ln\left(\frac{\theta e^{-x\lambda}}{a\cdot\left(e^{b\cdot(y-x)}-1\right)+1}+1\right)}{\ln(\theta+1)}1_{x<y} \tag{5.26}$$

is a survival function if $2ab \geq \lambda \geq b$ and $0 < a \leq 1$. The marginal survival function is concave in $x = 0$ if $a < \frac{\theta+1}{\theta+2}$. Setting $a = 0.55$, $b = 0.5$, $\lambda = 0.55$ and $\theta = 40$, the marginal expected value, the variance and the Kendall tau for different values of t are given in Table 5.3: as expected, since h is submultiplicative, the expected value of X_t decreases with t.

t	τ	$E[X_t]$	$VAR[X_t]$
0	0.3	5.46	11.9
1.5	0.28	4.79	9.97
3	0.27	4.17	8.4

Table 5.3: Kendall Tau, Marginal Expected Value and Variance for different values of t

The plot of the common marginal survival function for the same values of the parameters, with different values of t, is given in Figure 5.5: it can be seen that it is concave for small values of the independent variable.

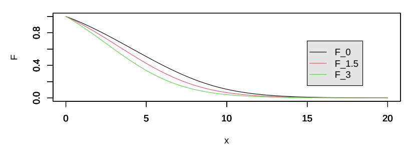


Figure 5.5: Marginal survival function of X and Y in the domain $(0, 20)$

The plots of the Kendall distribution function of the vector of residuals $\bar{X}_t$ and of the distortion functions $h_t(x)$ for different values of t are given in Figure 5.6: it can be seen that the two variables show positive association and that this dependence decreases slightly as t increases.

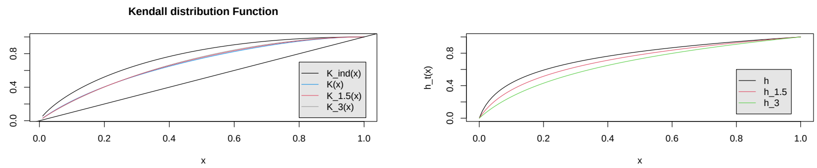


Figure 5.6: Kendall Distribution Function of $\bar{X}_t$ and Functions $h_t(x)$ for $t = 0$, $t = 1.5$ and $t = 3$

Using distribution (5.26), for the same values of the parameters a and b, the expected present value of joint and last survivor annuity for different values of λ, computing integrals (5.22) and (5.24) numerically, are shown in Table 5.4: the value of the joint survivor annuity decreases while the value of the last survivor annuity increases as λ increases due to singularity effect.

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X = Y)$
0.5	3.98	5.48	$\frac{1}{10}$
0.525	3.81	5.65	$\frac{1}{21}$
0.55	3.66	5.8	0

Table 5.4: Expected present value of joint and last survivor annuity for different values of λ, using distribution (5.26) with $\theta = 40$

The same computations are repeated for joint and last survivor annuities deferred of 1.5 *years and* 3 *years and results are shown in Tables 5.5 and 5.6.*

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X = Y)$
0.5	2.66	4.06	$\frac{1}{10}$
0.525	2.5	4.22	$\frac{1}{21}$
0.55	2.36	4.37	0

Table 5.5: Expected present value of joint and last survivor annuity deferred of 1.5 years for different values of λ, using distribution (5.26) with $\theta = 40$

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X = Y)$
0.5	1.69	2.84	$\frac{1}{10}$
0.525	1.55	2.98	$\frac{1}{21}$
0.55	1.42	3.1	0

Table 5.6: Expected present value of joint and last survivor annuity deferred of 3 years for different values of λ, using distribution (5.26) with $\theta = 40$

The plots of the Kendall distribution function of the vector of residuals $\bar{X}_t$ and of the functions $h_t(x)$ for different values of t and θ are given in Figure 5.7 and 5.8, setting $a = 0.9$, $b = 1.5$ and $\lambda = 2.7$: it can be noticed that dependence decreases when β increases and it increases slightly when t increases.

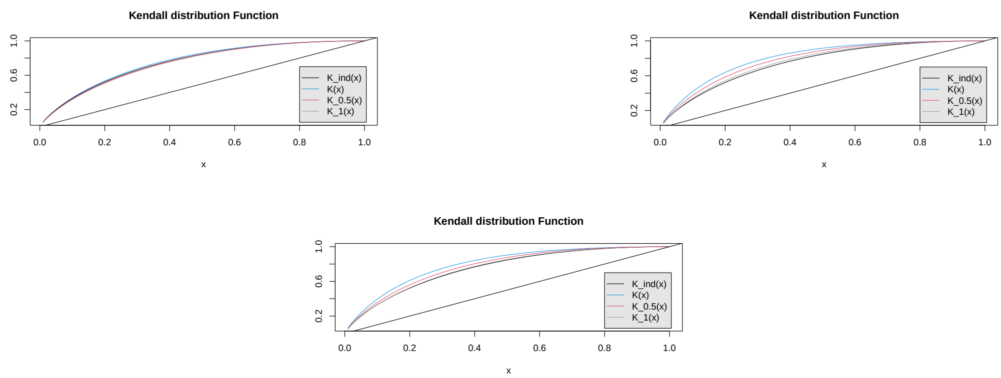


Figure 5.7: Kendall distribution function of of $\bar{X}_t$ for $t = 0$, $t = 0.5$ and $t = 1$. Top Left: $\theta = 1$. Bottom: $\theta = 10$. Top Right: $\theta = 20$.

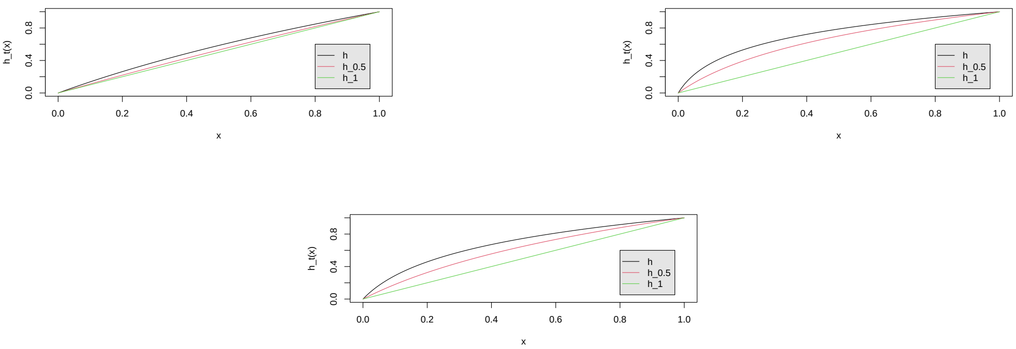


Figure 5.8: Distortion Functions $h_t(x)$ for $t = 0$, $t = 0.5$ and $t = 1$. Top Left: $\theta = 1$. Bottom: $\theta = 10$. Top Right: $\theta = 20$.

Example 5.4.11. *Let (X, Y) be a random vector satisfying bivariate pseudo-weak lack-of-memory property with generator $h(x) = e^{-\gamma(x^{-1}-1)}, \gamma > 0$ and distorted marginal survival functions $\bar{G}_i(x) = \left(1 + \frac{\xi}{\gamma}(e^{\beta x} - 1)\right)^{-1}, \ \xi, \beta > 0, \ i = 1, 2$. Then the function*

$$\bar{F}_{X,Y}(x, y) = \\ = \mathrm{e}^{-\xi \mathrm{e}^{y\lambda} \cdot \left(\mathrm{e}^{\beta \cdot (x-y)} - 1\right) - \gamma \mathrm{e}^{y\lambda} - \gamma} 1_{x \geq y} + \mathrm{e}^{-\xi \mathrm{e}^{x\lambda} \cdot \left(\mathrm{e}^{\beta \cdot (y-x)} - 1\right) - \gamma \mathrm{e}^{x\lambda} - \gamma} 1_{x < y} \tag{5.27}$$

is a survival function if and only if $\lambda \geq \beta$, $\beta(1 - \xi) \geq \lambda(1 - \gamma)$ and $2\beta\xi \geq \lambda\gamma$. This is the same distribution obtained in Marshall and Olkin (2015) with common marginal survival functions. The common marginal survival function is concave in 0 if $\xi < 1$. Setting $\beta = 0.35$, $\gamma = 0.17$, $\xi = 0.09$ and $\lambda = 0.37$, the marginal expected value, the variance and the Kendall tau for different values of t are shown in Table 5.7.

t	τ	$E[X_t]$	$VAR[X_t]$
0	-0.12	6	7.06
1.5	-0.09	4.77	5.58
3	-0.06	3.67	4.12

Table 5.7: Kendall Tau, Marginal Expected Value and Variance for different values of t

The plot of the common marginal survival function for the same values of the parameters, with different values of t, is given in Figure 5.9: it can be noticed that it is concave in the neighborhood of 0.

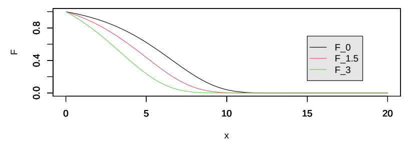
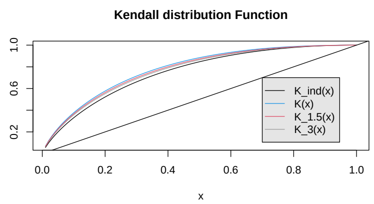


Figure 5.9: Marginal survival function of X and Y in the domain $(0, 20)$

The plots of the Kendall distribution function and of the distortion functions $h_t(x)$ for different values of t are given in Figure 5.10: it can be seen that the two random variables show positive association and that dependence increases slightly as t increases.



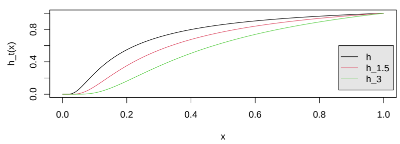


Figure 5.10: Kendall Distribution Function of $\bar{X}_t$ and Functions $h_t(x)$ for $t = 0$, $t = 1.5$ and $t = 3$

Using distribution (5.27), for the same values of the parameters β, γ and ξ, the expected present values of joint and last survivor annuity for different values of λ and of the deferment period are given in Tables 5.8, 5.9 and 5.10.

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X = Y)$
0.35	4.11	6.33	$\frac{1}{17}$
0.36	4	6.43	$\frac{1}{34}$
0.37	3.91	6.53	$\frac{1}{629}$

Table 5.8: Expected present value of joint and last survivor annuity for different values of λ, using distribution (5.27).

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X=Y)$
0.35	2.73	4.88	$\frac{1}{17}$
0.36	2.63	4.98	$\frac{1}{34}$
0.37	2.54	5.08	$\frac{1}{629}$

Table 5.9: Expected present value of joint and last survivor annuity with deferment period equal to 1.5 years for different values of λ, using distribution (5.27).

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X=Y)$
0.35	1.61	3.54	$\frac{1}{3}$
0.36	1.52	3.63	$\frac{8}{27}$
0.37	1.44	3.71	$\frac{29}{111}$

Table 5.10: Expected present value of joint and last survivor annuity with deferment period equal to 3 years for different values of λ, using distribution (5.27).

Example 5.4.12. *Let (X, Y) be a random vector whose distribution satisfies bivariate pseudo-weak lack-of-memory property with marginal survival functions $\bar{G}_i(x) = e^{-\mu x}$, $\mu > 0$, $i = 1, 2$ and with generator $h(x) = (1 - \theta + \theta x^{-1})^{-1}, \theta > 0$. Then the function*

$$\begin{aligned} &\bar{F}_{X,Y}(x,y) = \\ &= \frac{1}{\theta e^{\mu\cdot(x-y)+y\lambda} - \theta + 1} 1_{x\geq y} + \frac{1}{\theta e^{\mu\cdot(y-x)+x\lambda} - \theta + 1} 1_{x<y} \end{aligned} \tag{5.28}$$

is a survival function if and only if $\mu \leq \lambda \leq 2\mu$ and $\theta \geq \frac{1}{2}$. If, additionally, $\frac{1}{2} \leq \theta \leq 1$, we have that h is submultiplicative, as shown in Example 5.4.7. Unfortunately, the common marginal survival function is not concave in 0 for any values of the parameters. Setting $\mu = 0.2$, $\lambda = 0.4$ and $\theta = 0.51$, the marginal expected value, the variance and the Kendall tau for different values of t are given in Table 5.11.

t	τ	$E[X_t]$	$VAR[X_t]$
0	-0.18	6.87	81.14
1.5	-0.1	6.46	73.84
3	-0.06	6.13	68.13

Table 5.11: Kendall Tau, Marginal Expected Value and Variance for different values of t

The plot of the common marginal survival function for the same values of the parameters, with different values of t*, is given in Figure 5.11: as pointed out before, the marginal survival function is always convex.*

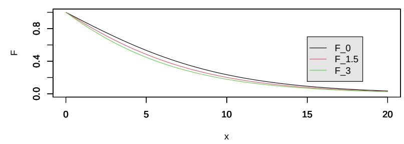


Figure 5.11: Marginal survival function of X and Y in the domain $(0, 20)$

The plots of the Kendall distribution function and of the distortion functions $h_t(x)$ *are shown in Figure 5.12: it can be seen that the two variables show negative dependence and that dependence increases slightly as* t *increases.*

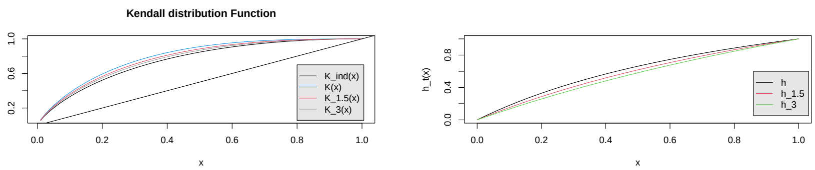


Figure 5.12: Kendall Distribution Function of $\bar{X}_t$ and Functions $h_t(x)$ for $t = 0$, $t = 1.5$ and $t = 3$

Using distribution (5.28), for the same values of the parameters μ *and* θ*, the expected present values of joint and last survivor annuity for different values of* λ*, computing integrals (5.22) and (5.24) numerically, are given in Table 5.12.*

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X=Y)$
0.35	3.46	7.63	$\frac{1}{7}$
0.375	3.25	7.83	$\frac{1}{15}$
0.4	3.07	8.01	0

Table 5.12: Expected present value of joint and last survivor annuity for different values of λ, using distribution (5.28) with $\theta = 0.51$

The same computations are repeated for joint and last survivor annuities deferred of 1.5 *years and* 3 *years and results are shown in Tables 5.13 and 5.14.*

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X=Y)$
0.35	2.19	6.2	$\frac{1}{7}$
0.375	2	6.39	$\frac{1}{15}$
0.4	1.83	6.56	0

Table 5.13: Expected present value of joint and last survivor annuity deferred of 1.5 years for different values of λ, using distribution (5.28) with $\theta = 0.51$

λ	Joint Survivor Annuity	Last Survivor Annuity	$P(X=Y)$
0.35	1.33	4.93	$\frac{1}{7}$
0.375	1.18	5.08	$\frac{1}{15}$
0.4	1.04	5.22	0

Table 5.14: Expected present value of joint and last survivor annuity deferred of 3 years for different values of λ, using distribution (5.28) with $\theta = 0.51$

The plots of the Kendall Distribution Function of (5.28) and the distortion functions $h_t(x)$ with $\lambda = 0.4$, $\mu = 0.2$, for different values of t and θ, are given in Figures 5.13 and 5.14: it can be seen that with $\theta = 0.5$ dependence is negative, while with $\theta = 3$ dependence is positive. Moreover, dependence slightly increases as t increases.

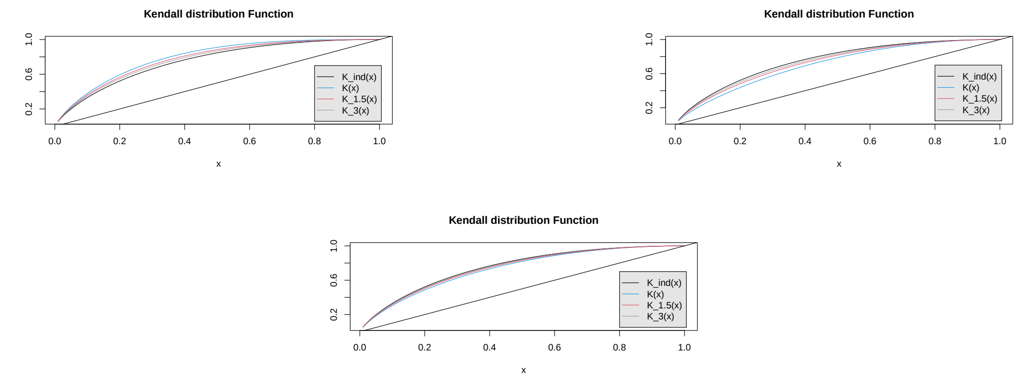


Figure 5.13: Kendall distribution function of $\bar{X}_t$ for $t = 0$, $t = 1.5$ and $t = 3$. Top Left: $\theta = 0.5$. Bottom: $\theta = 1.5$. Top Right: $\theta = 3$.

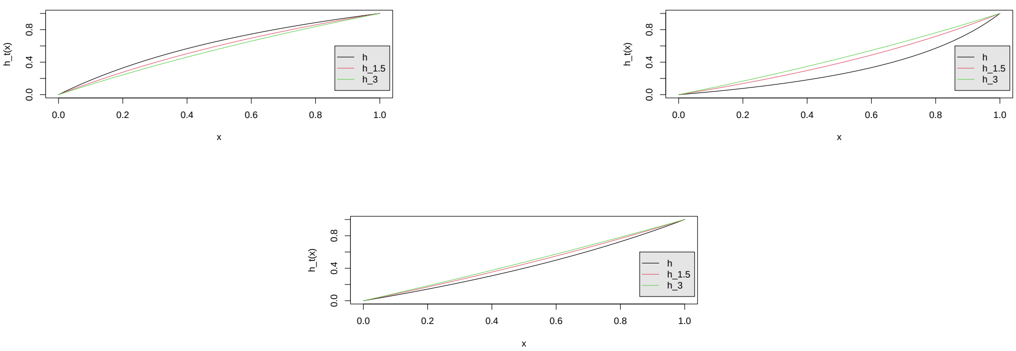


Figure 5.14: Distortion functions $h_t(x)$ for $t = 0$, $t = 1.5$ and $t = 3$. Top Left: $\theta = 0.5$. Bottom: $\theta = 1.5$. Top Right: $\theta = 3$.

5.5 Non-Life Insurance Applications

5.5.1 Modelling Bivariate Losses and Reinsurance Products

Let (X, Y) be a bivariate vector of losses for an insurance company. It can be noticed that, if the distribution of (X, Y) satisfies pseudo weak lack-of-memory property and if h is submultiplicative, then

$$P(X > x + t, Y > y + t) \leq P(X > x, Y > y)P(X > t, Y > t),$$

but, if h is supermultiplicative, then

$$P(X > x + t, Y > y + t) \geq P(X > x, Y > y)P(X > t, Y > t) :$$

the latter case corresponds to a riskier situation for the insurance company, since joint extreme losses may occur more frequently with respect to the former case.
Now we provide the expected value of different re-insurance products through which the insurance company may reduce its exposure.

First, we consider the following excess-of-loss reinsurance product in which the re-insurance company is asked to pay the excess-of-loss with respect to a given retention level: more precisely, the re-insurance function is given by

$$g(X, Y) = \begin{cases} X + Y - 2t \ \textit{if}\ X > t, Y > t \\ 0 \ \textit{otherwise} \end{cases}, \tag{5.29}$$

for some $t \geq 0$. The expected value of this product is given by:

$$\begin{aligned} &E\left[(X + Y - 2t)1_{X>t,Y>t}\right] = E\left[(X - t)1_{X>t,Y>t}\right] + \\ &+ E\left[(Y - t)1_{X>t,Y>t}\right] = (E[X - t|X > t, Y > t] + \\ &+ E[Y - t|X > t, Y > t])P(X > t, Y > t) = \\ &= h(e^{-\lambda t}) \int_0^\infty \left(\hat{h}_t(\bar{G}_1(z)) + \hat{h}_t(\bar{G}_2(z))\right) \, dz = \\ &= h(e^{-\lambda t}) \int_0^\infty \left(\hat{d}_t(\bar{F}_1(z)) + \hat{d}_t(\bar{F}_2(z))\right) \, dz, \end{aligned} \tag{5.30}$$

where

$$\hat{d}_t(x) = \frac{h(e^{-\lambda t} h^{-1}(x))}{h(e^{-\lambda t})}. \tag{5.31}$$

Moreover, we consider an insurance product that allows the insurer to recover the amount $\min(X,Y)-t$ provided that X and Y are larger than a retention level t, with reinsurance function given by

$$g(X,Y) = \begin{cases} \min(X,Y)-t \textit{ if } X>t, Y>t \\ 0 \textit{ otherwise} \end{cases} . \tag{5.32}$$

Since $P[\min(X,Y)-t>z|X>t,Y>t] = P[X-t>z, Y-t>z|X>t,Y>t] = \bar{F}_{\tilde{X}_t}(z,z)$, we have:

$$\begin{aligned} &E\left[(\min(X,Y)-t)1_{X>t,Y>t}\right] = \\ &= E[\min(X,Y)-t|X>t,Y>t]P(X>t,Y>t) = \\ &= h(e^{-\lambda t})\int_0^\infty \hat{h}_t(e^{-\lambda z})\; dz = \int_t^\infty h(e^{-\lambda z})\; dz. \end{aligned}$$

Finally, we consider an insurance product that allows the insurer to recover the amount $\max(X,Y)-t$ provided that X and Y are larger than a retention level t, with reinsurance function given by

$$g(X,Y) = \begin{cases} \max(X,Y)-t \textit{ if } X>t, Y>t \\ 0 \textit{ otherwise} \end{cases} . \tag{5.33}$$

Noticing that $\max(X,Y)-t = X-t+Y-t-\min(X-t,Y-t)$, we have that the expected value of (5.33) is given by

$$\begin{aligned} &E[(\max(X,Y)-t)1_{X>t,Y>t}] = \\ &= h(e^{-\lambda t})\int_0^\infty \left(\hat{h}_t(\bar{G}_1(z)) + \hat{h}_t(\bar{G}_2(z)) - \hat{h}_t(e^{-\lambda z})\right)\; dz = \\ &= h(e^{-\lambda t})\int_0^\infty \left(\hat{d}_t(\bar{F}_1(z)) + \hat{d}_t(\bar{F}_2(z)) - \hat{h}_t(e^{-\lambda z})\right)\; dz. \end{aligned}$$

The main advantage of pseudo weak lack-of-memory property with respect to the strong one is the flexibility in the choice of the marginal survival functions: the main disadvantage is the fact that the threshold t is the same for both components of the vector of losses. For this reason, since X and Y may be not homogeneous in scale, it is reasonable to assume that the vector $(X,\sigma Y)$ satisfies pseudo weak lack-of-memory property for a certain $\sigma>0$ with marginal survival functions $\bar{F}_1$ for X and $\bar{F}_2$ for Y: it can be shown that, in this case, the survival function of (X,Y) is given by

$$\bar{F}_{X,Y}(x,y) = \begin{cases} h\left(e^{-\lambda\sigma y}h^{-1}(\bar{F}_1(x-\sigma y))\right) \textit{ if } x \geq \sigma y \\ h\left(e^{-\lambda x}h^{-1}(\bar{F}_2(y-\frac{x}{\sigma}))\right) \textit{ if } x < \sigma y \end{cases} . \tag{5.34}$$

5.5.2 LOSS ALAE Application

Various processes in non-life insurance involve correlated pairs of variables, such as the claim Loss and the claim Allocated Loss Adjustment Expenses (ALAE), which include for example lawyers' fees and claims investigation expenses. Expensive claims generally need some time to be settled and induce considerable costs for the insurance company, so we expect some positive dependence between losses and their associated ALAE. The data we use in this section can be downloaded freely from "copula" package of the statistical software R and contain the LOSS and the ALAE of 1500 claims of a non-life insurance company: in this application, we consider only uncensored data, omitting 34 entries, and we will denote the Loss by X and the corresponding ALAE by Y, assuming that the distribution of the vector $(X, \sigma Y)$ satisfies pseudo weak lack-of-memory property. Estimation is performed in two steps: first, we estimate the best marginal distributions for X and Y, then we estimate the remaining parameters of the joint distribution of $(X, \sigma Y)$. Paramater estimations for some non-negative continuous distributions for X and Y are given in the following table.

Distribution	Parameters	BIC
Exponential	$\lambda = 2.43 * 10^{-5}$	34879
Gamma	$\alpha = 5.06 * 10^{-1}$, $\beta = 1.23 * 10^{-5}$	35296
Pareto	$\alpha = 1.24, s = 1.62 * 10^{5}$	35083
Weibull	$a = 6.35 * 10^{-1}, b = 2.83 * 10^{4}$	34907

Table 5.15: Results for marginal distribution of X

Distribution	Parameters	BIC
Exponential	$\lambda = 8.32 * 10^{-5}$	30476
Gamma	$\alpha = 0.68,\ \beta = 5.66 * 10^{-5}$	31324
Pareto	$\alpha = 2.35, s = 15876$	31050
Weibull	$a = 0.739, b = 8731$	31211

Table 5.16: Results for marginal distribution of Y

So, according to the BIC criterion, the best marginal distributions for X and Y are exponential distributions, with parameters $\lambda_1 = 2.43 * 10^{-5}$ and $\lambda_2 = 8.32 * 10^{-5}$ respectively. Substituting into equation (5.34) $\bar{F}_i(x) = e^{-\lambda_i x},\ i = 1, 2$, we have

$$\bar{F}_{X,Y}(x, y) = \begin{cases} h\left(e^{-\lambda\sigma y} h^{-1}(e^{-\lambda_1(x-\sigma y)})\right) & \text{if } x \geq \sigma y \\ h\left(e^{-\lambda x} h^{-1}(e^{-\lambda_2(y-\frac{x}{\sigma})}\right) & \text{if } x < \sigma y \end{cases}. \tag{5.35}$$

We already know that the function (5.35) is not always a bivariate survival function: however, for three different functional forms of h, we give sufficient conditions on the parameters such that (5.35) is a bivariate survival function, setting $\lambda = -G_1'(0) - \frac{G_2'(0)}{\sigma}$, where $\bar{G}_i(x) = h^{-1}(e^{-\lambda_i x}),\ i = 1, 2$ in order to make $P(X = Y) = 0$.

Example 5.5.1. *Let us consider the generator* $h(x) = \frac{e^{\theta x}-1}{e^{\theta}-1}$*: then* $\bar{G}_i(x) = h^{-1}(e^{-\lambda_i x}) = \frac{\log((e^{\theta}-1)e^{-\lambda_i x}+1)}{\theta}, \lambda_i > 0,\ i = 1, 2$*. The associated survival function* $\bar{F}$ *is given by*

$$\bar{F}(x, y) = \begin{cases} \frac{\exp\{e^{-\lambda\sigma y} \log((e^{\theta}-1)e^{-\lambda_1(x-\sigma y)}+1)\}-1}{e^{\theta}-1} & \text{if } x \geq \sigma y \\ \frac{\exp\{e^{-\lambda x} \log((e^{\theta}-1)e^{-\lambda_2(y-\frac{x}{\sigma})}+1)\}-1}{e^{\theta}-1} & \text{if } x < \sigma y \end{cases}.$$

The second mixed derivative of $\bar{F}$ *is non-negative in the region* $\{x \neq \sigma y,\ x \geq 0, y \geq 0\}$ *if* $\lambda \geq \max\left(\lambda_1, \frac{\lambda_2}{\sigma}\right)$*. Considering also singularity condition, we need that*

$$\left(\lambda_1 + \frac{\lambda_2}{\sigma}\right) \frac{(e^{\theta} - 1)}{\theta e^{\theta}} \geq \max\left(\lambda_1, \frac{\lambda_2}{\sigma}\right).$$

Example 5.5.2. *Let us consider* $h(x) = (1-\theta+\theta x^{-1})^{-1}, \theta > 0$*: then* $\bar{G}_i(x) = h^{-1}(e^{-\lambda_i x}) = \theta(e^{\lambda_1 x} - 1 + \theta)^{-1}$, $\lambda_i > 0$, $i = 1, 2$. *The associated survival function* $\bar{F}$ *is given by*

$$\bar{F}(x,y) = \begin{cases} (e^{\lambda\sigma y}(e^{\lambda_1(x-\sigma y)} - 1 + \theta) + 1 - \theta)^{-1} \text{ if } x \geq \sigma y \\ (e^{\lambda x}(e^{\lambda_2(y-\frac{x}{\sigma})} - 1 + \theta) + 1 - \theta)^{-1} \text{ if } x < \sigma y \end{cases}.$$

The second mixed derivative of $\bar{F}$ *is non-negative in the region* $\{x \neq \sigma y,\ x \geq 0, y \geq 0\}$ *if* $\lambda \geq \max(\lambda_1, \frac{\lambda_2}{\sigma})$ *and* $\theta \geq 1$. *Considering also singularity condition, we need that*

$$\frac{\lambda_1 + \frac{\lambda_2}{\sigma}}{\theta} \geq \max\left(\lambda_1, \frac{\lambda_2}{\sigma}\right),\ \theta \geq 1.$$

Example 5.5.3. *Let us consider* $h(x) = \frac{(x+1)^\theta - 1}{2^\theta - 1}$*: then* $\bar{G}_i(x) = h^{-1}(e^{-\lambda_i x}) = ((2^\theta - 1)e^{-\lambda_i x} + 1)^{\frac{1}{\theta}} - 1$, $\lambda_i > 0$, $i = 1, 2$. *The associated survival function* $\bar{F}$ *is given by*

$$\bar{F}(x,y) = \begin{cases} \frac{\left\{(e^{-\lambda\sigma y}[((2^\theta-1)e^{-\lambda_1(x-\sigma y)}+1)^{\frac{1}{\theta}}-1]+1\right\}^\theta - 1}{2^\theta - 1} \text{ if } x \geq \sigma y \\ \frac{\left\{(e^{-\lambda x}[((2^\theta-1)e^{-\lambda_2(y-\frac{x}{\sigma})}+1)^{\frac{1}{\theta}}-1]+1\right\}^\theta - 1}{2^\theta - 1} \text{ if } x < \sigma y \end{cases}.$$

The second mixed derivative of $\bar{F}$ *is non-negative in the region* $\{x \neq \sigma y,\ x \geq 0, y \geq 0\}$ *if* $\lambda \geq \max(\lambda_1, \frac{\lambda_2}{\sigma})$ *and* $\frac{\max(\lambda_1, \frac{\lambda_2}{\sigma})}{\lambda} \leq \theta \leq 1$. *Overall, we need that*

$$\left(\lambda_1 + \frac{\lambda_2}{\sigma}\right) 2^{1-\theta}(2^\theta - 1) \geq \max\left(\lambda_1, \frac{\lambda_2}{\sigma}\right),\ \theta \leq 1.$$

We compare the performance of the generators given above in the following table.

h	$\hat{\theta}$	$\hat{\sigma}$	BIC
$(1-\theta+\theta x^{-1})^{-1}$	1.61	2.8	61034
$\frac{(x+1)^\theta-1}{2^\theta-1}$	1	3.01	61209
$\frac{e^{\theta x}-1}{e^\theta-1}$	1.05	2.81	61113

Table 5.17: Log-likelihood for different choices of h

According to the results, the best generator h for our dataset is $h(x) = (1-\theta+\theta x^{-1})^{-1}$, with $\theta \approx 1.61$: for this value of θ, the generator is convex and, by Example 5.4.7, it is also super-multiplicative, meaning that

$$\bar{F}(x+t, y+t) \geq \bar{F}(x,y)\bar{F}(t,t).$$

Moreover, in the case of the second generator, applying constrained optimization of log-likelihood, we get $\theta = 1$, implying that standard lack-of-memory property holds true. From now on, we will consider only the generator $(1 - \theta + \theta x^{-1})^{-1}$, with $\theta = 1.61$ and $\sigma = 2.8$.

The Kendall Tau obtained simulating 10000 pair of observations is 0.309, very close to the actual sample value of 0.315. Moreover, let $K_t(x)$ the Kendall Distribution Function of the vector $X - t, Y - \frac{t}{\sigma} | X > t, Y > \frac{t}{\sigma}$: looking at the plot of the Kendall Distribution Function, we can see that the two variables show strong positive dependence and that dependence increases as the threshold t increases.

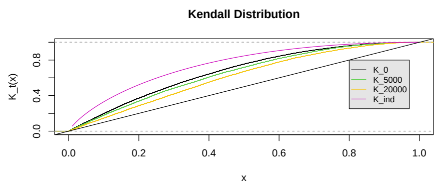


Figure 5.15: Kendall Distribution Function of $X - t, Y - \frac{t}{\sigma} | X > t, Y > \frac{t}{\sigma}$ for $t = 0$, $t = 5000$ and $t = 20000$

Now let us consider the vector $(X - t + Y - \frac{t}{\sigma} | X > t, Y > \frac{t}{\sigma})$. First, we can notice that

$$\bar{F}_{X-t,Y-\frac{t}{\sigma}|X>t,Y>\frac{t}{\sigma}}(x, y) = P(X > x + t, \sigma Y > t + \sigma y | X > t, \sigma Y > t) = \\ = \hat{d}_t(\bar{F}_{(X,\sigma Y)}(x, \sigma y)).$$

Moreover,

$$P\left(X + Y - t - \frac{t}{\sigma} > z \middle| X > t, Y > \frac{t}{\sigma}\right) = \\ = -\int_0^z \partial_1 \bar{F}_{X-t,Y-\frac{t}{\sigma}|X>t,Y>\frac{t}{\sigma}}(x, z - x)\ dx + P\left(X - t > z | X > t, Y > \frac{t}{\sigma}\right) = \\ = \hat{d}_t(\bar{F}_{X,\sigma Y}(x, 0)) - \int_0^z \hat{d}_t'(\bar{F}_{X,\sigma Y}(x, \sigma(z - x))\ \partial_1 \bar{F}_{X,\sigma Y}(x, \sigma(z - x))\ dx. \tag{5.36}$$

So we can recover implicitly the value-at-risk of the random variable $X + Y - t - \frac{t}{\sigma} | X > t, Y > \frac{t}{\sigma}$ as a function of the threshold t, as we show in Figure 5.16.

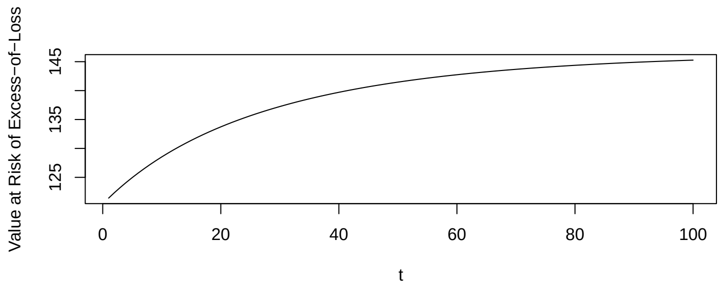


Figure 5.16: 95% value-at-risk of $X+Y-t-\frac{t}{\sigma}|X>t, Y>\frac{t}{\sigma}$ as a function of t (Values in Thousand of Euros)

Moreover, using (5.36), we can recover the mean-excess function of the random variable $X+Y-t-\frac{t}{\sigma}|X>t, Y>\frac{t}{\sigma}$: its graph is given in Figure 5.17.

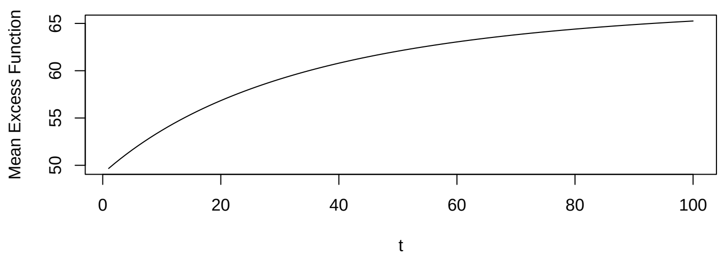


Figure 5.17: $E[X+Y-t-\frac{t}{\sigma}|X>t, Y>\frac{t}{\sigma}]$ as a function of t (Values in Thousand of Euros)

We can notice that the mean-excess function and the 95% value-at-risk increases as the threshold t increases, meaning that the distribution of $X+Y$ is heavy tailed.

Finally, using net equivalence principle, we compute the re-insurance premium Π for an excess-of-loss reinsurance with the following re-insurance function

$$\begin{cases} X+Y-t-\frac{t}{\sigma}, \ \textit{if } X>t, Y>\frac{t}{\sigma} \\ 0 \ \textit{otherwise} \end{cases} : \tag{5.37}$$

the plot of Π for different thresholds t is given in Figure 5.18.

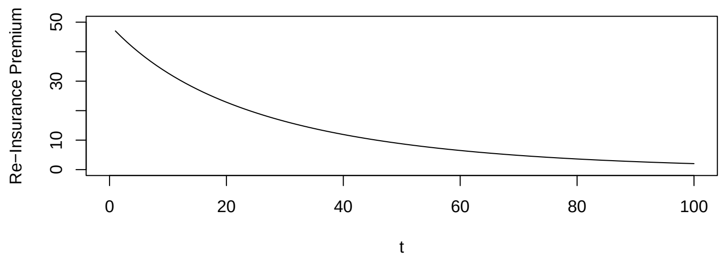


Figure 5.18: Re-Insurance Premium as a function of threshold t (Data in Thousands of Euros)

Similarly, we compute the reinsurance premiums for contracts of type (5.32) and (5.33) with threshold t for X and $\frac{t}{\sigma}$ for Y, comparing their values in Figure 5.19.

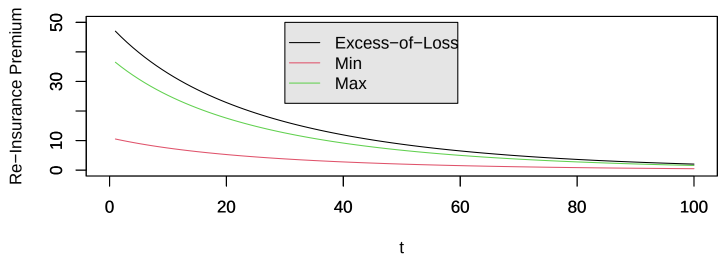


Figure 5.19: Re-Insurance Premium for different contracts as a function of threshold t (Data in Thousands of Euros)

As expected, the reinsurance premium for the excess-of-loss policy (5.37) is larger than the premium written on the maximum between $X - t$ and $Y - \frac{t}{\sigma}$, provided that $X > t$ and $Y > \frac{t}{\sigma}$; similar results hold for the policy written on the maximum with respect to the policy written on the minimum.

Chapter 6

Conclusions

The aim of this thesis was to generalize some well-known concepts in probability from the standard ring to the so called "g-semiring", where pseudo sum and pseudo product are written in terms of a function, called "generator".

In the first part of this thesis, assuming a measurable space provided with a pseudo-additive fuzzy measure, we have generalized the concept of independence (that we call pseudo-independence) substituting into the well-known definition of independence the standard product by the pseudo one. Then, we have given a generalization of univariate and bivariate moment generating function (that we call univariate and bivariate pseudo moment generating function) and we have shown that the properties of standard moment generating function valid for independent random variables still hold for the pseudo moment generating function in the case of pseudo-independent random variables.
Finally, we have considered a generalization of Schur-constant distribution (that we call pseudo Schur-constant distribution) substituting into its definition the standard sum by a non-commutative version of pseudo-sum: two characterizations of pseudo Schur-constant distributions, one in terms of distribution and the other in terms of bivariate pseudo moment generating function, have been given.

In the second part of the thesis, we have generalized lack-of-memory properties in strong and weak version (we call them pseudo strong and weak lack-of-memory properties) substituting into the associated functional equations the standard product by the pseudo one. After finding sufficient and necessary conditions under which the solutions of the new functional equations are bivariate survival functions, we have focused on the distribution satisfying pseudo weak lack-of-memory property: we have proved that it may have a singularity along the line $x = y$, determining its Kendall distribution function in full generality and studying tail dependence for specific choices of the generator and of the marginal survival functions.

Moreover, we have generalized Kaminsky (1983) and Marshall and Olkin (2015) strong and weak functional equations for residual lifetimes, showing that the solutions of these generalized functional equations coincide with the class of functions satisfying pseudo lack-of-memory properties. Furthermore, after studying the dependence structure of residual lifetimes, we have given sufficient conditions under which a generator is sub-multiplicative or super-multiplicative in the unit interval and we have analysed the impact that some distributions satisfying pseudo weak lack-of-memory property, generated by sub-multiplicative functions, have on the value of some well-known insurance contracts.

Finally, we have considered an application to the LOSS ALAE insurance modelling problem: assuming that the distribution of the vector of LOSS and of a suitable scalar transformation of ALAE follows pseudo weak lack-of-memory property, we have estimated the best joint distribution, we have analysed the dependence structure between LOSS and ALAE and we have determined the reinsurance premiums for different reinsurance contracts under net equivalence principle.

Chapter 7

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